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The covered name seems to be W. PARKINSON Gent.
NORWICH.

Sir ISAAC NEWTON's TWO TREATISES OF THE Quadrature of CURVES, AND ANALYSIS by Equations of an infinite Number of Terms, explained:

CONTAINING

The Treatises themselves, translated into *English*, with a large Commentary; in which the Demonstrations are supplied where wanting, the Doctrine illustrated, and the whole accommodated to the Capacities of Beginners, for whom it is chiefly designed.

By JOHN STEWART, A. M. Professor of Mathematicks in the Marishal College and University of *Aberdeen*.

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To his *GRACE*

THOMAS *Duke* of LEEDS.

MY LORD,

WHEN a ſtranger implores your GRACE's protection to a mathematical work, he is incouraged to it from the patronage, with which you are pleaſed to favour all the liberal ſciences; and not from any peculiar regard vouchſafed to this more than others. For among the many virtues ſo univerſally acknowledged in your GRACE's character, your zeal for the general advancement of learning is eminently conſpicuous, as being the *Preſident* and life of a *Society* founded ſolely for its incouragement. A *Society* ſo generous and diſintereſted in its inſtitution, that neither Greece nor Rome could ever boaſt of a nobler deſign; and ſo applauded by foreign nations, that they would gladly imitate it, were they bleſſed with ſuch a *Maecenas* to inſpire them with a like ardour for the promoting of ſcience. For whatever entertainment and delight knowledge affords the human mind, and how far ſoever it elevates thoſe, who are poſſeſſed of it, above the reſt of their ſpecies;

yet they only, who communicate it to others, have the merit of rendering it beneficial to mankind.

THE great regard, which your honoured *Society* retain for the works of the illuſtrious *Newton*, has ingaged them to condeſcend ſo far, as to countenance his *Commentator*. The ſame motive animated my indeavours to illuſtrate and explain thoſe principles, which are the foundation of the greateſt and moſt ſurpriſing diſcoveries, either of the preſent, or any preceding age. And this being the firſt book of the kind, which the *Society* have taken under their care, as the dignity of the author juſtifies their choice; ſo, I would hope, it may render the public more favourable to my performance. I ſhall however have this ſatisfaction, that while I offer a ſmall tribute to the memory of Sir *Iſaac Newton*; I have the honour to pay my ſincereſt acknowledgements of gratitude to your GRACE, and with the higheſt eſteem and veneration to ſubſcribe myſelf,

May it pleaſe your GRACE,

Your GRACE's moſt obliged,

Moſt devoted, and

Moſt obedient humble Servant,

JOHN STEWART.

The PREFACE.

SUCH as have duly read and confidered Sir *Iſaac Newton*'s Treatiſe of the *Quadrature of Curves*, and his other *Works* allied to it, muſt be ſenſible, that he has greatly extended the Limits of Geometry, by diſcovering a Method of inveſtigating Truth, different from all thoſe Methods, which Geometricians had made uſe of before his Time; and which leads to the Diſcovery of the moſt hidden and remote Truths in Mathematicks, and Philoſophy.

But his invaluable Diſcoveries, eſpecially ſuch of them as are contained in this Treatiſe, are delivered by himſelf in ſuch a conciſe Way (the Effect of his vaſt Genius), that they ſtill lye beyond the Reach of not a few, who have made ſome conſiderable Progreſs in the Mathematical Sciences: or at leaſt they find ſuch Stops and Difficulties, before they are well entered into this Work, that they give it over; either apprehending that thoſe Difficulties are ſo great, that they cannot ſurmount them at all; or, if they are able, that it will require more Time and Pains, than they can afford to beſtow upon ſuch abſtruſe and deep Speculations. Yet the ſame Perſons, moved by the Character of the Author and the Work, would willingly bestow any reasonable Time and Pains to understand these Discoveries, were they furnished with proper and sufficient Helps for that Purpose. This Consideration may justly make one wonder, why among the vast Number of Books and Performances, which have been sent abroad into the World, since the *Quadrature of Curves* was first published, in the year 1704, no one has ever obliged the publick (so far as I know) with a Commentary upon this Work, or any Explication of it; by which Means it happens, that it's great Worth and Excellency still lye in a great Measure concealed from the Bulk even of those, who have made some tolerable Proficiency in the Study of Mathematicks.

I do not pretend to assign all the Causes, that may have concurred to produce this Effect. It certainly cannot be owing to this, that learned Men are not sensible of the Necessity and Usefulness of such an Undertaking. One Reason perhaps may be, that they, who are best qualified for it, and could execute it to best Purpose, having once engaged so deep in the Study of Geometry, as the full understanding of our Author's Treatise requires, forthwith affect to be Authors themselves, rather than labour in the lower Office of explaining and illustrating the original Discoveries of others. Notwithstanding it must be owned, that one may be as usefully employed, to say no more, in this latter Office, as in the former, which is esteemed higher. Since it has been justly enough observed, that this Science is now brought to such a Degree of Perfection, that the Application of these abstruse and sublime Speculations to some useful Purposes, either in common Life or Philosophy, is much more wanted, and would tend more to the Good of Mankind; than the carrying them to yet a greater

greater Height, without making any Application of them for the Purposes mentioned, by which the Good of *Society* might be promoted, and the *wise* and *excellent* Constitution of Things in this material World better understood.

This naturally enough leads me to answer an Objection, that some perhaps may be ready to start, both against our Author's *Treatise*, and this *Attempt* to make it more universally known. *Cui usui* is the Word: Why spend so much Time and Labour upon such abstract geometrical Speculations? We acknowledge, say such, the Pursuit and Acquisition of Knowledge is an agreeable Exercise to the Mind of Man; but this alone is not sufficient to account for wasting away so much Time, which our Nature and Circumstances, as *reasonable* Beings and Members of *Society*, require to be otherwise employed. As such, others have an Interest in Us, and may justly demand a very considerable Share of our Time and Labour. *Non nobis solum nati sumus, ortusque nostri partem patria vindicat, partem amici.* Cic.

In answer to which, it is readily acknowledged, that other *rational* Beings have a very considerable *Interest* in our Time and Labours. But if the Matter be rightly considered, it will appear, that this *Interest* and *Concern* of theirs is by no Means overlooked or neglected; on the contrary, it is consulted and regarded. The *natural* Desire of Knowledge found in all Mankind is a great *Excitement* to the Pursuit and Acquisition of it, and at the same Time a sufficient *Indication*, that *Nature* designed it for a Part of our Employment; which is still confirmed by the great Satisfaction and Delight, the Mind of Man enjoys in the Discovery and Possession of Truth. *Ex quatuor autem locis, in quos honesti naturam vimque divisimus,*

divisimus, primus ille, qui in veri cognitione consistit, maxime naturam attingit humanam. Omnes enim trahimur & ducimur ad cognitionis et scientiæ cupiditatem, in qua excellere pulchrum putamus. Cic. Whoever therefore contributes to the Advancement of Knowledge in others, by bestowing some Time and Pains to make way for the Entrance of it into their Minds; so far *administers* to their *Happiness*, and consults their *Advantage*. And this holds especially in the present Case, if the natural Tendency and genuine Effect of mathematical Knowledge pursued and acquired be rightly considered, and attended to; as it strengthens our intellectual Powers, by forming in the Mind an Habit of thinking *closely*, and reasoning *accurately*, even upon other Subjects [1].

But further, every one, who is the least conversant in these Matters, knows, that natural Philosophy never was, nor can be successfully prosecuted or advanced, but by the

[1] Would you have a Man reason well, you must use him to it betimes, exercise his Mind in observing the Connection of Ideas, and following them in train. Nothing does this better than Mathematicks; which therefore, I think, should be taught all those, who have the Time and Opportunity, not so much to make them Mathematicians, as to make them reasonable Creatures. *Locke, On the Conduct of the Understanding.* And a little after: I have mentioned Mathematicks as a way to settle in the Mind an Habit of reasoning closely, and in Train; not that I think it necessary, that all Men should be deep Mathematicians; but that having got the way of reasoning, which that Study necessarily brings the Mind to, they might be able to transfer it to other Parts of Knowledge, as they shall have occasion.——I think the Study of Mathematicks of infinite Use even to grown Men; first by experimentally convincing them, that to make any one reason well, it is not enough to have Parts, wherewith he is satisfied, and that serve him well enough in his ordinary Course. A Man in those Studies will see, that however good he may think his Understanding; yet in many Things, and those very visible, it may fail him. This would take off that Presumption, which most Men have in that Part. ——The Study of Mathematicks would shew them the Necessity there is in reasoning to separate all the distinct Ideas, and see the Habitudes that all those concerned in the present Enquiry have to one another, and to lay by those which relate not to the Proposition in Hand, and wholly to leave them out of the reckoning. This is that, which in other Subjects besides Quantity, is absolutely necessary to just reasoning.

In geometria partem fatentur esse utilem teneris ætatibus: agitari namque animos, atque acui ingenia, & celeritatem percipiendi inde venire concedunt. Quintil.

Help

Help of Geometry and Arithmetick; for it is by Means of theſe Sciences, that we diſcover the *Conſtitution*, and inveſtigate the *Laws*, according to which this material World, and all Things in it, are ſo *wiſely* framed, maintained and preſerved, in that admirable and beautiful *Order* and *Harmony*, wherein we behold them: which is the true Notion and Deſign of natural Philoſophy. Accordingly you will find, that the greateſt Maſters in theſe Sciences have moſt ſucceſsfully laboured in the promoting and perfecting of natural Knowledge. Witneſs our noble Author, who in that admirable Performance, his *Mathematical Principles of natural Philoſophy*, hath happily ſhewn us, of what great Uſe abſtract Mathematical Knowledge may be, for inveſtigating the *Forces* of natural Bodies; and thence explaining and demonſtrating the Phænomena and Laws, which Nature obſerves in her Operations, both in the Solar Syſtem, and upon this our Earth. And as a great Part of the Diſcoveries contained in that Book, is owing to, and founded upon, the *Doctrine of Fluxions* (the Invention of the ſame happy Genius) as was obſerved ſome Time ago by the noble and learned Marquis *D'Hoſpital* [a]; ſo we find in particular, that he often proceeds upon the Quadrature of Curves as a *Poſtulatum*, or Principle already known and granted. See *Propoſ.* 46, 53, 54, 56, 81. *Book* I. and many other Places. By which he hath ſhewn, that the moſt ſublime Parts of Geometry, and particularly the Doctrine of Fluxions, and the Quadrature of Curves, are of infinite Uſe in true Philoſophy.

[a] C'eſt encore une Juſtice dûë au ſcavant Monſieur *Newton*, & que Monſieur *Leibnitz* lui a renduë lui-même: qu'il avoit auſſi trouvé quelque choſe de ſemblable au Calcul différentiel, comme il paroît par l'excellent livre intitulé, *Philoſophiæ naturalis Principia Mathematica*, qu'il nous donna en 1687: lequel eſt preſque tout de ce calcul. *Anal. des infiniment Petits.* Pref.

Wherefore by cultivating Geometry, and ſtudying even it's higheſt Parts, ſeveral noble and uſeful Purpoſes are ſerved and promoted. Our rational and intellectual Powers are greatly enlarged and ſtrengthened. Many Things uſeful and convenient in common Life are diſcovered. Natural Philoſophy in all it's Branches is happily advanced. And what is moſt conſiderable of all, the Mind is thereby led to perceive and obſerve the infinite Knowledge and Wiſdom, Power and Goodneſs of the *Almighty* Creator, and *bountiful* Preſerver and Governor of this Univerſe. The Knowledge and Contemplation of whoſe Works and Perfections is the moſt noble and delightful Exerciſe of our *reaſonable* Powers, and naturally leads the Mind to thank and praiſe, reverence and admire, love and imitate this ſupreme Being; which is the greateſt Happineſs and Perfection of any reaſonable Creature, and the true End of all our Knowledge, and of our very Exiſtence itſelf.

It is chiefly with a Deſign of promoting theſe noble Ends, that I publiſh to the World this *Commentary* upon Sir *Iſaac Newton*'s *Quadrature of Curves*, and his *Analyſis by Equations of an infinite Number of Terms:* And though my Attempt ſhould fail of the deſigned Effect, I ſhall at leaſt have the Pleaſure of deſigning well, which is ſomething to a virtuous Mind.

A TABLE OF CONTENTS.

Contents of Sir *Isaac Newton*'s Treatise of the Quadrature of Curves.

CONTENTS.

Cor.

CONTENTS.

Prop.

Contents of the Explication of the Quadrature of Curves.

SECTION I.

Notes on the Introduction to the Quadrature of Curves, Page 33.

If

The

SECT.

SECT. V.

Of the Position and Limits of the Areas of Curves, Page 133.

SECT. VI.

Notes on Article 32 *of the Quadrature of Curves.*

Demonstration of Prop. 6. *Quad. of Curves,* Page 194.

SECT. VII.

Notes on Articles 33—39 *of the Quadrature of Curves.*

Explication of Prop. 7. *Quad. of Curves,* Page 156.

SECT. VIII.

SECT. IX.

Notes on Articles 52—64 Quadrature of Curves, Page 185.

Explication

The

SECT. X.

The

SECT. XI.

SECT. XII.

Contents of the Analysis by Equations of an infinite Number of Terms.

The

Contents of Analysis by Equations &c. explained.

SECT. I.

SECT. II.

The

SECT. III.

Explication of the numeral Resolution of affected Equations contained in Art. 21—29, Page 388.

SECT. IV.

Explication of the Resolution of affected, specious or literal Equations, contained in Art. 30—34. 57.

SECT. V.

That

Sir

Sir *ISAAC NEWTON*'s TREATISE OF THE Quadrature of CURVES.

INTRODUCTION to the Quadrature of Curves.

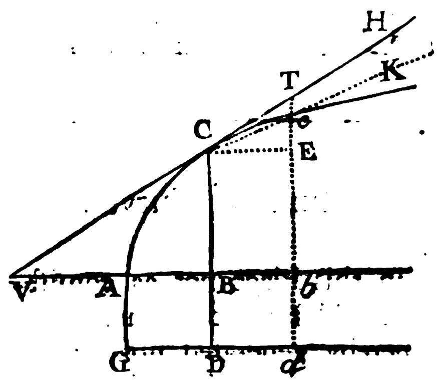
1. I Consider mathematical Quantities in this Place not as consisting of very small Parts; but as describ'd by a continued Motion. Lines are describ'd, and thereby generated not by the Apposition of Parts, but by the continued Motion of Points; Superficies's by the Motion of Lines; Solids by the Motion of Superficies's; Angles by the Rotation of the Sides; Portions of Time by a continual Flux: and so in other Quantities. These Geneses really take Place in the Nature of Things, and are daily seen in the Motion of Bodies. And after this Manner the Ancients, by drawing moveable right Lines along immoveable right Lines, taught the Genesis of Rectangles.

2. Therefore considering that Quantities, which increase in equal Times, and by increasing are generated, become greater or less according to the greater or less Velocity with which they increase and are generated; I sought a Method of determining Quantities from the Velocities of the Motions or Increments, with which they are generated; and calling these Velocities of the Motions or Increments *Fluxions*, and the generated Quantities *Fluents*, I fell by degrees upon the Method of Fluxions, which I have made use of here in the Quadrature of Curves, in the Years 1665 and 1666.

3. Fluxions are very nearly as the Augments of the Fluents generated in equal but very ſmall Particles of Time, and, to ſpeak accurately, they are in the *firſt Ratio* of the naſcent Augments; but they may be expounded by any Lines which are proportional to them.

4. Thus if the Area's ABC, ABDG be deſcribed by the Ordinates BC, BD moving along the Baſe AB with an uniform Motion, the Fluxions of theſe Area's ſhall be to one another as the deſcribing Ordinates BC and BD, and may be expounded by theſe Ordinates, becauſe that theſe Ordinates are as the naſcent Augments of the Area's.

5. Let the Ordinate BC advance from it's Place into any new Place *bc*. Complete the Parallelogram BCE*b*, and draw the right Line VTH touching the Curve in C, and meeting the two Lines *bc* and BA produc'd in T and V: and B*b*, E*c* and C*c* will be the Augments now generated of the Abſciſs AB, the Ordinate BC and the Curve Line AC*c*; and the Sides of the Triangle CET are in the *firſt Ratio* of theſe Augments conſidered as naſcent, therefore the Fluxions of AB, BC and AC are as the Sides CE, ET and CT of that Triangle CET, and may be expounded by theſe ſame Sides, or, which is the ſame thing, by the Sides of the Triangle VBC, which is ſimilar to the Triangle CET.

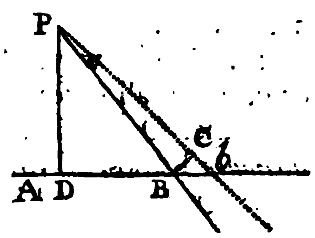


6. It comes to the ſame Purpoſe to take the Fluxions in the *ultimate Ratio* of the evaneſcent Parts. Draw the right Line C*c*, and produce it to K. Let the Ordinate *bc* return into it's former Place BC, and when the Points C and *c* coaleſce, the right Line CK will coincide with the Tangent CH, and the evaneſcent Triangle CE*c* in it's ultimate Form will become ſimilar to the Triangle CET, and it's evaneſcent Sides CE, E*c* and C*c* will be *ultimately* among themſelves as the Sides CE, ET and CT of the other Triangle CET, are, and therefore the Fluxions of the Lines AB, BC and AC are in this ſame Ratio. If the Points C and *c* are diſtant from one another by any ſmall Diſtance, the right Line CK will likewiſe be diſtant from the Tangent CH by a ſmall Diſtance. That the right Line CK may coincide with the Tangent CH, and the ultimate Ratios of the Lines CE, E*c* and C*c* may be found, the Points C and *c* ought to coaleſce and exactly coincide. The very ſmalleſt Errors in mathematical Matters are not to be neglected.

7. By the like way of reasoning, if a Circle describ'd with the Center B and Radius BC be drawn at right Angles along the Abscifs AB, with an uniform Motion, the Fluxion of the generated Solid ABC will be as that generating Circle, and the Fluxion of it's Superficies will be as the Perimeter of that Circle and the Fluxion of the Curve Line AC jointly. For in whatever Time the Solid ABC is generated by drawing that Circle along the Length of the Abscifs, in the same Time it's Superficies is generated by drawing the Perimeter of that Circle along the Length of the Curve AC. You may likewise take the following Examples of this Method.

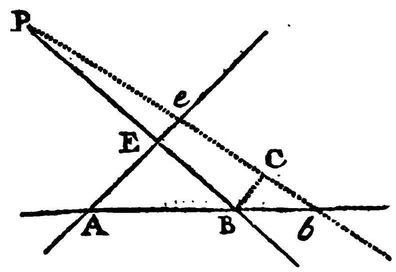
8. *Let the right Line* PB, *revolving about the given Pole* P, *cut another right Line* AB *given in Position: it is required to find the Proportion of the Fluxions of these right Lines* AB and PB.

Let the Line PB move forward from it's Place PB into the new Place P*b*. In P*b* take PC equal to PB, and draw PD to AB in such manner that the Angle *b*PD may be equal to the Angle *b*BC; and because the Triangles *b*BC, *b*PD are similar, the Augment B*b* will be to the Augment C*b* as P*b* to D*b*. Now let P*b* return into it's former Place PB, that these Augments may evanish, then the ultimate Ratio of these evanescent Augments, that is the ultimate Ratio of P*b* to D*b*, shall be the same with that of PB to DB, PDB being then a right Angle, and therefore the Fluxion of AB is to the Fluxion of PB in that same Ratio.



9. *Let the right Line* PB, *revolving about the given Pole* P, *cut other two right Lines given in Position, viz.* AB *and* AE *in* B *and* E: *the Proportion of the Fluxions of these right Lines* AB *and* AE *is sought.*

Let the revolving right Line PB move forward from it's Place PB into the new Place P*b*, so as to cut the Lines AB, AE in the Points *b* and *e*: and draw BC parallel to AE meeting P*b* in C, and it will be B*b* : BC :: A*b* : A*e*, and BC : E*e* :: PB : PE, and by joining the Ratios, B*b* : E*e* :: A*b* × PB : A*e* × PE. Now let P*b* return into it's former Place PB, and the evanescent Augment B*b* will be to the evanescent Augment E*e* as AB × PB to AE × PE; and therefore the Fluxion of the right Line AB is to the Fluxion of the right Line AE in the same Ratio.



10. Hence if the revolving right Line PB cut any curve Lines given in Position in the Points B and E, and the right Lines AB, AE now becoming moveable, touch these Curves in the Points of Section B and E: the Fluxion of the Curve, which the right Line AB touches, shall be to the Fluxion of the Curve, which the right Line AE touches, as AB × PB to AE × PE. The same thing would happen if the right Line PB perpetually touch'd any Curve given in Position in the moveable Point P.

11. *Let the Quantity x flow uniformly, and let it be proposed to find the Fluxion of* x^n.

In the same Time that the Quantity x, by flowing, becomes $x + o$, the Quantity x^n will become $\overline{x+o}|^n$, that is, by the Method of infinite Series's, $x^n + nox^{n-1} + \frac{n^2-n}{2}oox^{n-2} + \&c.$ And the Augments o and $nox^{n-1} + \frac{n^2-n}{2}oox^{n-2} + \&c.$ are to one another as 1 and $nx^{n-1} + \frac{n^2-n}{2}ox^{n-2} + \&c.$

Now let these Augments vanish, and their ultimate Ratio will be 1 to nx^{n-1}.

12. By like ways of reasoning, the Fluxions of Lines, whether right or curve in all Cases, as likewise the Fluxions of Superficies's, Angles and other Quantities, may be collected by the Method of *prime* and *ultimate* Ratios. Now to institute an Analysis after this manner in finite Quantities and investigate the *prime* or *ultimate* Ratios of these finite Quantities when in their nascent or evanescent State, is consonant to the Geometry of the Ancients: and I was willing to show that, in the Method of Fluxions, there is no necessity of introducing Figures infinitely small into Geometry. Yet the Analysis may be performed in any kind of Figures, whether finite or infinitely small, which are imagin'd similar to the evanescent Figures; as likewise in these Figures, which, by the Method of Indivisibles, use to be reckoned as infinitely small, provided you proceed with due Caution.

From the Fluxions to find the Fluents, is a much more difficult Problem, and the first Step of the Solution is equivalent to the Quadrature of Curves; concerning which I wrote what follows some considerable Time ago.

OF THE

QUADRATURE of CURVES.

13. IN what follows I consider indeterminate Quantities as increasing or decreasing by a continued Motion, that is, as flowing forwards, or backwards, and I design them by the Letters z, y, x, v, and their Fluxions or Celerities of increasing I denote by the same Letters pointed $\dot{z}$, $\dot{y}$, $\dot{x}$, $\dot{v}$. There are likewise Fluxions or Mutations more or less swift of these Fluxions, which may be call'd the second Fluxions of the same Quantities z, y, x, v, and may be thus design'd $\ddot{z}$, $\ddot{y}$, $\ddot{x}$, $\ddot{v}$: and the first Fluxions of these last, or the third Fluxions of z, y, x, v are thus denoted $\dddot{z}$, $\dddot{y}$, $\dddot{x}$, $\dddot{v}$: and the fourth Fluxions thus $\ddddot{z}$, $\ddddot{y}$, $\ddddot{x}$, $\ddddot{v}$. And after the same manner that $\dddot{z}$, $\dddot{y}$, $\dddot{x}$, $\dddot{v}$ are the Fluxions of the Quantities $\ddot{z}$, $\ddot{y}$, $\ddot{x}$, $\ddot{v}$, and these the Fluxions of the Quantities $\dot{z}$, $\dot{y}$, $\dot{x}$, $\dot{v}$; and these last the Fluxions of the Quantities z, y, x, v: so the Quantities z, y, x, v may be considered as the Fluxions of others, which I shall design thus $\acute{z}$, $\acute{y}$, $\acute{x}$, $\acute{v}$; and these as the Fluxions of others $\overset{\prime\prime}{z}$, $\overset{\prime\prime}{y}$, $\overset{\prime\prime}{x}$ $\overset{\prime\prime}{v}$; and these last still as the Fluxions of others $\overset{\prime\prime\prime}{z}$, $\overset{\prime\prime\prime}{y}$, $\overset{\prime\prime\prime}{x}$, $\overset{\prime\prime\prime}{v}$. Therefore $\overset{\prime\prime}{z}$, $\acute{z}$, z, $\dot{x}$, $\ddot{z}$, $\dddot{z}$, $\ddddot{z}$, $\overset{\therefore}{z}$, &c. design a Series of Quantities whereof every one that follows is the Fluxion of the one immediately preceding, and every one that goes before, is a flowing Quantity having that which immediately succeeds, for it's Fluxion. The like is the Series $\overset{\prime\prime}{\sqrt{az-zz}}$, $\overset{\prime}{\sqrt{az-zz}}$, $\sqrt{az-zz}$, $\overset{\cdot}{\sqrt{az-zz}}$, $\overset{\cdot\cdot}{\sqrt{az-zz}}$, $\overset{\cdot\cdot\cdot}{\sqrt{az-zz}}$; as likewise the Series $\overset{\prime\prime}{\frac{az+zz}{a-z}}$, $\overset{\prime}{\frac{az+zz}{a-z}}$, $\frac{az+zz}{a-z}$, $\overset{\cdot}{\frac{az+zz}{a-z}}$, $\overset{\cdot\cdot}{\frac{az+zz}{a-z}}$, $\overset{\cdot\cdot\cdot}{\frac{az+zz}{a-z}}$, &c.

14. And it is to be remarked that any preceding Quantity in these Series's is as the Area of a Curvilinear Figure of which the succeeding is the rectangular Ordinate, and the Abscis is z: as $\overset{\prime}{\sqrt{az\ \ zz}}$ the Area

of a Curve, whose Ordinate is $\sqrt{az - zz}$, and Abscis z. The Design of all these things will appear in the following Propopositions.

PROP. I. PROB. I.

15. *An Equation being given involving any Number of flowing Quantities, to find the Fluxions.*

SOLUTION.

Let every Term of the Equation be multiplied by the Index of the Power * of every flowing Quantity that it involves, and in every Multiplication change the Side or Root of the Power into it's Fluxion, and the Aggregate of all the Products with their proper Signs, will be the new Equation.

EXPLICATION.

16. Let a, b, c, d, &c. be determinate and invariable Quantities, and let any Equation be propos'd involving the flowing Quantities z, y, x, &c. as $x^3 - xy^2 + a^2z - b^3 = 0$. Let the Terms be first multiplied by the Indexes of the Powers of x, and in every Multiplication for the Root, or x of one Dimension write $\dot{x}$, and the Sum of the Factors will be $3\dot{x}x^2 - \dot{x}y^2$. Do the same in y, and there arises $-2xy\dot{y}$. Do the same in z, and there arises $aa\dot{z}$. Let the Sum of these Products be put equal to nothing, and you'll have the Equation $3\dot{x}x^2 - \dot{x}y^2 - 2xy\dot{y} + aa\dot{z} = 0$. I say the Relation of the Fluxions is defin'd by this Equation.

DEMONSTRATION.

17. For let o be a very small Quantity, and let o$\dot{z}$, o$\dot{y}$, o$\dot{x}$ be the Moments, that is the momentaneous synchronal Increments of the Quantities z, y, x. And if the flowing Quantities are just now z, y, x, then after a Moment of Time, being increas'd by their Increments o$\dot{z}$, o$\dot{y}$, o$\dot{x}$, these Quantities shall become $z + o\dot{z}$, $y + o\dot{y}$, $x + o\dot{x}$: which being wrote in the first Equation for z, y and x, give this

* The Word translated here *Power* is *Dignitas*, Dignity, by which must be understood not only perfect, but also imperfect Powers or surd Roots, which are express'd in the Manner of perfect Powers, as is well known, by fractional Indexes. In which Sense $x^{\frac{1}{2}}$, $x^{\frac{2}{3}}$, &c. are Powers; $\frac{1}{2}$ and $\frac{2}{3}$ their Indexes, and x the Side or Root. I use the Word Power, because Dignity is seldom us'd in *English* in this Sense.

Equation $x^3 + 3x^2 o\dot{x} + 3xoo\dot{x}\dot{x} + o^3\dot{x}^3 - xy^2 - o\dot{x}y^2 - 2xo\dot{y}y - 2\dot{x}o^2\dot{y}y - xo^2\dot{y}\dot{y} - \dot{x}o^3\dot{y}\dot{y} + a^2 z + a^2 o\dot{z} - b^3 = 0$.

Subtract the former Equation from the latter, divide the remaining Equation by o, and it will be $3\dot{x}x^2 + 3\dot{x}\dot{x}ox + \dot{x}^3 o^2 - \dot{x}y^2 - 2x\dot{y}y - 2\dot{x}o\dot{y}y - xo\dot{y}\dot{y} - \dot{x}o^2\dot{y}\dot{y} + a^2\dot{z} = 0$. Let the Quantity o be diminished infinitely, and neglecting the Terms which vanish, there will remain $3\dot{x}x^2 - \dot{x}y^2 - 2x\dot{y}y + a^2\dot{z} = 0$. Q. E. D.

A fuller Explication.

18. After the same manner if the Equation were $x^3 - xy^2 + aa\sqrt{ax - y^2} - b^3 = 0$, thence would be produced $3x^2\dot{x} - \dot{x}y^2 - 2xy\dot{y} + aa\dot{\overline{\sqrt{ax - y^2}}} = 0$. Where if you would take away the Fluxion $\dot{\overline{\sqrt{ax - y^2}}}$, put $\sqrt{ax - y^2} = z$, and it will be $ax - y^2 = z^2$, and by this Proposition $a\dot{x} - 2\dot{y}y = 2\dot{z}z$, or $\frac{a\dot{x} - 2\dot{y}y}{2z} = \dot{z}$, that is $\frac{a\dot{x} - 2\dot{y}y}{2\sqrt{ax - yy}} = \dot{\overline{\sqrt{ax - yy}}}$. And thence $3x^2\dot{x} - \dot{x}y^2 - 2xy\dot{y} + \frac{a^3\dot{x} - 2a^2\dot{y}y}{2\sqrt{ax - yy}} = 0$.

19. And by repeating the Operation, you proceed to second, third and subsequent Fluxions. Let $zy^3 - z^4 + a^4 = 0$ be an Equation propos'd, and by the first Operation it becomes $\dot{z}y^3 + 3z\dot{y}y^2 - 4\dot{z}z^3 = 0$; by the second $\ddot{z}y^3 + 6\dot{z}\dot{y}y^2 + 3z\ddot{y}y^2 + 6z\dot{y}^2 y - 4\ddot{z}z^3 - 12\dot{z}^2 z^2 = 0$, by the third, $\dddot{z}y^3 + 9\ddot{z}\dot{y}y^2 + 18\dot{z}\dot{y}^2 y + 3z\dddot{y}y^2 + 18z\ddot{y}\dot{y}y + 6z\dot{y}^3 - 4\dddot{z}z^3 - 36\ddot{z}\dot{z}z^2 - 24\dot{z}^3 z = 0$.

20. But when one proceeds thus to second, third and following Fluxions, it is proper to consider some Quantity as flowing uniformly, and for it's first Fluxion to write Unity, for the second and subsequent ones, nothing. Let there be given the Equation $zy^3 - z^4 + a^4 = 0$, as above; and let z flow uniformly, and let it's Fluxion be Unity: then by the first Operation it shall be $y^3 + 3z\dot{y}y^2 - 4z^3 = 0$; by the second $6\dot{y}y^2 + 3z\ddot{y}y^2 + 6z\dot{y}^2 y - 12z^2 = 0$; by the third $9\ddot{y}y^2 + 18\dot{y}^2 y + 3z\dddot{y}y^2 + 18z\ddot{y}\dot{y}y + 6z\dot{y}^3 - 24z = 0$.

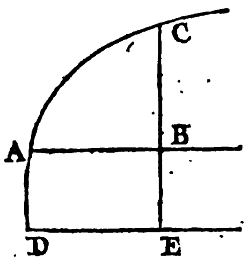
But in Equations of this Kind it must be conceived that the Fluxions in all the Terms are of the same Order, *i. e.* either all of the first Order $\dot{y}$, $\dot{z}$; or all of the second $\ddot{y}$, $\dot{y}^2$, $\dot{y}\dot{z}$, $\dot{z}^2$; or all of the third $\dddot{y}$, $\ddot{y}\dot{y}$, $\ddot{y}\dot{z}$, $\dot{y}^3$, $\dot{y}^2\dot{z}$, $\dot{y}\dot{z}^2$, $\dot{z}^3$, &c. And where the Case is otherwise the Order is to be completed by means of the Fluxions of a Quantity that flows uniformly, which Fluxions are understood. Thus the last Equation,

tion, by completing the third Order, becomes $9\dot{z}\ddot{y}y^2 + 18\dot{z}\dot{y}^2y + 3\ddot{z}\dot{y}y^2 + 18z\ddot{y}\dot{y}y + 6z\dot{y}^3 - 24z\dot{z}^3 = 0$.

PROP. II. PROB. II.

22. *To find such Curves as can be squared.*

Let ABC be the Figure to be found, BC the rectanglar Ordinate, and AB the Abscifs. Produce CB to E, so that $BE = 1$, and complete the Parallelogram ABED: and the Fluxions of the Areas ABC, ABED shall be as BC and BE. Assume therefore any Equation, by which the Relation of the Areas may be defined, and thence the Relation of the Ordinates BC and BE will be given by the first Proposition. Q. E. I.



The two following Propositions afford Examples of this.

PROP. III. THEOR. I.

23. If for the Abscifs AB and Area AE or $AB \times 1$ you write z promiscuously, and if for $e + fz^{\eta} + gz^{2\eta} + hz^{3\eta} + \&c.$ you write R: let the Area of the Curve be $z^{\theta}R^{\lambda}$, the Ordinate BC shall be equal to

$$\overline{\theta e \begin{array}{c}+\theta\\+\lambda\eta\end{array} \times fz^{\eta} \begin{array}{c}+\theta\\+2\lambda\eta\end{array} \times gz^{2\eta} \begin{array}{c}+\theta\\+3\lambda\eta\end{array} \times hz^{3\eta} + \&c.} \times z^{\theta-1}R^{\lambda-1}.$$

DEMONSTRATION.

24. For let $z^{\theta}R^{\lambda} = v$, it will be (by Prop. I.) $\theta\dot{z}z^{\theta-1}R^{\lambda} + \lambda z^{\theta}\dot{R}R^{\lambda-1} = \dot{v}$. For R^{λ} in the first Term of the Equation and z^{θ} in the second write $RR^{\lambda-1}$ and $zz^{\theta-1}$, and it will become $\overline{\theta\dot{z}R + \lambda z\dot{R}} \times z^{\theta-1}R^{\lambda-1} = \dot{v}$. But it was $R = e + fz^{\eta} + gz^{2\eta} + hz^{3\eta} + \&c$: and thence (by Prop. I.) it becomes $\dot{R} = \eta f\dot{z}z^{\eta-1} + 2\eta g\dot{z}z^{2\eta-1} + 3\eta h\dot{z}z^{3\eta-1} + \&c.$ which being substituted, and BE or 1 wrote for $\dot{z}$, it becomes

$$\overline{\theta e \begin{array}{c}+\theta\\+\lambda\eta\end{array} \times fz^{\eta} \begin{array}{c}+\theta\\+2\lambda\eta\end{array} \times gz^{2\eta} \begin{array}{c}+\theta\\+3\lambda\eta\end{array} \times hz^{3\eta} + \&c.} \times z^{\theta-1}R^{\lambda-1} = \dot{v} = BC.$$

Q. E. D.

PROP.

PROP. IV. THEOR. II.

25. If AB the Abſciſs of a Curve be z, and if for $e+fz^{n}+gz^{2n}$ + &c. you write R; and for $k+lz^{n}+mz^{2n}$ + &c. you write S: let the Area of the Curve be $z^{\theta}R^{\lambda}S^{\mu}$, the Ordinate BC will be equal to

$$\left.\begin{array}{llll}
\theta ek & \begin{smallmatrix}+\theta\\+\lambda n\end{smallmatrix}\times fkz^{n} & \begin{smallmatrix}+\theta\\+2\lambda n\end{smallmatrix}\times gkz^{2n}\ \text{- - - * - - - - - - * - - - -} & \\
& \begin{smallmatrix}+\theta\\+\mu n\end{smallmatrix}\times elz^{n} & \begin{smallmatrix}+\theta\\+\lambda n\\+\mu n\end{smallmatrix}\times flz^{2n} & \begin{smallmatrix}+\theta\\+2\lambda n\\+\mu n\end{smallmatrix}\times glz^{3n}\ \text{- - - * - - - -} \\
& & \begin{smallmatrix}+\theta\\+2\mu n\end{smallmatrix}\times emz^{2n} & \begin{smallmatrix}+\theta\\+\lambda n\\+2\mu n\end{smallmatrix}\times fmz^{3n}\ \begin{smallmatrix}+\theta\\+2\lambda n\\+2\mu n\end{smallmatrix}\times gmz^{4n}
\end{array}\right\}\times\left\{\begin{array}{l}z^{\theta-1}\\ R^{\lambda-1}\\ S^{\mu-1}\end{array}\right.$$

This Propoſition is demonſtrated after the Manner of the preceding.

PROP. V. THEOR. III.

26. If the Abſciſs AB of a Curve be z, and for $e+fz^{n}+gz^{2n}+hz^{3n}$ + &c. be wrote R: Let the Ordinate be $z^{\theta-1}R^{\lambda-1}\times\overline{a+bz^{n}+cz^{2n}+dz^{3n}+\&c}$: and put $\frac{\theta}{n}=r$, $r+\lambda=s$, $s+\lambda=t$, $t+\lambda=v$, &c. The Area ſhall be equal to

$$z^{\theta}R^{\lambda}\text{ into }+\frac{\frac{1}{n}a}{re}$$
$$+\frac{\frac{1}{n}b-sfA}{\overline{r+1}\times e}z^{n}$$
$$+\frac{\frac{1}{n}c-\overline{s+1}\times fB-tgA}{\overline{r+2}\times e}z^{2n}$$
$$+\frac{\frac{1}{n}d-\overline{s+2}\times fC-\overline{t+1}\times gB-vhA}{\overline{r+3}\times e}z^{3n}$$
$$+\frac{-\overline{s+3}\times fD-\overline{t+2}\times gC-\overline{v+1}\times hB}{\overline{r+4}\times e}z^{4n}$$
$$+\ \&c.$$

Where A, B, C, D, *&c.* denote the whole given Coefficients of the ſeveral Terms in the Series with their proper Signs + and —, *viz.*

A the Coefficient of the firſt Term $\dfrac{\frac{1}{n}a}{re}$

B the Coefficient of the ſecond Term $\dfrac{\frac{1}{n}b - sfA}{\overline{r+1} \times e}$

C the Coefficient of the third Term $\dfrac{\frac{1}{n}C - \overline{s+1} \times fB - tgA}{\overline{r+2} \times e}$

And ſo on.

DEMONSTRATION.

