A SHORT COURSE ON
DIFFERENTIAL EQUATIONS

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A SHORT COURSE ON

DIFFERENTIAL EQUATIONS

BY

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PREFACE

In many Colleges of Engineering, the need is felt for a text-book on Differential Equations, limited in scope yet comprehensive enough to furnish the student of engineering with sufficient information to enable him to deal intelligently with any differential equation which he is likely to encounter. To meet this need is the object of this book.

Throughout the book, I have endeavored to confine myself strictly to those principles which are of interest to the student of engineering. In the selection of problems, the aim was constantly before me to choose only those that illustrate differential equations or mathematical principles which the engineer may meet in the practice of his profession.

I have consulted freely the Treatises on Differential Equations of Boole, Forsyth, Johnson, and Murray. I am indebted to two of my colleagues, Professors N. C. Riggs and C. W. Leigh, for reading parts of the manuscript and verifying many of the answers to problems.

D. F. CAMPBELL.

CHICAGO, ILL.,
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PREFACE TO ENLARGED EDITION

This book as it first appeared consisted of the first eight chapters as here given. The kindly criticism by a number of those teachers for whose use it was intended on the need of a discussion of equations, that occur in investigations in Mathematical Physics, other than those given in these chapters has induced me to add Chapter IX to the book.

In the preparation of Chapter IX., I have drawn freely from Professor Byerly's Treatise on Fourier's Series and Spherical Harmonics, from Professor Bocher's pamphlet entitled Regular Points of Linear Differential Equations of the Second Order and from notes kindly loaned me by Professor Snyder of Cornell University. I have also consulted Heffter's Treatise on Linear Differential Equations with one Independent Variable.

To those teachers who have sent me their criticism of the book in its original form, as well as to others who have cordially received it, I am under the deepest obligations.

D. F. CAMPBELL.

CHICAGO, ILL.,
June, 1907.

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CHAPTER I

INTRODUCTION

1. There are various definitions given for a **function of one variable.** We shall here adopt the following :

If to every value of x there corresponds one or more values of $f(x)$, then $f(x)$ is said to be a function of x.

This definition includes a constant as a function of x, for if $f(x)$ is constant, then for every value of x, $f(x)$ has a value, namely, this constant.

A definition of a **function of two variables** is the following :

If to every pair of values of two variables x and y there corresponds one or more values of $f(x, y)$, then $f(x, y)$ is said to be a function of x and y.

This includes a constant or a function of one variable as a function of x and y.

A function $f(x)$ of one variable x is **single valued** when for every value of x there is one and only one corresponding value of $f(x)$.

A function $f(x)$ of one variable x is **continuous** for a value $x = a$ if $f(a)$ is finite, and

$$\underset{h \doteq 0}{\text{limit}}\left[f(a+h)\right] = \underset{h \doteq 0}{\text{limit}}\left[f(a-h)\right] = f(a).$$

A function $f(x, y)$ of two independent variables x and y is **single valued** when for every set of values for x and y there is one and only one corresponding value of $f(x, y)$.

A function $f(x, y)$ of two independent variables x and y is **continuous** for a set of values $x = a$, $y = b$ if $f(a, b)$ is finite, and

$$\lim_{\substack{h \doteq 0 \\ k \doteq 0}} \left[f(a + h, b + k) \right] = f(a, b)$$

no matter how h and k approach zero.

The following definitions are given in almost any work in calculus:

If $f(x)$ is a single valued and continuous function of x, given by the equation $y = f(x)$, then

Δx and Δy denote the increments of x and y respectively,

$$\frac{dy}{dx} = \lim_{\Delta x \doteq 0} \left[\frac{\Delta y}{\Delta x} \right],$$

$$dy = \frac{dy}{dx} dx.$$

If $f(x)$ is single valued and continuous, and dy/dx is continuous, then

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

In general, if $f(x)$ is single valued and continuous, and the preceding derivatives are all continuous, then

$$\frac{d^n y}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1} y}{dx^{n-1}}\right).$$

If $f(x, y)$ is a single valued and continuous function of two independent variables x and y, given by the equation $z = f(x, y)$, then $\partial z/\partial x$ is the derivative of z with respect to x when y is held constant; $\partial z/\partial y$ is the derivative of z with respect to y when x is held constant.

2. In a single valued and continuous function $f(x)$ of one variable x, given by the equation $y = f(x)$, whether x is the independent variable or a function of some other variable or variables, we have

$$d^2x = d(dx);\ d^3x = d(d^2x);\ \cdots;\ d^nx = d(d^{n-1}x);$$

$$d^2y = d(dy);\ d^3y = d(d^2y);\ \cdots;\ d^ny = d(d^{n-1}y).$$

Definitions. The differentials dx, d^2x, d^3x, $\cdots$, d^nx, or dy, d^2y, d^3y, $\cdots$, d^ny are called the first, second, third, $\cdots$, nth, differentials respectively.

3. Derivation of d^2y and d^3y when no assumption is made regarding x being independent or a function of some variable or variables.

$$d^2y = d(dy) = d\left(\frac{dy}{dx}dx\right) = \frac{d^2y}{dx^2}dx^2 + \frac{dy}{dx}d^2x.$$

$$d^3y = d(d^2y) = d\left(\frac{d^2y}{dx^2}dx^2 + \frac{dy}{dx}d^2x\right)$$

$$= \frac{d^3y}{dx^3}dx^3 + 3\frac{d^2y}{dx^2}dx d^2x + \frac{dy}{dx}d^3x.$$

By taking differentials in succession any differential may ultimately be found.

4. In the differentials of the preceding article, if x is an independent variable, it can be assumed without loss of generality, that Δx, or what is the same in this case, dx, is constant. That is, it can be assumed that x changes by equal increments. Under this supposition, therefore, d^2x and all higher differentials of x can be taken zero. Therefore, under this supposition,

$$d^2y = \frac{d^2y}{dx^2}dx^2,$$

$$d^3y = \frac{d^3y}{dx^3}dx^3.$$

The place which a derivative or differential occupies in the succession of derivatives or differentials indicates the **order** of the derivative or differential. Thus, a second derivative or differential is said to be of the second order, a third of the third order, and so on.

5. The only functions usually considered in elementary works in calculus are functions of a real variable. Such functions with one exception are the only ones considered in the following pages. The exception is e^z where z is a complex quantity.

The student is already familiar with the definition of e^x where x is real. He is, however, probably not familiar with the definition of e^z when z is a complex quantity. A definition of this function will now be given.

The infinite series

$$1 + z + \frac{z^2}{\lfloor 2} + \frac{z^3}{\lfloor 3} + \cdots,$$

where z is a complex quantity, can be shown to have a determinate, finite value for every value of z. It also reduces to the infinite series

$$1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$$

when z becomes real and equal to x, and this series, it will be remembered, is equal to e^x for all values of x. It therefore appears that the infinite series in z would be satisfactory as a definition of e^z. We shall define e^z by saying that it is equal to the infinite series

$$1 + z + \frac{z^2}{\lfloor 2} + \frac{z^3}{\lfloor 3} + \cdots$$

for all values of z.

From this definition, the following theorem can be established:

Theorem. If $z = x + yj$ where x and y are real, and $j = \sqrt{-1}$, then

$$e^z = e^x(\cos y + j \sin y).$$

Proof.

$$e^z = 1 + z + \frac{z^2}{\lfloor 2} + \frac{z^3}{\lfloor 3} + \cdots, \text{ by definition.}$$

$$= 1 + (x + yj) + \frac{(x + yj)^2}{\lfloor 2} + \frac{(x + yj)^3}{\lfloor 3} + \cdots.$$

Consider all the terms containing x^r. These are found from the terms

$$\frac{(x+yj)^r}{\underline{|r}} + \frac{(x+yj)^{r+1}}{\underline{|r+1}} + \cdots.$$

They are

$$\frac{x^r}{\underline{|r}}\left[1 + yj + \frac{(yj)^2}{\underline{|2}} + \frac{(yj)^3}{\underline{|3}} + \frac{(yj)^4}{\underline{|4}} + \frac{(yj)^5}{\underline{|5}} + \cdots\right],$$

or

$$\frac{x^r}{\underline{|r}}\left[1 + yj - \frac{y^2}{\underline{|2}} - \frac{y^3j}{\underline{|3}} + \frac{y^4}{\underline{|4}} + \frac{y^5j}{\underline{|5}} - \cdots\right].$$

Separate the real terms from the imaginary and there results,

$$\frac{x^r}{\underline{|r}}\left[1 - \frac{y^2}{\underline{|2}} + \frac{y^4}{\underline{|4}} - \cdots + j\left\{y - \frac{y^3}{\underline{|3}} + \frac{y^5}{\underline{|5}} - \cdots\right\}\right],$$

or

$$\frac{x^r}{\underline{|r}}[\cos y + j \sin y].$$

Let r take all positive integral values in succession from 0. In this way we get all the terms of the development e^z. Then

$$e^z = \left[1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \cdots\right][\cos y + j \sin y]$$

$$= e^x(\cos y + j \sin y).$$

The theorem is therefore proved.

Examples. $e^{-x+3xj} = e^{-x}(\cos 3x + j \sin 3x)$

EXERCISES

1. Given $y = \log x$, find dy, d^2y, d^3y:

(a), on the assumption that x is the independent variable;

(b), making no assumption with regard to x.

In the results of (b), substitute $x = \cos\theta$ and show that the results are the same as those obtained by first substituting the value of x in $\log x$ and then taking the differentials.

2. Given $y = e^x$ where $x = \cos\theta$, express dy, d^2y, d^3y in terms of θ without substituting the value of x in the equation $y = e^x$.

3. Given $y = \log x$ where $x = \sin\theta$, express dy, d^2y, d^3y in terms of θ without substituting the value of x in the equation $y = \log x$.

4. Prove that $e^{x-yj} = e^x(\cos y - j\sin y)$.

5. Prove that $e^{x+yj}e^{z+wj} = e^{x+z+(y+w)j}$, where x, y, z and w are real.

ANSWERS

1(*a*). $$dy = \frac{dx}{x};\ d^2y = -\frac{dx^2}{x^2};\ d^3y = \frac{2dx^3}{x^3}.$$

(*b*). $$dy = \frac{dx}{x};\ d^2y = \frac{xd^2x - dx^2}{x^2};\ d^3y = \frac{x^2d^3x - 3xdxd^2x + 2dx^3}{x^3}.$$

2. $$dy = -\sin\theta\, e^{\cos\theta}d\theta;$$
$$d^2y = -\sin\theta\, e^{\cos\theta}d^2\theta + (\sin^2\theta - \cos\theta)e^{\cos\theta}d\theta^2;$$
$$d^3y = -\sin\theta\, e^{\cos\theta}d^3\theta + 3(\sin^2\theta - \cos\theta)e^{\cos\theta}d\theta d^2\theta + \sin\theta(1 + 3\cos\theta - \sin^2\theta)e^{\cos\theta}d\theta^3.$$

3. $$dy = \cot\theta\, d\theta;\ d^2y = \cot\theta\, d^2\theta - \operatorname{cosec}^2\theta\, d\theta^2;$$
$$d^3y = \cot\theta\, d^3\theta - 3\operatorname{cosec}^2\theta\, d\theta d^2\theta + 2\operatorname{cosec}^2\theta\cot\theta\, d\theta^3.$$

6. Definition of differential equation. A differential equation is an equation involving derivatives or differentials with or without the variables from which these derivatives or differentials are derived.

The following are examples of differential equations:

$$\left(\frac{d^3y}{dx^3}\right)^2 = x^3\frac{dy}{dx}. \quad (1)$$

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0. \quad (2)$$

$$\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} + y^2 = 0. \quad (3)$$

$$3\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx}\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2 = 0. \quad (4)$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0. \tag{5}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0. \tag{6}$$

7. In examples (1) to (4) inclusive of the preceding article it will be noticed that differentials enter the equation only in derivatives. It is conceivable, however, that there might be an equation containing differentials other than those in the derivatives, as for example,

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + dy = x^2,$$

but there is no need of entering into a discussion of such equations, and we shall not do so. In what follows, we shall assume that if the equation is written in differential form, the differentials can all be converted into derivatives by the process of division.

8. Classes of differential equations. Differential equations are divided into two classes: **ordinary** and **partial.**

An ordinary differential equation is one in which all the derivatives involved have reference to a single independent variable.

A partial differential equation is one which contains partial derivatives and therefore indicates the existence of two or more independent variables with respect to which these derivatives have been formed.

Thus, in Art. 6, equations (1), (2), (3) and (4) are ordinary differential equations, and equations (5) and (6) are partial differential equations.

Chapters I to VII inclusive are devoted to a discussion of ordinary differential equations. Chapter VIII contains a short treatment of some partial differential equations.

9. Order and degree of a differential equation. The **order** of a differential equation is that of the highest derivative or differential in the equation.

Thus, in Art. 6, equations (1) and (4) are of the third order, and (2) and (3) of the second order.

The **degree** of a differential equation is the degree of the derivative or differential of highest order in the equation after the equation is freed from radicals and fractions in its derivatives.

Thus, in Art. 6, equation (1) is of the second degree, equations (2), (3) and (4) of the first degree.

10. Solutions of a differential equation. Let us consider the differential equation in each of the two following examples, and see if, from the equation, we can get a relation connecting x and y and not involving derivatives, such that, if the value of y in terms of x be substituted in the equation, the equation is satisfied.

EXAMPLE 1. $$\frac{dy}{dx} = x^2.$$

By integration, we get

$$y = \frac{x^3}{3} + c.$$

EXAMPLE 2. $$\frac{d^2y}{dx^2} + y = 0.$$

Multiply the equation by $2dy/dx$ and integrate.

$$\therefore \left(\frac{dy}{dx}\right)^2 + y^2 = c.$$

$$\therefore \frac{dy}{dx} = \pm\sqrt{c - y^2}.$$

$$\therefore \frac{dy}{\sqrt{c - y^2}} = \pm dx.$$

$$\therefore y = \pm\sqrt{c}\sin(x + c_1), \quad \text{or} \quad y = \pm\sqrt{c}\cos(x + c_2).$$

In example 1, if $\frac{1}{3}x^3 + c$ be substituted for y in the equation, there results $x^2 = x^2$. The equation is therefore satisfied.

In example 2, if $\pm\sqrt{c}\sin(x + c_1)$, or $\pm\sqrt{c}\cos(x + c_2)$ be substituted for y in the equation, there results, in the first case,

$\mp \sqrt{c} \sin (x + c_1) \pm \sqrt{c} \sin (x + c_1) = 0$, and in the second case, $\mp \sqrt{c} \cos (x + c_2) \pm \sqrt{c} \cos (x + c_2) = 0$. In either case the equation is satisfied.

Definition. A solution of a differential equation is a relation between the variables of the equation and not involving derivatives, such that if the value of the dependent variable be substituted in the equation, the equation is satisfied.

Thus, $y = \frac{1}{3}x^3 + c$ of example 1, and $y = \pm \sqrt{c} \sin (x + c_1)$ of example 2, are solutions of the equations.

In this book we shall not concern ourselves with the question of whether every differential equation has a solution but shall be content with finding solutions in the few special cases discussed here.

11. A solution of an ordinary differential equation may be one of three kinds: **general, particular** and **singular.**

A general solution is one which contains arbitrary constants equal in number to the exponent of the order of the equation.

Thus, in example 1, Art. 10, the number of arbitrary constants is one and the exponent of the order of the equation is 1, and in example 2 of the same article the number of arbitrary constants is two, and the exponent of the order of the equation is 2. In either case the solution is the general solution of the equation.

A particular solution of a differential equation is a solution obtained from the general solution by giving one or more of the constants particular values.

Thus

$$y = \frac{x^3}{3}, \quad y = \frac{x^3}{3} + 1 \quad \text{or} \quad y = \frac{x^3}{3} - 5,$$

of example 1, Art. 10, or $y = \sin x$, $y = 2 \sin x$, or $y = -3 \cos x$, of example 2 of the same article, are particular solutions of the equations.

A singular solution of a differential equation is a solution without arbitrary constants which cannot be derived from the general solution by giving the constants particular values.

Singular solutions will not be considered in this book.

12. A solution of a differential equation is not a general solution unless the constants are in number equal to the exponent of the order of the equation, and cannot be reduced to a fewer number of equivalent constants.

Thus, $y = ce^{x+a}$, c and a arbitrary constants, although it contains two arbitrary constants, is not the general solution of a differential equation of the second order, as can readily be shown. The equation $y = ce^{x+a}$ is the same as $y = c^a e^x$. Now c^a is equivalent to only one arbitrary constant because an arbitrary constant can have any value and thus all the particular solutions got by giving c and a all possible values can be obtained. Therefore $y = c^a e^x$ is equivalent to a solution $y = Ae^x$, A arbitrary, and cannot therefore be the general solution of a differential equation of the second order.

13. Let $y = f_1(x)$, $y = f_2(x)$, $\cdots$, $y = f_n(x)$ be solutions of a differential equation.

Definition. If the c's cannot be chosen, not all zero, such that $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)$ is identically zero, then the solutions are said to be **linearly independent.**

Thus, $y = \pm \sqrt{c} \sin (x + c_1)$ and $y = \pm \sqrt{c} \cos (x + c_2)$ of example 2, Art. 10, are such that no values c_3 and c_4, not both zero, can be chosen such that $\pm c_3 \sqrt{c} \sin (x + c_1) \pm c_4 \sqrt{c} \cos (x + c_2)$ is identically zero. The solutions are therefore linearly independent.

14. Derivation of an ordinary differential equation. Let

$$\phi(x, y, c_1) = 0 \tag{1}$$

be an equation containing x and y, and the arbitrary constant c_1.

By differentiation of (1) there results

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0. \tag{2}$$

Equation (2) will in general contain c_1. If between (1) and

(2), c_1 be eliminated, the result is a differential equation of the first order of which $\phi(x, y, c_1) = 0$ is the general solution.

EXAMPLE. Find the differential equation of which

$$y = \frac{m}{2} + c_1 e^{-x^2}$$

is the general solution.

$$\frac{dy}{dx} = -2c_1 x e^{-x^2}.$$

Eliminate c_1 between the equations. Therefore

$$\frac{dy}{dx} + 2xy = mx$$

is the differential equation of which

$$y = \frac{m}{2} + c_1 e^{-x^2}$$

is the general solution.

Sometimes the arbitrary constant is so involved that it disappears in the equation which results from the differentiation. In such a case this equation is the desired equation.

EXAMPLE. Find the differential equation of which $y^2 = 2c_1 x$ is the general solution.

Divide both sides of the equation by x.

$$\therefore \frac{y^2}{x} = 2c_1.$$

By differentiation there results

$$2x\frac{dy}{dx} - y = 0,$$

which is the desired differential equation.

Let

$$\phi(x, y, c_1, c_2) = 0 \qquad (1)$$

be an equation between x and y, and two arbitrary constants c_1 and c_2.

By differentiation of (1) there results

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0. \tag{2}$$

Equation (2) contains dy/dx and will in general contain c_1 and c_2 also. Eliminate one of the constants between the two equations. Suppose the constant c_1 to be eliminated. The resulting equation contains dy/dx and in general x, y and c_2. Call it

$$\psi\left(x, y, \frac{dy}{dx}, c_2\right) = 0.$$

By differentiation there results

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Equation (3) contains d^2y/dx^2 and will in general contain c_2. Eliminate c_2 between (2) and (3). The result is a differential equation of which $\phi(x, y, c_1, c_2) = 0$ is the general solution.

Example. Find the differential equation of which

$$y = c_1 x + \frac{c_2}{x}$$

is the general solution.

Differentiate $$y = c_1 x + \frac{c_2}{x}.$$

$$\therefore \frac{dy}{dx} = c_1 - \frac{c_2}{x^2}.$$

Eliminate c_1.

$$\therefore y - x\frac{dy}{dx} = \frac{2c_2}{x}.$$

Differentiate.

$$\therefore \frac{d^2y}{dx^2} = \frac{2c_2}{x^3}.$$

Eliminate c_2 between

$$\frac{d^2y}{dx^2} = \frac{2c_2}{x^3} \quad \text{and} \quad y - x\frac{dy}{dx} = \frac{2c_2}{x}.$$

$$\therefore x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0,$$

which is the desired differential equation.

15. It is seen from the preceding article that one constant can be removed after each differentiation. From this it would be expected that, starting with the differential equation, an arbitrary constant might be introduced every time the order of the differential equation was lowered by unity. Then, since lowering the order of a differential equation of the nth order by unity n times would result in a solution of the differential equation, it would be expected that a solution would contain not more than n arbitrary constants.

It is a theorem that a differential equation cannot contain a solution having more arbitrary constants than the exponent of the order of the equation unless the constants are such that they can be reduced to a fewer number of equivalent constants. This will be assumed without further discussion.

It is also a theorem that a differential equation cannot have more than one general solution. This theorem will be assumed without discussion.

16. A general solution may have various forms but there is always a relation between the constants of one form and those of another. Thus, the general solution of example 2, Art. 10, may be written $y = A\sin x + B\cos x$ instead of $y = \pm\sqrt{c}\sin(x + c_1)$. This latter form of solution is $y = \pm\sqrt{c}\cos c_1 \sin x \pm \sqrt{c}\sin c_1 \cos x$, so that $A = \pm\sqrt{c}\cos c_1$, and $B = \pm\sqrt{c}\sin c_1$.

EXERCISES

1. Determine the order and degree of the following equations.

(*a*). $\dfrac{d^2y}{dx^2} = c\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}$.

(*b*). $x^3\dfrac{d^3y}{dx^3} + x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$.

(*c*). $\dfrac{d^2y}{dx^2} = c\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$.

(*d*). $\dfrac{\partial u}{\partial t} = a^2\dfrac{\partial^2 u}{\partial x^2}$.

In each of the seven following exercises determine the differential equation of which the given equation is the general solution, given that c_1, c_2 and c are arbitrary constants.

2. $y = c_1 \sin mx + c_2 \cos mx.$

3. $v = c_1 \cos(mt + c_2).$

4. $(x - c_1)^2 + (y - c_2)^2 = m^2.$

5. $y = cx + c - c^3.$

6. $xy = c_1 e^x + c_2 e^{-x}.$

7. $y^2 - 2cx - c^2 = 0.$

8. $y(1 + x^2)^{\frac{1}{2}} = c_1 + \log \dfrac{x}{1 + (1 + x^2)^{\frac{1}{2}}}.$

9. Show that

$$y = c_1 x + \frac{c_2}{x} + c_3$$

is a solution of

$$\frac{d^3y}{dx^3} + \frac{3}{x}\frac{d^2y}{dx^2} = 0.$$

10. Show that

$$4y = \frac{1}{3x} + c_1 x^5 + c_2 x$$

is a solution of

$$x^2 \frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 5y = \frac{1}{x}.$$

11. Show that

$$y = \tfrac{2}{15} e^{2x} + c_1 e^{-x} + c_2 e^{-3x}$$

is a solution of

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 2e^{2x}.$$

12. Show that

$$v = \frac{c_1}{r} + c_2$$

is a solution of

$$\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0.$$

ANSWERS

2. $\dfrac{d^2y}{dx^2} = -m^2 y.$

3. $\dfrac{d^2v}{dt^2} = -m^2 v.$

4. $1+\left(\frac{dy}{dx}\right)^2 \pm \frac{m}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\frac{d^2y}{dx^2}=0.$

5. $y=x\frac{dy}{dx}+\frac{dy}{dx}-\left(\frac{dy}{dx}\right)^3.$

6. $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=xy.$

7. $y-2x\frac{dy}{dx}-y\left(\frac{dy}{dx}\right)^2=0.$

8. $\frac{dy}{dx}+\frac{x}{1+x^2}y=\frac{1}{x(1+x^2)}.$

CHAPTER II

CHANGE OF VARIABLE

17. Interchange of Variables. It is sometimes desirable to transform an expression involving derivatives of the function y, in $y = f(x)$ where x is the independent variable, into an equivalent expression involving derivatives of the function x, given by the same equation, where y is the independent variable.

The formulas for such a transformation can be readily established as follows:

$$\frac{dy}{dx} = \underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta y}{\Delta x} \right] = \underset{\Delta y \doteq 0}{\text{limit}} \left[\frac{1}{\frac{\Delta x}{\Delta y}} \right] = \frac{1}{\frac{dx}{dy}}. \qquad (1)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dy}\left(\frac{dy}{dx}\right) \cdot \frac{dy}{dx}, \qquad \text{since } \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}.$$

$$= \frac{d}{dy}\left(\frac{1}{\frac{dx}{dy}}\right) \cdot \frac{1}{\frac{dx}{dy}}, \qquad \text{by substitution from (1).}$$

$$= -\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2} \cdot \frac{1}{\frac{dx}{dy}} = -\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}. \qquad (2)$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dy}\left\{ -\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3} \right\} \cdot \frac{1}{\frac{dx}{dy}}$$

$$= \frac{-\left(\dfrac{dx}{dy}\right)^3 \dfrac{d^3x}{dy^3} + 3\left(\dfrac{d^2x}{dy^2}\right)^2 \left(\dfrac{dx}{dy}\right)^2}{\left(\dfrac{dx}{dy}\right)^7} = \frac{-\dfrac{d^3x}{dy^3}\dfrac{dx}{dy} + 3\left(\dfrac{d^2x}{dy^2}\right)^2}{\left(\dfrac{dx}{dy}\right)^5}. \quad (3)$$

The method of procedure for higher derivatives is evident.

