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Set-Valued Functions of Two Variables in Economic Theory

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ABSTRACT

Several properties of set-valued functions of two variables are studied. Specifically, we study the existence of : (i) Carathéodory-type selections, (ii) random fixed points and (iii) random maximal elements. An application to the problem of the existence of a random price equilibrium is also given.

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1. INTRODUCTION

Research in economic theory [see for example the classical treatise of Debreu (1959)] has necessitated the use of set-valued functions. The basic properties of set-valued which have been useful in economics are: The existence of fixed points, the existence of continuous selections and the existence of measurable selections. Before we discuss the main purpose of this paper, it may be instructive to outline a basic argument to describe the above properties.

Let X be a paracompact space, Y be a linear topological space and $\phi\colon X \to 2^Y$ (where 2^X denotes the set of all nonempty subsets of X) be a set-valued function such that for all $x \in X$, $\phi(x)$ is convex, nonempty and for each $y \in Y$ the set $\phi^{-1}(y) = \{x \in X\colon y \in \phi(x)\}$ is open in X. We will show that there exists a continuous function $f\colon X \to Y$ such that $f(x) \in \phi(x)$ for all $x \in X$, i.e., $\phi(\cdot)$ admits a *continuous selection.*

Since for each $y \in Y$, $\phi^{-1}(y)$ is open and $\phi(x)$ is nonempty for all $x \in X$, the collection $\mathbf{F} = \{\phi^{-1}(y)\colon y \in Y\}$ is an open cover of X. Since X is paracompact there is an open locally finite refinement $\mathbf{F}' = \{V_a\colon a \in A\}$ of $\mathbf{F}$. We can use a standard result [see for instance Dugundji (1966)] on the existence of a partition of unity subordinated to the above covering, i.e., we can find a set of continuous functions $\{g_a\colon a \in A\}$ such that $g_a\colon X \to [0, 1]$, $g_a(x) = 0$ for $x \notin V_a$ and $\sum_{a \in A} g_a(x) = 1$ for all $x \in X$. Since $\mathbf{F}'$ is a refinement of $\mathbf{F}$, for each $a \in A$ we may choose $y_a \in Y$ such that $V_a \subset \phi^{-1}(y_a)$. Define the function $f\colon X \to Y$ by $f(x) = \sum_{a \in A} g_a(x) y_a$. Given the fact that $\mathbf{F}'$ is locally finite and $\phi(\cdot)$ is convex valued, one can easily verify that $f\colon X \to Y$ in continuous and $f(x) \in \phi(x)$ for all $x \in X$, i.e., $\phi(\cdot)$ admits a continuous selection.

Note now that if X is a conpact, convex, nonempty subset of a locally convex linear topological space and $\phi\colon X \to 2^X$ is a set-valued function such that for all $x \in X$, $\phi(x)$ is convex, nonempty and for each $y \in X$, $\phi^{-1}(y)$ is open in X, then $\phi(\cdot)$ has a *fixed point* i.e., there exists an $x^* \in X$ such that $x^* \in \phi(x^*)$. The proof follows directly by combining the above result together with the Tychonoff fixed point theorem [see for instance Dugundji (1966, Theorem 2.2, p. 414)].

Finally, if (X, a) is a measurable space, Y is a complete separable metric space and $\phi: X \rightarrow 2^Y$ is a set-valued function such that for each closed subset V of Y the set $\{x \in X: \phi(x) \cap V \neq \emptyset\}$ belongs to a, and $\phi(\cdot)$ is nonempty closed valued, then there exists a measurable function $f: X \rightarrow Y$ such that $f(x) \in \phi(x)$ for all $x \in X$, i.e., $\phi(\cdot)$ admits a *measurable selection.*

To see this let $\{y_1, y_2, \ldots\}$ be a countable dense subset in Y. For each i, $(i = 1, 2, \ldots)$ and for each n, $(n = 1, 2, \ldots)$ let $B_n(i) = \{x \in Y: \operatorname{dist}(x, y_i) \leq \frac{1}{n}\}$, (where dist $\equiv$ distance). For each $x \in X$ set $\phi_0(x) = \phi(x)$ and define inductively $\phi_{n+1}: X \rightarrow 2^Y$ by $\phi_{n+1}(x) = \phi_n(x) \cap B_{n+1}(M_n(x))$, where $M_n(x) = \min\{i: \phi_n(x) \cap B_{n+1}(i) \neq \emptyset\}$. Then $\{\phi_n: n = 1, 2, \ldots\}$ is a decreasing sequence of nonempty closed subsets in X and the diameter of ϕ_n goes to zero. Define $f: X \rightarrow Y$ by $f(x) = \bigcap_{n=1}^{\infty} \phi_n(x)$. Then f is a selection from ϕ, and it is easily verified that f is measurable. In fact, it is easy to check that for each n, $(n = 1, 2, \ldots)$ and for each closed subset V of Y the set $\{x \in X: \phi_n(x) \cap V \neq \emptyset\}$ belongs to a. Since

$$f^{-1}(V) = \{x \in X: f(x) \in V\} = \bigcap_{n=0}^{\infty} \{x \in X: \phi_n(x) \cap V \neq \emptyset\}$$

we can conclude that $f^{-1}(V)$ is an element of a, i.e., f is measurable.

Recent work in economics and game theory [see for instance Yannelis (1987), Kim-Prikry-Yannelis (1989), Balder-Yannelis (1988) and Yannelis-Rustichini (1988)] has necessitated the use of set-valued defined either on the product space $T \times X$, (where T is a measure space and X is a topological space) or in an arbitrary subset U of $T \times X$. In particular, if $\phi: T \times X \rightarrow 2^Y$, (where Y is a linear topological space) is a set-valued function such that for each fixed $t \in T$, $\phi(t, \cdot)$ is lower semicontinuous, $\phi(\cdot, \cdot)$ is nonempty, convex, closed valued and lower measurable, one would like to know whether there exists a *Carathéodory-type selection from* ϕ, i.e., a function $f: T \times X \rightarrow Y$ such that $f(t, x) \in \phi(t, x)$ for all $(t, x) \in T \times X$ and for each fixed $t \in T$, $f(t, \cdot)$ is continuous and for each fixed $x \in X$, $f(\cdot, x)$ is measurable. Thus, the concept of a Carathéodory-type selection combines the notion of a continuous selection and measurable selection, via the setting of a product space.

It turns out that under appropriate conditions one can obtain several Carathéodory-type selection results adopting a similar argument with the one outlined to prove the existence of a continuous selection. Specifically, one can carry out a "parametrized" version of the above argument [which in turn is based on an idea of Michael (1956)] where the parameter t ranges over the measure space T.

Our main concern in this paper is to study several properties of set-valued maps of two variables. In particular, we will prove the existence of Carathéodory selections, the existence of random fixed points and the existence of random maximal elements. Finally, we will use the above results to prove the existence of a random price equilibrium.

2. PRELIMINARIES

2.1 Notation

2^A	denotes the set of all nonempty subsets of the set A.
con$A$	denotes the convex hull of the set A.
/	denotes the set of theoretic subtraction.

If ϕ: $X \to 2^Y$ is a correspondence (a correspondence is a set-valued function for which all image sets are non empty), then $\phi|_U$: $U \to 2^Y$ denotes the restriction of ϕ to U.

R^l	denotes the l-fold Cartesian product of the set of real numbers R.
$B(x, \epsilon)$	denotes the open ball centered at x of radius ϵ.
int$A$	denotes the interior of A.
cl$A$	denotes the closure of A.
dist	denotes distance
diam	denotes diameter.
proj	denotes projection.
$\emptyset$	denotes the empty set.

If X is a linear topological space, its dual is the space X^* of all continuous linear functionals on X. If $q \in X^*$ and $y \in X$ the value of q at y is denoted by $q \cdot y$.

2.2 Definitions

Let X and Y be sets. The graph G_ϕ of a correspondence ϕ: $X \to 2^Y$ is the set $G_\phi = \{(x, y) \in X \times Y: y \in \phi(x)\}$. If X and Y are topological spaces, ϕ: $X \to 2^Y$ is said to be *lower-semicontinuous* (l.s.c.) if the set $\{x \in X: \phi(x) \cap V \neq \emptyset\}$ is open in X for every open subset V of Y; ϕ: $X \to 2^Y$ is said to be *upper-semicontinuous* (u.s.c.) if the set $\{x \in X: \phi(x) \subset V\}$ is open in X for every open subset V of Y; if Y is a linear topological space. ϕ: $X \to 2^Y$ is said to be *upper-demicontinuous* (u.d.c.) if the set $\{x \in X: \phi(x) \subset V\}$ is open in X for every open half space V of Y. Obviously u.d.c. is a weaker requirement than u.s.c.

If (X, α) and (Y, β) are measurable spaces and $\phi\colon X \to 2^Y$ is a correspondence, ϕ is said to have a *measurable graph* if G_ϕ belongs to the product σ-algebra $\alpha \otimes \beta$. We are often interested in the situation where (X, α) is a measurable space, Y is a topological space and $\beta = \beta(Y)$ is the Borel σ-algebra of Y. For a correspondence ϕ from a measurable space into a topological space, if we say that ϕ has a measurable graph, it is understood that the topological space is endowed with its Borel σ-algebra (unless specified otherwise). In the same setting as above, i.e., (X, α) a measurable space and Y a topological space, ϕ is said to be *lower measurable* if $\{x \in X\colon \phi(x) \cap V \neq \emptyset\} \in \alpha$ for every V open in Y. Furthermore, ϕ is said to be *measurable* if $\{x \in X\colon \phi(x) \cap B \neq \emptyset\} \in \alpha$ for every closed B in Y.

We now define the concept of a Carathéodory selection which combines the notion of continuous selection and measurable selection.

Let (X, α) be a measurable space and Y and Z be topological spaces. Let $\phi\colon X \times X \to 2^Y$ be a (possibly empty-valued) correspondence. let $U = \{(x, z) \in X \times Z\colon \phi(x, z) \neq \emptyset\}$. A *Carathéodory-type selection from ϕ is a function $f\colon U \to Y$ such that $f(x, z) \in \phi(x, z)$; for each $x \in X$, $f(x, \cdot)$ is continuous on $U_x = \{z \in Z\colon (x, z) \in U\}$, and for each $z \in Z$ $f(\cdot, z)$ is measurable on $U_z = \{x \in X\colon (x, z) \in U\}$.*

If (X, α), (Y, β) and $(Z, \mathbf{F})$ are measurable spaces, $U \subseteq X \times Z$ and $f\colon U \to Y$, we call f *jointly measurable* is for every $B \in \beta$, $f^{-1}(B) = U \cap A$ for some $A \in \alpha \otimes \mathbf{F}$. It is a standard result that if Z is a separable metric space, Y is metric and $f\colon X \times Z \to Y$ is such that for each fixed $x \in X$, $f(x, \cdot)$ is continuous and for each fixed $z \in Z$, $f(\cdot, z)$ is measurable, then f is jointly measurable (where $\beta = \beta(Y)$, $\mathbf{F} = \beta(Z)$). It turns out, that in several instances U is a proper subset of $X \times Z$, and this situation is more delicate. However, in this more delicate situation it can be shown (see Proposition 3.1) that f is still jointly measurable.

2.3 Basic Theorems

We close this section by recalling some interesting results, [whose proofs can be found in Castaing-Valadier (1977)] which will be of fundamental importance in this paper:

PROJECTION THEOREM: Let (T, τ, μ) be a complete finite measure space and Y be a complete, separable metric space. If H belongs to $\tau \otimes \beta(Y)$, its projection $\text{proj}_T(H)$ belongs to τ.

AUMANN MEASURABLE SELECTION THEOREM: Let (T, τ, μ) be a complete finite measure space, Y be a complete, separable metric space and ϕ: $T \to 2^Y$ be a nonempty valued correspondence with a measurable graph, i.e., $G_\phi \in \tau \otimes \beta(Y)$. Then there is a measurable function f: $T \to Y$ such that $f(t) \in \phi(t)$ μ - a.e.

KURATOWKI AND RYLL-NARDZEWKI MEASURABLE SELECTION THEOREM: Let (T, τ) be a measurable space, Y be a separable metric space and ϕ: $T \to 2^Y$ be a lower measurable, closed, nonempty valued correspondence. Then there exists a measurable function f: $T \to Y$ such that $f(t) \in \phi(t)$ for all $t \in T$.

CASTAING REPRESENTATION THEOREM: Let (T, τ) be a measurable space, Y be a separable metric space and ϕ: $T \to 2^Y$ be a closed, nonempty valued correspondence. Consider the following statements:

(i) ϕ is lower measurable, and

(ii) there exist measurable functions f_n: $T \to Y$, $(n = 1, 2, \dots)$ such that $\text{cl}\{f_n(t)\text{: } n = 1, 2, \dots\} = \phi(t)$ for all $t \in T$.

