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SCIENTIFIC METHOD IN THE RECONSTRUCTION OF NINTH-GRADE MATHEMATICS

By

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and

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A COMPLETE REPORT OF THE INVESTIGATION OF THE ILLINOIS COMMITTEE ON STANDARDIZATION OF NINTH-GRADE MATHEMATICS
1913–1918

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CHAPTER I

NINTH-GRADE MATHEMATICS ON TRIAL

Ninth-grade mathematics in American high schools needs to be completely reconstructed.—What we teach and how we teach it—both must come in for most minute scrutiny and evaluation. More than one-half of the present subject-matter that we now teach our pupils will never be used by the vast majority; neither in subsequent courses in the high school, in occupational or other life-activities beyond the school, or even by that fraction of per cent of the population that engages in the various scientific professions. We make large claims for our instruction on the ground that we are training children "to think intelligently." It will be very difficult to substantiate these claims. The very organization of our course of study tends to inhibit this, even in the most intelligent of our departments. Eighty per cent of the ninth-grade course offers little or no opportunity to meet real "problem situations." Our textbooks emphasize habit formation and rote memory, and these textbooks almost exactly determine the course of study. Furthermore, it is relatively common to fail from 20 to 30 per cent of our students. The evidence shows that this cannot be justified. It shows furthermore that the difficulty can be attributed, not only to the dead wood in the curriculum, but to the fact that *not one of the currently used textbooks is thoroughly organized with respect to the "learning" of ninth-grade students.* The candid judgment, joint and several, of teachers of advanced mathematical and scientific subjects, of employers of labor, and of educational critics and reformers agrees with the conclusion of the scientific investigator who has measured results more objectively: ninth-grade mathematics cannot merely be rearranged or "reorganized." *It must be rebuilt from the ground up.*

This is a sweeping indictment of one of our most strongly intrenched subjects of study. The phraseology is necessarily dogmatic. The evidence for the statements will be supplied in detail throughout this report. The arraignment will be resented by the protagonists of "things as they are." We recognize full well that the condition that we paint, as well as the practical program for the future which we have been and are now working out, will be disposed of by *such* persons in the same way that the proverbial Mr. Perkins disposed of the giraffe. To *such* persons "no such animal *is* possible."

This monograph, however, has been written with a conviction that it is possible to reconstruct secondary mathematics from the ground up in a sound and scientific fashion. Our first chapter, only, is given over to painting a picture of the present status of things—to stating the indictment—the remaining ones contain the evidence. No comments have been made about the reconstruction of ninth-grade mathematics which do not rest upon the accumulation of quantitative evidence. This, in turn, is supported and checked at every step by a careful application of present-day psychology.

The school man recognizes the need for reconstruction.—For a number of years our educational conventions and our pedagogical journals have debated the "reorganization" of high-school mathematics; committees have been appointed for the consideration of ways and means of vitalizing courses of study, of "reorganizing" the content and arrangement of the subject-matter. The upshot of it all may be expressed in one word—*rearrangement, not* a complete rebuilding. "Educationist" critics have attacked our present mathematical situation on the grounds of lack of real purpose, of poorly selected and poorly organized content, of bad teaching, and of ineffective methods. These criticisms have been supplemented by continued attacks from the industrial and commercial employer, and from the general layman—who himself went through the mathematical curricula of our high schools of twenty years ago. The criticism generally has been, however, primarily without scientific foundation. Subjective opinion has largely determined the course of would-be reform in secondary mathematics.

Naturally nothing happened in the way of a sound reconstruction of the course of study. To cause us to do something about a condition under which we exist we must be told in terms so pertinent to our own needs that action will be inspired. We have been told that we have been doing a bad job in teaching mathematics in such general terms that the need of doing something about it has not been definitely felt. Mental inertia, in other words, has characterized our administration in secondary mathematics as it characterizes nearly every aspect of our daily life.

The quantitative method in studying such problems.—It is possible to substitute for prejudice and subjective opinion, concerning the effectiveness of our present scheme of things, a systematic and scientific measurement of results obtained from the present organization. Stock can be taken, relatively accurately, of courses of study, of the effectiveness of teaching methods, of the use of devices for improving the instruction of children in classes, and of methods of classifying and marking students. This *can* be done in such an intelligent way as to lead to concerted action on the part of our mathematical group toward progressive improvement. It is the purpose of this monograph to describe in detail the scientific study of the development of the efficiency and of the present organization of ninth-grade mathematics; to set forth a program for procedure and to canvass the results already available from the working out of that program in practice. Before turning to the detailed examination of the evidence concerning things as they are, let us first get a bird's-eye view, in the remainder of this chapter, of the present situation in the teaching of secondary mathematics.

THE RESULTS OBTAINED FROM THE PRESENT COURSE

There are three ways in which we may evaluate the effectiveness of the present scheme of mathematical teaching. We may consider, first, the general criticism that has come from the lay public outside the school and from our more intelligent critics within the school. Secondly, we may weigh critically the results obtained by a study of failures in secondary mathematics. Thirdly, we can judge the efficiency of our work by a careful consideration of the results of measuring, by means of standardized tests, achievements of children who have been educated under the present scheme.

The layman's comment.—The most striking characteristic of instruction in the public schools as shown by each of these three attacks upon the problem is the inefficiency revealed in the achievements of children trained under the present régime. For the past decade complaints from the lay public, concerning the content and methods of instruction in high-school mathematics, have come with increasing insistence. The industrial employer has said to us repeatedly, "Graduates of your courses have no 'mathematical intelligence'; they can, it is true, solve formal problems with some degree of rapidity and with a *moderate* assurance of accuracy, but as for *making use* of the formal skill which you have developed in them *in the handling of original problems*—they are not able to do it." This complaint with regard to mathematics is simply one aspect of the general criticism that has come from the social world. Children, they tell us, on leaving the public schools are not efficient in the various routine processes which they are supposed to have mastered. For a long time the business man has told us that children do not spell correctly, that they cannot figure rapidly and accurately, that they cannot read with understanding, that they cannot follow directions and reason about simple problems. The school man has been all too prone to regard this as a subjective judgment based upon prejudice rather than upon an intelligent knowledge of what is going on within the school.

More recently, however, a group of educational critics (who have been concerned primarily with the making over of courses of study from the standpoint of social utility, and who have frankly discarded, as they say, the doctrine of formal discipline as a basis for designing curricula) has reiterated the criticisms of the business man with added emphasis. Mr. Flexner in his *Modern School*, and Mr. Snedden in his various publications, represent two of the most active of the protagonists of the new movement for critical analysis of "things as they are" in the teaching of children. To recent times, however, this criticism from the educational reformer within the school, like that from the laity outside the school, has been unfounded from the standpoint of the use of scientific procedure. These men have insisted that this reorganization of the course of study must be primarily on the basis of social utility.

Their activity has been so great as to arouse even the academicians of the public-school world to more or less resentful denials of the validity of the present criticisms. Various mathematical associations, which have been compiling large numbers of individual "opinions" outside as well as inside the school, have tried to show the values and the effectiveness of the present scheme of things. The recent reports of the Mathematics Club of the city of Chicago and of the Association of Mathematics Teachers of New England are interesting examples of the awakening recognition on the part of teachers of high-school mathematics that their methods of teaching children how to use quantitative devices are under fire.

Evaluation of results through a study of "vital statistics."—No more effective method of beginning a study of school efficiency within a system or department can be adopted than that of studying *failures*. "*When a pupil fails*" might well be made the caption for a majority of our educational discussions in current high-school mathematics. That it has been a theme for common discussion in both pedagogical journals and educational meetings during the past few years is quite evident. *At least 20 per cent of the children who are required to take first-year mathematics "fail."* Careful compilations of non-promotion statistics for representative portions of the country show this to be true. When a school system is set up—when buildings are built, teachers are trained, superintendents and administrative officers are secured, courses of study are designed, and school machinery is established—all for the purpose of teaching children, certainly a grave educational and social problem is raised when the inventory at the end of the year shows that between one-fifth and one-third of the children as we know them are declared by those concerned with their education to be so lacking in ability as to necessitate the branding mark of "failure"! Teachers employed and trained by the state "to teach the boy" have said deliberately, "From 20 to 30 per cent of him is so bad that we have been unable to give him the essentials in the way of skill and reasoning ability embodied in the field of learning in question."

Mark, however, that in *exceptional* schools or departments less than 5 to 10 per cent of our children have been so failed. Within a state *exceptionally rare* school systems *may* have non-promotion rates in mathematics of less than 10 per cent, but at the same time that many other school systems have non-promotion rates in mathematics of more than 25 per cent. Mark well too the fact that within a school system or even within a department, teachers show great variability in their teaching of the present courses of study and in their setting up of and administration of a "marking system." This variation is represented by a non-promotion rate on the one hand of 5 per cent and on the other hand of 35 per cent—and this too not of single classes, in which the measure would be obviously invalid, but with the accumulation of large numbers of pupils as the succession of classes goes by.

It is quite proper to ask, Are human beings built that way? Is the distribution of ability throughout our general population so shaped as to imply that either the most common quality in our school population is "stupidity," or that on the other hand the most outstanding characteristic of public-school pupils is "talent"? For we find that in many cases does the method of marking children reveal that, instead of having failed 30 per cent of the entire student population, 30 per cent of them have been branded as of superior, or unusual, ability above normal. The proper reply to the painting of such a picture of inequalities in distribution of ability within different localities, within different school systems, or even within different departments of a school system is: "*Human beings are not built that way.*"

How then are they built? How does ability distribute in the general population? During the last few years many studies have been made in the attempt to answer this question. Measurements have been compiled, of varying degrees of accuracy, at first of a purely anthropometrical nature, and more recently of a psychological and social nature.

This is not the proper place to set forth in detail the evidence on this problem. In other publications one of the present writers has discussed it much more fully. Here we can merely summarize, in a more or less dogmatic way, the outstanding facts of the matter for the sake of applying them to our problem.

During the past century scientists have measured, with increasing accuracy, various physical and mental traits—for example, stature, cephalic index, strength of grip, intelligence, reaction time, errors in observation, etc. As the measurements have accumulated, several striking characteristics stand out in recurring fashion in the distributions which have been secured. First, relatively few measurements are found to fall at the extreme ends of the scale. That is, there are very few persons of unusual height, strength, intelligence, abilities in memory, etc., either above or below the normal. There are many persons, on the other hand, of average or mediocre ability; and there are more near the very middle of the scale of ability than at any other particular point. Finally, the numbers of persons diminish about equally rapidly on each side of the average. That is, there are about as many superior as there are inferior persons, and they are fewer in number than the mediocre on the one hand, and greater in number than the very exceptional on the other.

It is helpful to our thinking to use such a graphic device as is represented by Diagram I (page 75), in which points on the base line of the scale of the ability in question represent various amounts of the ability, and the vertical distances of points on the curve above such points on the base line represent the number of persons who have been found to possess the abilities named. This sort of curve (the specific type of which is known as the normal probability curve) has been shown to fit closely the distribution of actual anthropometrical measurements. It has also been found to resemble approximately enough the general shape of the distribution of mental abilities in school to permit us to make rough use of it.

To sum up our discussion, then, the best evidence that we have at the present time leads us to believe that mental abilities are so distributed as to resemble, at least closely enough for practical purposes in education, the curve known as the "normal probability curve." The practical application of all this means that ordinary folks are much more common than unusual ones; that it is extremely doubtful if we divide our children in the public schools into five groups in order of ability, that either the upper and the lowest group should contain more than 6 or 8 per cent of the entire group, that the second and fourth groups ought to contain more than 20 to 30 per cent of the entire group, and that the middle or mediocre portion ought to be larger than 35 to 45 per cent. Whether or not we accept the use of this kind of graphic device to help in thinking and in expressing our judgments on scholastic abilities, it is quite clear that a most casual examination of the non-promotion situation in high-school mathematics leads to a recognition of the fact that the present organization of things is not obtaining desirable results. At this point it will be objected, of course, that either general introspective criticism from the outside world or from the educational critic, or the measurements of results as shown by teachers' marks, is a very inadequate method of evaluating results obtained. The writers of this monograph agree with this position. In 1913 they began an investigation which has been carried on continuously for over four years, the primary object of which was to replace introspective and subjective opinion by careful, scientfic measurement of results, and by sound use of quantitative methods in the treatment and interpretation of data. This monograph is primarily a report of the procedure of the work of the persons involved in this investigation.

Evaluation of results by means of standardized tests.—During the last ten years the school man, by making a careful quantitative study of results of instruction in various courses of study, has confirmed the indictment of the business man, of the educational critic, and of the high non-promotion rate evidenced by vital statistics. It is shown that regardless of the specific subject of study, achievement in which is questioned, children do not make satisfactory records either in the formal aspects of school study or in the reasoning aspects. For example, the American school system is committed to the policy of accepting 70 per cent efficiency in spelling. The average achievement of children in spelling in eighty-four city school systems was shown in 1914 to be 73 per cent. It is rare indeed that this mark is exceeded. As objective tests have been given in arithmetic in the manipulation of the four fundamentals and of fractions, we find, almost without exception in the various school systems, that children are quite deficient in this purely formal skill. In the same way if we give tests in the routine processes of United States geography to children who are just completing the study of that subject, tests which will be agreed upon as adequate and fair measures of outcomes of instruction in that subject, we find that children on the average show an achievement of less than 40 per cent of what the teachers themselves will agree they ought to reveal. The careful conduct of such types of standardized tests in United States geography in 1916–17 in eight school systems in and around Chicago showed this to be true. It showed furthermore that in the application of the formal skills to reasoning situations in the United States geography children were almost hopelessly deficient—maintaining a percentile score of correct solution of less than one-third of the material attempted. In the same way a study of achievement in high-school Latin in twenty-one school systems in New Hampshire by Mr. H. A. Brown of the Bureau of Research of the State Department of Public Instruction shows that, at the end of two years of continuous study of high-school Latin, the typical child can translate only fifteen lines of simple connected prose in fifteen minutes, and that of these even with the use of a liberal scoring system less than five lines will be translated correctly. Furthermore it shows that the progress with succeeding years is so slight that the child will add not more than one line to his credit in either "attempts" or in "accuracy" with each succeeding year of training. Certainly such conclusions from the study of the measurement of results in high-school Latin call for a most intensive analysis of ways and means of improving the teaching of that subject.

The striking conclusions in regard to the present inefficiency in either formal or reasoning abilities are based, it must be noted, upon the use of tests which may be regarded as objective rather than subjective. We are becoming accustomed in these days to the use of the term "standardized tests." Little attempt has been made to set forth in detail sound methods of defining or of establishing principles for the design of a standardized test. In chapter IV we present a detailed discussion of the design and construction of standardized tests. At this point let us indicate, however, that the conclusion with regard to the results obtained in our present mathematical situation are based upon much more adequate measures of outcomes than are found in either of the preceding two attacks upon this problem.

They are based upon facts established by use of standardized tests; tests that are objective instead of subjective; tests that recognize clearly differences in the kinds of mental activity involved in the work; tests that measure adequately the separate types of subject-matter of the course; tests that are designed on carefully established principles; tests that are accompanied by a standardized method of scoring; tests for which the difficulty of each step in the tests has been definitely determined; finally, tests that have been given under conditions that were rigidly standardized as to time, directions, and classroom procedure.

With the same outcome, then, as in other subjects, if we give tests to measure the formal abilities involved in ninth-grade mathematics, we find that of fourteen such types only three (removal of parentheses, use of exponents, and simultaneous equations) have been satisfactorily *automatized;* in the others, children will do correctly less than half of the problems in a given amount of time which a class drilled carefully in this procedure has been shown to attain without an undue expenditure of class time.

To sum up, then, the indictment that is brought against the present organization of instruction in our subjects of study, mathematics included, and as studied from any one of our three points of view—from general criticism, from a study of failures, or by the use of standardized tests—this indictment that can be brought as a result of the application of any one of these methods is that of *decided inefficiency in both the routine and reasoning processes of mental work.*

WHEN A PUPIL FAILS

When a pupil fails a serious pedagogical and administrative problem is revealed. The location of its cause is a moral, administrative, and professional obligation automatically laid upon the school system. A miscalculation has occurred somewhere in the construction and operation of the educational machine. One or more of the four primary units in the organization has gone wrong: First, it is possible that it is "the boy." Secondly, perhaps the methods of classifying and marking pupils have been so designed and administered as to cause an *impasse* in the operation of the machine. Thirdly, it may be due, as the laity claim, to *poor teaching;* and finally, if we are to take the consensus of judgment from the criticism of the educational reformers, it must be partly due to the *course of study*. Let us, in concluding this introductory sketch, analyze the degree to which each one of these elements may be a contributing cause in the difficulty.

1. Is it the pupil?—The striking characteristics of the distribution of scholastic ability described under the caption "individual differences" have already been pointed out. We need not elaborate on the discussion here. All grades of ability are represented in our classes, very likely even in any one class. Even casual introspection on the make-up of our student population will convince us that the description that we gave in the preceding pages is relatively true to life. Careful testing of many classes shows that not only does the law of extreme individual differences fit classes very well for practical purposes, but that the use of a specific graphic device like the normal probability curve conduces to clarity of thinking and expression of our thoughts concerning such important problems.

Thus when more than 10 per cent (at the most) of our pupils fail, grave doubt is cast on the statement that the cause of failure is "ordinary stupidity," "mental incapacity," or "dullness," all of which terms are selected by teachers in their attempt to shift the responsibility for what may be a most important blocking of the smooth operation of our educational machinery.

2. Proper classification and marking of children.—But let us pass on. Even if it were traceable to the pupil—if the "failure to pass algebra" were caused by a defective nervous system, the location of the seat of the difficulty but makes the obligation on the operators of the machine greater and more direct. If the child really is defective he should never have been admitted to the regular upper-grade classes. Special ungraded forms of instruction should have been provided for him. If he is just slow, this trait should have been detected by the aid of tests designed for the purpose (more of such devices later), the results placing in the hands of the teacher information through which she directs the emphasis of her teaching and the degree to which she distributes her attention among the different members of the class. In view of the already determined facts concerning the wide variation of ability in classes, the all too inadequate 35-minute class hour, the large classes that prevail in our city systems, and the multitude of subjects and classes that have to be taught by our country school teachers and under conditions of ridiculously short class periods—in view of all this it is undoubtedly true that an imperative duty rests upon the administrator and teacher: *namely, that of using devices for classifying children in order of ability.* Furthermore, in those larger systems where concentration of pupils will permit it, this duty involves the segregation of children in classes of approximately equal ability. It is necessary that we cut down the span in ability in our individual classes as much as the limitations of school finance and geographic distribution of pupils will permit. We see here then a primary reason for designing and using effective devices for classifying and segregating children in order of ability. It will be pointed out later that one of the most important functions for tests for mathematical or quantitative intelligence is this very function of classification of students in order of ability.

In the same way it is incumbent upon us to so design the instruments by which we measure the outcomes of instruction that *performance is* adequately gauged and that from it ability *is* correctly inferred. Chapter IV, however, will discuss in much detail needed reforms in the conduct of these important types of educational machinery.

3. **Is it inefficient teaching?**—While the teachers are busily engaged in locating the cause of failure in the boy, the laity, at least of the older generation, is in goodly numbers placing the responsibility upon young untrained teachers and inefficient instruction. Look at the statistics of our teaching staff a moment and face frankly the typical situation in this country. Nearly one-fourth of our teachers leave the profession each year, the average teaching life being about four years; most high-school teachers of mathematics are not specifically trained to teach that particular subject (remember that we are discussing the *typical* situation *represented by the small town and city* and not by the large city system); teach-

ing hours generally run from five to seven class exercises per day (at least in all but our most "educationally minded" communities); classes are typically thirty in size and often over forty; there is wide variation of ability in classes; short teaching periods abound (thirty-five minutes is most common and the 35-minute period is all too short to do the things that must be done)—get the full perspective of the result of these causes working together, and you may well sum it up in *ineffective teaching*. Naturally this *is* one of the causes of the difficulty.

It is one of the fundamental theses of this monograph that "*first aid*" in a situation like this will be found in the design and use of important devices for improving class or *mass* instruction. It will not be possible in our generation to remedy most of the conditions just pointed out as conducive to ineffective teaching. To be specific, financial limitations and unenlightened state legislatures will inhibit the *complete and adequate training of teachers;* there is no distinct tendency to increase *rapidly* the *teaching life* of teachers. It is true that sporadic instances occur in progressive systems in which supervised study and other important administrative reasons are extending the *length of the teaching period* (to sixty or eighty minutes in rare cases), thus making the conditions of teaching much better. These instances are exceptions to the rule, however, and all signs point to their being that for years to come. A satisfactory inventory of the future tells us that in all probability we will continue to teach throughout our generation under about the same conditions as we do now. Such an inventory as we are making therefore should bring out in prominent relief the direction that our energies can take toward the early improvement of the material conditions of teaching.

It will be shown in chapter VI that one reason for inefficiency in high-school mathematics is *a lack of clear vision as to the proper differentiation of emphasis upon formal and reasoning work* and more in detail on the specific operations or phases of each of these types of work. It will be shown that we spend too much time on the formal aspects. Eighty per cent of the problem-material of first-year algebra now consists of matter which is primarily of a formal or automatic nature. A sane statement of the fundamental objectives of mathematics teaching certainly cannot be made to jibe with such a distribution of teaching emphasis. It will be shown later in detail that it is possible to shift completely the emphasis to the point where it ought to be—to cut down the total time devoted to formal drill work (from 80 to 20 per cent) by means of properly designed courses of study (that is, textbooks) and more directly by means of "*standardized practice* material"—by means, in other words, of mass-instruction devices which can aid in a material way even the untrained and inexperienced teacher of high-school mathematics.

4. **Is it the course of study?**—During the last fifteen years we have had developed in this country a movement for the quantitative study of school problems. School men have begun to use the scientific technique worked out by the physical scientists. They have, for example, studied by numerical methods the elimination and retardation of pupils in the public schools. They have also studied more or less scientifically the organization of promotion systems, and methods of classifying and grading children. And during the past ten years a very active propaganda has developed, leading toward the design and use of instruments for measuring results of instruction.

Regardless of the aspect of our school organization which is studied, the investigator, in revealing conditions of inefficiency, has without exception pointed to the *course of study as the fundamental cause*. Thorndike in his study of elimination of pupils from school, Ayres in reporting on laggards in our schools, Burk in sensationally attacking the "lock-step" in the public schools, Judd, Strayer, and others in analyzing non-promotion statistics in the recent school surveys—each and every one has called attention to the striking need for *reorganizing the "course of study."* The attack which has been made upon the course of study both from outside the school by the more intelligent laity and from within the school by the philosophic and social reformer has been supported and made definite by the contributions of scientific measurement.

We have already pointed out the general condition of inefficiency on the part of the pupils who pass through our courses of study. We should distinguish at this point the fact that this inefficiency is of two types. It relates to the formal work of the school but is also evidenced in the ability of children in handling "*problem situations*"—reasoning work—of the schools. It seems clear from an analysis of the situation that there are two fundamental reasons why children immediately at the close of a period of study in a given subject are unable to pass tests which groups of teachers agree involve fair requirements.

First, they have been taught too many things. They have been studying an overloaded curriculum. They have been unable, for example, to manipulate satisfactorily the important types of factoring—monomial factor and the general trinomial—largely because they have been learning to manipulate from eleven to seventeen types (depending on the textbooks on which they have been brought up), for the vast majority of which they will have absolutely no future use.

A *second* fundamental cause is that not only have they been studying too much, dissipating their energies over large masses of facts and principles and reasoning processes, but that *the teaching emphasis has not directed attention, first, to the most important phases of the material and, secondly, to the most difficult phases.* It will probably be agreed that an outstanding need at the present time is the careful determination of minimal essentials in each of our courses of study—the elimination of dead wood from our curricula. It will likewise undoubtedly be agreed that textbooks should be so organized and teaching programs so planned that each day's work should result in laying the emphasis of the class discussion upon the most important and the most difficult aspects of the subject-matter; that they are not, points to outstanding defects in the present course of study.

The *third* point grows out of the second: Not only have we had an improper teaching emphasis, but this has been due primarily to the fact that courses of study, in spite of our agreement that we ought to "psychologize learning," have not been minutely designed in terms of a step-by-step analysis of the ways in which children learn. Careful investigation of representative textbooks in first-year algebra, first by detailed introspective analysis and next by objective measurement of children's achievement in the classroom, shows that neither in their general arrangement of topics nor in their detailed organization of explanatory and problem-material in particular assignments have these books been designed with an adequate regard for the ways in which students learn mathematics.

Our *fourth* point is again related to the others. It is a trite comment to make, to say that the material in our courses of study has been abstract and remote from the pupils' experience. The comment, made as it has been for a generation, has been tacitly admitted by the designer and the teacher of the mathematics textbook. "Real" problems have engrossed the attention of both author and teacher more recently. It is sad to relate, however, that the "real" or the "practical" in problem work has generally meant "real" from the standpoint of the logically trained adult and not from the standpoint of the pupil. Another of the theses of this monograph will be found in the attempt to make clear and illustrate in detail the types of material which are real to ninth-grade students and which may properly be used for illustrative work in ninth-grade mathematics.

And finally, there is an evident need for a minute analysis of the course of study because the content of our courses of study is so directly an outgrowth always of the statement of the general aims and outcomes of instruction upon which the books are tacitly built. Lack of clear definition of aim and lack of specific itemization of outcomes of instruction always lead to "general," inadequately organized textbooks and a vague teaching program. The mathematics-teaching corps of this country must agree most definitely upon particular objectives toward which its teaching is aimed, and in a language pertinently related to the mental processes of children. It must lay down in completely itemized form a list of the mental outcomes from its instruction built in terms of specific processes of learning. Not until this is done, certainly, will the design of textbooks and the teaching of children lead to a satisfactory development of the ability in students to use intelligently the most powerful devices of quantitative thinking—the equation, the formula, the graph, and the properties and relations of the more important space forms. Chapters VII to IX of this monograph will state the writers' reaction to this basic problem.

The results of years of analysis and experimentation on the organization of a course of study, the initiation of a complete teaching program, and a clear delimitation of instruction in ninth-grade mathematics, together with a detailed analysis of the outcomes of teaching, will be set forth there.

Thus we have passed in review the outstanding facts cencerning the present status of the teaching of ninth-grade mathematics. The sketch is brief, the evidence has been but partial. We have aimed to orient the point of view of the reader to the scientific study of ways and means of improving, in a relatively permanent fashion, the teaching of the subject. Facts and figures, directed and intelligently applied by the new educational psychology and educational measurement, can be made to clarify our program for the eradication of the significant weaknesses which have been pointed out.

CHAPTER II

THE NINTH-GRADE COURSE IN MATHEMATICS: A DETAILED INVENTORY

What mathematics do ninth-grade children study?—Children who stay to the ninth grade of our public schools are forced to study "first-year" algebra. Statistical investigations compiled within the last three years have shown conclusively that more than 95 per cent of the schools of the country require one or more years of mathematics for graduation. We have at hand the evidence on the present status of mathematics courses of study in the very complete study of Professor L. V. Koos.[1] Mr. Koos did his work in his capacity as secretary of the *Committee on Reorganization of the Secondary School and the Definition of the Unit* of the North Central Association of Colleges and Secondary Schools. A carefully prepared question blank was filled in by 416 schools in fifteen states of the North Central Association. There can be little doubt that conclusions formed from a study covering such a wide territory will be representative of the more progressive practice of the city school systems of this country. Included in the study is a very careful question-blank analysis of current teaching of secondary-school mathematics. The points which were studied include: the extent of the mathematical offering; years in which courses appear; the time devoted to the teaching of mathematics and the number of years of mathematics required for graduation; important deviations made in the course from the plan of the text in use; various aspects of methods of teaching (such as disposition of the class period, types of method found most satisfactory, special devices used in teaching of algebra and trigonometry, the use of historical notes, the extent to which teachers make efforts to correlate algebra and geometry, and the extent to which teachers make efforts to meet current criticisms of high-school mathematics); the aims of teaching current among our teachers of

[1] *The Administration of Secondary-School Units* (Supplementary Educational Monographs, Vol. I, No. 3, Whole No. 3). Chicago: The University of Chicago Press, July, 1917.

mathematics; the extent to which the aims are fulfilled; and the point of view of teachers with regard to relative values of "content" and "discipline" in courses in mathematics. As already indicated, the evidence in Mr. Koos's report shows that nearly all children who continue to our ninth grade are required to study one or more years of mathematics before they may be graduated from the high schools. The reader is referred to Mr. Koos's report for the detailed tables supplying the itemized figures on this question.

At this point we shall do well, however, to quote his conclusions verbatim:

1. Elementary algebra is almost always a first-year high-school subject. Plane geometry is markedly a second-year subject, but is reported in some schools in the third year, or in the latter half of the second year and the first half of the third. Advanced algebra appears most commonly in the third and fourth years, but in a few schools in the second. Solid geometry appears in the third or fourth years and trigonometry in the fourth year.

2. Elementary algebra and plane geometry extend almost without exception through a full school year of 36 weeks or more. . . . . Some schools report periods of greater length.

3. Supervised study is reported in *a few* schools for elementary algebra or plane geometry, or both.

4. Most schools require two years of mathematics for graduation, while a small proportion each require none, two and one-half years, or three years. Still others vary the requirement with the high-school course taken.

5. Textbooks dominate content and organization of courses in mathematics.

6. There is no standard practice in the disposition of the class period as to recitation, study, teaching, and lesson assignment, except that *a very large* number of schools allow no class time, or a very small proportion of class time, for study.

7. The deductive, inductive, and analytic methods are most commonly used in class instruction.

8. Historical notes are introduced into courses in elementary algebra and plane geometry in somewhat more than half the schools and into the advanced courses in mathematics in approximately a third of the schools. They are reported as "humanizing," i.e., adding interest to, the work.

9. More than 60 per cent of the replies report efforts to correlate algebra and geometry. The values of such correlation are said to be: (1) making the subjects easier of comprehension, (2) teaching the unity of mathematics, and (3) increasing the interest in it.

10. Approximately 60 per cent of the replies report efforts to meet current criticisms of high-school mathematics. Such efforts are usually constituted of the introduction of "practical" problems, problems drawn from the vocations, or problems within the students' experiences.

A careful examination of the scope, methods, and treatment of the material in Professor Koos's investigation leaves in the mind of the reader confidence that we have here a fundamental and representative analysis of the present situation with regard to the content of our high-school courses of study. We believe that it establishes definitely the present status of ninth-grade mathematics as concerns its general organization and administration.

The content of the course determined by the textbook.—Probably no conclusion from Professor Koos's study is more important than is his fifth point, namely, "*textbooks dominate content and organization of courses in mathematics.*" We would generalize the status of the present situation by saying that it seems evident that *textbooks almost always completely determine the specific subject-matter that is taught to students in the course.* Let us examine this aspect of the situation a little more in detail.

We have already referred to the commonly recognized condition that teachers of ninth-grade mathematics are relatively untrained and inexperienced. Professor Josselyn's *Survey of Kansas High Schools* supplies very suggestive evidence for such conclusions. The author of that report showed clearly that of the 469 Kansas teachers of high-school mathematics who were teaching it in the year 1912–13, 163 of them were *entirely* unprepared for such teaching. Furthermore, since these data were obtained from the report of the teachers themselves, the actual training situation is much more acute than these figures show. Taken by and large, nearly one-half of our ninth-grade mathematics teachers are of less than two years' experience and have been almost entirely untrained for their work. Furthermore, the writers' contact with the situation leads to the conclusion that as far as *training in studying the mental life of pupils* goes, our mathematics teachers are *absolutely* untrained.

Such facts as those adduced in the Kansas study, typical as they certainly are for the country at large, make real the necessity for teachers to have the content of their subjects organized for them in the form of textbooks. We in this country have been committed to the practice of teaching children through the textbook. Our practice in this respect is in distinct contrast to that of European countries: Germany and France having a highly trained, mature, and experienced male teaching staff thrusts upon the child the burden of making his own textbooks. In this country, however, it will be agreed that an untrained, immature, inexperienced teaching staff has necessitated the employment of textbooks in the teaching of children regardless of the subject of study.

It should be noticed, furthermore, that, as intimated in the first chapter, the general introduction of new courses of study, whether in our colleges, high schools, or elementary schools, has always waited upon the production of a textbook in which the material has been laid out in such standardized form that it could be handled by teacher and pupil in continuously organized assignments. We should be reminded again too that this textbook-making or rather *course-of-study-making has always been done, down to our own generation, by college men.* More of that, however, in the next chapter.

Degree to which high-school teachers follow the textbook.— During the past few years the writers have, in the conduct of this investigation, canvassed the practices of high-school teachers of mathematics with respect to the degree to which they deviate from the textbook. High-school teachers of mathematics appear to fall rather sharply into three distinct groups.

First, there is the very large group, probably including the vast majority of mathematics teachers in our small town and city systems, many of whom have had thrust upon them the task of filling in the gaps in the largest department (either mathematics or English always), most of whom, again, have no specific training in mathematics beyond a one or two years' college course and no training in the *teaching* of mathematics. It is, moreover, a group whose contact with the secondary-mathematics classroom has been limited to one, two, or three years. There seems no doubt, since the typical school system of the country is the school system found in towns of from 10,000 to 25,000 people (82 per cent of our cities are less than 25,000 in population), that our *typical* mathematics teacher is the one we have just described.

But now for the *second* group: As one looks to the larger cities he occasionally finds teachers of longer experience, of fairly adequate training in college mathematics, *but almost no training in the teaching of mathematics;* teachers who have, however, certain original notions with respect to the most desirable order of presentation of material and certain well-fixed ideas with regard to what ought to be taught. With this relatively small group we are bound to find a tendency to make omissions from, additions to, or shifts in order as indicated by the standard textbook used. It certainly does not do injustice to the true situation, however, to say that the practices of this are exceptions to the rule rather than typical of ordinary practice.

Our *third* group of mathematics teachers includes that more intelligent fraction of 1 per cent in our secondary schools, who, either in partnership with the college men or on their own initiative, have gone into the business of writing textbooks themselves. We are seeing in our day a rapid accumulation of textbooks in secondary mathematics, mostly written by high-school men or with the collaboration of high-school men. It will be shown in succeeding pages that in these more recently written books the course of study of a generation ago has been modified but to a very slight extent, if at all. It will be shown, for example, that, in the nine most representative and currently used textbooks, the order of presentation of fundamental topics is almost identical. It will be shown, furthermore, that in the distribution of emphasis among the various topics *rearrangements* and slight modifications of former practices seem to be the order of the day rather than a complete systematic overhauling of the content.

In résumé, therefore, since the *typical teacher* is dependent upon someone else's organization of subject-matter, or, more specifically, another person's textbook—with teachers themselves confessing that the deviations that they make from the adopted textbooks are the exception rather than the rule—it seems quite clear that the content of the course in ninth-grade mathematics will be typified by the textbooks that have been found to be most representative of those currently used.

It may be well, however, before leaving this subject to quote Professor Koos's findings upon this question. We do so verbatim because of the pertinent way in which his data relate to our problem.

ORGANIZATION OF THE COURSES

The answers to the question, "What important deviations do you make in your course from the plan of the text you are using?" give full support to the conclusion that the content and organization of courses in mathematics are largely determined by the textbook used. It will be noted in Table XXVII [1] that a large proportion of the teachers report that they make no important deviations. To these, because of the conscientious way in which the teachers generally have responded to our inquiry, we may safely add practically all of those who make no answer to the question. Almost all deviations reported

TABLE 1*

DEVIATIONS FROM THE PLANS OF TEXTS USED REPORTED BY TEACHERS OF MATHEMATICS

Deviations from the Plan of the Text	Elementary Algebra
Omissions	21
Additions	6
Shifts of order	21
None	42
No answer	19
Total number of responses to the questionnaire	112

* This is only a part of Professor Koos's Table XXVII.

were readily classifiable under the categories "Omissions," "Additions," and "Shifts of Order" appearing in the table. This may be illustrated for all the divisions of the field by quotations from typical deviations reported by teachers of elementary algebra: "omit some theory," "add drill work," "shift order," "defer graphing," "omit graphing," "simpler problems added," "factoring before fractions," "much extra work," "give mimeographed lessons" in addition, "introduce transposition early," etc. Only a few report deviations of as much significance as "commence with equation and make all else subordinate to it" and "correlate various branches of mathematics."

To sum up: We have shown, therefore, the tremendous influence of the textbook on the course in mathematics—the powerful leverage of the small group of "adopted authors" on the educational careers of children in our public schools. It seems quite clear that a person who is desirous of really effecting permanent improvements in the subject-matter learned by children must embody his ideas in the form of a textbook. *The extent to which his course is based upon scientific investigation and is designed with reference to social and psychological needs, just so far will his course be successful in influencing human abilities for the better.*

The need for a foundational program for the design and construction of the course of study.—An intelligent discussion and evaluation of a current situation is always dependent upon a searching and quantitatively determined inventory of that situation. "Taking account of stock" is becoming almost as common in educational parlance as in commercial and industrial vocabularies.

In working at the general problem of the standardization of ninth-grade mathematics, the core of which was recognized as "educational measurement"—namely, the design and use of standardized tests for measuring the results of instruction—the writers recognized first the need for a minute analysis of the present course of study. Accordingly in 1915, a thoroughgoing program for the examination of the high-school mathematical curriculum was mapped out. This program, which will be set forth in detail in chapter VIII, included, in outline, the following steps:

1. *A detailed inventory of the present course.* This was to mean a tabular analysis of the content, order of topics, distribution of emphasis, proportioning of space to explanatory material, and organization of exercises and problems in textbooks which could be agreed upon as representative of those now in use in the country at large.

2. *A detailed analysis of the mathematical needs of children* as shown first by the subject-matter of other courses which they would take within the high school. It was decided to answer the question, "What specific mathematical operations, which are taught in ninth-grade mathematics, now are used, and to what extent are they used in subsequent courses—namely, advanced algebra, plane and solid geometry, trigonometry, physics, chemistry, general science, mechanical drawing, manual arts, etc.?" This criterion gave promise of being a most important one.

3. In answer to the insistent and recurring demand for the construction of courses of study in terms of occupational and

extra-occupational utility, in "life beyond the school," *our program necessitated the determination of the mathematical notions, principles, and operations*—of the types of mathematical reasoning and skills—*of which continued use is made in later life-activities.* It was recognized that to carry through this part of the program would necessitate canvassing courses of study in technical schools, "corporation schools," and "correspondence schools," and analyzing new types of "applied," "industrial," and "vocational" textbooks and courses of study. It was postulated, furthermore, that *this material would be appropriate curriculum material for the public schools only on the condition that it is found to represent the needs of a relatively large proportion of our student population.*

A MINUTE QUANTITATIVE ANALYSIS OF THE PRESENT COURSE OF STUDY

Leaving to chapter VIII the exposition of the general program and the underlying criteria and principles upon which we have proceeded, we now take up in detail the examination of our present course. This brings us back to the first step in the curriculum program, namely, the tabular analysis of currently used courses of study. At this point it should be stated that, under the direction of the writers, a detailed tabulation was made of the content of the most widely used textbooks in first-year algebra. Mr. E. C. Denny, a member of the Illinois Committee on the Standardization of First-Year Mathematics, undertook the tabulation of the problem-material in the texts which have been agreed upon. The present writers base their comments concerning the current situation upon an analysis and interpretation of these tabulations. It should be said that all the compilation of material was done with great care by Mr. Denny, who organized material in the different textbooks on the same basis and in conference with the committee agreed upon the specific procedure to follow in all conflicting cases. The writers, however, are responsible for the specific direction in which the investigation was carried out, for the organization of the program, and for the interpretation of the data. The study rests upon the assumption that, if a group of representative textbooks can be found which are now used in our ninth-grade classes, the tabulation of the problem-material within these books will lead to sound criticism of existing practice with regard to the course of study.

