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This book is intended as a sequel to the "First Lessons in Geometry," and, therefore, presupposes some acquaintance with that little treatise. I think it better, however, that some interval should elapse between the study of that book and of this, — during which time the child may be occupied in the study of Arithmetic.

Geometrical facts and conceptions are easier to a child than those of Arithmetic, but arithmetical reasoning is easier than geometrical. The true scientific order in a mathematical education would therefore be, to begin with the facts of Geometry, then take both the facts and reasoning of Arithmetic, and afterwards return to Geometry, not to its facts only, but to its proofs.

The object of "First Lessons in Geometry" is to develop the child's powers of imagination; the object of this book is to develop his powers of reasoning. That book I consider adapted to children from six to twelve years of age, this to children from thirteen to eighteen years old.

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SECOND BOOK IN GEOMETRY.

PART I.

CHAPTER I.

PRELIMINARY.

1. GEOMETRY is the science of form. We really begin to learn Geometry when we first begin to notice the forms of things about us. Some persons observe forms much more closely than others do; partly owing to their natural taste, and partly to their peculiar education. The study of plants, animals, and minerals, the practice of drawing, and the use of building blocks and geometrical puzzles, are good modes of leading one to notice, quickly and accurately, differences of form.

2. The second step in learning Geometry is to become able to imagine perfect forms, without seeing them drawn. The little book called "First Lessons in Geometry" was chiefly designed to help in the attainment of this power. It is filled with descriptions of forms that cannot be exactly drawn. This is especially true of many of the curves, which cannot be drawn so exactly as straight-lined figures and circles, but which we can, with equal ease, imagine perfect.

3. The third step in learning Geometry is to learn to reason about forms, and to prove the truth of the interesting facts which we think that we have observed. This is the only way in which we can become able to find out new truths, and to be certain that they are true. And the first part of this second book is written to teach the scholar how to reason out, or prove, geometrical truths.

4. After learning to reason out or prove geometrical truths, it is pleasant to know how to use them. This is not the only object of Geometry. It is worth while to know a truth, simply because it is true. But it is also pleasant to be able to apply that truth to practical use, for the benefit of our fellow-men. And the second part of this book is written to show in what way we can turn Geometry to practical use.

CHAPTER II.

DEFINITIONS.

5. Geometry is the science of *form*. Every form or shape is, in general, enclosed by a surface; every surface can be imagined as bounded, or else as divided, by lines; and in every line we can imagine an endless number of points.

6. A point is a place without any size. It has a position, but no dimensions; neither length, breadth, nor depth.

7. A line is a place having length, without breadth or depth. As we attempt to mark the position of a point by making a dot with the point of a pen or pencil, and the position of a line by moving the pencil point along the surface of the paper, we find it convenient to speak of a geometrical line as if it were made by the motion of a geometrical point. As the eye runs along the pencil line, so

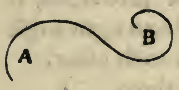
the eye of the mind runs along the geometrical line from point to point.

8. A surface is a place having length and breadth, without depth, that is, without thickness.

9. A solid is a place having length, breadth, and depth. A geometrical solid is not a solid body, but is simply the space that a solid body would occupy, if it were of that shape and in that place. In like manner a geometrical surface is not the surface of a solid body, but simply the surface of a geometrical solid.

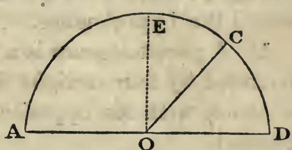
10. A straight line is a line that does not bend in any part. A point moving in it never changes the direction of its motion, unless it reverses its direction.

11. A curve is a line that bends imperceptibly at every point. It must not have any straight portion, nor any corners; that is to say, it must bend at every point, but the bend must be too small, at each place, to be measurable.



12. A plane is a geometrical surface, such that a point, moving in a straight line from any one point in the surface to any other, never leaves the surface. The common name of a plane is "a *flat* surface."

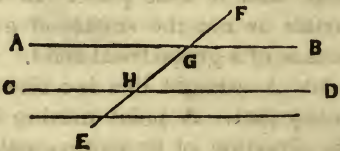
13. An angle is the difference of two directions in one plane. If the line C O should turn around the point O so as to make the arc D C grow longer, the difference of the directions of O C and O D would increase, and we should say that the angle D O C grew larger and larger until the point C arrived at A, so that the two lines O D and O C were opposite in direction.



14. If the point C were carried round half way to the opposite point A, that is, to the point E, the angle D O C

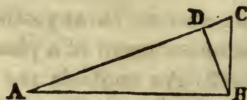
would be a right angle, as D O E is. A right angle is a difference of direction half as great as oppositeness of direction. The difference between an angle and a right angle is called the complement of the angle. The difference between an angle and two right angles is called the supplement of the angle. Thus C O E is the complement of D O C, and C O A is the supplement of D O C.

15. When two lines make no angle with each other, or make two right angles, they are called parallel lines. That is to say, parallel lines are those that lie in the same direction or in opposite directions. When two lines in a plane are not parallel, the point where they cross, or would cross if prolonged, is called the vertex of the angle. Lines making a right angle with each other are called perpendicular to each other.

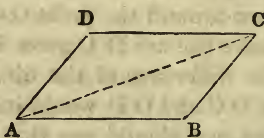


16. A triangle is a figure enclosed by three straight lines in one plane.

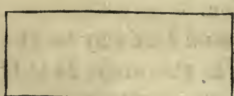
17. A right triangle is a triangle in which two of the sides make a right angle with each other. These sides are then called the legs of the triangle, while the third side is called the hypothenuse.



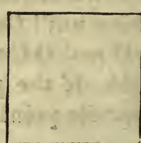
18. A parallelogram is a figure bounded by four straight lines in a plane, with its opposite sides parallel.



19. A rectangle is a parallelogram with its angles all right angles.



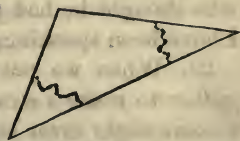
20. A square is a rectangle with its sides all equal.



CHAPTER III.

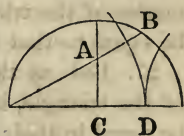
REASONING.

21. Suppose that we wished to make another person believe that the three angles of a triangle are, together, equal to two right angles. One way of convincing him would be to take a triangular piece of card, or of paper, cut off the corners by a waving line, and lay the three corners together, to show him that the outer edges will make a straight line, as two square corners put together will do.



22. Yet he might not be satisfied that the line was perfectly straight. Or perhaps he might say that if the angles of the triangle were in a different proportion, the corners put together would not make a straight line with their outer edges.

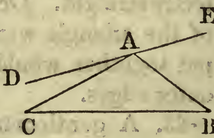
23. A gentleman once came to me and said, "I have found out that if you draw such and such lines, you will always find these two, A B and C D, equal. At least my most careful measurement shows no difference between them." I said to another gentleman, who knew something of Geometry, "Can you prove that these lines will be equal if the figure is drawn exactly as directed?" He said he would try, and in a few days he sent me what he called a proof. But on reading it I found it only amounted to saying that "if the lines are equal, they are equal." I then examined the matter myself, and found that the lines were, in reality, never equal, although the difference was always very small,— too small to be easily discovered by measurement.



24. Such errors, too small to be discovered by measurement, are sometimes large enough to do great mischief; and at any rate, however small, they are still errors, and it is best to get rid of errors, and to find the exact truth, whether the error is mischievous or not. In order to do this we must learn how to reason, how to prove truths. And in order to avoid such mistakes as that of my friend, who thought he had proved the false proposition of which I have been speaking, we must learn to reason correctly.

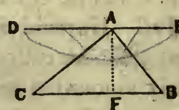
25. When we put the corners of a paper triangle together to make a straight line, we may say, Perhaps there is some slight error here, too small to be detected by measurement. How then shall we prove that there is no such error in a perfect geometrical triangle?

26. The first thought that occurs to us will be, that if any straight line be drawn through one vertex of a triangle, as D E is drawn through the point A, without passing through the triangle, the three angles on one side of the line, about the point A, are equal to two right angles, and if the sum of the three angles of the triangle is equal to two right angles, it must be equal to that of the three about the point A.



27. But as the central angle at A is already an angle of the triangle, it follows that the other two angles must be equal in their sum to the sum of the angles B and C.

28. Now, this will be true in whatever direction the line D E is drawn, only provided it does not pass through the triangle. Let us then imagine it to pass in such a direction as to make the angle B A E equal to the angle A B C, and it will only be necessary to prove that D A C is then equal to A C B. For if D A C is equal to A C B, then, since we have supposed B A E equal to A B C, and B A C is one of the



angles of the triangle, we shall have the three angles about A equal to the three angles of the triangle; and as the three angles about A are equal in their sum to two right angles, the three angles of the triangle will be equal to two right angles, which is what we wish to prove.

29. But to say that D A C is equal to A C B is equivalent to saying that A D and C B differ equally in their direction from the direction of A C; and since A C is a straight line and its direction from A is opposite to its direction from C, this is equivalent to saying that A D and C B go in opposite directions.

30. All that we have now to prove is, that the line A D or E D goes in the same direction as the line B C. But this needs no proof, because we have already supposed that E A makes the same angle with A B that B C does; and as A B is a straight line, the direction of E A and C B must be opposite. But as E A is part of the same straight line with A D, it has the same direction as A D. The proof is now complete.

31. And this mode of proof does not depend at all upon the particular shape of the triangle. We have made no supposition concerning the shape of A B C, except that it should be a triangle. We have, therefore, proved that the sum of the three angles of any triangle is equivalent to two right angles.

32. Thus we have analyzed the proposition that the sum of the three angles of a triangle is equivalent to two right angles, and found that it resolved itself at last into saying that two lines making equal angles on opposite sides at the end of a straight line must point in opposite directions — a proposition which is easily shown to be true.

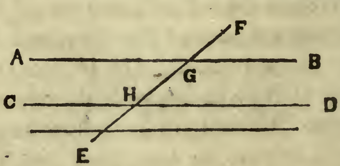
33. But this mode of analyzing is very tedious when stated in words. A geometer usually does not state it; he passes through it very rapidly in his own mind, and then

2 *

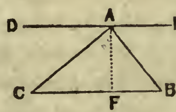
restates the process carefully in an inverted order, as follows in articles 34, 35, and 36.

34. When one straight line crosses another, the opposite or vertical angles are equal. For since each line has but one direction, the difference of direction on one side of the vertex must be the same as on the other side.

35. When one straight line crosses two parallel straight lines, the alternate internal angles are equal, or in the figure A G E is equal to D H F. For D H F is equal to B G F, having its sides pointing in the same direction as those of B G F, and A G E is equal to B G F by article 34.



36. Through the vertex of any triangle, as through A, draw a straight line D E parallel to the opposite side B C. Then E A B will be equal to its alternate internal angle A B C; and for the same reason D A C will be equal to A C B. So that the three angles of the triangle will be equal to the three angles about the point A, and their sum is plainly two right angles.



37. The mode of proving that the sum of the three angles of a triangle is equal to two right angles, by cutting a piece of card, is called experimental proof. It is of very little use in mathematics, but of great use in the study of physics, especially in mechanics and chemistry.

38. The mode of proof used in articles 26–31 is called, by metaphysicians and by writers on Arithmetic, *analysis*. But as geometers, in their writings, almost never use this method, they have no name for it; and when they speak of analysis, or of analytical methods, they usually refer to something else apparently of a very different character.

39. The mode of proof in articles 34–36, called by metaphysicians synthesis, by geometers demonstration or deduction, is that usually employed in stating geometrical results. This mode is chiefly applicable to mathematics, and must be used with very great caution in reasoning upon other subjects.

40. A proposition, which we wish to prove, may be compared to a mountain peak which we wish to show is accessible from the highway. The method of articles 26–31 may be compared to taking a flight by a balloon to the top of the peak, and then finding a path down to the highway; while the method of articles 34–36 may be compared to the direct ascent of the mountain. In either case we show that the peak is accessible, because we actually pass over all the steps of a connected pathway between the road and the mountain top.

Thus in geometrical demonstration we pass through every step connecting the simplest self-evident truths with the highest deductions of the science; while in the process which writers on Arithmetic call analysis, we pass over every step from the latter truths down to the simplest. In either case we prove that the higher truth really stands on the same basis as the simpler, and must, therefore, be true.

CHAPTER IV.

ANALYSIS AND SYNTHESIS.

41. WHAT I have called demonstration or deduction, but which is better called synthesis, because it is a putting together, one by one, of the parts of a complex truth, is the only mode of proof that you will usually find in works on

geometry. And if such works are carefully read they are always intelligible to a child of good geometrical reasoning powers.

42. But the study of such works does not always teach a child to reason for himself. The pupil says, "Yes, I understand all this, and yet I could not have done it without aid; I do not see how the writer knew where to begin; how he knew that by starting from these particular truths, and going in that particular path, he could reach that proposition." A pupil who had never studied geometry could not, for instance, tell why in articles 34–36 we should begin with showing that vertical angles are equal. He would not see any connection between that truth and the desired proof, and would not know that this synthesis had been preceded, in the mind of the writer, by a rapid analysis, such as that of articles 26–31.

43. It is as though a mountain guide, wishing to make for a child a path up to a mountain peak, should lead him along the highway until the peak was hidden by the lower hills about its base, and then begin boldly to clear a road, through the brush-wood and trees, until he reached the top. The child might say. "How did you dare begin at once to cut down the bushes and clear the path? How did you know that the road you were making would not lead you to the edge, or to the foot, of some precipice, or that it would not take you to a different peak from that which you wished to climb?" And if the child received no answer to his questions,—if he was not told that the guide had already climbed to the summit and again descended,—he would have learned little to help him in laying out paths for himself.

44. In like manner, although the descent from difficult propositions to more simple is more tedious than the ascent, it will be more useful to a learner, because it will show him the manner in which, by a mental process, we

discover the points from which we are to start in our ascent. That is to say, if we follow a good analysis, we shall learn how to perform synthesis for ourselves; but if we were simply to follow a writer's synthesis, we should not learn how to analyze, which must nevertheless always go before synthesis.

45. Among the first requisites in reasoning is a clear understanding of the object in view; that is, of the point to be proved; and next, a clear perception of each particular part of the demonstration, and of the connection of each part with the adjacent parts.

Thus, in laying out a path up a mountain, it is necessary to know exactly from what point you wish to start, and to what point you wish to go. It is also necessary to examine carefully each point of the road, for a single impassable place would destroy the value of the whole road.

46. Each step of the proof must be a simple step, and clearly true; that is, it must be so simple and self-evident as to be beyond all doubt.

47. The analysis must end, or the synthesis begin, with truths that are self-evident, or else that have been already proved. Your mountain path must begin on level, or at least on accessible ground.

48. Care must be taken not to introduce any thing as true which has not been proved. This would be like starting your mountain road in two places at once. You might afterwards find impassable barriers between the two parts of your road, and perhaps find that one of them could not be made to the top of the mountain, nor the other to its base. For example, in Art. 36, I drew a straight line through A, parallel to B C. This was very well — for no one can possibly doubt such a line might be drawn. But if, instead of that, I had said, Let us draw a straight line through A in such a manner as to make the angles on the two sides of A equal to the angles B and C, I should

have done what I had no right to do. For that would be taking for granted a thing which I must prove; namely, that a straight line can be thus drawn. It would be starting half way up my mountain, and taking for granted that the lower part of the path could be built afterwards. It would require a straight line to fulfil two conditions at once, without having shown that one condition does not exclude the other.

49. Whether we reason by synthesis or analysis, we must therefore reason very carefully, in order to connect the proposition which we wish to prove by a stairway of self-evident steps with a self-evident foundation.

50. By a self-evident truth, I mean a truth which cannot be made any plainer, and which is already perfectly plain to an intelligent person who looks steadily at it. For instance, that two straight lines can cross each other only in one place at once; that any curve can be cut by a straight line in at least two places; that either side of a triangle is shorter than the sum of the other two; that if three strings, and no more than three, come from one point, one of them must have an end at that point,— these are self-evident truths.

51. By a self-evident step in reasoning, I mean the statement of the relation of one truth to another, or of the dependence of one truth upon another, when that dependence or that relation is itself a self-evident truth. Self-evident steps in reasoning are simply the statement of self-evident truths of connection. For instance, when we have explained the meaning of "a straight line" by calling it a line that has in every part the same direction, and have explainedthe word "angle" to mean the difference of two directions in one plane, then it follows that the angle which two straight lines make with each other is the same in one part of the lines as in any other, and that the two different angles apparently made by two straight lines

cannot really be made, unless one of the lines goes in two opposite directions at the same time. No reasoning can make the connection between these definitions and the equality of vertical angles any more plain. It is a self-evident connection.

52. Or, suppose that we say that you cannot make one rope go from a centre post to the four corners of a square, and also around the square, and have but a single rope from post to post. We should prove it in this way. Let there be a rope around the square, and going also from each post to the centre. This of course can be imagined. It is a definite and allowable conception. But we will prove that

this rope must be in two pieces. For each of the four corners will have three lines coming from it, one towards each adjacent corner, and one towards the centre. Thus it follows by self-evident connection, from the conception of the rope going around the square and to each corner, that there will be four points, from each of which three lines come. But it is a self-evident truth that at each of these points there must be one end of a rope. Hence, by self-evident connection, there will be four ends of rope about the square. Hence, by self-evident connection with the self-evident truth that one piece of rope can have but two, and must have two, ends, it follows that there must be two pieces of rope, and cannot be only one. Now, the whole of this proof is simply the statement of self-evident connections between the proposition that one rope cannot go around a square and also from each corner to the centre without doubling, and the self-evident truths that a piece of rope must have two, and cannot have more than two, ends; and that when only three lines of rope come from one point, one of them must end at that point. The proof is simple; and yet intelligent men have spent hours in experimenting with a string and five posts thus arranged, or with a pencil and five dots representing posts.

53. Many self-evident truths are general, and self-evident steps are generally the recognition of general relations; and therefore most writers on reasoning say that reasoning consists simply in showing that a particular case comes under a general class, that is, that the only self-evident connection of propositions is the actual inclusion of one proposition in another. But in the mathematics, there are many self-evident truths which it is difficult to state in a general form; and I therefore think that the explanation which I have given of the process of reasoning will be of more use to you in your geometrical studies.

CHAPTER V.

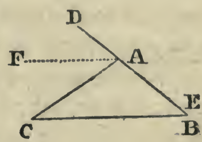
VARIETY OF PATHS.

54. As there are usually many paths by which we may ascend a hill, so there are usually many modes by which we may demonstrate a proposition. In the case of a simple proposition, it is not usually worth while to try more than one mode. But with more difficult problems, it is sometimes worth while to spend a great deal of labor in discovering the simplest mode of demonstration. There are geometrical truths which can be demonstrated in so simple a manner as to require only twenty lines to write down the demonstration; and yet some writers, from ignorance of this simple mode, have written more than twenty pages to prove the same truths.

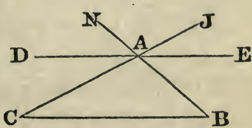
55. It will therefore be useful to you, to show you, by a simple example, such as that of the equality of the sum of the angles in a triangle to two right angles, the great variety of methods by which a single proposition can be proved.

56. In the proofs of this proposition, which I will now give you, I will not be careful to follow out every step. It will be enough, *for the purpose we now have in view*, simply to show you the general line of the paths, without taking you through every step of the way.

57. The line D E might be drawn so as to coincide in direction with one of the other sides of the triangle, as A B, which would give us the figure in the margin. And by imagining the dotted line A F parallel to C B, we should have F A C equal to A C B, and F A D equal to C B A, which would make the three angles at A equal to the three angles of the triangle, as in the former proof. (See Chapter III.)

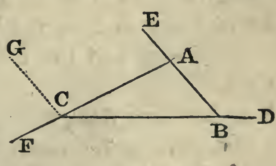


58. By prolonging the lines C A and B A through the point A, D E being parallel to B C, we should have N A D equal to A B C, J A E equal to A C B, and N A J equal to C A B. So that the three angles of the triangle will be equal to the three angles on the upper side of the line D E, which are manifestly equal to two right angles.



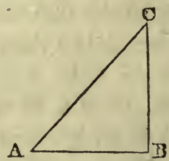
59. The three methods of proving this proposition that I have now given, are strictly geometrical. Others might be given, that are something more like algebraic reasoning.

60. Let us, for instance, imagine each side of a triangle prolonged at its right hand end, as in this figure, and also a line drawn from one vertex, as C, parallel to the opposite side B A. Now, the external angles F C B, D B A, E A F, are plainly equal to the three angles F C B, B C G, G C F, and these amount to four right angles. But each external angle is plainly



the supplement of one of the angles of the triangle; that is, it is equal to the difference between two right angles and one angle of the triangle. The sum of the three external angles must therefore be equal to the difference between six right angles and the angles of the triangle. But as this difference is four right angles, the three angles of the triangle must be equivalent to two right angles.

61. If we introduce the idea of motion, we can devise quite a different sort of demonstration. Suppose, for instance, that I stand at the point A, with my face towards C. Let me now turn to the right until I face towards B. I have now changed the direction of my face by an amount which is equal to the angle at A. Suppose that I now walk to B, without turning; I shall have my back towards A, and if standing still, I turn to the right, until I have changed my direction by an amount equal to the angle A B C; I shall have my back towards C. Let me now walk backward without turning, until I reach C, and I shall have my face towards B. I will now turn a third time to the right, until I face the point A. My three turnings, or changes of direction, have been equal to the three angles of the triangle; they have all been to the right; therefore my whole change of direction is equal to the sum of these angles; I am now looking in exactly the opposite direction to that from which I started; I am looking from C to A, instead of from A to C; I have turned half way round; that is, through two right angles. Whence, the sum of the three angles of the triangle is equivalent to two right angles.



62. Another demonstration, by means of motion, may be obtained as follows: Suppose an arrow, longer than either side of the triangle, to be laid upon the side A C, pointing in the direction from A to C. Taking hold of the pointed end beyond C, turn the arrow round upon the

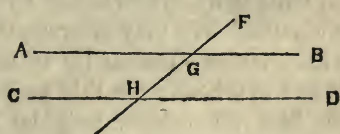
point A, as a pivot, until the arrow lies upon the line A B. Taking now hold of the further end, beyond A, turn the arrow upon B as a pivot, until the arrow lies upon the line B C. Using C as a pivot, turn it now until the point of the arrow is over A. The arrow has thus been reversed in direction, turned half way round, or through two right angles. It has been turned successively through the three angles of a triangle, and every time in the same direction, like the hands of a watch; so that its total change of direction, two right angles, is equivalent to the sum of the three angles.

