# SHORT INTRODUCTION TO GRAPHICAL ALGEBRA

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# A SHORT INTRODUCTION

TO

# GRAPHICAL ALGEBRA

BY

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*FOURTH EDITION REVISED AND ENLARGED*

London
MACMILLAN AND CO., LIMITED
NEW YORK: THE MACMILLAN COMPANY
1907

First Edition, November, 1902.
Reprinted December, 1902.
Second Edition, Revised and Enlarged, January, 1903.
Reprinted March, May, and September, 1903.
Third Edition, Revised and Enlarged, 1904.
Reprinted 1905.
Fourth Edition, Revised and Enlarged, 1907.



GLASGOW: PRINTED AT THE UNIVERSITY PRESS
BY ROBERT MACLEHOSE AND CO. LTD.

## PREFACE TO THE THIRD EDITION.

In this edition some re-arrangement has been made between pages 22 and 25 so as to introduce the text and illustrations of Arts. 27 and 28 before Examples V. A few more examples, taken from or suggested by recent examination papers, have been added in the last section. I have also inserted Tables of Logarithms, Antilogarithms, Square Roots and Cube Roots, which will diminish the labour of solution in many cases.

A Key, giving full solutions with diagrams of all the most important examples, is in the press. In preparing this, all the examples have been worked through again, and in a few instances slight alterations have been made in order to secure a better graph.

H. S. HALL.

*March*, 1904.

## PREFACE TO THE FOURTH EDITION.

In this edition I have added a section in which the leading principles of linear graphs have been applied to some miscellaneous problems illustrated by full-page diagrams. A set of miscellaneous examples illustrative of the types discussed in previous pages has also been inserted. I have endeavoured to exclude problems which can be more readily solved by easy Arithmetic or Algebra, and to retain only those in which a graphical solution possesses real advantage and interest. The growing fashion of introducing graphs into all kinds of elementary work, where they are not wanted, and where they serve no useful purpose—either in illustration of guiding principles or in curtailing calculation—cannot be too strongly deprecated.

H. S. HALL.

*March*, 1907.

# CONTENTS.

# GRAPHICAL ALGEBRA.

[*A considerable portion of this chapter may be taken at an early stage. For example, Arts.* 1–6 *may be read as soon as the student has had sufficient practice in substitutions involving negative quantities. Arts.* 7–14 *may be read in connection with Easy Simultaneous Equations. With the exception of a few articles the rest of the chapter should be postponed until the student is acquainted with quadratic equations. References to Hall and Knight's Elementary Algebra are given thus* : "E. A., Art. 100."]

**1.** Definition. Any expression which involves a variable quantity $x$, and whose value is dependent on that of $x$, is called a **function of x.**

Thus $3x+8$, $2x^2+6x-7$, $x^4-3x^3+x^2-9$ are functions of $x$ of the first, second, and fourth degree respectively.

**2.** The symbol $f(x)$ is often used to briefly denote a function of $x$. If $y=f(x)$, by substituting a succession of numerical values for $x$ we can obtain a corresponding succession of values for $y$ which stands for the value of the function. Hence in this connection it is sometimes convenient to call $x$ the **independent variable**, and $y$ the **dependent variable.**

**3.** Consider the function $x(9-x^2)$, and let its value be represented by $y$.

| | | |
|---|---|---|
| Then, when | $x=0$, | $y=0\times 9=0$, |
| " | $x=1$, | $y=1\times 8=8$, |
| " | $x=2$, | $y=2\times 5=10$, |
| " | $x=3$, | $y=3\times 0=0$, |
| " | $x=4$, | $y=4\times(-7)=-28$, |

and so on.

C

By proceeding in this way we can find as many values of the function as we please. But we are often not so much concerned with the actual values which a function assumes for different values of the variable as with *the way in which the value of the function changes.* These variations can be very conveniently represented by a **graphical** method which we shall now explain.

**4.** Two straight lines $XOX'$, $YOY'$ are taken intersecting at right angles in $O$, thus dividing the plane of the paper into four spaces $XOY$, $YOX'$, $X'OY'$, $Y'OX$, which are known as the first, second, third, and fourth quadrants respectively.

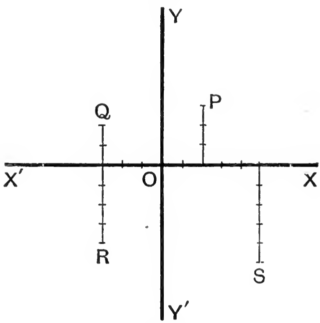



Fig. 1.

The lines $X'OX$, $YOY'$ are usually drawn horizontally and vertically; they are taken as lines of reference and are known as the **axis of x and y** respectively. The point $O$ is called the **origin.** Values of $x$ are measured from $O$ along the axis of $x$, according to some convenient scale of measurement, and are called **abscissæ,** *positive* values being drawn to the *right* of $O$ along $OX$, and *negative* values to the *left* of $O$ along $OX'$.

Values of $y$ are drawn (on the same scale) parallel to the axis of $y$, from the ends of the corresponding abscissæ, and are called **ordinates.** These are *positive* when drawn *above* $X'X$, *negative* when drawn *below* $X'X$.

**5.** The abscissa and ordinate of a point taken together are known as its **coordinates.** A point whose coordinates are $x$ and $y$ is briefly spoken of as "the point $(x, y)$."

The coordinates of a point completely determine its position in the plane. Thus if we wish to mark the point (2, 3), we take $x=2$ units measured to the right of $O$, $y=3$ units measured perpendicular to the $x$-axis and above it. The resulting point $P$ is in the first quadrant. The point $(-3, 2)$ is found by taking $x=3$ units to the left of $O$, and $y=2$ units above the $x$-axis. The resulting point $Q$ is in the second quadrant. Similarly the points $(-3, -4)$, $(5, -5)$ are represented by $R$ and $S$ in Fig. 1, in the third and fourth quadrants respectively.

This process of marking the position of a point in reference to the coordinate axes is known as **plotting the point.**

**6.** In practice it is convenient to use **squared paper**; that is, paper ruled into small squares by two sets of equidistant parallel straight lines, the one set being horizontal and the other vertical. After selecting two of the intersecting lines as axes (and slightly thickening them to aid the eye) one or more of the divisions may be chosen as our unit, and points may be readily plotted when their coordinates are known. Conversely, if the position of a point in any of the quadrants is marked, its coordinates can be measured by the divisions on the paper.

In the following pages we have used paper ruled to tenths of an inch, but a larger scale will sometimes be more convenient. See Art. 26.

*Example.* Plot the points $(5, 2)$, $(-3, 2)$, $(-3, -4)$, $(5, -4)$ on squared paper. Find the area of the figure determined by these points, assuming the divisions on the paper to be tenths of an inch.

Taking the points in the order given, it is easily seen that they are represented by $P$, $Q$, $R$, $S$ in Fig. 2, and that they form a rectangle which contains 48 squares. Each of these is *one-hundredth* part of a *square* inch. Thus the area of the rectangle is ·48 of a square inch.

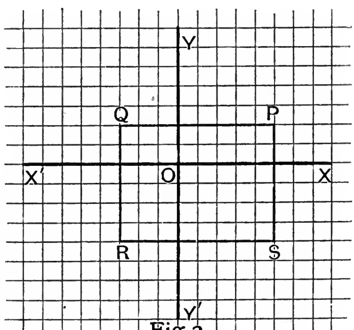


Fig. 2.

## EXAMPLES I.

[*The following examples are intended to be done mainly by actual measurement on squared paper; where possible, they should also be verified by calculation.*]

Plot the following pairs of points and draw the line which joins them:

**1.** $(3, 0)$, $(0, 6)$. **2.** $(-2, 0)$, $(0, -8)$.

**3.** $(3, -8)$, $(-2, 6)$. **4.** $(5, 5)$, $(-2, -2)$.

**5.** $(-2, 6)$, $(1, -3)$. **6.** $(4, 5)$, $(-1, 5)$.

**7.** Plot the points $(3, 3)$, $(-3, 3)$, $(-3, -3)$, $(3, -3)$, and find the number of squares contained by the figure determined by these points.

**8.** Plot the points $(4, 0)$, $(0, 4)$, $(-4, 0)$, $(0, -4)$, and find the number of units of area in the resulting figure.

**9.** Plot the points $(0, 0)$, $(0, 10)$, $(5, 5)$, and find the number of units of area in the triangle.

**10.** Shew that the triangle whose vertices are $(0, 0)$, $(0, 6)$, $(4, 3)$ contains 12 units of area. Shew also that the points $(0, 0)$, $(0, 6)$, $(4, 8)$ determine a triangle of the same area.

**11.** Plot the points $(5, 6)$, $(-5, 6)$, $(5, -6)$, $(-5, -6)$. If one millimetre is taken as unit, find the area of the figure in square centimetres.

**12.** Plot the points $(1, 3)$, $(-3, -9)$, and shew that they lie on a line passing through the origin. Name the coordinates of other points on this line.

**13.** Plot the eight points $(0, 5)$, $(3, 4)$, $(5, 0)$, $(4, -3)$, $(-5, 0)$, $(0, -5)$, $(-4, 3)$, $(-4, -3)$, and shew that they are all equidistant from the origin.

**14.** Plot the two following series of points:

(i) $(5, 0)$, $(5, 2)$, $(5, 5)$, $(5, -1)$, $(5, -4)$;

(ii) $(-4, 8)$, $(-1, 8)$, $(0, 8)$, $(3, 8)$, $(6, 8)$.

Shew that they lie on two lines respectively parallel to the axis of $y$, and the axis of $x$. Find the coordinates of the point in which they intersect.

**15.** Plot the points (13, 0), (0, $-13$), (12, 5), ($-12$, 5), ($-13$, 0), ($-5$, $-12$), (5, $-12$). Find their locus, (i) by measurement, (ii) by calculation.

**16.** Plot the points (2, 2), ($-3$, $-3$), (4, 4), ($-5$, $-5$), shewing that they all lie on a certain line through the origin. Conversely, shew that for *every* point on this line the abscissa and ordinate are equal.

## Graph of a Function.

**7.** Let $f(x)$ represent a function of $x$, and let its value be denoted by $y$. If we give to $x$ a series of numerical values we get a corresponding series of values for $y$. If these are set off as abscissæ and ordinates respectively, we plot a succession of points. If *all* such points were plotted we should arrive at a line, straight or curved, which is known as the **graph** of the *function* $f(x)$, or the **graph** of the *equation* $y=f(x)$. The variation of the function for different values of the variable $x$ is exhibited by the variation of the ordinates as we pass from point to point.

In practice a few points carefully plotted will usually enable us to draw the graph with sufficient accuracy.

**8.** The student who has worked intelligently through the preceding examples will have acquired for himself some useful preliminary notions which will be of service in the examples on simple graphs which we are about to give. In particular, before proceeding further he should satisfy himself with regard to the following statements :

(i) The coordinates of the origin are (0, 0).

(ii) The abscissa of every point on the axis of $y$ is 0.

(iii) The ordinate of every point on the axis of $x$ is 0.

(iv) The graph of all points which have the same abscissa is a line parallel to the axis of $y$. (*e.g.* $x=2$.)

(v) The graph of all points which have the same ordinate is a line parallel to the axis of $x$. (*e.g.* $y=5$.)

(vi) The distance of any point $P(x, y)$ from the origin is given by $OP^2=x^2+y^2$.

*Example* 1. Plot the graph of $y=x$.

When $x=0$, $y=0$; thus the origin is one point on the graph.

Also, when $x=1, 2, 3, \dots -1, -2, -3, \dots,$

$y=1, 2, 3, \dots -1, -2, -3, \dots$

Thus the graph passes through $O$, and represents a series of points each of which has its ordinate equal to its abscissa, and is clearly represented by $POP'$ in Fig. 3.

*Example* 2. Plot the graph of $y=x+3$.

Arrange the values of $x$ and $y$ as follows:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | 3 | 2 | 1 | **0** | −1 | −2 | **−3** | ... |
| $y$ | 6 | 5 | 4 | **3** | 2 | 1 | **0** | ... |

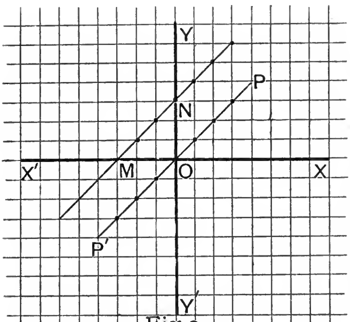



Fig. 3.

By joining these points we obtain a line $MN$ parallel to that in Example 1.

The results printed in larger and deeper type should be specially noted and compared with the graph. They shew that the distances $ON$, $OM$ (usually called the *intercepts on the axes*) are obtained by separately putting $x=0$, $y=0$ in the equation of the graph.

**Note.** By observing that in Example 2 each ordinate is 3 units greater than the corresponding ordinate in Example 1, the graph of $y=x+3$ may be obtained from that of $y=x$ by simply producing each ordinate 3 units in the positive direction.

In like manner the equations

$$y=x+5, \quad y=x-5$$

represent two parallel lines on opposite sides of $y=x$ and equidistant from it, as the student may easily verify for himself.

*Example* 3. Plot the graphs represented by the following equations:

(i) $y=2x$; (ii) $y=2x+4$; (iii) $y=2x-5$.

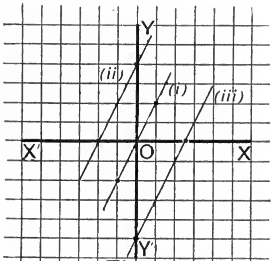



Fig. 4.

Here we only give the diagram which the student should verify in detail for himself, following the method explained in the two preceding examples.

## EXAMPLES II.

[*In the following examples Nos.* 1–18 *are arranged in groups of three; each group should be represented on the same diagram so as to exhibit clearly the position of the three graphs relatively to each other.*]

Plot the graphs represented by the following equations:

**1.** $y=5x$. **2.** $y=5x-4$. **3.** $y=5x+6$.

**4.** $y=-3x$. **5.** $y=-3x+3$. **6.** $y=-3x-2$.

**7.** $y+x=0$. **8.** $y+x=8$. **9.** $y+4=x$.

**10.** $4x=3y$. **11.** $3y=4x+6$. **12.** $4y+3x=8$.

**13.** $x-5=0$. **14.** $y-6=0$. **15.** $5y=6x$.

**16.** $3x+4y=10$. **17.** $4x+y=9$. **18.** $5x-2y=8$.

**19.** Shew by careful drawing that the three last graphs have a common point whose coordinates are 2, 1.

**20.** Shew by careful drawing that the equations

$$x+y=10, \quad y=x-4$$

represent two straight lines at right angles.

**21.** Draw on the same axes the graphs of $x=5$, $x=9$, $y=3$, $y=11$. Find the number of units of area enclosed by these lines.

**22.** Taking one-tenth of an inch as the unit of length, find the area included between the graphs of $x=7$, $x=-3$, $y=-2$, $y=8$.

**23.** Find the area included by the graphs of

$$y=x+6, \quad y=x-6, \quad y=-x+6, \quad y=-x-6.$$

**24.** With one millimetre as linear unit, find in square centimetres the area of the figure enclosed by the graphs of

$$y=2x+8, \quad y=2x-8, \quad y=-2x+8, \quad y=-2x-8.$$

**9.** The student should now be prepared for the following statements :

(i) For all numerical values of $a$ the equation $y=ax$ represents a straight line through the origin.

(ii) For all numerical values of $a$ and $b$ the equation $y=ax+b$ represents a line parallel to $y=ax$, and cutting off an intercept $b$ from the axis of $y$.

**10.** Conversely, since every equation involving $x$ and $y$ only in the first degree can be reduced to one of the forms $y=ax$, $y=ax+b$, it follows that *every simple equation connecting two variables represents a straight line.* For this reason an expression of the form $ax+b$ is said to be a **linear function** of $x$, and an equation such as $y=ax+b$, or $ax+by+c=0$, is said to be a **linear equation.**

*Example.* Shew that the points $(3, -4)$, $(9, 4)$, $(12, 8)$ lie on a straight line, and find its equation.

Assume $y=ax+b$ as the equation of the line. If it passes through the first two points given, their coordinates must satisfy the above equation. Hence

$$-4=3a+b, \quad 4=9a+b.$$

These equations give $\quad a=\frac{4}{3}, \quad b=-8.$

Hence $\quad y=\frac{4}{3}x-8, \quad$ or $\quad 4x-3y=24,$

is the equation of the line passing through the first two points. Since $x=12$, $y=8$ satisfies this equation, the line also passes through $(12, 8)$. This example may be verified graphically by plotting the line which joins *any two* of the points and shewing that it passes through the third.

## Application to Simultaneous Equations.

**11.** It is shewn [E. A., Art. 100] that in the case of a simple equation between $x$ and $y$, it is possible to find as many pairs of values of $x$ and $y$ as we please which satisfy the given equation. We now see that this is equivalent to saying that we may find as many points as we please on any given straight line. If, however, we have two simultaneous equations between $x$ and $y$, there can only be one pair of values which will satisfy both equations. This is equivalent to saying that two straight lines can have only one common point.

*Example.* Solve graphically the equations:

$$3x+7y=27, \quad 5x+2y=16.$$

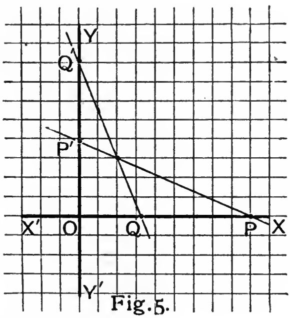


Fig. 5.

If carefully plotted it will be found that these two equations represent the lines in the annexed diagram. On measuring the coordinates of the point at which they intersect it will be found that $x=2$, $y=3$, thus verifying the solution given in E. A. Art. 103, Ex. 1.

**12.** It will now be seen that the process of solving two simultaneous equations is equivalent to finding the coordinates of the point (or points) at which their graphs meet.

**13.** Since a straight line can always be drawn by joining *any* two points on it, in solving *linear* simultaneous equations graphically, it is only necessary to plot two points on each line. The points where the lines meet the axes will usually be the most convenient to select.

**14.** Two simultaneous equations lead to no finite solution if they are inconsistent with each other. For example, the equations

$$x+3y=2, \quad 3x+9y=8$$

are inconsistent, for the second equation can be written $x+3y=2\frac{2}{3}$, which is clearly inconsistent with $x+3y=2$. The graphs of these two equations will be found to be two parallel straight lines which have no finite point of intersection.

Again, two simultaneous equations must be independent. The equations

$$4x+3y=1, \quad 16x+12y=4$$

are not independent, for the second can be deduced from the first by multiplying throughout by 4. Thus *any pair of values* which will satisfy one equation will satisfy the other. Graphically these two equations represent two coincident straight lines which of course have an unlimited number of common points.

