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Sequential Auctions of Non-Identical Objects

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Abstract:

Weber [1983] argues that the expected equilibrium prices of identical objects auctioned sequentially to expected profit maximizing bidders with symmetrically distributed privately-known values (with each bidder winning at most one object) should all be equal. In fact, in actual auctions, the prices seems to tend downwards. We show that for similar objects--objects having statistically identical, independent values--the trend will be upwards in some cases, but the overall trend will be downwards for any value distribution with bounded support.

We also consider non-similar (but still independent) objects and argue that the seller benefits from selling first the objects that contribute most to bidders' profits. To the extent that bidders most highly value those objects with high variance--high variance in bidders' values leads to high winner profits--sequentially auctioning the option to choose one of the remaining objects implements a good order.

Introduction:

In many actual sequential auctions of similar objects--auctions of farmland, of used restaurant equipment, of a bankrupt construction firm's inventory, of nursery stock, and of dairy cattle--the price tends to drop from one object to the next. This contrasts with the existing theory. In particular, Weber [1983] argues that in sequential auctions of identical objects to expected profit maximizing bidders with symmetrically-distributed privately-known values (with each bidder allowed to win at most object), the prices will be a martingale; ex ante, each object has the same expected price. More generally, with affiliated information, the prices will be a submartingale; the prices will tend to drift upwards. Of course, a particular realization of a submartingale might consist of a strictly decreasing sequence of numbers. But this would be an atypical outcome. And prices in actual auctions seem to trend downwards far too often to be explained as atypical outcomes.

Assuming the objects to be identical misses what turns out to be an essential element of the above mentioned auctions. The objects are not indeed identical. For example, the used restaurant tables that were auctioned sequentially were basically similar, but varied in structural soundness, in condition of the table top, and in condition of the base. Similarly, each dairy cow differed slightly in age, milk yield history, and genetic stock; all these factors affect bidders' values. So, instead of a bidder's value varies from object to object.

We model the values of different objects to a particular bidder as being independent draws from some fixed distribution. This may indeed be just as unrealistically extreme an assumption as is the assumption of identical objects--that is, an assumption that the values are perfectly correlated. But, the independent objects assumption provides an insightful contrast to the identical objects assumption. Going all the way to this extreme also simplifies the analysis and the interpretation of the results; in particular, the independence across objects avoids the effect of signalling (as studied, for example, by Ortega-Reichart [1968] and Engelbrecht-Wiggans and Weber [1987], and which underlies Weber's upward price trend result.)

Our results contrast with those for identical objects. In particular, the price trend now varies with the distribution of the bidders' values. For certain distributions of values, including the exponential, the expected prices increase strictly from one object to the next. But, for other distributions, including the uniform, the expected prices decrease strictly. And, in general for bounded distributions, for large enough numbers of objects, the overall price trend is downwards. That is, our model's predictions seem consistent with the sustained downwards sequences of prices in actual auctions.

The analysis also reveals the forces underlying the price trends in sequential auctions. A closer look at one of these--the effect of decreasing opportunities--has practical implications in the case of non-similar objects. In particular, the seller benefits from first selling the objects that contribute most to the bidders' profits. If the seller does not know which objects these are, then to the extent they are the objects that bidders consider most valuable, sequentially auctioning the right to choose one of the remaining objects implements the desired order.

<u>The Basic Model:</u>

We start by defining our model. In particular, imagine that n objects will be auctioned one after another without reserve. Initially, there are n+m (m>0) expected profit maximizing bidders. Each bidder may win at most one object; thus, the number of bidders drops by one in each auction and there will be $n-j+m+1$ bidders in the j^{th} auction.

To specify the informational assumptions, let $X_{i,j}$ denote a random variable with outcome $x_{i,j}$. Bidder i has a value of $x_{i,j}$ for object j; bidder i knows $x_{i,j}$ when bidding on object j, but does not yet know $x_{i,j+1}$, $x_{i,j+2}$, ... , $x_{i,n}$. (For notational simplicity, i always ranges from one to n+m, while j always ranges from one to n; bidders who have already won an object can simply ignore subsequent $x_{i,j}$'s.) Assume that for each i, $X_{i,1}$, $X_{i,2}$, ... $X_{i,n}$ are identically and independently distributed. Also, assume symmetry--but not necessarily independence--across bidders; more precisely, the joint distribution of $X_{1,j}$, $X_{2,j}$, ... , $X_{n+m,j}$ is symmetric in its arguments. Finally, assume that everyone knows the joint distribution of the bidders' values.