Then (i) is equivalent to (ii).

3. ELEMENTARY MEASURE THEORETIC FACTS

This Section contains several elementary results of measure theoretic character, which are going to be useful in the sequel.

PROPOSITION 3.1: Let (T, τ) be a measurable space, Z be a separatable metric space, Y be a metric space and $U \subseteq T \times Z$ be such that:

(i) for each $t \in T$ the set $U^t = \{x \in Z: (t, x) \in U\}$ is open in Z and

(ii) for each $x \in Z$ the set $U_x = \{t \in T: (t, x) \in U\}$ belongs to τ.

Moreover, let $f: U \to Y$ be such that for each $t \in T$, $f(t, \cdot)$ is continuous on U^t and for each $x \in Z$, $f(\cdot, x)$ is measurable on U_x. Then f is relatively jointly measurable with respect to the σ-algebra $\tau \otimes \beta(Z)$, i.e., for every V open in Y,

$$\{(t, x) \in U: f(t, x) \in V\} = U \cap A$$

for some $A \in \tau \otimes \beta(Z)$.

PROOF: Let x_n ($n = 0, 1, 2, \ldots$) be dense in Z. For $p \geq 1$ set $f_p(t, x) = f(t, x_n)$, for $(t, x) \in U$, if n is the smallest integer such that $x \in B(x_n, 1/p)$ and $(t, x_n) \in U$. It is easy to see that $f_p(t, x) = f(t, x_n)$ if (t, x) belongs to the set

$$[U_{x_n} \times (B(x_n, \frac{1}{p}) / \bigcup_{m<n} B(x_m, \frac{1}{p}))] \cap U.$$

Observe that by assumption (ii), $U_{x_n} \in \tau$. Note that f_p is defined everywhere on U. To see this, let $(t, x) \in U$. By (i), U^t is open. Thus, let $\epsilon > 0$ be such that $B(x, \epsilon) \subseteq U^t$. Since x_n ($n = 0, 1, 2, \ldots$) are dense in Z, there is some n such that $x_n \in B(x, \min(\epsilon, 1/p))$. Thus $x_n \in U^t$. Hence $x \in B(x_n, 1/p)$ and $(t, x_n) \in U$, and therefore, $f_p(t, x)$ is defined.

We will now show that f_p is relatively jointly measurable. To this end let V be open in Y and set

$$S_n = \{t \in U_{x_n}: f(t, x_n) \in V\}.$$

Since $U_{x_n} \in \tau$ and $f(\cdot, x_n)$ is measurable on U_{x_n}, it follows that $S_n \in \tau$. It can be easily checked that

$$f_p^{-1}(V) = \bigcup_{n=0}^{\infty} [S_n \times (B(x_n, \frac{1}{p}) / \cup_{m<n} B(x_m, \frac{1}{p}))] \cap U.$$

Thus, f_p is relatively jointly measurable.

Since for each $t \in T$, $f(t, \cdot)$ is continuous on U^t, we conclude that $f_p(t, x)$ converges to $f(t, x)$ as p goes to infinity. Thus $f(t, x)$ is relatively jointly measurable. The fact that a limit of relatively jointly measurable functions is relatively jointly measurable is clear, since relative joint measurability is just the ordinary measurability with respect to an appropriate σ-algebra; (in our case, with respect to the σ-algebra of subsets of U which are of the form $U \cap A$ where $A \in \tau \otimes \beta(Z)$.)

LEMMA 3.1: Let (T, τ) be a measurable space, X be a separable metric space and ϕ: $T \to 2^X$ be a set-valued function. Consider the following statements:

(a) $\phi(\cdot)$ is lower measurable,

(b) for each $x \in X$, the function $t \to \mathrm{dist}(x, \phi(x))$ is measurable in t, and

(c) the set-valued function ψ: $T \to 2^X$ defined by $\psi(t) = \mathrm{cl}\phi(t)$ has a measurable graph.

Then $a \iff b \iff c$.

PROOF: ($a \iff b$). Note that $\phi(\cdot)$ is lower measurable if for each open ball $B(x, \delta)$ in X the set $\phi^{-1}(B(x, \delta)) = \{t \in T\colon \phi(t) \cap B(x, \delta) \neq \emptyset\}$ belongs to τ. Also note that for each $x \in X$, the function $t \to \mathrm{dist}(x, \phi(t))$ is measurable in t if the set $\{t \in T\colon \mathrm{dist}(x, \phi(t)) < \delta\}$ belongs to τ for each $\delta > 0$. Since $\{t \in T\colon \phi(t) \cap B(x, \delta) \neq \emptyset\} = \{t \in T\colon \mathrm{dist}(x, \phi(t)) < \delta\}$, we can conclude that $a \iff b$.

($b \iff c$): Define the function f: $T \times X \to [0, \infty]$ by $f(t, x) = \mathrm{dist}(x, \phi(t))$. By Proposition 3.1, $f(\cdot, \cdot)$ is jointly measurable. Hence, we can conclude that:

$$f^{-1}(0) = \{(t, x): \text{dist}(x, \phi(t)) = 0\} = \{(t, x): x \in \text{cl}(\phi(t))\}$$

$$= G_{\psi} \in \tau \otimes \beta(X),$$

and this completes the proof of the Lemma.

LEMMA 3.2: Let (T, τ, μ) be a complete finite measure space, and Y be a complete separable metric space. Let X: $T \to 2^Y$ be a set-valued function with a measurable graph. Then there exist $\{f_k: k = 1, 2, ...\}$ such that:

(i) for all k, f_k is a measurable function from $\text{proj}_T(G_X)$ into Y, and

(ii) for almost all $t \in \text{proj}_T(G_X)$, $\{f_k(t): k = 1, 2, ...\}$ is a dense subset of $X(t)$.

PROOF: For each $n = 1, 2, ...$, let $\{E_i^n: i = 1, 2, ...\}$ be an open cover of Y such that $\text{diam}(E_i^n) < 1/2^n$. For each n, $i = 1, 2, \ldots$, define $T_i^n = \{t \in T: X(t) \cap E_i^n \neq \emptyset\}$. Since $T_i^n = \text{proj}_T\{(t, y) \in T \times Y: y \in X(t) \cap E_i^n\}$ and $X(\cdot) \cap E_i^n$ has a measurable graph in $T \times Y$, $T_i^n \in \tau$ by virtue of the projection theorem. It can be easily checked that $\cup_{i=1}^{\infty} T_i^n = \text{proj}_T(G_X) \equiv S$.

For each n, $i = 1, 2, ...$, define the set-valued function X_i^n: $T \to 2^Y$ by

$$X_i^n(t) = \begin{cases} X(t) \cap E_i^n & \text{if } t \in T_i^n \\ X(t) & \text{if } t \notin T_i^n \end{cases}.$$

Since the graph of X_i^n is $\{(t, y) \in T_i^n \times Y: y \in X(t) \cap E_i^n\} \cup \{(t, y) \in T / T_i^n \times Y: y \in X(t)\}$, the correspondence X_i^n has a measurable graph. Also, for each $t \in T$, $X_i^n(t) \neq \emptyset$ if and only if $X(t) \neq \emptyset$, hence the graphs of X_i^n and X have the same projection onto T. By the Aumann measurable selection theorem, for each n, $i = 1, 2, ...$, there exists a measurable function f_i^n: $S \to Y$ such that $f_i^n(t) \in X_i^n(t)$ for almost all $t \in T$. Fix t in T. Let $y \in X(t)$. Since for each n, $\{E_i^n: i = 1, 2, ...\}$ is an open cover of Y, for each n, there is some i such that $y \in X(t) \cap E_i^n$. Therefore, $\{f_i^n(t): n, i = 1, 2, \ldots\}$ is dense in $X(t)$. Hence, the sequence f_i^n, after a suitable reindexing, gives the desired sequence f_k. This completes the proof of the lemma.

LEMMA 3.3: Let (S_i, α_i) for $i = 1, 2,$ be measurable spaces, $h\colon\ S_1 \to S_2$ be a measurable function and $A \in \alpha_1 \otimes \alpha_2$. Then

$$\mathrm{proj}_{S_1}(G_h \cap A) \in \alpha_1.$$

PROOF: (a) If $A = A_1 \times A_2$, where $A_i \in \alpha_i,\ i = 1, 2,$ then $\mathrm{proj}_{S_i}(G_h \cap A) = A_1 \cap h^{-1}(A_2) \in \alpha_1$.

(b) If $\mathrm{proj}_{S_1}(G_h \cap A) \in \alpha_1$, then $\mathrm{proj}_{S_1}(G_h \cap A^c) \in \alpha_1$, where $A^c = S_1 \times S_2/A$. For, $\mathrm{proj}_{S_1}(G_h \cap A^c) = S_1/\mathrm{proj}_{S_1}(G_h \cap A)$.

(c) If $\mathrm{proj}_{S_1}(G_h \cap A_n) \in \alpha_1$ for all $n = 1, 2, \ldots$, then $\mathrm{proj}_{S_1}(G_h \cap (\cup_{n=1}^{\infty} A_n)) \in \alpha_1$. For, $\mathrm{proj}_{S_1}(G_h \cap (\cup_{n=1}^{\infty} A_n)) = \cup_{n=1}^{\infty}\mathrm{proj}_{S_1}(G_h \cap A_n)$. Therefore, $\mathrm{proj}_{S_1}(G_h \cap A) \in \alpha_1$ for all $A \in \alpha_1 \otimes \alpha_2$.

LEMMA 3.4: Let (T_i, τ_i) for $i = 1, 2, 3$ be measurable spaces, $y\colon\ T_1 \to T_3$ be a measurable function and $\phi\colon\ T_1 \times T_2 \to 2^{T_3}$ be a set-valued function with a measurable graph, i.e., $G_\phi \in \tau_1 \otimes \tau_2 \otimes \tau_3$. Let $W\colon\ T_1 \to 2^{T_2}$ be defined by

$$W(t) = \{x \in T_2\colon\ y(t) \in \phi(t, x)\}.$$

Then W has a measurable graph, i.e., $G_W \in \tau_1 \otimes \tau_2$.

PROOF: Define $h\colon\ T_1 \times T_2 \to T_3$ by $h(t, x) = y(t)$. Let $S_1 = T_1 \times T_2$, $\alpha_1 = \tau_1 \otimes \tau_2$, $S_2 = T_3$, $\alpha_2 = \tau_3$, and $A = G_\phi$. Then $h\colon\ S_1 \to S_2$ is a measurable function and $A \in \alpha_1 \otimes \alpha_2$. Hence, by Lemma 3.3,

$$G_W = \{(t, x)\colon\ (t, x, h(t, x)) \in A\} \in \alpha_1 = \tau_1 \otimes \tau_2.$$

LEMMA 3.5: Let (T, τ) be a measurable space, Z be an arbitrary topological space and W_n, $n = 1, 2, \ldots$ be correspondences from T into Z with measurable graphs. Then the correspondences $\cup_n W_n(\cdot)$, $\cap_n W_n(\cdot)$, and $Z/W_n(\cdot)$ have measurable graphs.

PROOF: Obvious.

LEMMA 3.6: Let (T, τ, μ) be a complete finite measure space, Z be a complete separable metric space , and $W: T \to 2^Z$ be a correspondence having a measurable graph. Then for every $x \in Z$, $\text{dist}(x, W(\cdot))$ is a measurable function, where $\text{dist}(x, \oslash) = \infty$.

PROOF: First observe that $S = \{t \in T: W(t) \neq \oslash\}$ belongs to τ by virtue of the projection theorem. Now let λ be a positive real number and note that $\{s \in S: \text{dist}(x, W(s)) < \lambda\} = \{s \in S: W(s) \cap B(x, \lambda) \neq \oslash\} = \text{proj}_T[G_W \cap (T \times B(x, \lambda))]$. Another application of the projection theorem concludes the proof.

LEMMA 3.7: Let (T, τ, μ) be a complete finite measure space, Z be a complete separable metric space, and $W: T \to 2^Z$ be a correspondence having a measurable graph. Then the correspondence $V: T \to 2^Z$ defined by

$$V(t) = \{x \in Z: \text{dist}(x, W(t)) > \lambda\},$$

(where λ any real number) has a measurable graph, i.e., $G_V \in \tau \otimes \beta(Z)$.

