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A Simultaneous Test of the Intertemporal Capital Asset Pricing Model, the Arbitrage Pricing Theory, and the Index Model

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Abstract

This paper extends Cheng and Grauer's [6] linear combination approach (LCA) to test the intertemporal CAPM, the APT and the index model. The LCA avoids the measurement problem and the unobservability of the market portfolio, the zero-beta portfolio and the k state variables (or factors). Given a sample of n asset returns which includes a subset of m asset returns having nonsingular covariance matrix, we show that by regressing (n-m) asset returns on a subset of the m asset returns, the sum of the slope coefficients must be insignificantly different from one for all of the asset pricing models. The intertemporal CAPM and the APT, however, can be distinguished from the index model in the intercept term, which must be equal to zero in the first two models. The empirical evidence from industry rates of return supports the hypothesis that stock returns during the 1973-1982 period are described by a two-factor APT or a one-state variable CAPM.

A Simultaneous Test of the Intertemporal Capital Asset Pricing Model, the Arbitrage Pricing Theory, and the Index Model

Cheng and Grauer [6] proposed a linear combination approach (LCA) to test the capital assest pricing model (CAPM) of Sharpe [26], Lintner [14], and Mossin [17]. The LCA of Cheng and Grauer avoids Roll's [18] critique on the traditional method of testing the CAPM which requires the identification of the market portfolio. The LCA has been extended by Jobson [13] to test the arbitrage pricing theory (APT)[1] of Ross [20, 21] thereby avoiding Shanken's [24, 25] criticism of testing the APT.[2]

The purpose of this paper is to develop a generalized LCA to simultaneously test all of the linear asset pricing models which have been examined in the finance literature. The models include the CAPM, the generalized CAPM of Merton [16] and Long [15], the APT, and the multi-index model. The generalized LCA proposed here is related to that of Cheng and Grauer, and Jobson, but it is different in some important aspects which will be explained later. Furthermore, we illustrate the strengths and weaknesses of the LCA and link our analysis to the mutual fund separation theory (MFST).[3] The LCA is an indirect method, which was originally designed only to test either the CAPM or the APT. It will be shown that the generalized LCA can be indirectly used to test Merton's intertemporal CAPM. The LCA also can be used to distinguish the intertemporal CAPM and the APT from the index model. In addition to avoiding the measurement errors, the LCA has the same purpose of alleviating estimation errors as proposed by Gibbons [10].

The analytic results explore relationship among a (k-1)-state variable CAPM, a k-factor APT, and a k-index model given a sample of n asset returns which includes a subset of m asset (or portfolio) returns having nonsingular covariance matrix.[4] The support for the models lies in the fact that the sum of the slope coefficients are insignificantly different from one in the multivariate regression of the (n-m) asset returns on the m asset returns (where $m > k$). Although the intercept term is required to be zero for both the generalized CAPM and the APT, it is permitted to be non-zero in the index model. To the best of our knowledge, requiring the sum of slopes coefficients to be insignificantly different from one is a new parameter restriction in testing the asset pricing model in financial economic research. Unless the regression restrictions on both the slope coefficients and the intercept are imposed in testing the generalized CAPM and the APT, incorrect conclusions regarding the tests of the models might occur. Furthermore, the parameter restrictions have additional interesting implications to the previous theoretical research on mutual fund separation theory. Merton [16] and Breeden [2] have shown that a k-state variable CAPM implies a (k+2) MFST, while Ross [22] and Connor [8] have shown that a k-factor APT implies a (k+1) MFST. The MFST can also be interpreted by the concept of spanning an asset return into a vector space as pointed out by Chamberlain and Rothschild [4]. It will be shown that the MFST theory implies restrictions on the regression slope coefficients and the intercept, or vice versa.

The remainder of the paper is organized as follows. The generalized linear combination approach to testing the linear asset pricing models and three related hypotheses are derived in section I. The investigation of the property of the Likelihood Ratio Test (LRT) is explored in section II. Section III tests the three hypotheses derived in section I to find which model (the generalized CAPM, the APT, or the multi-index model) best describes stock returns during the 1973-1982 period. Finally, a brief conclusion is contained in section IV.

I. The Generalized Linear Combination Approach

A. The Test of the Generalized CAPM

The generalized CAPM, originally introduced by Merton and Long and later extended by Breeden [2], allows more than one systematic risk in the pricing model. The vector form of a k-state variable CAPM can be written as follows:

$$E = E_0 \underline{1} + \beta_{ams} \begin{bmatrix} R_m - E_0 \\ \\ E^s - E_0 \underline{1} \end{bmatrix} \qquad (1)$$

where E = the nxl vector of expected returns on all risky assets,

E_0 = the expected return on a zero-beta portfolio, or the risk-free rate if it exists,

R_m = the expected return on the market portfolio,

E^s = the kxl expected returns on the assets perfectly correlated with the state variables,

$\underline{1}$ = the nxl column vector of unity, and

β_{ams} = the nx(k+1) matrix of "multiple regression" betas for all asset returns on the market return and on the asset returns which are perfectly correlated with the changes in the state variables, assuming that such assets exist.

