THE SCOPE OF FORMAL LOGIC

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THE NEW LOGICAL DOCTRINES EXPOUNDED, WITH SOME CRITICISMS.

BY

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CONTENTS

CHAPTER IV

General Logic and the Common Logical Doctrines

CHAPTER V

General Logic as the Basis of Arithmetical and of Geometrical Processes

CHAPTER VI

The Philosophical Treatment of Number

CHAPTER VII

The Philosophical Treatment of Space

INTRODUCTION

In my volume *The Development of Symbolic Logic* I traced the growth of Logic from the time of Boole to the time of Schröder, an exposition in which the writings of Venn, of the contributors to the Johns Hopkins *Studies in Logic,* and of certain other pre-Peanesque logicians received detailed consideration along with the writings of those two authors. At the close of my book I drew attention to a number of logicians who are in certain respects the successors of that group of writers, but who have presented such an extended treatment of the subject that they have almost created a new discipline. It is to the work of these last logicians that in the present pages I wish to give special consideration.

There is no doubt that this new view of Logic has already made a remarkable impression upon the philosophical world. But, with very few exceptions, those who have been impressed have been unable to apprehend the full significance of the doctrines that have been placed before them. It is my purpose to set forth the essential features of the new results in such a manner that this inability shall be removed·

I myself am satisfied with the importance of the work that has recently been done, and I hope that, before I have finished, the reader will be so too.

In the present work I do not propose to aim at the briefest exposition possible. In the opinion of certain thinkers, whose judgment I consider important, my former book might have had more in it of the nature of illustration. If I were to defend myself against this criticism I should say that I was then writing wholly for professed logicians. My aim was to point out for them the contributions that had been made to logical doctrine during the previous fifty years, and I wrote under the presumption that my readers were fairly familiar with the works on Symbolic Logic. But, whether I there erred on the side of concentration or not, I purpose on the present occasion both to offer detailed explanations and to give some illustrations. At the same time I hope that my work will not be wholly uncritical. The occupation of merely setting forth other persons' views is not one that is particularly attractive to me. In this subject, unless one can justify or reasonably reject an opinion it is hardly worth knowing the opinion at all. And, if it is unsatisfying merely to be aware of an opinion, it is as a rule equally unsatisfying merely to bring it before the attention of one's readers. But the criticism that I shall offer will be mostly in the nature of justification of the views set forth. On a few points, *e. g.*, that as to the employment of definitions in Symbolic Logic and the relation of definitions in this discipline to those in Philosophy, I have not fallen into line with the new exponents of Logic, but on the whole I am of opinion that the position occupied by the new thinkers is an eminently strong one.

The logicians whose work will be specially considered in the following pages are Frege, Peano and Russell. These three have contributed by far the greater share to the new doctrines. Frege's work began with his *Begriffsschrift* (1879), and has been continued in several of his publications. Peano's *Formulaire de Mathématiques* was published for the first time in the *Rivista di Matematica* in 1891. Since that date he has several times reproduced with additions his theories. Mr. Russell's doctrines are embodied in his *Principles of Mathematics* (1903), and in his important articles in *Mind* and in *The American Journal of Mathematics.*[1]

[1] And quite recently in his work performed in conjunction with Dr. Whitehead, viz., *Principia Mathematica.* This work will not be considered in the present volume, but may be commended to the early attention of the reader.

The questions to which attention will here be directed are the following. In the first place there are certain terms in the new treatment of Logic that require explanation. Such are the terms *propositional function, variable, indefinable, primitive proposition.* Secondly, there are to some extent differences in the symbols that are employed by the three logicians whom we are specially to consider. It will be well to make evident what the differences are: I shall give two chapters of typical proofs of propositions in the Calculus, explain the various literal symbols and symbols of operation, and so make manifest the peculiarities of the logicians in the use of symbols. This procedure will introduce us, among other things, to the important doctrine of Logical Types, and I shall take the opportunity of making certain critical observations upon that doctrine. In the immediately following chapters will be indicated in some detail the facts which lead us to hold that general Logic should be regarded as lying at the basis of the ordinary Formal Logic and of Pure Mathematics. And, finally, I shall direct attention to the philosophical assumptions that are involved in this view of generalized Formal Logic as fundamental in the conceptions of the latter of these two regions of knowledge.

Two observations concerning the title of the present work may here be made to prevent misunderstanding. I have spoken for two reasons of the Scope of *Formal* Logic. In the first place, if instead of this word the word "Symbolic" had been employed, the suggestion would have been conveyed that Symbolic Logic is some peculiar sort of Logic. But such is not the case. The symbolic logician employs symbols merely because the mechanical substitution of certain symbols for certain other symbols is a simpler process than the thinking out of the solution.[1] His *Logic* is the same as other people's, except that he presents a generalized discipline, whereas other people's Logic is a discipline that is unnecessarily restricted in character.

And, in the second place, I have spoken of the Scope of *Formal* Logic because I do not wish to raise the question whether there is any other than such Logic. In my earlier work I still held the view that there are certain branches of inquiry that are of such great importance that they should be gathered together under the designation "Inductive Logic." I now quite agree with Mr. Russell

[1] This is the *main* use of symbols. Some further remarks upon the nature of symbols will be found in Chap. VI.

that though such studies are important there is no sufficient reason for considering them a separate branch of Logic. In so far as such studies set forth methods of proof the studies are formal in character, and in so far as they refer to matters that are preliminary to the application of proof they are not Logic at all. But it is not my purpose to argue this question in the present volume. I have, therefore, designated this book not the Scope of "Logic" but merely of "Formal Logic." My object, in short, has not been to maintain that Formal Logic is the only Logic which is found in the so-called Inductive Logic, but that, so far as Syllogistic Logic is concerned, it is a special application of a truly general Logic, a Logic that lies also at the basis of mathematical reasoning, and that alone deserves the name of Formal Logic.

THE SCOPE OF FORMAL LOGIC

CHAPTER I

EXPLANATION OF TERMS

Propositional Function.—One of the most frequently recurring and of the most fundamental notions of modern generalized Logic is that of propositional function. This is a notion that finds no place in the ordinary treatments of syllogistic logic, and is foreign also to the generalized treatments of Boole and Venn. The meaning of the term *propositional function* comes to light from a consideration of such an expression as "x is a man." Here there is a reference to a *class* of propositions. Each member of that class has for its subject a different individual of a class of terms. The expression "x is a man" does not, however, refer to the *totality* of the individuals of the class of terms, but the expression refers to any one of the class. The reference is to any one of the individuals John Smith, William Jones, and the rest. In other words, one of these terms may be placed in the position of x, and the result will be a statement that either is true or is false. This last observation shows that the class to which reference is made is not the class of *men only*, but is the class of all possible individuals. Supposing that the individual in question is St. Paul's Cathedral, then, substituting this individual for x, we have " St. Paul's Cathedral is a man," which is a proposition that is false. A *propositional function* is thus an expression that contains one or more " variables "—here the variable is x—and a variable has reference to the class of all possible individuals.

There is no doubt that the notion of propositional function *should* enter into the formal treatment of thought. For the notion of propositional function both enters into certain propositions that are to be found in pure mathematics and is found in syllogistic logic. At first sight it might be thought that, though this is so in the former case, the doctrines that are unfolded in the syllogistic and Boolian logic do not admit of the presence of such a notion. But the truth is that all the valid processes unfolded in the earlier logical works may be so stated as to involve the notion in question.

In order to show that the notion of propositional function is found in mathematics we may refer to the propositions of Euclid.[1] And, in order to show that the ordinary logical processes may be so expressed as to involve such a notion, we may refer to the dictum of the first of the syllogistic figures.

The fourth proposition of Euclid, then, implies the fifth, *i. e.*, either the fourth is false or the fifth is true. Let p stand for the fourth proposition, and q for the fifth. Then "p implies q" is an expression that is a proposition. Next let us regard the proposition p. What it says is that, if x is a couple of triangles with two sides and the contained angles equal, x is a couple of triangles whose bases are equal. Now in this expression—also a genuine proposition—the antecedent is "x is a couple of triangles with two sides and the contained angles equal." This is a propositional function. For here not one particular couple is referred to, but any individual whatever. Thus, when we examine such an implication as those that subsist between the earlier and the later propositions of Euclid, we find that the formal elements of thought which are concerned include that of propositional function.

Secondly, in the syllogistic logic the *dictum de omni et nullo* is stated in some such form as the following :—" Whatever is predicated,

[1] *Cf.* Russell, *Principles of Mathematics*, p. 14.

whether affirmatively or negatively, of a term distributed may be predicated in like manner of everything contained under it."[1] Here there is reference to a class, to a predication, and to a portion of a class, whether that portion be a sub-class or whether it be an individual. But the truth embodied in the dictum may quite well be expressed in such language as that which is found in the formula that is adopted by Peano. This writer gives "$a, b, c \varepsilon K . a \supset b . b \supset c . \supset . a \supset c$."[2] The primitive proposition thus symbolized affirms that "if *a*, *b* and *c* are classes, and if *a* is included in *b*, and *b* is included in *c*, it follows that *a* is included in *c*." Here there are involved no fewer than three propositional functions, viz., "*a* is a class," "*b* is a class," "*c* is a class." The *a*, *b* and *c* refer to any individuals whatsoever. But the expressions "*a* is included in *b*" and "*a* is included in *c*" are not propositional functions, for the individual that is substituted for *a* in these expressions must be the same individual that is substituted for *a* in "*a* is a class." Similarly, "*b* is included in *c*" is not a propositional function.

It is to be noted that the Peanesque state-

[1] Keynes, *Studies and Exercises in Formal Logic*, 4th ed., p. 301.

[2] *Formulaire de Mathématiques*, Tome II, § 1.

ment of the dictum of the First Figure of the Syllogism avoids an error from which the common statement is not free. Underlying the common statement is the supposition that an individual and a class are on precisely the same footing in our processes of reasoning. That is to say, *a*, whether it is an individual or it is a class, is included in *c*, if *a* is included in *b*, and *b* is included in *c*. Such a supposition is unjustifiable. There is a relation, for instance,[1] between "Socrates" and "men" which is designated by the expression "Socrates is a man," and there is another relation between the class "men" and the class "classes," which is designated by the expression "men are a class"; we cannot, however, argue "Socrates is a man, men are a class, and, therefore, Socrates is a class." But when classes alone are concerned the relation of inclusion *is* transitive. Individuals and classes, in short, cannot be treated as being on an equal footing, and the ordinary statement of the *dictum de omni et nullo* is framed on the assumption that they can be so treated.

In the above determination of the meaning of the expression *propositional function* there was introduced a distinction which is of the greatest significance in logical doctrine. The

[1] See Russell, *Principles*, p. 19.

distinction is that between the mere consideration of a proposition and the " assertion " of a proposition. The example, for instance, that " the fourth proposition of Euclid implies the fifth " involves both the consideration and the assertion of a proposition. The complete statement is an assertion. But the antecedent " if a couple of triangles have two sides and the included angles equal, then the triangles are equal in every respect " is a proposition that is merely considered. This latter proposition, that is to say, is neither declared to be true nor declared to be false: it is simply an entity regarded by the mind as under certain circumstances implying the consequent. The conditions are that the antecedent be " asserted." In short, there is an assertion when a relation is declared to exist, while in an assertion that involves an antecedent and a consequent—these are not, as we saw, always involved, since such an expression as " Socrates is a man " is an assertion—these two elements are merely considered, the consequent being declared to be true if the antecedent is true.

This distinction is well brought out by both Frege and Russell, and they adopt practically the same symbols to designate asserted propositions. Frege's symbol is $\vdash\!\!-\!\!-\!\!-$. Thus

$\vdash$——— *a* means the proposition *a* is asserted.[1] Mr. Russell shortens the horizontal stroke; *e.g.*, $\vdash : p \,.\, \supset .\, q \supset p$, where *p* and *q* are propositions, the symbol $\supset$ means "implies," the two dots at the commencement indicate that the whole expression after the initial symbol is asserted, and the stops on the two sides of one of the implication-signs that it is *that* implication which is asserted. Peano keeps the distinction between assertion and consideration in mind, but does not adopt a special symbol to designate assertion. His primitive proposition corresponding to the above is given as $a, b \in K \,.\, \supset .\, ab \supset a$, *i. e.*, "if *a* and *b* are classes, then, if the class *ab* is found, the class *a* is found."

Variables.—In distinguishing what is meant by a propositional function it was necessary to employ a symbol that is called a Variable. This was the letter *x*. When we say "*x* is a man" or "*x* is a man implies *x* is a mortal" we are using a variable. What is meant by the latter expression is that, if in place of *x* we have an individual, and that individual is a man, then the individual is also a mortal. The second expression thus brings to the front with regard to the variable a fact that is also implied in the former expression, but is there not so

[1] See *Begriffsschrift*, p. 1.

obvious: this fact is that a variable has reference, as we saw above is the case, to any individual whatsoever. Thus, if in the second expression we take the individual "that tower," we have "if that tower is a man, that tower is a mortal." Here the antecedent contains a false proposition, but the entire statement is true, just as the original implication was true. Similarly, in the former of the two statements mentioned at the beginning of this section the x has reference to all individuals whatsoever. If for x in "x is a man" we substitute King George V we have a true proposition; if we substitute "that tree" we have a false proposition. As regards the implication "x is a man implies x is a mortal," it is to be noted that the variable occurs merely in the antecedent and in the consequent, not in the implication taken as a whole. In other words the implication here is a genuine proposition, and is not a propositional function.

The notion of a variable does not occur in the doctrines of Logic as they are ordinarily expounded. There we are always concerned with genuine propositions and their relations. But the processes of the common logic *may*, as we saw in considering propositional functions, all be so expressed as to introduce this idea. And it is certain that the notion of a variable occurs in pure mathematics. Hence the notion should appear in any logic that can rightly be designated "formal," that, in other words, is concerned with the formal elements of thought, whatever the matter thought about may be.

It must be carefully noted that though the statement which was made in the last paragraph but one is true, viz., that when a variable is used there is a reference to the totality of existing things (*i.e.*, any one of those things may be substituted for x, and propositions true or false result), this is not the same thing as saying, as is sometimes said, that every proposition is a statement about all reality—for instance, the proposition "Every A is B" is said to be equivalent to "Everything is either not A or B." The difference in the two doctrines is found in the fact that in the latter doctrine there is a reference to "all" the things that constitute reality, whereas, when the variable x is used, there is a reference to "any" of those things.

The conception of a variable is realized by Frege, who, however, does not discuss the subject with sufficient fullness.[1] He indicates how one portion of a statement may remain the same while another portion varies: the portion that remains the same is the "func-

[1] See *Begriffsschrift*, pp. 15–18.

tion," while each of the values of the varying portion is an "argument." Thus "Carbonic Acid Gas is heavier than Hydrogen" and "Carbonic Acid Gas is heavier than Oxygen" are two functions with the same argument, or two arguments with the same function, according respectively as we regard "Carbonic Acid Gas" as argument or as function. His notion of a variable best comes out in his statement that undetermined functions may be symbolized thus :—ϕ (A). Here A is an argument, *i. e.*, a definite individual, while ϕ represents a group of predicates whose members may be assigned to A. Thus ϕ is a variable, and each value of the variable when assigned to A will yield a proposition.

The characteristics of the Variable come well to the front in the logical thought of Peano, though he does not explicitly set them forth. From a consideration of his procedure it is made quite evident that he has fully realized the significance of this logical entity. In the first place, that such is the case is apparent in his presentation of those propositions that he deduces from the propositions that are primitive in character. Take, for instance, proposition 62 in the *Formulaire de Mathématiques* (1897). This reads :—

$$a, b \,\varepsilon\, \mathrm{K} \,.\, \supset \,.\!\cdot\!.\; a \supset b \,.\, = \,;\, c \,\varepsilon\, \mathrm{K} \,.\, c \supset a \,.\, \supset_c .\, c \supset b,$$

i. e., " if *a* and *b* are classes, then to say that *a* is *b* is equivalent to saying that, if *c* is a class, then *c*, whatever class it may be, is, if an *a*, then a *b*." Here both *a* and *b* are variables; the letters refer to any individuals whatever, but if they are classes then we may proceed to state the equivalence. And in this equivalence *c* has a similarly broad reference, but if *c* is a class, then, whatever class it may be, if it is an *a*, etc. The fact of his having realized the significance of the variable is also brought out, and in a more striking form, in the case of the examples that Peano gives both in the Notes at the conclusion of the *Formulaire* and in the *Introduction au Formulaire de Mathématique* (1894). For a symbolization of the statement " let *a* be a number, *b* a multiple of *a*, *c* a multiple of *b*; then *c* is a multiple of *a* " is given the following:—

$$a \,\varepsilon\, \mathrm{N} \,.\, b \,\varepsilon\, \mathrm{N} \times a \,.\, c \,\varepsilon\, \mathrm{N} \times b \,.\supset.\, c \,\varepsilon\, \mathrm{N} \times a.$$ [1]

In this symbolic statement the *a*, *b* and *c* again refer to any individuals whatsoever. For the *a* in $a \,\varepsilon\, \mathrm{N}$ an entity that is a positive whole number and an entity that is not a positive whole number may be substituted, and the result will be a true and a false proposition

[1] Notes, p. 38. In this formula the *a*, *b* and *c* are for abbreviation's sake understood as subscripts to the implication symbol.

respectively. And, as before, no difference is made, whichever be the substitution, to the truth of the *implication:* the fact will still remain that "if a is a positive whole number . . . then, etc."

The significance of the Variable is also to be gathered from Mr. Russell's incidental statements. For instance, it is pointed out (*Principles of Mathematics*, p. 5) that the doctrines of Euclid really belong to the region of Applied Mathematics, since in their case a particular kind of space is referred to. That is to say, a proposition of Pure Mathematics would merely state that "if x is a particular kind of space that has the properties signified by the Euclidean axioms, then x is a kind of space that has the properties set forth in the Euclidean propositions." Here, there is the employment of the variable x. The x might be anything whatsoever, but in case it is actual space, *i.e.*, a particular kind of space that has the properties signified by the Euclidean axioms, Euclid's propositions are to be accepted. And in the following page, when attention is called to the fact that in the expression $ax + by + c = 0$, the equation to a straight line, the a, b and c are really variables, the significance of the Variable becomes apparent. We have, "if a, b and c indicate the direction of a straight line, then the equation $ax + by + c = 0$ holds of that line": a, b and c might be anything else, but, if they are what is set forth in the hypothesis, then the consequent is true.

But the question as to the nature of the Variable is also *explicitly* referred to by Mr. Russell. His conclusion is that the Variable presupposes in addition to the notion of propositional function the notion of "any" and the notion of "denoting." When we take, that is to say, the expression "x is a man," a propositional function, it is possible to obtain a number of propositions by giving various values to the x. Now if we wish to define the x we shall say that x is that which is denoted by the term in any proposition whose form is determined by the words "x is a man." Or in general language the Variable is that which is denoted by the term in any proposition in the class of propositions referred to by a propositional function.[1] And in the same work the difference first so characterized by Peano[2] between a real and an apparent variable is drawn. A real variable is one that yields a different proposition for each value of the variable, while in the case of an apparent variable there is *not* a different proposition

[1] See *Principles of Mathematics*, p. 89.

[2] *Formulaire* (1897), p. 23.

for each value of x. In "x is a man" we have a real variable, but in "x is a man implies x is a mortal" we have not a series of propositions for each value of x: we have the same implication throughout.[1]

Indefinables.—The logician is concerned with the processes that regulate thought. Doubtless, as Sigwart points out,[2] it is not possible adequately to realize the significance of these processes unless certain other processes, *e. g.*, those of a psychological character, are to a certain extent considered; but the precise sphere—so far as his relation to Psychology is concerned—in which the logician works is definite enough. Now in laying down the principles that must be observed if thought is to be consistent the logician is inevitably led to the adoption of certain indefinable notions. He cannot start with nothing. It does not follow that the notions with which different logicians set out will be precisely the same. What is aimed at by the logician is to lay down principles through the observation of which

[1] It would thus appear that Mr. Russell regards the propositional function as prior to the variable. But probably he does not intend to maintain any such priority. The question of priority is certainly not a fundamental one: we can equally well speak of a propositional function by reference to a variable.

[2] *Logic*, Part I, Introd. § 5, sect. 4.

thought shall not fall short of the adopted standard, and so long as the principles laid down are capable of effecting this guidance it is unessential that exclusively one body of notions should be taken as indefinable. Hence it is that, though Frege, Peano and Russell have all laid down a list of indefinables, no two lists are the same.

In the calculus of propositions, *i. e.*, the calculus which sets forth the relations of propositions rather than the relations of classes, Mr. Russell takes as indefinable the notions of formal and material implication, together with any notions that may be involved in the former of these.[1] This is in the *Principles of Mathematics*. In the article in the *American Journal of Mathematics*, "Mathematical Logic as based on the Theory of Types," a later work,

[1] By "*p implies q*" is meant that either q is true or p is false, and *formal* and *material* implication are illustrated respectively by the statements "if x is a man, x is a mortal," and "if the fourth proposition of Euclid is true, then the fifth is true." The significance of implication is well brought out in the elucidation of the statement that of any two propositions one must imply the other. What we do in elucidating this statement is to take two propositions p and q and to assert that we must have either the truth of p or the falsity of p or the truth of q or the falsity of q. Then we change the order and reach the falsity of p or the truth of q or the falsity of q or the truth of p. And finally we assert that this is equivalent to "p implies q" or "q implies p."

there are mentioned seven. These are (1) any propositional function of a variable *x*, or of several variables *x*, *y*, *z* . . ., (2) the negation of a proposition, (3) the disjunction or logical sum of two propositions, (4) the truth of *any* value of a propositional function, (5) the truth of *all* values of a propositional function, (6) any predicative function of an argument of any type, and (7) assertion. Of these the first has already been explained. In the case of the third, disjunction is taken instead of implication—the latter was preferred in the *Principles*,—the change being made in order to reduce the number of indefinable notions. The fourth is intelligible by reference to what has been said of the Variable. Number (5) points to the fact that instead of speaking of any value of a variable we may wish to speak of the whole of the values. But the sixth is necessary because without it we should, in speaking of "all," sometimes be involved in self-contradiction. There is no such contradiction possible in speaking of, say, all animals, but when we speak, for instance, of all propositions the case is different.[1] Here there *will* be contradiction unless we distinguish

[1] I have argued later (at the close of Chap. III) that it is only under certain conditions that we must not speak of "all" propositions, but the question of conditions may for the moment be neglected.

between different orders of propositions. If, that is to say, we make an observation concerning propositions of a certain order, the observation that we make will not be of that order. When this notion of orders of propositions is introduced it is seen that a liar who says that each of the propositions which he affirms is false is not affirming a proposition which is true: the proposition he is uttering does not belong to the order of propositions that are affirmed. In other words, there may be predications concerning the statements that are of a certain order, and when predications are made of a statement of a given order they will each be a value of a "predicative function." In short, in our Logic we frequently wish to make statements concerning "all," and we do not wish thereby ever to be involved in contradiction. Hence, among our indefinables must find a place the notion of predicative function. The last indefinable, viz., assertion, has already been explained. In the calculus of classes Mr. Russell takes the notion of a class, the relation of an individual to its class, "such that," and propositional function.[1] If, that is to say, we start with the notion of the relation of propositions we must allow for the case where the propositions involve propositional

[1] *Principles*, pp. 1, 18, 19.

functions, and if we start with the notion of the relation of classes we must be able to express our class relationships by propositions that involve propositional functions.