27. According to the third Propoſition, let the following Expreſſions be

$$\begin{array}{llllll} & \text{Ordinates of Curves,} & & & & \text{Their Areas.} \\ 1^\circ & \theta eA \begin{smallmatrix}+\theta \\ +\lambda\eta\end{smallmatrix} \times fAz^{\eta} \begin{smallmatrix}+\theta \\ +2\lambda\eta\end{smallmatrix} \times gAz^{2\eta} \begin{smallmatrix}+\theta \\ +3\lambda\eta\end{smallmatrix} \times hAz^{3\eta} \;\&c. & & \times & & Az^{\theta}R^{\lambda} \\ 2^\circ & \text{- -} + \overline{\theta+\eta} \times eBz^{\eta} \begin{smallmatrix}+\overline{\theta+\eta} \\ +\lambda\eta\end{smallmatrix} \times fBz^{2\eta} \begin{smallmatrix}+\overline{\theta+\eta} \\ +2\lambda\eta\end{smallmatrix} \times gBz^{3\eta} \;\&c. & & z^{\theta-1} & & Bz^{\theta+\eta}R^{\lambda} \\ 3^\circ & \text{- - - - - - - - - - -} + \overline{\theta+2\eta} \times eCz^{2\eta} \begin{smallmatrix}+\overline{\theta+2\eta} \\ +\lambda\eta\end{smallmatrix} \times fCz^{3\eta} \;\&c. & & R^{\lambda-1} & & Cz^{\theta+2\eta}R^{\lambda} \\ 4^\circ & \text{- -} + \overline{\theta+3\eta} \times eDz^{3\eta} \;\&c. & & & & Dz^{\theta+3\eta}R^{\lambda} \end{array}$$

And if the Sum of the Ordinates be put equal to the Ordinate $a + bz^{\eta} + cz^{2\eta} + dz^{3\eta} +$ &c. into $z^{\theta-1} R^{\lambda-1}$, the Sum of the Areas $z^{\theta} R^{\lambda}$ into $A + Bz^{\eta} + Cz^{2\eta} + Dz^{3\eta} +$ &c. ſhall be equal to the Area of the Curve of which that is the Ordinate. Therefore let the correſpondent Terms of the Ordinates be put equal, and it will be

$$a = \theta eA$$
$$b = \overline{\theta + \lambda\eta} \times fA + \overline{\theta + \eta} \times eB$$
$$c = \overline{\theta + 2\lambda\eta} \times gA + \overline{\theta + \eta + \lambda\eta} \times fB + \overline{\theta + 2\eta} \times eC$$

&c.

And thence $A = \dfrac{a}{\theta e}$

$$B = \frac{b - \overline{\theta + \lambda\eta} \times fA}{\overline{\theta + \eta} \times e}$$

$$C = \frac{c - \overline{\theta + 2\lambda\eta} \times gA - \overline{\theta + \eta + \lambda\eta} \times fB}{\overline{\theta + 2\eta} \times e}$$

And ſo on *in infinitum.*

Now

Now put $\frac{\theta}{\eta} = r$, $r + \lambda = s$, $s + \lambda = t$, *&c.* and in the Area $z^{\theta} R^{\lambda} \times \overline{A + Bz^{\eta} + Cz^{2\eta} + Dz^{3\eta} + \&c.}$ write the Values of A, B, C, *&c.* just now found, and the Series proposed will come out. Q. E. D.

28. And it must be observed that every Ordinate is resolved into a Series two different ways. For the Index η may either be affirmative or negative.

Let the Ordinate $\frac{3k - lzz}{zz\sqrt{kz - lz^3 + mz^4}}$ be proposed: it may be wrote either thus $z^{-\frac{5}{2}} \times \overline{3k - lzz} \times \overline{k - lz^2 + mz^3}|^{-\frac{1}{2}}$
or thus $z^{-2} \times \overline{-l + 3kz^{-2}} \times \overline{m - lz^{-1} + kz^{-3}}|^{-\frac{1}{2}}$.

In the former Case you have $a = 3k$, $b = 0$, $c = -l$: $e = k$, $f = 0$, $g = -l$, $h = m$: $\lambda = \frac{1}{2}$, $\eta = 1$, $\theta - 1 = -\frac{5}{2}$, $\theta = -\frac{3}{2} = r$, $s = -1$, $t = -\frac{1}{2}$, $v = 0$.

In the latter it is $a = -l$, $b = 0$, $c = 3k$: $e = m$, $f = -l$, $g = 0$, $h = k$: $\lambda = \frac{1}{2}$, $\eta = -1$, $\theta - 1 = -2$, $\theta = -1$, $r = 1$, $s = 1\frac{1}{2}$, $t = 2$, $v = 2\frac{1}{2}$.

Both Cases must be tried. And if either of the Series's, by means of the Terms at length failing, break off and terminate, the Area of the Curve will be had in finite Terms. Thus in the first Case of this Example, by writing in the Series the Values of a, b, c, e, f, g, h, λ, θ, r, s, t, v, all the Terms after the first vanish *in inf.* and the Area of the Curve comes out $-2\sqrt{\frac{k - lz^2 + mz^3}{z^3}}$, and this Area, because of the negative Sign, is adjacent to the Abscis produced beyond the Ordinate. For every affirmative Area is adjacent both to Abscis and Ordinate; but a negative Area falls upon the opposite Side of the Ordinate, and is adjacent to the Abscis produced, the Sign of the Ordinate being understood to remain. By this means one of the Series's, and sometimes both of them, always terminates and becomes finite, if the Curve can be squared geometrically. But if the Curve don't admit of being thus squared, both Series's will be continued infinitely, and one of them will converge and give the Area by Approximation, except when r (because of the Area being infinite) is either nothing or a negative Integer; or when $\frac{z^{\eta}}{e}$ is equal to Unity. If $\frac{z^{\eta}}{e}$ be less than Unity, that Series will converge in which the Index η is affirmative: But if $\frac{z^{\eta}}{e}$ be greater than Unity, the other Series will converge. If in the one Case the Area be adjacent to the Abscis drawn as far as the Ordinate, in the other it is adjacent to the Abscis produced beyond the Ordinate.

29. Besides you are to observe, that if the Ordinate be a Product contain'd under a rational Factor Q and an irreducible surd Factor R^π, and the Side R of the surd Factor don't divide the rational Factor Q; it will be $\lambda - 1 = \pi$ and $R^{\lambda-1} = R^\pi$. But if the Side R of the surd Factor divide the rational Factor once, then it will be $\lambda - 1 = \pi + 1$, and $R^{\lambda-1} = R^{\pi+1}$: if it divide it twice, it will be $\lambda - 1 = \pi + 2$, and $R^{\lambda-1} = R^{\pi+2}$: if thrice, then $\lambda - 1 = \pi + 3$, and $R^{\lambda-1} = R^{\pi+3}$: and so on.

30. If the Ordinate be an irreducible rational Fraction having it's Denominator made up of two or more Terms: the Denominator must be resolved into all it's *prime* Divisors. And if there be any Divisor that has no equal, the Curve cannot be squared: but if there be two or more equal Divisors, one of them is to be rejected, and if there be yet other two or more equal to each other and unequal to the former, one of them is likewise to be rejected; and so you are to do in all the other equal Divisors, if more still remain: afterwards the Divisor which is left; or the Product contain'd under all the Divisors which are left (if there are more of them) must be put for R, and the Reciprocal of it's Square R^{-2} for $R^{\lambda-1}$, unless when that Product is a Square, or a Cube, or a Biquadrate, &c. in which Case it's Side or Root must be put for R, and the Index of the Power, *viz.* 2, or 3, or 4, taken negatively, for λ: and the Ordinate is to be reduced to the Denominator R^2, or R^3, or R^4, or R^5, &c.

Thus if the Ordinate were $\frac{z^5 + z^4 - 8z^3}{z^5 + z^4 - 5z^3 - z^2 + 8z - 4}$; because this Fraction is irreducible, and the Divisors of the Denominator have all of them Equals; for they are $z - 1$, $z - 1$, $z - 1$; and $z + 2$, $z + 2$, I reject one Divisor of each Magnitude, and the Product of the remaining Divisors $z - 1$, $z - 1$, $z + 2$, *viz.* $z^3 - 3z + 2$, I put for R, and the Reciprocal of it's Square $\frac{1}{R^2}$, or R^{-2}, for $R^{\lambda-1}$. Then I reduce the Ordinate to the Denominator R^2 or $R^{1-\lambda}$, and it becomes $\frac{z^6 - 9z^4 + 8z^3}{\overline{z^3 - 3z + 2}|^2}$, i. e. $z^3 \times \overline{8 - 9z + z^3} \times \overline{2 - 3z + z^3}|^{-2}$. And thence it is $a = 8$, $b = -9$, $c = 0$, $d = 1$, &c. $e = 2$, $f = -3$, $g = 0$, $h = 1$, $\lambda - 1 = -2$, $\lambda = -1$, $\eta = 1$, $\theta - 1 = 3$, $\theta = 4 = r$, $s = 3$, $t = 2$, $v = 1$. And these Values being inserted in the Series, the Area comes out $\frac{z^4}{z^3 - 3z + 2}$, all the Terms of the Series throughout, after the first, vanishing.

31. Finally,

31. Finally, if the Ordinate be an irreducible Fraction, and it's Denominator be a Product contained under a rational Factor Q and an irreducible surd Factor R^{π}, you must find all the *prime* Divisors of the Side R, and reject one Divisor of each Magnitude, and then multiply the rational Factor Q by the remaining Divisors, if there be any such; and if the Product thence arising be equal to the Side R, or any Power of it whose Index is a whole Number, let that Index be m, and it will be $\lambda - 1 = -\pi - m$ and $R^{\lambda-1} = R^{-\pi-m}$.

As if the Ordinate was $\frac{3q^5 - q^4x + 9q^3x^2 - q^2x^3 - 6qx^4}{\overline{q^2 - x^2} \times \overline{q^3 + q^2x - qx^2 - x^3}|^{\frac{1}{3}}}$ *, because R the Side of the surd Factor or $q^3 + q^2x - qx^2 - x^3$ has these Divisors $q + x$, $q + x$, $q - x$, which are of two different Magnitudes, I reject one Divisor of each Magnitude, and I multiply the rational Factor $q^2 - x^2$ by the Divisor which remains $q + x$. And because the Product $q^3 + q^2x - qx^2 - x^3$ is equal to the Side R, I put $m = 1$, whence, since $\pi = \frac{1}{3}$, it becomes $\lambda - 1 = -\frac{4}{3}$. Therefore I reduce the Ordinate to the Denominator $R^{-\frac{4}{3}}$, and it becomes

$$x^0 \times \overline{3q^6 + 2q^5x + 8q^4x^2 + 8q^3x^3 - 7q^2x^4 - 6qx^5} \times \overline{q^3 + q^2x - qx^2 - x^3}|^{-\frac{4}{3}}$$

Whence $a = 3q^6$, $b = 2q^5$, &c. $e = q^3$, $f = q^2$, &c. $\theta - 1 = 0$, $\theta = 1 = \eta$, $\lambda = -\frac{1}{3}$, $r = 1$, $s = \frac{2}{3}$, $t = \frac{1}{3}$, $v = 0$. And these being wrote in the Series, the Area comes out $\frac{3q^3x + 3qx^3}{\overline{q^3 + q^2x - qx^2 - x^3}|^{\frac{1}{3}}}$ †, all the Terms through the whole Series after the third vanishing.

PROP. VI. THEOR. IV.

32. If the Abscifs AB of a Curve be z, and R be put for $e + fz^{\eta} + gz^{2\eta} + hz^{3\eta} + \&c.$ and S for $k + lz^{\eta} + mz^{2\eta} + nz^{3\eta} + \&c.$ and if the Ordinate be $z^{\theta-1} R^{\lambda-1} S^{\mu-1}$ into $\overline{a + bz^{\eta} + cz^{2\eta} + dz^{3\eta} + \&c.}$; and the Rectangles of the Terms e, f, g, h, &c. and k, l, m, n, &c. be

ek	fk	gk	hk	&c.
el	fl	gl	hl	&c.
em	fm	gm	hm	&c.
en	fn	gn	hn	&c.

* In the former Impressions there was an Error in the Expression of this Ordinate, which stood thus $\frac{3q^5 - q^4x + 9q^2x^2 - q^2x^3 - 6x^3}{\overline{q^2 - x^2} \times \overline{q^3 + q^2x - qx^2 - x^3}|^{\frac{1}{3}}}$: as is easily collected from what follows. See the Notes Art. 136.

† Here also was an Error in the former Impressions, the Numerator of the Fraction being according to them $3q^2x + 3x^3$. See the Notes.

And if the numeral Coefficients of these Rectangles respectively be

$$\frac{\theta}{n} = r. \quad r + \lambda = s. \quad s + \lambda = t. \quad t + \lambda = v, \text{ \&c.}$$

$$r + \mu = s'. \quad s + \mu = t'. \quad t + \mu = v'. \quad v + \mu = w', \text{ \&c.}$$

$$s' + \mu = t''. \quad t' + \mu = v''. \quad v' + \mu = w''. \quad w' + \mu = x'', \text{ \&c.}$$

$$t'' + \mu = v'''. \quad v'' + \mu = w'''. \quad w'' + \mu = x'''. \quad x'' + \mu = y''', \text{ \&c.}$$

The Area of the Curve will be

$$z^{\theta} R^{\lambda} S^{\mu} \text{ into } + \frac{\frac{1}{n}a}{rek}$$

$$+ \frac{\frac{1}{n}b \begin{matrix} - sfk \\ - s'el \end{matrix} A}{\overline{r+1} \times ek} z^{n}$$

$$+ \frac{\frac{1}{n}c \begin{matrix} - \overline{s+1} \times fk \\ - \overline{s'+1} \times el \end{matrix} B \begin{matrix} - tgk \\ - t'fl \\ - t''em \end{matrix} A}{\overline{r+2} \times ek} z^{2n}$$

$$+ \frac{\frac{1}{n}d \begin{matrix} - \overline{s+2} \times fk \\ - \overline{s'+2} \times el \end{matrix} C \begin{matrix} - \overline{t+1} \times gk \\ - \overline{t'+1} \times fl \\ - \overline{t''+1} \times em \end{matrix} B \begin{matrix} - vhk \\ - v'gl \\ - v''fm \\ - v'''en \end{matrix} A}{\overline{r+3} \times ek} z^{3n}$$

$$+ \text{ \&c.}$$

Where A denotes the given Coefficient of the first Term $\frac{\frac{1}{n}a}{rek}$ with it's Sign + or —, B the given Coefficient of the second Term, C the given Coefficient of the third: and so on. And one or more of the Terms $a, b, c,$ &c. $e, f, g,$ &c. $k, l, m,$ &c. may be wanting.

The Proposition is demonstrated after the Manner of the preceding, and what Things are there remarked, likewise obtain here. Moreover the Series of such Propositions goes on infinitely, and the Progression of the Series is manifest.

PROP. VII. THEOR. V.

33. If for $e + fz^{n} + gz^{2n} +$ &c. be wrote R as above, and in the Ordinate of any Curve $z^{\theta \pm n\sigma} R^{\lambda \pm \tau}$, the given Quantities $\theta, \eta, \lambda, e, f, g,$ &c. remain the same; and for σ and τ be wrote successively any integral

Numbers: and if there be given the Area of one of these Curves which are design'd by innumerable Ordinates thus arising, when the Ordinates are Binomials under the Vinculum of the Root; or if the Areas of two of the Curves be given when the Ordinates are Trinomials under the Vinculum of the Root; or the Areas of three of the Curves, when the Ordinates are Quadrinomials under the Vinculum of the Root, and so on *in infinitum:* I say the Areas of all these Curves shall be given.

34. I reckon here for Names all the Terms under the Vinculum of the Root as well them that are deficient, as them that are full, whose Indexes of their Powers are in an arithmetical Progression. Thus the Ordinate $\sqrt{a^4 - ax^3 + x^4}$, because of two deficient Terms betwixt a^4 and $-ax^3$ ought to be esteemed a Quinquinomial. But $\sqrt{a^4 - x^4}$ is a Binomial and $\sqrt{a^4 + x^4 - \frac{x^8}{a^4}}$ a Trinomial, since the Progression goes on now by greater Differences. Now the Proposition is thus demonstrated.

CASE I.

35. Let the Ordinates of two Curves be $pz^{\theta-1}R^{\lambda-1}$ and $qz^{\theta+\eta-1}R^{\lambda-1}$ and their Areas pA and qB, R being a Trinomial $e + fz^{\eta} + gz^{2\eta}$. And since by Prop. 3. $z^{\theta}R^{\lambda}$ is the Area of a Curve, whose Ordinate is $\theta e + \overline{\theta + \lambda\eta} \times fz^{\eta} + \overline{\theta + 2\lambda\eta} \times gz^{2\eta}$ into $z^{\theta-1}R^{\lambda-1}$, subtract the former Ordinates and Areas from the latter Ordinate and Area, and there remains $\overline{\theta e - p \begin{matrix}+\theta\\+\lambda\eta\end{matrix} \times f - q \times z^{\eta} \begin{matrix}+\theta\\+2\lambda\eta\end{matrix} \times gz^{2\eta}} \times z^{\theta-1}R^{\lambda-1}$ the Ordinate of a new Curve, and $z^{\theta}R^{\lambda} - pA - qB$ it's Area. Put $\theta e = p$, and $\theta f + \lambda\eta f = q$, and the Ordinate becomes $\overline{\theta + 2\lambda\eta} \times gz^{2\eta} \times z^{\theta-1}R^{\lambda-1}$, and the Area $z^{\theta}R^{\lambda} - \theta eA - \theta fB - \lambda\eta fB$. Divide both by $\theta g + 2\lambda\eta g$, and call the Area thence arising C, and assuming any Quantity r, rC will be the Area of a Curve whose Ordinate is $rz^{\theta+2\eta-1}R^{\lambda-1}$. And by the same Method we have found the Area rC corresponding to the Ordinate $rz^{\theta+2\eta-1}R^{\lambda-1}$ from the Areas pA and qB, you may from the Areas qB and rC find a fourth Area, *viz.* sD corresponding to the Ordinate $sz^{\theta+3\eta-1}R^{\lambda-1}$, and so on *in inf.* And the like is the Rate of the Progression which proceeds the contrary way from the Areas B and A. If any of the Terms θ, $\overline{\theta + \lambda\eta}$, and $\theta + 2\lambda\eta$ be wanting and break off the Series, let the Area pA be assumed in the Beginning of the one Progression, and the Area qB in the Beginning of the other, and from these two Areas, all the Areas will be given in both Progressions. And contrarily, from any other two Areas

Areas aſſumed you can return by Analyſis to the Areas A and B, ſo that from two Areas given all the reſt are given. Q. E. O.

This is the Caſe of Curves in which θ the Index of z is increaſed or diminiſhed by the perpetual Addition or Subtraction of the Quantity η. The other Caſe reſpects Curves where the Index λ is increaſed or diminiſhed by Units.

CASE II.

36. If the Ordinates $pz^{\theta-1}R^{\lambda}$ and $qz^{\theta+\eta-1}R^{\lambda}$, to which the Areas pA and qB may now be ſuppoſed to anſwer, be multiplied by R or $\overline{e+fz^{\eta}+gz^{2\eta}}$, and then again be divided by R, they become $\overline{pe+pfz^{\eta}+pgz^{2\eta}}\times z^{\theta-1}R^{\lambda-1}$, and $\overline{qez^{\eta}+qfz^{2\eta}+qgz^{3\eta}}\times z^{\theta-1}R^{\lambda-1}$. And (by Prop. 3.) $az^{\theta}R^{\lambda}$ is the Area of a Curve, whoſe Ordinate is $\overline{\theta ae+\overline{\theta+\lambda\eta}\times afz^{\eta}+\overline{\theta+2\lambda\eta}\times agz^{2\eta}}\times z^{\theta-1}R^{\lambda-1}$, and $bz^{\theta+\eta}R^{\lambda}$ the Area of a Curve whoſe Ordinate is $\overline{\theta+\eta}\times bez^{\eta}+\overline{\theta+\eta+\lambda\eta}\times bfz^{2\eta}+\overline{\theta+\eta+2\lambda\eta}\times bgz^{3\eta}\times z^{\theta-1}R^{\lambda-1}$.

And the Sum of theſe four Areas is $pA+qB+az^{\theta}R^{\lambda}+bz^{\theta+\eta}R^{\lambda}$ and the Sum of the correſponding Ordinates

$$\overline{\begin{array}{llll}
\theta ae & +\overline{\theta+\lambda\eta}\times afz^{\eta} & +\overline{\theta+2\lambda\eta}\times agz^{2\eta} & +\overline{\theta+\eta+2\lambda\eta}\times bgz^{3\eta}\\
+pe & +\overline{\theta+\eta}\times be & +\overline{\theta+\eta+\lambda\eta}\times bf & +1\times qg\\
 & +1\times pf & +1\times pg & \\
 & +1\times qe & +1\times qf. &
\end{array}}\times z^{\theta-1}R^{\lambda-1}$$

If the firſt Term, the third and fourth be put equal to nothing ſeparately, by the firſt it will be $\theta ae+pe=0$, or $-\theta a=p$, by the fourth $-\theta b-\eta b-2\lambda\eta b=q$, and by the third (exterminating p and q) $\frac{2ag}{f}=b$. Whence the ſecond becomes $\frac{\lambda\eta aff-4\lambda\eta age}{f}$, and therefore the Sum of the four Ordinates is $\frac{\lambda\eta af^{2}-4\lambda\eta age}{f}z^{\theta+\eta-1}R^{\lambda-1}$, and the Sum of as many correſponding Areas is $az^{\theta}R^{\lambda}+\frac{2ag}{f}z^{\theta-\eta}R^{\lambda}-\theta aA+\frac{\overline{2\theta+2\eta+4\lambda\eta}}{f}agB$. Divide theſe Sums by $\frac{\lambda\eta af^{2}-4\lambda\eta age}{f}$, and if the laſt Quote be call'd D, then ſhall D be the Area of a Curve whoſe Ordinate is the firſt Quote $z^{\theta+\eta-1}R^{\lambda-1}$.

And by the ſame Method, if you put all the Terms of the Ordinate ſave the firſt equal each to nothing, you may find the Area of a Curve whoſe Ordinate is $z^{\theta-1}R^{\lambda-1}$. Let that Area be called C, and by the ſame Means that the Areas C and D have been found from the Areas A and

A and B, you may from these Areas C and D find other two Areas E and F answering to the Ordinates $z^{\theta-1}R^{\lambda-2}$ and $z^{\theta+\eta-1}R^{\lambda-2}$, and so on continually. And by a contrary Analysis you may return from the Areas E and F to the Areas C and D, and thence to the Areas A and B, and likewise to the other Areas which follow in the Progression. Therefore if the Index λ be increas'd or diminish'd by the continual Addition or Subtraction of Units, and of the Areas which answer to the Ordinates thus arising two of the most simple be given; all the others *in inf.* are given.

CASE III.

37. And by joining these two Cases together, if both the Index θ be any how increas'd or diminish'd by the continual Addition or Subtraction of η, and the Index λ by the continual Addition or Subtraction of Unity, the Areas corresponding to the several Ordinates thus arising, shall be given. Q. E. O.

CASE IV.

38. And by a like way of reasoning, if the Ordinate consist of a Quadrinomial under the radical Sign, and there be given three of the Areas, or if it consist of a Quinquinomial, and there be given four of the Areas, and so on: all the Areas shall be given which can be produced by the Addition or Subtraction of the Number η to the Index θ, or of Unity to the Index λ. And the like is the Reasoning with respect to the Curves, whose Ordinates are made up of the Binomials, and one Area of those which are not geometrically quadrable is given. Q. E. O.

PROP. VIII. THEOR. VI.

39. If for $e+fz^{\eta}+gz^{2\eta}+$ &c. and $k+lz^{\eta}+mz^{2\eta}+$ &c. be wrote R and S as above, and in the Ordinate of any Curve $z^{\theta\pm\eta\sigma}R^{\lambda\pm\tau}S^{\mu\pm\upsilon}$, the given Quantities θ, η, λ, μ, e, f, g, k, l, m, &c. remain the same; and for σ, τ and υ be wrote any whole Numbers successively: and if the Areas of two of the Curves be given which are design'd by the Ordinates thus arising, when the Quantities R and S are Binomials, or if the Areas of three of the Curves be given, when R and S together consist of five Nomes or Terms, or the Areas of four of the Curves when R and S both taken together consist of six Nomes or Terms, and so on *in infinitum:* I say the Areas of all the Curves shall be given. It is demonstrated after the Manner of the preceding Proposition.

PROP. IX. THEOR. VII.

40. The Areas of thofe Curves are equal among themfelves, whofe Ordinates are reciprocally as the Fluxions of their Abfciffes.

For the Rectangles contain'd under the Ordinates, and the Fluxions of the Abfciffes will be equal, and the Fluxions of the Areas are as thefe Rectangles.

COROL. I.

41. If there be affum'd any Relation betwixt the Abfciffes of two Curves, and thence by Prop. I. you feek the Relation of the Fluxions of the Abfciffes, and the Ordinates be made reciprocally proportional to thefe Fluxions, you may find innumerable Curves, whofe Areas are equal among themfelves.

COROL. II.

42. For every Curve whofe Ordinate is $z^{\theta-1}\times\overline{e+fz^{\eta}+gz^{2\eta}+\&c.}|^{\lambda}$, by affuming any Quantity for ν, and putting $\frac{\eta}{\nu}=s$ and $z^{s}=x$, paffes into another Curve equal to itfelf whofe Ordinate is $\frac{\nu}{\eta}x^{\frac{\nu\theta-\eta}{\eta}}\times\overline{e+fx^{\nu}+gx^{2\nu}+\&c.}|^{\lambda}$.

COROL. III.

43. And every Curve whofe Ordinate is $z^{\theta-1}\times\overline{a+bz^{\eta}+cz^{2\eta}+\&c.}\times\overline{e+fz^{\eta}+gz^{2\eta}+\&c.}|^{\lambda}$, if you affume any Quantity for ν, and put $\frac{\eta}{\nu}=s$, and $z^{s}=x$, paffes into another Curve equal to itfelf, whofe Ordinate is $\frac{\nu}{\eta}x^{\frac{\nu\theta-\eta}{\eta}}\times\overline{a+bx^{\nu}+cx^{2\nu}+\&c.}\times\overline{e+fx^{\nu}+gx^{2\nu}+\&c.}|^{\lambda}$

COROL. IV.

44. And every Curve whofe Ordinate is $z^{\theta-1}\times\overline{a+bz^{\eta}+cz^{2\eta}+\&c.}\times\overline{e+fz^{\eta}+gz^{2\eta}+\&c.}|^{\lambda}\times\overline{k+lz^{\eta}+mz^{2\eta}+\&c.}|^{\mu}$ by affuming any Quantity for ν, and putting $\frac{\eta}{\nu}=s$, and $z^{s}=x$, paffes into another Curve equal to it, whofe Ordinate is

$$\frac{\nu}{\eta}x^{\frac{\nu\theta-\eta}{\eta}}\times\overline{a+bx^{\nu}+cx^{2\nu}+\&c.}\times\overline{e+fx^{\nu}+gx^{2\nu}+\&c.}|^{\lambda}\times\overline{k+lx^{\nu}+mx^{2\nu}+\&c.}|^{\mu}$$

 COROL.

COROL. V.

45. And every Curve whose Ordinate is $z^{\theta-1} \times \overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda}$, by putting $\frac{1}{z}=x$, passes into another Curve equal to itself, whose Ordinate is $\frac{1}{x^{\theta+1}} \times \overline{e+fx^{-n}+gx^{-2n}+\&c.}|^{\lambda}$: that is $\frac{1}{x^{\theta+1+n\lambda}} \times \overline{f+ex^{n}}|^{\lambda}$, if the Quantity under the Vinculum of the Root be a Binomial; or $\frac{1}{x^{\theta+1+2n\lambda}} \times \overline{g+fx^{n}+ex^{2n}}|^{\lambda}$, if it be a Trinomial: and so for others.

COROL. VI.

46. And every Curve whose Ordinate is $z^{\theta-1} \times \overline{e+fz^{n}+gz^{2n}+\&c.}|^{\lambda} \times \overline{k+lz^{n}+mz^{2n}+\&c.}|^{\mu}$, by putting $\frac{1}{z}=x$, passes into another equal Curve, whose Ordinate is $\frac{1}{x^{\theta+1}} \times \overline{e+fx^{-n}+gx^{-2n}+\&c.}|^{\lambda} \times \overline{k+lx^{-n}+mx^{-2n}+\&c.}|^{\mu}$, that is $\frac{1}{x^{\theta+1+n\lambda+n\mu}} \times \overline{f+ex^{n}}|^{\lambda} \times \overline{l+kx^{n}}|^{\mu}$, if the Quantities under the Vinculums of the Roots be Binomials, or $\frac{1}{x^{\theta+1+2n\lambda+n\mu}} \times \overline{g+fx^{n}+ex^{2n}}|^{\lambda} \times \overline{l+kx^{n}}|^{\mu}$ if the Quantity under the Vinculum of the first Root be a Trinomial, and that under the second a Binomial: and so for others.

47. And observe, that the two equal Areas in these two last Corollaries, lye towards contrary Parts of the Ordinates. If the Area in the one Curve be adjacent to the Absciss, the Area equal to it in the other Curve, is adjacent to the Abscis produced.

COROL. VII.

48. If the Relation betwixt the Ordinate y of any Curve and the Abscis z be defin'd by any affected Equation of this Form,

$$y^{\alpha} \times \overline{e+fy^{n}z^{\delta}+gy^{2n}z^{2\delta}+hy^{3n}z^{3\delta}+\&c.} = z^{\beta} \times \overline{k+ly^{n}z^{\delta}+my^{2n}z^{2\delta}+\&c.}$$

this Figure, by assuming $s=\frac{n-\delta}{n}$, $x=\frac{1}{s}z^{s}$ and $\lambda=\frac{n-\delta}{\alpha\delta+\beta n}$, passes into another equal to itself, whose Abscis x, from the Ordinate v given, is determined by an Equation not affected, *viz.*

$$\frac{1}{s}v^{\alpha\lambda} \times \overline{e+fv^{n}+gv^{2n}+\&c.}|^{\lambda} \times \overline{k+lv^{n}+mv^{2n}+\&c.}|^{-\lambda} = x.$$

COROL. VIII.

49. If the Relation betwixt y the Ordinate of any Curve, and z it's Abscifs be defined by any affected Equation of this Form $y^{\alpha} \times$

$$\overline{e + fy^{n}z^{\delta} + gy^{2n}z^{2\delta} + \&c.} = z^{\beta} \times \overline{k + ly^{n}z^{\delta} + my^{2n}z^{2\delta} + \&c.}$$
$$+ z^{\gamma} \times \overline{p + qy^{n}z^{\delta} + ry^{2n}z^{2\delta} + \&c.}$$

This Figure, by taking $s = \frac{\eta - \delta}{\eta}$, $x = \frac{1}{s}z^{s}$, $\mu = \frac{\alpha\delta + \beta\eta}{\eta - \delta}$ and $\nu = \frac{\alpha\delta + \gamma\eta}{\eta - \delta}$, passes into another equal to itself, whose Abscifs x is determined from the Ordinate v given, by this Equation less affected $v^{\alpha} \times$

$$\overline{e + fv^{\eta} + gv^{2\eta} + \&c.} = s^{\mu}x^{\mu} \times \overline{k + lv^{\eta} + mv^{2\eta} + \&c.}$$
$$+ s^{\nu}x^{\nu} \times \overline{p + qv^{\eta} + rv^{2\eta} + \&c.}$$

COROL. IX.

50. Every Curve whose Ordinate is

$$\pi z^{\theta - 1} \times \overline{\nu e + \overline{\nu + \eta}fz^{\eta} + \overline{\nu + 2\eta}gz^{2\eta} + \&c.} \times \overline{e + fz^{\eta} + gz^{2\eta} + \&c.}|^{\lambda - \pi}$$
$$\times \overline{a + b \times \overline{ez^{\nu} + fz^{\nu+\eta} + gz^{\nu+2\eta} + \&c.}|^{\tau}}|^{\omega}$$

if $\theta = \lambda\nu$, and you assume $x = \overline{ez^{\nu} + fz^{\nu+\eta} + gz^{\nu+2\eta} + \&c.}|^{\pi}$, $\sigma = \frac{\tau}{\pi}$ and $\vartheta = \frac{\lambda - \pi}{\pi}$, passes into another equal to itself, whose Ordinate, is $x^{\vartheta} \times \overline{a + bx^{\sigma}}|^{\omega}$. And you may observe that the former Ordinate, in this Corollary, becomes more simple by putting $\lambda = 1$, or by putting $\tau = 1$, and so ordering it that the Root of the Power may be extracted whose Index is ω; or yet by putting $\omega = -1$, and $\lambda = 1 = \tau = \sigma = \pi$, to pass other Curves.

COROL. X.

51. For $\overline{ez^{\nu} + fz^{\nu+\eta} + gz^{\nu+2\eta} + \&c}$, $\overline{\nu ez^{\nu-1} + \overline{\nu+\eta}fz^{\nu+\eta-1} + \overline{\nu+2\eta}gz^{\nu+2\eta-1} + \&c}$, $\overline{k + lz^{\eta} + mz^{2\eta} + \&c}$, and $\overline{\eta lz^{\eta-1} + 2\eta mz^{2\eta-1} + \&c.}$ write R, $\dot{r}$, S and $\dot{s}$ respectively, and every Curve whose Ordinate is $\pi S\dot{r} + \phi R\dot{s} \times R^{\lambda-1}S^{\mu-1} \times \overline{aS^{\nu} + bR^{\tau}}|^{\omega}$, if it be $\frac{\mu + \nu\omega}{\lambda}$ * $= \frac{-\nu}{\tau} = \frac{\phi}{\pi}$, $\frac{\tau}{\pi} = \sigma$, $\frac{\lambda - \pi}{\pi} = \vartheta$, and $R^{\pi}S^{\phi} = x$, passes into another equal to itself, whose Ordinate

* There is in this Place an Error in the printed Copies, which I have corrected. See the Demonstration in the Notes, Art. 332—335.

is

is $x^{\vartheta} \times \overline{a + bx^{\sigma}}|^{\omega}$. And observe that the former Ordinate becomes more simple, by putting Unities for τ, υ, and λ or μ, and so ordering it that the Root of that Power may be extracted whose Index is ω, or by putting $\omega = -1$ or $\mu = \omega$.

PROP. X. PROB. III.

52. *To find the most simple Figures with which any Curve may be geometrically compared, whose Ordinate y, from the Abscis z given, is determined by an Equation not affected.*

CASE I.

53. Let the Ordinate be $az^{\theta-1}$, and the Area shall be $\frac{1}{\theta}az^{\theta}$, as is easily collected from Prop. 5. by putting $b = 0 = c = d = f = g = h$ and $e = 1$.

CASE II.

54. Let the Ordinate be $az^{\theta-1} \times \overline{e + fz^{\eta} + gz^{2\eta} + \&c.}|^{\lambda-1}$, and if the Figure can be geometrically compared with rectilinear Figures, it will be squared by Prop. 5, by putting $b = 0 = c = d$. But if otherwise, you must change it into another Curve equal to it, whose Ordinate is $\frac{a}{\eta}x^{\frac{\theta-\eta}{\eta}} \times \overline{e + fx + gx^2 + \&c.}|^{\lambda-1}$, by Cor. 2. Prop. 9. Afterwards if you reject the Units from the Indexes of the Powers, *viz.* $\frac{\theta-\eta}{\eta}$ and $\lambda - 1$ by the seventh Prop. till these Powers become as low as possible, you shall have arrived at the most simple Figures, which can be discovered in this way. Then every one of them gives another which is sometimes more simple by Cor. 5. Prop. 9. And from these compared together by Prop. 3. and Cor. 9. and 10. Prop. 9, more simple Figures as yet sometimes discover themselves. Finally, from the Assumption of the most simple Figures the Area sought may be computed by coming back through the same Steps.

CASE III.

55. Let the Ordinate be $z^{\theta-1} \times \overline{a + bz^{\eta} + cz^{2\eta} + \&c.} \times \overline{e + fz^{\eta} + gz^{2\eta} + \&c.}|^{\lambda-1}$, and this Figure, if it can be squared, will be squared by Prop. 5. If not, it's Ordinate must be distinguished into the Parts $z^{\theta-1} \times a \times \overline{e + fz^{\eta} + gz^{2\eta} + \&c.}|^{\lambda-1}$, $z^{\theta-1} \times bz^{\eta} \times \overline{e + fz^{\eta} + gz^{2\eta} + \&c.}|^{\lambda-1}$, &c. and then the most simple Figures must be found, with which the Figures answering to these Parts can

can be compared, by Caſe 2. For the Areas of the Figures anſwering to theſe Parts, join'd together with their proper Signs + and — make up the whole Area required.

CASE IV.

56. Let the Ordinate be $z^{\theta-1} \times \overline{a + bz^{n} + cz^{2n} + \&c.}^{\lambda-1} \times \overline{e + fz^{n} + gz^{2n} + \&c.}^{\lambda-1} \times \overline{k + lz^{n} + mz^{2n} + \&c.}^{\mu-1}$; and if the Curve can be ſquared, it will be ſquared by Prop. 6. If otherwiſe, you muſt convert it into a more ſimple one by Cor. 4. Prop. 9. and thence you may compare it with the moſt ſimple Figures by Prop. 8. and Cor. 6, 9 and 10. Prop. 9. as is done in Caſes ſecond and third.

CASE V.

57. If the Ordinate conſiſt of different Parts, all the Parts muſt be accounted as Ordinates of ſo many Curves; and theſe Curves, as many of them as can be ſquared, are to be ſquared ſingly, and their Ordinates to be taken away from the whole Ordinate. Then the Curve, which the remaining Part of the Ordinate deſigns, muſt be compared ſeparately (as in the ſecond, third and fourth Caſes) with the moſt ſimple Figures it can be compared with. And the Sum of all the Areas muſt be accounted the Area of the Curve propos'd.

COROL. I.

58. Hence alſo, every Curve whoſe Ordinate is the affected ſquare Root of it's own Equation, may be compared with the moſt ſimple Figures, either rectilinear, or curvilinear. For that Root always conſiſts of two Parts, which conſidered ſeparately, are not the affected Roots of Equations.

Let the Equation $a^2y^2 + z^2y^2 = 2a^3y + 2z^3y - z^4$ be propoſed; and the Root being extracted will be $y = \frac{a^3 + z^3 + a\sqrt{a^4 + 2az^3 - z^4}}{aa + zz}$: whoſe rational Part $\frac{a^3 + z^3}{a^2 + z^2}$, and irrational Part $\frac{a\sqrt{a^4 + 2az^3 - z^4}}{a^2 + z^2}$ are the Ordinates of Curves, which, according to this Propoſition, may either be ſquared; or elſe compared with the moſt ſimple Figures, with which they admit of a geometrical Compariſon.

COROL.

COROL. II.

59. And every Curve whoſe Ordinate is defin'd by any affected Equation, which, by Cor. 7. Prop. 9, paſſes into an Equation not affected, is either ſquared by this Propoſition, if it can be ſquared; or is compared with the moſt ſimple Figures with which it may be compared. And by this means every Curve is ſquared whoſe Equation conſiſts of three Terms. For that Equation, if it be affected, is changed into one not affected by Cor. 7. Prop. 9: and then, by Cor. 2[d] and 5[th], Prop. 9, by paſſing into the moſt ſimple Curve, it either gives the Quadrature of the Figure, if it admit of one; or gives the moſt ſimple Figure, with which it may be compared.

COROL. III.

60. And every Curve, whoſe Ordinate is defined by any affected Equation, which, by Cor. 8. Prop. 9, paſſes into an affected quadratic Equation, is either ſquared by this Prop. and Cor. 1, if it can be ſquared; or is compared with the moſt ſimple Figures with which it admits of a geometrical Compariſon.

SCHOLIUM.

61. When Figures are to be ſquared, it would be too troubleſom a Work to have recourſe always to theſe general Rules: it's much better once to ſquare the more ſimple and uſeful Kind of Figures, and record the Quadratures in a Table; afterwards conſult the Table, as often as you have any ſuch Curve to ſquare. And the two following Tables are of this Kind: in which z denotes the Abſciſs, y the rectangular Ordinate, and t the Area of the Curve to be ſquared; and d, e, f, g, h, η are given Quantities with their proper Signs $+$ and $-$.