Assuming that the following return generating processes for the k-state variable CAPM are linear:

$$R_t = E_0 + \beta_{am} r_t^m + B_1 r_t^1 + \dots + B_k r_t^k + e_t$$

$$= E_0 + \beta_{ams} \begin{bmatrix} r_t^m \\ \\ S_t \end{bmatrix} + e_t, \qquad (2)$$

where R_t = the nx1 vector of random returns on all assets per unit of time during time t,

r_t^m = the random return on the market portfolio in a deviation form, i.e., $R_{mt} - R_m$,

S_t = the kx1 vector of r_t^j,

r_t^j = the jth element of S_t, which is the return on the asset perfectly correlated with state variable j in a deviation form, and

e_t = the nx1 vector of random error terms.

Here r_t^m and r_t^j by definition have mean zero and are assumed to be independent of e_t. The elements of e_t are assumed to have mean zero. Notice that we do not assume e_t to be mutually independent of each other. Furthermore, we assume that there are n risky assets in the whole sample. For convenience, we define $B = \beta_{ams}$ and $A_t = [r_t^m, S_t]$. The set N is assumed to consist of the entire set of n assets and is divided into two mutually exclusive subsets I and J. The subset I consists of the first m assets which have nonsingular covariance matrix, while the subset J is composed of the remaining q assets $(m + q = n)$. That is, N is the union of subsets I and J. We further assume that m is equal to or greater than <u>k+2</u>. Let asset h be the reference asset (or portfolio) which is observable and is selected from subset I. Then, we define

$R^*_{it} = R_{it} - R_{bt}$, $E^*_i = E_i - E_b$, $B^*_i = B_i - B_b$, and $e^*_{it} = e_{it} - e_{bt}$, $i \epsilon I$ except $i \neq b$. Then, suppose that R_{It} is an (m-1)x1 vector of returns on all the assets in I except the reference asset b, while R_{Jt} is an qx1 vector of returns on all the assets in J. It is clear from the above definitions and assumption that the rank of B^*_I is <u>k+1</u>, where B^*_I is an (m-1)x(k+1) matrix. If we use the two-step approach or multivariate approach of Gibbons [10] to test the model, we must confront the measurment problem and unobservability of the market portfolio, the zero-beta portfolio (or the risk-free asset), and the k state variables. The question that follows is that if we can replace the market portfolio, the zero-beta portfolio (the risk-free asset), and the k state variables with other assets which do not have the measurement problem and are observable. Fortunately, we can use a linear algebraic concept to accomplish this end. The following analysis illustrates how to generalize the LCA of Cheng and Grauer, and Jobson.

To replace the unobservable variables, $A_t = [r^m_t, S_t]$, in equation (2) and the zero-beta portfolio by other observable asset returns, the model of (2) should be partitioned into three mutually exclusive equation systems as follows:

$$R_{bt} = E_b + B_b A_t + e_{bt}, \qquad b \epsilon I, \tag{3}$$

$$R_{It} = E_I + B_I A_t + e_{It}, \qquad i \epsilon I \text{ but } i \neq b, \text{ and}, \tag{4}$$

$$R_{Jt} = E_J + B_J A_t + e_{Jt}, \qquad j \epsilon J. \tag{5}$$

The asset in equation (3) is the reference asset, which is conceptually used to replace the zero-beta portfolio. The (m-1) assets in equation

(4) are used to replace the market portfolio and k state variables. The q assets in equation (5) are the remaining assets. Subtracting (3) from (4) and (5), respectively, we arrive at

$$R^*_{It} = E^*_I + B^*_I A_t + e^*_{It}, \text{ and} \tag{6}$$

$$R^*_{Jt} = E^*_J + B^*_J A_t + e^*_{Jt}. \tag{7}$$

Applying matrix operation to (6) and assuming that $(B^{*'}_I B^*_I)^{-1}$ exists, we have

$$A_t = (B^{*'}_I B^*_I)^{-1} B^{*'}_I [R^*_{It} - E^*_I - e^*_{It}] \tag{8}$$

For convenience, we define $H_I = (B^{*'}_I B^*_I)^{-1} B^{*'}_I$, and $H_{JI} = B^*_J H_I$. Substituting (8) into (7) and rearranging the terms, we have

$$R^*_{Jt} = [E^*_J - H_{JI} E^*_I] + H_{JI} R^*_{It} - H_{JI} e^*_{It} + e^*_{Jt}. \tag{9}$$