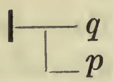
As regards Peano, he, starting with the relation of classes, adopts the notions of a class, the relation of a member to its class, any object, formal implication, the simultaneous affirmation of two propositions, symbolic definition, and negation.[1]

Frege's indefinables are the notions of negation, the relation involved when *q* is true or *p* is not true, and truth-value.[2] The second of these notions, Mr. Russell says, is not the same as his own "implication," maintaining that Frege does not limit the *p* and *q* to propositions. That is to say, it is urged that Frege's "*p* is not true" is not the same thing as the assertion "*p* is false": the latter involves the notion that *p* is a proposition, whereas the former does not. If it really is the case that Frege does not take "*p* implies *q*" to have meaning only when *p* and *q* are propositions, he certainly cannot define proposition in the way that Mr. Russell defines it, viz., to say that *p* is a proposition is equivalent to saying *p* implies *p*, or, in other words, "every pro-

[1] *Op. cit.* pp. 3, 7.

[2] *Begriffsschrift*, pp. 1, 2, 10; Russell, pp. 502, 519.

position implies itself, and whatever is not a proposition implies nothing."[1] In order to designate what he means by a proposition Frege would have to resort to the notion of truth-value, the third indefinable just mentioned. Similarly with Mr. Russell's interpretation of literal symbols it is possible to define negation: "not-p is equivalent to the assertion that p implies all propositions," or to the assertion "'r implies r' implies 'p implies r,' whatever r may be."[2] Such a definition would be inadmissible if p did not necessarily stand for a proposition. I think Mr. Russell is right in his observation concerning Frege's second indefinable. At first I thought Frege was here wrongly interpreted. He certainly says[3] "not every content can become a judgment by means of the symbol |—— set before it; for example, the content 'house' cannot." I concluded, therefore, he always intends such a combination as



to represent only the relationship of propositions; that is to say, whenever implication is involved the letters stand exclusively for propositions. But here it is evident the *impli-*

[1] *Principles*, p. 15. [2] *Ibid.* p. 18.
[3] *Begriffsschrift*, p. 2.

cation is asserted, not the propositions p and q. It may well, therefore, be that the whole statement expresses the relation of the kind that Mr. Russell says it does, and so does not signify the same as he signifies when using the word "implies."

Primitive Propositions.—Starting with these indefinable notions, and with certain other notions that are defined by means of them, it is possible to proceed at once to lay down the propositions of Symbolic Logic. But of the propositions laid down in that discipline it is found that some are and some are not deducible from others. Where a distinction is thus drawn between propositions the former are called Primitive Propositions. Peano and Russell indicate the propositions that they consider to be primitive. Frege does not specially call attention to such propositions, but he has two at the commencement of his *Begriffsschrift* which are of the kind in question: the truth of these two is, that is to say, based upon the truth of no more elementary proposition. The *choice* of the primitive propositions laid down by a logician will not necessarily vary with the indefinable notions with which he starts, for it is conceivable that the indefinable notions and the notions defined by means of these will give rise to the same list of primitive propositions; but there is a strong probability that, where the indefinables vary, the primitive propositions will vary also. The *number* of such propositions may also vary from the same cause, and also from the fact that different writers are not equally successful in effecting the reduction of propositions to simpler forms. For the sake of method it is eminently desirable that the list given of initial propositions shall include only propositions that are not deducible from others. But so far as the validity of the subsequent demonstrations is concerned it is not essential that the number of primitive propositions should be thus precisely ascertained.

The primitive propositions given by Peano are the following:—"if *a* is a class, then *a* implies *a*," "if *a* and *b* are classes, then *ab* is a class," "if *a* and *b* are classes, then *ab* implies *a*," "if *a* and *b* are classes, then *ab* implies *b*," "if *a* and *b* are classes and *a* implies *b*, and if *x* is an *a*, then *x* is a *b*," "if *a*, *b* and *c* are classes, and *a* implies *b*, and *b* implies *c*, then *a* implies *c*," "if *a*, *b* and *c* are classes, and *a* implies *b*, and *b* implies *c*, then *a* implies *bc*."[1] The next primitive, that which is given as prop. 72, is longer. Turning it out

[1] These propositions are numbers 21 to 27 in the *Formulaire*.

of Peano's characteristic symbols it reads :—"if a, b and c are classes, and if x is an a, and if from the fact that the couple (x, y) is a b it follows that the couple, whatever (x, y) may be, is a c, then from the fact that x, whatever it may be, is an a it may be concluded that if the couple (x, y) is a b, then, whatever y may be, the couple (x, y) is a c." And somewhat later we have three other primitives given: "not-a is a class," "not not-a is equivalent to a," and "if ab implies c, and x is an a while x is not a c, then x is not a b.[1]

These primitive propositions are, as we saw would probably be the case, neither in number nor in character precisely the same as those given by Mr. Russell. This writer mentions ten.[2] He does not assert, however, that these are incapable of being reduced to simpler propositions; but merely that he has been unable to make any such reductions. The letters that he uses represent propositions, and consequently none of his examples are *identical* with those of Peano, whose letters

[1] *Op. cit.* props. 105, 106, 107. It will be noticed that the primitive propositions here are introduced as they may be found to be required in the course of the demonstrations.

[2] In the *Principles of Mathematics.* In the article, "Mathematical Logic as based on the Theory of Types," above referred to, the number is given as fourteen.

stand for classes, but there is an *analogy* between certain members of the two sets. For instance, the principle of Syllogism, which was given above in Peano's list, appears in Mr. Russell's as "if p implies q and q implies r, then p implies r." Mr. Russell's seventh and eighth axioms are the principles of Importation and Exportation, and are to the effect respectively that "if p implies p and r implies r, and if p implies that q implies r, then pq implies r," and "if p implies p and q implies q, then, if pq implies r, then [1] p implies that q implies r." Of these the second is analogous to the longest of Peano's primitives. But Peano does not regard the principle of Importation as primitive; it can in his view be deduced from the principle of the Syllogism and the principle of Exportation.[2] On the other hand, both logicians admit the principle of "Composition"—it was the seventh in Peano's list. Where we are speaking of propositions the principle asserts that "a proposition which implies each of two propositions implies them both." [3] The principle of Simplification is also found in the list of each logician —Russell's reads, "if p implies p and q implies

[1] This second "then" is superfluous.

[2] See *Formulaire*, p. 6.

[3] *Principles of Mathematics*, p. 17.

q, then *pq* implies *p* "—but the remainder in each list is peculiar to the respective logician.

Definitions.—Before leaving this question as to the use of terms it is desirable to give some consideration to the subject of definition. We have seen that the symbolic logician must start with certain indefinable notions, and with a number of primitive propositions that involve these notions. But in the course of his procedure he makes use of symbols that represent neither indefinables nor primitive propositions: these symbols represent notions whose character is described in terms of indefinable notions. When a further notion is in such a relationship brought before the attention we have what is known as a definition. Thus Peano defines " *a* is *b* " and Russell defines negation by reference to their respective indefinables.[1] It will thus be observed that in a measure a definition is of the nature of a volition: we determine at the outset that a notion shall be marked off

[1] It may sometimes happen that some or all of the terms employed in a definition are *not* themselves indefinable, but it is always the case that the terms are either indefinables or such as may be defined by means of indefinables. The definition of " negation," for instance, may involve nothing but indefinables, or it may involve the term " proposition "; " proposition " is not itself an indefinable, but is definable by means of the indefinable notion of " implication."

by a certain selection of indefinable notions. Hence it is that Russell says:[1] "definitions have no assertion-signs, because they are not expressions of propositions, but of volitions." But we must here make a distinction which is of great importance. Definitions of this kind are not *arbitrary* volitions. We may, for instance, define negation by reference to our indefinable notions, but our definition must be such that no contradiction shall be involved when we bring our negative class or proposition into relation with the corresponding positive; our definition of negation must be, among other things, one that allows of the affirmation "not not-p implies p." In defining by means of our indefinable notions, though we have a choice, we must choose with a certain end in view, viz., the avoidance of subsequent contradictory statements.

On the other hand, there are certain definitions used in the Calculus that *are* wholly arbitrary. An instance of one of these is given by Frege in his *Begriffsschrift*, p. 55. He here gives an "equivalence" where it is intended to define the right-hand member by means of the left-hand member. Such a definition

[1] *American Journal of Mathematics*, art. "The Theory of Implication," vol. xxviii, No. 2, p. 176.

is an arbitrary one: the expression $\begin{array}{c}\delta\\ |\\ \alpha\end{array}\left(\begin{array}{l}F(\alpha)\\ f(\delta, \alpha)\end{array}\right.$ might be taken as equivalent to anything else whatsoever instead of being taken as an abbreviated form of $\begin{array}{l} \overset{\mathfrak{d}}{\smile}\!-\!\overset{\mathfrak{a}}{\smile}\!-\!\!\begin{array}{l} F(\mathfrak{a}) \\ \lfloor f(\mathfrak{d}, \mathfrak{a}) \end{array} \\ \lfloor F(\mathfrak{d}) \end{array}$.[1] In short, all definitions *are* volitions, but all definitions are not arbitrary volitions.

In the next place it is to be observed that though definitions in Symbolic Logic are in their nature marked off from assertions, all such definitions may be introduced into reasonings in precisely the same way as assertions may be. This fact is made quite evident by Frege both in so many words and in his method of demonstrating the truth of his 70th proposition.[2] This demonstration, as usual, is established because the truth of the hypothesis is already known. But what the hypothesis sets forth is the equivalence that has been determined upon in the 69th proposition. That is to say, what we do in the more complicated proofs is to take one of the primitive propositions, or one of the simpler propositions that are derived from them, and to sub-

[1] The meaning of the latter of these expressions is not relevant in the present argument.

[2] *Begriffsschrift*, p. 58.

stitute expressions of a complicated character for the symbols employed in such proposition. And it is quite irrelevant whether the substitution made in the hypothesis is of an assertion or of an equivalence it has been decided to adopt. The implication set forth in the consequent necessarily follows in either case.

The nature and treatment of definitions are up to a certain point well indicated by Frege. Mr. Russell quite clearly points out that definitions are of the nature of volitions, but he does not distinguish, so far as I have seen, between arbitrary and reasoned definitions, and he does not explain how it is that definitions are used in the same way as assertions. After the declaration of the volitional character of definitions and of the fact that in consequence of this character they have no assertion-signs, some explanation is needed why definitions *are* treated just like assertions. That Mr. Russell does so treat definitions is seen in many places. For example, in the article on "The Theory of Implication," already referred to, prop. 4·24 makes use of prop. 4·1 in precisely the same way as prop. (3) is used, where prop. 4·1 is a definition. It is indeed possible to say that here the definitions are *not* treated as assertions, but are merely reminders of equivalences that have been agreed upon. But I do not see that anything is gained by speaking of definitions in this way: it is less confusing to hold, as Frege holds, that when a definition is brought forward we have an assertion. And in certain proofs we *must* interpret our definitions as assertions. Frege's 75th proposition (in the *Begriffsschrift*), for instance, cannot be proved unless prop. 69 is known to be *true*. Frege, it may be noticed, signifies by a double assertion-sign those statements that are originally definitions: he uses $||\!—$ instead of $|\!—$.

And, lastly, between these definitions of Symbolic Logic and those of Philosophy there is a striking difference, but there are also some similarities. As regards the difference, in philosophical definitions we enumerate the attributes that are signified by the name, or we abbreviate this process by referring to the *genus* and *differentia* of the object. Now in this enumeration what we are doing is to refer in the case of external objects to the sensations that we receive from them, and in the case of mental processes to the simple modes of consciousness that are revealed by introspection. Here the ultimates that constitute the elements of our definitions are " naturally selected."[1] In Symbolic Logic, on the contrary, the ultimates at our disposal are ideas that are "artificially selected." We are not at the outset limited to a certain set of indefinables, but we make a choice from those available. And subsequently it is from the ultimates thus chosen that we make a selection for the purposes of definitions. Hence it is that Mr. Russell affirms that the distinction between the two kinds of definition consists in the fact that in philosophical definition we are, and in logical definition we are not, analyzing "the idea to be defined into constituent ideas."

[1] The whole of this discussion on Definition appeared in *Mind*, vol. xix, N.S., No. 75. A logician in the course of some appreciative remarks which he has sent me concerning the contribution suggests that the term "selective" would be preferable to "volitional" as applied to Definitions. I have no objection to the change: the word "volition" was used by me because it is the word used by Mr. Russell in the passage that led to my criticism. The other suggestion from the correspondent is one that I cannot adopt. He says that "ultimates 'naturally selected'" is an unfortunate expression, and prefers "indicated" to "selected." I do not think the change is an improvement. What we want to distinguish are the ultimates that the symbolic logician determines shall be those to which we refer, and the ultimates that in natural science and philosophy are determined not *by* us but *for* us, and I think the terms "artificial selection" and "natural selection" precisely fix this distinction.

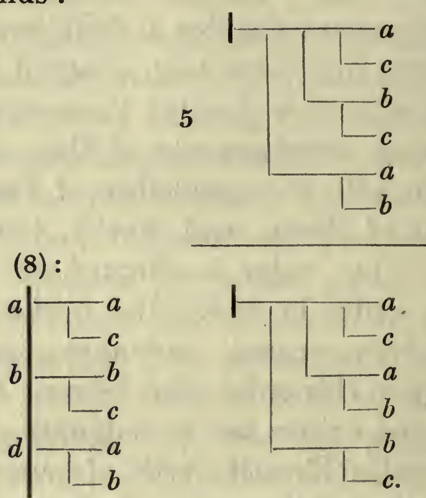
On the other hand, in both kinds of definition there is an artificial selection from among the ultimates thus respectively at our disposal. An external object such as an orange, or a mental process such as attention, may be defined by reference to more classes than one. And, in the same way, we are not restricted to one selection from our artificially-constituted ultimates in defining our non-ultimate notions in the Logical Calculus—the notion of disjunction, for instance, may be defined with or without reference to the notion of *such that*. And, in the second place, in both kinds of definition the ultimates are immediately presented. The notion of "implication," the notion, that is to say, which is involved when we say that the proposition p is false or the proposition q is true, is as immediate as the notion "blue": both notions are discernible by the mind as unanalyzable constituents of its experience.

CHAPTER II

VARIATIONS IN SYMBOLIC PROCEDURE

In the present chapter I shall consider the symbolisms that have been adopted by those logicians who have devoted themselves to the most recent developments of the science. I will begin with the symbolism of Frege, then take that of Peano, and, finally, that of Mr. Russell. This order is adopted not because it is the order in which the works of these logicians first appeared—such appearances, however, *were* in this order—but because the latest of the three writers has in elaborating his own system availed himself—with, of course, abundance of acknowledgments—of what was best in the work of his predecessors. Peano, it may be observed, though aware of Frege's work, and adopting certain of this writer's propositions, has not adopted Frege's assertion-signs, and so fails to emphasize the important distinction that is to be drawn between a proposition that is asserted and one that is merely considered.

It will be best first of all to take one of Frege's proofs that involve simple symbols, and then to take one that involves complicated symbols, and to explain in each case the significance of the various symbols that are employed. As an example of an argument that involves simple symbols prop. 9 in the *Begriffsschrift* may be taken. This proposition appears thus :—



Here the letters represent *propositions*. The small perpendicular stroke at the commencement of the first line indicates that the proposition represented by all that is on the right of this stroke is *asserted*. Frege, that is to say, draws, as we have seen, a distinction between propositions that are asserted and those that are merely considered, and he denotes the former in this manner. The separate propositions *a*, *b*, *c*, are thus merely considered. Those perpendicular lines that reach the uppermost horizontal line indicate what we generally describe as " antecedents " of the proposition mentioned on the right of that line. And, similarly, a perpendicular that reaches a horizontal line other than the top one indicates that the proposition on the right of the horizontal line from which the perpendicular starts is an antecedent of the proposition at the end of the horizontal line that is reached. The first part of this 9th proposition will thus read :—" If it is true that *b* implies *a*, and that *c* implies *b*, and if *c* is true, then *a* is true." The figure 5 at the commencement points to the fact that what is found on its right-hand is the conclusion of the 5th proposition. The number 8, followed by the colon, indicates that in the conclusion of the 8th proposition instead of *a*, *b* and *c* are to be substituted the expressions that stand on the right of those letters. When this substitution is made it will be found that the hypothesis is the expression following the figure 5, and thus the consequent is shown to be true. This consequent is the third member in the group of symbols. In a few cases it happens that the proposition adduced to show that the hypothesis is true is the proposition that has been proved immediately before; under such circumstances Frege does not consider it necessary to refer explicitly to such proposition. An instance of such omission is found in the proof of prop. 43. Finally, it occasionally happens that a double colon is used after the number of a preceding proposition. In such a case the proposition that is reached by making substitutions for the original symbols is found to be the hypothesis of the preceding proposition. The consequent of this proposition is thus discovered to be true, and forms the new proposition.

From this account it becomes evident that Frege's symbolism has certain decided advantages. In the first place, it distinctly indicates, as we saw above, what propositions are asserted and what propositions are merely considered. In the second place, it is possible with such a symbolism to observe the precise implications that are indicated : the horizontal and perpendicular lines carry one's attention immediately from the antecedent to the respective consequent. There is involved, that is to say, no such elaborate system of dots as that which Peano is compelled to use, a system which is scientific enough, but is one which presents some difficulty when an attempt is being made to realize the relations of the various implications that are symbolized. The great drawback to Frege's symbolic procedure is its want of compactness, or, as Venn has said, "the inordinate amount of space demanded for its display. Nearly half a page is sometimes expended on an implication which, with any reasonable notation, could be compressed into a single line." [1] On the whole, I think that for the special purpose that Frege has in mind in the *Begriffsschrift*, viz., the demonstration that arithmetical propositions are all illustrations of certain propositions whose validity is set forth by the symbolic logician, the symbolism that is given in that work is excellent, but that for the general purposes of Symbolic Logic it is better to have a less diagrammatic system of symbolism.

A good example of Frege's proofs that involve complicated symbols is prop. 77, a proposition of moderate difficulty. This is one of those proofs that show the way in which the propositions of arithmetic are but illustrations of truths that can be reached by a procedure that is exclusively logical in character. The proof is presented in the following manner :—

[1] *Symbolic Logic*, 2nd ed., p. 494.

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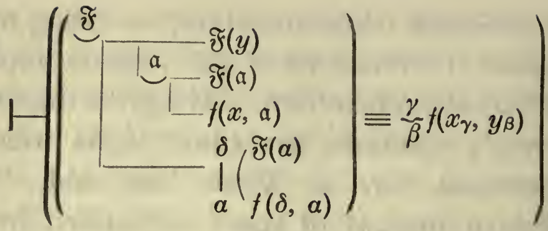
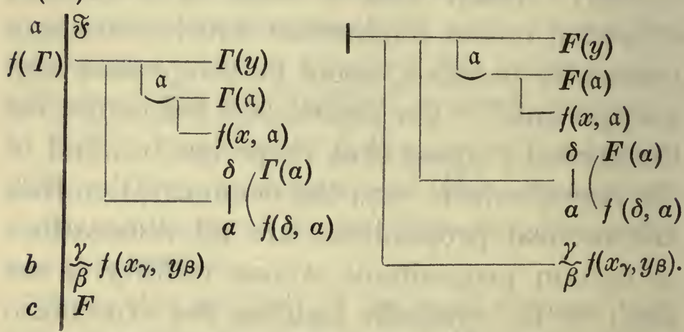


(68):



Of this the verbal expression is given as follows: "If y follows x in the f-series, if the peculiarity F is bequeathed to the f-series, and if each result of an application of the experience f to x has the peculiarity F, then y has the peculiarity F."

Here the numbers 76 and 68, the colon following the latter number, the small thick perpendicular strokes, the thin horizontal and perpendicular strokes, and the substitutions indicated by the long perpendicular line in the second group of symbols, have been already explained. In explaining the remaining symbols we will take first the proposition that is adduced in proof of the third proposition. The expression $\underset{\mathfrak{a}}{\overset{\mathfrak{d}}{|}}\left(\begin{matrix}\mathfrak{F}(\mathfrak{a})\\ f(\mathfrak{d}, \mathfrak{a})\end{matrix}\right.$ together with the small horizontal concave line indicates, when taken in isolation, that, if each $\mathfrak{d}$ has the peculiarity $\mathfrak{F}$, and $\mathfrak{a}$ is the result of an application of the experience f to $\mathfrak{d}$, then each $\mathfrak{a}$ has the peculiarity $\mathfrak{F}$, whatever $\mathfrak{F}$ may be. The expression has been declared to be equivalent to this by prop. 69, which is originally a definition, and is subsequently used as a statement that is true. The two lines above, viz., $\overset{\mathfrak{a}}{\smile}\!\begin{array}{l}\text{—}\mathfrak{F}(\mathfrak{a})\\ \text{—}f(x, \mathfrak{a})\end{array}$, affirm in isolation that if $\mathfrak{a}$ is the result of an application of the experience f to x, then each $\mathfrak{a}$ has the experience indicated by $\mathfrak{F}$. Finally, the whole expression on the left of the sign of equivalence ($\equiv$) signifies that, if both these implications hold, then y has the experience indicated by $\mathfrak{F}$. This implication is then stated to be equivalent to the expression $\overset{\gamma}{\underset{\beta}{\smile}} f(x_\gamma, y_\beta)$. This is one of those propositions that originally are definitions—the symbol ($\equiv$) indicates this original character of the statement.[1]

[1] Frege has two methods of indicating that an expression is originally a definition. He always uses the symbol of equivalence, and sometimes he adds a thick perpendicular stroke to the assertion-sign, while

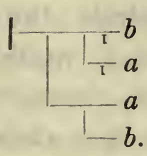
Coming now to the second group of symbols, that in which there is pointed out what substitutions are to be made in one of the less elaborate forms of proof, the substitution that calls for special attention is the second. In this substitution any term of which f is asserted is to occupy the position of Γ in the right-hand expression, unless, as here, such term is given equivalent to another, in which case there will be a double substitution. In the case before us we first, that is to say, replace Γ on the right-hand by c and then replace c by F.

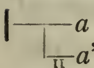
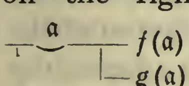
With regard to the third group of symbols it is to be noticed that the hypothesis which is known to be true is as usual omitted, and the conclusion alone is given. The hypothesis before us is the prop. 76. This hypothesis is another instance of the way in which definitions *must* be treated as assertions if the validity of the conclusion is to be established.

From the consideration of these two propositions it is possible to observe the signification of almost all the symbols that Frege employs. There is one other symbol of import-

at other times he contents himself with the ordinary assertion-sign. The former method is adopted when the definition is first of all set forth, while the latter method is that which is found when the definition comes actually to be used in the course of a demonstration.

ance to which attention must be called. It is that for negation. To denote this notion a short, perpendicular stroke is written beneath a horizontal stroke. For instance, prop. 28 is given as :—



This reads : " If *b* implies *a*, then, if *a* is not true, *b* is not true." It is convenient on such a method to indicate double negation. Thus , prop. 31, signifies that if it is false that *a* is false, then *a* is true. And no difficulty is experienced in indicating how far the negation is to extend : the proposition that is denied has reference to the proposition that is on the right of the sign of negation, and all the propositions whose perpendiculars terminate on the right of that sign. For instance,  reads : " It is not true that, if an object is *g*, such object is *f*." [1]

The suitability of Frege's symbolism for the special purpose that he has in hand in the *Begriffsschrift* is thus obvious. For making immediately evident what is the hypothesis

[1] See *Begriffsschrift*, prop. 59.

and what is the consequent in a complicated proposition that has to be proved his procedure is admirable. And in the case of the more complicated arguments, such as the second of the above two, though there is a cumbersomeness about his symbols, the process of proof is more evident than it would be if they were less diagrammatic in character, and appeared in long lines with the elaborate system of brackets that would then be necessary. This advantage of Frege's method in the case of complicated problems will, however, be more apparent when we have considered one of the linear methods of symbolic representation. To these we now proceed, and we begin, as we said, with that of Peano. It will be best, as before, first to take one of the comparatively simple forms of proposition, and then to take one of the more complex.