A TABLE

A TABLE

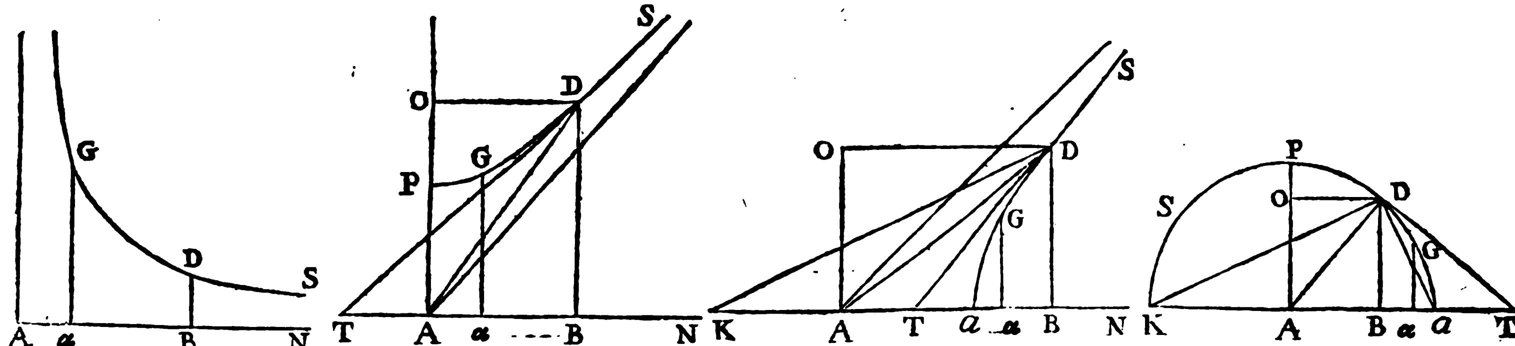
Of the more simple kind of Curves which may be squared.

		Forms of Curves.	Areas of the Curves.
I		$dz^{n-1}=y$	$\frac{d}{n}z^{n}=t.$
II		$\frac{dz^{n-1}}{e^2+2efz^n+f^2z^{2n}}=y$	$\frac{dz^n}{ne^2+nefz^n}=t.$ Or $\frac{-d}{nef+nf^2z^n}=t$
III	1	$dz^{n-1}\sqrt{e+fz^n}=y$	$\frac{2d}{3nf}R^3=t.$ Where $R=\sqrt{e+fz^n}$
	2	$dz^{2n-1}\sqrt{e+fz^n}=y$	$\frac{-4e+6fz^n}{15nf^2}dR^3=t$
	3	$dz^{3n-1}\sqrt{e+fz^n}=y$	$\frac{16e^2-24efz^n+30f^2z^{2n}}{105nf^3}dR^3=t$
	4	$dz^{4n-1}\sqrt{e+fz^n}=y$	$\frac{-96e^3+144e^2fz^n-180ef^2z^{2n}+210f^3z^{3n}}{945n^4}=t$
IV	1	$\frac{dz^{n-1}}{\sqrt{e+fz^n}}=y$	$\frac{2d}{nf}R=t.$
	2	$\frac{dz^{2n-1}}{\sqrt{e+fz^n}}=y$	$\frac{-4e+2fz^n}{3nf^2}dR=t$
	3	$\frac{dz^{3n-1}}{\sqrt{e+fz^n}}=y$	$\frac{16e^2-8efz^n+6f^2z^{2n}}{15nf^3}dR=t$
	4	$\frac{dz^{4n-1}}{\sqrt{e+fz^n}}=y$	$\frac{-96e^3+48e^2fz^n-36ef^2z^{2n}+30f^3z^{3n}}{105nf^4}dR=t$

A TABLE

Of the more simple kind of Curves which may be compared with the Ellipsis and Hyperbola.

Let aGD, or PGD, or GDS be a Conic Section whose Area is required to the Quadrature of the Curve proposed, and let it's Centre be A, Axis Ka, Vertex a, half Conjugate Axis AP, the given Beginning of the Abscis A, or a, or α, the Abscis AB, or aB, or $\alpha B = x$, the Rectangular Ordinate $BD = v$, and the Area ABDP, or $aBDG$, or $\alpha BDG = s$, αG being an Ordinate at the Point α. Let KD, AD, aD be joined; draw the Tangent DT meeting the Abscis AB in T; and let the Parallelogram ABDO be completed. And if perchance the Areas of two Conic Sections be required to the Quadrature of the Curve proposed, call the Abscis of the latter ξ, the Ordinate Υ, and the Area σ. And let $\div$ denote the Difference of two Quantities, when it is uncertain whether the latter should be subtracted from the former, or the former from the latter. And in the sixth Form write p for $\sqrt{ff - 4eg}$.



Forms of the Curves.		Conic Section, it's Abscis.	Ordinate.	Areas of the Curves.
I	1	$\frac{dz^{\eta-1}}{e+fz^{\eta}} = y$	$z^{\eta} = x$	$\frac{d}{e+fx} = v$	$\frac{1}{\eta}s = t = \alpha GDB$, Fig. 1.
	2	$\frac{dz^{2\eta-1}}{e+fz^{\eta}} = y$	$z^{\eta} = x$	$\frac{d}{e+fx} = v$	$\frac{d}{\eta f}z^{\eta} - \frac{e}{\eta f}s = t$
	3	$\frac{dz^{3\eta-1}}{e+fz^{\eta}} = y$	$z^{\eta} = x$	$\frac{d}{e+fx} = v$	$\frac{d}{2\eta f}z^{2\eta} - \frac{de}{\eta f^2}z^{\eta} + \frac{e^2}{\eta f^2}s = t$

Forms of Curves.		Conic Section, it's Abscifs.	Ordinate.	Areas of the Curves.
II	1 $\frac{dz^{\frac{1}{2}\eta-1}}{e+fz^{\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^{2}}=v$	$\frac{2xv\div 4s}{\eta}=t=\frac{4}{\eta}ADGa$, Fig. 3, 4.
II	2 $\frac{dz^{\frac{3}{2}\eta-1}}{e+fz^{\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^{2}}=v$	$\frac{2d}{\eta f}z^{\frac{1}{2}\eta}+\frac{4es-2exv}{\eta f}=t$
II	3 $\frac{dz^{\frac{5}{2}\eta-1}}{e+fz^{\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^{2}}=v$	$\frac{2d}{3\eta f}z^{\frac{3}{2}\eta}-\frac{2de}{\eta f^{2}}z^{\frac{1}{2}\eta}+\frac{2e^{2}xv-4e^{2}s}{\eta f^{2}}=t$
III	1 $\frac{d}{z}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x^{2}$	$\sqrt{f+ex^{2}}=v$	$\frac{4de}{\eta f}\times\overline{\frac{v^{3}}{2ex}-s}=t=\frac{4de}{\eta f}$ into $aGDT$ or into $APDB\div TDB$, Fig. 2, 3, 4.
III	Or thus	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^{2}}=v$	$\frac{8de^{2}}{\eta f^{2}}\times\overline{s-\frac{1}{2}xv-\frac{fv}{4e}+\frac{f^{2}v}{4e^{2}x}}=t=\frac{8de^{2}}{\eta f^{2}}$ into $aGDA+\frac{f^{2}v}{4e^{2}x}$ Fig. 3, 4.
III	2 $\frac{d}{z^{\eta+1}}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x^{2}$	$\sqrt{f+ex^{2}}=v$	$\frac{-2d}{\eta}s=t=\frac{2d}{\eta}APDB$ or $\frac{2d}{\eta}aGDB$, Fig. 2, 3, 4.
III	Or thus	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^{2}}=v$	$\frac{4de}{\eta f}\times\overline{s-\frac{1}{2}xv-\frac{fv}{2e}}=t=\frac{4de}{\eta f}\times aGDK$, Fig. 3, 4.
III	3 $\frac{d}{z^{2\eta+1}}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^{2}}=v$	$\frac{-d}{\eta}s=t=\frac{d}{\eta}\times -aGDB$ or $BDPK$, Fig. 4.
III	4 $\frac{d}{z^{3\eta-1}}\sqrt{e+fz^{\eta}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^{2}}=v$	$\frac{3dfs-2dv^{3}}{6\eta e}=t$
IV	1 $\frac{d}{z\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x^{2}$	$\sqrt{f+ex^{2}}=v$	$\frac{4d}{\eta f}\times\overline{\frac{1}{2}xv\div s}=t=\frac{4d}{\eta f}$ into PAD or into $aGDA$, Fig. 2, 3, 4.
IV	Or thus	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^{2}}=v$	$\frac{8de}{\eta f^{2}}\times\overline{s-\frac{1}{2}xv-\frac{fv}{4e}}=t=\frac{8de}{\eta f^{2}}$ into $aGDA$, Fig. 3, 4.
IV	2 $\frac{d}{z^{\eta+1}\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x^{2}$	$\sqrt{f+ex^{2}}=v$	$\frac{2d}{\eta e}\times\overline{s-xv}=t=\frac{2d}{\eta e}$ into POD or into $AODGa$, Fig. 2, 3, 4.
IV	Or thus	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^{2}}=v$	$\frac{4d}{\eta f}\times\overline{\frac{1}{2}xv\div s}=t=\frac{4d}{\eta f}$ into $aDGa$, Fig. 3, 4.
IV	3 $\frac{d}{z^{2\eta+1}\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^{2}}=v$	$\frac{d}{\eta e}\times\overline{3s\div 2xv}=t=\frac{d}{\eta e}$ into $3aDGa\div\Delta aDB$, Fig. 3, 4.
IV	4 $\frac{d}{z^{3\eta+1}\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^{2}}=v$	$\frac{10dfxv-15dfs-2dex^{2}v}{6\eta e^{2}}=t$

	Forms of the Curves.	Conic Section, it's Abscis.	Conic Section, it's Ordinate.	Areas of the Curves.
V. 1	$\frac{dz^{\eta-1}}{e+fz^{\eta}+gz^{2\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}+gz^{2\eta}}}=x$	$\sqrt{\frac{d}{g}+\frac{f^2-4eg}{4g^2}x^2}=v$	$\frac{xv-2s}{\eta}=t$
Or thus		$\sqrt{\frac{dz^{2\eta}}{e+fz^{\eta}+gz^{2\eta}}}=x$	$\sqrt{\frac{d}{e}+\frac{f^2-4eg}{4e^2}x^2}=v$	$\frac{2s-xv}{\eta}=t$
V. 2	$\frac{dz^{2\eta-1}}{e+fz^{\eta}+gz^{2\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}+gz^{2\eta}}}=x$ $fz^{\eta}+gz^{2\eta}=\xi$	$\sqrt{\frac{d}{g}+\frac{f^2-4eg}{4g^2}x^2}=v$ $\frac{1}{e+\xi}=\Upsilon$	$\frac{d\sigma+2fs-fxv}{2\eta g}=t$
VI. 1	$\frac{dz^{\frac{1}{2}\eta-1}}{e+fz^{\eta}+gz^{2\eta}}=y$	$\sqrt{\frac{2dg}{f-p+2gz^{\eta}}}=x$ $\sqrt{\frac{2dg}{f+p+2gz^{\eta}}}=\xi$	$\sqrt{d+\frac{-f+p}{2g}x^2}=v$ $d+\frac{-f-p}{2g}\xi^2=\Upsilon$	$\frac{2xv-4s-2\xi\Upsilon+4\sigma}{\eta p}=t$
VI. 2	$\frac{dz^{\frac{3}{2}\eta-1}}{e+fz^{\eta}+gz^{2\eta}}=y$	$\sqrt{\frac{2dez^{\eta}}{fz^{\eta}-pz^{\eta}+2e}}=x$ $\sqrt{\frac{2dez^{\eta}}{fz^{\eta}+pz^{\eta}+2e}}=\xi$	$\sqrt{d+\frac{-f+p}{2e}x^2}=v$ $\sqrt{d+\frac{-f-p}{2e}\xi^2}=\Upsilon$	$\frac{4s-2xv-4\sigma+2\xi\Upsilon}{\eta p}=t$
VII. 1	$\frac{d}{z}\sqrt{e+fz^{\eta}+gz^{2\eta}}=y$	$z^{\eta}=x$ $\frac{1}{z^{\eta}}=\xi$	$\sqrt{e+fx+gx^2}=v$ $\sqrt{g+f\xi+e\xi^2}=\Upsilon$	$\frac{4de^2\xi\Upsilon+2def\Upsilon-2dfgxv\,{+4deg \atop -2f^2d}\,v-8de^2\sigma+4dfgs}{4\eta g-\eta f^2}=t$
VII. 2	$dz^{\eta-1}\sqrt{e+fz^{\eta}+gz^{2\eta}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{d}{\eta}s=t=\frac{d}{\eta}\times\alpha GDB$, Fig. 2, 3, 4.
VII. 3	$dz^{2\eta-1}\sqrt{e+fz^{\eta}+gz^{2\eta}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{d}{3\eta g}v^3-\frac{df}{2\eta g}s=t$
VII. 4	$dz^{3\eta-1}\sqrt{e+fz^{\eta}+gz^{2\eta}}=y$	$z\eta=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{6dge-5df}{24\eta g^2}v^3+\frac{5df^2-4deg}{16\eta g^2}s=t$

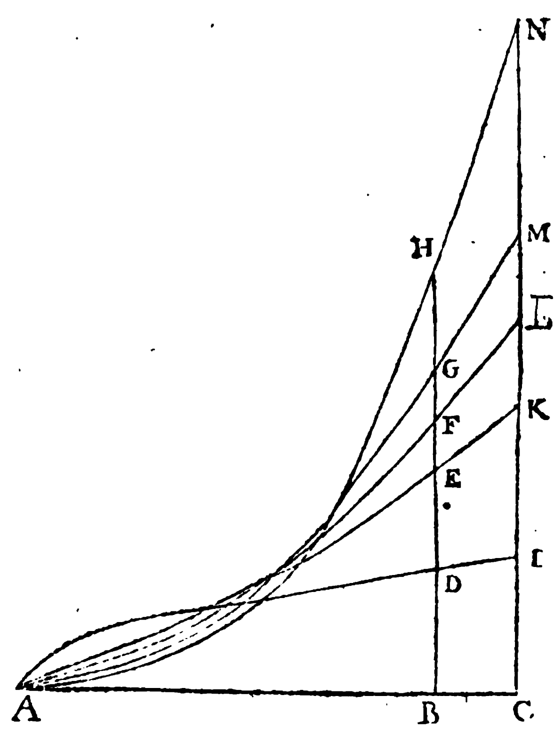
		Forms of the Curves.	Conic Section, its Abſciſs.	Ordinate.	Areas of the Curves.
VIII	1	$\frac{dz^{\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^{2}}=v$	$\frac{8dgs-4dgxv-2dfv}{4\eta eg-\eta f^{2}}=t=\frac{8dg}{4\eta eg-\eta f^{2}}\times\alpha GDB\pm\Delta DBA$, Fig. 2, 4.
	2	$\frac{dz^{2\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^{2}}=v$	$\frac{-4dfs+2dfxv+4dev}{4\eta eg-\eta f^{2}}=t$
	3	$\frac{dz^{3\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^{2}}=v$	$\frac{\begin{matrix}3df^{2}\\-4deg\end{matrix}s\ \begin{matrix}-2df^{2}\\+4deg\end{matrix}xv-2defv}{4\eta eg^{2}-\eta f^{2}g}=t$
	4	$\frac{dz^{4\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^{2}}=v$	$\frac{\begin{matrix}36defg\\-15df^{3}\end{matrix}s\ \begin{matrix}+8deg^{2}\\-2df^{2}g\end{matrix}x^{2}v\ \begin{matrix}+10df^{3}\\-28defg\end{matrix}xv\ \begin{matrix}+10def^{2}\\-16de^{2}g\end{matrix}v}{24\eta eg^{3}-6\eta f^{2}g^{2}}=t$
IX	1	$\frac{dz^{\eta-1}\sqrt{e+fz^{\eta}}}{g+hz^{\eta}}=y$	$\sqrt{\frac{d}{g+hz^{\eta}}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^{2}}=v$	$\frac{\begin{matrix}4fg\\-4eh\end{matrix}s\ \begin{matrix}-2fg\\+2eh\end{matrix}xv+2df\frac{v}{x}}{\eta fh}=t$
	2	$\frac{dz^{2\eta-1}\sqrt{e+fz^{\eta}}}{g+hz^{\eta}}=y$	$\sqrt{\frac{d}{g+hz^{\eta}}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^{2}}=v$	$\frac{\begin{matrix}4egh\\-4fg^{2}\end{matrix}s\ \begin{matrix}-2egh\\+2fg^{2}\end{matrix}xv+\frac{2}{3}dh\frac{v^{3}}{x^{3}}-2dfg\frac{v}{x}}{\eta fh^{2}}=t$
X	1	$\frac{dz^{\eta-1}}{g+hz^{\eta}\sqrt{e+fz^{\eta}}}=y$	$\sqrt{\frac{d}{g+hz^{\eta}}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^{2}}=v$	$\frac{2xv-4s}{\eta f}=t=\frac{4}{\eta f}ADGa$, Fig. 3, 4.
	2	$\frac{dz^{2\eta-1}}{g+hz^{\eta}\sqrt{e+fz^{\eta}}}=y$	$\sqrt{\frac{d}{g+hz^{\eta}}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^{2}}=v$	$\frac{4gs-2gxv+2d\frac{v}{x}}{\eta fh}=t$
XI	1	$dz^{-1}\sqrt{\frac{e+fz^{\eta}}{g+hz^{\eta}}}=y$	$\sqrt{g+hz^{\eta}}=x$ $\sqrt{h+gz^{-\eta}}=\xi$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^{2}}=v$ $\sqrt{\frac{fg-eh}{g}+\frac{e}{g}\xi^{2}}=\Upsilon$	$\frac{2dxv^{3}z^{-\eta}-4dfs-4de\sigma}{\eta fg-\eta eh}=t$
	2	$dz^{\eta-1}\sqrt{\frac{e+fz^{\eta}}{g+hz^{\eta}}}=y$	$\sqrt{g+hz^{\eta}}=x$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^{2}}=v$	$\frac{2d}{\eta h}s=t$
	3	$dz^{2\eta-1}\sqrt{\frac{e+fz^{\eta}}{g+hz^{\eta}}}=y$	$\sqrt{g+hz^{\eta}}=x$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^{2}}=v$	$\frac{dhxv^{3}\ \begin{matrix}-3dfg\\-deh\end{matrix}s}{2\eta fh^{2}}=t.$

62. In these Tables the Series's of Curves of each Form may be continued *in infinitum* towards both Parts. For in the first Table, with respect to the Numerators of the Areas of the third and fourth Forms, the numeral Coefficients of the initial Terms (2, —4, 16, —96, 868, *&c.*) are found by multiplying the Numbers —2, —4, —6, —8, —10, *&c.* continually into one another: and the Coefficients of the subsequent Terms are derived from those of the initial Terms, by multiplying them orderly, in the third Form, by $-\frac{3}{2}$, $-\frac{5}{4}$, $-\frac{7}{6}$, $-\frac{9}{8}$, $-\frac{11}{10}$, *&c.* in the fourth Form, by $-\frac{1}{2}$, $-\frac{3}{4}$, $-\frac{5}{6}$, $-\frac{7}{8}$, $-\frac{9}{10}$, *&c.* And the Coefficients of the Denominators 3, 15, 105, *&c.* arise by multiplying the Numbers 1, 3, 5, 7, 9, *&c.* continually into one another.

63. And in the second Table, the Series's of Curves of the first, second, fifth, sixth, ninth and tenth Forms, by the Help of Division alone; and the other Forms by the Help of the third and fourth Propositions, are drawn out both Ways *in infinitum.*

64. Moreover these Series's, by changing the Sign of the Number η, use to vary. For thus, to take an Example, the Curve $\frac{d}{z}\sqrt{e+fz^{\eta}}=y$ becomes $\frac{d}{z^{\frac{1}{2}\eta+1}}\sqrt{f+ez^{\eta}}=y$.

PROP. XI. THEOR. VIII.



65. Let ADIC be any Curve, having the Absciss AB $=z$, the Ordinate BD $=y$; and let AEKC be another Curve, whose Ordinate BE is equal to the Area of the former ADB applied to Unity; and AFLC a third Curve, whose Ordinate BF is equal to Area of the second AEB applied to Unity; and AGM a fourth Curve whose Ordinate BG is equal to the Area of the third Curve AFB applied to Unity; and AHNC a fifth Curve, whose Ordinate BH is equal to AGB the Area of the fourth applied to Unity: and so on *in infinitum.* And let A. B. C. D. E. *&c.* be the Areas of Curves,

Curves, having the Ordinates y, zy, z^2y, z^3y, z^4y, &c. and the common Absciss z.

66. Let there be any given Absciss AC $= t$, and let BC $= t - z$ $= x$: and let P, Q, R, S, T be the Areas of Curves having the Ordinates y, xy, x^2y, x^3y, x^4y, &c. and the common Absciss x.

67. And let all these Areas be terminated at the whole given Absciss AC, and likewise at the Ordinate CI given in Position, and infinitely produced.

And of the Areas mentioned at the Beginning:

The 1st, ADIC $= A = P$

2d, AEKC $= tA - B = Q$

3d, AFLC $= \frac{t^2A - 2tB + C}{2} = \frac{1}{2}R$

4th, AGMC $= \frac{t^3A - 3t^2B + 3tC - D}{6} = \frac{1}{6}S$

5th, AHNC $= \frac{t^4A - 4t^3B + 6t^2C - 4tD + E}{24} = \frac{1}{24}T$.

COROL.

68. Whence if the Curves, whose Ordinates are y, zy, z^2y, z^3y, &c. or y, xy, x^2y, x^3y, &c. can be squared; the Curves ADIC, AEKC, AFLC, AGMC, *&c.* may likewise be squared; and we shall have the Ordinates BE, BF, BG, BH, which are proportional to the Areas of the Curves.

SCHOLIUM.

69. We said formerly that there were first, second, third, fourth, *&c.* Fluxions of flowing Quantities. These Fluxions are as the Terms of infinite converging Series.

70. Thus if z^n be the flowing Quantity, and by flowing become $\overline{z+o}|^n$, and afterwards be resolved into the converging Series $z^n + noz^{n-1} + \frac{n^2-n}{2}ooz^{n-2} + \frac{n^3-3n^2+2n}{6}o^3z^{n-3} +$ &c. The first Term of this Series z^n will be that flowing Quantity; the second will be the first Increment or Difference, to which consider'd as nascent, it's first Fluxion is proportional: the third $\frac{n^2-n}{2}ooz^{n-2}$ will be * it's second Increment or Difference, to which consider'd as nascent the second Fluxion is proportional: the fourth Term $\frac{n^3-3n^2+2n}{6}o^3z^{n-3}$ will be * it's third Increment or Difference, to which considered as nascent the third Fluxion is proportional: and so on *in infinitum*.

71. But these Fluxions may be expounded by the Ordinates of Curves BD, BE, BF, BG, BH, &c. See the Fig.

* See the Notes upon the Place.

72. Thus

72. Thus if the Ordinate BE $(=\frac{ADB}{1})$ be the flowing Quantity, it's first Fluxion will be as the Ordinate BD.

73. If BF $(=\frac{AEB}{1})$ be the flowing Quantity, it's first Fluxion will be as the Ordinate BE; and it's second Fluxion as the Ordinate BD.

74. If BH $(=\frac{AGB}{1})$ be the flowing Quantity, it's first, second, third and fourth Fluxions will be as the Ordinates BG, BF, BE and BD respectively.

75. And hence in such Equations as involve only two unknown Quantities, of which one is a Quantity flowing uniformly, and the other any Fluxion of the other flowing Quantity, that other Fluent may be found by the Quadrature of Curves. For let it's Fluxion be expounded by BD, and if it be the first Fluxion, seek the Area ADB $=$ BE $\times 1$; if it be the second Fluxion, seek the Area AEB $=$ BF $\times 1$; if the third, seek the Area AFB $=$ BG $\times 1$; &c. and the Area so found will be the Exponent of the Fluent sought.

76. But moreover in Equations which involve a Fluent and it's first Fluxion without the other Fluent; or two Fluxions of the same Fluent, the first and second, or the second and third, or the third and fourth, &c. without either of the two Fluents: the Fluents may be found by the Quadrature of Curves. Let there be given the Equation $aa\dot{v} = av + vv$, while it is $v =$ BE, $\dot{v} =$ BD, $z =$ AB and $\dot{z} = 1$, and that Equation, by completing the Dimensions of the Fluxions, will become $aa\dot{v} = av\dot{z} + vv\dot{z}$, or $\frac{aa\dot{v}}{av+vv} = \dot{z}$.

77. Now let v flow uniformly, and let it's Fluxion $\dot{v} = 1$; and it will be $\frac{aa}{av+vv} = \dot{z}$, and by squaring the Curve whose Ordinate is $\frac{aa}{av+vv}$ and Abscis v, you'll have the Fluent z.

78. Moreover let the Equation be $aa\ddot{v} = a\dot{v} + \dot{v}\dot{v}$; while it is $v =$ BF, $\dot{v} =$ BE, $\ddot{v} =$ BD, and $z =$ AB; and by means of the Relation betwixt $\ddot{v}$ and $\dot{v}$ or BD and BE, you'll find the Relation betwixt AB and BE as in the immediately preceding Example. Afterwards by means of this Relation, you'll find the Relation betwixt AB and BF, by squaring the Curve AEB.

79. Such Equations as involve three unknown Quantities may sometimes be reduced to Equations which involve two only: and in these Cases the Fluents will be found from the Fluxions as above. Let the Equation $a - bx^m = cxy^n\dot{y} + dy^{2n}\dot{y}\dot{y}$ be proposed: put $y^n\dot{y} = \dot{v}$, and it will be $a - bx^m = cx\dot{v} + d\dot{v}^2$. This Equation, by squaring the Curve, whose Abscis is x and Ordinate $\dot{v}$, gives the Area v; and the other

other Equation, *viz.* $y^n\dot{y} = \dot{v}$, by returning back to the Fluents gives $\frac{1}{n+1}y^{n+1} = v$: whence the Fluent y is obtain'd.

80. But besides, in Equations which involve three unknown Quantities, although they can't be reduced to Equations which involve only two, the Fluents will be sometimes found by the Quadrature of Curves. Let the Equation $\overline{ax^m + bx^n}|^p = rex^{r-1}y^s + sex^r\dot{y}y^{s-1} - f\dot{y}y^t$ be given, while it is $\dot{x} = 1$; and the latter Part $rex^{r-1}y^s + sex^r\dot{y}y^{s-1} - f\dot{y}y^t$, by going back to the Fluents, becomes $ex^ry^s - \frac{f}{t+1}y^{t+1}$, which therefore is as the Area of a Curve whose Abscis is x, and Ordinate $\overline{ax^m + bx^n}|^p$: and thence the Fluent y is given.

81. Let the Equation $\dot{x} \times \overline{ax^m + bx^n}|^p = \frac{d\dot{y}y^{n-1}}{\sqrt{e+fy^n}}$ be given, and the Fluent whose Fluxion is $\dot{x} \times \overline{ax^m + bx^n}|^p$, will be as the Area of a Curve, whose Abscis is x, and it's Ordinate $\overline{ax^m + bx^n}|^p$. Likewise the Fluent whose Fluxion is $\frac{d\dot{y}y^{n-1}}{\sqrt{e+fy^n}}$ will be as the Area of a Curve whose Abscis is y, and it's Ordinate $\frac{dy^{n-1}}{\sqrt{e+fy^n}}$, that is (by Case 1. Form 4th, Tab. 1.) as the Area $\frac{2d}{nf}\sqrt{e+fy^n}$. Therefore I put $\frac{2d}{nf}\sqrt{e+fy^n}$ equal to the Area of the Curve whose Abscis is x and Ordinate $\overline{ax^m + bx^n}|^p$, and so the Fluent y will be had.

82. And let it be remarked that every Fluent which is collected from a first Fluxion, may be augmented or diminished by any Quantity which don't flow at all. That which is collected from a second Fluxion may be augmented or diminished by any Quantity whose second Fluxion is nothing. A Fluent collected from a third Fluxion may be increased or diminished by any Quantity that has no third Fluxion. And so on *in inf.*

83. After the Fluents are collected from the Fluxions, if there arise any Doubt about the Truth of the Conclusion, then the Fluxions of the Fluents found are again to be collected, and compared with the Fluxions proposed at the Beginning. For if they come out equal, the Conclusion is right: if otherwise, the Fluents must be so corrected, that their Fluxions may be equal to the Fluxions at first propos'd. For a Fluent may be assumed at pleasure, and the Assumption be corrected, by putting the Fluxion of the assumed Fluent equal to the Fluxion proposed, and comparing the homologous Terms with one another.

And thus, by these Beginnings, we are led to yet more important Truths.

Sir

Sir *ISAAC NEWTON's* TREATISE OF THE Quadrature of Curves explained.

SECT. I.

Containing Notes on the Introduction.

1. THE incomparable Sir *Isaac Newton*, having not only enriched Mathematics and Philosophy with many new and useful Discoveries in their various Parts; but also extended the *Limits* of Geometry, by the noble and sublime Invention of *Fluxions*, takes care first of all, in this Introduction to the Quadrature of Curves, to give us just Notions of his Method of Fluxions: because it is a Method entirely new, by which the most remote and useful Properties and Relations of Quantities are discovered, and demonstrated, which were in vain attempted by the geometrical Analysis of the Ancients; or the specious Analysis of the Moderns: and is the Method he made use of in the Doctrine of the Quadrature of Curves contained in this Treatise: as he himself informs us in this Place.

2. The Method of Fluxions is twofold: or rather, it may be reduced to two very general and comprehensive Problems. 1°. From the *Fluents* given to find the *Fluxions*. 2°. From the *Fluxions* given to find the *Fluents*. Which two Problems may be conceived as equivalent to these, 1°. The *Length* of the Space described by a moving

 Body

Body being continually or at all Times given, to find the *Velocity* of the Motion at any Time propofed. 2°. The *Velocity* of the Motion being at all Times given, to find the *Length* of the Space defcribed at any Time propofed. The firft of thefe Problems includes the *direct* Method of Fluxions; the other, the *inverfe* Method.

3. Both thefe Methods of Fluxions are capable of being applied; and accordingly have been applied to a vaft Number of ufeful Purpofes, by fuch as have treated of this Subject: of all which one of the moft noble and ufeful is the Doctrine of the Quadrature of Curves delivered in this Treatife. The Defign of which is to fhew how infinite Numbers of various Kinds of geometrical Curves may be fquared, *i. e.* their Areas exactly determined, by comparing them with right-lined Figures: or, if that cannot be done, it fhews what way their Areas may be nearly determined, by an infinite converging Series; and the Curves geometrically compared, either with the Circle, Ellipfe and Hyperbola; or elfe, with other the moft fimple Curves, with which they admit of a geometrical Comparifon. So that by what our Author delivers in this Treatife, and in his Analyfis by Equations of an infinite Number of Terms, (which I have fubjoined to this Performance) all geometrical Curves, *i. e.* fuch, the Relation of whofe Abfcifs to the Ordinate is defined by an algebraical Equation of any Degree, can be either exactly fquared; or, if that cannot be done, the Area may be expreffed by an infinite converging Series, of which a finite Number of Terms fhall make fuch a near Approach to the true Area, as to differ from it by lefs than any given Difference. And even in *mechanical* Curves, *i. e.* fuch Curves, whofe Property cannot be defined by any *finite* Equation of what Degree foever, he fhews how to exhibit approximate Values of their Areas; or to reduce them to geometrical Curves.

4. But although this be the primary Defign of this Treatife of our Author's; yet it may be confidered in a more general and extenfive View, as containing the Doctrine of Fluxions, both *direct* and *inverfe*, whatever the Nature of the flowing Quantities be. For the firft Prop. contains the Solution of the firft General Problem mentioned above *. And by means of the Quadrature of Curves, the fecond General Problem, containing the inverfe Method of Fluxions, may be folved in a great many Cafes, although not univerfally: as we fhall have Occafion to fhew, when we come to explain our Author's laft Scholium.

* Art. 2. of this Explication.

Art. 3. of the Quadrature of Curves.

5. After the true Meaning of *Fluents* and *Fluxions* is laid down by our Author, he proceeds next, in this Article, to fhew what *Proportion* the Fluxions of flowing Quantities bear to one another. Accordingly

ingly he tells us, " They are very nearly to one another as the *Aug-*
" *ments* of the Fluents generated in equal, but very ſmall Times: and
" that if we would ſpeak accurately, they are in the *firſt* Ratio of the
" *naſcent* Augments, but may be expounded by any Lines, which are
" proportional to them." He ſays afterwards *, " that it comes to * Art. 6.
" the ſame Purpoſe if the Fluxions be taken in the *laſt* Ratio of the
" *evaneſcent* Parts." Now, becauſe the Doctrine of *prime* and *ultimate* Ratios has been ſo much controverted of late [1], I ſhall here enquire whether we have any diſtinct Idea thereof: which I reckon to be the more neceſſary, becauſe the noble Inventor has laid down this Notion of Fluxions in this Place, as that by which we are to be directed in our Conceptions of them.

6. Before we enter upon the Diſcuſſion of this Point, it will be proper to obſerve, 1°, That the Fluxions of flowing Quantities, being the *Celerities* with which they are ſuppoſed to flow; and by which they are generated: the Fluxion of any variable Quantity, ought never to be conſidered *abſolutely*, or by itſelf; but *relatively*, or with Relation to the Fluxion of ſome other flowing Quantity of the ſame Kind. So that if at any Time, for Brevity's ſake, the Fluxion of a Quantity be mentioned abſolutely; yet there is always a *ſuppoſed* Relation to the Fluxion of ſome other Quantity with which it is underſtood to be compared: and ordinarily in the Compariſon, one of the flowing Quantities is ſuppoſed to flow *equably* and *uniformly*; ſo that it's Fluxion, being conſtant and invariable, is conſidered as a *Standard* or Meaſure, to which the Fluxions of the other Quantities are referred. 2°, That if the Fluxion of any flowing Quantity, or it's Celerity of flowing, be continually varying, *i. e.* continually accelerated or retarded, for any Time, the Fluxion in any one Place or at any one Inſtant of that Time, is different from the Fluxion of that Quantity in any other Place, or at any other Inſtant of Time. For whatever is continually varying, by Increaſe or Decreaſe, by the very Suppoſition, muſt be of a different Value in every different Place, or at every different Inſtant of Time, otherwiſe it would not be continually varying. To illuſtrate which in the Caſe of a falling Body: the Velocity with which it falls in any one Place, or at any one Inſtant of Time, during it's Fall, is different from the Velocity it has in any other Place, or at any other Inſtant of Time: and it is all one, whether the Motion be uniformly accelerated, or retarded; or whether it be accelerated or retarded according to any other Law.

[1] See the Analyſt publiſhed by the Biſhop of *Cloyne*, with an Anſwer to it by *Philalethes Cantabrigienſis*: likewiſe the Biſhop's Defence, and *Philalethes*'s Reply.

7. Before we proſecute this Affair any further, we ſhall here borrow from Sir *I. Newton* the following Lemma [1].

" Quantities, as likewiſe the Ratios of Quantities, which conſtantly " tend to Equality during any finite Time, and before the End of that " Time come nearer to one another than by any given Difference, at " laſt become equal.

" If you deny it; let them be at laſt unequal, and let their laſt " Difference be D. Therefore they cannot come nearer to Equality " than by the given Difference D: contrary to the Hypotheſis."

8. If it ſhould be alledged by any, that, notwithſtanding this Demonſtration, the Quantities and Ratios mentioned in the Lemma may differ at laſt, although that Difference be leſs than any given or aſſignable Difference: ſuch would do well to conſider, that ſuch a way of reaſoning, being admitted, would overturn ſome of the fineſt Demonſtrations of the moſt accurate Geometricians among the Ancients themſelves, ſuch as *Euclid* and *Archimedes*, whoſe Works have undergone the ſtricteſt Scrutiny and Examination of the beſt Geometricians, ſince their Time to this very Day. For Example, How does *Archimedes* demonſtrate that a Circle is equal to a rectangular Triangle, having it's Baſe equal to the Circumference, and it's perpendicular Altitude equal to the Radius of the Circle [2]? He does it by ſhewing that a Circle can neither be greater nor leſs than ſuch a Triangle. But how does he prove this? By ſhewing that the Circle is neither greater nor leſs than the Triangle by any given Space. Again, *Euclid* demonſtrates [3] that Circles are to each other as the Squares of their Diameters, by ſhewing that the Square of the Diameter of the one Circle is to the Square of the Diameter of the other, neither as the firſt Circle is to a Space greater, nor yet to a Space leſs than the other Circle. But let us ſee what he means by a Space greater or leſs than the other Circle. Why, he means a Space differing from it by a given or aſſignable Difference; *i. e.* according to the Lemma premiſed to that Propoſition, ſuch a Difference, as, repeated a certain Number of Times, may exceed that Circle, as appears to any one that reads that Prop. and Lemma: which Lemma is the Foundation of the Method of Exhauſtions, made uſe of by *Euclid* and *Archimedes* in theſe and many other Propoſitions.

9. Now if any one ſhould object, that, notwithſtanding of what *Archimedes* and *Euclid* have demonſtrated in theſe Propoſitions, the Circle may be greater or leſs than the rectangular Triangle; and the Ratio of the Squares of the Diameters may be greater or leſs than the

[1] Lem. 1. of his Principles.
[2] See his Κυκλου Μετρησις.
[3] See Prop. 2. B. 12. of his Elements.

Ratio of the Circles, although not by any given or aſſignable Difference; yet by a Difference leſs than any given Difference: is not this the very ſame Objection as that raiſed againſt Sir *I. Newton*'s Lemma? And therefore, if it be of no Weight againſt *Euclid* and *Archimedes*; no more is it againſt our Author. But the Truth of the Matter is, that a Difference leſs than any Thing aſſignable, is the ſame Thing as no Difference at all: for repeat it as often as you pleaſe, it can never be equal to any finite Quantity: and therefore can bear no Ratio to it, by Def. 4. B. 5. Elements, conſequently it can be of no Importance to make the Thing greater or leſs [1].

10. Def. 1. The ultimate Ratio of Quantities is the Limit to which the Ratio of variable Quantities continually approaches during any finite Time, and juſt attains to at the End of the Time.

11. Def. 2. The ultimate Ratio of evaneſcent Quantities is the Limit to which the Ratio of variable Quantities diminiſhing without Bound, continually approaches, to come nearer to it than by any given Difference; but which it never goes beyond: yet no ſooner attains to, than the Quantities being diminiſhed infinitely, vaniſh.

12. Def. 3. The prime Ratio of Quantities is the Limit from which the Ratio of variable Quantities continually recedes; during ſome finite Time, and with which it coincides at the very Beginning of the Time: and is only a different Way of conceiving the ſame Thing as is contain'd in Def. 1.

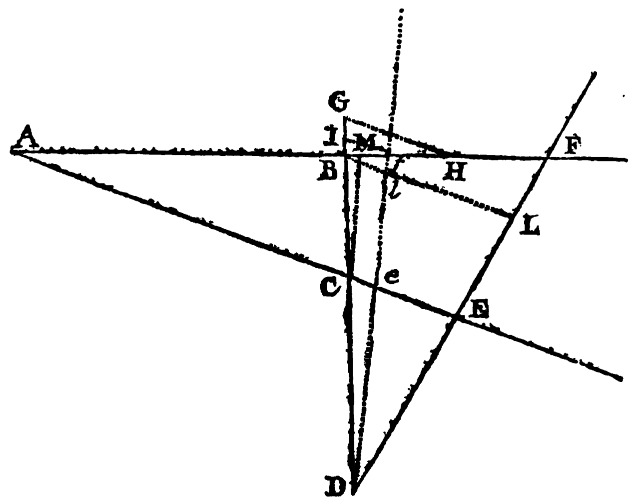
13. Def. 4. The prime Ratio of naſcent Quantities is the Ratio with which they ariſe; or it is the Limit from which their varying Ratio continually recedes; after the like Manner as the ultimate Ratio of evaneſcent Quantities was defined a Limit, to which their varying Ratio continually approaches in Def. 2. being only a different Way of conceiving the ſame Thing; although it cannot be ſo eaſily expreſs'd. The Whole is illuſtrated by what follows.

[1] *Archimedes* tells us in the Preface to his Treatiſe of the *Sphere* and *Cylinder*, that he aſſumes this for a Principle or Poſtulate.

Ἔτι δὲ τῶν ἀνίσων γραμμῶν, καὶ τῶν ἀνίσων ἐπιφανειῶν, καὶ τῶν ἀνίσων στερεῶν, τὸ μεταξὺ τοῦ ἐλάσσονος ὑπερέχειν τοιούτῳ, ὃ συντιθέμενον ἑαυτὸ ἑαυτῷ, δυνατόν ἐστιν ὑπερέχειν παντὸς τοῦ προτεθέντος τῶν πρὸς ἄλληλα λεγομένων.