By partitioning E in (1) into three mutually exclusive equation systems corresponding to equations (3) through (5), and manipulating, we obtain

$$E^*_J = H_{JI} E^*_I. \tag{10}$$

Clearly, the first term on the RHS of (9) has vanished. Now, solving (9) for the reference asset b explicitly and using the relation of (10), the result becomes[5]

$$R_{Jt} = H_{JI} R_{It} + [\underline{1} - H_{JI}\underline{1}]R_{bt} + e_{Jt} - H_{JI} e_{It} - [\underline{1} - H_{JI}\underline{1}]e_{bt}. \tag{11}$$

Summing up the slope coefficients on R_{It} and R_{bt} for each asset $j \varepsilon J$, we have

$$H_{JI}\underline{1} + [\underline{1} - H_{jI}\underline{1}] = \underline{1} \tag{12}$$

The implications of (10) and (12) are that if we run the following multiple regression

$$R_{jt} = a_{j0} + \sum_{i}^{p} b_{ji} R_{it} + u_{jt}, \quad j \varepsilon J,\ i \varepsilon I, \text{ and } t = 1,\ldots,T,\ k+2 \leq p \leq m \tag{13}$$

the intercept term is zero and the sum of the slope coefficients is one. Furthermore, the (10) and (12) imply that any asset can be constructed from a portfolio of other k+2 or more assets if the k-state variable CAPM is not rejected. The weight of the portfolio (which duplicates the return on asset j) on asset $i \varepsilon I$ is simply the "multiple regression" slope coefficient, b_{ji}. It is intuitively understandable and theoretically required for the restrictions on the sum of the slope coefficients to be one and the intercept to be zero from the k+2 mutual fund separation theory of Merton's k-state variable CAPM. The k-state variable CAPM implies that any risky asset can be spanned in a k+2 dimension vector space. (See Chamberlain and Rothschild [4].) Our restrictions on the slope coefficients and intercept have the same implications as that of Chamberlain and Rothschild, and the mutual fund separation theory. From equation (11), it is obvious that the error term u_{jt} in equation (13) is correlated with the independent variables (the explanatory assets). However, since the components of u_{jt} involve variables with mixed signs, the problem of heteroskedasticity appears to be minor.[6] If the residual terms of the assets in subset I were not minor, the OLS regression coefficients in (13) would be inconsistent and biased, and an instrumental variable approach or an errors-in-variables approach is required. However, using portfolios as the explanatory variables can alleviate the problem.[7] This is the reason we have chosen industry portfolios for the empirical study in section III. Since the problem of heteroskedasticity is not serious, the OLS method can be used to obtain asymptotically efficient and unbiased estimates.[8]

Let us assume that the usual assumptions on the ordinary least squares (OLS) are applied to equation (13), and that both the returns vector, and the covariance matrix of returns on assets and the state variable changes are stationary. The joint hypothesis tests of the k-state variables CAPM and the stationary assumption are equivalent to the tests of the following hypotheses from the OLS regression on equation (13).

$$H_0: \quad a_{j0} = 0 \text{ and } \sum_{j=1}^{p} b_{ji} = 1, \text{ for } m \geq p \geq k+2, \; j \varepsilon J, \text{ and } i \varepsilon I$$

$$H_1: \quad a_{j0} \neq 0 \text{ or } \sum_{j=1}^{p} b_{ji} \neq 1, \text{ for } m \geq p \geq k+2, \; j \varepsilon J, \text{ and } i \varepsilon I.$$

Under the null hypothesis, if the number of the explanatory assests is no less than k+2, the intercept term is insignificantly different from zero, and the sum of the "multiple regression" slopes is insignificantly different from one.

From equations (11) and (13), it is evident that the explanatory assets beyond the k+2 assets cannot further explain the dependent variables. The adjusted R-square from the multiple regression, therefore, should not significantly increase by increasing p provided $p \geq k+2$. Let $R_j^2(p)$ be the adjusted R-square for asset j regressed on p explanatory assets. According to the above argument, the following hypothesis should not be rejected:

$$R_j^2(p) = R_j^2(p+1) = \ldots, \quad j\varepsilon J, \quad \text{and } p \geq k + 2.$$

Note that Cheng and Gauer [6] have used this hypothesis to test the CAPM.

Because S-L-M's or Black's [1] CAPM is a special case of the generalized CAPM, (13) can still be used to test S-L-M's or Black's CAPM. The above approach generalizes Cheng and Grauer's in three ways: (1) The price in Cheng and Grauer method is replaced by the return. This is advantageous in that the vector of returns and the covariance matrix of returns are theoretically more stable than those of the price.[9] (2) The restriction on the sum of the slope coefficients in the multiple regression being insignificantly different from one is valuable in empirical testing. And, (3) Equation (13) can also be used to indirectly test Merton's generalized CAPM.

