A SHORT INTRODUCTION TO GRAPHICAL ALGEBRA

H. S. HALL

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TO

GRAPHICAL ALGEBRA

BY

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PREFACE TO THE SECOND EDITION.

THE first edition of this little book was undertaken at very short notice in order to meet a sudden demand. It was, in fact, rather hastily compiled during a seaside holiday, and I had neither time nor opportunity for adequately treating the practical side of graphical work. Consequently all questions dealing with statistics and physical formulae were deliberately omitted to enable me to present the analytical aspect of the subject in sufficient detail within the limits of a few pages.

The present edition has been very considerably enlarged. The additions are of two kinds : first, a further development of the illustrations arising out of graphs of known functions ; and secondly, the application to practical questions in which the graph has to be obtained by plotting a series of values determined by observation or experiment.

The subject is practically inexhaustible ; but it is hoped that a student who has worked intelligently through the following pages will have added something useful and interesting to his algebraical knowledge, and will find himself sufficiently equipped to pursue the study further in the laboratory or workshop.

I am indebted to several friends for advice and suggestions. In particular, I wish to express my thanks to Mr. D. Rintoul of Clifton College, and to my former pupil Mr. E. A. Price of Winchester.

H. S. HALL.

January, 1903.

CONTENTS.

GRAPHICAL ALGEBRA.

[*A considerable portion of this chapter may be taken at an early stage. For example, Arts.* 1–6 *may be read as soon as the student has had sufficient practice in substitutions involving negative quantities. Arts.* 7–14 *may be read in connection with Easy Simultaneous Equations. With the exception of a few articles the rest of the chapter should be postponed until the student is acquainted with quadratic equations. References to Hall and Knight's Elementary Algebra are given thus* : "E. A., Art. 100."]

1. Definition. Any expression which involves a variable quantity x, and whose value is dependent on that of x, is called a **function of x**.

Thus $3x+8$, $2x^2+6x-7$, $x^4-3x^3+x^2-9$ are functions of x of the first, second, and fourth degree respectively.

2. The symbol $f(x)$ is often used to briefly denote a function of x. If $y=f(x)$, by substituting a succession of numerical values for x we can obtain a corresponding succession of values for y which stands for the value of the function. Hence in this connection it is sometimes convenient to call x the **independent variable**, and y the **dependent variable.**

3. Consider the function $x(9-x^2)$, and let its value be represented by y.

Then, when $x=0$, $y=0\times 9=0$,

" $x=1$, $y=1\times 8=8$,

" $x=2$, $y=2\times 5=10$,

" $x=3$, $y=3\times 0=0$,

" $x=4$, $y=4\times(-7)=-28$,

and so on.

By proceeding in this way we can find as many values of the function as we please. But we are often not so much concerned with the actual values which a function assumes for different values of the variable as with *the way in which the value of the function changes.* These variations can be very conveniently represented by a **graphical** method which we shall now explain.

4. Two straight lines XOX', YOY' are taken intersecting at right angles in O, thus dividing the plane of the paper into four spaces XOY, YOX', $X'OY'$, $Y'OX$, which are known as the first, second, third, and fourth quadrants respectively.

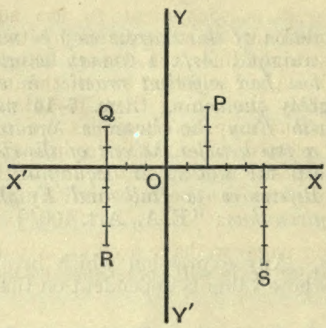

Fig. 1.

The lines $X'OX$, YOY' are usually drawn horizontally and vertically; they are taken as lines of reference and are known as the **axis of x and y** respectively. The point O is called the **origin.** Values of x are measured from O along the axis of x, according to some convenient scale of measurement, and are called **abscissæ**, *positive* values being drawn to the *right* of O along OX, and *negative* values to the *left* of O along OX'.

Values of y are drawn (on the same scale) parallel to the axis of y, from the ends of the corresponding abscissæ, and are called **ordinates.** These are *positive* when drawn *above* $X'X$, *negative* when drawn *below* $X'X$.

5. The abscissa and ordinate of a point taken together are known as its **coordinates**. A point whose coordinates are x and y is briefly spoken of as "the point (x, y)."

The coordinates of a point completely determine its position in the plane. Thus if we wish to mark the point (2, 3), we take $x=2$ units measured to the right of O, $y=3$ units measured perpendicular to the x-axis and above it. The resulting point P is in the first quadrant. The point $(-3, 2)$ is found by taking $x=3$ units to the left of O, and $y=2$ units above the x-axis. The resulting point Q is in the second quadrant. Similarly the points $(-3, -4)$, $(5, -5)$ are represented by R and S in Fig. 1, in the third and fourth quadrants respectively.

This process of marking the position of a point in reference to the coordinate axes is known as **plotting the point.**

6. In practice it is convenient to use **squared paper**; that is, paper ruled into small squares by two sets of equidistant parallel straight lines, the one set being horizontal and the other vertical. After selecting two of the intersecting lines as axes (and slightly thickening them to aid the eye) one or more of the divisions may be chosen as our unit, and points may be readily plotted when their coordinates are known. Conversely, if the position of a point in any of the quadrants is marked, its coordinates can be measured by the divisions on the paper.

In the following pages we have used paper ruled to tenths of an inch, but a larger scale will sometimes be more convenient. See Art. 26.

Example. Plot the points $(5, 2)$, $(-3, 2)$, $(-3, -4)$, $(5, -4)$ on squared paper. Find the area of the figure determined by these points, assuming the divisions on the paper to be tenths of an inch.

Taking the points in the order given, it is easily seen that they are represented by P, Q, R, S in Fig. 2, and that they form a rectangle which contains 48 squares. Each of these is *one-hundredth* part of a *square* inch. Thus the area of the rectangle is ·48 of a square inch.

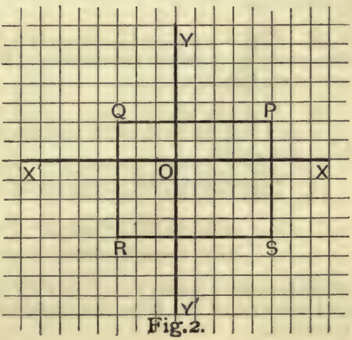

Fig. 2.

EXAMPLES I.

[*The following examples are intended to be done mainly by actual measurement on squared paper; where possible, they should also be verified by calculation.*]

Plot the following pairs of points and draw the line which joins them:

1. (3, 0), (0, 6).

2. $(-2, 0)$, $(0, -8)$.

3. $(3, -8)$, $(-2, 6)$.

4. (5, 5), $(-2, -2)$.

5. $(-2, 6)$, $(1, -3)$.

6. (4, 5), $(-1, 5)$.

7. Plot the points (3, 3), $(-3, 3)$, $(-3, -3)$, $(3, -3)$, and find the number of squares contained by the figure determined by these points.

8. Plot the points (4, 0), (0, 4), $(-4, 0)$, $(0, -4)$, and find the number of square units in the resulting figure.

9. Plot the points (0, 0), (0, 10), (5, 5), and find the number of square units in the triangle.

10. Shew that the triangle whose vertices are (0, 0), (0, 6), (4, 3) has an area of 12 square units. Shew also that the points (0, 0), (0, 6), (4, 8) determine a triangle of the same area.

11. Plot the points (5, 6), $(-5, 6)$, $(5, -6)$, $(-5, -6)$. If one millimetre is taken as unit, find the area of the figure in square centimetres.

12. Plot the points (1, 3), $(-3, -9)$, and shew that they lie on a line passing through the origin. Name the coordinates of other points on this line.

13. Plot the eight points (0, 5), (3, 4), (5, 0), $(4, -3)$, $(-5, 0)$, $(0, -5)$, $(-4, 3)$, $(-4, -3)$, and shew that they are all equidistant from the origin.

14. Plot the two following series of points:

(i) (5, 0), (5, 2), (5, 5), $(5, -1)$, $(5, -4)$;

(ii) $(-4, 8)$, $(-1, 8)$, (0, 8), (3, 8), (6, 8).

Shew that they lie on two lines respectively parallel to the axis of y, and the axis of x. Find the coordinates of the point in which they intersect.

15. Plot the points (13, 0), (0, −13), (12, 5), (−12, 5), (−13, 0), (−5, −12), (5, −12). Find their locus, (i) by measurement, (ii) by calculation.

16. Plot the points (2, 2), (−3, −3), (4, 4), (−5, −5), shewing that they all lie on a certain line through the origin. Conversely, shew that for *every* point on this line the abscissa and ordinate are equal.

Graph of a Function.

7. Let $f(x)$ represent a function of x, and let its value be denoted by y. If we give to x a series of numerical values we get a corresponding series of values for y. If these are set off as abscissæ and ordinates respectively, we plot a succession of points. If *all* such points were plotted we should arrive at a line, straight or curved, which is known as the **graph** of the *function* $f(x)$, or the **graph** of the *equation* $y=f(x)$. The variation of the function for different values of the variable x is exhibited by the variation of the ordinates as we pass from point to point.

In practice a few points carefully plotted will usually enable us to draw the graph with sufficient accuracy.

8. The student who has worked intelligently through the preceding examples will have acquired for himself some useful preliminary notions which will be of service in the examples on simple graphs which we are about to give. In particular, before proceeding further he should satisfy himself with regard to the following statements :

(i) The coordinates of the origin are (0, 0).

(ii) The abscissa of every point on the axis of y is 0.

(iii) The ordinate of every point on the axis of x is 0.

(iv) The graph of all points which have the same abscissa is a line parallel to the axis of y. (*e.g.* $x=2$.)

(v) The graph of all points which have the same ordinate is a line parallel to the axis of x. (*e.g.* $y=5$.)

(vi) The distance of any point $P(x, y)$ from the origin is given by $OP^2=x^2+y^2$.

Example 1. Plot the graph of $y=x$.

When $x=0$, $y=0$; thus the origin is one point on the graph.

Also, when $x=1, 2, 3, \ldots -1, -2, -3, \ldots,$

$$y=1, 2, 3, \ldots -1, -2, -3, \ldots$$

Thus the graph passes through O, and represents a series of points each of which has its ordinate equal to its abscissa, and is clearly represented by POP' in Fig. 3.

Example 2. Plot the graph of $y=x+3$.

Arrange the values of x and y as follows:

x	3	2	1	**0**	−1	−2	**−3**	...
y	6	5	4	**3**	2	1	**0**	...

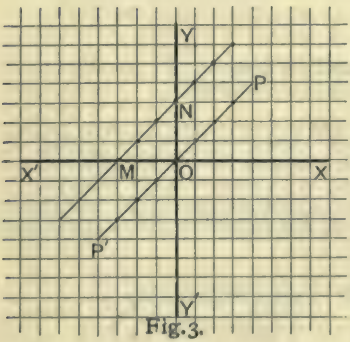

Fig. 3.

By joining these points we obtain a line MN parallel to that in Example 1.

The results printed in larger and deeper type should be specially noted and compared with the graph. They shew that the distances ON, OM (usually called the *intercepts on the axes*) are obtained by separately putting $x=0$, $y=0$ in the equation of the graph.

Note. By observing that in Example 2 each ordinate is 3 units greater than the corresponding ordinate in Example 1, the graph of $y=x+3$ may be obtained from that of $y=x$ by simply producing each ordinate 3 units in the positive direction.