First of all, however, it is desirable, in order to make the explanation of the propositions more direct, to give a general description of the system of brackets and dots that is adopted by Peano. So far as brackets are employed their use is the same as that which is found in algebra, *i. e.*, they keep together those expressions that are immediately connected. When dots occur they take the place of brackets; the replacement is made in order that the confusion which would arise if many of the latter were found in a small collection of literal symbols may be avoided. Sometimes in Peano's expressions all the brackets are replaced by dots; in other cases it is found more convenient to have a combination of the two methods of grouping. In general it may be said the greater the number of dots in a group of dots the more complicated are the expressions on its left or right.

As an instance of the employment of this method of grouping propositions the following expression may be taken: *ab . c : de .·. fg :: h.*[1] Here the single dot preceding the *c* indicates that this letter is to be taken with the preceding group. The two dots that follow the *c* unite the whole expression before them to the group *de*. That we are not to proceed further than this group in effecting the union is made manifest by two of the dots (.·.). When these two have thus guided us, there remains one, and this shows that to the whole preceding expression the group *fg* is to be united—that we are here not to take up more than this group is indicated by three of the dots (::). Finally, the remaining dot of the four (::) instructs us to combine the *h* with the whole expression that precedes. We have in this way effected a

[1] *Formulaire*, p. 23.

union that would by means of brackets and vincula appear in the following manner:— $[\{(\overline{ab}\,c)(\overline{de})\}\{fg\}]h$. The method of dots besides being neater is shorter, for there is found to be no loss of legibility if the dots corresponding to one member in a couple of brackets are omitted.

As an example of proofs that contain only the simpler form of symbol the proof of the Importation proposition may be taken. This proposition, as we saw, is taken by Russell as primitive in character, but is demonstrable on Peano's view by means of the principles of Exportation and Syllogism. The proposition to be proved (prop. 73) appears thus:—

$$a, b, c \,\varepsilon\, \mathrm{K} \therefore x \,\varepsilon\, a \,.\, \supset_x : (x, y) \,\varepsilon\, b \,.\, \supset_y .\, (x, y) \,\varepsilon\, c \therefore \supset x \,\varepsilon\, a \,.\, (x, y) \,\varepsilon\, b \,.\, \supset_{x, y} .\, (x, y) \,\varepsilon\, c.$$

This in words may be read: "Let a, b, and c be classes, and let x is an a, whatever x may be, imply that the couple (x, y), whatever y may be, is, if a b, then a c, then if x is an a, and (x, y) is a b, it will be the case that (x, y), whatever (x, y) may be, is a c." [1]

The signification of the dots will, from what was said above, be immediately apparent. Of the other symbols special attention must be given to ε, the subscript, and (x, y). The ε is a symbol that denotes the relation of an object to the class of which it is a member. Peano

[1] Cf. *Formulaire*, p. 38.

and Frege have the distinction of having first recognized this important relation. Previous logicians regarded the relation as equivalent to that of a class to an including class, but the two relations are quite distinct, and when they are regarded as equivalent it is impossible to observe the full scope of logical science.[1]

The subscript x indicates that it is quite a matter of indifference which of the particular objects that we have in mind is taken. If a in the expression $x \varepsilon a$ stands for "a man," then—taking the hypothesis of the whole implication in isolation—we have the fact that, whichever of the objects that we have in mind is taken, if that object is a man the implication bounded by the dots (:) will follow. Similarly, the subscript x, y, denotes that it is immaterial which of the couples that we have in mind is taken : the couple will, if it is a b, be also a c.

Finally, the expression (x, y) which we have just said indicates a "couple," limits the objects that we have in mind to those that consist of two individuals. Thus (0, 1) is a couple, (James Mill, John Stuart Mill) is a couple. A couple, as Peano observes,[2] is an object quite

[1] See what was said in Chap. I, when we were referring to the expression "propositional function."

[2] *Formulaire*, p. 36.

distinct from the two objects of which it is composed. And, just as we can substitute for x in " x is a man " and obtain a true or a false proposition, so we can substitute for the couple (x, y) in the expression in which it occurs, and obtain a similar result. For instance, if we substitute for (x, y) in " (x, y) is a couple satisfying the equation $x^2 + 2y^2 = 1$ " the value $(1, 0)$, we obtain a proposition that is true. As before, the original expression is a propositional function, and that which is obtained by substituting an individual for the variable is a proposition.

We may now proceed to an example that contains several symbols in addition to certain of the symbols that have just been mentioned. A proposition that introduces directly or indirectly quite a number of symbols is number 463. The conclusion to be reached is thus represented :—

$$\cup`(u \cup v) = (\cup`u) \cup (\cup`v).$$

The proof is as follows :—

$$[\text{P461}\,.\supset.\,\cup`(u\cup v) = \overline{x\,\varepsilon}\,\exists\{(u\cup v)\cap\overline{y\,\varepsilon}\,(x\,\varepsilon\,y)\}$$

$$\text{P217}\,.\supset.\quad ,, \quad = \overline{x\,\varepsilon}\,\exists\{u\cap\overline{y\,\varepsilon}\,(x\varepsilon y)\,.\cup.\,v\cap\overline{y\,\varepsilon}\,(x\varepsilon y)\}$$

$$\text{P410}\,.\supset.\quad ,, \quad = \overline{x\,\varepsilon}\,\{\exists\,u\cap\overline{y\,\varepsilon}\,(x\varepsilon y)\,.\cup.\,\exists v\cap\overline{y\varepsilon}\,(x\varepsilon y)\}$$

$$\text{P234}\,.\supset.\quad ,, \quad = \overline{x\varepsilon}\,\exists\{u\cap\overline{y\varepsilon}\,(x\,\varepsilon y)\}\cup\overline{x\,\varepsilon}\,\exists\{v\cap\overline{y\varepsilon}\,(x\varepsilon y)\}$$

$$\text{P461}\,.\supset.\,\text{P.}]$$

Taking the symbols of the conclusion, the $\cup$ joining the u and v signifies logical addition.

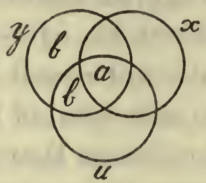
It is the same symbol as Venn's + and is defined by Peano by means of a double negation. That is to say, he defines $a \cup b$ by stating that it is equivalent to $-[(-a)(-b)]$, or that which is not both not-a and not-b.[1] The $\cup$ ' in the conclusion is a symbol that denotes the logical sum of the classes that compose a class. This symbol occurs three times. On the left the class, the logical sum of whose sub-classes is taken, is $u \cup v$, and on the right the classes whose sub-classes are summed are u and v respectively. The conclusion, then, states that the logical sum in the former case is equivalent to the sum of the two logical sums in the latter. Proceeding to the proof, prop. 461, which is said to imply the equivalence on the right, is the following :—

$$\cup ' u = \overline{x \varepsilon} \{\exists . u \cap \overline{y \varepsilon} (x \varepsilon y)\}.$$

This means that to take the logical sum of the classes that compose the class u is equivalent to taking certain existing objects (x), viz., those that are u and that are y, provided the y's are such that an x is a y. This notion of a logical sum is a difficult one, and I will, therefore, describe it further by reference to a diagrammatic illustration. Let the class u of classes, one of the contained classes, and x, the objects indicated by the variable, be repre-

[1] See prop. 201.

sented by three intersecting circles in the manner adopted by Dr. Venn. Then the pro-



visional statement in the definition forbids our making any reference to the existence of *b*. Hence the definition as a whole simply affirms the existence of *a*; in other words, there are some *x*'s, viz., the *y*'s that are *u*'s: it is those *x*'s that constitute the logical sum of *u*.

The ∃ that is used in the above expression is thus equivalent to the *v* or the >0 that are used by Venn: to say ∃ *a* is to say that there are *a*'s.[1] And the line above the $x \,\varepsilon$ is part of the symbolism that is sometimes employed in designating a class. Supposing *a* is a class, then by prefixing $x \,\varepsilon$ to *a* we obtain an expression that signifies "*x* is an *a*." If, then, to the $(x \,\varepsilon\, a)$ we prefix $\overline{x \,\varepsilon}$, we designate the *x*'s such that *x* is an *a*; in other words we reach again the class *a*.

Substituting, then, $(u \cup v)$ for *u* in prop. 461, we obtain the equivalence :—

$$\cup \text{‘} (u \cup v) = \overline{x \,\varepsilon}\, \exists \{(u \cup v) \frown \overline{y \,\varepsilon}\, (x \,\varepsilon\, y)\}.$$

[1] See prop. 400.

In the next line of the proof there is use made of prop. 217. This proposition sets forth the equivalence of $(a \cup b)\, c$ to $ac \cup bc$; in the expression before us the $\overline{y}\, \varepsilon\ (x\, \varepsilon\, y)$ takes the place of the c, and u and v respectively take the place of a and b. Prop. 410 is to the effect that $\exists\, (a \cup b)$ is equivalent to $\exists\, a\, .\cup .\, \exists\, b$. Hence $\exists$ in the expression that we have reached can be prefixed to each of the entities that are joined by $\cup$. Then by an application of prop. 234, which says that $\overline{x}\, \varepsilon\, (x\, \varepsilon\, a\, .\cup .\, x\, \varepsilon\, b) = a \cup b$ we are able to reach the proposition :—

$$\overline{x}\, \varepsilon\, \exists\, \{u \cap \overline{y}\, \varepsilon\, (x\, \varepsilon\, y)\} \cup \overline{x}\, \varepsilon\, \exists\, \{v \cap \overline{y}\, \varepsilon\, (x\, \varepsilon\, y)\}.$$

And, finally, these alternatives are by prop. 461 equivalent respectively to $\cup$ ‘ u and $\cup$ ‘ v.

These two propositions give a pretty complete conception of the symbols that are employed by Peano. He has, however, one or two others that need attention. We saw that $\exists$ signifies not-nothing. The symbol which Peano uses for 0 is Λ. For “ everything” he employs in place of the 1 found in Venn, and the ∞ in the work of Mrs. Ladd-Franklin, the symbol v. Again, there are some classes that contain only a single member; a special symbol is provided to designate such a class, namely, $\iota\, x$, where ι is the initial letter of the word ἴσος. And, inversely, just as $\overline{x}\, \varepsilon\, (x\, \varepsilon\, a)$

points to objects that satisfy the condition ($x \varepsilon a$), so, when the single object of a class is referred to, Peano makes use of the symbol $\bar{\iota}$. And, finally, he requires a symbol to indicate the operation of transforming one object into another. The symbols that indicate such operations are ɟ and f. When the former is used there is conveyed the further information that the letter which stands for the particular operation that effects the transformation is to be *after* the symbol for the object that gives place to the new object, while when f is employed such letter is to be *before* the symbol for the first object. Thus a ɟ b signifies those operations that will turn the a's into the b's, and that a letter which represents one of the operations is to be placed after the symbol a; while b f a signifies the operations that will effect a similar change, and that the letter representing one of the operations is to be before the symbol for the object that is to be transformed. Suppose u stands for one of the operations that will change objects of the class a into objects of the class b. Then in the two cases that have just been described we shall have respectively the following equivalences :—

$$u \varepsilon a \,\text{ɟ}\, b \,.=.\, x \varepsilon a \,.\supset_x.\, xu \varepsilon b$$

and

$$u \varepsilon b \,\text{f}\, a \,.=.\, x \varepsilon a \,.\supset_x.\, ux \varepsilon b.$$ [1]

[1] See prop. 500 and prop. 501.

We may now pass to the consideration of the symbolism that has been set forth by Mr. Russell. This shows a decided improvement on the two systems that have just been described. Mr. Russell avails himself of what is best in the work of his predecessors, and he has in many particulars introduced changes for the better in the courses of procedure that he has adopted. He has also originated certain symbolic methods that are extremely useful. We are occupied in this chapter merely in pointing out the symbolism that has been adopted by those who have realized the full scope of logical theory, and in setting forth in particular the differences that obtain in the symbols of the three chief exponents of the new doctrine; otherwise it would be necessary to state that Mr. Russell's excellence consists not merely in his having laid down a most effective system of symbols, but also in his having with great fullness discussed the philosophical implications of the notions that he has symbolized. His results in the latter region of inquiry far exceed in value those that had been reached either by Frege or by Peano. In a future chapter we shall have the opportunity of considering some of the philosophical implications that Mr. Russell has unfolded.

Confining ourselves for the present to the question of the signification of his symbols, we may refer to one of Mr. Russell's recent papers, viz., that entitled *The Theory of Implication.*[1] In that paper he first of all enumerates and proves the propositions that are the "most important in the theory of *material* implication."[2] After what has been said of the symbols of earlier logicians it will be sufficient if we take one only of Mr. Russell's proofs. We will take the proposition that is designated ✻5·43.[3] This proposition with the proof reads as follows :—

[1] *American Journal of Mathematics*, vol. xxviii, No. 2, p. 159.

[2] For the distinction between formal and material implication see *The Principles of Mathematics*, p. 14. "The proposition p implies the proposition q" is an instance of material implication; "x is a man implies x is a mortal" is an instance of formal implication. In such a statement as "if a, b, and c are points, then, whatever points they are, if $a\,b = a\,c$, then the angle $a\,b\,c =$ the angle $a\,c\,b$," there is a material implication within a formal implication. The material implication is, "if $a\,b = a\,c$, then $\angle\, a\,b\,c = \angle\, a\,c\,b$." It is changes that may be effected in such material as this that are set forth by Mr. Russell in the above article.

[3] This method of referring to a proposition Peano sometimes uses. The asterisk signifies the term "proposition." The 5 points to the fact that the proposition in question is related to other propositions, viz., those that commence with the same number. The 4 indicates a sub-class of such related propositions. And, finally, 3 points to the fourth member in this sub-class—the first is not designated by a third number, but appears merely as 5·4.

$\vdash :. p . \equiv : p \vee q . p \vee \sim q$

Dem.

$\vdash . * 4{\cdot}28 . \quad) \vdash : p .) . p \vee q : p .) . p \vee \sim q :$

[Comp] $\quad) \vdash : p .) . p \vee q . p \vee \sim q \qquad (1)$

$\vdash . * 3{\cdot}44 \frac{\sim p}{p} .) \vdash :. \sim p) q .) : \sim p) \sim q .) . p :.$

[Imp] $\quad) \vdash :. \sim p) q . \sim p) \sim q .) . p :.$

[(* 4·11)] $\quad) \vdash :. p \vee q . p \vee \sim q .) . p \qquad (2)$

$\vdash . (1) . (2) . \quad) \vdash$. Prop.

We will again commence with the symbols in the conclusion, and proceed to those in the proof. As is the case with Frege the literal symbols stand for propositions. The symbol $\vdash$ signifies precisely the same as Frege's |——, viz., that the proposition which follows is *asserted*, and not merely considered. The system of dots throughout the conclusion and the proof is that which is employed by Peano; the (:), for instance, indicates that the *whole* of what follows the sign of equivalence is identical with the p on the left. For the sign of equivalence Mr. Russell, with Frege, prefers $\equiv$ to the $=$ of Peano. The former is undoubtedly the better symbol, as it does not lead the attention away to quantitative considerations, but keeps it fixed on those that are strictly logical in character. Lastly, the symbol $\vee$ signifies disjunction or logical addition : $p \vee q$ is affirmed to be the same thing as $\sim p \ ) \ q$.[1]

[1] See prop. 4·11.

Coming to the demonstration—this is signified by *Dem.*—the first line is to the effect that prop. 4·28 is asserted, and that consequently it is asserted that p implies p or q, and p implies p or not-q. The proposition that is referred to is to the effect that $p\,.\supset.\,p \vee q$, so that a double use of the proposition is here involved. In the second line the word Comp appears in square brackets. This means that we are to bring the principle of Composition to bear upon the proposition that has just been asserted. The principle of Composition lays down that "if p implies each of two propositions, it implies their propositional product;"[1] or in symbols the principle is :—

$$\vdash :. p \supset q \,.\, p \supset r \,.\supset: p \,.\supset.\, q \,.\, r.$$

We thus reach in the second line the assertion :

$$p \,.\supset.\, p \vee q \,.\, p \vee \sim q.$$

In line three there is an assertion-sign and the number of a proposition, but there is no square bracket. We are in this way instructed simply to assert the proposition referred to. By the expression that immediately follows, viz., $\dfrac{\sim p}{p}$, it is pointed out, however, that the assertion is not to be made with the identical symbols that were originally employed, but with $\sim p$

[1] See page 181.

substituted for p. When in prop. 3·44, which appears as

$$\vdash :. p \supset q . \supset : p \supset \sim q . \supset . \sim p,$$

such substitution is made, we arrive at the expression on the right. In line four, as in line two, a proposition is cited that has to be brought to bear upon the assertion in the previous line. In the present case the utilized proposition is that which is known as the principle of Importation—a term borrowed from Peano. The principle in Mr. Russell's symbolism is the following :—

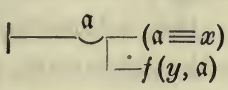
$$\vdash :. p . \supset . q \supset r : \supset : p . q . \supset . r.$$

That is to say, if p implies that q implies r, then $p\,q$ implies r. The expressions that in the third line take the place of p, q, and r are respectively $\sim p \supset q$, $\sim p \supset \sim q$, and p. In the last line but one there again occur square brackets; this time the proposition that is to modify the result just reached is not one of sufficient importance to receive a special name, but is referred to merely by a number. We are by such proposition enabled to substitute for the two implications in the fourth line certain symbols which they define, namely $p \vee q$ and $p \vee \sim q$—that is to say, we are here calling in the aid of a proposition that is a definition and not an assertion.[1] Finally, by taking

[1] I have already explained that in our proofs definitions always may be, and sometimes must be regarded as assertions. See the last few paragraphs of Chap. I.

results (1) and (2) together, we have that "p implies 'p or q' and 'p or not-q,'" and that this logical product implies p; hence the equivalence set forth in the conclusion has been established.

The above proposition introduces a large number of the symbols that are employed by Mr. Russell. When he comes to deal with *formal* implication he has another symbol, one that expresses a notion of great importance. In the proposition that we have considered there is no need to introduce any symbol to denote "all values." For in this case, as in the case of all the other propositions up to 6·71, the assertion holds of "any value" of the variables. It may happen, however, that we wish to speak of all values, or, in other words, of each value of the variable, and such a necessity may arise either at the outset or in the course of expressing an assertion. We may, for instance, in an implication involving p and q state that for all values of p and q the implication holds, or that for any value of p the implication holds for all values of q. To denote this notion of all values of x the symbol (x). is employed by Mr. Russell. This he prefixes to the portion of the assertion—or to the whole, if it is a case of that kind—that involves the introduction of the notion of all values.[1] To signify, that is to say, that $(C \between x)$ is true for all values of x he uses the expression $(x) . (C \between x)$. In adopting a general symbol to denote this notion of all values Mr. Russell resembles Frege. The latter employs for the purpose a depression in the horizontal line that points to the proposition which is asserted; for instance,



signifies that if each $\mathfrak{a}$ is the result of the application of an experience f to y, then an $\mathfrak{a}$ is identical with x. The $\mathfrak{a}$ with the depression in the horizontal line fulfils precisely the same function as the initial (x). that is employed by Mr. Russell. Peano also has a method, but not a general one, of referring to all of a class. He can symbolize "all a's are b's." This is done by utilizing a subscript x; the statement may, that is to say, be put into the form $x \varepsilon a . \supset_x . x \varepsilon b$.[2] But Peano has no method of stating that p is true, if p happens not to be an *implication*.[3] The symbolic procedures of Frege and Russell are, therefore, on this

[1] He gives a special name to that portion of the assertion which involves the introduction of the notion of all values: the portion in question constitutes the "range" of the variable.

[2] See prop. 12 in the *Formulaire*.

[3] See Russell, *ibid.* p. 194.

question superior to his. Of the two former the better is that of Mr. Russell, for it shows a compactness that is wanting in the symbolism of Frege.

From our consideration of the above problems it is observable what are the chief differences in the symbols that are employed by the three thinkers who have done most in recent years for the advancement of logical theory. To put the matter in short compass, the principal differences are the following. Both Frege and Russell adopt a symbol to denote the important fact that a proposition is "asserted." Peano has no symbol for such a conception. Hence, so far as can be gathered from his symbols, there is no difference between a proposition that is merely considered and one that is asserted: in other words, there is no difference in character between the hypothesis of a proposition and the proposition itself. Thus when he in a proof quotes a proposition p, the p cannot be regarded as a truth that has been established, and that carries with it the truth of another proposition, but the p must appear as part of a hypothesis: we shall have the statement "if the prop. p is true, then the consequent is true." Secondly, as we have just seen, Frege and Russell have at their disposal a symbol to signify that "all" the members of a class are involved in the whole or in a part of an assertion. Peano can in certain cases symbolize this notion, but he cannot do so in a general manner. On the other hand, Peano's symbol of implication and his brackets are for general purposes to be preferred to the lines and groupings of lines that are adopted by Frege. Both of these improvements have been adopted by Mr. Russell.[1] Among Mr. Russell's other selected symbols are $\sim$ to signify negation (on the employment of Peano's symbol there is a source of error in the associations of the minus sign),[2] $\equiv$ and not $=$, with its suggestions of quantitative relations, to denote equivalence,[3] and $\vee$ instead of $\smile$ to signify disjunction.[4] It may be observed also that, though he prefers (x) . as a rule to signify that all the x's are referred to, he employ's Peano's method of subscripts, when this is found to be the more convenient.

[1] His symbol of implication—it is rather more curved than that given in these pages—is certainly not an inverted c, but it was, I expect, suggested by this.

[2] Peano recognizes the advantage of using the symbol $\sim$, and in his manuscripts and some of his publications uses it. See *Formulaire*, p. 40.

[3] Frege also employs the symbol $\equiv$.

[4] Here Peano's $\smile$ is as suitable as Mr. Russell's symbol.

CHAPTER III

EXAMPLES OF PROOFS IN GENERALIZED LOGIC

OBSERVATIONS ON THE DOCTRINE OF LOGICAL TYPES

In order to give the reader the opportunity of becoming familiar with the symbolisms which have been described in the previous chapter I will in the present chapter set forth with the respective proofs a dozen other propositions that are laid down by one or other of the three logicians whose work we are specially considering. The propositions selected will for the most part be important ones, *i. e.*, such as form the support of several others. I shall quote at the conclusion of each proof those propositions that have been referred to in the course of the proof. But I shall not give the proofs of the cited propositions, and the proofs of the propositions that these presuppose, and so on, until we reach nothing but primitive propositions : this course is not necessary for our present purpose. Should occasion arise for any words of explanation in presenting the proof of a proposition, these will be given, but after what has been said in the previous chapter not many explanations will be necessary. I will take the logicians in the same order as before.