If the return premiums instead of total returns are employed to derive the testing model, then the equation corresponding to (11) would be

$$R_{Jt} - E_0\underline{1} = B_J(B_I^{'}B_I)^{-1}B_I^{'}(R_{It} - E_0\underline{1}) + B_J^{'}(B_I^{'}B_I)^{-1}B_I^{'}e_{It} + e_{Jt} \quad (14)$$

$$i\varepsilon I, \text{ and } j\varepsilon J.$$

Note that the intercept term in equation (14) is still equal to zero, but the sum of the slope coefficients is no longer equal to one. The implication of (14) is that any asset return premium can be expressed by a linear combination, but not a portfolio, of other k+1 (not k+2) asset premiums, since the sum of the weights is not equal to one. This testing model has two potential weaknesses: (1) The return on either the zero-beta portfolio or the risk-free asset is very difficult, if not impossible, to estimate. (2) The sum of the "multiple regression" slopes is now no longer equal to one. This implies that there does not exist any relationship between the LCA and the MFST.

B. The Test of the APT

Equation (13) can also be used to test Ross's APT. In Ross's APT, the market portfolio does not appear in equations (1) and (2). Consequently, everything will be the same as above, except that the number of assets in set I, m, is now only required to be equal to or greater than k+1 instead of k+2. Further, Ross's k-factor APT implies that any return on asset $j\varepsilon J$ can be constructed and fully explained by the portfolio of other k+1 (not k+2) asset returns or more. Comparing the above approach with Jobson's [13],[10] it can be seen that our approach has the following different implications: (1) We use rates of return on reference assest (R_b) instead of either the zero-beta portfolio or the risk-free asset.[11] (2) The number of explanatory assets is not required to be pre-determined. (3) Our model has a very important condition, requiring that the sum of the "multiple regression" slope coefficients is equal to one, provided the number of explanatory assets is no less than k+1. (4) Our constraints are consistent with Ross and Connor's k+1 MFST, but Jobson's constraints are not.

C. The Test of the Multi-index Model

In addition to the previously discussed applications, this linear combination approach can be used to test the index model. Let us explain it in the following. A k-index model can be stated as

$$R_t = \alpha + BI_t + \varepsilon_t \qquad (15)$$

where I_t is a kx1 index vector.

We partition R_t in equation (15) into three mutually exclusive equation systems as we have done in equations (3) to (5). However, the number of assets in I, m, is required to be no less than k+1 only. The indexes I_t can be replaced by the assets in subset I as shown in equation (8). Equation (11) is then expressed as

$$R_{Jt} = [\alpha_J^* - H_{JI}\alpha_I^*] + H_{JI}R_{It} + [\underline{1} - H_{JI}\underline{1}]R_{bt} \tag{16}$$

$$- H_{JI}e_{It} - [\underline{1} - H_{JI}\underline{1}]e_{bt} + e_{Jt}.$$

It can be seen that equation (16) is the same as equation (11), except that the intercept term in equation (16) may not be equal to zero, and $m \geq k+1$ instead of $m \geq k+2$. However, the sum of the "multiple regression" slopes is still equal to one, if the number of the explanatory assets is no less than k+1. This implies that any asset return can be constructed by a portfolio of at least k+1 asset returns plus a constant, if the k-index model is not rejected. Even though the intercept is no longer required to be zero, the restriction that the sum of the slope coefficients is equal to one is still necessary. In comparison with the generalized CAPM and the APT, the index model is less restrictive.

For easy comparison, the required conditions for the above three models to be acceptable are tabulated as follows:

Model	Minimum Explanatory Assets	$\Sigma b_{ji} = 1$	$a_{jo} = 0$	Equilibrium Model
(k-1)-state variable CAPM	k+1	yes	yes	yes
k-factor APT	k+1	yes	yes	?
k-index model	k+1	yes	no	no

D. Three Hypotheses Formulated to Test the Linear Asset Pricing Models

In sum, equation (13) can be used to test the generalized CAPM, the APT, and the index model simultaneously. Let p be the number of the explanatory assets and $S_j(p)$ be the sum of the "multiple regression" slope coefficients for asset j regressed on the p explanatory assets. The following hypotheses are formulated, based upon the regression of equation (13):

HYPOTHESIS 1: The intercepts equal zero, i.e., $a_{j0} = 0$, $j \epsilon J$.

HYPOTHESIS 2: The sum of the "multiple regression" betas equal unity for $p \geq p^*$; i.e, $S_j(p) = S_j(p+1) = \ldots = 1$, $j \epsilon J$, and $p \geq p^*$,

HYPOTHESIS 3: The adjusted coefficients of determination are constant for $p \geq p^*$; i.e., $R_j^2(p) = R_j^2(p+1) = \ldots$, $j \epsilon J$, and $p \geq p^*$.

where p^* is the minimum number of p to accept both HYPOTHESES 2 and 3. Now, we have the following situations:

1. If HYPOTHESES 1, 2 and 3 are all accepted for $p^* = 2$, then any of the following models is acceptable.
   a. S-L-M's CAPM,
   b. Black's CAPM,
   c. Ross's one-factor APT.
2. Everything is the same as 1, except for $p^* > 2$. Then any of the following models is acceptable.
   a. Ross's (p^*-1)-factor APT,
   b. Merton's (p^*-2)-state variable CAPM.
3. If HYPOTHESIS 1 is rejected, while both HYPOTHESES 2 and 3 are not rejected, then only the (p^*-1)-index model is acceptable.