63. You have thus seen how a single proposition may be proved in a variety of ways. We have shown what is the value of the sum of the angles in a triangle in six different ways; in three, by what is called rigid geometry; in one, by a partly algebraical process; and in two, by introducing the idea of motion. And I wish you to observe, that every one of the six ways is satisfactory. They are all proofs that are certain, because they lead you from self-evident truths by self-evident steps. One is not more certain than the other, because they are all absolutely certain. The only choice between them is, that some are more purely geometrical; some are better adapted to the peculiar tastes of different students; and some are neater, and more quickly perceived by untaught persons.

Examples.

By aid of the principles and methods of the five preceding chapters, the learner may demonstrate (sometimes in a variety of ways) the following simple propositions: —

I. When parallel lines are crossed by a third, the *external - internal* angles are equal; that is, F G B = G H D, &c.



II. If two lines, cut by a third, make the alternate-internal, the external-internal or the opposite-external angles equal, the lines are parallel.

III. If two lines, crossed by a third, make the adjacent internal angles (as B G H, D H G) supplements to each other, the lines are parallel.

IV. If two lines make the same angle with a third, they are parallel to each other.

V. State this proposition for the cases when the angle is zero, one right angle, and two right angles.

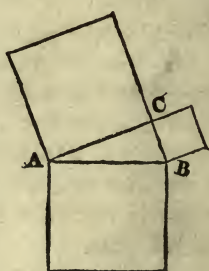
VI. Parallel lines can never meet. [NOTE. To prove a negative of this kind, the easiest mode is to show the absurdity of the affirmative. In the present case, grant that the lines met at a certain point, and show from the nature of the straight line, that the parallel lines must in this case be one line, which is absurd.]

VII. Only one perpendicular can pass through a given point to a given straight line. [Proof by VI.]

CHAPTER VI.

THE PYTHAGOREAN PROPOSITION.

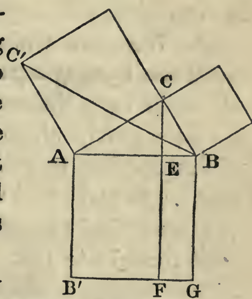
64. YOU recollect that the square built on the hypothenuse of a right triangle is equivalent in its area to the sum of the squares built upon its legs. This is one of the most useful of all geometrical truths. Let us first analyze it in one or two modes, and then build it up synthetically by the same paths. We may afterwards, if we like, devise other modes of analysis and synthesis; for this proposition, like all others, may be approached in various ways.



65. The Pythagorean proposition or theorem might be suggested in different ways. But in whatever way we were led to suspect that the square on the hypothenuse is equivalent to the sum of the squares on the legs, we should, in reflecting upon it, probably begin by drawing a right triangle with a square built upon each side.

66. We should inquire whether the square on the hypothenuse could be divided into two parts that should be respectively equal to the other two squares. And we should judge that these parts should be somewhat similar to each other in shape, because the legs do not differ in their relations to the hypothenuse, except in size, and in the angles they make with it.

67. But we cannot readily conceive of any division of the square into two somewhat similar parts, except into two rectangles. And then it is apparent that two rectangles, bearing respectively the same relations to the squares on the legs, may be formed by drawing a line from the vertex of the right angle at right angles with the hypothenuse, and continuing it through the square, as C F is here drawn.



68. It will now only be necessary to show that one of these rectangles is equivalent to its corresponding square; because the same mode of proof will obviously answer for the other rectangle and its square.

69. Now, if we know, or can prove, that the area of a rectangle is measured by the product of its sides, we shall have to prove that A E $\times$ A B′, or A E $\times$ A B, is equivalent to A C $\times$ A C′.

70. But by the doctrine of proportion it may be shown that this would be equivalent to saying that A E is to A C as A C is to A B.

71. Again, it may be shown by geometry that this proportion between the lines A B, A C, and A E, would be true if the triangle A E C were similar to A C B, and that A E stood in one to A C as A C stood to A B in the other; so that all that remains for us to do is to show that these triangles are similar.

72. But we can show by geometry that two triangles are similar when their angles are equal.

73. And it is easy to show that the angles of these triangles are equal to each other.

74. For C A B and C A E are the same angle; A C B and A E C are both right angles; and therefore A B C and A C E are each complements of C A E. Moreover, A C and A E are situated in the triangle A E C, in the same manner that A B and A C are situated in the triangle A B C.

75. We have thus, in articles 66–74, sufficiently analyzed the Pythagorean proposition to enable us to build it up again in a deductive form. This analysis, however, has been partly algebraical, as it has introduced the idea of multiplying two lines to produce a surface. Let us now begin and build up the proposition by the same road. We shall find 31 articles necessary; and I will number them from 76 to 106.

First Proof of the Pythagorean Proposition.

76. *Definition.* The comparative size of two quantities is called their ratio; thus, if one is twice as large as the other, they are said to be in the same ratio as that of 2 to 1; or to be in the ratio 2 to 1; or it is said, in a looser way, that their ratio equals 2.

77. *Notation.* Ratio is written by means of the marks :, ÷, and by writing one quantity over the other. Thus, $A : B$, $A \div B$, and $\frac{A}{B}$, are each used to signify the ratio

of A to B. These marks are the same as those used in arithmetic to signify Quotient, because the meaning of a quotient is "a number having the same ratio to 1 that the dividend has to the divisor." The ratio of A to B is not the quotient of A divided by B, but it is the ratio of that quotient to unity.

78. *Axiom.* If each of two quantities is multiplied or divided by the same number, the ratio of the products or quotients will be the same as that of the quantities themselves. Thus twenty inches is in the same ratio to twenty rods as one inch to one rod, or as the twentieth of an inch to the twentieth of a rod.

79. *Definition.* A proportion is the equality of two ratios. Thus (if we use the sign $=$ to signify "is equal to") $A : B = C : D$ is the statement of a proportion. It signifies that A is in the same ratio to B that C is to D.

80. *Definition.* When a proportion is written as in article 79, the first and last terms, that is, A and D, are called the extremes, and the others, that is, B and C, are called the means.

81. *Theorem.* In every proportion the product of the means is equal to that of the extremes. *Proof.* In any proportion, as $M : N = P : Q$, we wish to prove (using the mark $\times$ to signify "multiplied by") that $M \times Q = N \times P$. Now, in order to do this, we must use only self-evident truths. The only truth of this character that we have given above is that of article 78. But in order, by means of the multiplications of article 78, to change the first ratio $M : N$ into $M \times Q$, we must, whatever else we do, at least multiply each term by Q, and this will give us $M \times Q : N \times Q = P : Q$; and in order to change the second ratio $P : Q$ into $N \times P$, we must, at all events, multiply each term by N, and this will give us $M \times Q : N \times Q = N \times P : N \times Q$.

Thus, from the self-evident truth of article 78, we find that the product of the means bears the same ratio to the product $N \times Q$ that is borne to it by the product of the extremes. And as it is self-evident that two quantities, bearing the same ratio to a third, must be equal to each other, we have proved that the product of the means is equal to that of the extremes.

82. *Definition.* When both the means are the same quantity, that quantity is called a mean proportional between the extremes.

83. *Corollary.* It follows from article 81, that the product of the mean proportional multiplied by itself is equal to the product of the extremes.

84. *Definitions.* A unit of length is a line taken as a standard of comparison for lengths. Thus an inch, a foot, a pace, a span, &c., are units. The *length* of any line is its ratio to the unit of length.

85. *Definition.* A unit of surface is a surface taken as a standard of comparison. The most common unit of surface is a square whose side is a unit of length.

86. *Definition.* The area of a surface is the ratio of the surface to the unit of surface.

87. *Theorem.* Any straight line in the same plane with two parallel lines makes the same angle with one that it does with the other. *Proof.* For as the straight line has but one direction, and each of the parallel lines may always be considered as going in the same direction as the other, the difference of that direction from the direction of the third straight line must be the same for each of the parallel lines.

88. *Corollary.* If a straight line is parallel to one of two parallel lines, it is parallel to the other; if at right angles to one of the two, it is at right angles to the other.

89. *Theorem.* If a straight line makes on the same side of itself the same angle with two other straight lines in the same plane, those other straight lines must be parallel.

Scholium. The line must not be conceived as reversing its direction at any point. *Proof.* For if two directions differ equally from a third, they must be equal to each other.

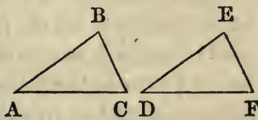
Second Scholium. If the straight line reverses its direction between the other lines, and makes equal angles with them, it shows that it crosses each at an equal distance from their point of mutual intersection.

90. *Axiom.* If the boundaries of one plane surface are similar to those of another in such a way that the two surfaces would coincide in extent if laid one upon the other, the two surfaces are equivalent.

CHAPTER VII.

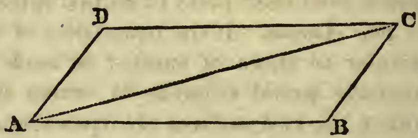
THE PYTHAGOREAN PROPOSITION CONTINUED

91. *Theorem.* If a triangle has one side and the adjacent angles equal respectively to a side and the adjacent angles in another triangle, the two triangles are equal. *Proof.* Let us suppose that, in the triangles A B C and D E F, we have the side A B equal to the side D E, the angle at A equal to the angle at D, and that at B equal to that at E.



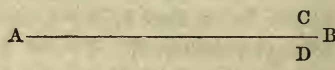
Let us imagine the triangle D E F to be laid upon A B C in such a manner as to place E upon B, and D upon A, which can be done, because A B is equal to D E. Now, as the angle A is equal to D, the line D F will run in the same direction as A C, and, as it starts from the same point, will coincide with it. Also, since the angle B is equal to E, the line E F will coincide with B C. The point (F) of intersection of D F and E F must therefore coincide with C, the point of intersection of A C and B C. Whence, by article 90, the triangles are equal.

92. *Theorem.* The opposite sides of a parallelogram are equal. *Proof.* Article 90 gives us the only test of geometrical equality. So that, in order to prove this theorem, we must show that in a parallelogram like A B C D, A B may be made to coincide with D C, and B C with A D. And this would evidently be done if we could show that



the triangle A B C is equal to A D C. But in these triangles the line A C is the same, and by article 87 the adjacent angles A C B and C A B are equal to the adjacent angles C A D and A C D; whence, by article 91, the two triangles are equal, and A D is equal to B C, and A B equal to D C.

93. *Axiom.* If one end of a straight line stands still while the other turns round, the end that moves will *begin* to move in a direction at right angles to that of the line itself. Thus if A B were to begin to turn about the point A, B would *begin* to move either towards C or towards D. [If this proposition is not acknowledged as an axiom, the proof is in Ex. XIX. at the close of the chapter.]

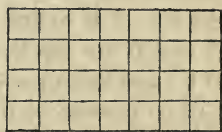


94. *Theorem.* The angles of a triangle cannot be altered without altering the length of the sides. *Proof.* If in any triangle, as A B C, the sides were unchangeable, any alteration of the angles A and B would, by article 93, make the point C move in two directions at once, (namely, at right angles to A C, and at right angles to B C,) which is impossible, and therefore the angles cannot be altered.

95. *Corollary.* If the three sides of a triangle are respectively equal to the three sides of another triangle, the angles of one must be equal to those of the other, and the equal angles are enclosed in the equal sides.

96. *Theorem.* If the opposite sides of a quadrangle are equal, the quadrangle is a parallelogram. *Proof.* If, in the quadrangle A B C D, the sides A B and C D are equal, and also the sides A D and B C are equal, then, by drawing the diagonal A C, we have the triangles A B C and A D C composed of equal sides, and, by article 95, the angle D A C must be equal to the angle A C B, and the angle D C A to the angle B A C; whence, by article 89, the figure is a parallelogram.

97. *Theorem.* The area of a rectangle is the product of its length by its breadth. *Proof.* By drawing lines, at a distance apart equal to the unit of length, parallel to the sides of the rectangle, we shall (articles 87–89) divide the rectangle into little squares, each of which is a unit of surface. Moreover, these squares

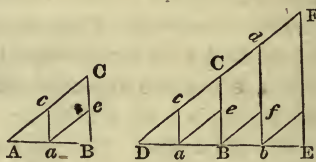


are arranged in as many rows as there are units of length in one side of the rectangle, each row containing as many squares as there are units of length in the other side; so that the whole number of squares is found by multiplying the length of the rectangle by its breadth.

98. *Scholium.* In the above proof it is taken for granted that the sides of the rectangle can be divided into units of length. This can usually be done by taking the units sufficiently short, that is to say, if the lines are not an even number of inches in length, we may take tenths of an inch as the unit; if they are not even tenths, we can divide them into hundredths, or thousandths, or even millionths, of an inch. If, after dividing each line into millionths of an inch, any thing less than the millionth of an inch were left at

either end, it would be too small to be taken into consideration. There would be no error, even in reasoning, from neglecting it. For as long as any thing is left at the ends of the lines, we can choose smaller units; but as long as the units are of any size at all, our reasoning holds good, and the rectangle is measured by the product of its dimensions.

99. *Theorem.* If the angles of one triangle are equal to those of another triangle, any two sides of one of the triangles have the same ratio to each other as that of the two sides including the same angle in the other triangle. *Proof.* Let the triangles A B C and D E F be equiangular with respect to each other. Place the vertex A upon the vertex D, and allow the side A B to fall upon the side D E. Since the angles A and D are equal, the line A C will fall upon the line D F; and since the angles C and F are equal, the line B C will lie parallel to the line E F.



Let the sides A B and D E be divided into units of length, A a, a B, B b, &c. Through the points of division draw lines a c, b d, &c., parallel to E F. Draw also the lines a e, B f, &c., parallel to D F. By article 91, the triangles A a c, a B e, &c., are equal. By article 92, a e is equal to c C, B f to C d, &c. Hence it is easy to see that A B is composed of the same number of times A a, that A C is of A c, and that in like manner D E is as many times D a, as D F is times D c. And thus, by article 78, D E : D F = A B : D C, because each of these ratios is equal to B a : a e.

100. *Scholium.* If the lines A B and D E do not consist of a certain number of times the first unit of length which we have chosen, we may choose a unit so small as to make the remainder small enough to be neglected.

101. *Definitions.* The right angle, right triangle, legs, and hypothenuse are defined in articles 14 and 17.

102. *Theorem.* The sum of the three angles of a triangle is equivalent to two right angles.

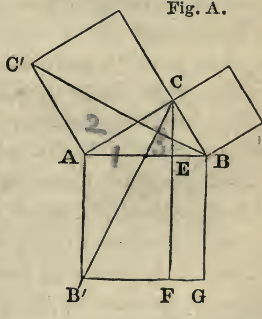
This proposition has been proved in articles 26–31, 34–36, and 57–62.

103. *Corollary.* The sum of the two angles opposite to the legs of a right triangle is equivalent to one right angle.

104. *Corollary.* If an angle opposite a leg in one right-triangle is equal to an angle in another right triangle, the two right triangles are equiangular with respect to each other.

105. *Theorem.* If from the vertex of the right angle in a right triangle, a line be drawn at right angles to the hypothenuse dividing the hypothenuse into two segments, each leg is a mean proportional between the whole hypothenuse and the segment nearest the leg. *Proof.* Let A B C be a right triangle with a right angle at C. Draw C F at right angles to A B. The triangle B E C is right angled at E, and has an angle at B equal that at B in the triangle A B C. Hence, by article 104, the triangle B E C has its angles equal to those of A B C. Hence, by article 99, B E : B C = B C : B A. In the same way A E : A C : : A C : A B.

Fig. A.



106. *Theorem.* The square on the hypothenuse is equivalent to the sum of the squares on the legs. *Proof.* Let A C B be a right triangle, with a right angle at C, and let a square be drawn on each side. Draw C F at right angles

to A B. The figure B F will be a rectangle, because all its angles will be right angles. It will, therefore, be measured by the product of B E into E F, or (since E F = B G, and B G = B A) by the product of B E × B A. But since B C is a mean proportional between the lines B E and B A, this product is equal to B C × B C, which is the measure of the square on B C. That is, the measure of the rectangle B F is the same as that of the square on B C. In the same manner it may be shown that the rectangle A F is equivalent to the square on A C. But the sum of these two rectangles is evidently equal to the square on the hypothenuse.

107. In these thirty-one articles I have given you a proof of the Pythagorean proposition in the usual synthetic form. Parts of the proof are not completely filled out; but the omitted steps are so short and easy, that I think you will have no difficulty whatever in supplying them. Do not be satisfied with understanding each of the thirty-one articles, but examine them closely from the 76th to the 106th, and see whether I have introduced any thing which is not necessary to the proof of 106. In making this examination, it will be most convenient for you to proceed backward.

These thirty-one articles have been here introduced as lemmas, i. e., preparatory propositions, for demonstrating the Pythagorean proposition. But they are also, each one, truths worth knowing, and will aid in establishing many theorems that have no connection with a right triangle.

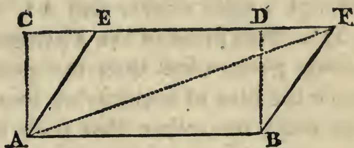
108. Another mode of analyzing this proposition would be suggested by our knowledge of the fact that any triangle is equivalent to half a rectangle of the same base and altitude. I will not lead you through this analysis, but will simply build up for you, by synthesis, a

Second Proof of the Pythagorean Proposition.

109. *Definitions.* Any side of a triangle or quadrangle may be called its base, and the altitude of the figure is the distance from the base to the most distant vertex of the figure. This distance is measured by a straight line at right angles to the base, and contained between the vertex and the base, prolonged if need be.

110. *Theorem.* Every parallelogram is equivalent to a rectangle of the same base and altitude. *Proof.* Let A B C D be a rectangle, and A B E F a parallelogram having the same base A B, and the same altitude B D. It is manifest if the triangle B D F by which the parallelogram overlaps the rectangle is equal to the triangle A E C by which the rectangle overlaps the parallelogram, the two quadrangles are equivalent. But A E and its adjacent angles are equal to B F and its adjacent angles, and therefore the triangles are equal (Art. 91), and the quadrangles equivalent.

Fig. C.



111. *Theorem.* Every triangle is equivalent to half a rectangle of the same base and altitude. *Proof.* Let A F B be a triangle, and A B C D a rectangle having the same base A B and the same altitude B D, (Fig. C.) Continue C D to F, and draw A E parallel to B F. The triangle A E F has its three sides equal to those of A B F; the triangles are, therefore, equal to each other (Art. 95); and each is equal to half the parallelogram A B E F, which is equivalent to the rectangle A B C D.

112. *Theorem.* The square on the hypothenuse is equivalent to the sum of the squares on the legs. *Proof.*

Having drawn the figure (Fig. A.), as for the former proof, draw the lines C′ B, and B′ C. The triangle A B′ C has the same base A B′, and the same altitude A E as the rectangle A F, and is equivalent to half that rectangle. The triangle A B C′ has the same base A C′ and the same altitude A C as the square C′ C, and is equivalent to half that square; so that if the triangles A B C′ and A B′ C are equal, the rectangle is equivalent to the square. But these triangles are equal, for if A B′ C were turned about the vertex A as on a pivot until the point C covered C′, then B′ would cover B, and the triangles would coincide. For A C would rotate through a right angle, and A B′ through a right angle; and A C = A C′, and A B = A B′.

113. This proof of the Pythagorean proposition is more strictly geometrical than the preceding, as it does not involve the idea of multiplying lines to measure areas. But you must remember that both are equally conclusive. I have here also omitted some of the shorter steps. You should not only be able to fill out these steps when the omission is pointed out to you, but also to discover the omission for yourselves. Take the proofs which I have written down and examine them step by step, asking at each step, Is that strictly self-evident? Can it be questioned? Can it be divided into two steps? Is there need of proof? If so, has the proof been given in a previous article? It is only by such an earnest study of the book and of the subject that you can make the process of mathematical reasoning become a sure and pleasant road for you to the discovery of truth.

Examples.

Demonstrate the theorems that follow.

VIII. If one triangle have two sides and the included angle, equal to two sides and the included angle in another triangle, the two triangles are equal.

IX. Lines drawn from a point in a perpendicular to points at equal distances from its foot are equal.

X. The perpendicular is the shortest line from a point to a straight line. [This may be proved from the Pythagorean proposition.]

XI. An isosceles triangle is one in which two sides are equal. Prove by VIII. (having first drawn one line in the triangle) that the angles opposite the equal sides are equal.

XII. An equilateral triangle is equiangular.

XIII. A line bisecting (dividing equally) the angle between two equal sides in a triangle, is perpendicular to the third side, and bisects it.

XIV. Two angles in a triangle being equal, the opposite sides are equal. [Use Art. 91, looking at the triangle from both sides of its plane, or conceiving it turned over.]

XV. An equiangular triangle is equilateral.

XVI. If one side of a triangle is prolonged, the external angle thus formed is equal to the sum of the opposite interior angles, and so greater than either of them.

XVII. If one side of a triangle is longer than another, the angle opposite the first is greater than that opposite the second. [Let A B C be the triangle, and A C be longer than A B. Put the point D on A C, making A D = A B. Join B D. Now prove XVII. by means of XI. and the last clause of XVI.]

XVIII. In an isosceles triangle either of the equal angles is the complement of half the third angle.

XIX. If a straight line, A B, revolves about the point A, the point B moves at right angles to B A. [Allow B to have moved, complete the triangle, apply XVIII., and then suppose the distance moved to be infinitesimal.]

XX. The square on the diagonal of a square is double that square.

XXI. If the four angles of a quadrangle consist of two pairs of opposite equal angles, the quadrangle is a parallelogram.

XXII. The sum of the angles of a quadrangle is equivalent to four right angles; of a pentagon, to six; of a heptagon, to ten.

XXIII. The perpendicular on the hypothenuse from the opposite vertex is a mean proportional between the segments of the hypothenuse.

XXIV. The areas of triangles having the same base are proportional to their altitudes; that is, have the ratio of their altitudes.

CHAPTER VIII.

THE MAXIMUM AREA.

114. I WILL prove only one more proposition; but I will select a difficult one, in order that it may require a number of preliminary proofs. I will select the proposition given in the "First Lessons in Geometry," Chap. XXIII. § 14: Of all isoperimetrical figures the circle is the very largest.

115. When we attempt to analyze this, we shall see that it implies that any regular polygon is less than a circle isoperimetrical with it, and that any other polygon is less than a regular one isoperimetrical with it.

116. Let us begin, however, by defining a few of the words we shall need to use.

117. A polygon is a plane figure bounded by straight lines.

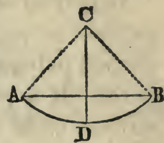
118. The perimeter of a polygon is the sum of the length of its sides.

119. Isoperimetrical polygons are those of equal perimeter.

120. Among quantities of the same kind, the largest is called a maximum.

121. A circle is a plane figure, bounded by one line that curves equally in every part. This line is called the circumference of the circle, and frequently the line itself is called the circle. Portions of the circumference are called arcs.