In like manner the equations

$$y=x+5, \quad y=x-5$$

represent two parallel lines on opposite sides of $y=x$ and equidistant from it, as the student may easily verify for himself.

Example 3. Plot the graphs represented by the following equations:

(i) $y=2x$; (ii) $y=2x+4$; (iii) $y=2x-5$.

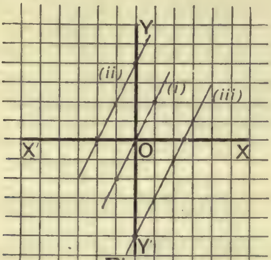

Fig. 4.

Here we only give the diagram which the student should verify in detail for himself, following the method explained in the two preceding examples.

EXAMPLES II.

[*In the following examples Nos.* 1–18 *are arranged in groups of three; each group should be represented on the same diagram so as to exhibit clearly the position of the three graphs relatively to each other.*]

Plot the graphs represented by the following equations:

1. $y=5x$. **2.** $y=5x-4$. **3.** $y=5x+6$.

4. $y=-3x$. **5.** $y=-3x+3$. **6.** $y=-3x-2$.

7. $y+x=0$. **8.** $y+x=8$. **9.** $y+4=x$.

10. $4x=3y$. **11.** $3y=4x+6$. **12.** $4y+3x=8$.

13. $x-5=0$. **14.** $y-6=0$. **15.** $5y=6x$.

16. $3x+4y=10$. **17.** $4x+y=9$. **18.** $5x-2y=8$.

19. Shew by careful drawing that the three last graphs have a common point whose coordinates are 2, 1.

20. Shew by careful drawing that the equations

$$x+y=10, \quad y=x-4$$

represent two straight lines at right angles.

21. Draw on the same axes the graphs of $x=5$, $x=9$, $y=3$, $y=11$. Find the number of square units enclosed by these lines.

22. Taking one-tenth of an inch as the unit of length, find the area included between the graphs of $x=7$, $x=-3$, $y=-2$, $y=8$.

23. Find the area included by the graphs of

$$y=x+6, \quad y=x-6, \quad y=-x+6, \quad y=-x-6.$$

24. With one millimetre as linear unit, find in square centimetres the area of the figure enclosed by the graphs of

$$y=2x+8, \quad y=2x-8, \quad y=-2x+8, \quad y=-2x-8.$$

9. The student should now be prepared for the following statements :

(i) For all numerical values of a the equation $y=ax$ represents a straight line through the origin.

(ii) For all numerical values of a and b the equation $y=ax+b$ represents a line parallel to $y=ax$, and cutting off an intercept b from the axis of y.

10. Conversely, since every equation involving x and y only in the first degree can be reduced to one of the forms $y=ax$, $y=ax+b$, it follows that *every simple equation connecting two variables represents a straight line.* For this reason an expression of the form $ax+b$ is said to be a **linear function** of x, and an equation such as $y=ax+b$, or $ax+by+c=0$, is said to be a **linear equation**.

Example. Shew that the points $(3, -4)$, $(9, 4)$, $(12, 8)$ lie on a straight line, and find its equation.

Assume $y=ax+b$ as the equation of the line. If it passes through the first two points given, their coordinates must satisfy the above equation. Hence

$$-4=3a+b, \quad 4=9a+b.$$

These equations give $a=\frac{4}{3}$, $b=-8$.

Hence $y=\frac{4}{3}x-8$, or $4x-3y=24$,

is the equation of the line passing through the first two points. Since $x=12$, $y=8$ satisfies this equation, the line also passes through $(12, 8)$. This example may be verified graphically by plotting the line which joins *any two* of the points and shewing that it passes through the third.

Application to Simultaneous Equations.

11. It is shewn [E. A., Art. 100] that in the case of a simple equation between x and y, it is possible to find as many pairs of values of x and y as we please which satisfy the given equation. We now see that this is equivalent to saying that we may find as many points as we please on any given straight line. If, however, we have two simultaneous equations between x and y, there can only be one pair of values which will satisfy both equations. This is equivalent to saying that two straight lines can have only one common point.

Example. Solve graphically the equations:

$$3x+7y=27, \quad 5x+2y=16.$$

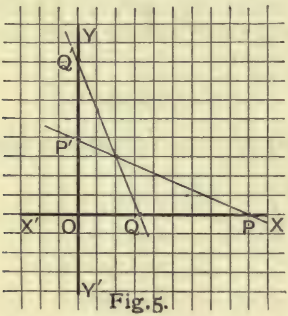

Fig. 5.

If carefully plotted it will be found that these two equations represent the lines in the annexed diagram. On measuring the coordinates of the point at which they intersect it will be found that $x=2$, $y=3$, thus verifying the solution given in E. A. Art. 103, Ex. 1.

12. It will now be seen that the process of solving two simultaneous equations is equivalent to finding the coordinates of the point (or points) at which their graphs meet.

13. Since a straight line can always be drawn by joining *any* two points on it, in solving *linear* simultaneous equations graphically, it is only necessary to plot two points on each line. The points where the lines meet the axes will usually be the most convenient to select.

14. Two simultaneous equations lead to no finite solution if they are inconsistent with each other. For example, the equations

$$x+3y=2, \quad 3x+9y=8$$

are inconsistent, for the second equation can be written $x+3y=2\frac{2}{3}$, which is clearly inconsistent with $x+3y=2$. The graphs of these two equations will be found to be two parallel straight lines which have no finite point of intersection.

Again, two simultaneous equations must be independent. The equations

$$4x+3y=1, \quad 16x+12y=4$$

are not independent, for the second can be deduced from the first by dividing throughout by 4. Thus *any pair of values* which will satisfy one equation will satisfy the other. Graphically these two equations represent two coincident straight lines which of course have an unlimited number of common points.

EXAMPLES III.

Solve the following equations, in each case verifying the solution graphically:

1. $y=2x+3$, $y=6x-1$.

2. $y=3x+4$, $y=x+8$.

3. $y=4x$, $y=3x+3$.

4. $2x-y=8$, $4x+3y=6$.

5. $3x+2y=16$, $5x-3y=14$.

6. $6y-5x=18$, $4x=3y$.

7. $2x+y=0$, $\frac{1}{2}y-3x=8$.

8. $2x-y=3$, $5x-3y=9$.

9. $2y=5x+15$, $3y-4x=12$.

10. Prove by graphical representation that the three points (3, 0), (2, 7), (4, −7) lie on a straight line. Where does this line cut the axis of y?

11. Prove that the three points (1, 1), (−3, 4), (5, −2) lie on a straight line. Find its equation. Draw the graph of this equation, shewing that it passes through the given points.

12. Shew that the three points (3, 2), (8, 8), (−2, −4) lie on a straight line. Prove algebraically and graphically that it cuts the axis of x at a distance $1\frac{1}{3}$ from the origin.

15. We shall now give some graphs of functions of higher degree than the first.

Example 1. Plot the graph of $2y = x^2$.

Corresponding values of x and y may be tabulated as follows :

x	...	3	2·5	2	1·5	1	0	-1	-2	-3	...
y	...	4·5	3·125	2	1·125	·5	0	·5	2	4·5	...

Here, in order to obtain a figure on a sufficiently large scale, it will be found convenient to take two divisions on the paper for our unit.

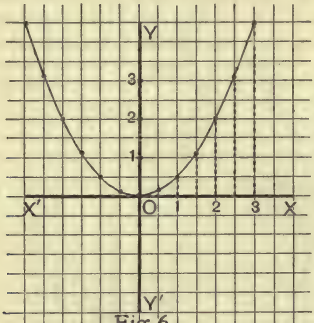

Fig. 6.

If the above points are plotted and connected by a line drawn freehand, we shall obtain the curve shewn in Fig. 6. This curve is called a **parabola**.

There are two facts to be specially noted in this example.

(i) Since from the equation we have $x = \pm\sqrt{2y}$, it follows that for every value of the ordinate we have two values of the abscissa, *equal in magnitude and opposite in sign.* Hence the graph is symmetrical with respect to the axis of y; so that after plotting with care enough points to determine the form of the graph in the first quadrant, its form in the second quadrant can be inferred without actually plotting any points in this quadrant. At the same time, in this and similar cases beginners are recommended to plot a few points in each quadrant through which the graph passes.

(ii) We observe that all the plotted points lie above the axis of x. This is evident from the equation; for since x^2 must be positive for all values of x, every ordinate obtained from the equation $y=\frac{x^2}{2}$ must be positive.

In like manner the student may shew that the graph of $2y=-x^2$ is a curve similar in every respect to that in Fig. 6, but lying entirely below the axis of x.

Note. Some further remarks on the graph of this and the next example will be found in Art. 21.

Example 2. Find the graph of $y=2x+\frac{x^2}{4}$.

Here the following arrangement will be found convenient:

x	3	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8
$2x$	6	4	2	0	−2	−4	−6	−8	−10	−12	−14	−16
$\frac{x^2}{4}$	2·25	1	·25	0	·25	1	2·25	4	6·25	9	12·25	16
y	8·25	5	2·25	0	−1·75	−3	−3·75	−4	−3·75	−3	−1·75	0

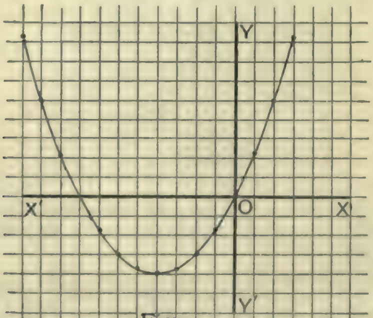

Fig. 7.

From the form of the equation it is evident that every positive value of x will yield a positive value of y, and that as x increases y also increases. Hence the portion of the curve in the first quadrant lies as in Fig. 7, and can be extended indefinitely in this quadrant. In the present case only two or three positive values of x and y need be plotted, but more attention must be paid to the results arising out of negative values of x.

When $y=0$, we have $\frac{x^2}{4}+2x=0$; thus the two values of x in the graph which correspond to $y=0$ furnish the roots of the equation $\frac{x^2}{4}+2x=0$.

16. If $f(x)$ represent a function of x, an approximate solution of the equation $f(x)=0$ may be obtained by plotting the graph of $y=f(x)$, and then measuring the intercepts made on the axis of x. These intercepts are values of x which make y equal to zero, and are therefore roots of $f(x)=0$.

17. If $f(x)$ gradually increases till it reaches a value a, which is algebraically greater than neighbouring values on either side, a is said to be a **maximum value** of $f(x)$.

If $f(x)$ gradually decreases till it reaches a value b, which is algebraically less than neighbouring values on either side, b is said to be a **minimum** value of $f(x)$.

When $y=f(x)$ is treated graphically, it is now evident that maximum and minimum values of $f(x)$ occur at points where the ordinates are algebraically greatest and least in the immediate vicinity of such points.

Example. Solve the equation $x^2-7x+11=0$ graphically, and find the minimum value of the function $x^2-7x+11$.

Put $y=x^2-7x+11$, and find the graph of this equation.

x	0	1	2	3	**3·5**	4	5	6	7
y	11	5	1	−1	**−1·25**	−1	1	5	11

The values of x which make the function $x^2-7x+11$ vanish are those which correspond to $y=0$. By careful measurement it will be found that the intercepts OM and ON are approximately equal to 2·38 and 4·62.