## EXAMPLES III.

Solve the following equations, in each case verifying the solution graphically:

**1.** $y=2x+3$, $y+x=6$.

**2.** $y=3x+4$, $y=x+8$.

**3.** $y=4x$, $2x+y=18$.

**4.** $2x-y=8$, $4x+3y=6$.

**5.** $3x+2y=16$, $5x-3y=14$.

**6.** $6y-5x=18$, $4x=3y$.

**7.** $2x+y=0$, $y=\frac{4}{3}(x+5)$.

**8.** $2x-y=3$, $3x-5y=15$.

**9.** $2y=5x+15$, $3y-4x=12$.

**10.** Prove by graphical representation that the three points (3, 0), (2, 7), (4, −7) lie on a straight line. Where does this line cut the axis of $y$?

**11.** Prove that the three points (1, 1), (−3, 4), (5, −2) lie on a straight line. Find its equation. Draw the graph of this equation, shewing that it passes through the given points.

**12.** Shew that the three points (3, 2), (8, 8), (−2, −4) lie on a straight line. Prove algebraically and graphically that it cuts the axis of $x$ at a distance $1\frac{1}{3}$ from the origin.

**15.** We shall now give some graphs of functions of higher degree than the first.

*Example* 1. Plot the graph of $2y = x^2$.

Corresponding values of $x$ and $y$ may be tabulated as follows :

| $x$ | ... | 3 | 2·5 | 2 | 1·5 | 1 | 0 | −1 | −2 | −3 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | ... | 4·5 | 3·125 | 2 | 1·125 | ·5 | 0 | ·5 | 2 | 4·5 | ... |

Here, in order to obtain a figure on a sufficiently large scale, it will be found convenient to take two divisions on the paper for our unit.

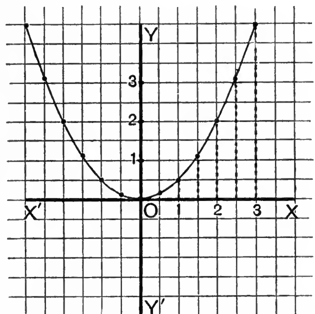


Fig. 6.

If the above points are plotted and connected by a line drawn freehand, we shall obtain the curve shewn in Fig. 6. This curve is called a **parabola**.

There are two facts to be specially noted in this example.

(i) Since from the equation we have $x = \pm\sqrt{2y}$, it follows that for every value of the ordinate we have two values of the abscissa, *equal in magnitude and opposite in sign*. Hence the graph is symmetrical with respect to the axis of $y$; so that after plotting with care enough points to determine the form of the graph in the first quadrant, its form in the second quadrant can be inferred without actually plotting any points in this quadrant. At the same time, in this and similar cases beginners are recommended to plot a few points in each quadrant through which the graph passes.

(ii) We observe that all the plotted points lie above the axis of $x$. This is evident from the equation; for since $x^2$ must be positive for all values of $x$, every ordinate obtained from the equation $y=\frac{x^2}{2}$ must be positive.

In like manner the student may shew that the graph of $2y=-x^2$ is a curve similar in every respect to that in Fig. 6, but lying entirely below the axis of $x$.

**Note.** Some further remarks on the graph of this and the next example will be found in Art. 21.

*Example* 2. Find the graph of $y=2x+\frac{x^2}{4}$.

Here the following arrangement will be found convenient:

| $x$ | 3 | 2 | 1 | 0 | −1 | −2 | −3 | −4 | −5 | −6 | −7 | −8 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $2x$ | 6 | 4 | 2 | 0 | −2 | −4 | −6 | −8 | −10 | −12 | −14 | −16 |
| $\frac{x^2}{4}$ | 2·25 | 1 | ·25 | 0 | ·25 | 1 | 2·25 | 4 | 6·25 | 9 | 12·25 | 16 |
| $y$ | 8·25 | 5 | 2·25 | 0 | −1·75 | −3 | −3·75 | −4 | −3·75 | −3 | −1·75 | 0 |

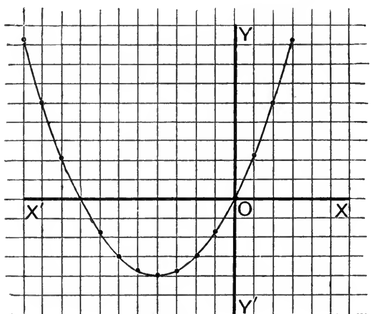


Fig. 7.

From the form of the equation it is evident that every positive value of $x$ will yield a positive value of $y$, and that as $x$ increases $y$ also increases. Hence the portion of the curve in the first quadrant lies as in Fig. 7, and can be extended indefinitely in this quadrant. In the present case only two or three positive values of $x$ and $y$ need be plotted, but more attention must be paid to the results arising out of negative values of $x$.

When $y=0$, we have $\frac{x^2}{4}+2x=0$; thus the two values of $x$ in the graph which correspond to $y=0$ furnish the roots of the equation $\frac{x^2}{4}+2x=0$.

**16.** If $f(x)$ represent a function of $x$, an approximate solution of the equation $f(x)=0$ may be obtained by plotting the graph of $y=f(x)$, and then measuring the intercepts made on the axis of $x$. These intercepts are values of $x$ which make $y$ equal to zero, and are therefore roots of $f(x)=0$.

**17.** If $f(x)$ gradually increases till it reaches a value $a$, which is algebraically greater than neighbouring values on either side, $a$ is said to be a **maximum value** of $f(x)$.

If $f(x)$ gradually decreases till it reaches a value $b$, which is algebraically less than neighbouring values on either side, $b$ is said to be a **minimum** value of $f(x)$.

When $y=f(x)$ is treated graphically, it is now evident that maximum and minimum values of $f(x)$ occur at points where the ordinates are algebraically greatest and least in the immediate vicinity of such points.

*Example.* Solve the equation $x^2-7x+11=0$ graphically, and find the minimum value of the function $x^2-7x+11$.

Put $y=x^2-7x+11$, and find the graph of this equation.

| $x$ | 0 | 1 | 2 | 3 | **3·5** | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 11 | 5 | 1 | −1 | **−1·25** | −1 | 1 | 5 | 11 |

The values of $x$ which make the function $x^2-7x+11$ vanish are those which correspond to $y=0$. By careful measurement it will be found that the intercepts $OM$ and $ON$ are approximately equal to 2·38 and 4·62.

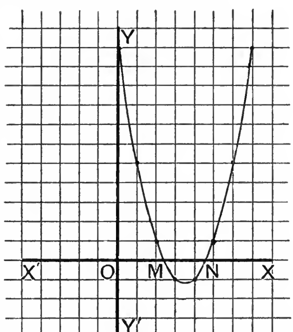



Fig.8.

The algebraical solution of

$$x^2-7x+11=0$$

gives $$x=\frac{1}{2}(7\pm\sqrt{5}).$$

If we take 2·236 as the approximate value of $\sqrt{5}$, the values of $x$ will be found to agree with those obtained from the graph.

Again, $x^2 - 7x + 11 = \left(x - \frac{7}{2}\right)^2 - \frac{5}{4}$. Now $\left(x - \frac{7}{2}\right)^2$ must be positive for all real values of $x$ except $x = \frac{7}{2}$, in which case it vanishes, and the value of the function reduces to $-\frac{5}{4}$, which is the least value it can have.

The graph shews that when $x = 3{\cdot}5$, $y = -1{\cdot}25$, and that this is the algebraically least ordinate in the plotted curve.

**18.** The following example shews that points selected for graphical representation must sometimes be restricted within certain limits.

*Example.* Find the graph of $x^2 + y^2 = 36$.

The equation may be written in either of the following forms:

$$\text{(i)}\quad y = \pm\sqrt{36 - x^2}\,; \qquad \text{(ii)}\quad x = \pm\sqrt{36 - y^2}.$$

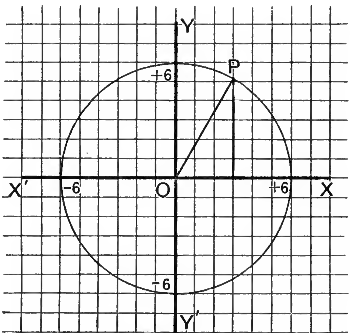


Fig. 9.

In order that $y$ may be a real quantity we see from (i) that $36 - x^2$ must be positive. Thus $x$ can only have values between $-6$ and $+6$. Similarly from (ii) it is evident that $y$ must also lie between $-6$ and $+6$. Between these limits it will be found that all plotted points will lie at a distance 6 from the origin. Hence the graph is a circle whose centre is $O$ and whose radius is 6.

This is otherwise evident, for the distance of any point $P(x, y)$ from the origin is given by $OP = \sqrt{x^2 + y^2}$. [Art. 8.] Hence the equation $x^2 + y^2 = 36$ asserts that the graph consists of a series of points all of which are at a distance 6 from the origin.

**Note.** To plot the curve from equation (ii), we should select a succession of values for $y$ and then find corresponding values of $x$. In other words we make $y$ the *independent* and $x$ the *dependent* variable. The student should be prepared to do this in some of the examples which follow.

## EXAMPLES IV.

**1.** Draw the graphs of $y=x^2$, and $x=y^2$, and shew that they have only one common chord. Find its equation.

**2.** From the graphs, and also by calculation, shew that $y=\frac{x^2}{8}$ cuts $x=-y^2$ in only two points, and find their coordinates.

**3.** Draw the graphs of

(i) $y^2=-4x$; (ii) $y=2x-\frac{x^2}{4}$; (iii) $y=\frac{x^2}{4}+x-2$.

**4.** Draw the graph of $y=x+x^2$. Shew also that it may be deduced from that of $y=x^2$, obtained in Example 1.

**5.** Shew (i) graphically, (ii) algebraically, that the line $y=2x-3$ meets the curve $y=\frac{x^2}{4}+x-2$ in one point only. Find its coordinates.

**6.** Find graphically the roots of the following equations to 2 places of decimals:

(i) $\frac{x^2}{4}+x-2=0$; (ii) $x^2-2x=4$; (iii) $4x^2-16x+9=0$;

and verify the solutions algebraically.

**7.** Find the minimum value of $x^2-2x-4$, and the maximum value of $5+4x-2x^2$.

**8.** Draw the graph of $y=(x-1)(x-2)$ and find the minimum value of $(x-1)(x-2)$. Measure, as accurately as you can, the values of $x$ for which $(x-1)(x-2)$ is equal to 5 and 9 respectively. Verify algebraically.

**9.** Solve the simultaneous equations

$$x^2+y^2=100, \qquad x+y=14;$$

and verify the solution by plotting the graphs of the equations and measuring the coordinates of their common points.

**10.** Plot the graphs of $x^2+y^2=25$, $3x+4y=25$, and examine their relation to each other where they intersect. Verify the result algebraically.

**19. Infinite and zero values.** Consider the fraction $\frac{a}{x}$ in which the numerator $a$ has a *certain fixed value*, and the denominator is a *quantity subject to change*; then it is clear that the smaller $x$ becomes the larger does the value of the fraction $\frac{a}{x}$ become. For instance

$$\frac{a}{\frac{1}{10}}=10a, \quad \frac{a}{\frac{1}{1000}}=1000a, \quad \frac{a}{\frac{1}{1000000}}=1000000a.$$

By making the denominator $x$ sufficiently small the value of the fraction $\frac{a}{x}$ can be made as large as we please; that is, if $x$ is made *less than any quantity that can be named*, the value of $\frac{a}{x}$ will become *greater than any quantity that can be named.*

A quantity less than any assignable quantity is called **zero** and is denoted by the symbol 0.

A quantity greater than any assignable quantity is called **infinity** and is denoted by the symbol $\infty$.

We may now say briefly

*when* $x=0$, *the value of* $\frac{a}{x}$ *is* $\infty$.

Again if $x$ is a quantity which gradually increases and finally becomes *greater than any assignable quantity* the fraction becomes *smaller than any assignable quantity.* Or more briefly

*when* $x=\infty$, *the value of* $\frac{a}{x}$ *is* 0.

**20.** It should be observed that when the symbols for zero and infinity are used in the sense above explained, they are subject to the rules of signs which affect other algebraical symbols. Thus we shall find it convenient to use a concise statement such as "when $x=+0$, $y=+\infty$" to indicate that when a *very small and positive* value is given to $x$, the corresponding value of $y$ is *very large and positive.*

**21.** If we now return to the examples worked out in Art. 15, in Example 1, we see that when $x=\pm\infty$, $y=+\infty$; hence the curve extends upwards to infinity in both the first and second quadrants. In Example 2, when $x=+\infty$, $y=+\infty$. Again $y$ is negative between the values 0 and $-8$ of $x$. For all negative values of $x$ numerically greater than 8, $y$ is positive, and when $x = -\infty$, $y = +\infty$. Hence the curve extends to infinity in both the first and second quadrants.

The student should now examine the nature of the graphs in Examples IV. when $x$ and $y$ are infinite.

*Example.* Find the graph of $xy = 4$.

The equation may be written in the form

$$y = \frac{4}{x},$$

from which it appears that when $x=0$, $y=\infty$ and when $x=\infty$, $y=0$. Also $y$ is positive when $x$ is positive, and negative when $x$ is negative. Hence the graph must lie entirely in the first and third quadrants.

It will be convenient in this case to take the positive and negative values of the variables separately.

(1) *Positive values:*

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | $\infty$ | 4 | 2 | $1\frac{1}{3}$ | 1 | ·8 | $\frac{2}{3}$ | ... | 0 |

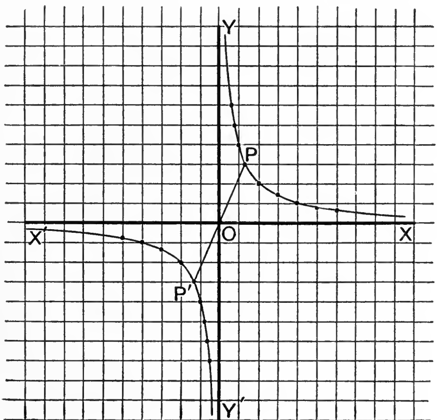



Fig. 10.

Graphically these values shew that as we recede further and further from the origin on the $x$-axis in the positive direction, the values of $y$ are positive and become smaller and smaller. That is the graph is continually approaching the $x$-axis in such a way that by taking a sufficiently great positive value of $x$ we obtain a point on the graph as near as we please to the $x$-axis but never actually reaching it until $x=\infty$. Similarly, as $x$ becomes smaller and smaller the graph approaches more and more nearly to the positive end of the $y$-axis, never actually reaching it as long as $x$ has any finite positive value, however small.

(2) *Negative values:*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | $-0$ | $-1$ | $-2$ | $-3$ | $-4$ | $-5$ | ... | $-\infty$ |
| $y$ | $-\infty$ | $-4$ | $-2$ | $-1\frac{1}{3}$ | $-1$ | $-\cdot 8$ | ... | $-0$ |

The portion of the graph obtained from these values is in the third quadrant as shewn in Fig. 10, and exactly similar to the portion already traced in the first quadrant. It should be noticed that as $x$ passes from $+0$ to $-0$ the value of $y$ changes from $+\infty$ to $-\infty$. Thus the graph, which in the first quadrant has run away to an infinite distance on the positive side of the $y$-axis, reappears in the third quadrant coming from an infinite distance on the negative side of that axis. Similar remarks apply to the graph in its relation to the $x$-axis.

**22.** When a curve continually approaches more and more nearly to a line without actually meeting it until an infinite distance is reached, such a line is said to be an **asymptote** to the curve. In the above case each of the axes is an asymptote.

**23.** Every equation of the form $y=\frac{c}{x}$, or $xy=c$, where $c$ is constant, will give a graph similar to that exhibited in the example of Art. 21. The resulting curve is known as a **rectangular hyperbola**, and has many interesting properties. In particular we may mention that from the form of the equation it is evident that for every point $(x, y)$ on the curve there is a corresponding point $(-x, -y)$ which satisfies the equation. Graphically this amounts to saying that any line through the origin meeting the two branches of the curve in $P$ and $P'$ is bisected at $O$.

**24.** In the simpler cases of graphs, sufficient accuracy can usually be obtained by plotting a few points, and there is little difficulty in selecting points with suitable coordinates. But in other cases, and especially when the graph has infinite branches, more care is needed. The most important things to observe are (1) the values for which the function $f(x)$ becomes zero or infinite; and (2) the values which the function assumes for zero and infinite values of $x$. In other words, we determine the *general character* of the curve in the neighbourhood of the origin, the axes, and infinity. Greater accuracy of detail can then be secured by plotting points at discretion. The selection of such points will usually be suggested by the earlier stages of our work.

The existence of symmetry about either of the axes should also be noted. When an equation contains no *odd* powers of $x$, the graph is symmetrical with regard to the axis of $y$. Similarly the absence of odd powers of $y$ indicates symmetry about the axis of $x$. Compare Art. 15, Ex. 1.

*Example.* Draw the graph of $y=\frac{2x+7}{x-4}$. [See fig. on next page.]

We have $y=\frac{2x+7}{x-4}=\frac{2+\frac{7}{x}}{1-\frac{4}{x}}$, the latter form being convenient for infinite values of $x$.

(i) When $y=0,\quad x=-\frac{7}{2}$,
,, $y=\infty,\quad x=4$;

$\therefore$ the curve cuts the axis of $x$ at a distance $-3{\cdot}5$ from the origin, and meets the line $x=4$ at an infinite distance.

If $x$ is positive and very little greater than 4, $y$ is very great and positive. If $x$ is positive and very little less than 4, $y$ is very great and negative. Thus the infinite points on the graph near to the line $x=4$ have positive ordinates to the right, and negative ordinates to the left of this line.

(ii) When $x=0,\quad y=-1{\cdot}75$,
,, $x=\infty,\quad y=2$;

$\therefore$ the curve cuts the axis of $y$ at a distance $-1{\cdot}75$ from the origin, and meets the line $y=2$ at an infinite distance.

By taking positive values of $y$ very little greater and very little less than 2, it appears that the curve lies above the line $y=2$ when $x=+\infty$, and below this line when $x=-\infty$.

The general character of the curve is now determined: the lines $PO'P'$ ($x=4$) and $QO'Q'$ ($y=2$) are asymptotes; the two branches of the curve lie in the compartments $PO'Q$, $P'O'Q'$. and the lower branch cuts the axes at distances $-3{\cdot}5$ and $-1{\cdot}75$ from the origin.