The transformations to which these formulas apply are called **change of the independent variable** or **interchange of variables.**

EXAMPLE. Change the independent variable from x to y in the equation

$$3\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx}\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2 = 0.$$

Substitute from (1), (2) and (3).

$$\therefore 3\left\{-\frac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3}\right\}^2 - \frac{1}{\dfrac{dx}{dy}} \cdot \frac{-\dfrac{d^3x}{dy^3}\dfrac{dx}{dy} + 3\left(\dfrac{d^2x}{dy^2}\right)^2}{\left(\dfrac{dx}{dy}\right)^5}$$

$$+ \frac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3} \cdot \frac{1}{\left(\dfrac{dx}{dy}\right)^2} = 0.$$

$$\therefore \left(\frac{d^3x}{dy^3} + \frac{d^2x}{dy^2}\right)\frac{dx}{dy} = 0.$$

18. Change of the dependent variable. Suppose that y is a function of x and at the same time is a function of some other variable z. The derivatives of y with respect to x can then be expressed in terms of derivatives of z with respect to x.

As a function of z, let $y = \phi(z)$. Denote differentiation with respect to z by primes. Then

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{d\phi(z)}{dz}\frac{dz}{dx} = \phi'(z)\frac{dz}{dx}. \quad (4)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d\phi'(z)}{dx}\frac{dz}{dx} + \phi'(z)\frac{d^2z}{dx^2} = \phi''(z)\left(\frac{dz}{dx}\right)^2 + \phi'(z)\frac{d^2z}{dx^2}. \quad (5)$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \phi'''(z)\left(\frac{dz}{dx}\right)^3 + 3\phi''(z)\frac{dz}{dx}\frac{d^2z}{dx^2} + \phi'(z)\frac{d^3z}{dx^3}.$$

Similarly for higher derivatives.

The above transformation is called **change of the dependent variable.**

EXAMPLE. In the equation

$$(1+y^2)\frac{d^2y}{dx^2} - (2y-1)\left(\frac{dy}{dx}\right)^2 + 3x(1+y^2)\frac{dy}{dx} = 0,$$

change the dependent variable from y to z where $y = \tan z$.

$$\frac{dy}{dx} = \sec^2 z\frac{dz}{dx}.$$

$$\frac{d^2y}{dx^2} = 2\sec^2 z\tan z\left(\frac{dz}{dx}\right)^2 + \sec^2 z\frac{d^2z}{dx^2}.$$

Substitute in the equation.

$$\therefore 2\sec^4 z\tan z\left(\frac{dz}{dx}\right)^2 + \sec^4 z\frac{d^2z}{dx^2} - (2\tan z - 1)\sec^4 z\left(\frac{dz}{dx}\right)^2 + 3x\sec^4 z\frac{dz}{dx} = 0.$$

$$\therefore \frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2 + 3x\frac{dz}{dx} = 0.$$

19. Change of the independent variable. Suppose that y is a function of x where x is a function of some other variable z. The derivatives of y with respect to x can then be expressed in terms of derivatives of y with respect to z.

As a function of z let $x = \phi(z)$. Denote differentiation with respect to z by primes. Then

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{\frac{dx}{dz}} = \frac{dy}{dz}\frac{1}{\phi'(z)}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dz}\left(\frac{dy}{dz}\frac{1}{\phi'(z)}\right)\frac{1}{\frac{dx}{dz}} = \frac{1}{\{\phi'(z)\}^2}\frac{d^2y}{dz^2} - \frac{\phi''(z)}{\{\phi'(z)\}^3}\frac{dy}{dz}.$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dz}\left[\frac{1}{\{\phi'(z)\}^2}\frac{d^2y}{dz^2} - \frac{\phi''(z)}{\{\phi'(z)\}^3}\frac{dy}{dz}\right]\frac{1}{\frac{dx}{dz}}.$$

$$= \frac{1}{\{\phi'(z)\}^3}\frac{d^3y}{dz^3} - 3\frac{\phi''(z)}{\{\phi'(z)\}^4}\frac{d^2y}{dz^2} - \frac{\phi'(z)\phi'''(z) - 3\{\phi''(z)\}^2}{\{\phi'(z)\}^5}\frac{dy}{dz}.$$

Similarly for higher derivatives.

The above transformation is called **change of the independent variable.**

Example. In the equation

$$\frac{d^2y}{dx^2} - \frac{x}{1-x^2}\frac{dy}{dx} + \frac{y}{1-x^2} = 0,$$

change the independent variable from x to z, where $x = \cos z$.

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = -\operatorname{cosec} z\frac{dy}{dz}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dz}\left(-\operatorname{cosec} z\frac{dy}{dz}\right)\frac{1}{\frac{dx}{dz}}$$

$$= -\operatorname{cosec}^2 z\cot z\frac{dy}{dz} + \operatorname{cosec}^2 z\frac{d^2y}{dz^2}.$$

Substitute in the equation.

$$\therefore \operatorname{cosec}^2 z\frac{d^2y}{dz^2} - \operatorname{cosec}^2 z\cot z\frac{dy}{dz} + \operatorname{cosec}^2 z\cot z\frac{dy}{dz} + \operatorname{cosec}^2 z\, y = 0.$$

$$\therefore \frac{d^2y}{dz^2} + y = 0.$$

When changing either the dependent or independent variable to a third variable, it is better to work out each derivative in the particular case considered rather than use the derivatives expressed in the general case as formulas.

EXERCISES

In each of the four following exercises, change the independent variable from x to y.

1. $\dfrac{d^2y}{dx^2} - 2y\dfrac{dy}{dx} = 0.$

2. $\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 - y\left(\dfrac{dy}{dx}\right)^3 = 0.$

3. $\dfrac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} = \rho.$

4. $\dfrac{d^3y}{dx^3}\dfrac{dy}{dx} - 3\left(\dfrac{d^2y}{dx^2}\right)^2 + 3\dfrac{d^2y}{dx^2}\left(\dfrac{dy}{dx}\right)^2 - 2\left(\dfrac{dy}{dx}\right)^4 - x\left(\dfrac{dy}{dx}\right)^5 = 0.$

In each of the two following exercises change the dependent variable from y to z.

5. $(1 + y^2)\dfrac{d^2y}{dx^2} - 2y\left(\dfrac{dy}{dx}\right)^2 - 2(1 + y^2)\dfrac{dy}{dx} = y^2(1 + y^2),$

where $y = \tan z$.

6. $$y^2\frac{d^3y}{dx^3} - \left(3y\frac{dy}{dx} + 2xy^2\right)\frac{d^2y}{dx^2} + \left\{2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx} + 3x^2y^2\right\}\frac{dy}{dx} + x^3y^3 = 0,$$

where $y = e^z$.

In each of the four following exercises, change the independent variable from x to z.

7. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0,$ where $x = e^z$.

8. $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 0,$ where $x = \sin z$.

9. $x^3\dfrac{d^3v}{dx^3} + 2x^2\dfrac{d^2v}{dx^2} + v = 0,$ where $x = e^z$.

10. $\dfrac{d^2v}{dx^2} + \dfrac{2x}{1 + x^2}\dfrac{dv}{dx} + \dfrac{v}{(1 + x^2)^2} = 0,$ where $x = \tan z$.

11. Transform the formula for radius of curvature,

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$

into polar coördinates, the equations of transformation being $x = r\cos\theta$, $y = r\sin\theta$.

ANSWERS

1. $\frac{d^2x}{dy^2} + 2y\left(\frac{dx}{dy}\right)^2 = 0.$

2. $\frac{d^2x}{dy^2} + \frac{dx}{dy} + y = 0.$

3. $\rho = -\frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}.$

4. $\frac{d^3x}{dy^3} + 3\frac{d^2x}{dy^2} + 2\frac{dx}{dy} + x = 0.$

5. $\frac{d^2z}{dx^2} - 2\frac{dz}{dx} = \sin^2 z.$

6. $\frac{d^3z}{dx^3} - 2x\frac{d^2z}{dx^2} + 3x^2\frac{dz}{dx} + x^3 = 0.$

7. $\frac{d^2y}{dz^2} + y = 0.$

8. $\frac{d^2y}{dz^2} = 0.$

9. $\frac{d^3v}{dz^3} - \frac{d^2v}{dz^2} + v = 0.$

10. $\frac{d^2v}{dz^2} + v = 0.$

11. $\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}.$

CHAPTER III

ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

20. An ordinary differential equation in one dependent variable, of the first order and first degree, may be represented by the equation

$$Mdx + Ndy = 0$$

where M and N are functions of x and y and do not contain derivatives.

The equation $Mdx + Ndy = 0$ cannot be integrated in the general form. There are certain particular forms of it, however, which can be integrated. Some of these will now be investigated.

21. LINEAR DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

Definition. An ordinary linear differential equation of the first order is an equation in the form

$$\frac{dy}{dx} + Py = Q$$

where P and Q are functions of x and do not contain y or derivatives.

The general solution of the equation

$$\frac{dy}{dx} + Py = Q$$

can be found as follows :

Multiply both sides of the equation by $e^{\int Pdx}$.

$$\therefore \frac{dy}{dx}e^{\int Pdx} + Pye^{\int Pdx} = Qe^{\int Pdx}.$$

If the substitution $u = ye^{\int P dx}$ be made, the left hand member of the equation reduces to du/dx.

$$\therefore \frac{du}{dx} = Qe^{\int P dx}.$$

$$\therefore u = \int Qe^{\int P dx} dx + c.$$

$$\therefore ye^{\int P dx} = \int Qe^{\int P dx} dx + c.$$

$$\therefore y = e^{-\int P dx} \int Qe^{\int P dx} dx + ce^{-\int P dx}, \qquad (1)$$

which is the general solution of the equation.

In the original equation, if P is zero, the equation reduces to the familiar form $dy/dx = Q$, and the general solution is

$$y = c + \int Q dx.$$

If Q is zero, the equation becomes

$$\frac{dy}{dx} + Py = 0,$$

and the general solution is $y = ce^{-\int P dx}$.

When Q, in the equation

$$\frac{dy}{dx} + Py = Q,$$

is zero, the equation is called the **ordinary linear differential equation of the first order with the right hand member zero.**

Example. Find the general solution of the equation

$$\frac{dy}{dx} + \frac{1}{x} y = x^2.$$

Multiply both sides of the equation by $e^{\int \frac{1}{x} dx}$.

$$\therefore \frac{dy}{dx} e^{\int \frac{1}{x} dx} + \frac{1}{x} e^{\int \frac{1}{x} dx} y = x^2 e^{\int \frac{1}{x} dx}.$$

Let $u = ye^{\int \frac{1}{x} dx}$.

$$\therefore \frac{du}{dx} = x^2 e^{\int \frac{1}{x} dx}. \quad \therefore u = \int x^2 e^{\int \frac{1}{x} dx} dx + c.$$

Now $e^{\int \frac{1}{x} dx} = e^{\log x} = x$.

$$\therefore u = \int x^3 dx + c = \frac{x^4}{4} + c. \quad \therefore y = \frac{1}{x}\left(\frac{x^4}{4} + c\right) = \frac{x^3}{4} + \frac{c}{x}.$$

It is usual to solve an ordinary linear differential equation of the first order by substituting directly in formula (1). Thus, in the above example, formula (1) becomes

$$y = e^{-\int \frac{1}{x} dx} \int x^2 e^{\int \frac{1}{x} dx} dx + ce^{-\int \frac{1}{x} dx}$$

$$= \frac{x^3}{4} + \frac{c}{x}.$$

22. EQUATIONS REDUCIBLE TO THE LINEAR FORM

A form easily reducible to the linear form is

$$\frac{dy}{dx} + Py = Qy^n$$

where P and Q are functions of x and do not contain y or derivatives.

Divide by y^n.

$$\therefore y^{-n} \frac{dy}{dx} + Py^{-n+1} = Q.$$

Let $y^{-n+1} = u$.

$$\therefore (-n + 1) y^{-n} \frac{dy}{dx} = \frac{du}{dx}.$$

$$\therefore \frac{1}{-n+1} \frac{du}{dx} + Pu = Q.$$

$$\therefore \frac{du}{dx} + (1 - n) Pu = Q(1 - n),$$

which is linear and can therefore be solved by the methods of Art. 21.

EXAMPLE. Find the general solution of the equation

$$\frac{dy}{dx} + \frac{1}{x}y = x^2y^3.$$

Divide by y^3.

$$\therefore\ y^{-3}\frac{dy}{dx} + \frac{1}{x}y^{-2} = x^2.$$

Let $y^{-2} = u$.

$$\therefore\ -2y^{-3}\frac{dy}{dx} = \frac{du}{dx}.$$

$$\therefore\ \frac{du}{dx} - \frac{2}{x}u = -2x^2.$$

$$\therefore\ u = -2e^{\int\frac{2}{x}dx}\int x^2e^{-\int\frac{2}{x}dx}dx + ce^{\int\frac{2}{x}dx} = -2x^3 + cx^2.$$

Therefore $-2x^3y^2 + cx^2y^2 = 1$ is the general solution of the equation.

23. VARIABLES SEPARABLE

Sometimes the equation $Mdx + Ndy = 0$ can be brought to the form $Xdx + Ydy = 0$ where X is a function of x alone and Y is a function of y alone. In such a case the general solution is evidently

$$\int Xdx + \int Ydy = c,$$

c being an arbitrary constant.

EXAMPLE. Find the general solution of the equation

$$x\sqrt{1-y^2}dx + y\sqrt{1-x^2}dy = 0.$$

Divide by $\sqrt{1-y^2}\sqrt{1-x^2}$.

$$\therefore\ \frac{x}{\sqrt{1-x^2}}dx + \frac{y}{\sqrt{1-y^2}}dy = 0.$$

$$\therefore\ \int\frac{xdx}{\sqrt{1-x^2}} + \int\frac{ydy}{\sqrt{1-y^2}} = c_1.$$

Therefore $\sqrt{1-x^2} + \sqrt{1-y^2} = c$ is the general solution of the equation.

The process of reducing the equation $Mdx + Ndy = 0$ to the form $Xdx + Ydy = 0$ is called **separation of the variables.**

24. EXACT DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

Definition. The ordinary differential equation $Mdx + Ndy = 0$ where M and N are functions of x and y, is said to be **exact** when there is a function $u(x, y)$ such that $du = Mdx + Ndy$.

EXAMPLE. The equation $2xydx + x^2dy = 0$ is said to be exact because $u = x^2y$ is such that $du = 2xydx + x^2dy$.

When there is a function $u(x, y)$ such that $du = Mdx + Ndy$, then $u = c$, where c is an arbitrary constant, is the general solution of the equation $Mdx + Ndy = 0$.

Condition that the equation $Mdx + Ndy = 0$ be exact. If the equation $Mdx + Ndy = 0$ be exact, then, by definition, there is a function $u(x, y)$ such that $du = Mdx + Ndy$. Now

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy,$$

from the definition of the differential of two independent variables.

$$\therefore M = \frac{\partial u}{\partial x}, \quad \text{and} \quad N = \frac{\partial u}{\partial y}.$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

That the equation $Mdx + Ndy = 0$ be exact, it is therefore necessary that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Conversely, the condition is sufficient. That is if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

then $Mdx + Ndy = 0$ is an exact differential equation.

Proof: Let $\int M dx = P, \quad \therefore \dfrac{\partial P}{\partial x} = M.$

$$\therefore \frac{\partial^2 P}{\partial y \, \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

$$\therefore \frac{\partial N}{\partial x} = \frac{\partial^2 P}{\partial y \, \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right). \quad \therefore N = \frac{\partial P}{\partial y} + F(y).$$

$$\therefore Mdx + Ndy = \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy + F(y)dy = d\{P + Q(y)\},$$

where $Q(y)$ is such that $dQ(y) = F(y)dy$.

Therefore, if

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y},$$

the left hand member of the equation $Mdx + Ndy = 0$ is an exact differential and therefore the equation is an exact differential equation.

To find the general solution of the equation $Mdx + Ndy = 0$ when the equation is exact.

Let $u(x, y)$ be a function whose differential is $Mdx + Ndy$.

Since $\dfrac{\partial u}{\partial x} = M,$

$$\therefore u = \int M dx + F(y).$$

Since $\dfrac{\partial u}{\partial y} = N,$

$$\therefore N = \frac{\partial}{\partial y}\int M dx + \frac{dF(y)}{dy}.$$

$$\therefore \frac{dF(y)}{dy} = N - \frac{\partial}{\partial y}\int M dx.$$

$$\therefore F(y) = \int\left(N - \frac{\partial}{\partial y}\int M dx\right) dy.$$

$$\therefore u = \int M dx + \int\left(N - \frac{\partial}{\partial y}\int M dx\right) dy.$$

The general solution of the equation is $u = c$ where c is an arbitrary constant.

EXAMPLE. Find the general solution of the equation

$$(x^3 + 2xy + y)dx + (y^3 + x^2 + x)dy = 0.$$

This is an exact differential equation. Therefore the general solution can be obtained by the above method.

Since $\dfrac{\partial u}{\partial x} = M$,

$$\therefore u = \int (x^3 + 2xy + y)dx + F(y) = \frac{x^4}{4} + x^2y + xy + F(y).$$

Now $\dfrac{\partial u}{\partial y} = N$.

$$\therefore \frac{\partial}{\partial y}\left\{\frac{x^4}{4} + x^2y + xy + F(y)\right\} = y^3 + x^2 + x.$$

$$\therefore x^2 + x + \frac{dF(y)}{dy} = y^3 + x^2 + x.$$

$$\therefore \frac{dF(y)}{dy} = y^3. \quad \therefore F(y) = \frac{y^4}{4}.$$

$$\therefore u = \frac{x^4}{4} + x^2y + xy + \frac{y^4}{4}.$$

Therefore $\dfrac{x^4}{4} + x^2y + xy + \dfrac{y^4}{4} = c$ is the required general solution of the equation.

25. INTEGRATING FACTORS

It sometimes happens that the differential equation

$$Mdx + Ndy = 0$$

is not exact but becomes so when it is multiplied by some quantity. Thus,

$$\frac{dy}{dx} + Py = Q$$

of Art. 21, is not exact but becomes so after multiplication by $e^{\int Pdx}$.

Definition. A factor which changes a differential equation into an exact differential equation is called an **integrating factor** of the equation.

Sometimes an integrating factor can be found by inspection.

EXAMPLE. Find the general solution of the equation

$$(x^2e^x - y^2)dx + 2xydy = 0.$$

The equation is not exact as it stands but becomes so on multiplication by $1/x^2$.

Multiply by $1/x^2$.

$$\therefore \frac{x^2e^x - y^2}{x^2}dx + \frac{2y}{x}dy = 0.$$

$$\therefore e^x dx - \frac{y^2}{x^2}dx + \frac{2y}{x}dy = 0.$$

$$\therefore de^x + d\left(\frac{y^2}{x}\right) = 0.$$

$$\therefore e^x + \frac{y^2}{x} = c. \quad \therefore y^2 = -xe^x + cx.$$

Therefore $y^2 = -xe^x + cx$ is the general solution of the equation.

Rules have been devised for finding integrating factors in many cases where they cannot be found by inspection. For a discussion of them, the student is referred to Boole's, Murray's, or Johnson's Differential Equations.

26. EQUATIONS HOMOGENEOUS IN x AND y

Definition. If M and N of the equation $Mdx + Ndy = 0$ are both of the same degree in x and y and are homogeneous, the equation is said to be **homogeneous.**

To find the general solution of the equation $Mdx + Ndy = 0$ when the equation is homogeneous.

$$\frac{dy}{dx} = -\frac{M}{N}.$$

Divide both numerator and denominator of $-\frac{M}{N}$ by x raised to the power indicated by the degree of M or N.

Then every term in M and N is constant or in the form of a coefficient multiplied by some power of $\frac{y}{x}$.

Then

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

Let $y = vx$.

$$\therefore\ x\frac{dv}{dx} + v = f(v).$$

Therefore

$$\frac{dx}{x} = \frac{dv}{f(v) - v},$$

an equation in which the variables are separated, and can therefore usually be integrated without difficulty.

EXAMPLE. Find the general solution of the equation $(x^2 + y^2)dx - xydy = 0$.

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}.$$

Let $y = vx$.

$$\therefore\ v + x\frac{dv}{dx} = \frac{1 + v^2}{v}. \quad \therefore\ x\frac{dv}{dx} = \frac{1 + v^2}{v} - v = \frac{1}{v}.$$

$$\therefore\ vdv = \frac{dx}{x}.$$

$$\therefore\ v^2 = 2 \log cx.$$

Therefore $y^2 = 2x^2 \log cx$ is the general solution of the equation.

27. EQUATIONS OF THE FORM $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$

The general solution of an equation in the above form can be found as follows :

Let $x = x' + x_0$, and $y = y' + y_0$, where x' and y' are new variables, and x_0 and y_0 are constants.

Change the variables to x' and y'.

$$\therefore \frac{dy'}{dx'} = \frac{a_1x' + b_1y' + a_1x_0 + b_1y_0 + c_1}{a_2x' + b_2y' + a_2x_0 + b_2y_0 + c_2}. \qquad (1)$$

Case I. If x_0 and y_0 can be determined such that

$$a_1x_0 + b_1y_0 + c_1 = 0, \quad \text{and} \quad a_2x_0 + b_2y_0 + c_2 = 0,$$

then, on determining them such, equation (1) becomes

$$\frac{dy'}{dx'} = \frac{a_1x' + b_1y'}{a_2x' + b_2y'},$$

which is homogeneous and can be solved by the method of Art. 26.

Case II. If x_0 and y_0 cannot be determined such that

$$a_1x_0 + b_1y_0 + c_1 = 0, \quad \text{and} \quad a_2x_0 + b_2y_0 + c_2 = 0,$$

then, as was seen in algebra,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{1}{m}.$$

By substitution, the original equation becomes

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{m(a_1x + b_1y) + c_2}.$$

Let $a_1x + b_1y = v$.

$$\therefore a_1 + b_1\frac{dy}{dx} = \frac{dv}{dx}.$$

Therefore

$$\frac{dv}{dx} = a_1 + b_1\frac{v + c_1}{mv + c_2},$$

an equation in which the variables are separable.

EXAMPLE 1. Find the general solution of the equation

$$(2x + 3y - 6)\frac{dy}{dx} = 6x - 2y - 7.$$

$$\frac{dy}{dx} = \frac{6x - 2y - 7}{2x + 3y - 6}.$$

Let $x = x_0 + x'$, $y = y_0 + y'$ where

$$6x_0 - 2y_0 - 7 = 0, \quad \text{and} \quad 2x_0 + 3y_0 - 6 = 0.$$

$$\therefore x = \tfrac{3}{2} + x',\ y = 1 + y'. \quad \therefore \frac{dy'}{dx'} = \frac{6x' - 2y'}{2x' + 3y'}.$$

$$\therefore x'\frac{dv}{dx'} = \frac{6 - 2v}{2 + 3v} - v = \frac{6 - 4v - 3v^2}{2 + 3v}.$$

$$\therefore \frac{(2 + 3v)dv}{6 - 4v - 3v^2} = \frac{dx'}{x'}.$$

$$\therefore -\log c_1x' = \log (3v^2 + 4v - 6)^{\frac{1}{2}}.$$

$$\therefore \frac{1}{c_1x'} = (3v^2 + 4v - 6)^{\frac{1}{2}}.$$

$$\therefore \frac{c_2}{(x - \frac{3}{2})^2} = 3\left(\frac{y - 1}{x - \frac{3}{2}}\right)^2 + 4\left(\frac{y - 1}{x - \frac{3}{2}}\right) - 6.$$

Therefore $3y^2 + 4xy - 6x^2 - 12y + 14x = c$ is the general solution of the equation.

Example 2. Find the general solution of the equation

$$(3x - y + 4)\frac{dy}{dx} = 6x - 2y - 7.$$

This comes under case II.

Let $6x - 2y = v$.

$$\therefore 6 - 2\frac{dy}{dx} = \frac{dv}{dx}.$$

$$\therefore 6 - 2\frac{v - 7}{\frac{1}{2}v + 4} = \frac{dv}{dx}.$$

$$\therefore \frac{dv}{dx} = \frac{2v + 76}{v + 8}.$$

$$\therefore \frac{(v + 8)dv}{v + 38} = 2dx.$$

$$\therefore \left(1 - \frac{30}{v + 38}\right)dv = 2dx.$$

$$\therefore v - 30 \log (v + 38) = 2x + c_1.$$

$$\therefore 4x - 2y - 30 \log (6x - 2y + 38) = c_1.$$

Therefore $2x - y - 15 \log (3x - y + 19) = c$ is the general solution of the equation.

EXERCISES

Find the general solution of each of the thirty-six following equations.