The algebraical solution of

$$x^2-7x+11=0$$

gives $$x=\frac{1}{2}(7\pm\sqrt{5}).$$

If we take 2·236 as the approximate value of $\sqrt{5}$, the values of x will be found to agree with those obtained from the graph.

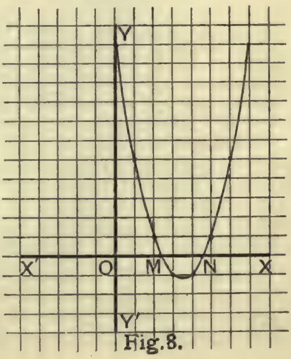

Fig. 8.

Again, $x^2-7x+11=\left(x-\frac{7}{2}\right)^2-\frac{5}{4}$. Now $\left(x-\frac{7}{2}\right)^2$ must be positive for all real values of x except $x=\frac{7}{2}$, in which case it vanishes, and the value of the function reduces to $-\frac{5}{4}$, which is the least value it can have.

The graph shews that when $x=3{\cdot}5$, $y=-1{\cdot}25$, and that this is the algebraically least ordinate in the plotted curve.

18. The following example shews that points selected for graphical representation must sometimes be restricted within certain limits.

Example. Find the graph of $x^2+y^2=36$.

The equation may be written in either of the following forms:

$$\text{(i)}\quad y=\pm\sqrt{36-x^2}\,;\qquad \text{(ii)}\quad x=\pm\sqrt{36-y^2}.$$

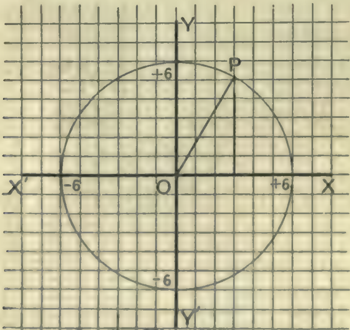

Fig. 9.

In order that y may be a real quantity we see from (i) that $36-x^2$ must be positive. Thus x can only have values between -6 and $+6$. Similarly from (ii) it is evident that y must also lie between -6 and $+6$. Between these limits it will be found that all plotted points will lie at a distance 6 from the origin. Hence the graph is a circle whose centre is O and whose radius is 6.

This is otherwise evident, for the distance of any point $P(x, y)$ from the origin is given by $OP=\sqrt{x^2+y^2}$. [Art. 8.] Hence the equation $x^2+y^2=36$ asserts that the graph consists of a series of points all of which are at a distance 6 from the origin.

Note. To plot the curve from equation (ii), we should select a succession of values for y and then find corresponding values of x. In other words we make y the *independent* and x the *dependent* variable. The student should be prepared to do this in some of the examples which follow.

EXAMPLES IV.

1. Draw the graphs of $y=x^2$, and $x=y^2$, and shew that they have only one common chord. Find its equation.

2. From the graphs, and also by calculation, shew that $y=\frac{x^2}{8}$ cuts $x=-y^2$ in only two points, and find their coordinates.

3. Draw the graphs of

(i) $y^2=-4x$; (ii) $y=2x-\frac{x^2}{4}$; (iii) $y=\frac{x^2}{4}+x-2$.

4. Draw the graph of $y=x+x^2$. Shew also that it may be deduced from that of $y=x^2$, obtained in Example 1.

5. Shew (i) graphically, (ii) algebraically, that the line $y=2x-3$ meets the curve $y=\frac{x^2}{4}+x-2$ in one point only. Find its coordinates.

6. Find graphically the roots of the following equations to 2 places of decimals:

(i) $\frac{x^2}{4}+x-2=0$; (ii) $x^2-2x=4$; (iii) $4x^2-16x+9=0$;

and verify the solutions algebraically.

7. Find the minimum value of x^2-2x-4, and the maximum value of $5+4x-2x^2$.

8. Draw the graph of $y=(x-1)(x-2)$ and find the minimum value of $(x-1)(x-2)$. Measure, as accurately as you can, the values of x for which $(x-1)(x-2)$ is equal to 5 and 9 respectively. Verify algebraically.

9. Solve the simultaneous equations

$$x^2+y^2=100, \qquad x+y=14;$$

and verify the solution by plotting the graphs of the equations and measuring the coordinates of their common points.

10. Plot the graphs of $x^2+y^2=25$, $3x+4y=25$, and examine their relation to each other where they intersect. Verify the result algebraically.

19. Infinite and zero values. Consider the fraction $\frac{a}{x}$ in which the numerator a has a *certain fixed value*, and the denominator is a *quantity subject to change*; then it is clear that the smaller x becomes the larger does the value of the fraction $\frac{a}{x}$ become. For instance

$$\frac{a}{\frac{1}{10}}=10a, \quad \frac{a}{\frac{1}{1000}}=1000a, \quad \frac{a}{\frac{1}{1000000}}=1000000a.$$

By making the denominator x sufficiently small the value of the fraction $\frac{a}{x}$ can be made as large as we please; that is, if x is made *less than any quantity that can be named*, the value of $\frac{a}{x}$ will become *greater than any quantity that can be named.*

A quantity less than any assignable quantity is called **zero** and is denoted by the symbol 0.

A quantity greater than any assignable quantity is called **infinity** and is denoted by the symbol ∞.

We may now say briefly

when $x=0$, *the value of* $\frac{a}{x}$ *is* ∞.

Again if x is a quantity which gradually increases and finally becomes *greater than any assignable quantity* the fraction becomes *smaller than any assignable quantity.* Or more briefly

when $x=\infty$, *the value of* $\frac{a}{x}$ *is* 0.

20. It should be observed that when the symbols for zero and infinity are used in the sense above explained, they are subject to the rules of signs which affect other algebraical symbols. Thus we shall find it convenient to use a concise statement such as "when $x=+0$, $y=+\infty$" to indicate that when a *very small and positive* value is given to x, the corresponding value of y is *very large and positive.*

21. If we now return to the examples worked out in Art. 15, in Example 1, we see that when $x=\pm\infty$, $y=+\infty$; hence the curve extends upwards to infinity in both the first and second quadrants. In Example 2, when $x=+\infty$, $y=+\infty$. Again y is negative between the values 0 and -8 of x. For all negative values of x numerically greater than 8, y is positive, and when $x=-\infty, y=+\infty$. Hence the curve extends to infinity in both the first and second quadrants.

The student should now examine the nature of the graphs in Examples IV. when x and y are infinite.

Example. Find the graph of $xy=4$.

The equation may be written in the form

$$y=\frac{4}{x},$$

from which it appears that when $x=0$, $y=\infty$ and when $x=\infty$, $y=0$. Also y is positive when x is positive, and negative when x is negative. Hence the graph must lie entirely in the first and third quadrants.

It will be convenient in this case to take the positive and negative values of the variables separately.

(1) *Positive values :*

x	0	1	2	3	4	5	6	...	∞
y	∞	4	2	$1\frac{1}{3}$	1	·8	$\frac{2}{3}$	...	0

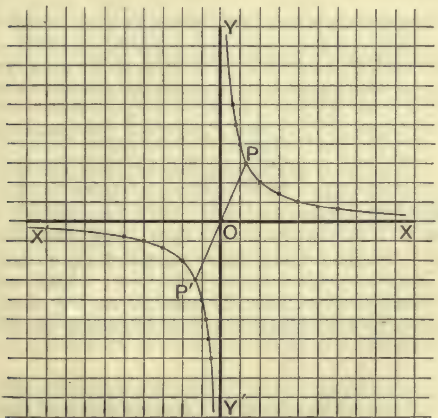

Fig. 10.

Graphically these values shew that as we recede further and further from the origin on the x-axis in the positive direction, the values of y are positive and become smaller and smaller. That is the graph is continually approaching the x-axis in such a way that by taking a sufficiently great positive value of x we obtain a point on the graph as near as we please to the x-axis but never actually reaching it until $x=\infty$. Similarly, as x becomes smaller and smaller the graph approaches more and more nearly to the positive end of the y-axis, never actually reaching it as long as x has any finite positive value, however small.

(2) *Negative values:*

x	-0	-1	-2	-3	-4	-5	...	$-\infty$
y	$-\infty$	-4	-2	$-1\frac{1}{3}$	-1	$-\cdot 8$	...	-0

The portion of the graph obtained from these values is in the third quadrant as shewn in Fig. 10, and exactly similar to the portion already traced in the first quadrant. It should be noticed that as x passes from $+0$ to -0 the value of y changes from $+\infty$ to $-\infty$. Thus the graph, which in the first quadrant has run away to an infinite distance on the positive side of the y-axis, reappears in the third quadrant coming from an infinite distance on the negative side of that axis. Similar remarks apply to the graph in its relation to the x-axis.

22. When a curve continually approaches more and more nearly to a line without actually meeting it until an infinite distance is reached, such a line is said to be an **asymptote** to the curve. In the above case each of the axes is an asymptote.

23. Every equation of the form $y=\frac{c}{x}$, or $xy=c$, where c is constant, will give a graph similar to that exhibited in the example of Art. 21. The resulting curve is known as a **rectangular hyperbola**, and has many interesting properties. In particular we may mention that from the form of the equation it is evident that for every point (x, y) on the curve there is a corresponding point $(-x, -y)$ which satisfies the equation. Graphically this amounts to saying that any line through the origin meeting the two branches of the curve in P and P' is bisected at O.

24. In the simpler cases of graphs, sufficient accuracy can usually be obtained by plotting a few points, and there is little difficulty in selecting points with suitable coordinates. But in other cases, and especially when the graph has infinite branches, more care is needed. The most important things to observe are (1) the values for which the function $f(x)$ becomes zero or infinite; and (2) the values which the function assumes for zero and infinite values of x. In other words, we determine the *general character* of the curve in the neighbourhood of the origin, the axes, and infinity. Greater accuracy of detail can then be secured by plotting points at discretion. The selection of such points will usually be suggested by the earlier stages of our work.

The existence of symmetry about either of the axes should also be noted. When an equation contains no *odd* powers of x, the graph is symmetrical with regard to the axis of y. Similarly the absence of odd powers of y indicates symmetry about the axis of x. Compare Art. 15, Ex. 1.

Example. Draw the graph of $y=\frac{2x+7}{x-4}$. [See fig. on next page.]

We have $y=\frac{2x+7}{x-4}=\frac{2+\frac{7}{x}}{1-\frac{4}{x}}$, the latter form being convenient for infinite values of x.

$$\left.\begin{array}{lll}\text{(i) When} & y=0, & x=-\frac{7}{2}, \\ \quad ,, & y=\infty, & x=4;\end{array}\right\}$$

$\therefore$ the curve cuts the axis of x at a distance $-3{\cdot}5$ from the origin, and meets the line $x=4$ at an infinite distance.

If x is positive and very little greater than 4, y is very great and positive. If x is positive and very little less than 4, y is very great and negative. Thus the infinite points on the graph near to the line $x=4$ have positive ordinates to the right, and negative ordinates to the left of this line.

$$\left.\begin{array}{lll}\text{(ii) When} & x=0, & y=-1{\cdot}75, \\ \quad ,, & x=\infty, & y=2;\end{array}\right\}$$

$\therefore$ the curve cuts the axis of y at a distance $-1{\cdot}75$ from the origin, and meets the line $y=2$ at an infinite distance.