And he mentions the ſame in the Preface to his Treatiſe of *Spiral Lines*. Likewiſe in the Preface to his *Quad.* of the *Parab.* he tells us, that in order to demonſtrate that a Segment of the Parabola is $\frac{4}{3}$ of a Triangle having the ſame Baſe and Altitude, he aſſumes this for a Foundation, that the Exceſs of unequal Spaces, whereby the greater exceeds the leſs, may be ſo often added to itſelf, as to exceed any propos'd finite Space. And he tells us, that Geometers before his Time had made uſe of the ſame Principle in demonſtrating, that Circles are to one another in the duplicate Ratio of their Diameters; that Spheres are in the triplicate Ratio of their Diameters; that a Pyramid is a third Part of a Priſm of the ſame Baſe and Altitude, &c.

14. Let the two right Lines AB, AC given in Position, and meeting in the Point A be interfected by a third right Line DCB, alfo given in Pofition, in the Points C and B: and let another right Line DEF revolve about the given Point D as a Pole, always cutting AC, AB in the Points E and F; and fuppofe it to approach DCB continually, until it at length fall in with it: then the Ratio of DC to CB is the ultimate Ratio of DE to EF. For the varying Ratio of DE to EF continually approaches to the given or invariable Ratio of DC to CB; fo as that before DEF fall in with DCB, the Ratio of DE to EF comes nearer to the Ratio of DC to CB than by any given Difference; or, which is the fame thing, than any other given or affignable Ratio does. For, produce DB unto G, fo as that the Ratio of DC to CG be as near the Ratio of DC to CB, as that of DE to EF can ever arrive to, if poffible: through G draw GH parallel to AC meeting AB in H: let DEF by revolving, come into the Pofition D*ef*, fo that *f* lye betwixt B and H: draw *f*I parallel to AC, meeting DB in I. Then, from parallel Lines, D*e* : *ef* :: DC : CI; but the Ratio of DC to CI is nearer to the Ratio of DC to CB, than that of DC to CG is; wherefore alfo the Ratio of D*e* to *ef* is nearer to the Ratio of DC to CB than that of DC to CG is, contrary to the Hyp. Whence it appears that the Ratio of DC to CB is the Limit to which the varying Ratio of DE to EF continually approaches, and comes nearer to than by any given Difference, and no fooner arrives at, than the Line DEF falls in with DCB; or, which is the fame, ceafes to move; therefore the Ratio of DC to CB, is * the ultimate Ratio of the variable Lines DE, EF.



* Art. 10. of this Explication.

15. If the right Line DEF be fuppos'd to move back again from DCB, where it is fuppos'd to begin to move; then the Ratio of DC to CB is the * prime Ratio of DE to EF.

* Art. 12. of this Explication.

16. Suppofe the fame Things as before: and let DEF, revolving upon D as a Pole, move towards DCB until it coincide with it, and the

the Points E and F coincide with the Points C and B, then I say the Ratio of the variable Lines CE, BF is continually changing: the compounded Ratio of DC to DB and AC to AB, is a Limit to which their Ratio continually approaches, and comes nearer to than by any given Difference; or than any other given Ratio doth; and that before the Points E and F fall in with C and B; but no sooner arrives at, than the variable Lines CE, BF being infinitely diminished, vanish; and the Points E and F fall in with C and B.

Through B draw BL parallel to AC meeting DF in L: then CE : BF * :: CE : BL [1] + BL : BF. Now CE : BL :: DC : DB *, and therefore constant or given; but BL : BF [2] is continually varying: for when DLF comes into any other Position D*lf*, if BL : BF :: B*l* : B*f*, LF would be parallel to *lf* *, which it never can be; and it's evident also that B*l* : B*f* is greater than BL : BF; wherefore CE : BF is continually varying and increasing until DEF fall in with DCB. 2°. If any given Ratio can be as near to the compounded Ratio of DC to DB and AC to AB as CE : BF comes to: let it be $DC \times AC : DB \times AH$ a less Ratio than $DC \times AC : DB \times AB$, because it has been shewn that CE : BF is always increasing; so that $AH > AB$. Assume the Point M any where in the Line AB, betwixt B and H, join CM: let DEF by revolving come into the Position D*ef* parallel to CM. Then C*e* : B*f* :: C*e* : B*l* + B*l* : B*f* :: DC : DB + AC : AM *; or C*e* : B*f* :: $DC \times AC : DB \times AM$; where for C*e* : B*f* > $DC \times AC : DB \times AH$, (because $DC \times AC : DB \times AM > DC \times AC : DB \times AH$,) contrary to the Hyp. 3°. CE : BF never goes beyond $DC \times AC : DB \times AB$; for it has been demonstrated that C*e* : B*f* :: $DC \times AC : DB \times AM < DC \times AC : DB \times AB$: But it approaches nearer to it than by any given Difference; and that, while CE and BF are yet of a finite Magnitude: therefore $DC \times AC : DB \times AB$ is the Limit at which CE : BF no sooner arrives, than CE and BF being infinitely diminished vanish; just while the Points E, F coincide with the Points C, B. So that $DC \times AC : DB \times AB$ is the ultimate Ratio of the continually diminishing and at last evanescent Lines CE, BF *.

* Def. 5. B.6.Elem.
* Prop. 4. B.6.Elem.
* Prop. 2. B.6.Elem.
* Prop. 4. B.6.Elem.
* Art. 11. of this Explication.

17\. If the right Line DEF be suppos'd to turn back again from DCB, where it's Motion begins, then the same Ratio, *viz.* $DC \times AC : DB \times AB$, is the prime Ratio of CE and BF considered as nascent *.

* Art. 13. of this Explication.

18\. If any one should object that there can be no ultimate Ratio of continually diminishing and at last evanescent Quantities: because

[1] By this way of Notation I mean a Ratio compounded of the two Ratios of CE to BL and of BL to BF, the same as that of $CE \times BL$ to $BL \times BF$.

[2] By this Notation I mean the Ratio of BL to BF.

before

before they vaniſh it is not the laſt; and after they vaniſh, they have no Ratio. The Anſwer is, that the ultimate Ratio is neither the Ratio of them before they vaniſh; nor after they vaniſh; but the Ratio wherewith they vaniſh, or the Limit to which their varying Ratio no ſooner arrives, than they vaniſh. If there was any Thing in this Objection, it would infer that when a falling Body is ſtopt in it's Motion, it has no laſt or ultimate Velocity: for the Velocity before it was ſtopt is not the laſt, and the Velocity, after it is ſtopt, is none at all. But every one may ſee, that by the laſt Velocity is meant neither the one nor the other of theſe; but the Velocity it has at that very Inſtant it ſtops, which it does not arrive at before it is ſtopt; and no ſooner arrives at but it is ſtopt. For ſince it moves with a continually accelerated Velocity, it has a different Velocity for every different Inſtant of Time, and therefore at that Inſtant it ſtops, it has acquired a Velocity different from the Velocity it had at any other Inſtant of Time. And the ſame is the Caſe of the ultimate Ratio of evaneſcent Quantities, whoſe Ratio is continually varying; it is that Ratio they have that very Inſtant they vaniſh: for, ſince they are ſuppoſed to have a different Ratio for every different Inſtant of Time, they muſt have a certain determinate Ratio, with which they vaniſh, otherwiſe they never can vaniſh, contrary to the Hypotheſis: and that is the ultimate Ratio of the evaneſcent Quantities.

19. It ſignifies nothing to ſay ultimate Quantities cannot be aſſigned, in regard Quantity is diviſible without End: for it is not the Quantities themſelves that are hereby determined, but only their Ratio: which is capable of being determined, as has been already ſhewn *.

* Art. 16. of this Explication.

20. As theſe are the only Objections that can be rais'd againſt this Doctrine of prime and ultimate Ratios, the noble Author was too clear ſighted not to perceive them: and accordingly he has fully anſwered them elſewhere [1], to ſo great Satisfaction to every intelligent and unprejudiced Reader, that the great Duſt which has been raiſed of late about the Whole of this Doctrine, muſt be owing to Weakneſs, or ſome worſe Principle.

21. Having endeavoured to clear up the Meaning of prime and ultimate Ratios, and remove the Objections that are brought againſt that Doctrine: that I may clearly demonſtrate the fundamental Propoſition concerning the Proportions of Fluxions, laid down by our Author, and formerly mentioned *, I ſhall premiſe ſome Lemmas reſpecting Motion.

* Art. 3. of the Quadrature of Curves. Art. 5. of this Explication.

[1] See Schol. to Sect. 1. of his Principles; and Lem. 2. B. 2.

LEMMA

LEMMA I.

22. If the right Line AC be described by the Motion of the Point B, with a Velocity continually accelerated, the Spaces described in equal Times will still increase, as the Time from the Beginning of it's Motion increases. For if the Motion was equable or uniform, the

Spaces describ'd in equal Times would be equal, therefore when it is continually accelerated, the Spaces describ'd in equal Times must still increase, the further the Time is from the Beginning of the Motion.

LEMMA II.

23. If the Motion of the Point B be continually retarded, the Spaces described in equal Times will still diminish.

This is also evident by the same way of reasoning.

LEMMA III.

24. If the Point B move with a Motion continually accelerated or continually retarded, it's Velocity at any one Instant of Time is different from what it is at any other Instant of Time.

This is evident from the very Notion of a Motion continually accelerated, or continually retarded: for such Motion cannot continue the same for any the least Time.

LEMMA IV.

25. If the Motion of the Point B be continually accelerated or continually retarded, in such manner, as that the variable Velocity receive finite Additions or Diminutions, only in finite Times; and the Velocities at any two Instants of Time be taken, during which the Point B moves, it must pass through all possible intermediate Degrees of Velocity lying betwixt the two Velocities assumed, in the Space of Time lying betwixt the two Instants assumed. For let V denote any intermediate Velocity whatsoever, through which, if possible, the variable Velocity mentioned in this Lemma doth not pass; let M and N denote two Velocities, one on each Side of V, and the nighest to it which the variable Velocity mentioned in the Lemma passes through, then the Difference betwixt M and N is a finite Difference, (since there is a Velocity V assigned betwixt them) consequently the variable Velocity receives a finite Addition or Diminution in a Time less than any assignable, contrary to the Hyp. Therefore, *&c.*

LEMMA V.

26. If the Space or Augment B*b* be deſcribed by the Point B, moving with a Velocity continually accelerated or continually retarded, in any Time, an uniform Velocity capable of deſcribing the Space B*b* in the ſame Time will fall in betwixt the two extreme Velocities, wherewith the Point B moves at the Beginning and End of the Space or Augment B*b*.

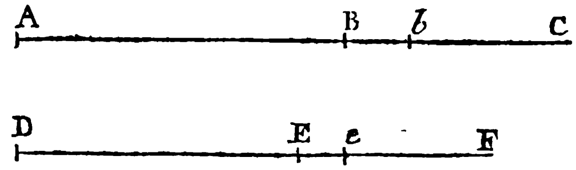
For when the Motion is accelerated, the Velocity with which the Point B moves at the Beginning of the Space B*b*, continued the ſame, would not be capable of deſcribing ſo great a Space as B*b*, in the ſame Time it is deſcribed by the accelerated Motion: but the Velocity with which B moves at the End of the Space B*b*, would produce a greater Space than B*b* in that Time. Therefore an equable or uniform Velocity capable of deſcribing B*b* in the ſame Time it is deſcribed by an accelerated Velocity, muſt fall in betwixt the two, *i.e.* it muſt be greater than the firſt and leſs than the ſecond, ſince, when uniform Velocities or equable Motions are compared, and the Time given, the Velocities are as the Spaces deſcribed directly. And it is evident, that the like way of reaſoning may be applied to the Caſe of the continually retarded Motion: whence the Truth of the Lemma appears.

LEMMA VI.

27. If the Point B move with an uniform Motion, along the Line AC, and the Point E move along the Line DF with a Motion continually accelerated or continually retarded: and B*b* and E*e* be Spaces deſcribed or Augments generated by the Motion of theſe Points in the ſame or equal Times: the leſs the Spaces or Augments B*b*, E*e* be taken, or, which is the ſame thing, the leſs the Time be in which they are deſcribed, the nearer will the Ratio of B*b*, E*e* approach to the Ratio of the Velocities with which the Points B and E move at the Beginning of the Spaces or Augments: and by diminiſhing B*b* and E*e* more and more, their Ratio ſhall come nearer to the Ratio of theſe Velocities than by any given or aſſignable Difference: and therefore the ultimate Ratio of theſe Spaces or Augments B*b*, E*e* as they vaniſh (which is the ſame with their prime Ratio, conſidered as naſcent) will be the Ratio of the Velocities mentioned.

The leſs the Space or Augment E*e* is, and conſequently, the leſs the Time is, in which it is deſcribed, the nearer will the Velocities, with which the Point E moves at the Beginning and End of that Space, approach

approach to one another: and by diminishing the Augment E*e*, or the Time continually, these extreme Velocities may be made to approach nearer to one another than by any given Difference, as appears by Lem. 3; therefore much more, the uniform Velocity capable of producing E*e* in the same Time, will still approach nearer and nearer to the Velocity with which the Point E moves at the Beginning of the Space E*e*, the less that Space, or the Time wherein it is described, is: and by continually lessening that Space or Time, the uniform Velocity shall differ from the Velocity, with which the Point E moves at the Beginning of the Augment E*e*, by less than any given Difference; since the uniform Velocity falls in betwixt the two extreme Velocities *. If then we put $\dot{B}$ for the Velocity with which the Point B moves, which is supposed to be constant or invariable; $\dot{E}$ for the Velocity with which Point E moves at the Beginning of the Space E*e*; and V for an uniform Velocity capable of describing E*e* in the same Time it is described by the variable Velocity $\dot{E}$: then from the Nature of uniform Motion, B*b* : E*e* :: $\dot{B}$: V; but the less that E*e* is taken, the less is the Difference betwixt V and $\dot{E}$ *; so that when the Augment E*e* is very small, it is nearly $\dot{B}$: V :: $\dot{B}$: $\dot{E}$, consequently, when the Spaces or Augments B*b*, E*e* are taken very small, or such as are describ'd in very small and equal Times, it will be nearly B*b* : E*e* :: $\dot{B}$: $\dot{E}$. Finally, since by continually diminishing E*e* and consequently B*b*, V approaches nearer to $\dot{E}$ than by any given Difference, it follows that B*b* : E*e* approaches nearer to $\dot{B}$: $\dot{E}$ than by any given Difference: and therefore the ultimate Ratio of the evanescent Augments or Spaces B*b*, E*e*, or, which is the same, the prime Ratio of them considered as nascent, is the Ratio of $\dot{B}$ to $\dot{E}$. * Which Things were to be demonstrated.



* Art. 26. of this Explication.

* Art. 26. of this Explication.

* Art. 7. of this Explication.

28. These Things being demonstrated, the Truth of the fundamental Proposition laid down by our Author *, will easily appear. For, Fluxions being defined *the Velocities of the Motions or Increments by which the flowing Quantities are generated*, suppose any one Quantity to flow uniformly, *i. e.* to have a constant or invariable Fluxion at all Times; and another Quantity of the same kind to flow according to any Law whatsoever, whether by an uniform Velocity, or a Velocity continually varying; and that at any rate, whether continually accelerated, or continually retarded: and let there be taken Increments of

* Art. 3. of the Quadrature of Curves.

 these

these flowing Quantities, generated in the same or equal Terms: then in the preceding Fig. AB and DE represent the flowing Quantities, whatever the Nature of these Quantities be (just as all kind of Quantities are represented by right Lines in the fifth Book of *Euclid*'s Elem.) B*b*, E*e* the cotemporaneous Increments. And therefore it appears from the Nature of Motion, 1°, That if the Fluents flow with constant or invariable Fluxions, the Fluxions are to one another as the Increments generated in any equal Times. And from what has been demonstrated, 2°, That if the one Fluent have a constant Fluxion, and the other a Fluxion continually varying, and their Augments be taken such as are produced in very small and equal Times, the Fluxions will be nearly as these Augments. 3°, In every Case accurately, the Fluxions are in the prime Ratio of the nascent Increments, or the ultimate Ratio of the evanescent Decrements, according as the flowing Quantities are supposed to increase or decrease. And further, because this is true when one of the Fluxions is given or constant, it follows that the same is true, when the Fluxions are all variable. For let P, Q denote any two homogeneous flowing Quantities, whose Fluxions are variable: A, any other Fluent of the same kind, which has an invariable Fluxion: let these Fluxions be denoted by the Symbols $\dot{P}$, $\dot{Q}$, $\dot{A}$ respectively; and let p, q, a denote the prime Ratios of their nascent Augments. Then, by what has been demonstrated $\dot{P} : \dot{A} :: p : a$ and $\dot{A} : \dot{Q} :: a : q$, therefore by Equality $\dot{P} : \dot{Q} :: p : q$, as was to be shewn.

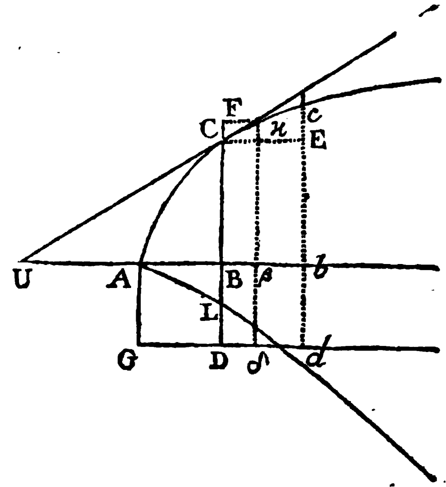
Art. 4. of the Quadrature of Curves.

29. After laying down what we have been about demonstrating, touching the Proportion of Fluxions, our Author proceeds to shew the Application thereof, in determining the Fluxions of curvilinear Areas, in this Art.; and the Fluxions of the Abscisses, Ordinates and Curve Lines in the two succeeding Art. Now, what is laid down in Art. 4. may be thus demonstrated. Supposing the same Things as are there mentioned, I say F.ABC [1] : F.ABDG :: BC : BD, when ABDG is a Parallelogram. To demonstrate which, we need only shew that the prime Ratio of the nascent Increments, when the Areas are supposed to flow by increase; or the ultimate Ratio of the evanescent Decrements, when they are supposed to flow by decrease, is the Ratio of BC to BD. Let BC, BD, constituting one right Line, move forward, and come into the new Position *bc*, *bd*; so that the Increments generated in the same Time be B*bc*C and B*bd*D. Then it is evident, that, when *cbd* has got to any finite Distance from CBD, B*bc*C : B*bd*D $>$ BC : BD; for by completing the Parallelogram BCE*b*, it is BC : BD :: BCE*b* : B*bd*D * $<$ B*bc*C : B*bd*D: but I say B*bc*C : B*bd*D may be nearer to BC : BD,

* Prop. 1. B.6. Elem.

[1] F.ABC denotes the Fluxion of the Area ABC: and so of other like Notations.

BD, than by any given Difference; or than any given or assignable Ratio, by supposing *bc* to be still nearer and nearer to BC. For if possible let BF : BD be as near to BC : BD as B*bc*C : B*bd*D can be : draw Fκ parallel to AB meeting the Curve in κ; complete the Parallelograms BβκF and BβδD. Then BβκF : BβδD > BβκC : BβδD, but BβκF : BβδD :: BF : BD, therefore BF : BD > BβκC : BβδD, wherefore BβκC : BβδD is nearer to BC : BD than BF : BD is (since BβκC : BβδD can never be less than BC : BD, as was shewn) contrary to the Hyp. Therefore BC : BD is the prime Ratio of the nascent Augments B*bc*C and B*bd*D *. And it is the same thing to consider B*bc*, B*bd*D as Decrements, by supposing *cbd* to return back again into the Position CBD: for then it is demonstrated the same way that BC : BD is the ultimate Ratio of B*bc*C and B*bd*D when vanishing. But the Fluxions of the Areas ABC and ABDG are to one another in the prime Ratio of the nascent Increments *, or ultimate Ratio of the evanescent Decrements B*bc*C, B*bd*D *. Therefore F.ABC : F.ABDG :: BC : BD, when ABDG is a Parallelogram, as was to be shewn.



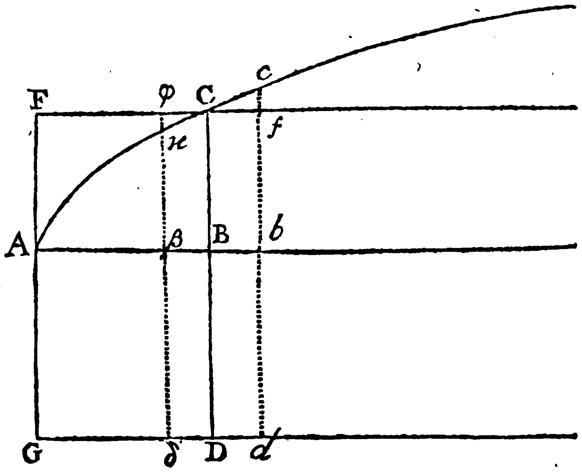
* Art. 13. of this Explication.

* Art. 3. and 6. of the Quadrature of Curves. Art. 28. of this Explication.

30. But further, although both Fluents were curvilinear Areas, I say their Fluxions shall be as the Ordinates, when the Ordinates do move with the same uniform Motion along the Absciss AB. For let ABL be another curvilinear Area described by the Ordinate BL, then, by what has been demonstrated, F.ABC : F.ABDG :: BC : BD and F.ABDG : F.ABL :: BD : BL, therefore by Equality, F.ABC : F.ABL :: BC : BL.

31. Although this Method of investigating and demonstrating the Proportions of the Fluxions of plain Figures, be laid down by Sir *I. Newton* in this Place; and, as I have shewn, is both certain and geometrical: yet such perhaps may be the Scruples of some; and such the Obstinacy of others, against the whole of this Doctrine of prime and ultimate Ratios of nascent and evanescent Quantities, that I hope it will not be unacceptable to the Reader, if I shall shew that the Proportions of the Fluxions of plain Figures (and consequently by Analogy of all other flowing Quantities) may be demonstrated in another manner, upon the most indisputable Principles of Geometry, without introducing either infinitely little Quantities, as has been done, without sufficient Caution,

by ſome; but purpoſely avoided by Sir *I. Newton* in this Treatiſe; or yet naſcent and evaneſcent Quantities, with their prime and ultimate Ratios, upon which Foundation he has built this Doctrine.



* Art. 4. of the Quadrature of Curves. and Art. 29. of this Explication.

Let ABC and ABDG be a curvilinear Area and Parallelogram deſcribed as before, the ſame things being ſuppos'd as formerly *: and further let ABCF be another Parallelogram deſcrib'd at the ſame time by a given Ordinate equal to AF, ſo that all the three Areas ABC, ABDG and ABCF be deſcribed by the uniform Motion of their reſpective Ordinates, lying always in the ſame right Line: then I ſay F.ABC : F.ABDG :: BC : BD.

For ſince the Ordinates of the Parallelograms ABDG, ABCF are given or invariable right Lines, and theſe Lines are ſuppoſed to move uniformly along the common Abſciſs AB, the Fluxions or Velocities of flowing of the Parallelograms muſt be conſtant and invariable, for the Spaces generated in equal Terms are evidently equal in the reſpective Parallelograms; and the Fluxions or Velocities of flowing in that Caſe are as the Increments generated in any the ſame times *, or by Compoſition as the whole Areas generated, that is F.ABCF : F.ABDG :: (ABCF : ABDG ::) BC : BD *, BC being conſidered as the invariable Ordinate of ABCF. But the Velocity with which the curvilinear Area ABC is deſcribed muſt be leſs than the Velocity with which the Parallelogram ABCF is deſcribed at any Time before the Ordinate BC arrive to it's preſent Poſition, as when it is in $\beta\kappa\phi$, becauſe $\beta\kappa < \beta\phi$: again, the Velocity with which ABC flows, at any time after BC has paſt it's preſent Poſition, as when it is in *bfc*, is greater than the Velocity with which ABCF flows, ſince $bc > bf$: and theſe things are true, let $\beta\kappa\phi$ and *bfc* be as near to BC as you pleaſe, becauſe the Curve AC and right Line FC interſect each other in one Point only: but the Velocity with which ABC flows, is continually or inceſſantly varying, ſince it's Ordinate BC never continues of the ſame Length, for any the leaſt Time. Wherefore, betwixt the two Inſtants of Time that the Ordinate BC has the Poſitions $\beta\kappa$ and *bc*, the variable Velocity

* Art. 28. of this Explication.

* Prop. 1. B.6.Elem.

with which the Area ABC flows muſt paſs through all poſſible intermediate Degrees *, therefore alſo through that Degree of Velocity with which ABCF flows, which is invariable and intermediate: But it has been demonſtrated, that this cannot happen at any Time, when the Ordinate BC is out of the preſent Poſition, therefore it muſt happen, when the Ordinate BC obtains the preſent Poſition: But the Fluxions of flowing Quantities are the Velocities with which they flow or increaſe *: therefore, when the Ordinate BC comes to the preſent Poſition, the Fluxions of the curvilinear and rectilinear Areas ABC and ABCF are equal: but, as was already ſhewn, F.ABCF : F.ABDG :: BC : BD, therefore F.ABC : F.ABDG :: Ord. BC : BD.

* Art. 25. of this Explication.

* Art. 2. of the Quadrature of Curves.

32. From this, we may conclude as formerly *, that when the flowing Areas are both Curvilinear, their Fluxions are to one another as the Ordinates.

* Art. 30. of this Explication.

33. And thus we have demonſtrated that the Fluxions of all plain Figures, deſcribed by the equable Motion of their Ordinates always lying in the ſame variable right Line, and moving along a common Abſciſs, are every where as their reſpective Ordinates; and that without introducing either infinitely ſmall Quantities, or yet the Doctrine of prime and ultimate Ratios of naſcent and evaneſcent Quantities. And therefore, the Foundation upon which the Doctrine of Fluxions is built, would ſtill remain unſhaken, although the Doctrine of prime and ultimate Ratios could be ſhewn to be uncertain and precarious; which yet all Sir *I. Newton*'s Adverſaries ſhall never be able to do.

From what has been ſaid the following Corollaries may be deduced.

34. Cor. 1. The Fluxion of any plain Figure, generated by the uniform Motion of it's Ordinate along the Abſciſs, is every where proportional to the Ordinate.

35. Cor. 2. Although our Author has hitherto mentioned no Motion of the Ordinates but what is uniform and the ſame along the Abſciſſes, or rather common Abſciſs: yet the Motion of the Ordinates along the Abſciſſes of different Curves, *i. e.* the Fluxions of the Abſciſſes of different Curves, may be ſuppoſed different; yea and the Fluxion of the Abſciſs of the ſame Curve may be ſuppoſed variable, by being accelerated or retarded, and that according to any Law. But I ſay in every Caſe, the Fluxions of the Areas of plain Figures, deſcribed by the parallel Motion of their Ordinates along the Abſciſſes, are always to one another as the Rectangles contained under the Ordinates and the Fluxions of the reſpective Abſciſſes. For by what has been demonſtrated, when the Fluxions of the Abſciſſes are uniform and the ſame in all, the Fluxions of the Areas are as the Ordinates directly: but if the Fluxion of

the Abſciſs be increaſed or diminiſhed in any Proportion, the Velocity with which the Area increaſes, *i. e.* it's Fluxion muſt be increaſed or diminiſhed in the ſame Proportion; ſince the Space that would be deſcribed with that Velocity in any given Time, would be increaſed or diminiſhed in that Proportion: and it is no matter, whether the Fluxion of the Abſciſs of the ſame Figure continue invariably the ſame, or vary according to any Law, by being accelerated or retarded; for it is only the preſent Fluxion of the Area: therefore it follows that the Fluxions of any plain Figures, are always as the Rectangles contained under the Ordinates of theſe Figures, and the Fluxions of their Abſciſſes, or right Lines proportional to and expounding the Fluxions of theſe Abſciſſes.

36. The ſame thing may be more ſhortly demonſtrated by means of Symbols thus, as is done by our Author elſewhere [1], in the foregoing Figure, let the Ordinates BC, BD move at right Angles along the Abſciſs AB; let AG or BD be equal to Unity; the common Abſciſs AB $= x$, the curvilinear Area ABC applied to BC $= z$, or ABC $= z \times 1 = z$, then is ABDG $= (x \times \text{BD} = x \times 1 =)\ x$. Now putting $\dot{x}$ and $\dot{z}$ to ſtand for the Fluxions of x and z reſpectively, by what has been demonſtrated, $\dot{z} : \dot{x} :: \text{BC} : \text{BD} = 1$: for it is all one whether the Fluxion of AB, *i. e.* $\dot{x}$ continue invariably the ſame, or vary according to any Law: therefore by reducing the Analogy to an Equation $\dot{z} = \left(\frac{\dot{x} \times \text{BC}}{\text{BD}} = \frac{\dot{x} \times \text{BC}}{1} =\right) \dot{x} \times \text{BC}$, *i. e.* F.ABC $=$ F.AB $\times$ BC, *viz.* the Fluxion of the Abſciſs multiplied into the Ordinate, *i. e.* in other Words, if the Ordinate of any plain Figure, continuing the ſame in Length, that it is in any given Place, were ſuppoſed to be carried perpendicularly along the Abſciſs with an uniform Velocity, equal to the Velocity with which the Abſciſs increaſes at that particular Place, the ſuperficial Space thereby produced in any given Time, would be equal to the ſuperficial Space produced or deſcribed in that ſame Time, by the Velocity with which the Area flows or increaſes at that Place, continuing uniformly the ſame.

37. But ſuppoſe the Ordinates not to move perpendicularly along their Abſciſſes; but in any given Angle, provided it be the ſame in all, the Fluxions of the flowing Areas will ſtill be proportional to the Rectangles contained under the Ordinates and Fluxions of the Abſciſſes. For in the Demonſtration of our Author's fourth Art. the Ordinates BC, BD may make any given Angle with the Abſciſs AB *: therefore it's eaſy to apply the Reaſoning at Art. 35. to this Caſe.

* Art. 35. of this Explication.

[1] See his Method of Fluxions and infinite Series, publiſhed by *J. Colſon*, pag. 23.

38. Cor.

38. Cor. 3. If two or more plain Figures begin to be deſcribed at the ſame Time, by their Ordinates moving along their Abſciſſes in a given Angle, and ſo be deſcribed in the ſame Time; or, which comes to the ſame purpoſe, if they be deſcribed in equal Times, and the Rectangles contain'd under the Ordinates and the Fluxions of the Abſciſſes or right Lines proportional to and expounding the Fluxions of the Abſciſſes, be equal in theſe Figures, at every the ſame or correſpondent Inſtant of their flowing or increaſing, throughout the whole Time of their Deſcription, the Areas deſcribed ſhall be equal. For theſe Rectangles are proportional to the Fluxions of the Areas *, and therefore the Fluxions of the Areas are always equal to each other, *i. e.* the Velocities with which the Figures are deſcribed, are every where equal to each other: and therefore the Areas produced or deſcribed with theſe Velocities in the ſame or equal Times muſt be equal, from the Nature of Motion.

* Art. 35. and 37. of this Explication.

It would be the ſame thing, if the Ordinates of the Curves and the Fluxions of their Abſciſſes were reciprocally proportional: for then would the Rectangles mentioned be equal *.

* Prop. 14. B. 6. Elem.

39. Cor. 4. Converſely if the flowing Areas be every where equal to each other, the Rectangles contain'd under the Ordinates and the Fluxions of the Abſciſſes, ſhall be every where equal to each other: for, if not, the Fluxions of the plain Figures, *i. e.* the Velocities with which they flow or increaſe, would not be equal: and if the Velocities of Deſcription were not every where equal, the Areas deſcribed could not be every where equal.

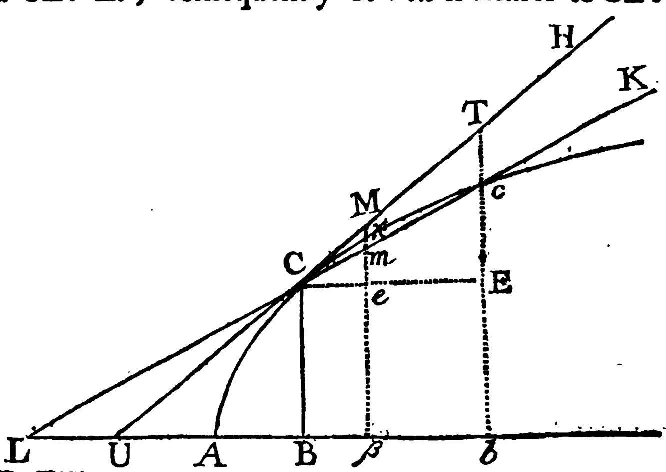
40. Schol. It may be here remarked, with reſpect to variable or flowing Quantities of any kind whatſoever, that, if they be at all times equal to each other, the Velocities of their Mutation, *i. e.* the Fluxions with which they flow at any the ſame Time, muſt alſo be equal: converſely, if the Velocities of Mutation or Fluxions of any variable Quantities be always equal, and moreover theſe Quantities ariſe together, when they flow by increaſe; or vaniſh together, when they flow by decreaſe, the variable Quantities ſhall always be equal at the ſame Time. For theſe variable or flowing Quantities may be repreſented and expounded by the flowing plain Figures; and their Velocities of Mutation, by the Fluxions of theſe Figures, mentioned in Cor. 3. and 4. Wherefore, ſince the Fluxions of theſe equal Figures are always equal; and converſely the Fluxions being equal, the flowing plain Figures are equal; hence the variable Quantities of any kind, and their Velocities of Mutation or Fluxions muſt be equal. Which is likewiſe evident in itſelf from the Nature of the Thing.

Moreover any variable or flowing Quantities, which are in a given or conſtant Ratio, muſt have their Fluxions in the ſame given Ratio: and converſely variable Quantities, whoſe Fluxions are always in a given Ratio; if ſo be the Quantities ariſe together, or vaniſh together as before, are in the ſame given Ratio themſelves: as is ſufficiently evident from what hath been ſaid.

* Art. 5, 6. of the Quadrature of Curves.

41. Suppoſing the ſame Things as are ſuppoſed by our Author, * I ſay the prime Ratio of the naſcent Augments B*b*, E*c* and C*c*, *viz.* of the Abſciſs, Ordinate and Curve-line AB, BC and AC; or the ultimate Ratio of theſe Lines conſidered as evaneſcent, is that of the Sides of the Triangle CET; or of the Triangle VBC ſimilar to it, *viz.* CE, ET and CT, or VB, BC and VC the Subtangent, Ordinate and Tangent.

For through any Point as *m* of the Subtenſe C*c*, lying within the Curve, draw the Ordinate, βx parallel to BC or *bc*, meeting CE in *e*; and produce it, till it meet the Tangent VC in M: then $Ce : ex > (Ce : eM ::)$ CE: ET; but $Ce : ex < (Ce : em ::)$ CE: E*c*: therefore $Ce : ex$ falls in betwixt CE: ET and CE: E*c*; conſequently $Ce : ex$ is nearer to CE: ET than CE: E*c* is. Wherefore the nearer *bc* is to BC, the nearer doth the Ratio of the Increments B*b* and E*c* approach to the Ratio of CE to ET or of VB to BC, but ſo as that, when the Points B and *b* are at any finite Diſtance from one another CE: E*c* $>$ CE: ET.



Yet the Ratio of theſe Augments may be nearer to the Ratio of CE to ET, than any other aſſignable Ratio. For let the Point L be aſſumed in AB produced beyond V, ſuch as that LB: BC be a Limit, if poſſible, nearer than which the variable Ratio of the Augments B*b*, E*c*, can never approach to CE: ET or VB: BC: where LB muſt be greater than VB, becauſe CE: E*c* $>$ (CE: ET ::) VB: BC, while there is yet any Diſtance betwixt BC and *bc*. Draw LC and produce it, beyond C; then muſt it enter within the Curve, otherwiſe a right Line would fall in betwixt the Curve and it's Tangent, which is impoſſible: through any Point *m*, of the Line LC produced, lying within the

the Curve draw the Ordinate $\beta m\kappa$, meeting CE in e; and VC in M. Then Ce : $e\kappa$ > (Ce : eM : :) * VB : BC. Again, Ce : $e\kappa$ < (Ce : em : :) * LB : BC: therefore Ce : $e\kappa$ falls in betwixt VB : BC and LB : BC, and consequently Ce : $e\kappa$ comes nearer to VB : BC or CE : ET than LB : BC does, cont. Hyp. Therefore there is no assignable Ratio, which is so near the Ratio of VB to BC or CE to ET, as the Ratio of the Augments Bb, Ec may sometime or another be. Wherefore CE : ET or VB : BC is the prime Ratio of the nascent Augments Bb, Ec; or, which is the same, it is the ultimate Ratio of these Parts, when vanishing. * Prop. 3. B. 6. Elem.

Now with respect to the curvilinear Augment, Cc; let the Ordinate bEc approach continually towards BC, and at length fall in with it, when the Points C, c and E, and likewise B and b coalesce; in the mean time let the Tangent CT and Chord Cc be supposed to be produced still to distant Parts H and K. Then 'tis evident the Angle HCK made by the Tangent and Subtense still diminishes, as the Arch Cc diminishes, or as bEc approaches to BC: because, as has been already demonstrated, the varying Ratio of Ec to ET is still increasing, and the Triangle CET is given in Species: but it has been shewn likewise, that the ultimate Ratio of CE to Ec is that of CE to ET; and that therefore the ultimate Ratio of Ec and ET is a Ratio of Equality; and consequently the ultimate Ratio of the three Triangles CcE, CκcE and CTE is a Ratio of Equality likewise; and as to their Form, they are at last similar. Therefore, since the Triangle CET is given in Species, and always similar to the Triangle VBC, it follows that the prime or ultimate Ratio of the nascent or evanescent Parts Bb, Ec and Cκc, whereby the Absciss, Ordinate and Curve-line are increased or diminished, is that of CE, ET and CT or VB, BC and VC. Consequently the Fluxions of the Abscis, Ordinate and Curve-line are proportional to these Lines *, and may be expounded by them: as our Author affirms. * Art. 28. of this Explication.

42. Schol. From the Relations of the Fluxions of the Abscis, Ordinate and Curve-line just now demonstrated, are derived two principal Branches of the general and extensive Doctrine of Fluxions, *viz.* the Method of drawing Tangents to Curves: and the Rectification of Curve-lines, or determining their Lengths. The first belongs to the *direct*; the other to the *inverse* Method of Fluxions. The one depends upon this, that, if you call the Abscis AB $= x$, the rectangular Ordinate BC $= y$, and Curve-line AC $= z$; and denote their Fluxions by the Symbols $\dot{x}$, $\dot{y}$ and $\dot{z}$, then $\dot{y} : \dot{x} :: y :$ VB, that is VB $= \frac{\dot{x}}{\dot{y}} \times y$: the other upon this that $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$ or $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$: both of which are

 evident

evident from what has been ſaid: ſince the evaneſcent Triangle CE*cx*C is ultimately ſimilar to the right-angled Triangle VBC: and the ultimate Ratio of the evaneſcent Augments is the ſame with the Ratio of the Fluxions.

* Art. 7. of the Quadrature of Curves.
* Of this Explication.