I. Frege's prop. 5 is employed in propositions 6, 7, 9, 16 and elsewhere, and is thus highly important. It and the proof are the following :—

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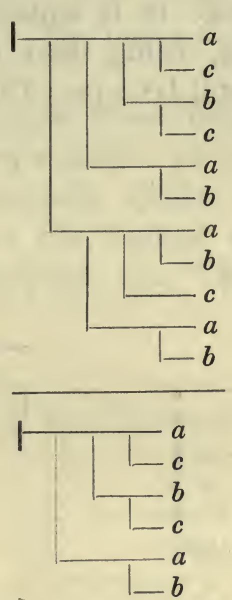
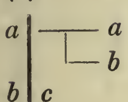
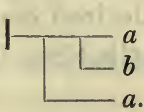


(1)::



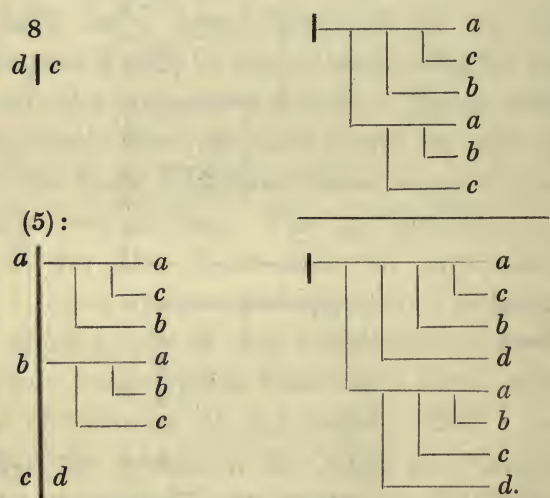
(5.

Here prop. (1) in which the indicated substitutions are made is

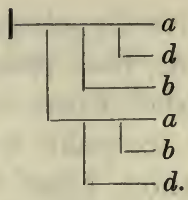


The course of procedure in prop. (5) is thus, firstly, to state the conclusion of prop. (4), secondly to make two substitutions in prop. (1), and thirdly to observe that since the implication indicated in this new form of prop. (1) is the hypothesis in the conclusion to prop. (4) the consequent in this conclusion is true, viz., the implication indicated in the third aggregation of symbols.

II. Prop. 12 is equally important with the preceding, being used in props. 13, 15, 16, 24, and still later on. The proof is as follows :—



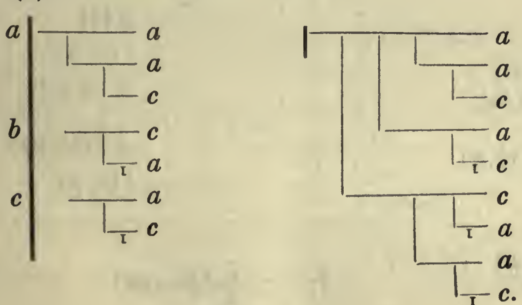
Here to provide the true hypothesis by means of which the desired conclusion shall be established a substitution is made in prop. (8), which is



The substitution made is of *c* for *d*. Then in prop. 5 the three substitutions indicated beneath that number on the left are made. And in the implication thus obtained, the hypothesis being true, the consequent, *i. e.*, prop. 12, is established.

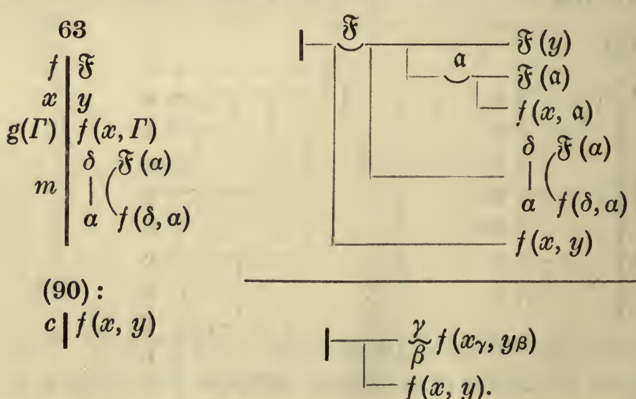
III. The third example from Frege shall be one that introduces negative propositions, and that does not actually quote the proposition which asserts the desired truth of the hypothesis. Take prop. 45. This is the following :—

(5) :

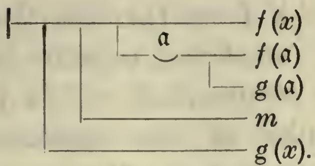


Here the full proposition that is reached when the substitutions indicated on the left are made has for hypothesis the conclusion of prop. 44. As that conclusion has only just been established Frege does not consider there is necessity to make special reference to it. There are three negative propositions introduced in the conclusion of 45. The conclusion may be read, "if it is the case that the truth of *c* follows from the suppositions that the falsity of *c* implies the truth of *a* and *a* is false, and if the truth of *a* follows from the falsity of *c*, and if *c* implies *a*, then *a* is true."

IV. As a last illustration of Frege's symbolism may be taken one of his more complicated propositions, one that introduces the notion of the variable. Prop. 91 is typical here, and is important as lying at the basis of prop. 92. Prop. 91 appears as follows :—

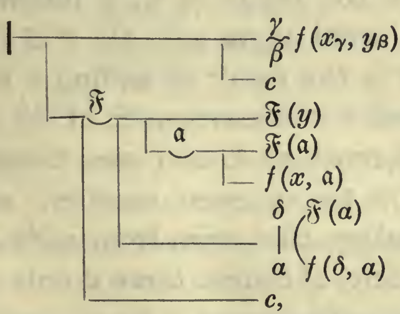


Prop. 63, in which the substitutions are made to obtain the true hypothesis, is



That is to say, "if x is a g, and if m is true, and if, whatever $\mathfrak{a}$ is, $\mathfrak{a}$ is a g implies that $\mathfrak{a}$ is an f, then x is an f." After the substitutions have been made the reading may be given as "if y is the result of the application of the experience f to x, and if in the case of all the results of the application of the experience f to x it can be said that they have the experience $\mathfrak{F}$, and if it is the case that the experience $\mathfrak{F}$ is bequeathed to the f-series,[1] then y has the experience $\mathfrak{F}$."

Prop. 90, in which $f(x, y)$ is substituted for c to obtain the combination of hypothesis and consequent, is



[1] This is the second part of the hypothesis: sometimes it is convenient in giving the words corresponding to symbols to alter the order of the parts of the hypothesis.

or in words "if from the fact that c is true, and that if, whatever experience $\mathfrak{F}$ is, this experience is bequeathed to the $\mathfrak{f}$-series, and that all results of the application of the experience $\mathfrak{f}$ to x have the experience $\mathfrak{F}$, it follows that y has the experience $\mathfrak{F}$, then, if c is true, y follows on x in the $\mathfrak{f}$-series." The reader may like to have some concrete examples of the hypothesis and consequent reached by making the substitution in question. Here are two such examples, one from what all would agree is the qualitative region, and the other from the region of arithmetic. (a) "If from the fact that y is the son of x, and that sons invariably exhibit the quality of dutifulness,[1] and that, if x is the father of $\mathfrak{a}$, then $\mathfrak{a}$, whoever he is, has the quality of dutifulness, it follows that y has that quality, then if x is the father of y, y follows in the fatherhood series upon x." (b) "If from the fact that 6 is the result of adding 2 to 4, and that evenness[2] is characteristic of the numbers that are formed as the 6 was formed, viz., by adding 2 to an even number, and that all the numbers that arise from adding 2 to 4 are even—here, of course, there is only one such

[1] I take this as representative of the qualities that (let us say) characterize sons.

[2] Evenness is taken as representative of the qualities that characterize the numbers in question.

individual—it follows that 6 is even, then, if 6 is the result of adding 2 to 4, it follows that 6 succeeds 4 in the series whose members are even and are constituted in each case by the addition of 2 to an even number."

Recognizing, then, that in the case of this combination of hypothesis and consequent the truth of the hypothesis is known, we are able to assert the truth of the consequent, *i. e.*,

$$\vdash\!\!\!\!-\!\!\!\!\!\!\!\!\quad \begin{array}{l} \frac{\gamma}{\beta} f(x_\gamma, y_\beta) \\ f(x, y), \end{array}$$

the proposition to be proved. This may be read, "if y is the result of an application of the experience f to x, then y follows x in the f-series."

V. Coming now to Peano, we will commence with prop. 112. This is known as the principle of Transportation.[1] The proposition is

$$a \supset b . = . -b \supset -a,$$

and the proof appears thus :—

$$\left[\text{P111} . \left(\frac{-b, -a}{a, \ b} \right. \text{P111} . \supset . \text{P} . \right]$$

This latter means that we twice utilize prop. 111. On the first occasion the original form in which prop. 111 appeared is adopted, viz.,

$$a \supset b . \supset . -b \supset -a,$$

[1] Or, rather, it is one of the propositions that go by this name. Prop. 109, viz.,

$$ab \supset c . = . a - c \supset - b,$$

is also asserted to be this principle (*Formulaire*, p. 7).

and on the second occasion the symbols $-b$ and $-a$ are substituted in the original form for a and b respectively. By this means we obtain two *implications* that allow of our asserting the *equivalence* to be proved. It may be noted that Frege has prop. 111, *i. e.*, the implication, but he does not take the further step of asserting prop. 112, the equivalence.

VI. A proposition in which is illustrated Peano's manner of dealing with negative terms is No. 215. In the proof of this proposition use is made of the earlier of the two propositions that are spoken of as principles of Transportation. Prop. 215 is as follows :—

$$a\,(b \cup c) \supset ab \cup ac.$$

And the proof appears thus :—

$$\begin{array}{lll} [ab \supset ab\,.\,\text{Transp}\,.\supset. & a\,[-(ab)] \supset (-b) & (1) \\ ac \supset ac\,. \quad ,, & .\supset.\ a\,[-(ac)] \supset (-c) & (2) \\ (1)\,(2)\,.\,\text{Cmp}\,.\supset. & a\,[-(ab)-(ac)] \supset (-b)(-c) & (3) \\ (3)\,.\,\text{Transp}\,.\supset. & \text{P}]\,. & \end{array}$$

Here the principle of Composition is quoted in the third line, viz.,

$$a, b, c\,\varepsilon\,\text{K}\,.\,a \supset b\,.\,a \supset c\,.\supset.\,a \supset bc.$$

But really prop. 34 is employed:

$$a \supset b\,.\,c \supset d\,.\supset.\,ac \supset bd.$$

Finally, Peano tacitly refers to prop. 201. In this last there is set forth the significance of a logical sum. The proposition is

$$a, b \varepsilon \mathrm{K} . \supset . a \cup b = -[(-a)(-b)],$$

or in words "if a and b are classes, the logical sum of a and b is that class which consists of objects that are not both not-a and not-b."

VII. A good instance of one of Peano's propositions that introduce the notion of the Variable is No. 256. This proposition is the following :—

$$x \varepsilon a - b . = : c \varepsilon \mathrm{K} . a \supset b \cup c . \supset . x \varepsilon c.$$

That is to say, "to maintain that x is a member of the class a not-b is equivalent to maintaining that if c is a class, then whatever class it is, if a is contained in the sum of b and c, x is a member of the class c." The proof is as follows :—

$$[\mathrm{P}61 . \supset .\because . x \varepsilon a - b . = : c \varepsilon \mathrm{K} . a - b \supset c . \supset_c . x \varepsilon c$$
$$\mathrm{P}254 . \supset . \mathrm{P}].$$

Here prop. 61, to which reference is made, is

$$a \varepsilon \mathrm{K} . \supset .\because . x \varepsilon a . = : b \varepsilon \mathrm{K} . a \supset b . \supset_b . x \varepsilon b.$$

Or in words, "if a is a class, then to say that x is a member of it is equivalent to saying that if b is a class, then whatever class b is, if a is contained in it, x is a b." And prop. 254 is

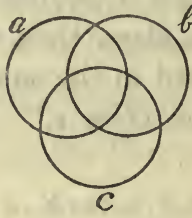
$$a - b \supset c . = . a \supset b \cup c,$$

or, "if the class $a - b$ is contained in c, then the

F 2

class *a* is contained in either *b* or *c*," and vice versa.

It sometimes happens in the case of a proposition that is proved in the usual way, that it is possible without difficulty to realize intuitively the truth of the proposition. Even with quite complicated propositions, such as those that are given by Frege relative to the members of a series, this is not infrequent. But in the case of the proposition before us it is not at all easy intuitively to realize the truth of the equivalence. The employment of the following diagram may, however, help



the reader over the difficulty. Let three classes, *a*, *b* and *c*, be represented by three circles, and let each class overlap each other class. Then there are eight compartments that may possibly have members. Now reading the proposition from left to right we have, " if *x* is a member of the compartment *a* not-*b*, and if the compartment *a* not-*b* not-*c* is erased, then *x* must be a member of the class *c*." Here there is no difficulty, since the existence of individuals that are *a* not-*b* not-*c* is clearly denied if *a* is asserted to be contained in *b* or *c*, and since if *x* is excluded from that portion of *a* not-*b* the individual must fall in the *c* portion. The difficulty arises in reading from right to left. Here we have, "if *a* is contained in *b* or *c*, then *x* is contained in *c*." At first sight it might be thought that in the case of the statement "*a* is *b* or *c*" the compartments *a* not-*b* *c* and *abc* are on an equal footing, *i. e.*, that it is a matter of indifference whether we say "*a* not-*b* is *c*," or we say "*abc* is *c*." But, though both of these class-inclusions follow if *a* is in *b* or *c*, they are not both identical with this statement. Only the former is identical with it. So that we cannot substitute for "$a \supset b \cup c . \supset_c . x \,\varepsilon\, c$" the implication "$abc \supset c . \supset_c . x \,\varepsilon\, c$." Hence we can never reach the conclusion "$x \,\varepsilon\, abc$." The only substitution that can be made for "$a \supset b \cup c . \supset_c . x \,\varepsilon\, c$" is "*a* not-*b* is $c . \supset_c . x \,\varepsilon\, c$," from which we reach the conclusion "$x \,\varepsilon\, a$ not-*b*." Then having two mutual implications the equivalence is established.

VIII. It was mentioned in the last chapter that Peano has a special symbol to designate a class that possesses one member only, viz., the symbol ι. An example in which this symbol figures conspicuously is prop. 422.

This proposition may be given as the fourth of those we select from Peano. The proposition is

$$a \varepsilon \mathrm{K} . \supset : x \varepsilon a . = . \iota x \supset a,$$

and the proof is

$$\begin{array}{lll} [\mathrm{P}84 \supset : & a \varepsilon \mathrm{K} . x \varepsilon a . y \varepsilon \iota x . \supset . y \varepsilon a . & (1) \\ (1)\ \mathrm{Export} . \supset : & a \varepsilon \mathrm{K} . x \varepsilon a . \supset : y \varepsilon \iota x . \supset_y . y \varepsilon a . & (2) \\ (2) . \supset : & a \varepsilon \mathrm{K} . x \varepsilon a . \supset . \iota x \supset a . & (3) \\ & a \varepsilon \mathrm{K} . \iota x \supset a . \supset . x \varepsilon \iota x . \iota x \supset a . \supset . x \varepsilon a & (4) \\ (3)\ (4) \supset \mathrm{P}]. & & \end{array}$$

The conclusion that is here established may be read, "if a is a class, then to say that x is a member of the class a is equivalent to saying that the class which is composed exclusively of the individual x is contained in a." In the proof prop. 84, to which reference is made, is to the effect that "if a is a class, and x is a member of it, and x is identical with y, then y is a member of a," and appears as

$$a \varepsilon \mathrm{K} . x \varepsilon a . x = y . \supset . y \varepsilon a.$$

So that $y \varepsilon \iota x$ is the same thing as $y = x$. The principle of Exportation mentioned in the next line is

$$a, b, c \varepsilon \mathrm{K} : x \varepsilon a . (x, y) \varepsilon b . \supset_{x,y} . (x, y) \varepsilon c :$$
$$\supset .\therefore x \varepsilon a . \supset_x : (x, y) \varepsilon b . \supset . (x, y) \varepsilon c .$$

This may be at once read by reference to the reading in the preceding chapter of the principle of Importation. We thus reach the fact that "if a is a class, and x is an a, it follows, if y is a member of the class x, a class with only one member, that, whatever y is, it is a member of the class a." From this statement line (3) immediately follows by prop. 12 : we can, that is to say, always express an implication, such as that which occurs at the end of line (2), as a class-inclusion. The implication that we have reached is, however, only an implication, not an equivalence. To obtain the latter we have to observe, as is done in line (4), in the first place that the expression $\iota x \supset a$ implies itself, and secondly that x is equivalent to x, or, in other words, that the relationship $x \varepsilon \iota x$ holds good. Then we know by prop. 25 that if we have $x \varepsilon \iota x$ and $\iota x \supset a$ we have $x \varepsilon a$. Thus we reach

$$\iota x \supset a . \supset . x \varepsilon a .$$

Then by combining this implication with that established in line (3) we have the equivalence required.

IX. Finally, coming to the propositions set forth by Mr. Russell in the *American Journal of Mathematics* in his two articles "The Theory of Implication" and "Mathematical Logic as based on the Theory of Types," we will take, to begin with, a proposition out of the former article, viz., No. 3·42. This proposition is important as lying at the basis of prop. 3·43, to the truth of which appeal is frequently made, *e.g.*, in props. 3·47, 3·5 and 4·44. The statement and proof of prop. 3·42 are as follows :—

$$\vdash :. p \supset q . \supset : \sim p \supset q . \supset . q$$

Dem.

$$\vdash . * 3{\cdot}22 \quad \supset \vdash :. p \supset q . \supset : \sim q \supset \sim p \qquad (1)$$

$$\vdash . \text{Syll} \frac{\sim q, \sim p, q}{p, q, r} . \supset \vdash :. \sim q \supset \sim p . \supset : \sim p \supset q . \supset . \sim q \supset q \qquad (2)$$

$$\vdash . * 3{\cdot}12 \frac{\sim p \supset q, \sim q \supset q, q}{p, \quad q, \quad r} . \supset$$

$$\vdash :: \sim q \supset q . \supset . q : \supset :. \sim p \supset q . \supset . \sim q \supset q : \supset : \sim p \supset q . \supset . q \qquad (3)$$

$$\vdash . (3) . * 3{\cdot}41 . \supset \vdash :. \sim p \supset q . \supset . \sim q \supset q : \supset : \sim p \supset q . \supset . q \qquad (4)$$

$$\vdash . (1) . (2) . (4) . \text{Syll} . \supset \vdash . \text{Prop.}$$

The proposition that is here established may be read, "if p implies q, then, if the falsity of p implies the truth of q, q is true." In the first line of the proof the significance of prop. 3·22 is set forth, viz.,

$$\vdash : p \supset q . \supset . \sim q \supset \sim p .$$

The principle of Syllogism next referred to, in which $\sim q$, $\sim p$ and q are to be substituted for p, q and r respectively, is

$$\vdash :. p \supset q . \supset : q \supset r . \supset . p \supset r .$$

In the third line prop. 3·12, in which also three substitutions are to be made for p, q and r, is

$$\vdash :. q \supset r . \supset : p \supset q . \supset . p \supset r .$$

Then in line (4) prop. 3·41 is brought to bear upon line (3). The proposition thus adduced is

$$\vdash : \sim p \supset p . \supset . p .$$

This proposition being true the proposition that follows the leading implication-sign in line (3) is asserted. Finally, the principle of Syllogism is again adduced. It is employed first in connection with the results reached in lines (1) and (2), and then in connection with the assertion which is established in line (4).

X. A proposition that introduces the symbols for identity and logical addition is No. 5·78, viz.,

$$\vdash :. p) q . \vee . p) r : \equiv : p .) . q \vee r .$$

This proposition is also noteworthy as making it obvious that in certain, at any rate, of Mr. Russell's propositions the hypothesis is understood "if the letters referred to represent *propositions*." For in this case the assertion does not hold good if the letters stand for classes. As a matter of fact, almost all the letters used in the article on "The Theory of Implication" as far as the middle of p. 192 stand for propositions, but in the case of some of the assertions that fact is not obvious. Any one seeing, for instance, without any explanations the statement of 2·9, viz.,

$$\vdash . \sim (\sim p)) p,$$

would not be able to tell whether this means "the contradictory of the contradictory of the proposition p implies the proposition p," or "the negation of the negative of the class

p is included in the class p." In the case of the proposition before us, however, as Mr. Russell here and elsewhere points out,[1] while it is true that "p implies q or p implies r" is identical with the statement "p implies q or r," it is not the case that "p is included in q, or p is included in r" is identical with "p is included in p or r"; this fact may at once be seen by substituting "English people," "men" and "women" for p, q and r respectively.

The demonstration of 5·78 is the following :—

$$\begin{array}{ll} \vdash . * 5{\cdot}55{\cdot}39 .) \vdash :. p) q . \vee . p) r : & \equiv : \sim p \vee q . \vee . \sim p \vee r : \\ {[* 5{\cdot}33]} & \equiv : \sim p . \vee . q \vee \sim p \vee r : \\ {[* 5{\cdot}31{\cdot}37]} & \equiv : \sim p . \vee . \sim p \vee q \vee r : \\ {[* 5{\cdot}33]} & \equiv : \sim p \vee \sim p . \vee . q \vee r : \\ {[* 5{\cdot}25]} & \equiv : \sim p . \vee . q \vee r : \\ {[* 5{\cdot}55]} & \equiv : p .) . q \vee r :. \\ &) \vdash . \text{Prop.} \end{array}$$

In the first line there is a reference to two propositions, viz. 5·55 and 5·39. These are respectively

$$\vdash : \sim p \vee q . \equiv . p) q$$

and $$\vdash :. p \equiv r . q \equiv s .) : p \vee q . \equiv . r \vee s .$$

In the second of these $p) q$, $\sim p \vee q$, $p) r$ and $\sim p \vee r$ are substituted for p, r, q and s respectively, and so the statement

$$\begin{array}{l} \vdash :. p) q \equiv \sim p \vee q . p) r \equiv \sim p \vee r . \\ \quad) : p) q . \vee . p) r : \equiv : \sim p \vee q . \vee . \sim p \vee r : \end{array}$$

is obtained. Then, since here the hypothesis

[1] See also my *Development of Symbolic Logic*, p. 202.

is seen to be true by means of a double application of the first proposition, we have the statement

$$\vdash :. p \supset q . \vee . p \supset r : \equiv : \sim p \vee q . \vee . \sim p \vee r.$$

Upon this assertion is brought to bear in the second line prop. 5·33, viz.,

$$\vdash : (p \vee q) \vee r . \equiv . p \vee (q \vee r).$$

In line (3) prop. 5·31 is

$$\vdash : p \vee q . \equiv . q \vee p,$$

and prop. 5·37 is

$$\vdash :. p \equiv q . \supset : p \vee r . \equiv . q \vee r.$$

In this case, analogously to the process above, we substitute $p \vee q$ and $q \vee p$ for p and q respectively in the second proposition, and obtain

$$\vdash :. p \vee q . \equiv . q \vee p . \supset : p \vee q . \vee r : \equiv : q \vee p . \vee r;$$

then the hypothesis being true by the first proposition the assertion is reached

$$\vdash : p \vee q . \vee r : \equiv : q \vee p . \vee r,$$

and hence the substitution of $\sim p . \vee . \sim p \vee q \vee r$ for $\sim p . \vee . q \vee \sim p \vee r$. In the fourth line prop. 5·33 is again employed, and in the fifth prop. 5·25, viz.,

$$\vdash : p . \equiv . p \vee p,$$

reduces the expression on the right to $\sim p . \vee . q \vee r$. Finally, a further use of 5·55 gives the implication $p . \supset . q \vee r$. That is to say, we have reached the desired assertion of the equivalence of $p \,)\, q \,.\, \vee \,.\, p \,)\, r$ and $p \,.\,) \,.\, q \vee r$.