1. $\dfrac{dy}{dx} + \dfrac{1}{x} y = 1 - x^2.$

2. $\dfrac{dy}{dx} + \cot x\, y = \operatorname{cosec}^2 x.$

3. $\dfrac{dy}{dx} = x - y.$

4. $(1 + x^2) \dfrac{dy}{dx} + x^2 y = x^3 - x^2 \tan^{-1} x.$

5. $\dfrac{dy}{dx} + \dfrac{x}{1 + x^2} y = \dfrac{1}{x(1 + x^2)}.$

6. $x(1 - x^2) \dfrac{dy}{dx} + (x^2 - 1) y = x^3.$

7. $\dfrac{dy}{dx} + \cos x\, y = \frac{1}{2} \sin 2x.$

8. $x(1 - x^2) \dfrac{dy}{dx} + (2x^2 - 1) y = ax^3.$

9. $\dfrac{dy}{dx} + \sin x\, y = y^2 \sin x.$

10. $(1 - x^2) \dfrac{dy}{dx} - xy = axy^2.$

11. $\dfrac{dy}{dx} + \cos x\, y = y^n \sin 2x.$

12. $3y^2 \dfrac{dy}{dx} + y^3 = x - 1.$

13. $\dfrac{dy}{dx} - \tan x\, y = y^4 \sec x.$

14. $y \sqrt{x^2 - 1}\, dx + x \sqrt{y^2 - 1}\, dy = 0.$

15. $(e^y + 1) \cos x\, dx + e^y \sin x\, dy = 0.$

16. $\sqrt{2ay - y^2} \operatorname{cosec} x\, dx + y \tan x\, dy = 0.$

17. $y(3 + y) \dfrac{dy}{dx} = x(2y + 3).$

18. $(x^3 - 3x^2 y + 5xy^2 - 7y^3) dx + (y^4 + 2y^2 - x^3 + 5x^2 y - 21xy^2) dy = 0.$

19. $(x^3 + 4xy + y^2)dx + (2x^2 + 2xy + 4y^3)dy = 0.$

20. $\sin x \cos y\, dx + \cos x \sin y\, dy = 0.$

21. $(x^2 + \log y)dx + \frac{x}{y}\, dy = 0.$

22. $x(x - 2y)dy + (x^2 + 2y^2)dx = 0.$

23. $5xydy - (x^2 + y^2)dx = 0.$

24. $(x^2 + 3xy - y^2)dy - 3y^2dx = 0.$

25. $(x^2 + 2xy)dy - (3x^2 - 2xy + y^2)dx = 0.$

26. $5xydy - (4x^2 + y^2)dx = 0.$

27. $(x^2 - 2xy)dy + (x^2 - 3xy + 2y^2)dx = 0.$

28. $3x^2dy + (2x^2 - 3y^2)dx = 0.$

29. $(3x + 2y - 7)\frac{dy}{dx} = 2x - 3y + 6.$

30. $(6x - 5y + 4)\frac{dy}{dx} = 2x - y + 1.$

31. $(5x - 2y + 7)\frac{dy}{dx} = x - 3y + 2.$

32. $(x - 3y + 4)\frac{dy}{dx} = 5x - 7y.$

33. $(x - 3y + 4)\frac{dy}{dx} = 2x - 6y + 7.$

34. $(5x - 2y + 7)\frac{dy}{dx} = 10x - 4y + 6.$

35. $(2x - 2y + 5)\frac{dy}{dx} = x - y + 3.$

36. $(6x - 4y + 1)\frac{dy}{dx} = 3x - 2y + 1.$

The following formulas, derived in almost any work in calculus, are inserted here for convenience of reference :

The subtangent and subnormal at a point (x, y) on a curve whose equation is expressed in rectangular coördinates are $y\dfrac{dx}{dy}$ and $y\dfrac{dy}{dx}$ respectively. The polar subtangent and polar subnormal at a point (r, θ) on a curve are $r^2\dfrac{d\theta}{dr}$ and $\dfrac{dr}{d\theta}$ respectively.

The angle between the radius vector to a point (r, θ) and the tangent line to the curve at the point is

$$\tan^{-1}\frac{r}{\dfrac{dr}{d\theta}}.$$

The equation of the tangent line to the curve $y = f(x)$ at the point (x_1, y_1) on the curve is

$$y - y_1 = \left.\frac{dy}{dx}\right|_{x=x_1}(x - x_1).$$

The area enclosed between the curve $y = f(x)$, the x-axis, and the ordinates whose abscissas are x_0 and x_1 respectively is

$$\int_{x_0}^{x_1} y dx$$

provided the curve does not cut the x-axis between x_0 and x_1.

The length of the arc of the curve $y = f(x)$ between the points (x_0, y_0) and (x_1, y_1) on the curve is

$$\int_{x_0}^{x_1}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx.$$

37. Determine the curve whose subtangent at a point on it is n-times the abscissa of the point. Find the particular curve that goes through the point $(3, 4)$. Plot the curve (a), for $n = 1$, (b), for $n = 2$.

38. Determine the curve whose subtangent at a point on it is n-times the subnormal at the point. Find the particular curve that goes through the point $(\sqrt{n}, 2)$. Plot the curve when $n = 4$.

39. Determine the curve whose subtangent is constant and equal to a. Plot the curve, (a), when $a = 1$, (b), when $a = 2$.

40. Determine the curve whose subnormal is constant and equal to a. Find the particular curve that goes through the point (1, 2).

41. Determine the curve which is such that the length of the perpendicular from the foot of the ordinate of any point on the curve to the tangent line at that point is constant and equal to a. Determine the particular curve when $c = a$. At what angle does this curve cut the y-axis?

42. Determine the curve which is such that the area between the curve, the x-axis, and two ordinates, is equal to the arc between the ordinates.

43. Determine the curve which is such that the perpendicular from the origin upon any tangent line is equal to the abscissa of the point of contact.

44. Determine the curve in which the angle between the radius vector and the tangent line is n-times the vectorial angle. Plot the curve when $n = \frac{1}{2}$.

45. Determine the curve in which the polar subnormal is proportional to the sine of the vectorial angle.

46. Determine the curve in which the polar subtangent is proportional to the length of the radius vector.

The equation for a circuit containing induction and resistance is

$$L\frac{di}{dt} + Ri = e$$

where e is the electromotive force [E.M.F.] impressed upon the circuit, R the resistance offered by the circuit, L the coefficient of induction, i the current, and t the time during which the circuit is in operation. In each of the four following exercises, determine the current in the circuit after a time t supposing that the resistance and induction are constant.

47. The E.M.F. is zero. Solve subject to the condition that $i = I$ when $t = 0$.

48. The E.M.F. is constant and equal to E.

49. The E.M.F. is a simple sine function of the time, $= E \sin \omega t$ where E is the maximum value of the impressed E.M.F., and ω is the angular velocity, equivalent to $2\pi n$ where n denotes the number of complete periods or alternations per second.

50. The E.M.F. is the sum of two components each following the sine law, that is, $e = E_1 \sin \omega t + E_2 \sin (b\omega t + \theta)$.

The equation for a circuit containing resistance and capacity is

$$\frac{di}{dt} + \frac{i}{RC} = \frac{1}{R}\frac{de}{dt}$$

where e is the E.M.F., R the resistance, C the capacity, i the current, and t the time during which the circuit is in operation.

In each of the two following exercises determine the current in the circuit after a time t, supposing that the resistance and capacity are constant.

51. The E.M.F. is constant and equal to E.

52. The E.M.F. is a simple sine function of the time, $= E \sin \omega t$.

The equation for a circuit containing resistance and capacity is

$$R\frac{dq}{dt} + \frac{q}{C} = e$$

where e is the E.M.F., R the resistance, C the capacity, q the quantity of charge in the conductor, and t the time during which the circuit is in operation. In each of the three following exercises determine the charge in the circuit after a time t, supposing that the resistance and capacity are constant.

53. The E.M.F. is zero. Solve subject to the condition that $q = Q$ when $t = 0$.

54. The E.M.F. is constant and equal to E.

55. The E.M.F. is a simple sine function of the time, $= E \sin \omega t$.

ANSWERS

1. $4xy = 2x^2 - x^4 + c$. 2. $y \sin x = \log \tan \frac{x}{2} + c$.

3. $y = x - 1 + ce^{-x}$.

4. $y = x - 1 - \tan^{-1}x + ce^{\tan^{-1}x - x}$.

5. $y(1 + x^2)^{\frac{1}{2}} = \log \dfrac{x}{1 + (1 + x^2)^{\frac{1}{2}}} + c$.

6. $y = -\dfrac{x}{2} \log (1 - x^2) + cx$.

7. $y = \sin x - 1 + ce^{-\sin x}$.

8. $y = ax + cx\sqrt{1 - x^2}$.

9. $\dfrac{1}{y} = 1 + ce^{-\cos x}$.

10. $\dfrac{1}{y} = -a + c\sqrt{1 - x^2}$.

11. $y^{1-n} = 2 \sin x - \dfrac{2}{1 - n} + ce^{-(1-n)\sin x}$.

12. $y^3 = x - 2 + ce^{-x}$.

13. $y^{-3} = -3 \sin x \cos^2 x - \sin^3 x + c \cos^3 x$.

14. $\sqrt{x^2 - 1} - \sec^{-1} x + \sqrt{y^2 - 1} - \sec^{-1} y = c$.

15. $(e^y + 1) \sin x = c$.

16. $\operatorname{cosec} x + \sqrt{2ay - y^2} - a \operatorname{vers}^{-1} \dfrac{y}{a} = c$.

17. $2y^2 + 6y - 9 \log (2y + 3) = 4x^2 + c$.

18. $\dfrac{x^4}{4} - x^3y + \frac{5}{2}x^2y^2 - 7xy^3 + \dfrac{y^5}{5} + \frac{2}{3}y^3 = c$.

19. $\dfrac{x^4}{4} + 2x^2y + xy^2 + y^4 = c$.

20. $\cos x \cos y = c$.

21. $\dfrac{x^3}{3} + x \log y = c$.

22. $x^2e^{\frac{2y}{x}} = c(x + y)^3$.

23. $(4y^2 - x^2)^5 = cx^2$.

24. $y^2 = c\left(\dfrac{y - x}{y + x}\right)^3$.

25. $c(y^2 + 3xy - 3x^2) = x \left\{\dfrac{2y + (3 - \sqrt{21})x}{2y + (3 + \sqrt{21})x}\right\}^{\frac{2}{\sqrt{21}}}$.

26. $(x^2 - y^2)^5 = cx^2$.

27. $y = x \log \dfrac{c}{x}$.

28. $cx = \left\{\dfrac{6y - (3 + \sqrt{33})x}{6y - (3 - \sqrt{33})x}\right\}^{\frac{\sqrt{3}}{\sqrt{11}}}.$

29. $(y - \frac{32}{13})^2 + 3(x - \frac{9}{13})(y - \frac{32}{13}) - (x - \frac{9}{13})^2 = c.$

30. $(5y - 2x - 3)^4 = c(4y - 4x - 3).$

31. $\dfrac{\{2(y - \frac{3}{13}) - (4 - \sqrt{14})(x + \frac{17}{13})\}^{1/\sqrt{14}}(2y^2 - \frac{148}{13}y - 8xy + x^2 + \frac{58}{13}x + \frac{715}{169})}{\{2(y - \frac{3}{13}) - (4 + \sqrt{14})(x + \frac{17}{13})\}^{1/\sqrt{14}}} = c.$

32. $(3y - 5x + 10)^2 = c(y - x + 1).$

33. $15y - 30x + c = 3 \log (5x - 15y + 17).$

34. $4x - 2y + c = 16 \log (5x - 2y + 23).$

35. $2y - x + c = \log (x - y + 2).$

36. $2y - x + c = \frac{1}{4} \log (12x - 8y + 1).$

37. $y^n = cx$; $y^n = \frac{4^n}{3}x.$

38. $y = \dfrac{1}{\sqrt{n}}x + c$; $y = \dfrac{1}{\sqrt{n}}x + 1.$

39. $y^a = ce^x.$

40. $y^2 = 2ax + c$; $y^2 = 2ax + 4 - 2a.$

41. $y = \dfrac{c}{2}\left(e^{\frac{x}{a}} + \dfrac{a^2}{c^2}e^{-\frac{x}{a}}\right)$; $y = \dfrac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$; zero.

42. $y = \dfrac{1}{2c}(e^{\mp x} + c^2e^{\pm x}).$

43. $x^2 + y^2 = cx.$

44. $r^n = c \sin n\theta.$

45. $r = c - k \cos \theta.$

46. $r = ce^{\theta/k}.$

47. $i = Ie^{-\frac{R}{L}t}.$

48. $i = \dfrac{E}{R} + ce^{-\frac{R}{L}t}.$

49. $i = \dfrac{E}{L\left(\dfrac{R^2}{L^2} + \omega^2\right)}\left(\dfrac{R}{L}\sin \omega t - \omega \cos \omega t\right) + ce^{-\frac{R}{L}t}.$

50. $$i = \frac{E_1}{L\left(\frac{R^2}{L^2} + \omega^2\right)}\left(\frac{R}{L}\sin \omega t - \omega \cos \omega t\right)$$
$$+ \frac{E_2}{L\left(\frac{R^2}{L^2} + b\omega^2\right)}\left[\frac{R}{L}\sin(b\omega t + \theta) - b\omega\cos(b\omega t + \theta)\right]$$
$$+ ce^{-\frac{R}{L}t}.$$

51. $i = ce^{-\frac{1}{RC}t}$.

52. $i = ce^{-\frac{1}{RC}t} + \frac{CE\omega}{1 + R^2C^2\omega^2}(\cos \omega t + RC\omega \sin \omega t)$.

53. $q = Qe^{-\frac{1}{RC}t}$.

54. $q = CE + ce^{-\frac{1}{RC}t}$.

55. $q = \frac{CE}{1 + R^2C^2\omega^2}(\sin \omega t - RC\omega \cos \omega t) + ce^{-\frac{1}{RC}t}$.

CHAPTER IV

ORDINARY LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

28. Definition. An ordinary linear differential equation is a differential equation in one dependent variable which is linear in the dependent variable and its derivatives.

We saw in Art. 21 that the type of an ordinary linear differential equation of the first order is

$$\frac{dy}{dx} + Py = Q$$

where P and Q are functions of x, and do not contain y or derivatives.

In general, the type of an ordinary linear differential equation is

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_n y = X$$

where P_1, P_2, $\cdots$, P_n, and X, are functions of x, and do not contain y or derivatives.

In this chapter the only cases considered are those where P_1, P_2, $\cdots$, P_n are constants and real. Two forms of this equation present themselves, namely, when the right hand member is zero, and when the right hand member is not zero.

RIGHT HAND MEMBER ZERO

29. We shall first prove a theorem used in the investigation of equations in this form. It is:

Theorem. If $y = y_1$, $y = y_2$, $\cdots$, $y = y_n$, are solutions of the equation

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_n y = 0,$$

then

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n, \qquad c_1, c_2, \cdots c_n,$$

arbitrary constants, is also a solution of the equation.

Proof. Substitute $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ in the equation.

$$\therefore c_1\left[\frac{d^ny_1}{dx^n} + P_1\frac{d^{n-1}y_1}{dx^{n-1}} + P_2\frac{d^{n-2}y_1}{dx^{n-2}} + \cdots + P_ny_1\right]$$

$$+ c_2\left[\frac{d^ny_2}{dx^n} + P_1\frac{d^{n-1}y_2}{dx^{n-1}} + P_2\frac{d^{n-2}y_2}{dx^{n-2}} + \cdots + P_ny_2\right]$$

$$+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$+ c_n\left[\frac{d^ny_n}{dx^n} + P_1\frac{d^{n-1}y_n}{dx^{n-1}} + P_2\frac{d^{n-2}y_n}{dx^{n-2}} + \cdots + P_ny_n\right] = 0.$$

Now each expression in brackets is zero, since

$$y = y_1,\ y = y_2,\ \cdots,\ y = y_n,$$

are solutions of the equation. Therefore

$$y = c_1y_2 + c_2y_2 + \cdots + c_ny_n$$

is a solution of the equation.

Cor. If $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is a solution of the equation, then $y = c_1y_1$, $y = c_2y_2$, $\cdots$, $y = c_ny_n$, are solutions of the equation.

If $y = y_1$, $y = y_2$, $\cdots$, $y = y_n$ are linearly independent solutions of the equation, then $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is the general solution of the equation (see Art. 13).

30. To find a solution of the equation

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_ny = 0, \tag{1}$$

in the form $y = e^{mx}$.

Let $y = e^{mx}$ and substitute in the equation. If $y = e^{mx}$ is a solution of the equation, then

$$e^{mx}(m^n + P_1m^{n-1} + P_2m^{n-2} + \cdots + P_n) = 0.$$

Since e^{mx} cannot be zero for any value of m, then must

$$m^n + P_1 m^{n-1} + P_2 m^{n-2} + \cdots + P_n = 0.$$

Therefore, if $y = e^{mx}$ is a solution of the equation, it is necessary that

$$m^n + P_1 m^{n-1} + P_2 m^{n-2} + \cdots + P_n = 0.$$

Conversely, if m has a value m_1 such that

$$m_1{}^n + P_1 m_1{}^{n-1} + P_2 m_1{}^{n-2} + \cdots + P_2 = 0,$$

then $y = e^{m_1x}$ is a solution of the equation. This is obvious because on substitution of $y = e^{m_1x}$, the equation reduces to

$$e^{m_1x}(m_1{}^n + P_1 m_1{}^{n-1} + P_2 m_1{}^{n-2} + \cdots + P_n) = 0.$$

Therefore the necessary and sufficient condition that equation (1) has a solution in the form $y = e^{mx}$ is that m be such that

$$m^n + P_1 m^{n-1} + P_2 m^{n-2} + \cdots + P_n = 0.$$

Definition. The equation

$$m^n + P_1 m^{n-1} + P_2 m^{n-2} + \cdots + P_n = 0$$

is called the **auxiliary equation** of

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_ny = 0.$$

31. To find the general solution of the equation

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_ny = 0.$$

When the auxiliary equation has distinct roots. Denote the roots by $m_1, m_2, \cdots, m_n$. Then n linearly independent solutions of the equation are $y = e^{m_1x}$, $y = e^{m_2x}$, $\cdots$, $y = e^{m_nx}$, and the general solution is $y = c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_ne^{m_nx}$ (see Art 29).

EXAMPLE 1. Find the general solution of the equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0.$$

Let $y = e^{mx}$.

$$\therefore e^{mx}(m^2 - 3m - 4) = 0. \quad \therefore (m-4)(m+1) = 0.$$

Therefore $y = c_1e^{4x} + c_2e^{-x}$ is the general solution of the equation.

EXAMPLE 2. Find the general solution of the equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + 4y = 0.$$

Let $y = e^{mx}$.

$$\therefore e^{mx}(m^2 + m + 4) = 0.$$

$$\therefore \left(m + \frac{1-\sqrt{15}j}{2}\right)\left(m + \frac{1+\sqrt{15}j}{2}\right) = 0, \quad j = \sqrt{-1}.$$

Therefore $y = \bar{c}_1e^{-\frac{1-\sqrt{15}j}{2}x} + \bar{c}_2e^{-\frac{1+\sqrt{15}j}{2}x}$ is the general solution of the equation.

This solution may be written as

$$y = c_1e^{-\frac{1}{2}x}\cos\frac{\sqrt{15}}{2}x + c_2e^{-\frac{1}{2}x}\sin\frac{\sqrt{15}}{2}x \quad \text{(see Art. 5).}$$

When the auxiliary equation has multiple roots. Suppose that the auxiliary equation $m^n + P_1m^{n-1} + P_2m^{n-2} + \cdots + P_n = 0$ has the roots $m_1, m_2, m_3, \cdots, m_n$.

At first suppose that two roots are equal. Suppose for definiteness that $m_2 = m_1$. Then a solution of the differential equation is

$$y = (c_1 + c_2)e^{m_1x} + c_3e^{m_3x} + c_ne^{m_nx}.$$

Since $c_1 + c_2$ is equivalent to only one constant, this solution contains only $n - 1$ arbitrary constants and is not therefore the general solution of the equation.

To find the general solution in this case :

Suppose that the differential equation is such that its auxiliary equation has the roots m_1, $m_1 + h$, m_3, $\cdots$, m_n. The general solution of this equation is

$$y = c_1 e^{m_1 x} + c_2 e^{(m_1+h)x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$
$$= c_1 e^{m_1 x} + c_2 e^{m_1 x} e^{hx} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}.$$

Expand e^{hx} by Maclaurin's Theorem to n terms and the remainder.

$$\therefore e^{hx} = 1 + hx + \frac{(hx)^2}{\underline{|2}} + \cdots + \frac{(hx)^{n-1}}{\underline{|n-1}} + \frac{(hx)^n}{\underline{|n}} e^{hx_1},$$
$$0 < x_1 < x.$$

Substitute in the above equation.

$$\therefore y = (c_1 + c_2) e^{m_1 x}$$
$$+ c_2 h x e^{m_1 x}\left[1 + \frac{hx}{2} + \frac{(hx)^2}{\underline{|3}} + \cdots + \frac{(hx)^{n-2}}{\underline{|n-1}} + \frac{(hx)^{n-1}}{\underline{|n}} e^{hx_1}\right]$$
$$+ c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}.$$

Since c_2 is arbitrary, h may be chosen such that $c_2 h$ is any constant B for all values of h. Since c_1 is arbitrary, $c_1 + c_2$ may be chosen such that $c_1 + c_2 = A$. Then

$$y = A e^{m_1 x} + B x e^{m_1 x}\left[1 + \frac{hx}{2} + \cdots + \frac{(hx)^{n-2}}{\underline{|n-1}} + \frac{(hx)^{n-1}}{\underline{|n}} e^{hx_1}\right]$$
$$+ c_3 e^{m_3 x} + \cdots c_n e^{m_n x}, \quad (1)$$

where A and B are arbitrary constants.

Let h approach zero. As h approaches zero, the assumed auxiliary and differential equations approach identity with the given ones, and (1) approaches the general solution of the given differential equation.

Now

$$\underset{h \doteq 0}{\text{limit}} \left[\frac{hx}{2} + \cdots + \frac{(hx)^{n-2}}{\underline{|n-1}} + \frac{(hx)^{n-1}}{\underline{|n}} e^{hx_1}\right] = 0.$$

Therefore the general solution of the differential equation is

$$y = (A + Bx) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x},$$

or, as we shall write it,

$$y = (c_1 + c_2x)e^{m_1x} + c_3e^{m_3x} + \cdots + c_ne^{m_nx}.$$

In a similar manner it can be shown that if three roots of the auxiliary equation are equal, the general solution of the differential equation is

$$y = (c_1 + c_2x + c_3x^2)e^{m_1x} + c_4e^{m_4x} + \cdots + c_ne^{m_nx}.$$

and, in general, if r roots are equal, the general solution of the equation is

$$y = (c_1 + c_2x + \cdots + c_rx^{r-1})e^{m_1x} + c_{r+1}e^{m_{r+1}x} + \cdots + c_ne^{m_nx}.$$

If a pair of imaginary roots occur twice, the part of the general solution derived from these roots is

$$\begin{aligned} y &= (c_1 + c_2x)e^{(a+\beta j)x} + (c_3 + c_4x)e^{(a-\beta j)x} \\ &= (c_1+c_2x)e^{ax}(\cos\beta x + j\sin\beta x) + (c_3+c_4x)e^{ax}(\cos\beta x - j\sin\beta x) \\ &= e^{ax}[(A_1 + B_1x)\cos\beta x + (A_2 + B_2x)\sin\beta x]. \end{aligned}$$

EXAMPLE 1. Find the general solution of the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0.$$

The auxiliary equation is $m^2 + 2m + 1 = 0$, or $(m + 1)^2 = 0$. The general solution is therefore $y = e^{-x}(c_1 + c_2x)$.

EXAMPLE 2. Find the general solution of the equation

$$\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0.$$

The auxiliary equation is $m^4 - 4m^3 + 8m^2 - 8m + 4 = 0$, or $\{(m - 1 - j)(m - 1 + j)\}^2 = 0$.

The general solution is therefore

$$y = e^x\{(A_1 + B_1x)\cos x + (A_2 + B_2x)\sin x\}.$$

32. As a physical application of the above principles, consider the following discussion (see Emptage, Electricity and Magnetism, page 180) :

In a galvanometer in which resistance is offered to the motion of the needle, the equation of motion of the needle for small oscillations may be written as

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \omega^2(\theta - \alpha) = 0, \qquad (1)$$

where θ is the angle through which the needle turns in the time t, k is a constant depending on the resistance offered to the motion of the needle, ω^2 is a constant depending on the moments of the restoring forces on the needle, and α is the angle which the needle at rest makes with the line from which angles are measured. Let $\theta - \alpha = \theta'$, and substitute in (1).

$$\therefore \frac{d^2\theta'}{dt^2} + 2k\frac{d\theta'}{dt} + \omega^2\theta' = 0.$$

This is a linear differential equation of the second order with constant coefficients and right hand member zero. The auxiliary equation is $m^2 + 2km + \omega^2 = 0$. The roots of the auxiliary equation are $m = -k \pm \sqrt{k^2 - \omega^2}$.

Case I. If $k > \omega$.

In this case, $\theta - \alpha = c_1 e^{(-k+\sqrt{k^2-\omega^2})t} + c_2 e^{(-k-\sqrt{k^2-\omega^2})t}$ is the general solution of (1).

Case II. If $k = \omega$.

In this case $\theta - \alpha = (c_1 + c_2 t)e^{-kt}$ is the general solution of (1).

Case III. If $k < \omega$.

In this case $\theta - \alpha = e^{-kt}[c_1 \cos\sqrt{\omega^2 - k^2}\, t + c_2 \sin\sqrt{\omega^2 - k^2}\, t]$ is the general solution of (1).

In cases I and II the motion is not oscillatory. The needle can go through the position of equilibrium for one value of t, after which it reaches a position of maximum deflection and then continually approaches but never reaches the position of equilibrium. In case III there are oscillations in equal times, the periodic time being

$$T = \frac{2\pi}{\sqrt{\omega^2 - k^2}}.$$

EXERCISES

Find the general solution of each of the following equations.

1. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0.$

2. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 2y = 0.$

3. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0.$

4. $2\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 3y = 0.$

5. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0.$

6. $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + \frac{dy}{dx} - 5y = 0.$

7. $2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0.$

8. $\frac{d^4y}{dx^4} - y = 0.$

9. $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0.$

ANSWERS

1. $y = c_1e^{-2x} + c_2e^{-x}.$

2. $y = c_1e^{-(1+\sqrt{3})x} + c_2e^{-(1-\sqrt{3})x}.$

3. $y = c_1e^{x} + c_2e^{2x} + c_3.$

4. $y = e^{-x}(c_1 + c_2x) + c_3e^{\frac{3}{2}x}.$

5. $y = e^{x}(c_1 + c_2x + c_3x^2).$

6. $y = c_1e^{x} + e^{-2x}(c_2 \cos x + c_3 \sin x).$

7. $y = c_1e^{-\frac{1}{2}x} + e^{x}(c_2 \cos x + c_3 \sin x).$

8. $y = c_1e^{x} + c_2e^{-x} + c_3 \cos x + c_4 \sin x.$

9. $y = (c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x.$

RIGHT-HAND MEMBER NOT ZERO

33. Symbolic form of equation. The equation, when the right hand member is not zero, is

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_ny = X, \qquad (1)$$

where P_1, P_2, $\cdots$ P_n, are constants, and X is a function of x but not of y.

Let

$$\frac{d}{dx} = D,$$

and the equation may be written,

$$D^n y + P_1 D^{n-1} y + P_2 D^{n-2} y + \cdots + P_n y = X.$$

Suppose that y is treated as an algebraic factor of the left hand member of the equation. On this supposition, the equation becomes

$$(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n) y = X.$$

Suppose also that $D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n$, factored as an algebraic expression in D, is

$$(D - m_1)(D - m_2) \cdots (D - m_n),$$

and that the equation is written

$$(D - m_1)(D - m_2) \cdots (D - m_n) y = X. \qquad (2)$$

Equation (2) is not equivalent to equation (1) except in a symbolic sense. Let us see what conventions must be made in order that equation (2) be equivalent to equation (1).

Let us make the convention that $(D - m)u$ where u is a function of x is equal to

$$\frac{du}{dx} - mu.$$

Also, let us agree that we shall begin at the right of the left hand member of (2) and work towards the left, evaluating according to the preceding convention at each step. Then

$$(D - m_n) y = \frac{dy}{dx} - m_n y,$$

$$(D - m_{n-1})(D - m_n) y = (D - m_{n-1}) \left(\frac{dy}{dx} - m_n y \right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dx} - m_n y \right) - m_{n-1}\left(\frac{dy}{dx} - m_n y \right)$$

$$= \frac{d^2y}{dx^2} - (m_{n-1} + m_n) \frac{dy}{dx} + m_{n-1} m_n y,$$

. . . .

and finally,

$$(D-m_1)(D-m_2)\cdots(D-m_n)y$$
$$=\frac{d^ny}{dx^n}-(m_1+m_2+\cdots+m_n)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+(-1)^n m_1m_2\cdots m_ny.$$

Now $-(m_1+m_2+\cdots+m_n)=P_1, \cdots, (-1)^n m_1m_2\cdots m_n=P_n$, since the factors of $D^n+P_1D^{n-1}+P_2D^{n-2}+\cdots+P_n$, treated as an algebraic expression in D are $D-m_1$, $D-m_2$, $\cdots$, $D-m_n$. This expression is therefore the same as the left hand member of equation (1). Therefore, with these conventions, equations (2) is equivalent to equation (1).

EXAMPLE. With the above conventions, the equation

$$\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y=X$$

may be written in the equivalent form $(D-m_1)(D-m_2)y=X$, where $D-m_1$ and $D-m_2$ are the factors of the expression $D^2+P_1D+P_2$ treated as an algebraic expression in D. For,

$$(D-m_2)y=\frac{dy}{dx}-m_2y,$$

$$(D-m_1)(D-m_2)y=(D-m_1)\left(\frac{dy}{dx}-m_2y\right)$$
$$=\frac{d^2y}{dx^2}-(m_1+m_2)\frac{dy}{dx}+m_1m_2y.$$

Now $-(m_1+m_2)=P_1$, and $m_1m_2=P_2$. Therefore the second form of the equation is equivalent to the first.

Definitions. When equations in the form (1) are expressed in the form (2), they are said to be expressed **symbolically**, or to be expressed by means of **symbolic factors.**

When a symbolic factor $D-m$ and a function u are applied to each other so as to give $(D-m)u$ or $\frac{du}{dx}-mu$, the function u is said to be **operated upon** by $D-m$, or the factor $D-m$ to be **multiplied symbolically** by u.

The factor $D - m$ is called the **symbolic operator**, or more briefly, **the operator.**

34. Theorem. The order in which the symbolic factors in the equation of the last article are taken is immaterial.

Consider in illustration the equation of the second order. Let the equation be taken in the form

$$(D - m_2)(D - m_1)y = X.$$

Then

$$(D - m_1)y = \frac{dy}{dx} - m_1 y,$$

and

$$(D - m_2)(D - m_1)y = (D - m_2)\left(\frac{dy}{dx} - m_1 y\right)$$

$$= \frac{d^2y}{dx^2} - (m_1 + m_2)\frac{dy}{dx} + m_1 m_2 y.$$

Therefore $(D - m_2)(D - m_1)y = X$ is equivalent to

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = X.$$

Also, $(D - m_1)(D - m_2)y = X$ is equivalent to

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = X. \qquad \text{(See Art. 33.)}$$

Therefore, in the case of the equation of the second order, the order in which the factors are taken is immaterial.

The proof in the general case is left as an exercise to the student.

35. First method of solution of the equation

$$(D - m_1)(D - m_2)y = X.$$

Let $(D - m_2)y = u$. The equation then becomes

$$(D - m_1)u = X \quad \text{or} \quad \frac{du}{dx} - m_1 u = X.$$

The general solution of the equation

$$\frac{du}{dx} - m_1 u = X$$

is (see Art. 21)

$$u = e^{m_1 x}\int e^{-m_1 x} X dx + \bar{c}_1 e^{m_1 x}.$$

$$\therefore (D - m_2)y = e^{m_1 x}\int e^{-m_1 x} X dx + \bar{c}_1 e^{m_1 x}.$$

$$\therefore y = e^{m_2 x}\int e^{(m_1 - m_2)x}\int e^{-m_1 x} X (dx)^2 + \bar{c}_1 e^{m_2 x}\int e^{(m_1 - m_2)x} dx + c_2 e^{m_2 x}.$$

$$= e^{m_2 x}\int e^{(m_1 - m_2)x}\int e^{-m_1 x} X (dx)^2 + c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

This is the general solution of the given equation.

Example. Find the general solution of the equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \cos x.$$

Write the equation as $(D - 1)(D - 2)y = \cos x$.
Let $(D - 2)y = u$. The equation then becomes

$$(D - 1)u = \cos x \quad \text{or} \quad \frac{du}{dx} - u = \cos x.$$

$$\therefore u = e^x \int e^{-x}\cos x\, dx + \bar{c}_1 e^x$$

$$= \tfrac{1}{2}(\sin x - \cos x) + \bar{c}_1 e^x.$$

$$\therefore (D - 2)y = \tfrac{1}{2}(\sin x - \cos x) + \bar{c}_1 e^x.$$

$$\therefore y = \tfrac{1}{2}e^{2x}\int e^{-2x}(\sin x - \cos x)dx + \bar{c}_1 e^{2x}\int e^{-x}dx + c_2 e^{2x}$$

$$= \tfrac{1}{10}\cos x - \tfrac{3}{10}\sin x + c_1 e^x + c_2 e^{2x}.$$

This is the general solution of the given equation.

36. To solve the equation

$$(D - m_1)(D - m_2)\cdots(D - m_n)y = X,$$

we may proceed as follows:

First, let $(D - m_2)\cdots(D - m_n)y = u$. The equation then becomes $(D - m_1)u = X$.

From this equation, u can be determined as in the case of the equation of the second order. Let

$$(D - m_3) \cdots (D - m_n)y = v.$$

Then

$$(D - m_2)v = u.$$

From this equation v can be determined in the same manner as u was determined before. After $n - 1$ such steps there results $(D - m_n)y = z$ where z is a known function of x.

The general solution of the equation $(D - m_n)y = z$ is the general solution of the original equation.

37. The following theorems concerning the symbolic operator will now be established:

Theorem I. A constant factor in a function may be written in front of the operator.

Proof: Let au be a function containing a constant a as a factor. Let $D - m$ be the operator. Then

$$(D - m)au = \frac{dau}{dx} - mau, \qquad \text{by definition}$$

$$= a\left(\frac{du}{dx} - mu\right)$$

$$= a(D - m)u.$$

Theorem II. The result when the operator is applied to the sum of a number of functions is equal to the sum of the results found when the operator is applied to each of the functions separately.

Proof: Let $u + v + w + \cdots + z$ be the sum of a number of functions. Let $D - m$ be the operator. Then

$$(D - m)(u + v + w + \cdots + z)$$

$$= \frac{d(u + v + w + \cdots + z)}{dx} - m(u + v + w + \cdots + z)$$

$$= \frac{du}{dx} - mu + \frac{dv}{dx} - mv + \frac{dw}{dx} - mw + \cdots + \frac{dz}{dx} - mz$$

$$= (D - m)u + (D - m)v + (D - m)w + \cdots + (D - m)z.$$

38. The equation $(D - m_1)(D - m_2) \cdots (D - m_n)y = X$ may be written in the form

$$y = \frac{1}{(D - m_1)(D - m_2) \cdots (D - m_n)} X.$$

In the first form the symbolic operators

$$D - m_1,\ D - m_2,\ \cdots,\ D - m_n$$

applied in succession give X. Moreover, by the theorem of Art. 34, the order in which the operators are applied is immaterial. If the second form, therefore, is to be the same as the first, the symbolic expression $\frac{1}{(D-m_1)(D-m_2) \cdots (D-m_n)} X$ must be such that, when operated upon by

$$D - m_1,\ D - m_2,\ \cdots,\ D - m_n,$$

in succession in any order, the result is X.

Definition. The symbol $\frac{1}{(D - m_1)(D - m_2) \cdots (D - m_n)}$ is called the **inverse symbolic operator,** or, more briefly, the **inverse operator.**

39. Let

$$y = \frac{1}{(D - m_1)(D - m_2)} X$$

be a linear differential equation where the symbolic factors viewed as algebraic factors are distinct. Break up $\frac{1}{(D - m_1)(D - m_2)}$ into partial fractions as if it were an algebraic expression in D. Then

$$\frac{1}{(D - m_1)(D - m_2)} \equiv \frac{1}{m_1 - m_2}\left(\frac{1}{D - m_1} - \frac{1}{D - m_2}\right).$$

Let

$$\frac{1}{m_1 - m_2}\frac{1}{D - m_1} X = u \quad \text{and} \quad -\frac{1}{m_1 - m_2}\frac{1}{D - m_2} X = v.$$

Theorem. The result of operating on $u + v$ with $(D - m_1)(D - m_2)$ is X.

Proof: Operate on $u + v$ with $(D - m_1)(D - m_2)$.

$(D - m_1)(D - m_2)(u + v) = (D - m_1)(D - m_2)u +$
$(D - m_1)(D - m_2)v$, by theorem II, Art. 37.
Now

$$(D - m_1)u = \frac{1}{m_1 - m_2}X, \quad \text{and} \quad (D - m_2)v = -\frac{1}{m_1 - m_2}X,$$

by definition and theorem I, Art. 37.

$$\therefore\ (D - m_1)(D - m_2)(u + v)$$

$$= (D - m_2)\frac{1}{m_1 - m_2}X + (D - m_1)\left(-\frac{1}{m_1 - m_2}X\right)$$

$$= \frac{1}{m_1 - m_2}[(D - m_2)X - (D - m_1)X] = X.$$

40. When the symbolic factors $D - m_1$ and $D - m_2$, viewed as algebraic factors, are distinct, the result of operating on

$$y = \frac{1}{m_1 - m_2}\frac{1}{D - m_1}X - \frac{1}{m_1 - m_2}\frac{1}{D - m_2}X$$

with $(D - m_1)(D - m_2)$ is X, by the preceding article, and the result of operating on

$$y = \frac{1}{(D - m_1)(D - m_2)}X$$

with the same factors is X, by definition. Therefore when the symbolic factors $D - m_1$ and $D - m_2$, viewed as algebraic factors, are distinct, the inverse operator of

$$y = \frac{1}{(D - m_1)(D - m_2)}X$$

may be broken up into partial fractions the same as if it were an algebraic expression in D, and the result of operating with $(D - m_1)(D - m_2)$ on the expression formed by multiplying each of the fractions symbolically by X, and taking the algebraic sum of the results, is X.

In general, when the symbolic factors

$$D - m_1,\ D - m_2,\ \cdots,\ D - m_n,$$

viewed as algebraic factors are distinct, the inverse operator of

$$y = \frac{1}{(D - m_1)(D - m_2)\cdots(D - m_n)} X$$

can be broken up into partial fractions the same as if it were an algebraic expression in D, and the result of operating with $(D - m_1)(D - m_2)\cdots(D - m_n)$, on the expression formed by multiplying each fraction symbolically by X and taking the algebraic sum of the results is X.

The proof of this theorem is left as an exercise to the student.

41. Second method of solution of the equation

$$y = \frac{1}{(D - m_1)(D - m_2)} X.$$

Break up $\frac{1}{(D - m_1)(D - m_2)}$ into partial fractions the same as if it were an algebraic expression in D.

$$\therefore \frac{1}{(D - m_1)(D - m_2)} \equiv \frac{1}{m_1 - m_2}\left(\frac{1}{D - m_1} - \frac{1}{D - m_2}\right).$$

Let

$$u = \frac{1}{m_1 - m_2}\frac{1}{D - m_1} X \quad \text{and} \quad v = -\frac{1}{m_1 - m_2}\frac{1}{D - m_2} X.$$

Operate on u with $D - m_1$.

$$\therefore \frac{du}{dx} - m_1 u = \frac{1}{m_1 - m_2} X.$$

$$\therefore u = \frac{1}{m_1 - m_2} e^{m_1 x}\int e^{-m_1 x} X dx + c_1 e^{m_1 x}.$$

Operate on v with $D - m_2$

$$\therefore \frac{dv}{dx} - m_2 v = -\frac{1}{m_1 - m_2} X.$$

$$\therefore v = -\frac{1}{m_1 - m_2} e^{m_2 x}\int e^{-m_2 x} X dx + c_2 e^{m_2 x}.$$

$$\therefore y = \frac{1}{m_1 - m_2} e^{m_1 x} \int e^{-m_1 x} X dx - \frac{1}{m_1 - m_2} e^{m_2 x} \int e^{-m_2 x} X dx$$
$$+ c_1 e^{m_1 x} + c_2 e^{m_2 x},$$

which is the general solution of the equation.

EXAMPLE. Find the general solution of the equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \cos x.$$

Write the expression in the form

$$y = \frac{1}{(D-1)(D-2)} \cos x.$$

Break up $\frac{1}{(D-1)(D-2)}$ into partial fractions the same as if it were an algebraic expression in D.

$$\therefore \frac{1}{(D-1)(D-2)} \equiv -\frac{1}{D-1} + \frac{1}{D-2}.$$

Let

$$-\frac{1}{D-1} \cos x = u \quad \text{and} \quad \frac{1}{D-2} \cos x = v.$$

Operate on u with $D - 1$.

$$\therefore \frac{du}{dx} - u = -\cos x.$$

$$\therefore u = \tfrac{1}{2} \cos x - \tfrac{1}{2} \sin x + c_1 e^x.$$

Operate on v with $D - 2$.

$$\therefore \frac{dv}{dx} - 2v = -\cos x.$$

$$\therefore v = -\tfrac{2}{5} \cos x + \tfrac{1}{5} \sin x + c_2 e^{2x}.$$

$$\therefore y = \tfrac{1}{10} \cos x - \tfrac{3}{10} \sin x + c_1 e^x + c_2 e^{2x},$$

which is the general solution of the equation.

This method does not apply when the symbolic factors viewed as algebraic factors are not distinct.

42. It will be noticed in the example of the preceding article that the result is the same as that found by applying the method of Art. 35 to the same equation. This will be the case in any linear differential equation with constant coefficients to which both methods apply.

The first method of solution will apply in all cases where the left hand member of the equation can be factored into linear factors in D. The second method will also apply if the linear factors in D are all distinct. If two or more factors are equal, and the inverse operator be broken up into partial fractions, the term or terms corresponding to these factors may be evaluated by the first method.

Usually the second method is easier of application than the first.

43. An examination of either method by which the general solution of a linear differential equation of the nth order with constant coefficients and second member not zero is derived shows immediately that the general solution consists of the sum of two parts, one containing terms not involving arbitrary constants, the other containing terms involving such constants. Moreover the arbitrary constants are involved so that when any one is zero, the term in which it appears vanishes.

Definition. The part of the general solution of a linear differential equation with constant coefficients and second member not zero which contains the arbitrary constants is called the **complementary function** of the general solution of the equation.

EXERCISES

Find the general solution of each of the fourteen following equations.

1. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 2e^{2x}$.

2. $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = x^2$.

3. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 2y = x$.

4. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = e^x$.

5. $\dfrac{d^3y}{dx^3} + 5\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} = x.$

6. $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = x.$

7. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x.$

8. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x.$

9. $\dfrac{d^2y}{dx^2} + y = \cos x.$

10. $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 4y = x.$

11. $\dfrac{d^2y}{dx^2} + y = \sin x.$

12. $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = \cos x.$

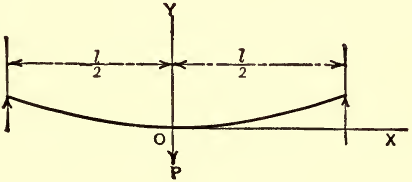
13. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = e^x.$

14. $\dfrac{d^4y}{dx^4} - y = x^4.$

In each of the six following exercises, find the equation of the elastic curve of the beam from the given differential equation, determining the constants of integration. Find also the deflection of the beam. In these equations, E is the modulus of elasticity, I is the moment of inertia of a cross section of the beam about a gravity axis in the section perpendicular to the applied forces, and l is the length of the beam.

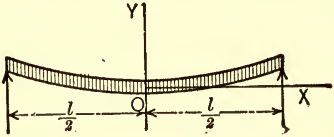
15. The beam rests on supports at its ends. It is supposed weightless with a weight P at its middle point.

$$EI\frac{d^2y}{dx^2} = \frac{P}{2}\left(\frac{l}{2} - x\right).$$



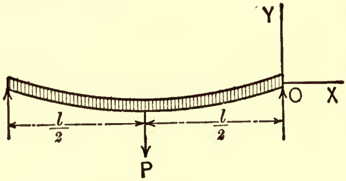
16. The beam rests on supports at its ends. It is supposed to be of uniform cross section and of weight w per unit of length.

$$EI\frac{d^2y}{dx^2} = \frac{w}{2}\left(\frac{l^2}{4} - x^2\right).$$



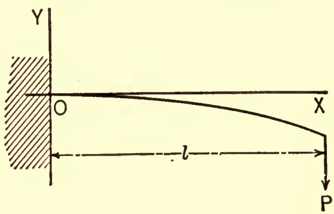
17. The beam rests on supports at its ends. It is supposed to be of uniform cross section and of weight w per unit of length, and to have a weight P at its middle point.

$$EI\frac{d^2y}{dx^2} = -\frac{wl + P}{2}x - \frac{w}{2}x^2.$$



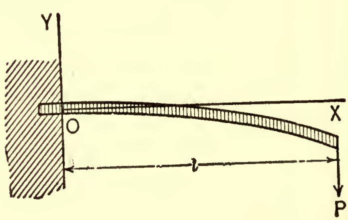
18. The beam is a cantilever fixed horizontally in the wall. It is supposed weightless with a weight P at its extremity.

$$EI\frac{d^2y}{dx^2} = -P(l - x).$$

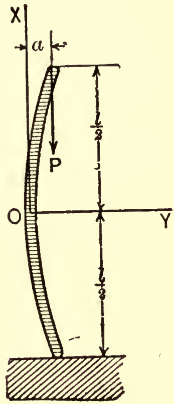


19. The beam is a cantilever fixed horizontally in the wall. It is supposed to be of uniform cross section and of weight w per unit of length, and to have a weight P at its extremity.

$$EI\frac{d^2y}{dx^2} = -Pl + (P + wl)x - \frac{w}{2}(l^2 + x^2).$$



20. The beam is vertical. It has rounded ends. It is supposed weightless. It is deflected a small amount a and a load P is applied at its upper end just sufficient to hold it in position.



$$EI\frac{d^2y}{dx^2} = P(a - y).$$

The equation for a circuit containing resistance, induction and capacity in terms of the current i is

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = \frac{1}{L}\frac{de}{dt};$$

in terms of the quantity of charge q is

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{1}{L}e$$

where e denotes the E.M.F., R the resistance, L the induction, C the capacity, and t the time during which the circuit is in operation. In each of the three following exercises, determine the current and quantity of charge in the circuit after a time t, supposing that the resistance, induction and capacity are constant.

21. The E.M.F. is equal to $f(t)$. Solve when $R^2C \neq 4L$.

22. The E.M.F. is constant and equal to E.

23. The E.M.F. is a simple sine function of the time, $= E\sin\omega t$. Solve when $R^2C \neq 4L$.

ANSWERS

1. $y = \frac{2}{15}e^{2x} + c_1e^{-x} + c_2e^{-3x}$.

2. $y = \frac{1}{9}x^3 - \frac{4}{9}x^2 + \frac{26}{27}x + c_1 + c_2e^{-x} + c_3e^{-3x}$.

3. $y = \dfrac{x+2}{2} + c_1e^{(2+\sqrt{2})x} + c_2e^{(2-\sqrt{2})x}$.

4. $y = \frac{1}{3}e^x + c_1e^{-\frac{(3-\sqrt{13})x}{2}} + c_2e^{-\frac{(3+\sqrt{13})x}{2}}$.

5. $y = \frac{1}{12}x^2 - \frac{5}{36}x + c_1 + c_2e^{-2x} + c_3e^{-3x}$.

6. $y = -\dfrac{6x+11}{36} + c_1e^x + c_2e^{2x} + c_3e^{3x}$.

7. $y = e^x(\frac{1}{2}x^2 + c_1 + c_2x)$. 8. $y = x + 2 + e^x(c_1 + c_2x)$.

9. $y = \dfrac{x \sin x}{2} + c_1 \cos x + c_2 \sin x$.

10. $y = -\frac{1}{4}x + \frac{1}{4} + c_1e^{-x} + c_2e^{-2x} + c_3e^{2x}$.

11. $y = -\dfrac{x \cos x}{2} + c_1 \cos x + c_2 \sin x$.

12. $y = \frac{1}{4}(\cos x - x \sin x - x \cos x) + c_1 \sin x + c_2 \cos x + c_3e^x$.

13. $y = e^x(\frac{1}{6}x^3 + c_1x^2 + c_2x + c_3)$.

14. $y = -x^4 - 24 + c_1e^x + c_2e^{-x} + c_3 \sin x + c_4 \cos x$.

15. $4EIy = \dfrac{Pl}{2}x^2 - \dfrac{P}{3}x^3$. Deflection $= \dfrac{1}{48}\dfrac{Pl^3}{EI}$.

16. $8EIy = \dfrac{wl^2}{2}x^2 - \dfrac{w}{3}x^4$. Deflection $= \dfrac{5}{384}\dfrac{wl^4}{EI}$.

17. $2EIy = -\dfrac{wl+P}{6}x^3 - \dfrac{w}{12}x^4 + \dfrac{(wl+P)l^2}{8}x - \dfrac{wl^3}{24}x$.

Deflection $= \dfrac{1}{48}\dfrac{Pl^3}{EI} + \dfrac{5}{384}\dfrac{wl^4}{EI}$.

18. $2EIy = -Plx^2 + \dfrac{P}{3}x^3$. Deflection $= \dfrac{1}{3}\dfrac{Pl^3}{EI}$.

19. $2EIy = \dfrac{P}{3}x^3 - \dfrac{w}{12}x^4 - Plx^2 + \dfrac{wl}{3}x^3 - \dfrac{wl^2}{2}x^2$.

Deflection $= \dfrac{1}{3EI}Pl^3 + \dfrac{1}{8EI}wl^4$.

20. $x = \sqrt{\dfrac{EI}{P}} \operatorname{vers}^{-1}\dfrac{y}{a}$.

Deflection $= a \operatorname{vers}\sqrt{\dfrac{P}{EI}}\dfrac{l}{2}$, and $\therefore P = \dfrac{EI\pi^2}{l^2}$.

21 $$i = \frac{C}{\sqrt{R^2C^2 - 4LC}}\left\{e^{-\frac{t}{T_1}}\int e^{\frac{t}{T_1}}f'(t)dt - e^{-\frac{t}{T_2}}\int e^{\frac{t}{T_2}}f'(t)dt\right\} + c_1 e^{-\frac{t}{T_1}} + c_2 e^{-\frac{t}{T_2}},$$

where

$$T_1 = \frac{2LC}{RC - \sqrt{R^2C^2 - 4LC}} \quad \text{and} \quad T_2 = \frac{2LC}{RC + \sqrt{R^2C^2 - 4LC}}.$$

The value for q differs from that for i only in having $f(t)$ instead of $f'(t)$.