To examine the lower branch in detail values of $x$ may be selected between $-\infty$ and $-3\cdot5$ and between $-3\cdot5$ and 4.

| $x$ | $-\infty$ | ... | $-16$ | $-8$ | $-6$ | $-3\cdot5$ | $-1$ | 0 | 2 | 3 | ... | 4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 2 | ... | 1·25 | ·75 | ·5 | 0 | $-1$ | $-1\cdot75$ | $-5\cdot5$ | $-13$ | ... | $-\infty$ |

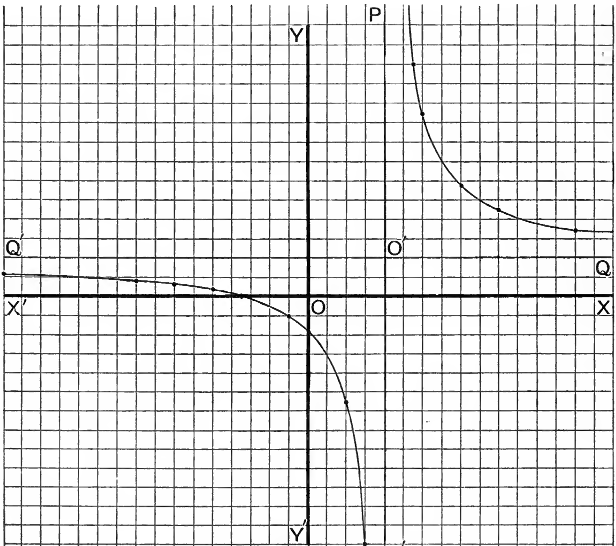



Fig. 11.

The upper branch may now be dealt with in the same way, selecting values of $x$ between 4 and $\infty$. The graph will be found to be as represented in Fig. 11.

**25.** When the equation of a curve contains the square or higher power of $y$, the calculation of the values of $y$ corresponding to selected values of $x$ will have to be obtained by evolution, or else by the aid of logarithms. We give one example to illustrate the way in which a table of four-figure logarithms may be employed in such cases.

*Example.* Draw the graph of $y^3 = x(9 - x^2)$.

For the sake of brevity we shall confine our attention to that part of the curve which lies to the right of the axis of $y$, leaving the other half to be traced in like manner by the student.

When $x=0$, $y=0$; therefore the curve passes through the origin. Again, $y$ is positive for all values of $x$ between 0 and 3, and vanishes when $x=3$; for values of $x$ greater than 3, $y$ is negative and continually increases numerically.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|---|---|
| $x^2$ | 0 | 1 | 4 | 9 | 16 | 25 | 36 | ... |
| $9 - x^2$ | 9 | 8 | 5 | 0 | $-7$ | $-16$ | $-27$ | ... |
| $y^3$ | 0 | 8 | 10 | 0 | $-28$ | $-80$ | $-162$ | ... |
| $\log y^3$ | | | 1 | | 1·4472* | 1·9031* | 2·2095* | ... |
| $\log y$ | | | ·3333 | | ·4824 | ·6344 | ·7365 | ... |
| $y$ | 0 | 2 | 2·15 | 0 | $-3·04$ | $-4·31$ | $-5·45$ | ... |

These points will be sufficient to give a rough approximation to the curve. For greater accuracy a few intermediate values such as $x=1·5$, $2·5$, $3·5$ ... should be taken, and the resulting curve will be as in Fig. 12, in which we have taken *two-tenths of an inch as our linear unit.*

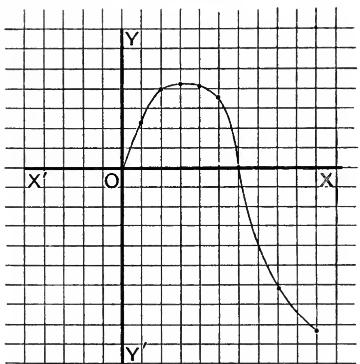



Fig. 12.

* In taking logarithms of the successive values of $y^3$, the negative sign is disregarded, but care must be taken to insert the proper signs in the last line which gives the successive values of $y$.

## Measurement on Different Scales.

**26.** For convenience on the printed page we have supposed the paper to be ruled to tenths of an inch, generally using one of the divisions as our linear unit. In practice, however, it will often be advisable to choose a unit much larger than this in order to get a satisfactory graph. For the sake of simplicity we have hitherto measured abscissæ and ordinates on the same scale, but there is no necessity for so doing, and it will often be found convenient to measure the variables on different scales suggested by the particular conditions of the question.

As an illustration let us take the graph of $y=\frac{x^2}{2}$, given in Art. 15. If with the same unit as before we plot the graph of $y=x^2$, it will be found to be a curve similar to that drawn on page 11, but elongated in the direction of the axis of $y$. In fact, it will be the same as if the former graph were stretched to twice its length in the direction of the $y$-axis.

**27.** Any equation of the form $y=ax^2$, where $a$ is constant, will represent a parabola elongated more or less according to the value of $a$; and the larger the value of $a$ the more rapidly will $y$ increase in comparison with $x$. We might have very large ordinates corresponding to very small abscissæ, and the graph might prove quite unsuitable for practical applications. In such a case the inconvenience is obviated by measuring the values of $y$ on a considerably smaller scale than those of $x$.

Speaking generally, whenever one variable increases much more rapidly than the other, a small unit should be chosen for the rapidly increasing variable and a large one for the other. Further modifications will be suggested in the examples which follow.

**28.** On the opposite page we give for comparison the graphs of

$$y=x^2 \text{ (Fig. 13), and } y=8x^2 \text{ (Fig. 14).}$$

In Fig. 13 the unit for $x$ is twice as great as that for $y$.

In Fig. 14 the $x$-unit is ten times the $y$-unit.

It will be useful practice for the student to plot other similar graphs on the same or a larger scale. For example, in Fig. 14 the graphs of $y=16x^2$ and $y=2x^2$ may be drawn and compared with that of $y=8x^2$.

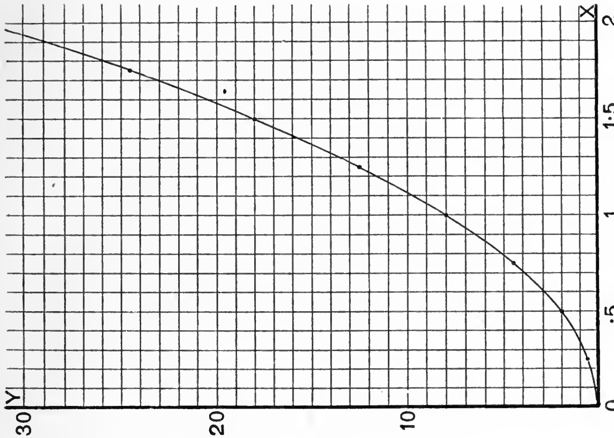



Fig. 14.

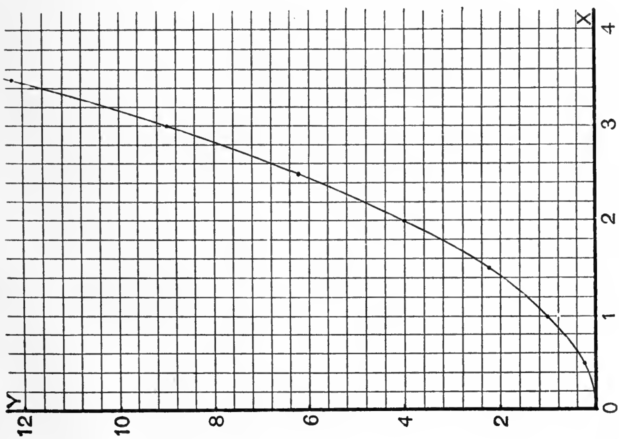



Fig. 13.

## EXAMPLES V.

**1.** Plot the graph of $y=x^3$. Shew that it consists of a continuous curve lying in the first and third quadrants, crossing the axis of $x$ at the origin. Deduce the graphs of

$$\text{(i)}\quad y=-x^3;\qquad \text{(ii)}\quad y=\frac{1}{2}x^3.$$

**2.** Plot the graph of $y=x-x^3$. Verify it from the graphs of $y=x$, and $y=x^3$.

**3.** Plot the graph of $y=\frac{1}{x^2}$, shewing that it consists of two branches lying entirely in the first and second quadrants. Examine and compare the nature and position of the graph as it approaches the axes.

**4.** Discuss the general character of the graph of $y=\frac{a}{x^2}$ where $a$ has some constant integral value. Distinguish between two cases in which $a$ has numerical values, equal in magnitude but opposite in sign.

**5.** Plot the graphs of

$$\text{(i)}\quad y=1+\frac{1}{x},\qquad \text{(ii)}\quad y=2+\frac{10}{x^2}.$$

Verify by deducing them from the graphs of $y=\frac{1}{x}$, and $y=\frac{10}{x^2}$.

**6.** Plot the graph of $y=x^3-3x$. Examine the character of the curve at the points $(1, -2)$, $(-1, 2)$, and shew graphically that the roots of the equation $x^3-3x=0$ are approximately $-1{\cdot}732$, 0, and $1{\cdot}732$.

**7.** Solve the equations:

$$3x+2y=16,\qquad xy=10,$$

and verify the solution by finding the coordinates of the points where their graphs intersect.

**8.** Plot the graphs of

$$\text{(i)}\quad y=\frac{15-x^2}{x},\qquad \text{(ii)}\quad x=\frac{10-y^2}{y},$$

and thus verify the algebraical solution of the equations $x^2+xy=15$, $y^2+xy=10$.

9. Trace the curve whose equation is $y=\frac{x}{2-x}$, shewing that it has two branches, one lying in the first and third quadrants, and the other entirely in the fourth. Find the equations of its asymptotes.

Plot the graphs of

10. $y=\frac{1+x}{1-x}$.

11. $y=\frac{1+x^2}{1-x}$.

12. $y=\frac{x^2-15}{x-4}$.

13. $y=\frac{(x-1)(x-2)}{x-3}$.

14. $y=\frac{x^2+x+1}{x^2-x+1}$.

15. $y=\frac{x^2+5x+6}{x^2+1}$.

16. $y=x^3-6x^2+11x-6$.

17. $10y=x^3-5x^2+x-5$.

18. $y=\frac{20}{x^2+2}$.

19. $y=\frac{40x}{x^2+10}$.

20. $y=\frac{x(8-x)}{x+5}$.

21. $y=\frac{(x-2)(x-3)}{x-5}$.

22. $y=\frac{(x-1)(x-2)(x+1)}{4}$.

23. $y^2=x^2-5x+4$.

24. $4y^2=x^2(5-x)$.

25. $y^2=\frac{x(3-x)(x-8)}{x^2+5}$.

26. $y^2=\frac{(x+7)(x-4)(x-10)}{x^2+5}$.

27. $y^2=\frac{x^2(49-x^2)}{50}$.

28. $y^2=\frac{(81-x^2)(x^2-4)}{100}$.

29. $5y^3=x(x^2-64)$.

30. $5y^3=x^2(36-x^2)$.

31. Plot the graphs of $y=x^3$, and of $y=2x^2+x-2$. Hence find the roots of the equation $x^3-2x^2-x+2=0$.

32. Find graphically the roots of the equation

$$x^3-4x^2-5x+14=0$$

to three significant figures.

**29.** Besides the instances already given there are several of the ordinary processes of Arithmetic and Algebra which lend themselves readily to graphical illustration.

For example, the graph of $y=x^2$ may be used to furnish numerical square roots. For since $x=\sqrt{y}$, each ordinate and corresponding abscissa give a number and its square root. Similarly cube roots may be found from the graph of $y=x^3$.

*Example* 1. Find graphically the cube root of 10 to 3 places of decimals.

The required root is clearly a little greater than 2. Hence it will be enough to plot the graph of $y=x^3$ taking $x=2{\cdot}1$, $2{\cdot}2$, ... The corresponding ordinates are $9{\cdot}26$, $10{\cdot}65$, ...

When $x=2$, $y=8$. Take the axes through this point and let the units for $x$ and $y$ be 10 inches and $\cdot5$ inch respectively. On this scale the portion of the graph differs but little from a straight line, and yields results to a high degree of accuracy.

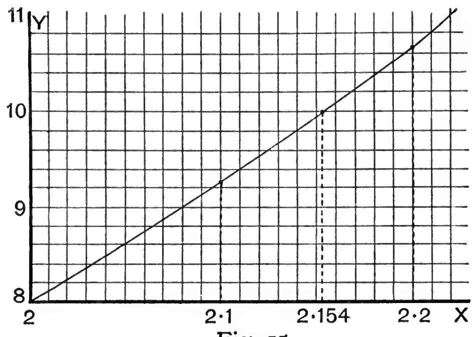



Fig. 15.

When $y=10$, the measured value of $x$ will be found to be $2{\cdot}154$.

*Example* 2. Shew graphically that the expression $4x^2+4x-3$ is negative for all real values of $x$ between $\cdot5$ and $-1{\cdot}5$, and positive for all real values of $x$ outside these limits. [Fig. 16.]

Put $y=4x^2+4x-3$, and proceed as in the example given in Art. 16, taking the unit for $x$ four times as great as that for $y$. It will be found that the graph cuts the axis of $x$ at points whose abscissæ are $\cdot5$ and $-1{\cdot}5$; and that it lies below the axis of $x$ between these points. That is, the value of $y$ is negative so long as $x$ lies between $\cdot5$ and $-1{\cdot}5$, and positive for all other values of $x$.

Or we may proceed as follows:

Put $y_1=4x^2$, and $y_2=-4x+3$, and plot the graphs of these two equations. At their points of intersection $y_1=y_2$, and the values of $x$ at these points are found to be ·5 and −1·5. Hence for these values of $x$ we have

$$4x^2=-4x+3, \quad \text{or } 4x^2+4x-3=0.$$

Thus the roots of the equation $4x^2+4x-3=0$ are furnished by the abscissæ of the common points of the graphs of $4x^2$ and $-4x+3$.

Again, between the values ·5 and −1·5 for $x$ it will be found graphically that $y_1$ is less than $y_2$, hence $y_1-y_2$, or $4x^2+4x-3$ is negative.

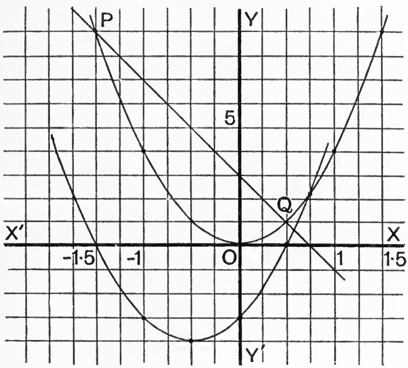



Fig. 16.

Both solutions are here exhibited.

The upper curve is the graph of $y=4x^2$; $PQ$ is the graph of $y=-4x+3$; and the lower curve is the graph of $y=4x^2+4x-3$.

**30.** Of the two methods in the last Example the first is the more direct and instructive; but the second has this advantage:

If a number of equations of the form $x^2=px+q$ have to be solved graphically, $y=x^2$ can be plotted once for all on a convenient scale, and $y=px+q$ can then be readily drawn for different values of $p$ and $q$.

Equations of higher degree may be treated similarly.

For example, the solution of such equations as

$$x^3 = px + q, \quad \text{or} \quad x^3 = ax^2 + bx + c$$

can be made to depend on the intersection of $y = x^3$ with other graphs.

*Example.* Find the real roots of the equations

$$\text{(i)}\ x^3 - 2{\cdot}5x - 3 = 0; \qquad \text{(ii)}\ x^3 - 3x + 2 = 0.$$

Here we have to find the points of intersection of

$$\text{(i)}\ y = x^3, \quad y = 2{\cdot}5x + 3; \qquad \text{(ii)}\ y = x^3, \quad y = 3x - 2.$$

Plot the graphs of these equations, choosing the unit for $x$ five times as great as that for $y$.

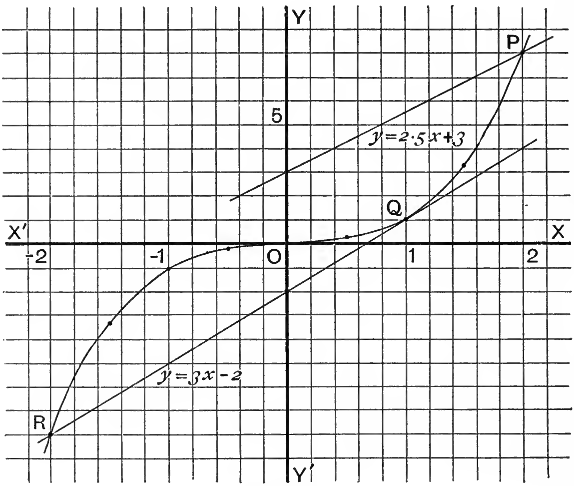



**Fig. 17.**

It will be seen that $y = 2{\cdot}5x + 3$ meets $y = x^3$ only at the point for which $x = 2$. Thus 2 is the only real root of equation (i).

Again $y = 3x - 2$ *touches* $y = x^3$ at the point for which $x = 1$, and cuts it where $x = -2$.

Corresponding to the former point the equation $x^3 - 3x + 2 = 0$ has two equal roots. Thus the roots of (ii) are 1, 1, $-2$.

**31.** Apart from questions of convenience with regard to any particular graph, we may observe that in many cases the variables whose values are plotted on the two axes denote magnitudes of different kinds, so that there is no necessary relation between the units in which they are measured.

A good illustration of this kind is furnished by tracing the variations of the Trigonometrical functions graphically.

*Example.* Trace the graph of $\sin x$.

In any work on Trigonometry it is shewn that as the angle $x$ increases from 0° to 90°, the value of $\sin x$ is positive, and increasing gradually from 0 to 1. From 90° to 180°, $\sin x$ is positive, and decreasing from 1 to 0. From 180° to 270°, $\sin x$ is negative, and increasing numerically from 0 to $-1$. And from 270° to 360°, $\sin x$ is negative, and decreasing numerically from $-1$ to 0.

(See Hall and Knight's *Elementary Trigonometry*, Art. 86.)

We shall here exhibit these variations independently by putting $y = \sin x$, and plotting the values of $y$ corresponding to values of $x$ differing by 30°.

By the aid of a table of sines we have:

| $x$ | 0° | 30° | 60° | 90° | 120° | 150° | 180° | 210° | 240° | 270° | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ or $\sin x$ | 0 | ·5 | ·866 | 1 | ·866 | ·5 | 0 | − ·5 | − ·866 | − 1 | ... |

The graph is represented by the continuous waving line shewn in Fig. 18.

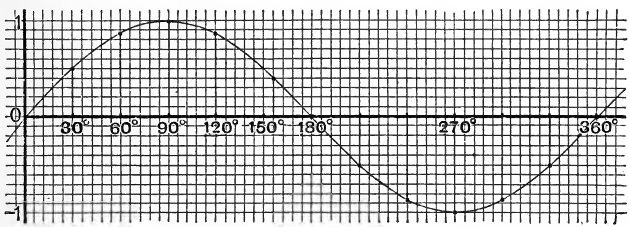



Fig. 18.

On the $x$-axis each division represents 6°, and on the $y$-axis ten divisions have been taken as the unit.

## EXAMPLES VI.

**1.** Draw the graph of $y=x^2$ on a scale twice as large as that in Fig. 13, and employ it to find the squares of ·72, 1·7, 3·4; and the square roots of 7·56, 5·29, 9·61.

**2.** Draw the graph of $y=\sqrt{x}$ taking the unit for $y$ five times as great as that for $x$.

By means of this curve check the values of the square roots found in Example 1.