Two preliminary question-blank investigations were made to enable us to determine which textbooks could be safely regarded as representative of existing practice. The nine given below in Table 2 were found to be used in 41 out of 51 high schools represented in the University of Illinois High-School Conference and in 92 out of 109 of the high schools co-operating in the North Central Association of Secondary Schools and Colleges. Thus in approximately 85 per cent of schools which can be regarded as among the more progressive high schools of the country the teaching of mathematics to ninth-grade children is determined by these nine textbooks. We give herewith a table showing the extent to which each of the different textbooks included in the investigation was used in these two districts.

TABLE 2

Text	Number of Cities Using Each Text in Illinois Conference	Text	Number of Cities Using Each Text in North Central Association
Wells and Hart	7	Wells and Hart	28
Rietz, Crathorne, and Taylor	1	Marsh	8
Slaught and Lennes	16	Slaught and Lennes	10
Collins	6	Collins	8
Hawkes, Luby, and Touton	10	Hawkes, Luby, and Touton	29
Wentworth-Smith	1	Wentworth-Smith	6
Wentworth (New School)	7	Wentworth (New School)	5
Young and Jackson	2	Milne	11
Comstock	1	Breslich	1
		Wells	3

It was assumed that the time which children give to first-year algebra will be represented approximately by the numerical distribution of problems of various kinds in these textbooks. Accordingly a careful count of the number of problems included in each book under each of three types was made. The types selected were (1) verbal problems, (2) formal problems, and (3) graphs. In preparing the tabulation problems were classified in terms of the following definition:

By the term "verbal problems" we mean those stated in the form of written expression as distinguished from those stated in algebraic form and of a mechanical or drill type. These latter, of the mechanical and drill type, usually given for the purpose of habitualizing such processes as the four fundamentals, finding special products, factoring, evaluating algebraic expressions, etc., we have designated as formal problems.

DISCUSSION OF THE FINDINGS OF THE INVESTIGATION

I. PROPORTIONATE OPPORTUNITY OFFERED FOR TRAINING IN MEETING DIFFERENT KINDS OF SITUATIONS

The emphasis on the formal.—If the teaching of children is determined by these textbooks an interpretation of Table 3, which

TABLE 3

PERCENTAGE OF ALL PROBLEMS FOUND IN NINE TEXTBOOKS IN FIRST-YEAR ALGEBRA DEVOTED TO VARIOUS KINDS OF OPERATIONS

	Verbal Problems	Formal Problems	Graphs
Textbook A	18.0	80.0	2.0
Textbook B	15.8	81.0	3.2
Textbook C	{12.0 {1.5	57.7 28.0	0.6
Textbook D	15.3	82.3	2.4
Textbook E	13.3	84.0	2.7
Textbook F	16.2	82.1	1.7
Textbook G	16.0	78.3	5.7
Textbook H	20.6	76.1	3.3
Textbook I	15.9	80.2	3.9
Arithmetic mean	16.0	81.4	2.6
Extremes	{20.6 {13.3	85.7 76.1	5.7 0.6
Range	7.3	9.6	5.1

shows the number and percentage of these problems found in nine textbooks of various kinds, will enable us to comment pertinently upon the organization of our material from the standpoint of these larger categories. Note, first, that the typical textbook devotes more than 80 per cent of its problems to material of a formal or automatic type, the type that requires *after its initial presentation* little or no thought. *Certainly less than one-third of our teaching time*, even if teaching time is only partially correlated with the distribution of problem-content, *is devoted to "thought" problems in which the pupil is forced to meet new thinking situations.*

We have here an anomalous situation. No group of antagonists to the application of quantitative methods in the improvement of educational practice has been more active in denouncing the assumed interest and emphasis (of the "standardizers") on the formal processes than have authors and teachers of mathematical textbooks. Yet *we find the authors themselves determining that children shall devote the vast majority of their energy to "manipulative" work in which little or no opportunity is provided for real, analytical, or organizing thinking*. Teachers of mathematics themselves are beginning to recognize the fallacy of such unsound educational practice as that which gives the preponderance of time and energy to purely routine, manipulative processes. Certainly it would be a *sound contention* to maintain that, if children are to be required to take one year of so-called secondary mathematics, in the course of that year's instruction *as much time and energy as possible should be devoted to the meeting of situations which may be thought of as providing opportunity for development of mathematical "thinking."* In this connection consideration of the results of such an analysis as we are now making raises grave doubts as to the validity of the assumption current among teachers of mathematics that the traditional organization provides opportunity for the pupil to meet and master the really important mathematical notions and tools. One of the fundamental theses growing out of this investigation is that if ninth-grade mathematics is to be the last required course it should be built about a core of fundamental notions and operations involved in the expression and determination of quantitative relationships. The perspective which we are now painting of the situation merely takes account of the larger outlines of the classification of our subject-matter. Succeeding sections of this chapter will give a more detailed perspective of the points which we are now making.

The reader, as were the writers, is undoubtedly astonished at *the meager opportunity provided* the pupil *for mastering* one of the two most important *methods* of representation, namely *graphic representation*. An analysis of these textbooks shows conclusively that the authors have either not recognized or have hesitated to incorporate in their texts this most important mathematical tool.

Not more than one week in forty is devoted on the average, even in those schools which teach "graphs" at all (as an isolated operation), *to giving an insight into and tool mastery of this most important pictorial METHOD of representing and describing relationships.* The lack of agreement among the more recent textbooks shows that as yet the traditional conservatism of mathematics teachers has not yielded to the demands of modern scientific and industrial need. The reader will probably recall that for a long time the desirability of introducing graphic methods in algebra was debated by mathematics teachers themselves. However, in the books which have been appearing recently there is a very slightly increased emphasis on this most important device. It will be established later, however, that "graphs" is still regarded as *subject-matter* and not as one of the basic METHODS of representing number and relationships.

II. THE OPPORTUNITY OFFERED FOR TRAINING IN "THINKING"

The organization and grouping of verbal-equation problems.—From the standpoint of the "thinking" involved in the solution of the problems found in the ordinary mathematics textbook, the verbal-equation problem is accepted by teachers of mathematics as furnishing the best instance of desirable mental activity on the part of the pupils. The solution of this type of problem involves careful, discriminating reading—the ability to recognize the real problem in the example and to express this in algebraic symbolism, as well as the specific skill involved in the solution of the algebraic statement.

Classification of verbal problems on the basis of degree and number of unknowns in the equation.—Our classification groups the verbal-equation problems under two categories: (1) the degree of the equation and number of unknowns involved in it; (2) the sources or situations from which these problems are drawn. An outstanding emphasis is placed upon equations of the first degree in one unknown and equations of the second degree in one unknown. Briefly summarized, Table 4 shows that the average text devotes three-fifths of its verbal-equation problems to the equation of the first degree in one unknown, a little less than one-fifth to the quadratics in one unknown, about one-seventh to first-degree equations in two unknowns, and a very small part to first-degree equations in three unknowns and quadratics in two unknowns.

Throughout these texts very little specific practice is offered the pupil in *translation* from verbal to algebraic language and none from algebraic to verbal language. Even though the solution of any one of these examples involves "translation," clear psychological analysis shows that preparatory to manipulation of worded problems there should be a carefully graded series of problems involving practice in translation alone. It might be pointed out here that we have found no evidence whatsoever of any translation from algebraic symbols to word statements. Again, it may be said that *from the standpoint of rationalizing and clarifying this algebraic symbolism, no exercise can be more profitable for the pupil than the translation of algebraic statements into word language.*

Sources or situations which give rise to verbal-equation problems.—It is interesting to note that about five-sixths of all the verbal-equation problems contained in these representative textbooks may be classified under eleven categories—those based upon percentage and its applications, geometry, physics, motion, ratio and proportion, coins, mixtures, digits, number relations, age, and clock relations. Number-relations problems predominate.

The percentage problems are based upon concepts which the pupil has carried over from his arithmetic. They include examples involving interest, profit and loss, discount, and partnership. The examples drawn from geometry are based chiefly upon angles, upon lines, and upon perimeters of common space forms and their areas and volumes. A few of these deal with the formal side of geometry. Evidently the authors assume that the pupils have certain intuitive, non-demonstrable facts concerning these space relations which will supply adequately the information needed in solving these examples. The physics problems are based chiefly upon temperature and thermometers, levers, falling bodies, horsepower engines, and formulas for light, sound, and electricity. The examples involving the concept of uniform motion are drawn chiefly from rates of rowing up and down stream, the movement of trains, and so forth. The examples involving ratio and proportion are drawn chiefly from similar triangles and levers. The remaining categories are sufficiently clear to necessitate no detailed description.

The evidence shows that there is a definite, conscious attempt on the part of authors to correlate rather closely this phase of the problem-work of algebra with arithmetic, mensurational geometry, and science. In another part of the study it is pointed out that a historical study of the texts that have appeared since 1892 points to the increasing tendency of textbook writers to draw their problems from geometry and science with a diminishing emphasis upon digit problems, mixtures, and clock problems. The rather large emphasis given to problems involving number relations can probably be explained by the relative ease with which these problems can be made and the fact that they do not involve situations as yet not understood by the pupil. It should be pointed out here that *a fundamental criterion* determining the selection of verbal problems is that *the situations and setting from* which these are drawn must not be so far removed from the pupil's experience and knowledge as to necessitate more time in the teaching of the situation itself than in the mathematics presented in that situation. Hence a great many applications of mathematics cannot be included in the first-year high-school course for the reason that it is impracticable to teach the situations from which the problems might be drawn.

III. HOW THE PUPIL SPENDS MOST OF HIS TIME: DEVELOPING SKILL WITH THE FORMAL OPERATIONS

A careful tabulation of these textbooks showing the proportional distribution of space given to various operations is abstracted in Table 4.

We shall state repeatedly that the core of our mathematical creed is that we teach mathematical subjects in the public schools to *develop in the pupil the ability to use intelligently the most powerful devices of quantitative thinking: the equation, the formula, the graph, and the properties of the more important space forms.* This really means the equation. Only in so far as habituation or automatic

TABLE 4

PERCENTAGE OF TOTAL PROBLEMS IN EACH OF NINE TEXTBOOKS DEVOTED TO EACH OF THE OUTSTANDING OPERATIONS IN FIRST-YEAR ALGEBRA

	Equations	Four Fundamentals	Special Products	Factoring	Fractions	Radicals	Exponents
Textbook A	15.3	17.3	8.1	11.6	7.9	2.8	9.6
Textbook B	17.3	22.4	6.1	10.7	8.0	6.9	3.4
Textbook C	16.6	15.9	6.7	12.9	5.9	16.0	10.6
Textbook D	13.2	18.6	8.1	16.0	10.2	5.8	5.5
Textbook E	11.1	9.7	4.5	10.5	9.1	12.6	9.6
Textbook F	17.5	13.5	5.0	8.6	7.3	10.0	3.6
Textbook G	19.5	16.7	4.0	9.7	5.6	11.5	5.4
Textbook H	13.6	14.5	4.8	16.6	5.9	16.4	10.4
Textbook I	15.9	20.3	5.6	11.3	5.7	16.4	3.4
Arithmetic mean	15.3	16.6	5.8	11.9	7.2	9.8	7.9
Extremes	19.5	22.4	8.1	16.0	10.2	16.4	10.6
	11.1	9.7	4.0	8.6	5.6	2.8	3.4
Range	8.4	12.7	4.1	7.4	4.6	13.6	7.2

efficiency in the formal processes contributes to efficiency in the solution of problems primarily of the reasoning or interpretative type, ought we to insist upon skill or habituation in them.

Check this criterion against the actual distribution of space in our texts: 15.3 per cent of the formal material devoted to the equation (13 per cent of the verbal problems apply to it also. If we add them together, we find that, roughly, one-fourth of the entire problem-material is devoted to the equation which nearly all mathematics teachers will openly embrace as the central operation of algebra). At least, serious doubt is raised as to whether textbook writers have even yet responded to this demand.

Emphasis on manipulation of four fundamentals, special products, and factoring.

On the other hand, note that about one-sixth of the formal examples of the book are based explicitly upon the four fundamentals with another one-sixth of the book devoted to special products and factoring! In other words, one-third, by and large, of our instructional attention in first-year algebra is devoted to these purely formal types of material. Careful analysis of the actual use of current ninth-grade mathematics in other high-school subjects and in occupational activities shows that it is almost impossible to defend the large amount of attention to this material, and absolutely impossible to defend this emphasis upon special products and factoring. The only use which is made of the latter is a very meager one found in connection with the solution of quadratics—and that use forms less than 2 per cent of all the algebraic material called into play—even in this further academic instruction in the high school!

Evidently textbook writers have not practiced what they preach concerning the use of the equation as the core about which the material shall be organized. In these fundamental operations a great deal of material has been included which has no relation to any equational work. For example, we find no instance in which polynomial multiplication or division (involving more than two terms) has any relation to the solution of equations. And yet they abound in the textbooks. Clearly a sane judgment on the matter would lead to an immediate elimination of such routine material. What criterion for curriculum-making—social, psychological, or what not—can legitimately justify the retention of practically seventy-five pages of such formal padding in secondary courses as the various "cases" of factoring? In truth, we are habit-minded, thinking in deep ruts of traditionalism, not to be able to evaluate the content of our course of study in terms of life-use, or in terms of real opportunity of training in thinking. Surely not one among us will rise to defend *this formal material* on the grounds of "thinking" value. If one is about to, perhaps chapter VII will throw light on the problem.

In the detailed analysis of formal problems, we find that *only* 2.3 per cent of the examples provide training in "evaluation" as a specific operation, yet we shall show in chapter VIII that evaluation is one of the four most important operations to be taught in ninth-grade mathematics.

But let us discuss special products and factoring a bit more in detail. The reader will note from the table that there two operations, which are usually taught together, occupy, roughly, 17 per cent of the problem-material in the average textbook. Under special products the emphasis is placed on the square of a binomial, the product of the sum and difference of two numbers and the product of two quantities having a common term. A more minute analysis of the importance of this topic will be made in chapter VIII. In the organization of special products and factoring, the typical procedure is to organize the material in accordance with a number of types, presenting under each one a very brief exposition and offering detailed practice by many examples in developing skill in the manipulation of each one. This gives the pupil an opportunity of acquiring a certain dexterity in the manipulation of one type at a time. Then he is introduced to a totally different type, the learning of which is inhibited by the learning of the former type, and proceeds to acquire the same degree of skill (or lack of skill as the tests show) in the manipulation of it. Thus this process is continued until the six or eight types of special products and the seven to seventeen types of factoring have been ground through. Then follows a miscellaneous list of examples providing the pupil with the opportunity of applying or generalizing his experience to the solution of these examples. Only in three of the texts examined is the pupil provided opportunity to generalize or apply the skill acquired in any one of these types until after all have been mastered. The writers raise the question here of the advisability of providing the pupil with more opportunity for the recognition of the type or the generalization of specific processes rather than restricting him so long to the manipulation of these specific types and operations organized and classified under some particular type.

Fractions. Our next topic is fractions. About 7 per cent of the examples are classified under this heading.

The large emphasis on fractions, 7.2 per cent of the total space, is doubtless due to the tendency of authors to exhibit examples in which the skill they have attempted to develop in factoring can be used. Evidently it is not due to the need for developing skill in these complicated fractional forms. From the standpoint of their use in the equation, the formula, the graph, or in the study of the properties of space forms, these complicated fractional expressions in the current texts cannot be justified. However, it is interesting to note a diminishing emphasis on this symbolic manipulative phase of algebra. Not more than two or three lessons in complex fractions are found in the average book, and these include none of the many-storied, stair-stepped kind which were an important part of the mathematical menu of high-school Freshmen a generation ago.

Ratio and proportion. The readers who are inclined to think that in the required courses in the public schools the pupil should have an opportunity to learn the *fundamental mathematical notions* of wide use and application will wonder at the slight emphasis given to ratio and proportion. This important quantitative concept gets *slightly* more attention than imaginary numbers, 1.2 per cent and .8 per cent, respectively! As a topic, ratio and proportion usually appears late in the year's work and may be omitted altogether without any hindrance to subsequent work! Is there anything more fundamental to give a student in high-school mathematics than a clear grasp of the principle of variation, of proportionality, of functionality? The present writers are more concerned about this than about any other foundational elements in the scheme of quantitative notions and devices of which children need to have a mastery on leaving the public school. Is there anything more fundamental in quantitative thinking than the ability to represent "*law*," than the ability to represent and to interpret statements of relationship—the very core of clear thinking itself? Yet notice the attention paid to it by our present courses! If the reader answers, "But I always organize *my* instruction throughout the year so as to bring these things out," the plain answer to it is, "*Most* teachers do not." The textbook and their limited training and experience will not permit them to do so. In order to get a *notion taught, a skill developed generally among the pupils in schools, a detailed exposition and a detailed opportunity for practice must be given through the textbook.* If, on the other hand, the reader remonstrates that these notions and devices of expressing relationship are too difficult for ninth-grade students, the plain answer is, "Our experimental results show that it can be done, and too in addition to developing efficient control over the essential formal operations that are utilized in the manipulation of a simple equation." Chapter IX presents the evidence.

Order of introduction of topics. A comparative study of the order of introduction of topics in the texts which were examined showed a very decided uniformity and agreement. The established order is literal number, simple equations, positive and negative numbers, addition and subtraction, parentheses, multiplication, division, linear equations, special products, factoring, quadratics by factoring, H.C.F. and L.C.M., fractional and literal equations, graphs, simultaneous equations, radicals, simultaneous quadratics, quadratic equations, ratio and proportion, variation, exponents, and binomial expansion. Six of the texts agree almost exactly in this order. Those topics about which there is the least agreement are H.C.F. and L.C.M., ratio and proportion, quadratic equations, simultaneous equations, and graphs. There is about an equal division of the texts between those writers who introduce graphs before simultaneous equations and those who introduce it after. At no point, on the whole, are authors more neglectful of both unifying mathematical and psychological principles than with this question of order of topics.

Teaching by rule. A study of the attempts made by the textbooks to aid the pupil through the development and statement of rules and principles shows first, that it is common practice to formulate rules for most of the processes in algebra, though such statements are not always labeled as rules; secondly, that these rules are developed inductively; and thirdly, that they are intended to serve as a model whose perfection in form the pupil should try to reach in the statement of and direction of his mathematical thinking. Nothing illustrates better the formalism in our teaching, at least from the psychological standpoint, than this practice of using rules to govern skill in the manipulation of formal material. Psychological analysis shows for example that the mental processes included in memorizing a rule and in setting up the habit of controlling a particular skill by facility in bringing the right rule to bear on the automatic reaction desired are quite different. Furthermore, the response is *not* automatic at all, if it involves the use of the rule. That is, the desired response does not automatically follow the presentation of the stimulus. Rules merely crystallize and generalize specific habit-formation. They are of service in *interpreting* habit-systems which we have developed, not in learning the skill itself. Our discussion of "economy of time in learning" in chapter IX will elaborate upon this important matter.

CHAPTER III

HOW ALGEBRA BECAME THE NINTH-GRADE COURSE: A HISTORICAL STUDY

In the previous chapter a detailed analysis of the content of the present course was made. There we showed a great emphasis on the formal, manipulative phase of algebra. How this came about and where it came from will be investigated in this chapter. In other words, we shall now trace the development of algebra as a school study, both with regard to its position in the curriculum and as to its developing content. To do this will necessitate first a study of the curricula of colleges and secondary schools from the Colonial period to the present time and, secondly, a careful examination of the content of typical textbooks used during this period. Let us first turn to the curricula of these schools to ascertain the *position* of algebra in the various types of schools, as well as to note the entrance requirements of the higher schools from their beginning to the present time.

I. THE POSITION OF ALGEBRA IN THE CURRICULUM

Mathematics at Harvard University.—The development of the mathematical curriculum of this oldest of American colleges has an interesting history. In 1643 the college course covered only three years. There were no *entrance* requirements in mathematics, no mathematics offered in the first two years, and in the Senior year arithmetic and geometry were studied. Algebra was an *unknown* science there. This course was practically unchanged till the beginning of the eighteenth century. Respecting it, Cajori says: "It appears that in 1700, algebra had not yet become a college study."[1] From the fact that Ward's *Mathematics* was used as a text in Harvard some time during the period 1726–38, it is probable that algebra (one part of this text is devoted to algebra) then became a part of the mathematical course. But no

[1] Florian Cajori, *Teaching and History of Mathematics in the United States*, U.S. Bureau of Education, 1890.

direct proof of its existence prior to 1786 is in evidence. It is known that in 1787 arithmetic was studied in the Freshman year and algebra in the Sophomore year. In 1802 arithmetic to the "Rule of Three" (proportion) was required for entrance.

Algebra required for admission to Harvard.—In 1820 algebra became for the first time a required subject for entrance. A statement from the catalogue of 1825, enumerating the requisites for admission, states that "arithmetic including vulgar and decimal fractions . . . . and algebra, to the end of simple equations, comprehending also the doctrine of roots, and powers, and arithmetical and geometrical proportion." Geometry was required for entrance after 1844. Little change occurred until 1872–73, at which time the following requirements were set up: arithmetic, algebra through quadratics, and plane geometry, including as much as in the first thirteen chapters of Pierce's *Geometry*. The requirements for entrance in the scientific school were more extensive, solid geometry, logarithms, and plane trigonometry being demanded.

Mathematics at Yale.—"As at Harvard, so at Yale the mathematics were studied at that time during the last year of the college course, and after the study of physics had been completed." In 1719 Alsted's *Geometry* and Gassendi's *Astronomy* were used in the Senior year. In 1742 elementary mathematics came to be removed from its august position in the curriculum as a Senior study. The rector of the college advised students to pursue a regular course of academic studies in the following order: "In the first year to study principally the tongues, arithmetic, and algebra; the second, logic, rhetoric, and geometry; the third, mathematics and natural philosophy; and the fourth, ethics and divinity." In 1777 the following texts were used: Freshman class, Ward's *Arithmetic;* Sophomore class, Hammond's *Algebra,* Ward's *Mathematics;* Junior class, Ward's *Trigonometry,* Atkinson and Wilson's *Trigonometry.*

Thus at Yale algebra occupied a place in the upper classes until about 1742, after which time it was introduced in the Freshman class.

Entrance requirements at Yale.—The college preparatory schools gradually took over the elementary mathematics formerly taught in the colleges. *In general the same courses that had been offered in the colleges came to be included in the curricula of the preparatory schools and academies.* Proof of this is not wanting. In 1824 the mathematical course at Yale was: Freshman, Day's *Algebra* during the first two terms, with no mathematics for the third term; Sophomores, six books of Playfair's *Euclid* during the first and part of the second term, and Day's *Mathematics* (including plane trigonometry, logarithms, mensuration of solids and surfaces, isopherimetry, navigation, and surveying) and Dutton's *Conic Sections* during the rest of the year; Juniors, Dutton's *Spherical Trigonometry*, Enfield's *Astronomy*, and Vince's *Fluxions;* no mathematics in the Senior year. In 1848 the Freshman course was Day's *Algebra*, Playfair's *Euclid.* But compare this with the entrance requirements of 1845: arithmetic and Day's *Algebra* to quadratics. In 1855 the entrance requirements were increased by the addition of two books of Playfair's *Euclid.*

It is *important* to note here that these courses were the *identical* ones previously taught in the college. The *same textbooks* were used. Again, in the requirements of 1887 we see that arithmetic, algebra as far as logarithms in Loomis, and all of plane geometry were included, *showing that both the algebra and geometry previously taught in the college have been dropped down bodily into the preparatory schools: the high schools and academies.*

A study of the mathematical courses and the entrance requirements at William and Mary, Dartmouth, and other of the early colleges only adds to the certainty of conclusions to be drawn from our foregoing study.

Conclusions concerning position of high-school algebra.—First, in the earliest history of the colleges algebra was *not even a college study.* Secondly, from its position as a later college subject it gradually became a Freshman subject, always studied prior to geometry. Thirdly, with the growth of the academies and secondary schools both algebra and geometry, but algebra *before* geometry, became college preparatory subjects. Fourthly, algebra was moved *bodily* from the college curriculum to the high-school curriculum; the *same texts* were used that had been used in the colleges. The authors of the college texts (in most cases college professors of mathematics, whose chief interest was in a logical, detailed exposition of the content) thus determined the course of study to be pursued by young and immature pupils in secondary schools. A study of the content and organization of these courses will indicate to what extent they were adapted to the capacities and mathematical needs of the "learner."

Algebra having become an entrance requirement for the colleges, we shall now try to find its position in the curriculum of the preparatory schools. Little evidence was available on this point. However, it is known that in the English Classical School in Boston in 1821 arithmetic was the only mathematics taught in the first year, and that both algebra and geometry were taught in the second year. Mathematics (the exact kind is not given) was also a part of the third or last year of that school. Thus it is certain that from its position as a Freshman college subject it did not drop immediately to the first-year high school.

These considerations lead us to conclude that we have today very much the same course in our ninth-grade mathematics that was formerly offered in the colleges. More evidence for this conclusion will now be brought to the reader's attention by an analysis to the *content* of these early courses.

II. THE DEVELOPMENT OF THE CONTENT OF ALGEBRA

It was the first aim of this chapter to determine the *position* of algebra in the curricula of the American schools. Our remaining task is to trace the change in the content of the course itself as it has gradually been sifted down to our present ninth grade from its former position as an upper-college subject. To do this we have selected a large number of algebras published at different periods since 1751. An attempt was made to select, as far as possible, texts that had been widely used and were therefore typical of the practice of their time. In an analysis of these texts, by determining the order of introduction of topics, the relative amount of space devoted to each topic, the relative number of *verbal* and *formal* problems, the *nature* of the presentation of new topics, etc., it is possible to follow the development of this subject. The texts with their dates of publication are listed in the Appendix.

Order of introduction of topics.—A study of Table 5 shows *marked* agreement among most texts on the order of introducing certain operations or topics, and an *equal degree* of disagreement on others. Almost invariably the "learner" gets his first glimpse of the subject of algebra by reading several pages of definitions, axioms, the use of notation, etc. Typical of these are:

> Algebra is a specious arithmetic, or an arithmetic in letters; it consists of addition, subtraction, multiplication, division, involution and evolution, etc., or it is the art of abstract reasoning upon quantity, by general and indefinite representations, in order to resolve problems, invent theorems, and to demonstrate both. Addition, Rule 1st: If quantities are alike and have the same sign before them, to be added together, put down the sum of the coefficients, with the common sign before them, and the common letter after them. . . . . [1]

or

> Mathematics is the science of quantity. Anything which can be *multiplied*, *divided*, or *measured* is called *quantity*. Thus a line is a quantity, because it can be doubled, trebled, or halved; and can be measured by applying it to another line, as a foot, a yard, or an ell. Time is a species of quantity, whose measure can be expressed in hours, minutes, and seconds. But color is *not* a quantity. It cannot be said, with propriety, that one color is twice as great as another. The operations of the mind, such as choice, thought, desire, hatred, etc., are not quantities. They are incapable of mensuration. Arithmetic is the science of numbers. Algebra is a method of computing by letters and other symbols. Geometry is that part of mathematics which treats of magnitude. . . . . [2]

This continues for twenty pages, including a discussion of *propositions*, *problems*, the influence of mathematics on the reasoning powers, and an explanation of most of the terminology of all of the operations of algebra!

There were, however, a few *notable* exceptions to this general practice. Let us examine the first page of Joseph Ray's *Elementary Algebra*. A series of "Intellectual Exercises" extending over twenty pages is introduced by:

> 1. I have 15 cents, which I wish to divide between William and Daniel, in such a manner that Daniel shall have twice as many as William; what number must I give to each?

[1] Carr, *Algebraist's Companion*.

[2] Day, *Introduction to Algebra*.

Or

Int. to eq for No..
Definition tion....
Add., sul div....
Fractions.
S. equatio
Involution
Evolution
Radicals.
Simult. eq
Quad. eq.,
Quad. eq.,
Cubic and
Imaginari
Arith. and
Binomial t
Theory of
Permut. a
Interest, a
Indetermi
Ratio, pro
Infinite se
Logarithm
Evaluatio
Number t
Generaliz relation
Negative
Constructi
Variation.
Mathemat
Special pro
Factoring.
G.C.D. an

If I give William a certain number, and Daniel twice that number, both will have 3 times that *certain number;* but both together are to have 15 cents; hence 3 times a certain number is 15.

Now, if 3 times a certain number is 15, one-third of 15, or 5, must be the number. Hence William received 5 cents, and Daniel twice 5, or 10 cents.

If, instead of a certain number, we represent the number of cents William is to receive, by x, then the number Daniel is to receive will be represented by $2x$, and what both receive will be represented by x added to $2x$, or $3x$. If $3x$ is equal to 15, then $1x$ or x is equal to 5.

The learner will see that the two methods of solving this question are the same in principle; but that it is more convenient to represent the quantity we wish to find, by a single letter than by one or more words.

In the same manner, let the learner continue to use the letter x to represent the *smallest* of the required numbers in the following questions.

NOTE.—x is read x, or one x, and is the same as $1x$, $2x$ is read two x or 2 times x. $3x$ is read three x, or 3 times x, and so on.

The reader should compare this first glimpse of, or introduction to, algebra with that of Day, previously quoted, or with the introductions found in many currently used texts. This innovation, made first by Warren Colburn, of introducing the pupil to algebra by a large number of "intellectual exercises"—a series of problems requiring reasoning from the pupil, and only that simple, "intuitive" use of the equation which pupils beginning algebra can actually make—was, from the standpoint of a real psychology of learning, *one of the most significant contributions to the teaching of algebra. It is even doubtful if present practice can compare favorably with the results of this remarkable insight.*

Equally certain is it also that the "learner" then had to devote himself to the four fundamental operations. Of all the texts examined, twelve introduce these operations next. Table 5 shows an increasing number of pages devoted to this work in the later books.

Involution and evolution formerly followed the four fundamentals, but later lost their early position and came to be introduced immediately before quadratic equations. *Fractions* were generally fifth in order of topics presented. In the earliest books *they were of a very simple kind, never requiring factoring, or a complicated lowest common denominator.* Following the next topic, radicals, came simple equations. Thus not until the "learner" had studied all *six* of the fundamental operations with their applications to fractions did he meet the EQUATION at all. Certainly neither the *equation* nor *functionality* was the "core about which the subject is organized." In this respect great progress has been made. Today the pupil meets the equation early in his course, and constantly recurs to it throughout his study. This, however, does not imply that even yet sufficient emphasis is placed upon the equation.

At this point we should notice the growth in the content of the course. Note from Table 5 how many topics are not treated at all in the earlier texts: evaluation, graphs, formulas, special products, factoring, etc. Again, some topics have disappeared altogether from the later texts: solution of cubic and higher equations by approximation methods, permutations and combinations, interest and annuities, etc.

Note the position of evaluation.—Ray's text (1848) was the first (of those examined) to recognize it as a specific process or operation. Present practice seems not to have recognized the importance of evaluation, either for its pedagogic value or for its occupational use. The report of our later investigation in chapter VIII shows this to be true. To be sure, it is found in most texts, but not in such form as to contribute to the mastery of the concept of functionality.

The reader may inquire how algebra could be presented without the presentation of negative numbers, for the table indicates that, as such, negative numbers were not generally recognized. We find the first reference to them in Colburn's *Introduction to Algebra* (1830): "They arise from the necessity of expressing subtraction by a sign, because it cannot actually be performed on dissimilar quantities. They are only positive quantities, *subtracted*, and in their nature they differ in nothing from positive quantities" (p. 112). Before this time no discussion was given of negative numbers. They were defined as *subtractive* numbers, and not as numbers representing opposite qualities. In chapter II it was seen that positive and negative numbers are now formally introduced and discussed early in the current texts.

It is interesting to note that *special products and factoring assumed their present proportions somewhat recently*. They first appeared as separate topics in Ray's text (1848), but received slight attention. *Beginning with Wentworth's "Elements of Algebra," 1881 (first edition, 1871), great emphasis has been placed on this type of manipulation.* In the preface of the foregoing book the author says, "Particular attention should be paid to the subject of factoring; for a thorough knowledge of this subject is requisite to success in common algebraic work." *In that edition sixteen types or cases are presented.* It appears from this study of texts that the present emphasis on factoring has developed since that time. A *good* reason for this extensive use we cannot give; certainly it is not a fundamental mathematical notion. Surely this emphasis cannot be justified.

The special characteristics of the most widely used of these texts as determined by the authors' prefaces.—In the preface of their texts the authors frequently set forth their conceptions of the chief *raison d'être.* Let us examine these to trace, if possible, any development of the content or method of presentation and organization of the subject. Bonnycastle (1806) writes:

> The following compendium is formed entirely upon the model of those writers [Newton, Maclaurin, Sanderson, Simpson, and Emerson] and is intended as a useful and necessary introduction to them. Almost every subject which belongs to pure algebra is concisely and distinctly treated of; and no pains have been spared to make the whole as easy and intelligible as possible. A great number of elementary books have already been written upon this subject; but there are none which I have yet seen but what appear to me to be extremely defective. Besides being totally unfit for the purpose of teaching, they are generally calculated to vitiate the taste and mislead the judgment. A tedious and inelegant method prevails through the whole, so that the beauty of the science is generally destroyed by the clumsy and awkward manner in which it is treated; and the learner, when he is afterwards introduced to some of our best writers, is obliged to unlearn and forget everything which he has been at so much pains in acquiring.

Evidently this author was primarily concerned in planning an introduction to algebra that would measure up to the standards of the great mathematicians. An entirely different point of view is that of Warren Colburn:

> The first object of the author of the following treatise has been to make the transition from arithmetic to algebra as gradual as possible. The book

therefore commences with practical questions in simple equations, such as the learner might readily solve without the aid of algebra. This requires the explanation of only the signs plus and minus, the mode of expressing multiplication and division, and the sign of equality, together with the use of a letter to express the unknown quantity. These may be understood by any one who has a tolerable knowledge of arithmetic. All of them except the use of the letter have been explained in arithmetic. To reduce such an equation requires only the application of the ordinary rules of arithmetic; and these are applied so simply that scarcely any one can mistake them, if left entirely to himself. One or two questions are solved first with little explanation in order to give the learner an idea of what is wanted, and he is then left to solve several by himself.

The most simple combinations are given first, then those which are more difficult. The learner is expected to derive most of his knowledge by solving the examples himself; therefore care has been taken to make the explanations as few and as brief as is consistent with giving an idea of what is required.

When the learner understands the purpose of *representing known quantities* as well as *unknown, by letters* or *general symbols*, he is considered as fairly introduced to the subject of algebra, and ready to commence where the subject is usually commenced in other treatises. Accordingly he is taught the fundamental rules, as applied to literal quantities. Much of this, however, is only a recapitulation in a general form, of what he has previously learnt in a particular form.

This same point of view is emphasized again by Sherwin (1855). Quoting:

The great difficulty, in the study of algebra, is to attain a clear comprehension of the earliest steps. The first principles should therefore be communicated to the learner gradually, and in the most simple and intelligible manner.

Experience proves that these principles are most successfully taught by means of easy problems. But even when this mode is pursued, a majority of pupils find trouble in expressing algebraically the conditions of the problems. The author has therefore placed at the commencement of his work a series of introductory exercises, designed to familiarize the learner with representing quantities and performing the simplest algebraic processes, also to prepare him for putting problems into equations.

Again, quoting from Davies' *Algebra* (1855):

Although algebra naturally follows arithmetic in a course of scientific studies, yet the change from the methods of reasoning on numbers to a system of reasoning entirely conducted by letters and signs is rather abrupt, and not unfrequently discourages the pupil.

In this work it has been the intention to form a connecting link between arithmetic and algebra, to unite and blend, as far as possible, the reasoning on numbers with the more abstruse methods of analysis.

This emphasis on a gradual, well-graded transition from arithmetic to algebra and upon the importance of specific practice in the use of letters for numbers is somewhat stronger at the present time. However, the currently used texts that succeed in doing this as well as those three early texts are certainly in the minority.

The text as teacher: or the introduction of new principles and rules.—Perhaps in no other respect has there been as marked improvement in the organization of subject-matter in the texts as in the manner in which new principles and rules are developed. Formerly the rules of procedure were stated abruptly and dogmatically, with no attempt to show their reasonableness or the way in which they were derived. Note, for instance, the *first statements* in the chapter on "Subtraction" in Ward's *Mathematics*.

Subtraction of whole quantities: Subtraction of whole quantities is performed by one general rule: Change all the signs of the subtrahend, or suppose them in your mind to be changed; then *add* all the quantities together, as before in addition, and their sum will be the true remainder, or difference, required.

This extremely unfortunate method of introducing new topics prevailed quite generally. Our records show that Warren Colburn's was the first text to employ an inductive development of rules of procedure. With the exception of Ray's, no other text used this improved technique of inductive development of rules as successfully as Colburn for nearly a half-century. In our present texts there is a more definite attempt to develop all rules inductively by studying a number of particular cases for the basis of the generalization. Thus, again, there has been some advance from the point of view of a real psychology of learning, in the way the texts help the pupil through the development of rules and principles.

The extent of the problem-material and the relative number of verbal and formal problems.—It is surprising to find that the earliest texts offered the "learner" no practice in the solution of examples, either of a formal or of a verbal nature. *They were reading courses.* The demands upon the mental activity of the pupil consisted in following explanations and illustrations of theorems and problems. Skill in the formal processes was no aim in the early texts. To show the extent of the "example" material for the pupil to master, and the relative emphasis on verbal and formal examples, Table 6 has been prepared and arranged in chronological order. It is assumed that the total number of "pages of examples" of each type (verbal and formal) will be roughly a measure of the relative emphasis on the two.

TABLE 6

NUMBER OF PAGES OF EXAMPLES TO BE SOLVED BY THE PUPIL IN NINETEEN ALGEBRA TEXTBOOKS

	Formal Examples	Verbal Examples	Per Cent of Formal Examples
Carr	0	0	0
Ward	0	0	0
Frend	0	0	0
Manning	0	0	0
Bonnycastle	40	16	71.5
Simpson and McClure	0	0	0
Ryan	65	24	73.0
Lacroix and Farrar	0	0	0
Colburn	45	37	54.9
Day	32	17	65.3
Pix	19	0	10.0
Ray	44	42	51.2
Loomis	46	24	65.2
Perkins	39	16	70.9
Davies	55	44	55.6
Sherwin	36	46	43.9
Stoddard and Henkle	53	51	51.0
Thompson and Day	49	25	66.2
Wentworth	100	47	68.1

The present-day emphasis on the formal example seems to date from Wentworth's "Elements." In that text the author states that about four thousand examples have been included. Conspicuous for their emphasis on the reasoning phases of algebra are Colburn's, Ray's, and Davies' texts.