4. If HYPOTHESIS 1 is not rejected, the index model is not different from either the generalized CAPM or the APT.[12]

5. If none of the above situations apply, then either the linear asset pricing models do not explain the historical data very well, or the stationary assumption has been violated, or both.

From the above situations, it can be seen that ambiguity is associated with the linear combination approach. The LCA is unable to distinguish the (p^*-1)-factor APT from the (p^*-2)-state variable CAPM. However, theoretically or intuitively, it is not easy to distinguish between these two models. Yet, the main purpose of this method is to determine the appropriate models as being the APT, the CAPM, etc. In addition, the approach can be used to test specific models, like the APT or the CAPM, against the non-specific model, like the multi-index model.

In order to test the above three hypotheses, the following four equation systems are formulated.

$$R_{jt} = a_{j0} + \Sigma b_{ji} R_{it} + u_{jt}. \qquad \text{(U)}$$

$$R_{jt} = a_{j0} + \Sigma b_{ji} R_{it} + u_{jt}, \quad \text{s.t.} \quad a_{j0} = 0. \qquad \text{(R1)}$$

$$R_{jt} = a_{j0} + \Sigma b_{ji} R_{it} + u_{jt}, \quad \text{s.t.} \quad \Sigma b_{ji} = 1. \qquad \text{(R2)}$$

$$R_{jt} = a_{j0} + \Sigma b_{ji} R_{it} + u_{jt}, \quad \text{s.t.} \quad a_{j0} = 0, \text{ and } \Sigma b_{ji} = 1. \qquad \text{(R3)}$$

where $t = 1,\ldots,T$, and $j \in J = \{1,\ldots,q\}$.

Suppose that all of the OLS assumptions hold in equations (U), (R1), (R2) and (R3). Let $|\Sigma_{R(p)}|(|\Sigma_{U(p)}|)$ be the determinant of the contemporaneous covariance matrix estimated from the residuals of the restricted (unrestricted) model with p explanatory assets by the OLS method. In this special case, the independent variables are the same across the firms, so the seemingly uncorrelated regression (SUR) method can not be expected to outperform the OLS method. From equations (U) to (R3), the following appropriate Likelihodd Rate Test (LRT) can be created to test the three hypotheses.

(a) The test of HYPOTHESIS 1

$$-2*\ln\lambda = T[\ln|\Sigma_{R1(p)}| - \ln|\Sigma_{U(p)}|] \sim \chi^2_q. \qquad (17)$$

(b) The test of HYPOTHESIS 2

$$-2*\ln\lambda = T[\ln|\Sigma_{R2(p)}| - \ln|\Sigma_{U(p)}|] \sim \chi^2_q. \qquad (18)$$

(c) The joint test of HYPOTHESES 1 and 2

$$-2*\ln\lambda = T[\ln|\Sigma_{R3(p)}| - \ln|\Sigma_{U(p)}|] \sim \chi^2_{2q}. \qquad (19)$$

(d) The test of HYPOTHESIS 3

$$-2*\ln\lambda = T[\ln|\Sigma_{U(p)}| - \ln|\Sigma_{U(p+1)}|] \sim \chi^2_q. \qquad (20)$$

The hypothesis tests are based upon the assumption that the multi-betas in the genralized CAPM or in the index model or the factor loadings in the APT are stationary over time. Therefore, it is a joint hypothesis test. That is, if the joint test of hypotheses 1, 2 and 3 is rejected, it does not mean that the APT or the generalized CAPM must be rejected. This implies that either (1) the generalized CAPM (or the APT) is rejected, or (2) the stationary assumption is violated, or (3) both.

II. The Power of the LRT: Simulation Analysis

This section investigates the power of the LRT in rejecting the null hypotheses described in the previous section. A simulated data set composed of 19 securities are generated from a 2-factor APT.[13] Since we already know that the data is generated from a 2-factor APT, the number of explanatory variables is run from 2 to 4. In order to have a wide dispersion of the factor loadings, four securities denoted by A, B, C, and D with different patterns of factor loadings are chosen as the explanatory variables. We run the multivariate simultaneous equation systems (U), (R1), (R2), and (R3), with all possible combinations of these four securities used as the independent variables. For example, for p=3, we have to run four separate simultaneous equations systems, the combinations of the independent variables being ABC, ABD, ACD, and BCD.