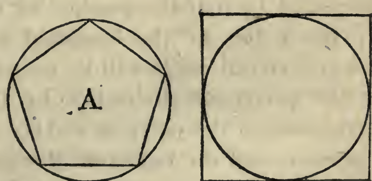
122. *Theorem.* There is a point within the circle equally distant from every point of the circumference. This point is called the centre of the circle.



Proof. Let A D and B D be equal adjacent arcs in a circumference. Through the points A, B, and D draw lines at right angles to the curve at those points. Now, since the circle curves uniformly at every point, B D is in all respects equal to D A. If the figure B C D were laid upon D C A the arcs would coincide; and also, D C would go in the direction of A C, while B C would go in the present direction of D C. The figures A C D and D C B would thus coincide, and A C = D C = B C. Hence the points A, B, and D are equally distant from C. But A and B may be taken any where in the circle, only provided they are equally distant from D; and hence every point in the circle is equally distant from C, the centre of the circle.

123. A straight line joining the centre to the circumference is called a radius. The straight line formed of two opposite radii is called a diameter.

124. All radii are of course equal to each other, and all diameters equal to each other.



125. A straight line joining the two ends of an arc is called a chord.

126. A straight line which, however much prolonged, touches the circle in one point only, is called a tangent.

127. It is manifest that the tangent coincides in direction with the arc at the point of contact.

128. A polygon formed wholly of chords in a circle is said to be inscribed in that circle.

129. A polygon formed wholly by tangents to a circle is said to be circumscribed about the circle.

130. The circle is said to be inscribed in the circumscribed polygon, and to be circumscribed about the inscribed polygon.

131. If a polygon, about which a circle can be circumscribed, or in which a circle can be inscribed, has its sides equal, one to the other, the polygon is called a regular polygon, and the centre of these circles is called also the centre of the polygon.

132. Let us now attempt to analyze the proposition that the circle is the maximum among isoperimetrical polygons. This is equivalent to saying that if an isoperimetrical circle and regular polygon are laid one over the other, the polygon will be the smaller. But we see that by laying them one on the other, a circle inscribed in the polygon would be smaller than the isoperimetrical circle. The question of course suggests itself, whether the area of a regular polygon is not proportional to the radius of the inscribed circle. Now, it is plain that it is. For by dividing the polygon into triangles, by lines from its centre to its vertices, we find the area of the polygon will be the sum of the areas of the triangles, and these areas will be measured by half the product of the perimeter multiplied by the radius of the inscribed circle. The area of the isoperimetrical circle will be measured by half the product of the perimeter multiplied by the radius. But as the perimeters of the polygon and the isoperimetrical circle are the same, and the radius of the inscribed circle is smaller than that of the isoperimetrical circle, it is evident that the area of the polygon is smaller than that of the isoperimetrical circle.

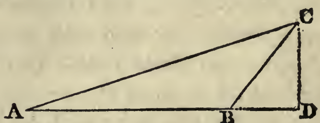
It will now remain to show that the area of a regular polygon is greater than that of an isoperimetrical irregular polygon. It is evident that this can be done, since a polygon of given sides is manifestly largest when most nearly circular, and a polygon of a given number of sides is manifestly largest when the sides are equal. We can surely have no difficulty in proving these two points, and then our proof will be complete.

133. Let us return, then, to the synthetic mode, and establish these propositions: First, that the maximum of polygons formed of given sides may be inscribed in a circle; secondly, that the maximum of isoperimetrical polygons having a given number of sides has its sides equal; and thirdly, that such a regular polygon is of smaller area than a circle isoperimetrical with it.

134. *Theorem.* The area of a triangle is found by multiplying the base by half the altitude. This theorem has been already proved. (Art. 111.)

135. We shall need the Pythagorean proposition, which implies all the propositions into which we have already analyzed it. (Arts. 64–113.)

136. *Theorem.* Of two unequal lines, from a point to a third straight line, the shorter is more nearly perpendicular to the third line. *Proof.* Let C be the given point, and A D the third straight line. Let C A and C B be two lines, of which C B is the shorter. Draw C D perpendicular to A D. We wish to prove that B D is shorter than A D. But this is manifest from the Pythagorean proposition, since the square on A D is the difference of the squares on A C and C D, and the square on B D is the difference of the squares on B C (which is smaller than A C) and the same C D.



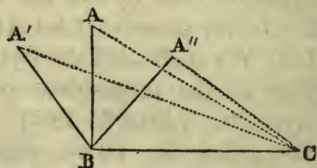
137. *Corollary.* A perpendicular is the shortest line from a point to a given straight line.

138. *Theorem.* The radius is perpendicular to the tangent at its extremity. *Proof.* For if not, then, by Art. 137, the tangent would pass inside the circle, which is contrary to its definition.

139. *Corollary.* The radius is perpendicular to the arc at its extremity.

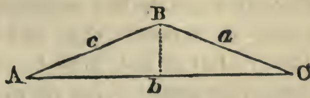
140. *Theorem.* Either side of a triangle is shorter than the sum of the other two. *Proof.* Upon either side, prolonged if necessary, drop from the opposite vertex a perpendicular. The sum of the distances from the foot of this perpendicular to the adjoining vertices cannot be less than the whole of the selected side, but must, by 137, be less than the sum of the other two. *Another proof.* The straight line is the shortest line between its extremities.

141. *Theorem.* The maximum of triangles having two sides given is formed when these two sides are at right angles. *Proof.* Let A′ B, A B, and A″ B be equal to each other. The area of A′ B C, A B C, or A″ B C, being found by multiplying B C into half the perpendicular height of A, A′, or A″, above B C, will be in proportion to that height. Let, then, A B be perpendicular to B C, and the height of A above the base will equal B A. But the height of A″ above the base must, by 137, be less than B A″, which is equal to B A.



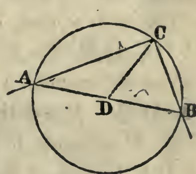
142. An angle is said to be measured by an arc of a circle such as would be intercepted by radii making that angle with each other. And, since the circumference curves equally in all parts, and the radii are at right angles to it, it is evident that this measure is just, and that the angle will bear the same ratio to four right angles that the arc bears to a whole circumference, whatever be the size of the circle.

143. *Theorem.* If two sides in a triangle are equal, the angles opposite those sides are equal. *Proof.* Let A B and B C be equal sides in a triangle. Imagine A C divided in the centre, at the point *b*. The triangles A B *b* and C B *b* will now be composed of equal sides, and we have already proved (Arts. 91–95) that they must have equal angles; that is, the angle at A is equal to that at C.



144. *Theorem.* If one side of a triangle is prolonged, the external angle is equal to the sum of the opposite internal angles. This has been proved in Art. 57.

145. Two chords starting from one point in a circumference intercept double the arc that would be intercepted by radii making the same angle; that is, the angle of the chords is measured by half the arc included between them. *Proof.* If one chord, as A B, passes through the centre D of the circle, it is plain that by drawing D C the angle C D B will be equal to the sum of the angles C A D and D C A. But since D A and D C are equal, these angles are equal, and C D B is equal to twice C A D.

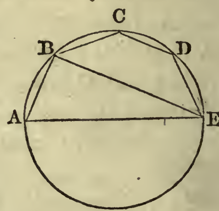


If neither chord passes through the centre of the circle, we can draw a third chord, starting from A, passing through the centre of the circle, and apply this reasoning to the two angles formed with this third chord by the other two. The angle of the other two chords will simply be the sum or the difference of these two angles.

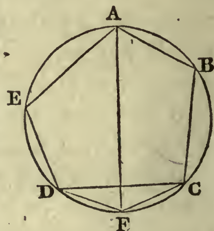
146. *Corollary.* If the vertex of a right angle be placed in the circumference, the sides will intercept a semicircle.

147. *Corollary.* If a circle be circumscribed about a triangle, and one side of the triangle passes through the centre of the circle, the opposite angle is a right angle.

148. *Theorem.* The maximum of polygons, having all the sides given but one, may have a circle circumscribed about it, having the unknown side for a diameter. *Proof.* Let A B C D E be the maximum polygon, formed of given sides A B, B C, &c., and the unknown side, A E. Join B E by a straight line. Now, since the polygon is a maximum, we cannot, leaving B E unaltered, by altering A E enlarge the triangle A B E, because that would enlarge the polygon. The angle A B E is therefore a right angle, by Art. 141, and a circumference, having A E for its diameter, would pass through the point B. In like manner it can be shown that a circumference having the same diameter would pass through each of the other points.



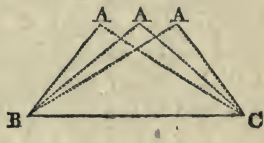
149. *Theorem.* The maximum of polygons formed with given sides can be inscribed in a circle. *Proof.* Let A B C D E be a polygon, formed of given sides, with a circle circumscribed about it. Draw the diameter A F, and join F C and F D. The polygons A B C F and A E D F are now maximum polygons, and therefore A B C D E must also be a maximum, since its enlargement would enlarge the sum of the other two.



We have thus proved the converse of the proposition, and the proposition is true, unless there is more than one maximum form of the polygon.

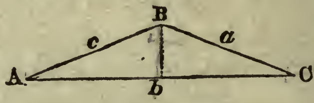
The converse is more easily proved than the proposition, and I therefore proved it, on the assumption that there is but one maximum form. That is, I have proved that a polygon of given sides, when inscribed in a circle, is a maximum; but that does not strictly *prove* that the maximum can always be inscribed in a circle; except on the assumption, which is, however, a safe one, that a polygon formed of given sides, arranged in a given order of succession, can have but one maximum form.

150. *Theorem.* Of isoperimetrical triangles with one side given, the maximum has the two undetermined sides equal. *Proof.* In order to prove this we have only to show that the point A is at its greatest distance from the base B C, when opposite the middle of it. This might seem scarcely to need proof. For when we use a string and stick to illustrate the problem, we can see that by sliding the finger from the middle of the string, it can be brought down into a line with the stick; and the greatest height from the stick is near the middle of the string. Further consideration shows it must be exactly at the centre of the string, because the finger and string have precisely the same relation to one end of the stick as to the other; and a motion towards either end must affect the height of the finger in a similar manner.



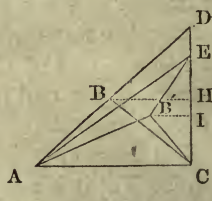
This reasoning is doubtless satisfactory to every fair mind. Yet it is not a good mathematical demonstration, and I have given it to you for the purpose of illustrating the peculiar nature of mathematical reasoning. The reasoning just given leaves no real doubt on the mind, but it is rather because we *see* with the eye that the finger is highest in the middle, than because we see with the mind that it must be. There is another step still lacking, to prove to us that the highest points are not on each side of the exact middle, as that would satisfy the conditions of symmetry and of declination towards each end. Let us then seek a proof which shall not force us to consider the whole motion of the finger, but which shall simply compare two forms of the triangle, one with the finger in the middle of the string, and one with the finger on one side.

151. *Theorem.* If a straight line be drawn from the vertex of two equal sides in a triangle, at right angles to the third side, it divides the third side into equal parts.



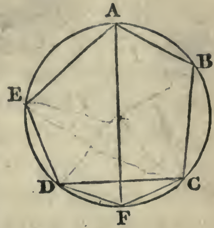
Proof. Let c and a be equal sides in a triangle A B C. Since the angles at A and C are equal, the angles b B C and b B A are also equal. If, therefore, the triangle B b C be folded over on the line B b, the line a will take the same direction as the line c, and, being of the same length, will coincide with it. Hence, b C will also coincide with b A, and the two lines must be of equal length.

152. *New proof of Art.* 150. Let A B C and A B′ C be isoperimetrical, and let A B and B C be equal. Continue A B to D, making B D = B A = B C, and join D C. Then, by Art. 147, the angle D C A is a right angle. Draw B′ E making it equal to B′ C. Join A E. A E will be less than the sum of A B′ and B′ E, that is, less than A B′ and B′C, that is, less than A B and B C, that is, less than A D. But if A E is less than A D, then C E must be less than C D, by Art. 136. Draw B H and B′ I at right angles to C D; we have C I, which is half C E, less than C H which is half C D. But C I and C H are the altitudes of the triangles A B C and A B′ C above their common base A C. The triangle with the undetermined sides equal has the greatest altitude, and must be the largest triangle.



153. *Theorem.* The maximum of isoperimetrical poly-

gons of a given *number* of sides is equilateral, that is, has equal sides. *Proof.* Let A B C E D be the maximum of isoperimetrical polygons of a given number of sides. Then A B must equal B C. For if it did not, then after joining A and C we could enlarge the triangle A B C by equalizing A B and A C, and thus enlarge the polygon without altering the number of sides of the perimeter, and the present form would not be the maximum.



In like manner we may prove that B C = C E, &c.

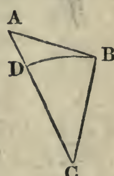
154. *Corollary.* The maximum of isoperimetrical polygons of a given number of sides is regular by Arts. 149 and 153.

155. *Axiom.* A circle may be considered as a regular polygon having an unlimited number of sides. And this regular polygon may be considered as either inscribed in or circumscribed about the real curve.

156. *Theorem.* The area of a regular polygon is measured by half the product of the perimeter into the radius of the inscribed circle. *Proof.* For if lines be drawn from each vertex of the polygon to the centre, the polygon will be divided into triangles having a common altitude equal to the radius of the inscribed circle, the sum of the bases of these triangles being equal to the perimeter of the polygon.

157. *Corollary.* The area of a circle is measured by half the product of the radius into the circumference.

158. *Theorem.* The perimeter of a circumscribed polygon is greater than the circumference of the circle. *Proof.* Let A B be half a side of a circumscribed polygon, and D B the portion of arc intercepted by lines drawn from A and B to the centre of the circle. Di-



vide D B into arcs so small that each may be considered as a short straight line. Through the points of division draw lines extending from the line A B to the point C. At the end B, the little arcs are equal to the corresponding pieces of the line A B; but as you approach A the divisions of the line grow longer than the corresponding divisions of the arc, for two reasons; first, the little arcs are at right angles to the radii, while the portions of the line are not (Art. 139); secondly, the little arcs are nearer to the point C, towards which the radii converge. The whole of A B must therefore be longer than the whole of D B. But it is manifest that the circumference consists of as many times D B as the perimeter does of the line A B.

159. *Corollary.* The circle inscribed in a regular polygon is smaller than a circle isoperimetrical with the polygon, and has a shorter radius.

160. *Corollary.* The circle is the maximum among isoperimetrical regular polygons.

161. *Corollary.* The circle is the maximum among isoperimetrical figures; a proposition towards which we have been directing our course through 48 articles, some of which are themselves complex propositions, referring to the preceding chapters. No other science requires any thing like such long trains of connected reasoning as those used in the mathematics. An argument in other matters usually consists of only a few steps — what are called long arguments being really a collection of shorter independent proofs of the same thing. In the mathematics, we are frequently required to take, as in the present instance, hundreds of consecutive steps to attain a single position.

162. *Scholium.* A slight modification of the reasoning in Arts. 158–160, would show, that of isoperimetrical polygons that is greatest which has the greatest number of sides.

163. He that really wishes to learn geometry must learn to work alone. I advise the learner now to take up "First Lessons in Geometry," and, beginning with the fourth chapter, go through to the twenty-sixth, trying how many of the facts he can prove. I think he can, if he sets himself to work with a good will, prove the greater part. Perhaps he will be obliged to ask some help of his teacher, but I think not much. He will, however, do well to show his demonstrations to his teacher for his criticism.

When he comes to Chap. XXVI. of the "First Lessons" he will be obliged to lay down the book again, as the propositions in the remainder of the book cannot be proved without the aid of higher branches of mathematics — Algebra, Trigonometry, and the Calculus.

164. In proving the Pythagorean proposition, and the proposition that the circle is the maximum among isoperimetrical plane figures, I have tried to give good examples of the mathematical mode of proof, — the analysis, in which the mind turns the proposition over in every form, trying all sorts of experiments upon it intellectually, to discover its vulnerable side, — and the synthesis, by which we then enter step by step into the very secret of the mystery.

Analysis consists in taking the proposition itself as the starting point, and going, step by step, to self-evident truths, or at least to truths already proved. Synthesis consists in starting with self-evident truths, or truths already proved, and going step by step to the truth which you would prove. But synthesis generally requires a previous rough analysis, by which you select the proper point of departure for your synthetical reasoning.

A species of analysis, called *reductio ad absurdum*, is often used in cases where true analysis or true synthesis is difficult. In this form of proof you assume that your proposition is not true, and by analysis show that this would lead you, step by step, to the denial of self-evident truth. This shows the proposition to be true, by simply showing that it cannot be false. Art. 138 gives an instance of this proof; also Example VI.

I think you will, after mastering this book thoroughly, be able to read any of the books on geometry which you will be at all likely to meet with.

PART II.

CHAPTER I.

GEOMETRICAL CONSTRUCTION.

165. The first reason for learning Geometry is, that it teaches us truth. This reason would be sufficient in itself. It is as important for us to learn truth as it is for us to eat food. But there is another reason for learning Geometry, and that is, the use which we may make of its truths.

166. The uses which can be made of Geometry are of two kinds. We can use it in the investigation of other kinds of truths, that is, in studying Astronomy, Mechanics, Chemistry, and other sciences; or we can use it in arts and trades. I shall not, however, attempt to keep up this distinction in the following pages.

167. *Definition.* We can frequently solve a mathematical question by representing given quantities as lines and angles, constructing or drawing the figure on paper, and then measuring the lines or angles representing the unknown quantities. This is called "solution by geometrical construction." I will explain it more fully a few pages farther on.

168. The first requisite for geometrical constructions, after a supply of drawing paper and pencils, and a plane table, is a straight ruler. It is not necessary that the table should be perfectly plane, but it must be smooth, and nearly plane. But the ruler should have a perfectly straight edge; at least as nearly so as the material of which it is made will allow.

The mechanical means of obtaining a straight edge will illustrate the uses of geometrical knowledge. The axiom that the shortest distance between two points is measured by a straight line, shows that a stretched thread will mark a straight line, and afford a guide for sawing out a tolerably straight ruler. The sawn edge will not, however, be smooth; and in the process of smoothing it may be brought more nearly into a perfectly straight line by the application of various tests.

The first method is by "sighting" the edge; that is, looking at it with one end very near one eye, and observing whether the farther end will, upon raising the near end a little higher, disappear at the same moment that the whole edge disappears. If any part of the edge remains in sight after any other part has disappeared, the edge cannot be perfectly straight. This assertion is founded on the assumption that light moves in straight lines. "Sighting" does not afford a very delicate means of testing a straight edge, partly on account of the impossibility of looking at a near and at a distant point at the same instant.

A second test is founded on the obvious truth, that two lines cannot coincide unless they are both straight or both have the same bendings. If two straight edges are placed together, and touch throughout their whole extent, the probability is very strong that they are perfectly straight. That probability is still further increased if they continue to touch in their whole extent when one is made to slide backwards and forwards upon the other. In that case they must be either straight, or else arcs of the same circumference. This may be finally tested by drawing a fine line by means of one of the suspected straight edges upon firm paper, and then applying the same straight edge to the opposite side of the line. If curved, this reversion will at once show it.

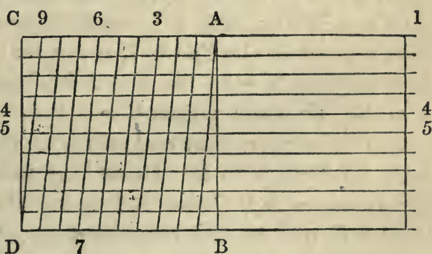
169. The second most important requisite for geomet-

rical construction is a pair of compasses, or dividers. These are made of various degrees of delicacy, and are of various prices. Some have merely steel points, by which circles can be scratched upon paper or upon wood; others are arranged to carry lead pencils, or to carry ink in a peculiar kind of pen. The best ink for such uses is, however, made by rubbing the solid "Indian ink" with water.

170. Compasses, or dividers, have two uses, as their two names imply. They can be opened to any width, and thus be made to measure the length of lines, and the distances between points. Or, having been opened to the width of the radius of a required circle, one point can be held still at the centre while the other traces the circumference. The joint should be firm enough to prevent the radius from readily changing its length.

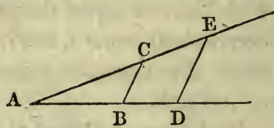
171. The third requisite is a scale. This is a piece of wood, bone, ivory, or metal, marked on its various sides with lines at equal distances (of an inch, half inch, or other convenient unity), having at one end also a diagonal network for measuring tenths, hundredths, and thousandths of the unit.

The lines A B and C D are parallel, and one unit apart. This space is divided into tenths on A C and also on B D, and each point of division on A C is joined to the next higher point on B D. The space between the line A C and the line D B is divided into tenths by parallel lines, such as 4, 4, and 5, 5. The modes of using this scale will



be obvious on the slightest reflection. The distance between A and 6 is six tenths of the units, between B and 7 is seven tenths. But the lines A B, and 6, 7, are, at any intermediate point, at an intermediate distance apart; as, for instance, their distance measured on the line 4, 4, would be 64 hundredths of a unit; measured on the line 5, 5, it would be 65 hundredths; and measured at three tenths of the way down from 4, 4, to 5, 5, it would be 643 thousandths of a unit. Thus, with very fine pointed dividers you can readily measure to the thousandths of a unit; or rather measure to the hundredths, and estimate very accurately to thousandths. If you have no such scale prepared, you can, with a very fine pencil point, sharpened flat, draw one for yourself on card, and make it durable by sizing it with a drop of gum water.

172. I can now give you an illustration of the definition in Art. 167. Suppose that you wish to know the product of 1.413 multiplied by .647. This would be the same as wishing to know what number bears the same ratio to 1.413 that .647 bears to unity. Draw then two lines, A D and A E, as long as you please, and making what angle you please. Open your compasses to the length of a unit, and putting one foot at A with the other make a dot at C.



Open then again to the width .647 (that is to say, in Fig., Art. 171, until they measure the distance from a point in the line 6, 7, seven tenths of the way from 4, 4, towards 5, 5, to the line A B), and set that off from A to B. Join the points C and B by a straight line. In like manner open your compasses to the width 1.413, and set it off from A to E. Draw E D parallel to C B, and as the triangles A B C and A D E are similar, it is manifest that A D will be the required product. Open your compasses till they just stretch from A to D, and you will find, on applying them to your scale, that the length is .916 — the product of 1.413 by .648.

This would be multiplying two numbers by geometrical construction. You would represent the numbers by lines, and unity by a line, and the product of the numbers will then be represented by a line bearing the same ratio to the multiplicand that the multiplier bears to unity. This line is found by drawing two similar triangles; and the line being measured gives the product in figures.

173. In the simple illustration I have given, there would be no advantage in a geometrical construction over the arithmetical process. But it is by no means always so. On the contrary, there are a vast variety of cases in which geometrical construction is by far the best method of solving practical questions of mathematics. For this reason, I recommend the scholar to make himself familiar with its processes. The whole of the second part of this volume is intended to assist him in gaining this knowledge.