Graphing. The reader is surprised to find no mention of "graphic representation" in any of these texts. The answer is that, even as an isolated operation, "graphing" did not appear until 1899. Cajori states that to George W. Evans, of Boston, belongs the credit for first introducing "graphing" into secondary-school algebra. (He notes, however, that it was even then introduced as an isolated operation.) Our first notice of it was in Well's *Algebra*, 1902, which contains a supplement on "graphs," prepared by Professor Robert J. Aley. Since that time there has been a steady growth in the emphasis placed on "graphing" as a component part of the *subject-matter* of first-year algebra. The emphasis has *not* recognized it as a fundamental *method* of algebra, however. The extreme conservatism of mathematics teachers is well illustrated by their reluctance to incorporate this graphical material in their classroom work. It is not unfair to say that the mathematics teachers are so thoroughly imbued with the thought that the chief value of mathematics is the discipline which is derived through an organization of material which satisfies the logical requirements of the adult trained mind that they have been loath to admit concrete and practical materials. It should be pointed out here that the first use of concrete materials to illustrate the laws of algebra was made by Stoddard and Henkle in 1861. They illustrated subtraction by finding the distance between two points, the minuend and the subtrahend, on a *distance scale*. These early algebras were unfortunately weak in the failure to use concrete material, such as space forms and motion, for their illustrative work. It goes without saying that great progress has been made in this phase of the work.

Formulas. Here too there has been a desirable growth in the content of high-school algebra. Our study of the earlier books shows no treatment of formulas. Again, from the point of view of occupational use and also as an excellent means of illustrating and building up the concept of dependence or functionality the formula is almost of first importance. We cannot account for the omission of formulas and graphs from these early books except by saying that the writers did not recognize the utility or social value of the subject-matter.

Summary.—To sum up, we have shown where algebra came from and by what route it got into the ninth grade. The college professor of mathematics, in the main, designed it; the college first provided it a setting. It gradually came to be regarded as a tool or preparatory subject and settled down bodily through the grades of the college and the secondary school. It has nearly always been taught prior to demonstrative geometry.

Thus the college, through its faculty, has designed the curriculum, not only for the college, but for the secondary and elementary schools as well. Now college men are primarily interested in "logic," not in "psychologic." In mathematics as well as in other subjects they have been interested to write books, to organize courses of study, from the standpoint of the adult trained mind, which sees in mathematics not the step-by-step "learning process" of the child (with his difficulties and groping methods of acquiring knowledge of, and facility in the use of, the operations in question), but logically arranged "fundamental concepts." Cataloguing and classification of these concepts have replaced his interests in the problem of "learning" of the child. Thus the standardization of mathematical courses of study in this country has been a logical one, seldom a psychological one.

The result of all this has been too that we have had formal, logical classification of subjects as well as of courses of study, and these claim prestige primarily on the grounds of logevity. This has led to what may fairly be called a traditional or a horizontal cross-sectional method of subdividing the content of mathematics. "Arithmetic," "algebra," "geometry," "trigonometry"—these are the cue notions that mean to the child mathematics, instead of "the equation," "ratio," "measurement," "congruence," "similarity," "functional dependence," "scale," "variation," etc. Our course is broken up into horizontal cross-sectional chunks of subject-matter, whose name implies to the child a distinct difference in content which is belied by careful mathematical as well as psychological analysis of the content itself. For example, we find that in arithmetic, algebra, analytical geometry, trigonometry, etc., "ratio" is a fundamental notion and tool which is utilized in each one. To the average child, however, even should he have been so fortunate as to have been taught ratio in each one of these subjects of study, the fact that it is the same, identical notion and operation, but made use of in new ways and related to new connections, has not been made evident. This illustration will serve to show that we have been more concerned with the cutting up of our subject-matter into portions that could be fitted into the graded scheme of the public and higher schools than with the seeking out of the fundamental notions and tool operations which are needed to facilitate thinking about quantitative things and the organizing of these in sequential courses reaching from the lowest levels of the elementary schools to the highest levels of our colleges and universities. Careful investigation has shown that public-school teaching of mathematics is characterized more by the former, and almost not at all by the latter, procedure.

This discussion must have by this time called the attention of the reader to the fact that a *fundamental issue* underlying the present organization of our courses of study is that concerned with *the need for a complete overhauling of the content of mathematics from the standpoint of a psychologically and sequentially worked-out scheme.*

CHAPTER IV

THE DESIGN AND CONSTRUCTION OF STANDARDIZED TESTS IN SECONDARY MATHEMATICS

The place of "tests" in our program.—If one wishes to make an inventory of the efficiency of instruction in any subject, he naturally will start with the construction of a *test* with which to measure the results obtained. The attempt to organize such a test will shortly make evident the need for classifying clearly the *subject-matter* of the course for which the test is being designed. That is, the very attempt to plan specific measures of the outcomes of instruction will raise in the mind of the investigator various issues concerning the forms of classification in which his subject-matter ought to be arranged. The attempt to systematize the content of the course, however, will reveal the imperative need for a clean-cut statement of the *aims and outcomes of instruction* in that subject of study.

This is exactly the order of development of the steps of the program upon which the writers began work five years ago. The *first* need was for *tests*. The design of the tests necessitated a classification of the component parts of the course of study, and the latter in turn compelled the enumeration of the objectives and outcomes of the course of study. With the investigation relatively complete, however, it is now possible, and probably wise from the standpoint of the reader, to show the *improvement of instruction* in any subject and the systematic analysis of the course of study by means of quantitative methods *must* necessarily *involve the utilization of a fivefold program. The writers believe that the carrying through of this entire fivefold program is necessary if any movement for improvement in our teaching situation is to be relatively permanent in our generation.* Furthermore they are interested to set forth the program at this point, and in some detail, because of the applicability of its various steps to the improvement of the situation in any subject of study in our public schools.

I. THE FIVEFOLD PROGRAM FOR THE IMPROVEMENT OF INSTRUCTION IN ANY SCHOOL SUBJECT

First step: The determination of the general aim and outcomes of the subject in question.—Chapter VII will be devoted to a rather detailed discussion of this necessary *first step* in the analysis of secondary mathematics. It will be our task at this time, however, to show the very great importance of a detailed statement of the "general aim" or objective underlying the instruction in each subject of study, and of the specific *statement of outcomes* from that instruction. At the start we must distinguish clearly the difference between "objective" and "outcome." Nothing is more important to the teacher, to the administrator, or to the "educationist" critic than the writing out of the general aim of the subject of study in question so as to include a very clear and minute analysis of what the instruction in the course is intended to do. The writing of this detailed statement is one of the most difficult tasks that teacher or administrator can be called upon to do.

For example, a careful analysis of the aims of instruction in each of the high-school subjects made by Professor Koos's committee of the North Central Association referred to in chapter I and supplemented by the analysis made by the writers in 1915–16 reveals clearly that one of the fundamental aims or objectives underlying instruction in each subject in the high school is the *generalized one* of *training in "logical thinking."* The specific wording may be different in the various statements, but they almost universally imply the development of abilities of generalization—of analysis, of discrimination, of comparison—an emphasis upon the formation of abilities of clear judgment, etc. This points out rather strikingly that the aims of *each* course, in addition to being partly designed to satisfy the social or occupational needs of children in later adult life, fundamentally must be determined in psychological terms. This calls to mind again the two important criteria upon which we shall debate the organization of a course of study: the sociological criterion and the psychological criterion. The former insists that in our high-school courses of study there shall be a definite recognition of the social needs—the specific life-needs of the boys and girls in each of the activities in which they are engaged. In addition there shall also be a clear recognition of the leisure-time needs. The latter criterion—the psychological one—requires that subject-matter shall be included and organized within the curriculum in terms of a careful psychological analysis of each of the mental processes called into play.

It is in this connection that we shall turn our attention in our more detailed discussion especially to the question of an analysis of the outcomes of instruction. Once having stated the general aim of the course, we need to follow this up by a careful classification of the outcomes of instruction made in psychological terms. This must be carried out to the extreme form of a detailed statement of the specific habits set up, of the facts memorized, of the principles which are to be fixed; also an analysis of the problem of new types of situations which the pupil is expected to be able to handle successfully at the end of the course. *Not until this detailed classification of habit-formation, memory-responses, powers of logical thinking, etc., is made for each of the subjects of study shall we be able to make a helpful and permanent contribution to the teaching of each one.*

Second step: The clear-cut classification of the subject-matter of the course.—In this brief outline of the fivefold program we shall pass over with scant mention this second step. The previous chapter has outlined the necessary steps in the standardization of the course of study, and has set forth in detail the conclusions from a careful investigation of the present status of our course in mathematics. The last two chapters in the monograph will outline the necessary procedure in the design of the course of study.

Third step: Design and conduct of tests which will adequately measure ability in each of the fundamental phases of the subject-matter agreed upon.—The statement of the aims and outcomes of instruction, when followed by a detailed analysis of the content of the course of study, paves the way for the design of tests for each of the fundamental operations. The teaching of the subject-matter in the course must be checked up from week to week and from month to month by perfectly objective tests. No step in the procedure of improving instruction in any subject of study is of more importance than this one of devising a method for measuring efficiency in the formal processes and in their application in reasoning situations. We shall give, in succeeding sections, a detailed *distinction* between various types of tests. Our classification will draw the line between *tests for intelligence* (for "central" abilities involving reasoning processes) and *tests which measure specifically the outcomes of training*. We shall distinguish between various types of tests for formal processes (for example, "rate" tests) and those for "reasoning" abilities. We shall discuss in detail the principles of design of the various kinds of tests. We must call attention here then merely to the importance of this third step in the procedure of standardizing the teaching of a subject of study.

Fourth step: The critical evaluation of the results of testing pupils' abilities.—In order to give a complete and differentiated statement of fundamental weaknesses in learning (for example, as revealed by the typical errors made by pupils), we shall make clear that standardized tests are of value to the teacher primarily in that they reveal, if properly designed, fundamental weaknesses in learning and mistaken emphases in teaching. The distinction will be drawn between *accidental errors and recurring errors*, both types of errors being of decided importance to the teacher. Definite suggestions should be made to show how accidental errors (those due to misreading, miswriting, hurry, carelessness, etc.) can be partially eliminated in class instruction. We have gone far enough in our analysis, however, to establish clearly that the fundamental need of the teacher is to have in her hands *before the instruction is given* a detailed list of typical recurring errors made by pupils, with each operation involved in the subject-matter. Knowing the errors that children most commonly make, she will organize her instruction from the initial presentation through the use of drill for the automatization of the formal process in such a way as to emphasize the difficult portions of the material. "*Differentiated teaching emphasis*" is a phrase which should be common to the educational thinking of school teachers generally. The writers wish to stress here the belief that the *determination of fundamental difficulties* (through the compilation of typical errors) which are encountered by children *in each subject of study is the primary business of investigators in this field*. The technique involved in this particular study clearly is applicable to any other subject of instruction. Not until school men generally put on record, and in the hands of the teacher, a clear discussion of the particular portions of the subject-matter which must be emphasized, and then supplement this by specific devices for the carrying out of the procedure will they be doing their duty to the young and untrained teacher.

The evaluation of the results of testing leads to a clear distinction between methods of individual instruction as contrasted with methods of class instruction. The use of tests has led us in this investigation to recognize clearly the different approaches which may be made to the problem of remedying pupils' mistakes. Certain types may be remedied by mass-instruction. *From the standpoint of economy of time* the teacher needs a formal device which, with classes revealing as great individual differences as do our public-school classes, will enable her to give, with a relatively small expenditure of time, wisely planned drill upon difficult operations. We shall show later the importance of taking advantage of our knowledge of pupils' mistakes by the design of *formal practice exercises which will concentrate practice upon the most important recurring errors.*

Fifth step: Experimental teaching.—If the investigation of the aims and outcomes of instruction, of the content of the course, of the design and use of standardized tests, of the evaluation of methods of learning, and of the design and use of practice exercises to improve the efficiency of learning has been carried on completely, the investigation should lead into a careful "experimental-teaching" program within the classroom. We should point out here that *never in the history of the teaching of any method of study has this fivefold program been carried out.* During the last fifteen years of the quantitative movement workers in the special fields have carried through one or more of these steps. They have designed tests, they have used them; there has been some design and use of practice exercises (for example, in arithmetic). But no detailed analysis of learning based upon all of these types of scientific material has ever been included in the systematic experimental teaching of the subject in question under controlled conditions.

There is a tendency to be noted throughout the country to question the validity of the measuring movement (at least that aspect of it which is stressing the design of standard tests), largely because of a feeling that the movement is not going to "eventuate." There certainly has been a tendency evident to design tests for the sake of designing tests, and it is clear that many of the protagonists of the movement have not seen the implications of the use of the test. It will be one of the important theses of this monograph to show that the four years of work which was done upon the *first three steps of this fivefold program were entirely preliminary to the critical evaluation of results and to the setting up of a program of experimental teaching, through which only it is possible to make complete use of standard tests to improve teaching.* We cannot emphasize too strongly, therefore, that our fifth step is after all our most important one. It is probably the one upon which school men will be engaged from generation to generation: careful critical analysis of each aspect of the teaching situation through the carrying on of controlled experiments in teaching and administration. It seems quite clear that this most promising movement, therefore, must be reinforced by a clear statement of the specific directions in which it can and must eventuate. It will be the business of the present writers to report in chapter IX in detail the setting up and conduct of such a teaching experiment. We may point out in brief at this point the extent to which this step of the program has already been carried through in our classes. The tests have been given and the results worked up in more than one hundred school systems. The remedying of the errors which were recognized in the evaluation of these results has been attacked from the standpoint of mass-instruction by the design and use of these formal "practice exercises." These in turn are now being used in more than two hundred school systems. Many of our larger cities have adopted them for use in all of their classrooms.

The most effective use that can be made of them, however, is in connection with the careful investigation of the best methods of presenting the materials of ninth-grade mathematics, of the proper introduction of drill exercises, of the most effective length of drills, and frequency of recurrence of such drills. We shall report later the detailed way in which six classes have been taught under carefully controlled conditions by the two writers, each observing the teaching of the other daily throughout the school year 1917–18. The report of the details of this program will reveal the validity of the conclusions to which we have come. The course of study which has come out of the result of this investigation of educational needs, of the presentation of the content of first-year mathematics, and of a careful psychological investigation of learning has been taught under varied conditions in these classes. Elaborate stenographic notes have been kept of the work of each class each day. The aims of the experimental work have been to investigate not only the optimum time for reviewing drill exercises with each of the various operations of ninth-grade mathematics, but also the best methods of presenting to students on a perfectly rational basis each phase of the learning of such subject-matter. Difficulties revealed in the conduct of one recitation were immediately experimented on in the next, with complete records kept of the various methods of trial, failure, and success.

Thus the *outcome* of the carrying on of the entire program ought to be, first, *a course of study* organized in terms of special needs and also in terms of a very careful psychological analysis of learning in the subject in question; second, a clear statement of the *objectives* underlying the instruction of this course *and* a detailed classification of the *outcomes of learning* in the form of habits to be established, skills to be set up, information to be acquired, reasoning abilities to be developed, etc.; third, a systematic "standardized" test with which the teacher can measure the outcomes of instruction, both of a purely formal nature and of a reasoning nature (both the capacity to react efficiently with a specific operation and an ability to use these operations successfully in meeting new situations; in this connection tests for the segregation of children in order of ability which will measure roughly the "quantitative intelligence" of the pupils in question); fourth, *corrective devices* for the improvement of purely formal abilities which can be shown to be economical methods of perfecting skills; fifth, and finally, a detailed statement of classroom method, aids of all sorts to the teacher in presenting material, in habitualizing the formal skills, and in rationalizing each aspect of the subject-matter.

THIS CHAPTER IS PRIMARILY A DISCUSSION OF STANDARDIZED TESTS—THEIR FUNCTION, PRINCIPLES OF DESIGN, AND METHODS OF CONSTRUCTION

II. WHY USE STANDARDIZED TESTS IN SCHOOL STUDIES?

A school system must have *machinery for classifying children* in order of ability, machinery of which it can make most effective use at the beginning of a school term; *machinery* likewise, *one of whose functions is the measuring* of the degree to which the pupils have mastered the materials which have been put before them. The latter function is important in that adequate diagnosis of their weaknesses and strengths can be made—in order that we can detect and act upon, accordingly, the difficulty of various aspects of our subject; *finally* that we have at hand an objective method of crediting or *marking student work.*

The improvement of technique in each of the sciences has always waited upon the design and construction of measuring instruments. Astronomy developed a factual basis and laws and principles only with the invention of the lens and the construction of the telescope. The science of physiology and the place of Harvey in the scientific hall of fame was made possible primarily by the microscope. The measurement of reaction time, and all the items in the long list of psychological functions which are now being *measured,* waited in the same way upon the design of *measuring* instruments. And so the improvement of the technique of teaching school processes is dependent upon the adequate design of *instruments with which to measure the results of teaching* and administering them.

Three outstanding functions do we find, therefore, for standardized tests: (1) at least in order of use, the function which they serve as devices for classifying or grading children for the purposes of mass-instruction; (2) a function of *diagnosis*—the location of children's weaknesses through the objective evidence of the errors in their own work; and (3) a "*supervising*" *and marking function*—that which leads to an objective method of evaluating the work of teacher and pupil. The "system" under which we live requires us to be continually checked up, measured, and credited for what we do. So in school is necessitated the establishment of an adequate system of "prods" and "pressures" to keep us speeded up to our optimum level of efficiency. That the system of crediting student work in current use is in need of an overhauling is becoming very clear to school men and critics of school practices alike.

1. The need of tests for the purpose of classifying children.—Let us discuss these a bit more in detail. Two groups of persons in the school world have a critical need for *classificatory* tests—the school administrator and the school teacher. The school administrator's need stands out in his evident lack of tools for "ordering," or classifying children in order of ability, in the various grades of the city schools. Already we are seeing important experiments carried on in our more progressive school systems in the "segregation" of children in order of ability. It has been pointed out in the previous chapter that among the striking reasons *why pupils fail* is found a combination of several conditions: large classes, short teaching periods, a *very wide range of ability in single classes*, and lack of sound methods of detecting grades of ability which are present in our classes, and of cataloguing the particular weaknesses of the various children. These experiments in segregation are quite properly based upon the point of view that we cannot hope to do much in our generation about the first two of these conditions, but that the third and fourth are quite amenable to rapid and economical revision. In other words, the wide range of ability within classes, that is, the present heterogeneity evident in our classes, can be transformed into a state of relative homogeneity provided we have placed at our command *adequate tools for measuring ability*. Thus the administrative argument for the establishment of standard tests for classification purposes appears to have much weight.

But for the teacher too the detection of "who's who" in her class is a first step toward the successful teaching of that class. Two things would be revealed by tests, whether it was her own subjective test or a carefully standardized test. First, she would "find out" her children and know which were the strong and which were the weak, and would know furthermore the particular strengths and weaknesses of each one. In addition, provided the test were properly designed so as to be representative of the various aspects of her subject-matter, she would know in general that in the handling of such and such kinds of material the difficulties which would be encountered by the typical pupil would be "thus and so."

In saying "tests for classification" we mean classification in order of *ability*. Recall our discussion in chapter I of capacity, ability, and performance, especially that our *school tests measure performance*. Now, to the extent that the test is an adequate measure, has been given under normal conditions, and has been repeated, to that extent shall we be able to *infer ability from performance*. Note, therefore, that these classification tests must *not* be performance tests to the extent that they are based primarily upon knowledge of the sort represented in the subject to be studied. In other words, we shall need a test in this connection for "central" abilities, not one dependent upon facts peculiar to the field of learning about to be studied, one furthermore that presents to the child "problems," and demands on his part generalizing and analytical functions.

2. Need for tests for diagnosis.—Classification tests—tests for mathematical intelligence—will place in our hands pertinent information concerning the individuals in our classes *prior* to the time of their beginning the study of ninth-grade mathematics. As far as the conduct of the course is concerned, however, the *diagnosis function is by far the most important.* The teacher is primarily interested in standardized tests because they reveal specifically difficulties in learning, and point out definitely needed changes in teaching emphasis.

We are primarily interested in ways and means of improving instruction in first-year algebra—only incidentally interested in establishing standards or norms of attainment. The norm of attainment is of value only as it spurs the teacher to a high standard of class teaching; the test is of value only as it points out principal typical difficulties that pupils face in learning algebra. If it is properly designed, that is, if it measures specific processes, it can be made to do this.

Pupils' errors in first-year algebra.—Pupils make two principal kinds of errors: (1) accidental errors (i.e., errors of reading, writing, following directions, arithmetic, etc.), and (2) recurring errors. Recurring errors supply a means of determining exactly which types of problems, operations, or processes pupils have not mastered, provided the tests are rigidly designed on the "cycle" or "rotation" principle. Thus *in diagnosing difficulties in "learning" it is the recurring errors that are significant.* They reveal which types of operation have not been mastered. The Standardized Tests submitted herewith, designed as they are on the principle of recurring types of operation, reveal exactly the typical difficulties which pupils encounter.

3. **The need for tests with which to set norms of attainment.**—The measuring movement, as it has developed to the present time, has caused to stand out in the minds of school people the standardizing, or norm of attainment, aim. It is probably true that most educational testers and users of tests recognize in the possibility of setting up "norms" or "standards" and of comparing teacher with teacher, school with school, and system with system, the primary end of the new technique. Undoubtedly this does represent a critical need, primarily to the superintendent and principal, in checking up teachers for the purposes of future supervision, promotion, readjustment of the salary schedule, etc. Likewise the administrator recognizes in the possibility of crediting the work of the pupil and of promoting him as a result of objective measurement a most important tool for the improvement of his practice.

The supervisory and marking "values" of tests.—The supervisory and marking "values" of tests are revealed, first, in the commonly felt need to exert artificial pressures or "drives" over human beings in order to keep them up to optimum efficiency. There is little doubt that each one of us needs to have quantitatively stated goals, professional hurdles, set before him against which his developing effectiveness shall be frequently checked up. As American schools are administered, superintendents exert very little influence over the classroom by means of direct supervision. It is rare indeed that a superintendent or even a principal observes the teaching of individual teachers on his corps often enough in a given school year to form an adequate judgment of its effectiveness. Lack of recognition on the part of boards of education of the need for a proper distribution of executive functions—the business and routine as distinguished from the educational or truly supervisory functions—has developed in this country what might be characterized "office" or "long-range" supervision. At best any supervisory influence that is exerted is indirect; hence any tool to make this indirect supervision more pertinent to the improvement of teaching in classrooms (which ordinarily never feel the direct influence of the superintendent) will be welcomed. Thus we are finding a rather ready acceptance of the use of standardized tests by administrative officers because of the larger degree of semi-direct supervision which they permit over the classroom work.

Likewise the teacher recognizes the supervisory values in tests. She is now able to replace her vague feeling of success or failure, most commonly of course the former, with definite knowledge as to whether her instruction has resulted in as large an achievement on specific aspects of the work as has that of a large group of "best schools" or classes. Instead of being told by a supervisor who has observed her teaching once or twice in a year that it, on the whole, is "poor," "good," or "indifferent" (and probably thereby resenting the comment as being founded upon a teaching experience and maturity of judgment that is not as good as her own), she is compelled to recognize that her students have equaled, exceeded, or fallen below a record of achievement that was agreed upon by herself and by a large number of other teachers as both possible and acceptable in progressive school teaching. One of the very important effects that has come from the *adoption of quantitative methods* in school practice has been that the use of standardizing methods, the establishment of quantitative norms, and the use of these in comparing our own achievements *make us "do something about it."* It seems quite evident, therefore, that to the teacher as well as to the superintendent there are important supervisory values in tests.

We have already referred at various points in our discussion to the improvement that is coming about in the operation of the marking system in public schools. We shall merely add a final comment to this brief enumeration of the functions of tests that are now making possible the adequate measurement of ability in children. We, in our generation, shall live to see it relatively uncommon to find teachers failing or assigning a mark of "90 per cent" to one-third, one-fourth, or even one-sixth of our student population. The distribution of scholastic ability as determined by school marks will clearly change from the badly skewed type which it is at present to one of a reasonable symmetry.

III. THE TYPES OF TESTS IN WHICH VARIOUS MENTAL PROCESSES ARE MEASURED

Accompanying our threefold classification of the purposes of standardized tests, we shall set forth next a brief enumeration of the various types of tests. It is convenient to classify our tests, first, from the standpoint of the kinds of mental processes involved in the measurement. The discussion of classification tests which was given above calls attention, first, to tests for "central intelligence." Accompanying these to satisfy the functions of diagnosis and marking are two important types of *school training tests* which we shall refer to from now on as (1) *formal tests* and (2) *reasoning tests*. In our discussion of the outcomes of instruction, as the second step in the fivefold program, we brought out the distinction between the immediate specific and preparatory outcomes and the immediate generalized outcomes. Paralleling this distinction in outcomes is the contrast in the two corresponding types of test. For the skill or automatic processes we have what we shall call *formal tests*. For the measurement of the generalized outcomes from instruction we have what we shall call *reasoning tests*. The reader should remember that these will measure *performance*, and that from the repeated use of them we are able to *infer ability*.

Not only should tests be distinguished in our thinking from the standpoint of the types of mental processes which they measure, but it would be helpful to classify tests from the standpoint of the principles upon which they are organized. For that reason let us distinguish between "rate" tests and "development" tests.

THE DISTINCTION BETWEEN RATE TESTS AND DEVELOPMENT TESTS

The rate test.—Courtis, in his testing work, distinguishes two kinds of tests—the rate test and the development test. It will be helpful to us to use his terminology. We shall, however, define from our own standpoint the connotation of these two words. By "*rate*" test we first imply a "time" test. It will be helpful to think of a rate test as being distinguished from a development test on the basis that the latter makes use of the "time" factor either only incidentally or not at all—for example, Woody's Arithmetic Scale, Thorndike's Algebra Scale, Starch's Arithmetic Scale, or Stone's Reasoning Tests in Arithmetic. We shall point out later that the time element is used only as a rough check upon the amount of material which can be worked by the student.

The second distinction between the rate test and the development test is found in connection with the grouping together of like or unlike kinds of subject-matter. For example, in Thorndike's Algebra Scale he has put together in one test containing only nine problems examples of a purely formal and automatic type (for example, if $a=4$ and $b=3$, what does $a+b=$?) and examples of a most complicated reasoning type—problems involving many steps, partly of an analytical and comparative nature and partly of a purely formal nature. By the rate test, on the other hand, we shall have in mind only those types of tests in which the ability involved in the working of any one problem is roughly the same as that involved in the working of any other. For example, any one of the writers' algebra tests involves a formal skill of manipulating a particular kind of algebraic operation (for example, evaluation, the removal of parentheses, clearing of fractions, solution of quadratic equations, etc.). It is true that there are included in any one test problems in which the ability involved in solution is not exactly the same as that involved in the solution of others in the test. But the same general type of mental process is involved in the working of all of these examples; hence this second fundamental distinction between the rate test and the development test.

The third distinction is found in the organization or arrangement of the parts of the test. In the rate test we always do one of two things: either we put together in one test operations or kinds of problems of exactly the same nature (for example, a series of problems all involving the addition of similar fractions, or a series of problems all involving factoring by completing the square, etc.), or we combine in one test a group of problems involving very closely related abilities. Whether we do one or the other in the construction of the *rate test* the difficulty of each example of the test has been determined, and the examples have been arranged in terms either of a rotating or of a "*cycle*" principle, or all problems of like difficulty have been put together. In the *development test*, on the other hand, although the difficulty of each example has been determined as in the case of the rate test, the examples are arranged in order of increasing difficulty. Furthermore *different kinds of abilities are measured in the same test.* For example, in the Arithmetic Scale designed by Woody it will be noted that the problems are arranged in such a fashion that the first is the easiest and receives the lowest score if correct, that the others are arranged in accordance with increasing difficulty, and that many kinds of operation are put together in the same test.

The pedagogical significance of the rate test and of the develment test.—To what degree does each type of test lead to helpful classroom use? Recall here that standardized tests are being designed and used for two purposes: first and primarily, they are being adopted by school people generally for use in their classes because they reveal very specifically the particular difficulties that pupils meet in the given subject-matter, and that thereby they enable the teacher to take advantage of such facts in the improvement of her own teaching. As has been pointed out in our earlier report, they *point to weaknesses in "teaching emphasis" as well as to particular difficulties that pupils meet.* The second and subordinate function of standard tests was pointed out to be an administrative one: that of being able to establish standards of achievement and norms of attainment against which the work of any teacher, any school, or any system of schools can be checked. In this case we are not immediately interested in particular problems of teaching except as we may use the standards which have been set up and the averages of achievement which have been determined for the pupils of different classes to trace back in the case of particular classrooms to the immediate causes for the degree of success or of failure which has been revealed.

Having in mind clearly, therefore, that *standardized tests* to be of real service to the school system should be used *primarily as* *measures of learning and for teaching diagnosis*, we can now distinguish again between the rate test and the development test. The rate test offers opportunity to the pupil to do the same particular kind of an operation several times. The development test, on the other hand, offers opportunity for the working of only one example of a particular type. In this connection, therefore, the rate test is seen to be far superior to the development test in the degree to which it enables us to put our fingers upon the specific difficulty of the pupil.

This can be made still clearer by emphasizing the distinction in the kinds of errors made by pupils. We have distinguished two classes: *accidental errors* and *recurring errors*. Pedagogically the most significant errors made by pupils are the *recurring ones*. It is these recurring errors which are revealed by the rate test and are not revealed by the development test.

IV. PRINCIPLES OF DESIGN OF STANDARDIZED TESTS IN MATHEMATICS

What are the essentials of a standardized test?—First, the test must be objective rather than subjective; that it must be relatively external to the individual judgment of a teacher or specialist in the field. It must be objective in the same sense that the use of a foot rule in measuring linear distance is objective as contrasted with the use of subjective judgment of length. Secondly, a standardized test *in "problem" courses* must be built with a clear recognition of distinct differences in learning various kinds of subject-matter. It will not, for example, confuse memoriter with reasoning work, habit-formation with more complex types of mental activity. Fundamentally it will take account of the fact that we have two large groups of outcomes from school studies—the formal or automatic group and the reasoning or generalized group. Thirdly, a standardized test is one that is built so specifically in accordance with the organization of the subject-matter that the parts of the test will fit definitely its clearly recognized and more important aspects. For example, each of the specific tests given in the Appendix measures ability with a given algebraic operation. In the fourth place, we will agree that a standardized test must have been designed upon carefully established principles. The subject-matter must not be put together in a haphazard fashion, as most teachers' tests or examinations are, but carefully, in terms of some predetermined principle of organization and arrangement. Fifthly, there must be worked out for it a standardized method of scoring, or giving credit to the results obtained by using it, which will be accepted as an adequate measure of performance in the skill in question. Sixthly, the relative difficulty of various portions of the test must have been determined by giving it to large numbers of children in many representative schools throughout the country. The weights or scores that are assigned to different portions of the test must therefore rest upon carefully determined principles and hypotheses concerning intellectual ability. Finally, in the seventh place, the test must have been given under carefully standardized conditions with regard to timing, uniformity of directions, and classroom procedure in general.

V. PRINCIPLES OF DESIGN OF FORMAL OR RATE TESTS IN ALGEBRA

Based upon the foregoing principles the writers have carried through to completion the design and construction of a set of 16 rate tests for the formal abilities, and have determined the difficulty of many verbal problems to enable teachers to build development "reasoning" tests. We recognize as a definite gap in our procedure thus far the fact that we have been unable to standardize a set of tests for "classification" purposes—tests for what might be called "quantitative intelligence." Such tests are being designed—in fact have been experimented on since September, 1917—and will be presented to teachers and students of education shortly.

To complete our discussion of design of the two types of tests we need a more detailed exposition of our procedure.

The formal tests, as designed and presented herewith, conform to the following requirements:

a) They are made up of a series of example tests each of which is designed as a specific test for a definite, formal operation in algebraic manipulation.

b) Each specific test is made up of a number of examples (10 to 28), each of an elemental nature and involving the operation in question, and for each of which the degree of difficulty has been determined carefully by experimentation.

c) *The cycle principle of design.* Because it is impossible to arrange separate tests for all kinds of operations involved (owing to lack of time in classroom handling, etc.), those examples which involve closely related operations are grouped in one test and arranged rigidly in rotation. Thus the student solving 20 examples may be compared with the one solving 10 examples.

d) Each test is designed as a time test, the time being so arranged (by experimentation) that no student can quite finish the test in the time given, but that all can do a considerable number—otherwise the measure of efficiency will be too coarse. Care is to be taken to see that all pupils start and stop the test at the same instant.

e) The directions are all given orally by the experimenters, so that differences in rate of reading and in comprehending directions will not complicate results.

f) Test examples are of the alternative sort, i.e., they are designed to give either right or wrong answers—otherwise careful evaluation and weighing of answers will be necessary.

Time tests of formal processes in any subject of study must be designed in accordance with the *cycle principle, or rotation of examples*. The various ways in which the symbols, letters, etc., may be arranged for a given type of operation should appear in exact rotation in the test. For example, in Test I there are six principal ways in which parentheses examples may be "arranged," i.e., considering the use of the + and − signs, ( ), letters, etc. These appear in Test I in such order that in the first, seventh, thirteenth, nineteenth, and twenty-fifth examples the signs and symbols occupy the same relative positions, i.e., the examples involve the same algebraic and mental processes. Our research shows that it is of the utmost importance that this "cycle principle" be followed in the most rigorous fashion. The importance of this point is indicated by the differences in difficulty that are revealed by Table I, in the original report.[1] This shows how a few failures

[1] "Standardized Tests and the Improvement of Teaching," H. O. Rugg and J. R. Clark, *School Review*, XXV (February and March, 1917).

to follow the principle exactly caused quite different percentages of failure on the part of pupils in solving the examples.

Attention should be called to the fact that the validity of other algebra tests which have been drawn up without regard to this principle should be seriously called in question. For example, the Indiana Algebra Tests (based upon the Standard Research Tests devised by W. S. Monroe), sent out from the University of Indiana, have been made up in such a way that the efficiency of pupils solving a given number of examples on any test cannot be validly compared with the efficiency of pupils solving half as many, a third as many, twice as many, etc. To illustrate this point we reproduce Test II of this series of tests.

1. $4(3x-4)=$
2. $-5x^3(4x-1)=$
3. $-7(2+3x)=$
4. $-5(-4+6x)=$
5. $3(-1+6x)=$
6. $-4a^2(8x+4a^5)=$
7. $-7(-5x+8=)$
8. $-8x(-3x-5a^3)=$
9. $6(2-4x)=$
10. $-x(5-6x)=$
11. $-3(9+x)=$
12. $-5(-7x+3)=$
13. $-5(-4x-6y)=$
14. $-6y^2(-9-7x)=$

Certainly no definite principle of design controls the placing of examples in this test. Our results show that *this is an essential step that must be followed if we are to have sound criticism of school practice in these matters.* It can be suggested from the results of our investigation that these problems, many of which are several times as difficult as others (example 1 compared with examples 2, 6, 8, or 14, to illustrate), are not put together in such a way as to lead to comparable results in testing pupils.

The construction of tests to measure the abilities of pupils should be based upon the most scientifically worked-out research principles. Recognizing the urgent need for care in such work, we have checked the "cycle principle" of rotating examples in two different years of experimentation. The results are given in Table I in our original report for readers interested in this phase of the work. This table gives the percentage of all of those pupils (over 2,500 took the various tests) who attempted each problem in each test, and who failed to work the example in question correctly. In this investigation it has not been possible to inquire in detail into the effect of "practice" in working the recurring examples of the cycle. We believe that it will operate in the more difficult tests to give a gradually decreasing percentage of failures on successive examples of a particular type. Table I has been made up and examined carefully for the purpose of discovering which examples in each test are not roughly equal in difficulty to corresponding examples in other cycles. An example in which a distinctly larger or smaller percentage of failures is found has been replaced by another. Careful study of such problems, in almost all cases, has revealed peculiarities in construction, *or in scoring*, that cause an example to be thrown out. For instance, in example 3 in Test VI the percentage of failures is 3.5 per cent as compared to from 30 per cent to 36 per cent in all other corresponding examples. Study shows that the example is so constructed that we cannot determine by an inspection of the pupil's answer whether the mental process is correct or not. The particular error made by students in this example is that due to "adding exponents instead of multiplying them"; $(n^2)^2$ gives the same answer regardless of the process, and thus the work of the pupil cannot be diagnosed to find out whether he is "right" or not. This method of analysis has been applied to each example in each test, and the accompanying list (Table I in the original report) of corrections is given. The tests were reprinted the next year with these corrections made. We believe that they now thoroughly justify the title "Standardized Tests in First-Year Algebra," and that they may be used by teachers to check up specifically the ability of their pupils in the formal operations.

VI. PRINCIPLES OF DESIGN UTILIZED IN THE CONSTRUCTION OF DEVELOPMENT TESTS

In connection with our problem of classifying the various types of tests and of distinguishing the mental processes involved in the outcomes of instruction which these tests are supposed to measure, we need at this point to draw carefully the distinction between the two fundamental methods upon which *development tests* are now being designed. The first method we shall call the *teacher-judgment method.* The second method we shall call the *proportion-of-pupils-solving method.*

A. The teacher-judgment method.—Thorndike, in 1910, made the first use of the teacher-judgment method in the design of his Handwriting Scale. He subsequently used the same method in the design of his Algebra Scale and of the Hillegas-Thorndike Drawing Scale. The method assumes that the difficulty of problems, or that the merit of the handwriting sample, of the English composition or what not, may be determined by the consensus of judgment of "competent judges." Competent judges are taken to be persons who have taught the subject in question, or who have supervised instruction, or who have made special studies of learning and teaching the subject. In general, the method regards teachers as competent judges. Hence the title, the "teacher-judgment method."

The method, furthermore, rests upon a very fundamental assumption, namely, that *equally-often-noted differences are equal.* The clear grasp of this assumption is essential to an understanding of the validity of design of standardized tests by this method. To explain it in detail, let us show the way in which Thorndike applied it in the construction of his Algebra Scale.

The procedure in this case involved the selection of a list of examples (25 in number, *made up of both formal and reasoning types*, and varying from very easy to very difficult). The list of 25 examples was submitted to a group of 200 teachers of mathematics with a request that each one arrange the problems in order of difficulty from easiest to hardest. That is, the problem that was regarded by the teacher as the least difficult was to be ranked 1, the next least difficult, 2, etc., the last problem in the list being regarded as the most difficult to solve. Note that the use of this method implies that the teacher can introspect on the mental processes utilized by pupils in solving examples—pupils whose "mental content" is widely different from that which she herself has developed—and that she can determine differences in difficulty for as large a number of examples as 25. It further assumes a group of judges who are very closely homogeneous with respect to ability to do such a thing.