PROOF: Define the function $g: T \times Z \to [0, \infty]$ by $g(t, x) = \text{dist}(x, W(t))$. By Lemma 3.6, $g(\cdot, x)$ is measurable for each x, and obviously $g(t, \cdot)$ is continuous for each t. By Proposition 3.1 we have that g is jointly measurable, i.e., measurable with respect to the product σ-algebra $\tau \otimes \beta(Z)$. Hence, $G_V = g^{-1}([\lambda, \infty]) \in \tau \otimes \beta(Z)$, i.e., $V(\cdot)$ has a measurable graph.

LEMMA 3.8: Let (S, α) be a measurable space, X be a separable metric space and $W: S \to 2^X$ be a lower measurable correspondence. Then the set-valued function $V: S \to 2^X$ defined by

$$V(s) = \{x \in X: \text{dist}(x, W(s)) < \lambda\},$$

(where λ is any real number,) has a measurable graph, i.e., G_V belongs to $\alpha \otimes \beta(X)$.

PROOF: Define the function $g: S \times X \to [0, \infty]$ by $g(s, x) = \text{dist}(x, W(s))$. Since $W(\cdot)$ is lower measurable, it follows that $g(\cdot, x)$ is measurable for every fixed x, for

$$\{s \in S:\ \mathrm{dist}(x, W(s)) < \lambda\} = \{s \in S:\ W(s) \cap B(x, \lambda) \neq \emptyset\}$$

and the latter set belongs to α by the assumption of lower measurability. Obviously, for each fixed $s \in S$, $g(s, \cdot)$ is continuous. Hence by Proposition 3.1, g is measurable with respect to the product σ-algebra $\alpha \otimes \beta(X)$. Therefore,

$$G_V = \{(s, x):\ x \in V(s)\} = g^{-1}((-\infty, \lambda)) \in \alpha \otimes \beta(X),$$

i.e., $V(\cdot)$ has a measurable graph.

FACT 3.1: Let (T, τ) be a measurable space, $S \subseteq T$, $S \in \tau$ and Y be a complete, separable metric space. Let ϕ: $T \to 2^Y$ be a lower measurable correspondence and f: $S \to Y$ be a measurable function. Then the set-valued function ψ: $T \to 2^Y$ defined by

$$\psi(t) = \phi(t) \cap (f(t) + B(0, \epsilon))$$

is lower measurable. (Here we understand that $f(t) + B(0, \epsilon) = \emptyset$ if $t \notin S$).

PROOF: We must show that $\{t \in T:\ \psi(t) \cap U \neq \emptyset\} \in \tau$ for every open subset U of Y. For each $t \in T$, let $\theta(t) = (\phi(t) \cap U) + B(0, \epsilon)$. Observe that

$$\{t \in T:\ \psi(t) \cap U \neq \emptyset\} = \{t \in T:\ (\phi(t) \cap U) \cap (f(t) + B(0, \epsilon)) \neq \emptyset\}$$

$$= \{t \in S:\ f(t) \in \theta(t)\} = \mathrm{proj}_T(G_f \cap G_\theta).$$

Since U is open, $\phi(t) \cap U$ is lower measurable, and since $\theta(t) = \{y \in Y:\ \mathrm{dist}(y, \phi(t) \cap U) < \epsilon\}$, $\theta(\cdot)$ has a measurable graph by Lemma 3.8. Therefore by Lemma 3.3, $\mathrm{proj}_T(G_f \cap G_\theta) \in \tau$. Therefore $\{t \in T:\ \psi(t) \cap U \neq \emptyset\} \in \tau$, and this completes the proof of the Fact.

LEMMA 3.9: Let (S, α) be a measurable space Y, be a separable metric space and ψ: $S \to 2^Y$ be a lower measurable correspondence. Then the correspondence θ: $S \to 2^Y$ defined by

$$\theta(s) = \{y \in Y:\ \mathrm{dist}(y, \psi(s)) = 0\},$$

has a measurable graph, i.e., $G_\theta \in \alpha \otimes \beta(Y)$.

PROOF: Consider the function $g: S \times Y \rightarrow [0, \infty]$ defined by $g(s, y) = \text{dist}(y, \psi(s))$. Since $\psi(\cdot)$ is lower measurable it follows that for each fixed $y \in Y$, $g(\cdot, y)$ is measurable, for

$$\{s \in S: \text{dist}(y, \psi(s)) < \epsilon\} = \{s \in S: \psi(s) \cap B(y, \epsilon) \neq \emptyset\}$$

and the latter set belongs to α by the assumption of lower measurability. Obviously for each fixed $s \in S$, $g(s, \cdot)$ is continuous. Therefore, by Proposition 3.1, $g(\cdot, \cdot)$ is jointly measurable, i.e., g is measurable with respect to the product σ-algebra $\alpha \otimes \beta(Y)$. It can be easily seen that:

$$G_\theta = \{(s, y) \in S \times Y: y \in \theta(s)\} = \{(s, y) \in S \times Y: g(s, y) = 0\}$$

$$= g^{-1}(0) \in \alpha \otimes \beta(Y).$$

Consequently, $\theta(\cdot)$ has a measurable graph as was to be shown.

LEMMA 3.10: Let (S, α) be a measurable space, Y be a separable metric space and $\phi: S \rightarrow 2^Y$ be a nonempty compact valued and lower measurable correspondence. Let X be a nonempty subset of Y. Define the correspondence $\theta: S \rightarrow 2^X$ by

$$\theta(s) = \{q \in X: q \cdot \phi(s) > 0\}.$$

Then $G_\theta \in \alpha \otimes \beta(X)$, i.e., $\theta(\cdot)$ has a measurable graph.

PROOF: Since $\phi(\cdot)$ is lower measurable and closed valued, there exist measurable functions (Castaing representation) $u_i : S \rightarrow Y, i \in I$ (where I is a countable set) such that

$$cl\{u_i(s) : i \in I\} = \phi(s) \qquad \text{for all } s \in S.$$

We then have that for all $s \in S$

$$\theta(s) = \bigcup_{n=1}^{\infty} g_n(s),$$

where

$$g_n(s) = \{q \in X:\ \text{for all}\ i \in I,\ q \cdot u_i(s) > \frac{1}{n}\}.$$

We now show that $G_{g_n} \in \alpha \otimes \beta(X)$. To this end, for each $i \in I$ define $h_i : S \times X \to [0, \infty]$ by $h_i(s, q) = q \cdot u_i(s)$. It is easily seen that for each $s \in S$, $h_i(s, \cdot)$ is continuous and for each $q \in X, h_i(\cdot, q)$ is measurable, and therefore by Proposition 3.1 $h_i(\cdot, \cdot)$ is jointly measurable. Consequently, $h_i^{-1}((\frac{1}{n}, \infty))$ belongs to $\alpha \otimes \beta(X)$ and so does $\bigcap_{i \in I} h_i^{-1}((\frac{1}{n}, \infty))$. It can be easily checked that $G_{g_n} = \bigcap_{i \in I} h_i^{-1}((\frac{1}{n}, \infty))$. Therefore, $g_n(\cdot)$ has a measurable graph, i.e., $G_{g_n} \in \alpha \otimes \beta(X)$. It follows from Lemma 3.5 that $\bigcup_{n=1}^{\infty} G_{g_n} \in \alpha \otimes \beta(X)$. Since $G_\theta = \bigcup_{n=1}^{\infty} G_{g_n}$, we conclude that $\theta(\cdot)$ has a measurable graph. This completes the proof of the Lemma.

LEMMA 3.11: Let (S, α) be a measurable space and $\phi: S \to 2^{R^l}$ be a nonempty compact convex valued and lower measurable correspondence. Let B be a compact, convex, nonempty subset of R^l. Define $\theta: S \to 2^B$ by

$$\theta(s) = \{q \in B:\ q \cdot \phi(s) > 0\}.$$

Then $\theta(\cdot)$ is lower measurable.

PROOF: By virtue of Theorem 4.4 in Himmelberg (1975, p. 59), it suffices to show that the correspondence $h: S \to 2^B$ defined by $h(s) = B/\theta(s) = \{q \in B:\ q \cdot \phi(s) \leq 0\}$ is measurable. Since $\phi(\cdot)$ is lower measurable and compact valued, it is also measurable [Himmelberg (1975, Theorem 3.1, p. 55)]. Hence, it follows from the Castaing representation theorem that there exist measurable functions $u_i: S \to R^l$, $i \in I$, (where I is a countable set) such that for all $s \in S$ $cl\{u_i(s): i \in I\} = \phi(s)$. We then have that $h(s) = \{q \in B:\ \text{for all}\ i \in I, q \cdot u_i(s) \leq 0\}$. It can be easily checked that $h(\cdot)$ is measurable, and so is $B/h(\cdot)$. Since $B/h(s) = \theta(s) = \{q \in B:\ q \cdot \phi(s) > 0\}$ we conclude that $\theta(\cdot)$ is measurable. Since measurability of $\theta(\cdot)$ implies lower measurability of $\theta(\cdot)$. [Proposition 2.1

in Himmelberg (1975, p. 55)] the proof of Lemma is now complete.

If F_n, $(n = 1, 2, ...)$ is a sequence of nonempty subsets of a metric space X, we will denote by LsF_n and LiF_n the set of its limit superior and limit inferior points respectively, i.e.,

$$LsF_n = \{x \in X\colon\ x = \lim_{k\to\infty} x_{n_k},\ x_{n_k} \in F_{n_k},\ k=1,2,...\},\ \text{and}$$

$$LiF_n = \{x \in X\colon\ x = \lim_{n\to\infty} x_n, x_n \in F_n,\ n=1,2,...\},$$

LEMMA 3.12: Let (T, τ, μ) be a complete finite measure space and X be a separable metric space. Let $\{F_n\colon\ n = 1, 2, ...\}$ be a sequence of nonempty valued and lower measurable correspondences. Then $LiF_n(\cdot)$ has a measurable graph, i.e., $G_{LiF_n} \in \tau \otimes \beta(X)$.

PROOF: First notice that $LiF_n(\cdot)$ is closed valued [recall from Kuratowski (1966, pp. 336-337), that if A_n is a sequence of sets, LiA_n and LsA_n are both closed sets]. By definition [see Kuratowski (1966, p. 335)], $LiF_n(t) = \{f \in X\colon \lim_{n\to\infty} \text{dist}(f, F_n(t)) = 0\}$. Since by assumption the sequence of set-valued functions $F_n(\cdot)$ have a measurable graph and (T, τ, μ) is a complete measure space, $F_n(\cdot)$ are lower measurable. It follows from Lemma 3.1 that $\text{dist}(f, F_n(t))$ is continuous in f and measurable in t, i.e., $\text{dist}(\cdot, \cdot)$ is jointly measurable with respect to the σ-algebra $\tau \otimes \beta(X)$. Hence, $\lim_{n\to\infty} \text{dist}(f, F_n(t))$ is jointly measurable with respect to the σ-algebra $\tau \otimes \beta(X)$. Notice that

$$G_{LiF_n} = \{(t, f) \in T \times X :\ \lim_{n\to\infty} \text{dist}(f, F_n(t)) = 0\}.$$

Since $\lim_{n\to\infty} \text{dist}(f, F_n(t))$ is jointly measurable, the set G_{LiF_n} belongs to $\tau \otimes \beta(X)$, i.e., LiF_n has a measurable graph. This completes the proof of the Lemma.

REMARK 3.1: Under the assumptions of Lemma 3.12, $LsF_n(\cdot)$ has a measurable graph as well. Simply notice that [see Kuratowski (1966, p. 337)] $LsF_n(t) = \{f \in X\colon\ Li\ \text{dist}(f, F_n(t)) = 0\}$.

Bibliographical Notes: Lemma 3.1 is due to Debreu (1967). Proposition 3.1 generalizes an earlier result of Kuratowski (1966). The argument is in essence that of Kuratowski, [see also Castaing-Valadier (1977) and Himmelberg (1975) for similar arguments]. Lemmata 3.2-3.8 are taken from Kim-Prikry-Yannelis (1987, 1988) and Lemmata 3.8-3.11 are new. Lemma 3.12 is taken from Yannelis (1989).

4. CARATHÉODORY-TYPE SELECTION THEOREMS

Below we state three Carathéodory-type selection theorems. The reader can easily see that neither theorem implies the other.

THEOREM 4.1: Let (T, τ, μ) be a complete finite measure space, Y be a separable Banach space and Z be a complete separable metric space. Let $X: T \to 2^Y$ be a nonempty-valued correspondence having a measurable graph, i.e., $G_X \in \tau \otimes \beta(Y)$, and $\phi: T \times Z \to 2^Y$ be a convex valued correspondence (possibly empty-valued) with a measurable graph, i.e., $G_\phi \in \tau \otimes \beta(Z) \otimes \beta(Y)$, satisfying the following conditions:

(i) for each $t \in T$, $\phi(t, x) \subset X(t)$ for all $x \in Z$.