The generalized linear combination approach described in the previous section is robust in tests of the linear asset pricing models. However, it is tricky in doing empirical tests, due to the correlations among the explanatory portfolios and the random property of the returns. Therefore, we borrow the dynamic programming concept to select the optimal portfolio combination for each given p. The objective is to minimize the determinant of the residual covariance matrix (or diagonal matrix if the residuals are assumed to be mutually independent) of the unconstrained equation system (U). For p=4, we have already chosen A, B, C and D as the 4-security combination. For p=3, we select the 3-security combination having the minimum value of $|\Sigma(3)|$ from the 4 possible combinations. Suppose they are securities A, B, and C. Then for p=2, we select the 2-security combination having minimum value of $|\Sigma(2)|$ from the 3 possible 2-security combinations of securities A, B, and C. Suppose they are security A and B. Since the data is generated from a two-factor model APT, the LRT should show that a model with $p^*=3$ is the most appropriate one to explain the data. The LRT result for the simulated data is shown in Table 1.

[Put Table 1 here]

The first figure shown in each cell is the chi-square value calculated from the full covariance matrix of the residuals (denoted F), while the second figure is the chi-square value calculated from the diagonal matrix of the residuals (denoted D). The LRT based on the full covariance matrix of the residuals rejects all four hypotheses for all p. However, the chi-square drops a lot, when p goes from 2 to 3. For all 4 cases, the chi-squares stay almost the same when p goes from 3 to 4. The LRT based upon the diagonal matrix of the residuals (D) shows the same conclusion. However, the data are generated from an exact two-factor APT. The p^* should be three. Therefore, a test similar to the "scree" test in the factor analysis[14] would be the appropriate method to determine the p^* in the likelihood ratio test. When the returns of 19 industry portfolios are used to test the above hypotheses in the next section, we will employ this criteria to determine an appropriate p^*.

III. Some Empirical Results

This section tests the linear asset pricing models, such as the CAPM, the generalized CAPM, the APT and the multi-index model, using the generalized linear combination approach described in section I. The objective is to investigate whether or not the above joint hypotheses are accepted. Nineteen industry common stock portfolios are formulated in the same manner used by Schipper and Thompson [23] and Stambaugh [27]. Using monthly returns from period 1973-1982, portfolios are formed, because they provide a convenient way to limit the computational dimensions of the linear combination method and to alleviate the heterskedasticity problem previously mentioned. The number of explanatory portfolios run ranges from 2 to 4. Food and beverage industry (#2), stone, clay and glass industry (#7), other transportation industry (#15) and the utilities industry (#16) are chosen as the explanatory portfolios. These four industries were selected as the explanatory assets in order to insure that data contained a wide range of the beta coefficients.[15] Cheng and Grauer [6] also used this approach to test their hypotheses. We find the optimal combination for p=3 is (2,15,16) and for p=2 is (2,15). Table 2 indicates the LRT results for the tests of the linear asset pricing models for 19 industry portfolios from 1973 to 1982.

[Table 2 here]

From Table 2, the hypothesis (HP1) that the intercept term is insignificantly different from zero is not rejected in all p values. This implies that the APT is insignificantly different from the factor model.[16] Namely, if the hypothesis that returns are generated from a linear factor model is not rejected, the APT is also not rejected. Similarly, the generalized CAPM (or the CAPM if p=2) is also insignificantly different from the multi-index model.[17] From HP2, the hypothesis that the sum of regression slope coefficients is equal to one is rejected for all p values. This implies that if the LRT does not over-reject the null hypotheses either (1) the linear asset pricing models--the generalized CAPM, the APT and the multi-index model--are rejected, or (2) the stationary assumption is violated, or (3) both. The results from HP3 indicate the same conclusions as those from HP2. However, the hypothesis (HP4) that the R-square does not significantly increase by increasing p is not rejected at p=3. This implies that the number of explanatory portfolios beyond three cannot explain additional variation in the dependent variables. Therefore, a 2-factor APT or a one-state variable CAPM is not rejected according to the R-square criterion.

From the previous section, we know that the LRT always over-rejects the null hypothesis and that it is necessary to use the "scree" test to determine the proper p^*. We find that the chi-square for both HP2 and HP3 are almost unchanged based on the full matrix of residuals and are increased based upon the diagonal matrix of residuals, when p goes from 3 to 4. Therefore, p^* is three, and this implies that the 2-factor APT (or a one-state variable CAPM) explain the data better than other models (p=2 or 4). Together these four hypothesis tests, we can reasonably conclude that the 1973-1982 stock returns can be described by a 2-factor APT or a one-state variable CAPM, but not the S-L-M's or the zero-beta CAPM. This confirms Chen's [5], and Roll and Ross's [19] findings that the APT outperforms the CAPM.

VI. Conclusion

In this paper, we generalized the linear combination approach to the simultaneous test of alternative linear asset pricing models. The LCA avoids Roll's and Shanken's criticisms. Three hypotheses based upon the approach are formulated to test the generalized CAPM, the APT and the multi-index model. Since the LRT seems to over-reject the null hypothesis, we use a simulation analysis to investigate the property of the LRT. We found that the "scree" test is an appropriate criterion to determine the p^*.