Having secured the judgment of each of 200 teachers on the order of the problems as to difficulty, Thorndike now brings in the principle of "equally-often-noted-differences." He applies it in the following way: He searches through his table and selects those examples which, in the opinion of approximately the same percentage of the group of teachers, were more difficult than one another. To make the matter clearer he found that example A, for instance, was ranked by approximately 75 per cent of the judges as more difficult than example B, that example B was also ranked by approximately 75 per cent of the judges as more difficult than example C, example C was also ranked by approximately 75 per cent of the judges as more difficult than example D, etc. In other words, about the same percentage of the judges, namely, 75 per cent (the difference recognized by 75 per cent is used by Thorndike as the *unit difference*), recognize differences between A and B, B and C, C and D, etc. The use of the principle of "*equally-often-noted-differences*" implies that A is as much more difficult than B, as B is than C, as C is than D, etc., *because the same percentage of competent judges recognize the difference.* This principle, which has been used in the construction of a number of educational tests, was first enunciated in an article by Fullerton and Cattell in 1898, and to the writers' knowledge had not been made use of in the construction of mental or educational tests until Thorndike made this specific application.

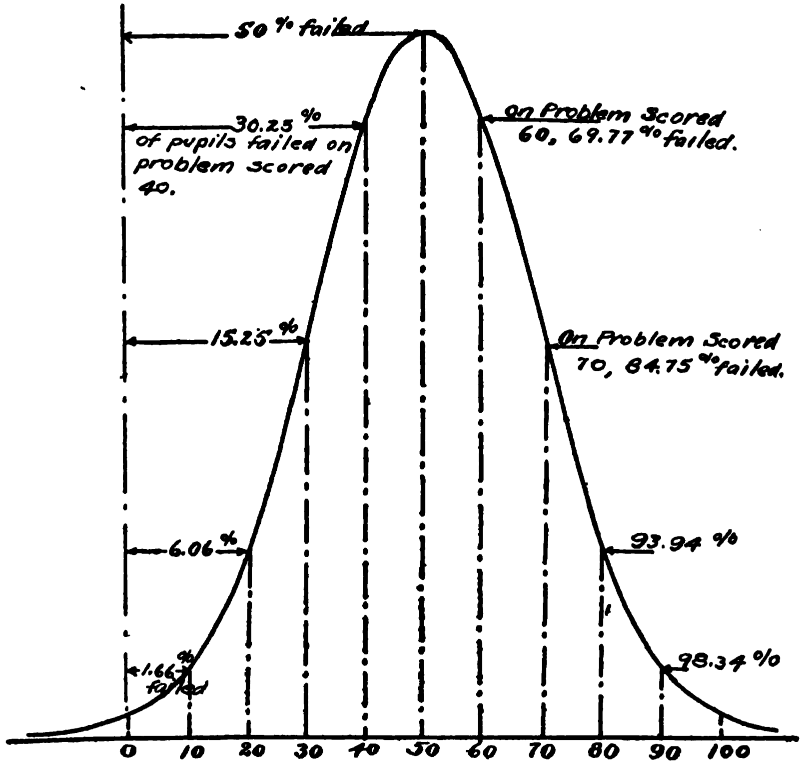
Using this method results then in a series of examples or samples of student work (in handwriting and in composition, for example), the intervals between the examples and samples being regarded as approximately equal. The method is, of course, inapplicable in the case of those examples which either nearly all the judges recognized as being more difficult than others, or almost no judges recognized as being more difficult than others, and in the case in which exactly half of the judges recognized one example as being more difficult than another, and the other half reversed the decision.

B. The proportion-of-pupils-solving method.—The name of this other method indicates what it is, namely, that the difficulty of examples or the merit of any types of student work will be determined approximately by the proportion of pupils solving the example or doing the work. For instance, instead of determining the difficulty by asking some teacher (a competent judge) to tell how hard the example is for the child, we turn to the child himself and ask him to show us how hard it is by working it. We assume at this point that in a very large group of children equality in the difficulty of the example will be represented by equality in the proportion of the children who are able to solve it successfully. One other important assumption is necessary at this point, namely, the one concerning the form of the distribution of ability among the pupils involved in the investigation. The specific way in which this method is used in the design of development tests has been shown in detail in our original report. Recall here that the form of intellectual ability ascribed to the general pupil population is that which conforms roughly to the normal probability curve.

We are now able to put the two assumptions together and apply our method. Assuming that a given proportion of the children, for example, 90 per cent, are able to solve a given example successfully, and that another proportion, for example, 45 per cent, are able to solve another example correctly, the inference does not follow that the former is twice as difficult as the latter. On the other hand, these percentages must be referred to the normal probability curve which was assumed to represent the distribution of intellectual ability. The base line of this curve, it must be remembered, now represents the scale of difficulty (or of merit in the case of handwriting or English composition) upon which we are going to distribute the problems in our scale. The position of any one problem then is determined by finding out how far over on the base line one has to go from the zero-point at the left to include a percentage of the area of the curve equal to the percentage of children who solved the example. The distance from the zero-point over to this point on the base line is regarded as representing graphically and numerically the score that will be assigned to that example. We reproduce Diagram I from the original report to make clearer the procedure.

The relative validity of the teacher-judgment method and the proportion of pupils-solving method were checked up during the first year of the investigation on the standardization of first-year mathematics by making a comparison of the results obtained by

Thorndike's method with those obtained by having high-school students solve the same list of examples. The 1916 report[1] gives



Scale of Algebraic Difficulty

DIAGRAM I.—Distances on the base line represent, to scale, relative difficulty of problems. Area under the curve represents total number of pupils that were tested for ability to translate verbal problems. 0 and 100 points are arbitrarily at $2.5 \times \sigma$ from the mean. Mean is set arbitrarily at 50. Area of the curve between 0 and any point on base line represents percentage of pupils who failed the problem placed at that point.

the results of this comparison. Space will not permit their repetition here. Suffice it to say that teacher judgments of the relative difficulty of examples, even when aided by quantitative enumerations of like and unlike processes, cannot be regarded as valid for

[1] H. O. Rugg, "The Experimental Determination of Standards in First-Year Algebra," *School Review*, XXIV (1916), 37–66.

the construction of "example-scales" of definitely evaluated units of difficulty.

SAMPLE DEVELOPMENT TEST

Standardized Tests in First-Year Algebra (Verbal Problems)

Rugg and Clark

(Graded on an absolute scale of difficulty from 0 to 100, in accordance with two assumptions: (1) among first-year high-school students the distribution of algebraic ability approximates the "normal" probability curve; (2) the difficulty of problems varies as the percentage of pupils able to solve the problems correctly, superimposed on the base line of this curve.)

Score	Verbal List A
85	A train running from Chicago to Denver at average speed of 40 miles an hour takes 3 hours longer to make the run than one running at 45 miles an hour. What is the distance from Chicago to Denver?
80	If a cistern can be filled by one pipe in x minutes and emptied by another in $x+5$ minutes, what part of the cisternful runs in one minute if both pipes are open?
75	Find two numbers whose sum is 51, such that if the greater is divided by their difference, the quotient is $3\frac{1}{2}$.
70	Twice the width of the Pennsylvania station in New York exceeds its length by 80 feet. 4 times the length exceeds the perimeter by 700 feet. Find dimensions.
65	If a boy $4\frac{3}{4}$ feet tall casts a shadow $4\frac{1}{2}$ feet long at the same time that a school buildng casts a shadow $57\frac{1}{2}$ feet long, how high is the school building?
60	A father 54 years old has a son aged 9 years. In how many years will the age of the father be just 4 times that of the son?
55	Two boys play at teeter. One weighs 100 pounds and sits 6 feet from the point of support. The other weighs 120 pounds. How far from the point of support must he sit in order to make the board balance?
50	What number has the property that when multiplied by $\frac{1}{3}$ the result is greater by 1 than when multiplied by $\frac{1}{4}$?
45	If the width of a rectangle is W increased by 10 and its length L increased by 20, write the equation for its perimeter.
40	8 times a certain number equals 45 diminished by the number. State the equation by which you would find the number.
35	If W and L are the width and length of a rectangle, write the equation for its area in terms of W and L.
30	If you represent a number by x, how will you represent 5 more than 5 times the number?
25	Express the following verbal statement in algebraic form: the square of a side plus five.

CHAPTER V

WHAT THE STANDARDIZED TESTS REVEALED

I. SKETCH OF PROGRESS OF ENTIRE INVESTIGATION

The investigation of methods of standardizing first-year mathematics, now in its fifth year, began at the 1913 meeting of the Mathematics Section of the University of Illinois High-School Conference. At that time a committee was instructed to begin the study of establishing standards by which the results of one year's high-school algebra might be measured. The problem as attacked by the chairman of that committee (Mr. R. L. Modesitt, of the Eastern Illinois Normal School) consisted of an attempt to determine the relative degree of difficulty inhering in a set of mixed algebra examples (i.e., a list containing both formal and verbal problems) as shown by the varying proportions of a group of 50 pupils who were asked to solve the examples. This attempt, resulting in an adaptation of Thorndike's Mixed Scale, was to be extended during the next school year. Other duties compelling Mr. Modesitt to give up the work, the carrying on of the investigation was turned over to one of the present writers.

The investigation of 1914–15.—The procedure of the preliminary investigation during the year 1914–15 may be outlined briefly as follows:

1. A preliminary statement of the aims and outcomes of the teaching of high-school algebra.

2. The determination of the basic method of designing and constructing tests for measuring efficiency in first-year algebra (the investigation of the validity of the assumptions underlying Thorndike's Mixed Scale concerning (*a*) the relative difficulty of examples, and (*b*) the validity of testing pupils with a scale composed of both formal and reasoning examples).

3. Classification of the subject-matter of first-year algebra and the determination of a list of specific operations whose efficiency should be tested. It should be remembered that during the first

two years of this investigation the organization of the subject-matter which gave the operations for which tests were being designed was based entirely upon the *currently accepted* "fundamental" operations, that is, those represented by currently used textbooks. It was only with the beginning of our third year that we were able to turn our attention to the thoroughgoing quantitative investigation of the curriculum—of what *ought* to be taught.

4. A preliminary statement of principles which should govern the selection of test examples. The statement made in chapter IV is our final statement based in large part upon the working out of the preliminary plan in this year of 1914–15.

5. Selection of example material with which to build each test with preliminary individual experimentation to standardize the arrangement and timing of these examples. This first set of tests included only eight operations. In the preliminary statement of principles the plan known as the "*cycle principle*" was tentatively agreed upon. The arrangement of the examples only made partial use, however, of this principle. *In the subsequent work tests have been designed rigidly in accord with it, however.*

6. Final organization of test sheets and the giving of tests in 8 school systems in Illinois according to standard directions. This work was entirely voluntary on the part of the co-operating schools, was done by progressive school teachers under the direction of one of the present writers, and made a large contribution to the early improvement of the testing situation.

7. The correction of the test papers, the original tabulation of scores obtained, the statistical treatment of results with an attempt to state typical conditions, and a minute analysis of the specific errors made in each example. A table was prepared in this early report showing the detailed results in numbers of examples "attempted" and "right" for each of the 8 schools. No attempt was made with this year's work to set up standards, the inquiry being regarded as decidedly preliminary to such an endeavor. Consideration of the results obtained, however, showed clearly the condition of inefficiency that prevailed in practically all school systems in instruction in first-year algebra. Neither in the development of skill with the purely formal operations nor in the develop-

ment of ability in pupils to meet new reasoning situations could the attainment of the pupils be regarded as satisfactory. It was recognized at this time that the succeeding years of research must be directed at improving a critical condition revealed under the caption "*a vague teaching emphasis.*" The test papers were scrutinized minutely, and the errors made by the pupils were classified somewhat roughly in terms of mental processes involved in the different operations. The study made still clearer the need for the use of scientific principles in designing tests, in order to be able to form a clear and complete statement of the typical weaknesses in learning first-year algebra to which children are subject.

The investigations of 1915–16 and 1916–17.—A new statement was made of the fundamental operations, ability in which should be tested. These were now extended to include three more than had been used in the previous year's work, making eleven in all. Remember that the list was still built to fit existing conditions in the course of study as determined by the textbooks. This year, however, saw the important step taken by the committee of extending the program for determining what *ought* to be taught in a one-year "required" course in mathematics. The writers saw clearly that testing and evaluation of results was not the sole crux of the problem of teaching first-year mathematics, but that a complete quantitative program would have to be laid down for the building of a course of study on a scientific basis. The program that was initiated at this time included the careful determination of the extent to which the operations of first-year algebra as currently taught were used in other high-school subjects. It was also planned at that time, with the compilations in this portion of the study, to extend the investigation to include the definite analysis of the occupational and leisure-time mathematical needs of boys and girls in their later life. The complete report of this aspect of the investigation, however, will be given in chapter VIII.

The new statement of fundamental formal operations led to the redesigning of the tests. The "cycle principle" was elaborated and applied rigidly to the building of each test. Three new tests were added, making 11. This third revision of the standardized algebra tests was submitted to teachers of algebra during the spring of 1916. Twenty-seven school systems co-operated by giving the tests in their classes at the end of the school year under carefully standardized conditions. As a result another minute analysis was made of the content and organization of each test. To the specific arrangement of signs, symbols, and number data in each test was fitted rigorously the "cycle principle."

In 1916–17 still further revisions in these examples were reported to the public on pages 11 to 15 of our 1917 report. These changes were all made, and 3 new formal tests were added, making 14 in the regular list and 2 supplementary tests. This fourth revision and the third reprinting of 20,000 copies was published and distributed by the School of Education of the University of Chicago in May, 1917. This distribution to co-operating schools has since that time been done at wholesale cost. During the months of May and June, 1917, more than 10,000 sets were distributed. This 1917–18 edition has now been given to more than seventy-five school systems. Scores have been made up, and standards of attainment have been established. In a short report[1] published after the original report, appeared the following information regarding these tests:

> In their complete form the tests [which are reprinted in full in Appendix A] cover sixteen formal operations. In addition there are two preliminary tests which can be given prior to the formal testing to familiarize pupils with the routine of taking "time" tests. The formal tests are printed in compact, easily handled form, as follows: Tests I–IX inclusive and X–XIV inclusive are printed in two $4\frac{1}{2}\times 7$-inch booklets, and each booklet is planned so as to fit a class exercise of forty minutes. Tests XV ("Graphs") and XVI ("Quadratic Equations with Irrational Roots") are supplementary tests on single sheets that can be given to those pupils who have completed that work. The verbal tests are in the form of lists of "translation" problems, the difficulty of each of which has been scored, and which can be put together easily, in a test, by the teacher. The tests have now been made to include each of the fundamental operations, shown in list on page 347 [of the May, 1917, *School Review*], which have to be mastered in the present first-year algebra.
>
> The price of the tests has now been set at 4 cents per set. This price merely covers the expense of printing and handling. The tests have been designed primarily in the interest of improving the teaching of algebra. No

[1] H. O. Rugg and J. R. Clark, "A Co-operative Investigation in the Testing and Experimental Teaching of First-Year Algebra," *School Review*, XXV (1917), 346–49.

profit is being made on them. In ordering tests, order a set for each pupil. A set includes (1) Books I and II, containing Tests I–XIV; (2) Supplementary Tests XV and XVI on single sheets; (3) Preliminary Tests A and B. With each order will be included in sufficient quantities (1) complete printed directions for giving and scoring the tests; (2) printed score sheets for records of individual pupils; (3) printed class-record sheets on which class averages can be computed easily; (4) tentative standard scores obtained with these tests in 27 school systems; (5) lists of verbal problems which have been scored for difficulty.

Book I		Book II	
Test No.	Operation	Test No.	Operation
1	Collecting terms	10	Fractional equations
2	Substitution	11	Practical formulae
3	Subtraction	12	Quadratic equations
4	Simple equations	13	Simultaneous equations
5	Parentheses	14	Radicals
6	Special products		Supplementary Tests
7	Exponents	15	Graphs
8	Factoring	16	Quadratic equations (irrational roots)
9	Clearing fractions		

II. ESTABLISHMENT OF NORMS OF ATTAINMENT IN FORMAL EFFICIENCY AND EVALUATION OF RESULTS

That teachers and administrative officers need norms of attainment in school processes for the purposes of supervision and marking has been made clear in chapter IV. The conduct of our

TEST NO. 4

(Illustrative of the Set)

SIMPLE EQUATIONS

1) $2x+3=11$	Answer......	15) $21x-12-26x=14$	Answer......
2) $4c=6c+12$	Answer......	16) $4x+5=17$	Answer......
3) $5x-3=-20$	Answer......	17) $11s=13s+20$	Answer......
4) $13=2x-8$	Answer......	18) $9x-6=-40$	Answer .....
5) $12x-7-15x=10$	Answer......	19) $23=6x-9$	Answer......
6) $3x+4=16$	Answer......	20) $15x-11-21x=18$	Answer......
7) $8p=15p+14$	Answer......	21) $6x+3=33$	Answer......
8) $7x-6=-29$	Answer......	22) $12y=15y+12$	Answer .....
9) $17=3x-8$	Answer......	23) $10x-7=-33$	Answer......
10) $9x-8-13x=7$	Answer......	24) $21=8x-5$	Answer......
11) $5x+2=27$	Answer......	25) $18x-3-23x=9$	Answer......
12) $6t=9t+21$	Answer......	Number attempted................	
13) $8x-7=-35$	Answer......	Number right....................	
14) $19=5x-9$	Answer......		

investigation has now made it possible to set up norms that will be adequately representative for the schools of the country. From the detailed records of the results of the testing we give in the Appendix five typical sets of scores, namely, that of (1) the best school, (2) the average of the upper third, (3) the ninth school, (4) the average of all the schools, and (5) the lowest school. These five scores of each test both for accuracy ("rights") and speed ("attempts") show the range of attainment for representative algebra instruction, the best and the poorest attainment, and a norm which can be checked against by teachers of algebra.

The average attainment of the upper third of the schools as the norm of attainment in first-year algebra.—There have been many suggestions of late that the average of the entire group be taken as a standard practice. The writers are convinced that the average of the entire group sets up too low a standard. It is contributed to by many schools of a very low instructional efficiency. An "optimum" attainment, which, after all, is what our norm is, should be contributed to only by schools which, working under normal conditions, have maintained a high level of instructional efficiency. Thus in using the average of some upper group like the upper third we set a standard contributed to only by schools that have maintained this high level. To take the average of the upper third means to discount the *best* attainment (that is, of the highest school by allowing for possible unusual conditions which might lead to the very best score), and yet to set a standard that has actually been exceeded by many schools. In case any teacher wishes to compare her scores, however, with those of schools whose attainment falls at points lower down on the scale, she is able to do so by comparing with the score of the ninth school or with that of the poorest school.

III. EVALUATION OF RESULTS

The detailed tables showing the scores made by each school make clear the striking differences in the speed and accuracy with which pupils manipulate different formal operations. With only two of the fourteen operations can it be said that present methods of instruction lead to satisfactory automatism of these operations.

In the case of the removal of parentheses, in the use of exponents, and in the solution of simultaneous equations pupils have evidently reduced the manipulation to a high state of efficiency. Even in the case of exponents, however, typical practice reveals not more than an 85 per cent efficiency.

On the other hand, the score of the poorest school was but slightly more than half of the average of the upper nine. In fact, in all remaining tests (except those noted above) the score of the poorest school is less than half that of the average of the upper third. At some it is only one-fourth to one-sixth of the standard score. Such low levels, combined as they generally are with low efficiency in "translation" processes (see Table II in Appendix), *certainly call for closer supervision of instruction.* As pointed out in our 1916 report, they indicate, in some cases at least, a lack of teaching emphasis on certain operations with a correspondingly greater emphasis on others. This question of teaching emphasis in the case of very good and very poor schools is further elaborated by tracing a record of individual schools in each of the different operations. If this is done it will be found that the three poorest schools fall in the lowest third in all but two tests. The two best schools fall in the upper third in all but two of the eleven tests.

The scores for all the remaining tests, therefore, point out a startling deficiency in making automatic a skill necessary to success in the use of "algebraic" mathematics in applied problems. Note, for example, the original scores for Test 10 (fractions), for Test 12 (quadratics), Test 16 (graphs), Test 14 (radicals), Test 11 (practical formula), Test 13 (simultaneous equations). The upper group in each case shows what may be regarded as a fair degree of speed, but almost universally displays a woeful lack of accuracy. *It certainly will be agreed by teachers of algebra that instruction in these operations should lead to greater efficiency. Under such a condition recurring mistakes in a class would be impossible; accidental mistakes would occur, but would be very rare.*

The necessity for a high degree of efficiency in the formal processes.—*Teachers agree* that pupils must have *automatic* skill in manipulating the "tables" in arithmetic; that the spelling of "common words" shall be absolutely mastered (*automatized*), so that pupils will almost *never* make a mistake in spelling them; that a certain quality of handwriting shall be written by our pupils at a definite speed, and that it shall be done *automatically;* that pupils shall perfect certain "*automatic*" *habits* of "reading" in the early years, so that lessons in history and geography and literature shall not be turned into lessons in the development of the formal skill of getting meaning from the printed page.

In the same way, in order that the pupil may use successfully algebraic methods in the solution of verbally stated problems, he must have absolute mastery of the tool operations he is going to use in that solution. At least, immediately at the close of a year's instruction, a pupil should remove parentheses, factor, solve simple equations, use special products, exponents, radicals, etc., just as he uses the multiplication table, writes, spells, or gets meaning from written language—*in a word, automatically.* It is not economic or expedient to force pupils to raise to "thinking" or "reasoning" levels the formal manipulation of these purely tool operations. The writers are therefore insisting on thoroughness in the formal operations, in the interest of "*economy of time*" in *ninth-grade mathematics;* in other words, *in order that a larger amount of time may be spent in developing skill with the formal operations in solving "original" problems.* This report will show methods by which this may be done.

The reader should be cautioned that "automatism" in the more complex processes (e.g., fractional equations) does not necessarily imply the instantaneous reaction of the pupil with the completely worked-out "answer" to the example. In the case of problems containing but one or two steps, the automatic response *should* be the "answer." But in the cases of problems involving several steps, *automatism means the continuous unbroken reaction of the pupil with the proper steps in the solution.* The steps in the procedure of manipulation of the operation in question should be made completely a part of his automatic system of habits.

The relation between efficiency in the automatic process and efficiency in reasoning processes.—Teachers and administrators who have been loath to adopt "measuring methods" in their school practice and who have hesitated to introduce practice exercises do so on the ground that such a procedure will "mechanize" teaching. This same skepticism has been commonly expressed as a result of the 1915 report on algebra—the point of view of the writer and the implications of the report certainly having been misunderstood. The 1916 testing was therefore planned to take account of this question. Seventeen school systems took the eleven formal tests and also the verbal or reasoning tests. Table IV in our 1917 report gives the comparative ranks of each of the 17 schools in both types of test. To permit a composite comparison of ability of the two types, the rank of each of the 17 schools has been determined for accuracy in each of the eleven tests. A tabulation was then made to give the aggregate ranks of each school. The list was then reranked, calling the school that had the lowest number of aggregate ranks 1, the next lowest 2, etc., throughout the list. The final comparison of ranks is shown in Table IV.

TABLE IV

COMPARISON OF THE RANK OF 17 SCHOOLS IN EFFICIENCY IN MANIPULATION OF THE FORMAL OPERATIONS WITH RANK OF SAME SCHOOLS IN EFFICIENCY IN REASONING TESTS

School System	Rank in Aggregate Scores Made In		School System	Rank in Aggregate Scores Made In	
	Reasoning Test	All Formal Tests		Reasoning Test	All Formal Tests
A	1	1	[illegible]	3	6
C	9	9	M	8	11
F	11	5	N	4	14
Y	16	16	P	13	2
G	14	10	Q	6	4
I	10	13	R	12	7
J	2	3	V	15	17
Z	7	8	W	17	15
K	5	12			

The first, ninth, sixteenth, tenth, second, seventh, third, eighth, sixth, fifteenth, and seventeenth schools in the "reasoning tests" occupy ranks in the "formal tests" equal to these or not displaced by more than 3 places in 17. That is, 11 of the 17 schools stand almost in the same relative position in reasoning efficiency as in formal efficiency. Of the whole 17 schools only 2 show a distinct reversal of position in the two types, schools N and P. In view of the preponderance of evidence showing high positive correlation between these two types of processes we are unable to account for the status of the data on these two schools. Of the remaining 4 schools, G is displaced 4 places, R, 5 places, and K, 7 places. For those interested in correlations we may say that by Spearman's foot-rule method (with 17 cases any coefficient has little validity, hence we may as well use a rough approximate method as the more accurate product-moment method) $r=0.59$, which may be regarded as a high correlation.

Careful study of our complete data leads us to the conviction that those teachers who have been most successful in developing skill in manipulating the formal processes have also been most successful in developing ability to "translate" verbally stated problems into algebraic symbolism. We regard this as pertinent to the question of "mechanization" of the teaching process. . . .

We believe that it emphasizes the main point we are trying to make: namely, that with carefully planned practice periods, using scientifically built-up practice material, teachers will be still more effective in developing "original abiltiy."

Certainly, without the aid of scientifically planned practice devices, teachers who have been most successful in doing the one, have been, very generally, most successful in doing the other. We have no reason to believe, from our present detailed data, that teachers who drill their pupils to a relatively high state of "formal efficiency" do so by neglecting the higher thought processes in algebraic solution. The teachers, furthermore, have done this without the aid of carefully differentiated drill exercises. They have followed, in the main, the traditionally planned textbook organization of theoretical presentation and "problem exercises."

We take the position that much greater correlation might come about, then, through the aid of carefully designed practice exercises which put the chief emphasis in practice upon the difficult processes, saving time for more detailed training in those types of solution that call for analysis, discrimination, and other higher thought processes. Testing pupils' ability to manipulate algebraic operations does not lead to a general blind "mechanization" of the teaching process. It leads to an intelligent understanding of *what* operations ought to get the most drill and relatively how much they ought to be drilled—in a word, *differentiated drill takes the place of blind wholesale drill such as will ensue if one follows the textbook organization as the sole guide.*

The relation between speed and accuracy in "formal efficiency."—Table V in our 1917 report gives the rank of each of the schools in number of problems attempted and number right. The high degree of relationship between speed (represented by "number of problems attempted") and accuracy (represented by "number of problems right") will be evident to the reader. The schools ranking high in speed are very generally the schools ranking high in accuracy. In Test V, which we will take at random, the foot-rule correlation gives $R=0.70$ or $r=0.89$ by Spearman's transmutation formula. By supplying the detailed data in Table V we do not need to discuss the material in detail here. The conclusion is clear that speed and accuracy in these formal operations go hand in hand.

We feel that this point is of great importance in the methodology of teaching. From individual experimentation in the classroom, the writers are convinced that *speeding up class work conduces to greater accuracy in the manipulation of formal operations.* The results of testing pupils in 27 schools confirm our classroom experience.

Thus the teaching program that we would lay before teachers is contributed to by the findings of the last sections: First, *drill pupils daily in the formal operations by stressing the most difficult and the most important processes.* Secondly, drill pupils under "timed" conditions. *It should be made "second nature" to manipulate the different formal operations under the pressure of "time." Life-situations continually demand it—the school should constantly prepare for it.*[1]

[1] H. O. Rugg and J. R. Clark, "Standardized Tests and the Improvement of Teaching in First-Year Algebra," *School Review*, XXV (1917), 196–213.

CHAPTER VI

THE DEVELOPMENT OF FORMAL ABILITIES THROUGH DIFFERENTIATED DRILL: TEACHING SKILLS UNDER "TIMED" CONDITIONS

Why such inefficiency as an outcome of instruction as has been shown by the giving of tests? We have had occasion at various points in this report to point out two outstanding reasons: (1) an overloaded course of study and (2) an undifferentiated teaching emphasis. An important element in the latter is a lack of psychological insight into the "learning process" in secondary mathematics.

To refer briefly to the first reason in an illustrative way: Pupils cannot factor rapidly and accurately the more important types primarily because they have been forced to distribute their attention and energy over "17 types" or "11 types" (if they have been brought up on either one of two well-known texts), each operation of which has been so constructed as to inhibit the learning of any other! Furthermore the teacher of elementary algebra will agree, we are quite sure, that only two of these, namely, the monomial factor and the general trinomial, are really basic to the learning process in factoring. In addition, it should be pointed out that the learning of these two types makes the learning of all the remaining types, separately, a pure waste of time. However, chapter VIII will take up that issue more in detail.

In the second reason, an *undifferentiated teaching emphasis*, we are confronted with the primary question—a more or less blind adherence to the organization of the textbook. Most teachers make few or no omissions from the textbook. Addition to, or shifts in order of presentation of, the material are relatively rare. Explanatory material and exercises are assigned in the main just as the author has arranged the material in the book. Clearly this is the crux of the problem, for skill in manipulation is in direct proportion to the kind and amount of practice offered in the various types. Surely sanity in assigning problem-work must take definite account of the difficulty of the specific examples on the lesson. It is not too much to assert that no "exercise" of examples in mathematics ought to be printed in the textbook without the "proportion-of-pupils-solving" having previously been studied most carefully, and without the examples arranged specifically in accordance with such findings.

I. THE DISCOVERY OF IMPORTANT TYPES OF ERRORS

We wish to make clear in this report the great importance of recognizing clearly two outstanding types of errors, and furthermore of distinguishing clearly between them. The four years of investigational work showed that it would be very helpful to the teaching process to classify such errors of pupils as (1) *accidental errors* and (2) *recurring errors*. The improvement of instruction in a given subject naturally hinges on the problem of eliminating these types of errors.

1. Accidental errors. What they are and how to get rid of them.—An accidental error is probably due to intermittent concentration, to lapses of attention, to instances of mind-wandering—probably seldom to fatigue. It will be helpful, it is true, to image the distribution of susceptibility to accidental error in the pupils in our classes in terms of a distribution like that of the normal probability curve. Likewise it is true that there will be a large range of variability in the extent to which different pupils are subject to these accidental errors, the degree to which are developed attitudes of accuracy and stick-to-it-iveness, habits of continuous concentration, and tendencies to the overcoming of mental inertia, which causes so many of the failures in our school studies.

There can be no doubt that one of the fundamental causes of failure in any school study is mental inertia and its accompanying weakness—mind-wandering, day-dreaming. Our years of experimentation in the classroom teaching of first-year mathematics have shown clearly that one of the fundamental changes needed is that involved in the transforming of our class exercise from that of a "sluggish meandering" into that of spirited, continuous attentiveness to each aspect of the work carried on in the class hour. There are two fundamental steps in this process:

a) The thoroughgoing utilization of the practice of "*timing*" *all of our formal exercises.* No more clearly effective device for improving teaching has come out of the quantitative movement in education than that found in the recent adoption of the practice of *timing* the class exercises. Our investigations show clearly that all formal work, all work organized for the purpose of developing automatic skill in school studies, whether at the blackboard, in the working of problems from the textbook, or in the use of printed practice material, should be done under the stimulating pressure of "time." The practice of "holding the watch" on rapid-fire work should be relatively common in our classrooms. A direction from teachers which should be commonly heard is, "I will give you —— minutes," etc.

b) It will not be effective, however, to turn our class exercises into spirited, timed drills unless we recognize that in each class exercise, even as short as they now are, we must have variety of mental activity. The commonly heard criticism that the measuring movement is trying to "mechanize" instruction, that it is emphasizing the "formal at the expense of the "reasoning" element in our school studies, would have much justification if we rested our argument at this point. Our investigation again has shown that variety in planning of class periods is fundamentally necessary to get the most effective results. This means, for example, devoting *part* of the time to explanations of "something new" (and few class hours should be allowed to go by without the pupils having been introduced to something new), *part* to a discussion of home work in which the pupil is given practice in the manipulation of some recently acquired processes, *and part to the timed drilling of the pupils in operations which have been thoroughly rationalized, and skill in which is now being developed by formal practice material.* Thus both the general implications of psychological research during the past two decades and the specific results from our own classroom investigations place in our hands two very important principles for the *development of a "spirited" atmosphere in the classroom: first, through the timing of all formal activities, and second, enhancing the possibility of keeping the class up to optimum efficiency all the time by desirable, rapid shifts in the type of mental processes involved.*

The thorough application of these two principles will do much to eliminate accidental errors.

2. **Recurring errors.**—We pointed out in the previous chapter that in diagnosing weaknesses in learning it is the recurring errors of pupils that are significant to the teacher who is trying to perfect the pupil's manipulation of the formal operations. There seems little doubt that these *recurring errors exhibit a startling tendency prevalent among our mathematics teachers toward a "clouded teaching emphasis."* We deduce that the cause of a poor teaching emphasis is traceable to one of three things: (*a*) the teacher is teaching too many operations, is trying to crowd too much material into her course; or (*b*) she does not know enough about the *ways in which pupils learn* these different operations to be perfectly clear as to which *are* the difficult ones; or (*c*) there simply is not enough time in a 35-minute class hour to teach all the operations which are laid down in our standard textbooks.

a) It is probably true that the average teacher does teach too many things—in algebra, in geography, in arithmetic, in spelling, in United States history, etc. Our stock example, of course, is the "17 (or 8) types of factoring"—fine material for an exhibition of mental gymnastics, but clearly material the inclusion of which we would be hard pressed to defend in a present-day course of study. If the text were properly designed, the teacher at this point would not need a supplementary device to enable her to pick out and drill in proportion to their difficulty the most important operations. *Clearly we do need in the field of secondary mathematics a clean-cut statement based upon scientific investigation as to what are the important and the difficult operations.* But more of that in chapter VIII on the course of study.

b) If the fault in an undifferentiated teaching emphasis is that the teacher does not know which are the difficult operations (and many teachers have written the present writers after giving standardized tests that it was the first time in their lives that they had ever known specifically which *are* the most difficult operations), the suggestion as to the remedy is simple and direct. The use of standardized tests can be made to lead to a detailed and helpful classification of the errors of pupils. It can be illustrated by an analysis of errors made in working the writers' Standardized Tests.

II. A DETAILED ANALYSIS OF PUPILS' ERRORS

We shall describe here the typical errors made by pupils as revealed by an evaluation of the tests given. The tabulation showing the frequency and percentage of the recurring errors is printed in the Appendix.

Test 1: Collecting terms. The error common to practically all algebraic work appears in the first test: mistake in *signs* in combining similar terms. The terms are recognized as similar, but before all are combined the pupil loses the sign, e.g., in $5x^2+3-3x^2-2-7x^2-5$, he combines $5x^2$ and $-3x^2$, but in combining this result with $-7x^2$ the sign of the $2x^2$ (which is positive) drops from the focus of his attention, and, with the stimulus of the $-7x^2$, he combines a $-2x^2$ and a $-7x^2$, giving $-9x^2$. We note in passing that elasticity or flexibility of attention which will permit the student to hold the result of one operation ($+2x^2$) while he is adjusting to a new one ($-7x^2$) is an important problem for the study of "learning."

The next most frequent error is that of adding the coefficients of similar terms without regard to their sign; e.g., in the foregoing exercise the pupil gets $15x^2$. Less frequent than these are: adding exponents of similar terms; failure to recognize similar terms at all (combines all terms); equating terms, evidently traceable to the influence of the equation; and accidental errors in omission of letters, exponents, signs, etc.

Test 2: Evaluation or substitution. Relatively few errors are made in this operation. Of the most frequent it will be noted that *nearly one-third are due to squaring the product of literal factors instead of the one factor designated by the exponent.* Thus the evaluation of ab^2 is many times more difficult than $2ab$. Here as in most of the subsequent operations the prevalence of a great amount of inaccuracy of a purely arithmetical nature is in evidence. Practically half of the errors in this operation are of this kind.

Test 3: Simple equations. By far the greatest difficulty in solving simple equations is that illustrated by problems of the type $4c=6c+12$. The pupil finds it very, very difficult to combine properly the similar terms $4c$ and $-6c$. He forgets the sign of $2c$, obtaining for his result $c=6$. Evidently he knows how to "*transpose*," or to get rid of, the $6c$ in the right member. He either does not think "$-2c$" at all, or, when his attention is immediately on the division of 12 by the coefficient of x, the sign drops from his mind. Following in frequency comes the ever-present arithmetical difficulty. Other errors of lesser difficulty are: error in signs in division, e.g., $-3x=12, x=4$; incomplete solution, e.g., leaving the work in the form $5x=25$; the tendency to combine dissimilar terms, e.g., $3x+4=14$, giving $x=2$; and the inverted result in division, e.g., $5x=13, x=\frac{5}{13}$.

Test 5: Parentheses. A study of the table shows that a rather high degree of efficiency has been secured in the teaching of this operation. As would be expected, about *one-half of the mistakes were in the use of signs.* Of these, twice as many were made when the minus sign was used outside the parentheses as when the minus sign preceded the first letter inside. The writers wish to explain here that no examples involving a complexity of symbols of aggregation have been included in this test. The importance of this kind of manipulation did not seem to warrant its inclusion.

Test 6: Special products. The chief difficulty here is that involved in obtaining the *sum of the cross-products, one-fourth of all the recurring errors being of this type.* This difficulty may be explained by improper emphasis in teaching and by a temporary lapse of attention between the first and the last term of the result obtained. A very large group of pupils fail to square the literal factor when the exponent is greater than one, and another considerable group use the product of two numbers instead of twice the product in squaring binomials. The error marked No. 8, "inability in the operation tested," is relatively large in all these tests. We give in Table III in the Appendix the percentage of all errors that were made which revealed *positive inability* to use the operation in question in each test. This table reveals that there is a very large number of pupils who are absolutely unable to handle certain manipulations of the following operations: special products, factoring, fractions, exponents, quadratics, and radicals. Certainly this degree of inefficiency cannot be explained in terms of lack of "ability" on the part of the pupil. In these particular instances the performance of the pupil is an entirely inadequate measure of his ability, for subsequent investigation has shown that when teachers recognize the relative extent of these various difficulties this degree of "inability" is greatly reduced. Hence the pupil is not primarily at fault.

Test 7: Exponents. There are three prominent errors in the use of exponents. Reference to the table will indicate that the *addition of exponents in involution involves the greatest difficulty.* Following this is (1) the error of squaring the exponent instead of multiplying (as in $[x^3]^2=x^9$), and (2) the failure to raise all the factors of the product to the required power (as in $[ab^2]^2=ab^4$). A bit of psychological analysis at this point will show that specific knowledge of these errors will tend to increase markedly the teacher's effectiveness. Our experimentation proves this conclusively.

Test 8: Factoring. It is significant that *39 per cent of all the recurring errors indicate positive inability in particular types of factoring.* Aside from this the chief difficulty is in *finding* such a combination of factors that the *sum of the cross-products* will produce the middle term of the trinomial in question. For example, $5x^2+16x+3=(5x+3)\times(x+1)$. Next to this the failure to recognize the monomial factor or the highest monomial factors are the most frequent. For example, $6c^4-18c^6=6c(c^3-3c^5)$.

But the greatest difficulty encountered in learning to handle the operation is that of recognizing the necessity of continuing the process of factoring until the *prime* factors have been found. At the root of the matter, however, is the development of an attitude of distributing attention between two equally important things, namely, finding the monomial factor and continuing the factoring until the prime factors are found. Students are outstandingly weak with this particular type of manipulation. They *may* have perfect mastery over either type separately, but they fail repeatedly with examples requiring them to do the two things. Why? Purely and simply because they have not been given practice in manipulating the processes together. Our psychological analysis shows clearly that *skill in the manipulation of an isolated operation will not* *operate effectively when the operation is associated with other operations in new situations, unless practice is given in the application.* In this particular case it merely means that we *must establish the habit* of looking first for the common factor and second for the prime factor. Our experiment shows that it is the only aspect of the process of factoring really difficult to make clear to the student.