22. $$i = c_1 e^{-\frac{RC - \sqrt{R^2C^2-4LC}}{2LC}t} + c_2 e^{-\frac{RC+\sqrt{R^2C^2-4LC}}{2LC}t},$$

when $R^2C > 4L$.

$$i = c_1 e^{-\frac{R}{2L}t}\cos\frac{\sqrt{4LC - R^2C^2}}{2LC}t + c_2 e^{-\frac{R}{2L}t}\sin\frac{\sqrt{4LC - R^2C^2}}{2LC}t,$$

when $R^2C < 4L$.

$$i = c_1 e^{-\frac{R}{2L}t} + c_2 t e^{-\frac{R}{2L}t},$$

when $R^2C = 4L$.

$$q = CE + c_1 e^{-\frac{RC-\sqrt{R^2C^2-4LC}}{2LC}t} + c_2 e^{-\frac{RC+\sqrt{R^2C^2-4LC}}{2LC}t},$$

when $R^2C > 4L$.

$$q = CE + c_1 e^{-\frac{R}{2L}t}\cos\frac{\sqrt{4LC - R^2C^2}}{2LC}t + c_2 e^{-\frac{R}{2L}t}\sin\frac{\sqrt{4LC - R^2C^2}}{2LC}t,$$

when $R^2C < 4L$.

$$q = CE + c_1 e^{-\frac{R}{2L}t} + c_2 t e^{-\frac{R}{2L}t},$$

when $R^2C = 4L$.

23. $$i = \frac{E\omega^2 R}{R^2\omega^2 + \left(\frac{1}{C} - L\omega^2\right)^2}\sin\omega t + \frac{E\omega\left(\frac{1}{C} - L\omega^2\right)}{R^2\omega^2 + \left(\frac{1}{C} - L\omega^2\right)^2}\cos\omega t + c_1 e^{-\frac{t}{T_1}} + c_2 e^{-\frac{t}{T_2}},$$

where T_1 and T_2 have the values given in exercise 21.

$$q = \frac{-E\omega R}{R^2\omega^2 + \left(\frac{1}{C} - L\omega^2\right)^2} \cos \omega t + \frac{E\left(\frac{1}{C} - L\omega^2\right)}{R^2\omega^2 + \left(\frac{1}{C} - L\omega^2\right)^2} \sin \omega t$$

$$+ c_1 e^{-\frac{t}{T_1}} + c_2 e^{-\frac{t}{T_2}}.$$

CHAPTER V

HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS. EXACT LINEAR DIFFERENTIAL EQUATIONS

HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

44. Definition. A homogeneous linear differential equation is an equation of the form

$$x^n \frac{d^ny}{dx^n} + p_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1} x \frac{dy}{dx} + p_n y = X$$

where $p_1, p_2, \cdots, p_{n-1}, p_n$ are constants, and X is a function of x but not of y.

This equation can be transformed into an ordinary linear differential equation with constant coefficients by changing the independent variable from x to z, the equation of transformation being $x = e^z$. The equation that results from the transformation may be solved by the methods of the last chapter. If a solution is $y = f(z)$, the corresponding solution of the original equation is $y = f(\log x)$.

45. The transformation and general solution of a homogeneous linear differential equation in the general case will not be considered here. We shall merely consider them in a particular example.

EXAMPLE. Find the general solution of the equation

$$x^3 \frac{d^3y}{dx^3} + 4x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = \log x.$$

Let $x = e^z$. $\therefore z = \log x$,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz},$$

$$\frac{d^2y}{dz^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right),$$

$$\frac{d^3y}{dz^3} = \frac{d}{dx}\left\{\frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\right\} = \frac{1}{x^3}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right).$$

$$\therefore x\frac{dy}{dx} = \frac{dy}{dz}.$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}.$$

$$x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}.$$

Substitute in the equation.

$$\therefore \frac{d^3y}{dz^3} + \frac{d^2y}{dz^2} - 4\frac{dy}{dz} - 4y = z.$$

The general solution of this equation can be found by the methods of the last chapter. It is

$$y = -\tfrac{1}{4}z + \tfrac{1}{4} + c_1e^{-z} + c_2e^{-2z} + c_3e^{2z}.$$

The general solution of the original equation is therefore

$$y = -\tfrac{1}{4}\log x + \tfrac{1}{4} + \frac{c_1}{x} + \frac{c_2}{x^2} + c_3x^2.$$

EXACT LINEAR DIFFERENTIAL EQUATIONS

46. Definition. A linear differential equation

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-3}\frac{d^3y}{dx^3} + P_{n-2}\frac{d^2y}{dx^2} + P_{n-1}\frac{dy}{dx} + P_ny = X$$

is said to be **exact,** when, if the left hand member be represented by V, the expression Vdx is the exact differential of some function U which does not contain an integral of y.

The expression U is evidently an expression actually containing a derivative of order $n - 1$.

47. To find the necessary and sufficient condition that the equation of the preceding article be exact, and a method of solution of such an equation.

Multiply each term by dx and take the integral of each term.

$$\therefore \int P_0 \frac{d^n y}{dx^n} dx + \int P_1 \frac{d^{n-1}y}{dx^{n-1}} dx + \cdots + \int P_{n-3} \frac{d^3 y}{dx^3} dx$$

$$+ \int P_{n-2} \frac{d^2 y}{dx^2} dx + \int P_{n-1} \frac{dy}{dx} dx + \int P_n y dx = \int X dx + c.$$

Now

$$\int P_n y dy = \int P_n y dy \qquad \text{identically.}$$

And, by integration by parts,

$$\int P_{n-1} \frac{dy}{dx} dx = - \int P'_{n-1} y dx + P_{n-1} y,$$

$$\int P_{n-2} \frac{d^2 y}{dx^2} dx = \int P''_{n-2} y dx - P'_{n-2} y + P_{n-2} y',$$

$$\int P_{n-3} \frac{d^3 y}{dx^3} dx = - \int P'''_{n-3} y dx + P''_{n-3} y - P'_{n-3} y' + P_{n-3} y'',$$

$$\cdots\cdots\cdots\cdots,$$

where the primes denote differentiation with respect to x.

$$\begin{aligned} \therefore \int X dx + c = \int (P_n - P'_{n-1} + P''_{n-2} - P'''_{n-3} + \cdots) y dx. \\ + (P_{n-1} - P'_{n-2} + P''_{n-3} - \cdots) y \\ + (P_{n-2} - P'_{n-3} + \cdots\cdots) y' \\ + (P_{n-3} - \cdots\cdots\cdots) y'' \\ + \cdots\cdots\cdots\cdots \end{aligned}$$

Write the expression in brackets as Q_n, Q_{n-1}, $\cdots$, Q_0 respectively.

$$\therefore \int X dx + c = \int Q_n y dx + Q_{n-1} y + Q_{n-2} y' + \cdots + Q_0 \frac{d^{n-1}y}{dx^{n-1}}. \quad (1)$$

Now in order that the equation be integrable there must be no term in the right hand member of (1) containing an integral of y. The necessary and sufficient condition for this is that $Q_n = 0$.

Therefore the necessary and sufficient condition that the equation be exact is that $Q_n = 0$ or

$$P_n - P'_{n-1} + P''_{n-2} - P'''_{n-3} + \cdots = 0. \qquad (2)$$

When this condition is satisfied the equation reduces to

$$Q_0 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + Q_{n-2}\frac{dy}{dx} + Q_{n-1}y = \int X dx + c. \qquad (3)$$

If the coefficients in (3) satisfy a relation in Q similar to (2) in P, equation (3) is exact and the above process may be repeated.

EXAMPLE. Solve the equation

$$x(1-x^2)\frac{d^3y}{dx^3} + (1-5x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 6x.$$

Here $P_n = 2,\ -P'_{n-1} = 2,\ P''_{n-2} = -10,\ -P'''_{n-3} = 6.$

$$\therefore P_n - P'_{n-1} + P''_{n-2} - P'''_{n-3} = 0.$$

The necessary and sufficient condition that the equation be exact is therefore satisfied.

$$Q_{n-1} = -2x + 10x - 6x \equiv 2x, \quad Q_{n-2} = 1 - 5x^2 - 1 + 3x^2 \equiv -2x^2,$$
$$Q_{n-3} = x(1-x^2).$$

The equation therefore reduces to

$$x(1-x^2)\frac{d^2y}{dx^2} - 2x^2\frac{dy}{dx} + 2xy = 3x^2 + c_1.$$

In this equation, $P_n = 2x,\ -P'_{n-1} = 4x,\ P''_{n-2} = -6x.$

$$\therefore P_n - P'_{n-1} + P''_{n-2} = 0.$$

The necessary and sufficient condition that this equation be exact is therefore satisfied.

$$Q_{n-1} = -2x^2 + 3x^2 - 1 \equiv x^2 - 1, \quad Q_{n-2} = x(1-x^2).$$

The equation therefore reduces to

$$x(1-x^2)\frac{dy}{dx} + (x^2-1)y = x^3 + c_1x + c_2.$$

This equation is not exact. It is however an ordinary linear differential equation of the first order, and can therefore be solved by the method of Art. 21. The general solution is

$$y = -\tfrac{1}{2}x\log(1-x^2) + \frac{c_1}{2}x\log\frac{x^2}{1-x^2} + c_2\left(-1+\tfrac{1}{2}x\log\frac{1+x}{1-x}\right) + c_3x,$$

which is therefore the general solution of the original equation.

EXERCISES

Find the general solution of each of the following equations.

1. $x^3\dfrac{d^3y}{dx^3} + 7x^2\dfrac{d^2y}{dx^2} + 8x\dfrac{dy}{dx} = (\log x)^2.$

2. $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} - 8y = x.$

3. $x^3\dfrac{d^3y}{dx^3} - 3x^2\dfrac{d^2y}{dx^2} + 6x\dfrac{dy}{dx} - 6y = x^3.$

4. $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 4y = \log x.$

5. $x^3\dfrac{d^3y}{dx^3} + 4x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 0.$

6. $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = 2x.$

7. $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = \log x.$

8. $(x^2-1)\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = 2x.$

9. $(1+x^2)\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = x.$

10. $\dfrac{d^2y}{dx^2} - \cot x\dfrac{dy}{dx} + \operatorname{cosec}^2 x\, y = \cos x.$

11. $(x^2-x)\dfrac{d^2y}{dx^2} + (3x-2)\dfrac{dy}{dx} + y = 0.$

12. $(x + 3x^2)\frac{d^2y}{dx^2} + 2(1 + 6x)\frac{dy}{dx} + 6y = \sin x.$

13. $(x^3 + x^2 - 3x + 1)\frac{d^3y}{dx^3} + (9x^2 + 6x - 9)\frac{d^2y}{dx^2}$
$+ (18x + 6)\frac{dy}{dx} + 6y = x^3.$

14. $x^2\frac{d^3y}{dx^3} + 5x\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = -\frac{1}{x^2}.$

ANSWERS

1. $y = \frac{1}{9}(\log x)^3 - \frac{4}{9}(\log x)^2 + \frac{26}{27}\log x + c_1 + \frac{c_2}{x} + \frac{c_3}{x^3}.$

2. $y = -\frac{1}{5}x + c_1x^2 + \frac{c_2}{x^4}.$

3. $y = \frac{1}{2}x^3\log x - \frac{3}{4}x^3 + c_1x + c_2x^2 + c_3x^3.$

4. $y = \frac{1}{4}\log x + \frac{1}{4} + c_1x + c_2x^2 + \frac{c_3}{x^2}.$

5. $y = \frac{c_1}{x} + \frac{c_2}{x}\log x + c_3x.$ 6. $y = \frac{1}{3}x^2 + \frac{c_1}{x} + c_2.$

7. $y = \log x + 2 + c_1x\log x + c_2x.$

8. $y = \frac{1}{3}\frac{x^3}{x^2 - 1} + \frac{c_1x}{x^2 - 1} + \frac{c_2}{x^2 - 1}.$

9. $y = \frac{1}{6}\frac{x^3}{1 + x^2} + \frac{c_1x}{1 + x^2} + \frac{c_2}{1 + x^2}.$

10. $y = x\sin x + c_1\sin x\log(\operatorname{cosec} x - \cot x) + c_2\sin x.$

11. $y = \frac{c_1}{x}\log(x - 1) + \frac{c_2}{x}.$

12. $(x + 3x^2)y = -\sin x + c_1x + c_2.$

13. $(x^3 + x^2 - 3x + 1)y = \frac{1}{120}x^6 + c_1 + c_2x + c_3x^2.$

14. $y = \frac{1}{2x}(\log x)^2 + c_1 + \frac{c_2}{x}\log x + \frac{c_3}{x}.$

CHAPTER VI

CERTAIN PARTICULAR FORMS OF EQUATIONS

48. An equation in the form $\frac{d^n y}{dx^n} = f(x)$.

An equation in this form is exact and can therefore be integrated by the methods of the preceding chapter. It can also be integrated by direct integration.

The first integration gives

$$\frac{d^{n-1}y}{dx^{n-1}} = \int f(x)\,dx + a_1$$

where a_1 is an arbitrary constant.

The second gives

$$\frac{d^{n-2}y}{dx^{n-2}} = \int\int f(x)\,(dx)^2 + a_1 x + a_2$$

where a_2 is an arbitrary constant.

After n integrations there results

$$y = \int\int\int \cdots \int f(x)(dx)^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n$$

where $c_1, c_2, \cdots, c_n$ are arbitrary constants.

49. An equation in the form $\frac{d^n y}{dx^n} = f(y)$.

An equation in this form can in general be integrated only when $n = 1$ and $n = 2$.

When $n = 1$ the equation is

$$\frac{dy}{dx} = f(y).$$

To integrate, separate the variables.

$$\therefore \frac{dy}{f(y)} = dx.$$

$$\therefore x + c = \int \frac{dy}{f(y)}.$$

When $n = 2$ the equation is in the form

$$\frac{d^2y}{dx^2} = f(y).$$

To integrate, multiply by $2\,\frac{dy}{dx}$.

$$\therefore 2\,\frac{dy}{dx}\,\frac{d^2y}{dx^2} = 2f(y)\,\frac{dy}{dx}.$$

Now

$$2\,\frac{dy}{dx}\,\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)^2.$$

$$\therefore \frac{d}{dx}\left(\frac{dy}{dx}\right)^2 = 2f(y)\,\frac{dy}{dx}.$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = 2\int f(y)dy + c_1.$$

Suppose that $2\int f(y)dy = \psi(y)$.

$$\therefore \frac{dy}{dx} = \pm\,\sqrt{\psi(y) + c_1}.$$

$$\therefore \pm\,\frac{dy}{\sqrt{\psi(y) + c_1}} = dx.$$

$$\therefore x + c_2 = \pm\int \frac{dy}{\sqrt{\psi(y) + c_1}}.$$

50. An equation that does not contain x directly. Such an equation is of the form

$$F\left(y, \frac{dy}{dx}, \cdots, \frac{d^ny}{dx^n}\right) = 0.$$

Let

$$\frac{dy}{dx} = p.$$

Then

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy};$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(p\,\frac{dp}{dy}\right) = \frac{d}{dy}\left(p\,\frac{dp}{dy}\right)\frac{dy}{dx}$$

$$= p^2\frac{d^2p}{dy^2} + p\left(\frac{dp}{dy}\right)^2;$$

and so on.

The equation then becomes a differential equation in p and y of order $n - 1$. Suppose that it can be solved and that the solution is $p = f(y)$. Then a solution of the original equation is

$$x + c = \int \frac{dy}{f(y)}.$$

51. An equation which does not contain y directly.

Such an equation is of the form

$$F\left(x, \frac{dy}{dx}, \cdots, \frac{d^ny}{dx^n}\right) = 0.$$

Let

$$\frac{dy}{dx} = p.$$

The equation then becomes a differential equation in p and x of order $n - 1$. If the equation can be solved for p and the solution is $p = f(x)$, a solution of the original equation is

$$y + c = \int f(x)dx.$$

52. An equation of the first order solvable for y.

In such a case, when solved for y, the equation becomes

$$y = F(x, p). \tag{1}$$

Differentiate with respect to x.

$$\therefore p = \phi\left(x, p, \frac{dp}{dx}\right).$$

This equation does not contain y explicitly. It is an equation of the first order in p and x. If it can be integrated as an equation in p and x, there results on integration an equation between x, p and an arbitrary constant. From the resulting equation and (1), if p can be eliminated, there results an equation between x, y and an arbitrary constant, which will be the general solution of the equation.

53. An equation of the first order solvable for x.

In such a case, when solved for x, the equation becomes $y = F(y, p)$. Differentiate with respect to y.

$$\therefore \frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right).$$

The method of procedure from this point is similar to that in the preceding article.

EXERCISES

Find the general solution of each of the twelve following equations.

1. $\dfrac{d^ny}{dx^n} = x.$

2. $\dfrac{d^2y}{dx^2} = \cos x.$

3. $x^2\dfrac{d^2y}{dx^2} = \log x.$

4. $\dfrac{d^2y}{dx^2} = -a^2y.$

5. $\dfrac{d^2y}{dx^2} = \dfrac{1}{y^2}.$

6. $y\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 = 0.$

7. $y\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 = 1.$

8. $(1 + x^2)\dfrac{d^2y}{dx^2} - 1 - \left(\dfrac{dy}{dx}\right)^2 = 0.$

9. $x\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} = 3x.$

10. $x = \dfrac{dy}{dx} + \dfrac{d^2y}{dx^2}.$

11. $x = y + \left(\dfrac{dy}{dx}\right)^2.$

12. $y = x\dfrac{dy}{dx} - \left(\dfrac{dy}{dx}\right)^2.$

13. Find the curve whose curvature is constant and equal to κ.

14. If a sphere of radius R_1 is surrounded by a concentric shell of radii R_2 and R_3, the potential function, V, at a point either in the space between the conductors or outside the outer, satisfies the equation

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0,$$

where r is the distance of the point from the center of the sphere.

Solve the equation given that V_1 is the potential on the sphere and V_2 on the spherical shell.

15. If a circular cylinder of radius R_1 is surrounded by a circular cylindrical shell of radii R_2 and R_3, both of very great length, the potential function, V, in the space between the conductors, is such that

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = 0,$$

where r is the distance from the point to the axis of the cylinder.

Solve the equation given that V_1 is the potential on the cylinder and V_2 on the spherical shell.

ANSWERS

1. $y = \dfrac{x^{n+1}}{(n+1)!} + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_{n-1}.$

2. $y = -\cos x + c_1x + c_2.$

3. $y = -\log x - \frac{1}{2}(\log x)^2 + c_1x + c_2.$

4. $y = c_1 \sin(ax + c_2).$

5. $\pm x + c_2 = \frac{\sqrt{y}}{c_1}\sqrt{c_1 y - 2} + \frac{2}{c_1^{\frac{3}{2}}}\log(\sqrt{c_1 y} + \sqrt{c_1 y - 2})$.

6. $y = c_2 e^{c_1 x}$.

7. $\pm x + c_1 = \frac{1}{c}\log(cy + \sqrt{c^2 y^2 - 1})$.

8. $y = -\frac{1}{c_1}x - \frac{c_1^2 + 1}{c_1^2}\log(1 - c_1 x) + c_2$.

9. $y = \frac{3}{8}x^2 + \frac{c_1}{x^2} + c_2$.

10. $y + c_1 = \frac{1}{2}x^2 - x + ce^{-x}$.

11. $\frac{1}{2}x + c_1 = \mp\sqrt{x - y} - \log(1 \mp \sqrt{x - y})$.

12. $y = cx - c^2$.

13. A circle of radius $\frac{1}{\kappa}$.

14. $V = \frac{R_1 R_2}{R_2 - R_1}\,\frac{V_1 - V_2}{r} + \frac{R_2 V_2 - R_1 V_1}{R_2 - R_1}$.

15. $V = \frac{V_1 - V_2}{\log\frac{R_1}{R_2}}\log r + \frac{V_2 \log R_1 - V_1 \log R_2}{\log\frac{R_1}{R_2}}$.

CHAPTER VII

ORDINARY DIFFERENTIAL EQUATIONS IN TWO DEPENDENT VARIABLES

54. So far, the differential equations considered consisted of two variables, one independent and one dependent. We shall now consider equations in three variables. These may be divided into two classes: those in which there is only one independent variable, and those in which there is only one dependent variable. The first comes under the class called **ordinary** or **total differential equations**: the second, **partial differential equations.** This chapter is taken up with a discussion of a few forms of ordinary differential equations. The next chapter is devoted to partial differential equations.

55. If $f(x, y)$ is a single valued and continuous function of the two independent variables x and y, given by the equation $z = f(x, y)$, and $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are continuous, then, by definition,

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy,$$

or

$$dz = \frac{\partial f(x, y)}{\partial x} dx + \frac{\partial f(x, y)}{\partial y} dy. \tag{1}$$

If $f(x, y, z)$ is a single valued and continuous function of the three independent variables x, y and z, given by the equation $u = f(x, y, z)$, and $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ are continuous, then, by definition,

$$du = \frac{\partial f(x, y, z)}{\partial x} dx + \frac{\partial f(x, y, z)}{\partial y} dy + \frac{\partial f(x, y, z)}{\partial z} dz. \tag{2}$$

56. Equation (1) of the preceding article has, as a special case when $z = 0$, the equation

$$\frac{\partial f(x, y)}{\partial x} dx + \frac{\partial f(x, y)}{\partial y} dy = 0.$$

That is, the equation is true for the equation $f(x, y) = 0$ where x and y are independent variables. If y in $f(x, y) = 0$ is a single valued and continuous function of x, the equation holds true for all values of x for which y is a single valued and continuous function, for in this case y is merely restricted to values which it could assume, as well as others, in the more general case where it is independent.

This can be seen more clearly perhaps by a consideration of the geometrical representations of the equations.

The equation $z = f(x, y)$ when x and y are independent variables represents a surface. If $z = 0$, the surface is the xy-plane, and the equation

$$\frac{\partial f(x, y)}{\partial x} dx + \frac{\partial f(x, y)}{\partial y} dy = 0$$

holds true for every point in the plane. If y is a single valued and continuous function of x, the equation $f(x, y) = 0$ represents a curve in the xy-plane in which the equation expressed in the form $y = \phi(x)$ gives a single valued and continuous function of x, and since

$$\frac{\partial f(x, y)}{\partial x} dx + \frac{\partial f(x, y)}{\partial y} dy = 0$$

holds true for all sets of values of x and y in the plane, it holds true for all sets of values which together determine a point on the curve in the plane.

57. Equation (2) of Art. 55 has as a special case when $z=0$, the equation

$$\frac{\partial f(x, y, z)}{\partial x} dx + \frac{\partial f(x, y, z)}{\partial y} dy + \frac{\partial f(x, y, z)}{\partial z} dz = 0.$$

By reasoning similar to that employed in the preceding article in the case of two dependent variables, it may be seen that this equation holds true when z is a single valued and continuous function of x and y.

58. An integral relation in x, y and z, equated to an arbitrary constant c, say $\phi(x, y, z) = c$, can always be expressed in the form

$$Pdx + Qdy + Rdz = 0,$$

where P, Q and R are functions of x, y and z, and do not contain the arbitrary constant c.

For, the result of taking the differential of each member of the equation $\phi(x, y, z) = c$ is, by the preceding article,

$$\frac{\partial\phi(x, y, z)}{\partial x}dx + \frac{\partial\phi(x, y, z)}{\partial y}dy + \frac{\partial\phi(x, y, z)}{\partial z}dz = 0,$$

and this equation is in the specified form.

Example. The result of taking the differential of each member of the equation $x^2y - xz^2 + y^2z = c$ where c is arbitrary, is

$$(2xy - z^2)dx + (x^2 + 2yz)dy + (y^2 - 2xz)dz = 0.$$

This equation is in the form $Pdx + Qdy + Rdz = 0$.

The resulting equation $Pdx + Qdy + Rdz = 0$ is such that P, Q and R are proportional to

$$\frac{\partial\phi}{\partial x}, \ \frac{\partial\phi}{\partial y} \ \text{and} \ \frac{\partial\phi}{\partial z},$$

respectively.

Conversely, however, an equation of the form

$$Pdx + Qdy + Rdz = 0$$

where P, Q and R are functions of x, y and z, does not necessarily give rise to a solution of the form $\phi(x, y, z) = c$. This can be seen immediately because an equation of the form

$$Pdx + Qdy + Rdz = 0$$

which gives rise to a relation $\phi(x, y, z) = c$ must be such that P, Q and R are proportional to

$$\frac{\partial \phi}{\partial x}, \quad \frac{\partial \phi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial z},$$

respectively, and these relations cannot hold for all values of P, Q and R.

59. To determine when an equation of the form

$$Pdx + Qdy + Rdz = 0$$

has a solution of the form $\phi(x, y, z) = c$.

If it be assumed that $Pdx + Qdy + Rdz = 0$ has a solution $\phi(x, y, z) = c$, then P, Q and R must be proportional to

$$\frac{\partial \phi}{\partial x}, \quad \frac{\partial \phi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial z},$$

respectively, or

$$\mu P = \frac{\partial \phi}{\partial x}, \quad \mu Q = \frac{\partial \phi}{\partial y}, \quad \mu R = \frac{\partial \phi}{\partial z},$$

where μ is a certain unknown function. From the first two of these equations there results

$$\frac{\partial}{\partial y}(\mu P) = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial}{\partial x}(\mu Q),$$

or

$$\mu \frac{\partial P}{\partial y} + P \frac{\partial \mu}{\partial y} = \mu \frac{\partial Q}{\partial x} + Q \frac{\partial \mu}{\partial x}.$$

$$\therefore \mu \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = Q \frac{\partial \mu}{\partial x} - P \frac{\partial \mu}{\partial y}, \qquad (1)$$

Similarly, by using the first and third equations we get

$$\mu \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) = P \frac{\partial \mu}{\partial z} - R \frac{\partial \mu}{\partial x}, \qquad (2)$$

and by using the second and third,

$$\mu \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) = R \frac{\partial \mu}{\partial y} - Q \frac{\partial \mu}{\partial z}. \qquad (3)$$

Multiply equations (1), (2) and (3) by R, Q and P respectively, and add.

$$\therefore P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0. \quad (4)$$

Therefore, if the equation $Pdx + Qdy + Rdz = 0$ has a solution $\phi(x, y, z) = c$, equation (4) must be satisfied.