CHAPTER II.

POSTULATES.

174. In the same manner that there are certain truths too plain to need proof, which are called axioms, so there are processes of mechanical construction too simple and easy to make it doubtful whether they can be performed. These are sometimes called postulates, that is, things asked, because you are asked to take it for granted that they can be done.

175. *Postulate. It is possible to have a plane sheet of paper.* — Of course no surface of paper can be perfectly plane; but it is easy to obtain a table so nearly plane, and

paper so nearly plane in its surface, that no appreciable error can arise in using them as plane.

176. *Postulate. A straight line can be drawn from one point to another on plane paper.*—This again cannot, of course, be done with perfect exactness; but when the points are marked by fine dots, we can bring the straight edge of a ruler up to the two points, and draw a line so nearly through them, and so nearly straight, that there shall be no appreciable error.

177. *Postulate. Any straight line may be continued at either end for any distance.*—This requires, of course, the same limitation as to accuracy as the preceding postulate. The edge of the ruler is to be applied to the line already drawn, as a guide in prolonging it farther.

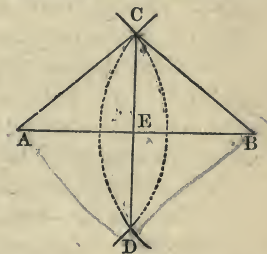
178. *Postulate. Around any point as a centre a circle may be drawn of any radius required.*—This postulate, also, must be limited to mean, that this can be done without appreciable error. But no man can put down one foot of the compasses exactly in the centre of a dot, nor open a pair exactly to the length of a given line.

179. In the writings of geometers, usually these postulates are not limited, and are used only as the foundation of theoretical solutions. But as I do not see the value of putting a theorem into the form of a problem, unless for practical use in geometrical constructions, I have added these limitations.

CHAPTER III.

STRAIGHT LINES AND ANGLES.

180. *Problem. To divide a line into equal parts.* — If the parts are to be two in number, draw from the ends of the line, as centres, arcs of equal radii, the radius being long enough to cause the arcs to intersect each other at two points. Join the points of intersection by a straight line, and this line will intersect the given line in the



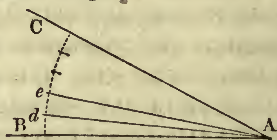
middle. For, since B C and A C are equal, the angles C A E and C B E are (Ex. XI.) equal; and since the triangles D A C and D B C are composed of sides of the same length, the angles B C E and A C E are (by Art. 95) equal. Hence, by Art. 91, the triangle B E C and A E C are equal, and A E = E B.

If the parts are to be more in number than two, other methods of division may readily be devised, which shall only require the postulates of Chap. II.

But a slight extension of the postulate of Art. 178 renders all methods of division practically unnecessary. The ability to draw a circle around any centre implies your ability to put one foot of the compasses on any point you choose. And the postulate, that you can draw it with any radius, implies that the compasses may be opened to any width desired. But if we can open the compasses to extend from A to B, it is practically just as true that we can open them to extend half way, so that two "steps" shall take us from A to B, or one third, one fourth, &c.;

the way, so that three or four steps may carry us from one end to the other. If the first effort does not succeed, the width of the compasses is to be altered in proportion to the number of steps we have made. In dividing a line into sevenths, if our seventh step left us .14 of an inch from the end of the line, the compasses are to be opened only one fiftieth of an inch wider than before.

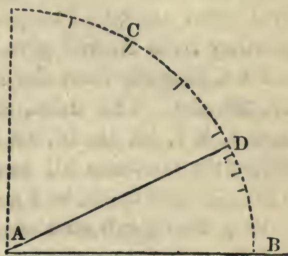
181. *To divide any angle into equal parts.* — A similar extension of the postulate in Art. 178 will show us that we can, without appreciable error, find a chord, which, being applied a given number of times to a given arc, will coincide at its extremities with the extremities of the arc; in other words, we can open the compasses to a width which will step over an arc in a given number of steps.



Let, then, B A C be the angle to be divided. With any radius, taking A as the centre, draw an arc B C between the sides of the angle. Step, with the compasses, over this arc in as many steps as the parts into which the angle is to be divided. Let *d* and *e* be two points of division thus determined nearest B. Draw *d* A and *e* A, and the angles *e* A *d* and *d* A B are two of the required parts of the angle B A C. For if we imagine chords *e d* and *d* B to be drawn, then the triangles *e* A *d* and *d* A B will be composed of sides of the same length, that is, equilateral with respect to each other, and of course equal in all their parts (Art. 95); whence the angles *e* A *d* and *d* A B will be equal.

182. *Corollary.* If the figure *e* A *d* were turned over on the line A *d* as a hinge, the line A *e* would coincide with A B, and, as all parts of the arcs are at equal distances from A, the arcs *e d* and *d* B would coincide. That is, equal chords in the same circle subtend equal arcs, and arcs can be equally divided by a pair of dividers, in the same manner as straight lines and angles.

183. *Problem. To draw an angle of a given number of degrees.* — From any point A, as centre, in a straight line A B, with any radius A B, describe an arc B C. Keeping the compasses open at the same width, place one foot at B, and with the other mark the point C. The arc B C is then (Ex. XII.) an arc of 60°. Closing the compasses until they will



pass over the same arc in four steps, you obtain (by Art. 182) arcs of 15°. Selecting either of these, according to the degrees required, close the compasses, until they divide it into arcs of 5°. By dividing one of these arcs of 5° into five equal parts, you can obtain the required degree, counting from B up to D. Join D A, and you manifestly have the angle required. Thus, if the given number of degrees were twenty-seven, we should take the second arc of 15°, the third arc of 5° in that arc of 15°, and the second degree of those five.

184. *The formation of a protractor.* — Take a piece of hard, smooth card, draw a fine, straight line, as A B (see fig. above), and with a convenient radius, say three inches, draw the arc B C. Measure carefully the arc B of 60° by having the compasses, while yet unaltered from the radius with which you drew the arc, step from B to C. Divide the arc as accurately as possible into four equal arcs of 15° each, and set off two such arcs beyond C, so as to make the whole arc 90°. Divide each arc of 15° carefully into three equal parts, which will each be 5°. Divide each also into five parts, each of which will be 3°. By stepping over the whole arc with the compasses open for three degrees, first stepping over it lightly to make sure that twenty steps will exactly make 60°, and then with a heavier step, so as

to leave footprints; repeating this heavier stepping from each point of division of the 5° arcs, you can divide the prolonged arc into 90 equal degrees. The first divisions, starting from B, will give 3, 6, 9, 12, 15, 18, 21, &c. The second, starting from the 5° point, will give 2, 5, 8, 11, 14, 17, 20, &c. The third, starting from the 10° point, will give 1, 4, 7, 10, 13, 16, 19, 22, &c.; and these three series evidently embrace all numbers. Mark each fifth point with a longer mark, and number them from B towards C.

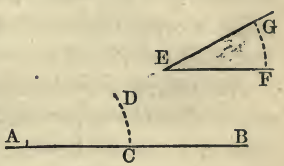
185. The graduated arc and its centre, described in Art. 183, is called a protractor, and may be found for sale, engraved on wood, ivory, or brass. It is used for measuring angles, and also for drawing angles of a given size. There are two ways in which it can be finished and used. The first way of measuring an angle, is to draw an arc between its sides, prolonged if necessary, with the same radius as that of your protractor, its centre being exactly at the vertex. Set the compasses so as to reach exactly across this arc from side to side of the angle; then, placing one foot of the compasses at the point B of the protractor, the other will mark out on the graduated arc the size of the angle. The reverse process of drawing a given angle consists in drawing an arc of the same radius as that of the protractor, and then with the compasses taking the chord of the given number of degrees from the protractor and setting it upon the arc; lines drawn from these two points of the arc to the centre from which it was drawn, will make the required angle.

The second method of using the protractor is to cut off all the card below the line A B and all outside the graduated arc B C. Placing then the point A over the vertex of the angle, and making A B coincide with one side of the angle, the other side prolonged if necessary, will pass out under the graduated edge of the card, and the degree of the angle can be at once read. Or, if you wish to draw an angle of a given size, having placed the edge A B as just directed, make a dot on the paper, at the right degree on the graduated edge, and then, removing the protractor, join the dot by a straight line with the vertex that was under A.

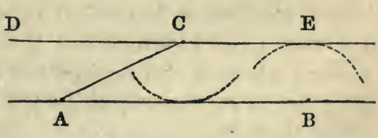
Protractors may be purchased having graduated arcs of 180°, or of 360°. In the latter case, the central part of the plate is removed, and a piece of transparent mica inserted, with a fine dot upon it to mark the exact centre.

186. *To draw an angle equal to a given angle.*—If the given angle is given in degrees, the required angle may be drawn by Art. 185. But if the given angle is one simply drawn on paper, as A B C, then from the vertex of the given angle as a centre, with any radius, draw an arc between the sides of the angle; and with the vertex of the required angle as a centre, with the same radius, describe an arc of equal length. (Art. 182.) Lines drawn through the extremities of this arc to the vertex will make the required angle.

Thus, if it be required to draw a line from the point A, making the angle E with the line A B, draw with any radius the arc F G, and with the same radius the arc C D, making C D equal to F G. A line drawn through the points A and D will make the required angle.



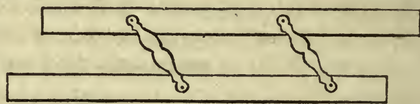
187. *To draw through a given point, as* C, *a line parallel to a given line, as* A B.—Join C to any point in the given line by a straight line, as C A. Make the angle D C A equal to C A B, and the line D C will be manifestly parallel to A B. To draw through C a line making any angle



with A B, we need only draw, from any point in A B, a line making the required angle with A B, and then draw through C a line parallel with the line so drawn.

188. A simpler mode of doing the same thing, though not allowed by the postulates of Chap. II., is to open the compasses until, with one foot on the point C, the other will describe an arc touching the line A B, but not cutting it. With the same radius and one foot at B, describe another arc at E. Draw a line through C, touching, but not cutting, the arc E, and it will be parallel to A B. The proof may be readily discovered by the learner.

189. The instrument called a parallel ruler is simply two rulers with parallel edges, joined by two strips of brass, riveted to the



rulers, but the rivets allowing motion in the plane of the paper on which it is laid. Great care must be taken to have the rivet holes in the two pieces of brass at equal distances, and also those in the rulers at equal distances. If this is done, then, while one ruler is held still and the other moved, the moving ruler must remain parallel to its first position.

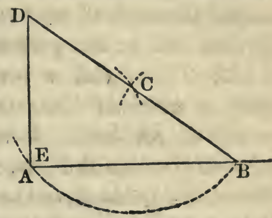
Additional care is usually taken, in making the instrument, that this position shall be parallel to that of the stationary ruler, by having the holes in each ruler on a line parallel with its edges. Another kind of parallel ruler is made by simply mounting a ruler on rollers. This is less accurate.

The readiest and most accurate mode of drawing parallel lines is, however, to use a flat triangle of wood, one side of which is slid against the edge of a straight ruler, held firmly stationary, while the other sides remain parallel to their first position.

190. The drawing of a parallel line is simply the draw-

ing of an angle equal to zero. Another angle of peculiar interest is the right angle, and there are better ways than that of Art. 185 for drawing a right angle.

191. *To raise a perpendicular at a given point* A *upon a straight line* A B. — *First Method.* From any point B, in the line A B, with any radius B C, describe an arc, and from the point A, with the same radius, describe an arc cutting the first at C. Draw the line B C, prolonging C D to equal C B. Join D A by a straight line, and it will be perpendicular to A B, by Art. 147.



Second Method. Opening the compasses to any convenient width, step off five equal portions of A B, beginning at A. Let B be the fourth point of division. From B as a centre, with a radius equal to five of these parts, draw an arc above A, and from A as a centre, with a radius equal to three parts, draw a second arc intersecting the first at D. Join D to A by a straight line, and it will be at right angles to A B, by Art. 106.

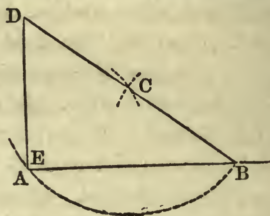
Third Method. Measure with the compasses equal distances on each side of A, and bisect the line thus measured off by the method of Art. 180, and you have a line passing through A at right angles to A B.

Fourth Method. Visiting or business cards are usually cut very exactly at right angles. By applying one corner of a card at A, and making one edge coincide with A B, the other edge will be at right angles to A B. The accuracy of this right angle may be tested by drawing perpendiculars on opposite sides of A.

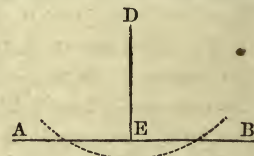
Fifth Method. If you have neither card nor compasses, fold a piece of writing paper carefully, and then double the folded edge carefully on itself. This corner of four thicknesses of paper will be a square corner, to be used as the card.

Sixth Method. From any point outside the line, as C, with a radius equal to C A, draw an arc cutting A B, say in B, and prolonged to a point where the radius B C, prolonged through C, may cut it, say in D. D A will then, by Art. 147, be the perpendicular required.

192. *To let fall a perpendicular from a point, as* D, *upon a straight line, as* A B. — *First Method.* Join any point of the line A B, as, for instance, the point B, to the point D, by a straight line D B. From C, the middle of D B, with a radius equal to C B or C D, draw an arc cutting A B at E, and D E will be (by Art. 147) the perpendicular required.



Second Method. From D as a centre, describe any arc cutting the line A B in two places. The middle point between these places will be E, the foot of the perpendicular from D.



Third Method. Make one side of the square card coincide with A B, and slip the card along until the end passes through D. Or if the triangle of Art. 189 be made with angles of 30°, 60°, and 90°, its square corner may be used.

Fourth Method. If the perpendicular is to be drawn merely for the sake of measuring its length, that is, for finding the distance of D from the line A B, it need not be drawn; but you may simply place one foot of the compasses in D, and then open them wide enough to describe an arc touching, but not cutting, A B. Although the possibility of doing this is not claimed in the postulates, yet it is practically equivalent to the postulate of Art. 178.

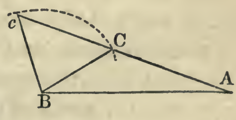
CHAPTER IV.

TRIANGLES.

193. *To draw a triangle of three given sides.*—If the sides are not drawn, but are given in numbers, open the compasses to extend upon the scale to a number corresponding to one of the sides. If the lines are drawn, open the compasses to the length of one of them. Set the dividers on paper with sufficient pressure to mark the points where the feet touch. From these points as centres, with radii equal to the other sides of the triangle, draw arcs cutting each other. This point of intersection and the points used as centres will be the vertices of the required triangle, and must be joined by straight lines.

194. *To draw a triangle when two sides and one angle are given.*—*First Case. When the angle is included between the given sides.* Draw two lines making the required angle; and upon each line set off with the compasses, from the vertex, the length of the given sides; join their extremities by a straight line, and you evidently have the required triangle.

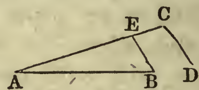
Second Case. When the angle is opposite one of the given sides. Draw two lines, A B and A C, making the given angle. Set off from the vertex A the given adjacent side A B, and from B as a centre, with the other given side B C as a radius, draw an arc cutting A C in C or *c*. Either A B C or A B *c* will be the required triangle. If B C is greater than A B, only one triangle can be formed.



195. *To draw a triangle when one side and two angles*

are given. — *First Case. When the side lies between the angles.* Draw a line equal in length to the given side, and draw at the ends of it lines making the given angles with the given side. These lines being produced far enough to meet, will give the required triangle.

Second Case. When one angle is opposite the given side. If the angles are given in degrees, the simplest way is to add the two angles together, and subtract the sum from 180°; this will give the third angle, and reduce this case to the first case. But if the angles are given by being drawn, it will be better to draw the given side A B, and at one end raise the line A C, making the given adjacent angle. At any point, as C, draw C D, making A C D equal to the given opposite angle.



Through B draw B E parallel to C D, and B E A is evidently the required triangle. Such a line as C D should be drawn lightly, so that, if necessary, it can be erased.

196. It is manifest that, in all the problems of this chapter, if the sides are given in numbers, any convenient unit may be taken to represent unity in the numbers. That is to say, if the original numbers represent feet, yards, or miles, they may in your drawing be taken as inches, tenths of inches, twentieths, or hundredths, as you please; only remembering that the same quantity must be taken as the unit in all parts of any one figure.

In drawing profiles, or vertical sections, however, two units are usually employed. Thus, in drawing a sketch of the elevations and depressions of a railroad 100 miles long, in which the greatest elevation attained was 500 feet, you might represent the length on a scale of one mile to an inch, but the elevations and depressions on a scale of 400 feet or 500 feet to an inch.

197. The problem of Art. 193 is impossible if either of the given sides is greater than the sum of the other two.

198. In the second case of Art. 194, the problem is impossible if the side opposite the given angle is too short to reach the side not given.

199. In Art. 195, the problem is impossible if the sum of the angles given equals or exceeds 180°.

Examples.

200. Draw a triangle, with sides of 19, 33, and 41 feet. Draw one with sides of 41, 33, and 19 miles. Draw triangles with sides of 18, 13, and 27 ; of 341, 263, 501 ; of 76.8, 54, 43.7 ; of 673, 321, 352 ; of 67, 32, 34 ; of 71, 39, 43 ; of 67, 29, 47.

201. Measure, by Art. 185, the angles of each triangle in Art. 200, and test the accuracy of your measures by adding the angles of each triangle together ; the sum should, of course, be 180°.

202. Draw a map of a triangular building lot, whose sides are 97 and 73 feet, the angle between these sides being 57°. Draw a map of another triangular lot, with sides of 84 and 77 feet, and the angle opposite the side of 77 feet equal to 43°. Draw a triangle with a side of 81, another of 41, and the angle opposite 41 equal to 23°. Try the same with the angle 37°.

203. Measure the other angles of the triangles of Art. 202, and the third sides, testing the angles as in Art. 201.

204. One side of a triangular lot being 83, what is the size of the opposite angle, and the length of the other two sides, the adjacent angles being 67° and 93°. Answer the same question when the angles are 33° and 111°. When the angles are 61° and 119°.

205. If we have only the angles of a triangle given, we of course cannot discover the length of the sides.

206. It is plain that, having two sides given, we must also have an angle given, in order to draw the triangle.

207. If one side is given, it is plain that we must have two angles given, or, having but one actually given, must have some condition given which will determine another; such, for instance, as the ratio which the unknown angles bear to each other.

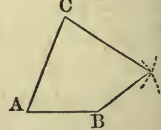
208. A ship at anchor finds that a round lighthouse-tower, known to be 17 feet in diameter, covers a degree and a half of the horizon; in other words, lines drawn from the ship to opposite sides of the tower make an angle of 1° 30′; and those lines make, with one diameter of the tower, equal angles of 89° 15′ each. Draw this triangle, and find from it the distance of the ship from the lighthouse.

What is the moon's diameter, if her distance from the earth is 240,000 miles, and her apparent diameter 30′?

CHAPTER V.

QUADRANGLES.

209. *To draw a quadrangle when all the sides and one angle are given; the sides, including the angle, being named.*—Draw two lines, making the given angle, A, and measure upon them the sides, including the angle. From the extremities, B, C, of these sides, as centres, with radii equal to the other sides, draw arcs intersecting each other, and the point of intersection will be the fourth vertex of the quadrangle.

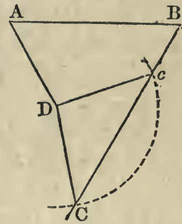


210. It is manifest that the arcs will intersect in two places, and also that the third and fourth sides can change

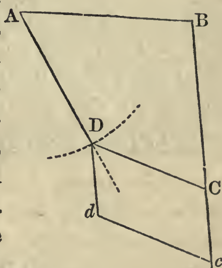
places with each other, so that four quadrangles can sometimes be drawn satisfying the conditions of Art. 209.

211. *To draw a quadrangle when three sides and two angles are given, the order of the sides and angles being named.*—If both the angles are included between given sides, draw the middle side and raise the other two sides, making the proper angles with the middle side. It will only remain to join their extremities by a straight line.

If one angle is adjacent to the unknown side, as, for instance, if A B, A D, and D *c* are given sides, and A and B given angles, draw A B and A D of the given lengths and making the given angle. From B draw B C of indefinite length, making B of the given size. From D as a centre, with the given length D C as a radius, describe an arc cutting B C in C and *c*. Join either D C or D *c*, and it completes the quadrangle.



But if both given angles are adjacent to the unknown side, that is, suppose B *c* the unknown side, and B and C the given angles, then at B, and at any other point on B *c*, as *c*, raise at the proper angles the sides B A and *c d* of the given length. From the point A, with the radius A D, draw an arc. From *d* draw, parallel to *c* B, a line cutting this arc at D. Draw D C parallel to *d c*, and the quadrangle is manifestly completed.



212. *To draw a quadrangle when two sides and three angles are given, their order of position being named.*—The three angles being subtracted from 360° will give the fourth angle.

If the given sides are adjacent, as A B, A D, draw those sides of the given length, making the given angle, and from the extremities, B and D, draw lines making the given angles B and D. The intersection of these lines will complete the quadrangle.

If the given sides are not adjacent, as A B and D C, from the extremities of A B, draw lines A D and B C, of any length, but making the given angles A and B. From any point *c* in B *c* draw *c d*, making the angle B *c d* equal to the given angle C. Take *c d* equal to the given side D *c*. Draw *d* D parallel to *c* B. The point at which *d* D crosses A D will be one corner of the quadrangle. From this point D draw D C parallel to *d c*, and the quadrangle is manifestly completed.

213. My garden is an irregular quadrangle, the sides being 150, 207, 315, and 97 feet. The sides are placed in that order, and the angle between the first and second is 96°. Draw me a plot.

The south front of a lot is 37 feet, the east side 63, the west 52 feet, and the south-west corner is 92°, the south-east 62°. Draw a map.

The south front of a lot of land being 31 feet, the east side 53, the west 46 feet, the south-west corner is a right angle, the north-east corner measures 78°. Draw a map.

The sides of another lot, and its south front, measure the same as in the last example, but the two corners in the rear are square corners. Draw a map.

CHAPTER VI.

CIRCLES.

214. *To draw an arc of a circle, the radius being given.* — As a practical question this is one of great importance, as it concerns not only the process of geometrical con-

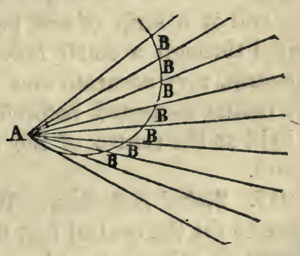
struction, but many processes of mechanical construction also. In geometrical construction the compasses usually afford the readiest means of drawing arcs and circumferences; but in mechanical construction of machines, roads, and other things, the compasses are frequently of no value.