By taking positive values of y very little greater and very little less than 2, it appears that the curve lies above the line $y=2$ when $x=+\infty$, and below this line when $x=-\infty$.

The general character of the curve is now determined: the lines $PO'P'$ $(x=4)$ and $QO'Q'$ $(y=2)$ are asymptotes; the two branches of the curve lie in the compartments $PO'Q$, $P'O'Q'$, and the lower branch cuts the axes at distances $-3{\cdot}5$ and $-1{\cdot}75$ from the origin.

To examine the lower branch in detail values of x may be selected between $-\infty$ and $-3{\cdot}5$ and between $-3{\cdot}5$ and 4.

x	$-\infty$	...	-16	-8	-6	$-3{\cdot}5$	-1	0	2	3	...	4
y	2	...	1·25	·75	·5	0	-1	$-1{\cdot}75$	$-5{\cdot}5$	-13	...	$-\infty$

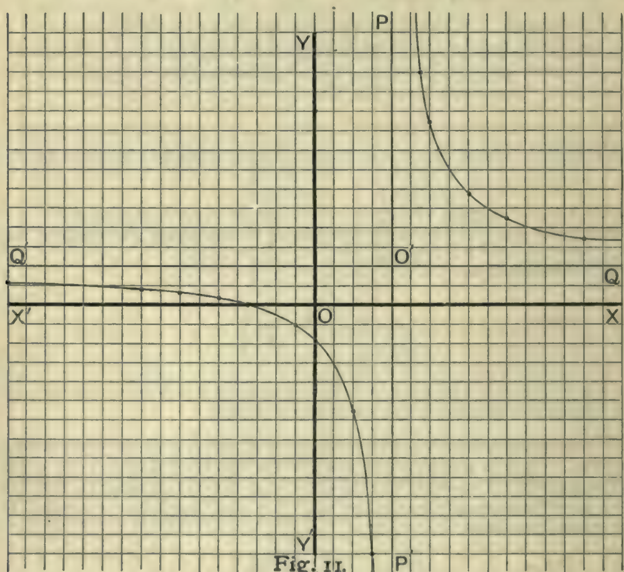

Fig. 11.

The upper branch may now be dealt with in the same way, selecting values of x between 4 and ∞. The graph will be found to be as represented in Fig. 11.

25. When the equation of a curve contains the square or higher power of y, the calculation of the values of y corresponding to selected values of x will have to be obtained by evolution, or else by the aid of logarithms. We give one example to illustrate the way in which a table of four-figure logarithms may be employed in such cases.

Example. Draw the graph of $y^3 = x(9 - x^2)$.

For the sake of brevity we shall confine our attention to that part of the curve which lies to the right of the axis of y, leaving the other half to be traced in like manner by the student.

When $x=0$, $y=0$; therefore the curve passes through the origin. Again, y is positive for all values of x between 0 and 3, and vanishes when $x=3$; for values of x greater than 3, y is negative and continually increases numerically.

x	0	1	2	3	4	5	6	...
x^2	0	1	4	9	16	25	36	...
$9 - x^2$	9	8	5	0	-7	-16	-27	...
y^3	0	8	10	0	-28	-80	-162	...
$\log y^3$			1		1·4472*	1·9031*	2·2095*	...
$\log y$			·3333		·4824	·6344	·7365	...
y	0	2	2·15	0	$-3·04$	$-4·31$	$-5·45$	...

These points will be sufficient to give a rough approximation to the curve. For greater accuracy a few intermediate values such as $x=1·5$, $2·5$, $3·5$ ... should be taken, and the resulting curve will be as in Fig. 12, in which we have taken *two-tenths of an inch as our linear unit.*

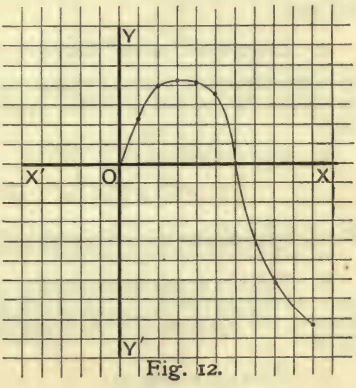

Fig. 12.

* In taking logarithms of the successive values of y^3, the negative sign is disregarded, but care must be taken to insert the proper signs in the last line which gives the successive values of y.

26. For convenience on the printed page we have supposed the squared paper to be ruled to tenths of an inch, generally using one of the divisions on the paper as our linear unit. In practice, however, it will often be advisable to choose a unit much larger than this, especially in cases where one of the variables increases or decreases much more rapidly than the other. Attention is directed to this point in the examples which follow. The student will find it difficult to get a satisfactory graph unless a suitable unit of measurement is chosen.

EXAMPLES V.

1. Plot the graph of $y=x^3$. Shew that it consists of a continuous curve lying in the first and third quadrants, crossing the axis of x at the origin. Deduce the graphs of

$$\text{(i)}\quad y=-x^3; \qquad \text{(ii)}\quad y=\frac{1}{2}x^3.$$

2. Plot the graph of $y=x-x^3$. Verify it from the graphs of $y=x$, and $y=x^3$.

3. Plot the graph of $y=\frac{1}{x^2}$, shewing that it consists of two branches lying entirely in the first and second quadrants. Examine and compare the nature and position of the graph as it approaches the axes.

4. Discuss the general character of the graph of $y=\frac{a}{x^2}$ where a has some constant integral value. Distinguish between two cases in which a has numerical values, equal in magnitude but opposite in sign.

5. Plot the graphs of

$$\text{(i)}\quad y=1+\frac{1}{x}, \qquad \text{(ii)}\quad y=2+\frac{10}{x^2}.$$

Verify by deducing them from the graphs of $y=\frac{1}{x}$, and $y=\frac{10}{x^2}$.

6. Plot the graph of $y=x^3-3x$. Examine the character of the curve at the points $(1, -2)$, $(-1, 2)$, and shew graphically that the roots of the equation $x^3-3x=0$ are approximately $-1{\cdot}732$, 0, and $1{\cdot}732$.

7. Solve the equations:

$$3x+2y=16, \qquad xy=10,$$

and verify the solution by finding the coordinates of the points where their graphs intersect.

8. Plot the graphs of

$$\text{(i)}\quad y=\frac{15-x^2}{x}, \qquad \text{(ii)}\quad x=\frac{10-y^2}{y},$$

and thus verify the algebraical solution of the equations $x^2+xy=15$, $y^2+xy=10$.

9. Trace the curve whose equation is $y=\dfrac{x}{2-x}$, shewing that it has two branches, one lying in the first and third quadrants, and the other entirely in the fourth. Find the equations of its asymptotes.

Plot the graphs of

10. $y=\dfrac{1+x}{1-x}$.

11. $y=\dfrac{1+x^2}{1-x}$.

12. $y=\dfrac{x^2-15}{x-4}$.

13. $y=\dfrac{(x-1)(x-2)}{x-3}$.

14. $y=\dfrac{x^2+x+1}{x^2-x+1}$.

15. $y=\dfrac{x^2+5x+6}{x^2+1}$.

16. $y=x^3-6x^2+11x-6$.

17. $10y=x^3-5x^2+x-5$.

18. $y=\dfrac{20}{x^2+2}$.

19. $y=\dfrac{40x}{x^2+10}$.

20. $y=\dfrac{x(8-x)}{x+5}$.

21. $y=\dfrac{(x-2)(x-3)}{x-5}$.

22. $y=\dfrac{(x-1)(x-2)(x+1)}{4}$.

23. $y^2=x^2-5x+4$.

24. $4y^2=x^2(5-x)$.

25. $y^2=\dfrac{x(3-x)(x-8)}{x^2+5}$.

26. $y^2=\dfrac{(x+7)(x-4)(x-10)}{x^2+5}$.

27. $y^2=\dfrac{x^2(49-x^2)}{50}$.

28. $y^2=\dfrac{(81-x^2)(x^2-4)}{100}$.

29. $5y^3=x(x^2-64)$.

30. $5y^3=x^2(36-x^2)$.

31. Plot the graphs of $y=x^3$, and of $y=x^2+11x-3$. Hence find the roots of the equation $x^3-x^2-11x+3=0$ to two decimal places.

32. Find graphically the roots of the equation

$$x^3-4x^2-5x+14=0$$

to three significant figures.

Measurement on Different Scales.

27. Attention has already been drawn to the necessity for care in selecting suitable units of measurement in graphical work. In some of the practical applications we are about to give this consideration is of special importance.

Although for the sake of simplicity we have hitherto measured abscissæ and ordinates on the same scale, there is no necessity for so doing, and it will often be found convenient to measure the variables on different scales suggested by the particular conditions of the question.

As an illustration let us take the graph of $y=\frac{x^2}{2}$, given in Art. 15. If with the same unit as before we plot the graph of $y=x^2$, it will be found to be a curve similar to that drawn on page 11, but elongated in the direction of the axis of y. In fact, it will be the same as if the former graph were stretched to twice its length in the direction of the y-axis.

28. Any equation of the form $y=ax^2$, where a is constant, will represent a parabola elongated more or less according to the value of a; and the larger the value of a the more rapidly will y increase in comparison with x. We might have very large ordinates corresponding to very small abscissæ, and the graph might prove quite unsuitable for practical applications. In such a case the inconvenience is obviated by measuring the values of y on a considerably smaller scale than those of x.

Speaking generally, whenever one variable increases much more rapidly than the other, a small unit should be chosen for the rapidly increasing variable and a large one for the other. Further modifications will be suggested in the examples which follow.

On the opposite page we give for comparison the graphs of

$$y=x^2 \text{ (Fig. 13)}, \quad \text{and } y=8x^2 \text{ (Fig. 14)}.$$

In Fig. 13 the unit for x is twice as great as that for y.

In Fig. 14 the x-unit is ten times the y-unit.

It will be useful practice for the student to plot other similar graphs on the same or a larger scale. For example, in Fig. 14 the graphs of $y=16x^2$ and $y=2x^2$ may be drawn and compared with that of $y=8x^2$.

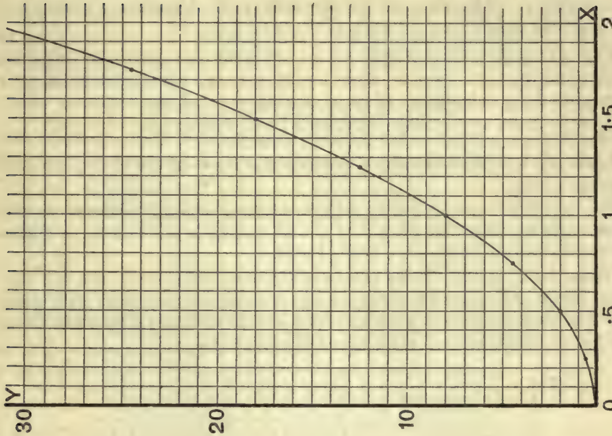

Fig. 14.