Conversely, if equation (4) is satisfied, the equation

$$Pdx + Qdy + Rdz = 0$$

has a solution $\phi(x, y, z) = c$. The proof of this theorem is somewhat long and will not be given in this book.* The theorem however will be assumed in the subsequent work.

Definition. Equation (4) is called the **condition of integrability** of the equation $Pdx + Qdy + Rdz = 0$.

60. To solve the equation $Pdx + Qdy + Rdz = 0$ when the condition of integrability is satisfied.

Suppose at first that z is constant so that the equation becomes $Pdx + Qdy = 0$. Solve this equation. Suppose that the solution is $f(x, y, z) =$ a constant. Let $u = f(x, y, z)$. Find a quantity μ such that

$$\mu P = \frac{\partial u}{\partial x}.$$

$$\therefore \mu Q = \frac{\partial u}{\partial y}.$$

Multiply the equation $Pdx + Qdy + Rdz = 0$ by μ.

$$\therefore \mu(Pdx + Qdy + Rdz) = 0.$$

This equation may be written in the form $du + Sdz = 0$ where u and S are in general functions of x, y and z. In the equation $du + Sdz$, change the variables from x, y and z to x, u and z by means of the relation $u = f(x, y, z)$. The equation then be-

* For a proof of this theorem and also that S' of Art. 60 does not contain x, the student is referred to Forsyth, *A Treatise on Differential Equations*, Art. 152.

comes $du + S'dz = 0$. It can be shown that S' does not contain x. Assuming that it does not, the equation $du + S'dz$ can be integrated as an equation in u and z. The general solution of the equation is the general solution of the original equation.

EXAMPLE. Solve the equation

$$\frac{yz}{x^2+y^2}\,dx - \frac{xz}{x^2+y^2}\,dy - \tan^{-1}\frac{y}{x}\,dz = 0.$$

Suppose that z is constant. The equation then becomes

$$\frac{yz}{x^2+y^2}\,dx - \frac{xz}{x^2+y^2}\,dy = 0 \quad \text{or} \quad y\,dx - x\,dy = 0.$$

The solution of this equation is

$$\frac{x}{y} = \text{a constant}.$$

Let $u = \frac{x}{y}$.

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{y}.$$

Let $\mu P = \frac{1}{y}$.

$$\therefore \mu = \frac{x^2+y^2}{y^2 z}.$$

Multiply the original equation by $\frac{x^2+y^2}{y^2z}$.

$$\therefore \frac{1}{y}\,dx - \frac{x}{y^2}\,dy - \frac{x^2+y^2}{y^2z}\tan^{-1}\frac{y}{x}\,dz = 0.$$

Now

$$du = \frac{1}{y}\,dx - \frac{x}{y^2}\,dy.$$

$$\therefore du - \frac{x^2+y^2}{y^2z}\tan^{-1}\frac{y}{x}\,dz = 0.$$

Substitute

$$y = \frac{x}{u}$$

in this equation, y being derived from the equation

$$u = \frac{x}{y}.$$

$$\therefore du - \frac{u^2+1}{z}\tan^{-1}\frac{1}{u}\,dz = 0.$$

Separate the variables.

$$\therefore -\frac{du}{(u^2+1)\tan^{-1}\dfrac{1}{u}} + \frac{dz}{z} = 0.$$

Let $\tan^{-1}\dfrac{1}{u} = v.$

$$\therefore \frac{dv}{v} + \frac{dz}{z} = 0.$$

$$\therefore vz = c.$$

$$\therefore z\tan^{-1}\frac{y}{x} = c.$$

Therefore

$$z\tan^{-1}\frac{y}{x} = c$$

is the general solution of the original equation.

61. Suppose that in the equation

$$Pdx + Qdy + Rdz = 0$$

the condition of integrability is not satisfied. Then there is no relation $\phi(x, y, z) = c$ which satisfies the equation. In such a case a relation

$$\psi(x, y, z) = 0$$

is assumed arbitrarily and a relation $\phi(x, y, z) = c$ is sought which, together with $\psi(x, y, z) = 0$, will satisfy the equation. By differentiation of $\psi(x, y, z) = 0$ there results

$$\frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial z}dz = 0.$$

From this equation and (1) suppose that z and dz be eliminated. Then there will result an equation of the form $P'dx + Q'dy = 0$

where P' and Q' are functions of x and y the values of which depend upon $\psi(x, y, z)$. Suppose that a solution of this equation containing an arbitrary constant is found and is $\phi(x, y, z) = c$. Then this solution and $\psi(x, y, z) = 0$ together give a solution of the equation.

As an illustration consider the following example :

The equation

$$L\frac{di}{dt} + Ri = e,$$

considered in exercises 47 to 50 inclusive, Chapter III, for special cases of e, does not satisfy the condition of integrability if e, i and t are variables independent of each other. For, the equation may be written as

$$Ldi + (Ri - e)dt + 0 \cdot de = 0.$$

By application of the condition of integrability there results

$$L\left\{\frac{\partial(Ri-e)}{\partial e} - \frac{\partial o}{\partial t}\right\} + (Ri - e)\left(\frac{\partial o}{\partial i} - \frac{\partial L}{\partial e}\right) + 0\left\{\frac{\partial L}{\partial t} - \frac{\partial(Ri-e)}{\partial i}\right\} = 0,$$

or

$$-L = 0.$$

Since L is not zero, the equation does not satisfy the condition of integrability. Assume $e = f(t)$, however, and the equation becomes an ordinary linear differential equation of the first order. The solution is

$$i = \frac{e^{-\frac{R}{L}t}}{L}\int e^{\frac{R}{L}t}f(t)dt + ce^{-\frac{R}{L}t}.$$

From this solution the results of exercises 47 to 50 inclusive, Chapter III, may be found by substitution.

62. The cases considered thus far consisted of one equation in two dependent variables. Another important class of equations is the case of two total differential equations in two dependent variables where each equation is of the first degree with constant coefficients. The method of solution of this class of equations is as follows :

By differentiation and elimination, obtain one equation in one unknown. This equation may be solved by methods previously discussed. The solution found must be a solution of the original equations. Another solution is found by substituting the one just found in the equations. The complete solution consists of two linearly independent relations between the variables.

EXAMPLE. Solve the equations

$$\frac{dz}{dx} - 2y + z = 0. \tag{1}$$

$$\frac{dy}{dx} - 6y + 5z = 0. \tag{2}$$

Differentiate (2) with respect to x.

$$\therefore \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5\frac{dz}{dx} = 0. \tag{3}$$

Multiply (1) by -5, and add to (2) and (3).

$$\therefore \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0.$$

This is a linear differential equation of the second order with constant coefficients and right hand member zero. It can therefore be solved by the methods of Art. 31.

$$\therefore y = c_1e^x + c_2e^{4x}. \tag{4}$$

Substitute this value of y in (2) and solve for z.

$$\therefore z = c_1e^x + \frac{2c_2}{5}e^{4x}. \tag{5}$$

Equations (4) and (5) together constitute a set of solutions of the given equations.

EXERCISES

In each of the seven following equations, show that the condition of integrability is satisfied. Solve the equation.

1. $(y + z)dx + (z + x)dy + (x + y)dz = 0.$

2. $(2x^2y + 2xy^2 + 2xyz + 1)dx + (x^3 + x^2y + x^2z + 2xyz + 2y^2z + 2yz^2 + 1)dy + (xy^2 + y^3 + y^2z + 1)dz = 0$

3. $(2xy + z^2)dx + (x^2 + 2yz)dy + (y^2 + 2xz)dz = 0.$

4. $(a + z)ydx + (a + z)xdy - xydz = 0.$

5. $(y + b)(z + k)dx + (x + a)(z + k)dy + (x + a)(y + b)dz = 0.$

6. $(yz + 2x)dx + (xz + 2y)dy + (xy + 2z)dz = 0.$

7. $(2xyz + y^2z + yz^2)dx + (x^2z + 2xyz + xz^2)dy + (x^2y + xy^2 + 2xyz)dz = 0.$

Solve the following sets of equations.

8. $\frac{dz}{dx} + 7y - 3z = 0, \quad 7\frac{dy}{dx} + 63y - 36z = 0.$

9. $\frac{dz}{dx} + 2\frac{dy}{dx} + 3y = 0, \quad \frac{dy}{dx} + 3y - 2z = 0.$

10. $\frac{dy}{dx} + 3y + z = 0, \quad \frac{dz}{dx} + 3y + 5z = 0.$

11. $\frac{dy}{dx} + 3y + 2z = 0, \quad \frac{dz}{dx} + 2y - 4z = 0.$

12. $\frac{dy}{dx} - 3y - 2z = 0, \quad \frac{dz}{dx} + y - 2z = 0.$

13. $\frac{dy}{dx} + \frac{dz}{dx} + 6y = 0, \quad \frac{dz}{dx} + 5y + z = 0.$

14. $\frac{dz}{dx} + \frac{dy}{dx} + 5y - 3z = x + e^x, \quad \frac{dy}{dx} + 2y - z = e^x.$

15. $\frac{dz}{dx} + y + 3z = e^x, \quad \frac{dy}{dx} + 3y + 4z = e^{2x}.$

16. $\frac{dz}{dx} - 3y + 2z = e^x, \quad \frac{dy}{dx} + 2y - z = e^{3x}.$

17. $\frac{dz}{dx} + 5y - 2z = x, \quad \frac{dy}{dx} + 4y + z = x.$

18. $\frac{dz}{dx} + 7y - 9z = e^x, \quad \frac{dy}{dx} - y - 3z = e^{2x}.$

19. $\frac{dy}{dx} - 2y - 2z = e^{3x}, \quad \frac{dz}{dx} + 5y - 2z = e^{4x}.$

ANSWERS

1. $xy + yz + zx = c.$

2. $x^2y + y^2z + \log(x + y + z) = c.$

3. $x^2y + y^2z + z^2x = c.$ 4. $xy = c(a + z).$

5. $(x + a)(y + b)(z + k) = c.$ 6. $xyz + x^2 + y^2 + z^2 = c.$

7. $xyz(x + y + z) = c.$

8. $y = c_1e^{-3x} + c_2xe^{-3x}, \quad z = \frac{7c_1}{6}e^{-3x} + \frac{7c_2}{36}e^{-3x} + \frac{7c_2}{6}xe^{-3x}.$

9. $y = c_1e^{-x} + c_2e^{-6x}, \quad z = c_1e^{-x} - \frac{3c_2}{2}e^{-6x}.$

10. $y = c_1e^{-6x} + c_2e^{-2x}, \quad z = 3c_1e^{-6x} - c_2e^{-2x}.$

11. $y = c_1e^{\frac{1+\sqrt{65}}{2}x} + c_2e^{\frac{1+\sqrt{65}}{2}x},$

$$z = -\frac{c_1}{4}(7 + \sqrt{65})e^{\frac{1+\sqrt{65}}{2}x} - \frac{c_2}{4}(7 + \sqrt{65})e^{\frac{1+\sqrt{65}}{2}x}.$$

12. $y = c_1e^{\frac{5}{2}x}\cos\frac{\sqrt{7}}{2}x + c_2e^{\frac{5}{2}x}\sin\frac{\sqrt{7}}{2}x,$

$$z = \left(\frac{\sqrt{7}c_2}{4} - \frac{c_1}{4}\right)e^{\frac{5}{2}x}\cos\frac{\sqrt{7}}{2}x - \left(\frac{c_2}{4} + \frac{\sqrt{7}c_1}{4}\right)e^{\frac{5}{2}x}\sin\frac{\sqrt{7}}{2}x.$$

13. $y = c_1 e^{-x} \cos \sqrt{5}x + c_2 e^{-x} \sin \sqrt{5}x,$

$z = -c_1 \sqrt{5} e^{-x} \sin \sqrt{5}x + c_2 \sqrt{5} e^{-x} \cos \sqrt{5}x.$

14. $y = -x - \frac{1}{2}xe^x + c_1 e^x + c_2 e^{-x},$

$z = -1 - 2x - \frac{3}{2}e^x - \frac{3}{2}xe^x + 3c_1 e^x + c_2 e^{-x}.$

15. $y = \frac{5}{21}e^{2x} - \frac{1}{3}e^x + c_1 e^{-x} + c_2 e^{-5x},\quad z = -\frac{1}{21}e^{2x} + \frac{1}{3}e^x - \frac{c_1}{2}e^{-x} + \frac{c_2}{2}e^{-5x}.$

16. $y = \frac{5}{22}e^{3x} + \frac{1}{6}e^x + c_1 e^{-(2-\sqrt{3})x} + c_2 e^{-(2+\sqrt{3})x},$

$z = \frac{3}{22}e^{3x} + \frac{1}{2}e^x + c_1 \sqrt{3}\, e^{-(2-\sqrt{3})x} - c_2 \sqrt{3}\, e^{-(2+\sqrt{3})x}.$

17. $y = -\frac{7}{169} + \frac{3}{13}x + c_1 e^{-(1-\sqrt{14})x} + c_2 e^{-(1+\sqrt{14})x},$

$z = -\frac{11}{169} + \frac{1}{13}x - c_1(3 + \sqrt{14})e^{-(1-\sqrt{14})x}$

$- c_2(3 - \sqrt{14})e^{-(1+\sqrt{14})x}.$

18. $y = \frac{1}{7}e^x - \frac{1}{2}e^{2x} + c_1 e^{5x} \cos \sqrt{5}x + c_2 e^{5x} \sin \sqrt{5}x.$

$$z = -\frac{1}{2}e^{2x} + \left(\frac{4c_1}{3} + \frac{\sqrt{5}c_2}{3}\right)e^{5x} \cos \sqrt{5}x$$

$$+ \left(\frac{4c_2}{3} - \frac{\sqrt{5}c_1}{3}\right) e^{5x} \sin \sqrt{5}x.$$

19. $y = \frac{1}{11}e^{3x} + \frac{1}{7}e^{4x} + c_1 e^{2x} \cos \sqrt{10}x + c_2 e^{2x} \sin \sqrt{10}x.$

$$z = -\frac{5}{11}e^{3x} + \frac{1}{7}e^{4x} + \frac{\sqrt{10}c_2}{2} e^{2x} \cos \sqrt{10}x - \frac{\sqrt{10}c_1}{2} e^{2x} \sin \sqrt{10}x.$$

CHAPTER VIII

PARTIAL DIFFERENTIAL EQUATIONS.

63. So far we have considered differential equations in which there is only one independent variable. We shall now consider equations involving two independent variables. Such equations belong to the class called partial differential equations.

In this book, the independent variables will be denoted by x and y, and the dependent variable by z. The partial derivative of z with respect to x and with respect to y will be denoted by p and q, respectively.

Definition. A linear partial differential equation of the first order is an equation of the form

$$Pp + Qq = R,$$

where P, Q and R are functions of x, y and z, and do not contain p or q.

64. If there are two equations containing x, y and z, p and q, which can be solved for p and q, the result may be substituted in

$$dz = pdx + qdy$$

thus giving an ordinary differential equation. Usually, however, there is only a single differential equation given.

65. Derivation of a partial differential equation.

(*a*) By the elimination of constants. Let $\phi(x, y, z, c_1, c_2) = 0$ be a relation between x, y, z and two arbitrary constants c_1 and c_2. By differentiation of $\phi(x, y, z, c_1, c_2) = 0$ with respect to x holding y constant there results

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z} p = 0.$$

By differentiation with respect to y holding x constant there results

$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} q = 0.$$

By means of these two equations and $\phi(x, y, z, c_1, c_2) = 0$, c_1, and c_2 can be eliminated. The result is an equation

$$F(x, y, z, p, q) = 0$$

which is a partial differential equation of the first order.

EXAMPLE. Let $x^2 + y^2 + z^2 + c_1 x + c_2 y = 0$ be an equation between x, y and z, and two arbitrary constants c_1 and c_2. By differentiation with respect to x holding y constant there results

$$2x + c_1 + 2zp = 0.$$

By differentiation with respect to y holding x constant there results

$$2y + c_2 + 2zq = 0.$$

By elimination of c_1 and c_2 between the three equations there results

$$x^2 + y^2 - z^2 + 2xzp + 2yzq = 0.$$

This is a partial differential equation of the first order.

(b) By the elimination of an arbitrary function. Suppose that u and v are functions of the variables x, y and z, and that $\phi(u, v) = 0$ where $\phi(u, v)$ is an arbitrary function of u and v.

The differential of $\phi(u, v) = 0$ is

$$\frac{\partial \phi}{\partial u} du + \frac{\partial \phi}{\partial v} dv = 0.$$

Now

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial z} d_x z$$

when y is constant, and

$$du = \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} d_y z$$

when x is constant, and similarly for v.

Therefore the partial derivatives of the equation $\phi(u, v) = 0$ with respect to x and y, respectively, are

$$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right] + \frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right] = 0$$

and

$$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right] + \frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right] = 0.$$

Eliminate $\dfrac{\partial\phi}{\partial u}$ and $\dfrac{\partial\phi}{\partial v}$ from these equations.

$$\therefore \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right)\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right)\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right).$$

When arranged in powers of p and q and the coefficients expressed as determinants, the equation becomes

$$\begin{vmatrix} \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \end{vmatrix} p + \begin{vmatrix} \dfrac{\partial u}{\partial z} & \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial z} & \dfrac{\partial v}{\partial x} \end{vmatrix} q = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}.$$

This may be written in the form

$$Pp + Qq = R \tag{1}$$

where

$$\frac{P}{\begin{vmatrix} \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \end{vmatrix}} = \frac{Q}{\begin{vmatrix} \dfrac{\partial u}{\partial z} & \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial z} & \dfrac{\partial v}{\partial x} \end{vmatrix}} = \frac{R}{\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}}.$$

This is a partial differential equation of the first order. Therefore from the equation $\phi(u, v) = 0$ a partial linear differential equation of the first order can be formed which does not contain the arbitrary function $\phi(u, v)$.

EXAMPLE. Suppose that $u = x + y + z$ and $v = x^2 + y^2 + z^2$. Let $\phi(u, v) = 0$ be an equation connecting u and v where $\phi(u, v)$ is an arbitrary function of u and v.

By differentiation of $\phi(u, v) = 0$ with respect to x and with respect to y there result

$$\frac{\partial\phi}{\partial u}(1 + p) + \frac{\partial\phi}{\partial v}(2x + 2zp) = 0$$

and

$$\frac{\partial\phi}{\partial u}(1 + q) + \frac{\partial\phi}{\partial v}(2y + 2zq) = 0,$$

respectively. By elimination of $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$ from these equations there results

$$\begin{vmatrix} 1 & 1 \\ z & y \end{vmatrix} p + \begin{vmatrix} 1 & 1 \\ x & z \end{vmatrix} q = \begin{vmatrix} 1 & 1 \\ y & x \end{vmatrix},$$

or

$$(y - z)p + (z - x)q = x - y.$$

This is a partial linear differential equation of the first order which does not contain the arbitrary function

$$\phi(x + y + z, x^2 + y^2 + z^2).$$

66. We have seen that a differential equation with two independent variables can be derived from an expression containing two arbitrary constants or from an expression containing an arbitrary function of two independent functions of the variables. We see therefore that a differential equation with two independent variables may involve in its solutions, arbitrary constants or an arbitrary function of the variables.

Definitions. A relation between the variables of a differential equation with two independent variables which includes two arbitrary constants is called a **complete integral** of the equation.

A relation between the variables of a differential equation with two independent variables which involves an arbitrary function of two independent functions of these variables is called a **general integral** of the equation.

There is another class of solutions called **singular integrals** but these will not be considered here.

67. Consider the two equations $u = c_1$ and $v = c_2$ where u and v are functions of x, y and z, and c_1 and c_2 are arbitrary constants. By differentiation of $u = c_1$ and $v = c_2$, there result

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \qquad (1)$$

and

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0, \qquad (2)$$

respectively.

Multiply (1) by $\frac{\partial v}{\partial z}$, (2) by $\frac{\partial u}{\partial z}$, and subtract.

$$\therefore \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} dx - \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} dy = 0.$$

Multiply (1) by $\frac{\partial v}{\partial y}$, (2) by $\frac{\partial u}{\partial y}$, and subtract.

$$\therefore \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} dx - \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} dz = 0.$$

$$\therefore \frac{dx}{\begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix}} = \frac{dy}{\begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix}} = \frac{dz}{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}}.$$

Now $\phi(u, v) = 0$ is a general integral of the equation

$$Pp + Qq = R,$$

if

$$\frac{P}{\begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix}} = \frac{Q}{\begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix}} = \frac{R}{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}}.$$ See Art. 65.

Therefore $\phi(u, v) = 0$ is a general integral of the equation $Pp + Qq = R$ if $u = c_1$ and $v = c_2$ are solutions of the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

68. From the investigations of Arts. 65 and 67 the following rule for finding a general integral of the linear partial differential equation $Pp + Qq = R$ is determined.

Solve the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

Suppose that $u = c_1$ and $v = c_2$ are two independent integrals of these equations. Then $\phi(u, v) = 0$ where $\phi(u, v)$ is an arbitrary function of u and v is a general integral of the equation $Pp + Qq = R$.

Definition. The equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

are called the **subsidiary equations** of $Pp + Qq = R$. They are also sometimes called **Lagrange's equations.**

69. As illustrations of the method of solution of a linear partial differential equation of the first order, consider the following examples.

EXAMPLE 1. Solve the equation $x^2p + xyq + y^2 = 0$.

Write the subsidiary equations

$$\frac{dx}{x^2} = \frac{dy}{xy} = -\frac{dz}{y^2}.$$

Solve the equation

$$\frac{dx}{x^2} = \frac{dy}{xy}. \quad \therefore \frac{x}{y} = c_1.$$

Solve the equation

$$\frac{dy}{xy} = -\frac{dz}{y^2}.$$

From

$$\frac{x}{y} = c_1,$$

substitute the value of x, and the equation becomes

$$\frac{dy}{c_1} = -dz.$$

$$\therefore \frac{y}{c_1} = -z + c_2.$$

$$\therefore \frac{y^2}{x} + z = c_2.$$

A general integral of the original equation is therefore

$$\phi\left(\frac{x}{y}, \frac{y^2}{x} + z\right) = 0.$$

EXAMPLE 2. Solve the equation $(y - z)p + (z - x)q = x - y$. Write the subsidiary equations

$$\frac{dx}{y - z} = \frac{dy}{z - x} = \frac{dz}{x - y}.$$

From a familiar theorem of algebra, if

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f},$$

then $la + mc + ne = lb + md + nf$ where l, m and n are any multipliers whatsoever. Application of this theorem to the subsidiary equations gives

$$dx + dy + dz = 0, \tag{1}$$

when $l = m = n$, and

$$xdx + ydy + zdz = 0, \tag{2}$$

when $l = x$, $m = y$, $n = z$.

Solve equations (1) and (2). Therefore $x + y + z = c_1$ and $x^2 + y^2 + z^2 = c_2$ are solutions of equations (1) and (2), and therefore of the subsidiary equations. A general integral of the original equation is therefore $\phi(x + y + z, x^2 + y^2 + z^2) = 0$.

EXERCISES

Determine the partial differential equations of which the four following equations are complete solutions, c_1 and c_2 being arbitrary constants.

1. $z = c_1x + c_2y.$

2. $z^2 = c_1x^2 + c_2y^2.$

3. $z = (x + c_1)(y + c_2).$

4. $\dfrac{x^2}{c_1^{\,2}} + \dfrac{y^2}{c_2^{\,2}} + \dfrac{z^2}{a^2} = 1.$

Eliminate the arbitrary function from each of the four following equations.

5. $\phi(x+y-z,\ x^2+y^2-z^2)=0.$

6. $\phi(x + y + z,\ z) = 0.$

7. $z = e^y\phi(x + y).$

8. $z = f(x^2 + y^2).$

Find a general integral of each of the following equations.

9. $xzp - yzq = xy.$

10. $x^2p + y^2q - z^2 = 0.$

11. $x^2yp + yq = x^2z.$

12. $xp - yq = x - y.$

13. $(y^2 - z^2)p + (z^2 - x^2)q + (y^2 - x^2) = 0.$

14. $(2z - 3y)p + (3x - 4z)q = 4y - 2x.$

ANSWERS

1. $xp + yq = z.$

2. $xp + yq = z.$

3. $pq = z.$

4. $xzp + yzq - z^2 + a^2 = 0.$

5. $(y-z)p+(z-x)q=y-x.$

6. $p - q = 0.$

7. $p - q = - z.$

8. $yp - xq = 0.$

9. $\phi\left(xy,\ \log y + \dfrac{z^2}{2xy}\right) = 0.$

10. $\phi\left(\dfrac{1}{x} - \dfrac{1}{y},\ \dfrac{1}{y} - \dfrac{1}{z}\right) = 0.$

11. $\phi\left\{y + \dfrac{1}{x},\ \dfrac{x^2}{(xy+1)^2}\log xy + \dfrac{x^2}{xy+1} - \log z\right\} = 0.$

12. $\phi(xy,\ x + y - z) = 0.$

13. $\phi(x + y + z,\ x^3 + y^3 + z^3) = 0.$

14. $\phi(4x + 2y + 3z,\ x^2 + y^2 + z^2) = 0.$

CHAPTER IX

APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS. INTEGRATION IN SERIES

APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

70. It is shown in the Analytical Theory of Heat that the change of temperature in any solid at a point (x, y, z) within the solid is given by the equation

$$\frac{\partial u}{\partial t} = c^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right], \qquad (1)$$

where u represents the temperature at the point and t denotes time.