(ii) for each t, $\phi(t, \cdot)$ has open lower sections in Z, i.e., for each $t \in T$, and each $y \in Y$, $\phi^{-1}(t, y) = \{x \in Z: y \in \phi(t, x)\}$ is open in Z.

(iii) for each $(t, x) \in T \times Z$, if $\phi(t, x) \neq \emptyset$, then $\phi(t, x)$ has a nonempty interior in $X(t)$.

Let $U = \{(t, x) \in T \times Z: \phi(t, x) \neq \emptyset\}$ and for each $x \in Z$, $U_x = \{t \in T: (t, x) \in U\}$ and for each $t \in T$, $U^t = \{x \in Z: (t, x) \in U\}$. Then there exists a Carathéodory-type selection from ϕ, i.e., there exists a function $f: U \to Y$ such that $f(t, x) \in \phi(t, x)$ for all $(t, x) \in U$ and for each $x \in Z$, $f(\cdot, x)$ is measurable on U_x and for each $t \in T$, $f(t, \cdot)$ is continuous on U^t. Moreover, $f(\cdot, \cdot)$ is jointly measurable.

PROOF: Let $\phi_x(t) \equiv \phi(t, x)$ for all $x \in Z$. Notice that for each $x \in Z$, $\phi_x(\cdot)$ has a measurable graph in $T \times Y$. Observe that

$$U_x = \{t \in T: \phi_x(t) \neq \emptyset\} = \text{proj}_T(G_\phi).$$

It follows from the Projection Theorem that $U_x \in \tau$. By Lemma 3.2 there exist measurable functions $\{y_n(\cdot): n = 1, 2, \ldots\}$ such that for each t, $\{y_n(t)\}$ is a countable dense subset of $X(t)$. For each $t \in T$, let $W_n(t) = \{x \in Z: y_n(t) \in \phi(t, x)\}$. By assumption (ii) $W_n(t)$ is open in Z.

Since by (iii) for each $(t, x) \in U$, $\phi(t, x)$ has nonempty interior in $X(t)$ and $\{y_n(t) : n=1,2, ...\}$ is dense in $X(t)$, it follows that $\{W_n(t) : n=1,2, ... \}$ is a cover of the set U^t. By Lemma 3.5, $W_n(\cdot)$ has a measurable graph. For each $m = 1, 2, ...$ define the operator $(\ )_m$ by

$$(W)_m = \{w \in W: \ \text{dist}(w, Z/W) \geq 1/2^m\}.$$

For each $n = 1, 2, \ldots$ and t in T let $V_n(t) = W_n(t)/\cup_{k=1}^{n-1}(W_k(t))_n$. Obviously, $V_n(t)$ is open in Z. It can be easily checked that $\{V_n(t)\colon \ n=1,2, ... \}$ is a locally finite open cover of the set U^t. Since for each n, $W_n(\cdot)$ has a measurable graph, so does $V_n(\cdot)$ by Lemmata 3.5 and 3.4. Let $\{g_n(t, \cdot) : \ n=1,2, ... \}$ be a partition of unity subordinated to the open cover $\{V_n(t)\colon \ n=1,2, ... \}$; for instance, for each $n = 1, 2, ...,$ let

$$g_n(t, x) = \frac{\text{dist}(x, Z/V_n(t))}{\sum_{k=1}^{\infty} \text{dist}(x, Z/V_k(t))}.$$

Then $\{g_n(t, \cdot) : n = 1, 2, \ ... \}$ is a family of continuous functions $g_n(t, \cdot)$: $U^t \to [0, 1]$ such that $g_n(t, x) = 0$ for $x \notin V_n(t)$ and $\sum_{n=1}^{\infty} g_n(t, x) = 1$ for all $(t, x) \in U$. Define $f\colon U \to Y$ by $f(t, x) = \sum_{n=1}^{\infty} g_n(t, x) y_n(t)$. Since $\{V_n(t) : \ n = 1, 2, ... \}$ is locally finite, each x has a neighborhood N_x which intersects only finitely many $V_n(t)$. Hence, for each $t \in T$, $f(t, \cdot)$ is a finite sum of continuous functions on N_x and it is therefore continuous on N_x. Consequently, $f(t, \cdot)$ is continuous. Furthermore, for any n such that $g_n(t, x) > 0$, $x \in V_n(t) \subset W_n(t) = \{x \in Z\colon \ y_n(t) \in \phi(t, z)\}$, i.e., $y_n(t) \in \phi(t, x)$. So $f(t, x)$ is a convex combination of elements $y_n(t)$ from the convex set $\phi(t, x)$. Consequently, $f(t, x) \in \phi(t, x)$ for all $(t, x) \in U$. Since $V_n(\cdot)$ has a measurable graph, dist$(x, Z/V_n(\cdot))$ is a measurable function by Lemmata 3.5 and 3.6. Hence, for each n and x, $g_n(\cdot, x)$ is a measurable function. Since for each n, $y_n(\cdot)$ is a measurable function, it follows that $f(\cdot, x)$ is measurable for each x, i.e., $f(t, x)$ is a Carathéodory-type selection from $\phi|_U$. Finally, it follows from Proposition 3.1 that $f(\cdot, \cdot)$ is jointly measurable. This completes the proof of the theorem.

THEOREM 4.2: Let (T, τ, μ) be a complete finite measure space, Y be a separable Banach space and Z be a complete, separable metric space. Let $\phi\colon T \times Z \to 2^Y$ be a convex, closed (possibly empty-) valued correspondence such that:

(i) $\phi(\cdot, \cdot)$ is lower measurable with respect to the σ-algebra $\tau \otimes \beta(Z)$ and

(ii) for each $t \in T$, $\phi(t, \cdot)$ is l.s.c.

Then there exists a jointly measurable Carathéodory-type selection from ϕ.

PROOF: We begin by proving the existence of an approximate Carathéodory selection. To this end, let $U = \{(t, x) \in T \times Z\colon \phi(t, x) \neq \emptyset\}$. For each $x \in X$, let $U_x = \{t \in T\colon (t, x) \in U\}$ and for each $t \in T$, let $U^t = \{x \in Z\colon (t, x) \in U\}$. We will show that there exists an approximate or ϵ- Carathéodory-type Selection from ϕ, i.e., given $\epsilon > 0$, there exists a function $f^\epsilon\colon U \to Y$ such that $f^\epsilon(t, x) \in \phi(t, x) + B(0, \epsilon)$, and for each $x \in Z$, $f^\epsilon(\cdot, x)$ is measurable on U_x and for each $t \in T$, $f^\epsilon(t, \cdot)$ is continuous on U^t. Since Y is separable we may choose $\{y_n : n = 1, 2, \dots\}$ to be a countable dense subset of Y. For each $t \in T$ and $\epsilon > 0$, let $W_n^\epsilon(t) = \{x \in Z\colon y_n \in [\phi(t, x) + B(0,\epsilon)]\}$. It follows from (ii) that for each $t \in T$ and $n = 1, 2, \dots, W_n^\epsilon(t)$ is open in Z. Since for each $(t, x) \in U$, $\phi(t, x) \neq \emptyset$, the set $\{W_n^\epsilon(t) : n = 1, 2, \dots\}$ is an open cover of U^t. Note that $\phi(t, x) + B(0,\epsilon) = \{y \in Y\colon \text{dist}(y, \phi(t, x)) < \epsilon\}$. Setting $S = T \times Z$, $X = Y$, $\alpha = \tau \otimes \beta(Z)$ and $W(s) = \phi(t, x)$ for $s = (t, x) \in S$ in Lemma 3.8 we conclude that $\phi(\cdot, \cdot) + B(0, \epsilon)$ has a measurable graph. By Lemma 3.4, $W_n^\epsilon(\cdot)$ has a measurable graph. As in the previous theorem for each $m = 1, 2, \dots$, define the operator $(\ )_m$ on subsets of Z by

$$(W)_m = \{ w \in W\colon \text{dist}(w, Z/W) \geq \frac{1}{2^m}\}.$$

For $n = 1, 2, \dots$, let $V_n^\epsilon(t) = W_n^\epsilon(t) = W_n^\epsilon(t)/\cup_{k=1}^{n-1}(W_k^\epsilon(t))_n$. It can be easily checked that $\{V_n^\epsilon(t) : n = 1, 2, \dots\}$ is a locally finite open cover of the set U^t. Since for each n, $W_n^\epsilon(\cdot)$ has a measurable graph, by Lemmata 3.5 and 3.7, $V_n^\epsilon(\cdot)$ has a measurable graph. Let

$\{g_n^\epsilon(t, x)$: $n = 1, 2, \cdots\}$ be a partition of unity subordinated to the open cover $\{V_n^\epsilon(t)$: $n = 1, 2, \dots\}$, for instance, for each $n = 1, 2, \dots$, let

$$g_n^\epsilon(t, x) = \frac{\mathrm{dist}(x, Z/V_n^\epsilon(t))}{\sum_{k=1}^{\infty} \mathrm{dist}(x, Z/V_k^\epsilon(t))}.$$

Then $\{g_n^\epsilon(t, \cdot)$: n 1, 2, ... $\}$ is a family of continuous functions $g_n^\epsilon(t, \cdot): U^t \to [0, 1]$ such that $g_n^\epsilon(t, x) = 0$ for $x \notin V_n^\epsilon(t)$ and $\sum_{n=1}^{\infty} g_n^\epsilon(t, x) = 1$ for all $(t, x) \in U$. Define $f^\epsilon: U \to Y$ by $f^\epsilon(t, x) = \sum_{n=1}^{\infty} g_n^\epsilon(t, x) y_n$. Since $\{V_n^\epsilon(t)$: $n = 1, 2, \dots\}$ is locally finite, each x has a neighborhood N_x which intersects only finitely many $V_n^\epsilon(t)$. Hence, $f^\epsilon(t, \cdot)$ is a finite sum of continuous functions on N_x and it is therefore continuous on N_x. Consequently, $f^\epsilon(t, \cdot)$ is a continuous function on U^t. Moreover, for any n such that $g_n^\epsilon(t, x) > 0$, $x \in V_n^\epsilon(t) \subset W_n^\epsilon(t) = \{z \in Z\colon y_n \in [\phi(t, z) + B(0, \epsilon)]\}$, i.e., $y_n \in \phi(t, x) + B(0, \epsilon)$. So $f^\epsilon(t, x)$ is a convex combination of elements from the convex set $\phi(t, x) + B(0, \epsilon)$. Therefore, $f^\epsilon(t, x) \in \phi(t, x) + B(0, \epsilon)$ for all $(t, x) \in U$. Since $V_n^\epsilon(\cdot)$ has a measurable graph, by Lemmata 3.5 and 3.6, $\mathrm{dist}(x, Z/V_n^\epsilon(\cdot))$ is a measurable function for every $x \in Z$. Hence, for each n, x, $g_n^\epsilon(\cdot, x)$ is a measurable function. Consequently, $f^\epsilon(\cdot, x)$ is measurable for each x. Therefore f^ϵ is an approximate or ϵ- Carathéodory-type selection from $\phi|_U$. Now we can construct inductively, functions f_l: $U \to Y$, $l = 1, 2, \dots$, such that

(a) $f_l(t, \cdot)$ is continuous on U^t and $f_l(\cdot, x)$ is measurable on U_x,

(b) $f_l(t, x) \in \phi(t, x) + B(0, 1/2^l)$, $l = 1, 2, \dots$,

(c) $f_l(t, x) \in f_{l-1}(t, x) + 2B(0, 1/2^{l-1})$, $l = 2, 3, \dots$.

The existence of f_l satisfying (a) and (b) for $l = 1$, is guaranteed by the above argument. Suppose that we have $f_1, \dots, f_k$ satisfying (a), (b), and (c) for $l = 1, 2, \dots, k$. We must find f_{k+1}: $U \to Y$ which satisfies (a), (b), and (c) for $l = k + 1$. Now define $\phi_{k+1}(t, x) = \phi(t, x) \cap (f_k(t, x) + B(0, 1/2^k))$. Then $\phi_{k+1}(t, x)$ is nonempty, by the induction hypothesis, and it can be easily checked that for each $t \in T$, $\phi_{k+1}(t, \cdot)$ is l.s.c. It follows from Fact 3.1 that $\phi^{k+1}(\cdot, \cdot)$ is lower measurable. By the above argument (the existence of an approximate Carathéodory-type selection) there exists $f_{k+1}: U \to Y$ such that $f_{k+1}(t, x) \in \phi_{k+1}(t, x) + B(0, 1/2^{k+1}))$. But then $f_{k+1}(t, x) \in (f_k(t, x) + B(0, 1/2^k)) + B(0, 1/2^{k+1}) \subset (f_k(t, x) + 2B(0, 1/2^k))$ which is (c) and $f_{k+1}(t, x) \in \phi(t, x) + B(0, 1/2^{k+1})$ which is (b). By (c), $\{f_l: l = 1, 2, \ldots\}$ is uniformly Cauchy, and therefore converges uniformly to $f: U \to Y$. Since ϕ is closed valued $f(t, x) \in \phi(t, x)$ for all $(t, x) \in U$. Furthermore, for each $t \in T$, $f(t, \cdot)$ is continuous in U^t and for each $x \in Z$, $f(\cdot, x)$ is measurable on U_x and therefore, by Proposition 3.1, $f(\cdot, \cdot)$ is jointly measurable. This completes the proof of the theorem.