In the previously mentioned examples, the auctions followed the common ascending-price oral format. In our model, we use Vickrey [1961] auctions--sealed-bid sales in which the highest bidder wins, but pays only an amount equal to the highest losing bid. Given the private values nature of our model, the Vickrey auction provides a simple, seemingly plausible approximation to actual oral auctions.

An Equilibrium:

To derive an equilibrium bidding strategy for our sequential auctions, we will start with the last auction and work toward the first. To derive the condition applied iteratively in this process, focus on any one auction, say that of object j ($j<n$). Imagine that the strategies used in subsequent auctions will be independent of how others bid in this auction (but not necessarily independent of who wins this auction); given the independence across objects, this will be true in the equilibrium that we identify for our model. Then fixing the strategies used by bidders in subsequent auctions fixes the expected profit to each bidder in subsequent auctions (conditional on the outcome of this auction).

We will repeated refer to these profits. So, let $L_{i,j}$ denote the expected profit to bidder i from auctions j+1, j+2, ... , n conditional on bidder i losing object j. For the moment, also consider the case in bidders may win more than one object--in this case, the value $x_{i,j}$ may depend on what objects i won previous to the auction of object j--and let $W_{i,j}$ denote the expected profit to bidder i conditional on winning object j. Since object n is the last object, define $L_{i,n}$ and $W_{i,n}$ to be zero for all bidders i.

This allows us to consider the auction of object j in isolation from other auctions. In particular, in auction j, bidder i in effect has a privately-known net value of $x_{i,j} + W_{i,j} - L_{i,j}$ for winning. So, the standard argument for Vickrey auctions with privately-known values (see for example, Engelbrecht-Wiggans [1991]) yields the following:

Proposition 1: If bidder i knows $x_{i,j}$, $L_{i,j}$, and $W_{i,j}$ (and the latter are independent of how bidder i bids in this auction) when bidding on object j, then bidder i should truthfully bid equal to the net value that would be gained by winning this auction. More precisely, bidder i has the dominant strategy in this auction of bidding equal to $x_{i,j} + W_{i,j} - L_{i,j}$.

If bidders may win any number of objects, and each bidder's value for a set of objects equals the sum of the individual values, then $L_{i,j}$ equals $W_{i,j}$, and each bidder i has the dominant strategy of bidding equal to $x_{i,j}$. That is, bidders can bid in each auction as if it were the only auction. We later refer to this as the case of "unrestricted, independent auctions."

Repeated application of Proposition 1 defines an equilibrium strategy for a sequence of auctions. Specifically, in the last auction, let bidders bid truthfully (that is, equal to their $x_{i,n}$'s), just as they might be assumed to do if this were the only auction. Then, calculate the $L_{i,n-1}$'s and the $W_{i,n-1}$'s resulting from such bidding. Next, let bidders in the next to last stage bid truthfully as defined by the proposition. Then calculate the $L_{i,n-2}$'s and the $W_{i,n-2}$'s resulting from this two-stage strategy. Iterate this process; the subsequent calculation of equilibrium profits will be for the equilibrium defined by this process. (The equilibrium strategies so defined are not dominant strategies; in particular, $L_{i,j}$ and $W_{i,j}$ depend on how other bidders other bidder i bid on objects subsequent to object j, and if for some reason of their own, each bidder other than i bid extremely large amounts on each object, then bidder i's $L_{i,j}$ and $W_{i,j}$ would be zero for all j, and i should bid differently than if they were non-zero.).

Return now to the case in which each bidder may win at most one object. Thus, $W_{i,j}$ equals zero for all i and j. And $L_{i,j}$ will be independent of i; thus, we hereafter simply write L_j.

Expected Equilibrium Profits:

To derive the expected profits at the equilibrium resulting from iteratively applying Proposition 1, start by looking at the contribution to the bidders' profits made by the auction of any one object, say object j. In particular, let π_j denote the expected profit in auction j (ex ante to the bidders seeing their values for object j) to each bidder; since there will be n-j+m+1 bidders on object j, the bidders together have an expected profit of $(n-j+m+1)\pi_j$ in this auction, all of which profit goes to the one bidder who actually wins.