XI. Towards the end of his article on "The Theory of Implication," Mr. Russell has a proposition, viz., No. 7·15, that introduces his symbolism for "all." This proposition, that is to say, is one of those whose literal symbols do not stand exclusively for propositions. We will consider this example. The proposition is

$$\vdash :. (x) . p \,)\, (C \between x) . \equiv : p .) . (x) . (C \between x),$$

and the proof is as follows :—

$\vdash . * 7{\cdot}12 . \;) \vdash :. (x) . p \,)\, (C \between x) .) : p .) . (x) . (C \between x)$ (1)
$\vdash . * 3{\cdot}12 . \;) \vdash :: (x) . (C \between x) .) . (C \between y) :$
$\quad) :. p .) . (x) . (C \between x) :) : p .) . (C \between y)$ (2)
$\vdash . (2) . * 7{\cdot}1 .) \vdash :. p .) . (x) . (C \between x) :) : p .) . (C \between y)$
$[* 7{\cdot}11] \quad) \vdash :. (x) :. p .) . (x) . (C \between x) :) : p .) . (C \between x) :.$
$[* 7{\cdot}12] \quad) \vdash :. p .) . (x) . (C \between x) :) : (x) : p .) . (C \between x)$ (3)
$\vdash . (1) . (3) . \;) \vdash$. Prop.

The assertion that is here established may be read, "to say that in the case of all x's the truth of p implies that x is a C is equivalent to saying that if p is true, then, in the case of every x, x is a C." In line (1) of the proof the proposition which is symbolized, viz., 7·12, may be read, "in the case of every x if the truth of p implies that x is a C, then, if p is true, it follows that, in the case of every x, x is a C," and in the second line prop. 3·12 is to the effect that "from the fact that, in the case of every x, if x is a C then y is a C, it follows that, if the truth of p implies that, in the case of every x, x is a C, then, if p is true, y is a C." In the third line prop. 7·1 is

$$\vdash :(x).(C\between x).).(C\between y),$$

that is to say, "if of all x's it can be said that x is a C, then this can be said of any x, *e. g.*, of y." This proposition being asserted, the consequent in line (2) is seen to be true. In the fourth line 7·11 is to the effect that what is true of any is true of all. Hence we can proceed from line 3—this line may be read "whatever y may be, if p implies that, in the case of all x's, x is a C, then if p is true y is a C"—to the statement "in the case of all x's if p implies for all x's that x is a C, then p implies that x is a C." It is to be noted that here x has two significations. The first x has for its range the whole proposition, while in the case of the second x the range is $(C\between x)$ only. In general it would be better, as Mr. Russell points out, not to have the same letter where different ranges are concerned. In the line numbered (3) there is a second resort to prop. 7·12, and thus is reached an implication which is the reciprocal of that in line (1). The assertion of the equivalence in question is therefore established.

XII. Our last example from Mr. Russell

shall be from the second of the above-mentioned articles, viz., that entitled "Mathematical Logic as based on the Theory of Types."[1] We will take the following inference, an immediate inference from a definition. On p. 254 of the *American Journal of Mathematics* there is given this definition :—

$$C = \hat{\gamma}\hat{R}\{\gamma = \hat{x}((\exists y) : xRy . \vee . yRx)\}, \qquad \text{Df.},$$

and from this there springs immediately the assertion

$$\vdash . C`R = \hat{x}\{(\exists y) : xRy . \vee . yRx\}.$$

The consideration of this example is highly important, not indeed as illustrating a method of *proof*—if the inference is immediate there is obviously no method of proof involved—but (1) as bringing to the front the original method adopted by Mr. Russell for removing certain contradictions which have always perplexed logicians and mathematicians, and (2) as indicating the similarity which exists between classes and relations, a similarity which enables us to apply the same calculus to both those entities.

We will first of all point out the significance of the letters that are employed in the above

[1] The author of this article has expressed his views in less technical and in an eminently lucid manner in a recent number of the *Revue de Métaphysique et de Morale*, the article there being entitled "La Théorie des Types Logiques."

two expressions. The letter C at the commencement of the first stands for "campus" or "field" of a relation. The right-hand member of the expression defines the field, a fact that is indicated by the combination of (=) with Df. Reading this definition we have, "a relation R and certain terms x, which are such that a term y exists and the x's stand in the relation R to it, or it stands in the relation R to them." The assertion that springs from this definition is to the effect that "those terms that stand in the relation of field to R consist of the x's such that there exists a y, and the x's stand in the relation R to y, or the y stands in the relation R to them." Now in the remarks which he makes upon the definition Mr. Russell asserts that the relation R must be "homogeneous," *i. e.*, x and y must be of the same "type." We are thus introduced to the original notion that figures so largely in the later writings of this author.

By reference to the theory of Types we are able to explain the origin of and to avoid the above-mentioned mathematical and logical contradictions. Of the logical contradictions the *Epimenides* is the best known, and of those of a mathematical character Burali-Forti's to the effect that "a certain ordinal is and is not the ordinal number of all ordinals" is a good example. The *Epimenides* may be expressed thus :—

All the statements of Cretans are lies,

" All the statements of Cretans are lies " is the statement of a Cretan,

therefore,

" All the statements of Cretans are lies " is false,

which is absurd by the first premise.

The contradiction here arises from our regarding a type as including among its members a member of a higher type. That is to say, the first premise is a statement about " all the statements of Cretans," while the second premise is one about the statement " all the statements of Cretans are lies." The individuals about which information is given are thus of different types. To avoid contradiction we must not universally identify a member of a higher type with one of a lower. Or, to put the facts in other words, we must not with the predicate " lies " speak about " *all* the statements of Cretans." For if we do, and a Cretan makes the observation concerning all such statements, his statement will be at the same time a statement of a Cretan (and so false) and a statement about all Cretans which is true.

Here the notion of Type for the explanation and avoidance of a contradiction is certainly useful. But I think Mr. Russell goes too far when he says that "'all propositions' must be a meaningless phrase."[1] What may safely be said is that the expression "all propositions" is a meaningless phrase when the predicate is "are lies." For in certain cases we *may* have premises about "all propositions." For instance, let us, ignoring the existence of propositions that express relations, take the following:—

> All propositions consist of subject and predicate,
>
> "All propositions consist of subject and predicate" is a proposition,

therefore,

> "All propositions consist of subject and predicate" consists of subject and predicate.

There is here no contradiction: the conclusion is not at variance with anything in the premises. We must not, therefore, say *in general* that "all propositions" is a meaningless phrase. But where the propositions referred to are the utterances of liars we are undoubtedly precluded from referring to the totality. And the statement of this fact is

[1] "Mathematical Logic as based on the Theory of Types," p. 224.

quite sufficient to indicate how the *Epimenides* has arisen, and how the contradiction may be avoided.

The mathematical contradiction pointed out by Burali-Forti concerning the ordinal of all ordinals may also be removed by means of the doctrine of logical types. The contradiction is the following. Let the ordinal of the last member of a series of well-ordered series be α. Then the ordinal number of this series of well-ordered series will be $\alpha + 1$. Hence, where α is the ordinal number of the last member of the series of all well-ordered series, the ordinal number of that series will be $\alpha + 1$. And the series of all well-ordered series is a well-ordered series.[1] Hence the ordinal number of the last member of all well-ordered series is α and $\alpha + 1$, which is absurd. Here the unjustifiable procedure is the mention of the series of *all* well-ordered series. For the

[1] This is self-evident if such series is determined by reference to only one quality. But it is not necessary that there should be restriction to one quality. Suppose, for instance, that the position of a well-ordered series in the series of well-ordered series were determined by the numbers of individuals involved. It might well happen that there would be several well-ordered series with the same number of individuals. In that case the order of those series that possessed the same number of individuals would be established by reference to some other quality. Thus, the totality of well-ordered series would be still well-ordered.

totality " all well-ordered series " constitutes a well-ordered series, and will, therefore, find a place in the totality of well-ordered series. But no such place can be found for the new series, since in the constitution of this series every well-ordered series has already been taken into account.

In both of the above cases the notion of Type is undoubtedly of great value. In our logical processes we wish to avoid every form of contradiction, and in certain cases contradiction is inevitable if we speak of " all " of a certain class. Hence in these cases we must not speak of " all."

As regards the example which has led us to refer to this important doctrine of logical types it was said that Mr. Russell affirms that x and y must be of the same type. Such is certainly sometimes the case. For, supposing x stands for John Smith, y stands for " William Brown is a Liberal," and R stands for " maintains that." It is clear that in the expression $xRy \,.\vee .\, yRx$, while " John Smith maintains that William Brown is a Liberal " has a meaning, there would be no meaning if in the expression we exchanged the position of " John Smith " and " William Brown is a Liberal." But I do not think we may go on to say *in general* that x and y must be objects of the same type. For it is quite possible to have a meaning to both alternatives in the above disjunctive expression when x and y are of different types. For instance, let x stand for "the South Magnetic Pole is discovered," y for "it is believed that the South Magnetic Pole is discovered," and R for "was published at the same time as." Then the two expressions in the disjunctive are "'the South Magnetic Pole is discovered' was published at the same time as 'it is believed that the South Magnetic Pole is discovered,'" and the latter statement was published at the same time as the former. The two statements in each alternative are certainly of different orders: "the South Magnetic Pole is discovered" is a first-order proposition, and "it is believed that the South Magnetic Pole is discovered" is a second-order proposition. And, even in some cases where x is an individual of the first type, *i. e.*, an individual about which something is stated in a first-order proposition, and y is an individual of a higher type, there may be a meaning to each member of the disjunctive. Take, for instance, the statement "the person a came into existence the same year as 'Lycidas is dead.'" Here there are related two objects of different types, and there is a meaning if the positions of the two objects are interchanged.

Thus only under certain conditions is it essential that the two objects in the alternatives under consideration should be of the same type. If, therefore, we wish to make as general a statement as we can with regard to the types of these objects we must set forth what the conditions are. It is quite true, as Mr. Russell points out, that in the above explanation of Burali-Forti's contradiction the objects x and y must be of the same type.[1] For here what we state so far as the field of relations is concerned is that a certain well-ordered series is on one supposition, and is not on another supposition, identical with one of the members of a series of well-ordered series. Clearly, therefore, the objects between which there is *identity* are of the same type. This case, however, is not an instance of the fact that x and y must *always* be, but of the fact that x and y must *under certain conditions* be of the same type. The explanation of Burali-Forti's contradiction affords, that is to say, an illustration of a truth that has less generality than the one laid down by Mr. Russell. The less general may, but the broader statement may not, be accepted as true.

[1] This, I take it, is the fact to which Mr. Russell is referring when he says that the observation concerning the homogeneity of R has a "connection" with Burali-Forti's contradiction.

In the second place, the example before us is useful as calling attention to the similarity that exists between classes and dual relations.[1] We have in the example a field of *relations*, and the question naturally arises what a relation is, and how it is to be treated. The answer to this question is that the definition and treatment of relations are similar to the definition and treatment of classes. That is to say, just as a class is defined as the a's such that there exists a function φ, and the a's are identical with the z's which are such that z is an argument to φ, so we may define a relation as those R's which are such that there exists a function φ, and the R's are identical with the couples (x, y), these being such that φ can be predicated of each one of them. And, as regards treatment, just as we have in the case of the product and the sum of two classes, and the negative of a class, the following :—

$$\alpha \frown \beta = \hat{x}(x \,\varepsilon\, \alpha \,.\, x \,\varepsilon\, \beta) \quad \text{Df.},$$
$$\alpha \smile \beta = \hat{x}(x \,\varepsilon\, \alpha \,.\vee .\, x \,\varepsilon\, \beta) \quad \text{Df.},$$
$$-\alpha = \hat{x}\{\sim (x \,\varepsilon\, \alpha)\} \quad \text{Df.},$$

i. e., the product of two classes α and β is defined as the x's such that x is an α and x is a β, the sum as the x's such that x is an α or

[1] See Russell, "Mathematical Logic as based on the Theory of Types," p. 252.

x is a β, and the negative of a class α as the x's such that it is not true that x is an α, so we have :—

$$R \mathbin{\dot\frown} S = \hat{x}\hat{y}\,(xRy \,.\, xSy) \quad \text{Df.},$$
$$R \mathbin{\dot\smile} S = \hat{x}\hat{y}\,(xRy \,.\vee.\, xSy) \quad \text{Df.},$$
$$\dot{-}R = \hat{x}\hat{y}\,\{\sim(xRy)\} \quad \text{Df.},$$

i. e., the product of the relations R and S consists of the couples (x, y) such that x stands in the relation R to y, and x stands in the relation S to y, the sum consists of the couples (x, y) such that x stands in the relation R to y, or x stands in the relation S to y, and the negative of the relation R consists of those couples (x, y) which are such that it is not true of them that x stands in the relation R to y.

As an instance of the appearance of both a relation and a class in one of the implications of general formal logic we may take the truth referred to in the fourth proposition of Euclid. Here we should say, "if x is a kind of space that is indicated by the Euclidean axioms, then x is a space such that, if the points in the lines AB and AC and the rays in the angle BAC have respectively a one-one relation to the points in the lines DE, DF and the rays in the angle EDF, the points in BC have a one-one relation to the points in EF." When here we speak of x as "a space" we are referring to a class, and when we assert the correspondence of a point in *A B* to one in *D E* we are referring to a relation. And as an instance of the way in which a relation takes the place of a class without restricting another class, we may take "x is a father of y implies x is a benefactor of y." In both these cases the relations are to be treated just as classes are treated. In the former example the relation in question restricts a class—in this case it is a class with only one member, viz., one of the members of the class "spaces"—just as one class may restrict another class, and in the latter example the relations occupy the positions of subject and predicate respectively, just as these positions may be occupied by classes of individuals only.[1]

[1] This doctrine concerning Relations, it may be pointed out, is not quite in agreement with the view to which I called attention in the sixth chapter of my *Development of Symbolic Logic*. I there showed that a Logic of Relatives, in the sense of a general treatment of copulæ, is impossible, but that the doctrine expounded by Mr. Johnson concerning the synthesis of multiply-quantified propositions may undoubtedly be accepted. In adopting the above view of the nature of Relations, *i. e.*, that these are really classes, determined by intension, we certainly make no attempt to introduce into our logical doctrine any general treatment of copulæ. For instance, if we know that the distance of *A* and of *B* from *C* is a mile, we do not, unless we have additional data, assert that we know the distance of *A* from *B*. And, on the other hand, we do admit the possibility of the synthesis of multiply-quantified pro-

positions. The proposition "all m's love all n's," for example, will be expressed as "if x is an object with m qualities, then x is an individual in one of the couples signified by the term 'lover,' the other individual in the couple being any one of those loved individuals which has n qualities," and this is readily combined with the similar expression of "all m's serve all n's." But in admitting this possibility of the synthesis of multiply-quantified propositions we do not, it is clear from this example, appeal for justification to the case of the singular or molecular proposition. As to which is here the preferable procedure there can be, I think, no doubt. If appeal is made to the singular proposition we should assert that, because this, that, and the other examined m is a lover and a servant, therefore all m's are lovers and servants; here the notion of "all m's" is reached by a process of summation. But if we hold that "all m's love all n's" and "all m's serve all n's" are equivalent respectively to "if anything is an m it loves each n" and "if anything is an m it serves each n," we can proceed *immediately* to "if anything is an m it loves and serves each n"; in other words, to "all m's love and serve each n." And inasmuch as an immediate inference is simpler than a process of summation our procedure is the preferable one. Whether or not in reaching the given multiply-quantified propositions we always start with singular propositions—a question that we need not here discuss—it is certain that, when we come to combine multiply-quantified propositions with one another, the better way is not to refer to singulars for justification, but to employ the notion of the variable, and to regard classes as constituted by reference to intension.

CHAPTER IV

GENERAL LOGIC AND THE COMMON LOGICAL DOCTRINES

It is possible now to indicate the manner in which Symbolic Logic, or, in other words, general logic, deals with the processes of the common logic and the processes of quantitative mathematics. The former of these inquiries has not received sufficient attention from writers, and I shall, therefore, treat the subject in some detail. This investigation will occupy us in the present chapter. In Chap. V I shall show how the processes of arithmetic and of geometry may be replaced by processes whose validity has been established by the logician. These two departments of mathematics are chosen as types of the application of logic to quantities for two reasons. In the first place, arithmetic and geometry are the best known and the most fundamental of the mathematical doctrines; and, in the second place, arithmetical and geometrical processes have been explicitly stated by Kant *not* to be susceptible of replacement by logical processes but to owe their validity to intuition exercised upon two kinds of objects.

In order to make manifest the manner in which general logic deals with the processes of ordinary logic we will commence with the doctrine of Opposition, then proceed to the processes of Immediate Inference, next to the process of Syllogism, and, finally, to those forms of reasoning of a more complicated character that are treated in certain text-books.

With reference to the manner in which the doctrine of opposition of propositions is dealt with by the logician who employs truly general symbols, Frege has pointed out all that it is necessary to observe.[1] He shows, that is to say, how the notions of a variable and a function may be introduced in every case where ordinary logic sets forth the relations of opposed propositions. The common logic states that a universal affirmative and a particular negative with the same subject and predicate cannot both be true. This fact appears in generalized logic as an assertion that $-\overset{\mathfrak{a}}{\smile}\!\!\begin{array}{l}\!\!-\!\!\!\top P(\mathfrak{a})\\ \;\;\lfloor X(\mathfrak{a})\end{array}$ and $\top\!\overset{\mathfrak{a}}{\smile}\!\!\begin{array}{l}\!\!-\!\!\!\top P(\mathfrak{a})\\ \;\;\lfloor X(\mathfrak{a})\end{array}$ cannot be true together; in other words, that if it is true that in the case of each $\mathfrak{a}$ if it is an X it is a P, then it is false that it is false that in the case of each $\mathfrak{a}$ if it is an X

[1] *Begriffsschrift*, pp. 23, 24.

it is a P. Here the $\mathfrak{a}$ is a variable, and the X and P are functions. Similarly $-\overset{\mathfrak{a}}{\smile}\begin{array}{l}\top P(\mathfrak{a})\\ \llcorner X(\mathfrak{a})\end{array}$ and $\top\overset{\mathfrak{a}}{\smile}\begin{array}{l}\top P(\mathfrak{a})\\ \llcorner X(\mathfrak{a})\end{array}$ are representations in the symbols of general logic of propositions that cannot be true together: the propositions in question are respectively the E and I of the Square of Opposition.[1] And, just as Frege holds that the two propositions as thus expressed cannot be true together, so he holds that if one is not true, the other must be true. Whereas, however, in the ordinary logic this truth is taken to be intuitively obvious, Frege implicitly *proves* the proposition: the case in question is an application of prop. 31.[2] It may be observed that Frege's symbolism precisely indicates what in the common logic "some" is taken to mean. The popular meaning of some as "more than one" and the Hamiltonian meaning as "not all" are at once excluded. In the case of the proposition $\top\overset{\mathfrak{a}}{\smile}\begin{array}{l}- P(\mathfrak{a})\\ \llcorner X(\mathfrak{a})\end{array}$, for instance, it is suffi-

[1] In the case of the above four implications the assertion-sign does not occur, since the propositions are merely considered: the possibility of each proposition's being true at the same time that the other member of the couple is true is considered, and that possibility is perceived to be unrealizable.

[2] *Begriffsschrift*, p. 44.

cient that one α that is X should not be P, and the case is not excluded where not any α that is X is P.

In considering the treatment by the symbolic logician of the processes of Immediate Inference we will take Conversion first, and we will begin with the universal affirmative. In most of the text-books on Formal Logic it is maintained that an A proposition is converted by means of an I proposition, but Keynes has pointed out that such a process is illegitimate unless universal affirmatives carry with them the presupposition of the existence of objects corresponding to the subject-term.[1] Now this is precisely the view of the symbolist concerning the conversion of the universal affirmative. He certainly does not discuss the question, but he purposely excludes such process of inference. For instance, Peano and Russell realizing that the only method of expressing the particular proposition is by means of a symbol for existence—they both employ the symbol $\exists$: Peano writes $\exists\, a$ for "there are some a's," and Russell $\exists\, x, y$ for "there is a couple x, y."[2]—and that in the implication which corresponds to the A proposition there is *no* presupposition

[1] *Formal Logic*, pp. 223–226.

[2] "Mathematical Logic as based on the Theory of Types," *American Journal of Mathematics*, vol. xxx, No. 3, p. 246.

of existence, offer no implication corresponding to Conversion *per accidens*. Or, to put the matter concretely, Peano's expression $x \varepsilon a . \supset_x . x \varepsilon b$ carries with it no information that there exist x's that are a's, and so are ab's; consequently, he cannot conclude $\exists\, b\, a$, the symbolism of the proposition "some b's are a's." In the case of Frege the particular proposition certainly is expressed not in an existential manner, but in a manner analogous to that in which the universal is expressed. But there is an *implicit* reference to existence, and so the conversion of A is recognized as not permissible. Frege's assertion $\vdash\!\overset{\mathfrak{a}}{\smile}\!\!\begin{array}{l} P(\mathfrak{a}) \\ X(\mathfrak{a}) \end{array}$ simply says that if each $\mathfrak{a}$ is an X, then each $\mathfrak{a}$ is a P: we are not told that there are such things as $\mathfrak{a}$ which are X. Hence from this implication none can be reached concerning $\mathfrak{a}$'s that are P.[1]

On the other hand, all of these logicians implicitly symbolize the conversion of the universal negative and of the particular affirmative. Take, for instance, Peano's symbolism.

[1] It may be observed that in his previous article, that on "The Theory of Implication," Mr. Russell leans to Frege's symbolism rather than Peano's. Prop. 7·25, which is a genuine particular, is $\vdash . \sim(s) . s$, *i.e.*, it is not true in the case of every s that s is true, or "some s's are not true."

In his notes on prop. 400 he observes that the ways of expressing the universal negative and particular affirmative are respectively $xy = \Lambda$[1] and $\exists\, xy$; that is to say, where x is subject and y is predicate we assert in the case of the universal negative that there do not exist things that are x and at the same time y, and in the case of the particular affirmative that there do exist things that are both x and y. And in prop. 30 Peano proves that $ab \supset ba$; consequently, by reversing the order of the symbols, $ba \supset ab$, and so by prop. 16 these products are equivalents. Hence in $xy = \Lambda$ and $\exists\, xy$ we can exchange the positions of the x and y. And when the expressions are then read in words we have the converses of E and I respectively. As regards Mr. Russell's view it might at first sight be thought that he would prove that "q implies not-p," may be deduced from "p implies not-q," and would maintain that the legitimacy of the conversion of E has thus been demonstrated. The deduction of the second implication from the first may certainly be effected. By substituting $\sim p$ for p and $\sim q$ for q in prop. 4·11—$p \vee q \,.\! = .\, \sim p \supset q$—and bringing to bear prop. 3·2[2] upon the definition thus altered, we obtain

[1] He also admits the forms $x \supset -y$ and $- \exists\, x\, y$.

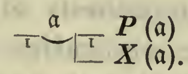
[2] $\vdash . \, p \supset \sim(\sim p)$.