Sometimes a thread, string, or rope is fastened by one end to the spot selected for the centre of the arc, while the other end is carried round. It is plain that if the line is kept stretched, and equally stretched, its moving end remains always at the same distance from the stationary end, and the curve must be the circumference of a circle.

Sometimes the thing on which you wish to describe a circle is turned round, as in a lathe.

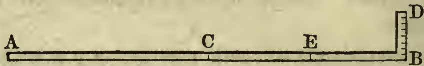
Sometimes, at the blacksmith's shop, a circle is made by bending a strip of iron equally at every part. This is done by passing it between three rollers. The amount of bending at each point must be inconceivably small, because any perceptible bend at any one point would make an angle there. Yet small as the bending at each point must be, it must be practically measured by a screw that raises or depresses the middle roller, and thus alters the curvature of the tire as it passes between them.

In laying out railroads, arcs of circles are drawn by measuring off equal angles from one point, as A, and setting off equal chords between the sides of the angles, beginning at the point A. That these will be chords to a circle may be proved from Art. 145. Engineers have tables prepared, telling them what the angle must be in order to have chords of 100



feet each from circles of 300 feet, 400 feet, or any other radius.

In laying out garden paths and walks, it is convenient to have a wooden "square," such as A B D. A B should be straight, and have a mark in the middle at C. B D should



be divided into small divisions. Putting down two stakes, one at A and one at C, a third one may be placed on D, at such a distance from B as is desirable. Taking up the square, now place the end A at the stake which was at C, and place the point C at the stake which was at D. Put down a fourth stake on the side D as far from B as the third stake was placed. The size of the circle will depend upon the distance from B at which the stake on the side D is placed. For a small circle, divide C B in the middle at E, and use the instrument as though it had been cut off at C. By using the whole for a small circle, D is carried too far from B, and thus C D becomes longer than A C, which will make the first few stakes irregular.

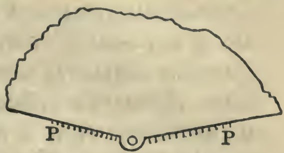
215. *First Solution of Art.* 214. Open the compasses to the given radius, and draw the arc as usual.

216. *Second Solution.* If the radius is too long for the compasses to be opened to that width, we may use a string, a strip of wood or of paper. If a pin be thrust through one end of a strip of stiff paper, and a pencil point be inserted through a small hole at the required distance on the strip, very accurate arcs of circles may be drawn, and the radius measured beforehand with accuracy. The pin is held at the centre of the circle, and the pencil carried round.

217. *Third Solution.* When there is no convenient place to set the central foot of the compasses, or the pin in the paper strip, upon, other plans may be adopted.

Draw two lines on a piece of stiff paper at an angle of

165° 31′. Trim the paper off to these lines, leaving a small piece about the intersection. At the point of intersection make a small hole in the paper to insert a lead pencil. Placing two pins P P at a distance apart equal to half the radius, a neat arc may thus be drawn. Other angles may be used with corresponding parts of the radius. For instance, 170° 24′, with pins at a distance apart equal to one third the radius. Of course, in drawing these angles it is easier to measure the supplement of the angle, that is, the remainder after subtracting it from 180°, and then simply let the lines cross.

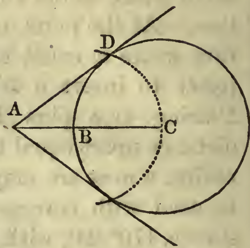


218. *Fourth Solution.* If we draw a tangent to a circle, and measure off upon it, from the point of contact, distances equal to one tenth the radius, we shall find the distance of the tangent to the circumference at the first three points of division equal to .005, .020, .046 of the radius. Hence we may draw an arc of a given radius by drawing a straight line, and marking upon it half a dozen spots equally distant, at a distance equal to one tenth the radius. If over any one of these we make a dot at the distance of five thousandths of the radius, over the next a dot at the distance of twenty thousandths, &c., a curve drawn carefully through these dots will be an arc as required. If we take twentieths of the radius instead of tenths, the distance of the curve will be .001, .005, .011, .020, .031, .046.

219. *To draw through a given point a tangent to a given circle.— First Case.* If the point is inside the circle, the problem is insoluble.

Second Case. If the point is in the circumference, draw a radius to the circumference at that point. Draw a line at right angles to the end of the radius, and it will be a tangent.

Third Case. When the point is outside the circle, let A be the point and C the centre of the circle. Join the point A to the centre C by a straight line. From the middle point B, of that line, with a radius B C, equal to half the line, draw an arc which will cut the circumference at the points through which the tangents from A must pass. For it is manifest that if the angle A D C were drawn, it would be a right angle, by Art. 147.

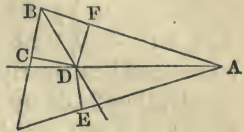


Practically, it is only necessary to lay the ruler with its edge upon A, and touching the circumference without cutting. This is not, however, practically useful when A is very nearly touching the circumference, and not at all practical when A is actually on the circumference, as in that case the angle of the line with the radius to the point of contact might fail to be a right angle.

220. Draw a circle of two inches radius. A semicircle of three inches diameter. Draw, by Art. 217, an arc three inches long, with a radius of sixteen inches. Draw, by Art. 218, an arc five inches long, with a radius of ten inches, also of twenty inches. What angle does a tangent through A make with a straight line drawn from A to the centre of the circle, the radius being 1.5 inches, and the distance from A to the centre 3 inches? Over how many degrees of latitude could you then look from a balloon at a height of four thousand miles above the sea, supposing it possible to rise that height? Over how many degrees of latitude could you look from a balloon at the height of one thousand miles?

221. *To inscribe a circle in a triangle.*—Bisect two of the angles of the triangle; that is, divide each angle into two equal parts by a straight line. The point where these

two lines intersect each other will be the centre of the required circle, and the radius will be the length of a perpendicular to either side of the triangle. That these perpendiculars will be equal in length, may readily be shown from the equality of the triangles which they form. Thus the triangles A F D and A E D are equal, because they have the same line A D for hypothenuse, and equal angles, by the construction of the figure. Therefore D F = D E. In like manner, D G may may be shown to be equal to D F.



222. *To circumscribe a circle about a triangle.* In other words, *To draw a circumference that will pass through three given points, as the vertices of a triangle.* — As the centre of the circle must be equally distant from each of the points, it must be found on a line perpendicular to the middle of a line joining any two points. (Art. 136.) In other words, if we draw lines perpendicular to two sides of the triangle, at the middle of those sides, the centre will be in both these lines; that is, will be found at their intersection.

223. *To find the centre of a given arc.* — It is manifest from Art. 222, that we only need draw any two chords in the arc, and erect perpendiculars at the middle of each chord.

224. *To find the radius of a circumference that will pass through three given points when they lie nearly in a straight line.* In other words, *When the radius is large, to draw the arc without finding the centre.* — The arc may be drawn by setting up pins at the extreme points, and cutting a piece of paper with straight edges at the angle that is made by lines to the intermediate point, then proceeding as in Art. 217. The length of the radius may be found by measuring the supplement of this angle.

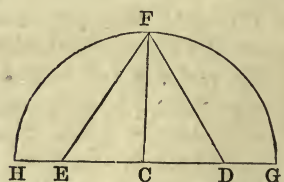
According as it measures 1°, 2°, 3°, 4°, 5°, 6°, 7°, 8°, 9°, or 10°, so must you multiply the distance between the extreme points by the number 28.65, 14.33, 9.55, 7.17, 5.74, 4.78, 4.12, 3.59, 3.20, or 2.88, to obtain the radius. It will be noticed how nearly any of these numbers can be obtained by dividing 28.65 by the number of degrees in the supplement; this will enable you to obtain it for angles not consisting of whole degrees. Thus for the angle $2\frac{1}{2}$ degrees, you will divide 28.65 by five halves.

225. Make three dots upon paper nearly in a straight line, and discover by Art. 224 the radius of a circle that will pass through them.

226. *To describe an equilateral triangle in a circle.*— Step round the circumference with the radius, and you will require six steps (since by joining two of the points thus marked with each other and with the centre of the circle you form an equilateral triangle, whose angles must each equal $\frac{1}{3}$ of 180°, or $\frac{1}{6}$ of 360): by joining the alternate points of division with straight lines you draw the triangle required.

227. *To describe a hexagon in a circle.* — Step round the circumference with the radius, and join each point, thus marked, with the adjacent points, by straight lines: the hexagon is drawn.

228. *To describe a square in a given circle.*—Bisect the arcs to which two opposite sides of the hexagon are chords, join the points of bisection with the two vertices of the hexagon that are at 90° from them, and you will manifestly have drawn a square.



229. *To describe a regular pentagon or five-sided figure in a circle.*— Draw a diameter H G, and erect a radius C F perpendicular to it. From D, at the bisection of C G, measure D E equal to

D F. Join E to F by a straight line, and it will be equal in length to one side of a pentagon.

The proof of this proposition is somewhat intricate, and would require more use of algebraic language than is consistent with the design of this little book. Supposing the radius C F to be 1; then C D would equal $\frac{1}{2}$. From this we should find the value of D F by the Pythagorean proposition. E C would then be found by subtracting $\frac{1}{2}$ from D F. Hence, by the Pythagorean proposition, we could find E F. Next, supposing a chord of 72° (the side of a pentagon) drawn in a circle of radius of unity, we could show that its length would be precisely the same as that of E F.

But the simplest way to draw a pentagon is to open the compasses to, as nearly as you can estimate it, the fifth of a circumference, and after stepping round once, alter their width, as nearly as you can estimate it, the fifth part of the resulting error. This really conforms to the spirit of the postulate, that one can open the compasses to a given radius; and the preceding method is given simply to show a way of drawing a pentagon in conformity with the letter of the postulate. If the compasses are opened so as to step round five times without any apparent resulting error, their real error is probably but one fifth as great as it usually is in measuring a radius.

CHAPTER VII.

AREAS.

230. Areas are the numbers which measure surfaces; that is, which express the ratio of the surfaces to a unit surface, or surface adopted as a unit, or standard of reference.

231. The usual unit of surface is a square whose side is a linear unit; for instance, a square inch, square mile, &c.

232. *To find the area of a rectangle.*—Multiply the length of a side by that of an end, and the product will be the area. (Art. 97.)

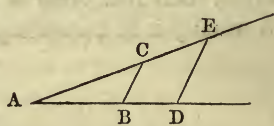
233. *To find the area of a parallelogram.*—Multiply the length of a side by the distance to the opposite side; the product will be the area. (Art. 110.)

234. *To find the area of a triangle.*—Multiply the length of either side by the distance from the opposite vertex; the product will be twice the area. (Art. 111.)

235. *To find the area of any polygon.*—Divide the polygon by diagonals—that is, by lines drawn through vertices not adjacent—into triangles, and measure these triangles.

236. *To multiply two numbers by geometrical construction.*—This problem is not commonly practically useful, and yet to one who wishes only approximative results, and dislikes numerical computation, it may be made to yield good results, especially if care be taken in using the parallel ruler.

Have prepared, on a piece of hard, smooth paper, two lines at an angle of 30° or 40°, and on one of them a unit, A B, measured from the vertex A, and permanently marked. Lay off either number from A to C, the other from A to D. Join B C, and draw D E parallel to B C.

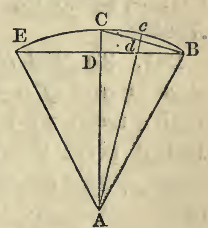


A E is the required product. That is, if A B were an inch, A C $1\frac{1}{2}$ inches, and A D 2 inches, then A E would be found to be 3 inches.

Proof. The triangles A B C and A D E given, by Art. 99 the proportion A E is to A D as A C is to A B, that is, as A C is to unity. But this is the definition of a product, that it is a quantity bearing the same ratio to the multiplicand that the multiplier does to unity.

237. *To find the area of a circle.*—If the circumference, instead of being a curve, consisted of many millions of short, straight sides, it is plain that the circle could be divided into many millions of little triangles, by means of many millions of radii. Now, the circumference may be thus conceived of, and the area of these triangles may be found, according to Art. 234, by multiplying the circumference (which is the sum of the short sides) by the radius (the distance to the opposite vertex), and dividing by two. The circumference or the radius may be divided by two before they are multiplied. (Art. 157.)

238. *To find the circumference of a circle from knowing its radius.*—It is manifest from Arts. 237 and 99 that the ratio of the circumference to the radius is the same in all circles, and we have only to find the circumference of the circle whose radius is unity. Suppose, then, that B E is a chord of 60°, bisected at D, and that A is the centre of the circle. Knowing D B is equal to $\frac{1}{2}$, and A B to 1, we can, by the Pythagorean proposition, calculate the length of A D. Subtracting this from 1 gives us D C. Then, knowing D C



and D B, we can, by the same proposition, calculate B C. Bisecting B C at d, we know A C and C d, and can therefore calculate A d, and thus find $d\,c$. Hence we get the chord C c, or chord of $\frac{1}{24}$ of a circle. By continuing this process of applying the Pythagorean proposition we can find the chord of the 48th, or 96th, or 192d of a circumference. Multiplying these chords by 48, or 96, or 192, gives us nearly the length of the circumference; and the greater the number of times that we bisect the arc of 60°, the more nearly will we attain the exact length. The length of the semi-circumference, with a radius of unity, or of a circumference with a diameter of unity, is called π.

Vastly more rapid ways of calculating π have been found by the Differential Calculus. Its exact value cannot be obtained, because the diameter and circumference are not in the ratio of any two numbers whatever. For ordinary calculations π may be taken as equal to 3.1416, and for the most accurate calculations 3.14159265.

239. As the area of a circle is, by Art. 237, equal to the radius multiplied by the semi-circumference, and as the semi-circumference is, by Art. 238, equal to the radius multiplied by π, it follows that the area is equal to the radius multiplied by itself and then by π. In other words, the area of a circle is 3.1416 times as large as the square on the radius.

240. As the square on a radius is one fourth the square on the diameter, we may find the area of a circle by multiplying the square of the diameter by one fourth of 3.1416, that is, by .7854; or, for very accurate calculations, by .785398.

241. *To find the length of an arc of any number of degrees, and of a given radius.*—The semi-circumference with a radius of 1 is 3.14159265, and if we divide this by 180 it gives the length of one degree, .017453. If we multiply this decimal (.017453) by the number of degrees in the given arc, it will give the length of that arc in a circumference whose radius is 1. Multiplying this by the given radius will give the required arc.

242. *To find the area of a sector of a circle; that is, of a figure included between two radii and an arc.*—Multiply the arc by the radius, and the product will be twice the area. Or find the area of the circle, divide it by 360, and multiply the quotient by the number of degrees in the arc.

243. *To find the area of a segment of a circle; that is, of a figure included between an arc and its chord.*—Find the area of the sector, having the same arc, and also of the triangle included between the radii and the chord. If the

arc is less than 180°, subtract the triangle from the sector; if the arc is more than 180°, add the triangle to the sector.

EXAMPLES.

244. With a radius of 7.3 inches what is the length of the circumference? Of an arc of 79°? of 53°? of $58\frac{1}{2}$°? What is the area of the circle? Of a sector of 51°? of 37°? Of a segment of 63°? of 79°? of 176°? of 183°?

245. In measuring the altitude of triangles, or the distance of any point from a line, the simplest mode, justified by the spirit, although not the letter, of the usual postulates, is to place one foot of the compasses on the point, and open them until the other foot, swinging near the line, will touch it without crossing it.

246. What is the area of a triangle whose sides are 17, 23, 31? What is the area of the four lots in Art. 213? Of the triangles in Arts. 204, 202, 200.

CHAPTER VIII.

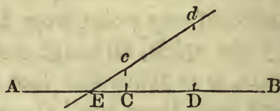
DOUBLE POSITION.

247. In many practical problems there may be no direct method of solution, and nevertheless there may be direct modes of testing the accuracy of a solution. In these cases the arithmetical rule of "Double Position" is a most valuable means of obtaining the number sought. It proceeds upon the simple supposition that the errors of a result are in proportion to the errors of the data from which the result is obtained.

248. To solve a question by double position, you must first discover a mode of testing an answer by subjecting it to calculations which, if the answer is correct, will yield a

given number. Make two "positions," or supposed answers, test them, and note the errors of the results. Then the difference of the results is to the difference of the positions as the error of either result is to the error of its position, and the solution of this problem in the "Rule of Three" will enable you to correct your "position."

249. *To solve the question of double position by geometrical construction.* — Draw a straight line A B of any length. Mark upon it, at any convenient place, a point C, to represent the smaller number of your two positions. Measure C D equal to the difference of your positions, taken



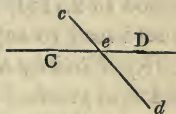
on the scale. Above or below C and D, at right angles to A B, at a distance equal to the error of their results, mark the dots *d* and *c* measured on the same scale as C D, or on a different scale, and join them by a straight line, cutting A B in E. Measure C E on the same scale as that on which C D was measured, and subtract it from the smaller position. This will give a new position, more correct than C or D. Try now two new positions, nearly equal to this corrected one, and employ a larger scale in constructing them, and thus proceed until you find a position nearly enough exact for your purpose. If either result is too small, the corresponding point *c* or *d* will be below the line A B, and E will fall between C and D, so that C E must be added to the position C. If D *d* is less than C *c*, then E will fall beyond D, and it will be better to measure D E, and add it to the position D. If the line *c d* is too nearly parallel to A B, then the distance C D must be measured on a smaller scale, or D *d* and C *c* on a larger scale.

In many cases a more exact result can be more rapidly attained by making three positions, plotting three points like *d* and *c*, and then drawing an arc instead of a straight line through them. The intersection of this arc with the line A B will then show most exactly the true point for a "position" which will stand the test.

250. A few examples will show more clearly the meaning of the above directions.

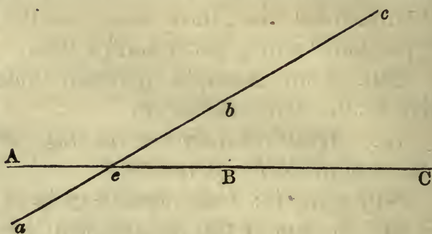
(*a.*) *What two numbers are they whose sum is* 11 *and the sum of their squares* 76?

Supposing the least number to be 3, the greater will be 8, and the sum of the squares will be 9 + 64 = 73. It is therefore plain that 3 is too large, or rather that 8 is too small. Supposing, therefore, the least number to be 2.5, the greater number will be 8.5, and the sum of the squares will be 6.25 + 72.25 = 78.5. On a straight line A B, I now make two dots, C and D, at the distance of .5 of an inch apart, because my positions for the smallest number were 2.5 and 3, and 3 — 2.5 = .5. Over C, I put a dot, *c*, at the distance of .25 of an inch, because 78.5 — 76 = 2.5, and I reduce it to one tenth the scale, for convenience sake. Under D, I put the dot *d*, at the distance .3 of an inch, because 76 — 73 = 3, which I reduce to one tenth the scale. Joining *c* to *d* by a straight line, I measure the distance from C to *e*, and find it .23 of an inch. Adding this to 2.5 gives me 2.73 for a new "position."



Supposing successively the smaller number to be 2.71, 2.72, and 2.73, the greater number would become 8.29, 8.28, and 8.27; and the sums of the squares 76.068, 75.957, and 75.846. The differences of these numbers from the required sum, 76, are + .063, — .043, and — .154.

Drawing, now, the straight line A B C, I put the points A B C at equal distances, one inch apart; that is, I map the differences between my positions, multiplying each by 100. I next put the points *a b c* below and above the line, at the distances .315, .215, and .77; that is, I map the differences of my results, multiplying each by 5. Holding, now, a straight edge from A to C, I mark the point *e* where I judge by my eye that an arc through *a b c* would cut the line A B. Measuring A *e* I find it .61, and as A B is magnified 100 times, I add .0061 to my first position, giving me 2.7161 for a new position. Then, taking 2.7161 and 2.7162 for new positions, I might construct a new one on a scale of 10,000 for 1, instead of 100 for 1, and this would give me a result still more accurate; that is, I should find that .000018 is to be added to 2.7161, giving 2.716118. This process continued would lead to any desired degree of accuracy.



(*b.*) *Find by this process two numbers whose difference is* 1 *and product* 11. — Suppose the smaller number 3 and larger 4, the product will be 12, giving an error of 1. Suppose the smaller number 2.9 and larger 3.9, the product will be 11.31, giving an error of .31. Map the differences of positions multiplied by 10, and the difference of results from 11, without changing the scale.

(*c.*) *Find two numbers whose difference is* 1 *and the difference of their squares is* 6. — Suppose the numbers to be 2 and 3, the difference of their squares is 5, an error of 1. Suppose them to be 2.2 and 3.2, the difference of their squares is 5.4, an error of .6. Suppose 3.5 and 2.5, the difference of the squares will be 6. Hence 3.5 and 2.5 is the exact answer. Double position frequently thus leads, by a fortunate guess, directly to an exact result.

(*d.*) *What number is that, the difference between whose second and third powers is* 12? — Suppose 2, and the resulting error is 8. Suppose 3, and the resulting error is — 6. Suppose 2.5, and the resulting error is 2.6. Suppose 2.6, and the resulting error is 1.18. Suppose 2.7, and the resulting error is —.39. Hence 2.5, 2.6, and 2.7

should be mapped with their difference of .1 represented by one inch, and the error of the results mapped as decimals of an inch. A curve drawn through the three points would give the result, with an error less than .001.

(*e.*) *What two numbers are they whose difference is* 1, *and the difference of their third powers* 7 ?

(*f.*) Solve the last question when you have written 8 for 7.

(*g.*) *Find two numbers such that their sum added to the square root of their sum will equal* 12, *and the sum of their cubes will equal* 189. — Suppose 3 for one number. The cube of 3 is 27, which subtracted from 189 leaves 162. The cube root of 162 is about 5.6. The sum of 3 and 5.6 is equal to 8.6, the square root of which is about 2.9. Add 2.9 to 8.6 gives 11.5, and it should give 12, so that the error of result is .5. A second position will probably give you an exact result, so that there will be no need of construction.

(*h.*) *Perform the same example, substituting* 176 for 189.

CHAPTER IX.

INTERPOLATION AND AVERAGE.

251. Suppose that you wish to know what is the highest point to which the thermometer rises on a given day, what it is at half past 10 o'clock, and what is the average heat, that is, the mean temperature of the day, and at what time the heat is greatest. But suppose that you are only able to observe the thermometer at 4 o'clock, 6 o'clock, $8\frac{1}{2}$ o'clock, and so on, at irregular hours during the day. How shall you, from these observations, find the answers to your four questions ?

You cannot obtain perfectly accurate answers in any way; but the simplest way of getting tolerably correct answers is by geometrical construction.