THEOREM 4.3: The statement of Theorem 3.1 remains true without closed valueness of ϕ: $T \times Z \to 2^Y$ if either

(i) Y is finite dimensional or

(ii) $\phi(t, x)$ has a nonempty interior for all $(t, x) \in U$.

PROOF: We begin by proving the following claim:

CLAIM: Under the conditions of Theorem 4.3 there exists a countable collection **F** of Carathéodory-type selections from ϕ such that for every $(t, x) \in U$, $\{f(t, x) : f \in \mathbf{F}\}$ is dense in $\phi(t, x)$.

PROOF: Let $\{E^n: n = 1, 2, \ldots\}$ be a convex open basis of Y. For each $n = 1, 2, \ldots$, $U^n = \{(t, x) \in T \times Z : \phi(t, x) \cap E^n \neq \emptyset\} \in \tau \otimes \beta(Z)$. For each $t \in T$ and each n, set $U^n(t) = \{x \in Z : (t, x) \in U^n\}$. Note that for each $t \in T$, $U^n(t)$ is open in Z. Moreover, $U^n(\cdot)$ has a measurable graph. For each $k = 1, 2, \ldots$, and $t \in T$, let $A_k^n(t) = \{x \in Z: \text{dist}(x, Z/U_n(t)) \geq 1/2^k\}$. By Lemma 3.7, $A_k^n(\cdot)$ has a measurable graph. Note that $\cup_{k=1}^{\infty} A_k^n(t) = U^n(t)$, and for each $t \in T$, A_k^n is closed in Z. Define $\phi_k^n: T \times Z \to 2^Y$ by

$$\phi_k^n(t,x) = \begin{cases} \mathrm{cl}(\phi(t,x) \cap E^n) & \text{if } x \in A_k^n(t) \\ \phi(t,x) & \text{if } x \notin A_k^n(t). \end{cases}$$

Since for each $t \in T$, $A_k^n(t)$ is closed in Z, $\phi_k^n(t,\cdot)$ is l.s.c. Moreover, since for every open subset V of Y, $\{(t,x): \phi_k^n(t,x) \cap V \neq \emptyset\} = \{(t,x): \mathrm{cl}(\phi(t,x) \cap E^n) \cap V \neq \emptyset,$ $x \in A_k^n(t)\} \cup \{(t,x): \phi(t,x)$: $\phi(t,x) \cap V \neq \emptyset,\ x \notin A_k^n(t)\} \in \tau \otimes \beta(Z)$, it follows that $\phi_k^n(\cdot,\cdot)$ is lower measurable. By Theorem 4.2 there exist Carathéodory-type selection $f_k^n(\cdot,\cdot)$ from $\phi_k^n(\cdot,\cdot)$. Let **F** be the collection of all $f_k^n(\cdot,\cdot)$, $k, n = 1, 2, \ldots$. Then **F** is a countable collection of Carathéodory-type selections from ϕ, and it can be easily seen that $\{f(t,x)$: $f \in \mathbf{F}\}$ is dense in $\phi(t,x)$ for all $(t,x) \in U$. This completes the proof of the claim.

We will now need the following notions. If K is a closed, convex subset of a normed linear space, then a *supporting set* of K is a closed convex subset S of K, $S \neq K$, such that if an interior point of a segment in K is in S, then the whole segment is in S. The set of all elements of K which are not in any supporting set of K will be denoted by $I(K)$. The following facts below are due to Michael (1956, p. 372).

FACT 4.1: If any convex subset K of Y is either closed or has an interior point or is finite dimensional, then $I(\mathrm{cl}\, K) \subset K$.

FACT 4.2: Let K be a nonempty, closed, convex separable subset of a Banach space Y, and $\{y_i = 1, 2, \ldots\}$ be a dense subset of K. If

$$z_i = y_i + \frac{(y_i - y_1)}{\max(1, \|y_i - y_1\|)} \quad \text{for all } i \text{ and } z = \sum_{i=1}^{\infty} (\frac{1}{2})^i z_i,$$

then $z \in I(K)$.

We are now ready to complete the proof of Theorem 4.3:

Define ψ: $T \times Z \to 2^Y$ by $\psi(t,x) = \mathrm{cl}\, \phi(t,x)$. Since for each $t \in T$, $\phi(t,\cdot)$ is l.s.c. so is $\psi(t,\cdot)$. Moreover, ψ is lower measurable. By the above claim there exist Carathéodory-type selections $\{g_k(t,x)$: $k = 1, 2, \ldots\}$ dense in $\psi(t,x)$ for all $(t,x) \in U$. For each $k = 1, 2,$

... , let

$$f_k(t, x) = g_1(t, x) + \frac{g_k(t, x) - g_1(t, x)}{\max(1, ||g_k(t, x) - g_1(t, x)||)} ,$$

$$f(t, x) = \sum_{k=1}^{\infty} \frac{1}{2^k} f_k(t, x).$$

By Fact 4.2, $f(t, x) \in I(\psi(t, x))$ for all $(t, x) \in U$. Since the series defining f converge uniformly, it follows that for each $t \in T$, $f(t, \cdot)$ is continuous and for each $x \in X$, $f(\cdot, x)$ is measurable. By Fact 4.1, $f(t, x) \in I(\psi(t, x)) \subset \phi(t, x)$ if either (i) or (ii) of Theorem 4.3 are satisfied. This completes the proof of the theorem.

Bibliographical Notes: Theorems 4.1 - 4.3 are due to Kim-Prikry-Yannelis (1987, 1988). Less general versions of Theorem 4.2 are given by Castaing (1975), Fryszkowski (1977) and Rybiński (1985). Applications of these theorems in economics and game theory can be found in Yannelis (1987), Kim-Prikry-Yannelis (1989), Yannelis-Rustichini (1988), Balder-Yannelis (1988) and Yannelis (1989b).

5. RANDOM FIXED POINT THEOREMS

Let (T, τ) be a measurable space, X be a metric space and $\phi: T \times X \to 2^X$ be a correspondence. If there exists a measurable function $f: T \to X$ such that $f(t) \in \phi(t, f(t))$ for all $t \in T$, then we say that ϕ has a *random fixed point.*

We begin by proving a random version of the Kakutani-Fan-Glicksberg fixed point theorem.

THEOREM 5.1: Let (T, τ, μ) be a complete finite measure space and X be a compact, convex, nonempty subset of *a* locally convex, separable, metrizable linear topological space. Let ϕ: $T \times X \to 2^X$ be a nonempty, convex, closed valued correspondence such that:

(i) $\phi(\cdot, \cdot)$ is lower measurable, i.e., for every open subset V of X the set $\{(t,x) \in T \times X: \phi(t, x) \cap V \neq \emptyset\}$ belongs to $\tau \otimes \beta(X)$,

(ii) for each fixed $t \in T$, $\phi(t, \cdot)$ is u.s.c.

Then ϕ has a random fixed point.

PROOF: Define the correspondence $F: T \to 2^X$ by

$$F(t) = \{x \in X: \text{dist}(x, \phi(t, x)) = 0\}.$$

Setting $S = T \times X$, $X = Y$, $\alpha = \tau \otimes \beta(X)$ and $\psi(s) = \phi(t, x)$ for $s = (t, x)$ in Lemma 3.9, we conclude that $F(\cdot)$ has a measurable graph, i.e., $G_F \in \tau \otimes \beta(X)$. It can be easily checked that for each fixed $t \in T$, the correspondence $\phi(t, \cdot)$: $X \to 2^X$ satisfies all the conditions of the Fan-Glicksberg fixed point theorem [see for instance Glicksberg (1952)]. Hence, for all $t \in T$, $F(t) \neq \emptyset$. Consequently, the correspondence $F: T \to 2^X$ satisfies all the conditions of the Aumann measurable selection theorem and therefore there exists a measurable function $\bar{x}$: $T \to X$ such that $\bar{x}(t) \in F(t)$ for almost all $t \in T$, i.e., $\text{dist}(\bar{x}(t), \phi(t, \bar{x}(t))) = 0$ for almost all $t \in T$. Since $\phi(\cdot, \cdot)$ is closed valued we conclude that $\bar{x}(t) \in \phi(t, \bar{x}(t))$ for almost all $t \in T$, i.e., $\phi(\cdot, \cdot)$ has a random fixed point. This completes the proof of the Theorem.

The result below is a random version of Fan's Coincidence Theorem, [Fan (1969)].

THEOREM 5.2: Let X be compact convex nonempty subset of a locally convex separable and metrizable linear topological space Y and let (T, τ, ν) be a complete finite measure space. Let $\gamma: T \times X \to 2^Y$ and $\mu: T \times X \to 2^Y$ be two nonempty, convex, closed and at least one of them compact valued correspondences such that:

(i) $\mu(\cdot, \cdot)$ and $\gamma(\cdot, \cdot)$ are lower measurable,

(ii) for each fixed $t \in T$, the correspondences $\mu(t, \cdot): X \to 2^Y$ and $\gamma(t, \cdot): X \to 2^Y$ are u.s.c.

(iii) for every $t \in T$ and every $x \in X$, there exist three points $y \in X$, $u \in \gamma(t, x)$, $z \in \mu(t, x)$ and a real number $\lambda > 0$ such that $y - x = \lambda(u - z)$.

Then there exists a measurable function $x^*: T \to X$ such that $\gamma(t, x^*(t)) \cap \mu(t, x^*(t)) \neq \emptyset$ for almost all $t \in T$.

PROOF: Define the correspondence $W: T \times X \to 2^Y$ by $W(t, x) = \gamma(t, x) \cap \mu(t, x)$. Since $\gamma(\cdot, \cdot)$ and $\mu(\cdot, \cdot)$ are closed valued and lower measurable and at least one of them is compact valued, it follows from Theorem 4.1 in Himmelberg (1975) that $W(\cdot, \cdot)$ is lower measurable. Define the correspondence $\phi: T \to 2^X$ by

$$\phi(t) = \{x \in X: W(t, x) \neq \emptyset\}.$$

Observe that

$$G_\phi = \{(t, x) \in T \times X: x \in \phi(t)\} = \{(t, x) \in T \times X: W(t, x) \neq \emptyset\}$$

$$= \{(t, x) \in T \times X: W(t, x) \cap Y \neq \emptyset\},$$

and the latter set belongs to $\tau \otimes \beta(X)$ since $W(\cdot, \cdot)$ is lower measurable. Therefore, $G_\phi \in \tau \otimes \beta(X)$. It follows from Fan's Coincidence Theorem, that for each $t \in T$, $\phi(t) \neq \emptyset$. Thus, the correspondence $\phi: T \to 2^X$ satisfies all the conditions of the Aumann Measurable Selection Theorem and consequently, there exists a measurable function $x^*: T \to X$ such that

$x^*(t) \in \phi(t)$ for almost all t in T, i.e., $\gamma(t, x^*(t)) \cap \mu(t, x^*(t)) \neq \emptyset$ for almost all t in T. This completes the proof of the Theorem.

An immediate corollary of the above result is Theorem 5.1.

COROLLARY 5.1: Let X be a compact, convex, non-empty subset of a locally convex separable and metrizable linear topological space Y and let (T, τ, ν) be a complete finite measure space. Let γ: $T \times X \to 2^X$ be a nonempty, convex, closed valued correspondence such that for each fixed $t \in T$, $\gamma(t, \cdot)$ is u.s.c. and $\gamma(\cdot, \cdot)$ is lower measurable. Then $\gamma(\cdot, \cdot)$ has a random fixed point.

PROOF: Define the correspondence μ: $T \times X \to 2^X$ by $\mu(t, x) = \{x\}$. Clearly for each fixed $t \in T$, $\mu(t, \cdot)$ is u.s.c. and $\mu(\cdot, \cdot)$ is convex, lower measurable, nonempty, compact valued. Let $x \in X$ and $t \in T$. By choosing $u \in \gamma(t, x)$, $z = x \in \mu(t, x)$ and $\lambda \in (0, 1)$ assumption (iii) of Theorem 5.2 is satisfied (simply notice that since X is convex $y = x + \lambda(u - z) = \lambda u + (1 - \lambda)x \in X$). Hence, by the previous theorem there exists a measurable function x^*: $T \to X$ such that $\gamma(t, x^*(t)) \cap \mu(t, x^*(t)) \neq \emptyset$ for almost all $t \in T$, i.e., $x^*(t) \in \gamma(t, x^*(t))$ for almost all $t \in T$.