Next quantify π_j. Clearly, if each bidder i bids a constant--in particular, L_j--more than the value $x_{i,j}$, then the winner's profit will be exactly this constant amount greater than if everyone bid equal to their corresponding $x_{i,j}$. But Engelbrecht-Wiggans [1991] establishes that when bidders bid equal to their true values, each bidder has an expected profit equal to the expected marginal contribution this bidder makes to the expected total social value. More precisely, if each bidder i were to bid $x_{i.j}$ on object j, then π_j would equal $v(n-j+m+1) - v(n-j+m)$, where $v(k)$ denotes the expected social value $E[\max\{X_{1.j}, X_{2.j}, \ldots, X_{k.j}\}]$ of an object sold in an auction with k bidders. In fact, each bidder i bids $x_{i.j} + L_j$, and so the n-j+m+1 bidders together actually have an expected profit of $(n-j+m+1)\ [v(n-j+m+1)-v(n-j+m)] + L_j$.

Now, to get L_{j-1}--the total expected profit to each bidder from auctions j, j+1, ..., n just before bidders see their value for object j--consider two cases. If i loses object j, i gets nothing from auction j. So, conditional on losing object j, i has an expected profit of L_j from auctions j and j-1, j-2, ... , n. Alternatively, if i wins object j, i gets nothing from subsequent auctions. So, conditional on winning, i has an expected profit of $(n-j+m+1)\ [v(n-j+m+1)-v(n-j+m)] + L_j$. And, given the ex ante symmetry of all bidders, each of the bidders has a $1/(n-j+m+1)$ chance of winning the auction. Thus, L_{j-1} equals $[1/(n-j+m+1)]\ \{(n-j+m+1)\ [v(n-j+m+1)-v(n-j+m)] + L_j\} + [1 - 1/(n-j+m+1)]\ L_j$. Simplifying this expression yields $L_{j-1} = L_j + [v(n-j+m+1)-v(n-j+m)]$. This together with $L_n = 0$ yields the following:

Proposition 2: If for each object, each bidder on that object bids truthfully (as defined in Proposition 1), then each bidder's total ex ante expected profit equals the sum of each bidder's expected profits from individual unrestricted independent auctions with appropriate numbers of bidders. More precisely, $L_{j-1} = \Sigma_{k\geq j}\ [v(n-k+m+1)-v(n-k+m)]$ $(= v(n-j+m+1) - v(m))$.

Expected Equilibrium Prices:

Now we compute the expected equilibrium prices $p_1, p_2, \ldots, p_n$. Clearly, the expected price of an object auctioned to k bidders must be the expected social value $v(k)$ minus the k bidders' expected profit from this object. But, we previously found that n-j+m+1 bidders together have an expected profit of

$(n-j+m+1)[v(n-j+m+1)-v(n-j+m)] + L_j$ on object j, and that L_j equals $v(n-j+m)-v(m)$. So, appropriate substitutions and simplifications yield the following:

Proposition 3: The equilibrium price p_j of object j is given by the expression $v(m) - (n-j+m)[v(n-j+m+1)-v(n-j+m)]$.

Let us calculate the expected equilibrium price p_j for several distributions. One, if the bidders' values are independent samples from the uniform distribution on the interval [a,b] (with a large enough so that all bids exceed zero, or any other specified reservation price), then $v(k) = b - (b-a)/(k+1)$, and thus $p_j = b - (b-a)[1/(m+1) + (n-j+m)/(n-j+m+2)(n-j+m+1)]$. Clearly, as n-j increases, this increases toward $b - (b-a)/(m+1)$. So, as j increases, p_j decreases; for the uniform distribution, the expected prices form a strictly decreasing sequence.

Two, if each value equals a constant c (large enough so that all bids exceed any reservation price) plus an independent sample from the exponential distribution with mean u, then $v(k)-v(k-1) = u/k$, and thus $p_j = v(m) - (n-j+m)u/(n-j+m+1)$. Clearly, as n-j increases, this decreases toward $v(m)-u$. So, as j increases, p_j increases; in this case, the expected prices form a strictly increasing sequence.

Even though the sequence of prices may go in either direction, we can make a more specific, potentially practical statement. In particular, if the support of the distribution is bounded above--this rules out the second example above, but would be a plausible assumption for actual auctions--the expected prices will tend to decrease.