$\sim p \vee \sim q . = . p) \sim q$. Then by prop. 5·31[1] the disjunction in this last expression may be written $\sim q \vee \sim p$; and this by a second employment of 4·11 gives $q) \sim p$, or "if q, then not-p," which is the converse of the original proposition. But such a deduction does not touch the question of the conversion of the universal negative. For there is introduced here no notion of "all"; in other words there is no reference to the variable. If Mr. Russell admits, as he doubtless would, the validity of the conversion of E, it must be on the ground that the validity is intuitively obvious. From, that is to say, the statement $x \varepsilon a)_x x \varepsilon$ not-b the statement $x \varepsilon b)_x x \varepsilon$ not-a immediately follows. The implication embodying the conversion of the particular affirmative is in Mr. Russell's symbolism also implicitly recognized as legimate, but the legitimacy varies in obviousness according as we take the later or the earlier representation that he would have suggested for this proposition. If we take $\exists\, x y$ as his reading of the particular it is clear, as in the case of Peano, that this may be written $\exists\, y x$, an expression which, read in words, is the converse of the original proposition. If, on the other hand, we read "some x's are y" analogously to "some s's are not true" in the

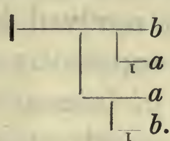
[1] $\vdash : p \vee q . \equiv . q \vee p$.

note at the close of the last paragraph, and say $\vdash . \sim (x).x) \sim y$, it is rather suggested that there are x's and that these are y's; hence there are y's which are x's, and that is all that is needed to establish the converse.

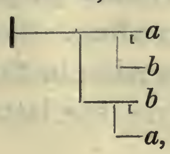
This last argument also lies at the basis of Frege's implicit assertion of the possibility of conversion in the case of the particular affirmative, a proposition which, as we saw, is by him symbolized thus:



So far as the universal negative is concerned, Frege could also reach intuitively an implication corresponding to the conversion of E; but he could not affirm that he has, if we make certain substitutions in his letters, demonstrated the validity of the process, viz., in prop. 33 of the *Begriffsschrift*. This proposition is the following :—



It might at first sight be thought that if in this we substitute not-*b* for *a*, and obtain



and then introduce the notion of the variable, and read "if 'x is an a' implies that it is not the case with any x that it is a b, then 'x is a b' implies that it is not the case with any x that it is an a," and, finally, read this last as "if no a is b then no b is a" we have shown that the validity of the conversion of the universal negative rests upon proof. But this procedure is not valid. When the notion of the variable is introduced the implication that constitutes the hypothesis of the whole proposition, and the implication that constitutes the consequent of the whole proposition have as their consequents *propositional functions*, and the notion of truth does not, as we have seen, attach to these: "x is a not-a" cannot be conceived either as true or as false.

Having considered the manner in which the generalized logic treats the process of Conversion it is not difficult to observe the treatment that is received by the process of Obversion. Frege apparently regards Obversion either as a form of inference too obvious to require explicit consideration, or as having reference merely to the possibility of substituting for each of the four propositions A, E, I, O, an alternative reading. Peano, on the other hand, regards Obversion in three of these cases as having exclusively the latter characteristic;

in the fourth case there is nothing to show whether he regards Obversion as merely supplying an alternative reading of the original proposition, or as a process of inference. He represents the four propositions in question thus :—

A by $x = xy$, or $x - y = \Lambda$, or $- \exists (x - y)$,
E by $xy = \Lambda$, or $x \supset - y$, or $- \exists (xy)$,
I by $\exists xy$,
O by $\exists x - y$.[1]

Here in the first two cases Peano clearly regards the forms $x - y = \Lambda$ and $x \supset - y$, which would ordinarily be taken to represent the obverses, as merely alternative readings for $x = xy$ and $xy = \Lambda$ respectively. In symbolizing O he restricts himself to the form which represents the obverse of O rather than that proposition itself. In the case of I there is no reference to an obverse form, so that it cannot positively be decided whether Peano here regards the obverse as an inference or as an alternative reading. But as he gives the proposition O in the form $\exists x - y$, that is to say, in the form of the obverse, the presumption is that he regards the obverse of I merely as an alternative reading.

The process of Obversion is in Mr. Russell's work not explicitly recognized. He has a

[1] See *Formulaire*, p. 48.

proposition that is *analogous* to the obversion of E, and three other fundamental implications would be illegitimate unless the generalized logic admitted the validity of a process analogous to the obversion of A. The proposition that is analogous to the obversion of E is number 3·34, viz., $\vdash : \sim (p \,\boldsymbol{)}\, q) . \boldsymbol{)} . p \,\boldsymbol{)} \sim q$. The three propositions that would be invalid unless the process analogous to the obversion of A be admitted are numbers 2·92, 3·21 and 3·22. These are analogous to what is ordinarily known as Contraposition, or to what Keynes would speak of as "full contraposition,"[1] and, as is well known, the contraposition of A is invalid unless the obversion of A is valid. It will be observed that in the case of this last logician I have spoken of processes "analogous to" the process of Obversion in the ordinary logic. The reason is that Mr. Russell lets his symbols stand for propositions and not for classes. So that in 3·34, for instance, we are not excluding on the left hand two classes from one another, but we are denying the truth of a certain implication. A representation of the obversion of the propositions A, E, I, O, is, however, possible with Mr. Russell's symbols. The proposition A is symbolized—in his earlier paper—by

[1] *Formal Logic*, p. 135.

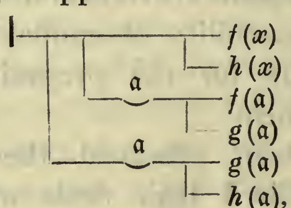
$(x):(\mathrm{A} \between x).\supset.(\mathrm{B} \between x)$. The particular negative would be represented by means of the negative symbol prefixed to this expression. The propositions I and E are represented by means of the symbol $\exists$, a symbol which is introduced in the paper on "Mathematical Logic as based on the Theory of Types." Thus the four propositions and their obverses may be represented as follows :—

	Original.	Obverse.
A.	$(x):(A \between x).\supset.(B \between x)$	$\sim \exists A$ not-B
E.	$\sim \exists AB$	$(x):(A \between x).\supset.(\text{not-}B \between x)$
I.	$\exists AB$	$\sim (x):(A \between x).\supset.(\text{not-}B \between x)$
O.	$\sim (x):(A \between x).\supset.(B \between x)$	$\exists A$ not-B.

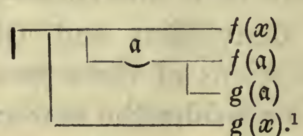
It should, however, be observed that Mr. Russell has not actually used the symbols $\exists$ and $\sim \exists$ respectively in the representation of I and E, and that, if we confine ourselves to the symbolism of his earlier paper, we must say that he regards Obversion merely as pointing to the possibility of employing an alternative reading for the propositions of the traditional scheme.

Having, then, observed the manner in which generalized logic deals with the processes of Immediate Inference—the processes other than Conversion and Obversion are merely applications of these two processes—we proceed to consider the new symbolism for Syllogism. The process that the ordinary logician has here in mind can be well dealt with by the thinker who takes the broader view of the scope of logic; indeed the modern treatment of Syllogism is, as we saw in Chap. I, more accurate than the customary expositions. The three symbolists whom we are specially considering have not, however, treated syllogistic doctrine in an equally satisfactory manner: rather it is observable how each succeeding symbolist shows an improvement upon the work of his predecessor. Frege deals with the facts of the case, but he deals with some of them twice over. Peano symbolizes just what should be symbolized, but in a somewhat clumsy fashion. And Mr. Russell's treatment shows neither tautology nor want of succinctness.

In place of the ordinary symbolism for *Barbara* Frege supplies two forms, viz.,



and



[1] See props. 65 and 62 respectively.

The former of these may be read, " if in the case of each $\mathfrak{a}$, if it is an h, it is a g, and in the case of each $\mathfrak{a}$, if it is a g, it is an f, then, if a particular $\mathfrak{a}$, viz., x, be taken which is an h, the x is an f." And the latter proposition may be read, " if a particular individual x is a g, and in the case of each $\mathfrak{a}$, if it is a g, it is an f, then x is an f." The former of these expressions is intended to cover the case where the ordinary logician has in mind classes only, and the latter to cover the case where the subject of the minor premise is an individual. Now there is nothing inaccurate in Frege's symbolism. But the second of these forms is unnecessary. For in the first it may well happen that the class of $\mathfrak{a}$'s that is referred to as being h possesses only a single member.

Peano's symbolism for *Barbara* shows, on the other hand, no tautology. What he does is to supply two forms, one of which corresponds precisely with that found in ordinary logic, and the other covers the case where the minor premise sets forth the relation of an individual to a class. These forms are,

$$a, b, c \,\epsilon\, \mathrm{K} \,.\, a \supset b \,.\, b \supset c \,.\supset.\, a \supset c,$$

and

$$a, b \,\epsilon\, \mathrm{K} \,.\, a \supset b \,.\, x \,\epsilon\, a \,.\supset.\, x \,\epsilon\, b,$$

which are respectively props. 26 and 25. Clearly there is here no superfluity of statement. But it is unnecessarily cumbersome to have one form for classes, and one where an individual is concerned. We cannot truly retain the ordinary logician's single representation of the syllogism, for that representation treats an individual as identical with a class, but we *can* represent the transitive inclusion of classes by means of a variable, and so speak exclusively of propositions that set forth the relation of an individual to a class. This is what Mr. Russell has done. His symbolism for *Barbara* is the following:—

$$\vdash :.(x).(A \between x))(B \between x):(x).(B \between x))(C \between x):):(x): (A \between x))(C \between x).$$

Here all reference to class-inclusion is excluded, and so the inaccuracy of the common representation, and the cumbersomeness of that of Peano, are avoided. And Mr. Russell in excluding all reference to classes does not offer *two* forms, as does Frege,[1] but covers all the facts in one statement: as Mr. Russell says, this is the "general" form of *Barbara*.

[1] There is a difference, but not an important one, between Mr. Russell's symbolism of *Barbara* and that given by Frege. Mr. Russell's consequent says, "in the case of all x's if x is an A, then that x is a C," while Frege's consequent says, "if a particular individual, viz., x be taken, which is an h, that individual will be an f." Frege, that is to say, mentions by name the individual which is selected, while Mr. Russell refers to the individual merely as one of a certain class.

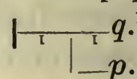
It will be seen that symbolic logicians are here exclusively concerned with the representation of a particular mood of a particular figure. But it is quite possible with the general symbols to represent all the other moods of the syllogism. That such is the case is obvious, since it is possible, as we have seen, for these symbols to represent each of the propositions A, E, I and O.

In considering the symbolist's relation to the syllogism the questions that arise in connexion with the process of Reduction must not be overlooked. These questions offer no peculiar difficulty to the symbolist. He may, that is to say, either consider that the moods of the last three figures are each intuitively obvious, or he may hold that each of these three figures has a dictum of its own, or his view may be that all the moods of such figures must be proved by being brought to the corresponding mood of the first figure. Supposing we take the usual view of Reduction, namely, that the moods of the so-called imperfect figures must be brought to the corresponding moods of the first figure, the symbolist can effect such reduction, if he can find place for the doctrine of Opposition, for simple and accidental Conversion, and for Transposition of premises. We have already seen that he finds no difficulty in dealing with Opposition and Conversion. Equally certain is it that he is able to transpose his premises. It is true that Peano has no proposition that sets forth the legitimacy of this process, for his prop. 30, viz., *ab* Ɔ *ba*, sets forth a relation of classes. But Russell explicitly and Frege implicitly give a justification for the process of Metathesis. Prop. 4·22 in "The Theory of Implication" is the following:—

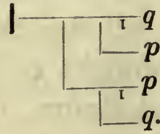
$$\vdash : p \cdot q \cdot) \cdot q \cdot p.$$

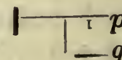
Similarly,

$$\vdash : q \cdot p \cdot) \cdot p \cdot q.$$

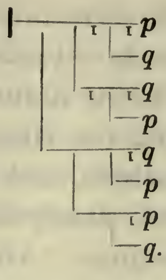
Hence prop. 5·3, viz., $\vdash : p \,.\, q \,.\equiv .\, q \,.\, p$, is established. In the *Begriffsschrift* that such an implication is true becomes apparent from the following considerations. The product *p q* is representable by the implication [1]

Then in prop. 33, substituting not-*p* for *a*, and not-*q* for *b*, we get

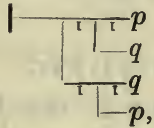


Also in prop. 33, substituting  for not-*b*, and  for *a*, we get

[1] See p. 12.



Here by what was obtained in the former substitution the hypothesis is seen to be true; hence the consequent is true. That is to say, we get



or "*pq* implies *qp*." Similarly, "*qp* implies *pq*," and so *pq* is equivalent to *qp*. Thus there is nothing to prevent Frege and Russell from proving the validity of the moods of the imperfect figures by processes analogous to or identical with[1] those which are usually employed in Reduction.

Proceeding now to the symbolist's treatment of the Conditional, Hypothetical, and Hypothetico-categorical Syllogisms [2] we come upon a matter of fundamental importance. With regard to Conditional and Hypothetical

[1] Metathesis is the only process in which there is *identity* of treatment.

[2] See Keynes, p. 348.

Syllogisms, these involve nothing but implications, and so can be symbolized in general logic. That such is the nature of Conditionals is at once evident, for they can readily be turned into Categoricals, and Categoricals have been shown to be susceptible of being expressed as implications. And a reasoning of the form "if q is true, r is true, and if p is true, q is true; therefore, if p is true, r is true," the form known as the Hypothetical Syllogism, is also merely an implication: we can say in symbols:—

$$\vdash : q \supset r . p \supset q) : p \supset r.$$

But, when we come to the Hypothetico-categorical or Mixed Hypothetical Syllogism we have to distinguish this very carefully from a proposition with which it is likely to be confused. There is no doubt whatever that the following syllogism *can* be expressed as an implication, viz., "if p implies q, and if p is true, then q is true." But the following statement, which is what is commonly known as the Hypothetico-categorical or Mixed Hypothetical Syllogism, cannot be expressed as an implication: "p implies q, and p is true, therefore q is true." This argument does not state in the second premise "if p is true" but states "p is true"—the possibility of p's being false does not occur. And inasmuch as the symbolist can deal only with implications this proposition cannot be symbolized by him. Nevertheless he employs this very proposition in every proof that he establishes. Take, for instance, the proof of Russell's "principle of assertion."[1] This proposition is the following:—

$$\vdash : p . p \supset q . \supset . q.$$

In proof of this the propositions 3·1 and that known as "Imp." are employed. The former of these says,

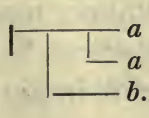
$$\vdash :. p . \supset : p \supset q . \supset . q.$$

The latter, if we substitute $p \supset q$ for q, and q for r, gives

$$\vdash :. p . \supset : p \supset q . \supset . q \supset :. p . p \supset q \supset : q.$$

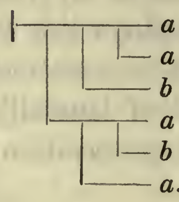
That is to say, in the latter we assert that, if p implies that "p implies q" implies q, then, if p is true and p implies q, it is implied that q is true. And in the former we assert that the hypothesis here *is* true. Hence we conclude that the consequent—our proposition to be proved—is true.

Or take Frege's proposition 26. This is

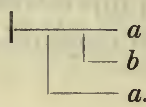


[1] "The Theory of Implication," prop. 4·35.

What we do is to substitute *a* for *d* in prop. 8. This gives us



And by prop. 1 we know that the following implication holds :—



Hence we reach the assertion that we set out to establish.

Here again Russell shows a stronger grasp of the situation than his contemporary workers. Frege without referring—so far as I have been able to discover—to this proposition that is fundamental in all inferences, just makes use of such proposition. Mr. Russell, on the other hand, not merely employs the proposition, but he recognizes that he is employing it : in his view it is one of the primitives, though not one that can be symbolized.[1]

These considerations show that certain statements which are commonly made with respect to the Mixed Hypothetical Syllogism are incorrect : such statements imply that the Mixed

[1] See " The Theory of Implication," p. 164.

Hypothetical Syllogism *can* be expressed as an implication. Keynes, who may here be taken to be representative of the exponents of the common logic, says, for instance, that the *modus ponens* and the *modus tollens* fall into line respectively with the first and second figures of the Categorical Syllogism, and that the Mixed Hypothetical Syllogism can be reduced to the form of a Pure Hypothetical Syllogism. So that, since the Categorical Syllogism and the Pure Hypothetical Syllogism can be expressed as implications, it would follow that the Mixed Hypothetical Syllogism can also be expressed as an implication.

The more complex forms of reasoning, such as the Dilemma, place before the symbolist no particular difficulty. As a matter of fact, though this argument is ignored by symbolists, Russell has two propositions which, if they be subjected to a quite simple process, give exactly the Simple and the Complex Constructive Dilemma. The propositions in question are numbers 4·44 and 4·48. The former of these is

$$\vdash :. q \supset p . r \supset p . \supset : q \vee r . \supset . p,$$

and the latter is

$$\vdash :. p \supset r . q \supset s . \supset : p \vee q . \supset . r \vee s.$$

Now in each of these cases the principle of

Importation may be applied,[1] and we get respectively

$$\vdash :. q \supset p . r \supset p . q \vee r . \supset : p$$

and

$$\vdash :. p \supset r . q \supset s . p \vee q . \supset : r \vee s,$$

or "if q implies p, and r implies p, and either q or r is true, then p is true," and "if p implies r, and q implies s, and either p or q is true, then either r or s is true." The Simple and the Complex Destructive Dilemma may also be dealt with by the symbolic logician. I will take the Simple Destructive Dilemma, and will show how it may be proved by means of propositions which are laid down by Russell. This thinker's prop. 4·36 in "The Theory of Implication" is the following:—

$$\vdash : \sim q . p \supset q . \supset . \sim p.$$

In this substitute qr for q, and we get

$$\vdash : \sim (qr) . p \supset qr . \supset . \sim p.$$

But by prop. 4·43, the principle of Composition, we have

$$\vdash :. p \supset q . p \supset r . \supset : p . \supset . q . r,$$

and by prop. 5·6 we have

$$\vdash : \sim (q . r) . \equiv . \sim q \vee \sim r.$$

Hence we reach the conclusion

$$\vdash : \sim q \vee \sim r . p \supset q . p \supset r . \supset . \sim p,$$

[1] The significance of this principle was explained in the second chapter.

or, changing the order in the hypothesis by means of prop. 4·22,

$$\vdash : p \supset q . p \supset r . \sim q \vee \sim r . \supset . \sim p .$$

And this is the well-known Simple Destructive Dilemma.

In the present chapter we have been considering those inferences that have for a long time—some of them for a very long time—been regarded as constituting the body of Formal Logic. Since the time of Boole, however, inferences of a more complicated character have found their way into the logician's exposition. Keynes, for instance, assigns Part IV[1] for the treatment of these complicated arguments. Now on the modern view of the scope of logic it is quite possible to deal with all such forms of inference: they at once fall, that is to say, into line with the processes of Immediate Inference, Syllogism and Dilemma in being susceptible of expression in those symbols that are essential if we would attain to a truly general logic, a logic that is at the basis of all reasoning whatsoever. Take, for instance, a proposition of four terms that is not found in pre-Boolian logic. Suppose that we desire to prove that, if *a*, *b*, *c*, and *d* are classes, and *a* is included in *b*, *a* in *c*, and

[1] In his earlier editions. In his 4th edition he places these arguments in an Appendix.

bc in *d*, it is the case that *a* is included in *d*. This deduction, which is Peano's prop. 35, would be established by Keynes without reference to anything but the relation of classes to one another. But we can, instead of speaking of such a relation, speak of the relation of propositional functions, *i. e.*, of expressions that introduce the notion of a variable and of ε. We shall then get the proof that is offered by Peano. At first sight Peano's proof appears to be precisely that which would be offered by Keynes. But really Peano's procedure is not one in which reference is made to class-relationships, but is one in which the expressions that are used are contractions for propositional functions. As he says,[1] the expression "$N \times 6 \supset N \times 2$"[2] "is an abbreviation of "$x \varepsilon N \times 6 . \supset_x . x \varepsilon N \times 2$." The above proposition when expressed in truly general symbols will be

$$x \varepsilon a . \supset_x . x \varepsilon b . x \varepsilon a . \supset_x . x \varepsilon c . x \varepsilon bc . \supset_x . x \varepsilon d \supset : x \varepsilon a . \supset_x . x \varepsilon d.$$

And the proof of this implication will be as follows:—

$$[\text{Hp} . \supset . x \varepsilon a . \supset_x . x \varepsilon b . x \varepsilon a . \supset_x . x \varepsilon c . \text{Cmp.} \supset : x \varepsilon a . \supset_x . x \varepsilon bc$$

$$\text{Hp} . \supset . x \varepsilon bc . \supset_x . x \varepsilon d$$

$$\text{Hp} . \supset . x \varepsilon a . \supset_x . x \varepsilon bc . x \varepsilon bc . \supset_x . x \varepsilon d .$$

$$\text{Syll} . \supset . \text{Ths}].$$

[1] *Formulaire*, p. 29.

[2] By N he means a positive whole number.

And in a precisely similar manner may arguments involving more than four terms be dealt with by means of the new symbols. Thus Boole's proposition

$$(ae \cup b - e)(ce \cup d - e) \equiv ace \cup bd - e,$$

which is number 271 in Peano, may be transformed into the following :—

$$(-a \cup -e : \supset . b - e)(-c \cup - e . \supset . d - e) \equiv - a \cup - c \cup - e . \supset . bd - e,$$

i. e., $(a \supset - e . \supset . b - e)(c \supset - e . \supset . d - e) \equiv ac \supset - e . \supset . bd - e,$

or $(x \varepsilon a . \supset_x . x \varepsilon - e \supset : x \varepsilon b - e)(x \varepsilon c . \supset_x . x \varepsilon - e \supset : x \varepsilon d - e) \equiv x \varepsilon ac . \supset_x . x \varepsilon - e \supset : x \supset bd - e,$

where we have not class-relationships but implications either between propositional functions or between a proposition and a propositional function.

CHAPTER V

GENERAL LOGIC AS THE BASIS OF ARITHMETICAL AND OF GEOMETRICAL PROCESSES

WHEN we leave the region of the common logic and observe the manner in which application of the laws of general logic may take the place of the processes of Arithmetic, we find ourselves on ground that has been well described by Peano. Frege has both given the deductions [1] and discussed [2] the philosophical nature that must be ascribed to Number, if we would thus bring arithmetical processes into relation with general logic. Up to the present time [3] Mr. Russell has here confined himself to a discussion of the philosophical question. This question we are postponing to a subsequent chapter. In the first part of the present chapter (1) we shall take some well-known *propositions* in Arithmetic, and shall show how they may be expressed in the symbols

[1] In his *Grundgesetze der Arithmetik.*

[2] In his *Grundlagen der Arithmetik.*

[3] That is to say, previous to the publication of the *Principia Mathematica.*

of general logic, and (2) we shall observe the manner in which arithmetical *processes* are but substitutes for the applications of laws which are found in the propositional calculus.