REMARK 5.1: Theorem 5.2 and Corollary 5.1 remain true if we replace the assumption that (T, τ, ν) is a complete finite (or σ-finite) measure space, by the fact that (T, τ) is simply a measurable space. In this case one only needs to observe that in the proof of Theorem 5.2 for each fixed $t \in T$, $W(t, \cdot)$ is u.s.c. (as it is the intersection of two u.s.c. correspondences) and therefore, the correspondence ϕ: $T \to 2^X$ is closed valued. Since $\phi(\cdot)$ is closed valued and it has a measurable graph by Lemma 3.1, $\phi(\cdot)$ is lower measurable. One can now appeal to the Kuratowski and Ryll-Nardzewski measurable selection theorem to complete the proof of Theorem 5.2.

THEOREM 5.3: Let (T, τ, μ) be a complete finite measure space, and X be a nonempty compact convex subset of a separable Banach space Y. Let ϕ: $T \times X \to 2^X$ be a nonempty convex, closed valued correspondence such that:

(i) $\phi(\cdot,\cdot)$ is lower measurable and

(ii) for each $t \in T$, $\phi(t,\cdot)$ is l.s.c.

Then ϕ has a random fixed point.

PROOF: It follows from Theorem 4.2 that there exists a function $f: T \times X \to X$ such that $f(t,x) \in \phi(t,x)$ for all $(t,x) \in T \times X$, and for each $x \in X$, $f(\cdot,x)$ is measurable and for each $t \in T$, $f(t,\cdot)$ is continuous. Moreover, $f(\cdot,\cdot)$ is jointly measurable.

Define the set-valued function $F: T \to 2^X$ by $F(t) = \{x \in X: g(t,x) = 0\}$. where $g(t,x) = f(t,x) - x$. It follows from the Tychonoff fixed point theorem that for each fixed $t \in T$, the function $f(t,\cdot)$: $X \to X$ has a fixed point. Therefore, for each $t \in T$, $F(t) \neq \emptyset$. Since g is jointly measurable, F has a measurable graph. Hence by the Aumann measurable selection theorem there exists a measurable function x^*: $T \to X$ such that for almost all t in T, $x^*(t) \in F(t)$, i.e., $x^*(t) = f(t, x^*(t)) \in \phi(t, x^*(t))$. This completes the proof of the theorem.

REMARK 5.2: The statement of Theorem 5.3 remains true without the closed valuedness of ϕ: $T \times X \to 2^X$ if either

(i) Y is finite dimensional or

(ii) $\phi(t,x)$ has a nonempty interior for all $(t,x) \in T \times X$.

The argument is similar to that adopted in the proof of Theorem 5.3 except that one must use now Theorem 4.3 instead of Theorem 4.2.

REMARK 5.3: The statement of Theorem 5.3 remains true if we replace (i) and (ii) by:

(i) for each fixed $t \in T$, $\phi(t,\cdot)$ has an open graph in $X \times X$, and

(ii) ϕ has a measurable graph, i.e., $G_\phi \in \tau \otimes \beta(X) \otimes \beta(X)$.

The argument is the same with that adopted for the proof of Theorem 5.3 except that one must now appeal to Theorem 4.1.

We conclude this Section by proving a random fixed point theorem for weakly u.s.c. (w-u.s.c.) set-valued functions, which has found useful applications in mathematical economics. However, before we state our result we will need some notation.

Let (T, τ, μ) be a finite measure space, X be a separable Banach space and let $L_1(\mu, X)$ denote the space of equivalence classes of X-valued Bochner integrable function on (T, τ, μ). We denote by S_F^1 the set of all Bochner integrable selections from the set-valued function $F: T \to 2^X$, i.e., $S_F^1 = \{x \in L_1(\mu, X): x(t) \in F(t) \;\; \mu\text{-}a.e.\}$.

<u>THEOREM 5.4</u>: Let $\phi : T \times X \to 2^X$ be a nonempty, convex, weakly compact valued correspondence such that :

(i) $\phi(\cdot, \cdot)$ is lower measurable,

(ii) for each $t \in T$, $\phi(t, \cdot)$ has a weakly closed graph, i.e., $w\text{-}Ls\; \phi(t, x_n) \subset \phi(t, x)$, (where $w\text{-}Ls$ denotes weak limit superior) whenever the sequence $\{x_n : n = 1, 2, \ldots\}$ converges to x,

(iii) $\phi(t, x) \subset F(t) \;\; \mu\text{-}a.e.$, where $F : T \to 2^X$ is a lower measurable, integrably bounded, weakly compact, convex and nonempty valued correspondence.

Then $\phi(\cdot, \cdot)$ has a random fixed point.

<u>PROOF</u>: Define the set-valued operator $\psi : S_F^1 \to 2^{S_F^1}$ by $\psi(x) = S^1_{\phi(\cdot, x(\cdot))}$. In view of assumption (iii) it follows from Diestel's theorem [see for instance Yannelis (1989b), Theorem 3.1] that S_F^1 is a weakly compact subset of $L_1(\mu, X)$. Obviously S_F^1 is convex and by virtue of the Kuratowski and Ryll-Nardzewski measurable selection theorem we can conclude that S_F^1 is nonempty. We now show that ψ is w-u.s.c. i.e., the set $\{x \in S_F^1 : \psi(x) \subset V\}$ is open in S_F^1 for any weakly open subset V of S_F^1. Since S_F^1 is weakly compact and $\psi(\cdot)$ is weakly closed valued, it suffices to show that $\psi(\cdot)$ has a weakly closed graph. To this end let $\{x_n(\cdot) : n = 1, 2, \ldots\}$ be a sequence in S_F^1 converging in the $L_1(\mu, X)$ norm to $x(\cdot) \in S_F^1$, we must show that

(5.1) $w\text{-}Ls\ S^1_{\phi(\cdot, x_n(\cdot))} = w\text{-}Ls\ \psi(x_n) \subset S^1_{\phi(\cdot, x(\cdot))} = \psi(x)$.

(By passing to a subsequence if necessary if may assume that $x_n(t)$ converges to $x(t)$ μ-*a.e.*). Let $z \in w\text{-}Ls\ \psi(x_n)$,i.e., there exists $\{z_{n_k} : k = 1, 2, \ldots\}$ in S^1_F such that z_{n_k} converges weakly to $z \in S^1_F$ and $z_{n_k} \in \psi(x_{n_k}) = S^1_{\phi(\cdot, x_{n_k}(\cdot))}$, i.e., $z_{n_k}(t) \in \phi(t, x_{n_k}(t))$ μ-*a.e.* We must show that $z \in \psi(x)$. It follows from Theorem 4.1 in Yannelis (1989b) that $z(t) \in \overline{con}\ w\text{-}Ls\{z_{n_k}(t)\}$ μ-*a.e.* and therefore

(5.2) $z(t) \in \overline{con}\ w\text{-}Ls\ \phi(t, x_n(t))$ μ-*a.e.*

Since by assumption (ii) for each $t \in T$, $\phi(t, \cdot)$ has a weakly closed graph we have that:

(5.3) $w\text{-}Ls\ \phi(t, x_n(t)) \subset \phi(t, x(t))$ μ-*a.e.*

Combining now (5.2) and (5.3) and taking into account that ϕ is convex valued we conclude that $z(t) \in \phi(t, x(t))$ μ-*a.e.* Since ϕ is weakly compact valued we have that $z \in S^1_{\phi(\cdot, x(\cdot))} = \psi(x)$, and this proves (5.1). Hence, $\psi : S^1_F \to 2^{S^1_F}$ satisfies all the conditions of the Fan-Glicksberg fixed point theorem and consequently there exists $x^* \in S^1_F$ such that $x^* \in \psi(x^*)$, i.e., $x^*(t) \in \phi(t, x^*(t))$ μ-*a.e.* This completes the proof of the theorem.

Bibliographical Notes: Theorems 5.1 and 5.4 are new. Theorem 5.2 is taken from Yannelis-Rustichini (1988). Theorem 5.3 is a random version of a result in Yannelis-Prabhakar (1983) and it is taken from Kim-Prikry-Yannelis (1987). The literature on random fixed points is growing rapidly, and perhaps one of the most complete references is Itoh (1979). Applications of random fixed points in game theory can be found in Yannelis-Rustichini (1988).

6. RANDOM MAXIMAL ELEMENTS AND RANDOM EQUILIBRIA

6.1 Random Maximal Elements

Let X be a nonempty subset of a linear topological space. Let $P: X \to 2^X$ be a *preference correspondence*. We read $y \in P(x)$ as "y is strictly preferred to x". For instance if > is a binary relation on X one may define $P: X \to 2^X$ by $P(x) = \{y \in X: y > x\}$. The correspondence $P: X \to 2^X$ is said to have a *maximal element* if there exists $\bar{x} \in X$ such that $P(\bar{x}) = \emptyset$. Several results on the existence of maximal elements with applications to equilibrium theory have been given in the literature [see for instance Sonnenschein (1971) and Yannelis-Prabhakar (1983) among others.] Notice that the above preference correspondences need not be representable by utility functions. We will now allow our preference correspondence to depend on the states of nature, i.e., we allow for random preferences.

let (T, τ, μ) be a complete finite measure space. We interpret T as the states of nature of the world, and suppose that T is large enough to include all the events that we consider to be interesting. τ will denote the σ-algebra of events. A *random preference correspondence* P is a mapping from $T \times X$ into 2^X. We read $y \in P(t, x)$ as "y is strictly preferred to x at the state of nature t". We now can introduce the concept of a random maximal element which is the natural analogue of the ordinary (deterministic) notion of a maximal element. The correspondence $P: T \times X \to 2^X$ is said to have a *random maximal element* if there exists a measurable function $\bar{x}: T \to X$ such that $P(t, \bar{x}(t)) = \emptyset$ for almost all t in T.

The following two theorems on the existence of random maximal elements below generalize the ordinary (deterministic) maximal elements results given in Sonnenschein (1971), and Yannelis-Prabhakar (1983). These theorems will also play a key role in proving random price equilibrium theorem in Section 6.2.

<u>THEOREM 6.1</u>: Let (T, τ, μ) be a complete finite measure space and X be a compact, convex, nonempty subset of R^l. Let $P: T \times X \to 2^X$ be a correspondence (possibly empty-

valued) such that:

(i) for every open subset V of X, $\{(t, x) \in T\colon \text{con}\, P(t, x) \cap V \neq \emptyset\}$ belongs to $\tau \otimes \beta(X)$.

(ii) for each $t \in T$, $P(t, \cdot)$ is l.s.c.

(iii) for every measurable function $x\colon T \to X$, $x(t) \notin \text{con}\, P(t, x(t))$ for almost all $t \in T$.

Then there exists a measurable function $\bar{x}\colon T \to X$ such that $P(t, \bar{x}(t)) = \emptyset$ for almost all t in T.