**3.** From the graph of $y=x^3$ (on the scale of the diagram of Art. 29) find the values of $\sqrt[3]{9}$ and $\sqrt[3]{9 \cdot 8}$ to 4 significant figures.

**4.** A boy who was ignorant of the rule for cube root required the value of $\sqrt[3]{14 \cdot 71}$. He plotted the graph of $y=x^3$, using for $x$ the values 2·2, 2·3, 2·4, 2·5, and found 2·45 as the value of the cube root. Verify this process in detail. From the same graph find the value of $\sqrt[3]{13 \cdot 8}$.

**5.** Find graphically the values of $x$ for which the expression $x^2-2x-8$ vanishes. Shew that for values of $x$ between these limits the expression is negative and for all other values positive. Find the least value of the expression.

**6.** From the graph in the preceding example shew that for any value of $a$ greater than 1 the equation $x^2-2x+a=0$ cannot have real roots.

**7.** Shew graphically that the expression $x^2-4x+7$ is positive for all real values of $x$.

**8.** On the same axes draw the graphs of

$$y=x^2, \quad y=x+6, \quad y=x-6, \quad y=-x+6, \quad y=-x-6.$$

Hence discuss the roots of the four equations

$$x^2-x-6=0, \quad x^2-x+6=0, \quad x^2+x-6=0, \quad x^2+x+6=0.$$

**9.** If $x$ is real, prove graphically that $5-4x-x^2$ is not greater than 9; and that $4x^2-4x+3$ is not less than 2. Between what values of $x$ is the first expression positive?

**10.** Solve the equation $x^3=3x^2+6x-8$ graphically, and shew that the function $x^3-3x^2-6x+8$ is positive for all values of $x$ between $-2$ and 1, and negative for all values of $x$ between 1 and 4.

**11.** Shew graphically that the equation $x^3+px+q=0$ has only one real root when $p$ is positive.

**12.** Trace the curve whose equation is $y=2^x$. Find the approximate values of $2^{4\cdot75}$ and $2^{5\cdot25}$. Express 12 as a power of 2 approximately.

Prove also that $\log_2 26\cdot9+\log_2 38=10$.

**13.** By repeated evolution find the values of $10^{\frac{1}{2}}$, $10^{\frac{1}{4}}$, $10^{\frac{1}{8}}$, $10^{\frac{1}{16}}$. By multiplication find the values of $10^{\frac{3}{16}}$, $10^{\frac{5}{16}}$, $10^{\frac{6}{16}}$, $10^{\frac{7}{16}}$, $10^{\frac{9}{16}}$. Use these values to plot a portion of the curve $y=10^x$ on a large scale. Find correct to three places of decimals the values of $\log 3$, $\log 1\cdot68$, $\log 2\cdot24$, $\log 34\cdot3$. Also by choosing numerical values for $a$ and $b$, verify the laws

$$\log ab=\log a+\log b\,; \quad \log\frac{a}{b}=\log a-\log b.$$

[*By using paper ruled to tenths of an inch, if* 10 *in. and* 1 *in. be taken as units for x and y respectively, a diagonal scale will give values of x correct to three decimal places and values of y correct to two.*]

**14.** Calculate the values of $x(9-x)^2$ for the values 0, 1, 2, 3, ... 9 of $x$. Draw the graph of $x(9-x)^2$ from $x=0$ to $x=9$.

If a very thin elastic rod, 9 inches in length, fixed at one end, swings like a pendulum, the expression $x(9-x)^2$ measures the tendency of the rod to break at a place $x$ inches from the point of suspension. From the graph find where the rod is most likely to break.

**15.** If a man spends 22*s*. a year on tea whatever the price of tea is, what amounts will he receive when the price is 12, 16, 18, 20, 24, 28, 33, and 36 pence respectively? Give your results to the nearest quarter of a pound. Draw a curve to the scale of 4 lbs. to the inch and 10 pence to the inch, to shew the number of pounds that he would receive at intermediate prices.

**16.** Draw the graphs of $\cos x$ and $\tan x$, on a scale twice as large as that in Art. 31.

**17.** Draw the graph of $\sin x$ from the following values of $x$:

$$5°,\ 15°,\ 30°,\ 45°,\ 60°,\ 75°,\ 85°,\ 90°.$$

Find the value of $\sin 37°$, and the angle whose sine is $\cdot8$.

**18.** Find from the tables the value of $\cos x$ when

$$x=0°,\ 10°,\ 20°,\ 30°,\ 40°,\ 50°,\ 60°.$$

Draw a curve on a large scale shewing how $\cos x$ varies as $x$ increases from 0° to 60°.

Find from the curve the values of $\cos 25°$ and $\cos 45°$. Verify by means of the tables.

**19.** Draw on the same diagram the graphs of the functions $\sin x$, $\cos x$, and $\sin x+\cos x$.

Derive from the figure the general solution of $\sin x+\cos x=0$.

**20.** The range of a certain gun is $1000 \sin 2A$ yards, where $A$ is the elevation of the gun. Find from the tables the value of $1000 \sin 2A$ when $A$ has the values

$$10^\circ,\ 15^\circ,\ 20^\circ,\ 25^\circ,\ 30^\circ,\ 35^\circ,\ 40^\circ,\ 45^\circ,\ 50^\circ,$$

and draw a curve shewing how the range varies as $A$ increases from $10^\circ$ to $50^\circ$.

**21.** From the tables find the values of $\tan 10x - 2 \tan 9x + 1$ for the following values of $x$: $0^\circ, 1^\circ, 2^\circ, \ldots 9^\circ$. Draw a curve shewing how $\tan 10x - 2\tan 9x + 1$ varies with $x$ when $x$ lies between $0^\circ$ and $9^\circ$. Find to the nearest tenth of a degree a value of $x$ for which the given expression vanishes.

## Practical Applications.

**32.** In all the cases hitherto considered the equation of the curve has been given, and its graph has been drawn by first selecting values of $x$ and $y$ which satisfy the equation, and then drawing a line so as to pass through the plotted points. We thus determine accurately the position of as many points as we please, and the process employed assures us that they all lie on the graph we are seeking. We could obtain the same result without knowing the equation of the curve provided that we were furnished with a sufficient number of corresponding values of the variables *accurately calculated.*

Sometimes from the nature of the case the form of the equation which connects two variables is known. For example, if a quantity $y$ is directly proportional to another quantity $x$ it is evident that we may put $y = ax$, where $a$ is some constant quantity. Hence in all cases of direct proportionality between two quantities the graph which exhibits their variations is a straight line through the origin. Also since two points are sufficient to determine a straight line, it follows that in the cases under consideration we only require to know the position of one point besides the origin, and this will be furnished by any pair of simultaneous values of the variables.

*Example* 1. Given that 5·5 kilograms are roughly equal to 12·125 pounds, shew graphically how to express any number of pounds in kilograms. Express $7\frac{1}{2}$ lbs. in kilograms, and $4\frac{1}{4}$ kilograms in pounds.

Here measuring pounds horizontally and kilograms vertically, the required graph is obtained at once by joining the origin to the point whose coordinates are 12·125 and 5·5.

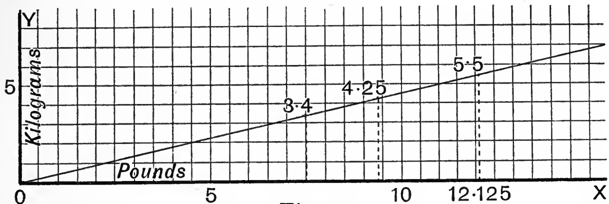



Fig. 19.

By measurement it will be found that $7\frac{1}{2}$ lbs. $=3·4$ kilograms, and $4\frac{1}{4}$ kilograms $=9·37$ lbs.

*Example* 2. The expenses of a school are partly constant and partly proportional to the number of boys. The expenses were £650 for 105 boys, and £742 for 128. Draw a graph to represent the expenses for any number of boys; find the expenses for 115 boys, and the number of boys that can be maintained at a cost of £710.

If the expenses for $x$ boys are represented by £$y$, it is evident that $x$ and $y$ satisfy a linear equation $y=ax+b$, where $a$ and $b$ are constants. Hence the graph is a straight line.

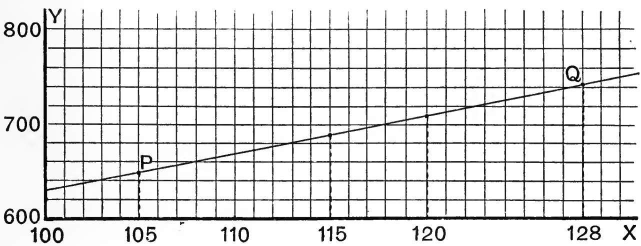



Fig. 20.

As the numbers are large, it will be convenient if we begin measuring ordinates at 600, and abscissæ at 100. This enables us to bring the requisite portion of the graph into a smaller compass. The points $P$ and $Q$ are determined by the data of the question, and the line $PQ$ is the graph required.

By measurement we find that when $x=115$, $y=690$; and that when $y=710$, $x=120$. Thus the required answers are £690, and 120 boys.

**33.** Sometimes corresponding values of two variables are obtained by observation or experiment. In such cases the data cannot be regarded as free from error; the position of the plotted points cannot be absolutely relied on; and we cannot correct irregularities in the graph by plotting other points selected at discretion. All we can do is to draw a curve to lie as evenly as possible among the plotted points, passing through some perhaps, and with the rest fairly distributed on either side of the curve. As an aid to drawing an even continuous curve a thin piece of wood or other flexible material may be bent into the requisite curve, and held in position while the line is drawn.* When the plotted points lie approximately on a straight line, the simplest plan is to use a piece of tracing paper or celluloid on which a straight line has been drawn. When this has been placed in the right position the extremities can be marked on the squared paper, and by joining these points the approximate graph is obtained.

*Example* 1. The following table gives statistics of the population of a certain country, where $P$ is the number of millions at the beginning of each of the years specified.

| Year | 1830 | 1835 | 1840 | 1850 | 1860 | 1865 | 1870 | 1880 |
|---|---|---|---|---|---|---|---|---|
| $P$ | 20 | 22·1 | 23·5 | 29·0 | 34·2 | 38·2 | 41·0 | 49·4 |

Let $t$ be the time in years from 1830. Plot the values of $P$ vertically and those of $t$ horizontally and exhibit the relation between $P$ and $t$ by a simple curve passing fairly evenly among the plotted points. Find what the population was at the beginning of the years 1848 and 1875.

The graph is given in Fig. 21 on the opposite page. The populations in 1848 and 1875, at the points $A$ and $B$ respectively, will be found to be 27·8 millions and 45·3 millions.

*Example* 2. Corresponding values of $x$ and $y$ are given in the following table:

| $x$ | 1 | 4 | 6·8 | 8 | 9·5 | 12 | 14·4 |
|---|---|---|---|---|---|---|---|
| $y$ | 4 | 8 | 12·2 | 13 | 15·3 | 20 | 24·8 |

Supposing these values to involve errors of observation, draw the graph approximately and determine the most probable equation between $x$ and $y$. [See Fig. 22 on p. 36.]

* One of "Brooks' Flexible Curves" will be found very useful.

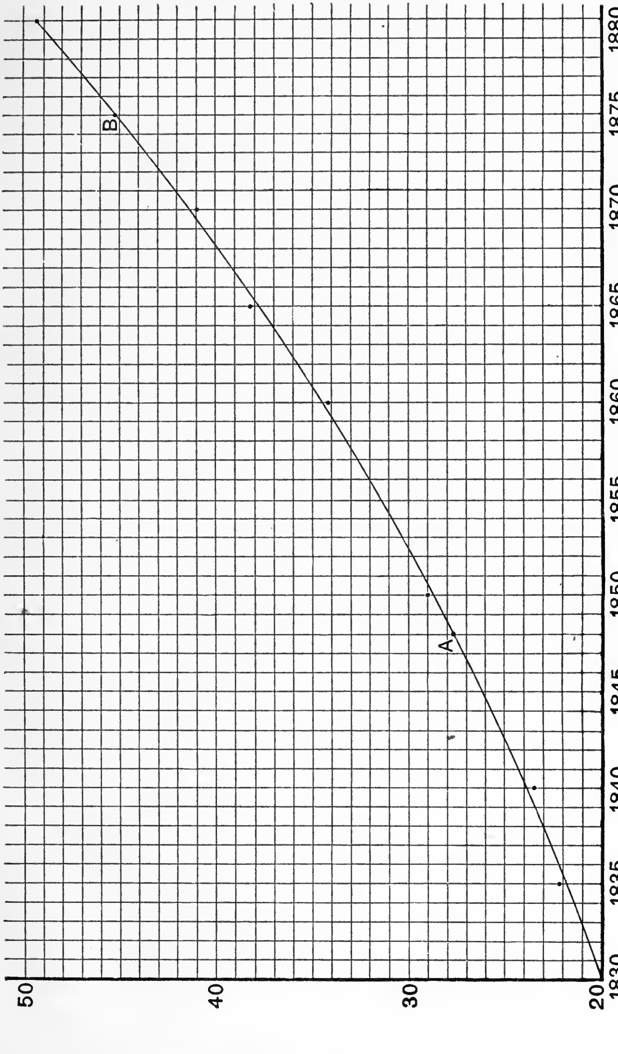

1880
1875
1870
1865
1860
1855
1850
1845
1840
1835
1830
50
40
30
20
A
B

Fig. 21.

After carefully plotting the given points we see that a straight line can be drawn passing through three of them and lying evenly among the others. This is the required graph.

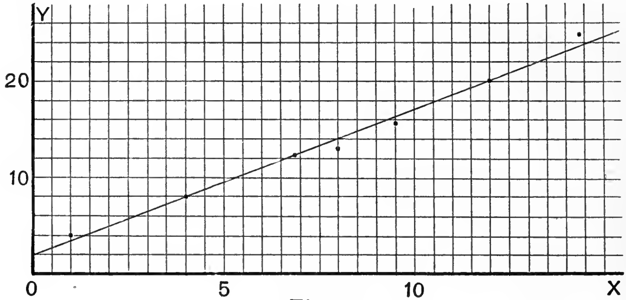



Fig. 22.

Assuming $y=ax+b$ for its equation, we find the values of $a$ and $b$ by selecting two pairs of simultaneous values of $x$ and $y$.

Thus substituting $x=4$, $y=8$, and $x=12$, $y=20$ in the equation, we obtain $a=1{\cdot}5$, $b=2$. Thus the equation of the graph is $y=1{\cdot}5x+2$.

**34.** In the last Example as the graph is linear it can be produced to any extent within the limits of the paper, and so any value of one of the variables being determined, the corresponding value of the other can be read off. When large values are in question this method is not only inconvenient but unsafe, owing to the fact that any divergence from accuracy in the portion of the graph drawn is increased when the curve is produced beyond the limits of the plotted points. The following Example illustrates the method of procedure in such cases.

*Example.* In a certain machine $P$ is the force in pounds required to raise a weight of $W$ pounds. The following corresponding values of $P$ and $W$ were obtained experimentally :

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $P$ | * 3·08 | 3·9 | 6·8 | 8·8 | 9·2 | * 11 | 13·3 |
| $W$ | 21 | 36·25 | 66·2 | 87·5 | 103·75 | 120 | 152·5 |

By plotting these values on squared paper draw the graph connecting $P$ and $W$, and read off the value of $P$ when $W=70$. Also determine a linear law connecting $P$ and $W$; find the force necessary to raise a weight of 310 lbs., and also the weight which could be raised by a force of 180·6 lbs.

As the page is too small to exhibit the graphical work on a convenient scale we shall merely indicate the steps of the solution, which is similar in detail to that of the last example.

Plot the values of $P$ vertically and the values of $W$ horizontally. It will be found that a straight line can be drawn through the points corresponding to the results marked with an asterisk, and lying evenly among the other points. From this graph we find that when $W=70$, $P=7$.

Assume $P=aW+b$, and substitute for $P$ and $W$ from the values corresponding to the two points through which the line passes. By solving the resulting equations we obtain $a=\cdot08$, $b=1\cdot4$. Thus the linear equation connecting $P$ and $W$ is $P=\cdot08W+1\cdot4$.

This is called the **Law of the Machine.**

From this equation, when $W=310$, $P=26\cdot2$, and when $P=180\cdot6$, $W=2240$.

Thus a force of 26·2 lbs. will raise a weight of 310 lbs.; and when a force of 180·6 lbs. is applied the weight raised is 2240 lbs. or 1 ton.

**Note.** The equation of the graph is not only useful for determining results difficult to obtain graphically, but it can always be used to check results found by measurement.

**35.** The example in the last article is a simple illustration of a method of procedure which is common in the laboratory or workshop, the object being to determine the law connecting two variables when a certain number of simultaneous values have been determined by experiment or observation.

Though we can always draw a graph to lie fairly among the plotted points corresponding to the observed values, unless the graph is a straight line it may be difficult to find its equation except by some indirect method.

For example, suppose $x$ and $y$ are quantities which satisfy an equation of the form $xy=ax+by$, and that this law has to be discovered.

By writing the equation in the form

$$\frac{a}{y}+\frac{b}{x}=1, \quad \text{or } au+bv=1;$$

where $u=\frac{1}{y}$, $v=\frac{1}{x}$, it is clear that $u$, $v$ satisfy the equation of a straight line. In other words, if we were to plot the points corresponding to the reciprocals of the given values, their linear connection would be at once apparent. Hence the values of $a$ and $b$ could be found as in previous examples, and the required law in the form $xy=ax+by$ could be determined.

Again, suppose $x$ and $y$ satisfy an equation of the form $x^n y = c$, where $n$ and $c$ are constants.

By taking logarithms, we have

$$n \log x + \log y = \log c.$$

The form of this equation shews that $\log x$ and $\log y$ satisfy the equation to a straight line. If, therefore, the values of $\log x$ and $\log y$ are plotted, a linear graph can be drawn, and the constants $n$ and $c$ can be found as before.