The monthly returns of 19 industry portfolios were used to test the three hypotheses. It was found that the generalized CAPM and the APT were insignificantly different from the index model. A two-factor APT or a one-state variable CAPM was the best model to describe the 1973-1982 industry common stock returns according to this new robust test method. The S-L-M's or zero-beta CAPM was rejected for the 1973-82 common stock returns. This finding further supports the Roll and Ross [19] and Chen [5] argument in which the APT empirically outperforms the CAPM. Errors-in-variables models of equation (13) in terms of individual security rates of return will be done in the future research.

Footnotes

[1]The APT has been extended by Huberman [11], Chamberlain and Rothschild [4], and Ingersoll [12]. See Dybvig and Ross [9] for a complete bibliography.

[2]Dybvig and Ross [9] have argued that the APT is testable. In his reply, however, Shanken [25] insisted that Ross's APT is still not testable in principle.

[3]The constraints proposed either by Cheng and Grauer or by Jobson are unable to link to the mutual fund separation theory.

[4]In the original version of this paper, "m independent assets" rather than "m asset returns having nonsingular covariance matrix" was used. The authors are grateful to an anonymous referee for pointing out the former unnecessarily strong assumption. Actually, Jobson [13] used "linearly independent assets" instead of "independent assets" assumption. The linearly independent assets assumption is equivalent to the nonsingular covariance matrix assumption. The purpose of this assumption regarding the m asset returns is to rule out the multi-collinearity problem when the (n-m) asset returns are regressed on the subset of the m asset returns. If the m asset returns are assumed to be independent, the "multiple regression" slope must be equal to the "simple regression" slope on each of the m explanatory asset returns.

[5]If the set I is so chosen that m is exactly equal to k+2, then $(B_I^{*'} B_I^{*})^{-1} B_I^{*'} = (B_I^{*})^{-1}$, and (11) can be simplified as follows:

$$R_{Jt} = B_J^{*}(B_I^{*})^{-1}R_{It} + [\underline{1} - B_J^{*}(B_I^{*})^{-1}\underline{1}]R_{bt}$$

$$- B_J^{*}(B_I^{*})^{-1}e_{It} - [\underline{1} - B_J^{*}(B_I^{*})^{-1}\underline{1}]e_{bt} + e_{Jt}.$$

[6]In their footnotes (13), Cheng and Grauer [7] point out this argument.

[7]In an equal-weighted random portfolio formed from n individual securities having mutually independent error terms, the residual variance will be one nth of the individual residual variance.

[8]If the heteroskedasticity problem were not minor (empirically it is minor), the following procedure can be used to obtain asympotically unbaised and efficient estimators. Rewrite equation (13) for asset j as

$$R_{jt} = a_{j0} + \sum_{i=1}^{p} b_{ji}R_{it} + u_{jt}, \tag{A.1}$$

$$= a_{j0} + \sum_{i=1}^{p} b_{ji}R_{it} - \sum_{i=1}^{p} b_{ji}e_{it} + e_{jt}. \tag{A.2}$$

The variance of u_{jt} in (A.2) is as follows:

$$Var(u_j) = \Sigma b_{ji} \, Var(e_i) + Var(e_j), \tag{A.3}$$

where we assume that e_{jt} and e_{it} are independent of each other. Therefore, we can use the procedures proposed in Theil [28, p. 614] to estimate b.

First of all, we select one security from set J, and p securities from set I. Then, we use this p+1 securities to run the OLS regression of equation (A.2). Each time, one of the p+1 securities as the dependent variable, and the remaining as the dependent variables, the OLS can be used to estimate b_{ji} and $Var(u_j)$). Next, we can use p+1 simultaneous equations system of (A.3) to estimate $Var(e_i)$, $i \varepsilon I$.

Secondarily, we can use the following equation to obtain the unbiased estimator of b_{ji}.

$$\underline{b}_j = (R'_{It} R_{It} - T \hat{\Sigma})^{-1} R'_{It} R_{Jt} \tag{A.4}$$

where $\hat{\Sigma}$ is a diagonal matrix with the i-th entry as $Var(e_i)$ estimated from (A.3), and T is the time period.

[9]Since the autocorrelation problem using prices is severe, Cheng and Grauer [6] used the Cochrane and Orcutt method to correct the estimated coefficient and their variances. However, this correlation procedure may destroy the original structure relationship.

[10]Jobson's result is similar to equation (14) except that it is required that p=k+1 in Jobson's model, but it is only required that $p \geq k+1$ in our model.

[11]Because the lending rate is generally different from the borrowing rate, the measure of the risk-free rate is a difficult problem.

[12]Compared situation 4 with situation 2 (or situation 1 if p=2), both situations have the same null hypothesis; namely, the stock returns are described by the generalized CAPM or the APT. However, they have different alternative hypothesis. The alternative hypothesis in situation 2 (or situation 1) is not specified, but it is in situation 4 as the index model. The multivariate test of Gibbons is similar to that of situtation 4.