PROOF: Define the correspondence $\psi\colon T \times X \to 2^X$ by $\psi(t, x) = \text{con}\, P(t, x)$. By Proposition 2.6 in Michael (1956) for all $t \in T$, $\psi(t, \cdot)$ is l.s.c. and by assumption (i) $\psi(\cdot, \cdot)$ is lower measurable. Let $U = \{(t, x) \in T \times X\colon \psi(t, x) \neq \emptyset\}$. By Theorem 4.3 there exists a Carathéodory selection from ψ, i.e., there exists a function $f\colon U \to X$ such that $f(t, x) \in \psi(t, x)$ for all $(t, x) \in U$ and for each $t \in T$, $f(t, \cdot)$ is continuous on $U^t = \{x \in X\colon (t, x) \in U\}$ and for each $x \in X$, $f(\cdot, x)$ is measurable on $U_x = \{t \in T\colon (t, x) \in U\}$. Notice that for each $t \in T$, $U^t = \{x \in X\colon \psi(t, x) \neq \emptyset\} = \{x \in X\colon \psi(t, x) \cap X \neq \emptyset\}$ is open in the relative topology of X, since for each $t \in T$, $\psi(t, \cdot)$ is l.s.c. Furthermore, it follows at once from the lower measurability of $\psi(\cdot, \cdot)$ that the set $U = \{(t, x) \in T \times X\colon \psi(t, x) \cap X \neq \emptyset\}$ belongs to $\tau \otimes \beta(X)$. By virtue of the Projection Theorem we have that

$$\text{proj}_T(U \cap (T \times \{x\})) = \text{proj}_T(\{(t, x) \in T \times X\colon \psi(t, x) \neq \emptyset\} \cap (T \times \{x\}))$$

$$= \{t \in T\colon \psi(t, x) \neq \emptyset\}$$

$$= U_x \in \tau.$$

Hence, by Proposition 3.1, $f(\cdot, \cdot)$ is jointly measurable. Define the correspondence $\theta\colon T \times X \to 2^X$ by

$$\theta(t,x)=\begin{cases}\{f(t,x)\} & \text{if} \quad (t,x)\in U\\ X & \text{if} \quad (t,x)\notin U.\end{cases}$$

By Lemma 6.1 in Yannelis-Prabhakar (1983) we have that for each $t \in T$, $\theta(t,\cdot)$: $X \to 2^X$ is u.s.c.. Clearly, θ is convex and nonempty valued and it can be easily seen that $\theta(\cdot,\cdot)$ is lower measurable. Hence by Corollary 5.1, θ: $T \times X \to 2^X$ has a random fixed point, i.e., there exists a measurable function $\bar{x}$: $T \to X$ such that $\bar{x}(t) \in \theta(t, \bar{x}(t))$ for almost all t in T. Suppose that for a non-null subset S of T, $(t, \bar{x}(t)) \in U$. Then by the definition of θ, $\bar{x}(t) = f(t, \bar{x}(t)) \in \psi(t, \bar{x}(t)) = \text{con } P(t, \bar{x}(t))$ for all $t \in S$, a contradiction to assumption (iii). Hence, for almost all t in T, $(t, \bar{x}(t)) \notin U$ and consequently $\psi(t, \bar{x}(t)) = \emptyset$ for almost all t in T which implies that $P(t, \bar{x}(t)) = \emptyset$ for almost all t in T. This completes the proof of the Theorem.

Theorem 6.1 can be extended to separable Banach spaces by strengthening the continuity assumption (ii). More formally we can state the following extension of Theorem 6.1.

THEOREM 6.2: Let (T, τ, μ) be a complete finite measure space and X be a compact, convex, nonempty subset of a separable Banach space. Let P: $T \times X \to 2^X$ be a correspondence (possibly empty-valued) such that

(i) $\{(t, x, y) \in T \times X \times X\colon\ y \in \text{con } P(t, x)\} \in \tau \otimes \beta(X) \otimes \beta(X)$,

(ii) for each $t \in T$ and each $y \in X$ the set $P^{-1}(t, y) = \{x \in X\colon\ y \in P(t, x)\}$ is open in the relative norm topology of X,

(iii) for each $(t, x) \in T \times X$, if $P(t, x) \neq \emptyset$ then $P(t, x)$ has a nonempty interior in the relative norm topology of X,

(iv) for every measurable function x: $T \to X$, $x(t) \notin \text{con } P(t, x(t))$ for almost all $t \in T$.

Then there exists a measurable function $\bar{x} : T \rightarrow X$, such that $P(t, \bar{x}(t)) = \emptyset$ for almost all $t \in T$.

PROOF: The proof is almost identical with the proof of Theorem 6.1. Define $\psi: T \times X \rightarrow 2^X$ by $\psi(t, x) = \text{con } P(t, x)$. By virtue of Lemma 5.1 in Yannelis-Prabhakar (1983) for each $t \in T$ and each $y \in X$ the set $\psi^{-1}(t, y) = \{x \in X: y \in \psi(t, x)\}$ is open in the relative norm topology of X. By Theorem 4.1 there exists a Carathéodory selection from ψ. One can now proceed as in the proof of Theorem 6.1 to complete the proof.

Below we indicate how versions of Theorems 6.1 and 6.2 can be easily proved by combining the deterministic maximal elements results given in Yannelis-Prabhakar (1983) with the Aumann measurable selection theorem.

THEOREM 6.1′: Theorem 6.1 remains true if one replaces assumption (i) by

(i′) $P(\cdot, \cdot)$ is lower measurable.

PROOF: Define the correspondence $M: T \rightarrow 2^X$ by $M(t) = \{x \in X: P(t, x) = \emptyset\}$. It can be easily checked that for each fixed t in T, the correspondence $P(t, \cdot): X \rightarrow 2^X$ satisfies all the assumptions of Theorem 5.2 in Yannelis-Prabhakar (1983, p. 239) and so for each fixed t in T, the correspondence $P(t, \cdot): X \rightarrow 2^X$ has a maximal element, i.e., there exists $\bar{x}_t \in X$ such that $P(t, \bar{x}_t) = \emptyset$ for all t in T. Therefore, for each $t \in T$, $M(t) \neq \emptyset$. Since by assumption $P(\cdot, \cdot)$ is lower measurable, the set

$$A = \{(t, x) \in T \times X: P(t, x) \neq \emptyset\} = \{(t, x) \in T \times X: P(t, x) \cap X \neq \emptyset\},$$

belongs to $\tau \otimes \beta(X)$, and so does the complement of the set A which is denoted by A^c. Observe now that

$$G_M = \{(t, x) \in T \times X: x \in M(t)\} = \{(t, x) \in T \times X: P(t, x) = \emptyset\}$$

$$= \{(t, x) \in T \times X: P(t, x) \neq \emptyset\}^c = A^c.$$

and the latter set belongs to $\tau \otimes \beta(X)$ as it was noted above. Thus, $M(\cdot)$ has a measurable graph. We can not appeal to the Aumann measurable selection theorem to ensure the existence of a measurable function $\bar{x}$: $T \to X$ such that $\bar{x}(t) \in M(t)$ for almost all t in T, i.e., $P(t, \bar{x}(t)) = \emptyset$ for almost all t in T. This completes the proof of the Theorem.

THEOREM 6.2′: Theorem 6.2 remains true if assumption (iii) is dropped and assumption (i) is replaced by

(i′) $P(\cdot, \cdot)$ is lower measurable.

PROOF: The proof is similar with that of Theorem 6.1′. Define M: $T \to 2^X$ by $M(t) = \{x \in X\colon P(t, x) = \emptyset\}$. Using Theorem 5.1 in Yannelis-Prabhakar (1983, p. 239) we can conclude that $M(t) \neq \emptyset$ for all t in T. Adopting the argument of the previous Theorem one can show that $G_M \in \tau \otimes \beta(X)$. Appeal now to the Aumann measurable selection theorem to complete the proof.

REMARK 6.1: Theorems 6.1′ and 6.2′ remain true if we replace the assumption that (T, τ, μ) is a complete finite measure space by the fact that (T, τ) is a measurable space. The proofs remain the same provided that one observes that the correspondence M: $T \to 2^X$ defined by $M(t) = \{X \in X\colon P(t, x) = \emptyset\}$ is closed valued since for each $t \in T$, $P(t, \cdot)$ is l.s.c. (this is also true if for each $t \in T$ and each $y \in X$, $P^{-1}(t, y)$ is open in X). Since $M(\cdot)$ is closed valued and it has a measurable graph, it is also lower measurable (recall Lemma 3.1). By virtue of the Kuratowski and Ryll-Nardzewski measurable selection theorem, one can assure the existence of a measurable function x^*: $T \to X$ such that $x^*(t) \in M(t)$ for all $t \in T$, i.e., $P(t, x^*(t)) = \emptyset$ for all $t \in T$.

6.2 Random Equilibria

An *exchange economy* $\bar{E} = \{(X_i, P_i, e_i)\colon i = 1, 2, \ldots, N\}$ is a family of ordered triples (X_i, P_i, e_i) where,

(i) $X_i \subset R^l$ is the *consumption set* of agent i ,

(ii) P_i: $X_i \to 2^{X_i}$ is the *preference correspondence* of agent i, and

(iii) e_i is the *initial endowment* of agent i, where $e_i \in X_i$ for all i.

The pair (e_i, P_i) is the characteristic of agent i, i.e., his/her initial endowment and preference correspondence. The interpretation of the preference correspondence P_i is as in Section 6.1, i.e., we read $y_i \in P_i(x_i)$ as "agent i strictly prefers the consumption vector y_i to x_i".

Let $\Delta = \{q \in \Omega: \sum_{i=1}^{l} q_i = 1\}$ (where Ω denotes the positive cone of R^l). For $p \in \Delta$, $B_i(p) = \{x \in X_i: p \cdot x \leq p \cdot e_i\}$ denotes the *budget set* of agent i, and $D_i(p) = \{x_i \in B_i(p): P_i(x_i) \cap B_i(p) = \emptyset\}$ denotes the *demand set* of agent i.

Define the *aggregate excess demand* $\varsigma: \Delta \to 2^{R^l}$ for the economy $\bar{\mathbf{E}}$ by $\varsigma(p) = \sum_{i=1}^{N} D_i(p) - \sum_{i=1}^{N} e_i$.

As in Debreu (1959) a *free disposal price equilibrium* is a vector $\bar{p} \in \Delta$ such that $\varsigma(\bar{p}) \cap (-\Omega) \neq \emptyset$. A *price equilibrium* is a vector $\bar{p} \in \Delta$ such that $0 \in \varsigma(\bar{p})$.

Conditions which guarantee the existence of either a free disposal price equilibrium or price equilibrium are by now well-known in the literature, see for instance Debreu (1959) and his references. We now amend the deterministic economy described above by introducing randomness.

Let (T, τ, μ) be a complete finite measure space.

A *random exchange economy* $\mathbf{E} = \{(X_i, P_i, e_i): i = 1, 2, \cdots, N\}$ is a family of ordered triples (X_i, P_i, e_i), where

(i) $X_i \subset R^l$ is the *consumption set* of agent i,

(ii) P_i: $T \times X_i \to 2^{X_i}$ is the *random preference correspondence* of agent i,

(iii) e_i: $T \to R^l$ is the *random initial endowment* of agent i, where $e_i(t) \in X_i$ for all $t \in T$.

Notice that now each agent's characteristics, i.e., preferences and endowments depend on the state of nature. Hence, randomness is explicitly introduced into agents' characteristics. In this framework, $y_i \in P_i(t, x_i)$ means that "agent i strictly prefers y_i to x_i at the state of nature t".

For $p \in \Delta$ and $t \in T$ define the *random budget set* of agent i by $B_i(t, p) = \{x \in X_i\text{: } p \cdot x \leq p \cdot e_i(t)\}$ and the *random demand set* of agent i by $D_i(t, p) = \{x_i \in B_i(t, p)\text{: } P_i(t, x_i) \cap B_i(t, p) = \emptyset\}$. Define the *aggregate random excess demand* ς: $T \times \Delta \to 2^{R^l}$ for the economy **E** by $\varsigma(t, p) = \sum_{i=1}^{N} D_i(t, p) - \sum_{i=1}^{N} e_i(t)$. We now define the natural analogues of the ordinary concepts of price equilibrium.

A *free disposal random price equilibrium* is a measurable function $\bar{p}$: $T \to \Delta$ such that

$$\varsigma(t, \bar{p}(t)) \cap (-\Omega) \neq \emptyset \qquad \text{for almost all } t \text{ in } T.$$

A *random price equilibrium* is a measurable function $\bar{p}$: $T \to \Delta$ such that

$$0 \in \varsigma(t, \bar{p}(t)) \qquad \text{for almost all } t \text{ in } T.$$

Notice that now the equilibrium price (or the market clearing price) depends on the states of nature. Hence, in this framework the market clearing price will change from one state of the environment to another.

The concept of random price equilibria which is obviously a generalization of the ordinary (deterministic) notion of price equilibrium is not new. It can be traced to Bhattacharya-Majumdar (1973 Section IV, p. 45), Hildenbrand (1971, p. 427) and more recently to Weller (1982, p. 75).

Below we provide conditions which guarantee the existence of either a free disposal random price equilibria or random price equilibria.

THEOREM 6.3: Let $\varsigma: T \times \Delta \to 2^{R^l}$ be a random aggregate excess demand correspondence, satisfying the following assumptions:

(i) For each $t \in T$, $\varsigma(t, \cdot)$ is u.d.c.,

(ii) $\varsigma(\cdot, \cdot)$ is lower measurable, i.e., for every open subset V of R^l, $\{(t, p) \in T \times \Delta: \varsigma(t, p) \cap V \neq \emptyset\} \in \tau \otimes \beta(\Delta)$,

(iii) for all $(t, p) \in T \times \Delta$, $\varsigma(t, p)$ is convex, compact and nonempty,

(iv) for all measurable $p: T \to \Delta$ there exists $z \in \varsigma(t, p(t))$ such that $p(t) \cdot z \leq 0$ for all $t \in T$.

Then there exists a free disposal random equilibrium, i.e., there exists a measurable function $\bar{p}: T \to \Delta$ such that $\varsigma(t, \bar{p}(t)) \cap (-\Omega) \neq \emptyset$ for almost all t in T.