Tests 9 and 10: Fractions. Errors in the use of signs predominate, for example, in addition and in multiplication, failure to change signs if the numerator is preceded by the minus sign. Errors in the use of signs in these more complex problems indicate that *this very simple process of collecting signs and numbers has not been properly habituated earlier in the course.* Next to the failure to change the signs, if the fraction is preceded by a minus sign the most frequent error is the failure to multiply each side of the equation by the lowest common multiple of the denominators. Many other minor errors are listed in the table.

Test 11: Formulas. The error most in evidence in this operation is that involved in selecting the coefficient of the unknown in the solution of fractional equations. For example, in $L=\frac{Mt-g}{t}$, it is common to find $M=\frac{Lt}{t-g}$, showing that the pupil considers $t-g$ as the coefficient of M. Then follows the tendency in division of each side of the equation to get the result inverted, for example, $V=LWH$, $W=\frac{HL}{V}$. The next most important error is interchanging factors on both sides of the equation, for example, $E=\frac{PL}{K}$, $P=\frac{EL}{K}$. The fact of the recurrence of this error certainly indicates a lack of clean-cut practice in habituation to this type of work.

Test 12: Quadratic equations. Here again the greatest difficulty is that of finding factors such that the sum of the cross-products will produce the middle term of the trinomial to be factored, e.g., $n^2-7n=12$; $n=6$, or 2. In addition to this the failure to find both roots gives considerable difficulty.

Test 13: Simultaneous equations. An examination of the table shows a rather high degree of efficiency in the learning of this operation. The arithmetic errors and *mistakes in signs* in addition and subtraction are the most frequent. These errors are common to most of the operations of algebra and are admittedly difficult to eliminate.

Test 14: Radicals. The chief difficulty, aside from absolute inability to handle radicals, is that of not extracting the required root of any factor of the corresponding degree in the radicant. The mistakes in factoring the radicant, the perfect square factor being correct, and the other one incorrect, is also noticeable. As would be expected the failure to multiply the numerator and the denominator by the rationalizing factor is also a source of difficulty.

A good-sized book could be written on the psychology of these errors. In the judgment of the authors this, together with the detailed exposition of methods of eradicating the errors, is very much needed by teachers. It is impossible to go into greater detail in this monograph. This discussion must be of an illustrative sort only. During the coming half-year a complete report of this sort will be made in a teachers' manual covering all aspects of the teaching of secondary mathematics.

III. THE NEED FOR FORMAL DEVICES TO PERMIT THE ECONOMICAL DEVELOPMENT OF "SKILL" BY MEANS OF MASS-INSTRUCTION

If pupils show a lack of mastery of the formal skills in first-year algebra owing to the fact that as the subject is now taught there is not time in the year to give optimum skill in the manipulation of these operations, the implication is clear that *devices are needed to supplement the textbook.* The data of the investigation of the use of our current course of study in other high-school subjects show that at the present time more practice is offered in the use of "imaginaries" than in the use of "graphic representation"; or, for example, that the very important process of "factoring the sum and the difference of two cubes" receives more attention than the processes of "formulas" and "evaluation" combined! *In other words, the "teaching emphasis" of the textbook is thoroughly uncritical.* Our report of the use of these operations in other high-school subjects as given in chapter VIII will show that the *manipulation of formulas and the process of evaluation are two of the most important* *processes*, from the standpoint of use, included in the first-year mathematics course.

The conclusion is direct. With our textbooks organized as they are at the present time teachers have a crying need for *devices for conducting drills economically*. Frankly, the solution of the problem at the present time is *the printed practice exercise*. It should be stressed that any *such device is fundamentally only an accessory built to accompany a well-planned textbook*. But *from the standpoint of the psychological criterion* as well as the social criterion, from the standpoint of effectiveness in teaching children the essential quantitative notions and giving them basic skills in ninth-grade mathematics, *such textbooks are not available. Hence, the "Practice Exercise."*

A word of caution should be stated at this point. *We insist, as we did in a preceding chapter, that*

> *the fundamental aim of instruction in first-year algebra is to develop in the pupil the ability to use intelligently the most powerful devices of quantitative thinking:* the EQUATION, the FORMULA, the GRAPH, and the PROPERTIES OF THE MOST COMMON SPACE FORMS. Only in so far as habituation or automatic efficiency in the formal processes contributes to efficiency in the solution of problems primarily of the reasoning or interpretive type do we insist upon skill or habituation in them.

There is evidence, however, both of expert opinion and of experimentation, that ability in the solution of new and original problems is accompanied by, and varies roughly with, *the degree to which the pupil has habituated these essential tool operations*. We recognize, therefore, that this phase of the teaching process is subsidiary to, although a *necessary* correlate of, the more fundamental aim stated above.

Carefully controlled tests of ability of pupils in first-year algebra, covering more than fifty school systems, have revealed: first, a very decided lack of skill in the manipulation of the formal operations; second, evident inability to use these operations successfully in their applications. It should be stressed that we recognize that the most important outcomes of the study of algebra are thought outcomes. We insist, therefore, that in order to give proper emphasis to this thought aspect we must devise ways and means of reducing the undue amount of time, now being given to the tool processes, in order to spend more time upon their application. That is, our fundamental problem is a problem both of teaching emphasis and of proper distribution of time on the various aspects of teaching. Thus our immediate aim is to organize a teaching program that will result in increasing the pupil's efficiency in the use of the necessary tool operations with the expenditure of such a small amount of time that by far the larger proportion of time and energy can be put on the interpretive aspect of teaching.

The results of the testing and of the preliminary classroom experimentation of the past year have shown clearly that mass-instruction of the practice or drill types can be effectively given by means of formal practice exercises. Also that it will result in a very satisfactory proficiency in the tool operations with the expenditure of a relatively small proportion of the total class time. It should be stressed that formal drill can be given by means of this material without the use of more than ten to fifteen minutes a day. Having experimented carefully with the use of tests and practice exercises, the writers now wish to put these before teachers of algebra in the hope that they may aid in solving one of the important problems of mathematics teaching. Two such exercises, Nos. 9 and 10, are printed herewith to illustrate the composition of the entire set.

The purposes of the practice exercises should be made clear at the start. They may be listed definitely as follows:

PURPOSES OF PRACTICE EXERCISES

1. In general, to make the pupil more efficient in his skill in manipulating the "tool" operations of first-year algebra.

2. To concentrate practice upon those operations that have been shown to be particularly difficult for the pupil, i.e., those types of problems in which the pupil makes the greatest number of mistakes.

3. To set up for the pupil and the teacher a definite, objective goal of achievement to be attained, after which further practice will probably not yield results commensurate with the time required.

4. To employ as a teaching device one of the most influential factors of learning, namely, short, spirited, frequent, "timed," practice periods.

5. To stimulate the pupil to compete against his own previous record, made definite and objective, and kept by himself as a measure of his own progress in the mastery of the subject-matter.

STANDARDIZED PRACTICE EXERCISES
FIRST-YEAR ALGEBRA

Set No. 10. Quadratic Equations
Standard: 12 "rights" in 3 min.

1. $x^2-2x=8$ ....
2. $b^2-b=6$ ....
3. $x^2-36=0$ ....
4. $n^2-5n-6=0$ ....
5. $a^2+4a=21$ ....
6. $b^2-4b=45$ ....
7. $y^2-y=20$ ....
8. $r^2-49=0$ ....
9. $t^2-7t-8=0$ ....
10. $x^2+6x=40$ ....
11. $n^2-8n=33$ ....
12. $p^2-p=12$ ....
13. $a^2-25=0$ ....
14. $y^2-9y-10=0$ ....
15. $a^2+3a=54$ ....
16. $x^2-7x=44$ ....
17. $c^2-c=30$ ....
18. $m^2-64=0$ ....

Set No. 9. Factoring continued
Standard: 12 "rights" in 6 min.

1. $3x^2-6x-24$ ....
2. $9x^4-25y^6$ ....
3. p^4-p^2-20 ....
4. $p^2+5p+10$ ....
5. $9x^2+12xy+4y^2$ .....
6. $2y^2-6y-20$ ....
7. $16r^6-49y^4$ ....
8. x^6-x^3-12 ....
9. c^2+3c+8 ....
10. $25d^2+30dc+9c^2$ ....
11. $5a^2-20a-60$ ....
12. z^4-81w^6 ....
13. y^8-y^4-30 ....
14. $d^2+7d+11$ ....
15. $49p^2+28pq+4q^2$ ....
16. $3b^2-15b-150$ ......
17. $4n^{10}-121x^8$ ....
18. x^2-x-56 ....

(Answers are printed on the back of each set to enable the pupil to correct his own work.)

6. To help reduce, as much as possible, the great amount of time now being given to class drills which careful measurement and individual experience show to be ineffective.

7. To provide a means by which individual differences may be practically recognized and met.

GENERAL PRINCIPLES OF DESIGN OF PRACTICE EXERCISES

1. We wish to make it clear that these practice exercises have been designed after detailed study of the errors of pupils, as discussed above. This study has enabled us to design exercises that will offer practice on the various operations somewhat closely in accordance with their relative difficulty, as shown by the table of errors in the Appendix. Furthermore the degree of difficulty has been checked by examination of the achievements of large numbers of pupils on the writers' Standardized Algebra Tests. That is, they have been designed to give the pupil practice on particular algebraic processes roughly in proportion to their difficulty, as shown by the number of pupils unable to solve them.

2. The standards of attainment are tentative, but they are sufficiently low that approximately 50 per cent of the class will attain them on the fourth or fifth trial. They have been set after a careful study of the record cards of three large classes. These records show the rapidity with which various types of pupils perfect skill with each operation. Sometimes "optimum" skill (beyond which further practice brings diminishing returns) is attained in two or three trials (as, for example, with parentheses). In other cases it takes five trials—never more than this. The trials should come no oftener than once or twice a week.

3. More problems have been included on each card than the number in the standard. They are arranged on a definite rotation scheme and can be used as tests if desired. This makes it feasible to begin with any problem in the list and work either way. By so doing the pupils are much less likely to remember the answer to particular problems.

4. Types of formal problems which are not necessary to the realization of the primary aim of instruction in first-year algebra, as stated above, are not included here for practice. The writers question the advisability of including such types of problems as the division of polynomials by polynomials, factoring the sum and difference of two cubes, factoring by grouping terms, finding the highest common factor, solving complicated combinations of fractions, etc.

SUGGESTIONS FOR THE USE OF THE PRACTICE EXERCISES

1. The foregoing discussion should make it clear that the practice exercises are in no wise intended to take the place of, nor to be given simultaneous with, the most thorough rationalization and explanation of the particular types of problems under consideration. They should be given after the class has clearly grasped the meaning and significance of the operation. The length of time devoted to getting this clear understanding of the various operations must necessarily vary with the teacher. Thus the time to begin the use of formal practice on any operation cannot be standardized for large groups of teachers. Each one must determine at what period he should turn to the use of the formal material.

2. We suggest that practice should be given on a particular exercise (e.g., the one on collecting terms, if that operation has just been explained and discussed thoroughly) each day until 75 per cent of the class attain the standard number of problems set for that exercise, two days in succession. That is, excuse any individual pupil from practice on a particular exercise as soon as he has made the standard score twice in succession. Remember that these standard scores have come from the actual attainments of three large classes. They are set at the point attained by 50 per cent of the class on the third, fourth, or fifth trial, the trial depending on the number of the exercise. If after trying the scheme the teacher is convinced that the standards are too low or too high, he may change them to suit his own purposes. The writers are convinced that skilful use of these exercises will tend to raise the standards in the future rather than to lower them. The general principle underlying the establishment of the standards is that practice should not be extended beyond the point at which there ceases to be a marked increase in efficiency except for purposes of permanence or retention.

3. The presence of rapid, average, and slow pupils in the same class necessitates a decision concerning the best use to make of the time of the rapid pupil. Acting on the principle that we want the "optimum" amount of formal skill, not the extreme amount of which the pupil is capable, we suggest excusing the pupil from the class to work on problems of application or interpretation con-

cerning those operations which have thus far been explained. That is, experience with both arithmetic and algebra exercises is proving that it is better to give the fast pupil extensive training with each operation, both "formal" and "applied," rather than to permit him to move on at once to the learning of other types of subject-matter. In this way the members of the class are held together as regards formal skill (which we emphasize as a fundamental principle), but are permitted to get as much facility as they are capable of in the use of algebraic devices in "thinking" situations. Thus the fundamental doctrine is: cut off training in formal skills at "optimum" in order that the utmost possible opportunity may be given the bright pupil for developing his power to meet "problem" situations.

In order to excuse the rapid pupil from the practice exercise the teacher must have "problem" work of some kind at hand from which to make assignments. Teachers will do well to make a detailed study of this aspect of the general procedure, and will be amply repaid if they carefully collect and classify problems which will give varying opportunity for application and interpretation. In the book on the teaching of secondary mathematics this need will be filled.

4. With the standard skill once attained on a given operation the drill time of the class goes largely to a new operation. The teacher, however, faces the problem of giving "review" practices to "hold the skill." The writers' manual will make detailed suggestions as to the best intervals to leave between the practices. In the meantime the decision must be left largely to the initiative of the individual teacher. It is probable, however, that it will be helpful to review a given exercise under time conditions as often as once a week, and not much oftener. On the whole it is important to make use of the exercises very flexible, and to fit their use to the needs and abilities of the individual class.

CLASSROOM PROCEDURE

In the specific classroom handling of the practice material the writers have found it helpful to keep in mind the following suggestions:

1. Economize time in passing and collecting cards. Let the first pupil in each row place the card for the day's practice on the desk of each pupil, with the "problem" side up. Immediately at the close of the practice period let the last pupil in the row collect the cards in that row. Systematize the procedure from the first day.

2. Make all directions before the cards are passed. From the start impress it on the pupils that any study of the exercise on their desks before the teacher's signal "go" is unfair. The work of the pupil is to be done on a blank sheet of paper against which the practice card is placed. Caution the pupils not to write on the card itself.

3. To prevent memorizing answers to examples on successive days, change the number of the example with which the class begins, i.e., if they begin with the first example on the card today, tomorrow they may begin with the fourth, the sixth, or with the last one on the card and work backward toward the first.

4. With all pupils ready, start all to working at the signal (e.g., "go"). To be effective, the exercise must be a "time drill." It is of the utmost importance that the teacher time the exercise accurately. Always start the class at the even minute or half-minute by your watch. Long experience has shown us that it takes the greatest care to conduct these exercises effectively. Make sure that each pupil stops exactly at the signal "stop."

5. One of the most valuable outcomes of the use of the practice material is the part taken by the pupil in correcting the work, in noting his mistakes, in recording his own achievement, and in comparing it with the standard and with the achievements of other pupils in the same class. Thus systematize from the start the scoring, recording, and criticizing of the work done. If you feel that the pupils should not correct their own work, have them exchange papers immediately at the word "stop." The writers let the pupils do it themselves. The answers to the examples are printed on the back of each card in the exact order of the examples themselves. The pupil turns his card over, lays the "answer side" against his neighbor's work, and checks each answer. He then writes the number of "Attempts" and "Rights" on the paper and passes it back to the owner. The latter then records his score in

"Attempts" and in "Rights" on his record card. At irregular intervals, and always when a pupil has made the standard score, the teacher will need to check the accuracy of the pupil's work.

6. The teacher can stimulate interest in self-improvement by various devices. A helpful one is: have those who made the standard score hold up their hands, commend their work, and raise the question with the rest as to how many can do better the next day. The teacher will find that a human interest on her part in the pupils' scores will be a great stimulus to increased effort on their part.

7. Teachers should keep the practice cards and the record cards except during practice periods.

Devices to enhance the economical development of skill by means of class instruction—formal practice exercises—have thus been described. *Their effectiveness in "economizing" time in learning* will be shown in chapter IX.

The writers believe that practice material in any experimental stage should not be commercialized. Their practice exercises have been designed, after careful experimentation and great labor, in the firm belief that they will aid in improving the teaching of elementary algebra. The financial charge for these to the teacher has been set, therefore, at a minimum figure which will barely cover the cost to the writers for (1) printing, (2) handling, and (3) the expense of conducting the co-operative study next year. The charge worked out on this cost basis has been set at 9 cents per set of 7 practice cards each, and ½ cent per pupil record card. A teacher in ordering will need: (1) as many sets of practice cards as she has pupils in her largest class, with perhaps a few more to cover wastage; (2) a pupil record card for each different pupil in her classes. Thus, if a teacher's largest class is 25 and she has 60 different pupils, the entire cost to her will be $2.55. It is undoubtedly true that a set of cards will last two or three years, perhaps more. The foregoing charge can be reduced with future revised editions in case the sale of the exercises justifies a large printing. The present cost, small as it is, could have been reduced if it had been possible to venture a large printing of the set. It will be noted that these cards have been copyrighted. This has been done to protect them from premature commercialization and strictly in the interest of the scientific study of methods of improving teaching in algebra.

NOTE.—The "Practice Exercises" are distributed by H. O. Rugg, School of Education, University of Chicago.

The "Tests" are distributed by the School of Education, University of Chicago.

CHAPTER VII

THE OPPORTUNITY FOR TRAINING IN "LOGICAL THINKING" OFFERED BY SECONDARY MATHEMATICS

ANALYSIS OF THE SUBJECT-MATTER FROM THE STANDPOINT OF LEARNING

Failure by pupils to grasp *mathematical principles* (as well as to develop satisfactory formal skill) has been traced in this monograph to a *clouded teaching emphasis*. The root of the difficulty is that *we have not analyzed the learning process in secondary mathematics*. We have not clearly marked out the general aim and the various outcomes from teaching, and have not focused the latter sharply on the former at each step of the instruction.

Now it is one of the main theses of the writers that the teaching of a subject of study cannot be justified unless it can be made to develop problem-solving abilities, abilities which permit a student to meet a novel situation in an effective manner. And it clearly is the task of this report at this point to analyze, at least briefly, the extent to which various abilities enter into progressive living, and to show the extent to which instruction in mathematics can provide opportunity for the development of such abllities. In fulfilling this important function the writers wish to call attention to what seems to them to be a needed reply to the uncritical assertions of the educational critics that the design of a mathematical course shall not rest upon what they choose to call "mental discipline."

Mental abilities demanded for success in various life-activities. —What ability is it that distinguishes the organizer, the executive, the analyst—the leader who can visualize all the ramifications of a big problem—from the habit-minded automatic worker? Clearly *the ability to see relationships, the ability to pick out essentials, the ability to analyze and organize.*

Take the worker in the unskilled trades, for example, who makes chief use, throughout the day's work, of coarse, manual skills.

Gross motor co-ordination engrosses his attention in the vast majority of the activities in which he takes part; practically no abilities of organization and planning are required of him; he is required to take directions orally and to follow them precisely, being expected to do only what others have shown him how to do. His life is made up, in other words, very largely of manual imitation.

Or take the worker in the semi-skilled occupations, the so-called automatic worker in agriculture, industry, and business. His success, it is true, depends on greater skill in co-ordination, or greater co-ordination in mechanical details, and he makes some use of the three R's. However, no leadership is required of him, and almost no initiative. If directions are expressed specifically, either orally or in writing, he is required to be able to follow them out rigorously. In other words, his chief assets are skill of a coarse sort, very moderate abilities in the three R's, and the ability of following directions consistently.

Or take the skilled worker of whom is demanded a finer control over processes of motor co-ordination; considerable organizing ability for definitely set mechanical jobs; the ability to take directions relating to his specific field, either orally or in writing—directions which may not be expressed in such consecutive and detailed form, but in which he is called upon to supplement the general directions with the knowledge of the proper types of technique to bring in at various points. Ability is of this sort, whether found in the construction of a machine part or the laying out of a stairway or a floor, in the arrangement of a delivery of goods over a grocery route or the manipulation and care of various types of machinery, or what not. Such occupations at least are differentiated from the two below them in their requirements for higher levels of "thinking abilities," primarily in the degree to which they offer opportunity for, and demand skill in, the analysis of problem data, the selection of better methods of procedure, and the putting together of a few easily recognized major elements in the situation.

Or consider the various head-work occupations which may be thought of as requiring intellectually these organizing and analytical abilities in a larger degree. Here a more definitely organized equipment of an intellectual sort is required: systems of habits of dealing with intellectual material, the three R's perfected to a degree not required on the lower occupational levels, abilities of organizing intellectual materials, more readily typified perhaps by those engaged in the clerical occupations, in the teaching occupations, and on the higher level in the various professional, medical, legal, engineering, and designing occupations. Here again we find *the greatest desideratum for advancement* from one level of occupation to another to be *ability in analyzing and organizing data*, regardless of the types of situation in which they are set.

The important element in analytical and organizing abilities: the ability to see relationships. As we have surveyed crudely each of these occupations, what is it that distinguishes the foreman from the worker, the superintendent from the boss, the manager from the clerk, the chief clerk from the stenographer, typist, or biller, the principal and superintendent from the teacher, the chief executive from the department head? Fundamentally—presupposing, in all of these, qualities of steadiness and industry—analytical and organizing abilities accompanied by mental initiative of a preponderant sort. More in detail, *the ability to see the relations in complex problems*, to comprehend unaided the *relations* of one component to another, to trace the ramification of *related* "causes and effects" or, conversely, taking a mass of unorganized data, to see the connections, the *relationships*, between the parts, to systematize them, and to fit them together into a smoothly working whole.

With any one of the occupational levels to which we have referred (at least for all but the extreme "automatic worker") life-tasks are set up in a continuous stream of "problems" to be solved. It is true that the most effective mode of meeting the *commonly recurring* problems is to automatize certain methods of attacking them, to habitualize the mode with which we respond to them. In this way most of the daily life, both physical and mental, of our lower-level worker is "habitual"—he thinks and feels and acts in ruts, it is only too true. But surely the function of any educational agency, in addition to the development of "skills" needed in life, is to equip us with attitudes and possibilities of response so effective that we shall see and shall be able to meet successfully these "problems" of daily living. *It is this problem-solving attitude that is fundamental to success. We should therefore visualize the opportunity and the function of the school as that of providing situations, most economically organized, for training in the development of problem-solving attitudes and of skill in the meeting of problem-situations.*

We stress problem-solving attitudes because our attitudes contribute immediately to the efficiency with which we react to the situations we meet in daily life. An attitude is a state of mind which results from our prior responses to situations which were similar to that which we now confront, at least which were ostensibly like it. It seems quite clear that our attitudes determine the manner in which we respond to given situations, the way in which we interpret them—that is, the way in which they are made meaningful to us. Our intelligent, our constructive, our rational responses are always dependent upon the degree to which we recognize causal relationships in the elements of the present situation. Thus the setting up of effective problem-solving attitudes depends primarily on the extent to which the school has provided the pupil with genuine opportunity to feel, to analyze, to compare, to organize—in a word, to recognize and set out clearly *relationships. Continuous opportunity for problem-solving then, just as it is the mental core of progressive living, likewise should be the mental core of an intelligently organized course of study.* The "problem" may be the planning of the most efficient way to ship a bill of goods, or the delivering of a wagonload over a route in the most economical manner. It may be the laying out of some clerical task, or the getting of the greatest efficiency out of a machine. Whatever it is, the extent of the worker's contribution and of its recognition by society in the long run depends primarily on his ability to see relationships and to overcome the mental inertia which continually operates to prevent us from acting upon this. *Fundamental to his success is the development of ability to analyze a problem, to compare intelligently the various possible methods of solution, to test critically the various hypotheses to which the problems give rise, to generalize the conclusions, and to recognize the field of their application.*

WHAT IS LOGICAL THINKING?

But this, after all, is the mental ability which appears in the vocabulary of the average mathematics teacher under the caption "logical thinking." Teachers almost universally state as one of the fundamental aims underlying their teaching (regardless, too, of which subject of study theirs is) "the development of abilities of logical thinking." The tables from Professor Koos's recent study offer detailed evidence on the far-reaching extent to which this aim is held by teachers of mathematics and other subjects in the high school. The phraseology may not be the same in all cases, but the connotation is. For example, "logical thinking" may be replaced by "development of powers of observation," "powers of discrimination," etc. The English teacher, for example, finds in the organizing of material and the giving of a title to a "theme" or an essay the same large opportunity for training in powers of "organization," "comparison," and "analysis" that the mathematics teacher does in the more specifically set algebraic "problem" or geometrical proposition. In the same way the chemist finds in the determination of his "unknown" the same provision for calling into play powers of analysis, comparison, discrimination, and logical, sequential organization that the teacher of manual arts does in his arrangement of "projects" in which the pupil is required systematically to lay out the construction of a utensil or a machine. The latter too is believed to call into play similar processes of planning, organizing, analyzing, comparing, putting together, picking out the essentials, etc.

It seems quite clear then that no discussion of the reorganization and standardization of a high-school subject of study can really make its contribution without a detailed offering upon the question of the fundamental aims and outcomes of instruction in that subject of study. *Inefficiency in secondary mathematics has been shown to be due largely to a lack of the specific delimitation of aims and outcomes.* It is one of the theses of the present writers that further progress in secondary mathematics demands insistently a complete statement of the aims and outcomes of instruction in that subject—one, furthermore, that will be explained in terms which relate specifically to the learning of pupils, and which will have a distinct bearing upon classroom teaching. The classification of "outcomes," to be most helpful, must be organized in such a way as to take definite account of the kind of outcome that the teacher will recognize as important. In other words, it must be organized and classified in such terms that the teacher can use it specifically in her classroom practice. She wants to be told that as a result of teaching a particular "operation" these habits are to be established, that this material is to be memorized, that the proper approach to these problems should be made "rational," should be reasoned about, and that this "reasoning-about" process can be done specifically thus and so—that it should take account of such and such phases of the learning process in children.

The writers have found that the classification of outcomes which was used in chapter IV, and in terms of which they have done most of their thinking in the past years of investigational work, has been definitely helpful in this way. The classification makes use of three types:

1. *The immediate, specific, and preparatory outcomes*, including the comprehension and manipulation of the tool and automatic phases of the subject-matter, certain of these to be used in many specific problems and in the solution of other mathematical problems; to be mastered as tools preparatory to the taking of other mathematical courses as well as for use as automatic tools for the solution of all types of mathematical problems.

2. *Immediate generalized outcomes*, involving recourse to selective, analytic, and conceptualizing abilities; ability to apply principles in addition to ability to remember rules and to make certain fundamental habitual adjustments in the solution of practical and applied problems and of problems belonging primarily to other mathematical fields.

3. *Remote and less tangible outcomes*, involving the development of ability, for example, to deal with general number concepts and to "think quantitatively": (*a*) the development of attitudes of orientation in algebraic or general mathematical fields which contain problem-situations; (*b*) confidence in one's ability to use algebraic symbols successfully in meeting new situations; (*c*) a broadened intellectual background or perspective for the general

cultural comprehension and interpretation of the scientific methods by which technical problems may be solved, that is, the *development of a scientific attitude.* Having characterized these in this brief way, let us take them up more in detail.

THE OUTCOMES OF INSTRUCTION IN SECONDARY MATHEMATICS

I. IMMEDIATE AND SPECIFIC OUTCOMES

First, standing out is (*a*) *a large group of specific habits* involved in the *manipulation* of the fundamental algebraic operations. These specific habits involve ability in manipulation of a score or more of typical modes of solution. The operations included are the fundamental arithmetic operations, various special products, the removal of parentheses, the process of evaluation, the solution of equations of various types, and the use of radicals and exponents. The aim should be to reduce this phase of learning to the automatic basis as early as possible. This must necessarily constitute one of the definitely recognized tasks of the teacher. It will be shown later that our chief problem in this connection is to devise ways and means of developing optimum skill with the least expenditure of energy and time.

b) *A large body of memorized responses* to familiar "cues" of solutions or proofs; for example, memory of figures and of the proper steps to use in proving the truth of certain statements—of specific devices which may be called into play to aid in the development of a method of procedure in algebra, for instance, "letting x or y represent the unknown," great masses of facts about consecutive numbers and other types of number relations, a vast group of relations of simple phenomena which are used as the source of problems, for example, (1) distance equals rate × time, or (2) weight × lever-arm equals weight × lever-arm; or arbitrarily agreed-upon conventions concerning the use of a negative number to represent oppositeness, for example, distance to the right of, and above, a datum regarded as "positive." Success in solving problems (irrespective of the extent of one's skill in analysis and organization) clearly is dependent upon absolute mastery of these tool devices—*memory of conventional terminology, space relations,* and *algebraic procedure.*

c) *Ability to deal correctly with a large number of rather definite number and space concepts.* Correlative with the establishment of these concepts must be the development of a vocabulary of space terminology used in their expression. Thus an enlarged technical and practical vocabulary is developed as an immediately specific outcome; for example, "equality," "consecutive number," "reciprocal," "ratio," "dependence," "variation," "proof," "reasons," "rectangle," etc.

d) Another specific and immediate outcome from instruction in mathematics as well as our other school subjects is the *abilities developed in specific oral expression.* This expression may be of memorized statements—demonstrations, axioms, definitions—or of definitely formulated modes of reasoning which are involved in original solution. Such outcomes arise largely from specific drill in the correct association of ideas and "speaking habits." Teachers' meetings throughout the country seldom fail to point out the duty that devolves upon each department, outside of English, to take advantage of the opportunities provided in their courses of study to develop these habits of correct speech. In this connection a fundamental problem arises: Shall pupils be permitted freedom or even looseness of expression in the continuous development of solutions, or proofs, in order that the trend of thought may not be broken, or that the favorite attitude of the pupil may not be upset? Important issues are evidently involved in such problems. To what extent shall mathematical instruction result in drill in accuracy of oral expression? To what extent is continuous thinking "on one's feet" to be displaced by constant attention to correct habits of speech? Certainly this problem merits the most careful study on the part of teachers of this subject as well as of others.

Our psychological analysis of most effective ways of organizing class discussions in ninth-grade mathematics makes very clear a sound procedure. Clear it must be that the *primary object of class discussion in secondary mathematics is the development of skill in continuous "thinking on one's feet."* True it is, however, that skill must always be developed through practice in a variety of situations. Correctness in oral expression in the English class does not imply correctness in oral speech in the chemical laboratory, in the forgeroom, at the blackboard solving algebra problems, at the breakfast table, or in the midst of the "gang." It is true that we need constant checking up—that "ideas" of correct speech must be habitually held to the conscious level in each of the various types of intercourse in which we take part in our daily lives. However, instruction is so organized in school that apart from the English class the development of habits of correct speech certainly is a secondary function and not a primary one. Thus we would say that our analysis of the situation makes it quite clear that no pupil should be interrupted in the course of such continuous thinking for the correction of his modes of expression. Our practice has shown, however, that a large "incidental" outcome of mechanical instruction is the ability to check up constantly one's methods of oral expression. The definiteness of the terminology, of the methods of attack, of the vocabulary absolutely necessary for the correct interpretation of the situations presented in mathematics, offers unusual opportunity for practice in this direction.

II. IMMEDIATE GENERALIZED OUTCOMES FROM THE STUDY OF MATHEMATICS

Outstanding throughout the discussions of this monograph has been an emphasis upon the training of children in "*logical thinking*." In spite of the common use that is made of this term by teachers of all school subjects and by educators generally, an examination of the literature and detailed discussions with many teachers of mathematics shows that there is a need for a new statement of what is "logical thinking" in mathematics. Let us introspect carefully through the process of "reasoning about," or "thinking through," a problem-situation. In other words, let us enumerate the *steps of "logical thinking."*

We find them to be, first, the grasp of the fundamental issue involved in the situation which is presented to us, the conceptualizing or the "getting of the meaning" of a problem, the recognizing of the outstanding components which must be taken apart and put together in order to solve the problem. Second, the "problem" once recognized, progress waits upon the recognition in the present situation of similarities to situations which have been met and dealt with before. Third, the recalling of correct modes of attack or of solution of just such past experiences—the bringing of them in review in rapidly changing order. Probably "in a flash" would perhaps be a better description of the actual mental processes, of the ways in which we review the method by which such problems have been attacked before. Fourth, the instantaneous, "intuitional" selection from such modes of response which are recalled from past experiences, of responses that give promise of being appropriate to the "cues" which are presented to us in the present situation. Fifth, the fitting in of these methods or modes of solution to the present problem, trying one and then the other, weighing the possible results of the use of each. Sixth, the concentration upon, and the holding in mind of, these steps as they are tried, one after another, in this flashlike scheme, in a more or less continuous train of thought until the end is reached. Seventh, critically reserving complete acceptance of the explanation just arrived at, or of the correctness of the "answer" just obtained, with the accompanying development of habits of testing or checking the validity of the same. It seems probable that *one of the most important outcomes* from the study of mathematics is this setting up of continually present *attitudes of checking or testing our own conclusions. Clearly the form in which our material is set provides opportunity for practice in the doing of this to a degree that no other subject in the curriculum is found to offer.*

Examples of practical "problem-solving."—A more concrete picture of the ways in which the human mind actually behaves when it is confronted with a "general" problem-situation, and again with a mathematical situation, may not be amiss here. The former type first:

The carrying out of my afternoon's work, let us suppose, requires that I lecture twice to classes—at 1:30 and at 2:30 with a 10-minute intermission between the classes. Shortly before the beginning of the first class I walk from my house to the lecture-hall, a distance of three blocks. On entering the building three minutes before the beginning of the first class-hour, in a flash I "feel" that some notes which are absolutely necessary for the complete discussion to be carried on in the second lecture are at home reposing in the third right-hand drawer (or is it the second ?) of my desk. The *problem* flashes before me. The clear recognition of the effect upon the second lecture of the lack of the notes forces in "flashlike" fashion a review of possible ways of acting to have them at hand at the desired moment. Automatically, seemingly, because the response follows so immediately upon the recognition of the problem, my hand takes out my watch, notes that there are only three minutes, runs in quick perspective over the situation, and comes to the almost instantaneous conclusion that the time left is insufficient for me to turn about, walk the three blocks, open two doors with that perplexing Yale key, rummage through the materials in the third drawer of that desk to find the notes desired—with the bare possibility of their not being there also recognized, thereby forcing a search through other receptacles. In a flash, I say, comes a response, "There is not enough time left to go after the notes now." Successively there "flashes up" the possibility of going to the telephone, calling my home, and asking for the material to be sent. Instantaneously comes the reaction, "There is no one available to send with them." Next, following in rapid succession, comes the suggestion to get someone in the building to go after them, with the accompanying parallel reaction that the success of this plan depends upon the rather slim chance of finding someone available for that purpose, and also upon the possibility of giving directions which can locate the material. Finally, having discarded in rapid-fire order all of these ways of solving this practical and "real" problem, the final solution comes to me: "Of course—a 10-minute intermission between the two classes will provide ample time for the getting of the notes either by a personal trip or by sending someone else." The recognition of the problem, the trying of various methods of solution one after another, each one demanding in its turn clear recall of such past experiences, correct images of elapsed time necessary, and of adequate perspectives of possible solutions by the various methods—all of these proposed solutions succeeded one another with the final and acceptable solution presented probably in the course of a few seconds.

Let us consider another illustration of logical thinking—as it appears in mathematics. Suppose I have the problem of finding the factors of $6x^2-7x-20$. Instantly I recall that the first term, $6x^2$, was obtained by multiplying two terms (the first term of each factor). The decision follows at once that these terms must be either (1) $3x$ and $2x$, or (2) $6x$ and x. Automatically comes the schematic form, with which I react habitually:

$$(3x \qquad\qquad)\ (2x \qquad\qquad).$$

I examine the last term of the trinomial to find the probable second term of each factor; the recognition follows that the possibilities are 10 and 2, or 4 and 5, or 20 and 1. I test my first hypothesis: that 10 and 2 with the proper signs may be the number sought. To do this I complete the schematic form stated a giving

$$(3x \qquad 10)\ (2x \qquad 2).$$

But through the visual imagery suggested by the curved lines I see that this arrangement could not give as the sum of the cross-products the middle term $-7x$. Instantly I interchange the 10 and the 2, giving

$$(3x \qquad 2)\ (2x \qquad 10).$$

Similarly checking shows that the trial of this lead is false too. I try again, visualizing the factors of the second term, as 4 and 5, with the proper signs attached. Checking this again by reference to the visual scheme

$$(3x \qquad 4)\ (2x \qquad 5),$$

I recognize that the sum of these cross-products could be $-7x$ if the $15x$ were negative and the $8x$ positive. Thus I affix the signs, giving

$$(3x+4)(2x-5)$$

as the solution of my problem. The analysis thus reveals all the steps in "thinking" enumerated above. To emphasize this *general type of factoring* will be shown later to be making use of the results of the study of "economy of time in learning."

With these two illustrations of what is *logical thinking*, consideration of the *subject-matter of mathematics* will show that it *is unusually well adapted for bringing about this outcome;* in the first place, its material is most systematically organized in the form of "problems"—demonstrations, statements of relations to be proved; in the second place, it is unique in that it has organized in a very definite order a logical "step-by-step" system of "cues" to response. No point in our psychological analysis of the provision offered for training in secondary mathematics ought to be more stressed than this one: *the possibility of an ordered systematic* *scheme of "cues,"* and one which offers training in response to new situations. In the third place, therefore, we find it unusually well adapted to training in thinking because of the practice that it offers in responding to these ordered systems of "cues" and in holding the series in mind until the end of the proof is reached.

No more important aspect of this whole question of the development of generalizing abilities will be found than that concerned with the development of definite *concepts of method* in solving problem-situations. Certainly chief in importance in our discussion is the large twofold aspect that this reveals. We note first, the ability to state any problem in equational form, that is, to recognize equivalent elements in our problems or, as in the case of geometry, the selection and discrimination of the known relationships from those which are to be established; second, the accompanying ability to represent the correct quantities and their relations to known quantities by algebraic symbolism. But this all implies that habits of discrimination, analysis, and comparison have been formed; *a habitual "looking for" the unknown quantities, letting x and y or what not represent them; picking out the identical or equivalent elements in the problems in order that when these are organized in "equation" form identity may be expressed.* It is believed that these critical abilities of analysis and selection, the *"picking-out-the-essentials" abilities*, form one of the most important outcomes from the study of high-school mathematics.

In the foregoing references to the conceptualizing of proper methods the steps in their *initial* stages are clearly to be regarded as *reasoned* processes. Our view of *reasoning* is such, however, that we are constantly building up a larger and better-ordered response-system, and that practice is continually reducing much of the response to the automatic level. Therefore the stages of, first, the recognition of "how" to use the equation with the given problem, and, second, the "picking-out" of unknowns are probably always in part a reasoned process. Our purpose, however, should well be to short-circuit as much of this as possible, and to habitualize a method of solving problems expressed in algebraic symbolism.