*Example.* The weight, $y$ grammes, necessary to produce a given deflection in the middle of a beam supported at two points, $x$ centimetres apart, is determined experimentally for a number of values of $x$ with results given in the following table:

| $x$ | 50 | 60 | 70 | 80 | 90 | 100 |
|---|---|---|---|---|---|---|
| $y$ | 270 | 150 | 100 | 60 | 47 | 32 |

Assuming that $x$ and $y$ are connected by the equation $x^n y = c$, find $n$ and $c$.

| $\log x$ | $\log y$ |
|---|---|
| 1·699 | 2·431 |
| 1·778 | 2·176 |
| 1·845 | 2·000 |
| 1·903 | 1·778 |
| 1·954 | 1·672 |
| 2·000 | 1·519 |

From a book of tables we obtain the annexed values of $\log x$ and $\log y$ corresponding to the observed values of $x$ and $y$. By plotting these we obtain the graph given in Fig. 23, and its equation is of the form

$$n \log x + \log y = \log c.$$

To obtain $n$ and $c$, choose *two extreme points through which the line passes.* It will be found that when

$$\log x = 1{\cdot}642, \quad \log y = 2{\cdot}6$$

and when

$$\log x = 2{\cdot}1, \quad \log y = 1{\cdot}21.$$

Substituting these values, we have

$$2{\cdot}6 + n \times 1{\cdot}642 = \log c \text{ ..................(i)},$$

$$1{\cdot}21 + n \times 2{\cdot}1 = \log c \text{ ..................(ii)};$$

$$\therefore \ 1{\cdot}39 - 0{\cdot}458n = 0;$$

whence

$$n = 3{\cdot}04.$$

$$\therefore \text{ from (ii) } \log c = 6{\cdot}38 + 1{\cdot}21$$

$$= 7{\cdot}59;$$

$$\therefore \ c = 39 \times 10^6, \text{ from the tables.}$$

Thus the required equation is $x^3 y = 39 \times 10^6$.

The student should work through this example in detail on a larger scale. The adjoining figure was drawn on paper ruled to tenths of an inch and then reduced to half the original scale.

**7.** At different ages the mean after-lifetime ("expectation of life") of males, calculated on the death rates of 1871-1880, was given by the following table:

| Age | 6 | 10 | 14 | 18 | 22 | 26 | 27 |
|---|---|---|---|---|---|---|---|
| Expectation | 50·38 | 47·60 | 44·26 | 40·96 | 37·89 | 34·96 | 34·24 |

Draw a graph to shew the expectation of any male between the ages of 6 and 27, and from it determine the expectation of persons aged 12 and 20.

**8.** In the Clergy Mutual Assurance Society the premium (£*P*) to insure £100 at different ages is given approximately by the following table:

| Age | 20 | 22 | 25 | 30 | 35 | 40 | 45 | 50 | 55 |
|---|---|---|---|---|---|---|---|---|---|
| *P* | 1·8 | 1·9 | 2·0 | 2·3 | 2·7 | 3·1 | 3·6 | 4·4 | 5·5 |

Illustrate the same statistics graphically, and estimate to the nearest shilling the premiums for persons aged 34 and 43.

**9.** If *W* is the weight in ounces required to stretch an elastic string till its length is *l* inches, plot the following values of *W* and *l*:

| *W* | 2·5 | 3·75 | 6·25 | 7·5 | 10 | 11·25 |
|---|---|---|---|---|---|---|
| *l* | 8·5 | 8·7 | 9·1 | 9·3 | 9·7 | 9·9 |

From the graph determine the unstretched length of the string, and the weight the string will support when its length is 1 foot.

**10.** In the following table *P* and *A* (expressed in hundreds of pounds) represent the Principal and corresponding Amount for 1 year at 3 per cent. simple interest.

| *P* | 2·3 | 2·7 | 3·0 | 3·5 | 3·9 | 5·2 | 7·6 |
|---|---|---|---|---|---|---|---|
| *A* | 2·369 | 2·781 | 3·090 | 3·605 | 4·017 | 5·356 | 7·828 |

Plot the values of *P* and *A* on a large scale, and from the graph determine the Principal which will amount to (i) £329. 12*s.*; (ii) £597. 8*s*.

**11.** The highest and lowest marks gained in an examination are 297 and 132 respectively. These have to be reduced in such a way that the maximum for the paper (200) shall be given to the first candidate, and that there shall be a range of 150 marks between the first and last. Find the equation between $x$, the actual marks gained, and $y$, the corresponding marks when reduced.

Draw the graph of this equation, and read off the marks which should be given to candidates who gained 200, 262, 163 marks in the examination.

**12.** A body starting with an initial velocity, and subject to an acceleration in the direction of motion, has a velocity of $v$ feet per second after $t$ seconds. If corresponding values of $v$ and $t$ are given by the annexed table,

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $v$ | 9 | 13 | 17 | 21 | 25 | 29 | 33 | 37 | 41 | 45 |
| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

plot the graph exhibiting the velocity at any given time. Find from it (i) the initial velocity, (ii) the time which has elapsed when the velocity is 28 feet per second. Also find the equation between $v$ and $t$.

**13.** The connection between the areas of equilateral triangles and their bases (in corresponding units) is given by the following table:

| | | | | | | |
|---|---|---|---|---|---|---|
| Area | ·43 | 1·73 | 3·90 | 6·93 | 10·82 | 15·59 |
| Base | 1 | 2 | 3 | 4 | 5 | 6 |

Illustrate these results graphically, and determine the area of an equilateral triangle on a base of 2·4 ft.

**14.** A body falling freely under gravity drops $s$ feet in $t$ seconds from the time of starting. If corresponding values of $s$ and $t$ at intervals of half a second are as follows:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $t$ | ·5 | 1 | 1·5 | 2 | 2·5 | 3 | 3·5 | 4 |
| $s$ | 4 | 16 | 36 | 64 | 100 | 144 | 196 | 256 |

draw the curve connecting $s$ and $t$, and find from it

(i) the distance through which the body has fallen after 1·8″.

(ii) the depth of a well if a stone takes 3·16″ to reach the bottom.

**15.** A body is projected with a given velocity at a given angle to the horizon, and the height in feet reached after $t$ seconds is given by the equation $h=64t-16t^2$. Find the values of $h$ at intervals of $\frac{1}{4}$th of a second and draw the path described by the body. Find the maximum value of $h$, and the time after projection before the body reaches the ground.

**16.** The keeper of a hotel finds that when he has $G$ guests a day his total daily profit is $P$ pounds. If the following numbers are averages obtained by comparison of many days' accounts determine a simple relation between $P$ and $G$.

| $G$ | 21 | 27 | 29 | 32 | 35 |
|---|---|---|---|---|---|
| $P$ | $-1\cdot8$ | 2 | $3\cdot2$ | $4\cdot5$ | $6\cdot6$ |

For what number of guests would he just have no profit?

**17.** A man wishes to place in his catalogue a list of a certain class of fishing rods varying from 9 ft. to 16 ft. in length. Four sizes have been made at prices given in the following table:

| 9 ft. | 11 ft. 9 in. | 14 ft. 4 in. | 16 ft. |
|---|---|---|---|
| 15*s*. | 22*s*. | 31*s*. | 38*s*. |

Draw a graph to exhibit prices for rods of intermediate lengths, and from it determine the probable prices for rods of 13 ft. and 15 ft. 8 in.

**18.** The following table gives the sun's position at 7 A.M. on different dates:

| Mar. 23 | Ap. 3 | Ap. 20 | May 8 | May 27 | June 22 | July 18 | Aug. 5 | Aug. 25 |
|---|---|---|---|---|---|---|---|---|
| 80° E. | 82° E. | 85° E. | 89° E. | 92° E. | 95° E. | 94° E. | 91° E. | 85° E. |

Shew these results graphically, and estimate approximately the sun's position at the same hour on June 8th.

**19.** At a given temperature $p$ lbs. per square inch represents the pressure of a gas which occupies a volume of $v$ cubic inches. Draw a curve connecting $p$ and $v$ from the following table of corresponding values:

| $p$ | 36 | 30 | $25\cdot7$ | $22\cdot5$ | 20 | 18 | $16\cdot4$ | 15 |
|---|---|---|---|---|---|---|---|---|
| $v$ | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |

**20.** Plot on squared paper the following measured values of $x$ and $y$, and determine the most probable equation between $x$ and $y$:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | 3 | 5 | 8·3 | 11 | 13 | 15·5 | 18·6 | 23 | 28 |
| $y$ | 2 | 2·2 | 3·4 | 3·8 | 4 | 4·6 | 5·4 | 6·2 | 7·25 |

**21.** The following table refers to aqueous solution of ammonia at a given temperature; $x$ represents the specific gravity of the solution, and $y$ the percentage of ammonia:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | ·996 | ·992 | ·988 | ·984 | ·980 | ·976 | ·968 |
| $y$ | ·91 | 1·84 | 2·80 | 3·80 | 4·80 | 5·80 | 7·82 |

Draw a graph shewing the variations of $x$ and $y$, and find its equation.

**22.** Corresponding values of $x$ and $y$ are given in the following table:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | 1 | 3·1 | 6 | 9·5 | 12·5 | 16 | 19 | 23 |
| $y$ | 2 | 2·8 | 4·2 | 5·3 | 6·6 | 8·3 | 9 | 10·8 |

Supposing these values to involve errors of observation, draw the graph approximately, and determine the most probable equation between $x$ and $y$. Find the correct value of $y$ when $x=19$, and the correct value of $x$ when $y=2·8$.

**23.** The following corresponding values of $x$ and $y$ were obtained experimentally:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | 0·5 | 1·7 | 3·0 | 4·7 | 5·7 | 7·1 | 8·7 | 9·9 | 10·6 | 11·8 |
| $y$ | 148 | 186 | 265 | 326 | 388 | 436 | 529 | 562 | 611 | 652 |

It is known that they are connected by an equation of the form $y=ax+b$, but the values of $x$ and $y$ involve errors of measurement. Find the most probable values of $a$ and $b$, and estimate the error in the measured value of $y$ when $x=9·9$.

**24.** In a certain machine $P$ is the force in pounds required to raise a weight of $W$ pounds. The following corresponding values of $P$ and $W$ were obtained experimentally:

| $P$ | 2·8 | 3·7 | 4·8 | 5·5 | 6·5 | 7·3 | 8 | 9·5 | 10·4 | 11·75 |
|---|---|---|---|---|---|---|---|---|---|---|
| $W$ | 20 | 25 | 31·7 | 35·6 | 45 | 52·4 | 57·5 | 65 | 71 | 82·5 |

Draw the graph connecting $P$ and $W$, and read off the value of $P$ when $W=60$. Also determine the law of the machine, and find from it the weight which could be raised by a force of 31·7 lbs.

**25.** The following values of $x$ and $y$, some of which are slightly inaccurate, are connected by an equation of the form $y=ax^2+b$.

| $x$ | 1 | 1·6 | 3 | 3·7 | 4 | 5 | 5·7 | 6 | 6·3 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 3·25 | 4 | 5 | 6·5 | 7·4 | 9·25 | 10·5 | 11·6 | 14 | 15·25 |

By plotting these values draw the graph, and find the most probable values of $a$ and $b$.

Find the true value of $x$ when $y=4$, and the true value of $y$ when $x=6$.

**26.** The following table gives corresponding values of two variables $x$ and $y$:

| $x$ | 2·75 | 3 | 3·2 | 3·5 | 4·3 | 4·5 | 5·3 | 6 | 7 | 8 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 11 | 9·8 | 8 | 6·5 | 6·1 | 5·4 | 5 | 4·3 | 4·1 | 4 | 3·9 |

These values involve errors of observation, but the true values are known to satisfy an equation of the form $xy=ax+by$. Draw the graph by plotting the points determined by the above table, and find the most probable values of $a$ and $b$. Find the correct values of $y$ corresponding to $x=3·5$, and $x=7$.

**27.** Observed values of $x$ and $y$ are given as follows:

| $x$ | 100 | 90 | 70 | 60 | 50 | 40 |
|---|---|---|---|---|---|---|
| $y$ | 30 | 31·08 | 33·5 | 35·56 | 37·8 | 40·7 |

Assuming that $x$ and $y$ are connected by an equation of the form $xy^n=c$, find $n$ and $c$.

**28.** The following values of $x$ and $y$ involve errors of observation:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | 66·83 | 63·10 | 58·88 | 51·52 | 48·53 | 44·16 | 40·36 | 37·15 |
| $y$ | 144·5 | 158·5 | 177·8 | 208·9 | 236·0 | 264·9 | 309·0 | 346·7 |

If $x$ and $y$ satisfy an equation of the form $x^n y = c$, find $n$ and $c$.

**29.** In the following table the values of $C$ and $C'$ represent the calculated and observed amounts of water, in cubic feet per second, flowing through a circular orifice for different heads of water represented by $H$ feet.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $H$ | 60 | 69·12 | 82 | 92·16 | 106 | 115·2 | 134 |
| $C$ | ·0133 | ·0141 | ·0154 | ·0163 | ·0175 | ·0182 | ·0197 |
| $C'$ | ·0133 | ·0141 | ·0153 | ·0162 | ·0173 | ·0180 | ·0194 |

Plot the graph of $C$ and $H$ and also that of $C'$ and $H$, and deduce the probable error in the observed flow for a head of 120 feet.

**30.** The following table gives the pressures (in lbs. per sq. in.) and corresponding Fahrenheit temperatures at which water boils:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $P$ | 29·7 | 24·54 | 17·53 | 14·7 | 12·25 | 9·80 | 7·84 |
| $t$ | 249·6 | 239·0 | 221·0 | 212·0 | 203·0 | 192·3 | 182·0 |

Shew graphically the relation between temperature and pressure of boiling water.

**31.** It is known that the relation of pressure to volume in saturated steam under certain conditions is of the form $pv^n = \text{constant}$. Find the value of the index $n$ from the following data:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $p$ | 10·2 | 14·7 | 20·8 | 24·5 | 33·7 | 39·2 | 45·5 |
| $v$ | 37·5 | 26·6 | 19·2 | 16·4 | 12·2 | 10·6 | 9·2 |

where $p$ is measured in lbs. per sq. in., and $v$ is the volume of 1 lb. of steam in cub. ft.

**32.** The following table gives the speed and corresponding indicated horse-power of the engines of a ship:

| Speed in knots | 11 | 12·4 | 13·3 | 14·25 | 14·8 | 15·5 |
|---|---|---|---|---|---|---|
| I.H.P. | 1000 | 1500 | 2000 | 2500 | 3000 | 3500 |

At what speed will she go when she develops 4000 I.H.P.?

**33.** In testing a steam-engine when steam was expanded to 4·8 times its original volume, the following quantities of steam per indicated horse-power per hour were used:

| Steam per I.H.P. per hr. in lbs. | 16·9 | 17 | 17·2 | 18 | 20·3 |
|---|---|---|---|---|---|
| I.H.P. | 40·5 | 33 | 25·5 | 19 | 11 |

When the ratio of expansion in the engine was 10 instead of 4·8, the steam used was as follows:

| Steam per I.H.P. per hr. in lbs. | 15 | 15·5 | 16 | 18 | 26·5 |
|---|---|---|---|---|---|
| I.H.P. | 33 | 27·2 | 23 | 15 | 5 |

At what H.P. will the consumption of steam be the same in the two cases, and what is the consumption of steam at that H.P.?

**34.** The power required to produce a given speed in the case of each of two ships is given in the following tables:

(i)

| Speed | 8 | 10·7 | 12·7 | 14 | 16 | 16·2 |
|---|---|---|---|---|---|---|
| I.H.P. | 500 | 1000 | 1500 | 1950 | 2800 | 3000 |

(ii)

| Speed | 8 | 10 | 12 | 12·5 | 13·5 | 14·5 | 16·1 | 16·7 |
|---|---|---|---|---|---|---|---|---|
| I.H.P. | 200 | 400 | 920 | 1100 | 1500 | 2000 | 3000 | 3500 |

At what speed will they generate the same H.P.?

**35.** £$A$ represents the amount at compound interest of £$P$ in $n$ years at $r$ per cent.; the following table gives corresponding values of $A$ and $n$:

| $A$ | 218·5 | 238·6 | 268·5 | 310·5 | 360·6 | 417·8 |
|---|---|---|---|---|---|---|
| $n$ | 3 | 6 | 10 | 15 | 20 | 25 |

By plotting the values of $n$ and $\log A$, determine a simple algebraical relation between them. Thence find the numerical values of $P$ and $r$; also from the graph find in how many years £200 will amount to £301. 6*s*. at 3 per cent.

**36.** The keeper of a restaurant finds that when he has $G$ guests a day his total daily expenditure is $E$ pounds, and his total daily receipts amount to $R$ pounds. The following numbers are averages obtained by comparison of his books on many days:

| $G$ | 210 | 270 | 320 | 360 |
|---|---|---|---|---|
| $E$ | 16·7 | 19·4 | 21·6 | 23·4 |
| $R$ | 15·8 | 21·2 | 26·4 | 29·8 |

By plotting these values find $E$ and $R$ when he has 340 guests. What number of guests per day gives him (i) no profit, (ii) £6 profit? Find simple algebraical relations between $E$ and $G$, $R$ and $G$, $P$ and $G$, where £$P$ is the daily profit.

**37.** A manufacturer finds that when he is employing $W$ workmen his total weekly expenditure (including wages, material, coal, gas, insurance of premises, etc.) amounts to $E$ pounds, and his receipts amount to $R$ pounds. After carefully balancing his books for many weeks, the following table of average results was drawn up:

| $W$ | 25 | 30 | 35 | 45 | 50 | 60 |
|---|---|---|---|---|---|---|
| $E$ | 30·2 | 35·4 | 40·1 | 48·0 | 53·1 | 59·8 |
| $R$ | 27·5 | 39·1 | 49·8 | 75·0 | 84·8 | 109·2 |

From these data determine simple algebraical relations between $E$ and $W$, $R$ and $W$, $P$ and $W$, where £$P$ represents his weekly profits. Also find graphically (i) the number of workmen necessary to ensure a weekly profit of £18. 10*s*., (ii) the smallest number of men that will enable the manufacturer to pay expenses.

**38.** At the following draughts in sea water a particular vessel has the following displacements:

| | | | | |
|---|---|---|---|---|
| Draught $h$ feet - | 15 | 12 | 9 | 6·3 |
| Displacement $T$ tons | 2098 | 1512 | 1018 | 586 |

By plotting $\log T$ and $\log h$ on squared paper, obtain a simple relation between $T$ and $h$. If one ton of sea water measures 35 cubic feet find the relation between $V$ and $h$, if $V$ is the displacement in cubic feet.

**39.** The following quantities are thought to follow a law of the form $pv^n = c$.

| | | | | | |
|---|---|---|---|---|---|
| $v$ | 1 | 2 | 3 | 4 | 5 |
| $p$ | 205 | 114 | 80 | 63 | 52 |

Ascertain if this is the case, and find the most probable values of $n$ and $c$.

**40.** In some experiments in towing a canal boat the following observations were made; $P$ being the pull in pounds and $v$ the speed of the boat in miles per hour:

| | | | | | |
|---|---|---|---|---|---|
| $P$ | 76 | 160 | 240 | 320 | 370 |
| $v$ | 1·68 | 2·43 | 3·18 | 3·60 | 4·03 |

By plotting $\log P$ and $\log v$, shew that $P$ and $v$ approximately satisfy an equation of the form $P = bv^a$, and find the best values for $a$ and $b$.

**41.** In the following table $h$ represents the draught in feet of a certain vessel when her displacement is $D$ tons.

| | | | | |
|---|---|---|---|---|
| $h$ | 6 | 10 | 12 | 15 |
| $D$ | 407 | 998 | 1343 | 2070 |

By plotting the values of $\log h$ and $\log D$ determine an algebraical relation between $D$ and $h$. Find the displacement for a draught of 8 ft.