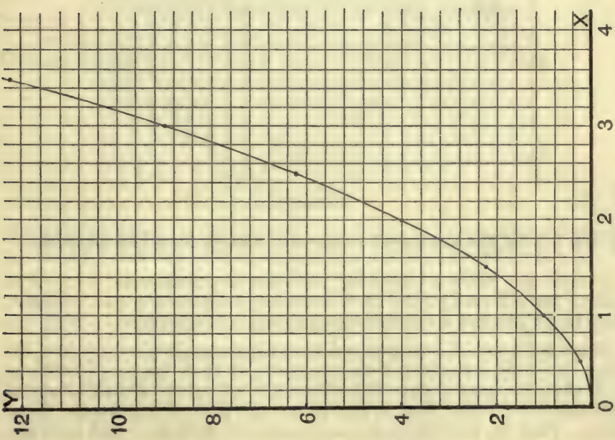

Fig. 13.

29. Besides the instances already given there are several of the ordinary processes of Arithmetic and Algebra which lend themselves readily to graphical illustration.

For example, the graph of $y=x^2$ may be used to furnish numerical square roots. For since $x=\sqrt{y}$, each ordinate and corresponding abscissa give a number and its square root. Similarly cube roots may be found from the graph of $y=x^3$.

Example 1. Find graphically the cube root of 10 to 3 places of decimals.

The required root is clearly a little greater than 2. Hence it will be enough to plot the graph of $y=x^3$ taking $x=2{\cdot}1$, $2{\cdot}2$, ... The corresponding ordinates are $9{\cdot}26$, $10{\cdot}65$, ...

When $x=2$, $y=8$. Take the axes through this point and let the units for x and y be 10 inches and ·5 inch respectively. On this scale the portion of the graph differs but little from a straight line, and yields results to a high degree of accuracy.

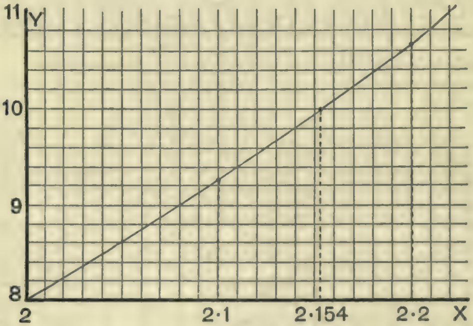

Fig. 15.

When $y=10$, the measured value of x will be found to be $2{\cdot}154$.

Example 2. Shew graphically that the expression $4x^2+4x-3$ is negative for all real values of x between ·5 and $-1{\cdot}5$, and positive for all real values of x outside these limits. [Fig. 16.]

Put $y=4x^2+4x-3$, and proceed as in the example given in Art. 16, taking the unit for x four times as great as that for y. It will be found that the graph cuts the axis of x at points whose abscissæ are ·5 and $-1{\cdot}5$; and that it lies below the axis of x between these points. That is, the value of y is negative so long as x lies between ·5 and $-1{\cdot}5$, and positive for all other values of x.

Or we may proceed as follows:

Put $y_1 = 4x^2$, and $y_2 = -4x+3$, and plot the graphs of these two equations. At their points of intersection $y_1 = y_2$, and the values of x at these points are found to be ·5 and −1·5. Hence for these values of x we have

$$4x^2 = -4x+3, \quad \text{or} \quad 4x^2+4x-3=0.$$

Thus the roots of the equation $4x^2+4x-3=0$ are furnished by the abscissæ of the common points of the graphs of $4x^2$ and $-4x+3$.

Again, between the values ·5 and −1·5 for x it will be found graphically that y_1 is less than y_2, hence $y_1 - y_2$, or $4x^2+4x-3$ is negative.

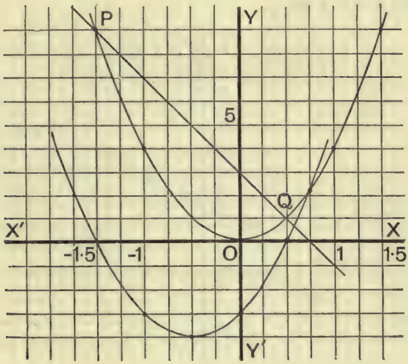

Fig. 16.

Both solutions are here exhibited.

The upper curve is the graph of $y=4x^2$; PQ is the graph of $y=-4x+3$; and the lower curve is the graph of $y=4x^2+4x-3$.

30. Of the two methods in the last Example the first is the more direct and instructive; but the second has this advantage:

If a number of equations of the form $x^2=px+q$ have to be solved graphically, $y=x^2$ can be plotted once for all on a convenient scale, and $y=px+q$ can then be readily drawn for different values of p and q.

Equations of higher degree may be treated similarly.

For example, the solution of such equations as

$$x^3 = px + q, \quad \text{or} \quad x^3 = ax^2 + bx + c$$

can be made to depend on the intersection of $y = x^3$ with other graphs.

Example. Find the real roots of the equations

(i) $x^3 - 2{\cdot}5x - 3 = 0$; (ii) $x^3 - 3x + 2 = 0$.

Here we have to find the points of intersection of

(i) $y = x^3$, $y = 2{\cdot}5x + 3$; (ii) $y = x^3$, $y = 3x - 2$.

Plot the graphs of these equations, choosing the unit for x five times as great as that for y.

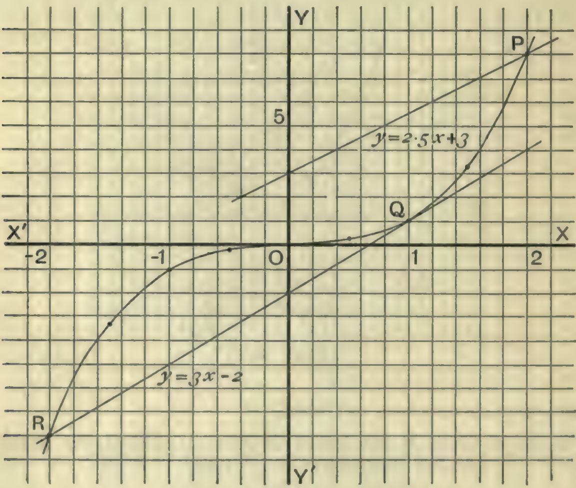

Fig. 17.

It will be seen that $y = 2{\cdot}5x + 3$ meets $y = x^3$ only at the point for which $x = 2$. Thus 2 is the only real root of equation (i).

Again $y = 3x - 2$ *touches* $y = x^3$ at the point for which $x = 1$, and cuts it where $x = -2$.

Corresponding to the former point the equation $x^3 - 3x + 2 = 0$ has two equal roots. Thus the roots of (ii) are 1, 1, -2.

31. Apart from questions of convenience with regard to any particular graph, we may observe that in many cases the variables whose values are plotted on the two axes denote magnitudes of different kinds, so that there is no necessary relation between the units in which they are measured.

A good illustration of this kind is furnished by tracing the variations of the Trigonometrical functions graphically.

Example. Trace the graph of $\sin x$.

In any work on Trigonometry it is shewn that as the angle x increases from 0° to 90°, the value of $\sin x$ is positive, and increasing gradually from 0 to 1. From 90° to 180°, $\sin x$ is positive, and decreasing from 1 to 0. From 180° to 270°, $\sin x$ is negative, and increasing numerically from 0 to -1. And from 270° to 360°, $\sin x$ is negative, and decreasing numerically from -1 to 0.

(See Hall and Knight's *Elementary Trigonometry*, Art. 86.)

We shall here exhibit these variations independently by putting $y = \sin x$, and plotting the values of y corresponding to values of x differing by 30°.

By the aid of a table of sines we have:

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	...
y or $\sin x$	0	·5	·866	1	·866	·5	0	$-$ ·5	$-$ ·866	-1	...

The graph is represented by the continuous waving line shewn in Fig. 18.

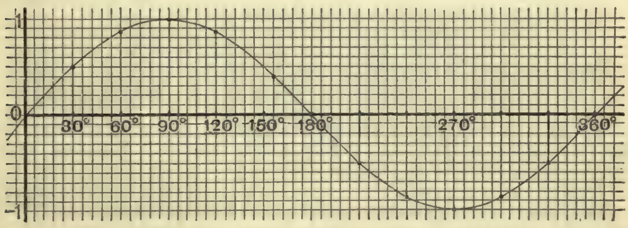

Fig. 18.

On the x-axis each division represents 6° and on the y-axis ten divisions have been taken as the unit.

EXAMPLES VI.

1. Draw the graph of $y=x^2$ on a scale twice as large as that in Fig. 13, and employ it to find the squares of ·72, 1·7, 3·4; and the square roots of 7·56, 5·29, 9·61.

2. Draw the graph of $y=\sqrt{x}$ taking the unit for y five times as great as that for x.

By means of this curve check the values of the square roots found in Example 1.

3. From the graph of $y=x^3$ (on the scale of the diagram of Art. 29) find the values of $\sqrt[3]{9}$ and $\sqrt[3]{9 \cdot 8}$ to 4 significant figures.

4. A boy who was ignorant of the rule for cube root required the value of $\sqrt[3]{14 \cdot 71}$. He plotted the graph of $y=x^3$, using for x the values 2·2, 2·3, 2·4, 2·5, and found 2·45 as the value of the cube root. Verify this process in detail. From the same graph find the value of $\sqrt[3]{13 \cdot 8}$.

5. Find graphically the values of x for which the expression x^2-2x-8 vanishes. Shew that for values of x between these limits the expression is negative and for all other values positive. Find the least value of the expression.

6. From the graph in the preceding example shew that for any value of a greater than 1 the equation $x^2-2x+a=0$ cannot have real roots.

7. Shew graphically that the expression x^2-4x+7 is positive for all real values of x.

8. On the same axes draw the graphs of

$$y=x^2, \quad y=x+6, \quad y=x-6, \quad y=-x+6, \quad y=-x-6.$$

Hence discuss the roots of the four equations

$$x^2-x-6=0, \quad x^2-x+6=0, \quad x^2+x-6=0, \quad x^2+x+6=0.$$

9. If x is real, prove graphically that $5-4x-x^2$ is not greater than 9; and that $4x^2-4x+3$ is not less than 2. Between what values of x is the first expression positive?

10. Solve the equation $x^3=3x^2+6x-8$ graphically, and shew that the function x^3-3x^2-6x+8 is positive for all values of x between -2 and 1, and negative for all values of x between 1 and 4.

11. Shew graphically that the equation $x^3+px+q=0$ has only one real root when p is positive.

12. Trace the curve whose equation is $y=2^x$. Find the approximate values of $2^{4\cdot75}$ and $2^{5\cdot25}$. Express 12 as a power of 2 approximately.

Prove also that $\log_2 26\cdot9+\log_2 38=10$.

13. By repeated evolution find the values of $10^{\frac{1}{2}}$, $10^{\frac{1}{4}}$, $10^{\frac{1}{8}}$, $10^{\frac{1}{16}}$. Thence by multiplication by 10 find the values of $10^{\frac{3}{2}}$, $10^{\frac{5}{4}}$, $10^{\frac{9}{8}}$, $10^{\frac{17}{16}}$. Use these values to plot a portion of the curve $y=10^x$ on a large scale. Find correct to three places of decimals the values of $\log 3$, $\log 5$, $\log 3\cdot25$, $\log 15\cdot36$. Also, by choosing numerical values for a and b, verify the laws

$$\log ab=\log a+\log b\,;\quad \log\frac{a}{b}=\log a-\log b.$$

[*By using paper ruled to tenths of an inch, if* 10 *in. and* 1 *in. be taken as units for x and y respectively, a diagonal scale will give values of x correct to three decimal places and values of y correct to two.*]

14. Calculate the values of $x(9-x)^2$ for the values 0, 1, 2, 3, ... 9 of x. Draw the graph of $x(9-x)^2$ from $x=0$ to $x=9$.