Supposing, then, we have the numerical statement " Prime numbers greater than 3 are of the form 6N + 1 or of the form 6N – 1." This law of Arithmetic is transformable into an implication that introduces the symbols for logical multiplication and logical addition. Let Np stand for "prime numbers," and N for " positive whole numbers," and we have, that is to say,

$$x \, \varepsilon \, \mathrm{Np} \cap (3 + \mathrm{N}) \, . \supset_x . \, x \, \varepsilon \, (6\mathrm{N} + 1) \cup (6\mathrm{N} - 1).$$

This is read, " If x is a prime number and greater by 3 than a certain positive whole number, then x is either greater or less by one than six times a certain positive whole number."[1] Here we have on the left of the sign of implication the variable x, the letter ε which indicates the relation in which an individual stands to the class of which the individual is a member, and a class, this last consisting of the individuals common to prime numbers and to positive whole numbers that are respectively greater by 3 than certain positive whole numbers. And on the right of the sign of implication we again have the variable and ε and a class of objects,

[1] *Formulaire*, p. 41.

the class consisting in this case of either the numbers that are one greater than six times certain positive whole numbers or the numbers that are one less than six times such numbers. That is to say, we have an implication subsisting between propositional functions. The numerical statement has thus been transposed into a logical statement.

As a second instance of a numerical truth that may be read as a purely logical statement the following may be taken. It is somewhat more complicated than the preceding and introduces additional logical symbols. "Let *a*, *b* and *c* be three quantities of which the first is positive. Let there be a given positive number *h* of any magnitude. It is possible to determine a positive number *k*, which is such that whatever be the value of *x* higher than *k*, the trinomial $ax^2 + bx + c$ is always greater than *h*." This statement may, as Peano points out,[1] be turned into an implication thus:—

$$a \,\varepsilon\, \mathrm{Q} \,.\, b, c \,\varepsilon\, \mathrm{q} \,.\, h \,\varepsilon\, \mathrm{Q} \,.\, \supset .\, \exists\, \mathrm{Q} \frown \overline{k\,\varepsilon}\,[x \,\varepsilon\, k + \mathrm{Q} \,.\, \supset_x .\, a\,x^2 + b\,x + c > h].$$

Here the letter Q stands for "a positive real number," q for "a real number," and *b*, *c* for "a couple." Thus the left-hand expression reads, "if *a* is a positive real number, *b* and *c* are real numbers, and *h* is a positive real

[1] *Formulaire*, p. 47.

number." On the right we have the symbol ∃ followed by Q, and the latter letter is joined by means of the symbol for logical multiplication to the remaining expression. We are thus informed that there "exist" certain objects, viz., those that are thus logically multiplied together. Such objects are declared to be positive real numbers, and k's "such that"—this is the signification of the vinculum—if x is a number greater than k then the expression $ax^2 + bx + c$ is greater than h. We have, consequently, expressed the given proposition in terms that involve the conceptions and only the conceptions of general logic.

And, lastly, as an instance of the manner in which arithmetical expressions may be stated in exclusively logical language, take the definition of a logarithm, viz., "a logarithm is the characteristic of a real function of a positive variable." We let, as before, q stand for a real number, and Q stand for a positive real number. And we employ the symbol f in such a manner that if we have the expression $u \varepsilon b$ f a we mean that u is an operation which, if brought to bear upon a, will yield b. We then get as the definition of a logarithm log ε q f Q, *i.e.*, a logarithm is a member of a certain class, and that class consists of the individuals which are signs of operations of such a kind that, if they be brought to bear upon a positive real number, the result is a real number.[1]

Proceeding to the consideration of the manner in which logical *processes* may take the place of arithmetical *processes*, we have two good examples stated in the *Formulaire* in the notes on props. 72 and 218. These examples we will examine in turn. Supposing, then, we start with the arithmetical statement that "if a is a positive whole number, b is a multiple of a, and c is a multiple of b, then c is a multiple of a," and we desire to prove that "if a is a positive whole number, and b is a multiple of a, then every multiple of b is also a multiple of a." The truth of this general proposition must be established, according to Jevons,[2] by a process of deduction from certain known laws of quantities. The logician can, however, reach the conclusion by utilizing exclusively the notions and laws of the logical calculus. For the original statement may be represented thus :—

$$a \varepsilon \mathrm{N} \,.\, b \varepsilon \mathrm{N} \times a \,.\, c \varepsilon \mathrm{N} \times b \,.\, \supset_{a,b,c} .\, c \varepsilon \mathrm{N} \times a.$$

Upon this statement may then be brought to bear the principle of Exportation, and we reach the following :—

$$a \varepsilon \mathrm{N} \,.\, b \varepsilon \mathrm{N} \times a \,.\, \supset_{a,b} .\, c \varepsilon \mathrm{N} \times b \,.\, \supset_{c} .\, c \varepsilon \mathrm{N} \times a.$$

[1] See Peano, *Formulaire*, p. 54.

[2] See *The Principles of Science*, p. 230.

Here, inasmuch as in the thesis the *c* is an apparent[1] variable, we may omit that letter, and we obtain

$$a \varepsilon \mathrm{N} \,.\, b \varepsilon \mathrm{N} \times a \,.\, \supset_{a,\,b} .\, \mathrm{N} \times b \supset \mathrm{N} \times a.$$

That is to say, if *a* is a positive whole number and *b* is a multiple of *a*, then, whatever *a* and *b* may be, all multiples of *b* are multiples of *a*. We have, that is to say, from a purely logical statement reached by means of laws laid down in the logical calculus another purely logical statement. In this way an arithmetical process may give place to one that is throughout logical in character.

The example just considered shows how a series of arithmetical processes may be replaced by a series of logical processes in the solution of a problem. The second of the above-mentioned notes is useful at only one particular point in illustrating the manner in which logical processes may take the place of arithmetical processes. The whole series of algebraical operations in question might undoubtedly be replaced, as was the case in the previous example, by a series of logical operations, but Peano's object here is merely to illustrate the application of prop. 218; he, therefore, allows some of the algebraical processes to remain in their ordinary form.

[1] For the meaning of this term see Chap. I.

The problem which is taken is to find the solution of the simultaneous equations

$$x^2 + y^2 = 25 \,.\, \frown \,.\, xy = 12.$$

Peano points out that from the two equations by means of simple algebraical laws we get

$$(x + y)^2 = 49 \,.\, \frown \,.\, (x - y)^2 = 1,$$

and that then, by extracting the square roots, there is reached a logical product of logical sums :—

$$x + y = 7 \,.\, \smile \,.\, x + y = -7 : \frown : x - y = 1 \,.\, \smile \,.\, x - y = -1.$$

It is at this point that the application of strictly logical law may take the place of algebraical procedure. The law which is applicable is No. 218, viz.,

$$(a \smile b)(c \smile d) = ac \smile ad \smile bc \smile bd.$$

This at once gives us

$$x + y = 7 \,.\, x - y = 1 : \smile : x + y = 7 \,.\, x - y = -1 : \smile : x + y = -7 \,.\, x - y = 1 : \smile : x + y = -7 \,.\, x - y = -1.$$

Having reached this logical sum of logical products, we may again apply the rules of Algebra and so obtain the required solution, *i. e.*,

$$x = 4 \,.\, y = 3 : \smile : x = 3 \,.\, y = 4 : \smile : x = -3 \,.\, y = -4 : \smile : x = -4 \,.\, y = -3.$$

We have thus shown how arithmetical statements may be expressed as logical statements, and how arithmetical processes may have their places taken by logical processes. It will be convenient in the next place to leave the question of the logical foundations of Arithmetic, and to observe how the notions and laws of logic may take the place of the notions and deductions of Geometry. Having made this observation we shall return to the question of Arithmetic, and shall show that the substitution of logical for arithmetical notions and procedure is not a matter of indifference. It will be pointed out, for instance, how certain propositions in Arithmetic which have hitherto resisted all attempts at justification may by the logician be rigorously proved to be true. The scientific superiority of the substitution in question will thus be established.

To begin with, then, we may express in logical language the notions that are found in Geometry. Take the notions of the properties of a given plane. These properties would ordinarily be described in the language of Geometry. But suppose that we regard the totality of straight lines that pass through a certain point in the plane. Then it is quite possible to affirm that the plane consists of the logical sum of such straight lines. Here by the logical sum of the straight lines is meant the logical sum of the class of classes, each of which is composed of the points that constitute the respective line—the logical sum of a class of classes is "the smallest class that contains all the classes *u*," where *u* is a class of classes. Similarly, the centre of this plane is identical with the logical product of such straight lines *i. e.*, with the logical product of the class of classes just mentioned, the logical product of a class of classes being—where *u* is a class of classes—"the largest class contained in each of the classes *u*." Or in symbols, if *u* is the aggregate of all the straight lines which pass through a certain point *a*, and are contained in a plane, we have that the plane is identical with $\cup$ ‘ *u*, and the centre with $\cap$ ‘ *u*.[1]

Coming to the question of processes we may take the example which is mentioned in Peano's note to prop. 109. This example shows how Euclid's proposition I. 19 may be established by means of the application of the principles of Transportation and Multiplication. In this proof the propositions from which we set out are Euclid I. 5 and I. 18, both logically interpreted. That is to say, we have the following :—

$$a, b, c \,\varepsilon\, \text{Points} \,.\, \text{side}\,(a, c) = \text{side}\,(a, b) \,.\, \supset \,.\, \text{angle}\,(a, b, c) = \text{angle}\,(a, c, b),$$

$$a, b, c \,\varepsilon\, \text{Points} \,.\, \text{side}\,(a, c) > \text{side}\,(a, b) \,.\, \supset \,.\, \text{angle}\,(a, b, c) > \text{angle}\,(a, c, b).$$

Then in each of these propositions we apply

[1] See Peano, *Formulaire*, p. 52.

the principle of Transportation, which is as follows:—

$$ab \supset c . = . a - c \supset - b,$$

and we get

$$a, b, c \; \varepsilon \; \text{Points} . \text{angle}\,(a, b, c) - = \text{angle}\,(a, c, b) . \supset . \text{side}\,(a, c) - = \text{side}\,(a, b),$$

$$a, b, c \; \varepsilon \; \text{Points} . \text{angle}\,(a, b, c) - > \text{angle}\,(a, c, b) . \supset . \text{side}\,(a, c) - > \text{side}\,(a, b).$$

Finally we apply the principle of Multiplication, which is

$$a \supset b . c \supset d . \supset . ac \supset bd,$$

and we observe that $x - = y . x - > y$ is equivalent to $x < y$. So we obtain

$$a, b, c \; \varepsilon \; \text{Points} . \text{angle}\,(a, b, c) < \text{angle}\,(a, c, b) . \supset \text{side}\,(a, c) < \text{side}\,(a, b),$$

which is the expression in logical language of the truth set forth in Euclid, prop. I. 19.[1]

In this way, then, the notions and processes of Arithmetic and of Geometry may be replaced by the notions and processes of general logic.

[1] We said above that Euclid's props. I. 5 and I. 18 were both logically interpreted. A word is needed to explain this. In the first of these what we have is "the fact that a, b and c belong to the class points, and that the class of points from a to c has a one-one relation to the class of points from a to b, implies that the class of points extending from a point in the class $a \,..\, b$ to a point in the class $b \,..\, c$ has a one-one relation to the class of points extending from the corresponding points in the classes $a \,..\, c$ and $c \,..\, b$. And similarly with the interpretation of prop. I. 18.

We may now take the important step of showing that it is not a matter of indifference which kind of notions and processes is adopted, but that the resort to general logic is accompanied by the greatest advantages. These advantages vary according as it is the Kantian or the experiential method of regarding mathematical truths that is replaced. If general logic is regarded as taking the place of the Kantian procedure, we effect a scientific simplification of high significance: instead of regarding Formal Logic and Mathematics as two distinct sciences, each proceeding from its own axioms, we embrace the two disciplines in a single science. If, on the other hand, it is Mathematics as resting on an experiential basis that is replaced by logical notions and processes, then not only is a scientific simplification effected, but the whole science of Mathematics, instead of being problematical, is transformed into a science whose truths are *certain*—the method of regarding Mathematics that has always been adopted by common sense. In order to illustrate the manner in which such simplification or certainty is realized we will take the Association law—what is generally regarded as the "axiom"—of Cardinal Addition, and will by means of exclusively logical notions and procedure prove that the law is true.

In the course of the demonstration we shall have the opportunity of explaining one or two symbols that were passed over in the second and third chapters.

The law of Cardinal Addition is expressed as follows :—

$$(a + b) + c = a + (b + c).$$

For the cardinal numbers indicated by the letters on the two sides of this equation we proceed to substitute classes, and for the symbol of equality the symbol of identity.

To commence with c on the left, the substitution for this number consists of the classes which are similar to the class γ, where γ is a class with c members; in symbols

$$c = \bar{y}\{c = \mathrm{Nc}`\gamma \,.\, y = \gamma\}.$$ [1]

The substitution for $a + b$ is as follows :—

$$\bar{\xi}\{(\exists\, \alpha, \beta) \,.\, a = \mathrm{Nc}`\alpha \,.\, b = \mathrm{Nc}`\beta \,.\, \alpha \cap \beta = \Lambda \,.\, \xi = \alpha \cup \beta\}.$$

That is to say, $a + b$ is equivalent to the classes ξ, which classes are respectively identical with (=) the parts α and β ($\xi = \alpha \cup \beta$), these parts existing ($\exists\, \alpha, \beta$), being mutually exclusive ($\alpha \cap \beta = \Lambda$), and consisting respectively of a and b members

[1] *Cf.* Russell, "Mathematical Logic as based on the Theory of Types," *American Journal of Mathematics*, vol. xxx, No. 3, p. 256. The justification of this definition, so far as it implies that Number is conceptual in character, will be considered when we come to discuss the philosophical foundations of the new treatment of mathematical notions.

($a = \mathrm{Nc}'\alpha$ and $b = \mathrm{Nc}'\beta$). So that $(a + b) + c$ consists of the classes the constitution of each of which is $(\alpha \cup \beta) \cup \gamma$. Similarly, $a + (b + c)$ may be replaced by the classes the constitution of each of which is $\alpha \cup (\beta \cup \gamma)$. But by the Associative Law in *Logic*[1] the group $(\alpha \cup \beta) \cup \gamma$ is equivalent to the group $\alpha \cup (\beta \cup \gamma)$. Thus all the classes whose constitution is $(\alpha \cup \beta) \cup \gamma$ are equivalent to all the classes whose constitution is $\alpha \cup (\beta \cup \gamma)$. Hence the group $(a + b) + c$ is equivalent to the group $a + (b + c)$. In this manner the law of Cardinal Addition is proved to be true,[2] the law which by the Kantians is taken for granted, and by the experientialists is regarded as admitting only of a high degree of probability. The treatment of the fundamental propositions of Geometry and of the other branches of Mathematics is analogous to this treatment of the fundamental notions of Arithmetic. It is thus

[1] See Russell, "The Theory of Implication," p. 186. Peano in the note on prop. 207 of the *Formulaire* speaks of the proposition thus numbered, viz.,

$$a \cup b \cup c = a \cup c \cup b = b \cup a \cup c,$$

as the Associative Law of logical addition. That is an error. The Associative Law is, however, derivable from props. 205, 206, 207.

[2] This proof well indicates the kind of work upon which Frege is occupied in his *Grundgesetze der Arithmetik*. In the *Grundlagen* his object is merely to make it *probable* that Arithmetic is a branch of logic.

apparent not merely that mathematical propositions *may* be expressed as propositions involving exclusively logical notions, but that this transformation *should* be effected.

A few words of explanation may be added concerning the substitution for $Nc{}^{\mathfrak{c}}\alpha$. We have said that $Nc{}^{\mathfrak{c}}\alpha$, the cardinal number of α, signifies the class of classes similar to α. By "similar to α" is meant that there exists a one-one relation between the members of the class α and the members of every other class in the group of classes, a one-one relation being defined as a relation such that if it holds between x and y, between x' and y, and between x and y', then x is identical with x', and y is identical with y', whatever objects x, y, x', y' may represent. We may express, if we so desire, both a one-one relation and the notion of "similar to" in symbols. We have, if $1 \rightarrow 1$ signifies the class of one-one relations,

$$1 \rightarrow 1 = \hat{R}\{xRy \,.\, x'Ry \,.\, xRy'\,.)_{x,y,x',y'} \,.\, x = x' \,.\, y = y'\}.$$[1]

And the notion of "similar to" is thus symbolized :—

$$\text{Sim} = \hat{\alpha}\hat{\beta}\{(\exists R) \,.\, R \,\varepsilon\, 1 \rightarrow 1 \,.\, D{}^{\mathfrak{c}}R = \alpha \,.\, \text{ɑ}{}^{\mathfrak{c}}R = \beta\}.$$

Here the interpretation of $\hat{\alpha}\hat{\beta}$ and of $\exists$ is as before. The latter part of the right-hand member signifies that R is a one-one relation,

[1] See Russell, "Mathematical Logic as based on the Theory of Types," p. 256.

and that α and β are respectively the domain and the converse domain of that relation, the "domain" being the class of referents, *i. e.*, of terms that have the relation *R*, and the converse domain being the class of relata, *i. e.*, of terms *to* which the relation *R* proceeds.[1]

In this chapter we have shown (1) that arithmetical notions and processes may be replaced by logical notions and processes, (2) that geometrical notions and processes may be similarly replaced, and (3) that general logic *ought* for scientific purposes, or to enable us to reach conclusions that have always been supported by common sense, to be regarded as lying at the basis of Pure Mathematics. We now proceed to set forth the philosophical justification of this treatment of numerical and geometrical doctrines, and we commence with a discussion of the nature of Number.

[1] Russell, *The Principles of Mathematics*, pp. 96, 97.

CHAPTER VI

THE PHILOSOPHICAL TREATMENT OF NUMBER

It will be convenient in dealing with Number from the philosophical standpoint to establish first of all the positive characteristics of Number; secondly, to consider certain negative characteristics of it—those of which the corresponding positives have by certain philosophers been assigned to it;[1] and thirdly, to indicate in a concrete manner the fact that the treatment of Number in the preceding chapters implicitly rests upon that conception of the nature of Number which is here set forth. The outcome of this discussion will be the *complete* justification of the logical treatment of Number, since justification from a scientific point of view or from the point of view of common sense has already been demonstrated.

We will commence, then, by setting forth and establishing the positive characteristics of

[1] In dealing with these two features I have found Frege's work, *Die Grundlagen der Arithmetik*, exceedingly valuable.

Number. These are three. Any number—say 4—is in character (1) conceptual, (2) single, and (3) objective. These characteristics must be considered in turn.

By the assertion that a number is conceptual is meant in the first place the fact that it admits of our asking whether there exist any objects corresponding to it. All concepts admit of our asking such a question: the answer in the case of the concept animal is "yes," and in the case of the concept centaur "no." The conceptual character of Number is particularly brought to the front in connexion with the number 0. Here the answer to the question whether there exist objects corresponding to the number is in the negative. In the case of most numbers the answer is in the affirmative. Again, it is observable that Number is conceptual, since we can make assertions about a number without being in perceptual contact with any particular instances of it. Just as we can say "all men are animals" without our being in actual contact with men, so we can say that 4^2 equals 16 without our having before us any instances of the number 4. And, thirdly, we are able to deal with numbers in propositions without our being able either to perceive or to have a mental picture of any corresponding entities

We can make scientific statements concerning the earth, though we cannot see the earth as a whole or form a picture of it—the image of a globe is not the image of the earth, but is a substitute for such an image,—and in the same way numbers consisting of many figures may be treated with accuracy though we cannot see or imagine the corresponding objects. In all such cases it is neither percepts nor images but concepts that we have under consideration.

The question may, however, be raised whether concepts *have* the three defining characteristics just ascribed to them.[1] This question deserves a careful answer. The first characteristic ascribed to concepts would be generally admitted: whereas in the case of a proper name there would be no sense in asking whether there exist things corresponding to the connotation of the name—a proper name, I here assume, does not signify any peculiarities at all—it is always possible to ask in the case of general names, and of singular names other

[1] If a concept possesses these three attributes I shall, since no other species of mental entity or act of attention possesses them, take the three to constitute the *definition* of a concept. A concept, that is to say, is a mental entity or act of attention which (1) is such that we can ask concerning it if there exist corresponding objects, (2) is not necessarily accompanied by corresponding perceptual objects, and (3) may exist without the possibility of there being corresponding percepts or images.

than those which are proper, whether there exist things possessing the attributes that the names embody. As regards the second peculiarity of concepts, viz., that there may be concepts without there being present to the senses any corresponding objects, this is manifest from the fact that, if we are given a statement with a concept as subject, we cannot immediately assign the predicate to any present object unless we have a further statement asserting that the present object is included under the concept: "all men are animals" allows us to say nothing of anything present to the senses unless we have the further proposition "this object is an animal." And, in the third place, some thinkers have affirmed that at any rate there *could* always be a percept and an image corresponding to a concept. But this possibility is certainly not universal. We may argue concerning the physical features of the objects on the other side of the moon, though we can never perceive those features. Similarly, the concept of the distance from the earth to the sun may be without difficulty employed in our astronomical reasonings without our being able to perceive that distance: if the distance were perceptible it would have been one of the most familiar perceptions of mankind, and would have required no calculation. And in both these cases there is also not possible an adequate image. If "image" is taken in the sense of memory-image, it is clear there can be no image at all of these features and this distance, since there is no possible perception of them. If, on the other hand, "image" is taken to mean a product of constructive imagination, there can here be no adequate image. It is true that we can form a mental picture in the case of the hidden surface of the moon and of the distance of the sun, but such image is a substitute for an adequate image: the superficial and linear measurements are respectively quite beyond the capacity of imagination.

We have shown that concepts are not always accompanied by corresponding objects, and that in some instances the corresponding objects are neither perceptible nor imaginable. If such is the case it is not unnaturally asked how it is that we employ concepts with such readiness in our propositions, and with such certainty in our reasonings. It is quite clear that if we must not in general rely upon perception or imagination as a justification for our use of concepts in these ways, there must be some other justification for such use. What we have is that certain of the simpler relations of concepts are intuitively obvious, and the more complicated relations are deduced from those which are simple : immediate apprehension of simple conceptual relations, and deduction of other relations take the place of perception and imagination. These conceptual relations may in a few instances be dealt with without external aid. But generally in dealing with such relations we have to make liberal use of symbols. The nature of these last calls for a moment's further notice.[1] In the first place, though they are perceptual or imaginative in character, they are something quite different from the objects that are embraced under the concept. And, in the second place, though the symbols help us to realize the simpler relations of concepts, the symbols afford no ground for our setting forth such relations. In a word, the relations of concepts are established by reason of their own nature, and not by reference to percepts or images, but conceptual relations are generally more readily apprehended if we resort to perception or imagination in one particular, viz., in the employment of symbols.

Number, then, is conceptual in character, concepts being entities possessing the three

[1] I pointed out in the Introduction that by means of symbols a mechanical process may take the place of reflection upon conceptual relations.

attributes just described. The second peculiarity of a number is singleness. In considering this peculiarity it is necessary to show that it really does attach to a number, and, in the second place, to distinguish the name of a number from a proper name. On the supposition that the former point has been demonstrated, the latter does not present much difficulty. For since a number, as we have described, is of the nature of a concept, the name of a number will be akin to a general name, that is to say will be a singular but not a proper name.[1] We come, then, to the first of the above problems. Here we have to show that singleness *is* a quality attaching to Number.