In polar or spherical coordinates the equation becomes

$$\frac{\partial u}{\partial t} = \frac{c^2}{r^2} \left[\frac{\partial \left(r^2 \frac{\partial u}{\partial r} \right)}{\partial r} + \frac{1}{\sin\theta} \frac{\partial \left(\sin\theta \frac{\partial u}{\partial \theta} \right)}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2 u}{\partial \phi^2} \right], \qquad (2)$$

and in cylindrical coordinates,

$$\frac{\partial u}{\partial t} = c^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} \right]. \qquad (3)$$

If the solid is a rectangular plate so thin that the thickness need not be taken into account, equation (1) becomes

$$\frac{\partial u}{\partial t} = c^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]. \qquad (4)$$

If the solid is a wire of infinite extent so thin that the breadth or thickness need not be taken into account, equation (1) becomes

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}. \qquad (5)$$

In the case of a sphere when the temperature u depends merely on the distance of the point from the center, equation (1), as can be seen from (2), reduces to

$$\frac{\partial (ru)}{\partial t} = c^2 \frac{\partial^2 (ru)}{\partial r^2}. \tag{6}$$

In the problem of permanent states of temperature, $\partial u/\partial t = 0$, and the equation becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, \tag{7}$$

an equation known as Laplace's Equation, and sometimes written $\nabla^2 u = 0$. This equation also figures in the Theory of Potential.

In polar or spherical coordinates, equation (7) becomes the right-hand member of equation (2) set equal to zero, and in cylindrical coordinates, it becomes the right-hand member of (3) set equal to zero.

In the Theory of Acoustics, in considering for instance the transverse vibrations of a stretched elastic string, there occurs the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{8}$$

and if the resistance of the air be taken into account, the equation

$$\frac{\partial^2 y}{\partial t^2} + 2k \frac{\partial y}{\partial t} = c^2 \frac{\partial^2 y}{\partial x^2}. \tag{9}$$

In the problem of the vibrations of a stretched elastic membrane, there occurs the equation

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right], \tag{10}$$

which in cylindrical coordinates becomes

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left[\frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \frac{\partial z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \phi^2} \right]. \tag{11}$$

71. As an illustration of transformation of coordinates, consider the transformation of Laplace's Equation in two dimensions,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

from rectangular to polar coordinates.

The equations of transformation are $x = r\cos\theta$, $y = r\sin\theta$.

Now u is a function of x and y and therefore of r and θ. Therefore, as seen in calculus,

$$du = \frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial \theta}d\theta.$$

If y is held constant, this equation becomes,

$$d_x u = \frac{\partial u}{\partial r}d_x r + \frac{\partial u}{\partial \theta}d_x\theta.$$

Divide by Δx, or what is the same, dx, and there results the equation

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}. \qquad (1)$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}. \qquad (2)$$

Since $x = r\cos\theta$ and $y = r\sin\theta$, therefore $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1} y/x$.

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} = \cos\theta,$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r} = \sin\theta,$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{y}{r^2} = -\frac{\sin\theta}{r},$$

and

$$\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{x}{r^2} = \frac{\cos\theta}{r}.$$

Now

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r},$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial r}\left[\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right]\cos\theta - \frac{\partial}{\partial \theta}\left[\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right]\frac{\sin\theta}{r}$$

$$= \left[\frac{\partial^2 u}{\partial r^2}\cos\theta - \frac{\partial^2 u}{\partial r\,\partial\theta}\frac{\sin\theta}{r} + \frac{\partial u}{\partial\theta}\frac{\sin\theta}{r^2}\right]\cos\theta$$

$$- \left[-\frac{\partial u}{\partial r}\sin\theta + \frac{\partial^2 u}{\partial r\,\partial\theta}\cos\theta - \frac{\partial^2 u}{\partial\theta^2}\frac{\sin\theta}{r} - \frac{\partial u}{\partial\theta}\frac{\cos\theta}{r}\right]\frac{\sin\theta}{r}.$$

Similarly

$$\frac{\partial^2 u}{\partial y^2} = \left[\frac{\partial^2 u}{\partial r^2}\sin\theta + \frac{\partial^2 u}{\partial r\,\partial\theta}\frac{\cos\theta}{r} - \frac{\partial u}{\partial\theta}\frac{\cos\theta}{r^2}\right]\sin\theta$$

$$+ \left[\frac{\partial^2 u}{\partial r\,\partial\theta}\sin\theta + \frac{\partial u}{\partial r}\cos\theta + \frac{\partial^2 u}{\partial\theta^2}\frac{\cos\theta}{r} - \frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right]\frac{\cos\theta}{r}.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}.$$

Therefore the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

in polar becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0.$$

72. A method of determining particular solutions of those of the above equations with constant coefficients is illustrated in the following example.

EXAMPLE. Find particular solutions of the equation

$$\frac{\partial^2 z}{\partial t^2} = c^2\left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right].$$

Assume that there is a particular solution in the form $z = e^{\alpha x+\beta y+\gamma t}$ where α, β and γ, are constants.

Substitute in the equation.

$$\therefore \gamma^2 e^{\alpha x+\beta y+\gamma t} = c^2(\alpha^2+\beta^2)e^{\alpha x+\beta y+\gamma t}.$$

Now $e^{\alpha x+\beta y+\gamma t}$ cannot be zero for any values of x, y and t.

$$\therefore \gamma^2 = c^2(\alpha^2+\beta^2).$$

Therefore $z = e^{ax+\beta y \pm c\sqrt{a^2+\beta^2}\,t}$ is a particular solution of the equation where a and β are arbitrary constants.

The above solution can be put into another form as follows: Let $a = aj$ and $\beta = \beta j$ where $j = \sqrt{-1}$.

Then

$$z = e^{(ax+\beta y \pm ct\sqrt{a^2+\beta^2})j}.$$

Therefore

$$z = \sin\,(ax + \beta y \pm ct\sqrt{a^2 + \beta^2}), \qquad (1)$$

and

$$z = \cos\,(ax + \beta y \pm ct\sqrt{a^2 + \beta^2}), \quad \text{(See Art. 5.)} \qquad (2)$$

are particular solutions of the equation.

From these can be found particular solutions in the forms

$$z = \sin ax \sin \beta y \sin ct\sqrt{a^2 + \beta^2},$$

$$z = \sin ax \sin \beta y \cos ct\sqrt{a^2 + \beta^2},$$

and six others. The determination of these six is left as an exercise to the student.

73. Consider the equation

$$\frac{\partial\left(r^2\dfrac{\partial u}{\partial r}\right)}{\partial r} + \frac{1}{\sin\theta}\,\frac{\partial\left(\sin\theta\dfrac{\partial u}{\partial\theta}\right)}{\partial\theta} = 0, \qquad (1)$$

which is Laplace's Equation in spherical coordinates where u is independent of ϕ.

Let $u = r^m P$, where P is a function of θ alone, and m is a positive integer. On substitution there results the equation

$$m(m+1)r^m P + \frac{r^m}{\sin\theta}\,\frac{\partial\left(\sin\theta\dfrac{\partial P}{\partial\theta}\right)}{\partial\theta} = 0.$$

Change the independent variable from θ to x where $x = \cos\theta$.

$$\therefore\ (1 - x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + m(m+1)P = 0. \qquad (2)$$

The solutions of equation (1) are known when P is determined from equation (2). Equation (2), not only when m is a positive integer but for all values of m, is called Legendre's Equation. Its solutions are discussed in Arts. 76, 85 and 86.

74. To find particular solutions of the equation

$$\frac{\partial^2 z}{\partial t^2} = c^2\left[\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r}\right], \tag{1}$$

which is equation (11), Art. 70, when z is independent of ϕ, let $z = R \cdot T$ where R is a function of r alone and T is a function of t alone. Substitute in the equation.

$$\therefore R\frac{d^2 T}{dt^2} = c^2 T\left[\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right],$$

or

$$\frac{1}{c^2 T}\frac{d^2 T}{dt^2} = \frac{1}{R}\left[\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right]. \tag{2}$$

The right-hand member of (2) does not involve t. Therefore the left-hand member does not. The left-hand member does not involve r. Therefore the right-hand member does not. Therefore each member is constant. Call the constant $-\mu^2$.

$$\therefore \frac{d^2 T}{dt^2} + \mu^2 c^2 T = 0, \tag{3}$$

and

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \mu^2 R = 0. \tag{4}$$

A particular solution of (1) is therefore $R \cdot T$ where T is determined by equation (3) and R by (4).

Particular solutions of equation (3) are $T = \cos \mu ct$ and $T = \sin \mu ct$. (See Art. 31.)

To solve equation (4), let $r = x/\mu$ and substitute in the equation.

$$\therefore \frac{d^2 R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + R = 0. \tag{5}$$

Equation (5) is a special case of the more general equation

$$x^2\frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0,$$

known as Bessel's Equation. Its solutions are considered in Arts. 79 to 84 inclusive.

INTEGRATION IN SERIES

75. It will be noted that as yet in this book no equations with variable coefficients, of higher order than the first, have been considered except a few very special cases discussed in Chaps. V and VI. The remainder of this chapter is devoted to a discussion of linear differential equations of the second order with coefficients rational integral functions of x, and second member zero. To such a set belong Legendre's and Bessel's Equations mentioned above.

Not all differential equations, not even all in the comparatively simple form of linear differential equations of the first order, are capable of solution in finite form. When solutions cannot be found in finite form, recourse is had to integration in series. In the set about to be considered, some equations have solutions in finite form and some have not.

We shall attempt here to find solutions only in the form of infinite, convergent, power series.

If an equation be capable of solution in finite form, this form is found when a solution is attempted in the form of a power series. For instance, in exercise 11, page 120, the solution found as if it were made up of infinite series is in reality in finite form.

Sometimes the series that make up the solution of an equation may be recognized as those of familiar functions. In such cases, the solution can be written in terms of those functions. For instance, in the answer given on page 122 for exercise 12, page 120, if A be taken equal to 2 and B to 1, and the two particular solutions be added, there results the series

$$x^{-2}\left[1+(2x)+\frac{(2x)^2}{\lfloor 2}+\frac{(2x)^3}{\lfloor 3}+\cdots+\frac{(2x)^n}{\lfloor n}+\cdots\right],$$

which is $x^{-2}e^{2x}$. If the second solution be subtracted from the first, there results the series which is $x^{-2}e^{-2x}$. The general solution is therefore

$$y = c_1 x^{-2} e^{2x} + c_2 x^{-2} e^{-2x}.$$

76. Let us attempt to find a solution of Legendre's Equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + m(m+1)y = 0,$$

in the form of a power series in x.

At first, assume that there is a power series

$$y = g_0 x^\kappa + g_1 x^{\kappa+1} + \cdots + g_r x^{\kappa+r} + \cdots \equiv \sum_{\nu=0}^{\nu=\infty} g_\nu x^{\kappa+\nu}$$

where g_0, g_1, $\cdots$, κ are constants, which will formally, i. e., without regard to whether the series converges or not, satisfy the equation. It is no restriction to assume, as we shall, that $g_0 \neq 0$, because, if there is any solution at all, one at least of the g's is not zero, and we assume that the series begins with the term containing the first g which does not vanish.

Since

$$y = \sum_{\nu=0}^{\nu=\infty} g_\nu x^{\kappa+\nu},$$

$$\therefore \frac{dy}{dx} = \sum_{\nu=0}^{\nu=\infty} (\kappa+\nu) g_\nu x^{\kappa+\nu-1},$$

and

$$\frac{d^2y}{dx^2} = \sum_{\nu=0}^{\nu=\infty} (\kappa+\nu)(\kappa+\nu-1) g_\nu x^{\kappa+\nu-2}.$$

Substitute in the equation.

$$\therefore \sum_{\nu=0}^{\nu=\infty} [(1-x^2)(\kappa+\nu)(\kappa+\nu-1) g_\nu x^{\kappa+\nu-2} - 2x(\kappa+\nu) g_\nu x^{\kappa+\nu-1} + m(m+1) g_\nu x^{\kappa+\nu}] = 0,$$

or

$$\sum_{\nu=0}^{\nu=\infty} [(\kappa+\nu)(\kappa+\nu-1) g_\nu x^{\kappa+\nu-2} - \{(\kappa+\nu)(\kappa+\nu+1) - m(m+1)\} g_\nu x^{\kappa+\nu}] = 0. \qquad (1)$$

If

$$y = \sum_{\nu=0}^{\nu=\infty} g_\nu x^{\kappa+\nu}$$

is to satisfy the equation, the coefficients of each power of x in (1) must be zero. Therefore there results the following series of equations:

$$\kappa(\kappa-1)g_0 = 0, \quad (\kappa+1)\kappa g_1 = 0,$$

$$(\kappa+2)(\kappa+1)g_2 - \{\kappa(\kappa+1) - m(m+1)\}g_0 = 0,$$

$$(\kappa+3)(\kappa+2)g_3 - \{(\kappa+1)(\kappa+2) - m(m+1)\}g_1 = 0,$$

$$\cdots \cdots \cdots \cdots \qquad (2)$$

$$(\kappa+2r)(\kappa+2r-1)g_{2r} - \{(\kappa+2r-2)(\kappa+2r-1) - m(m+1)\}g_{2r-2} = 0.$$

From the first of these equations, since $g_0 \neq 0$, therefore $\kappa = 0$, or $\kappa = 1$. At first, take $\kappa = 1$.

Substitute in equation (2) and calculate the g's in succession.

$$g_1 = 0, \quad g_2 = -\frac{(m-1)(m+2)}{\lfloor 3}g_0, \quad g_3 = 0,$$

$$g_4 = \frac{(m-1)(m-3)(m+2)(m+4)}{\lfloor 5}g_0, \quad \cdots, \quad g_{2r-1} = 0,$$

$$g_{2r} = (-1)^r \times \frac{(m-1)(m-3)\cdots(m-2r+1)(m+2)(m+4)\cdots(m+2r)}{\lfloor 2r+1}g_0,$$

$$\cdots \cdots \cdots \cdots$$

$$\therefore y = g_0 x\left[1 - \frac{(m-1)(m+2)}{\lfloor 3}x^2 + \frac{(m-1)(m-3)(m+2)(m+4)}{\lfloor 5}x^4 + \cdots + \frac{g_{2r}}{g_0}x^{2r} + \cdots\right], \qquad (3)$$

where g_0 is arbitrary, and g_{2r} has the value given above, formally satisfies the equation. Since this series is convergent, (3) is a particular solution of the equation.

Next, take $\kappa = 0$.

Substitute in equations (2). g_1 is arbitrary. Call it zero.

$$g_2 = -\frac{m(m+1)}{\lfloor 2} g_0,\ g_3 = 0,\ g_4 = \frac{m(m-2)(m+1)(m+3)}{\lfloor 4} g_0,$$

$$\cdots,\ g_{2r-1} = 0,$$

$$g_{2r} = (-1)^r \frac{m(m-2)\cdots(m-2r+2)(m+1)\cdots(m+2r-1)}{\lfloor 2r} g_0,$$

. . . .

$$\therefore y = g_0\left[1 - \frac{m(m+1)}{\lfloor 2} x^2 + \frac{m(m-2)(m+1)(m+3)}{\lfloor 4} x^4 - \cdots + \frac{\overline{g}_{2r}}{g_0} x^{2r} + \cdots\right] \quad (4)$$

where g_0 is arbitrary, and $\overline{g}_{2r}$ has the value given above, formally satisfies the equation. Since the series is convergent, (4) is a particular solution of the equation.

If solution (3) be denoted by y_1 and solution (4) by y_2, the general solution of the equation is $y = Ay_1 + By_2$ where A and B are arbitrary constants. (See Art. 11.)

77. The general form of a linear differential equation of the second order with right hand member zero is

$$q_0(x)\frac{d^2y}{dx^2} + q_1(x)\frac{dy}{dx} + q_2(x)\cdot y = 0. \quad (1)$$

It will be assumed here that $q_0(x)$, $q_1(x)$, $q_2(x)$ are rational integral functions of x.

If a solution is to be found in the form of a power series in $x - a$, it will be convenient to write the equation in the form

$$(x-a)^2 p_0(x)\frac{d^2y}{dx^2} + (x-a)p_1(x)\frac{dy}{dx} + p_2(x)\cdot y = 0, \quad (2)$$

where $p_0(x)$, $p_1(x)$, $p_2(x)$ are rational integral functions of x. The equation can be written in this form in an unlimited number of ways by multiplying it through by a suitable power of $x - a$ and a rational fraction neither the numerator nor denominator of which contains $x - a$.

Definition. The point a is a **regular point** of equation (2) if $p_0(a) \neq 0$.

Without at first making any assumption with regard to the point a, substitute

$$y = \sum_{\nu=0}^{\nu=\infty} g_\nu (x-a)^{\kappa+\nu}$$

in equation (2) and attempt to determine the g's so that the equation is formally satisfied.

$$\sum_{\nu=0}^{\nu=\infty} \Big[(\kappa+\nu)(\kappa+\nu-1)p_0(x) + (\kappa+\nu)\,p_1(x) + p_2(x) \Big] g_\nu (x-a)^{\kappa+\nu} = 0. \quad (3)$$

Call the expression in square brackets $f(x, \kappa+\nu)$.

Develop $f(x, \kappa+\nu)$ into a power series in $x-a$ by Taylor's Theorem.

$$\therefore f(x, \kappa+\nu) = f(a, \kappa+\nu) + f'(a, \kappa+\nu)\frac{x-a}{\lfloor 1} + \cdots + f^{(n)}(a, \kappa+\nu)\frac{(x-a)^n}{\lfloor n}.$$

Substitute this development in (3), equate each power of $x-a$ to zero and there results the following series of equations:

$$\begin{aligned} &g_0 f(a, \kappa) = 0, \\ &g_1 f(a, \kappa+1) + g_0 f'(a, \kappa) = 0, \\ &g_2 f(a, \kappa+2) + g_1 f'(a, \kappa+1) + g_0 \frac{f''(a, \kappa)}{\lfloor 2} = 0, \quad (4) \\ &\cdots\cdots\cdots\cdots \\ &g_n f(a, \kappa+n) + g_{n-1} f'(a, \kappa+n-1) + \cdots + g_0 \frac{f^{(n)}(a, \kappa)}{\lfloor n} = 0, \\ &g_{n+1} f(a, \kappa+n+1) + g_n f'(a, \kappa+n) + \cdots + g_1 \frac{f^{(n)}(a, \kappa+1)}{\lfloor n+1} = 0, \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

Now $g_0 \neq 0$. Therefore $f(a, \kappa) = 0$.

And $\qquad f(a, \kappa) \equiv \kappa(\kappa-1)p_0(a) + \kappa p_1(a) + p_2(a).$

$$\therefore \kappa(\kappa - 1)p_0(a) + \kappa p_1(a) + p_2(a) = 0.$$

From this equation can be determined the value or values of κ which are to be used in the subsequent equations (4). If $p_0(a)$ is not zero, the equation is of the second degree. If $p_0(a)$ is zero, the equation is of lower degree than the second.

The necessary and sufficient condition that the equation

$$\kappa(\kappa - 1)p_0(a) + \kappa p_1(a) + p_2(a) = 0$$

is of the second degree is therefore that the point a be a regular point of the differential equation.

Definition. The equation $\kappa(\kappa - 1)p_0(a) + \kappa p_1(a) + p_2(a) = 0$ is called the **indicial equation** of the differential equation (2).

If the point a is regular, the indicial equation gives two values of κ, say κ' and κ'', and from equations (4), for either value of κ, the values of g_1, g_2, $\cdots$, may be computed, in general, in terms of g_0.

Therefore in general there are two series in ascending powers of $x - a$, namely,

$$y = \sum_{\nu=0}^{\nu=\infty} g_\nu (x - a)^{\kappa' + \nu},$$

and

$$y = \sum_{\nu=0}^{\nu=\infty} g_\nu (x - a)^{\kappa'' + \nu},$$

where g_0 is arbitrary in either series, which will formally satisfy equation (2).

78. The following theorems with regard to the solutions of the differential equation

$$(x - a)^2 p_0(x)\frac{d^2y}{dx^2} + (x - a)p_1(x)\frac{dy}{dx} + p_2(x) \cdot y = 0$$

in a power series in $x - a$ have been established. The proofs are too long to be given in this book. For a discussion of these theorems the student is referred to a pamphlet entitled "Regular Points of Linear Differential Equations of the Second Order" by Professor Maxime Bôcher, published by Harvard University.

Theorem I. If a is a regular point of the differential equation, and the difference of the roots of the indicial equation is not zero or a positive integer, two solutions in the form of a power series in $x - a$, viz.,

$$y = \sum_{\nu=0}^{\nu=\infty} g_\nu (x - a)^{\kappa'+\nu},$$

and

$$y = \sum_{\nu=0}^{\nu=\infty} g_\nu (x - a)^{\kappa''+\nu},$$

where κ' and κ'' are the roots of the indicial equation, exist, and these series are convergent. In each of these series g_0 is arbitrary.

In this case, if y_1 denotes one of the series and y_2 the other, the general solution of the equation is $y = Ay_1 + By_2$ where A and B are arbitrary constants.

A case to which this theorem applies is Bessel's equation when n is not zero nor an integer, discussed in Art. 80.

Note. By the difference of the roots of the indicial equation being a positive integer is meant that the greater minus the less is a positive integer.

Theorem II. If a is a regular point of the equation and the difference of the roots of the indicial equation is a positive integer n, the necessary and sufficient condition that two solutions of the form under Theorem I exist is that

$$g_{n-1} f'(a, \kappa'' + n - 1) + \cdots + g_0 \frac{f^{(n)}(a, \kappa'')}{\underline{|n}} = 0,$$

(see equations (4), Art. 77), where κ'' is the smaller of the roots, and when this condition is fulfilled, the series are convergent.

In this case the series corresponding to the larger value of κ, say κ', can be found as before. In the series corresponding to κ'', g_0 and g_1 are arbitrary, but if g_1 be chosen zero a particular solution in terms of g_0 is found. Then if y_1 denotes the first series, and y_2 the second, the general solution of the equation is $y = Ay_1 + By_2$ where A and B are arbitrary constants.

A case to which this theorem applies is Legendre's Equation discussed in Art. 76.

Theorem III. If a is a regular point of the equation, and the difference of the roots of the indicial equation is either zero, or a positive integer n where

$$g_{n-1}f'(a, \kappa'' + n - 1) + \cdots + g_0 \frac{f^{(n)}(a, \kappa'')}{\underline{|n}} \neq 0,$$

two solutions are found, one being

$$y = \sum_{\nu=0}^{\nu=\infty} g_\nu (x-a)^{\kappa'+\nu},$$

the other being

$$y = \log(x-a) \sum_{\nu=0}^{\nu=\infty} g_\nu (x-a)^{\kappa'+\nu} + \sum_{\nu=0}^{\nu=\infty} \bar{g}_\nu (x-a)^{\kappa''+\nu},$$

where κ' is the larger root of the indicial equation and κ'' the smaller, and these series are convergent. In the first of these series g_0 is arbitrary. In the second, g_0 is arbitrary and $\bar{g}_\nu$ is determined in terms of g_0.

If y_1 denotes the first series and y_2 the last term of the second, the general solution of the equation is

$$y = [A + B\log(x-a)]\, y_1 + By_2,$$

where A and B are arbitrary constants.

A case to which this theorem applies is Bessel's Equation when $n = 0$, or an integer, discussed in Arts. 82 and 83.

Theorem IV. If the point a is not a regular point of the equation there are not two solutions of the equation in any of the forms under Theorems I and III, and if any series in one of these forms is found it is usually not convergent.

Cases to which this theorem would apply will not be considered in this book.

BESSEL'S EQUATION

79. We shall now consider the solutions of the equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0,$$

in the form of a power series in x.

The equation as it stands is in the form (2) of Art. 77, where

$$p_0(x) = 1, \quad p_1(x) = 1, \quad p_2(x) = x^2 - n^2.$$

Since $p_0(x)$ cannot be zero for any value of x, all points of this equation are regular. Therefore the solutions of the equation for all values of x, and in particular when $x = 0$, will come under one or other of the forms mentioned in the first three theorems of Art. 78.

Substitute

$$y = \sum_{\nu=0}^{\nu=\infty} g_\nu x^{\kappa+\nu}$$

in the equation.

$$\therefore \sum_{\nu=0}^{\nu=\infty} [(\kappa+\nu)(\kappa+\nu-1) + (\kappa+\nu) + (x^2 - n^2)] g_\nu x^{\kappa+\nu} = 0,$$

or

$$\sum_{\nu=0}^{\nu=\infty} [(\kappa+\nu)^2 + x^2 - n^2] g_\nu x^{\kappa+\nu} = 0.$$

The equations for the determination of the g's are therefore :

$$\begin{aligned}
(\kappa^2 - n^2) g_0 &= 0, \\
[(\kappa+1)^2 - n^2] g_1 &= 0, \\
[(\kappa+2)^2 - n^2] g_2 + g_0 &= 0, \\
[(\kappa+3)^2 - n^2] g_3 &= 0, \qquad (1) \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot \\
[(\kappa+2r-1)^2 - n^2] g_{2r-1} &= 0, \\
[(\kappa+2r)^2 - n^2] g_{2r} + g_{2r-2} &= 0, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot
\end{aligned}$$

Since $g_0 \neq 0$, from the first equation there results $\kappa = \pm n$. The difference of the roots of the indicial equation is therefore $\pm 2n$. If $n = 0$, this difference is zero. If $n = \pm \frac{p}{2}$, where p is an odd integer, or if n is an integer, this difference is a positive integer. If n is neither zero nor $\pm \frac{p}{2}$ nor an integer, the difference is neither zero nor a positive integer.