With so much, both teachers of mathematics and critics of educational theory will doubtless be agreed. With the next statement, however, namely, that the development of "concepts of method" in solving problem-situations, both in other mathematical and scientific fields of study, in non-mathematical or scientific fields, and in so-called practical or non-academic situations (those entered into in daily living apart from the school)—with this statement, we say, doubtless there will be less agreement. The very statement itself centers attention upon one of the most fundamental educational issues—that expressed in the phrase "*transfer of training.*"

Application of the present status of thought on the place of mental discipline in school studies.—For more than twenty years educationists have debated, on both introspective a priori grounds and from the standpoint of somewhat meager scientific investigation, the possibility that the effect of training received in one type of mental activity will spread or "transfer" so as to increase the achievement in fields partially related. We say partially related, because certainly no one at the present time implies transfer of ability gained in one field or learning to another field to which it is absolutely unrelated. We say this furthermore with a complete recognition of the contribution that has been made by more than thirty attempts to answer the question experimentally. There seems to be no doubt that a careful study of the thirty investigations in question indicates that *there is distinct evidence for basing the design of curricula and the teaching of children in a belief in the possibility of transference of "training."* The experimental training of the abilities of either adults or school children in either laboratory or schoolroom certainly will result in an increased efficiency on the part of the subjects in other abilities which are in some way related to the trained abilities. Transfer is an accepted experimental fact, but as to the extent to which training transfers, and what are the most favorable conditions for its transfer, specialists are not always agreed; in fact, we may say that almost no experiments have been conducted as yet in which the opportunity for the determination of methods by which transfer spreads has been provided on clean-cut quantitative grounds. However, the assembled data on this fundamental question of the agencies of transfer indicate a distinct tendency on the part of specialists to stress certain similar features of their experimental results. One of the present writers has reported in a previous publication the results of three years of study and experimentation in this specific field. We shall condense here a summary statement, adapted to the explanation of our present problem, from that publication.[1]

That the various investigators in this field maintain that there are distinct *generalizing outcomes* is seen by the use of such explanatory terms as abilities in the "organization of methods of procedure," "conceptualizing abilities," "the development of improved methods of learning," "increased power of concentration," "grouping tricks," "improved technique in learning," "ideals of method of analysis or attack," "more effective methods of distributing attention," of concentrating upon or of extending the range of attention. The fundamental importance of attitudinal factors is stressed; for example, the development of attitudes of orientation in the general fields of reaction, of adjustment to experimental conditions, of confidence in success, of a "central predisposition" to meet new situations. Thus it seems clear that the weight of the argument is to the effect that "transfer" is possible through the generalization of various central functions. We may note that this of itself might well explain the slight amount of transfer that has been found. For most of the experimentation has been upon so-called "peripheral" functions, reaction time, speed, and the like, in which motor co-ordinations have been of the simplest sort—those in which "central" modes of improvement are least effective.

The experiments indicate that the central improvement consists in devising methods of learning, for example, tricks and short cuts for meeting problem-situations; as one writer has put it: "Our instruments do not improve; we only learn to use them better. Those who do not learn to use their instruments . . . . from practice, show little or no transfer of improvement through practice." It is pointed out that improvement should be spoken of as "central" or "peripheral" rather than as "general" or "special"—as a better understanding of how to use the organs of sense rather than as a change in the constitution of these organs. The experiments show

[1] H. O. Rugg, *The Experimental Determination of Mental Discipline in School Studies*. Baltimore, Md.: Warwick & York, 1916. Pp. 132.

that we must distinguish between the ideational possibilities of transferred improvement and the vain hope o the "spreading" function of rigidly developed sensory, perceptual, and motor adjustments. These latter have to be taken over into new situations unchanged and can operate with increased efficiency only as the conscious utilization of them in combination has been made more effective through experience.

We have thus made a somewhat detailed reference to the important issue of transfer of training because of its pertinency to the emphasis that we are laying upon abilities in *generalization* as outcomes from the study of mathematics. Aside from the specific reaction to definite cues, or systems of cues, the largest part of the outcome from the training obtained in school studies is to be found in the development of ideas of "how" to meet new situations. The most efficient reaction can thus be brought about by a high degree of accuracy and immediacy in practice in automatically responding to many specific cues. These in turn are largely enhanced by the variety and the adaptability of the concepts of method which the individual has acquired. It is believed that instruction in mathematics can be made to form a definite contribution to the development of concepts of method. The separate demonstrations and solutions of problems develop in the pupil, day after day, through continued drill, ability in the formation of ideas of "how" to solve the problem. *Adequate textbooks and adequate teaching will imply that due emphasis is laid upon giving the pupil a conscious recognition of "methods" of meeting new situations.*

It may be helpful to add that among these immediately generalized outcomes from the study of secondary mathematics are critical abilities in observation, comparison, and discrimination. The successful grasp of geometrical proof, for example, and its use in original solution, necessitates the constant calling into play of habits of *analysis*—the recognition of similarities and differences, etc. There seems to be no doubt that the subject-matter of mathematics offers to an unusual degree opportunity for training in accuracy in "reading." It is the constant complaint of teachers of our higher grades and of educational investigators in the quantitative movement that pupils cannot "read," that is, that they have not built up the ability of obtaining meanings rapidly and accurately from the printed page. Because of the specific connotation of the terminology of mathematics its mastery exerts a pressure, a leverage, over the pupil's reading habits to a degree not found in the content of other subjects of study. Carelessness, lapses of attention, and distraction in the reading of mathematical problems are so fatal to the pupil's success in mathematics that he is forced to recognize the importance of an intensive degree of concentration and care in its reading. The very "taking," therefore, of a course in mathematics promotes the development of critical abilities in observation, comparison, and discrimination which, after all, are involved in meaningful reading.

III. RELATIVELY REMOTE AND LESS TANGIBLE OUTCOMES FROM HIGH-SCHOOL MATHEMATICS

Detailed psychological analysis of the principles of learning in descriptive geometry, in elementary geometry, has made prominent the primary importance of the development of right attitudes to the same extent that sound problem-solving attitudes have come out of the learning of school studies. So, too, other types of attitudes contribute immediately to success in the mastery of the material. For example, the development of the attitudes of orientation with mathematical or quantitative material. This implies attitudes of familiarity with the elements of new situations as expressed in the form of "issues"—something to be solved or proved with reference to quantitative material. Tendencies to adjustment or response, another way of defining attitudes after all, are at once set up with practice in responding; attitudes, the relations leading up to response, are necessary accompaniments of the activity, and determine the clearness of meanings. Thus practice in the solution of mathematical problems involving step-by-step *analysis*, *comparison*, and *discrimination* and culminating in the successful completion of the solution or proof develops constantly more active and more pertinent attitudes. These in turn contribute to more immediate acquisition of meanings and thereby further the efficiency of the learning process. Thus we see the outstanding importance of the development of attitudes of orientation.

In addition we find as important outcomes the growth of attitudes of confidence in one's ability to successfully meet situations which involve in any way a quantitative setting. The successful mastery of consecutive steps in the original solution of a problem leads to just such attitudes of confidence. The implication is direct and important that exercises and problems should thus be introduced to the pupil with but gradually increasing difficulty, and that sufficient practice should be given in each step in the process to develop confidence in one's ability to solve new problems of a similar kind—*through the recognition of one's own skill as shown by actual performance.* The chief purpose of such practice should be a lessening of the emotional disturbances in the response-system which will leave the individual free to work critically and analytically through his problem to a successful solution.

But there are other still more generalized attitudes which have an importance as outcomes from the study of high-school mathematics. They include, for example, the trained perspectives which one gets of an important field of knowledge and of economical methods of intellectual analysis which men have developed. It is difficult to distinguish between attitudes of the sort discussed above and these generalized perspectives, but it may be helpful to distinguish them in terms of specificness of application. In this sense perspectives or mental backgrounds, of both mathematical knowledge and various phases of abstraction and method which are efficacious in contributing to the interpretation of situations, are to be thought of as distinct outcomes from the learning process in mathematics. These will become larger and aid more effectively in the correct interpretation of new situations as the practice in using correct analytical forms and working with many types of mathematical solution is varied.

There is in addition to the above group of purely intellectual outcomes a definite group of emotionalized outcomes which undoubtedly contribute to the efficiency of the learning process. There is, for example, the "set" of the learner, both physiological and mental, the particular make-up of which at any special time will determine the efficiency of his response, in spite of the readiness of the individual from the standpoints discussed above. A careful study of the learning process in various manual skills, for example, by one of the present writers indicates that the degree to which the pupil will succeed in handling given problem-material in a particular class exercise will be determined by these mental "sets," and by the "emotional tone" of the learner on a given day. With this large and important group of outcomes from school studies, the writers do not propose to deal in the present monograph except to make clear their belief in their existence, in their efficacy, and in the fact that they themselves are largely outgrowths of successful responses which are set up by definite training in the specific aspects of the course of study.

PRELIMINARY ANALYSIS OF ILLUSTRATIVE MATHEMATICAL SUBJECT-MATTER[1]

Having stated the aims and outcomes of instruction in secondary mathematics, and having analyzed "thinking" or "problem-solving" in mathematical courses, we are led to inquire how much opportunity is provided the pupil for securing training in this process. With the emphasis on the formal processes in the typical course in algebra (80 per cent of the currently used textbook space is devoted to them), it is evident that memory and habit formation of a purely manipulatory and "skill" kind really have the greatest claim on the pupil's time. This leaves little opportunity for contact with situations providing genuine opportunities for training in problem-solving. Thus, "algebra," as it is now organized, contains very little real mathematics if mathematics is essentially a subject that primarily involves logical thinking. For example, there is serious doubt as to whether the pupil who survives the present course has any adequate concept of *functional relation* or *dependence*, even though this is recognized by the most able critics of mathematical education as the *central organizing principle of algebra*. In the case of geometry, however, this training value is much more prominent. Here the pupils, through the original problems or "exercises," get abundant practice in that most important mental activity: *the analytical study of a group of data or*

[1] The writers will shortly submit their complete psychological analysis of the subject-matter of high-school mathematics in a book: *The Psychology and Teaching of Elementary Mathematics*.

relationships resulting in the discovery and establishment of a truth. From the point of view of this psychological analysis of learning in mathematics the writers are constrained to protest here against the practice of many teachers in emphasizing the propositions of geometry (completely proved in the textbook in most cases) to the exclusion of the original exercises. It is more than probable that in the future the demonstrations of space relations will emphasize, more and more, the method of proof—analytical reasoning—by which the conclusions are established rather than the synthetic organization or syllogistical proofs as is now the general practice.

To contribute facility in stating and determining functional relationships "algebra" is organized about the equation. Its method is essentially that of the science of the equation, and *this method does not change* from the beginning to the end of the course. Similarly, to provide simple concrete material about which pupils may reason —analyze and organize—geometry is organized about common space forms. The latter material, especially, has a few pronounced characteristics, it is compact, made up of relatively few types of matter. The material, in other words, is decidedly homogeneous. It is always set in specific form. Thus, geometry is essentially a "form" study, it gives relatively little "information." *Like algebra, its method does not change* from the first to the last proposition to be proved or problem to be solved.

What might be "algebra's" contribution.—Algebra provides an opportunity for training in reasoning—in thinking—both through the verbal problem and in the initial stages of learning in the formal operations. It provides this opportunity in the following specific ways:

1. It sets problems definitely, each one consisting of a specific numerical situation arranged in verbal form and offering opportunity for training in "translation." From the standpoint of the analysis of "reasoning" in verbal problems *"translation" is the critical point in learning* in algebra. It involves the fundamental steps pointed out above which are comprised in logical thinking. It has two outstanding phases: first, the recognition of equality between two quantities or magnitudes, and second, the ability to express these in symbolic language.

If we agree that mathematics consists primarily in recognizing, stating, and determining the definite relationship between quantities and magnitudes, *translation* clearly becomes a necessary basic element in the procedure of solving a mathematical problem. Let us recall at this point that in chapter II *it was shown that almost no specific training is given the pupil in "translation" as such* under the present organization of the course. It is doubtful if a pupil studying the traditional text would recognize the importance of this process since little or no conscious instruction is offered in it. Yet it is clear that it is one of the two basic steps.

2. The material of algebra can be *organized to present an ordered system of "cues" for response*—not so definitely organized, however, as in the case of geometry. Furthermore this first step in solution is perfectly generalized—more so than like steps in other subjects; for example, the analysis of the problem leading to the recognition of the proper use of the equation in the problem and the correct representation of the unknown quantities and their relations to known quantities.

As a specific illustration of the degree to which the *material of algebra can be organized as an "ordered system of cues,"* let us take an illustration. The pupil can be made much more efficient if he is led to see the value of "ordering" his procedure in organizing systematically the steps in the solution of his equation; for example, in the simpler work of the following equation:

$$2(x-5)-8=10-(x-3)$$

First step: Removal of parentheses (R.P.) $2x-10-8=10-x+3$
Second step: Collecting terms (C.T.) $2x-18=13-x$
Third step: Separating knowns from unknowns $3x=31$
Fourth step: Dividing by coefficient of unknown $x=10\frac{1}{3}$
Fifth step: Checking (Ck.)

3. Simple examples of the opportunity for reasoning are found in the first portions of the present course, especially in those texts which introduce the equation and general principles early. The equation axioms themselves and their application in the solution of simple equations involve reasoning for the beginner but are soon short-circuited and made habitual. In the same way it is possible

and desirable to introduce the laws of operation of signed numbers in a reasoned rather than in a dogmatic way. This raises pertinently a most important and much-debated issue in learning school subjects, namely, "Shall the various operations be taught carefully upon a thoroughly rational basis, or shall the pupil be permitted the early use of the rule-of-thumb method of attacking problem-situations?" *In other words, is habituation prior to rationalization desirable?* We shall return shortly to the more detailed consideration of this problem.

It is extremely difficult to indicate the point at which formal problems that are solved by reasoning processes become subject to more or less automatic reaction—to well-ordered systems of cues. It is probably much less rapid than in geometry; the material is so varied in nature. In the initial stages of the learning process, all verbally stated problems, even of the simplest sort, afford opportunity for training in reasoning. For example, in dealing with the idea of "literal number" we find: "If b represents the number of square feet in a rectangle, what does $\frac{2}{3}b$ represent?" There is no opportunity for short-circuiting the process by substituting in a formula. The procedure involves "thought" even of the one-step type. Even if solved by roundabout methods, the process would gradually reduce itself to habitual reaction. Algebra teachers will do well not to provide the "short-cuts," the formulas and rules of procedure, too early. The fundamental tools—for example, symbolism, drill in using it, formulas, the equation, etc.—have to be given early, but beyond these simple working tools detailed opportunity ought to be given the pupil to *reason*.

CHAPTER VIII

CURRICULUM-MAKING IN SECONDARY MATHEMATICS: CURRENT TENDENCIES AND PRELIMINARY INVESTIGATIONS

A. A FOUNDATIONAL PROGRAM

In chapter II we sketched a program for the construction of a course of study in ninth-grade mathematics. This program was to be made up of four important steps:

1. An analysis of the present content of the course. (The complete report of this has been made in chapter II.)

2. A statistical determination of the *use of the topics* of present ninth-grade mathematics *in other high-school subjects*.

3. The design of a course of study upon a clear-cut determination of *occupational needs* of high-school mathematics; those mathematical materials found to be of use in particular life-occupations (providing also for those activities outside either the school or the occupation). It was pointed out that a tabular analysis of both non-academic and non-occupational needs would have to be made also.

4. Psychological foundation. This fourth step is very important, for the course of study should be built upon a psychological criterion: organized in terms of gradually increasing difficulty, and in harmony with the ways in which children learn.

In this connection we should recall that the *fundamental principle underlying the construction of courses of study in the public schools of America* is that *subject-matter shall be general and not specialized*. The design of our courses of study is diametrically opposite in thesis to that, for example, of the German Empire. We have adopted the principle of putting into our courses only those types of information, those fundamental skills, those types of reasoning situations, which ought to be *common* to the adult lives of a relatively large proportion of the boys and girls who pass through them. Thus, American curricula are fundamentally general and not specialized.

Hence, if mathematics beyond arithmetic is to be *required* of relatively all children who pass through the public schools, it must at least contain and ought to emphasize those fundamental quantitative *notions* and tool *operations* which are necessary to successful handling of either tool- or problem-situations in later life. These tool- or problem-situations may be found in either (1) *further study in the public-school course;* (2) the *specific occupational activity* of adults; or (3) the *daily activities outside either school or occupation* (including the leisure-time activities).

Criteria for the design of a course of study.—Therefore, the construction of a one-year course of study in mathematics *which shall be required* of relatively all children who pass through the public schools necessitates the satisfaction of two criteria:

1. The *social criterion.* This implies that we must determine the mathematical needs of adults in their occupational and non-occupational activities. A propos of situations found in the trade school, the "corporation" or "apprentice" school, and the "correspondence" school—all are desirable for inclusion in public high-school courses *only on condition, first,* that they are common to the occupations of a reasonably large proportion of the adults, who as children pass through the grades in question; and *secondly,* that they are not so specialized as to be relatively unintelligible to children. The latter point implies that there must be in the situations which we include in our course the possibility of grading the subject-matter minutely and economically *in terms of the learning of children.* This points to:

2. The *psychological criterion.* Whereas considerable progress has been made in the improvement of the mathematics course of study from the standpoint of adapting it to social needs, almost no advance has been made in constructing courses strictly in accordance with a psychological analysis of learning the various types of subject-matter which are represented. *Our second important principle, therefore, is that of organizing each element of the mathematics course of study completely in terms of the facts of learning which have been established for the kinds of subject-matter in question.* To the satisfaction of this criterion our classroom experimentation is contributing in an important way. Each item of subject-matter

which demands admittance to the course of study on social grounds must be carefully studied from the standpoint of learning. *This cannot be done introspectively by adult-trained judgment—it must be done by careful classroom experimentation.*

Clearly our analysis of the present content shows the glaring inadequacy of the present course in "thought" materials. Throughout this report, the reader has been brought face to face with the extreme emphasis that our school course in high-school mathematics now lays upon the formal and manipulatory aspect of the subject-matter. Furthermore, the historical development shows that this emphasis upon the formal has not been characteristic of the teaching of high-school mathematics through its entire course, but that it is a relatively recent development. The past discussions have shown, furthermore, that the course has not only been unsatisfactory from the standpoint of the relative degree to which formal and reasoning elements have been emphasized, but also that only within the past few years has there been any recognition at all of the need for designing textbooks on the basis of a thoroughgoing social program.

More recently, however, the sociological emphasis in education has brought about several distinct movements which affect the reorganization of the mathematics course of study. The first has revealed a distinct vocational demand for the modification of the subject-matter and has shown three outstanding aspects—the agricultural, the commercial, and the industrial. The second movement has revealed itself in an insistent demand for economy of time: first, economy of time by establishing minimal essentials of the subject-matter, and second, economy of time in teaching and learning. The third movement has called for the rearrangement of the subject-matter of mathematics in the intermediate grades to fit the "junior high school," which has resulted from the reorganization of the grades.

B. RECENT ATTEMPTS TO MODIFY THE PRESENT CONTENT OF THE HIGH-SCHOOL MATHEMATICS CURRICULUM

1. **The extreme vocational demand.**—During the past decade there has been a distinct movement from without the school,

among those engaged in managing the industrial and commercial enterprises of the country, to fill in the gaps which have been evident in the training of children in the public schools along vocational lines. This has been well shown in the movement to organize associations of so-called "corporation" schools. In 1911 the National Association of Corporation Schools was formed, made up of representatives of something more than a hundred leading industrial and manufacturing concerns, covering a variety of types of occupations. The delegate in nearly every case is the employment manager or the representative of the company who has immediate charge of the employing of labor. These men have recognized that, due in part to the early elimination of boys and girls from school and in part to immigration conditions, there has been a distinct and rapidly growing need for extra public-school education. It has been assumed that many, at least one-third, of these corporation schools have included in their curricula some instruction in mathematics. Our investigation shows later that this instruction generally is merely an application of arithmetic together with, in the case of the manufacturing industries, the larger engineering concerns, etc., the applications of algebra, trigonometry, and geometry. In the main this instruction has been distinctly vocational and applied, and has broken away most decidedly from the academic course of study that is taught in the conventional mathematics course in the public schools. The subsequent discussions of this chapter will point out clearly the facts concerning the corporation courses of study in mathematics.

Accompanying this innovation we have seen for some time the rapid growth of the so-called "correspondence" schools. These, too, have come to fulfil a demand for education of a semi-professional and applied sort for those who have had a partial public-school education and who need to have their mathematical and scientific principles applied specifically to the trades and engineering.

In addition, there has been a rapid development of "extension" and "short courses," largely in state universities and in agricultural and mechanical arts colleges, but also in connection with normal schools and public-school systems. These institutions, too, are exhibiting distinct tendencies to organize, for the person whose interest in mathematics is largely of the utilitarian sort, a course quite thoroughly unacademic and made up largely of "handbook" material. In the same way, business and commercial colleges have extended their interests to include some brief reference to and study of applied mathematics of the more elementary sort.

2. The degree to which the traditional books and mathematics courses have been modified.—The movements discussed above, however, have all been initiated by persons more or less dissatisfied with the contribution of the public school and are in the direction of a real socializing of secondary mathematics. The criticism from the outside and from within the public school has taken shape more recently in the recognition, on the part of school men themselves, of the evident degree to which the public school fails to satisfy the social criterion. There are four distinct aspects of this partial reformation of the content of our traditional mathematics books from within.

A. Books on "applied mathematics."—Since 1910 nearly a score of textbooks have come from the press, written in the main by authors of traditional "algebras" and "geometries," and in evident acquiescence to the demand for the introduction of more applied material in our high-school mathematics books. They are represented, for example, by Wentworth and Smith's *Vocational Algebra*. Here we have a traditional organization of content from the standpoint of the topics which are included from the conventional course, with an emphasis upon industrial applications.

B. Movement for "combined" or "unified" mathematics.—In the past few years we have had striking examples, from the contributions of Professors Myers and Breslich, of the movement for the combining of the essentials of geometry with the essentials of algebra. Our analysis of the content of such books is shown in detail later. Suffice it to say here that these books have been based upon the point of view that there are fundamental operations in geometry and algebra—distinct bodies of material in the two traditional subjects which ought in some way to be merged or unified into a coherent course. The best example of that at the present time is Mr. Breslich's *First-Year*, *Second-Year*, and *Third-Year Mathe-*

matics. It should be said that this movement, which has affected a considerable number of schools aside from the University High School of the University of Chicago, was originally and now is primarily an outgrowth of the training of preparatory-school students. The very direction in which the work is developing, namely, the issuing of textbooks organized by "years"—one to fit each year of the traditionally organized high school—indicates that protagonists of this movement are primarily interested in the preparation of students for college.

The point of view that will be taken throughout this discussion is that ninth-grade mathematics will certainly come to be regarded by school men in our generation as the last year of mathematics that will be required. There are marked evidences of the tendency to lead mathematical instruction through a reorganized intermediate school mathematics into a last year required course in the ninth grade. So we would point out that this movement for combined mathematics is evidently looking upward into the college instead of downward into the grades. However, the movement is a distinct exhibit of the recognition of our more progressive school men that algebra and geometry, as traditionally organized, need to be carefully canvassed with a view to the working out of continuous sequential courses.

C. The tendency toward a thoroughgoing modification of content in "general" mathematics.—Another illustration of the changes going on in the content of the textbooks used in our public-school classes is shown by two recently issued books, Evans and Marsh, *First Year Mathematics*, and Keal and Phelps, *Secondary Mathematics, I*. The Keal and Phelps textbook is a distinct attempt of the school men to work out "a course in mathematics that will enable the student to recognize fundamental principles and to apply them in shop, drawing-room, and laboratory; and, second, to so develop the course that each year's work will be a unit." It should be pointed out that earlier books which have applied mathematics to the later occupational needs of children have not been founded on the detailed investigation of these later occupational needs. Keal and Phelps's textbook was the first to be so constructed, although only in a partial way. It places in the hands of the teacher a systematic presentation of fundamental quantitative devices, the latter having been compiled with some reference to the needs of certain mechanical occupations. Just what a radical innovation the book is will be shown by our tabulations later in this chapter. The book, however, is an interesting exhibit of the one-sided point of view that school men are now taking in their construction of textbooks. The Keal and Phelps book is organized about certain studies which are so highly specialized that certain chapters are almost "non-teachable" to the average student. For example, it is extremely difficult to defend the detailed content of the chapter on "pulleys, gears, and speeds." In the remainder of the book there is not revealed such an unusual emphasis upon special studies, but it is clear that from the standpoint of the psychology of learning such books must be said to be traditional. By this we mean that in it there has been very little consideration of the step-by-step process by which children learn to handle intelligently the fundamental devices for quantitative thinking. These books, however, provide a striking confirmation of the changes in the attitude of high-school teachers of mathematics toward the content of their subject.

D. The "junior high school" mathematics movement.—During the past year several books have appeared which were issued in an attempt to satisfy the demand for a unified course in mathematical instruction in the seventh, eighth, and ninth years. We already have heard of a number of attempts to combine the essentials of algebra and geometry in the seventh and eighth grades. To do this successfully, of course, the schools need systematic textbook material. Several such collations have already been published—Vosburgh and Gentleman, *Junior High School Mathematics*, and the Wentworth-Smith-Brown volume, for example. These present tendencies, as exhibited by such books as the Vosburgh-Gentleman and the Wentworth-Smith-Brown, show clearly that high-school men are working toward a two-year or a two-and-one-half year course which will cover the essential principles and applications of arithmetic, algebra, and geometry. The detailed scheme of the books, as well as of the others which we have discussed in this monograph, reveals, however, that the *fundamental psychological weak-*

ness in mathematics textbooks centers around the poor grading of problems. Furthermore, the subject-matter of these books is organized in such a way as to be distinctly remote from the pupils' experience. It is too abstract.

We may sum up, then, by commenting upon the improvement of the content of the subject-matter of first-year mathematics, upon the elimination of much dead wood in the curriculum, and upon the tendency on the part of school men themselves to adapt mathematics more closely to social needs. *It is clearly evident that important advances are being made.* On the other hand, *from the standpoint of the psychological criterion*, analyses of textbooks on high-school mathematics show that *little or no advance has been made up to the present time.*

C. THE DEGREE TO WHICH THE TOPICS OF FIRST-YEAR ALGEBRA ARE USED IN SUBSEQUENT HIGH-SCHOOL SUBJECTS

In the evaluation of the social criterion for the building of the course of study the most important single principle of use is that of future use in later high-school subjects. *Later curriculum needs* form a most important principle upon which to work in analyzing the mathematical needs of students. We have seen in our day the rapid development of high-school populations. More and more children are continuing through the tenth, eleventh, and twelfth grades. The holding power of the school is constantly becoming greater.

Let us add another principle upon which to organize our discussion. Administrative developments of the past five years show clearly that the seventh, eighth, and ninth grades will be regarded in our generation as a "unit"—the *intermediate school.* Courses of study will be designed more and more with a view to satisfying the needs of this new type of school. Add to the recognition of this situation clear insight into the tendency to require less mathematics of all pupils and we have a most important principle upon which to work, namely, *ninth-grade mathematics will without doubt*, in our generation, *be the last required year of mathematics which all children will have to take.*

Furthermore, we must recognize that we are dealing with a critical situation in first working on the ninth-grade problem. Personally, the writers feel that the proper place to begin the reorganization is with the seventh grade. The critical need, however, that faces us has to do with the present ninth-grade situation. Hence the emphasis upon ninth-grade mathematics in this monograph. In the future publications of the writers the extension of the course of study, which is being evolved as a result of our analysis of ninth-grade mathematics, will be made downward into the seventh and eighth grades.

Note the contrast, therefore, in the tendencies among the writers of high-school textbooks of mathematics. Nearly every writer in the field issues a "First Course in Mathematics," followed by a "Second Course" or "Second Year." Some of them too are committed to a "Third Course" and a "Fourth Course." Two things stand out clearly from a consideration of this tendency. In the *first* place, these writers are interested in the preparation of students for college. The primary value in their first course, therefore, is found in its preparatory function—either preparing for college or preparing for another year of mathematics. In the *second* place, it is quite clear that these writers are overlooking the tendency toward extending the analysis of the present organization of mathematics downward into the elementary school.

Hence the importance of including in this last year the most important mathematical notions and devices. The determination of what these notions and devices are cannot be done introspectively—at the desk or even in the classroom. If we accept the social criterion in constructing courses of study, we will make careful scientific studies of the more general and more common mathematical needs. This necessitates a careful analysis of the actual use of mathematics in other curriculum studies as well as in later mathematical courses. We shall find out, e.g., whether polynomial division and the various types of factoring are emphasized in later high-school courses as they are in the present course. If they are not, clearly we may be unable to defend their inclusion in the present course. Be that as it may, we will at least pile up basic data upon which we can design intelligently a last year required course—furthermore one that will have adequate preparatory values for later mathematical courses.

A TABULAR ANALYSIS OF THE USE OF FIRST-YEAR ALGEBRA IN OTHER HIGH-SCHOOL COURSES

During the year 1916–17 the writers, collaborating with Miss Marjory Miller, canvassed in minute detail the extent to which the operations of "first-year algebra" were used in each of the remaining high-school subjects. Koos showed that in other subjects, as well as in mathematics, the textbook determines the course of study. He also established for the North Central Association of Secondary Schools and Colleges the frequency of use of each of the outstanding textbooks. In preparing our tabulations, therefore, we selected on a basis of his figures the one textbook which clearly had the most representative frequency of use. The frequency of use of each book listed is given in Table 7. The table makes clear

TABLE 7

LIST OF TEXTBOOKS AND MANUALS WHICH WERE ANALYZED, TOGETHER WITH STATEMENT OF RELATIVE FREQUENCY OF THEIR USE IN NORTH CENTRAL HIGH SCHOOLS

Subject	Text	Fractional Part of North Central High Schools in Which Text is Used
First-Year Algebra	Wells-Hart	28/92
Plane Geometry, Solid Geometry	Wentworth-Smith	47/122
Commercial Arithmetic	Schneck	
Physics	Millikan and Gale (Text and Manual)	40/113 and 28/113
Chemistry	McPherson and Henderson (Text and Manual)	33/94
General Science	Hessler (Text and Manual)	
Physiography	Dryer—Text; Darling—Manual	

the fact that these textbooks typify the present status of offerings of subject-matter in the various courses in question. Furthermore, reasoning from the status of the nine books which we tabulated in first-year algebra, we assume that the tabulation of the use of mathematics in the one most representative book will lead to a determination of use that is fairly representative of all of them.

In this tabulation three distinct questions have been considered: (1) To what extent is each of the sixteen major operations with their thirty-three subdivisions actually needed to handle the subsequent topics in first-year algebra itself? (2) To what extent is each one needed in later mathematical courses—plane and solid geometry and advanced algebra? (3) To what extent is each one needed in other high-school subjects?

The same classification of topics (thirty-three specific ones) was used as in the case of the 1915–16 tabulation made by Mr. Denny. Frequency of use was determined by a detailed count of single problems—the count in every case being made for the entire book and being carried through and checked with painstaking care. Miss Miller's tabulations are very accurate, indeed. For our purposes they portray, definitely, the present status of the extent to which pupils actually use the material of first-year algebra. The care with which the study was conducted may be shown by a brief statement by Miss Miller: "Our exact problem was the determination, in just how many of the 3067 problems, a knowledge of the operation of addition, subtraction was necessary to the solution of the problem, and of the total number of problems in which addition and subtraction, in how many of these was it used in multiplying and dividing; in how many in equations, in how many in addition of fractions and so on through the list of the 31 other operations. When this tabulation had been followed through for addition and subtraction 31 similar trips were made for each of the 31 other processes."

Limitations of space will prevent us from giving a detailed definition of each of the operations listed. The report of this part of the study is being published in more detailed form elsewhere. In analyzing the use of algebra in plane and solid geometry, we found its use first in the actual proof of theorems, corollaries, and exercises which required rigid demonstration; second, in numerical problems or exercises which were based on the various propositions.

Let us next summarize the important statistical results of this investigation and then bring sharply in review the outstanding conclusions. To present them in brief space, we will tabulate the material given below. First, however, let us state in condensed form the outstanding conclusions from the study of the use of various topics of first-year algebra within first-year algebra itself.

TABLE 8

PERCENTAGE OF TOTAL NUMBER OF PROBLEMS (IN WHICH ALGEBRAIC TOPICS ARE USED) IN WHICH USE IS MADE OF PARTICULAR OPERATIONS

	Plane Geometry	Solid Geometry
Addition and subtraction	25.7	7.5
Multiplication and division	57.6	34.1
Exponents	21.2	27.7
Equation 1 *D.* 1 *U.*	33.4	20.2
Removal of ( )	17.5	16.2
Evaluation	62.4	81.8
Radicals	32.5	28.9
$(a+b)$	.3	....
$(a+b)(a-b)$	.8	....
$(x+a)(x+b)$	....	....
$(ax+b)(cx+d)$	....	....
$(x+y)(x+xy+y)$	....	....
Factor $ab+ac$	1.4	....
" $a^2+2ab+b$	.6	.3
" a^2-b^2	.6	....
" x^2+px+q	....	....
" ax^2+bx+c	....	....
" x^3+y^3	....	....
Recognition of type	2.5	....
Equation 2 *D.* 1 *U.*	16.4	19.6
" 1 *D.* 2 *U.*	3.9	....
" 1 *D.* 3 *U.*	....	....
L.C.M. and H.C.F.	....	8.1
L.C.D. and lowest terms	.3	7.8
Complex fractions	....	.6
Add. and sub. of fractions	.8	2.3
Mult. and div. of fractions	13.3	6.9
Fractional equations	11.0	5.8
Literal equations	2.5	11.6
Literal formulas	.3	9.2
Imaginaries	....	....
Graphs	....	4.0
Ratio and proportion	12.5	6.9

1. *Relative use of certain topics of first-year algebra in first-year algebra itself*
   a) Thirty per cent of the problems in the book examined, which is typical of other first-year books, involve some kind of *factoring*. More than 40 per cent of these *factoring* problems occur as formal exercises for the *learning of one or more particular kinds of factoring*. More than 75 per cent of these factoring problems are used *only* in some type of factoring. There are but two operations, aside from forms of factoring,

in which factoring is used as an *economical tool*, namely, combinations of fractions and solution of fractional equations.

b) *Graphic representation* is introduced as an isolated operation—shown by the fact that 70 per cent of its use is confined to formal exercises in learning to make graphs.

c) There is more provision made for practice in working "*imaginaries*" than for graphic representation or for formulas.

d) There is more practice given to *reducing fractions to lowest terms* than to

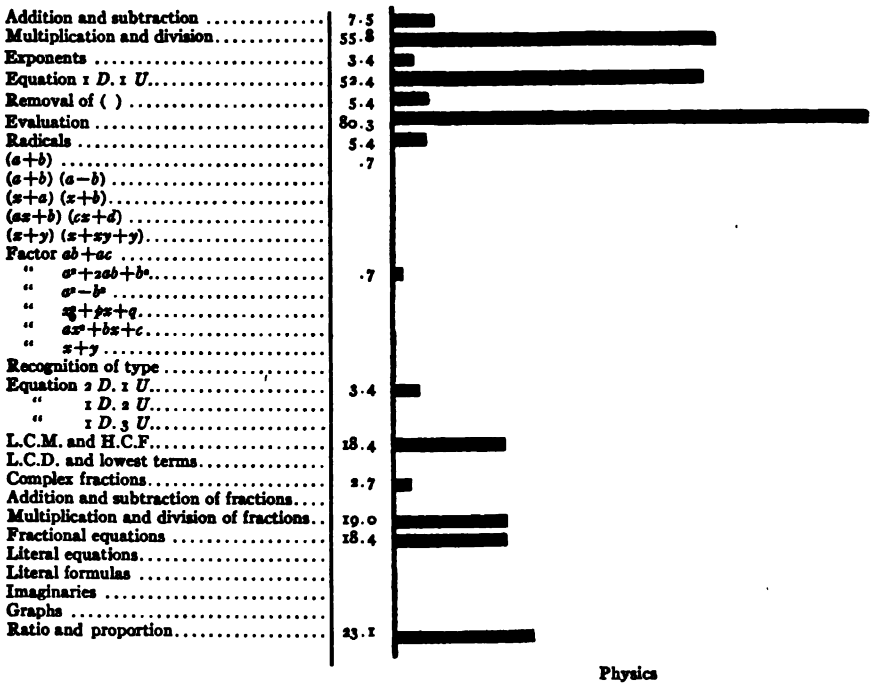


DIAGRAM II.—Percentage of total number of problems (in which algebraic topics are used) in which use is made of particular operations.

graphic representation, literal formulas, and literal equations combined.

e) Only 9 per cent of the problems of the book involve the use of *evaluation* (even this small amount includes "substitution" in solution of simple equations with *two* unknowns).

f) The use of exponents occurs in half the problems of the book.

2. *The use of operations of first-year algebra in advanced algebra*

a) While evaluation gets little emphasis in first-year algebra, at the time when it is necessarily learned, it begins to reveal its importance in advanced algebra and other courses (20 per cent of the problems of the book make use of it).

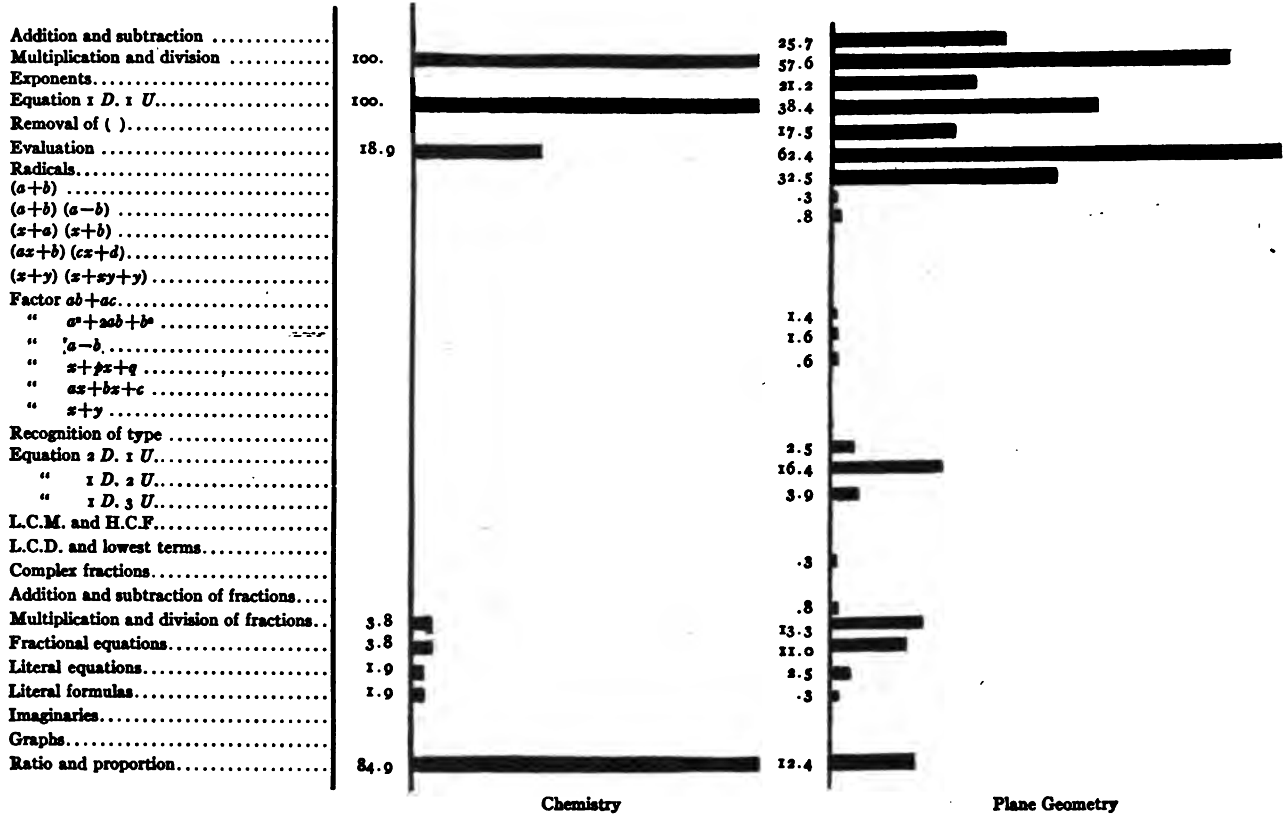


DIAGRAM III—Percentage of total number of problems (in which algebraic topics are used) in which use is made of particular operations *b*) *Ratio and proportion* and *graphic representation* get almost no attention.

c) There are more examples on *imaginaries* than on formulas and graphic representation combined.

d) There are more examples on factoring of "*sum and difference of two cubes*" than on graphic representation and on formulas combined.

e) Classifying the operations of first-year algebra in order of frequency of use in advanced algebra, we find (1) *of great importance:* parentheses, radicals, and evaluation; (2) *showing considerable use:* equations in the first degree with one unknown, exponents, multiplication and division of fractions, logarithms; (3) *revealing little or no use: the remaining twenty-five operations which are brought into elementary algebra.*

3. *The use of operations of first-year algebra in plane geometry and solid geometry*

We may summarize the findings briefly by classifying the operations in terms of "wide use," "considerable use," and "little or no use." (The tables read as follows: "Of the total number of problems in the physics text, *which involved the use of algebra whatever*, 52.4 per cent of them required a knowledge of "equations of the first degree, one unknown," etc.)