REMARK 6.1: Observe that Theorem 6.3 gives as a Corollary the ordinary (deterministic) Gale-Nikaido-Debreu (G-N-D) excess demand theorem [see for instance Debreu (1959, p. 82)] simply by fixing $t \in T$ and considering the correspondence $\varsigma(t, \cdot): \Delta \to 2^{R^l}$. Also it is important to note that the argument adopted to prove Theorem 6.3 does not use the G-N-D theorem. The proof we give is direct (starts from "scratch") and provides an alternative way to prove the ordinary G-N-D result.

PROOF OF THEOREM 6.3: Define the correspondence $F: T \times \Delta \to 2^{\Delta}$ by $F(t, p) = \{q \in \Delta: q \cdot z > 0 \text{ for all } z \in \varsigma(t, p)\}$.

We will show that the correspondence $F: T \times \Delta \to 2^{\Delta}$ satisfies all the properties of Theorem 6.1 and therefore it has a random maximal element. By construction the random maximal element, will turn out to be a random price equilibria.

(i) *The correspondence* $F: T \times \Delta \to 2^{\Delta}$ *is convex valued and for all measurable* $p: T \to \Delta$, $p(t) \notin F(t, p(t))$, *for almost all* t *in* T..

It can be easily checked that for all $(t, p) \in T \times \Delta$, $F(t, p)$ is convex. Moreover, it follows directly from assumption (iv) that $p(t) \notin con\ F(t, p(t)) = F(t, p(t))$ for all measurable $p: T \to \Delta$ and all t in T.

(ii) *For each fixed* $t \in T$, $F(t, \cdot)$ *is l.s.c.*

By virtue of Proposition 4.1 in Yannelis-Prabhakar (1983) it suffices to show that for each $t \in T$ and each $q \in \Delta$ the set $F^{-1}(t, q) = \{p \in \Delta\colon\ q \in F(t, p)\}$ is open in Δ. To this end, let $V_q = \{x\colon\ q \cdot x > 0\}$ be an open half space in R^l. Since for each t in T, $\varsigma(t, \cdot)$ is u.d.c., the set $W = \{p \in \Delta\colon\ \varsigma(t, p) \subset V_q\}$ is open in Δ. It can be easily checked that $W = F^{-1}(t, q)$. Therefore, for each $t \in T$ and each $q \in \Delta$ the set $F^{-1}(t, q)$ is open in the relative topology of Δ.

(iii) *The correspondence* $F\colon\ T \times \Delta \to 2^{\Delta}$ *is lower measurable.*

Setting $S = T \times \Delta$, $\alpha = \tau \otimes \beta(\Delta)$, $\phi(s) = \varsigma(t, p)$ and $\theta(s) = F(t, p)$ for $s = (t, p)$ in Lemma 3.10, we conclude that $F(\cdot, \cdot)$ is lower measurable.

Therefore, the correspondence, $F\colon\ T \times \Delta \to 2^{\Delta}$ satisfies all the assumptions of Theorem 6.1 and consequently, there exists a measurable function $\bar{p}\colon\ T \to \Delta$ such that $F(t, \bar{p}(t)) = \emptyset$ for all t in T, i.e.,

(6.1) for all $q \in \Delta$ there exists $z \in \varsigma(t, \bar{p}(t))$ such that $q \cdot z \leq 0$ for almost all t in T.

We now show that (6.1) implies that

(6.2) $\varsigma(t, \bar{p}(t)) \cap (-\Omega) \neq \emptyset$ for almost all t in T.

Suppose otherwise, then for all $t \in A$, where A is a non-null subset of T, $\varsigma(t, \bar{p}(t)) \cap (-\Omega) = \emptyset$. Since $\varsigma\colon\ T \times \Delta \to 2^{R^l}$ is convex and compact valued and $-\Omega$ is a closed convex cone, the sets $\varsigma(t, \bar{p}(t))$ and $-\Omega$ can be strictly separated, i.e., there exist $r \in R^l$, $r \neq 0$ and $b \in R$ such that

(6.3) $\sup_{y \in -\Omega} r \cdot y < b < \inf_{x \in \varsigma(t, \bar{p}(t))} r \cdot x$

Notice that $b > 0$ and $r \geq 0$. Without loss of generality we may assume that $r \in \Delta$. It follows from (6.3) that $r \cdot z > 0$ for all $z \in \varsigma(t, \bar{p}(t))$ and for all $t \in A$, a contradiction to (6.1). Hence, (6.2) holds and this completes the proof of the Theorem.

Notice that the dimensionality of the commodity spaces in Theorem 6.3 is finite. We now provide an extension of Theorem 6.3 to infinite dimensional commodity space and in particular to a separable Banach space whose positive cone has a nonempty norm interior. The Theorem below may be seen as a generalization of the deterministic equilibrium results of Florenzano (1983) and Yannelis (1985), but only if the underlying commodity space is separable.

THEOREM 6.4: Let Y be a separable Banach space, C the closed convex cone of Y, having an interior point u, $C^* = \{p \in Y^*: p \cdot x \leq 0 \text{ for all } x \in C\} \neq \{0\}$ the dual cone of C and $\tilde{\Delta} = \{p \in C^*: p \cdot u = -1\}$. Let $\varsigma: T \times \tilde{\Delta} \to 2^Y$ be an aggregate random excess demand correspondence satisfying the following conditions:

(i) For each $t \in T$, $\varsigma(t, \cdot): \tilde{\Delta} \to 2^Y$ is u.d.c., in the weak* topology, (i.e., $\varsigma(t, \cdot): (\tilde{\Delta}, w^*) \to 2^Y$ is u.d.c.),

(ii) $\varsigma(\cdot, \cdot)$ is lower measurable, i.e., for every open subset V of Y, $\{(t, p) \in T \times \tilde{\Delta}: \varsigma(t, p) \cap V \neq \emptyset\} \in \tau \otimes \beta_{w^*}(\tilde{\Delta})$, where $\beta_{w^*}(\tilde{\Delta})$ is the Borel σ-algebra for the weak* topology on $\tilde{\Delta}$,

(iii) $\varsigma(t, p)$ is convex, compact and nonempty for all $(t, p) \in T \times \Delta$,

(iv) for all measurable $p: T \to \tilde{\Delta}$, there exists $x \in \varsigma(t, p(t))$ such that $p(t) \cdot x \leq 0$, for all $t \in T$.

Then there exists a measurable function $\bar{p}: T \to \tilde{\Delta}$ such that $\varsigma(t, \bar{p}(t)) \cap C \neq \emptyset$ for almost all t in T.

PROOF: We begin by proving an elementary fact.

<u>FACT 6.1</u>: Let X be a Hausdorff linear topological space, C a closed convex cone of X having an interior point u and $C^* = \{p \in X^*: p \cdot x \leq 0$ for all $x \in C\} \neq \{0\}$ the dual cone of C. Then $r \cdot u < 0$ for any $r \in C^*$.

<u>PROOF</u>: Suppose by way of contradiction that for some $r \in C^*$, $r \cdot u = 0$. Pick a symmetric neighborhood V of zero with $u + V \subseteq C$. If $x \in X$, then for some $\lambda \in R$, $\lambda > 0$ we have that $\pm \lambda x \in V$ and consequently $\pm \lambda r \cdot x = r(u \pm \lambda x) \leq 0$. Hence, $r \cdot x = 0$ for each $x \in X$, i.e., $r = 0$, a contradiction. Therefore, $r \cdot u < 0$ for any $r \in C^*$, and this completes the proof of the Fact.

We now proceed with the proof of Theorem 6.4, whose idea is essentially the same with that of Theorem 6.3. Define the correspondence $F\colon T \times \tilde{\Delta} \to 2^{\tilde{\Delta}}$ by

$$F(t, p) = \{q \in \tilde{\Delta}: q \cdot z > 0 \qquad \text{for all } z \in \varsigma(t, p)\}.$$

First notice that by Alaoglu's theorem $\tilde{\Delta}$ is weak* compact [Jameson (1970, Theorem 3,8, p. 123)]. Moreover, since Y is a separable Banach space, $\tilde{\Delta}$ is a compact metric space. Adopting the arguments used in the proof of Theorem 6.3 one can easily see that $F\colon T \times \tilde{\Delta} \to 2^{\tilde{\Delta}}$ satisfies all the properties of Theorem 6.2 (of course, one now has to use Lemma 3.9 to show that $G_F \in \tau \otimes \beta_{w^*}(\tilde{\Delta}) \otimes \beta_{w^*}(\tilde{\Delta})$). Hence, there exists a measurable function $\bar{p}\colon T \to \tilde{\Delta}$ such that $F(t, \bar{p}(t)) = \emptyset$ for almost all t in T, i.e.,

(6.4) for all $q \in \tilde{\Delta}$ there exists $z \in \varsigma(t, \bar{p}(t))$ such that $q \cdot z \leq 0$ for almost all t in T.

We show that (6.4) implies that

(6.5) $\varsigma(t, \bar{p}(t)) \cap C \neq \emptyset$ for almost all t in T.

Suppose otherwise, then for all t in a non-null subset A of T, $\varsigma(t, \bar{p}(t)) \cap C = \emptyset$. By the separating hyperplane theorem there exist $r \in Y^*/\{0\}$ and $b \in R$ such that

(6.6) $\sup\limits_{y \in C} r \cdot y < b < \inf\limits_{x \in \varsigma(t, \bar{p}(t))} r \cdot x.$

Notice that $b > 0$ and $r \in C^*$. Without loss of generality we may assume that $r \in \tilde{\Delta}$. In fact, if $r \notin \tilde{\Delta}$ then $u \in \text{int}\, C$ implies (recall Fact 6.1) that $r \cdot u < 0$ and we can replace

r by $\frac{u}{-r \cdot u}$. It follows from (6.6) that $r \cdot z > 0$ for all $z \in \varsigma(t, \bar{p}(t))$ for all $t \in A$, a contradiction to (6.4) Hence, (6.5) holds and this completes the proof of Theorem 6.4.

REMARK 6.2: As we noted earlier Theorem 6.4 may be seen as a generalization of the deterministic equilibrium results of Florenzano (1983) and Yannelis (1985). Moreover, our arguments adopted for the proof of Theorem 6.4 provide an alternative way to prove the above deterministic equilibrium results of the above authors. We do wish however to indicate that a version of Theorem 6.4 can be easily obtained by combining the deterministic result in Yannelis (1985) with the Aumann measurable selection theorem as follows:

THEOREM 6.4′: Replace assumption (ii) in Theorem 6.4 by

(ii′) $\varsigma(\cdot,\cdot)$ is measurable, i.e., for every closed subset V of Y, the set $\{(t,p) \in T \times \tilde{\Delta}: \varsigma(p,t) \cap V \neq \emptyset\}$ belongs to $\tau \otimes \beta_{w^*}(\tilde{\Delta})$.

Suppose that conditions (i), (iii) and (iv) of Theorem 6.4 are satisfied. Then the conclusion of Theorem 6.4 holds.

PROOF: Define the correspondence $W: T \to 2^{\tilde{\Delta}}$ by

$$W(t) = \{p \in \tilde{\Delta}: \varsigma(t,p) \cap C \neq \emptyset\}.$$

By Theorem 3.1 in Yannelis (1985, p. 597) for each fixed $t \in T$ there exists $\bar{p}_t \in \tilde{\Delta}$ such that $\varsigma(t, \bar{p}_t) \cap C \neq \emptyset$. Therefore, $W(t) \neq \emptyset$ for all $t \in T$. Observe that

$$G_W = \{(t,p) \in T \times \tilde{\Delta}: p \in W(t)\} = \{(t,p) \in T \times \tilde{\Delta}: \varsigma(t,p) \cap C \neq \emptyset\}.$$

It follows, at once from the measurability of $\varsigma(\cdot,\cdot)$, (assumption (ii′)) that $G_W \in \tau \otimes \beta_{w^*}(\tilde{\Delta})$, i.e., $W(\cdot)$ has a measurable graph. Appeal now to the Aumann measurable selection theorem to ensure the existence of a measurable function $\bar{p}: T \to \tilde{\Delta}$ such that $\bar{p}(t) \in W(t)$ for almost all t in T, i.e., $\varsigma(t, \bar{p}(t)) \cap C \neq \emptyset$ for almost all t in T.

Bibliographical Notes: All the results in this section are new. They generalize the deterministic results on the existence of maximal elements of Sonnenschein (1971), and

Yannelis-Prabhakar (1983) as well as the excess demand equilibrium existence theorems of Debreu (1959), Aliprantis-Brown (1983), Florenzano (1983) and Yannelis (1985), among others.

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