Draw a straight line A B, and on it mark points corresponding in their distances from each other to the intervals of time between your observations. The scale may be a quarter of an inch to an hour, one tenth of an inch to an hour, or any other scale you please. Now, from the degrees of the thermometer, at each observation, subtract the greatest number of tens contained in the lowest degree. Set the remainders in any convenient scale, say one tenth of an inch to a degree, over the corresponding point on A B. Connect the points thus obtained, by drawing through them as easy and natural a curve as possible. The highest point of the thermometer for the day will be found by measuring the distance of the highest point of the curve from the straight line A B. The time when the thermometer was highest will be found by measuring the distance from A to the point upon A B under the highest point of the curve. The temperature at $10\frac{1}{2}$ o'clock will be found by measuring the height of the curve over the point on A B corresponding to $10\frac{1}{2}$ o'clock. The question of the mean temperature, or average height of the thermometer, will be a little more difficult to answer.

Draw, from those points on the line A B between which you wish to find the mean temperature, perpendicular lines, long enough to pass through the curve, and a little more. Draw a fine silk thread tight, hold it on the paper parallel to A B, and move it nearer or farther from A B until the area between the curve and the thread, on one side the thread, seems equal to the area between the thread, the curve, and the perpendiculars on the other side the thread; the distance of the thread from the line A B will then indicate the mean temperature during the time included between the perpendiculars.

252. It is plain that the same method can be applied

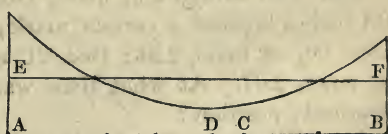
to any similar questions concerning the barometer, or dew-point, or other meteorological phenomena; or to the force of steam, or any thing varying by unknown laws.

253. A similar method may also be employed in calculating things that alter by known, but complicated laws, when you wish to arrive at an approximate result without arithmetical labor. Thus examples which we have given in the chapter on Double Position may be solved by this mode of interpolation.

Let us take, for example, the question, What two numbers differ by 2, and form the product 5? This may be solved by double position, as in Art. 250, Example (*b*), or it may be done as directed in Art. 251, by using one of the numbers as the hour, and the product as the temperature. If the number subtracted from the products, to make them easier to plot, be taken equal to the required product, 5, this process of Art. 251 becomes exactly the same as that of Art. 250. Art. 251 is therefore simply a wider and more useful application of the method which you had used in double position.

254. I found one morning the thermometer at 6 o'clock ten above zero, at $6\frac{1}{2}$ o'clock it was seven above, at $6\frac{3}{4}$ o'clock six above, at 20 minutes before 8 it was seven above, and at 8 nine above. Now, when was it coldest, what was the greatest degree of cold, what was the mean temperature for the two hours, and what was the temperature at sunrise, namely, at 20 minutes past 7?

I draw a line A B, two inches long, to represent the two hours, and mark dots upon it at distances corresponding to the intervals between the observations. Over the ends of the line,



and over the dots (finding the perpendicular by a squarecornered card), I mark points as many tenths of an inch above A B as the temperature at each observation was above 4°, six tenths above A, five tenths above B, and so on. Joining these points by a curve which looks easy and natural, I find it approaches A B most nearly at the point D, 1.08 inches from A, and is there .13 of an inch above A B. Hence I know that the greatest cold was about five minutes past 7, and that the thermometer was then at 5.3 above zero. At C, corresponding to sunrise, the distance of the curve from A B is .16 of an inch, and therefore at sunrise the temperature was 5.6 above zero. The thread E F appeared to leave a space between itself and the curve below equal to that between itself, the curve above, and the perpendiculars from A and B, when it was .29 of an inch above A B, and therefore the mean temperature of the two hours was 6.9 above zero.

255. The barometer stood, Monday morning, at 29.53 inches, Monday evening 29.45, Tuesday evening 29.61, Wednesday morning 29.73, Wednesday evening 29.71, Thursday evening 29.45, Friday morning 29.46. When was it highest, and when lowest, and what was the average height during the four days? What was the difference between its lowest and highest points?

In plotting this, the four days might be drawn four inches or four half inches. The height of the barometer, with 29.40 subtracted, would be in hundredths of an inch, 13, 5, 21, 33, 31, 5, 6. These may be mapped either as hundredths, or fiftieths, or even as tenths of an inch.

256. The sunlight at noon, Dec. 18, came on my floor 2.1 inches beyond a certain mark; Dec. 19, at noon, 2.2; Dec. 20, at noon, 2.21; Dec. 21, at noon, 2.17; Dec. 22, at noon, 2.07. At what time was the sun at his most southerly position?

257. Whenever, in mapping observations, you map on an enlarged scale, as when, in plotting the degrees of the

thermometer, you represent them by tenths of an inch, or, in plotting the example of Art. 256, you represent hundredths of an inch by tenths, you will probably find that no curve drawn through the points will look natural and easy; or at least none can be drawn without some undulation. If, then, as in Art. 256, we know that the real curve must be without waves, we may conclude that our mapping by magnifying the errors of observation has produced this appearance. We ought, in such a case, to draw a curve conforming as nearly to all the observations as a regular curve can — as little above one point or below another as possible.

CHAPTER X.

SURVEYING.

258. ENGINEERS and surveyors require nice and costly instruments for measuring angles and measuring lines; but a boy who chooses to make for himself a circle to measure angles, and a pole to measure lengths, can readily do so at a trifling cost, and find a great deal of pleasure in doing it.

259. In measuring lengths, the best instrument for a boy is a pole ten feet long, divided into feet, and each foot divided into tenths. The tenths of a foot will be rather larger than inches, and I recommend them in preference to inches simply for the ease of calculation. Feet and tenths of a foot can be written down like any other decimal fractions.

260. For measuring angles, the simplest thing for a boy's use is an instrument which he can make for himself, as follows: On a smooth piece of board, about ten inches

square, draw a circle with a radius of five inches. Through the centre of the circle draw a diagonal across the board. Starting from one of the points where this diagonal crosses the circumference of the circle, divide the circumference into 360 equal degrees. Make every fifth mark rather longer than the others, and number from 0 around to 180, each way from your starting point, where the diagonal crosses.

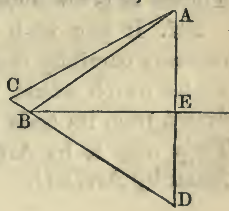
In the diagonal line outside the circumference, and near the point marked 180°, drive in a pin, which must stand perpendicular to the surface of the board. Over the other end of the line, near the zero, tack a piece of card or of zinc, with a narrow slit in it, in such manner that part of the zinc or card may stand up at right angles to the surface of the board, and the slit stand perpendicular over the diagonal. If now, looking with one eye through this opening in the zinc, the pin is made apparently to cover any small distant object, it is manifest that the diagonal line will point towards that object. Prepare now a flat stick, ten inches long, to turn freely about a screw passed through a hole in its middle, and screwed into the exact centre of the circle. One end of this stick must carry a card, or piece of zinc, perforated with a narrow opening, similar to that in the stationary piece. Next, bringing the two pieces of zinc, by turning the stick, as near to each other as possible, look through both slits at once, and insert a pin in the opposite end of the stick, so as to hide the pin already in the board. A mark of some kind (either a pencil mark or the cutting of a notch) must be made upon the end of the stick to show the exact place of the zero division when the stick is in this position. Now, it is plain that if the board is held steady and immovable, so that the stationary pin, as seen through the stationary slit, shall hide one distant object, while the revolving pin, as seen through the revolving slit, shall hide another distant object, the zero mark on the end of the stick will show upon the graduated circle what angle is made by straight lines drawn from the centre of the board to the two objects.

This simple piece of apparatus, if made with care, will serve very well for measuring the angular distance between any two stationary objects.

261. We sometimes wish to measure the angle which a line drawn from us to an object makes with a perpendicular or horizontal line. One mode of effecting this is by an artificial horizon. The surface of any fluid standing at rest is, except near the sides of the vessel, exactly level. Light is reflected from a surface at the same angle as that at which it falls upon it. For a very high or distant object, such as the heavenly bodies, we may consider the light that comes to our eye as parallel to that which falls upon a saucer of ink and water placed near the eye. For all the purpose of a boy's surveying, we may consider the top of a church steeple, or of a hill, as sufficiently distant to make the same assumption.

262. A saucer of water, colored by any thing which will prevent your seeing the inside of the saucer, may be called an artificial horizon. If you hold your head in such position as to see in the artificial horizon the image of a star or the vane of a steeple, and then, by means of the circle, measure the angle between the star and its image, half that angle will be the *altitude* of the star, or the angle which a line drawn from the vane to your eye makes with a level line.

Thus let A be a vane, B an artificial horizon, and C the eye of the beholder. D will be the apparent image of the vane. B E is a level line, and the angle A B E is half the angle A B D. But if A is so distant from B and C as to make the angle B A C insignificantly small,



A C D, which is the angle measured by the circle, may be considered as equal to A B D.

263. Instead of an artificial horizon, a plumb line may be used; but then you will require another instrument, as it would be difficult to make a plumb line hang from the exact centre of the circle, already occupied by the screw.

264. Upon a piece of board draw a quarter of a circle, and graduate it as for a protractor. At the centre of the arc drive in a pin perpendicular to the surface of the board. At the point marked 90° fasten a piece of card or zinc, pierced by a narrow slit. To the pin tie a piece of fine silk, long enough to reach beyond the farthest corner of the board, and to the end of the silk fasten a small weight. This instrument is called a quadrant. It is plain that, if the quadrant be held so as to allow the silk to play freely over the surface of the board, it will mark upon the graduated arc the measure of the angle which a line drawn from your eye (applied to the slit) to the object hid by the pin makes with the silk thread.

265. With these four simple instruments, a ten-foot pole, a circle, an artificial horizon, and a quadrant, you will be able to perform a great many interesting feats in surveying, and in the measurement of heights and distances.

266. It is easier to measure angles than to measure distances, although it requires much more care and accuracy in order to obtain good results. Men are therefore accustomed to make instruments to measure angles as perfect as possible, and then in surveying to measure only as many lengths as is absolutely necessary.

267. If you wish to make a map of a piece of land, measure carefully the length of one side, and then all the angles which would be made with this side by diagonals drawn from its ends to the different corners of the field. Then map it by Art. 195. The area of the lot may be found by Art. 234.

268. It may sometimes be more convenient to measure the length of more than one side. I only wish you to remember that the measurement of only one length is necessary, and that you will find it easier to measure an angle than a line. You can, if you please, after making a map by the measurement of one side and the angles, measure the other sides on the map, and then test the accuracy of your work by measuring them afterwards on the ground with the pole.

269. In measuring a line, you must remember that if the ground is uneven you will make the line too long if you follow the unevennesses of the ground. You must hold the pole level, and mark the spot directly under the elevated end. The area of a hill-side is not usually considered as that of its surface, but as that of its base.

CHAPTER XI.

HEIGHTS AND DISTANCES.

270. *To find the height of a perpendicular object on level ground, for instance, a building or a tree.*—From a point directly under the top of the object, measure off any convenient distance, and then with the quadrant take the *altitude*, or angle of elevation. Then, by Art. 195, draw the triangle, whose vertices are your own eye, the top of the object, and a point under the top at the level of your eye. Drawing beneath the base a line to represent the ground (the distance between this line and the base being the height of your eye), it is manifest that you can measure the height of the object, and also, if you choose, the distance from your feet to the top of the object.

271. At a distance of 60 feet from the centre of a church tower, I find the angle of elevation of the vane to be 56°. What is the height of the vane, my eye being 5 feet above the ground?

At the distance of 15 feet from a wall, the angle of elevation of its top is 63°. If my eye is 5 feet from the ground, how long a ladder will I need to climb the wall?

At the estimated distance of 5 miles, on a level, from the centre of a mountain, the image of its summit in an artificial horizon is 13° below the real summit. What is the probable height of the mountain above the point where I stand?

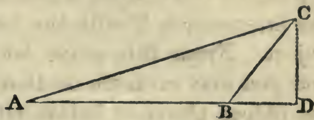
272. *From a known height to find the distance of an object on level ground.* — If we look in the opposite direction on the quadrant, that is, apply our eye near the centre, and make the pin appear in the centre of the slit at the moment when we can see the given object through the slit, it is manifest that the thread will mark the angle of elevation which we should have if viewed from the place of the object. Therefore, in this problem, the construction is the same as that of Art. 269, except that the height is given, and the angle to be used is found by subtracting the observed angle from 90°.

From the top of Prospect Hill, which I know to be 420 feet above the level of a plain below, I observe, with the quadrant, a stone wall on that plain to be 7° below a level. What is the distance of the wall from the top of the hill in a straight line, and what is its distance on a level?

Another wall is seen directly between me and the first wall, at a depression of 13°. What is the distance between these two walls?

273. *To solve Art.* 270 *when a point under the top of the object is inaccessible.* — This is frequently the case, especially with mountains and hills. The simplest mode of solution is, to measure on level ground a straight line going

directly towards the hill, and take the angle of elevation at each end of this line; then drawing a straight line, A D, measure off at one end a portion A B to represent your measured line. Drawing next the two lines, A C and B C, at the proper angles, to represent rays of light coming from the object, their intersection, C will represent the place of the object, and the perpendicular distance from the line A D can be easily measured.

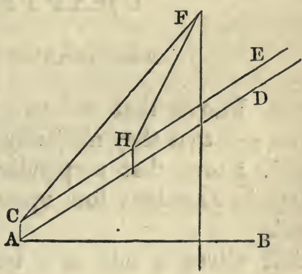


What is the perpendicular height of a hill when the measured level line is 310 feet, and the angles 27° and 15°?

274. If level ground cannot be obtained for the measurement, of Arts. 270 and 271, it is only necessary to measure by the quadrant the line of elevation of the ground on which the measurement is made. This can be done by placing at the highest end of the measured line a stake, whose top shall be on a level with your eye when you stand at its side. The angle of elevation of the top of this stake, taken from the other end of the line, will be the elevation of the ground.

275. The case supposed in Art. 270, and that in Art. 273, are in geometry the same, only that in Art. 270 one of the angles of elevation is 90°.

276. In constructing a case under Art. 274, draw a line A B to represent a level drawn through the station most distant from the object. Next draw a line A C perpendicular to A B, to represent the height of your eye. Draw now A D and C E parallel to each other, A D representing the ground, and the angle B A D being the angle of elevation of the ground.



Draw a line through C, making an angle with A B equal to the elevation of the object from C. Through H, the top of the stake, draw a line at the proper elevation, and the intersection F with the former line will represent the object. From this point let fall a perpendicular on A B, and you may measure on that perpendicular the height of the object, above or below, either the ground, or the level A B.

277. Measuring down a steep hill-side a line 220 feet long, directly towards a church steeple, which I knew to be 123 feet high, I found the angle of depression of the vane, at the upper end of the line, to be 10°, and at the lower end 4°. The depression of the line on the hill-side was 23°. How high was the upper end of the line above the foot of the steeple?

278. At the foot of a hill, which rises at an angle of 17°, the top of a tree on the hill-side has an angle of elevation equal to 37°. On measuring my distance to the foot of the tree, I find it about 417 feet. What is the height of the tree?

279. A house or tower may be taken as the side of the hill on which the base line is measured, in which case the angle of its elevation is 90°.

CHAPTER XII.

MISCELLANEOUS EXAMPLES.

280. Prove that the two diagonals of a rectangle are equal, and that they mutually bisect each other.

281. Prove that perpendiculars let fall from any two points in a straight line upon a parallel straight line are equal.

282. Given a side of a triangle 21 feet, the opposite

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angle 20°, and the ratio of the adjacent angles 3 : 1, what are the other sides? and the area?

283. One side of a triangle is 40 feet, the opposite angle is obtuse (greater than 90°), and the three angles are in the proportion of 31, 20, and 9. Required the other sides, and the area.

284. Prove that a square, circumscribed about a circle, is twice as large as an inscribed square.

285. Prove that the angle between two chords that do not touch each other is measured by half the difference of the arcs between them. (Prolong them until they meet, join their alternate extremities by a third chord, and apply Arts. 144 and 145.) Prove it also in another mode.

286. Prove that the angle made by a chord and a tangent is measured by half the difference of the arcs between them. Show what this becomes when the point of tangency is at one end of the chord, and when the chord also becomes a diameter.

287. When two chords cross, what is the measure of their angle?

288. Parallel lines intercept equal arcs. Prove it by Art. 285.

289. Prove that a perpendicular, let fall from the centre of a circle, on a chord, bisects it.

290. Two circles intersect. Given their radii, 15 and 10, and the length of their common chord, 7, what is the distance of their centres?

291. Prove that a quadrangle, whose north-east angle measures 70°, and south-west angle measures 110°, can be inscribed in a circle. (Divide it by a north-west and south-east diagonal, and prove that the circle which is circumscribed about one triangle is circumscribed about the other.)

292. Substitute 85° and 95° in Art. 291; add that a third angle is 100°, and determine, by construction, the ratio of each side to the diameter of the circle.

293. Two chords intersect. The segments nearer the centre of the circle are equal; prove that the chords intercept equal arcs.

294. Prove that parallel lines intercepted between parallel lines are equal.

295. How is the angle between two tangents to be measured?

296. Two sides of a triangle being given, and the length of a perpendicular let fall on one from the opposite vertex, to construct the triangle?

297. Two sides being given, 41 and 53, and the altitude of the triangle from the side 53 being 30°, what is the third side?

298. My house lot is a triangle, with a front of 100 feet. The perpendicular distance to the back corner is 67 feet. The perpendicular distance from one end of the front line to the opposite side is 79 feet. Draw a plot.

299. Given the base of a triangle, and its altitude, also the altitude when another side is taken as base. Construct the triangle.

300. One side of a triangle, and the perpendiculars let fall from its extremities on the other sides, being given, to construct the triangle.

301. One angle of a triangle is 70°, and the perpendiculars let drop from the other vertices on the sides are 40 and 30 feet. Construct the triangle.

302. Announce problem 300 in general form, and give a written exact rule for its solution.

303. One angle of a triangle is given, and the length of a perpendicular let fall from its vertex on the opposite side; also the perpendicular let fall from another vertex. Construct the triangle.

304. Given the base of a triangle, an adjacent angle, and its altitude. Construct the triangle.

305. On a given line, as a chord, construct an arc of a

given number of degrees. Use Arts. 289 and 286 to find the centre of the arc.

306. Given the base of a triangle, its opposite angle, and the altitude. Construct the triangle, using Art. 305.

307. On the three sides of any triangle describe squares exterior to the triangle. Connect the outermost corners of the squares by right lines, and prove that each of the three triangles thus formed is equivalent in area to the original triangle.

308. A quadrilateral, with diagonals equal to 12 and 14, is inscribed in a circle whose radius is 8. The diagonals make an angle of 78° with each other. Construct the quadrilateral. Give a general rule. Show when the problem would be impossible.

309. Given one angle of a triangle, and the segments of the opposite side made by a perpendicular let fall from the vertex. Construct the triangle.

310. A dressed piece of timber is 8 inches by 6, and is 3 feet long. What is the diagonal on the end, on the side, and on the edge? What is the longest straight line that could be passed through the timber?

311. A solid of six rectangular faces, like that of Art. 310, is a rectangular parallelopiped. Given its three dimensions, find by construction its diagonal.

312. Find by construction the value of such surds as the square root of 19, or of 29, or 39, or 79, or 17, or 24, &c.

313. The base of a triangle is 20 feet, the opposite angle 30°, and the distance from the middle of the base to the opposite vertex is 18 feet. Construct the triangle.

314. Announce Art. 313 in general form, give a general solution, and state the cases of impossibility.

315. One angle of a triangle, adjacent to the base, is 30°; the altitude is 20 feet; the diameter of the circumscribed circle is 40 feet. Construct the triangle.

316. With the same elements given as in Art. 315, when would the problem be impossible?

317. When two circles are tangent to each other, prove that the point of tangency is in a straight line, joining the centres.

318. Prove that a perpendicular from any point in a circumference, let fall on a diameter, is a mean proportional between the segments of the diameter.

319. Find by construction a mean proportional between two given lines.

320. A chord prolonged until it reaches a point without a circle may be called a secant. Prove that if two secants are drawn to one point, the entire secants are in the inverse ratio of the parts outside the circle.

321. If a tangent and secant are drawn to one point, prove that the tangent is a mean proportional between the whole secant and the part outside the circle.

322. Prove that three points fix the position of a plane. (Imagine the plane rotating on two of the points. Can it have more than one position, and still include the third point?)

323. The intersection of two planes is a straight line. Prove by Art. 12.

324. A sphere is a solid whose surface is at every point equidistant from a point called the centre. Prove that a section by a plane passing through the centre is a circle. Also prove that a section of a sphere by any plane is a circle.

325. The railroad starts due north from Tipton, and runs straight one mile, then curves to the east with a radius of 1320 feet until it bears 37° east of north, and runs straight in that direction $1\frac{1}{2}$ miles; then curves westerly, with a radius of 660 feet, until it bears 17° west of north, and runs straight in that direction $3\frac{1}{2}$ miles; then curves easterly on a radius of 2640 feet, until it runs 3° east of north, and runs straight 2 miles, to Haworth. Draw a map, determine the length of the railroad, and tell

the distance and direction, in an air line, of Haworth from Tipton.

326. The first straight line starting from Tipton rises 40 feet; the first curve rises at the rate of 30 feet to the mile; the second straight line runs level for five eighths of a mile, and rises the rest of the way at the rate of 25 feet to the mile; the second curve is level; the third straight line rises the first mile 45 feet, descends for three fourths of a mile at the rate of 48 feet a mile, and rises the rest of the way at the rate of 30 feet to the mile; the third curve rises for two thirds its length, at the rate of 35 feet a mile, and the other third descends at the rate of 40 feet a mile; the fourth straight line is descending all the way, at 30 feet to the mile. Draw a profile on a horizontal scale of 1 mile to an inch, and vertical scale of 100 feet to an inch, and tell the difference of level between Tipton and Haworth.

PART III.

SOLID GEOMETRY.

PREFATORY NOTE.

In this Third Part we have removed the figures from the text, in order to give the student *the opportunity to use his geometric imagination.* By steadily fixing the points named in his imagination, he may frequently dispense with the figure. But in recitation he should be able to draw his own figure. And in private study let him draw his own figure if he does not clearly conceive it without drawing. In unsolved problems he will frequently be obliged to use his pencil, in order to gain a clearer conception of the problem. If he can neither dispense with a figure, nor draw one himself, let him turn to the plate at the end of the volume.

In some cases the notation of the points saves the necessity of a figure. Thus, in reasoning, in Art. 412, concerning two solid bodies, each with four solid corners, we have denoted the corners of one by a, b, b', c', and of the other by a', b, b', c'; and thus the notation shows that three corners in one coincide with three in the other, and that the fourth, a, is analogous in position to the fourth, a', in the other.

CHAPTER I.

RATIO AND PROPORTION.