## Miscellaneous Applications of Linear Graphs.

**36.** When two quantities $x$ and $y$ are so related that a change in one produces a proportional change in the other, their variations can always be expressed by an equation of the form $y=ax$, where $a$ is some constant quantity. Hence in all such cases the graph which exhibits their variations is *a straight line through the origin*, so that in order to draw the graph it is only necessary to know the position of one other point on it. Such examples as deal with work and time, distance and time (when the speed is uniform), quantity and cost of material, principal and simple interest at a given rate per cent., may all be illustrated by linear graphs through the origin.

EXAMPLE 1. *At* 8 a.m. *A starts from* P *to ride to* Q *which is* 48 *miles distant. At the same time B sets out from* Q *to meet A. If A rides at* 8 *miles an hour, and rests half an hour at the end of every hour, while B walks uniformly at* 4 *miles an hour, find graphically*

(i) *the time and place of meeting*;

(ii) *the distance between A and B at* 11 *a.m.*;

(iii) *at what time they are* 14 *miles apart.*

In Fig. 24, on the opposite page, let the position of P be chosen as origin; let time be measured horizontally from 8 a.m. (1 inch to 1 hour), and let distance be measured vertically (1 inch to 20 miles).

In 1 hr. $A$ rides 8 mi.; therefore the point D (1, 8) marks his position at 9 a.m. In the next half-hour he makes no advance towards Q; therefore the corresponding portion of the graph is DE. The details of $A$'s motion may now be completed by the broken line PDEFGHKX.

On the vertical axis mark PQ to represent 48 mi. and mark the hours on the horizontal line through Q. At 9 a.m. $B$ has walked 4 mi. towards P. Measuring a distance to represent 4 mi. *downwards* we get the point R, and QR produced is the graph of $B$'s motion. It cuts $A$'s graph at X. Hence the point of meeting is X, which is 28 mi. from P, and the time is 1 p.m.

The distance between $A$ and $B$ at any time is shewn by the difference of the ordinates. Thus at 11 a.m. their distance apart is MG, which represents 20 mi.

Lastly, NT represents 14 mi.; thus $A$ and $B$ are 14 mi. apart at 11.30 a.m.

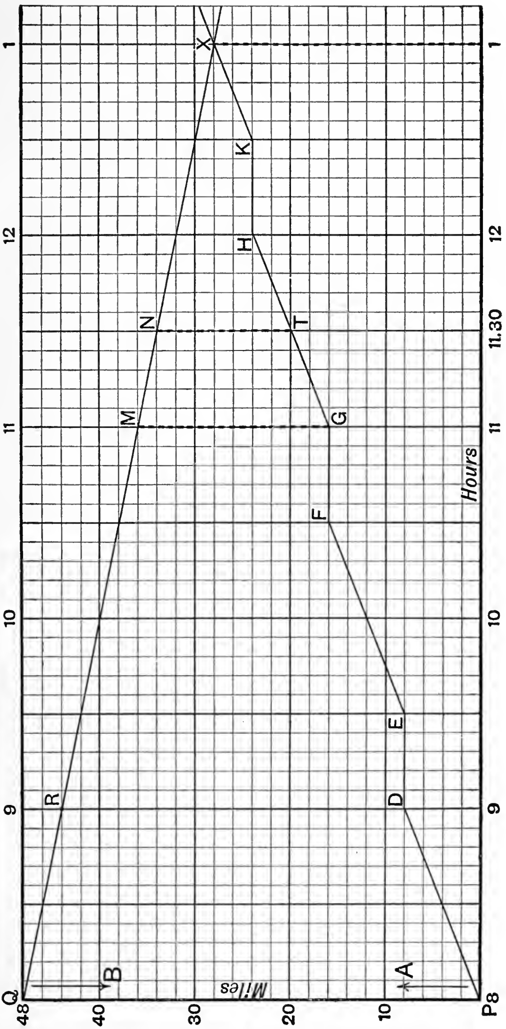



Fig. 24.

EXAMPLE 2. *A, B, and C run a race of* 300 *yards. A and C start from scratch, and A covers the distance in* 40 *seconds, beating C by* 60 *yards. B, with* 12 *yards' start, beats A by* 4 *seconds. Supposing the rates of running in each case to be uniform, find graphically the relative positions of the runners when B passes the winning post. Find also by how many yards B is ahead of A when the latter has run three-fourths of the course.*

In Fig. 25 let time be measured horizontally (0·5 inch to 10 seconds), and distance vertically (1 inch to 60 yards). O is the starting point for $A$ and $C$; take OP equal to 0·2 inch, representing 12 yards, on the vertical axis; then P is $B$'s starting point.

$A$'s graph is drawn by joining O to the point which marks 40 seconds. From this point measure a vertical distance of 1 inch downwards to Q. Then since 1 inch represents 60 yards, Q is $C$'s position when $A$ is at the winning post, and OQ is $C$'s graph.

Along the time-axis take 1·8 inch to R, representing 36 seconds; then PR is $B$'s graph.

Through R draw a vertical line to meet the graphs of $A$ and $C$ in S and T respectively. Then S and T mark the positions of $A$ and $C$ when $B$ passes the winning post.

By inspection RS and ST represent 30 and 54 yards respectively.

Thus $B$ is 30 yards ahead of $A$, and $A$ is 54 yards ahead of $C$.

Again, since $A$ runs three-fourths of the course in 30 seconds, the difference of the corresponding ordinates of $A$'s and $B$'s graphs after 30 seconds will give the distance between $A$ and $B$. By measurement we find VW = 0·45 inch, which represents 27 yards.

The student is recommended to draw a figure for himself on a scale twice as large as that given in Fig. 25.

**37.** When a variable quantity $y$ is partly constant and partly proportional to a variable quantity $x$, the algebraical relation between $x$ and $y$ is of the form $y = ax + b$, where $a$ and $b$ are constant. The corresponding graph will therefore be a straight line; and since a straight line is completely determined when the positions of two points are known, it follows that, in all problems which can be illustrated by linear graphs, it is sufficient if the data furnish for each graph two independent pairs of simultaneous values of the variable quantities.

Some easy examples of this kind have already been given on page 33 and in Examples VII. We shall now work out two more examples.

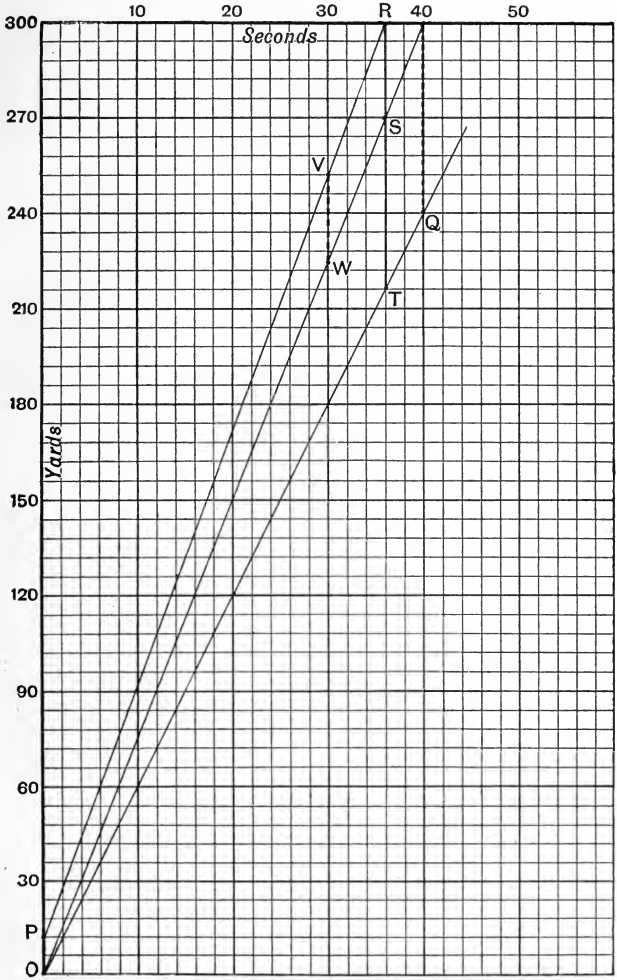

10
20
30
R
40
50
Seconds
300
270
240
210
180
150
120
90
60
30
P
O
Yards
V
S
W
Q
T

FIG. 25.

EXAMPLE 1. *In a certain establishment the clerks are paid an initial salary for the first year, and this is annually increased by a fixed bonus, the initial salary and the bonus being different in different departments. A receives* £130 *in his* 10th *year, and* £220 *in his* 19th. *B, in another department, receives* £140 *in his* 5th *year and* £180 *in his* 13th. *Draw graphs to shew their salaries in different years. In what year do they receive equal salaries? Also find in what year A earns the same salary as that received by B for his* 21st *year.*

In Fig. 26 let each horizontal division represent 1 year; and let the salaries be measured vertically, beginning at 130, with 1 division to represent £2.

If the salary at the end of $x$ years is denoted by £$y$, it is evident that in each case we have a relation of the form $y = ax + b$, where $a$ and $b$ are constant. Thus the variations of time and salary may be represented by linear graphs.

Since no bonus is received for the first year, $x = 9$, when $y = 130$, and $x = 18$, when $y = 220$. Thus the points P and Q are determined, and by joining them we have the graph for $A$'s salary. Similarly the graph for $B$'s salary is found by joining P′ (4, 140) and Q′ (12, 180).

These lines have the same ordinate and abscissa at L, where $x = 16$, $y = 200$. Thus $A$ and $B$ have the same salary when each have served 16 years, that is in their 17th year. Again $B$'s salary at the end of 20 years is given by the ordinate of M, which is the same as that of Q which represents $A$'s salary after 18 years.

Thus $A$'s salary for his 19th year is equal to $B$'s salary for his 21st year.

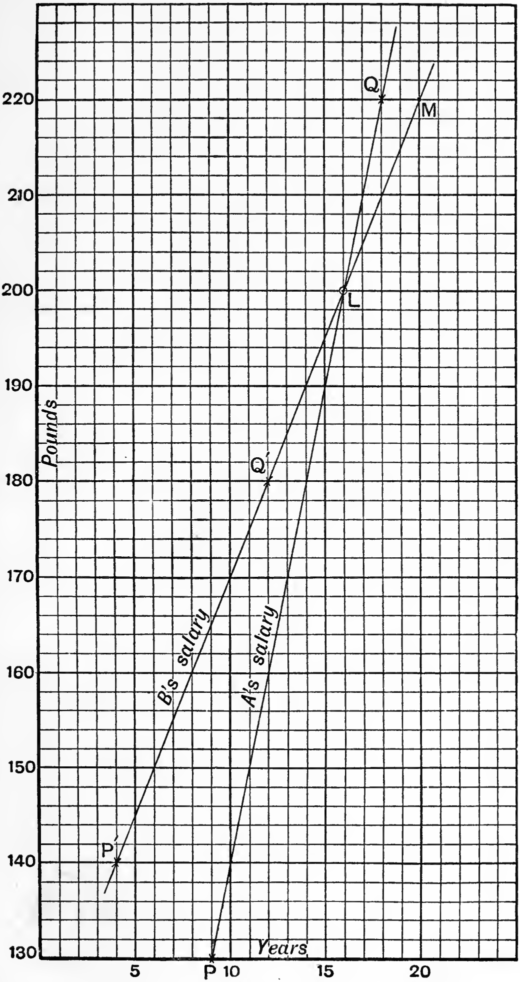
EXAMPLE 2. *Two sums of money are put out at simple interest at different rates per cent. In the first case the Amounts at the end of* 6 *years and* 15 *years are* £260 *and* £350 *respectively. In the second case the Amounts for* 5 *years and* 20 *years are* £330 *and* £420. *Draw graphs from which the Amounts may be read off for any year, and find the year in which the Principal with accrued Interest will amount to the same in the two cases. Also from the graphs read off the value of each Principal.*

When a sum of money is at simple interest for any number of years, we have

$$\text{Amount} = \text{Principal} + \text{Interest},$$

where 'Principal' is constant, and 'Interest' varies with the number of years. Hence the variations of Amount and Time may be represented by a linear graph in which $x$ is taken to denote the number of years, and $y$ the number of pounds in the corresponding Amount.

Here as the diagram is inconveniently large we shall merely indicate the steps of the solution which is similar in detail to that of the last example. The student should draw his own diagram.



220
210
200
190
180
170
160
150
140
130
Pounds
Q
M
L
Q'
B's salary
A's salary
P'
Years
5
P 10
15
20


FIG. 26.

Measure time horizontally (1 inch to 10 years), and Amount vertically (1 inch to £40) beginning at £260.

The first graph is the line joining L (6, 260) and M (15, 350). The second graph is the line joining L′ (5, 330) and M′ (20, 420). In each of these lines the ordinate of any point gives the Amount for the number of years given by the corresponding abscissa.

Again LM, L′M′ intersect at a point P where $x=25$, $y=450$. Thus each Principal with its Interest amounts to £450 in 25 years.

When $x=0$ there is no Interest; thus the Principals will be obtained by reading off the values of the intercepts made by the two graphs on the $y$-axis. These are £200 and £300 respectively.

**Note.** To obtain the result $y=200$ it will be necessary to continue the $y$-axis downwards sufficiently far to shew this ordinate.

## EXAMPLES VIII.

**1.** At noon $A$ starts to walk at 6 miles an hour, and at 1.30 p.m. $B$ follows on horseback at 8 miles an hour. When will $B$ overtake $A$? Also find

(i) when $A$ is 5 miles ahead of $B$;

(ii) when $A$ is 3 miles behind $B$.

[Take 1 inch horizontally to represent 1 hour, and 1 inch vertically to represent 10 miles.]

**2.** By measuring time along OX (1 inch for 1 hour) and distance along OY (1 inch for 10 miles) shew how to draw lines

(i) from O to indicate distance travelled towards Y at 12 miles an hour;

(ii) from Y to indicate distance travelled towards O at 9 miles an hour.

If these are the rates of two men who ride towards each other from two places 60 miles apart, starting at noon, find from the graphs when they are first 18 miles from each other. Also find (to the nearest minute) their time of meeting.

**3.** Two bicyclists ride to meet each other from two places 95 miles apart. $A$ starts at 8 a.m. at 10 miles an hour, and $B$ starts at 9.30 a.m. at 15 miles an hour. Find graphically when and where they meet, and at what times they are $37\frac{1}{2}$ miles apart.

**4.** $A$ and $B$ start at the same time from London to Blisworth, $A$ walking 4 miles an hour, $B$ riding 9 miles an hour. $B$ reaches Blisworth in 4 hours, and immediately rides back to London. After 2 hours' rest he starts again for Blisworth at the same rate. How far from London will he overtake $A$, who has in the meantime rested $6\frac{1}{4}$ hours?

**5.** At what distance from London, and at what time, will a train which leaves London for Rugby at 2.33 p.m., and goes at the rate of 35 miles an hour, meet a train which leaves Rugby at 1.45 p.m. and goes at the rate of 25 miles an hour, the distance between London and Rugby being 80 miles?

Also find at what times the trains are 24 miles apart, and how far apart they are at 4.9 p.m.

**6.** $A$, $B$, and $C$ set out to walk from Bath to Bristol at 5, 6, and 4 miles an hour respectively. $C$ starts 3 minutes before, and $B$ 7 minutes after $A$. Draw graphs to shew (1) when and where $A$ overtakes $C$; (ii) when and where $B$ overtakes $A$; (iii) $C$'s position relative to the others after he has walked 45 minutes.

[Take 1 inch horizontally to represent 10 minutes, and 1 inch to the mile vertically.]

**7.** X and Y are two towns 35 miles apart. At 8.30 p.m. $A$ starts to walk from X to Y at 4 miles an hour; after walking 8 miles he rests for half an hour and then completes his journey on horseback at 10 miles an hour. At 9.48 a.m. $B$ starts to walk from Y to X at 3 miles an hour; find when and where $A$ and $B$ meet. Also find at what times they are $6\frac{1}{2}$ miles apart.

**8.** $A$ can beat $B$ by 20 yards in 120, and $B$ can beat $C$ by 10 yards in 50. Supposing their rates of running to be uniform, find graphically how much start $A$ can give $C$ in 120 yards so as to run a dead heat with him. If $A$, $B$, and $C$ start together, where are $A$ and $C$ when $B$ has run 80 yards?

**9.** $A$, $B$, and $C$ run a race of 200 yards. $A$ gives $B$ a start of 8 yards, and $C$ starts some seconds after $A$. $A$ runs the distance in 25 seconds and beats $C$ by 40 yards. $B$ beats $A$ by 1 second, and when he has been running 15 seconds, he is 48 yards ahead of $C$. Find graphically how many seconds $C$ starts after $A$. Shew also from the graphs that if the three runners started level they would run a dead heat.

[Take 1 inch to 40 yards, and 1 inch to 10 seconds.]

**10.** A cyclist has to ride 75 miles. He rides for a time at 9 miles an hour and then alters his speed to 15 miles an hour covering the distance in 7 hours. At what time did he change his speed?

**11.** $A$ and $B$ ride to meet each other from two towns X and Y which are 60 miles apart. $A$ starts at 1 p.m., and $B$ starts 36 minutes later. If they meet at 4 p.m., and $A$ gets to Y at 6 p.m., find the time when $B$ gets to X. Also find the times when they are 22 miles apart. When $A$ is half-way between X and Y, where is $B$?

**12.** The distance from London to Bristol is 119 miles; if I were to set out at noon to cycle from London, riding 23 miles the first hour and decreasing my pace by 3 miles each successive hour, find graphically how long it would take me to reach Bristol. Also find approximately the time at which I should reach Faringdon, which is 48 miles from Bristol.

**13.** At 8 a.m. $A$ begins a ride on a motor car at 20 miles an hour, and an hour and a half later $B$, starting from the same point, follows on his bicycle at 10 miles an hour. After riding 36 miles, $A$ rests for 1 hr. 24 min., then rides back at 9 miles an hour. Find graphically when and where he meets $B$. Also find (i) at what time the riders were 21 miles apart, (ii) how far $B$ will have ridden by the time $A$ gets back to his starting point.

**14.** I row against a stream flowing $1\frac{1}{2}$ miles an hour to a certain point, and then turn back, stopping two miles short of the place whence I originally started. If the whole time occupied in rowing is 2 hrs. 10 mins. and my uniform speed in still water is $4\frac{1}{2}$ miles an hour, find graphically how far upstream I went.

[Take 1·2 of an inch horizontally to represent 1 hour, and 1 inch to 2 miles vertically.]

**15.** One train leaves Bristol at 3 p.m. and reaches London at 6 p.m.; a second train leaves London at 1.30 p.m. and arrives at Bristol at 6 p.m.; if both trains are supposed to travel uniformly, at what time will they meet? Shew from a graph that the time does not depend upon the distance between London and Bristol.