A. OPERATIONS WHICH SHOW FREQUENT USE IN "PROBLEMS" IN PLANE AND SOLID GEOMETRY

Plane Geometry		Solid Geometry	
Evaluation	62.4%	Evaluation	81.8%
Multiplication and division	57.6%	Multiplication and division	34.1%
Equations of the first degree, one unknown	38.4%	Equations of the first degree, one unknown	20.2%
Exponents	21.2%	Exponents	27.7%
Radicals	32.5%	Radicals	28.9%

B. OPERATIONS WHICH ARE USED TO A NOTICEABLE EXTENT

Plane Geometry		Solid Geometry	
Removal of parentheses	17.5%	Removal of parentheses	16.2%
Equations of the second degree, one unknown	16.4%	Equations of the second degree, one unknown	19.6%
Multiplication of fractions	13.3%	Multiplication of fractions	6.9%
Fractional equations	11.0%	Fractional equations	11.6%
Ratio and proportion	12.4%	Ratio and proportion	6.9%
		Literal formulas	9.2%

C. OPERATIONS WHICH HAVE LITTLE OR NO USE IN PLANE AND SOLID GEOMETRY

Plane Geometry	Solid Geometry
The remaining operations	The remaining operations

4. *The use of operations of first-year algebra in physics*

Using the same classification as in geometry, we find:

A. OPERATIONS WHICH SHOW FREQUENT USE IN PHYSICS

	Text	Manual
Multiplication and division	55.8%	47.2%
Equations of the first degree, one unknown	52.4%	47.2%
Evaluation	80.3%	72.0%

B. OPERATIONS WHICH ARE USED TO A NOTICEABLE EXTENT

	Text	Manual
Ratio and proportion	23.1%	21.3%
Fractions (multiplication and division)	19.0%	21.4%
Fractional equations	18.4%	16.8%
L.C.M. and H.C.F.	18.4%	16.8%

C. OPERATIONS WHICH HAVE LITTLE OR NO USE IN PHYSICS

The remaining twenty-six operations

5. *The use of operations of first-year algebra in high-school chemistry*
Using the same classification as above:

A. OPERATIONS SHOWING FREQUENT USE

	Text
Equations of the first degree, one unknown	100.0%
Multiplication and division	100.0%
Ratio and proportion	84.9%

B. OPERATIONS WHICH ARE USED TO A NOTICEABLE EXTENT

Evaluation	18.9%

C. OPERATIONS WHICH HAVE LITTLE OR NO USE IN CHEMISTRY

The remaining twenty-eight operations

Is it not clear that at least one-half of the material of first-year algebra cannot be defended in terms of academic use, that is, of use in other high-school subjects? How can we reconcile the spending of five to six weeks of the pupil's time (15 per cent of the whole school year) on such material, for example, as special products and factoring, when even in academic subjects such as physics and chemistry (in which if anywhere there would be a utilitarian need for training in the use of such material) we find absolutely no use for manipulatory skill? Surely, if such material is not demanded as a tool for effective work in other subjects *it cannot be included in a course of study on the grounds of representing a definite human need* of at least a reasonably large proportion of the children who go through our schools. The people of this country will shortly demand a course of study in each of our subjects which is built specifically for the "ninety and nine" (actually, of course, the "thirty-five to fifty") whose education is completed with the ninth to the twelfth grades and not with the "one" who goes through the college and enters the profession. Our thesis is this: for those slight uses of the present topics of first-year algebra which we know are actually needed by the fraction of 1 per cent who go on into higher mathematics in college, *skill in the manipulation of those types* (such as factoring [$x^3 \pm y^3$]) *should be given then and there.* Furthermore, brief psychological analysis will show clearly that that is the most economical place to teach the skill itself, namely in connection with its application.

But canvass the matter more fully. Our detailed tables on physics show that for eighteen of the thirty-three topics now taught in first-year algebra there is absolutely no call. Similarly, chemistry has no need for twenty-five of them and plane geometry for seventeen of them. Of the remaining operations, factoring of the types $a^2+2ab+b^2$, $ab+ac$, and a^2-b^2 receives .6, 1.4, and .6 of 1 per cent of use, respectively. Even quadratic equations are seldom met by the pupil in other subjects of study after he leaves first-year mathematics. Note that in the geometry the use of algebra stands out in the numerical problems far more than in the geometric proofs of the propositions. In both sorts of proofs addition, subtraction, multiplication, and division are necessary to a large extent. Beyond the use of the four fundamentals the equation of the first degree in one unknown and to some extent of the second degree stands out as essentially important.

The high percentage found for exponents, radicals, and square root is due to the fact that wherever a quadratic equation was found, even of the types of the pure quadratic, $x^2=25$, the tabulator considered that a knowledge of both exponents and square root was necessary in the solution of the equation. *In practically 100 per cent of the cases a radical or square root operation means simply the ability to take the square root of a number and of an even power of the unknown.* Exponents run high in the propositions because each time a proposition demanded the squaring of a letter, standing for the side of a triangle or other polygon, the tabulator implied that this required a knowledge of exponents. Hence it seems clear that *only a rudimentary knowledge of exponents and radicals is required for the geometry work.*

The high percentage under H.C.F., L.C.M., and L.C.D. needs explanation also. In all the cases in plane geometry, L.C.M. demands merely the ability to clear a fractional equation in which there is a single numerical denominator. In solid geometry, in only twelve out of twenty-six cases does it mean ability to manipulate other kinds of denominators. *Hence the simplicity of knowledge of L.C.M. or L.C.D. which is required.*

The manipulatory skill in quadratics is also of the most elemental kind, for in fifty-six of the fifty-eight problems classified as of second degree with one unknown, *we find them to be pure quadratics.* The table shows clearly the great importance of evaluation. However, as to the importance of ratio and proportion in plane geometry, there is considerable doubt. The instances which are recorded are not proportions which are solved, but merely represent the statement in terms of four letters which represent four lines of a proportional relation between the lines or of a pure proportion in the propositions or corollaries under which the theory of proportion is taught. Hence, we find a very limited use for ratio and proportion which implies an algebraic statement with an unknown to be found.

Nothing can show more clearly, however, how closely we have had our attention centered on the task of preparing for college mathematics than this comparative analysis which we are making of the emphasis upon manipulation as contrasted with the opportunity provided for problem-solving. It seems very clear that the textbook writers have had college mathematics in mind. The primary interest has been in the logical organization and cataloguing of these operations for use in later manipulation. It has not been in the child's experiences or in his use for *real* mathematics. But most important of all *we have filled up our course with this manipulatory material and have left almost no real mathematics.* Of all the operations which are now taught, *the only one which has value as real mathematics is the "equation of the first degree"* and *that only as a tool for representing relationships.* Note furthermore that of the materials demanded for use in advanced work the emphasis on ratio and proportion made by the academicians themselves is hardly an emphasis upon the expression of a scientific law.

D. THE USE OF FIRST-YEAR ALGEBRA IN THE INDUSTRIAL OCCUPATIONS

But our program for evaluating the present course includes an analysis of other types of use, namely, that of occupational use in industrial life.

Let us not lose our perspective in this study of minute details. *Public-school courses of study must be built to include the materials which are common to the life-needs of a large proportion of our population.* We must seek a new orientation in this work which will relegate the point of view of the protagonist of "preparation-for-college" to its proper subordinate place. The extent to which the teacher of college mathematics determines the mathematical careers of high-school children is diminishing rapidly and will do so still more rapidly in our generation.

Now, retardation statistics prove clearly that not more than half of the students stay long enough in school to study algebra, even if it is offered in the seventh and eighth grades as is being done in our more progressive school systems. Of those who leave early a large proportion are driven, by their limited intellectual training, into the industrial occupations—unskilled, semi-skilled, and skilled. We do not know how many of the ninth-grade students go into these industrial occupations. It is very difficult, almost impossible, to find out. The writers have canvassed the larger cities of the country in an effort to get complete statistics on the ages at which children leave school, the grades from which they enter occupations, the occupations into which they go, how long they stay in these initial jobs, and the degree to which they change their occupation during the first few years after leaving school. It has been practically impossible to get such data. We are trying at the present time to establish this point through the vocational bureaus of the cities of the country.

Nevertheless, it seems clear, although we have no detailed figures, that relatively fewer of the boys and girls who stay through the tenth, eleventh, and twelfth grades go into the industrial occupations than of those who leave in the lower grades. Probably the number is very small. In case, however, it may be large, there is a definite need of determining to what extent algebraic operations are used by the workers in these fields who are doing the higher types of designing and organizing mechanical activities. For the *worker* himself our studies enable us to deduce that absolutely no algebra is used in occupational work.

As we have pointed out, during recent years the industrial world has organized various kinds of educational agencies of its own to fill in the more obvious gaps in our public-school training. For example, it is becoming relatively common for the larger industrial enterprises of the country to organize schools in which English is taught to immigrants. Accompanying this kind of development there have been marked tendencies since 1911 to increase the degree to which our textbooks have been applied to industry, agriculture, commerce, etc. Thus, we have two sources to which to turn in making our last study.

1. We can tabulate the content of various applied mathematics textbooks which have been designed in the main by teachers of industrial evening classes; in continuation classes in institutions like Pratt and Armour institutes; in such extension courses as those offered in the University of Wisconsin. The list of such representative "applied" textbooks and also the content of the new "combined" or "general" mathematics is given in Table 10.

2. Then we can analyze the content of the courses best represented by the handbooks, textbooks, and problem-books of the "corporation schools" mentioned previously. The writers have communicated with each of the 104 corporation schools. Eighty responded, showing great interest in our undertaking. It was found that only seventeen offer instruction in mathematics, and that the instruction of a goodly proportion of these is arithmetical in nature. However, a tabulation is made of all the material contained in such sources. The writers believe that these two types of material really typify the instruction that is offered by agencies outside the school.

The analysis of these books has been collaborated in by the writers, Miss Florence Morgan and Mr. P. L. Wise, in the planning, organizing of procedure, and decision as to material. Miss Morgan did the detailed tabulation for the "corporation school" and shop-books, Mr. Wise for the "applied" or semi-traditional books. For the present interpretation, however, the writers are alone responsible. Each of these two studies will be published in detail shortly. The tabulation has been done with great care, elaborate statistical forms being used in which the data have been minutely classified. We accept unreservedly for the purposes of the interpretations of this report the validity of the final form of these tabulations.

Three kinds of material were tabulated—algebraic, geometric, and trigonometric. The algebraic material was tabulated on a cross-sectional form, listing on one side the thirty-three specific operations. It should be remembered that during the three years of tabular work reported in this monograph the same organization of algebraic topics was used. The "problem" and the "page" were used again as the units of tabulation, the aim being to find the number and percentage of problems in which each of the thirty-three operations was used, together with the distribution of expository material by pages. On the other side of the cross-sectional form these operations were classified as formal or verbal (translation). Each in turn was subdivided so that we can locate the emphasis placed on decimal and common fractions and on integral material. This is very important, for one of our besetting sins has been the utter neglect of decimal material.

In the same way, the geometric material was tabulated under twenty-six type forms, circle, rectangle, triangle, cyclinder, pyramid, etc. The material under each form was classified, according to whether it was a "type of mensuration," or a "type of proof," or whether it was "construction." Under types of mensuration the emphasis on perimeters, areas, and volumes was determined. Types of proof were split up into proof by "empirical definition," "semi-empirical definition," "graphic definition," and "demonstrative proof." Tables 10 to 14 give the statistical conclusions from this minute classification. They make clear some very striking truths about the extent to which the supervisors, designers, and others who plan the mechanical details of our industrial enterprises make use of the material of our present ninth-grade courses in mathematics.

If the judgment of the designers of trade-school and corporation school texts can be taken as a criterion of the need for specific types of mathematics, we have here a striking confirmation of the findings of our investigation of use in high-school subjects. Standing out

TABLE 9

EXTENT TO WHICH CORPORATION SCHOOLS TEACH MATHEMATICS BEYOND ARITHMETIC

Number of firms addressed	104
Number of firms who replied	80
Number having no "school"	16
Number teaching no mathematics	51
Number using a standard school text	17
Number using special problem-books, handbooks, loose-leaf sheets, etc.	17*

*For 4 of these 17 the special material supplements the regular text.

TABLE 10

DISTRIBUTION OF ALGEBRAIC AND GEOMETRICAL SUBJECT-MATTER IN EIGHT TRADE-SCHOOL TEXTS AND NINE CORPORATION SCHOOL TEXTS

NAME OF TEXT	ALGEBRAIC MATERIAL			GEOMETRIC MATERIAL	
	Total Number of Problems	Formal Problems	Verbal Problems	Problems	Proofs
I. TRADE-SCHOOL TEXTS					
Palmer	1333	1082	251	322	92
Norris and Craigo	192	142	50	46	63
Breckenridge, M and M.	243	175	68	233	17
Hale (*Prac. Mech.*)	44	44	0	116	0
Hale (*Prac. Appd. Math.*)	213	130	83	311	29
Burnham	7	7	0	87	8
Dale	10	0	10	39	55
Johnson	6	0	6	59	4
II. CORPORATION SCHOOL MATERIAL					
Carnegie Steel	471	330	141	185	14
Fore River Ship Bldg.	3	3	0	14	40
Warner-Swasey	7	0	7	11	0
American Rolling Mill	82	82	0	0	0
Westinghouse Machine	353	309	44	61	0
Westinghouse Electric	36	0	36	37	0
National Commercial Gas	56	40	16	23	58
Sante Fe Railway	0	0	0	26	11
Commonwealth Steel	43	11	32	8	0

TABLE 11

PROBLEMS ON TYPES OF MENSURATION CLASSIFIED AS TO WHETHER THEY INVOLVE PERIMETERS, AREAS, OR VOLUMES*

Name of Text	Perimeters		Areas		Volumes	
	Algebraic	Verbal	Algebraic	Verbal	Algebraic	Verbal
I. TRADE-SCHOOL TEXTS						
Palmer	23	0	108	4	133	0
Norris and Craigo	18	0	13	9	16	0
Breckenridge, M and M	12	0	75	0	122	0
Hale (*Prac. Mech.*)	6	0	104	1	5	0
Hale (*Prac. Appd. Math.*)	53	0	117	7	6	82
Burnham	0	7	0	45	0	13
Dale	......	......	......	......	......	......
Johnson	0	10	14	12	0	22
II. CORPORATION SCHOOL TEXTS						
Carnegie Steel	0	43	0	62	0	17
Fore River Ship Bldg	0	8	0	1	0	5
Warner-Swasey	0	6	0	2	0	3
American Rolling Mill	0	0	0	0	0	0
Westinghouse Machine	0	4	0	24	0	9
Westinghouse Electric	0	8	0	20	0	8
National Commercial Gas	0	3	0	11	0	9
Santa Fe Railway	0	0	0	21	0	1
Commonwealth Steel	0	2	0	5	0	1

* Data tabulated from seven textbooks on applied mathematics and from nine problem-books of corporation schools.

foremost in a consideration of the relative use of algebra, geometry, and trigonometry is the fact that *no geometry beyond that given in the upper grades of the elementary school is given.* With the exception of literal notation there is no additional geometrical information or skill beyond that which the pupil has on entering the ninth-grade course. Furthermore, there is almost no tendency to give anything in the way of proof but a purely empirical or semi-empirical definition. There is practically no demonstration in these texts. A good illustration of the tendency is that we were unable to find a single example of congruency or similarity in either the trade-school or corporation school texts. In general we find that the most outstanding usage in these books is toward an emphasis on algebra. Trigonometry comes in also for definite recognition.

The algebraic material in the practical texts. The most striking characteristic of the algebraic material is that *almost no complicated* *forms* of the various operations are given. That is, in multiplication we find nothing more complicated than the multiplication of two binomials, e.g. $(3x+2)(5x-7)$, (no polynomial division and multiplication, for example).

Only two of the trade-school texts give any *factoring* (one of these is Palmer's, which we must characterize as a thoroughly traditional book). Norris and Craigo (the other text to give factoring) include only the very simple types. Only two corporation texts utilize factoring, e.g., one of these is Carnegie Steel (which leaves the impression of having been designed by a person having to do with traditional school practice). It contains nothing but monomial and trinomial factoring at that. *Special products* is included in but two corporation school books and in one trade-school book.

Graphs are used but rarely. Little use is made of the graphing of a line or the positive or negative idea. This all confirms our discussion of graphical representation as a *method* and not as a *type of subject-matter*. The corporation school man finds no occasion to make use of graphs as an isolated operation. Naturally he does not psychologize his subject-matter in mathematics enough to see clearly its true place as a fundamental method of representing numbers.

Ratio and proportion is common to all texts. Many very simple examples occur, however, which could have been just as well solved by arithmetic as by algebra. Lever problems abound.

Evaluation stands out as one of the three most-used operations—confirming our conclusions made in the other portions of this study. Compare with this the fact that almost no use is made of *literal formulas*. Thus we find that the shop man needs to be able to evaluate a formula but not to be skilful in solving such types.

Radicals appear only in the Palmer text. *Quadratics* occur there likewise. Certainly from the standpoint of actual use in the very occupations in which the organizers and designers would make use of the subject-matter they had been taught in the secondary school the emphasis upon such operations must be scrutinized closely. Compare here again the conclusion from the study of the use of the various formal operations in other high-school subjects and one gets a clearer perspective on the waste of time in our present secondary instruction.

The verbal problem-material. The verbal material reveals itself as much more difficult than the corresponding formal subject-matter in these books. On the whole it is technical and one finds that the setting almost always is that which would preclude its being taught in the secondary school. Recall the discussion of the necessity that subject-matter be taught without an undue amount of time spent on the *teaching of the situation itself*. These situations are almost all foreign to either the pupil or the teacher of ninth-grade mathematics. A consideration of the verbal material shows that in general it emphasizes *evaluation*, the *simple equation*, *ratio and proportion*, the *literal equation*, and the *literal formula*. The equation comes in for a large amount of attention. There is almost no attention paid to special products, factoring, fractions, or fractional equations, H.C.F. or L.C.M.

We are now in a position to make some striking comparisons. Tables 12 and 13 compare the relative emphasis upon formal and translation material for various operations. The tables show that the movement for combined or general mathematics has not led to the important changes in emphasis on various algebraic topics that the vocational mathematics movement has. The attention devoted to the four fundamentals by the former books is practically as large as that found in the traditional textbooks discussed in chapter II. A slightly smaller emphasis is made on special products and factoring, but this is offset by the unusually large degree of space devoted to them by a book like Evans and Marsh.

Graphic representation continues to be treated as an isolated operation. So clear are the authors that this represents its real function that the applied books do not even refer to it. It seems that these textbook writers are missing the point in treating the graph as an operation. *It is not an operation. It is one of the two fundamental methods of representing numbers and relationships, an essential tool, not an algebraic operation at all.* Both the school man and the designer of industrial textbooks resent the foisting upon algebra of a new kind of *content*, and quite properly so. However, we must be perfectly clear on this—that the graph is not subject-matter;

TABLE 12

DISTRIBUTION OF FORMAL PROBLEMS IN TRADITIONAL MATHEMATICS TEXTBOOKS COMPARED WITH THOSE WHICH REPRESENT THE ATTEMPT TO MODIFY THE COURSE FROM WITHIN THE SYSTEM WITH THOSE OF A STRICTLY VOCATIONAL NATURE*

	Ratio and Proportion	Graphs	Four Fundamentals	Evaluation	Radicals	Special Products	Factoring	Equations	Literal Formulas
I. FORMAL PROBLEMS									
Traditional texts........	1.2	2.4	16.6	2.3	9.8	5.8	11.9	18.3	.9
"Combined "mathematics texts............	.68	1.6	14	4.5	3.8	3.3	8.7	21.7	.09
Vocational texts........	30.10	0	8.05	10.3	7.46	5.39	5.13	19.79	0
II. VERBAL PROBLEMS									
"Combined" mathematics texts............	1.95	1.68	4.58	1.15	.28	7.31	.33	23.21	1.23
Vocational texts........	16.11	3.91	3.61	6.53	0	0	0	8.14	8.07

*Table gives percentage of total formal problems and verbal problems devoted to each operation.

TABLE 13

PERCENTAGES OF PROBLEMS IN EACH OF ELEVEN "COMBINED," "GENERAL," OR "APPLIED" MATHEMATICS TEXTS IN VARIOUS FORMAL OPERATIONS

	Four Fundamentals	Evaluation	Radicals	Special Products	Factoring	Equations	Literal Formulas	Graphs	Ratio and Proportion
I. "COMBINED" OR "GENERAL" TEXTS									
Breslich...............	13.3	3.3	.28	2.45	6.89	15.05	.09	.75	.14
Evans and Marsh.......	10.94	1.55	......	4.23	12.48	26.04	......	.37	......
Keal and Phelps........	17.82	8.82	5.88	3.31	6.74	24.25	......	3.64	1.23
Average...........	16.54	7.28	3.08	3.79	6.57	20.77	.09	1.58	10.51
II. JUNIOR HIGH SCHOOL TEXTS									
Wentworth-Smith-Brown J.H.S. Math. Book I	......	......	......	......	......	4.21	......	......	22.2
J.H.S. Math. Book II	24.10	16.9	......	5.20	.20	16.80	......	......	......
Vosburgh and Gentleman J.H.S. Math. Book I	......	5.86	......	......	......	34.7	......	......	18.5
III. VOCATIONAL TEXTS									
Marsh Technical Algebra	7.02	1.62	5.58	2.08	6.69	28.95	......	......	......
Wentworth-Smith Vocational Algebra........	16.06	11.7	......	......	3.57	22.06	......	......	......
Dooley Vocational Mathematics............	2.18	13.8	......	8.70	......	......	......	......	......
Roray Industrial Arithmetic for Girls......	6.96	14.10	......	......	......	8.38	......	......	33.00
Stratton and Remmick Agricultural Arithmetic............	......	......	9.34	......	......	......	......	......	2.72
Average.........	8.05	10.30	7.46	5.39	5.13	19.79	0	0	30.10

it is a method. In fact, historically it was the earliest method. The pictorial or concrete-symbolic antedated the literal or abstract-symbolic.

TABLE 14

PERCENTAGES OF FORMAL PROBLEMS IN ELEVEN TEXTBOOKS DEVOTED TO (1) DECIMAL FRACTIONS, (2) COMMON FRACTIONS, AND (3) INTEGERS

	Total No. of Problems	Decimal Fractions	Common Fractions	Integers	Graphs
I. "COMBINED" OR "GENERAL" TEXTS					
Breslich	2,122	4.95	19.4	75.8	3.39
Evans and Marsh	1,607	17.0	11.9	71.0	2.62
Keal and Phelps	1,872	9.89	28.4	61.8	3.8
II. JUNIOR HIGH SCHOOL TEXTS					
Wentworth-Smith-Brown					
J.H.S. Math. Book I	95	1.07	44.7	55.4	....
J.H.S. Math. Book II	1,001	6.8	26.4	62.0	6.2
Vosburgh and Gentleman					
J.H.S. Math. Book I	222	12.2	28.4	59.5	4.96
III. VOCATIONAL TEXTS					
Marsh Technical Algebra	2,350	26.9	33.8	39.2	....
Wentworth-Smith Vocational Algebra	1,013	17.8	19.3	63.4	1.49
Dooley Vocational Mathematics	138	20.3	18.9	60.9	3.63
Roray Industrial Arithmetic for Girls	417	12.2	8.88	78.9	....
Stratton and Remmick Agricultural Arithmetic	257	37.7	17.5	44.8	6.62

TABLE 15

COMPARISON OF TOTAL NUMBER OF PROBLEMS OF AN ALGEBRAIC, GEOMETRIC, AND TRIGONOMETRIC CHARACTER FOUND IN THREE GENERAL MATHEMATICS TEXTBOOKS, THREE JUNIOR HIGH SCHOOL MATHEMATICS TEXTBOOKS, AND SIX TEXTBOOKS IN VOCATIONAL MATHEMATICS

	Algebraic	Geometric	Trigonometric
Breslich	2,122	277	....
Evans and Marsh	1,607	45	11
Keal and Phelps	1,872	85	....
Wentworth-Smith-Brown			
J.H.S. Math. Book I	95	300	....
J.H.S. Math. Book II	1,001	88	....
Vosburgh and Gentleman			
J.H.S. Math. Book I	222	228	....
Marsh Technical Algebra	2,324	0	262
Wentworth-Smith Vocational Algebra	1,013	23	....
Dooley Vocational Mathematics	138	113	25
Roray Industrial Arithmetic for Girls	417	394	....
Stratton and Remmick Agricultural Arithmetic	257	241	....

Note, however, that *evaluation* comes to be *recognized* by the applied mathematics writers *as an essential tool*, whereas the literal formula does not. That is, the application merely demands skill in evaluating the formula, not in manipulating. Again, an emphasis upon the literal formula in the future will evidently have to be defended, if at all, in terms of learning and in terms of the added facility that one gets with the simple equation. *Ratio and proportion* is receiving a constantly increasing amount of attention. This is especially true in the case of vocational mathematics. The treatment of radicals and of simple equations occupies about the same proportion of textbook space as in traditional books.

Table 13 shows the relative attention devoted to decimal fractions, common fractions, and to integers. We must not forget that our traditional textbooks offer almost no training in the handling of decimalized material. We find that our general mathematics books devote about one-sixth of their problems to decimals (the combined mathematics, however, recognize it much more meagerly; in fact hardly at all). On the other hand, the *vocational texts devote between 20 and 25 per cent of their problem-material to training in handling decimals.* A considerably smaller emphasis is placed upon the common fraction and a much smaller emphasis on integers than in the case of either the traditional or the combined mathematics. Whereas we find a conventional textbook devoting three-fourths of its problems to integers, between one-half and three-fifths of the problems of this sort are found in the vocational mathematics. The junior high school mathematics continues to emphasize the common fractions and gives little attention to decimals.

In the detailed report of this tabular analysis a complete comparison is made of the extent to which each textbook organizes its geometrical proofs as (1) empirical definition, (2) semi-empirical definition, (3) graphic proof, and (4) demonstrative proof. We find that the tendency is to give no demonstrative geometry in the ninth year. The combined mathematics books are the exceptions to the rule. There is, however, a considerable amount of use of empirical definition, that is, of verbal statements of a relation which might be illustrated graphically but is not demonstrated.

The tabular analyses which we have carried on to date enable us to make some very striking comparisons concerning what the pupil gets in his present ninth-grade course and what it is possible to give him in the course of one year of sequentially organized mathematics. Paralleling this comparison of what he gets and what he might well get, there must be a clear accounting of the relative opportunity for training in problem-solving—in logical thinking. We need to do this latter because those who have misinterpreted the previous 150 pages will tell us that we have neglected the criterion of "training the mind." No; far from it. We believe in it. It is a central part of our pedagogical doctrine. The weight of psychological judgment, however, shows that the greatest "transfer of training" comes with material of a problem-solving nature—of a generalizing nature. We have shown that it is this kind of material in which our present course is lacking. Hence we need a radical revision, not only on the basis of the criterion of utility, but also on the basis of "thinking" outcomes. *There are tool outcomes and thinking outcomes from school studies.* Surely the greater of these two types is the "thinking" outcome.

Is it not clear, however, that to get a thinking outcome the content of our course must be organized about "*problem*" material? Now problem-material is of two sorts in mathematics: (1) It may be artificial, verbally expressed material, as is the typical "word" problem of our algebra textbook. This type of material clearly requires thinking to a much larger extent than the formal types which are emphasized so much. (2) But *the best thinking material is the problem-material that is built around real mathematics*—in other words, material that has to do with the stating and determining of relationships.

E. THE FUNCTION OF MATHEMATICAL INSTRUCTION

Our thesis is this: **The central element in human thinking is seeing relationships clearly. In the same way the primary function of a secondary course in mathematics is to give ability to recognize verbally stated relationships between magnitudes, to represent such relationships economically by means of symbols, and to determine such relationships.** To give practice in doing the first and second of these things—recognizing and expressing relationships—the content of our courses must emphasize the verbally stated problem (just as the earlier algebras did) which leads to equational form. To do the second much practice in representation, in translation of verbal statements, must be given—practice which will utilize the two fundamental methods of representing numbers, literal and graphic. To provide practice in doing the third thing—determining relationships—the course must be so organized as to give thorough facility in the manipulation of the equation, simple and quadratic.

But to determine and express relationships between magnitudes—the expression of "law"—demands measurement: measurement in terms of fundamental categories, categories of space and of time. Now "categories of space" implies training with a few agreed-upon forms. The expression of the results of measurement gives rise to statements of equalities and inequalities. It demands furthermore the treatment of ratio as necessary to effective comprehension and manipulation. The fundamental expression of "law" *demands, likewise, a thoroughgoing grasp of the principle of functional dependence.* This in turn can be facilitated by a study of trigonometric functions and of simple angle relations. We are interested in trigonometric functions, not only because of their expression of relationship, but also as a tool device in the determination of distances, linear and angular.

But to build a course of study on this scheme will necessitate an emphasis upon material that we do not now include in the ninth-grade course. It will demand furthermore that we utterly disregard a large proportion of the subject-matter of the present course. Contrast, for example, what the student receives in the way of training with what he needs. In addition to the evidence which we have already developed in chapter II, what the student now receives in first-year algebra is shown for fifteen Indiana school systems by the following quotation from Mr. Lee Brinton's question-blank study, *Relative Importance of the Topics of Algebra* (J. Sterling Morton High School, Cicero, Ill.).

Do we need to comment further on the present emphasis of the course? Note the 12.5 class hours on addition and subtraction of algebraic expressions, the 19.6 class hours on multiplication and division, the 31.4 on special products and factoring, as contrasted with the 8.3 on simple equations, the 16.6 on fractions, and the 15.8 on radicals. Note that we give as much time to square root and quadratic surds (7.0) as to graphical representation (6.7).

TABLE 16

NUMBER OF CLASS HOURS DEVOTED TO PRINCIPAL FORMAL ALGEBRAIC OPERATIONS BY TEACHERS IN FIFTEEN HIGH SCHOOLS*

	No. Schools Reporting	Min. No. Hours	Max. No. Hours	Average No. Hours
Literal numbers	15	2	15	8.7
Positive and negative numbers	15	2.5	10	4.9
Addition and subtraction of algebraic expressions	15	10	17	12.5
Parentheses	15	5	8	6.3
Multiplication	15	7.5	19	13.1
Division	15	4	12.5	6.5
Simple equations	15	5	10	8.3
Special products and factoring	15	22.5	40	31.4
H.C.F. and L.C.M	15	2	10	4.3
Fractions	15	5	25	16.6
Simple equations	15	10	25	17.3
Graphical representation	15	3	15	6.7
Simultaneous linear equations	15	5	35	17.7
Square root and quadratic surds	15	5	10	7.0
Quadratic equations	15	8	30	16.3
Special products and factoring	14	5	20	12.8
Quadratic equations having two variables (graphic)	13	3	7	4.6
Simultaneous equations	14	9	19	13.5
Theory of quadratic equation exponents	14	6	20	10.4
Radicals	14	7	25	15.8
Logarithms	10	7	15	9.9
Progressions	11	5	15	8.4
Binomial theorem	9	2	7	4.1
Ratio, proportion, and variation	8	5	10	7.5
Supplementary topics	3	2	5	4.0

* After L. V. Brinton, 1915.

Note that there is no mention of offering practice in translation as such in evaluation or in the construction or manipulation of practical formulas.

Summary on economy of time through elimination of nonessentials.—To build a course on the criteria which we have discussed in detail throughout this monograph will necessitate a radically different organization. It will include, for example, thoroughgoing training on rational grounds in the representation of numbers by symbols. Much practice will be offered on this as a specific operation. Clear recognition of the need for giving pupils detailed training in translation. Translation from verbally stated material to algebraic symbolism with the converse very necessary practice in translation from algebraically stated material into its English equivalents. This latter point is especially important from the standpoint of effective learning. The treatment of positive and negative numbers on thoroughly rational grounds, if necessary using eight to ten class periods, to enable the pupil to reason through (by means of literal and graphical representation) each step in the process of determining the laws of signed numbers. Evaluation will be recognized as one of the most important operations of ninth-grade mathematics, and constant practice in many types will be offered throughout the course. Naturally the core of the course is the equation—primarily the simple equation, but also the quadratic. Nothing but automatic skill will be acceptable. To obtain this, however, thorough automatism in collecting terms, removal of parentheses, simple multiplication, and the handling of fractions (clearing and solution of equations with single numerical denominators) must be secured.

With the manipulation of the equation as a fundamental formal element the construction of formulas and skill in their manipulation must also engross the attention of this required course. Measurement will stand out as an underlying element of the course—the use of trigonometric functions, the sine, cosine, and tangent. Graphical representation will take its place as one of the two fundamental methods of representing numbers. From the beginning of the course pupils will deal with the scale and the unit, and will reason through most of their mathematical problems by making use of this essential method of representing relationships. Finally, the year's instruction will lead to a clear grasp of the underlying principles of functional dependence and to effective skill in the manipulation of the necessary tools.

To build such a course clearly demands the elimination of one-third to one-half of the present ninth-grade course. Note the economy of time through elimination of nonessentials. We shall save at least four weeks on special products and factoring, two to four weeks on the formal manipulation of addition, subtraction, multiplication, and division, a week on highest common factor and least common multiple, at least one week (probably two) on fractions, between one and two weeks on quadratic equations, between one and two weeks on radicals.

To many teachers of experience in first-year algebra the presentation of this proposed procedure will sound utopian. The proposal is made by the writers, however, as a result of careful experimentation with just such elimination and just such an organization of subject-matter. Chapter IX will discuss the results of experimentation in the first half-year with this course.[1]

[1] In the foregoing discussion of desirable content there is no attempt to sketch the course in the fashion in which it would be given to students. This has been done by the writers in the textbook on secondary mathematics which will shortly be published.

CHAPTER IX

EXPERIMENTAL TEACHING IN NINTH-GRADE MATHEMATICS

Thus we have carried on three distinct programs as a part of our investigation: a testing program, a course-of-study program, and a program of experimental teaching. Our testing and course-of-study programs revealed two outstanding weaknesses in the course: (1) a large amount of dead wood which ought to be eliminated; (2) an unpsychological organization of material, topics poorly arranged—not organized around a unifying mathematical principle and a unifying psychological principle; in other words, not organized with respect to the *learning of students.*

Our course-of-study program, furthermore, enabled us to show tentatively how much subject-matter actually is worth retaining in the ninth-grade course. This provided us with a list of foundational topics from which to build a course that would satisfy our criteria. Furthermore it gave us a systematic and continuous course made up of these elements and organized around important unifying principles.

The experimental program. We are now ready to discuss the conduct of our experimental program during its first half-year. This has distinguished clearly between classroom needs in rationalization and in habituation—between the degree to which our teaching provides opportunity, first for original thinking, for problem-solving, and second for the perfecting of skills. The latter point recognized that classes can be taught effectively by using mass devices—practice exercises—to develop skill economically.

Having laid out a tentative course on a scientific basis as indicated in the previous chapters, our last and most important step in the general program of improving teaching is to set up a detailed teaching experiment of this tentative course. This teaching experiment, begun in a preliminary way in 1916–17, has been carried through the first half-year of 1917–18 under carefully controlled experimental conditions. The investigation recognizes a number of definite próblems, the first of which is: *The determination of the most direct approach to the learning of such algebraic and geometrical material as is necessary to use the equation, the formula, and the graph in the stating of relationships.* The equation has been recognized, therefore, as the core of the first half-year of work. The experiment has been carried out on the principle that instruction must leave absolutely no gaps between the pupil's experience, that is, between the content of the pupil's mind and the mental content of the subject-matter of the course.

This necessitated a decision, *first*, as to which operations were mathematically basic and yet most easily grasped. Ten years of experience in the teaching of high-school mathematics and careful analysis of the learning process pointed clearly to the processes of "*representation*" and "*translation*" as these basic operations. It necessitated, *second*, the determination of the order of transition from topic to topic, worked out by a careful study of *mathematical transition* and supplemented by a minute analysis of the proper *psychological transition*. With this part of the study the writers are not yet ready to go on record. A detailed report of the entire first year of experimentation, covering all these points, will be published at the end of the year. The *third* decision that was necessary rested upon the experimental teaching of the organization of topics to which the foregoing analysis led.

The carrying on of this teaching experiment has embraced the following procedure:

1. The critical observation of daily classroom teaching by each of the present writers. During the past year each writer has taught two or more classes of first-year high-school pupils in the Parker High School, Chicago, under normal public-school conditions. The classes have approximated thirty children each. No change has been made in the method of classifying children from that which obtained in the other classes of the department. The teaching of each of the writers has been under the immediate observation of the other daily throughout the year.

2. The keeping of detailed records: (*a*) of important difficulties met by pupils with both formal and reasoning work; (*b*) of successful and unsuccessful methods of presentation of each type of

subject-matter; (*c*) of daily practice in ability to handle formal operations as shown, for example, by the scores on the Practice Exercises; and (*d*) of general reasoning abilities with quantitative material.

3. Daily conferences on appropriate content, on the proper distribution of time and emphasis, both within a class period and in successive periods; on best methods of rationalizing new material; on best methods of reviewing, both with formal material and with subject-matter which is set in verbal form.

4. Daily stenographic record of the conferences and the classroom work. The results of this latter step have shown quite clearly that the greatest improvement in teaching results from comparisons of critical judgments (both of the teacher and of the trained observer) which have been recorded immediately following the class periods. Our results show furthermore that such judgments, to be of value, must be discussed and recorded almost immediately after the class exercise. This has been done daily throughout the year, and will be continued in successive years.