327. In algebraic notation letters are used to represent numbers, either known or unknown, and the results of arithmetical operations on those numbers are represented by signs.

328. The sum of the two numbers a and x is written $a + x$, and is called a plus x.

329. The difference between a and x is written $a - x$, and is called a minus x.

330. The product of a by x is written either $a \times x$, or $a \cdot x$, or simply $a\,x$, and is called simply a, x.

331. The quotient of a divided by x is written $a \div x$, or $a : x$, or $\frac{a}{x}$.

332. A power or root is written either by means of exponents or of the radical sign. Thus, the xth power of a is a^x, and the xth root of a is either $a^{\frac{1}{x}}$ or $\sqrt[x]{a}$.

333. A bar over two quantities indicates that they are to be considered together, and a parenthesis is used for the same purpose. Thus, $\sqrt{a} + x$ is the sum of x and of the square root of a; but $\sqrt{a + x}$ is the square root of the sum of a and x.

334. The notation thus far explained may be illustrated by an example; such as $\sqrt{\left((a\,x^2 + (a\,y)^2 - t)^{\frac{2}{3}} \div t\right)}$. Here the number a must be multiplied by the second power of x, and the product added to the second power of the product a times y. From this sum we must subtract the number t. The cube root of this difference must

be raised to the second power, and then divided by the number t, and finally the square root of the quotient must be extracted.

335. The sign $=$ signifies that the sums of the quantities on either side of it are numerically equal. Thus, $P = nt + m$ signifies that the number P is equal to the sum of m and the product n times t.

336. The signs $>$ and $<$ are signs of inequality. Thus, $P > Q$ and $Q < P$ signify that P is greater than Q, and Q is less than P.

337. The first letters of the alphabet usually signify known, and the last letters unknown, quantities.

338. The signs $=$, $>$, $<$, are the only verbs in algebraic language, so that each sentence must contain one of them. Such a sentence is called an equation. Equations containing the sign $>$, or $<$, are sometimes called inequalities.

339. An equation may be transposed in any form whatever, if we are but careful to preserve the equality of the two members; that is, to add or subtract from one side of the sign precisely what we add or subtract from the other, &c. Thus, suppose we have the equation

$$(a^2 - ta)^2 = (x^2 - t)^2,$$

and wish to find the value of x in terms of t and a. We may first extract the square root of each member, which simply gives $a^2 - ta = x^2 - t.$

We may now add t to each member, producing the equation $a^2 - ta + t = x^2.$

From this we may at once infer that

$$x^2 = a^2 - ta + t;$$

and extracting the square of each member, obtain

$$x = \sqrt{a^2 - ta + t}.$$

340. The quantity under the radical sign may be put into other forms, thus:—

$$x = \sqrt{a^2 + t - a t}.$$
$$x = \sqrt{a^2 + (1 - a) t}.$$
$$x = \sqrt{a^2 - (a - 1) t}.$$
$$x = \sqrt{a (a - t) + t}.$$
$$x = \sqrt{t - (t - a) a}.$$

341. The equality of two ratios is called a proportion. Thus, the proportion a is to c as A is to C is announced also by saying that the ratio of a to c is the same as that of A to C. Writing this as an equality between two quotients, we obtain, —

(1.) $$\frac{a}{c} = \frac{A}{C}$$

Multiplying both members by the quantity C c, we get, —

(2.) $$a C = A c.$$

Dividing by A C will then give us

(3.) $$\frac{a}{A} = \frac{c}{C}$$

Adding A a to both members of (2.), we obtain, —

(4.) $$A a + C a = A a + A c.$$

Multiplying each member of (2.) by 2, and subtracting the product, member by member, from (4.), gives, —

(5.) $$A a - C a = A a - A c.$$

Equations (4.) and (5.) may be divided into factors, and written as in Art. 340.

(6.) $$a (A + C) = A (a + c).$$

(7.) $$a (A - C) = A (a - c).$$

Dividing (6.) by $(A + C)$ $(a + c)$, and (7.) by $(A - C)$ $(a - c)$, will give us, —

(8.) $$\frac{a}{a + c} = \frac{A}{A + C}$$

(9.) $$\frac{a}{a - c} = \frac{A}{A - C}$$

Hence, by equation (3.), we obtain, —

(10.) $$\frac{a}{A}=\frac{a+c}{A+C}=\frac{c}{C}$$

(11.) $$\frac{a}{A}=\frac{a-c}{A-C}=\frac{c}{C}$$

Adding $a\,c$ to both members of (2.), gives us, —

(12.) $$a\,c+a\,C=a\,c+A\,c.$$

Which, divided into factors, becomes

(13.) $$a\,(c+C)=c\,(a+A).$$

And this divided by $c\,(c+C)$ furnishes us with the equation, —

(14.) $$\frac{a}{c}=\frac{a+A}{c+C}$$

A great variety of results may thus be obtained from the primitive equation (1.), all of which may evidently be of use in treating the subject of similar figures in Geometry.

CHAPTER II.

PLANES AND ANGLES.

342. If of two straight lines, having a point in common, and at right angles to each other, the first remain stationary as an axis, while the second revolves, the second will generate a surface called a plane, and the first will be a line at right angles to the plane; that is, a perpendicular to the plane.

343. It is manifest that all points in the plane will be equidistant from any two points in the perpendicular taken equidistant from its foot. A plane may be defined from the converse; that is, a surface, every point of which is equidistant from two given points, is a plane.

344. Any two points in a plane being joined by a

straight line, every part of that line is in the plane. *Proof.* Let the foot of a perpendicular be P, the given points be B and C, and let A and A′ be points in the perpendicular equidistant from P. The triangles A B C and A′ B C are equal, because their sides are equal. Let D be any point in the line B C. Then the triangles A′ B D and A B D have the sides A B = A′ B, and B D common, and the angles at B equal; whence A D = A′ D, and the point D is in the plane.

345. Three points fix the position of a plane. For if a plane, passing through two of the points, be swung round upon the line joining them, as an axis, it can evidently take but one position, including the third point.

346. Two parallel lines are of necessity in one plane. For if through any point in the second line we draw a line parallel to the first line, and in the same plane with it, it must coincide with the second line; therefore, the second line is in that plane.

347. If a straight line move in such manner that any two points in it move in parallel straight lines, it generates a plane.

348. Two lines, having a point in common, lie in one plane. For a straight line from any point in the second line drawn through its intersection with the first line, and in the same plane with the first line, must coincide with the second line.

349. A perpendicular to two lines at their point of intersection is perpendicular to their plane; that is, by Art. 1, is perpendicular to every line in the plane drawn through its foot. *Proof.* Using the notation of Art. 344, let B and C be chosen in the given lines, and we have only to show that the angle D P A is a right angle. But this follows at once from the fact that in the triangle A A′ D we have A D = A′ D, and the base A A′ is bisected at P.

350. When a line is perpendicular to a plane, the plane is also said to be perpendicular to the line.

351. It is manifest from the Pythagorean proposition, that a perpendicular measures the shortest distance from a point to a plane, and that of two lines from a point to a plane, that more nearly perpendicular is shorter.

352. The intersection of two planes is a straight line. For, by Art. 344, the straight line joining two points of the intersection, lies wholly in both planes.

353. A second plane including a perpendicular to the first is said to be perpendicular to the first.

354. If a straight line be drawn in the first plane, from the foot of the perpendicular, at right angles to the intersection, it will be at right angles to two lines in the second plane, and be a perpendicular to it. Hence, each plane is perpendicular to the other, and they are said to be at right angles to each other.

355. The angle made by two planes may be called a diedral angle. A diedral angle is measured by the angle made by two lines, one in each plane, each perpendicular to the intersection of the planes. For it is manifest, that if the planes be brought to coincidence, these lines coincide, and that if the planes be then swung open the angle of these lines is generated with exactly the velocity of the motion of the planes.

356. Parallel lines, making the angle zero, may be conceived as meeting at an infinite distance in either direction. In like manner, when a diedral angle is zero, the planes do not of necessity coincide, but are parallel, having their intersection at an infinite distance.

357. As the intersection of parallel planes may be at an infinite distance in any direction, any two parallel lines, of which one is in either plane, may be considered as measuring their angle.

358. A line parallel to a line in a plane is said to be parallel to the plane.

359. A straight line, neither perpendicular to a plane

nor parallel to it, makes an angle with it, said to be equal to the angle which the line makes with the intersection of the plane by a perpendicular plane, including the given line.

360. When two parallel planes are cut by a third, the intersections are two parallel lines. They are straight lines by Art. 352, and although in the same plane, cannot approach each other in either direction, because the intersection of the parallel planes is at an infinite distance, in any direction.

361. A straight line makes the same angle with either of two parallel planes, whether the angle be zero, a right angle, or of intermediate value.

362. Parallel lines intercepted between parallel planes are equal. For, joining the points of interception by straight lines in the planes, gives, by Art. 360, a parallelogram, whose sides are of course equal.

Hence it is evident that parallel planes are every where equidistant.

363. When two lines neither intersect nor are parallel, it may be made evident by Art. 351 that the shortest distance between them is the distance of a point in one from a plane parallel to it drawn through the other. Hence, when two lines neither intersect nor are parallel, it is evident that the right line joining their points of nearest approach is perpendicular to each.

364. If two right lines are intercepted between parallel planes, a third parallel plane will divide the intercepts in the same proportion. *Proof.* Let A, C, and A′, C′, be the points of intersection of the right lines with the first planes, and B, B′, the points of intersection with the third plane. Draw $A\,B''\,C''$ parallel to $A'\,C'$, and we have, by Art. 362, $A\,B'' = A'\,B'$ and $B''\,C'' = B'\,C'$. Completing the similar triangles $A\,B\,B''$ and $A\,C\,C''$ gives us $A\,B : B\,C = A\,B'' : B''\,C'' = A'\,B' : B'\,C'$.

365. The intersection of three planes produces triedral angles, the point common to the three planes being called their vertex. In English, the vertex of a diedral angle is usually called an edge, that of a triedral angle a solid corner.

366. If a third plane is perpendicular to each of two planes, it is perpendicular to their intersection. *Proof.* From the triedral vertex raise a perpendicular to the third plane, and as it must be in both the other planes, it will coincide with their line of intersection.

367. The vertex of a triedral angle is the vertex of three plane angles, constituting the faces (*edrai*) of the triedral angle.

368. The sum of any two of the angles of the faces is greater than the third. For, if they were simply equal, the two faces would be brought into the same plane with the third, and thus reduce the solid corner to one plane; and if the sum of two were less than the third, the solid corner would be impossible.

369. If two triedral angles have the same angles on the faces, the diedral angles between equal plane angles are equal. *Proof.* Let A be the vertex in one, A′ in the other, and the plane angles B A C, B A D, D A C, be respectively equal to B′ A′ C′, B′ A′ D′, and D′ A′ C′. Make A′ B′ = A B, and the angles A B C, A B D, A′ B′ C′, A′ B′ D′ all right angles. We have now to prove that the angle D B C equals the angle D′ B′ C′, — which is done if we prove that the triangle D B C equals the triangle D′ B′ C′. But the triangles A B C and A′ B′ C″ have the side A B and its adjacent angles equal, by construction, to the side A′ B′ and its adjacent angles. Hence, B C = B′ C′, and A C = A′ C′. In like manner B D = B′ D′, and A D = A′ D′. Then, in the triangles A C D and A′ C′ D′, we have two sides in one, with their included angle, equal to two sides of the other with their included

angle. Hence, the third sides are equal, that is, D C = D′ C′. The three sides of the triangle D B C being thus proved equal to the three sides of D′ B′ C′, the angles are equal, and the diedral angle on the line A B is equal to that on A′ B′. By proper changes in the figure the same may be proved of the other diedral angles.

370. The two triedral angles may in this case be either equal or symmetrically equivalent. Place one plane, say that of A B C, and A′ B′ C′ horizontal with the vertices from you and the lines A B, and A′ B′ on your left. If, now, the lines A D and A′ D′ are both above, or both below, the horizontal plane, the triedral angles are equal; but if one is above and one below, they are merely equivalent.

371. A solid corner made of several planes may be called a polyedral angle. If the polygon produced by a new plane, cutting off a solid piece from this corner, has no reëntering angles, the corner is called a convex polyedral angle.

372. If the sum of the plane angles about a convex polyedral angle is zero, the polyedral angle becomes a needle-point, a line; and if the sum of the angles is 2π, that is four right angles, the polyedral angle becomes a plane. The sum is always, therefore, less than 2π.

CHAPTER III.

POLYEDRONS.

373. The least number of planes that can enclose a space is four. The solid thus enclosed has four triangular faces, and is called a tetraedron.

374. If two tetraedrons have each a solid angle enclosed in three triangles, equal and similarly arranged in one and

in the other, the tetraedrons are equal. For, if one solid angle be imagined laid in the other, so as to have one of the three triangles in one coincide with the corresponding triangle in the other, the other two will coincide, by Art. 369 and by hypothesis; and the boundaries of the fourth triangle in each thus coinciding, the fourth triangles themselves will coincide. The entire surface of one solid thus coinciding with that of the other, the two solids are equal.

375. If two triangles in one tetraedron are equal to two in another, and similarly disposed, and enclose the same diedral angle, then the two tetraedrons are equal. For it is manifest that the two triangles of the one may be imagined superimposed upon the two triangles of the other, and will coincide. Two sides of each of the unknown triangles in one tetraedron will then coincide with two sides in the unknown triangles of the other, and thus the whole surfaces will coincide.

376. Polyedrons, like polygons, are called similar when their homologous angles are equal and their homologous sides are proportional. It follows, by induction from the preceding sections, that polyedrons are similar when their homologous faces are similar polygons, similarly arranged.

377. Two tetraedrons are similar if a triedral angle in one and its homologous angle in the other are composed of similar triangles, similarly arranged. For, if these two angles are superimposed, they will coincide, by Arts. 369 and 374, and the fourth planes will be parallel to each other. Hence follows, by Arts. 360 and 374, the similarity of the fourth triangles, and the equality of ratios in the homologous sides.

378. It will also be easy to show that, if two triangles in one tetraedron are similar to two in another, and similarly arranged, and enclose an equal diedral angle, the two tetraedrons are similar.

379. If all the planes of a polyedron except one have a

common point of intersection, the polyedron is called a pyramid; the common point of intersection is called the vertex of the pyramid; the face, which does not reach the vertex, is called the base.

380. A pyramid is called triangular, quadrangular, &c., from the shape of its base. The other faces are, of course, always triangles.

381. A tetraedron is, therefore, a triangular pyramid, any face of which may be taken as its base.

382. If two faces of a polyedron are equal, and their homologous sides are parallel, and if each of the other faces is a plane joining a pair of these parallel sides, the polyedron is called a prism. The parallel faces are called the bases of the prism. The other faces are evidently parallelograms. A section parallel to the base of a prism is readily shown to be a polygon equal to the base.

383. When the bases of a prism are parallelograms, the prism is called a parallelopipedon.

384. A right parallelopipedon is a prism of which every face is a rectangle. When each face is a square, the prism is called a cube.

385. When a pyramid is intersected by a plane parallel to the base, the part intercepted between the bases is called the frustum of the pyramid. The part above the cutting plane is easily shown to be a pyramid, with all its angles equal to those of the given pyramid, and therefore similar to it.

386. Let a be the length of one side of the base of a pyramid, a' that of the homologous line on the upper end of the frustum; h the height of the pyramid, h' that of the similar pyramid cut off; b and b' the slant heights of the pyramids on the edge at the left end of a and a'.

By similar triangles we have $a : a' = b : b'$. Also $h : h' = b : b'$. Whence $h : h' = a : a'$. Whence, by the theory of proportions, $a - a' : a = h - h' : h$. Thus the total dimensions of the pyramid are obtained from that of the frustum, since $h - h'$ is simply the vertical height of the frustum; and $h = \dfrac{a(h - h')}{a - a'}$

Example. What is the height of a pyramid whose base has sides of 3, 6, $4\frac{1}{2}$, and 6 inches, and at the perpendicular height of two inches the sides of the frustum are 4, 8, 6, and 8 inches? What is the slant height on the corner, on which the slant height of the frustum is 3 inches?

387. Any polyedron can be divided into pyramids by simply selecting a point within the polyedron for a common vertex, and taking the faces of the polyedron as bases for the pyramids. By taking the common vertex for the pyramids in the surface of the polyedron, the number of the pyramids may be reduced. Thus a right parallelopipedon may be divided into six pyramids; but by bringing the common vertex up to one of the faces, the pyramid of which that face was base becomes zero, and the pyramids are reduced to five; on moving the vertex to one of the edges, a second pyramid becomes zero, reducing the number to four; and on taking a vertex of the parallelopipedon as the common vertex, the pyramids are reduced to three.

388. Any polyedron can be divided into triangular pyramids by simply dividing each base in Art. 387 into triangles.

389. Two bodies which are composed of equal and similarly arranged triangular pyramids are evidently equal.

CHAPTER IV.

AREAS.

390. When two polygons have all the angles of one equal to those of the other, and similarly arranged, and their homologous sides proportional, — i. e., each pair having the same ratio to each other, the polygons are called similar.

391. Similar polygons may evidently be divided into similar triangles by diagonals from homologous vertices.

392 Lines drawn in a similar manner, in two similar polygons, may evidently be made the sides of similar triangles, and shown to have the same ratio as homologous sides of the polygons.

393. Hence the altitudes of similar triangles have the same ratio as their bases.

394. Let b be the base and h the altitude of a triangle, and x be their ratio to the base and altitude of a similar triangle. The base of the second triangle will then be $b\,x$ and its altitude $h\,x$. The area of the first will be $\frac{1}{2}\,h\,b$, and of the second $\frac{1}{2}\,h\,b\,x^2$. The ratio of these areas will therefore be x^2.

Calling now the bases and altitudes b and B, h and H, and the areas s and S, we have

$$\mathrm{B} : b = \mathrm{H} : h = x\,;$$
$$\mathrm{S} : s = x^2 = \mathrm{B}^2 : b^2 = \mathrm{H}^2 : h^2.$$

395. It may easily be shown, by help of equation (10.) (Art. 341), in the theory of proportions, that the areas of similar polygons, and of any homologous areas in or about similar polyedrons, are in the same ratio; in other words, that in similar figures homologous lines have all the same

ratio, and that this ratio multiplied by itself will give the ratio of homologous surfaces; or, in other words, that in similar figures homologous surfaces are in the ratio of squares on homologous lines, or of circles on homologous lines as diameters.

396. The similar polyedrons spoken of in Arts. 395, 385, 376, may be defined as polyedrons capable of being divided into similar tetraedrons similarly arranged.

397. By the reasoning alluded to in Art. 395, it may be shown that the external surfaces of similar polyedrons are in the ratio of squares built upon their homologous edges, or upon any homologous lines; also that any pair of homologous faces are in the same ratio.

398. If any two pyramids be cut by a plane passing at equal distances from their summits, the areas of the sections have a fixed ratio, whatever be that distance. *Proof.* Let the plane pass first at the distance h, and secondly at the distance h' from the summits, and the areas of the sections be in the first case A and B, in the second a and b. We have, by Art. 397,—

$$\frac{A}{a}=\frac{h^2}{h'^2} \text{ and } \frac{B}{b}=\frac{h^2}{h'^2}$$

Hence, $\frac{A}{a}=\frac{B}{b}$, and by (3.), in the theory of proportions, we have $\frac{A}{B}=\frac{a}{b}$, which is what we wish to prove.

CHAPTER V.

VOLUMES.

399. THE volume of a solid is the ratio which it bears to a solid unit. The solid unit generally employed is a cube, whose faces are units of area, and whose edges are units of length.

400. It will readily be perceived that the volume of a right parallelopiped is the product of the lengths of its three edges. If each edge is commensurable with the unit of length, the right parallelopiped may be divided into cubes by three series of planes parallel to its faces, evidently equal in number to the product of the numbers into which each edge is divided. If the edges are incommensurable, we can choose a unit as small as we please, and so multiply our planes that the parallelopipedon shall need but an infinitesimal change to render it capable of being divided into cubes. If the right parallelopided cannot be divided into cubic inches, it may be into cubic tenths, or hundredths, or thousandths, &c., of an inch.

401. Any parallelopipedon is equivalent in volume to any other parallelopipedon of equivalent base and equal altitude. *Proof.* Set the two parallelopipeds upon one plane, and move a second plane, parallel to the bases, from above steadily down until the two planes coincide. This moving plane moves with equal velocity through each parallelopiped, and the sections of the two are constantly equivalent from first to last. The two solids therefore pass equal volumes through the moving plane in equal times, and the volumes of the two are equal when the two planes coincide.

402. *Corollary.* Any parallelopipedon is equivalent to a right parallelopiped on of equivalent base and altitude.

403. Let the six planes of a parallelopiped be A and A′, B and B′, C and C′; similar letters denoting parallel faces. Cut the solid by a plane, A″, perpendicular to B and to C. Transpose the parts, so that A shall coincide with A′, making the surfaces on A″ new bases for the solid. Cut this new solid by a plane, C″, perpendicular to A″ and to B, and transpose so that C shall coincide with C′. The solid is thus, without loss or addition, converted into a right parallelopiped; two bases, B and B′, remaining, in altered form, of the same size as before, and their distance apart, as measured on the intersection of A″ with C″, being unchanged. From Art. 402 thus proved, Art. 401 may be drawn as a corollary. But if the plane A″ cuts A or A′, this proof needs modification, by resections.

404. The diagonal of B being parallel to that of B′, a plane may be passed through these two diagonals, dividing every section made by a plane parallel to B into two equal triangles, and the parallelopiped into two triangular prisms, equivalent to each other. Conversely a triangular prism may be considered as one half a parallelopiped.

405. Two triangular prisms of equivalent base and equal altitude are, by Arts. 401 and 404, equivalent in volume.

406. The volume of a parallelopiped, or of a prism, is, by Arts. 400, 402, 404, 405, found by multiplying the area of its base by its altitude.

407. Pyramids of equivalent base and equal altitude are equal. *Proof.* Let the pyramids be set upon a plane, and a second plane, parallel to the first, move steadily from the vertex of the pyramids to their base. As the sections of the pyramids made by the second plane are, by Art. 398, at each instant equivalent, the pyramids must be passing with equal velocity through the second plane, and the total amounts passed through at any instant are equivalent, and the whole amounts, when the two planes coincide, will be equal.

408. Every triangular prism is divisible into three

equivalent tetraedrons. *Proof.* Let the ends of the prism be the equal triangles $a\ b\ c$ and $a'\ b'\ c'$, a and a' being at the ends of the same edge, &c. Pass two planes, $a\ b'\ c'$ and $a\ b\ c'$, through the prism, and you have manifestly divided it into three triangular pyramids. But of these, two have the common vertex a, and the equal bases $b\ b'\ c'$ and $b\ c\ c'$, and are therefore equal; while the third has a common vertex c', with the first of the others, and a base $a\ a'\ b'$ equal to its base $a\ b\ b'$, and is therefore equal to either of the others.