43. The Deſign of this Art. is to ſhew how the Fluxions of ſolid Figures; and alſo the Fluxions of their Curve-ſurfaces may be inveſtigated and determined. In order to which, ſuppoſe, in the Fig. at Art. 29 *, a Circle deſcribed with BC, the Ordinate of the Curve ABC, for Radius; and that that Circle moves along the Abſciſs AB with an uniform Motion, keeping always perpendicular to it: the moving Circle will generate a ſolid Figure of ſuch a Nature as correſponds to the plain Figure, of which the right Line BC, the Radius of the generating Circle, is the Ordinate perpendicularly applied to the Abſciſs AB: *i. e.* a Solid, which may be ſuppoſed to be otherwiſe generated by the Rotation of the plain Figure ABC about it's Axis AB. Now, to illuſtrate and demonſtrate what our Author lays down in this Place, ſuppoſe another Circle, having the given or invariable Line AG or BD for it's Radius, to be drawn along the ſame Abſciſs AB, and by that means generate a Cylinder: let the generating Circles begin to move from A at the ſame Time, and be found ever after in the ſame Plane at the ſame Time; juſt as the right Lines BC and BD were formerly ſuppoſed to lye always in the ſame Direction. Then by what was formerly demonſtrated *, the Fluxions of the Solid ABC and Cylinder ABDG are in the prime Ratio of their naſcent Increments. Let B*bc*C and B*bd*D be the ſolid Augments generated in the ſame Time: and then, as it was formerly demonſtrated, that the prime Ratio of the Augments B*bc*C and B*bd*D, *in ſtatu naſcenti*, when belonging to the plain Figures, was that of BC to BD; ſo now, by the like way of reaſoning, we may demonſtrate that the prime Ratio of the Augments B*bc*C and B*bd*D, *in ſtatu naſcenti*, when conſidered as belonging to the ſolid Figures, is that of the Circles having BC and BD for their Radius's; or which is the ſame, that of BC^q to BD^q (ſince Circles are as the Squares of their Radius's). For the varying Ratio of the ſolid Augments B*bc*C and B*bd*D, by ſuppoſing the Diſtance B*b* ſtill leſs and leſs, comes nearer to the Ratio of BC^q to BD^q, than any other given or aſſignable Ratio; but can never go beyond it. For let BC be produced to F, ſo that, if poſſible, $BF^q : BD^q$ may be as near to $BC^q : BD^q$ as B*bc*C : B*bd*D can in any caſe be. And, having made the ſame Conſtruction as at Art. 29, ſuppoſe B*b*EC and B$\beta\kappa$F to repreſent two ſmall Cylinders, having the right Lines BC or *b*E and BF or $\beta\kappa$ for the

* Art. 28. of this Explication.

the Radius's of their circular Bases. Now 'tis evident that when B*b* is of any finite Length, B*bc*C : B*bd*D > (B*b*EC : B*bd*D : :) * BC*q* : BD*q*; therefore I have supposed BF greater than BC. But since Bβ*κ*C < Cyl. Bβ*κ*F, we have Bβ*κ*C: Bβ*δ*D < (Bβ*κ*F : Bβ*δ*D : :) * BF*q* : BD*q*: Therefore Bβ*κ*C : Bβ*δ*D is less than BF*q* : BD*q*; but it was shewn that it cannot be less than BC*q* : BD*q*, wherefore it comes nearer to BC*q* : BD*q* than BF*q* : BD*q* is, contrary to the Hyp. Therefore the prime Ratio of the nascent Augments B*b*C and B*bd*D is that of BC*q* to BD*q* *; or, which is the same, that of the Circles having BC and BD for their Radius's. Wherefore the Fluxions of the Solid ABC and Cylinder ABDG are in the same Ratio *. Whence it's evident (by considering what was said in Art. 30.) that the Fluxions of all solid Figures, generated after this manner, the Fluxions of whose Abscisses are every where equal and uniform, are to one another in every Place, as the Circles, by whose Motion they are generated, are in that Place; or as the Squares of their Radius's: as our Author affirms.

* Prop. 2. B. 12. Ele.

* Art. 11. and 13. of this Explication.

* Art. 28. of this Explication.

44. Cor. Hence it will follow, that, although the Motion of the generating Circles along the Abscisses be different and unequal, *i. e.* although the Fluxions of the Abscisses be different in different Solids; and even variable in the same Solid, yet universally, the Fluxions of Solids, generated by drawing the Circles along the Abscisses, are in a compounded Ratio of the generating Circles, and the Fluxions of the Abscisses: as may easily be demonstrated, by considering what was said at Art. 35. *

* Of this Explication.

45. The other Thing to be demonstrated in this Art. is, that the Fluxion of the Curve-surface is as the Periphery of the generating Circle, having it's Radius BC, and the Fluxion of the Curve-line AC conjunctly; or in a compounded Ratio of these two. To illustrate and demonstrate which: suppose the Curve-line AC, and the Circumference of the Circle of which BC is Radius, to be both extended into right Lines; so as the latter, being made to move at right Angles along the former, may thereby generate or describe a plain Figure having the right Line equal to AC for it's Absciss; and the other, equal to the Perimeter of the Circle, whose Radius is BC, for it's Ordinate: and suppose the Velocity with which the rectilinear Abscis flows, or it's Fluxion, to be always the same with the Velocity or Fluxion of the Curve-line AC; and it's Ordinate in every Place, equal to the Circumference of the Circle, having for it's Radius BC, in the corresponding Place; then it's evident that the Curve and plain Surfaces, generated in the same time, will be equal; and that therefore their Fluxions, at the

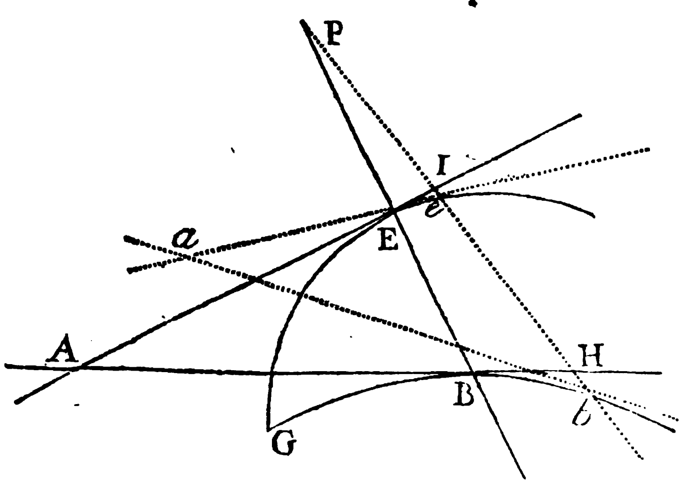
Art. 40. of this Explication. Art. 35. of this Explication.

the same Time, will be equal *. But the Fluxion of the plain Surface is always as the Ordinate and Fluxion of the Abscis jointly *: therefore also the Fluxion of the Curve-surface is as the Perimeter of the Circle, having the Line BC for Radius, and the Fluxion of the Curve-line AC jointly. The plain meaning of which is this: that if there were any Number of Curve-surfaces, belonging to Solids, generated after the manner that has been explained, the Velocities with which the several Curve-surfaces flow, are such, that the superficial Spaces, which would be described by these Velocities in any given time, supposing these Velocities to continue invariable for that time, are as the Rectangles contain'd under the Perimeters of the several generating Circles, and right Lines proportional to the Fluxions of the several Curve-lines, corresponding to AC.

46. Schol. Upon these Properties of the Fluxions of Solids; and their Curve-surfaces, are founded two principal Branches of the Doctrine of Fluxions: *viz.* the Cubature of Solids; or the Method of finding their solid Content: and the Plaining of Curve-surfaces; or the Method of finding the superficial Content: both belonging to the inverse Method of Fluxions.

Art. 8, 9, 10. of the Quadrature of Curves.

47. These three Articles contain some more Examples, for the further Illustration of the Method of prime and ultimate Ratio's, both in the Case of Right and Curve Lines, considered as flowing Quantities. The Example contained in Art. 8, is so plain as to need no further Illustration: that in Art. 9. is coincident with the one formerly adduced and explained Art. 16. * From which Example our Author's 10th Art. is deduced by way of Corollary: and is the only thing remaining in these Articles, that needs any further Explication. In order to which, let the right Line PB, revolving about the given Point P as a Pole, cut the Curves GB, GE given in position, in the Points B and E, and let the moveable right Lines AB and AE touch these Curves always in the moveable Points of Intersection B and E,



* Of this Explication.

 I say

I ſay the Fluxion of the Curve GB is to the Fluxion of the Curve GE in a Ratio compounded of AB : AE and PB : PE; that is F.GB : F.GE :: AB × PB : AE × PE.

For let the revolving Line PB move forward from it's Place PB into the new Place P*b*, while at the ſame time the Tangent-lines AB and AE, ſtill touching the Curves come into the new Poſition A*b* and A*e*, ſo as *b* and *e* be the new Points of Contact and Interſection: and let the Line P*b* cut AB and AE in the moveable Points H and I. Now ſuppoſe the Line P*b* to return to it's former Place PB; by which means the Lines A*b* and A*e* will return to their former Places AB and AE, and the Points A and *a*, B, *b* and H; E, *e* and I at laſt coincide. In which Caſe, the ultimate Ratio of the evaneſcent Augments B*b* and BH; E*e* and EI is a Ratio of Equality, as may eaſily be collected from what was demonſtrated the latter Part of Art. 41. *: for the ultimate Ratio of the Arch and Tangent, when vaniſhing, is a Ratio of Equality, whatever be the Angle, which the ſecant Line makes with the Curve or Tangent. Wherefore B*b*: E*e* :: BH : EI; when vaniſhing: but in that Caſe, BH : EI :: AB × PB : AE × PE *. Therefore alſo the ultimate Ratio of the evaneſcent Augments B*b* and E*e* is AB × PB : AE × PE. Therefore F.GB : F.GE :: AB × PB : AE × PE *. As was to be proved. * Of this Explication. * Art. 16. of this Explication. * Art. 28. of this Explication.

48. Moreover if the Line PB, inſtead of revolving about the immovable Point P as a Pole, ſhould perpetually touch a Curve given in Poſition in the movable Point P, the Ratio of the Fluxions of the Curves GB and GE, will be the ſame as before: ſince the ultimate Ratio of the evaneſcent Augments will be the ſame as before; as will eaſily appear by the ſame way of reaſoning: ſo that there is not any neceſſity to inſiſt particularly upon it.

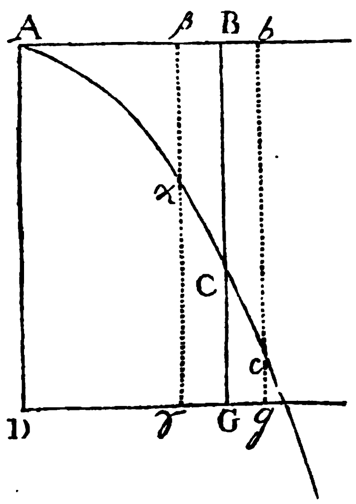
49. This Article teaches us how to find the Fluxion of any Power of a flowing Quantity, *i. e.* to find the Relation betwixt the Fluxion of any Power of a flowing Quantity, and the Fluxion of the Root or Quantity itſelf *. Art. 11. of the Quadrature of Curves. * Art. 6. of this Explication.

Let x repreſent any flowing Quantity, ſuppoſed to flow uniformly, and x^n any Power of that Quantity: let x be increaſed by the ſmall Augment o, ſo that in any ſhort Space of Time, it may become $x + o$: then in the ſame Time, x^n will become $\overline{x+o}|^n$; for into whatever new State or Value x comes, x^n muſt come into ſuch a new State or Value, as that ſtill, in every Place, or at every Inſtant of Time, it muſt be the nth Power of the Value of x in that Place or at that Inſtant of Time. Now by Sir *Iſaac Newton*'s binomial Theorem, $\overline{x+o}|^n = x^n$

+

$+ nx^{n-1}o + n \times \frac{n-1}{2} x^{n-2}oo + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}o^3 +$ &c. Wherefore, in the same Time that x becomes $x + o$, x^n becomes $x^n + nx^{n-1}o + n \times \frac{n-1}{2} x^{n-2}oo + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}o^3 +$ *&c.* From which two new Values subtracting the former Values x and x^n respectively, the Remainders, *viz.* o and $nx^{n-1}o + n \times \frac{n-1}{2} x^{n-2}oo + n \times \frac{n-1}{2} \times \frac{n-2}{2} x^{n-3}o^3 +$ *&c.* are the synchronal Increments: and if both be divided by o, we shall have 1 to $nx^{n-1} + n \times \frac{n-1}{2} \times x^{n-2}o + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}oo +$ *&c.* for the Ratio of the Augments generated in any the same Time, and that whether the Increment o be big or small. Now the Fluxions of Quantities are to one another in the prime Ratio of the nascent or ultimate Ratio of the diminishing, and at length evanescent Augments *. But by diminishing o more and more continually, the prime Ratio of the nascent, or ultimate Ratio of the evanescent Augments will be that of 1 to nx^{n-1}, because the varying Ratio of 1 to $nx^{n-1} + n \times \frac{n-1}{2} x^{n-2}o + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}oo +$ *&c.* (which is the Ratio of the Augments) by diminishing o continually; before o vanish, comes nearer to the Ratio of 1 to nx^{n-1} than by any given Difference. Therefore $F.x : F.x^n :: 1 : nx^{n-1}$.

* Art. 28. of this Explication.



50. The same thing might be demonstrated with greater Form, according to the same Principles: but waving that at this Time, I shall shew how it may be demonstrated by a geometrical Construction: in which the Quantities x and x^n are represented or expounded by plain Figures.

Suppose then the plain Figure ABC and Rectangle AG to be described in the same Time, by the uniform Motion of the right Line BCG along the common Abscis AB. Let $AB = x$, and BG ($= AD$) be the Ordinate of the flowing Rectangle AG, and let the given right Line AD or $BG = 1$: then is $AG = (1 \times x =)\ x$. Let BC be the Ordinate of the Figure ABC, whose Area is supposed equal to x^n. Now let the plain Figures ABC and AG be supposed first to flow by Increase, so that after a short Time, the right Line BCG may come into the new Place *bcg*: again let them be supposed to flow backwards, by decreasing, and let BCG come into the new Place $\beta\kappa\gamma$. Then let *bcg* and $\beta\kappa\gamma$ return again to BCG, and fall in with it at the same Time, so that B*b* and Bβ may be always equal. And, by what was formerly demonstrated *, the ultimate Ratio of the evanescent Augments B*bc*C and B*bg*G is BC : BG; and, for the same Reason, the ultimate Ratio of the evanescent Decrements B$\beta\kappa$C and B$\beta\gamma$G, is also BC : BG. Therefore, when these Increments and Decrements are just vanishing, we have B*bc*C : B*bg*G :: B$\beta\kappa$C : B$\beta\gamma$G: but because Bβ is always equal to B*b*, and therefore also when vanishing, hence B*bg*G $=$ B$\beta\gamma$G, therefore also the ultimate Ratio of B*bc*C and B$\beta\kappa$C is a Ratio of Equality. Put then B*b* $=$ o $=$ Bβ, and consequently A*b* $= x + o$ and A$\beta = x - o$: and from the Property of the Figure ABC, we have A*bc* $= \overline{x+o}|^n =$ (by Sir *Isaac Newton*'s binomial Theorem) $x^n + nx^{n-1}o + n \times \frac{n-1}{2}x^{n-2}oo + n \times \frac{n-1}{2} \times \frac{n-2}{3}x^{n-3}o^3 + \&c$; and $A\beta\kappa = \overline{x-o}|^n = x^n - nx^{n-1}o + n \times \frac{n-1}{2}x^{n-2}oo - n \times \frac{n-1}{2} \times \frac{n-2}{3}x^{n-3}o^3 + \&c.$ From the former subtract ABC $= x^n$; and subtract the latter from ABC $= x^n$, and you'll have $nx^{n-1}o + n \times \frac{n-1}{2}x^{n-2}oo + n \times \frac{n-1}{2} \times \frac{n-2}{3}x^{n-3}o^3 + \&c. =$ the Augment B*bc*C; and $nx^{n-1}o - n \times \frac{n-1}{2}x^{n-2}oo + n \times \frac{n-1}{2} \times \frac{n-2}{3}x^{n-3}o^3 - \&c. =$ the Decrement B$\beta\kappa$C. Now F.ABC : F.AG :: B*bc*C : B*bg*G when vanishing *: and likewise F.ABC : F.AG :: B$\beta\kappa$C : B$\beta\gamma$G, when vanishing. Wherefore, I say, it follows that F.ABC : F.AG :: $nx^{n-1}o$: o :: nx^{n-1} : 1. For if F.ABC : F.AG $<$ $nx^{n-1}o$: o, then by diminishing of o, $nx^{n-1}o + n \times \frac{n-1}{2}x^{n-2}oo + n \times \frac{n-1}{2} \times \frac{n-2}{3}x^{n-3}o^3 + \&c.$: o might become at length less than $nx^{n-1}o$: o, which can never be: again, if F.ABC : F.AG $>$ $nx^{n-1}o$: o, then $nx^{n-1}o - n \times \frac{n-1}{2}x^{n-2}o^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3}x^{n-3}o^3 - \&c.$: o, by diminishing o continually, might at length become greater than $nx^{n-1}o$: o, which can never be either, because in the Series $nx^{n-1}o - n \times \frac{n-1}{2}x^{n-2}oo + n \times \frac{n-1}{2} \times \frac{n-2}{3}x^{n-3}o^3 - \&c.$ (where the Terms are the same with those of the 11th Power of the Residual $x - o$, wanting the first Term, with the Signs changed) by diminishing o sufficiently, each Term will be greater than the succeeding one, and therefore the Terms being affected with the Signs $+$ and $-$ alternately, the whole Series, by diminishing o *in infinitum*, can never become

* Art. 29. of this Explication.

* Art. 28. of this Explication.

come greater than $nx^{n-1}o$. Wherefore, since it is neither F.ABC : F.AG $< nx^{n-1}o : o$; nor yet F.ABC : F.AG $> nx^{n-1}o : o$, it follows that F.ABC : F.AG :: $nx^{n-1}o : o :: nx^{n-1} : 1$.

51. Schol. In this Demonstration, the Index n of x^n is positive; but supposing it to be negative, we can prove, by the same way of Reasoning, that the Fluxion of x is to the Fluxion of x^{-n} as 1 to $-nx^{-n-1}$. Which shews that when x flows by Increase, x^n flows by Increase also; but x^{-n} flows by Decrease, it's Velocity or Fluxion being negative. But farther, it appears, 1°, That, when x flows uniformly by Increase, having it's Fluxion constant and invariable, if n be positive and greater than Unity, the Velocity with which x^n increases, or it's Fluxion is continually accelerated: for such a Number being substituted for n, in the Expression nx^{n-1}, makes the Index $n-1$ positive: and therefore it is evident the greater x is, the greater will the Quantity nx^{n-1} be, which expresses the Fluxion of x^n, when 1 expresses the Fluxion of x. 2°. If n be positive, but less than Unity, the Index $n-1$, in the Expression of the Fluxion nx^{n-1}, becomes negative in the Numerator; or positive in the Denominator, therefore the greater x is, the less will the Value of the Expression nx^{n-1} be: wherefore the Velocity with which x^n increases, or it's Fluxion is continually lessened or retarded. 3°. If n be negative, whatever be it's Value, the Expression $-nx^{-n-1}$ or $-\frac{n}{x^{n+1}}$, having the Index of x negative in the Numerator; or positive in the Denominator, shews the greater x is, the less will the Value of the negative Expression $-nx^{-n-1}$ or $-\frac{n}{x^{n+1}}$ be. Therefore it appears from thence, that when x flows uniformly by Increase, the Quantity x^{-n} flows backward or by Decrease, with a Velocity continually lessened or retarded.

52. Cor. By what has been demonstrated F.AG : F.ABC :: 1 : nx^{n-1}; likewise F.AG : F.ABC :: BG : BC *; hence BG : BC :: 1 : nx^{n-1}: but BG $=1$, therefore BC $= nx^{n-1}$. Whence it follows, that if the Abscis, perpendicular Ordinate and Area of a Curve be called x, y, and z respectively; and $z = x^n$ then $y = nx^{n-1}$: and conversely if $y = nx^{n-1}$ express the Relation betwixt the Abscis and Ordinate, then $z = x^n$ expresses the Relation betwixt the Area applied to Unity and the Abscis: or, if the Value of x be taken from the Equation $y = nx^{n-1}$, and insert in place of it, in the Equation $z = x^n$, it becomes $z = \overline{\frac{y}{n}}\Big|^{\frac{n}{n-1}}$ for the Relation betwixt the Area applied to Unity and the Ordinate: or lastly by taking the Value of x^n, we will have $z = \frac{1}{n}yx$ for the Relation betwixt the Area of the Curve, and the Rectangle contain-

* Art. 29. and 31. of this Explication.

ed under the Abſciſs and Ordinate. And this is the Foundation of the Quadrature of ſimple Curves [1].

From this it appears that if $nax^{n-1} = y$ expreſs the Relation of the Abſciſs and Ordinate of a Curve; where a denotes any given Quantity: then $ax^n = z$ ſhall exhibit the Relation betwixt the Abſciſs and Area applied to Unity: and converſely, if $z = ax^n$, then $y = nax^{n-1}$.

For ſince nx^{n-1} and nax^{n-1} are in a given or conſtant Ratio, *viz.* that of 1 to a: it follows, by Compoſition, that the Areas of the Curves which have nx^{n-1} and nax^{n-1} for their Ordinates, and x for their common Abſciſs, will be in the ſame given Ratio: but when $y = nx^{n-1}$, it is $z = x^n$ (as has been ſhewn) therefore when $y = nax^{n-1}$, it will be $z = ax^n$. Wherefore alſo converſely, if $z = ax^n$, it follows that $y = nax^{n-1}$. The ſame Thing may be eaſily demonſtrated by the Help of Art. 38, 39, 40 *. * Of this Explication.

53. After our Author has obſerved in this Article, that by like ways of reaſoning, in the Method of prime and ultimate Ratio's, the Fluxions of right and curve Lines in all Caſes whatſoever; the Fluxions likewiſe of Surfaces, Angles and other Quantities may be inveſtigated; he deſires it may be remarked, that the manner of proceeding in the Doctrine of Fluxions, by means of prime and ultimate Ratio's, is perfectly agreeable to the Analyſis of the Ancients: and that there is not any neceſſity of introducing into Geometry, by the Method of Fluxions, Figures infinitely ſmall. Art. 12. of the Quadrature of Curves.

This is carefully to be obſerved: and the Truth of it will appear by conſidering, that the Method which he takes in the ſeveral Examples, contained in this Introduction to his Treatiſe of the Quadrature of Curves, is this: firſt he determines the Ratio's of the cotemporaneous or ſynchronal Increments, by ſuch general Expreſſions, as are capable of being applied to theſe Increments whether taken great or ſmall; or whether generated in a greater or leſs, only the ſame Time: then, from theſe general Expreſſions, he finds in particular, what muſt be the prime or ultimate Ratio's of theſe Increments, conſidered as juſt ariſing or vaniſhing. Which is certainly a juſt and geometrical way of proceeding; to give an univerſal Solution of a Problem, applicable to all poſſible Caſes, and then apply it to one particular determinate Caſe. This Method of Solution is the moſt elegant; as well as moſt uſeful of any other, and what is commonly made uſe of both in the Analyſis of the Ancients; and that of the Moderns. Thus in deter-

[1] See our Author's admirable Treatiſe, *Analyſis by Equations of an infinite Number of Terms*, Rule firſt, and it's Demonſtration towards the End of that Treatiſe: likewiſe his *Method of Fluxions and infinite Series*, Prob. 9.

mining the Fluxion of x^n, or rather the Ratio of the Fluxion of x^n to the Fluxion of x: he firſt ſhews that the Ratio of their ſynchronal Increments, whether theſe Increments be great or ſmall, or, which is the ſame, whatever the Time be, wherein theſe cotemporaneous Increments are generated, is that of $nx^{n-1} + \frac{n^2 - n}{2}x^{n-2}o + \frac{n^3 - 3n^2 + 2n}{6}x^{n-3}oo$ + *&c.* to 1. Which expreſſes the Ratio of theſe Increments at all Times and in every Caſe: and therefore alſo, when theſe Increments are juſt ariſing; or juſt vaniſhing, as well as at any other Time: and then he ſhews that the foreſaid Ratio, in that particular Caſe, when the Increments are juſt ariſing or vaniſhing, becomes the ſame with $nx^{n-1} : 1$ [1]. The ſame way of reaſoning may be obſerved, in the other Examples propoſed in this Introduction: thus in the Example at Art. 9, in which our Author propoſes to determine the Ratio of the Fluxions of the flowing Lines AB and AE: he firſt demonſtrates, that their ſynchronal Increments B*b* and E*e*, in all Caſes, whether theſe Increments be great or ſmall, are as A*b* × PB to A*e* × PE: then he ſhews what that variable Ratio, ſtill expreſſed by finite Lines, will be, in that particular Caſe, when the Augments are juſt ariſing; or juſt vaniſhing: and he demonſtrates that it will, in that Caſe, become the ſame with the Ratio of AB × PB to AE × PE, and therefore he infers, in conſequence of what was formerly demonſtrated, that the ſame is the Ratio of the Fluxions of the Lines AB and AE. And thus it is evident, by this way of proceeding, as our Author obſerves, that the Analyſis is carried on, and the Fluxions of flowing or variable Quantities are expounded by finite Quantities: and although, as he obſerves, the Analyſis may be performed by means of Figures either finite or infinitely ſmall, which are ſuppoſed, ſimilar to the evaneſcent Figures; or yet by Figures, which, by the Method of Indiviſibles, are accounted as infinitely ſmall; yet ſince the Method of Indiviſibles is reckoned not ſo agreeable to the Principles of ſtrict Geometry; and our Ideas of infinite-

[1] Hence it appears that there is no Fallacy or Paralogiſm in this Method of Reaſoning, made uſe of by the ingenious Inventer, for finding the Fluxion of x^n, as is alledged by the Author of the *Analyſt*. See Art. 13, 14, 15, 16 of that Piece. For it is moſt evident that $nx^{n-1} + \frac{n^2-n}{2}x^{n-2}o$ + &c : 1, expreſſes the Ratio of the ſynchronal Increments of x^n and x in every Caſe: but when theſe Increments are ſuppoſed to be diminiſhed continually until they vaniſh, there muſt be ſome laſt Ratio, with which they vaniſh, otherwiſe they never could vaniſh: and ſince it is demonſtrated that the ultimate Ratio with which the Augments vaniſh can neither be leſs nor greater than the Ratio of nx^{n-1} to 1, it follows that the ultimate Ratio of the evaneſcent Increments of x^n and x, and conſequently the Ratio of their Fluxions is that of nx^{n-1} to 1. For a further Anſwer, ſee a Letter to the Author of the *Analyſt* by *Philalethes Cantabrigienſis*, *pag.* 54.

ly little Quantities are less clear and distinct, therefore I think these infinitely little Quantities; or such as are esteemed so, are not so proper to be introduced into Geometry, when we can obtain our End without them. And Sir *Isaac* seems to be of the same Mind, when he subjoins at the End of this Art. Provided one proceed cautiously.

54. The whole Doctrine of Fluxions, as was observed already, is reduced to two Problems: *viz. From the Fluents to find the Fluxions:* and conversely, *From the Fluxions to find the Fluents.* The last is by far the most difficult Problem: in the Solution of which there are different Steps and Cases, according to the different Suppositions made in the State of the Problem; and the Number of flowing Quantities, and Fluxions, which enter into it. The first Step of the Solution is equivalent to the Quadrature of Curves: and is in effect the same with this. *An Equation being given including one flowing Quantity, and the Fluxion of another uniformly flowing Quantity, to find this other in Terms made up of the first mentioned flowing Quantity and known Quantities.* Where such Value can be found in finite Terms, the Curve is geometrically quadrable, *i. e.* it's Area may be compared with a right-lined Figure: such as the Parabola. But where such Value cannot be had in finite Terms, the Curve cannot be squared geometrically: such are the Circle and Hyperbola. What our incomparable Author hath done in this Affair will appear from what follows.

Notes on the Quadrature of Curves.

SECT. II.

Containing Notes on Articles 13 *and* 14.

55. OUR Author in these two Articles gives us a short Account of the Origin and Notation of Fluxions; and of their Gradation or different Orders. The first two Things stand in no need of any further Illustration: but in regard the Doctrine of second, third and superior Orders of Fluxions is represented as altogether chimerical by a late Author [1] already mentioned, without any sufficient Foundation, I judge it will be necessary, especially for the sake of such as have been less conversant about these abstract Ideas, for whom these Notes are principally designed, to explain and illustrate at some Length the different Orders of Fluxions, mentioned but slightly by our Author in this Place.

[1] See the Analyst, Art. 36–48.

56. If

56. If a variable or flowing Quantity flow uniformly, ſo as to acquire equal Increments in equal Times, then it is evident it's Fluxion or Celerity of flowing being conſtant and invariable, that Fluxion muſt be the ſame in every Place, and at every Time: wherefore it can have no Fluxion or Mutation: Fluxion being only applicable to that which is variable, or flows from one Value to another. So that in this Caſe there is no Fluxion of the Fluxion, or no ſecond Fluxion of the flowing Quantity.

57. But if the flowing Quantity don't flow equably or uniformly; but be either continually accelerated or retarded, then it is evident that it's Fluxion, or Celerity of flowing, being different at different Times, is itſelf a variable, indeterminate or flowing Quantity; and therefore admits of a Fluxion: and this is called the Fluxion of the Fluxion, or the ſecond Fluxion of the firſt flowing Quantity.

58. Again if this laſt or ſecond Fluxion be conſtant and invariable, there is no third Fluxion: but if it be variable and inconſtant, then that variable Fluxion or Celerity, may be itſelf conſidered as a flowing Quantity: and therefore does admit of a Fluxion, as well as any other variable Quantity, which is alſo called the ſecond Fluxion of the firſt Fluxion; and the third Fluxion of the firſt flowing Quantity. And ſo we may proceed to ſuperior Orders of Fluxions, and that without end. For the Fluxion of any flowing Quantity, being nothing elſe but it's Celerity of flowing: and Celerity being itſelf a Quantity, there is no Reaſon why, when it is variable, it ſhould not be conſidered in the ſame Light with any other flowing Quantity, in this Reſpect: *i. e.* why it may not be conſidered as having a Fluxion, which expreſſes the ſwifter or ſlower Mutation, with which that Celerity flows or changes: and that notwithſtanding of what has been ſaid, by the Author of the Analyſt, of the Abſurdity of ſuppoſing a Celerity of a Celerity. But becauſe it is ſo ſtrenuouſly inſiſted upon, that the Conception of any Fluxion, particularly of a ſecond, third or any ſubſequent Fluxion, is altogether impoſſible, I ſhall endeavour fully to explain this Affair, in what follows.

59. The Fluxion of any variable or flowing Quantity, being the Velocity or Swiftneſs with which it flows or changes, by Increaſe or Decreaſe, according as it is ſuppoſed either to be continually increaſing or diminiſhing, the moſt natural and eaſy way of repreſenting and explaining a Fluxion, will be by ſuch Quantities as Space may be properly applied to, or predicated of: ſuch are geometrical Quantities, otherwiſe called *Magnitudes*, *viz.* Lines, Surfaces and Solids, to which one or more Dimenſions are applicable.

60. Now

60. Now as the Velocity with which any geometrical Quantity flows, by augmenting or diminiſhing, includes Time and Space in the Notion of it, whereby it may be meaſured and determined; it will be proper to obſerve, that, when the variable or flowing Magnitude is a Line, it is ſuppoſed to be generated or produced by the continual Motion or Flowing of a Point: and then, the Fluxion of the flowing Line, at any Time, is moſt naturally meaſured by the Length which the moving Point would run over in a certain given Time, ſuppoſing the Velocity, with which the Point moves, continued invariably the ſame, during the given Time, that it was at that Inſtant, or at that particular Place, when and where it's Fluxion is ſpoke of and enquired after. If the flowing Quantity be a Superficies, it is ſuppoſed to be generated or produced by the continual Motion of a Line: and the Fluxion of the flowing ſuperficial Space at any Place, or Inſtant of Time, during the flowing, is naturally meaſured by the Quantity of ſuperficial Space, that would be deſcribed in a certain given Time, by ſuppoſing the generating Line, continuing invariably the ſame, to be carried along another Line, with an uniform Motion, and thereby deſcribe a Space, which increaſes equably, and juſt as faſt through the whole given Time, as the flowing ſuperficial Space increaſed at that Place or Inſtant of Time. Laſtly, if the flowing Quantity be a Solid, it is ſuppoſed to be generated by the Motion of a plain Figure: and the Fluxion of the Solid at any Time or Place, is naturally meaſured by the Quantity of ſolid Space that would be deſcribed by the generating plain Figure continuing invariably the ſame, and drawn along a right Line, with an uniform Motion, ſo as to make the ſolid Space increaſe, through the whole given Time, juſt as faſt as the flowing Quantity increaſes at that particular Place or Inſtant of Time: when it's Fluxion is ſought.

61. This is the moſt eaſy and natural Way to conceive the Meaſure of the Fluxion of flowing Magnitudes: but notwithſtanding of this, the Fluxion of any Magnitude may be repreſented or expounded, by a Magnitude of any kind, provided it be always proportional to that Fluxion: thus the Fluxions of Superficies's and Solids; as well as of Lines, may be expounded by Lines, which are the moſt ſimple kind of Magnitudes, provided theſe Lines, by which they are expounded and repreſented, be always proportional to theſe Fluxions: juſt as *Euclid* has repreſented all Magnitudes by right Lines, in the 5th Book of the Elements. Thus, as has been ſhewn already, when a plain Figure is generated by the uniform Motion of the Ordinate along the Abſciſs, the Fluxion of the plain Figure is always as the Ordinate: and therefore may be expounded thereby.

62. When

62. When Fluxion or Velocity of flowing is applied to other Quantities, it muſt be underſtood in an analogical Senſe, it being applied to them, becauſe of the Similitude there is betwixt the increaſing and diminiſhing of any Quantity whatſoever, which is ſuppoſed to be continually paſſing from one State or Value to another; and the increaſing and diminiſhing of flowing or variable Magnitudes.

Now the whole Buſineſs of the different Orders of Fluxions may be explained in the following manner.

63. Let ABC be a plain Figure generated or deſcribed, by the uniform or equable Motion of it's Ordinate BC, along the Baſe or Abſciſs AB, always at right Angles to each other: and putting the Abſciſs $AB = x$, let the Nature of the Figure be ſuch, that it's Area $ABC = ax^n$; where a is any given or invariable Quantity; and n any indefinite Index to x: and let $AG = 1$. Now ſuppoſe the Fluxion of the Abſciſs AB, *i. e.* the Velocity with which the Point B moves, which is equable and uniform, to be ſuch that in a given Time, ſuppoſe a Second of Time, it moves through a Space equal to $AG = 1$. Draw AH perpendicular to AG and equal to it: and let the Rectangle HB be deſcribed, at the ſame Time with ABC, by it's invariable Ordinate equal to AH or Unity. Then 'tis evident, that in the ſame Time, the Point B moves through a linear Space equal to AG or 1, the Ordinate of the Rectangle BH will move through or deſcribe a ſuperficial Space equal to $AG \times AH$ or the Square of 1: that is, it will do it in a Second of Time. But, if the Velocity with which the Area ABC increaſes, were continued invariably the ſame, for a ſecond of Time, or which is the ſame *, if the Ordinate BC continued invariably the ſame, moving along AB, the ſuperficial Space deſcribed by BC, in a Second of Time, would be $BC \times AG = BC \times 1$. Therefore $AG \times AH$ or a ſuperficial Unite, being the Meaſure of the Fluxion of the Rectangle $HB = AB \times 1$; $BC \times AG = BC \times 1$ will be the Meaſure of the Fluxion of the flowing Area ABC. But ſince $ABC = ax^n$, it appears from what was formerly demonſtrated *, that the Ordinate $BC = nax^{n-1}$, therefore the ſuperficial Space, which is the Meaſure of the Fluxion of ABC, may likewiſe be expreſſed by $nax^{n-1} \times 1$ or nax^{n-1}.

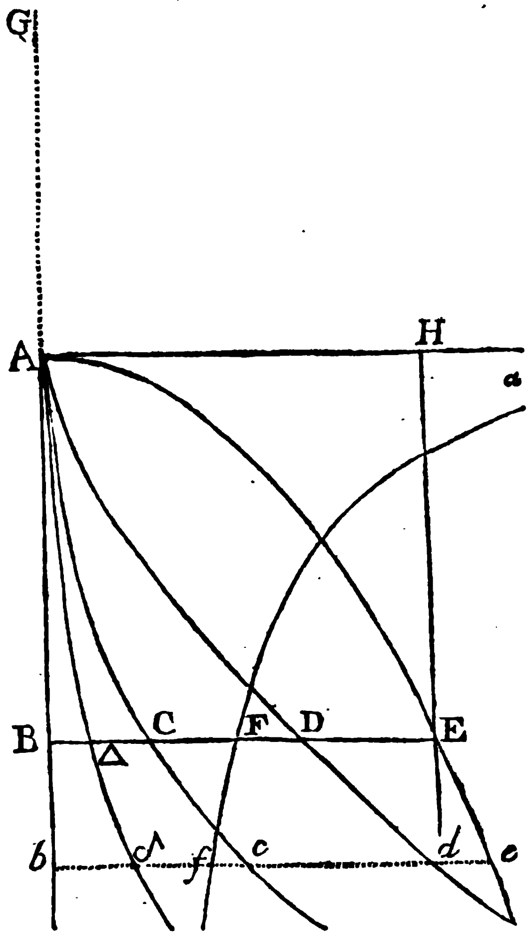
* Art. 31. of this Explication.

* Art. 47. of this Explication.

64. Now the Expreſſion nax^{n-1}, when $AB = x$ flows, denotes a variable or flowing Quantity, unleſs when $n = 1$: and a Quantity flowing by Increaſe, when $n - 1$ is poſitive *, and x increaſes: therefore the Fluxion of ABC is itſelf a variable or flowing Quantity, *i. e.* the Celerity with which ABC flows, is continually flowing or varying as to it's Value, which, in this Caſe, grows ſtill greater and greater. Suppoſe

* Art. 46. of this Explication.

poſe then another plain Figure ABD, whoſe Area ABD $=$ BC $\times$ AG or $nax^{n-1}\times 1 = nax^{n-1}$, deſcribed by it's Ordinate BD, moving along AB, at the ſame Time ABC is deſcribed by it's Ordinate BC: then the Fluxion of ABC is meaſured and expounded by ABD, which continually flows by Increaſe. But the Fluxion of ABD, or the Velocity with which it encreaſes, is ſuch, as would be ſufficient to deſcribe a ſuperficial Space equal to BD $\times$ AG $=$ BD $\times 1$, in the ſame Time the Point B deſcribes the linear Unite; the Ordinate of the Rectangle HB deſcribes the ſuperficial Unite; or the Ordinate BC of the Figure ABC, continuing invariably the ſame, would deſcribe the ſuperficial Space BC $\times 1$; *viz.* in a Second of Time; as eaſily appears from what has been ſaid with reſpect to the flowing of ABC: therefore BD $\times 1 =$ BD $\times$ AG is the Meaſure of the Velocity, with which ABD flows or increaſes: conſequently BD $\times$ AG or BD $\times 1$, is proportional to, and therefore may be the Exponent of the Celerity, with which the Fluxion of ABC changes, *i.e.* it is proportional to, or the Exponent of the Fluxion of the Fluxion of ABC, which is called the ſecond Fluxion of ABC. But, ſince the Area ABD $= nax^{n-1}\times 1$ or nax^{n-1}, the Ordinate BD $= n\times\overline{n-1}\times ax^{n-2}$ or $\overline{n^2-n}\times ax^{n-2}$ *, therefore $\overline{n^2-n}\times ax^{n-2}\times 1 = \overline{n^2-n}\times ax^{n-2}$ is the Exponent of the ſecond Fluxion of ABC.



* Art. 52. of this Explication.