It can be seen that this procedure built up a set of records as to how children learn that are invaluable in the teaching of mathematics. These records are being worked over in detail and will be published in the form of a systematic manual for teachers. The material will point out very specifically the particular difficulties encountered by pupils on each operation, the most economical method of rationalizing the operations, and the most effective ways of developing skill. Forming, as it does, more than four hundred pages of records, the material cannot be incorporated in full in this report, which has already exceeded its limits. The data collected, however, substantiate every statement made about the wastefulness of present methods. To make this clear, however, we ought to discuss here some typical problems of economy of time in learning.

TYPICAL PROBLEMS OF ECONOMY OF TIME IN LEARNING

One of our two fundamental theses has been that there is in the curriculum much dead wood that must come out; the other that probably one-third to one-half of the time spent in the first-year course can be saved, and at the same time greater mental efficiency secured (both with formal skills and with reasoning situations) by proper methods of teaching. This raises an important issue.

To illustrate scientifically the *methods of determining most economical teaching procedure*, let us take two typical examples. Recall that economy of time in learning may be, on the one hand, economy secured by such effective psychological organization of subject-matter as will enable one, in the *initial* presentation of the material, to save a considerable amount of time. On the other hand, economy of time in learning may imply that in the initial presentation much *more* time is needed than is given in traditional practice. This is because the added effect of a more gradual and *rational initial presentation* will result in economy of time in the later developments of the course owing to the accumulation of reasoning ability with problem-situations. We shall report two striking illustrations which have grown out of our investigation. The first has to do with the saving of from three to four school weeks in the teaching of special products and factoring. The other has to do with economy of time through teaching "signed numbers" on a thoroughgoing rational basis.

I. ECONOMY OF TIME IN TEACHING FACTORING

The results of "psychologizing learning" showed us that the issue in teaching factoring is: Shall we teach the "types" or "cases" separately, or is it possible to find one underlying process which if taught will lead to economical learning of all of the others? The traditional method which is followed by practically every textbook is to teach special products and factoring by types or cases. For example, we have before us a book used commonly among the nine which were investigated. Of the 275 pages in the first-year text 75 pages are devoted to special products and factoring. We find, e.g., that the pupils begin by learning Case I, the type form for which is $a(b+c)=ab+ac$. This is the monomial factor, really one of the two distinct operations in factoring. It takes about two days to present this type and to habituate and develop in the pupils a reasonable amount of skill. They then proceed to Case II of the type form as follows: $(a+b)(a+b)=a^2+2ab+b^2$. To master this type (the square of a binomial) and to perfect skill requires from two to three days. They are then ready for Case III, $(a+b)(a-b)=a^2-b^2$, the product of the sum and the difference of two numbers, and factoring the difference of two perfect squares. This can probably be done in two or three days more. We need not canvass the remaining types in detail. The process, however, is the same, day after day, until the seven cases have been completed. Note one very significant thing, however. *These cases are arranged in such a fashion that the learning of any one tends to inhibit the learning of any other.*

For example, the pupil having been habituated to, say, five cases, he is given a problem. It is necessary for him to "classify" it. If it is already classified, if the pupil knows to what type it belongs, then his skill with the particular type enables him to work the problem successfully. If the problem, however, occurs in a miscellaneous list of examples, the experience of practically all mathematics teachers proves that he is unable to generalize his previous training or to classify the problem correctly. Hence in such cases the pupil fails. In other words, the skill which had been set up in learning one case is sufficiently different from the learning of any other as not only to be of no avail, but even to inhibit efficiency with the other.

But really to apply psychology to our instruction we search through the various types to see if there is one general type, the learning of which by a given method can be made to fit all the others. Psychological analysis proved that there is one: the general trinomial of the form ax^2+bx+c. Again "psychology" and teaching sense tell one to analyze the difficulties in manipulating such an operation to discover the mental processes which are involved in its working, and to work out devices by which the processes can be aided. In this particular case we find that visual imagery is predominant, and to facilitate its use in the handling of this operation we make a device to aid the pupil in fixing cross-products.

Now pupils can be taught to factor rapidly and effectively all types but the monomial factor by just one method, the teacher using a procedure about as follows: first, present to the pupil the product of two general binomials; for example, he would be taught to expand $(3x+5)(2x+4)$, emphasizing all the time the importance of the correct middle term. If this type of special products is introduced as a tool or device in finding the areas of rectangles, pupils have an adequate motive. Considerable experience with this kind of work will soon result in the pupil himself making certain classifications or groupings of these examples. For instance, he will recognize that when the binomials are alike, except for the signs, the product will have only two terms, and that it will be the square of the first term minus the square of the second number. This type of procedure is of very great importance for the reason that in the habituation of this kind of work the pupil does not build up any habits which will inhibit learning to expand the product of *any* two binomials. On the other hand, if the material is organized in terms of its logical subdivisions, that is, by the case method, the skill which is set up in any one of these types tends to inhibit success in any different type. It is common for pupils to say: "I could do this if I knew which type it was, or which rule applied." This shows that they have acquired a kind of skill which they cannot generalize. It has been pointed out many times that the natural method of learning is that which proceeds from the undifferentiated, vague, complex mass of experience to the differentiated, detailed, classified, and logically related. In other words, we do not recognize the details, classifications, and logical divisions and subdivisions until after we have had considerable experience with the larger class or type which has been subdivided and classified. It seems that in the organization of this part of algebra, textbook writers have been so obsessed with the desire for a complete, logical, detailed classification of subject-matter as to forget the pupil and the methods by which he learns.

To compare the relative efficiency of the two methods of presentation the *writers have taught in six lessons* all the special products and factoring presented in the standard textbook (excepting the factorization of the sum and difference of two cubes). Ordinarily if one follows the textbook classification and organization it requires from four to six weeks. Brinton's study showed that 31.5 class-hours are devoted to it on the average in his 15 Indiana school systems.

II. ECONOMY OF TIME IN TEACHING "SIGNED NUMBERS"

As a second illustration let us discuss the two alternatives in the development of the use of signed numbers. Here the teacher is confronted with a question of prime importance, namely, shall this subject-matter be presented in a dogmatic, "take-it-from-me" way, or shall it be a carefully reasoned presentation? If it is the former, one would tell the class that in subtraction the sign of the subtrahend must be changed, and that we proceed as in addition, or that in multiplication the product of an even number of negative factors is positive. This would require not more than two days' time. To be sure, the pupils will be able to "give the answer," for the time being, and may even profess some interest in the successful manipulation of this kind of work. How much meaning this has for the pupil and how much opportunity for practice he has had for real thinking are serious questions. The story is told of a little boy who asked his father for help on his algebra. The father inquired, "What is this about, what does it mean?" And the boy replied, "Father, you are not supposed to know what it is about or what it means. You are just supposed to do it."

The second alternative for the teacher is to take a far greater length of time to develop a genuinely reasoned comprehension of negative numbers. She may review the numerous situations or experiences common to the pupils, the interpretation of which necessitates the distinction of "oppositeness" of qualities. For example, she may base a series of pertinent questions on thermometer readings at different times, or at different places at the same time. The pupil can be led to formulate and generalize these particular problems into a law of signs in subtraction. Similarly, by distinguishing between the distances east and west of a certain point, and the time before and after a certain fixed time, the pupil by determining the location of a train moving at some uniform rate can be led to see that negative times negative must be positive, and that the product of unlike numbers must be negative. This latter procedure, for all the laws of signed numbers, requires approximately two weeks' time. At its conclusion, however, the pupils do not leave the room with the characteristic remarks to each other of "when you add you subtract, and when you subtract you add, don't you?" That these remarks are common among pupils to whom the subject-matter has been presented in the usual dogmatic, textbook way can be verified by listening to the conversation of the pupils during this period of their training.

Thus, by making a very close relation between the common experience of the pupils and the laws of algebra, greater clearness and added meaningfulness will accrue to the pupil. Furthermore the practice of pointing out to the pupil the advantage of algebra as a tool devised for saving time in writing and in solving problems tends to create in him an attitude which facilitates the learning process.

Miscellaneous examples of economy-of-time studies to be reported later.—We are now keeping detailed records to show scientifically the effect of teaching rules, and the effect of much emphasis on training in translation and in constructing examples of their own. Our preliminary experiment has not offered records to substantiate quantitatively conclusions which are rather obviously apparent. We are finding, for example, that pupils learn more readily if the definition and rules of procedure are not presented until after a certain amount of informal work and experience has been given. For example, pupils can solve the easier examples in simple equations long before they can state formally the axioms used in their solution. This is because they have a certain amount of "intuitive," informal knowledge which can be presupposed and used before it is necessary to state formally and explicitly these axioms and rules of procedure. The same is true of subtraction. It is more effective to postpone the formal mathematical definition of subject-matter of subtraction of "signed numbers" until the pupil has had considerable experience involving those simpler uses.

The experimentation of the next half-year will take more complete account in our study of economy of time in learning of the value of practice in *translation* both from verbal to algebraic language and from algebraic to verbal language. We have found it very important, for example, to have such expressions as $a+b$, $a-b$, $\frac{a}{b}$, and ab translated into words, primarily because it emphasizes the idea that letters really represent numbers, and that they are not vague, intangible things to be blindly manipulated.

Probably the proper gradation of problems in the textbook in terms of gradually increasing difficulty will contribute more than any other factor to the developing of power on the part of the pupil to meet successfully new "problem-situations." The study of economy of time in learning will rest on no more important analysis than that of the proper organization of problem-material in textbooks.

Finally experimental analyses of the most economical methods of learning and teaching must study the advisability of offering more practice in the construction of examples by the pupils themselves. The limited amount of practice we have already been able to give in the experimental classes has been found to give them insight into the real nature of the problems, and an interest in their solution which is not to be had by restricting the examples to those which are formally set in the book. For example, the practice of having pupils make expressions to be factored by the rest of the class makes most meaningful what an expression is that can be factored. Or again, the illustrations of problems that pupils can give of functional dependence will clarify and supplement any textbook solution.

III. ECONOMY OF TIME IN LEARNING THROUGH THE USE OF PRACTICE EXERCISES

For our third example in economizing time in learning we have but to turn to the discussion of the use of practice-material. All formal drill on the various operations was given the class by means of the writers' Standardized Practice Exercises. Table 19 gives the dates showing the intervals of time that elapsed between successive trials with each exercise, and the total number of drills which the pupils had with each one. The table makes clear the strikingly small amount of time devoted to drill. It should be said here that in addition to this formal, rapid-fire drill there was a small amount of incidental use of the various operations in the classroom either in the seats or at the blackboard. However, this was distinctly incidental and did not give practice in the specific operation in question as such.

IV. COMPARATIVE ACHIEVEMENTS OF PUPILS TAUGHT "EXPERIMENTALLY" AS CONTRASTED WITH THOSE TAUGHT "TRADITIONALLY"

Our most important illustration of economy of time in learning, however, is shown by the comparative achievements of two sets of classes, the first set consisting of two experimental classes (55 pupils) taught by the writers, and the other set of four classes (112 pupils) taught by other members of the department. In the experimental classes the course of study suggested in chapter VIII was used in place of the material presented in the regular text. In the other four classes the regular text was taught page by page with practically no omissions, additions, or deviations. The regular teachers of the department taught those classes with no knowledge of the work done in the other classes, and used their usual line of procedure following the textbook closely. We present next a number of important comparisons: The first is a comparison of the achievements of pupils in the two sets of classes in formal skill in manipulation of four operations—simple equations, special products, factoring, and exponents. These were regarded as the four most typical formal operations which had been taught in the first half-year. Furthermore, since the textbook emphasizes the formal aspect of learning to such a degree, this comparison, if any, ought to be detrimental to the success of the experimental classes. In other words, we have here a comparison of achievement, in a purely formal skill, in which the experimental classes have spent about 20 per cent of their time on formal instruction and drill, while the other classes clearly have spent more than 70 per cent of their time on the same work.

The second means of comparison is still more important than the first. Not only can we check up the *formal skill* of the two sets of classes, but we can compare definitely the extent of the material covered by the two sets of classes in mathematics in general, setting forth the amount of contact that each group had with *real mathematical material*. In other words, we have, as indicated in Table 18, a list of the routine operations with the relative amount of time which has been spent in the traditional classes and in the experimental classes, together with the real mathematical operations to which each group of students has been introduced.

For the pupils in the experimental classes, however, we have another very helpful set of statistical records, namely, their achievement on reasoning tests as compared with the achievements of pupils in 17 school systems who had completed a year of algebra.

These statistics are completed by a fourth set in which we give the results of testing our students in the degree to which they have really mastered the new types of subject-matter, such as the principle of functionality, and construction of formulas. We print herewith the tests which were given to the experimental classes to show the reader the extensiveness of the instruction. Tables 17 to 19 give the comparative results found for each set of tests.

I. TESTS FOR ABILITY TO DEAL WITH VARIABLES

Test A. Recognition of Variables

1. Draw a circle around each variable and a rectangle around each constant:

a) $C = 2\pi R$
b) $D = 30t$
c) $y = 8x - 7$
d) $x = y$
e) $5.4 = 10.2$

Test B. Recognition of Relations between Variables

State whether the following variables are directly or inversely proportional:

a) The area of a rectangle of constant width and its height.
b) The time a train needs to run ten miles and its rate.
c) The distance traveled by a train running at a constant rate, and the time.
d) The cost of a railroad ticket at three cents a mile, and the distance.
e) The base and height of a rectangle of constant area.

Test C. Graphic Representation of Variables

Show graphically the relation between the cost of a railroad ticket at four cents *per* mile, and the number of miles traveled.

Test D. Ability to Translate Verbal Problems into Algebraic Form

No.	Score	
1	25	Express the following verbal statement in algebraic form: the square of a side plus five.
2	30	State the equation for: 4 increased by 3 times a certain number equals 19.
3	35	Express the following verbal statement in algebraic form: Four times the length of a side subtracted from 8.

4	40	Eight times a certain number equals 45 diminished by the number. State the equation by which you would find the number.
5	45	A rectangular lot is 10 rods longer than it is wide. Express in algebra that the distance around the lot is 80 rods.
6	55	Find the number whose third and fourth parts added together make 14.
7	60	A father is twice as old as his son. Twenty years ago the father was six times as old as his son was. Express in algebra.
8	65	Two trains approach each other, leaving stations 149 miles apart at the same time. One goes 10 miles an hour faster than the other, and they meet in 2 hours. What is the rate of each?
9	70	Twice the width of the Pennsylvania Station in New York exceeds its length by 80 feet. Four times the length exceeds the perimeter by 700 feet. Find the dimensions.

Test E. Ability to Construct Formulas

No.

1 Make a formula for the area of a square.

2 " " " " " total surface of a cube.

3 " " " " " volume of a rectangular box.

4 " " " " " " " " cube.

5 " " " " " volume of a box in cubic inches when dimensions are in feet.

6 Make a formula for the number of gallons in a rectangular tank whose dimensions are in feet (assume $7\frac{1}{2}$ gallons to one cubic foot).

7 Make a formula for the number of revolutions made by a wheel whose diameter is D feet, in going a mile.

8 Make a formula for the number of square feet in the surface of a circular smokestack.

TABLE 17

Comparison of Ability of Two "Experimental" Classes and Four "Traditional" Classes as Shown by Class Achievements on Four of the Standardized Algebra Tests

Test	Median Number of Examples Correct					
	"Experimental" Classes		"Traditional" Classes			
Operation	A	B	C	D	E	F
No. 6. Special products...	13	12	8	8	9	8
No. 7. Exponents........	16	20	8	9	8	8
No. 8. Factoring.........	8	9	5	6	5	5
No. 4. Simple equations...	6	7	6	6	6	5

Results of preliminary experimentation.—These four sets of records show four striking facts:

1. The teaching of the experimental classes emphasizing problem-solving, as they did, nevertheless developed a greater skill

TABLE 18

COMPARISON OF APPROXIMATE NUMBER OF CLASS EXERCISES DEVOTED TO DIFFERENT TOPICS OF THE FIRST HALF-YEAR OF HIGH-SCHOOL MATHEMATICS IN (1) 15 INDIANA HIGH SCHOOLS, WITH (2) TWO RUGG-CLARK EXPERIMENTAL CLASSES, PARKER HIGH SCHOOL, CHICAGO, 1917–18

Topics	15 Indiana High Schools (after L. V. Brinton)	Experimental Classes
"Literal number"	8.7	8
Signed numbers	4.9	10 (Including collecting terms)
Addition and subtraction of algebraic expressions	12.5	0
Parentheses	6.3	(Taught incidentally with others; approximately two class-periods)
Multiplication of algebraic expressions	13.1	(Incidental use of about four class exercises)
Division of algebraic expressions	6.5	0
Simple equations	8.3	11
Special products and factoring	31.4	6
H.C.F. and L.C.M	4.3	0 (Formally) (Incidental use of one class exercise)
Fractions	16.6	4
"Graphic representation"	6.7	(Constantly used from first week of course) (Relationship = 5)
"Translation" (specific training)	0	8 (estimated)
Formulas and evaluation	0	6
Variation	0	17
Second half-year includes in addition to customary material (quadratics, simultaneous equations, etc.) such types of material as:		
Angular measurement and scale drawing	....	5
Ratio (initial presentation)	....	2
Direct and inverse variation	....	4
Use of similar figures in finding unknown distances	....	6
Use of the right triangle in finding unknown distances (the sine, cosine, and tangent); Statistics, plotting of data, averages, etc.	....	12
Graphic solution of simultaneous equation; co-ordinates, plotting, etc	....	8

in manipulation of these four operations than the traditional classes in whose instruction the emphasis was on the manipulatory side. In three of these operations this superiority is quite marked indeed.

TABLE 19

ABILITY OF THE TWO EXPERIMENTAL CLASSES AS SHOWN BY TESTS A TO E INCLUSIVE

Class	Test A. Recognition of Variables	Test B. Recognition of Relation between Variables	Test C. Graph Test
	Average Percentage Correct	Average Percentage Correct	Percentage of Class That Worked Test Correctly
A	84.5	71.8	82.0
B	95.5	74.0	89.0

Class	Test D. Ability to Translate into Algebra — Percentage of Pupils in Each Experimental Class That Worked Each Translation Problem Correctly — Example No.									Test E. Construction of Formulas	
	1	2	3	4	5	6	7	8	9	Average (Median) Number of Problems Right	Average Time Required in Minutes
A	92.3	96.3	66.7	96.3	62.9	96.3	22.2	25.9	44.4	5	8
B	91.3	95.6	64.2	73.9	26.1	87.0	17.4	14.0	21.7	5	10
Average of 17 schools	93.4	88.6	81.3	72.7	61.9	38.1	27.3	18.3	11.4	....	....

TABLE 20

NUMBER OF DRILLS AND DATES ON WHICH EXPERIMENTAL CLASSES WERE DRILLED WITH THE STANDARDIZED PRACTICE EXERCISES

Exercise No.	Operation	Total Number of Drills	Dates of Exercises
1	Coll. terms	7	Oct. 3, 5, 10, 11, 15, 26, Jan. 24
2	....	4	Oct. 24, 26, Nov. 1, Jan. 24
3	Evaluation	5	Sept. 24, 27, Oct. 11, 15, Jan. 24
4	....	2	Oct. 24, Jan. 24.
6	....	2	Jan. 24, 25
7	....	2	Nov. 12, Jan. 25
8	....	3	Nov. 16, 20, Jan. 25

In the other it is slight, but noticeable. It will be objected that the use of the practice exercises was responsible for the greater amount of automatic skill in the handling of the formal material, and that these classes had an undue amount of practice. Table 20, however, shows that this is not true, for on only one operation did the experimental classes have more than three practice drills. Furthermore no drill was given to either class on any operation from November 20 to January 8, the time between being devoted to the study of literal and graphic representation of functionality.

2. A larger percentage of the experimental classes solved the reasoning or translation problems than did the average of the pupils in 17 school systems at the end of a full year of training. Note the results of Test D, in which in 6 of the 9 tests the experimental classes were superior to the average of the 17 school systems. It should be remembered, however, that it is in the second half-year of training in which the accumulation of the results of the careful rationalization work will begin to show up. In other words, the reader must remember that we are comparing a group whose training consists of a half-year with another that had a full year of training.

3. More than three-fourths of the experimental pupils had mastered something more than the rudiments of functional dependence, as shown by their skill in recognizing variables and the relations between variables, and in the graphic representation of such variables (see tests A, B, and C).

4. In addition, they understood clearly the principle of constructing formulas to the extent that each class on the average could construct correctly five like those in Test E, with an average total time of from eight to ten minutes.

To sum up, these classes have been taught with such a redistribution of time and emphasis, such a reorganization of teaching procedure that they not only excel in the formal aspects of ninth-grade mathematics, but show evidence that a full year of training will give the pupils a real grasp and insight into fundamental mathematical notions and devices.

We regard the outcomes of this preliminary experiment as definitely indicating the possibility of economizing time in learning by a thorough application of psychology to the learning process in secondary mathematics. There can be no doubt that in the teaching of factoring the accomplishment with the pupils of introducing on rational grounds the principle underlying factoring, and of perfecting skill in the way in which the experimentation did, greater efficiency has been secured with a parallel economizing of time.

There can be little doubt that pupils in the ninth grade can be taught in one year the fundamental notions and devices which all children should have on leaving school. Examination of the evidence of the first half-year of experimentation leads to this conclusion. It leads to the conclusion also that in addition to this most desirable outcome from school study in mathematics they can be given such skill in the manipulation of the algebraic tools—the formal operations—that their facility in problem-solving will not be hampered by an inability to use the equation and its subordinate tool elements.

CHAPTER X

A PROGRAM FOR THE RECONSTRUCTION OF NINTH-GRADE MATHEMATICS

Thus we have canvassed the present situation in ninth-grade mathematics: the course of study, how it is made up and what it emphasizes in the teaching of children; how it developed, where it came from, and who is responsible for the present condition. That the need for constructive criticism and for thorough reconstruction is great cannot be doubted. The content of one-third to one-half of the present course will never be made use of in any way, even by those who continue through the upper grades of the secondary school. Certainly this condition cannot be tolerated if we accept the social criterion for curriculum building. Likewise, we have canvassed the claims that have been made for the mathematics course on the grounds of "logical thinking." It is difficult if not utterly impossible to substantiate them, if we leave the emphasis as it is now on the formal side of the subject—in other words, if we continue to commit our teaching to "manipulation" rather than to "problem-solving." Testing the product of this teaching—in more than 50 school systems—leads to but one conclusion: our development of the formal skills as well as of "thinking" abilities has not operated to give a satisfactory efficiency.

Two causes stand out clearly: one, we teach too much—there is dead wood in the curriculum; second, our teaching has not always been so directed, through clear insight into the learning of children, as to give a proper teaching emphasis.

Two remedies suggest themselves: (1) to revise the course of study and eliminate the nonessentials, thus providing an emphasis on the types of notions and devices for which pupils will have a need later in life, providing a larger opportunity for training in "thinking" with verbal material, and furthermore providing an opportunity for more "real" mathematics; (2) to canvass the psychology of learning in secondary mathematics and set forth clearly the typical errors that pupils are prone to make; to design and make wise use of formal mass-instruction devices for teaching the "skills" —practice exercises—thus again leaving more time for "thinking" work. Through our critical analysis of "learning" we shall get clear thinking concerning the best organization of subject-matter in our future courses. Finally we shall clarify the teaching process through real insight into economy of time in learning.

Thus fundamental to the *permanent improvement* of the present situation are two important steps:

1. The thorough overhauling of the course of study in ninth-grade mathematics, the elimination of those types of material indicated in chapter VIII, the addition of the types of real mathematics not now a part of the ninth-grade course, and the construction of a continuous mathematical course, worked out around two basic principles, one mathematical and the other psychological.

2. The improvement of methods of teaching mathematics to ninth-grade students. Ideally this demands better training of mathematics teachers. In this direction advance in our own time will be slow and the effect for years to come will be inconsiderable. Hence to bring about an improvement in teaching we need to carry out a fourfold program:

a) We need a new type of textbook, one that will tell the pupil practically everything that a well-trained and intelligent teacher tells the pupil in the classroom. That is, *we need wordy textbooks.* Naturally old teachers will take issue with this statement. They will say that this will leave nothing for the teacher to do! Such a statement, however, is based upon a lack of real acquaintance with the psychology of learning the elements of real mathematics. We need wordy textbooks, therefore, organized in terms of a most careful fitting of the developing stages of the subject-matter to the growing mathematical content of the child's mind.

b) The fundamental need of the teacher, however, is a real, applied psychology. She cannot master the detailed treatises of general psychology, and the present-day applied psychologies are not applied specifically to her problem. Hence the *second* outstanding step in this program is the formulation of detailed manuals of method founded on the best psychology and on minute analyses of learning. Thus our future procedure ought to be to train the beginning teacher, or the teacher of no psychological equipment, by putting in her hands a real psychology of secondary mathematics.

c) A well-planned textbook and a detailed manual of method must be accompanied by completely standardized tests with which to check up at intervals throughout the year the results obtained from the instruction.

d) Finally the teacher's equipment must be rounded out by means of practice devices perfecting the formal skills: hence the practice exercise as an important accessory to the effective textbook.

This is the writers' program for the improvement of instruction in ninth-grade mathematics in the public school. It gives promise of offering definite solution for some very pressing problems. The standardized tests and the practice exercises are now available. The writers, among others it is to be hoped, will embody the results of their experimentation in teachers' manuals and textbooks to be presented to the consideration of the mathematics group very shortly. One or two individuals working alone, however, can do little in the way of effecting permanent reform. Teachers and supervisors of mathematical instruction throughout the country must co-operate to make such a program effective.

The foremost immediate need is *discussion—constructive criticism* of existing conditions. This monograph has pictured the present situation, has shown the first points of attack, and has suggested a definite procedure for revising what and how we teach. School administrators and teachers of mathematics can help immediately by discussing this proposed program, by suggesting a better one, or by carrying out new ones. Let us, whatever we do, *do something* to improve a very serious situation. *Experimentation, observation of others' work, the keeping of records, the consultation and co-operation of teachers in the department*—all these steps should become part and parcel of the routine of every department of secondary mathematics in the country. Teachers' associations, the best agency for organizing discussions of a program or a situation, can render great good by making use of the evidence which is reported in these pages. Let us substitute on the programs of our mathematical conventions the discussion of *quantitative evidence concerning what we do* for the traditional a priori and "personal testimony" types of discussion which are most common in our local and state meetings.

APPENDIX

TABLE I

THE TEXTS EXAMINED IN THE HISTORICAL STUDY (CHAPTER III)

Edition of	Author	Title of Text
1751........	Carr............	The Algebraist's Companion
1758........	Ward...........	The Young Mathematician's Guide
1796........	Frend...........	The Principles of Algebra
1796........	Manning........	An Introduction to Algebra and Arithmetic
1806........	Bonnycastle.....	....
1821........	Simpson-McClure	A Treatise of Algebra
1824........	Ryan...........	Elementary Treatise on Algebra
1825........	Lacroix-Farrar...	Elements of Algebra
1830........	Colburn.........	Introduction to Algebra upon the Inductive Method of Instruction
1841........	Day............	Introduction to Algebra
1846........	Ray............	Elementary Algebra
1850........	Loomis..........	Treatise on Algebra
1851........	Perkins.........	Elements of Algebra
1855........	Davies..........	Algebra
1855........	Sherwin.........	....
1861........	Stoddard-Henkle.	An Elementary Algebra
1869........	Thompson-Day..	Elements of Algebra
1881........	Wentworth......	Elements of Algebra

TABLE II

APPROXIMATE MEDIAN NUMBER OF PROBLEMS *Attempted* AND NUMBER OF PROBLEMS *Right* PER MINUTE FOR EACH TEST

	Test Number															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	Number of Problems Attempted															
Score of most efficient school	15	4.2	16	15	14.5	6.3	13.4	4.6	10	3.0	4.2	3.5	1.1	7.3	1.4	.....
Average score of best third of 27 schools	.....	3.5	.....	.....	11.6	5.8	12.5	4.1	.....	1.5	3.2	2.9	1.0	5.8	2.5	.....
Score of the ninth school	.....	3.2	.....	.....	10.7	5.5	11.5	3.9	.....	1.2	2.8	2.6	0.8	5.2	3.9	.....
Average score of the 27 schools	10.3	3.0	10.3	10	10.4	4.9	11.2	3.6	8.3	1.1	2.7	2.4	0.8	4.9	3.1	.....
Score of the poorest school	3.0	2.2	2	3	6.5	2.8	7.9	2.4	3	0.5	1.5	1.4	0.5	2.9	.....	.....
	Number of Problems Right															
Score of most efficient school	14	3.2	12	12	13.2	5.4	11.8	4.1	8	1.4	2.8	2.8	0.9	5.3	2.0	.....
Average score of best third of 27 schools	.....	2.7	.....	.....	11.0	4.8	10.7	3.5	.....	0.8	1.6	2.0	0.7	3.9	4.2	.....
Score of the ninth school	.....	3.2	.....	.....	10.0	4.4	9.0	2.9	.....	0.5	1.2	1.7	0.6	3.4	5.9	.....
Average score of 27 schools	9.0	2.2	6.4	7.6	9.7	3.8	9.2	2.9	4.4	0.5	1.1	1.4	0.6	2.8	7.9	.....
Score of the poorest school	2.0	1.2	1.4	1.4	6.1	2.2	4.3	1.7	1.4	0.2	0.1	0.5	0.3	0.1	.....	.....

TABLE III

COMPLETE LIST OF "RECURRING ERRORS" FOR EACH TEST, TOGETHER WITH FREQUENCY AND PERCENTAGE OF OCCURRENCE

(Each error in this list has occurred more than once upon the paper of some pupil. The numbers in the first column opposite each error indicate the *number of pupils* [out of a "random selection" of 100] who made that particular error *one or more* times in the test. The numbers in the second column indicate the percentage of all pupils who make "recurring errors" who made that particular error.)

No. of Error	Tests	I No. Pupils Making Error One or More Times	II Percentage of All "Recurring Errors"
	(REMOVAL OF PARENTHESES)		
1	Arithmetic errors: $24.8=182$; $18.0=18$........	15	35.7
2	Omission of symbols, signs, or terms: $6(3x+8)=18+48$........................	6	14.3
3	Errors of reading or writing: $(2x-3)^2=4y^2-12x+9$....................................	....	....
4	Errors in use of signs in multiplication:		
	a) Minus sign preceding ( ): $-4(3x-4)=-12x-16$..............................	14	33.3
	b) Minus sign preceding first term within ( ): $9(-7x-1)=63x-9$.................	7	16.7
	(SPECIAL PRODUCTS)		
1	Arithmetic errors...........................	5	6.6
2	Omission of symbols, signs, or terms...........	1	1.3
4	Errors in use of signs in multiplication.........	10	13.4
5	Errors in use of signs in addition: $(x+3)(x-4)=x^2+x-12$..................	2	2.6
6	Errors in cross-products: $(a-4)(a+5)=a^2-20$ (NOTE.—It may be due to carrying over rule for product of sum and difference; made by pupils who got product of sum and difference correctly.)	20	26.7
7	Use of product of two numbers instead of twice the product: $(2x-3)^2=4x^2-6x+9$..........	9	12.0
8	Inability in specific operation tested...........	15	20.0
9	Failure to square literal factor with exponent greater than 1: $(a^2+3)(a^2-3)=a^2-9$........	13	17.4
	(EVALUATION)		
1	Arithmetic errors...........................	33	47.8
8	Inability to grasp principle of substitution......	3	4.4
16	Squaring product of literal factor instead of the one designated: $ab^2=(3\cdot2)^2=36$ if $a=3$, $b=2$	22	31.8
11	Using exponent as coefficient and vice versa: $x^2=10$ if $x=5$............................	2	2.9
12	Adding factors instead of multiplying them: $3xy=3+5+2$ if $x=5$, $y=2$................	9	13.1

TABLE III—*Continued*

No. of Error	Tests	I No. Pupils Making Error One or More Times	II Percentage of All "Recurring Errors"
	(FACTORING)		
2	Omission of factors	4	4.3
3	Errors in writing	7	7.4
4	Errors in use of signs in multiplication	2	2.1
8	Positive inability to factor	37	39.4
13	Incorrect division by monomial factor: $5x^2+15x^3=5x^2(3x)$	1	1.1
14	Factors such that the sum of the cross-products are incorrect: $5x^2+16x+3=(5x+3)(x+1)$	20	21.3
15	Factors give correct first and second term, but incorrect third: $a^2+10a+24=(a+8)(a+2)$	4	4.3
16	Failure to recognize monomial factor: $6a^2+9a^3=(2a+3)(3a+3a^2)$	14	14.8
17	Failure to get *highest* monomial factor: $12y+18y^4=2(y+9y^4)$	5	5.3
	(FRACTIONAL EQUATIONS)		
1	Arithmetic errors	28	18.4
1a	Arithmetic error of this type: $-n\cdot 0=n$	8	5.2
3	Error in writing	3	1.9
4	Errors in use of signs in multiplication	4	2.6
5	Errors in use of signs in addition	28	18.3
8	Inability in specific operation tested	27	17.7
18	Failed to change signs if numerator preceded by − sign: $\frac{4x-2}{3}-\frac{x-3}{4}=0$; $16x-8-3x-9=0$	27	17:7
19	Multiplied only one side of equation by L.C.M.: $\frac{3x}{4}+\frac{3x}{2}=5$; $3x+6x=5$	16	10.5
20	Error in use of sign in transposition: $3x+2=6$; $3x=6+2$	4	2.6
22	Error in division (result inverted, e.g., $13x=-1$, $x=\frac{-13}{1}$)	1	.6
23	Multiplied numerator by L.C.D. (e.g., $\frac{2}{2-x}-\frac{5}{2+x}=0$; $4-2x-20+5x^2=0$)	7	4.6
	(EXPONENTS)		
1	Arithmetic errors	11	8.0
3	Error in writing	6	4.4
8	Inability in specific operation tested	21	15.3
25	Incorrect use of exponents in multiplication:		
	a) Multiplying exponents instead of adding them: $a^3\cdot a^5=a^{15}$	9	6.5
	b) Subtracting exponents instead of adding them: $x^7\cdot x=x^6$	4	2.9
	c) Failure to add $x^7\cdot x=x^7$	1	0.7

TABLE III—*Continued*

No. of Error	Tests	I No. Pupils Making Error One or More Times	II Percentage of All "Recurring Errors"
	(EXPONENTS)—*Continued*		
26	Incorrect use of exponents in involution:		
	a) Adding instead of multiplying: $(x^3)^4 = x^7$...	45	32.4
	b) Subtracting instead of multiplying: $(x^3)^4 = x$	1	0.7
	c) Dividing instead of multiplying: $(x^3)^4 = x^{\frac{3}{4}}$	3	2.2
	d) Raising to a power instead of multiplying: $(x^3)^2 = x^9$............................	18	13.1
	e) Failure to raise all possible factors to required power: $(ab^2)^2 = (ab^4)$...............	16	11.6
27	Incorrect use of exponents in division:		
	a) Adding instead of subtracting: $\frac{c^3}{c^2} = c^5$......	1	0.7
	b) Multiplying instead of subtracting: $\frac{y^5}{y^4} = y^{20}$	2	1.5
	(QUADRATIC EQUATIONS)		
8	Inability in specific operation tested...........	32	21.6
28*a*	Find only one root: $x^2 - 81 = 0$; $x = 9$...........	28	19.0
b	One root correct; other incorrect: $y^2 + y = 6$; $y = -3$ or -2..............................	2	1.3
14	Incorrect solution due to: factors such that sum of cross-products is incorrect: $n^2 - 7n = 12$; $n = 6$ or 2..............................	71	48.0
29	Incomplete, e.g., $x^2 - 81 =$, $x - 9 = 0$, $x + 9 = 0$..	9	6.0
30	Incorrect use of formula method of solution....	5	3.4
31	Solution by factoring left member; right member not 0: $n^2 - 7n = 12$, $n(n-7) = 12$............	1	0.7
	(RADICALS)		
1	Arithmetic error............................	14	9.7
8	Inability in specific operation tested...........	88	60.6
24	Incomplete: $\sqrt{8} = \sqrt{4} \times \sqrt{2}$..................	2	1.4
32	Left factor of same power as degree of radical: $\sqrt{32} = 2\sqrt{8}$; $\sqrt{a^3b^4} = b^2\sqrt{a^3}$................	23	15.9
33	Error in factoring; perfect square factor correct, other incorrect: $\sqrt{32} = 4\sqrt{4}$; $\sqrt{50} = 5\sqrt{5}$.....	4	2.8
34	Does not multiply numerator by rationalizing factor: $\sqrt{\frac{2}{3}} = \frac{1}{3}\sqrt{3}$..........................	8	5.5
35	Failed to take square root although factoring correctly: $\sqrt{x^9y^8} = x^4y^8\sqrt{x}$....................	2	1.4
36	Factored correctly, took square root correctly; placed radical sign over wrong factor: $\sqrt{a^5b^4} = \sqrt{a^2b^2} \cdot a$..............................	4	2.8

TABLE III—*Continued*

No. of Error	Tests	I No. Pupils Making Error One or More Times	II Percentage of All "Recurring Errors"
	("PRACTICAL" FORMULAS)		
3	Error in writing	4	2.5
8	Inability in specific operation tested	12	7.5
20	Errors in use of signs in transposition	4	2.5
22	Error in division, result inverted: e.g., $V=lwh$, $w=\frac{hl}{V}$	44	27.3
24	Incomplete	1	0.6
37	Transposing factors instead of terms: $V=Lwh$; $w=V-Lh$	12	7.5
38	Interchanged factors of both sides of equation: $E=\frac{PL}{K}$; $P=\frac{EL}{K}$	30	18.6
39	Error in selecting coefficient of the unknown in $L=\frac{Mt-g}{t}$, $M=\frac{Lt}{t-g}$ (considers $(t-g)$ as coefficient of M)	51	31.6
40	Divides a term of numerator by the denominator: e.g., $L=\frac{Mt-g}{t}=M-g$	3	1.9
	(SIMULTANEOUS EQUATIONS)		
1	Arithmetic errors: $3\cdot 21=73$	25	50
1*a*	Arithmetic errors of the type $n\cdot 0=0$	1	2
3	Errors in writing	4	8
8	Inability in specific operation tested	3	6
19	Multiply only one side of equation by L.C.M.	2	4
20	Error in use of signs in transposition: $6x-4=10$; $6x=6$	3	6
23	Multiply numerator by denominator in clearing fraction	1	2
24	Incomplete (e.g., $-y=-3$)	2	4
41	Errors in use of signs in subtraction: $\begin{cases}10x-15y=65\\10x+\ 6y=78\end{cases}$; $-9y=-63$	4	8
42	Errors in use of signs in division: $-12x=24$; $x=2$	1	2
43	Adds left member, subtracts right member or vice versa: $\begin{cases}4x+2y=20\\3x-2y=1\end{cases}$; $7x=19$	3	6
44	Adds corresponding members when he should subtract: $\begin{cases}2p+10q=54\\2p-\ 3q=\ 2\end{cases}$; $7q=52$	1	2

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