The argument by which it is established that this quality belongs to a number may be expressed as follows. If it is not true that a number (say 4) is single, then either there does not exist such a number or else there are many such numbers. The former conclusion would be generally rejected. Supposing, then, that we accept the latter conclusion, there must be some difference between the numbers 4, for otherwise they would be inconceivable. But

[1] By a singular name other than a proper name is to be understood such a name as "the present King of England"; here clearly only one object is involved, and we can ask whether there does or does not exist an object possessing the qualities signified by the name.

if there exists a variety of 4's, there is no certainty in the procedure of Arithmetic: we could never be sure that a process performed upon one 4 would yield the same result as would be yielded by the performance of that process upon another 4. But it is generally admitted that there is no such uncertainty in the performance of arithmetical processes. Hence there are not many 4's, but only one number 4. The argument is a Mixed Hypothetical Syllogism of the *modus tollens* description. The same argument applies in the case of numbers other than 4. And so in general language we may say that all numbers are singular, or in other words that singleness is a characteristic of Number.

The third characteristic of Number is objectivity. By this is meant the fact that Number does not vary with different individuals, but is the same for all. As Frege says,[1] if we attribute 10,000 square miles to the North Sea, the number 10,000 is as much objective as is the North Sea. It is quite true that persons frequently form mental pictures of numbers, and that these mental pictures vary in character, but the truth of that assertion does not carry with it the truth of the assertion that Number is subjective in nature. At the same

[1] *Die Grundlagen der Arithmetik*, p. 34.

time it must be understood that by "objective" here is meant nothing more than "the same for all." Nothing spatial in character is signified. The North Sea possesses both this attribute and that of objectivity. "10,000" possesses objectivity only. That this last is an attribute of Number may be proved in the same manner as was the characteristic of singleness. If numbers varied from individual to individual, we should never be certain that after performing correctly certain processes upon a number we should reach the same conclusion as other persons after the same procedure would reach. But we *are* certain in such cases of reaching the same conclusion. Hence Number does not vary from individual to individual, but is common to all, *i. e.*, is objective. There is no difference between finite and infinite numbers in respect of this characteristic. The infinite number of finite numbers,[1] for instance, is not something that varies with different persons, but is something that is the same for all.

We have thus shown that a number is conceptual and not perceptual or imaginative in character; is single, *i. e.*, does not admit of many

[1] Concerning the infinite character of the number of finite numbers see Russell, *The Principles of Mathematics*, p. 309.

examples; and, finally, is objective, does not, that is, exhibit variety in the case of different individuals. Our second duty is to indicate certain negative attributes of Number, the corresponding positives of which have sometimes been assigned to it. In the first place, Number is not attributable only to external things. If number were of the nature (say) of colour, it is quite clear that objects that are not external would have no number. There would thus be no number of concepts, of images, of selves, which is absurd. Secondly, Number is not something which always attaches to an *aggregate* of things, and is constituted by the special manner in which we regard that aggregate. For, to begin with, an aggregate may be regarded in many ways, and so may have many numbers: the House of Commons consists of 4 or of 670, according as we refer to parties or to members. And, as regards the invariable presence of an *aggregate* where Number is concerned, it is to be observed that Number attaches to such things as thoughts and actions, and neither of these can be said, as the leaves of a tree or the individuals of a city can be said, to constitute an aggregate. This second view of Number—Mill's view [1]—

[1] "What, then, is that which is connoted by a name of number? Of course, some property belonging to the

must, therefore, be rejected. In the third place, Number does not consist in abstraction from the differences of objects, and attention merely to the presence of the objects.[1] This doctrine of Jevons is liable to two insuperable objections. On the one hand, both 0 and 1 are numbers, and in neither case is there any abstraction involved from the differences of objects. And, on the other hand, Jevons would admit that, in the case of numbers other than these two, each of the objects from whose differences there is abstraction is *one*. *One*'s are thus susceptible of differences, and the self-evident truths of Arithmetic must be rejected, which is absurd. And, fourthly, Number does not consist in some symbols "1," joined together by the symbol for addition. The symbol 1 + 1 + 1 may plausibly *represent* the number 3, but such symbol does not *constitute* that number. Even the representation of numbers cannot always assume this form, for in the

agglomeration of things which we call by the name; and that property is, the characteristic manner in which the agglomeration is made up of, and may be separated into, parts."—*A System of Logic*, vol. ii, p. 151.

[1] "When I speak of *three men* I need not at once specify the marks by which each may be known from each . . . the abstract number *three* asserts the existence of marks without specifying their kind."—Jevons, *The Principles of Science*, p. 158.

case of the numbers 0 and 1 there is no opening for the symbol of addition.

We have now considered what it is that philosophical reflection leads us to regard as the nature of Number, and we have indicated certain views that cannot with self-consistency be held concerning that nature. We conclude this chapter by demonstrating that the manipulation of Number in the preceding chapters is in accordance with the conclusions concerning the nature of Number that are reached by philosophical reflection.

Take, then, the following equation, $\sqrt{2} \times \sqrt{2} = 2$. This equation when expressed in logical form appears as

$$x \varepsilon (\sqrt{2} \times \sqrt{2}) \,.\supset.\, x \varepsilon 2 \text{ and } x \varepsilon 2 \,.\supset.\, x \varepsilon (\sqrt{2} \times \sqrt{2}).$$

Here, since ε signifies the relation in which an individual stands to the class of which it is a member, it is clear that both the $(\sqrt{2} \times \sqrt{2})$ and the 2 are regarded as *classes*, that is as each *a* class. A number is thus taken to be singular. Again, though the number 2 may plausibly be said to have both a perceptual and an imaginative basis, it is clear that the number $\sqrt{2}$ cannot have such a basis, and hence that the process of squaring this quantity cannot have such a basis. In general, Number is thus taken to be neither perceptual nor imaginative in character: it is treated as conceptual. And, thirdly, neither 2 nor ($\sqrt{2} \times \sqrt{2}$) is taken to be something that varies from man to man. What is asserted is that if x is a member of *the* class ($\sqrt{2} \times \sqrt{2}$) then x is a member of *the* class 2. There is here nothing uncertain: a fixed set of class-members—in this case the members are classes—is referred to, something, that is to say, which is objective. In this example, then, of the logical treatment of an arithmetical equivalence there is found nothing which is opposed to the propositions which Philosophy asserts concerning the nature of Number.

Two other examples may be adduced to show that there is nothing involved in the modern treatment of arithmetical notions which is opposed to the views concerning Number that are unfolded by Philosophy, but rather that the modern treatment of Number is that which philosophical reflection indicates should be adopted. These illustrations shall be taken from the writings of Peano and Frege respectively. By way of elucidating the notion of a couple Peano brings forward the following:—

$$(1, 0)\, \varepsilon\, \overline{(x, y)}\, \varepsilon\, (x + 2y^2 = 1).$$[1]

This may be read (1, 0) is one of the couples

[1] *Formulaire de Mathématiques*, p. 36.

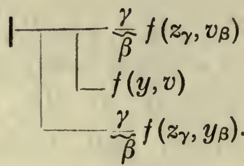
(x, y) that satisfy the equation $x^2 + 2y^2 = 1$. Here (1, 0) is an object composed of the object 1 and the object 0. Since, then, Peano here speaks of two objects as composing the couple, he clearly regards a number as something "singular." Again, (*a*) we may ask of both objects if there exist corresponding entities. (*b*) The object 0 may be present without any corresponding percepts or images. And (*c*) the object 0 can never have any corresponding percepts or images: the object in question is, as Russell, following Frege, holds,[1] a class having no members. Peano's proposition, that is to say, merely implies that we can symbolize the object 0, and can treat the object in precisely the same way as we can treat those classes that do have members: he does *not* mean that we can perceive or imagine entities corresponding to 0. And, in the third place, it is quite clear that the above assertion of the relation of a couple to a class of which it is a member involves something objective: Peano does not mean that the assertion varies in character according to the mental state of the individual who is making the assertion. He means that the equation $x^2 + 2y^2 = 1$ is an entity the same for every-

[1] *The Principles of Mathematics*, p. 517.

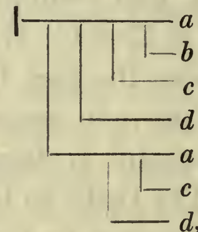
body, that the couple (1, 0) is another such entity, that the objects composing this couple may be substituted in the equation for x and y respectively, and that the resulting identity is one which everybody must recognize.

The example in point which we will take from Frege shall be one of his moderately complicated propositions. Here, as on a previous occasion, we shall have the opportunity of explaining one or two symbols that were not taken account of in Chapter II. A proposition that well illustrates the fact that the modern logician's procedure is in accordance with the dictates of Philosophy is number 111 in the *Begriffsschrift.* The truth to be proved is thus expressed: If y belongs to the f-series, whose first term is z, then each result of an application of the process f to y will belong to the f-series that commences with z, or else will precede z in the f-series. And the method of proof is as follows.

By proposition 108 it has been established that if v belongs to the f-series commencing with z, and if v is the result of an application of the process f to y, then v belongs to the f-series commencing with z. This in symbols appears thus :—



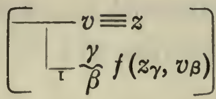
We, therefore, proceed to obtain a hypothetical statement in which this proposition is the antecedent, and the proposition to be proved is the consequent. Such a statement is reached if in prop. 25, which is



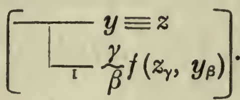
we make the following substitutions :—

a	$\frac{\gamma}{\tilde{\beta}} f(z_\gamma, v_\beta)$
c	$f(y, v)$
d	$\frac{\gamma}{\tilde{\beta}} f(z_\gamma, y_\beta)$
b	$\top \frac{\gamma}{\tilde{\beta}} f(v_\gamma, z_\beta)$.

Here and in prop. 108 the expressions $\frac{\gamma}{\tilde{\beta}} f(z_\gamma, v_\beta)$ and $\frac{\gamma}{\tilde{\beta}} f(z_\gamma, y_\beta)$ are new. They are respectively equivalent to the following :—

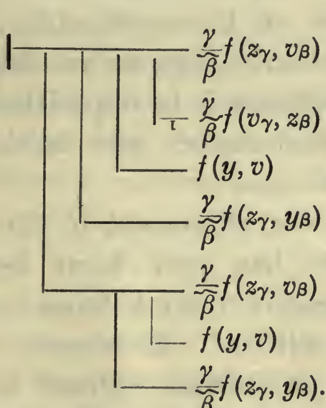


and

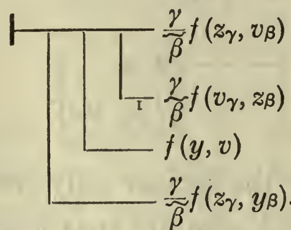


That is to say—the former only need be interpreted—$\underset{\beta}{\overset{\gamma}{\eqsim}} f(z_\gamma, v_\beta)$ signifies that v succeeds z in in the f-series, or v is identical with z in that series. This expression is, therefore, different from the expression $\underset{\beta}{\overset{\gamma}{\sim}} f(z_\gamma, v_\beta)$. The latter, as we have had occasion elsewhere to observe, and as we have just seen, signifies merely that *v succeeds z* in the f-series.

Making the substitution in question, we obtain



Then we are able to say that since in this hypothetical statement the antecedent is true, the consequent is true, viz.,



The proposition that is thus proved, and the proposition whose truth in the proof is taken as established, are both intended to refer quite generally to objects that are related to one another in a series. The propositions in question cover, for instance, the case of *persons* and the relation of *lover* and the case of *numbers* and the relation *greater by two than*. Confining our attention to the applicability of Frege's propositions to numbers, we are able to perceive that his treatment is in opposition to nothing which the philosopher sets forth to be the nature of Number.

Take the first statement in the proposition whose truth has just been demonstrated. That statement is, " if y belongs to the f-series, that begins with z." It is quite clear that y here is an *individual*. Hence any number which is substituted for y will be an individual; in other words, this logician regards numbers as singular in character. Again, though it is quite possible that y may indicate something which is obvious to the senses, *e. g.*, a father or a brother, it is also possible that the letter may indicate an object that is not thus apprehensible: mental objects such as faith, hope and love may take their places in a series quite as well as may objects that are presentable to the senses. The logical procedure is thus on this point not in opposition to the doctrines of Philosophy: the letter y (and similarly the letter z) covers both the case where the objects referred to are percepts and the case where the objects referred to are not percepts, *e. g.*, where they are emotions, or where they are concepts. And, in the third place, the objects y and z are regarded by Frege as entities that are objective, objects that are the same for all. At first sight it might be thought that the function (f) is regarded by him as something which is subjective in character, something which fluctuates with different individuals. But even this *relation* between the two objects is not regarded as something subjective: it is thought of as apprehended by mankind generally. The statement does not mean " if y *is regarded* by me as related in a certain way to z," but, " if the f-characteristic is universally recognized as appertaining to the couple (z, y) ": the relation signified by *f* is objective. So far as the *z* and *y* are concerned there can be no doubt at all that Frege regards them as objective. If the series is declared to commence with *z*, then *z* is obviously thought of as not varying with individuals, but as something which is the same for all. In short, Frege's proposition and the proposition which he adduces in proof of it, propositions which he intends to apply to series of numbers as much as to any other series, imply nothing at variance with the teaching that is unfolded by Philosophy concerning the qualities of Number.

CHAPTER VII

THE PHILOSOPHICAL TREATMENT OF SPACE

In the preceding chapter we considered the view which Philosophy reveals concerning the nature of Number, and we demonstrated that the procedure of modern Formal Logic here involves nothing that is at variance with philosophical teaching. In the present chapter we enter upon a similar consideration and demonstration with reference to the nature of Space. Here our business will be to discuss the question whether Space is absolute or relative in character, and to show that the modern formal logician's procedure is in accordance with the view that must be accepted as a result of this discussion.[1]

At first sight it might be thought that we should also have to discuss the question whether the notion of Space is *à priori* or is *à posteriori*. But, important as the settlement of this point is, we are not called upon here to

[1] I have followed Mr. Russell (*Principles of Mathematics*, Chap. LI) closely in the first part of this chapter.

effect the settlement. For the formal logician wholly ignores this question. Take, for instance, the case which has been referred to in a previous chapter, and which will be mentioned a few pages later on, viz., the case of the statements concerning the centre of a plane. There is no hint, when this is described as a logical product, as to whether the positions of the points constituting the respective lines are determined by the exercise of an intuitive capability or through accumulated experiences. The points in the lines are found in a certain position, and one of the points is common to many groups, but the question how the points came to be regarded as being in these positions does not arise. On the question of the *à priority* of the notion of Space, whether the Kantian or the experiential view or neither of them is correct is not a matter with which the formal logician is at all concerned. The only subject in which his doctrine is here capable of agreement or of disagreement with philosophical doctrine is whether Space is absolute or relative in character.

We proceed, then, to show that Space consists of points situated eternally with reference to one another, and does not consist of material points now having one and now having another relation to one another. Or, in other words, it will be shown that Space consists of relations between points and does not consist of variations of a certain quality of points. The method of demonstration that is available is that which sets forth the contradictions involved in the doctrine of material points, and the absence of contradiction in the doctrine of absolute position.

In the first place, then, those who assert the existence of material points hold—if Lotze may be taken as representative of such thinkers—that the position of points is determined at any one moment by the *interaction* of points. To this it must be replied that interaction of the kind suggested presupposes absolute position, since one interaction can be distinguished from another only by reference to such position. If it were our business here to improve the argument of those who maintain the doctrine of material points it might be remarked that there is no justification whatever for attributing the relation of such points to interactions: the correct view would be that the relation of material points is an ultimate fact and is not resolvable into interaction or into any other kind of action. But this recognition of the position of material points at any one moment as an ultimate fact would not involve justification for the resolution of Space into such points: the distinction of one arrangement of material points from another could be recognized only by reference to the fact of absolute position.

Secondly, it is held that there exist material points, since all propositions consist of the assigning of a predicate to a subject, and only *material* points could have attributes to be predicated. The answer to this is that it is absurd to hold that all propositions are of the kind in question. No doubt there are subject-predicate propositions. But there are also propositions that possess three terms, *i.e.*, that express a relation between two objects. "A is on the right of B," "A is the father of B," "A is less than B," are instances of such relational propositions. With such propositions in constant use it is absurd to hold that all propositions are of the subject-predicate kind. Moreover, those who hold that all propositions are of the subject-predicate kind also hold that only subjects exist—or, more strictly speaking, that only one subject exists. Such a doctrine leads us to ask what it is that takes place in predication. If predicates do not exist, then in predication nothing is assigned to the subject, and if predicates do exist then something other than subjects exists. The theory that only subjects exist is thus disproved. The conclusion of the dilemma is "either in predication nothing is assigned to the subject or something other than subjects exists," and the former alternative being absurd the latter is established. The relative theory of space is based on untenable ground if based on the doctrine that all propositions are of the subject-predicate kind, and only subjects exist.

A third argument in favour of the resolution of space into material points is that only such a resolution is compatible with the self-evident doctrine that it is impossible for new points to appear. The answer to this argument is that it is not impossible for new points to appear. Some things have being only and some have existence and being,[1] and it is quite possible for things to come out of the region of being only into the region of existence. A centaur, for instance, has being, for the creature may be the subject of a proposition. At the present moment the centaur certainly does not exist, but there is no absurdity in

[1] This distinction is brought out by Russell in the work above referred to. The distinction is also well treated by Case (*Ency. Brit.*, vol. xxx, p. 330), but he does not supply a term to indicate the region of reference that lies outside the region of existence—or rather it should be said that he does not supply such a term until the discussion is over: in his last sentence, p. 331, the term "being" *is* given in contradistinction to the term "existence."

thinking of it as coming into existence. The scientific facts established by the polar explorer have being before the expedition sets out: the existence of the facts is established as the expedition proceeds. And so it is with the case of points. There is no absurdity in conceiving all points as having being, and some only as having existence, and in conceiving some of those that have not possessed existence as coming into possession of it. The argument, therefore, that the doctrine of material points is established because it is the only one compatible with the self-evident truth that points cannot come into existence will not suffice: there is no such self-evident truth as the one here stated to exist.

On the other hand, there are no such contradictions involved in the doctrine of absolute position. This doctrine asserts in the first place simply that points are fixed. It does not assert, that is to say, that the position of points is determined by a process of interaction. The doctrine is content merely to take the position of points as it is found, and does not attempt to explain this position by means of a notion that presupposes some other theory of space. Secondly, the doctrine of absolute position does not absurdly identify relational propositions with subject-predicate propositions. It not only does not ignore the distinction in question but expresses itself exclusively by means of a relational proposition: the doctrine asserts that both points and the relations subsisting between them are eternally fixed. And, lastly, the doctrine of absolute position appeals for confirmation to no such proposition as the one which sets forth that points cannot come into existence. In the assertion that points are fixed absolutely with reference to one another the question of coming into existence does not necessarily arise. But if the absolutist were pressed for an opinion upon the possibility in question he would affirm that, instead of maintaining that the possibility does not exist, he maintains that of the points absolutely fixed some do and some do not exist, and that it is quite possible for the non-existing points to come into existence. In short, the absolutist adopts no such untenable propositions as those set forth in two of the above arguments, and he does not go against experience, as is done in the remaining statement: he simply takes the facts as he finds them, and does not seek either to explain them or to show their necessity. And until some philosopher makes evident that the facts are *not* as thus set forth, the theory of absolute position may well be retained.

Having, then, determined what is the view that is indicated by Philosophy as to the nature of Space, we proceed to show that the modern treatment of problems that involve spatial relations is in accordance with this view. To effect this demonstration we will take first of all two of Peano's propositions that were in an earlier chapter quoted for another purpose. And, in the next place, in order to emphasize the agreement of the modern treatment with philosophical teaching, we will illustrate the fact that the ordinary treatment of spatial problems is *not* carried out in accordance with that teaching.

Take, then, Peano's proof of the conclusion of *Euclid* I. 19. The two propositions with which we commence are, "the fact that *a*, *b* and *c* are points, and the side *ac* is equal to the side *ab*, implies that the angle *abc* is equal to the angle *acb*," and, "the fact that *a*, *b* and *c* are points and the side *ac* is greater than the side *ab*, implies that the angle *abc* is greater than the angle *acb*." And the two propositions that are obtained by the process of Transportation are similarly expressed. Here there is no suggestion that the position of the points in question is due to a process of interaction, or even that they are capable of movement. Nor does the mention of the points imply that they have any properties; on the contrary, the points that constitute the line *ac* merely stand in a one-one relation to the points that constitute the line *ab*. And, thirdly, in neither of the expressions is there any reference whatsoever to the question of the *existence* of the points: the proposition that constitutes the hypothesis in each case does not imply that the points in question exist. The statement that is made is, that is to say, merely to the effect that "if *a*, *b* and *c* are points . . . then . . ." So far as the question of existence is concerned, the points *a*, *b* and *c* may have been in existence and ceased to exist, or they may be in existence, or they may be going to exist. There is, in short, no suggestion here that points cannot come out of the region of mere being into the region of existence, and proceed out of the latter into the former. Thus the modern treatment of spatial problems as illustrated by this proposition proceeds along the lines which Philosophy sets forth as those which should be followed.

As a second instance of the fact that the modern treatment of spatial problems is in accordance with the results of philosophical investigation, take the proposition that sets forth a characteristic of the centre of a plane. This proposition states that the centre of every plane is the logical product of straight lines that pass through a point in the plane. Or in symbols $\frown\text{`}u = \iota a$.[1] Now, in this reference to a characteristic of the centre of a plane it is certain that the notion of interaction between points does not occur and is not suggested. Rather, in speaking of the logical product of classes of points, the points that constitute the classes are taken simply as given. Nor is it required that these points have any properties—except that of following one another in series, and in one case that of being the logical product of certain classes. The points can well enough be spoken of: it is not by reason of their properties that this is possible, but by reason of the relations in which they stand to one another and to the points in other classes. And, thirdly, it is quite possible that the plane and the centre of the plane to which reference is being made may not still or yet be in existence, or may exist. The description applies to any plane, whether that plane exists, or whether it no longer exists, or whether it does not yet exist. There

[1] This symbolic statement is Peano's, but it is not strictly correct. The sign of equality signifies that we have here a *definition* of the centre of a plane. What, however, we really have is merely a statement concerning the centre: other points in the plane may mark the intersection of certain lines.

is here, that is to say, nothing which suggests that points may not come out of the region of mere being into that of existence.

Lastly, in order to emphasize the fact that the modern treatment of spatial problems is in accordance with the results of philosophical reflection, we will show that the common method of proof in the case of such problems is *not* in accordance with those results. And to effect this demonstration we cannot do better than take one of the Euclidean proofs that is mentioned by Mr. Russell,[1] namely, the proof of the fourth proposition of the first Book. Here, I think, we must credit Euclid with having started by holding that the points of space *are* absolutely fixed, and with entertaining this view when his demonstration is concluded. But in effecting the demonstration there is no doubt that he departs from this view. He says, "if the triangle ABC be applied to the triangle DEF." To speak in this way is to make the points of space material in character. The points of the triangle ABC are conceived as *moving*, and nothing can be conceived as moving but that which is material. Not perceiving that to regard space as consisting of moving points is to presuppose a space of fixed positions,

[1] *The Principles of Mathematics*, pp. 405–407.

past which material entities can move, and not perceiving that the establishment of the equality of his material triangles is really due to the fact that two absolutely fixed triangles are under certain conditions recognized as equal, Euclid, for the time being, wholly discards the absolute and adopts the relative theory. It is manifest that only unsatisfactory results can be reached by one who thus shifts his ground on the question as to the nature of space: the general perplexity that Euclid has caused his readers is a good example of such results. The modern treatment of spatial problems does *not* endeavour to combine two such theories. It bases itself upon one only, the absolute theory, that which, as we have seen, a careful philosophical investigation leads us to adopt.

INDEX

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