**16.** At 7.40 a.m. the ordinary train starts from Norwich and reaches London at 11.40 a.m.; the express starting from London at 9 a.m. arrives at Norwich at 11.40 a.m.: if both trains travel uniformly, find when they meet. Shew, as in Ex. 15, that the time is independent of the distance between London and Norwich, and verify this conclusion by solving an algebraical equation.

**17.** A boy starts from home and walks to school at the rate of 10 yards in 3 seconds, and is 20 seconds too soon. The next day he walks at the rate of 40 yards in 17 seconds, and is half a minute late. Find graphically the distance to the school, and shew that he would have been just in time if he had walked at the rate of 20 yards in 7 seconds.

**18.** The annual expenses of a Convalescent Home are partly constant and partly proportional to the number of inmates. The expenses were £384 for 12 patients and £432 for 16. Draw a graph to shew the expenses for any number of patients, and find from it the cost of maintaining 15.

In a rival establishment the expenses were £375 for 5, and £445 for 15 patients. Find graphically for what number of patients the cost would be the same in the two cases.

**19.** A body is moving in a straight line with varying velocity. The velocity at any instant is made up of the constant velocity with which it was projected (measured in feet per second) diminished by a retardation of a constant number of feet per second in every second. After 4 seconds the velocity was 320, and after 13 seconds it was 140. Draw a graph to shew the velocity at any time while the body is in motion.

A second body projected at the same time under similar conditions has a velocity of 450 after 5 seconds, and a velocity of 150 after 15 seconds. Shew graphically that they will both come to rest at the same time. Also find at what time the second body is moving 100 feet per second faster than the first, and determine from the graphs the velocity of projection in each case.

**20.** To provide for his two infant sons, a man left by his will two sums of money as separate investments at different rates of interest, on the condition that the principal sums with simple interest were to be paid over to his sons when the amounts were the same. After 5 years the first sum amounted to £451, and after 15 years to £533. After 10 years the second sum amounted to £432, and after 20 years to £544. Draw graphs from which the amounts may be read off for any year, and find after how many years the sons were entitled to receive their legacies.

Also determine from the graphs what the original sums were at the father's death.

**21.** In a certain examination the highest and lowest marks gained in a Latin paper were 153 and 51. These have to be reduced so that the maximum (120) is given to the first candidate, and the minimum (30) to the lowest. This is done by reducing all the marks in a certain ratio, and then increasing or diminishing them all by the same number. In a Greek paper the highest and lowest marks were 161 and 56; after a similar adjustment these become 100 and 40 respectively. Draw graphs from which all the reduced marks may be read off, and find the marks which should be finally given to a candidate who scored 102 in Latin and 126 in Greek.

Shew also that it is possible in one case for a candidate to receive equal marks in the two subjects both before and after reduction. What are the original and reduced marks in this case?

## Miscellaneous Graphs.

**1.** Plot the graphs of

$$2y=3(x-4), \qquad 3y=1-5x,$$

obtaining at least five points on each graph. Find the coordinates of the point where they meet.

**2.** Draw the graphs represented by

$$y=5-3x, \qquad y=\tfrac{1}{3}(x+5);$$

and find the coordinates of their point of intersection.

**3.** By finding the intercepts on the axes draw the graphs of

$$\text{(i) } 15x+20y=6; \qquad \text{(ii) } 12x+21y=14.$$

In (i) take 1 inch for unit, and in (ii) take six tenths of an inch as unit. In each case explain why the unit is convenient.

**4.** Solve $y=10x+8$, $7x+y=25$ graphically.
[Unit for $x$, one inch; for $y$, one-tenth of an inch.]

**5.** From the graph of the expression $11x+6$, find its value when $x=1{\cdot}8$. Also find the value of $x$ which will make the expression equal to 20.

**6.** With the same units as in Ex. 4 draw the graph of the function $\dfrac{36-5x}{3}$. From the graph find the value of the function when $x=1{\cdot}8$; also find for what value of $x$ the function becomes equal to 8.

**7.** Shew that the straight lines given by the equations

$$9y=5x+65, \quad 5x+2y+10=0, \quad x+3y=11,$$

meet in a point. Find its coordinates.

**8.** Draw the triangle whose sides are given by the equations.

$$3y-x=9, \quad x+7y=11, \quad 3x+y=13;$$

and find the coordinates of its vertices.

**9.** Shew graphically that the values of $x$ and $y$ which satisfy the equations

$$5x=2y-18, \qquad 5y=6-7x,$$

also satisfy the equation $x+y=2$.

**10.** Draw the graphs of (i) $y=x^2$, (ii) $y=8x^2$.
In (i) take 0·4″ as unit for $x$, 0·2″ as unit for $y$.
In (ii) ...... 1″ ........... $x$, 0·1″ ........... $y$.

**11.** On the same scale as in Ex. 10. (ii) draw the graph of $y=16x^2$. Shew that it may also be simply deduced from the graph of Ex. 10. (ii).

**12.** Plot the graph of $y=x^2$, taking 1 inch as unit on both axes, and using the following values of $x$.

$$-0{\cdot}4,\quad -0{\cdot}3,\quad -0{\cdot}2,\quad -0{\cdot}1,\quad 0,\quad 0{\cdot}1,\quad 0{\cdot}2,\quad 0{\cdot}3,\quad 0{\cdot}4.$$

**13.** Draw the graph of $x=y^2$, from $y=0$ to $y=5$, and thence find the square roots of 7 and 3·6.

[Take 0·2″ as unit for $x$, 1″ as unit for $y$.]

**14.** Draw the graph of $y=5+x-x^2$ for values of $x$ from $-2$ to $+3$, and from the figure obtain approximate values for the roots of the equation $5+x-x^2=0$.

[Take 1″ as unit for $x$, 0·2″ as unit for $y$.]

**15.** Draw the graphs of

$$\text{(i) } 5x+6y=60, \quad \text{(ii) } 6y-x=24, \quad \text{(iii) } 2x-y=7;$$

and shew that they represent three lines which meet in a point.

**16.** If 1 cwt. of coffee costs £9. 12*s*., draw a graph to give the price of any number of pounds. Read off the price (to the nearest penny) of 13 lbs., 21 lbs., 23 lbs.

**17.** If 60 eggs cost 4*s*., find graphically how many can be bought for half-a-crown, and the cost of 26 eggs to the nearest penny.

**18.** If 1 cwt. of sugar costs £1. 6*s*. 8*d*., draw a graph to find the price of any number of pounds. Find the cost of 26 lbs. How many pounds can be bought for 4*s*. 10*d*. ?

**19.** Solve the following equations graphically.

(i) $x^2+y^2=53$, $y-x=5$;  (ii) $x^2+y^2=100$, $x+y=14$;

(iii) $x^2+y^2=34$, $2x+y=11$;  (iv) $x^2+y^2=36$, $4x+3y=12$.

[Approximate roots to be given to one place of decimals.]

**20.** Solve the equation $3+6x=x^2$ graphically, and find the maximum value of the expression $3+6x-x^2$.

**21.** A basket of 65 oranges is bought for 4*s*. 2*d*. Draw a graph to shew the price for any other number. How many could be bought for 3*s*. 4*d*. ? Find the price (to the nearest penny) which must be paid for 36 and for 78 oranges respectively.

**22.** If the wages for a day's work of 8 hours are 4*s*. 6*d*., draw a graph to shew the wages for any fraction of a day, and find (to the nearest penny) what ought to be paid to men who work $2\frac{1}{2}$, $3\frac{1}{2}$, $6\frac{1}{2}$ hours respectively. How many hours' work might be expected for 2*s*. 10*d*.?

[Take 1 inch to represent 1 hour, and one-tenth of an inch to represent 1 penny.]

**23.** Draw the graphs of $x^2$ and $3x+1$. By means of them find approximate values for the roots of $x^2-3x-1=0$.

**24.** If 24 men can reap a field of 29 acres in a given time, find roughly by means of a graph the number of acres which could be reaped in the same time by 15, 33, and 42 men respectively.

**25.** The highest marks gained in an examination were 136, and these are to be raised so that the maximum is 200. Shew how this may be done by means of a graph, and read off, to the nearest integer, the final marks of candidates who scored 61 and 49 respectively.

**26.** Draw a graph which will give the square roots of all numbers between 25 and 36, to three places of decimals.

[Plot the graph of $y=x^2$, beginning at the point (5, 25), with 10″ and 0·5″ as units for $x$ and $y$ respectively.]

**27.** I want a ready way of finding approximately 0·866 of any number up to 10. Justify the following construction. Join the origin to a point P whose coordinates are 10 and 8·66 (1 inch being taken as unit); then the ordinate of any point on OP is 0·866 of the corresponding abscissa. Read off from the diagram,

$$0{\cdot}866 \text{ of } 3,\ 0{\cdot}866 \text{ of } 6{\cdot}5,\ 0{\cdot}866 \text{ of } 4{\cdot}8, \text{ and } \frac{1}{0{\cdot}866} \text{ of } 5.$$

**28.** *A* starts from London at noon at 8 miles an hour; two hours later *B* starts, riding at 12 miles an hour. Find graphically at what time and at what distance from London *B* overtakes *A*. At what times will *A* and *B* be 8 miles apart? If *C* rides after *B*, starting at 3 p.m. at 15 miles an hour, find from the graphs

(i) the distances between *A*, *B*, and *C* at 5 p.m.;

(ii) the time when *C* is 8 miles behind *B*.

**29.** If O and Y represent two towns 45 miles apart, and if *A* walks from Y to O at 6 miles an hour while *B* walks from O to Y at 4 miles an hour, both starting at noon, find graphically their time and place of meeting.

Also read off from the graphs

(i) the times when they are 15 miles apart;

(ii) *B*'s distance from Y at 6.15 p.m.

**30.** At 8 a.m. $A$ starts from P to ride to Q which is 48 miles distant. At the same time $B$ sets out from Q to meet $A$. If $A$ rides at 8 miles an hour, and rests half an hour at the end of every hour, while $B$ walks uniformly at 4 miles an hour, find graphically

(i) the time and place of meeting;

(ii) the distance between $A$ and $B$ at 11 a.m.;

(iii) at what time they are 14 miles apart.

**31.** The following table gives statistics of the population of a certain country, where $P$ is the number of millions at the beginning of each of the years specified.

| Year | 1830 | 1835 | 1840 | 1845 | 1850 | 1855 | 1860 |
|---|---|---|---|---|---|---|---|
| $P$ | 20 | 22 | 24·5 | 28 | 31 | 36 | 41 |

Let $t$ be the time in years from 1830. Plot the values of $P$ vertically and those of $t$ horizontally and shew the relation between $P$ and $t$ by a simple curve passing fairly evenly among the plotted points. Find what the population was at the beginning of the years 1847 and 1858.

**32.** The salary of a clerk is increased each year by a fixed sum. After 6 years' service his salary is raised to £128, and after 15 years to £200. Draw a graph from which his salary may be read off for any year, and determine from it (i) his initial salary, (ii) the salary he should receive for his 21st year.

**33.** Draw the graphs of $y=x^2$ and $2y=x+3$ on the same diagram. Deduce the roots of the equation $2x^2-x-3=0$.

**34.** Taking 1 inch as unit, plot the graph of $y=x^3-3x$, taking the following values of $x$:

$0,\ \pm\cdot 2,\ \pm\cdot 4,\ \pm\cdot 6,\ \pm\cdot 8,\ \pm 1,\ \pm 1\cdot 2,\ \pm 1\cdot 4,\ \pm 1\cdot 6,\ \pm 1\cdot 8,\ \pm 2.$

Find the turning points, and the value of the maximum or minimum ordinates between the limits given.

**35.** From the graph in Ex. 34 find to two places of decimals the roots of $x^3-3x=0$.

**36.** Solve the following pairs of equations graphically:

(i) $x+y=15$, $xy=36$; (ii) $x-y=3$, $xy=18$; (iii) $x^2+y^2=13$, $xy=6$.

**37.** An india-rubber cord was loaded with weights, and a measurement of its length was taken for each load as tabulated. Plot a graph to shew the relation between the length of the cord and the loads.

| | | | | | | |
|---|---|---|---|---|---|---|
| Load in pounds - - | 10 | 12 | 17 | 21 | 23 | 25 |
| Length in centimetres | 36·4 | 37·7 | 40·5 | 43·0 | 44·3 | 45·4 |

What was the length of the cord unloaded?

**38.** A manufacturer has priced a certain set of lathes; the largest sells at £176, and the smallest at £40. He wishes to increase his prices so that the largest will sell at £200 and the smallest at £50. By means of a graph find an algebraical relation between the new price ($P$) and the old price ($Q$), and find to the nearest pound the new prices of lathes originally priced at £150, at £125. 10s., and at £78.

**39.** The mean temperature on the first day of each month, on an average of 50 years, had the following values:

| | | |
|---|---|---|
| Jan. 1, 37°; | May 1, 50°; | Sept. 1, 59°; |
| Feb. 1, 38°; | June 1, 57°; | Oct. 1, 54°; |
| Mar. 1, 40°; | July 1, 62°; | Nov. 1, 46°; |
| April 1, 45°; | Aug. 1, 62°; | Dec. 1, 41°. |

Represent these variations by means of a smooth curve.

[The difference of length of different months may be neglected.]

**40.** The price in pence of a standard Troy ounce of silver on January 1st in each of the ten years 1891–1900 was

45, 40, 36, 29, 30, 31, 28, 27, 27, 28.

Draw a smooth curve shewing its value approximately at any time during these ten years.

**41.** A manufacturer wishes to stock a certain article in many sizes; at present he has five sizes made at the prices given below:

| | | | | | |
|---|---|---|---|---|---|
| Length in inches | 20 | 27 | 33 | 45 | 54 |
| Price in shillings | 11 | 14·5 | 20 | 35 | 48·5 |

Draw a graph to shew suitable prices for intermediate sizes, and find what the prices should be when the lengths are 30 in. and 46 in.

## Antilogarithms.

| Log. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ·50 | 3162 | 3170 | 3177 | 3184 | 3192 | 3199 | 3206 | 3214 | 3221 | 3228 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| ·51 | 3236 | 3243 | 3251 | 3258 | 3266 | 3273 | 3281 | 3289 | 3296 | 3304 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| ·52 | 3311 | 3319 | 3327 | 3334 | 3342 | 3350 | 3357 | 3365 | 3373 | 3381 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| ·53 | 3388 | 3396 | 3404 | 3412 | 3420 | 3428 | 3436 | 3443 | 3451 | 3459 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| ·54 | 3467 | 3475 | 3483 | 3491 | 3499 | 3508 | 3516 | 3524 | 3532 | 3540 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| ·55 | 3548 | 3556 | 3565 | 3573 | 3581 | 3589 | 3597 | 3606 | 3614 | 3622 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| ·56 | 3631 | 3639 | 3648 | 3656 | 3664 | 3673 | 3681 | 3690 | 3698 | 3707 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| ·57 | 3715 | 3724 | 3733 | 3741 | 3750 | 3758 | 3767 | 3776 | 3784 | 3793 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| ·58 | 3802 | 3811 | 3819 | 3828 | 3837 | 3846 | 3855 | 3864 | 3873 | 3882 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| ·59 | 3890 | 3899 | 3908 | 3917 | 3926 | 3936 | 3945 | 3954 | 3963 | 3972 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| ·60 | 3981 | 3990 | 3999 | 4009 | 4018 | 4027 | 4036 | 4046 | 4055 | 4064 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 7 | 8 |
| ·61 | 4074 | 4083 | 4093 | 4102 | 4111 | 4121 | 4130 | 4140 | 4150 | 4159 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·62 | 4169 | 4178 | 4188 | 4198 | 4207 | 4217 | 4227 | 4236 | 4246 | 4256 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·63 | 4266 | 4276 | 4285 | 4295 | 4305 | 4315 | 4325 | 4335 | 4345 | 4355 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·64 | 4365 | 4375 | 4385 | 4395 | 4406 | 4416 | 4426 | 4436 | 4446 | 4457 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·65 | 4467 | 4477 | 4487 | 4498 | 4508 | 4519 | 4529 | 4539 | 4550 | 4560 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·66 | 4571 | 4581 | 4592 | 4603 | 4613 | 4624 | 4634 | 4645 | 4656 | 4667 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 10 |
| ·67 | 4677 | 4688 | 4699 | 4710 | 4721 | 4732 | 4742 | 4753 | 4764 | 4775 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| ·68 | 4786 | 4797 | 4808 | 4819 | 4831 | 4842 | 4853 | 4864 | 4875 | 4887 | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 9 | 10 |
| ·69 | 4898 | 4909 | 4920 | 4932 | 4943 | 4955 | 4966 | 4977 | 4989 | 5000 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| ·70 | 5012 | 5023 | 5035 | 5047 | 5058 | 5070 | 5082 | 5093 | 5105 | 5117 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 9 | 11 |
| ·71 | 5129 | 5140 | 5152 | 5164 | 5176 | 5188 | 5200 | 5212 | 5224 | 5236 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| ·72 | 5248 | 5260 | 5272 | 5284 | 5297 | 5309 | 5321 | 5333 | 5346 | 5358 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| ·73 | 5370 | 5383 | 5395 | 5408 | 5420 | 5433 | 5445 | 5458 | 5470 | 5483 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| ·74 | 5495 | 5508 | 5521 | 5534 | 5546 | 5559 | 5572 | 5585 | 5598 | 5610 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| ·75 | 5623 | 5636 | 5649 | 5662 | 5675 | 5689 | 5702 | 5715 | 5728 | 5741 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 10 | 12 |
| ·76 | 5754 | 5768 | 5781 | 5794 | 5808 | 5821 | 5834 | 5848 | 5861 | 5875 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| ·77 | 5888 | 5902 | 5916 | 5929 | 5943 | 5957 | 5970 | 5984 | 5998 | 6012 | 1 | 3 | 4 | 5 | 7 | 8 | 10 | 11 | 12 |
| ·78 | 6026 | 6039 | 6053 | 6067 | 6081 | 6095 | 6109 | 6124 | 6138 | 6152 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 13 |
| ·79 | 6166 | 6180 | 6194 | 6209 | 6223 | 6237 | 6252 | 6266 | 6281 | 6295 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| ·80 | 6310 | 6324 | 6339 | 6353 | 6368 | 6383 | 6397 | 6412 | 6427 | 6442 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| ·81 | 6457 | 6471 | 6486 | 6501 | 6516 | 6531 | 6546 | 6561 | 6577 | 6592 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| ·82 | 6607 | 6622 | 6637 | 6653 | 6668 | 6683 | 6699 | 6714 | 6730 | 6745 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| ·83 | 6761 | 6776 | 6792 | 6808 | 6823 | 6839 | 6855 | 6871 | 6887 | 6902 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| ·84 | 6918 | 6934 | 6950 | 6966 | 6982 | 6998 | 7015 | 7031 | 7047 | 7063 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 13 | 15 |
| ·85 | 7079 | 7096 | 7112 | 7129 | 7145 | 7161 | 7178 | 7194 | 7211 | 7228 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| ·86 | 7244 | 7261 | 7278 | 7295 | 7311 | 7328 | 7345 | 7362 | 7379 | 7396 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| ·87 | 7413 | 7430 | 7447 | 7464 | 7482 | 7499 | 7516 | 7534 | 7551 | 7568 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 16 |
| ·88 | 7586 | 7603 | 7621 | 7638 | 7656 | 7674 | 7691 | 7709 | 7727 | 7745 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| ·89 | 7762 | 7780 | 7798 | 7816 | 7834 | 7852 | 7870 | 7889 | 7907 | 7925 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 14 | 16 |
| ·90 | 7943 | 7962 | 7980 | 7998 | 8017 | 8035 | 8054 | 8072 | 8091 | 8110 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| ·91 | 8128 | 8147 | 8166 | 8185 | 8204 | 8222 | 8241 | 8260 | 8279 | 8299 | 2 | 4 | 6 | 8 | 9 | 11 | 13 | 15 | 17 |
| ·92 | 8318 | 8337 | 8356 | 8375 | 8395 | 8414 | 8433 | 8453 | 8472 | 8492 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| ·93 | 8511 | 8531 | 8551 | 8570 | 8590 | 8610 | 8630 | 8650 | 8670 | 8690 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| ·94 | 8710 | 8730 | 8750 | 8770 | 8790 | 8810 | 8831 | 8851 | 8872 | 8892 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| ·95 | 8913 | 8933 | 8954 | 8974 | 8995 | 9016 | 9036 | 9057 | 9078 | 9099 | 2 | 4 | 6 | 8 | 10 | 12 | 15 | 17 | 19 |
| ·96 | 9120 | 9141 | 9162 | 9183 | 9204 | 9226 | 9247 | 9268 | 9290 | 9311 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| ·97 | 9333 | 9354 | 9376 | 9397 | 9419 | 9441 | 9462 | 9484 | 9506 | 9528 | 2 | 4 | 7 | 9 | 11 | 13 | 15 | 17 | 20 |
| ·98 | 9550 | 9572 | 9594 | 9616 | 9638 | 9661 | 9683 | 9705 | 9727 | 9750 | 2 | 4 | 7 | 9 | 11 | 13 | 16 | 18 | 20 |
| ·99 | 9772 | 9795 | 9817 | 9840 | 9863 | 9886 | 9908 | 9931 | 9954 | 9977 | 2 | 5 | 7 | 9 | 11 | 14 | 16 | 18 | 20 |