65. Again if $n=2$, then $\overline{n^2-n}\times ax^{n-2} = \overline{4-2}\times ax^0 = 2a$ is a conſtant or invariable Quantity: in which Caſe the Fluxion of ABD, or the ſecond Fluxion of ABC is conſtant and invariable, ſince it's Exponent is ſo: conſequently ABC will have no third Fluxion, which

denotes the Celerity of the Mutation of the ſecond Fluxion. This will happen, when ABC is a right angled Triangle, having it's Baſe equal to $2ax$: and then ABD is a Rectangle, having the Abſciſs AB for it's variable Altitude, and the invariable Line denoted by $2a$ for it's Baſe. But if n be not equal to 2, the Expreſſion $\overline{n^2-n}\times ax^{n-2}$ is variable: and if $n > 2$, it is always increaſing, as x increaſes *. In which Caſe, ſuppoſe ABE another plain Figure, having it's Area equal to BD × AG or BD × 1 $= \overline{n^2-n}\times ax^{n-2}\times 1$ or $\overline{n^2-n}\times ax^{n-2}$: and let it be deſcribed by the Motion of it's Ordinate BE along the common Abſciſs AB, at the ſame Time ABC and ABD are deſcribed by their Ordinates BC and BD. Then it appears, by the ſame way of reaſoning as before, that the Area ABE is always proportional to, or the Exponent of the ſecond Fluxion of ABC: and further, that the Velocity with which the flowing Area ABE increaſes is meaſured by BE × AG = BE × 1: Therefore the Velocity with which ABE increaſes or it's Fluxion will be always proportional to, or the Exponent of the ſecond Fluxion of ABD; and of the third Fluxion of ABC; or, which is the ſame thing BE × AG = BE × 1 is always proportional to and conſequently the Exponent of the third Fluxion of ABC. Moreover ſince ABE $= \overline{n^2-n}\times ax^{n-2}$, the Ordinate BE $= \overline{n^2-n}\times\overline{n-2}\times ax^{n-3}$ * $= \overline{n^3-3n^2+2n}\times ax^{n-3}$, therefore $\overline{n^3-3n^2+2n}\times ax^{n-3}$ is likewiſe the Exponent of the firſt Fluxion of ABE; of the ſecond Fluxion of ABD; and of the third Fluxion of the firſt flowing Area ABC. Now if, in the Expreſſion ax^n, which is the Value of the Area ABC, we ſuppoſe $a = \frac{8}{105}$ and $n = \frac{7}{2}$, we ſhall have $BE = (\overline{n^3-3n^2+2n}\times ax^{n-3} =)\ x^{\frac{1}{2}}$: which ſhews that ABE is the Apollonian or common Parabola, having AG = 1 for the Parameter of the Axis.

* Art. 51. of this Explication.

* Art. 52. of this Explication.

Again if ABFα be a plain Figure, deſcribed by the uniform Motion of it's Ordinate BF along AB, together with the other plain Figures; ſuch, that it's Area ABFα = BE × AG = BE × 1 $= \overline{n^3-3n^2+2n}\times ax^{n-3}\times 1$, then the Area ABF$\alpha$ will alſo be proportional to, or the Exponent of the firſt Fluxion of ABE; of the ſecond Fluxion of ABD; and of the third Fluxion of the firſt flowing Area ABC.

66. And thus we might proceed to ſubſequent Orders of Fluxions; which will run on infinitely, if the indefinite Index of x, *viz.* n being poſitive, be not an Integer; or whatever it be, if negative: becauſe in every ſuch Caſe, the Exponent of x in the Expreſſion of any of the flowing Areas, in the Series of plain Figures, can never become nothing: and therefore the Exponents of their Fluxions will always be variable: and conſequently no Fluxion of any Order, in that Caſe, can ever

ever be a conſtant Quantity. Only it may be obſerved, that when the Exponent of the Power of x, in the Expreſſion of the Area of any Curve, in the Progreſſion of Curvilinear Areas, becomes negative, the Fluxion of that Area muſt likewiſe be negative *: for that Area diminiſhes as x increaſes; and therefore is poſited below the Ordinate; conſequently it is ſo far from having any Velocity of Increaſe, that upon the contrary, it decreaſes or diminiſhes, and ſo the Velocity with which it diminiſhes, is call'd a negative Fluxion. Thus, if we ſuppoſe $a = \frac{8}{105}$ and $n = \frac{7}{2}$, as formerly, and conſequently ABE a Parabola, then the Curve αF will be of the hyperbolical Kind, approaching the Aſſymptotes AB and AH continually: and it's Area ABFα being equal to $BE \times 1 = x^{\frac{1}{2}} \times 1 = x^{\frac{1}{2}}$, it's Ordinate $BF = \frac{1}{2}x^{-\frac{1}{2}}$ or $\frac{1}{2\sqrt{x}}$ *, and therefore it's Fluxion will be expounded by a Curvilinear Area lying below the Ordinate, for according to the Order of the Progreſſion, that Area muſt be $\frac{1}{2\sqrt{x}} \times 1$ or $\frac{1}{2}x^{-\frac{1}{2}}$, the Fluxion of which muſt be negative.

* Art. 51. of this Explication.

* Art. 52. of this Explication.

67. The Progreſſion of Curvilinear Areas, as Exponents of the various Orders of Fluxions and Fluents, may be ſuppoſed to be carried on likewiſe the contrary way. Thus if ABΔ be a Curvilinear Figure, having it's Ordinate BΔ equal to the Area ABC applied to AG or Unity: and if there were another Curve, whoſe Ordinate is equal to ABΔ applied to AG, and ſo on; that would conſtitute a Series of Curves running the contrary way, ſuch, that every Curvilinear Area in this Series will be a Fluent, of which the immediately preceding is the Exponent of it's Fluxion; that two Steps before it, is the Exponent of it's ſecond Fluxion, &c. And thus the Progreſſion of Curves, reckoning from ABC, may be ſuppoſed to be carried on both ways *in inf.*

68. And thus, I have endeavoured to explain and illuſtrate the whole Affair of the different Orders of Fluxions, ſo as to ſhew and demonſtrate that they have a real Foundation in Nature; and may be diſtinctly conceived.

69. Since the Curvilinear Areas ABD, ABE, ABFα, &c. are equal to the Ordinates BC, BD, BE, &c. multiplied by AG or 1 reſpectively: therefore the Ordinates BC, BD, BE, &c. will alſo be the Exponents of the firſt, ſecond, third, &c. Fluxions of the flowing Area ABC.

70. But it may be proper here to caution the Beginner, that he don't underſtand me, as if I ſaid, that the Areas ABD, ABE, ABFα, &c. or the Ordinates BC, BD, BE, &c. were really the firſt, ſecond, third, &c. Fluxions of the flowing Area ABC. They are only Exponents of theſe Fluxions, as being proportional to them: for a Fluxion,

which denotes the Celerity of Flowing or Mutation, is one thing; and a Surface, or a Line is another: although a Surface or a Line may be proportional to, and therefore the Exponent of a Celerity. So that the true Meaning of the thing is this: Suppose BCFDE, and *bfcde*, to be two different Positions of the Ordinates of the Curves already mentioned, generated and related to one another as above, the first Fluxion of ABC is to the first Fluxion of A*bc* as ABD to A*bd*; or as BC to *bc*: the second, third, &c. Fluxions of ABC are to the second, third, &c. Fluxions of A*bc* respectively, as ABE, ABFα, &c. to A*be*, A*bf*α, &c. or as BD, BE, &c. to *bd*, *be*, &c. respectively.

71. From what has been said, the Truth of what our Author advances in Art. 14. will easily appear, *viz.* If $\overset{''}{z}.\ \overset{'}{z}.\ z.\ \dot{z}.\ \ddot{z}.\ \dddot{z}.$ &c. $\overset{''}{\sqrt{az-zz}}.\ \overset{'}{\sqrt{az-zz}}.\ \sqrt{az-zz}.\ \dot{\overline{\sqrt{az-zz}}}.\ \ddot{\overline{\sqrt{az-zz}}}$, &c. $\overset{''}{\frac{az+zz}{a-z}}.\ \overset{'}{\frac{az+zz}{a-z}}.\ \frac{az+zz}{a-z}.\ \dot{\overline{\frac{az+zz}{a-z}}}.\ \ddot{\overline{\frac{az+zz}{a-z}}}$, &c. be Series's of Quantities, so related to each other, that every subsequent Quantity in any of these Series's, is the Fluxion of the preceding Quantity; and conversely every preceding Quantity being considered as Fluent have the succeeding one it's Fluxion: I say if this be the Relation of the Quantities to each other, then every preceding Quantity is as the Area of a Curvilinear Figure, whereof the immediately following is the rectangular Ordinate and z the Abscis.

For it has been demonstrated * that the Fluxion of any plain Figure, as ABC (see the preceding Fig.) is such, that when it is compared with the Fluxion of AB the Abscis; or rather with the Fluxion of the Rectangle AB × 1, there arises this Analogy F.AB × 1 : F.ABC :: 1 : BC. Wherefore if the Fluxion of AB × 1 or AB be denoted by 1 (which is supposed) then the Fluxion of ABC will be denoted by BC, *i. e.* the rectangular Ordinate. Consequently if any Quantity such as $\overset{'}{\sqrt{az-zz}}$ express the Area of a Curve, whose Abscis is z: and $\dot{z}=1$, as is supposed, then F.$\overset{'}{\sqrt{az-zz}}$ *i. e.* $\sqrt{az-zz}=$ the rectangular Ordinate, as our Author affirms: and so of others.

* Art. 29, 31, 63. of this Explication.

SECT. III.

Containing Notes on Art. 15——24.

Explication of Prop. I.

72. ANY Difficulty there is in our Author's Demonstration of this Proposition, arises from assuming $o\dot{x}$, $o\dot{y}$ and $o\dot{z}$ for the synchronal Increments of the flowing Quantities x, y and z: and neglecting these Terms, which involve o at the Result of the Operation.

2 73. If

73. If the synchronal Increments of x, y and z, be supposed to have already acquired any finite Magnitude, it must be acknowledged that $o\dot{x}$, $o\dot{y}$ and $o\dot{z}$ cannot express such Increments: because $o\dot{x}$, $o\dot{y}$ and $o\dot{z}$ are proportional to $\dot{x}$, $\dot{y}$ and $\dot{z}$, *viz.* to the Fluxions of x, y and z; or the Velocities with which they flow or increase, just as the Increments are beginning to arise: but finite Increments are not always proportional to the Fluxions of the flowing Quantities, since the Velocities, with which these finite Increments are generated, may vary during the Time they are generated: therefore $o\dot{x}$, $o\dot{y}$, $o\dot{z}$ cannot properly express finite Increments. But it was shewn formerly *, that the less the synchronal Increments of flowing Quantities are; or the less the Time be, in which they are produced, the nearer will their Ratio approach to the Ratio of the Fluxions: so as that, by diminishing these Increments; or the Time in which they are generated, continually more and more, the Ratio of the Increments, shall approach nearer to the Ratio of the Fluxions, than by any given Difference; and consequently their ultimate Ratio be the same with the Ratio of the Fluxions. Therefore by $o\dot{x}$, $o\dot{y}$, $o\dot{z}$ we must understand synchronal Increments in this Sense, *viz.* such as are indefinitely small, called by our Author *Moments* or *synchronal momentaneous Increments.* And surely, if there can be no Increments how small soever assigned, but there are Increments yet less and less, assign as small ones as you please; he may be allowed to assume a Notation denoting nascent or evanescent Increments, which are indefinitely small, although we have no positive adequate Idea of such Increments, or Moments. For a negative Idea is sufficient, in many Cases, for determining the Properties and Proportions of Quantities. What other than a negative Idea have I of the infinite Decimal ,6666, &c. Yet I can demonstrate that it is exactly equal to $\frac{2}{3}$. What other than a negative and very imperfect Idea has any one of $\sqrt{2}$? Yet it can be shewn that it is a mean Proportional betwixt 1 and 2. And although all Surds are negative Ideas; and involve some kind of Infinity in them, yet they admit of arithmetical Operations: and can be compared with one another. Thus though none can tell how much $\sqrt{2}$ and $\sqrt{8}$ are separately: yet we know that their Product is 4; and that the latter is double the former. After the same manner, though the momentaneous Increments of flowing Quantities cannot be distinctly and adequately conceived by the Mind, yet we may make use of some kind of Notation, by way of *Symbol* or *Representation* of them: and their Relations may be determined and express'd by finite Quantities, which are distinctly conceived: and this is all this Proposition requires or proposes.

* Art. 27. of this Explication.

74. The

74. The other thing I mentioned, as having any Difficulty, in the Demonſtration given by Sir *Iſaac* of this 1ſt Prop. is the neglecting the Terms of the Equation, at the Reſult of the Operation, wherein the Quantity o is found. The Reaſon of it is aſſigned briefly by himſelf: *viz.* becauſe the Quantity o is infinitely diminiſhed and vaniſhes. For the Equation, expreſſing the Relation of the ſynchronal Augments, will only then expreſs the Relation of the Fluxions, when the Quantity o is juſt vaniſhing: but when that Quantity vaniſhes, all the Terms wherein it is found muſt vaniſh likewiſe; which is the Reaſon of their being neglected and thrown out.

75. But in regard this Propoſition is a fundamental one; and contains the Subſtance of the direct Method of Fluxions, I ſhall ſhew how it may be demonſtrated at fuller Length. In order to which, it will be neceſſary to ſhew how the Fluxion of any ſimple Quantity, which contains the Product of any Number of flowing Quantities may be inveſtigated.

76. Caſe 1. When the flowing Quantity is the Rectangle or Product of two flowing Quantities, as xy; ſuppoſe $\dot{x}$ and $\dot{y}$ to denote the Fluxions of the Factors x and y, I ſay $F.xy = \dot{x}y + \dot{y}x$.

Put $z = xy$; and let $\underset{\cdot}{x}$, $\underset{\cdot}{y}$, $\underset{\cdot}{z}$ denote any ſynchronal Increments, great or ſmall of the flowing Quantities x, y, z: ſo that in the ſame Time that x becomes $x + \underset{\cdot}{x}$, y and z may become $y + \underset{\cdot}{y}$ and $z + \underset{\cdot}{z}$ reſpectively. Then becauſe the Equation $xy = z$ exhibits the Relation of the flowing Quantities x, y and z indifferently, or at all times, into whatever new Values they paſs, therefore $z + \underset{\cdot}{z} = \overline{x + \underset{\cdot}{x}} \times \overline{y + \underset{\cdot}{y}} = xy + \underset{\cdot}{x}y + x\underset{\cdot}{y} + \underset{\cdot}{x}\underset{\cdot}{y}$: take away z from the one Side of the Equation and xy equal to it, from the other; and we have $\underset{\cdot}{z} = \underset{\cdot}{x}y + x\underset{\cdot}{y} + \underset{\cdot}{x}\underset{\cdot}{y}$ an Equation defining the Relation of the Increments generated in the ſame Time, great or ſmall: and by dividing by $\underset{\cdot}{x}$, it is $\frac{\underset{\cdot}{z}}{\underset{\cdot}{x}} = y + \frac{\underset{\cdot}{y}}{\underset{\cdot}{x}}x + \underset{\cdot}{y}$. Now ſuppoſe the Increments $\underset{\cdot}{x}$, $\underset{\cdot}{y}$ and $\underset{\cdot}{z}$ to be diminiſhed infinitely, ſo as to be juſt vaniſhing, and in their ultimate Ratio: in that Caſe the Ratio of the evaneſcent Increments, is the ſame with the Ratio of the Fluxions *, *i. e.* $\frac{\underset{\cdot}{y}}{\underset{\cdot}{x}} = \frac{\dot{y}}{\dot{x}}$ and $\frac{\underset{\cdot}{z}}{\underset{\cdot}{x}} = \frac{\dot{z}}{\dot{x}}$: which Values being inſerted the Equation becomes $\frac{\dot{z}}{\dot{x}} = y + \frac{\dot{y}}{\dot{x}}x + \underset{\cdot}{y}$; where the Term $\underset{\cdot}{y}$, being infinitely diminiſhed, vaniſhes and goes out: conſequently we have $\frac{\dot{z}}{\dot{x}} = y + \frac{\dot{y}}{\dot{x}}x$, for the Relation of the Fluxions, *i. e.* $\dot{z} = \dot{x}y + x\dot{y}$. But ſince $z = xy$, $\dot{z} = F.xy$ *, therefore $F.xy = \dot{x}y + x\dot{y}$.

* Art. 28. of this Explication.

* Art. 40. of this Explication.

77. Cor. 1. $F.ax = a\dot{x}$. For ſuppoſe $ax = z$: then by what has been ſaid $a \times \overline{x + \underset{.}{x}}$ or $ax + a\underset{.}{x} = z + \underset{.}{z}$: ſubtract $ax = z$, there remains $a\underset{.}{x} = \underset{.}{z}$ or $a = \frac{\underset{.}{z}}{\underset{.}{x}}$: therefore the Increments of z and x are in a given Ratio: conſequently their Fluxions are in the ſame given Ratio *, *i.e.* $a = \frac{\dot{z}}{\dot{x}}$, whence $a\dot{x} = \dot{z}$: therefore $F.ax = a\dot{z}$. The ſame thing eaſily appears from Art. 40 *.

* Art. 28. of this Explication.

* Of this Explication.

78. Cor. 2. $F.xx = 2x\dot{x}$: for it is $\dot{x}x + x\dot{x} = 2x\dot{x}$.

79. Caſe 2. When the Fluent is the Product of three flowing Quantities as xyz, I ſay $F.xyz = \dot{x}yz + x\dot{y}z + xy\dot{z}$. For put $xy = u$, then $xyz = uz$, therefore $F.xyz = F.uz$ * $= \dot{u}z + u\dot{z}$ *: but ſince $xy = u$, $\dot{x}y + x\dot{y} = \dot{u}$. Wherefore if we inſert $\dot{x}y + x\dot{y}$ for $\dot{u}$ and reſtore xy for u, in the Equation $F.xyz = \dot{u}z + u\dot{z}$, we have $F.xyz = \dot{x}yz + x\dot{y}z + xy\dot{z}$.

* Art. 40.
* Art. 76. of this Explication.

80. Cor. $F.axz = a\dot{x}z + ax\dot{z}$. $F.ax^2 = 2ax\dot{x}$. For the flowing Quantities xz, axz; and x^2, ax^2, are in a given or conſtant Ratio, *viz.* that of 1 to a: therefore their Fluxions are in the ſame given Ratio *: but $F.xz = \dot{x}z + x\dot{z}$, and $F.x^2 = 2x\dot{x}$, therefore $F.axz = a\dot{x}z + .ax\dot{z}$ and $F.ax^2 = 2ax\dot{x}$. Likewiſe $F.x^3 = 3\dot{x}x^2$: for it is $\dot{x}xx + x\dot{x}x + xx\dot{x} = 3\dot{x}x^2$.

* Art. 40. of this Explication.

81. Caſe 3. When the Fluent is the Product of any Number of flowing Quantities, the general Rule for finding the Fluxion in every Caſe is this. Multiply the Fluent by the Index of the Power of every flowing Quantity it involves; and in every Multiplication, change the Root of the Power into it's Fluxion: and the Aggregate of all theſe Products is the Fluxion required.

For when the Fluent conſiſts of four Dimenſions, and as many different Letters, as $xyzu$, it's Fluxion is $\dot{x}yzu + x\dot{y}zu + xy\dot{z}u + xyz\dot{u}$: which is demonſtrated by reducing it to three, *viz.* by putting $t = xy$, *i. e.* making t a 4th Proportional to 1, x and y: and then $\overline{xy}zu = tzu$, therefore $F.xyzu = F.tzu = \dot{t}zu + t \times \overline{\dot{z}u + z\dot{u}}$ *. But ſince $t = xy$, $\dot{t} = \dot{x}y + x\dot{y}$ *, therefore by Reſtitution, $F.xyzu = \overline{\dot{x}y + x\dot{y}} \times zu + xy \times \overline{\dot{z}u + z\dot{u}} = \dot{x}yzu + x\dot{y}zu + xy\dot{z}u + xyz\dot{u}$. And thus by means of a Quantity of four Dimenſions, you may diſcover the Fluxion of a Fluent of five Dimenſions, and ſo on: in ſuch manner, that he who rightly conſiders the Proceſs, will ſee this Truth demonſtrated, *viz.* That the Fluxion of any Fluent, which is itſelf a ſimple Quantity, *i. e.* a Quantity of one Term, be it's Dimenſions what they will, is found by ſetting down the Quantity as often as there are flowing prime Diviſors

* Art 79. of this Explication.
* Art. 40. of this Explication.

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in it, only changing each prime Divisor into it's Fluxion, in order, one after another, and taking the Sum of all, after the Manner of the preceding Examples. Which is coincident with the general Rule prescribed under this 3[d] Case: or at least manifestly included in it.

However that the general Rule may be more clearly demonstrated, take the Quantity $x^ny^mz^p$ with indefinite Exponents: and I say $F.x^ny^mz^p = n\dot{x}x^{n-1}y^mz^p + mx^n\dot{y}y^{m-1}z^p + px^ny^m\dot{z}z^{p-1}$. For let it be $1 : x^n : r$ ∴∷, $1 : y^m : s$ ∷, and $1 : z^p : t$ ÷, or $x^n = r$, $y^m = s$ and $z^p = t$: then $F.x^ny^mz^p = F.rst = \dot{r}st + r\dot{s}t + rs\dot{t}$: but because $x^n = r$; $y^m = s$ and $z^p = t$, therefore $F.x^n = nx^{n-1}\dot{x} = \dot{r} : F.y^m = my^{m-1}\dot{y} = \dot{s}$: and $F.z^p = pz^{p-1}\dot{z} = \dot{t}$. Substitute for $\dot{r}$, $\dot{s}$ and $\dot{t}$ these their Values, and replace x^n, y^m and z^p, for r, s and t in the Equation $F.x^ny^mz^p = \dot{r}st + r\dot{s}t + rs\dot{t}$, and we have $F.x^ny^mz^p = nx^{n-1}\dot{x}y^mz^p + mx^ny^{m-1}\dot{y} + px^ny^mz^{p-1}\dot{z}$, according to the general Rule.

If the Fluent have any constant or invariable Quantity among it's Factors, as $a^2xy^3z^4$ the general Rule still holds. For $F.a^2xy^3z^4 = a^2\dot{x}y^3z^4 + 3a^2x\dot{y}y^2z^4 + 4a^2xy^3\dot{z}z^3$, as appears from what has been said already, since xy^3z^4 and $a^2xy^3z^4$ are in a given Ratio, and therefore their Fluxions in that same given Ratio *: But that likewise is the Fluxion of $a^2xy^3z^4$ according to the general Rule above, since $\dot{a} = 0$, and therefore the Term which involves $\dot{a}$ equal to nothing likewise, being $2\dot{a}axy^3z^4$.

* Art. 40. of this Explication.

82. And now these things being understood, the Solution of our Author's first Prop. will be very plain and easy, *viz.* thus: An Equation being given involving any number of flowing Quantities, transpose all the negative Terms to the opposite Side, that all may become positive, then find the Fluxions of both Sides, by taking the Aggregate of the Fluxions of all the Terms, found by the general Rule above: put the Fluxions of both Sides equal to each other: and so you shall have an Equation exhibiting the Relation of the Fluxions. Thus if the first Equation mentioned in this Prop. were given, *viz.* $x^3 - xy^2 + a^2z - b^3 = 0$, make it by Transposition stand thus $x^3 + a^2z = xy^2 + b^3$. Then take the Fluxions of both Sides, which, being the Sum of the Fluxions of all the Terms, I do, by taking the Fluxion of each Term, and adding them together: whereby we will have $3\dot{x}x^2 + a^2\dot{z} = \dot{x}y^2 + 2x\dot{y}y$; or by transposing all to one Side, $3\dot{x}x^2 - \dot{x}y^2 - 2x\dot{y}y + a^2\dot{z} = 0$, for the Fluxion of b^3 is nothing. Which is the same with that discovered by our Author: and defines the Relation of the Fluxions, in as far as that Relation can be determined from one Equation only including the flowing Quantities.

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The Demonstration of this Solution is founded upon Art. 40 *. Where it was shewn that the Fluxions of equal Quantities are equal. * Of this Explication.

83. It is evident that the Solution will be the same, if all the Quantities being brought to one Side, and made equal to nothing, the Fluxion of the whole be taken and put equal to nothing, it being understood that every Part of the Fluxion of a negative Term, must have an opposite Sign to what it would have, if the Term were positive. And so the Solution will be the same with our Author's.

84. But because fractional and surd Quantities may often occur in the fluential [1] Equation proposed, as in our Author's 2[d] Example *, it will be proper to shew more particularly how the Fluxions of such Quantities are to be found. And first with Respect to fractional Quantities. Let $\frac{x}{y}$ represent any fractional Quantity, having both Num[r]. and Den[r]. flowing Quantities. Put $y : x :: 1 : v$; or $\frac{x}{y} = v$ or yet $x = vy$: then $\dot{x} = \dot{v}y + v\dot{y}$ *: and by Transposition and Division $\frac{\dot{x} - v\dot{y}}{y} = \dot{v}$. Restore $\frac{x}{y}$ for v, and it is $\frac{\dot{x} - \frac{x}{y}\dot{y}}{y} = \frac{\dot{x}y - x\dot{y}}{yy} = \dot{v}$, *i.e.* (because $\dot{v} = \dot{\overline{\frac{x}{y}}}$) $\dot{\overline{\frac{x}{y}}} = \frac{\dot{x}y - x\dot{y}}{y^2}$. If either Numerator or Denominator was a constant Quantity, it is only putting $\dot{x}$ or $\dot{y}$ equal to nothing, and the Fluxion will be investigated the same way. By which way of reasoning, we deduce this general Rule for finding the Fluxion of a Fraction, *viz.* Multiply the Fluxion of the Numerator into the Denominator; subtract from that Product the Fluxion of the Denominator multiplied into the Numerator: and divide the Difference by the Square of the Denominator. Thus $\dot{\overline{\frac{a^2}{x^3}}} = \frac{0 - 3a^2\dot{x}x^2}{x^6} = -\frac{3a^2\dot{x}x^2}{x^6}$ or $-\frac{3a^2\dot{x}}{x^4}$ or $-3a^2\dot{x}x^{-4}$. Which Expressions of the Fluxion are all the same in effect: the first being the Notation arising from the Rule for finding the Fluxion of a Fraction; the last Notation arises immediately from the Application of the general Rule above *: for $\frac{a^2}{x^3} = a^2x^{-3}$. Likewise $\dot{\overline{\frac{ax + x^2}{b - x}}} = \frac{\overline{a\dot{x} + 2\dot{x}x} \times \overline{b - x} - \overline{ax + x^2} \times -\dot{x}}{b^2 - 2bx + xx} = \frac{ab\dot{x} + 2b\dot{x}x - \dot{x}x^2}{b^2 - 2bx + x^2}$. By the same way of reasoning $\dot{\overline{\frac{1}{x^n}}} = \frac{-nx^{n-1}\dot{x}}{x^{2n}} = -nx^{-n-1}\dot{x}$ which is a further Demonstration of Art. 51 *.

* Art. 18. of the Quadrature of Curves.

* Art. 40. of this Explication.

* Art. 81. of this Explication.

* Of this Explication.

[1] By a *fluential* Equation is meant an Equation containing flowing Quantities, whereby their Relation is determined: the Equation thence deduced according to this first Prop.; or even any other proposed, which contains Fluxions, and thereby determines the Relation of the Fluxions, is called a *fluxional* Equation.

 85. Now

85. Now with respect to the Fluxions of Surds: let $\sqrt[n]{ax - x^2}$ or $\overline{ax - x^2}|^{\frac{1}{n}}$ be a Surd with an indefinite Index. In order to find it's Fluxion, suppose $\overline{ax - x^2}|^{\frac{1}{n}} = v$ or $ax - x^2 = v^n$. Take the Fluxions of both, and we have $a\dot{x} - 2\dot{x}x = n\dot{v}v^{n-1}$: or, by dividing both by nv^{n-1}, $\frac{a\dot{x} - 2\dot{x}x}{nv^{n-1}} = \dot{v}$. Whence, by inserting the Value of v in place of it, we have $\dot{v} = \frac{a\dot{x} - 2\dot{x}x}{n \times \overline{ax - x^2}|^{1 - \frac{1}{n}}}$ or $\frac{1}{n} \times \overline{ax - xx}|^{\frac{1}{n} - 1} \times \overline{a\dot{x} - 2\dot{x}x}$. Which is in effect the same with what it ought to be, according to the Rule for finding the Fluxion of any Power of a flowing Quantity, Art. 49 and 50 *: or by the general Rule at Art. 81 *. Likewise F. $x^{\frac{n}{m}} = \frac{n}{m} x^{\frac{n}{m} - 1} \dot{x}$. For $x^{\frac{n}{m}} = \overline{x^n}|^{\frac{1}{m}}$, whose Fluxion, by what has been now said, is $\frac{1}{m} \overline{x^n}|^{\frac{1}{m} - 1} \times nx^{n-1}\dot{x} = \frac{1}{m} x^{\frac{n}{m} - n} \times nx^{n-1}\dot{x} = \frac{n}{m} x^{\frac{n}{m} - 1} \dot{x}$. And F. $x^{-\frac{n}{m}} = -\frac{n}{m} x^{-\frac{n}{m} - 1} \dot{x}$. For $x^{-\frac{n}{m}} = \frac{1}{x^{\frac{n}{m}}}$, whose Fluxion, by the last Art. and by what has been just now shewn, is $\frac{-\frac{n}{m} x^{\frac{n}{m} - 1} \dot{x}}{x^{\frac{2n}{m}}} = -\frac{n}{m} x^{-\frac{n}{m} - 1} \dot{x}$. Wherefore universally the Rule for finding the Fluxion of any Power of a flowing Quantity, be it's Exponent positive or negative; integral or fractional, and whether the Quantity affected by the Exponent be simple or compound, is this, multiply the flowing Quantity by the Exponent of the Power, diminish the Exponent by a Unity, and multiply further by the Fluxion of the Quantity or Root: agreeably to Art. 50 and 51 *.

* Of this Explication.

* Of this Explication.

* Of this Explication.

86. Suppose now the Equation $x^3 - xy^2 + aa\sqrt{ax - y^2} - b^3 = 0$, our Author's 2[d] Ex. was proposed, to find the Fluxions: By taking the Fluxions of all the Terms, according to the preceding Rules, it gives $3\dot{x}x^2 - \dot{x}y^2 - 2x\dot{y}y + \frac{1}{2}aa \times \overline{ax - y^2}|^{-\frac{1}{2}} \times \overline{a\dot{x} - 2\dot{y}y} = 0$, or, which is the same thing, $3\dot{x}x^2 - \dot{x}y^2 - 2x\dot{y}y + \frac{a^3\dot{x} - 2a^2\dot{y}y}{2\sqrt{ax - y^2}} = 0$. Which exhibits the Relation of $\dot{x}$ and $\dot{y}$. And so of others.

87. Schol. When Fractions and Radicals, involving the flowing Quantities, enter the fluential Equation, you may investigate the Equation exhibiting the Relation of the Fluxions, by a Substitution; thus, let the Equation $x^3 - ay^2 + \frac{by^3}{a+y} - xx\sqrt{ay + x^2} = 0$ express the Relation

tion of x and y: put $\frac{by^3}{a+y} = z$ and $xx\sqrt{ay+x^2} = v$: and so you will have three Equations, *viz.* $x^3 - ay^2 + z - v = 0$, $az + yz - by^3 = 0$, and $ax^4y + x^6 - v^2 = 0$: the first of which gives $3\dot{x}x^2 - 2a\dot{y}y + \dot{z} - \dot{v} = 0$, the second gives $a\dot{z} + \dot{z}y + z\dot{y} - 3b\dot{y}y^2 = 0$, and the third, $4a\dot{x}x^3y + ax^4\dot{y} + 6\dot{x}x^5 - 2\dot{v}v = 0$, for the Relations of $\dot{x}$, $\dot{y}$, $\dot{z}$, $\dot{v}$. Now by Reduction of Equations from the two last, you may find $\dot{z} = \frac{3b\dot{y}y^2 - z\dot{y}}{a+y}$ and $\dot{v} = \frac{4a\dot{x}x^3y + ax^4\dot{y} + 6\dot{x}x^5}{2v}$: which Values of $\dot{z}$ and $\dot{v}$ being substituted in place of them, in the first of the three fluxionary Equations, there arises $3\dot{x}x^2 - 2a\dot{y}y + \frac{3b\dot{y}y^2 - z\dot{y}}{a+y} - \frac{4a\dot{x}x^3y + ax^4\dot{y} + 6\dot{x}x^5}{2v} = 0$, *i.e.* by restoring the Values of z and v, $3\dot{x}x^2 - 2a\dot{y}y + \frac{3ab\dot{y}y^2 + 2b\dot{y}y^3}{a^2 + 2ay + yy} - \frac{4a\dot{x}xy + ax^2\dot{y} + 6\dot{x}x^3}{2\sqrt{ay+xx}}$: which expresses the Relation of $\dot{x}$ and $\dot{y}$. And after the same manner you may proceed, when there are other more complicate Terms, in the fluential Equation proposed.

88. The Fluxions we have been speaking of hitherto, under this first Prop. are first Fluxions. But our Author proceeds at Art. 19.* to shew how second, third and following Fluxions may be investigated. And he tells us, it is only by repeating the same Operation, by which the first Fluxions are found.

* Of the Quadrature of Curves.

Thus if we take the fluential Equation $zy^3 - z^4 + a^4 = 0$ of Art. 19.*: from it we deduce this fluxional Equation $\dot{z}y^3 + 3z\dot{y}y^2 - 4\dot{z}z^3 = 0$, for the Relation of the first Fluxions $\dot{z}$ and $\dot{y}$. Which Equation may be reduced to an Analogy $\frac{\dot{z}}{\dot{y}} = \frac{3zyy}{4z^3 - y^3}$, or $\dot{z} : \dot{y} :: 3zy^2 : 4z^3 - y^3$. Again from the Equation $\dot{z}y^3 + 3z\dot{y}y^2 - 4\dot{z}z^3 = 0$, by considering $\dot{z}$ and $\dot{y}$ as flowing Quantities as well as z and y, we deduce this other Equation $\ddot{z}y^3 + 3\dot{z}\dot{y}y^2 + 3\dot{z}\dot{y}y^2 + 3z\ddot{y}y^2 + 6z\dot{y}\dot{y}y - 4\ddot{z}z^3 - 12\dot{z}\dot{z}z^2 = 0$, or $\ddot{z}y^3 + 6\dot{z}\dot{y}y^2 + 3z\ddot{y}y^2 + 6z\dot{y}^2y - 4\ddot{z}z^3 - 12\dot{z}^2z^2 = 0$, containing the second Fluxions of z and y. After the same manner, by considering $\ddot{z}$ and $\ddot{y}$, as well as $\dot{z}$, $\dot{y}$, z and y, as flowing Quantities; and applying the general Rule contained in this Prop. we shall have $\dddot{z}y^3 + 3\ddot{z}\dot{y}y^2 + 6\ddot{z}\dot{y}y^2 + 6\dot{z}\ddot{y}y^2 + 12\dot{z}\dot{y}\dot{y}y + 3\dot{z}\ddot{y}y^2 + 3z\dddot{y}y^2 + 6z\ddot{y}\dot{y}y + 6\dot{z}\dot{y}^2y + 12z\dot{y}\ddot{y}y + 6z\dot{y}^2\dot{y} - 4\dddot{z}z^3 - 12\ddot{z}\dot{z}z^2 - 24\ddot{z}\dot{z}z^2 - 24\dot{z}^3z = 0$, or, by adding similar Terms $\dddot{z}y^3 + 9\ddot{z}\dot{y}y^2 + 9\dot{z}\ddot{y}y^2 + 18\dot{z}\dot{y}^2y + 3z\dddot{y}y^2 + 18z\ddot{y}\dot{y}y + 6z\dot{y}^3 - 4\dddot{z}z^3 - 36\ddot{z}\dot{z}z^2 - 24\dot{z}^3z = 0$, an Equation containing the third Fluxions of the flowing Quantities x and y. And so we might proceed to Equations containing the subsequent Orders of Fluxions. The Reason of which Operations is plainly contained in what

* Of the Quadrature of Curves.

what has been ſaid already, with reſpect to the finding an Equation containing the Relation of the firſt Fluxions from a fluential Equation propoſed. For although the fluential Equation contain three or more flowing Quantities, a fluxional Equation, may notwithſtanding, be deduced from it, exhibiting the Relation of the Fluxions: by proceeding according to the general Rule.

89. If we reſume the fluential Equation at the Beginning of the laſt Art. *viz.* $zy^3 - z^4 + a^4 = 0$, and the Relation of $\dot{z}$ and $\dot{y}$ thence derived, *viz.* $\frac{\dot{z}}{\dot{y}} = \frac{3zy^2}{4z^3 - y^3}$, the firſt expreſſes the Relation of the flowing Quantities x and y at all Times, or in every State; and the other, the Relation of their Fluxions, at all Times, in Terms made up of z and y. Wherefore if we aſſume a particular determinate Value for z or y, the correſponding Value of the other may be found, from the firſt Equation and thereby the Ratio of $\dot{z}$ and $\dot{y}$ or $\frac{\dot{z}}{\dot{y}}$ will be wholly known. Thus ſuppoſe that at any Time $z = 2a$: then by ſubſtituting $2a$ for z, in the Equation $zy^3 - z^4 + a^4 = 0$, we have $2ay^3 - 15a^4 = 0$, whence $y^3 = \frac{15}{2}a^3$ or $y = a\sqrt[3]{\frac{15}{2}}$. Subſtitute theſe Values of z and y in the Expreſſion of the Ratio of $\dot{z}$ and $\dot{y}$, and it becomes $\frac{\dot{z}}{\dot{y}} = \frac{6a^3\sqrt[3]{\frac{225}{4}}}{32a^3 - \frac{15a^3}{2}} = \frac{12\sqrt[3]{\frac{225}{4}}}{49} = \frac{\sqrt[3]{97200}}{\sqrt[3]{117649}}$ wholly a known Quantity. Which ſhews that, at what Time or Place, the variable Quantity z becomes equal to the known determinate Quantity $2a$, the Velocity *with which* z flows, at that Time, is to the Velocity with which y flows at the ſame Time, in the ſubtriplicate Ratio of 97200 to 117649; or as the Cube Roots of theſe Numbers. And if z be taken of any other Value, another known Relation of $\dot{z}$ and $\dot{y}$ will ariſe.

90. Hence it appears, that when a fluential Equation is propoſed, containing two variable or flowing Quantities only, the Relation of the firſt Fluxions of theſe two flowing Quantities, may always be had in Terms containing the two variable Quantities and known Quantities only: and therefore by aſſuming one of the flowing Quantities at pleaſure, the fluential Equation, by the Rules of Algebra, will give a correſponding known Value of the other flowing Quantity: wherefore the Relation which the Fluxions of theſe two Quantities, bear to one another at that Time, will be fully known, and determined, by ſubſtituting the particular determined Values of the flowing Quantities, in place of them, in the fluxionary Equation.

91. Take another Example. Let the fluential Equation $px - y^2 = 0$ be proposed: which belongs to the common Parabola: x being the Abſciſs; y, the Ordinate; and p, the Parameter of the Diameter. The fluxionary Equation thence ariſing, is $p\dot{x} - 2\dot{y}y = 0$, whence $\dot{x} : \dot{y} :: 2y : p$. *i. e.* the variable Ratio of $\dot{x}$ and $\dot{y}$, will be at all Times, and in every Place as twice the Ordinate in that Place to the Parameter. Now ſuppoſe I would know, what that Ratio would be when $x = p$, I inſert p for x, in the Equation $px - y^2 = 0$, and it becomes $p^2 - y^2 = 0$ or $p = y$: ſo that when the Abſciſs is equal to the Parameter; ſo is the Ordinate alſo. Wherefore inſert p for y, in the Value of the Ratio of $\dot{x}$ to $\dot{y}$, and it is $\dot{x} : \dot{y} :: 2p : p :: 2 : 1$, ſo that, at that Time, the Velocity, with which the Abſciſs increaſes, is double the Velocity with which the Ordinate increaſes. And ſo by aſſuming other Values of x; or of y, other known Relations of $\dot{x}$ and $\dot{y}$ will ariſe.