[13]For the simulated data, the risk free rate is set to be 0.5%, factor loadings are randomly drawn from a uniform distribution of (-1,2), and residuals are drawn from a normal distribution with a mean of zero and a variance of the sum of the squared factor loadings. The above distribution assumptions come from the preliminary empirical

results of testing the APT. The time period is 120. The reason to use 19 simulated securities and a time period of 120 are to match the real data used in the next section.

[14]Actually, p* is equivalent to the number of factors determined by the scree test in factor analysis plus one. See Cattell [3] for the scree test in factor analysis.

[15]The industry portfolio SEC codes, number of firms and estimated betas are as follows.

Portfolio description	SEC Code	# of firms 12/72	# of firms 12/82	Estimated betas 1973-1982
1. Mining	10-14	56	71	0.922
2. Food & beverages	20	75	51	0.803
3. Textile & apparel	22,23	58	45	1.081
4. Paper products	26	30	30	0.910
5. Chemical	28	87	83	0.847
6. Petroleum	29	28	22	0.745
7. Stone, clay, glass	32	43	31	1.045
8. Primary metals	33	56	49	0.932
9. Fabricated metals	34	45	46	1.102
10. Machinery	35	93	104	1.104
11. Appliance & elec. equip.	36	87	82	1.179
12. Transpor. equip.	37	64	50	1.150
13. Misc. manufacturing	38,39	64	59	1.197
14. Railroads	40	18	11	0.899
15. Other transport.	41,42,44 45, 47	34	35	1.203
16. Utilities	49	138	152	0.564
17. Departments stores	53	35	28	1.125
18. Other retail trades	50-52, 54-59	103	97	1.123
19. Banking, finance, real estate	60-67	184	240	1.069

[16]Roll and Ross call the return generating equation of the APT the factor model.

[17]The method proposed by Gibbons is designed to test the CAPM against the single index model as the alternative hypothesis.

Table 1

Likelihood Ratio Tests of the Linear Asset Pricing Models:
Simulated Data From a 2-factor Model

$$R_{jt} = a_{j0} + \sum_{i}^{p} b_{ji} R_{it} + e_{jt}, \quad j = 1,\ldots,15 \varepsilon J \text{ and } t = 1,\ldots,120.$$

HYPOTHESIS 1: $a_{J0} = \underline{0}$ (or $a_{j0} = 0$, $j \varepsilon J$)

HYPOTHESIS 2: $b_{JI}\underline{1} = \underline{1}$ (or $\Sigma b_{ji} = 1$, $j \varepsilon J$)

HYPOTHESIS 3: $a_{J0} = \underline{0}$ and $b_{JI}\underline{1} = \underline{1}$

HYPOTHESIS 4: $R_j^2(p) = R_j^2(p+1)$, $j \varepsilon J$

p	Residual covariance	chi-square value			
		HP1	HP2	HP3	HP4
2	(F)	177.0*	113.7*	120.0*	61.8*
	(D)	244.4*	217.6*	227.2*	176.4*
3	(F)	87.7*	92.1*	99.2*	45.6*
	(D)	132.8*	164.0*	182.8*	89.5*
4	(F)	87.0*	95.0*	103.8*	----
	(D)	143.2*	163.3*	176.4*	----
Degree of freedom		15	15	30	15

*Significant at the 1% level

Table 2

Likelihood Ratio Tests of the Linear Asset Pricing Models
for the 19 Industry Portfolios: 1973-1982

$$R_{jt} = a_{j0} + \sum_{j=1}^{p} a_{ji} R_{it} + e_{jt}, \quad j = 1,\ldots,15 \varepsilon J \text{ and } t = 1,\ldots,120.$$

HYPOTHESIS 1: $a_{J0} = \underline{0}$ (or $a_{j0} = 0, \quad j \varepsilon J$)

HYPOTHESIS 2: $b_{JI}\underline{1} = \underline{1}$ (or $\Sigma b_{ji} = 1, \quad j \varepsilon J$)

HYPOTHESIS 3: $a_{J0} = \underline{0}$ and $b_{JI}\underline{1} = \underline{1}$

HYPOTHESIS 4: $R_j^2(p) = R_j^2(p+1) \quad j \varepsilon J$

p	Residual covariance	chi-square value			
		HP1	HP2	HP3	HP4
2	(F)	1.37	97.3*	97.4*	154.6*
	(D)	0.75	144.2*	144.2*	374.0*
3	(F)	1.38	76.0*	76.4*	1.96
	(D)	0.84	91.6*	91.8*	12.4
4	(F)	1.39	75.7*	76.1*	----
	(D)	0.97	123.6*	123.8*	----
Degree of freedom		15	15	30	15

*Significant at the 1% level

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