To obtain the desired result, define $k=n-j+m+1$ and rearrange the expression for p_j to get $p_{n-k+m+1} = v(m) - k[v(k)-v(k-1)] + [v(k)-v(k-1)]$. The first term of this expression is independent of k. The negative of the second term equals the k bidders' combined expected profit at the dominant strategy equilibrium in a single-object Vickrey auction, and therefore equals the difference between the largest and the second largest of k identically (but not necessarily independently) distributed samples. (See Proposition 1 of Engelbrecht-Wiggans for a formal derivation.) Clearly, for a bounded (symmetric) distribution, this difference must eventually go to zero as k goes to infinity, and thus the second term eventually disappears. (In contrast, for the exponential distribution, it remains constant, equal to the mean of the distribution.) Since the second term is always negative and eventually goes to zero, and since the sum of the second and third terms is clearly always negative, the sum of the second and third terms must eventually go to zero. And so, for large enough k, the expected price will be greater than for smaller k. But large k correspond to auctions early in long enough sequences. So, overall, the prices tend downwards.

Discussion:

Three phenomena affect prices in sequential auctions. One effect--that of decreasing opportunities--works to raise prices on later objects. In particular, in contrast to losing in the last auction, losing in an earlier auction still leaves a bidder with opportunities to make a profit before the end of the sequence; not all is lost if you lose in an early auction. So, bidders might bid more aggressively in later auctions.

The other two effects arise from the fact that in our model--and, perhaps, quite typically--later auctions have fewer bidders. One, with fewer bidders, bidders might bid less aggressively. Two, at least in our model, the expected social value generated by an auction drops as the number of bidders drops. Together these two effects work to lower prices on later objects. Perhaps, indeed, this last effect--the dependence of the social value on the number of bidders--drives our results. But even so, except in the unrealistically extreme case of pure common values, this effect exists to some degree.

<u>Setting an Order for Selling the Objects:</u>

In the case of identical objects, each object has the same expected price ex ante. The expected total revenue to the seller(s) is unaffected by the order in which the objects are sold. Even if the auction consists of objects owned by different sellers, ex ante, the order in which the objects are sold makes no difference to any one seller's expected revenue.

In contrast, the order does make a difference for non-identical objects. In the case of statistically identical objects, the order makes no difference to the expected total revenue to the sellers. But, to the extent that expected prices decrease from one object to the next, each seller should like to see his or her own objects offered for sale early in the sequence.

In the case of dissimilar objects, the order also affects the expected total revenues. To illustrate, consider a simple example with two objects. Imagine that all bidders have very nearly the same value for object A, but have quite different values for object B. Consider the two possible orders for these two objects; assume that bidders bid truthfully as defined in Proposition 1.

First, sell object A last. Then all bidders will bid essentially the same amount in the last auction, and this last auction generates essentially no profit for its winner. So, in the first auction--the auction of object B--bidders bid very nearly truthfully. And thus, in total, the bidders' combined profit in the two auctions barely exceeds the profit they could expect from a sale of object B alone.

Second, sell object B last. Now the last auction generates a substantial expected profit. So, in the first auction, bidders now shade their values by a substantial amount. Even though all bidders will be bidding very nearly the same amount, this amount will be substantially less than their values for object A. Thus the first auction also generates a substantial expected profit. Together, the two auctions in this order generate nearly twice the expected profit for the bidders as does the other order.

Also consider the effect of the order on the expected total social value. Changing the number of bidders hardly affects the expected social value of object A. Not so for object B; increasing the number of bidders increases the expected social value. Thus, selling object B first--that is, selling object B in the auction with the larger number of bidders--generates a greater expected total social value for the two objects than does the other order.

Both effects work in the same direction. Selling object B first yields both a higher expected value for the objects and lower expected profits for the bidders. Clearly, the seller--or the sellers together, if there is more than one--benefit from selling object B first.

It is the variance, not the mean, of the distribution that matters. In particular, increasing the variance in the bidders' values for an object tends to increase both the bidders' profits and the effect that changing the number of bidders has on the expected social value. So, in general, roughly speaking, objects with high variance in the bidders' values should be sold first. (In our example, even if, with probability one, everyone values object A much more than object B, selling object B first generates the greater expected total revenue.)

Finally, consider the option auction--a sequential auctioning of the right to chose one of the remaining objects. If the bidders' values vary more from one object to another than from one bidder to another, then such option auctions tend to sell the objects in order of decreasing value; "value" here could be defined as the average of the means of the bidders' marginal distributions for the value of the object. And, to the extent that high values correspond to high variances, the option auction tends to implement a desirable order of sale. So, an auctioneer unfamiliar with bidders' preferences for a particular collection of dissimilar objects may prefer the option auction. Indeed, in the sale of restaurant equipment, the auctioneer turned to the option auction when faced with selling four very used, very different major appliances.

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