92. But if there be three variable or flowing Quantities in a fluential Equation propoſed, another Equation including at leaſt two of them, ought alſo to be given, that the Relation of their Fluxions may be fully determined; and alſo their Relation among themſelves. Thus let the fluential Equation $ax + by^2 - cxz = 0$ including three flowing Quantities x, y and z be given. The fluxional Equation thence deduced is $a\dot{x} + 2b\dot{y}y - c\dot{x}z - cx\dot{z} = 0$, which gives the Relation of the Fluxions $\dot{x}$, $\dot{y}$ and $\dot{z}$, as far as that Relation can be determined from one fluential Equation only. But the Relation of the Fluents in the fluential Equation, and of their Fluxions in the fluxionary Equation, will not be fully determined, unleſs another fluential Equation be given. As if it were ſuppoſed that $x - ay + z = 0$. From whence we deduce $\dot{x} - a\dot{y} + \dot{z} = 0$, for another Relation of the Fluxions, beſides the former. Therefore by comparing the two fluential Equations together; and the two fluxional Equations, thence deduced, together, you may extreminate any one of the flowing Quantities; and alſo any one of the Fluxions, and thereby you may obtain an Equation, which will entirely determine the Relation of the other two, whether flowing Quantities or Fluxions. Thus if I wanted to have the Relation of $\dot{x}$ and $\dot{y}$, fully determined: The firſt fluxional Equation gives $\dot{z} = \frac{a\dot{x} + 2b\dot{y}y - c\dot{x}z}{cx}$: the other, which is deduced from the 2[d] fluential Equation, *&c.* either given; or aſſumed, if not given, will give $\dot{z} = a\dot{y} - \dot{x}$: therefore, by equating theſe Values of $\dot{z}$, we have $\frac{a\dot{x} + 2b\dot{y}y - c\dot{x}z}{cx} = a\dot{y} - \dot{x}$. Again by means of one of the fluential Equations, as of $x - ay + z = 0$ take a Value of z, *viz.* $z = ay - x$, and put that in place of z in the Equation $\frac{a\dot{x} + 2b\dot{y}y - c\dot{x}z}{cx}$ =

$= a\dot{y} - \dot{x}$, and it becomes $a\dot{x} + 2b\dot{y}y - ac\dot{x}y - acx\dot{y} + 2c\dot{x}x = 0$, or $\dot{x} : \dot{y} :: acx - 2by : a - acy + 2cx$. Where if you take x at pleasure, the fluential Equations will give y; and so the Relation of $\dot{x}$ and $\dot{y}$ will be entirely determined.

More shortly thus. From the fluential Equation $ax + by^2 - cxz = 0$, proposed; and the other $x - ay + z = 0$, either given or assumed, find another Equation free of z; which will be $ax + by^2 = acxy - cx^2$: and from that you'll have $a\dot{x} + 2b\dot{y}y - ac\dot{x}y - acx\dot{y} - 2c\dot{x}x$, as formerly.

93. It may be observed here, that, when there are three flowing Quantities, and but one Equation to determine their Relation, the Fluxions, as well as the Quantities themselves, admit of an infinite Variety of different Relations: and therefore when in that Case, I assume another Equation, as above, I only thereby determine one of these Relations: which, from the Nature of the Thing, are indefinitely many.

94. If there are four, five or any other Number of flowing Quantities, there ought to be given as many Equations save one, as there are Quantities, in order to the full Determination of the Relations of the first Fluxions of these flowing Quantities: as plainly appears from what has been said.

95. It was observed formerly *, that the Fluxions of homogeneous Quantities are still to be considered in relation to one another: for without such a Consideration, we can make nothing of the Doctrine of Fluxions. And therefore, since every fluxionary Equation contains the Fluxions of two flowing Quantities at least, either express'd or understood; and thereby determines only the Relation of these Fluxions, it is left at liberty, to suppose one of these flowing Quantities to flow or change at any rate, either equably and uniformly; or according to any Law of Acceleration or Retardation, we think fit to frame or suppose: because such a Supposition never alters the Relation of the Fluxions or Velocities of flowing. Hence it appears, that our Conception of the Relation of the Fluxions, will then be most clear and distinct, when we suppose one of the flowing Quantities, to flow with an uniform and invariable Velocity, and call it Unity. For thereby that uniform Fluxion is made a common Standard, by which to measure the rest. Thus in the Equation $px - y^2 = 0$, mentioned above, to a Parabola, from which the fluxionary Equation $p\dot{x} - 2\dot{y}y = 0$ is deduced, I suppose the Absciss x to flow uniformly, and put it's Fluxion $\dot{x} = 1$. By which means the fluxional Equation becomes $p - 2\dot{y}y = 0$, *i. e.* $\dot{y} = \frac{p}{2y}$: so that the Fluxion of the Abscis being always 1; the Fluxion

* Art. 6. of this Explication.

of

of the Ordinate, will always be expreſs'd by the Quantity $\frac{p}{2y}$. I might have likewiſe ſuppoſed the Parabola to be ſo deſcribed, by the Motion of the Ordinate along the Axis, that the Velocity, with which the Ordinate increaſes, is always invariably the ſame, and call it Unity: then $p\dot{x} - 2\dot{y}y = 0$ becomes $p\dot{x} - 2y = 0$, or $\dot{x} = \frac{2y}{p}$: *i.e.* in ſuch a Caſe, the Velocity, with which the Abſciſs flows will be always expreſs'd by $\frac{2y}{p}$. And ſo of others.

96. This Suppoſition is of further Uſe, when we have occaſion to proceed to 2[d], 3[d] and other ſuperior Orders of Fluxions. For in all the Equations, for determining the Relations of theſe ſuperior Orders of Fluxions, the Symbol, for expreſſing the firſt Fluxion of the uniformly flowing Quantity, may be daſhed out, wherever it is found, being Unity; and all theſe Terms thrown out, in which a ſecond, third or any ſubſequent Fluxion of the uniformly flowing Quantity, would be involved: for ſince it's firſt Fluxion is Unity, a conſtant and invariable Quantity; every one of the ſubſequent Fluxions is nothing. Which our Author obſerves in Art. 20. * and by this means the Equations, for determining the Relations of the Fluxions, are conſiderably abridg'd, as is there to be ſeen.

* Of the Quadrature of Curves.

97. But hence it muſt happen, that a fluxionary Equation, will ſometimes appear not to have the Symbols of the Fluxions of the ſame Dimenſions, through all the Terms: which yet it ought to have, if we conſider the Manner of forming any fluxional Equation from it's fluential. Fluxions of the ſame Order, or whoſe Symbols are of the ſame Dimenſions, are ſuch as theſe, $\dot{x}$. $\dot{y}$. $\dot{z}$ of one Dimenſion; $\ddot{x}$. $\dot{x}^2$. $\dot{x}\dot{y}$; of two Dimenſions; $\dddot{x}$. $\ddot{x}\dot{y}$. $\dot{x}^3$. $\dot{x}^2\dot{y}$. $\dot{x}\dot{y}\dot{z}$ of three Dimenſions, &c. That the Symbols of the Fluxions, in each Term of any fluxionary Equation, as it is formed from the correſponding Fluential, whether it be a fluxionary Equation of the 1[ſt], 2[d], 3[d], or any other Order, muſt always be of the ſame Dimenſions, will eaſily appear to any one that conſiders, how the ſeveral Orders of fluxional Equations are formed above *. And it is neceſſary it ſhould be ſo, ſince Fluxions are Quantities of a different kind from the Quantities, whereof they are the Fluxions. Thus taking the preceding Example $px - yy = 0$: hence ariſes $p\dot{x} - 2\dot{y}y = 0$, where the Symbols of the Fluxions, in both Times, are of the ſame Dimenſion. But if we put $\dot{x} = 1$, it becomes $p - 2\dot{y}y = 0$, where one Term contains no Symbol of a Fluxion at all. Thus alſo, when we have the Equation $zy^3 - z^4 + a^4 = 0$ mentioned Art. 88 *, the third fluxional Equation, thence deduced, was ſhewn to be $\dddot{z}y^3 + 9\ddot{z}\dot{y}y^2 + 9\dot{z}\ddot{y}y^2$ +

* Art. 88. of this Explication, and Art. 18. and 19. of the Quadrature of Curves.

* Of this Explication.

$+ 18\dot{z}\dot{y}^2y + 3z\dot{\dot{y}}y^2 + 18z\dot{y}\ddot{y}y + 6z\dot{y}^3 - 4\dot{z}z^3 - 36\dot{z}\dot{z}z^2 - 24\dot{z}^3z = 0$: where all the Terms contain Symbols of Fluxions of three Dimensions. But if we suppose $\dot{z} = 1$, it appears in another Form, thus $9\ddot{y}y^2 + 18\dot{y}^2y + 3z\dot{\dot{y}}y^2 + 18z\dot{y}\ddot{y}y + 6z\dot{y}^3 - 24z = 0$.

98. Therefore, whenever a fluxionary Equation is proposed, where the Symbols of the Fluxions, in all the Terms, are not of the same Dimensions, these Dimensions must be completed, or supposed to be completed, by multiplying the lower Terms, by the Fluxion of some flowing Quantity, whose Fluxion is understood to be Unity, until the Symbols of the Fluxions arise to the same Number of Dimensions, in all the Terms. Thus the fluxionary Equation, just now mentioned, is to be completed, by multiplying the two first Terms $9\ddot{y}y^2$ and $18\dot{y}^2y$ by $\dot{z}$; and the last Term $-24z$, by $\dot{z}^3$: $\dot{z}$ being supposed equal to Unity. And so the Equation will appear thus $9\dot{z}\ddot{y}y^2 + 18\dot{z}\dot{y}^2y + 3z\dot{\dot{y}}y^2 + 18z\dot{y}\ddot{y}y + 6z\dot{y}^3 - 24\dot{z}^3z = 0$. Where the Dimensions are completed, by multiplying the lower Terms as often as is necessary by $\dot{z}$, which is the Fluxion of a flowing Quantity z, *viz.* such a one as is found in the given fluxional Equation, while in the mean Time, it's Fluxion is not found.

99. But if there be no variable or flowing Quantity in the given fluxionary Equation, but what has it's Fluxion in it likewise, then the Order of the Fluxions in the lower Terms, must be completed, by supposing some other variable or flowing Quantity, which flows uniformly, whose Fluxion must be understood to be Unity, and multiplying the lower Terms by that Fluxion, so often, until the Dimensions of the fluxionary Symbols be the same through all the Terms. Thus if the Equation $\dot{x} + \dot{x}\dot{y}x - ax^2 = 0$ were proposed, we are to suppose some third Quantity as z, which flows uniformly, having it's Fluxion $\dot{z} = 1$: and then multiply the first Term by $\dot{z}$, and the last Term by $\dot{z}\dot{z}$. And so we must conceive of the given fluxional Equation $\dot{x} + \dot{x}\dot{y}x - ax = 0$, as if it had been derived from this one $\dot{x}\dot{z} + \dot{x}\dot{y}x - a\dot{z}\dot{z}x^2 = 0$, by putting $\dot{z} = 1$. Likewise if $aa\dot{v} - av + vv = 0$ were given, I complete the Dimensions, by supposing another flowing Quantity z, whose Fluxion $\dot{z} = 1$. As if the Equation had been originally $aa\dot{v} - av\dot{z} - v^2\dot{z} = 0$. And so of others.

100. This first Prop. of our Author's, which I have been explaining, lays the Foundation of the whole Doctrine of Fluxions: particularly it contains the Solution of the first of the two general Problems mentioned at the Beginning*, *viz. the Length of the Space described, being continually or at all Times given, to determine the Velocity of the Motion at any Time proposed:* which in effect amounts to the same thing, with the

* Art. 2. of this Explication.

the Propofition as here exprefs'd: and is called the *direct* Method of Fluxions. The Application of thefe two general Problems is very extenfive: particularly by them we are enabled to folve the following ufeful Problems, *viz.*

1°. To determine the *Maxima* and *Minima* of Quantities.

2°. To draw Tangents to all Sorts of Curves, whether geometrical or mechanical.

3°. To determine the Points of contrary Flexure, and Retrogreffion in Curves.

4°. To determine the Quantity of Curvature at a given Point of a given Curve: *i.e.* to find the Length of the Radius of an equicurve Circle.

Under this may be comprehended the finding the Points, where a Curve has any given Degree of Curvature: where it has the greateft or leaft Curvature: and the Determination of the Locus of the Center of Curvature.

5°. To determine the Quality of Curvature at a given Point of a given Curve: *i.e.* it's Form, as it is more or lefs inequable; or as it varies more or lefs in it's Progrefs through different Parts of the Curve.

6°. To find as many Curves as we pleafe, which may be fquared: *i.e.* to find the Nature and Property of all fuch Curves, whofe Areas are exhibited by finite Equations *. * Prop. 2.

7°. To find as many Curves as we pleafe, the Relation of whofe Areas to the Area of any given Curve is affignable by finite Equations.

8°. To find as many Curves as we pleafe, whofe Lengths may be exprefs'd by finite Equations. To this Problem and the 4th belongs the determining the Evolutes of Curves.

9°. To find as many Curves as you pleafe, whofe Lengths may be compared with the Length of any Curve propofed; or with it's Area applied to a given Line.

10°. To find the Cauftics to all Sorts of Curves, whether by Reflection or Refraction.

11°. The *Quadrature* of Curves; or finding their Areas.

12°. The *Rectification* of Curve-lines; or finding their Lengths.

13°. The *Plaining* of Curve-furfaces; or reducing them to plain Surfaces.

14°. The *Cubature* of Solids; or finding their folid Contents.

15°. The finding the Centers of *Gravity*: and of *Ofcillation* or *Percuffion* of all Sorts of *Lines*, *Superficies's* and *Bodies*.

101. Thefe Problems for moft part belong immediately to the Geometry of Curves; but by their means, ferve to other excellent Purpofes

in Mathematics and natural Philosophy: witness that inimitable Work, the mathematical Principles of natural Philosophy: a great Part of which is founded upon the Geometry of Curves: and Principles of the same or like Nature with those of Fluxions.

102. To go through all these Problems would take up a great deal of Time; and be foreign to our Design: however some of them that come in most naturally, shall be explain'd and resolved in the Progress of this Work: particularly the 1st, 2d, 3d, 6th, 7th, 8th, 12th, 13th, 14th. The 11th is that whose Explication and Solution takes up the great Part of this Work.

103. Among geometrical Problems, one of the most noble and useful is the *Quadrature* of Curves: and accordingly the greatest Geometricians, in all Ages, have laboured this Point with all the Force of Invention; and exerted themselves to the utmost upon this Head. Yet there was very little done in that way, till after the Middle of the last Century, that the Doctrine of infinite Series was introduced; founded upon the Arithmetic of Infinites, made use of by Dr. *Wallis*. Which Method of infinite Series was improved and perfected by the Discovery of the Method of Fluxions by our celebrated Author about the Year 1666[1]: whereby the Quadrature of Curves is rendered universal.

104. This is a new kind of Analysis: by means of which the Areas of all Sorts of Curves are investigated and determined, either to a perfect Exactness; or, when the Nature of the Curve will not admit of that, to any Degree of Exactness, one can possibly require. And such is the Subject of this Treatise.

105. But if any one should think that the Quadrature of Curves, being accounted but one Branch of the Application of the Doctrine of Fluxions, scarcely deserves to be insisted upon at so great length: it must be known, that, the Quadrature of Curves, is not only among the most sublime and useful Discoveries in Geometry, considered in itself: but in effect includes under it, the finding the Length of Curvelines; the Solidity of Bodies, the Contents of their Surfaces, and Innumerable other Problems. Add to this that the original Principles, and

[1] See Dr. *Wallis*'s *Treatise of Algebra*, Chap. 82, 85, 91, and the *Commercium Epistolicum de varia re Mathematica*, particularly a Letter from Sir *I. Newton* to Mr. *Oldenburg*, dated *October* 24, 1676. Some time after Sir *I. Newton* had discovered his *Doctrine of infinite Series and Fluxions*, Mr. *James Gregory* Professor of Mathematics at *St. Andrews*, about the End of the Year 1670, fell upon a like general Method, serving for the same Purposes; having discovered many particular Series's long before that Time: as appears from the *Commercium Epist.* See Mr. *Gregory*'s Letters to Mr. *Colins*, dated *Sept.* 5. *Nov.* 23. *Dec.* 24. Anno 1670. *Feb.* 15. Anno 167$\frac{0}{1}$, likewise Mr. *Colins*'s Letter to Dr. *David Gregory*: and Sir *I. Newton*'s second Letter to Mr. *Oldenburg* already mentioned.

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justest Conceptions of the Method of Fluxions, are deduced from the Consideration of the Description of curvilinear Areas, after the Manner already explained. Upon which account, the Quadrature of Curves, here laid down by our Author, ought to be fully understood by all such as would have a just and comprehensive Notion of Fluxions.

Explication of Prop. 2.

106. By this Proposition, the Relation betwixt the Area of any Curve and it's Abscis or rather the Rectangle contain'd under the Abscis and an invariable right Line, supposed to be Unity, being exhibited, you can always determine the Nature of the Curve, *i.e.* the Relation betwixt the Abscis and Ordinate: or you may assume or feign any Relation you please, betwixt the curvilinear and rectilinear Areas ABC and ABED [1], and thence determine the Relation of the corresponding Ordinates AB and BC.

107. Although the two following Prop. be Examples of this, yet, for the sake of a further *Illustration* of this Prop. I shall shew it's Application, in some Examples, which are more simple and easy: by which, at the same Time, the Demonstration of it will be made more evident.

Ex. 1. Calling the Abscis AB $= z$; the Area ABC $= t$; the Ordinate BC $= y$, and the Ordinate BE $=$ (AD $=$) 1: and consequently the Rectangle ABED $= z \times 1 = z$: Let the Relation of the two Areas ABC and ABED, *i.e.* of t and z, be express'd by this Equation $z^2 = t$: then taking the Fluxions by Prop. 1, we have $2\dot{z}z = \dot{t}$ *: *i.e.* by reducing to an Analogy, $1 : 2z :: \dot{z} : \dot{t}$: but $\dot{z} : \dot{t} ::$ EB : BC *: therefore by Equality, $1 : 2z ::$ (EB : BC ::) $1 : y$: whence $2z = y$: Which defines the Relation of the Abscis and Ordinate of the Figure ABC: and shews it to be a right-angled Triangle, having the Base BE double the Altitude AB.

* Art. 40. of this Explication.
* Art. 29. of this Explication.

Ex. 2. Suppose we assume $\frac{2}{3}z\sqrt{az} = t$, or $\frac{2}{3}a^{\frac{1}{2}}z^{\frac{3}{2}} = t$, for the Relation of the Areas ABED and ABC: then, by taking the Fluxions, we have $a^{\frac{1}{2}}\dot{z}z^{\frac{1}{2}} = \dot{t}$, *i.e.* $1 : a^{\frac{1}{2}}z^{\frac{1}{2}} ::$ ($\dot{z} : \dot{t} ::$) EB $= 1$: BC $= y$: Whence $\sqrt{az} = y$ or $az = y^2$: which shews the Curve to be the common Parabola.

Ex. 3. Let $az^3 = 3b^2t + 3bcz^3t$ exhibit the Relation of the Areas: thence, by taking the Fluxions, we have $3a\dot{z}z^2 = \underline{3bb\dot{t} + 9bc\dot{z}z^2t}$ $\underline{+ 3bcz^3\dot{t}}$; or, by dividing by 3 and transposing $\overline{az^2 - 3bcz^2t} \times \dot{z}$ $= \overline{bb + bcz^3} \times \dot{t}$, *i.e.* $b^2 + bcz^3 : az^2 - 3bcz^2t ::$ ($\dot{z} : \dot{t} ::$) $1 : y$:

[1] See the Fig. belonging to this Prop.

 whence,

whence, by multiplying the Extremes and Means, we have $b^2y + bcz^3y = az^2 - 3bcz^2t$ for the Relation of z, y and t. Now that t may be exterminated: from the Equation expressing the Relation of the Areas, *viz.* $az^3 = 3b^2t + 3bcz^3t$, find the Value of t, which is $t = \frac{az^3}{3b^2 + 3bcz^3}$; substitute this Value in the Equation just now found, *viz.* $b^2y + bcz^3y = az^2 - 3bcz^2t$, and it becomes $b^2y + bcz^3y = az^2 - 3bcz^2 \times \frac{az^3}{3b^2 + 3bcz^3}$, *i.e.* by Reduction, $y = \frac{abz^2}{b^2 + 2bcz^3 + c^2z^6}$; which shews the Relation of z and y or AB and BC, and so determines the Nature of the Curve ABC sought.

Ex. 4. Let the Equation $2a^2t - 4zt - a^4 = tt$ be assumed for the Relation of z and t: by taking the Fluxions, we have $2a^2\dot{t} - 4\dot{z}t - 4z\dot{t} = 2\dot{t}t$; or $a^2\dot{t} - 2z\dot{t} - \dot{t}t = 2\dot{z}t$, *i. e.* $a^2 - 2z - t : 2t ::$ $(\dot{z} : \dot{t} ::)$ $1 : y$, whence $y = \frac{2t}{a^2 - 2z - t}$: and in order to exterminate t, let it's Value be found from the Equation $2a^2t - 4zt - a^4 = tt$, expressing the Relation of the Areas: which will be $t = a^2 - 2z \pm 2\sqrt{z^2 - a^2z}$: insert this Value of t in place of it, in the Equation $y = \frac{2t}{a^2 - 2z - t}$, and we shall have $y = \left(\frac{2a^2 - 4z \pm 4\sqrt{z^2 - aaz}}{a^2 - 2z - a^2 + 2z \mp 2\sqrt{z^2 - aaz}} = \frac{a^2 - 2z \pm 2\sqrt{z^2 - aaz}}{\mp\sqrt{z^2 - aaz}} =\right) \pm \frac{2z - a^2}{\sqrt{z^2 - a^2z}} - 2$ for defining the Nature of the Curve.

Ex. 5. Let the Equation $\frac{-2ab + acz^2}{3cc}\sqrt{b + cz^2} = t$ be given or assumed for expressing the Relation of z and t: then by taking the Fluxions, we have $\frac{2a\dot{z}z}{3c}\sqrt{b + cz^2} + \frac{c\dot{z}z}{\sqrt{b + cz^2}} \times \frac{-2ab + acz^2}{3cc} = \dot{t}$; or by due Reduction $\frac{a\dot{z}z^3}{\sqrt{b + cz^2}} = \dot{t}$: whence it is $1 : \frac{az^3}{\sqrt{b + cz^2}} :: (\dot{z} : \dot{t} ::)$ $1 : y$; or $y = \frac{az^3}{\sqrt{b + cz^2}}$.

* Art. 29, 31. of this Explication.

* Art. 95. of this Explication.

108. Schol. Since we have always $\dot{t} : \dot{z} :: y : 1$ *, thence $\dot{z}y = \dot{t}$, is an Equation expressing the Relation of $\dot{z}$ and $\dot{t}$ in every Case: in which Equation, if we suppose $\dot{z} = 1$, as we may do *, it becomes $y = \dot{t}$. Whence the Solution of this Problem may be a little more expeditious, thus: From the Equation propos'd or assum'd, expressing the Relation of t and z, find the Equation expressing the Relation of the Fluxions $\dot{t}$ and $\dot{z}$: in which Equation, put 1 for $\dot{z}$; or dash out $\dot{z}$ every where, and then the resulting Value of $\dot{t}$, will be the Value also of y: therefore substitute y for $\dot{t}$: and so you shall have an Equation expressing the Nature of the Curve. Thus in the 2d Ex. $\frac{2}{3}a^{\frac{1}{2}}z^{\frac{3}{2}} = t$, the Relation of the

the Fluxions is $a^{\frac{1}{2}}\dot{z}z^{\frac{1}{2}} = \dot{t}$, *i. e.* by ſubſtituting 1 for $\dot{z}$ and y for $\dot{t}$, $\sqrt{az} = y$. Likewiſe in the 4th Ex., *viz.* $2a^2t - 4zt - a^4 = tt$, the Relation of the Fluxions is $a^2\dot{t} - 2\dot{z}t - 2z\dot{t} = \dot{t}t$, *i. e.* $a^2y - 2t - 2zy = yt$, by means of which Equation, and the propoſed Equation, t being exterminated, we obtain, as formerly $y = \pm \frac{2z - aa}{\sqrt{z^2 - aaz}} - 2$ for defining the Nature of the Curve. And ſo in all other Caſes.

Our Author's 3d Prop. is an Example of the ſame thing: and is ſo plainly demonſtrated by himſelf, that, if what hath been ſaid in the foregoing Schol. be conſidered, there cannot remain any Difficulty in it: and therefore I ſhall paſs on to the next Propoſition.

Demonſtration of Prop. 4.

109. The Demonſtration of this Prop. is omitted by our Author, becauſe it may be eaſily ſupplied: if the Demonſtration of Prop. 3d be duly conſidered: and it is thus. Let us ſuppoſe the Area $z^{\theta}R^{\lambda}S^{\mu} = v$: then by taking the Fluxions according to Prop. 1, we have $\theta\dot{z}z^{\theta-1}R^{\lambda}S^{\mu} + \lambda z^{\theta}\dot{R}R^{\lambda-1}S^{\mu} + \mu z^{\theta}R^{\lambda}\dot{S}S^{\mu-1} = \dot{v}$.

Inſtead of $R^{\lambda}S^{\mu}$, in the firſt Term of the Equation, ſubſtitute $RR^{\lambda-1}SS^{\mu-1}$; inſtead of $z^{\theta}S^{\mu}$, in the ſecond Term, $zz^{\theta-1}SS^{\mu-1}$: and inſtead of $z^{\theta}R^{\lambda}$, in the third Term, ſubſtitute $zz^{\theta-1}RR^{\lambda-1}$: which Expreſſions are evidently equal to the others: and it becomes

$\theta\dot{z}z^{\theta-1}RR^{\lambda-1}SS^{\mu-1} + \lambda zz^{\theta-1}\dot{R}R^{\lambda-1}SS^{\mu-1} + \mu zz^{\theta-1}RR^{\lambda-1}\dot{S}S^{\mu-1} = \dot{v}$

Or $\overline{\theta\dot{z}RS + \lambda z\dot{R}S + \mu zR\dot{S}} \times z^{\theta-1}R^{\lambda-1}S^{\mu-1} = \dot{v}$.

But, by Hyp. $R = e + fz^{\eta} + gz^{2\eta} + \&c.$ and $S = k + lz^{\eta} + mz^{2\eta} + \&c.$

From which two laſt Equations, find the Relations of the Fluxions by Prop. 1, and you'll have $\dot{R} = \eta f\dot{z}z^{\eta-1} + 2\eta g\dot{z}z^{2\eta-1} + \&c.$ And $\dot{S} = \eta l\dot{z}z^{\eta-1} + 2\eta m\dot{z}z^{2\eta-1} + \&c.$

Subſtitute theſe Values of $\dot{R}$ and $\dot{S}$, in place of them in the preceding fluxional Equation; and likewiſe the Values of R and S in place of them: and put $\dot{z} = 1$. And ſo you'l have $\theta \times \overline{e + fz^{\eta} + gz^{2\eta} + \&c.} \times \overline{k + lz^{\eta} + mz^{2\eta} + \&c.} + \lambda z \times \overline{\eta fz^{\eta-1} + 2\eta gz^{2\eta-1} + \&c.} \times \overline{k + lz^{\eta} + mz^{2\eta} + \&c.} + \mu z \times \overline{e + fz^{\eta} + gz^{2\eta} + \&c.} \times \overline{\eta lz^{\eta-1} + 2\eta mz^{2\eta-1} + \&c.}$ into $z^{\theta-1}R^{\lambda-1}S^{\mu-1} = \dot{v} =$ [1] $(y=)$ BC * That is, if you actually multiply the ſeveral Factors; and obſerve to rank the Terms according to the Dimenſions of the variable Quantity z,

* Art. 108. of this Explication.

[1] y is ſuppoſed to ſtand for the Ordinate of the Curve.

θek

$$\left.\begin{array}{llll} \theta e k \begin{array}{l}+\theta\\+\lambda\eta\end{array}\times fkz^{\eta} & \begin{array}{l}+\theta\\+2\lambda\eta\end{array}\times gkz^{2\eta} \text{- - - -} & * \text{- - - - - - - -} & * \text{- - - - - - - -} \\ \begin{array}{l}+\theta\\+\mu\eta\end{array}\times elz^{\eta} & \begin{array}{l}+\theta\\+\lambda\eta\\+\mu\eta\end{array}\times flz^{2\eta} & \begin{array}{l}+\theta\\+2\lambda\eta\\+\mu\eta\end{array}\times glz^{3\eta} \text{- -} & * \text{- - - - - - -} \\ & \begin{array}{l}+\theta\\+2\mu\eta\end{array}\times emz^{2\eta} & \begin{array}{l}+\theta\\+\lambda\eta\\+2\mu\eta\end{array}\times fmz^{3\eta} & \begin{array}{l}+\theta\\+2\lambda\eta\\+2\mu\eta\end{array}\times gmz^{4\eta} \end{array}\right\} \times z^{\theta-1}R^{\lambda-1}S^{\mu-1} = BC$$

110. Cor. 1. Since θ and λ, the Exponents of the Powers of z and R, in the 3^d Prop. and θ, λ, μ, the Exponents of the Powers of z, R and S, in the 4th Prop. may be any Numbers poſitive or negative; integral or fractional: for the reaſoning is the ſame, whatever they be: hence it follows, that, when any of theſe Exponents is negative, in the Expreſſion of the Area of the Curve, all the Change that it brings upon the Expreſſion of the correſponding Ordinate, is this, that, wherever that negative Exponent, or theſe negative Exponents occur, it, or they muſt be affected with the negative Sign. Thus in the 3^d Prop. the following Areas and Ordinates correſpond.

Areas.	Ordinates.
111. $z^{-\theta}R^{\lambda}$	$\overline{\theta e \begin{array}{l}-\theta\\+\lambda\eta\end{array}\times fz^{\eta} \begin{array}{l}-\theta\\+2\lambda\eta\end{array}\times gz^{2\eta}\ \text{\&c.}}\times z^{-\theta-1}R^{\lambda-1}$
112. $z^{\theta}R^{-\lambda}$	$\overline{\theta e \begin{array}{l}+\theta\\-\lambda\eta\end{array}\times fz^{\eta} \begin{array}{l}+\theta\\-2\lambda\eta\end{array}\times gz^{2\eta}\ \text{\&c.}}\times z^{\theta-1}R^{-\lambda-1}$
113. $z^{-\theta}R^{-\lambda}$	$\overline{\theta e \begin{array}{l}-\theta\\-\lambda\eta\end{array}\times fz^{\eta} \begin{array}{l}-\theta\\-2\lambda\eta\end{array}\times gz^{2\eta}-\text{\&c}}\times z^{-\theta-1}R^{-\lambda-1}$

And the like are the Changes that will ariſe, in the Expreſſion of the Ordinate in Prop. 4.

Moreover it is evident that, if the Exponent of the Power of z without the Vinculum, in the Expreſſion of the Areas of the Curves belonging to theſe two Prop. were made $\theta \pm \eta\varpi$; ſo that the Area in Prop. 3, ſtood thus $z^{\theta\pm\eta\varpi}R^{\lambda}$: and that in Prop. 4. ſtood thus $z^{\theta\pm\eta\varpi}R^{\lambda}S^{\mu}$, where ϖ is ſuppoſed to denote any integer Number: all the Change, thereby brought, upon the Form of the Expreſſion of the correſponding Ordinates, is this, that $\theta \pm \eta\varpi$ muſt be ſubſtituted in place of θ, wherever it is found. So that the following Areas and Ordinates will correſpond.

Areas.

Areas. **Ordinates.**

114. $z^{\theta\pm\eta\sigma}R^{\lambda}$ -- $\overline{\theta\pm\eta\sigma}\times e \;+\overline{\theta\pm\eta\sigma+\lambda\eta}\,fz^{\eta} \;+\overline{\theta\pm\eta\sigma+2\lambda\eta}\,gz^{2\eta} \;+\overline{\theta\pm\eta\sigma+3\lambda\eta}\,hz^{3\eta} \;+\&c.\times z^{\theta\pm\sigma\eta-1}R^{\lambda-1}$

Or $\overline{\theta\pm\eta\sigma}\,ez^{\pm\eta\sigma} + \overline{\theta\pm\eta\sigma+\lambda\eta}\,fz^{\eta\pm\sigma\eta} + \overline{\theta\pm\eta\sigma+2\lambda\eta}\,gz^{2\eta\pm\sigma\eta} + \overline{\theta\pm\eta\sigma+3\lambda\eta}\,hz^{3\eta\pm\sigma\eta} + \&c.\times z^{\theta-1}R^{\lambda-1}$

115. $z^{\theta\pm\sigma\eta}R^{\lambda}S^{\mu}$ --

$$\left.\begin{array}{llll}
\overline{\theta\pm\sigma\eta}\times ek + \overline{\theta\pm\sigma\eta+2\lambda\eta}\times fkz^{\eta} & +\overline{\theta\pm\sigma\eta+2\lambda\eta}\times gkz^{2\eta} & \ast & \ast \\
+\overline{\theta\pm\sigma\eta+\mu\eta}\,elz^{\eta} & +\overline{\theta\pm\sigma\eta+\lambda\eta+\mu\eta}\times flz^{2\eta} & +\overline{\theta\pm\sigma\eta+2\lambda\eta+\mu\eta}\times glz^{3\eta} & \ast \\
 & +\overline{\theta\pm\sigma\eta+2\mu\eta}\times emz^{2\eta} & +\overline{\theta\pm\sigma\eta+\lambda\eta+2\mu\eta}\times fmz^{3\eta} & +\overline{\theta\pm\sigma\eta+2\lambda\eta+2\mu\eta}\times gmz^{4\eta}
\end{array}\right\}\times z^{\theta\pm\sigma\eta-1}R^{\lambda-1}S^{\mu-1}$$

Or

$$\left.\begin{array}{llll}
\overline{\theta\pm\sigma\eta}\times ekz^{\pm\sigma\eta} + \overline{\theta\pm\sigma\eta+2\lambda\eta}\times fkz^{\eta\pm\sigma\eta} & +\overline{\theta\pm\sigma\eta+2\lambda\eta}\times gkz^{2\eta\pm\sigma\eta} & \ast & \ast \\
+\overline{\theta\pm\sigma\eta+2\mu\eta}\times elz^{\eta\pm\sigma\eta} & +\overline{\theta\pm\sigma\eta+\lambda\eta+\mu\eta}\times flz^{2\eta\pm\sigma\eta} & +\overline{\theta\pm\sigma\eta+2\lambda\eta+\mu\eta}\times glz^{3\eta\pm\sigma\eta} & \ast \\
 & +\overline{\theta\pm\sigma\eta+2\mu\eta}\times emz^{2\eta\pm\sigma\eta} & +\overline{\theta\pm\sigma\eta+\lambda\eta+2\mu\eta}\times fmz^{3\eta\pm\sigma\eta} & +\overline{\theta\pm\sigma\eta+2\lambda\eta+2\mu\eta}\times gmz^{4\eta\pm\sigma\eta}
\end{array}\right\}\times z^{\theta-1}R^{\lambda-1}S^{\mu-1}$$

And it is further evident that, if any of the Signs of the Letters entering the Values of R or S, in the Expreſſion of the Areas, be negative, there muſt be a correſponding Change made upon the Signs in the Expreſſion of the Ordinates: which is eaſily known from the common Rules of Algebra. Which will be clear by applying the general Expreſſions to a particular Example. Wherefore, let $\frac{3a^2x-3x^3}{\sqrt[3]{b^3+ccx-dx^2}}$ be the Area of a Curve whoſe Abſciſs is x: that it may be compared with the general Form of Areas in Prop. 4, I bring it to ſtand thus $x^1\times\overline{b^3+c^2x-dx^2}|^{-\frac{1}{3}}\times\overline{3a^2\ast-3x^2}|^1$: by comparing which with the general Expreſſion $z^{\theta}R^{\lambda}S^{\mu}$, we have $z=x$. $R=b^3+c^2x-dx^2$, $S=3a^2-3x^2$. $\theta=1$. $\lambda=-\frac{1}{3}$, $\mu=1$. $e=b^3$, $f=c^2$, $g=-d$. $k=3a^2$, $l=0$, $m=-3$. Inſert theſe particular Values in the general Form of the Ordinate in Prop. 4, and you'll have $\overline{\begin{array}{l}3a^2b^3+2a^2c^2x-a^2dx^2-8c^2x^3+7dx^4\\ \qquad\qquad\qquad\qquad\quad -9b^3\end{array}}$ $\times\overline{b^3+c^2x-dx^2}|^{-\frac{4}{3}}$ for the Ordinate of the Curve whoſe Area is $\frac{3a^2x-3x^3}{\sqrt[3]{b^3+ccx-dx^2}}$.

The laſt Factor, correſponding to S, is repreſented as wanting the 2d Term, *viz.* thus $3a^2\ast-3x^2$, becauſe, ſince the Powers of z belonging

longing to R and S, in the general Form, are suppos'd to have their Indexes going on by the same Differences, being η, 2η, 3η, &c. and since the Value of η, in this particular Case, must be Unity: therefore it is evident why the second Term of the Value of S must be nothing, and therefore $l = 0$.

And moreover by the same way of reasoning it appears that, if it be $R = e + fz^{-\eta} + gz^{-2\eta} + hz^{-3\eta} + \&c.$ in Prop. 3: and $R = e + fz^{-\eta} + gz^{-2\eta} + \&c.$ and likewise $S = k + lz^{-\eta} + mz^{-2\eta} + nz^{-3\eta} + \&c.$ in Prop. 4: where the Indexes of the Powers of z under the Vinculums are all negative: then the Sign of η must be chang'd every where into it's Opposite in the Expressions of the Ordinates of the Curves, belonging to these Prop. respectively.

115. It may likewise be observed that, if the general Forms of the Areas, *viz.* $z^{\theta}R^{\lambda}$ and $z^{\theta}R^{\lambda}S^{\mu}$ be multiplied by any constant or given Quantity, as A, and so become $Az^{\theta}R^{\lambda}$ and $Az^{\theta}R^{\lambda}S^{\mu}$, the corresponding Ordinates of the Curves must be multiplied by the same given Quantity A. For the Fluents being increased or diminished in a given Ratio, the Fluxions are increased or diminished in the same given Ratio *: and the Areas being the Fluents, the Ordinates are as the Fluxions in this Case *, because the Fluxions of the Abscisses are the same.

* Art. 40. of this Explication.

* Art. 32. of this Explication.

116. By the same way of reasoning as that made use of in these two Prop. it appears that, if the Area of a Curve be $z^{\theta}R^{\lambda}S^{\mu}T^{\nu}$, where ν is any given Number, and $T = p + qz^{\eta} + rz^{2\eta} + \&c.$ and the other Symbols stand for the same things as before, the Ordinate of such Curve may be found: and so for other Areas more complex.

117. It follows from Prop. 3, and what has been said that if

$$\overline{\theta \pm \sigma\eta}\, eAz^{\pm\sigma\eta} + \overline{\theta \pm \sigma\eta + \lambda\eta} \times fAz^{\eta \pm \sigma\eta} + \overline{\theta \pm \sigma\eta + 2\lambda\eta} \times gAz^{2\eta \pm \sigma\eta} + \overline{\theta \pm \sigma\eta + 3\lambda\eta} \times hAz^{3\eta \pm \sigma\eta} + \&c. \times Az^{\theta - 1}R^{\lambda - 1}$$

be the Ordinate of a Curve, the Area of the Curve is $Az^{\theta \pm \sigma\eta}R^{\lambda}$: for if not, the Area $Az^{\theta \pm \sigma\eta}R^{\lambda}$ would belong to a Curve having a different Ordinate, contrary to what has been demonstrated. And the like is to be understood with respect to the Area and Ordinate contain'd in the general Expressions in Prop. 4.

SECT.

SECT. IV.

Containing Notes on Art. 27——31.

Some Lemmas serving to demonstrate several Things contain'd in the Author's Remarks upon Prop. 5.

LEMMA. I.

119. AN infinite Series of Powers of a proper Fraction, whose Indexes constitute an arithmetical Progression ascending, having finite Coefficients [1], will converge: *i. e.* a finite Number of Terms of the Series, will differ from the Sum of the whole Series by less than any given Difference.