409. *Corollary.* The volume of a triangular pyramid is found by multiplying the area of its base by one third its altitude.

410. *Corollary.* The volume of any pyramid is found by multiplying its base by one third its altitude.

411. When a prism is divided by a plane not parallel to, nor intersecting, the ends, each part of the prism is called a truncated prism.

412. The volume of a truncated triangular prism is equivalent to that of three pyramids having each a base equal to that of the prism, and altitudes equal to the altitudes of the three vertices of the prism. *Proof.* Using the notation of Art. 408, consider $a'\ b'\ c'$ as the base, and $a\ b\ c$ as the unequal triangle at the truncated end. It is manifest that of the three tetraedrons into which the planes $a\ b'\ c'$ and $a\ b\ c'$ divide the prism, one, viz., $a\ a'\ b'\ c'$, is one of the required three. A second, viz., $c'\ a\ b\ b'$, may be proved equivalent to a second of the required three, viz., to $b\ a'\ b'\ c'$. For they may be considered as having a common base, $b\ b'\ c'$, and their vertices a and a' are in a line parallel to the plane of that base. Finally, the third pyramid, $a\ b\ c\ c'$, is equivalent to the third required pyramid, $c\ a'\ b'\ c'$, because their vertices a and a' are at equal altitudes from the bases $b\ c\ c'$ and $b'\ c\ c'$, and those bases are equivalent triangles having the side $c\ c'$ in common, and the vertices b and b' in a line parallel to it.

413. The volumes of similar triangular pyramids are proportioned to the cubes built upon homologous lines. *Proof.* Let b be one edge of the base, h the altitude of that base as a triangle, and l the altitude of the pyramids. For a similar pyramid these lines become $b\,x$, $h\,x$, and $l\,x$. The areas of the bases will be $\frac{1}{2}\,b\,h$ and $\frac{1}{2}\,b\,h\,x^2$. The volumes will be $\frac{1}{6}\,b\,h\,l$ and $\frac{1}{6}\,b\,h\,l\,x^3$.

Using b, B, h, H, l, L, v, and V, as in Art. 394, we have

$$\frac{V}{v} = x^3 = \frac{B^3}{b^3} = \frac{H^3}{h^3} = \frac{L^3}{l^3}$$

That is, the volumes are in the ratio of the cubes of homologous lines; or the ratio of the volumes is the third power of that of the lines.

414. Any polyedron being decomposable into triangular pyramids, the last theorem may be extended to any similar polyedral figures.

CHAPTER VI.

THE CONE.

415. If one point in a straight line be held fast, while the line turning freely on that point be caused to glide through a plane curve, the line passing freely in space marks out a surface called the surface of a cone, whose vertex is the fixed point.

416. When the curve is a circle, and a line through its centre, perpendicular to its plane, passes through the vertex of the cone, the cone is called a circular cone; and this is the cone usually spoken of as the cone. The right line through the vertex and the centre of the circle is called the axis of the cone.

417. The base of a cone is the plane figure intercepted by its surface on any plane which cuts entirely across it.

418. The two parts of a cone lying on opposite sides of its vertex are called its nappes. In ordinary geometry we confine our attention to one nappe, and to the part intercepted between the vertex and the base.

419. As any curve may be considered a polygon of an infinite number of sides, a cone may be considered a pyramid, with a polygon for its base.

420. Hence the volume of a cone is one third the area of its base, multiplied by the altitude of its vertex above the base.

421. Hence, also, sections by planes parallel to the base will be figures similar to the base. The altitude of such a section above the base, and the lengths of two homologous lines in the section and the base, give, by Art. 386, data to determine the altitude of the cone.

422. When a circular cone has a circular base, it is called a right cone.

423. The right cone may be imagined as generated by the revolution of a right triangle about one leg as axis.

424. The convex surface of a right cone may be conceived as composed of infinitesimal triangles, all with a common vertex (that of the cone), and all with equidistant bases, forming the circumference of the base of the cone; so that this surface is measured by the product of half this circumference into the length of the side.

425. Let r be the radius of the base of a right cone, and then $2\,\pi\, r$ will be its circumference. Let l be its altitude, and then $\sqrt{r^2+l^2}$ will, by the Pythagorean proposition, be its slant height. We have, therefore, for the area of the base, $s = \pi\, r^2$, and for the area of the convex surface $S = \pi\, r\,(r^2+l^2)^{\frac{1}{2}}$; also, the volume, $v = \frac{1}{3}\,\pi\, r^2\, l$.

426. Pyramids and cones having equal altitudes will evidently be proportioned in their volumes to their bases.

427. When the vertex of a cone is infinitely distant from the base, the cone is called a cylinder; and if the cone is a right cone, the cylinder is a right cylinder. But in the consideration of a cylinder we usually limit ourselves to a part enclosed between parallel bases. Thus, a cylinder is simply a prism, with its parallelogram faces infinitely numerous and infinitely narrow.

428. The sections of a cylinder by planes parallel to its base, are evidently figures equal to the bases.

429. If the walls of the cylinder are perpendicular to its base, it is evident that the convex surface is measured by the product of the height of the cylinder into the periphery of the base.

430. The volume of the cylinder of Art. 429 is evidently the product of the area of the base by the altitude of the cylinder.

431. Hence, the volume of a cone is one third that of a cylinder of equivalent base and equal altitude.

CHAPTER VII.

OF THE SPHERE.

432. A SURFACE which curves equally in all directions, for all distances, from any initial point, is called the surface of a sphere.

433. There is a point within a sphere equally distant from all parts of its surface, which may be called the centre of the sphere. For, by definition, if we run round the sphere in any direction, keeping in the plane in which we start, perpendicular to the curved surface, we make a circle of a given size. Moreover the planes of these circles are perpendicular to the tangent plane at the initial point. Hence,

their diameters coincide, and they have a common centre — the centre of the sphere.

434. The sphere may be conceived as generated by the revolution of a semicircle about its diameter as axis.

435. The words *diameter* and *radius* may be used of a sphere in a sense analogous to that in which they are used of a circle.

436. All radii of a sphere are, then, equal.

437. Every section of a sphere made by a plane is a circle. *Proof.* Let fall from the centre a perpendicular upon the plane. Join the centre to all parts of the intersection of the surface with the plane by straight lines. As these lines are equal, they strike the plane at equal distances from the foot of the perpendicular. That foot is therefore the centre, and that line of intersecting surfaces the circumference of a circle.

438. When the plane passes through the centre of the sphere, the circle is called a great circle. All great circles, in the same sphere, are manifestly equal. All other circles on the sphere are called small circles.

439. A spherical triangle is a spherical surface enclosed between three arcs of great circles.

440. The sides of the spherical triangle measure the plane angles of the triedral angle made by their planes at the centre of the sphere. They are therefore not expressed in units of length, like the sides of plane triangles, but in degrees and minutes, which express simply their ratio to the circumference of the sphere.

441. The angles of a spherical triangle are the same as the diedral angles made by the planes of their sides.

442. Any two great circles bisect each other; for the intersection of their planes must pass through the centre of the sphere, and be a diameter in each circle.

443. Any side of a spherical triangle is, by Arts. 440 and 368, less than the sum of the other two.

444. The shortest path from point to point on the surface of a sphere is the arc of a great circle.

This is assumed as axiomatic in Arts. 432, 433. But if we assume other definitions of a sphere, we may prove this proposition thus: —

(1.) Two great circles, having two points of their circumferences in common, coincide; for these two points and the centre fix their planes as coincident.

(2.) A great circle and small circle cannot coincide for a finite distance; for, if two arcs coincide, their planes must coincide, and their radii be equal, and the circles be one and the same.

(3.) Let a and b be points on the surface of a sphere, connected by the arc $a\,b$ of a great circle, and the arc $a\,m\,b$ of a small circle. Draw the arcs of great circles $a\,m$ and $m\,b$. Then, by Art. 443, $a\,b < a\,m + m\,b$. But by drawing arcs of great circles from a, m, and b, to intermediate points on the arc of the small circle, it may be shown that $a\,m + m\,b$ is less than the sum of the four arcs; and by redivision, that these four are less than the sum of eight arcs of great circles, placed as consecutive chords in the small circle. This process may be continued until the polygon is undistinguishable from the arc of the small circle. A similar process may evidently be applied to any curve, connecting a and b, other than the arc $a\,b$.

445. The sum of the sides of a convex spherical polygon can evidently not exceed 360°, i. e., the circumference of a great circle.

446. The sum of the angles of a spherical triangle cannot, by Art. 441, exceed $3\,\pi$, or be less than π.

447. When the sum of the angles of a spherical triangle is increased to six right angles, the triangle becomes a hemisphere.

448. When the sum of the angles of a spherical triangle is decreased to two right angles, the triangle becomes infin-

itesimal in comparison with the sphere. Thus the triangles of ordinary surveying are considered as plane triangles, infinitesimal in comparison with the surface of the earth; but the triangles in surveying for maps, with sides of miles in length, are spherical triangles, the sum of whose angles exceeds π.

449. If a diameter of the sphere be drawn perpendicular to the plane of a circle, it passes through the centre of the circle, and its extremities are called poles of the circle.

450. The poles of a circle may be called its surface centres, and the arcs of great circles from the pole to the circumferences may be called surface radii.

451. The angle of two arcs of great circles (being in fact a diedral angle) may either be measured by the angle made by the intersections of the planes of the circles with the plane tangent to the sphere at the vertex of the two arcs (as is done in geodetic surveying), or by an arc struck with a surface radius of 90°, and the vertex as pole, and intercepted between the two arcs, prolonged if need be.

452. A plane perpendicular to the end of a radius is evidently a tangent plane.

453. With the vertices of a triangle as centres, and with radii each of 90°, draw three arcs intercepting. The triangle thus formed is called the polar triangle of the first.

454. Each vertex of the polar triangle is by hypothesis at 90° from each extremity of one side of the original triangle, and is, therefore, the pole of that side. For the great circle with that pole would pass through those extremities, and therefore coincide with that side. The original triangle is, therefore, a polar triangle to its polar triangle; and the two triangles may simply be called polar triangles.

455. If the arcs of Art. 453 be limited so that the pole of each arc be on the same side of it, as the triangle is, then the sides of a triangle are supplements to the angles

of its polar triangle. *Proof.* Let $a\ b\ c$ and A B C be the vertices of the two triangles. Let $b'\ c'$ be the points at which $a\ b$ and $a\ c$, prolonged if necessary, intercept B C. Now we have, by construction, —

$$B\,c' = 90°,\ C\,b' = 90.$$

But $\quad B\,c' = B\,b' + b'\,c'$, and $B\,b' + C\,b' = B\,C$.

Hence, $\quad B\,b' + b'\,c' + C\,b' = 180° = B\,C + b'\,c'$.

But $b'\ c'$ is the measure of the angle a. Hence, the angle a and the side B C are supplements of each other; their sum is $\pi = 180°$.

456. If the three sides of a triangle are respectively equal to the three sides of another, the two triangles are said to be equilateral with respect to each other.

457. If two triangles on the same sphere, or on equal spheres, are equilateral with respect to each other, they are also equiangular with each other; which follows from Arts. 440 and 369.

458. Place the centres of the equal spheres together, and bring one side of one triangle, say A B, into coincidence with the equal homologous side A′ B′ on the other triangle. If, now, the equal angles A and A′ lie on the same side of A B, the triangles evidently coincide. But if the angles A and A′ lie on opposite sides of the common side, these triangles, equilateral and equiangular with respect to each other, are called symmetrical.

459. If two triangles are equiangular with respect to each other, they are also mutually equilateral. *Proof.* For their polar triangles are, by Art. 455, equilateral, and therefore, by Art. 457, equiangular with respect to each other. Hence, by Art. 455, the triangles themselves are equilateral with respect to each other.

460. Spherical triangles, having one side and the adjacent angles, or two sides and the included angle, in one, equal to the like parts in the other, are either equal or symmetrical. For it may readily be shown that one would

coincide with the other, or with a triangle symmetrical with the other.

461. In measuring spherical surfaces a peculiar unit is sometimes used, namely, the surface of a triangle whose three sides are 90°, 90°, and 1°. This surface is called one degree of surface, and is $\frac{1}{720}$th of the surface of a sphere.

462. The surface included between two great semicircles is called a lune.

463. The surface of a lune is double the angle of the lune. In other words, the surface of the lune is to that of the sphere as the angle of the lune is to 360°, or double the angle is to 720°.

464. Symmetrical spherical triangles are equivalent in area. *Proof.* Place the vertex of one angle upon the vertex of the equal angle in the other triangle, giving the sides the same direction. Part of the triangles will coincide. The non-coincident parts will be new triangles, mutually equiangular, and therefore symmetrically equilateral. The same operation may be repeated upon them, and upon their non-coincident parts, until, finally, the non-coincident parts are infinitesimal symmetrical triangles, or zeros, mutually equiangular and equilateral, whose difference will differ nothing from zero.

465. *To measure the surface of a spherical triangle.* — *Solution.* Prolong one side, say A C, into a great circle. Prolong A B into a semicircle meeting A C prolonged in A′; and in like manner prolong C B to C′.

Now, the triangle A′ B C′ is symmetrical or equivalent to the triangle required to complete the triangle A B C into a lune with the angle B.

And the surface of the hemisphere (= 360°) is equal to the lune C A B C′ plus the triangle A′ B C′, plus the triangle A′ B C. The triangle A′ B C is, however, the lune A B C A′ minus the triangle A B C. Substituting for each lune the double of its angle gives us, —

$360° = 2\,C + 2\,B -$ triangle $A\,B\,C + 2\,A -$ triangle $A\,B\,C$.
$360° = 2\,(A + B + C) - 2$ triangle $A\,B\,C$.
$180° = A + B + C -$ triangle $A\,B\,C$.
Triangle $A\,B\,C = A + B + C - 180°$.
That is, the surface of a triangle is to that of its sphere as the excess of the sum of its angles over 180° is to 720°.

466. *To measure the surface of a frustum of a right cone.* — Let S be the area of the curved surface of the frustum, S$''$ that of the cone required to fill the deficiency of the frustum, S$'$ that of the cone thus completed; and let s, s'', and s' be the slant heights of the three bodies. Also, let r and r' be the radii of the two bases of the frustum. We then have, by Art. 424, and by the evident relations of the bodies, —

$$S' = \pi\, r\, s', \quad S'' = \pi\, r'\, s'',$$
$$S = S' - S'' = \pi\,(r\, s' - r'\, s'').$$

But, on passing a plane through the apex of the cone and the centre of the base, it will be evident, from similarity of triangles, that

$$r : r' = s' : s''; \text{ whence, } r'\, s' = r\, s'';$$

and it cannot alter the value of any quantity to add to it $r'\, s' - r\, s''$. We may, therefore, write, remembering that $s = s' - s''$,

$$S = \pi\,(r\, s' - r'\, s'' + r'\, s' - r\, s'') = \pi\,(r + r')\, s.$$

The quantity $\pi\,(r + r')$ is the circumference of a circle, whose radius is one half the sum of the radii of the bases; and it may readily be shown, by Art. 421, and reasoning similar to that of Art. 364, that this circle is a section of the frustum, parallel to the bases, bisecting the distance between them. Hence, the convex surface of the frustum of a right cone is measured by the product of its slant height into the circumference of a section midway between the bases.

467. Pass a plane through the centres of the bases of the frustum, and calling the length of half the sections of

the bases r and r', as in the preceding article, and the section of convex side s, let fall a perpendicular of the length h from the extremity of r' upon r. From the mid-point of s draw a perpendicular to the axis of the cone, and its length is readily shown to be $\frac{1}{2}(r + r')$. From the same point draw a perpendicular to s, until it touches the axis of the cone, prolonged if need be, and call the length of this line R. We have now two right triangles, of which the sides h and s in one are respectively perpendicular to two sides $\frac{1}{2}(r + r')$ and R in the other. They therefore enclose equal angles, and the two right triangles are similar, which gives

$$s : \mathrm{R} = h : \tfrac{1}{2}(r + r'), \qquad \mathrm{R}\,h = \tfrac{1}{2}(r + r')\,s,$$
$$\mathrm{S} = 2\,\pi\,\mathrm{R}\,h.$$

That is, the convex surface of a frustum of a right cone is measured by the product of its altitude into the circumference of a circle whose radius is a perpendicular to the slant surface reaching from its mid-point to the axis of the cone.

468. A portion of a sphere enclosed between two parallel planes — that is, a spherical segment with parallel bases — may, if the bases are brought infinitely near together, be considered as an infinitesimal frustum of a right cone, whose axis passes through the centre of the segment and of the sphere, and whose slant surface is perpendicular to the radii of the sphere touching the surface.

469. A hemisphere may thus be cut, by an infinite number of planes parallel to its base, into an infinite number of frustums of different cones. But all these frustums agree in having their axis pass through the centre of the sphere, and in having also perpendiculars to their slant surface pass through the centre of the sphere. Moreover, the sum of their altitudes is the radius of the sphere. Hence, by Art. 467, the surface of a hemisphere with the radius R is

$$\mathrm{S} = 2\,\pi\,\mathrm{R}^2.$$

But as the area of the base is $2\pi R \times \frac{1}{2} R = \pi R^2$, it follows that the convex surface of a hemisphere is double the plane surface of its base.

470. The surface of a sphere is therefore equivalent to that of four great circles.

471. As a sphere may evidently be divided into pyramids, with their common apex at the centre, and the sum of their bases constituting the surface of the sphere, we have, for its solidity, by Art. 410, —

$$V = \tfrac{4}{3}\pi R^3.$$

472. Using $D = 2R$ as the diameter of the circle and sphere, we have for the area of a circle, —

$$A = \pi R^2 = \tfrac{1}{4}\pi D^2 = .7854 \times D^2.$$

$$V = \tfrac{4}{3}\pi R^3 = \tfrac{1}{6}\pi D^3 = .5236 \times D^3.$$

CHAPTER VIII.

PROBLEMS AND THEOREMS.

473. Given the angle which a straight line makes with a plane. Find, by geometrical construction, the altitude from the plane of a point in the line at a given distance from its intersection with the plane. For example, what is the height of one end of a yardstick, the other end resting on the floor at an angle of 37°?

474. Find, by geometrical construction, the angle which a line makes with a plane, having given its altitude above two given points in the plane, vertically under it; and find the place of its intersection with the plane. For instance, let two posts on level ground be seven feet apart, and be, one four, the other six feet high. What is the inclination to the horizon of a line joining their summits, and where would it strike the ground?

475. Given the altitude of a second plane above three given points in a first plane, to find by geometrical construction the line of intersection of the planes, and their diedral angle.

476. A right line drawn from one vertex of a parallelopipedon to the vertex, which has no plane in common with the first, is called a diagonal of the parallelopipedon. Prove that a plane containing one diagonal may be rotated upon it until it contains a second. Prove that all four diagonals have a common point of intersection. Prove that each diagonal is bisected at this point.

477. The convex surface of the frustum of a sphere is called a zone. Prove that the area of a zone is the product of the altitude of the frustum into the circumference of a great circle.

478. Prove that the two tangents from a given point to a circle are equal. Prove that a right line from the given point to the centre of the circle bisects the angle between the tangents.

479. Prove that when a chord is bisected by a diameter, the semichord is a mean proportional between the segments of the diameter. Find a mean proportional between two given lines.

480. Prove — what is assumed in Art. 468 — that the line of tangency of surfaces, when a sphere is enclosed in a hollow cone, is the circumference of a circle whose plane is at right angles to the axis of the cone.

481. Prove that, if two chords are prolonged until they meet, the entire lines thus produced are inversely proportioned to the parts without the circle. Calling these entire lines secants, prove that if, from a given point, a tangent and a secant be drawn to a circle, the tangent will be a mean proportional between the secant and the part without the circle.

482. Find the centres, by construction, of the two cir-

cles whose circumferences pass through two given points and are tangent to a given right line in one plane with them. When would this problem become absurd?

483. Find the centres of the two circles whose circumferences pass through one given point and are tangent to two given right lines all in one plane. Remember that a right line passing through the centres is readily found, and that a perpendicular upon it from the given point is a semi-chord in both circles.

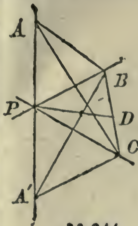
484. Find the centres of the four circles which are tangent to three given right lines in one plane. In what case would the problem be impossible?

485. It is evident that circles concentric with those of Art. 483 would pass equally near the given points and the given right lines. Find, then, the centres of the four circles which are tangent to a given circle and to two given right lines.

486. Given the radius of a sphere and distance at which a plane passes from its centre. Find the radius of the section. When does this radius equal that of the sphere? when become zero? and when become impossible?

487. A given sphere has its centre on the axis of a given cone, at a given distance from the vertex. Find the radii of the two circles made by the intersection of the surfaces. In what cases will they be equal? In what case will the two circles coincide? When will they coincide with a great circle?

488. Given a great circle in a sphere, and the radii of two small circles parallel to it. Find the hollow cones tangent to the sphere on the small circles. Find also the cones whose intersections with the sphere would give the circumferences of both circles. Four cones are required in each case.



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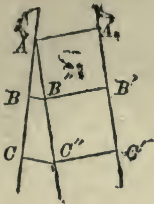

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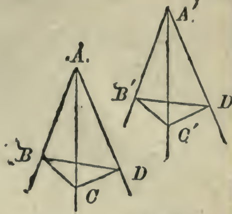

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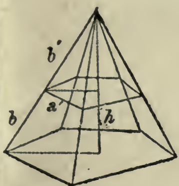

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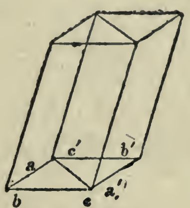

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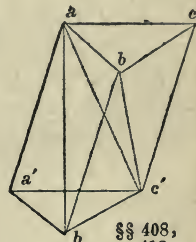

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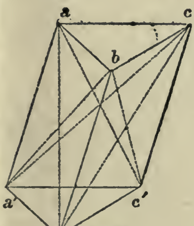

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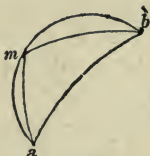

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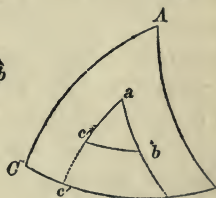

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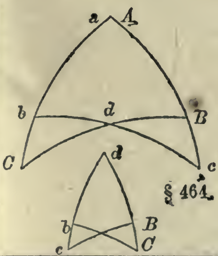

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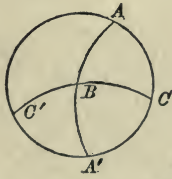

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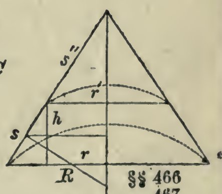

§§ 466, 467