If a very thin elastic rod, 9 inches in length, fixed at one end, swings like a pendulum, the expression $x(9-x)^2$ measures the tendency of the rod to break at a place x inches from the point of suspension. From the graph find where the rod is most likely to break.

15. If a man spends 22*s*. a year on tea whatever the price of tea is, what amounts will he receive when the price is 12, 16, 18, 20, 24, 28, 33, and 36 pence respectively? Give your results to the nearest quarter of a pound. Draw a curve to the scale of 4 lbs. to the inch and 10 pence to the inch, to shew the number of pounds that he would receive at intermediate prices.

16. Draw the graphs of $\cos x$ and $\tan x$, on a scale twice as large as that in Art. 31.

17. Draw the graph of $\sin x$ from the following values of x:

$$5^\circ,\ 15^\circ,\ 30^\circ,\ 45^\circ,\ 60^\circ,\ 75^\circ,\ 85^\circ,\ 90^\circ.$$

Find the value of $\sin 37^\circ$, and the angle whose sine is $\cdot8$.

18. Find from the tables the value of $\cos x$ when

$$x=0^\circ,\ 10^\circ,\ 20^\circ,\ 30^\circ,\ 40^\circ,\ 50^\circ,\ 60^\circ.$$

Draw a curve on a large scale shewing how $\cos x$ varies as x increases from 0° to 60°.

Find from the curve the values of $\cos 25^\circ$ and $\cos 45^\circ$. Verify by means of the tables.

19. Draw on the same diagram the graphs of the functions $\sin x$, $\cos x$, and $\sin x+\cos x$.

Derive from the figure the general solution of $\sin x+\cos x=0$.

20. The range of a certain gun is $1000 \sin 2A$ yards, where A is the elevation of the gun. Find from the tables the value of $1000 \sin 2A$ when A has the values

$$10°, 15°, 20°, 25°, 30°, 35°, 40°, 45°, 50°,$$

and draw a curve shewing how the range varies as A increases from 10° to 50°.

21. From the tables find the values of $\tan 10x - 2 \tan 9x + 1$ for the following values of x: 0°, 1°, 2°, ... 9°. Draw a curve shewing how $\tan 10x - 2 \tan 9x + 1$ varies with x when x lies between 0° and 9°. Find to the nearest tenth of a degree a value of x for which the given expression vanishes.

Practical Applications.

32. In all the cases hitherto considered the equation of the curve has been given, and its graph has been drawn by first selecting values of x and y which satisfy the equation, and then drawing a line so as to pass through the plotted points. We thus determine accurately the position of as many points as we please, and the process employed assures us that they all lie on the graph we are seeking. We could obtain the same result without knowing the equation of the curve provided that we were furnished with a sufficient number of corresponding values of the variables *accurately calculated.*

Sometimes from the nature of the case the form of the equation which connects two variables is known. For example, if a quantity y is directly proportional to another quantity x it is evident that we may put $y = ax$, where a is some constant quantity. Hence in all cases of direct proportionality between two quantities the graph which exhibits their variations is a straight line through the origin. Also since two points are sufficient to determine a straight line, it follows that in the cases under consideration we only require to know the position of one point besides the origin, and this will be furnished by any pair of simultaneous values of the variables.

Example 1. Given that 5·5 kilograms are roughly equal to 12·125 pounds, shew graphically how to express any number of pounds in kilograms. Express $7\frac{1}{2}$ lbs. in kilograms, and $4\frac{1}{4}$ kilograms in pounds.

Here measuring pounds horizontally and kilograms vertically, the required graph is obtained at once by joining the origin to the point whose coordinates are 12·125 and 5·5.

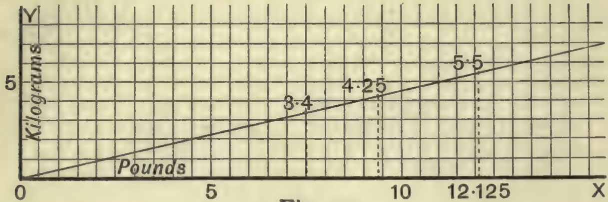

Fig. 19.

By measurement it will be found that $7\frac{1}{2}$ lbs. = 3·4 kilograms, and $4\frac{1}{4}$ kilograms = 9·37 lbs.

Example 2. The expenses of a school are partly constant and partly proportional to the number of boys. The expenses were £650 for 105 boys, and £742 for 128. Draw a graph to represent the expenses for any number of boys; find the expenses for 115 boys, and the number of boys that can be maintained at a cost of £710.

If the expenses for x boys are represented by £y, it is evident that x and y satisfy a linear equation $y = ax + b$, where a and b are constants. Hence the graph is a straight line.

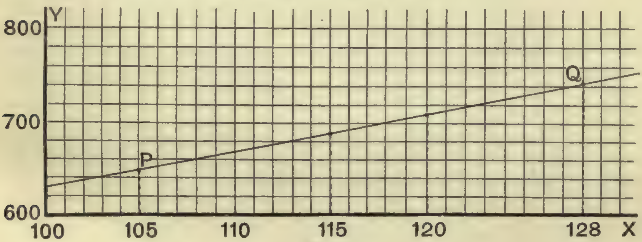

Fig. 20.

As the numbers are large, it will be convenient if we begin measuring ordinates at 600, and abscissæ at 100. This enables us to bring the requisite portion of the graph into a smaller compass. The points P and Q are determined by the data of the question, and the line PQ is the graph required.

By measurement we find that when $x = 115$, $y = 690$; and that when $y = 710$, $x = 120$. Thus the required answers are £690, and 120 boys.

33. Sometimes corresponding values of two variables are obtained by observation or experiment. In such cases the data cannot be regarded as free from error; the position of the plotted points cannot be absolutely relied on; and we cannot correct irregularities in the graph by plotting other points selected at discretion. All we can do is to draw a curve to lie as evenly as possible among the plotted points, passing through some perhaps, and with the rest fairly distributed on either side of the curve. As an aid to drawing an even continuous curve a thin piece of wood or other flexible material may be bent into the requisite curve, and held in position while the line is drawn.* When the plotted points lie approximately on a straight line, the simplest plan is to use a piece of tracing paper or celluloid on which a straight line has been drawn. When this has been placed in the right position the extremities can be marked on the squared paper, and by joining these points the approximate graph is obtained.

Example 1. The following table gives statistics of the population of a certain country, where P is the number of millions at the beginning of each of the years specified.

Year	1830	1835	1840	1850	1860	1865	1870	1880
P	20	22·1	23·5	29·0	34·2	38·2	41·0	49·4

Let t be the time in years from 1830. Plot the values of P vertically and those of t horizontally and exhibit the relation between P and t by a simple curve passing fairly evenly among the plotted points. Find what the population was at the beginning of the years 1848 and 1875.

The graph is given in Fig. 21 on the opposite page. The populations in 1848 and 1875, at the points A and B respectively, will be found to be 27·8 millions and 45·3 millions.

Example 2. Corresponding values of x and y are given in the following table:

x	1	4	6·8	8	9·5	12	14·4
y	4	8	12·2	13	14·8	20	24·8

Supposing these values to involve errors of observation, draw the graph approximately and determine the most probable equation between x and y. [See Fig. 22 on p. 36.]

* One of "Brooks' Flexible Curves" will be found very useful.

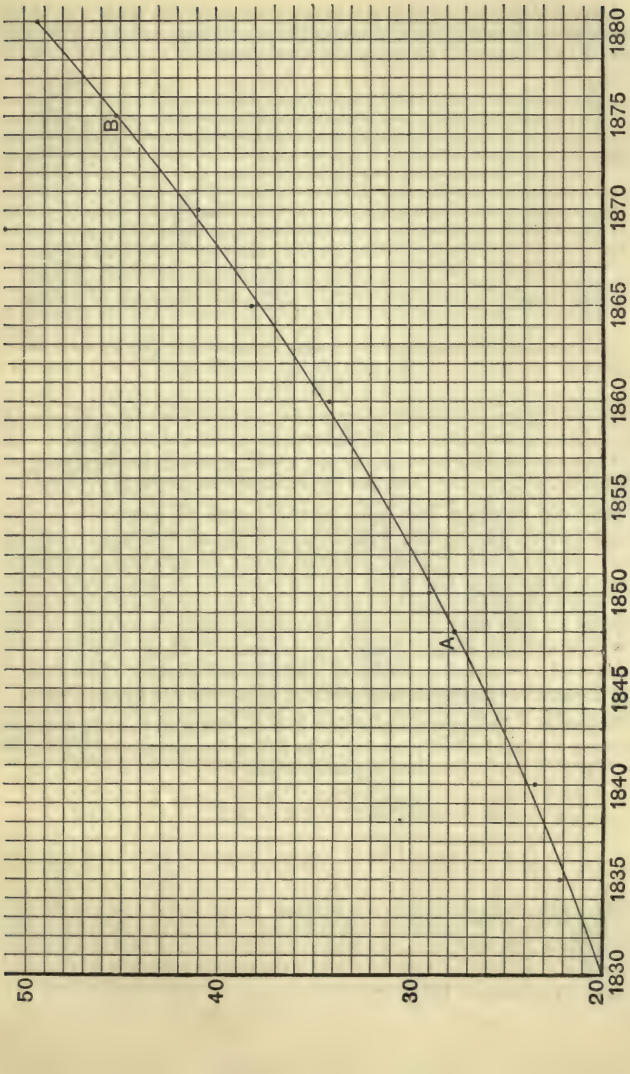
1830
1835
1840
1845
1850
1855
1860
1865
1870
1875
1880
20
30
40
50
A
B

Fig. 21.

After carefully plotting the given points we see that a straight line can be drawn passing through three of them and lying evenly among the others. This is the required graph.

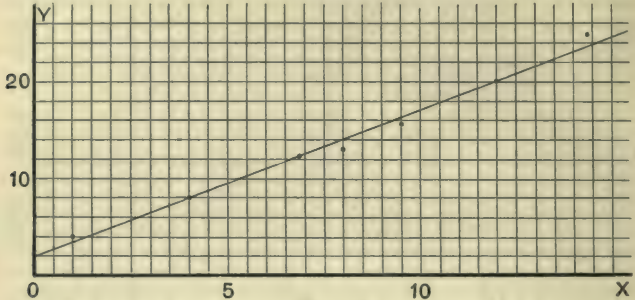

Fig. 22.

Assuming $y=ax+b$ for its equation, we find the values of a and b by selecting two pairs of simultaneous values of x and y.

Thus substituting $x=4$, $y=8$, and $x=12$, $y=20$ in the equation, we obtain $a=1{\cdot}5$, $b=2$. Thus the equation of the graph is $y=1{\cdot}5x+2$.

34. In the last Example as the graph is linear it can be produced to any extent within the limits of the paper, and so any value of one of the variables being determined, the corresponding value of the other can be read off. When large values are in question this method is not only inconvenient but unsafe, owing to the fact that any divergence from accuracy in the portion of the graph drawn is increased when the curve is produced beyond the limits of the plotted points. The following Example illustrates the method of procedure in such cases.