## A Table of Square Roots and Cube Roots of Numbers from 1 to 150.

| Square Root. | No. | Cube Root. | Square Root. | No. | Cube Root. | Square Root. | No. | Cube Root. |
|---|---|---|---|---|---|---|---|---|
| 1·000 | **1** | 1·000 | 7·141 | **51** | 3·708 | 10·050 | **101** | 4·657 |
| 1·414 | **2** | 1·260 | 7·211 | **52** | 3·733 | 10·100 | **102** | 4·672 |
| 1·732 | **3** | 1·442 | 7·280 | **53** | 3·756 | 10·149 | **103** | 4·688 |
| 2·000 | **4** | 1·587 | 7·348 | **54** | 3·780 | 10·198 | **104** | 4·703 |
| 2·236 | **5** | 1·710 | 7·416 | **55** | 3·803 | 10·247 | **105** | 4·718 |
| 2·449 | **6** | 1·817 | 7·483 | **56** | 3·826 | 10·296 | **106** | 4·733 |
| 2·646 | **7** | 1·913 | 7·550 | **57** | 3·849 | 10·344 | **107** | 4·747 |
| 2·828 | **8** | 2·000 | 7·616 | **58** | 3·871 | 10·392 | **108** | 4·762 |
| 3·000 | **9** | 2·080 | 7·681 | **59** | 3·893 | 10·440 | **109** | 4·777 |
| 3·162 | **10** | 2·154 | 7·746 | **60** | 3·915 | 10·488 | **110** | 4·791 |
| 3·317 | **11** | 2·224 | 7·810 | **61** | 3·936 | 10·536 | **111** | 4·806 |
| 3·464 | **12** | 2·289 | 7·874 | **62** | 3·958 | 10·583 | **112** | 4·820 |
| 3·606 | **13** | 2·351 | 7·937 | **63** | 3·980 | 10·630 | **113** | 4·835 |
| 3·742 | **14** | 2·410 | 8·000 | **64** | 4·000 | 10·677 | **114** | 4·849 |
| 3·873 | **15** | 2·466 | 8·062 | **65** | 4·021 | 10·724 | **115** | 4·863 |
| 4·000 | **16** | 2·520 | 8·124 | **66** | 4·041 | 10·770 | **116** | 4·877 |
| 4·123 | **17** | 2·571 | 8·185 | **67** | 4·062 | 10·817 | **117** | 4·891 |
| 4·243 | **18** | 2·621 | 8·246 | **68** | 4·082 | 10·863 | **118** | 4·905 |
| 4·359 | **19** | 2·668 | 8·307 | **69** | 4·102 | 10·909 | **119** | 4·919 |
| 4·472 | **20** | 2·714 | 8·367 | **70** | 4·121 | 10·954 | **120** | 4·932 |
| 4·583 | **21** | 2·759 | 8·426 | **71** | 4·141 | 11·000 | **121** | 4·946 |
| 4·690 | **22** | 2·802 | 8·485 | **72** | 4·160 | 11·045 | **122** | 4·960 |
| 4·796 | **23** | 2·844 | 8·544 | **73** | 4·180 | 11·091 | **123** | 4·973 |
| 4·899 | **24** | 2·884 | 8·602 | **74** | 4·198 | 11·136 | **124** | 4·987 |
| 5·000 | **25** | 2·924 | 8·660 | **75** | 4·217 | 11·180 | **125** | 5·000 |
| 5·099 | **26** | 2·962 | 8·718 | **76** | 4·236 | 11·225 | **126** | 5·013 |
| 5·196 | **27** | 3·000 | 8·775 | **77** | 4·254 | 11·269 | **127** | 5·027 |
| 5·292 | **28** | 3·037 | 8·832 | **78** | 4·273 | 11·314 | **128** | 5·040 |
| 5·385 | **29** | 3·072 | 8·888 | **79** | 4·291 | 11·358 | **129** | 5·053 |
| 5·477 | **30** | 3·107 | 8·944 | **80** | 4·309 | 11·402 | **130** | 5·066 |
| 5·568 | **31** | 3·141 | 9·000 | **81** | 4·327 | 11·446 | **131** | 5·079 |
| 5·657 | **32** | 3·175 | 9·055 | **82** | 4·344 | 11·489 | **132** | 5·092 |
| 5·745 | **33** | 3·208 | 9·110 | **83** | 4·362 | 11·533 | **133** | 5·104 |
| 5·831 | **34** | 3·240 | 9·165 | **84** | 4·380 | 11·576 | **134** | 5·117 |
| 5·916 | **35** | 3·271 | 9·220 | **85** | 4·397 | 11·619 | **135** | 5·130 |
| 6·000 | **36** | 3·302 | 9·274 | **86** | 4·414 | 11·662 | **136** | 5·143 |
| 6·083 | **37** | 3·332 | 9·327 | **87** | 4·431 | 11·705 | **137** | 5·155 |
| 6·164 | **38** | 3·362 | 9·381 | **88** | 4·448 | 11·747 | **138** | 5·168 |
| 6·245 | **39** | 3·391 | 9·434 | **89** | 4·465 | 11·790 | **139** | 5·180 |
| 6·325 | **40** | 3·420 | 9·487 | **90** | 4·481 | 11·832 | **140** | 5·192 |
| 6·403 | **41** | 3·448 | 9·539 | **91** | 4·498 | 11·874 | **141** | 5·205 |
| 6·481 | **42** | 3·476 | 9·592 | **92** | 4·514 | 11·916 | **142** | 5·217 |
| 6·557 | **43** | 3·503 | 9·644 | **93** | 4.531 | 11·958 | **143** | 5·229 |
| 6·633 | **44** | 3·530 | 9·695 | **94** | 4·547 | 12·000 | **144** | 5·241 |
| 6·708 | **45** | 3·557 | 9·747 | **95** | 4·563 | 12·042 | **145** | 5·254 |
| 6·782 | **46** | 3·583 | 9·798 | **96** | 4·579 | 12·083 | **146** | 5·266 |
| 6·856 | **47** | 3·609 | 9·849 | **97** | 4·595 | 12·124 | **147** | 5·278 |
| 6·928 | **48** | 3·634 | 9·899 | **98** | 4·610 | 12·166 | **148** | 5·290 |
| 7·000 | **49** | 3·659 | 9·950 | **99** | 4·626 | 12·207 | **149** | 5·301 |
| 7·071 | **50** | 3·684 | 10·000 | **100** | 4·642 | 12·247 | **150** | 5·313 |

# ANSWERS.

## I. Page 4.

**7.** 36. **8.** 32. **9.** 25. **11.** 1·2 sq. cm.

**12.** $y=3x$. Any point whose ordinate is equal to three times its abscissa.

**14.** The lines are $x=5$, $y=8$. The point (5, 8).

**15.** A circle of radius 13 whose centre is at the origin.

## II. Page 7.

**21.** 32 units of area. **22.** 1 sq. in.

**23.** 72 units of area. **24.** 0·64 sq. cm.

## III. Page 10.

**1.** $x=1$, $y=5$. **2.** $x=2$, $y=10$. **3.** $x=3$, $y=12$.

**4.** $x=3$, $y=-2$. **5.** $x=4$, $y=2$. **6.** $x=6$, $y=8$.

**7.** $x=-2$, $y=4$. **8.** $x=0$, $y=-3$. **9.** $x=-3$, $y=0$.

**10.** At the point (0, 21). **11.** $3x+4y=7$.

## IV. Page 15.

**1.** $y=x$. **2.** (0, 0), (−4, 2). **5.** (2, 1).

**6.** (i) 1·46, −5·46; (ii) 3·24, −1·24; (iii) 3·32, 0 68.

**7.** −5; 7. **8.** $-\frac{1}{4}$; 3·79, −0·79; 4·62, −1·62.

**9.** $x=8$, or 6; $y=6$, or 8.

**10.** The straight line $3x+4y=25$ *touches* the circle $x^2+y^2=25$ at the point (3, 4).

## V. Page 24.

**3.** Each axis is an asymptote to the curve, which approaches the axis of $y$ much less rapidly than it does the axis of $x$.

**7.** $x=2, \frac{10}{3}$; $y=5, 3$. **8.** $x=3, -3$; $y=2, -2$.

**9.** $x=2$; $y=-1$. **31.** −1, 1, 2. **32.** −2, 4·41, 1·59.

## VI. Page 30.

**1.** 0·52, 2·9, 11·6; 2·75, 2·3, 3·1. **3.** 2·080, 2·140. **4.** 2·4.

**5.** −2, 4; −9. **9.** −5 and 1. **10.** −2, 1, 4.

**12.** 26·9, 38, 3·58. **13.** 0·477, 0·225, 0·350, 1·538.

**14.** 3 in. from the point of suspension.

**15.** 22 lbs., $16\frac{1}{2}$ lbs., $14\frac{3}{4}$ lbs., $13\frac{1}{4}$ lbs., 11 lbs., $9\frac{1}{2}$ lbs., 8 lbs., $7\frac{1}{4}$ lbs. The curve is a rectangular hyperbola whose equation is $xy=22\times 12$.

**17.** 0·602. 53°. **18.** 0·906. 0·707. **19.** $x=n\pi+\frac{3\pi}{4}$.

**20.** The range varies from 342 yards, when $A=10°$, to 985 yards when $A=40°$. It reaches its maximum of 1000 yards when $A=45°$, and is again equal to 985 yards, when $A=50°$.

**21.** 5·9°, 7·7°.

## VII. Page 40.

**2.** (i) 54·1 grains; (ii) 0·2. **3.** 39·3; 91·6; $y=0{\cdot}393x$.

**4.** 3·85 in.; 17·6 in. **5.** 54·5° F. 86·9° F. $F=32+\frac{9}{5}C$.

**6.** $y=100+\frac{x}{10}$; £350; 4250. **7.** 45·96; 39·40.

**8.** £2. 12*s.*; £3. 8*s.* **9.** 8·1 in.; 24·375 oz.

**10.** (i) £320; (ii) £580. **11.** $y=\frac{10}{11}x-70$. 112; 168; 78.

**12.** 5 ft. per sec.; $5\frac{3}{4}$ secs.; $v=5+4t$. **13.** 2·49 sq. ft.

**14.** (i) 52 ft.; (ii) 160 ft. **15.** max. height = 64 ft.; 4 secs.

**16.** $P=0{\cdot}6\,G-14{\cdot}4$; 24. **17.** 26*s.*; 36*s.* 6*d.*

**18.** 93·5° E. **20.** $y=0{\cdot}21x+1{\cdot}37$.

**21.** $y=249{\cdot}8-250x$. **22.** $y=0{\cdot}4x+1{\cdot}6$; 9·2; 3.

**23.** $a=45{\cdot}7$, $b=118$. Error = 8·43 in defect.

**24.** 8·6; $P=0{\cdot}14\,W+0{\cdot}2$; 225 lbs.

**25.** $a=\frac{1}{4}$; $b=3$. 2; 12. **26.** $a=3$, $b=2$. 7; 4·25.

**27.** $n=3$, $c=27\times 10^5$. **28.** $n=1{\cdot}5$, $c=79500$.

**29.** 1·3 per cent. in defect. **31.** $n=\frac{17}{16}$.

**32.** 16 knots. **33.** H.P. = 6·9. 23 lbs.

**34.** 15·15 knots; 2420 H.P. **35.** $P=200$, $r=3$; 14 years.

**36.** £22. 10*s.*; £28. (i) 230; (ii) 350. $1100E=49G+8140$; $550R=52G-2255$; $20P=G-230$.

**37.** $E=0{\cdot}84\,W+10{\cdot}2$; $R=2{\cdot}32\,W-30{\cdot}5$; $P=1{\cdot}48\,W-40{\cdot}7$. (i) 40; (ii) 28.

**38.** $T=36{\cdot}31h^{1\cdot5}$; $V=1271h^{1\cdot5}$.

**39.** $n=0{\cdot}86$, $c=207$. **40.** $a=1{\cdot}78$, $b=31{\cdot}5$.

**41.** $D=15{\cdot}85h^{1\cdot8}$ : 669 tons.

## VIII. PAGE 56.

**1.** 6 p.m.; (i) 3.30 p.m.; (ii) 7.30 p.m. **2.** (i) 2 p.m.; 2.52 p.m.
**3.** 47 mi. from $A$'s starting place at 12.42 p.m.
11.12 a.m. and 2.12 p.m. **4.** 27 mi.
**5.** 35 mi. from London at 3.33 p.m. 3.9 p.m.; 3.57 p.m.; 36 mi.
**6.** (i) 15 mi. after $C$'s start, 1 mi. from Bath;
(ii) 45 ........................ $3\frac{1}{2}$ mi. ..............;
(iii) half a mile behind $A$ and $B$.
**7.** 9 mi. from Y at 12.48 p.m. 12.18 p.m. and 1.18 p.m.
**8.** 40 yds. $A$ 16 yds. ahead, $C$ 16 yds. behind.
**9.** 5 secs. **10.** 5 hours from the start.
**11.** 7.36 p.m.; 3 p.m. and 5 p.m.; 19 mi. from Y.
**12.** 7 hours. 4.47 p.m. **13.** 12.12 p.m. (i) 11 a.m.; (ii) 57 mi.
**14.** 5 mi. **15.** 4.12 p.m. **16.** 10.4 a.m.
**17.** 400 yds. **18.** £420. 20 for £480.
**19.** After 10 secs. 400 ft. per sec.; 600 ft. per sec.
**20.** 30 years. £410; £320.
**21.** 75 in Latin; 80 in Greek. 74 and 50.

## Miscellaneous Graphs. PAGE 60.

**1.** $(2, -3)$. **2.** $(1, 2)$. **4.** $x=1$, $y=18$. **5.** 26; 1·28 (approx.).
**6.** 9; 2·4. **7.** $(-4, 5)$. **8.** $(3, 4)$, $(4, 1)$, $(-3, 2)$.
**13.** 2·65, 1·91. **14.** 2·79, $-1$·79. **15.** The pt. $(6, 5)$.
**16.** 22*s.* 3*d.*; 36*s.*, 39*s.* 5*d.* **17.** 37; 1*s.* 9*d.* **18.** 6*s.* 2*d.*; 20·3.
**19.** (i) $x=2$, or $-7$; $y=7$, or $-2$. (ii) $x=8$, or 6; $y=6$, or 8.
(iii) $x=3$, or $-5·8$; $y=5$, or $-0·6$. (iv) $x=5·2$, or $-1·3$;
$y=-2·9$, or 5·7.
**20.** 6·46, $-0$·46; 12. **21.** 52; 2*s.* 4*d.*, 5*s.*
**22.** 1*s.* 5*d.*, 2*s.*, 3*s.* 8*d.*; 5 hrs. **23.** 3·30, $-0$·30.
**24.** 18, 40, 51. **25.** 90, 72. **27.** 2·60, 5·63, 4·16, 5·77.
**28.** 6 p.m., 48 mi. from London. At 4 and 8 p.m.
(i) $B$ 4 mi. behind $A$; $C$ 6 mi. behind $B$. (ii) 4.21 p.m.
**29.** 4.30 p.m., 18 mi. from O. (i) At 3 and 6 p.m. (ii) 20 mi.
**30.** (i) 1 p.m., 28 mi. from P; (ii) 20 mi.; (iii) 11.30 a.m.
**31.** $29\frac{1}{4}$ and 39 millions. **32.** £80; £240.
**33.** 1·5, $-1$. **34.** Max. ordinates $(=2)$ at the points $(-1, 2)$, $(1, -2)$.

**35.** $-1 \cdot 73$, 0, $1 \cdot 73$.

**36.** (i) $\left.\begin{array}{l} x=12, \text{ or } 3 \\ y=3, \text{ or } 12 \end{array}\right\}$ (ii) $\left.\begin{array}{l} x=6, \text{ or } -3 \\ y=3, \text{ or } -6 \end{array}\right\}$ (iii) $\left.\begin{array}{l} x=2,\ 3,\ -3,\ -2 \\ y=3,\ 2,\ -2,\ -3 \end{array}\right\}$

**37.** 31·4 cm.

**38.** $P=1 \cdot 1Q+6$; £171; £144; £92.

**41.** 17*s.*; 26*s.* 6*d.*

**42.** $4\frac{1}{2}$ mi. per hr.

**43.** 1.30 p.m.; 3 : 1.

**44.** 24 min.

**45.** At Northampton. 48·4 mi., 84·8 mi. The quickest run is from Willesden to Northampton.

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