80. At first assume n neither zero nor $\pm \frac{p}{2}$ nor an integer.

This is the case covered by Theorem I. There are therefore two solutions

$$y = \sum_{\nu=0}^{\nu=\infty} g_\nu x^{n+\nu}$$

and

$$y = \sum_{\nu=0}^{\nu=\infty} g_\nu x^{-n+\nu}.$$

To determine the g_ν substitute $\kappa = n$ in equations (1).

$$\therefore g_1 = 0, \quad g_2 = -\frac{g_0}{2(2n+2)}, \quad g_3 = 0,$$

$$g_4 = -\frac{g_2}{4(2n+4)} = \frac{g_0}{2\cdot 4(2n+2)(2n+4)},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$g_{2r-1} = 0,$$

$$g_{2r} = (-1)^r \frac{g_0}{2\cdot 4\cdots 2r(2n+2)(2n+4)\cdots(2n+2r)}.$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\therefore y = g_0 x^n \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} - \cdots + \frac{g_{2r}}{g_0} x^{2r} + \cdots \right] \quad (2)$$

where g_0 is arbitrary and g_{2r} has the value given above is a particular solution of the equation.

Similarly, on substituting $\kappa = -n$ in equations (1) there results

$$y = g_0 x^{-n} \left[1 + \frac{x^2}{2(2n-2)} + \frac{x^4}{2\cdot 4(2n-2)(2n-4)} + \cdots + \frac{g_{2r}}{g_0} x^{2r} + \cdots \right] \quad (3)$$

where g_0 is arbitrary and

$$g_{2r} = \frac{g_0}{2\cdot 4\cdots 2r(2n-2)(2n-4)\cdots(2n-2r)},$$

is a particular solution of the equation.

If y_1 denotes the first series and y_2 the second, the general solution of the equation is $y = Ay_1 + By_2$ where A and B are arbitrary constants.

81. Next, assume $n = \pm\frac{p}{2}$. Assume, for definiteness, that p is positive. In this case the difference of the roots of the indicial equation is a positive integer, viz. p. From an examination of equations (1) it is seen when $n = -\frac{p}{2}$ that both g_0 and g_p are arbitrary. Choose $g_p = 0$, and there results the same equation as (3) of the preceding article when $+\frac{p}{2}$ is substituted for n.

Therefore in this case there are two particular solutions of the equation which are the same as the solutions in the case of the preceding article when $\frac{p}{2}$ is substituted for n.

82. Next, assume n an integer. Since n appears only in the form of a square in the differential equation, it is sufficient to suppose it a positive integer.

In this case the difference of the roots of the indicial equation is the positive integer $2n$.

For the root $\kappa' = n$, the series is the same as (2) of Art. 80.

For the root $\kappa'' = -n$, the equation

$$[(-n+2n)^2 - n^2]g_{2n} + g_{2n-2} = 0$$

is such that the coefficient of g_{2n} is zero. Therefore, since $g_{2n-2} \neq 0$, this case comes under that mentioned in Theorem III, Art. 78.

To get a solution corresponding to κ'', let

$$y = \log x \sum_{\nu=0}^{\nu=\infty} g_\nu x^{\kappa'+\nu} + \sum_{\nu=0}^{\nu=\infty} \bar{g}_\nu x^{\kappa''+\nu}.$$

For the purpose of determining the coefficients $\bar{g}_\nu$, write the series in the form

$$y = \sum_{\nu=0}^{\nu=\infty} (g_{\nu-2n} \log x + \bar{g}_\nu) x^{\kappa''+\nu},$$

where $g_{-2n} = g_{-2n+1} = \cdots = g_{-1} = 0$.

Substitute in the equation.

$$\therefore \sum_{\nu=0}^{\nu=\infty} [(\kappa''+\nu)^2 + x^2 - n^2] g_{\nu-2n} \log x \cdot x^{\kappa''+\nu}$$
$$+\{2(\kappa''+\nu)g_{\nu-2n} + [(\kappa''+\nu)^2+x^2-n^2]\bar{g}_\nu\}x^{\kappa''+\nu} = 0. \quad (1)$$

Now in the coefficients of $\log x \cdot x^{\kappa''+\nu}$, g_{-2n}, g_{-2n+1}, $\cdots$, g_{-1}, are zero, and the remaining ones are the same as the left hand members of equations (1), Art. 79, with $\kappa'' + 2n$ substituted for κ. Now $\kappa'' + 2n = \kappa'$, and equations (1), Art. 79, hold for $\kappa = \kappa'$. Therefore all the coefficients of $\log x \cdot x^{\kappa''+\nu}$ vanish.

From the second set of terms in (1) are found the equations from which to determine $\bar{g}_\nu$. These equations are:

$$(\kappa''^2 - n^2)\bar{g}_0 = 0,$$
$$[(\kappa''+1)^2 - n^2]\bar{g}_1 = 0,$$
$$[(\kappa''+2)^2 - n^2]\bar{g}_2 + \bar{g}_0 = 0,$$
$$\cdots\cdots\cdots$$
$$[(\kappa''+2n)^2 - n^2]\bar{g}_{2n} + \bar{g}_{2n-2} + 2(\kappa''+2n)g_0 = 0,$$
$$[(\kappa''+2n+1)^2 - n^2]\bar{g}_{2n+1} + 2(\kappa''+2n+1)g_1 = 0,$$
$$[(\kappa''+2n+2)^2 - n^2]\bar{g}_{2n+2} + \bar{g}_{2n} + 2(\kappa''+2n+2)g_2 = 0,$$
$$\cdots\cdots\cdots\cdots\cdots$$

Since in these equations, $\kappa'' = -n$, therefore $\bar{g}_0$, an arbitrary constant, satisfies the first equation. Also,

$$\bar{g}_1 = 0, \quad \bar{g}_2 = \frac{\bar{g}_0}{2(2n-2)},$$

$$\bar{g}_3 = 0, \quad \bar{g}_4 = \frac{\bar{g}_0}{2\cdot 4(2n-2)(2n-4)}, \quad \cdots,$$

$$\bar{g}_{2r-1} = 0, \quad r<n, \quad \bar{g}_{2r} = \frac{\bar{g}_0}{2\cdot 4\cdots 2r(2n-2)(2n-4)\cdots(2n-2r)}.$$

Since the coefficient of $\bar{g}_{2n}$ is zero, this equation introduces no new $\bar{g}$. The equation, however, gives a means of determining the hitherto undetermined constant $\bar{g}_0$,

$$\bar{g}_{2n-2} = -2ng_0.$$

Also

$$\bar{g}_{2n-2} = \frac{\bar{g}_0}{2^{2n-2}(\lfloor n-1)^2}.$$

$$\therefore \bar{g}_0 = -2^{2n-1}\lfloor n \lfloor n-1\, g_0,$$

$$\bar{g}_1 = 0,\ \bar{g}_2 = \frac{\bar{g}_0}{2(2n-2)},\ \cdots,\ \bar{g}_{2n-2} = \frac{\bar{g}_0}{\{2\cdot 4\cdot 6\cdots(2n-2)\}^2},$$

where

$$\bar{g}_0 = -2^{2n-1}\lfloor n \lfloor n-1\, g_0.$$

Choose $\bar{g}_{2n} = 0$. Therefore

$$\bar{g}_{2n+1} = 0,$$

$$\bar{g}_{2n+2} = -\frac{(n+2)g_2}{2n+2} = \frac{(n+2)g_0}{2(2n+2)^2} = \frac{1}{2(2n+2)}\left\{\frac{1}{2} + \frac{1}{2n+2}\right\}g_0 \cdots,$$

$$\bar{g}_{2n+2r-1} = 0,$$

$$\bar{g}_{2n+2r} = (-1)^{r-1}\frac{1}{2\cdot 4\cdots 2r(2n+2)(2n+4)\cdots(2n+2r)}g_0 \times \sum_{r=1}^{r=r}\left(\frac{1}{2r} + \frac{1}{2n+2r}\right),$$

. . . .

$$\therefore y = \log x \sum_{\nu=0}^{\nu=\infty} g_\nu x^{\kappa'+\nu} - 2^{2n-1}\lfloor n \lfloor n-1\, g_0 x^{-n}\left[1 + \frac{x^2}{2(2n-2)} + \frac{x^4}{2\cdot 4(2n-2)(2n-4)} + \cdots + \frac{x^{2n-2}}{\{2\cdot 4\cdot 6\cdots(2n-2)\}^2}\right]$$

$$+ \frac{x^{n+2}}{2(2n+2)}\left\{\frac{1}{2} + \frac{1}{2n+2}\right\} + \cdots + \bar{g}_{2n+2r}x^{n+2r} + \cdots,$$

where g_0 is arbitrary and $\bar{g}_{2n+2r}$ has the value given above, is a particular solution of the equation

If the first solution be denoted by y_1 and all terms not involving $\log x$ in the second by y_2, the general solution of the equation is $y = (A + B\log x)y_1 + By_2$ where A and B are arbitrary constants.

83. Next, assume $n = 0$. In this case the difference of the roots of the indicial equation is zero. This case is covered by Theorem III.

The two infinite series when $n = 0$ can be found from equations derived as in the preceding case. They can also be found by letting n be zero in the results in that case. Two particular solutions are

$$y_1 = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \cdots + (-1)^r \frac{x^{2r}}{2^2 \cdot 4^2 \cdots (2r)^2} \cdots$$

and

$$y_2 = y_1 \log x + \frac{x^2}{2^2} - (1 + \tfrac{1}{2}) \frac{x^4}{2^2 \cdot 4^2} + \cdots + (-1)^{r-1} \frac{x^{2r}}{2^2 \cdot 4^2 \cdots (2r)^2} \sum_{r=1}^{r=r} \frac{1}{r}.$$

84. As will appear in applications to physical problems, when n is a positive integer it is convenient to take, not y_1 and y_2, but the quotients of these by $2^n \underline{|n}$ where g_0 is unity. These special solutions are written $J_n(x)$ and $W_n(x)$ so that

$$J_n(x) = \frac{x^n}{2^n \underline{|n}} \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4(2n+2)(2n+4)} \cdots \right]$$

$$= \sum_{r=0}^{r=\infty} \frac{(-1)^r x^{n+2r}}{2^{n+2r} \underline{|r} \, \underline{|n+r}},$$

and

$$W_n(x) = J_n(x) \log x - \left\{ 2^{n-1} \underline{|n-1}\, x^{-n} + \frac{2^{n-3} \underline{|n-2}\, x^{-n+2}}{\underline{|1}} \right.$$

$$\left. + \frac{2^{n-5} \underline{|n-3}\, x^{-n+4}}{\underline{|2}} + \cdots + \frac{x^{n-2}}{2^{n-1} \underline{|n-1}} \right\}$$

$$+ \sum_{r=1}^{r=\infty} \left\{ \frac{(-1)^{r-1} x^{n+2r}}{2^{n+2r} \underline{|r} \, \underline{|n+r}} \sum_{r=1}^{r=r} \left(\frac{1}{2r} + \frac{1}{2n+2r} \right) \right\}.$$

LEGENDRE'S EQUATION

85. Returning now to the equation

$$(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + m(m+1) y = 0,$$

considered in Art. 76, we see that it can be transformed into the form (2) of Art. 77 by multiplying through by x^2. The transformed equation is

$$x^2(1-x^2)\frac{d^2y}{dx^2} - 2x^3\frac{dy}{dx} + m(m+1)x^2y = 0,$$

where $p_0(x) = 1 - x^2$, $p_1(x) = -2x^2$, $p_2(x) = m(m+1)x^2$. Since $p_0(x) = 0$ when $x = -1$ and $x = 1$, the points -1 and 1 are not regular points of the equation. All other points are regular.

The indicial equation when $x = 0$ is

$$\kappa(\kappa - 1) = 0.$$

This equation has the roots 0 and 1. The differential equation in this case comes under the case mentioned in Theorem II. This case was already discussed in Art. 76.

86. It is convenient when m is a positive integer as it was in the illustration of Art. 73 to have a solution of Legendre's Equation as a series in descending powers of x. In this case we shall take the equation, as in Art. 73, in the form

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + m(m+1)P = 0.$$

Let

$$P = \sum_{\nu=0}^{\nu=\infty} g_\nu x^{n-\nu}$$

and substitute in the equation.

$$\therefore \sum_{\nu=0}^{\nu=\infty} (n-\nu)(n-\nu-1)g_\nu x^{n-\nu-2} - [(n-\nu)(n-\nu+1) - m(m+1)]g_\nu x^{n-\nu} = 0.$$

$$\therefore [n(n+1) - m(m+1)]g_0 = 0,$$

$$[(n-1)n - m(m+1)]g_1 = 0,$$

$$n(n-1)g_0 - [(n-2)(n-1) - m(m+1)]g_2 = 0,$$

. . . .

$$(n-2r-2)(n-2r-3)g_{2r-2} - [(n-2r)(n-2r+1) - m(m+1)]g_{2r} = 0.$$

. . . .

From the first equation, since $g_0 \neq 0$,

$$\therefore n(n+1) - m(m+1) = 0.$$

$$\therefore n = m \quad \text{or} \quad n = -m-1.$$

At first take $n = m$.

$$\therefore g_1 = 0, \quad g_2 = -\frac{(m-1)m}{2(2m-1)} g_0, \quad g_3 = 0, \quad \cdots, \quad g_{2r-1} = 0,$$

$$g_{2r} = (-1)^r \frac{(m-2r-3)(m-2r-2)\cdots(m-1)m}{2r\cdots 4\cdot 2(2m-2r+1)\cdots(2m-1)} g_0.$$

. . . .

A series in descending powers of x which satisfies the equation is therefore

$$P = g_0 x^m \left[1 - \frac{(m-1)m}{2(2m-1)} x^{-2} + \frac{(m-3)(m-2)(m-1)m}{4\cdot 2(2m-3)(2m-1)} x^{-4} + \cdots + \frac{g_{2r}}{g_0} x^{2r} + \cdots\right] \quad (1)$$

where g_0 is arbitrary and g_{2r} has the value given above.

By taking $n = -m-1$, there results the solution

$$P = g_0 x^{-m-1} \left[1 + \frac{(m+2)(m+1)}{2(2m+3)} x^{-2} + \frac{(m+4)(m+3)(m+2)(m+1)}{4\cdot 2(2m+3)(2m+5)} x^{-4} + \cdots + \frac{g_{2r}}{g_0} x^{2r}\right] \quad (2)$$

where g_0 is arbitrary and

$$g_{2r} = \frac{(m+2r)(m+2r-1)\cdots(m+1)}{2r\cdots 2(2m+3)\cdots(2m+2r+1)} g_0.$$

When m is a positive integer, solutions (3) and (4) of Art. 76 is a finite series according as m is even or odd, and in either case, equation (1) above is finite differing from (3) or (4) of Art. 76 only by a constant factor.

87. If series (1) of Art. 86 be multiplied by $\dfrac{(2m-1)(2m-3)\cdots 1}{\underline{|m}}$ or $\dfrac{\underline{|2m}}{2^m(\underline{|m})^2}$, the resulting integral is called the **Legendrian Coefficient** of the mth order, and is denoted by $P_m(x)$.

The successive values of $P_m(x)$ are readily found to be

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \tfrac{3}{2}x^2 - \tfrac{1}{2}, \quad \cdots.$$

EXERCISES

1. Find the remaining six particular solutions of the equation considered in Art. 72.

2. Find two particular solutions of the equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

3. Find two particular solutions of the equation

$$\frac{\partial (ru)}{\partial t} = c^2 \frac{\partial^2 (ru)}{\partial r^2}.$$

4. Find four particular solutions of the equation

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

5. Find four particular solutions of the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

6. Find four particular solutions of the equation

$$\frac{\partial^2 y}{\partial t^2} + 2k \frac{\partial y}{\partial t} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

7. Show that equation (1) of Art. 70, in rectangular coordinates, becomes equation (2) when transformed to polar or spherical coordinates. The equations of transformation are $x = r \cos \theta \sin \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \phi$.

Suggestion. $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r} \dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial \theta} \dfrac{\partial \theta}{\partial x} + \dfrac{\partial u}{\partial \phi} \dfrac{\partial \phi}{\partial x}$, and similarly for y and z.

Find the general solutions of the following equations:

8. $2x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + (1 + x^2)y = 0.$

9. $9x^2 \frac{d^2y}{dx^2} + (9x - x^2)\frac{dy}{dx} - (x + 1)y = 0.$

10. $4x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + (1 + x^2)y = 0.$

11. $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$

12. $x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + (2 - 4x^2)y = 0.$ 13. $\frac{d^2y}{dx^2} - xy = 0.$

14. $(x - x^2)\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0.$

15. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y = 0.$ 16. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2xy = 0.$

17. Show that,

$$J_0'(y) = -J_1(y),$$

$$2J_n'(y) = J_{n-1}(y) - J_{n+1}(y),$$

$$\frac{2n}{y}J_n(y) = J_{n-1}(y) + J_{n+1}(y).$$

ANSWERS

1. $z = \sin \alpha x \cos \beta y \sin ct\sqrt{\alpha^2 + \beta^2},$

$z = \sin \alpha x \cos \beta y \cos ct\sqrt{\alpha^2 + \beta^2},$

$z = \cos \alpha x \sin \beta y \sin ct\sqrt{\alpha^2 + \beta^2},$

$z = \cos \alpha x \sin \beta y \cos ct\sqrt{\alpha^2 + \beta^2},$

$z = \cos \alpha x \cos \beta y \sin ct\sqrt{\alpha^2 + \beta^2},$

$z = \cos \alpha x \cos \beta y \cos ct\sqrt{\alpha^2 + \beta^2}.$

2. $u = e^{-c^2\alpha^2 t}\cos \alpha x, \quad u = e^{-c^2\alpha^2 t}\sin \alpha x.$

3. $u = \frac{1}{r} e^{-c^2a^2t} \cos ar, \quad u = \frac{1}{r} e^{-c^2a^2t} \sin ar.$

4. $u = e^{-c^2(a^2+\beta^2)t} \cos ax \cos \beta y, \quad u = e^{-c^2(a^2+\beta^2)t} \cos ax \sin \beta y,$

$u = e^{-c^2(a^2+\beta^2)t} \sin ax \sin \beta y, \quad u = e^{-c^2(a^2+\beta^2)t} \sin ax \cos \beta y.$

5. $y = \cos ax \cos cat, \quad y = \sin ax \sin cat,$

$y = \sin ax \cos cat, \quad y = \cos ax \sin cat.$

6. $y = e^{-kt} \sin ax \cos t\sqrt{c^2a^2 - k^2}, \quad y = e^{-kt} \sin ax \sin t\sqrt{c^2a^2 - k^2},$

$y = e^{-kt} \cos ax \cos t\sqrt{c^2a^2 - k^2}, \quad y = e^{-kt} \cos ax \sin t\sqrt{c^2a^2 - k^2}.$

8. $$y = Ax\left[1 - \frac{1}{2\cdot 5}x^2 + \frac{1}{2\cdot 4\cdot 5\cdot 9}x^4 - \frac{1}{2\cdot 4\cdot 6\cdot 5\cdot 9\cdot 13}x^6 + \cdots + g_{2r}x^{2r} + \cdots\right] + Bx^{\frac{1}{2}}\left[1 - \frac{1}{2\cdot 3}x^2 + \frac{1}{2\cdot 4\cdot 3\cdot 7}x^4 - \frac{3}{2\cdot 4\cdot 6\cdot 3\cdot 7\cdot 11}x^6 + \cdots + \bar{g}_{2r}x^{2r}\right],$$

where

$$g_{2r} = (-1)^r \frac{1}{2\cdot 4\cdots 2r\cdot 5\cdot 9\cdots (2r+1)}$$

and

$$\bar{g}_{2r} = (-1)^r \frac{1}{2\cdot 4\cdots 2r\cdot 3\cdot 7\cdots (4r-1)}.$$

9.

$$y = Ax^{\frac{1}{3}}\left[1 + \frac{4}{3^2\cdot 5}x + \frac{4\cdot 7}{3^4\cdot 5\cdot 16}x^2 + \frac{4\cdot 7\cdot 10}{3^6\cdot 5\cdot 16\cdot 33}x^3 + \cdots + g_r x^r + \cdots\right] + Bx^{-\frac{1}{3}}\left[1 + \frac{2}{3^2}x + \frac{2\cdot 5}{3^4\cdot 8}x^2 + \frac{2\cdot 5\cdot 8}{3^6\cdot 21}x^3 + \cdots + \bar{g}_r x^r + \cdots\right],$$

where

$$g_r = \frac{4\cdot 7\cdots (1+3r)}{3^{2r}\cdot 5\cdots (2r+3r^2)} \quad \text{and} \quad \bar{g}_r = \frac{2\cdots (3r-1)}{3^{2r}\cdot 1\cdots (3r^2-2r)}.$$

10. $$y = Ax^{\frac{1}{2}}\left[1 + \frac{x^2}{8\cdot 3} + \frac{x^4}{8^2\cdot 3\cdot 10} + \cdots + g_{2r}x^{2r} + \cdots\right] + Bx^{-\frac{1}{2}}\left[1 + \frac{x^2}{8\cdot 1} + \frac{x^4}{8^2\cdot 1\cdot 6} + \cdots + \bar{g}_{2r}x^{2r} + \cdots\right],$$

where

$$g_{2r} = \frac{1}{8^r \cdot 3 \cdot 10 \cdots (2r^2 + r)} \quad \text{and} \quad \bar{g}_{2r} = \frac{1}{8^r \cdot 1 \cdot 6 \cdots (2r^2 - r)}.$$

11. $y = Ax^2 + Bx.$

12. $$y = Ax^{-1}\left[1 + \tfrac{4}{6}x^2 + \frac{4^2}{6 \cdot 20}x^4 + \cdots + g_{2r}x^{2r} + \cdots\right]$$

$$+ Bx^{-2}\left[1 + \tfrac{4}{2}x^2 + \frac{4^2}{2 \cdot 12}x^4 + \cdots + \bar{g}_{2r}x^{2r} + \cdots\right],$$

where

$$g_{2r} = \frac{4^r}{6 \cdot 20 \cdots 2(2r^2 + r)} \quad \text{and} \quad \bar{g}_{2r} = \frac{4^r}{2 \cdot 12 \cdots 2(2r^2 - r)}.$$

13. $$y = Ax\left[1 + \tfrac{1}{12}x^3 + \frac{1}{12 \cdot 42}x^6 + \cdots + g_{3r}x^{3r} + \cdots\right]$$

$$+ B\left[1 + \tfrac{1}{6}x^3 + \frac{1}{6 \cdot 30}x^6 + \cdots + \bar{g}_{3r}x^{3r} + \cdots\right],$$

where

$$g_{3r} = \frac{1}{12 \cdot 42 \cdots 3(3r^2 + r)} \quad \text{and} \quad \bar{g}_{3r} = \frac{1}{6 \cdot 30 \cdots 3(3r^2 - r)},$$

14. $y = A[1 - \tfrac{2}{3}x + \tfrac{1}{6}x^2] + Bx^{-2}[1 - 4x].$

15. $$y = [A + B \log x]$$

$$\times \left[1 - \frac{2}{1^2}x + \frac{2^2}{1^2 \cdot 2^2}x^2 - \frac{2^3}{1^2 \cdot 2^2 \cdot 3^2}x^3 + \cdots + g_r x^r + \cdots\right]$$

$$+ B\left[\frac{2^2}{1^2}x - \frac{2^3}{1^2 \cdot 2^2}x^2(1 + \tfrac{1}{2}) + \frac{2^4}{1^2 \cdot 2^2 \cdot 3^2}x^3[1 + \tfrac{1}{2} + \tfrac{1}{3}]\right.$$

$$\left. - \frac{2^5}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2}x^4[1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}] + \cdots + \bar{g}_r x^2 + \cdots\right],$$

where

$$g_r = (-1)^r \frac{2^r}{1^2 \cdot 2^2 \cdots r^2} \quad \text{and} \quad \bar{g}_r = (-1)^r \frac{2^{r+1}}{1^2 \cdot 2^2 \cdots r^2} \sum_{r=1}^{r=r} \frac{1}{r}.$$

16. $y=(A+B\log x)\left[1-\frac{2}{2^2}x^2+\frac{2^2}{2^2\cdot 4^2}x^4-\cdots+g_{2r}x^{2r}+\cdots\right]$

$$+B\left[\frac{2}{2^2}x^2-\frac{2^2}{2^2\cdot 4^2}x^4(1+\tfrac{1}{2})+\frac{2^3}{2^2\cdot 4^2\cdot 6^2}x^6(1+\tfrac{1}{2}+\tfrac{1}{3})\right.$$

$$\left.+\ldots+\bar{g}_{2r}x^{2r}+\ldots\right],$$

where

$$g_{2r}=\frac{2^r}{2^2\cdot 4^2\cdots(2r)^2}\quad\text{and}\quad \bar{g}_{2r}=\frac{2^r}{2^2\cdot 4^2\cdots(2r)^2}\sum_{r=1}^{r=r}\frac{1}{r}.$$