Example. In a certain machine P is the force in pounds required to raise a weight of W pounds. The following corresponding values of P and W were obtained experimentally :

P	2·48*	3·9	6·8	8·8	9·2	11*	13·3
W	21	36·25	66·2	87·5	103·75	120	152·5

By plotting these values on squared paper draw the graph connecting P and W, and read off the value of P when $W=70$. Also determine a linear law connecting P and W; find the force necessary to raise a weight of 310 lbs., and also the weight which could be raised by a force of 180·6 lbs.

As the page is too small to exhibit the graphical work on a convenient scale we shall merely indicate the steps of the solution, which is similar in detail to that of the last example.

Plot the values of P vertically and the values of W horizontally. It will be found that a straight line can be drawn through the points corresponding to the results marked with an asterisk, and lying evenly among the other points. From this graph we find that when $W=70$, $P=7$.

Assume $P=aW+b$, and substitute for P and W from the values corresponding to the two points through which the line passes. By solving the resulting equations we obtain $a=\cdot 08$, $b=1\cdot 4$. Thus the linear equation connecting P and W is $P=\cdot 08W+1\cdot 4$.

This is called the **Law of the Machine.**

From this equation, when $W=310$, $P=26\cdot 2$, and when $P=180\cdot 6$, $W=2240$.

Thus a force of 26·2 lbs. will raise a weight of 310 lbs.; and when a force of 180·6 lbs. is applied the weight raised is 2240 lbs. or 1 ton.

Note. The equation of the graph is not only useful for determining results difficult to obtain graphically, but it can always be used to check results found by measurement.

35. The example in the last article is a simple illustration of a method of procedure which is common in the laboratory or workshop, the object being to determine the law connecting two variables when a certain number of simultaneous values have been determined by experiment or observation.

Though we can always draw a graph to lie fairly among the plotted points corresponding to the observed values, unless the graph is a straight line it may be difficult to find its equation except by some indirect method.

For example, suppose x and y are quantities which satisfy an equation of the form $xy=ax+by$, and that this law has to be discovered.

By writing the equation in the form

$$\frac{a}{y}+\frac{b}{x}=1, \quad \text{or } au+bv=1;$$

where $u=\frac{1}{y}$, $v=\frac{1}{x}$, it is clear that u, v satisfy the equation of a straight line. In other words, if we were to plot the points corresponding to the reciprocals of the given values, their linear connection would be at once apparent. Hence the values of a and b could be found as in previous examples, and the required law in the form $xy=ax+by$ could be determined.

Again, suppose x and y satisfy an equation of the form $x^n y = c$, where n and c are constants.

By taking logarithms, we have

$$n \log x + \log y = \log c.$$

The form of this equation shews that $\log x$ and $\log y$ satisfy the equation to a straight line. If, therefore, the values of $\log x$ and $\log y$ are plotted, a linear graph can be drawn, and the constants n and c can be found as before.

Example. The weight, y grammes, necessary to produce a given deflection in the middle of a beam supported at two points, x centimetres apart, is determined experimentally for a number of values of x with results given in the following table:

x	50	60	70	80	90	100
y	270	150	100	60	47	32

Assuming that x and y are connected by the equation $x^n y = c$, find n and c.

$\log x$	$\log y$
1·699	2·431
1·778	2·176
1·845	2·000
1·903	1·778
1·954	1·672
2·000	1·519

From a book of tables we obtain the annexed values of $\log x$ and $\log y$ corresponding to the observed values of x and y. By plotting these we obtain the graph given in Fig. 23, and its equation is of the form

$$n \log x + \log y = \log c.$$

To obtain n and c, choose *two extreme points through which the line passes.* It will be found that when

$$\log x = 1·642, \quad \log y = 2·6$$

and when $\log x = 2·1, \quad \log y = 1·21.$

Substituting these values, we have

$$2·6 + n \times 1·642 = \log c \text{ ....(i)},$$

$$1·21 + n \times 2·1 \quad = \log c \text{ ....(ii)};$$

$$\therefore \; 1·39 - 0·458n = 0;$$

whence $n = 3·04.$

$$\therefore \text{ from (ii) } \log c = 6·38 + 1·21$$

$$= 7·59;$$

$$\therefore \; c = 39 \times 10^6, \text{ from the tables.}$$

Thus the required equation is $x^3 y = 39 \times 10^6$.

The student should work through this example in detail on a larger scale. The adjoining figure was drawn on paper ruled to tenths of an inch and then reduced to half the original scale.

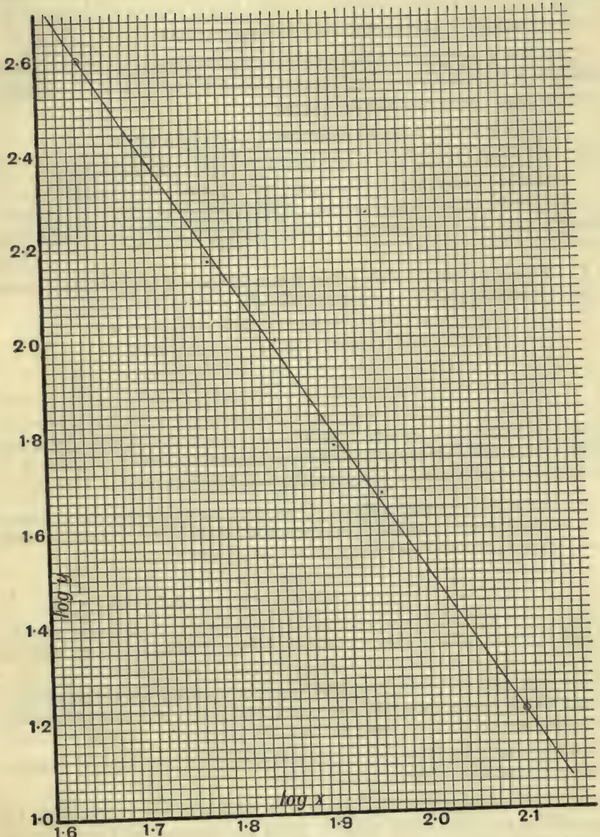

Fig. 23.

EXAMPLES VII.

1. Given that 6·01 yards = 5·5 metres, draw the graph shewing the equivalent of any number of yards when expressed in metres.

Shew that 22·2 yards = 20·3 metres approximately.

2. Draw a graph shewing the relation between equal weights in grains and grammes, having given that 10·8 grains = 1·17 grammes.

Express (i) 3·5 grammes in grains.

(ii) 3·09 grains as a decimal of a gramme.

3. If 3·26 inches are equivalent to 8·28 centimetres, shew how to determine graphically the number of inches corresponding to a given number of centimetres. Obtain the number of inches in a metre, and the number of centimetres in a yard. What is the equation of the graph?

4. The following table gives approximately the circumferences of circles corresponding to different radii:

C	15·7	20·1	31·4	44	52·2
r	2·5	3·2	5	7	8·3

Plot the values on squared paper, and from the graph determine the diameter of a circle whose circumference is 12·1 inches and the circumference of a circle whose radius is 2·8 inches.

5. For a given temperature, C degrees on a Centigrade are equal to F degrees on a Fahrenheit thermometer. The following table gives a series of corresponding values of F and C:

C	−10	−5	0	5	10	15	25	40
F	14	23	32	41	50	59	77	104

Draw a graph to shew the Fahrenheit reading corresponding to a given Centigrade temperature, and find the Fahrenheit readings corresponding to 12·5° C. and 31° C.

By observing the form of the graph find the algebraical relation between F and C.

6. For a certain book it costs a publisher £100 to prepare the type and 2s. to print each copy. Find an expression for the total cost in pounds of x copies. Make a diagram on a scale of 1 inch to 1000 copies, and 1 inch to £100 to shew the total cost of any number of copies up to 5000. Read off the cost of 2500 copies, and the number of copies costing £525.

7. At different ages the mean after-lifetime ("expectation of life") of males, calculated on the death rates of 1871-1880, was given by the following table:

Age	6	10	14	18	22	26	27
Expectation	50·38	47·60	44·26	40·96	37·89	34·96	34·24

Draw a graph to shew the expectation of any male between the ages of 6 and 27, and from it determine the expectation of persons aged 12 and 20.

8. In the Clergy Mutual Assurance Society the premium (£P) to insure £100 at different ages is given approximately by the following table:

Age	20	22	25	30	35	40	45	50	55
P	1·8	1·9	2·0	2·3	2·7	3·1	3·6	4·4	5·5

Illustrate the same statistics graphically, and estimate to the nearest shilling the premiums for persons aged 34 and 43.

9. If W is the weight in ounces required to stretch an elastic string till its length is l inches, plot the following values of W and l:

W	2·5	3·75	6·25	7·5	10	11·25
l	8·5	8·7	9·1	9·3	9·7	9·9

From the graph determine the unstretched length of the string, and the weight the string will support when its length is 1 foot.

10. In the following table P and A (expressed in hundreds of pounds) represent the Principal and corresponding Amount for 1 year at 3 per cent. simple interest.

P	2·3	2·7	3·0	3·5	3·9	5·2	7·6
A	2·369	2·781	3·090	3·605	4·017	5·356	7·828

Plot the values of P and A on a large scale, and from the graph determine the Principal which will amount to (i) £329. 12*s.*; (ii) £587. 8*s.*

11. The highest and lowest marks gained in an examination are 297 and 132 respectively. These have to be reduced in such a way that the maximum for the paper (200) shall be given to the first candidate, and that there shall be a range of 150 marks between the first and last. Find the equation between x, the actual marks gained, and y, the corresponding marks when reduced.

Draw the graph of this equation, and read off the marks which should be given to candidates who gained 200, 262, 163 marks in the examination.

12. A body starting with an initial velocity, and subject to an acceleration in the direction of motion, has a velocity of v feet per second after t seconds. If corresponding values of v and t are given by the annexed table,

v	9	13	17	21	25	29	33	37	41	45
t	1	2	3	4	5	6	7	8	9	10

plot the graph exhibiting the velocity at any given time. Find from it (i) the initial velocity, (ii) the time which has elapsed when the velocity is 28 feet per second. Also find the equation between v and t.

13. The connection between the areas of equilateral triangles and their bases (in corresponding units) is given by the following table:

Area	·43	1·73	3·90	6·93	10·82	15·59
Base	1	2	3	4	5	6

Illustrate these results graphically, and determine the area of an equilateral triangle on a base of 2·4 ft.

14. A body falling freely under gravity drops s feet in t seconds from the time of starting. If corresponding values of s and t at intervals of half a second are as follows:

t	·5	1	1·5	2	2·5	3	3·5	4
s	4	16	36	64	100	144	196	256

draw the curve connecting s and t, and find from it

(i) the distance through which the body has fallen after 1 min. 48 secs.

(ii) the distance through which it drops in the 4th second.

15. A body is projected with a given velocity at a given angle to the horizon, and the height in feet reached after t seconds is given by the equation $h = 64t - 16t^2$. Find the values of h at intervals of $\frac{1}{4}$th of a second and draw the path described by the body. Find the maximum value of h, and the time after projection before the body reaches the ground.

16. The keeper of a hotel finds that when he has G guests a day his total daily profit is P pounds. If the following numbers are averages obtained by comparison of many days' accounts determine a simple relation between P and G.

G	21	27	29	32	35
P	−1·8	1·11	3·2	4·5	6·6

For what number of guests would he just have no profit?

17. A man wishes to place in his catalogue a list of a certain class of fishing rods varying from 9 ft. to 16 ft. in length. Four sizes have been made at prices given in the following table:

9 ft.	11 ft. 9 in.	14 ft. 4 in.	16 ft.
15*s.*	22*s.*	31*s.*	38*s.*

Draw a graph to exhibit prices for rods of intermediate lengths, and from it determine the probable prices for rods of 13 ft. and 15 ft. 8 in.

18. The following table gives the sun's position at 7 A.M. on different dates:

Mar. 23	Ap. 3	Ap. 20	May 8	May 27	June 22	July 18	Aug. 5	Aug. 25
80° E.	82° E.	85° E.	89° E.	92° E.	95° E.	94° E.	91° E.	85° E.

Shew these results graphically, and estimate approximately the sun's position at the same hour on June 8th.

19. At a given temperature p lbs. per square inch represents the pressure of a gas which occupies a volume of v cubic inches. Draw a curve connecting p and v from the following table of corresponding values:

p	36	30	25·7	22·5	20	18	16·4	15
v	5	6	7	8	9	10	11	12

20. Plot on squared paper the following measured values of x and y, and determine the most probable equation between x and y:

x	3	5	8·3	11	13	15·5	18·6	23	28
y	2	2·2	3·4	3·8	4	4·6	5·4	6·2	7·25

21. The following table refers to aqueous solution of ammonia at a given temperature; x represents the specific gravity of the solution, and y the percentage of ammonia:

x	·996	·992	·988	·984	·980	·976	·968
y	·91	1·84	2·80	3·80	4·80	5·80	7·82

Draw a graph shewing the variations of x and y, and find its equation.

22. Corresponding values of x and y are given in the following table:

x	1	3·1	6	9·5	12·5	16	19	23
y	2	2·8	4·2	5·3	6·6	8·3	9	10·8

Supposing these values to involve errors of observation, draw the graph approximately, and determine the most probable equation between x and y. Find the correct value of y when $x=19$, and the correct value of x when $y=2·8$.

23. The following corresponding values of x and y were obtained experimentally:

x	0·5	1·7	3·0	4·7	5·7	7·1	8·7	9·9	10·6	11·8
y	148	186	265	326	388	436	529	562	611	652

It is known that they are connected by an equation of the form $y=ax+b$, but the values of x and y involve errors of measurement. Find the most probable values of a and b, and estimate the error in the measured value of y when $x=9·9$.

24. In a certain machine P is the force in pounds required to raise a weight of W pounds. The following corresponding values of P and W were obtained experimentally:

P	2·8	3·7	4·8	5·5	6·5	7·3	8	9·5	10·4	11·75
W	20	25	31·7	35·6	45	52·4	57·5	65	71	82·5

Draw the graph connecting P and W, and read off the value of P when $W=60$. Also determine the law of the machine, and find from it the weight which could be raised by a force of 31·7 lbs.

25. The following values of x and y, some of which are slightly inaccurate, are connected by an equation of the form $y=ax^2+b$.

x	1	1·6	3	3·7	4	5	5·7	6	6·3	7
y	3·25	4	5	6·5	7·4	9·25	10·5	11·6	14	15·25

By plotting these values draw the graph, and find the most probable values of a and b.

Find the true value of x when $y=4$, and the true value of y when $x=6$.

26. The following table gives corresponding values of two variables x and y:

x	2·75	3	3·2	3·5	4·3	4·5	5·3	6	7	8	10
y	11	9·8	8	6·5	6·1	5·4	5	4·3	4·1	4	3·9

These values involve errors of observation, but the true values are known to satisfy an equation of the form $xy=ax+by$. Draw the graph by plotting the points determined by the above table, and find the most probable values of a and b. Find the correct values of y corresponding to $x=3·5$, and $x=7$.

27. Observed values of x and y are given as follows:

x	100	90	70	60	50	40
y	30	31·08	33·5	35·56	37·8	40·7

Assuming that x and y are connected by an equation of the form $xy^n=c$, find n and c.

28. The following values of x and y involve errors of observation :

x	66·83	63·10	58·88	51·52	48·53	44·16	40·36
y	144·5	158·5	177·8	208·9	236·0	264·9	309·0

If x and y satisfy an equation of the form $x^n y = c$, find n and c.

29. In the following table the values of C and C' represent the calculated and observed amounts of water, in cubic feet per second, flowing through a circular orifice for different heads of water represented by H feet.

H	60	69·12	82	92·16	106	115·2	134
C	·0133	·0141	·0154	·0163	·0175	·0182	·0197
C'	·0133	·0141	·0153	·0162	·0173	·0180	·0194

Plot the graph of C and H and also that of C' and H, and deduce the probable error in the observed flow for a head of 120 feet.

30. The following table gives the pressures (in lbs. per sq. in.) and corresponding Fahrenheit temperatures at which water boils :

P	29·7	14·7	12·25	9·80	7·84	6·86
t	249·6	212·0	203·0	192·3	182·0	176·0

Shew graphically the relation between temperature and pressure of boiling water.

31. It is known that the relation of pressure to volume in saturated steam under certain conditions is of the form $pv^n = \text{constant}$. Find the value of the index n from the following data :

p	10·2	14·7	20·8	24·5	33·7	39·2	45·5
v	37·5	26·6	19·2	16·4	12·2	10·6	9·2

where p is measured in lbs. per sq. in., and v is the volume of 1 lb. of steam in cub. ft.

32. The following table gives the speed and corresponding indicated horse-power of the engines of a ship:

Speed in knots	11	12·4	13·3	14·25	14·8	15·5
I.H.P.	1000	1500	2000	2500	3000	3500

At what speed will she go when she develops 4000 I.H.P.?

33. In testing a steam-engine when steam was expanded to 4·8 times its original volume, the following quantities of steam per indicated horse-power per hour were used:

Steam per I.H.P. per hr. in lbs.	16·9	17	17·2	18	20·3
I.H.P.	40·5	33	25·5	19	11

When the ratio of expansion in the engine was 10 instead of 4·8, the steam used was as follows:

Steam per I.H.P. per hr. in lbs.	15	15·5	16	18	26·5
I.H.P.	33	27·2	23	15	5

At what H.P. will the consumption of steam be the same in the two cases, and what is the consumption of steam at that H.P.?

34. The power required to produce a given speed in the case of each of two ships is given in the following tables:

(i)

Speed	8	10·7	12·7	14	16	16·2
I.H.P.	500	1000	1500	1950	2800	3000

(ii)

Speed	8	10	12	12·5	13·5	14·5	16·1	16·7
I.H.P.	200	400	920	1100	1500	2000	3000	3500

At what speed will they generate the same H.P.?

ANSWERS.

I. Page 4.

7. 36. **8.** 32. **9.** 25. **11.** 1·2 sq. cm.

12. $y=3x$. Any point whose ordinate is equal to three times its abscissa.

14. The lines are $x=5$, $y=8$. The point (5, 8).

15. A circle of radius 13 whose centre is at the origin.

II. Page 7.

21. 32 sq. units. **22.** 1 sq. in.

23. 72 sq. units. **24.** ·64 sq. cm.

III. Page 10.

1. $x=1$, $y=5$. **2.** $x=2$, $y=10$. **3.** $x=3$, $y=12$.

4. $x=3$, $y=-2$. **5.** $x=4$, $y=2$. **6.** $x=6$, $y=8$.

7. $x=-2$, $y=4$. **8.** $x=0$, $y=-3$. **9.** $x=-3$, 0.

10. At the point (0, 21). **11.** $3x+4y=7$.

IV. Page 15.

1. $y=x$. **2.** (0, 0), (−4, 2). **5.** (2, 1).

6. (i) 1·46, −5·46; (ii) 3·24, −1·24; (iii) 3·32, ·68.

7. −5; 7. **8.** $-\frac{1}{4}$; 3·79, −·79; 4·62, −1·62.

9. $x=8$, or 6; $y=6$, or 8.

10. The straight line $3x+2y=25$ *touches* the circle $x^2+y^2=25$ at the point (3, 4).

V. Page 22.

3. Each axis is an asymptote to the curve, which approaches the axis of y much less rapidly than it does the axis of x.

7. $x=2, \frac{10}{3}$; $y=5, 3$. **8.** $x=\pm 3$; $y=\pm 2$.

9. $x=2$; $y=-1$. **31.** −3, 3·73, ·27. **32.** −2, 4·41, 1·59.

VI. Page 30.

1. ·52, 2·9, 11·6 ; 2·75, 2·3, 3·1. **3.** 2·080, 2·140. **4.** 2·4.

5. $-2, 4$; -9. **9.** -5 and 1. **10.** $-2, 1, 4$.

12. 26·9, 38, 3·58. **13.** ·477, ·699, ·512, 1·86.

14. 3 in. from the point of suspension.

15. 22 lbs., 16½ lbs., 14⅔ lbs., 13¼ lbs., 11 lbs., 9½ lbs., 8 lbs., 7⅓ lbs. The curve is a rectangular hyperbola whose equation is $xy = 22 \times 12$.

17. ·602. 53°. **18.** ·906. ·707. **19.** $x = n\pi + \frac{3\pi}{4}$.

20. The range varies from 342 yards, when $A = 10°$, to 984·8 yards when $A = 40°$. It reaches its maximum of 1000 yards when $A = 45°$, and is again equal to 984·8 yards, when $A = 50°$.

21. 5·9°, 7·7°.

VII. Page 40.

2. (i) 53·7 grains ; (ii) ·2 **3.** 39·3 ; 91·6 ; $y = {\cdot}393x$.

4. 3·85 in.; 17·6 in. **5.** 54·5° F. 86·9° F. $F = 32 + \frac{9}{5}C$.

6. $y = 100 + \frac{x}{10}$; £350 ; 4250. **7.** 45·96 ; 39·40.

8. £2. 12*s.*; £3. 8*s.* **9.** 8·1 in.; 24·375 oz.

10. (i) £320 ; (ii) £580. **11.** $y = \frac{10}{11}x - 70$. 112 ; 168 ; 78.

12. 5 ft. per sec.; 5 min. 45 secs.; $v = 5 + 4t$. **13.** 2·49 sq. ft.

14. (i) 52 ft.; (ii) 112 ft. **15.** max. height = 64 ft.; 4 secs.

16. $P = {\cdot}6\,G - 14{\cdot}4$; 24. **17.** 26*s.* ; 36*s.* 6*d.*

18. 93·5°E. **20.** $y = {\cdot}21x + 1{\cdot}37$.

21. $y = 249{\cdot}8 - 250x$. **22.** $y = {\cdot}4x + 1{\cdot}6$; 9·2 ; 3.

23. $a = 45{\cdot}7$, $b = 119$. Error = 9·43 in defect.

24. 8·6 ; $P = {\cdot}14\,W + {\cdot}2$; 225 lbs.

25. $a = \frac{1}{4}$; $b = 3$. 2 ; 12. **26.** $a = 3$, $b = 2$. 7 ; 4·25.

27. $n = 3$, $c = 27 \times 10^5$. **28.** $n = 1{\cdot}5$, $c = 79500$.

29. 1·3 per cent. in defect. **31.** $n = \frac{17}{16}$.

32. ·16 knots. **33.** H.P. = 6·9. 23 lbs.

34. 15·15 knots ; 2420 H.P.