Let $\frac{N}{D}$ represent any proper Fraction, $\overline{\frac{N}{D}}\big|^{n} . \overline{\frac{N}{D}}\big|^{n+p} . \overline{\frac{N}{D}}\big|^{n+2p} . \overline{\frac{N}{D}}\big|^{n+3p}$ *&c. in inf.* an infinite Series of Powers of that Fraction, whose Exponents are $n . n+p . n+2p . n+3p$, *&c.* then it is evident, the Series is an infinite geometrical Progression descending, having $\overline{\frac{N}{D}}\big|^{p}$ for the common Multiplier: wherefore, according to the Nature of such a Progression, the Sum of the whole is $\frac{\overline{\frac{N}{D}}|^{n}}{1-\overline{\frac{N}{D}}|^{p}}$. Let q be any given Quantity how small soever; and T a Term of the Series less than $q \times \overline{1-\overline{\frac{N}{D}}|^{p}}$ (for, it is evident, such a Term may be found) so that $T < q \times \overline{1-\overline{\frac{N}{D}}|^{p}}$. Now, the Sum of all the Terms after T inclusive, from the Property of a geometrical Progression, is $\frac{T}{1-\overline{\frac{N}{D}}|^{p}}$: therefore that Sum is less than $\frac{q \times \overline{1-\overline{\frac{N}{D}}|^{p}}}{1-\overline{\frac{N}{D}}|^{p}} = q$. Now, supposing the several Terms to have any finite Coefficients prefix'd, so that none of them be greater than A: then you need only assume any Term of the Series, which

[1] By finite Coefficients in this Lem. I mean such as the greatest of them is not greater than any assignable Quantity.

without it's Coefficient, is less than $\frac{q}{A} \times \overline{1 - \frac{N}{D}}|^{p}$: and then all the subsequent Terms from that Term inclusive, without the Coefficients, will be less than $\frac{\frac{q}{A} \times \overline{1 - \frac{N}{D}}|^{p}}{\overline{1 - \frac{N}{D}}|^{p}} = \frac{q}{A}$, as has been shewn: wherefore, since no Coefficient is greater than A, the Sum of all these Terms with their Coefficients prefixt, cannot be so great as $\frac{q}{A} \times A = q$, *i. e.* the Sum of them all is less than any given Quantity.

120. Schol. Hence it appears that an infinite Series of Powers of any Quantity greater than Unity, whose Indexes decrease in an arithmetical Progression, without Coefficients, or with finite Coefficients, will converge. For if $A \times N^{n} . B \times N^{n-p} . C \times N^{n-2p}$ &c. *in inf.* represent such a Series: it is equivalent to this other $AN^{n} \times \overline{\frac{1}{N}}|^{0} . BN^{n} \times \overline{\frac{1}{N}}|^{p} . CN^{n} \times \overline{\frac{1}{N}}|^{2p}$ &c. which converges by the foregoing Lemma.

LEMMA II.

121. An infinite Series, such as these just now mentioned, which converges when all the Terms are affected with the same Sign, will also converge when the Terms are affected with different Signs.

For since a Term may be found, when all are positive, or all negative, such, as that the Sum of it with all that follow shall be less than any given Quantity; much more will the Amount of that Term with all that follow be less than any given Quantity, when the Terms are affected with different Signs: and therefore the Series converges: since the Amount of the whole Series is supposed not to be less than a finite Quantity. Q. E. D.

LEMMA. III.

122. If any rational integral Fluent, including only one indeterminate or flowing Quantity x: and it's Fluxion, have any common prime Divisor, which includes x: I say the Fluent shall contain such common prime Divisor oftener by once and no more, than it's Fluxion doth.

Let $a + bx^{n} + cx^{2n} + dx^{3n} +$ &c. be a Fluent, where the Quantities a, b, c, d, &c. are constant or invariable; and any of them may be equal to nothing. It's Fluxion is $nb\dot{x}x^{n-1} + 2nc\dot{x}x^{2n-1} + 3nd\dot{x}x^{3n-1}$

+

$+$ &c *. Now it is demonſtrated by Writers upon Algebra [1], that, if the ſeveral Terms of the Quantity $a+bx^n+cx^{2n}+dx^{3n}+$ &c. be multiplied by the Exponents of the Powers of x in theſe Terms reſpectively, and the Products thence ariſing be divided by x: and if the Quantity which reſults, *viz.* $nbx^{n-1}+2ncx^{2n-1}+3ndx^{3n-1}+$ &c. have any common prime Diviſor with the Quantity propoſed $a+bx^n+cx^{2n}+dx^{3n}+$ &c. that prime Diviſor ſhall be contain'd juſt once oftener in the Quantity $a+bx^n+cx^{2n}+dx^{3n}+$ &c. than in the Quantity $nbx^{n-1}+2ncx^{2n-1}+3ndx^{3n-1}+$ &c: therefore if the flowing Quantity propoſed $a+bx^n+cx^{2n}+dx^{3n}+$ &c. and it's Fluxion $nb\dot{x}x^{n-1}+2nc\dot{x}x^{2n-1}+3nd\dot{x}x^{3n-1}+$ &c. have any common prime Diviſor, including the variable Quantity x, ſuch Diviſor muſt be contain'd juſt once oftener in the Fluent than in the Fluxion. Q.E.D. * Art. 82. of this Explication.

Explication of Prop. 5, and of the Author's Remarks upon it.

Although our Author's Demonſtration of this Propoſition may be evident enough to any one that carefully conſiders it, and underſtands what hath been already taught: yet for the ſake of a farther Illuſtration; and ſome Obſervations to be made upon it, I ſhall exhibit the Demonſtration at ſome further Length.

123. Wherefore, ſuppoſing the ſame things as are ſuppoſed by Sir *Iſaac*; and moreover that the Factor $a+bz^n+cz^{2n}+dz^{3n}$ &c. continued one Term further, is $a+bz^n+cz^{2n}+dz^{3n}+ez^{4n}+$ &c, ſo that the Ordinate of the Curve may be $z^{\theta-1}R^{\lambda-1}\times\overline{a+bz^n+cz^{2n}+dz^{3n}+ez^{4n}+\&c}$; likewiſe that $R=e+fz^n+gz^{2n}+hz^{3n}+iz^{4n}+$ &c; as alſo that $v+\lambda=u$: then by what has been demonſtrated *, the following Ordinates and Areas ſet oppoſite to them, will belong to the ſame Curves. * Art. 114, 116. of this Explication.

	Ordinates.		Areas.
1°.	$\theta eA {+\theta \atop +\lambda n} \times fAz^n {+\theta \atop +2\lambda n} \times gAz^{2n} {+\theta \atop +3\lambda n} \times hAz^{3n} {+\theta \atop +4\lambda n} \times iAz^{4n} + \&c.$		$Az^{\theta}R^{\lambda}$
2°.	$* \overline{\theta+n} \times eBz^n {+\overline{\theta+n} \atop +\lambda n} \times fBz^{2n} {+\overline{\theta+n} \atop +2\lambda n} \times gBz^{3n} {+\overline{\theta+n} \atop +3\lambda n} \times hBz^{4n} + \&c.$		$Bz^{\overline{\theta+n}}R^{\lambda}$
3°.	$* \overline{\theta+2n} \times eCz^{2n} {+\overline{\theta+2n} \atop +\lambda n} \times fCz^{3n} {+\overline{\theta+2n} \atop +2\lambda n} \times gCz^{4n} + \&c.$	$\times z^{\theta-1}R^{\lambda-1}$	$Cz^{\overline{\theta+2n}}R^{\lambda}$
4°.	$* \overline{\theta+3n} \times eDz^{3n} {+\overline{\theta+3n} \atop +\lambda n} \times fDz^{4n} + \&c.$		$Dz^{\overline{\theta+3n}}R^{\lambda}$
5°.	$* \overline{\theta+4n} \times eEz^{4n} + \&c.$		$Ez^{\overline{\theta+4n}}R^{\lambda}$

[1] See *Reyneau's Analyſe demontrée*, B. 4. Sect. 4.

Therefore if the Sum of these Ordinates be suppos'd equal to the given Ordinate $\overline{a+bz^n+cz^{2n}+dz^{3n}+\varepsilon z^{4n}+\&c.}\times z^{\theta-1}R^{\lambda-1}$, the Sum of the corresponding Areas, *viz.* $\overline{A+Bz^n+Cz^{2n}+Dz^{3n}+Ez^{4n}+\&c.}\times z^{\theta}R^{\lambda}$ shall be equal to the Area of the Curve, whereof the Ordinate propos'd, is the Ordinate.

Wherefore that the Values of A, B, C, D, E, *&c.* which are determinate Quantities, may be discovered, let the corresponding Terms of the two Ordinates be compared, by putting them equal: and thence arise the following Equations.

1°. $\theta eA = a$

2°. $\overline{\theta+\lambda n}\times fA+\overline{\theta+n}\times eB = b$

3°. $\overline{\theta+2\lambda n}\times gA+\overline{\theta+n+\lambda n}\times fB+\overline{\theta+2n}\times eC = c$

4°. $\overline{\theta+3\lambda n}\times hA+\overline{\theta+n+2\lambda n}\times gB+\overline{\theta+2n+\lambda n}\times fC+\overline{\theta+3n}\times eD = d$

5°. $\overline{\theta+4\lambda n}\times iA+\overline{\theta+n+3\lambda n}\times hB+\overline{\theta+2n+2\lambda n}\times gC+\overline{\theta+3n+\lambda n}\times fD+\overline{\theta+4n}\times eE = \varepsilon$.

From which, by Reduction of Equations, we obtain the following Equations, expressing the Values of A, B, C, *&c. viz.*

1°. $A = \frac{a}{\theta e}$

2°. $B = \frac{b-\overline{\theta+\lambda n}\times fA}{\overline{\theta+n}\times e}$

3°. $C = \frac{c-\overline{\theta+2\lambda n}\times gA-\overline{\theta+n+\lambda n}\times fB}{\overline{\theta+2n}\times e}$

4°. $D = \frac{d-\overline{\theta+3\lambda n}\times hA-\overline{\theta+n+2\lambda n}\times gB-\overline{\theta+2n+\lambda n}\times fC}{\overline{\theta+3n}\times e}$

5°. $E = \frac{\varepsilon-\overline{\theta+4\lambda n}\times iA-\overline{\theta+n+3\lambda n}\times hB-\overline{\theta+2n+2\lambda n}\times gC-\overline{\theta+3n+\lambda n}\times fD}{\overline{\theta+4n}\times e}$

&c. in inf.

Hence it appears, that, if the given Ordinate $\overline{a+bz^n+cz^{2n}+dz^{3n}+\varepsilon z^{4n}+\&c.}\times z^{\theta-1}R^{\lambda-1}$ be suppos'd equal to the Sum of the Ordinates above; and consequently the Area of the Curve, whose Ordinate is given, equal to the Sum of the corresponding Areas, the Values of A, B, C, *&c.* must be those express'd by the last Equations just now mentioned. But the Sum of the Areas is $\overline{A+Bz^n+Cz^{2n}+Dz^{3n}+Ez^{4n}+\&c.}\times z^{\theta}R^{\lambda}$. Wherefore, by inserting the Values of A, B, C, *&c.* found above, we shall have

$z^{\theta}R^{\lambda}$

$$z^{\theta}R^{\lambda} \times + \frac{a}{\theta e}$$
$$+ \frac{b - \overline{\theta + \lambda\eta} \times fA}{\overline{\theta + \eta} \times e} z^{\eta}$$
$$+ \frac{c - \overline{\theta + 2\lambda\eta} \times gA - \overline{\theta + \eta + \lambda\eta} \times fB}{\overline{\theta + 2\eta} \times e} z^{2\eta}$$
$$+ \frac{d - \overline{\theta + 3\lambda\eta} \times hA - \overline{\theta + \eta + 2\lambda\eta} \times gB - \overline{\theta + 2\eta + \lambda\eta} \times fC}{\overline{\theta + 3\eta} \times e} z^{3\eta}$$
$$+ \frac{\varepsilon - \overline{\theta + 4\lambda\eta} \times iA - \overline{\theta + \eta + 3\lambda\eta} \times hB - \overline{\theta + 2\eta + 2\lambda\eta} \times gC - \overline{\theta + 3\eta + \lambda\eta} \times fD}{\overline{\theta + 4\eta} \times e} z^{4\eta}$$
$$+ \&c.$$

for the Value of the Area of the Curve, having the propos'd Ordinate $z^{\theta-1}R^{\lambda-1} \times \overline{a + bz^{\eta} + cz^{2\eta} + dz^{3\eta} + \varepsilon z^{4\eta} + \&c.}$

124. Which Area, by putting $\frac{\theta}{\eta} = r$ or $\theta = r\eta$, $r + \lambda = s$, $s + \lambda = t$, $\overline{t + \lambda} = v$, $v + \lambda = u$; and inverting the Order of all the Terms ſave the firſt, in the ſeveral Numerators, ſtands thus

$$z^{\theta}R^{\lambda} \times + \frac{\frac{1}{\eta}a}{re}$$
$$+ \frac{\frac{1}{\eta}b - sfA}{\overline{r + 1} \times e} z^{\eta}$$
$$+ \frac{\frac{1}{\eta}c - \overline{s + 1} \times fB - tgA}{\overline{r + 2} \times e} z^{2\eta}$$
$$+ \frac{\frac{1}{\eta}d - \overline{s + 2} \times fC - \overline{t + 1} \times gB - vhA}{\overline{r + 3} \times e} z^{3\eta}$$
$$+ \frac{\frac{1}{\eta}\varepsilon - \overline{s + 3} \times fD - \overline{t + 2} \times gC - \overline{v + 1} \times hB - uiA}{\overline{r + 4} \times e} z^{4\eta}$$
$$+ \&c.$$

The ſame with the Expreſſion of the Area delivered by our Author; ſave only that he hath not ſet down the firſt and laſt Terms of the laſt Numerator, *viz.* $\frac{1}{\eta}\varepsilon$ and $-uiA$; which I have inſerted.

And the Law of Progreſſion of the Series is manifeſt.

125. But that the Properties of it; and the Laws or Conditions, according to which it either terminates; or is continued infinitely, may the

the more evidently appear: and that the Expression of the Area may be wholly in known Terms, let us insert instead of the Capitals A, B, C, D, &c. their Values at length, which are easily deduced from the foregoing Equations*. Which done, the Series expanded at full Length, is

*Art. 123. of this Explication.

$$z^{\theta}R^{\lambda}\times+\frac{\frac{1}{n}a}{re}z^{0}=A.$$

$$+\frac{\frac{1}{n}b}{\overline{r+1}\times e}z^{n}-\frac{sf\times\frac{1}{n}a}{r\times\overline{r+1}\times e^{2}}z^{n}=Bz^{n}$$

$$+\frac{\frac{1}{n}c}{\overline{r+2}\times e}z^{2n}-\frac{\overline{s+1}\times f\times\frac{1}{n}b}{\overline{r+1}\times\overline{r+2}\times e^{2}}z^{2n}+\frac{\overline{s+1}\times sf^{2}\times\frac{1}{n}a}{r\times\overline{r+1}\times\overline{r+2}\times e^{3}}z^{2n}-\frac{tg\times\frac{1}{n}a}{r\times\overline{r+2}\times e^{2}}z^{2n}$$
$$=Cz^{2n}$$

$$+\frac{\frac{1}{n}d}{\overline{r+3}\times e}z^{3n}-\frac{\overline{s+2}\times f\times\frac{1}{n}c}{\overline{r+2}\times\overline{r+3}\times e^{2}}z^{3n}+\frac{\overline{s+2}\times\overline{s+1}\times f^{2}\times\frac{1}{n}b}{\overline{r+1}\times\overline{r+2}\times\overline{r+3}\times e^{3}}z^{3n}$$
$$-\frac{\overline{s+2}\times\overline{s+1}\times sf^{3}\times\frac{1}{n}a}{r\times\overline{r+1}\times\overline{r+2}\times\overline{r+3}\times e^{4}}z^{3n}+\frac{\overline{s+2}\times tfg\times\frac{1}{n}a}{r\times\overline{r+2}\times\overline{r+3}\times e^{3}}z^{3n}$$
$$-\frac{\overline{t+1}\times g\times\frac{1}{n}b}{\overline{r+1}\times\overline{r+3}\times e^{2}}z^{3n}+\frac{\overline{t+1}\times sfg\times\frac{1}{n}a}{r\times\overline{r+1}\times\overline{r+3}\times e^{3}}z^{3n}-\frac{vh\times\frac{1}{n}a}{r\times\overline{r+3}\times e^{2}}z^{3n}$$
$$=Dz^{3n}$$

$$+\frac{\frac{1}{n}e}{\overline{r+4}\times e}z^{4n}-\frac{\overline{s+3}\times f\times\frac{1}{n}d}{\overline{r+3}\times\overline{r+4}\times e^{2}}z^{4n}+\frac{\overline{s+3}\times\overline{s+2}\times f^{2}\times\frac{1}{n}c}{\overline{r+2}\times\overline{r+3}\times\overline{r+4}\times e^{3}}z^{4n}$$
$$-\frac{\overline{s+3}\times\overline{s+2}\times\overline{s+1}\times f^{3}\times\frac{1}{n}b}{\overline{r+1}\times\overline{r+2}\times\overline{r+3}\times\overline{r+4}\times e^{4}}z^{4n}+\frac{\overline{s+3}\times\overline{s+2}\times\overline{s+1}\times sf^{4}\times\frac{1}{n}a}{r\times\overline{r+1}\times\overline{r+2}\times\overline{r+3}\times\overline{r+4}\times e^{5}}z^{4n}$$
$$-\frac{\overline{s+3}\times\overline{s+2}\times tf^{2}g\times\frac{1}{n}a}{r\times\overline{r+2}\times\overline{r+3}\times\overline{r+4}\times e^{4}}z^{4n}+\frac{\overline{s+3}\times\overline{t+1}\times fg\times\frac{1}{n}b}{\overline{r+1}\times\overline{r+3}\times\overline{r+4}\times e^{3}}z^{4n}$$
$$-\frac{\overline{s+3}\times\overline{t+1}\times sf^{2}g\times\frac{1}{n}a}{r\times\overline{r+1}\times\overline{r+3}\times\overline{r+4}\times e^{4}}z^{4n}+\frac{\overline{s+3}\times vfh\times\frac{1}{n}a}{r\times\overline{r+3}\times\overline{r+4}\times e^{3}}z^{4n}$$
$$-\frac{\overline{t+2}\times g\times\frac{1}{n}c}{\overline{r+2}\times\overline{r+4}\times e^{2}}z^{4n}+\frac{\overline{t+2}\times\overline{s+1}\times fg\times\frac{1}{n}b}{\overline{r+1}\times\overline{r+2}\times\overline{r+4}\times e^{3}}z^{4n}$$
$$-\frac{\overline{t+2}\times\overline{s+1}\times sf^{2}g\times\frac{1}{n}a}{r\times\overline{r+1}\times\overline{r+2}\times\overline{r+4}\times e^{4}}z^{4n}+\frac{\overline{t+2}\times tg^{2}\times\frac{1}{n}a}{r\times\overline{r+2}\times\overline{r+4}\times e^{3}}z^{4n}$$
$$-\frac{\overline{v+1}\times h\times\frac{1}{n}b}{\overline{r+1}\times\overline{r+4}\times e^{2}}z^{4n}+\frac{\overline{v+1}\times sfh\times\frac{1}{n}a}{r\times\overline{r+1}\times\overline{r+4}\times e^{3}}z^{4n}-\frac{ui\times\frac{1}{n}a}{r\times\overline{r+4}\times e^{2}}z^{4n}$$
$$=Ez^{4n}$$
$$+\ \&c.$$

 The

The same with the former Series, by inserting the Values of the Capitals A, B, C, *&c.* And that it may be reduced to regular Series's, begin at the first Term $\frac{\frac{1}{n}a}{re}z^0$, and running through all the rest in order, set down in one Series, all these Parts of each complex Term, which will constitute a Series of geometrical Proportionals going on by the common Multiplier $\frac{z^n}{e}$, with their proper Coefficients: then beginning with the first Part of the second complex Term, *viz.* $\frac{\frac{1}{n}b}{\overline{r+1}}\times\frac{z^n}{e}$, and running through all the subsequent Terms in order, setting down in another Series, all those Parts of each complex Term that will constitute a second geometrical Progression, going on by the same common Multiplier as before $\frac{z^n}{e}$, with their proper Coefficients prefix'd: then beginning with the first Part of the third complex Term, *viz.* $\frac{\frac{1}{n}c}{\overline{r+2}}\times\frac{z^{2n}}{e}$ take out of the subsequent Terms, as before, such Parts as constitute a third geometrical Progression proceeding by the same common Multiplier still, *viz.* $\frac{z^n}{e}$ with their proper Coefficients: and so do continually. And thence the Area of the Curve, having the propos'd Ordinate $z^{\theta-1}R^{\lambda-1}\times\overline{a+bz^n+cz^{2n}+dz^{3n}+ez^{4n}+\&c.}$ shall be express'd as follows,

$$z^{\theta}R^{\lambda}\times+\frac{\frac{1}{n}a}{r}\times\frac{z^0}{e}-\frac{sf\times\frac{1}{n}a}{r\times\overline{r+1}}\times\frac{z^n}{e^2}+\frac{\overline{s+1}\times sf^2\times\frac{1}{n}a}{r\times\overline{r+1}\times\overline{r+2}}\times\frac{z^{2n}}{e^3}-\frac{\overline{s+2}\times\overline{s+1}\times sf^3\times\frac{1}{n}a}{r\times\overline{r+1}\times\overline{r+2}\times\overline{r+3}}\times\frac{z^{3n}}{e^4}$$

$$+\frac{\overline{s+3}\times\overline{s+2}\times\overline{s+1}\times sf^4\times\frac{1}{n}a}{r\times\overline{r+1}\times\overline{r+2}\times\overline{r+3}\times\overline{r+4}}\times\frac{z^{4n}}{e^5}-\&c.$$

$$+\frac{\frac{1}{n}a}{r+1}\times\frac{z^n}{e}\left.\begin{matrix}-\frac{\overline{s+1}\times f\times\frac{1}{n}b}{\overline{r+1}\times\overline{r+2}}\\[2ex]-\frac{tg\times\frac{1}{n}a}{r\times\overline{r+2}}\end{matrix}\right\}\frac{z^{2n}}{e^2}\begin{matrix}+\frac{\overline{s+2}\times\overline{s+1}\times f^2\times\frac{1}{n}b}{\overline{r+1}\times\overline{r+2}\times\overline{r+3}}\\[2ex]+\frac{\overline{s+2}\times tfg\times\frac{1}{n}a}{r\times\overline{r+2}\times\overline{r+3}}\\[2ex]+\frac{\overline{t+1}\times sfg\times\frac{1}{n}a}{r\times\overline{r+1}\times\overline{r+3}}\end{matrix}\frac{z^{3n}}{e^3}\begin{matrix}-\frac{\overline{s+3}\times\overline{s+2}\times\overline{s+1}\times f^3\times\frac{1}{n}b}{\overline{r+1}\times\overline{r+2}\times\overline{r+3}\times\overline{r+4}}\\[2ex]-\frac{\overline{s+3}\times\overline{s+2}\times tf^2g\times\frac{1}{n}a}{r\times\overline{t+2}\times\overline{r+3}\times\overline{r+4}}\\[2ex]-\frac{\overline{s+3}\times\overline{t+1}\times sf^2g\times\frac{1}{n}a}{r\times\overline{r+1}\times\overline{r+3}\times\overline{r+4}}\\[2ex]-\frac{\overline{t+2}\times\overline{s+1}\times sf^2g\times\frac{1}{n}a}{r\times\overline{r+1}\times\overline{r+2}\times\overline{r+4}}\end{matrix}\frac{z^{4n}}{e^4}+\&c.$$

+

$$+\frac{\frac{1}{n}c}{r+2}\times\frac{z^{2n}}{e}-\frac{\overline{s+2}\times f\times\frac{1}{n}c}{\overline{r+2}\times\overline{r+3}}\frac{z^{3n}}{e^2}+\frac{\overline{s+3}\times\overline{s+2}\times f^2\times\frac{1}{n}c}{\overline{r+2}\times\overline{r+3}\times\overline{r+4}}\frac{z^{4n}}{e^3}-\&c.$$

$$-\frac{\overline{t+1}\times g\times\frac{1}{n}b}{\overline{r+1}\times\overline{r+3}}\qquad+\frac{\overline{s+3}\times\overline{t+1}\times fg\times\frac{1}{n}b}{\overline{r+1}\times\overline{r+3}\times\overline{r+4}}$$

$$-\frac{vb\times\frac{1}{n}a}{r\times\overline{r+3}}\qquad+\frac{\overline{s+3}\times vfb\times\frac{1}{n}a}{r\times\overline{r+3}\times\overline{r+4}}$$

$$+\frac{\overline{t+2}\times\overline{s+1}\times fg\times\frac{1}{n}b}{\overline{r+1}\times\overline{r+2}\times\overline{r+4}}$$

$$+\frac{\overline{t+2}\times tg^2\times\frac{1}{n}a}{r\times\overline{r+2}\times\overline{r+4}}$$

$$+\frac{\overline{v+1}\times sfb\times\frac{1}{n}a}{r\times\overline{r+1}\times\overline{r+4}}$$

$$+\frac{\frac{1}{n}d}{r+3}\times\frac{z^{3n}}{e}-\frac{\overline{s+3}\times f\times\frac{1}{n}d}{\overline{r+3}\times\overline{r+4}}\frac{z^{4n}}{e^2}+\&c.$$

$$-\frac{\overline{t+2}\times g\times\frac{1}{n}c}{\overline{r+2}\times\overline{r+4}}$$

$$-\frac{\overline{v+1}\times b\times\frac{1}{n}b}{\overline{r+1}\times\overline{r+4}}$$

$$-\frac{vi\times\frac{1}{n}a}{r\times\overline{r+4}}$$

$$+\frac{\frac{1}{n}e}{r+4}\times\frac{z^{4n}}{e}-\&c.$$

&c. in inf.

Which Series exhibits the Value of the Area; and is the ſame with the immediately preceding one; only that it is reduced into Order; and divided into ſo many diſtinct Series's: the Progreſſion of which, when duly conſidered, will be manifeſt. For the Coefficients of the firſt Terms of each diſtinct Series, conſtituting the firſt perpendicular Row, are $\frac{\frac{1}{n}a}{r}$, $\frac{\frac{1}{n}b}{r+1}$. $\frac{\frac{1}{n}c}{r+2}$. $\frac{\frac{1}{n}d}{r+3}$ *&c.* then the Coefficients of the ſubſequent perpendicular Rows, are all formed from the preceding in order: the ſecond

second perpendicular Row from the first; the third from the second; the fourth from the third, *&c.* after such an easy and natural way, as will easily appear to any one that considers it. Moreover the Progression of the uppermost horizontal Rows, in each particular Series is evident.

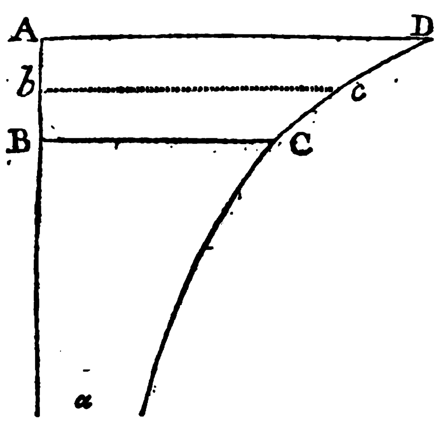
126. And now by considering these two Forms of the Expression of the Area, *viz.* that exhibited by our Author; and the immediately preceding one, we shall be enabled to discover several Properties, and Laws or Conditions, according to which, the Series will terminate or stop, after a certain Number of Terms from the Beginning; and when it will run out infinitely: and in what Cases it may or may not converge. Of all which by and by.

127. Our Author observes * that every Ordinate of a Curve is capable of being resolv'd into a Series two different Ways: *viz.* by making the Index η either positive, or negative. Thus let the Ordinate $\frac{3k - lz^2}{z^2\sqrt{kz - lz^3 + mz^4}}$ be proposed: it is reduced to the general Form mentioned in this Prop. 1°. by bringing the Quantity $z^2\sqrt{kz - lz^3 + mz^4}$ or $z^2 \times \overline{kz - lz^3 + mz^4}|^{\frac{1}{2}}$ out of the Denominator into the Numerator: which is done by changing the Signs of the Indexes of the Factors z^2 and $\overline{kz - lz^3 + mz^4}|^{\frac{1}{2}}$ into their Opposites: by which the Ordinate is express'd thus $z^{-2} \times \overline{kz - lz^3 + mz^4}|^{-\frac{1}{2}} \times \overline{3k - lz^2}$. 2°. By purging the radical Expression of $z^{-\frac{1}{2}}$: which is done by dividing it by $z^{-\frac{1}{2}}$, and multiplying the other Factor z^{-2} by $z^{-\frac{1}{2}}$, which two Operations compensate each other: for the radical Expression must be made as simple as may be: and so the whole Expression of the Ordinate propos'd, is reduc'd to this $z^{-\frac{5}{2}} \times \overline{k - lz^2 + mz^3}|^{-\frac{1}{2}} \times \overline{3k - lz^2}$; or rather $z^{-\frac{3}{2}-1} \times \overline{k - lz^2 + mz^3}|^{\frac{1}{2}-1} \times \overline{3k - lz^2}$, capable of being compared with the general Form $z^{\theta-1}R^{\lambda-1} \times \overline{a + bz^{\eta} + cz^{2\eta} + dz^{3\eta} + \&c.}$ i. e $z^{\theta-1} \times \overline{e + fz^{\eta} + gz^{2\eta} + hz^{3\eta} + \&c.}|^{\lambda-1} \times \overline{a + bz^{\eta} + cz^{2\eta} + dz^{3\eta} + \&c.}$ This is one Form of the Ordinate, where the Indexes of z, in the two Factors $\overline{k - lz^2 + mz^3}|^{-\frac{1}{2}}$ and $3k - lz^2$ are positive. But it may be brought to another Form, in which these Indexes are negative, *viz.* thus: divide the radical or surd Factor $\overline{k - lz^2 + mz^3}|^{-\frac{1}{2}}$ by $\overline{z^3}|^{-\frac{1}{2}}$, and multiply the other Factor $z^{-\frac{5}{2}}$ by the same $\overline{z^3}|^{-\frac{1}{2}}$ or $z^{-\frac{3}{2}}$, and then the Ordinate stands thus $z^{-4} \times \overline{m - lz^{-1} + kz^{-3}}|^{-\frac{1}{2}} \times \overline{3k - lz^2}$: further, divide the Factor $3k - lz^2$ by z^2, and multiply z^{-4} by the same z^2, and so you'll have

* Art. 28. of the Quadrature of Curves.

have $z^{-2} \times \overline{m - lz^{-1} + kz^{-3}}|^{-\frac{1}{2}} \times \overline{l - 3kz^{-2}}$, or, which is the same, $z^{-1-1} \times \overline{m - lz^{-1} + kz^{-3}}|^{\frac{1}{2}-1} \times \overline{l - 3kz^{-2}}$, where the Indexes of the Powers of z in the two last Factors, which were formerly positive, are made negative. And after the like manner, every Ordinate of a Curve may be expreſs'd after theſe two different ways. Accordingly, when any Ordinate is propoſed, it is to be reduced to theſe two different Forms: and both Caſes are to be tried. Which is done by applying the general Form of the Area, exhibited in this Prop. to the particular Caſe propoſed, in ſuch manner as our Author ſhews, *viz.* by ſubſtituting the particular Values inſtead of the correſponding general Expreſſions. By doing of which, the Series expreſſing the Area, puts on two different Forms: one or both of which will terminate or become finite, by the Terms vaniſhing at length; or elſe both of them will run out into infinite Series's. If one or both of the Expreſſions of the Area terminate, you have the Area exactly, and ſo the Curve is quadrable: but if both the Expreſſions of the Area run out infinitely, one of them will converge, and give the Area by Approximation, when the Curve is not quadrable.

128. Let DCα be a Curve, deſcrib'd by the Ordinate BC moving perpendicularly along the Abſciſs AB.



Let AB $= z$, BC $= y$, and a, a given Line: and let the Equation to the Curve be $y = \frac{a}{\overline{1+z}|^{2}}$. This Ordinate brought to due Form is $z^{0} \times \overline{1+z}|^{-2} \times a$ or $z^{1-1} \times \overline{1+z}|^{-1-1} \times a$, or yet, by the Method ſhewn above*, $z^{-2} \times \overline{1+z^{-1}}|^{-2} \times a$, *i. e.* $z^{-1-1} \times \overline{1+z^{-1}}|^{-1-1} \times a$. By comparing the firſt Form of the Ordinate, *viz.* $z^{1-1} \times \overline{1+z}|^{-1-1} \times a$ with the general Form $z^{\theta-1}R^{\lambda-1} \times \overline{a + bz^{\eta} + \&c.}$ we find $\theta = 1$, $\eta = 1$. $\lambda = -1$. $r = (\frac{\theta}{\eta} =)\ 1$. $s = (r + \lambda =)\ 0$. $e = 1$. $f = 1$. $g = 0 = h$, &c. $a = a$. $b = 0 = c$, &c. Whence, by Subſtitution, the general Expreſſion for the Area, $z^{\theta}R^{\lambda} \times \frac{\frac{1}{\eta}a}{re} + \frac{\frac{1}{\eta}b - sfA}{\overline{r+1} \times e}z^{\eta} + \&c.$ becomes $z \times \overline{1+z}|^{-1} \times a$ or $\frac{az}{1+z}$: All the Terms except the firſt vaniſhing.

*Art. 127. of this Explication.

nishing. But the other Form of the Ordinate proposed is $z^{-1-1} \times \overline{1 + z^{-1}}|^{-1-1} \times a$: which compar'd with the general Form, gives $\theta = -1 . \eta = -1 . \lambda = -1 . r = (\frac{\theta}{\eta} =) 1 . s = (r + \lambda =) 0$, the rest as formerly. Whence the Area, corresponding to this Form of the Ordinate, will be found, by a due Substitution of Values, in the general Expression for the Area, $z^{-1} \times \overline{1 + z^{-1}}|^{-1} \times -a$, or $\frac{-a}{1+z}$: all the Terms save the first vanishing in this Case also.

129. Hence it appears that we obtain two different Areas, both belonging to one and the same Curve DCα. But the first $\frac{az}{1+z}$ is the Area ABCD lying above the Ordinate BC, and comprehended betwixt it and the same Line AD drawn parallel to BC through the Beginning of the Abscis A: whereas the other Area, *viz.* $\frac{-a}{1+z}$ is represented by the Space αBC, lying below the Ordinate BC, and adjacent to the Abscis infinitely produced towards α. Now that there is nothing strange in this; but on the contrary, that it is what ought naturally to happen, will appear, if we consider that the two Areas ABCD and αBC have one and the same common Fluxion: because the Velocity with which the one flows by Increase, is the same with that whereby the other flows by Decrease: or the positive Fluxion of the one is the negative Fluxion of the other; accordingly while the one increases, the other decreases, and contrarily. And since a positive Fluxion produces a positive Fluent; and a negative Fluxion, a negative Fluent, hence AB ($=z$) and consequently the Area ABCD, being both supposed to flow by Increase, that Area is represented positive, being $\frac{az}{1+z}$; but the other αBC is represented negative, being $\frac{-a}{1+z}$. And so it always happens, that two Areas, terminating at the same Ordinate of a Curve, and situate upon different Sides of that Ordinate; but upon the same Side of the Abscis or Base, are affected with opposite Signs. And therefore since the Area above the Ordinate, because of it's positive Fluxion is positive; the Area below the Ordinate, if it lie upon the same Side of the Abscis produc'd, must be negative.

130. There are several other Things respecting the Position and true Values of curvilinear Areas, of which more afterwards. Let it suffice just now to shew how one may determine the initial Limit of any Area, or the Term whence it commences. To do this, all that is necessary, is to suppose the Expression of the Area equal to nothing, and the Value of the Abscis z, agreeing to that Supposition, will shew the initial

 Limit.

Limit. Thus in the Example above, the first of the two Areas was $\frac{az}{1+z}$: wherefore put $\frac{az}{1+z}=0$: hence $az=0$, *i. e.* $z=0$, since a is a given Quantity: therefore, when the Area $\frac{az}{1+z}$ is nothing, the Absciss z is nothing; and in no other Circumstances can the Expression $\frac{az}{1+z}$ be nothing, but when the Absciss z vanishes. And this shews that the initial Limit of the Area is at the Beginning of the Abscis, or is to be computed from the Line AD, drawn through A, parallel to BC: whence it follows that it is the Area ABCD. Again put the other Expression of the Area, *viz.* $\frac{-a}{a+z}=0$: and in that Case, z must be infinitely great, since otherwise the Expression $\frac{-a}{a+z}$ could not be equal to nothing: which shews that the Area designed thereby, lies along the Abscis infinitely produced below the Ordinate BC: and therefore denotes the Area αBC, as was said.

131. Now, although the initial Limit, from whence any curvilinear Area commences, may be different from either of these; and fall in any other Place; yet wherever it falls, it may be determin'd the same way. Thus taking the Area $-2\sqrt{\frac{k-lz^2+mz^3}{z^3}}$, which is shewn by our Author to belong to the Curve, whose Ordinate is $\frac{3k-lz^2}{z^2\sqrt{kz-lz^3+mz^4}}$: put $-2\sqrt{\frac{k-lz^2+mz^3}{z^3}}=0$, and thence you'll have this Equation $k-lz^2+mz^3=0$, by the Construction of which cubical Equation the Length of z the Abscis is determined, through the Extremity of which an Ordinate being drawn gives the initial Limit of the Area. And thus far with respect to the Effect of the two-fold Expression or Form of the Ordinate, in determining the initial Limit and Position of the Area of the Curve, in order to explain and illustrate what is said by our *Author* in Art. 28 *.

* Of the Quadrature of Curves.

132. Therefore when an Ordinate of any Curve is proposed, both Expressions or Forms of the Ordinate must be tried: and sometimes, both the corresponding Areas will be finite, and exactly determined, as in the Example above *: sometimes, one of them only can be exactly found, as in the Example adduced by Sir *Isaac* *. One or other of these will happen, if the Curve be quadrable, *i. e.* admit of a geometrical Comparison with a rectilinear Figure. But if the Curve be not quadrable, both Expressions of the Area will run out into an infinite Series: yet one or other of them will converge, and thereby approximate to the

* Art. 128. of this Explication.

* Art. 28. of the Quadrature of Curves.

 Value

Value of the Area. To which notwithſtanding there are ſome Exceptions. 1°. When the Value of r in both Series's expreſſing the Area is o or a negative Integer; although it be not o in both; nor a negative Integer in both at the ſame Time. For ſince r, $r+1$, $r+2$, $r+3$, &c. are Factors in the Denominators of the Terms of the Series expreſſing the Area of the Curve, according to the Author's Form of it *, hence it follows that if $r=0$ or a negative Integer, ſome Term muſt become infinitely great, the Denominator vaniſhing: and therefore if this be the Caſe with reſpect to both Forms of the Series, both that in which the Powers of z have poſitive Exponents; and that alſo in which they have negative Exponents, the Value of the Area in both Caſes is infinitely great, *i.e.* the Area lying on both Sides of the Ordinate. Whereas if r be o or a negative Integer in one of the Caſes only, you conclude from this that the Area upon the one Side of the Ordinate is infinite; but finite upon the other: which muſt be obſerved, leſt the Ambiguity in the Manner it is expreſs'd by Sir *Iſaac*, ſhould lead one into a Miſtake. Thus taking the Equation to the equilateral Hyperbola $\frac{1}{1+z}=y$: which reduced to Form is either $z^{1-1}\times\overline{1+z}|^{0-1}=y$, or $z^{0-1}\times\overline{1+z^{-1}}|^{0-1}=y$; by comparing the laſt of which with the general Form of Ordinates $z^{\theta-1}\times\overline{e+fz^{\eta}}|^{\lambda-1}=y$, you have $\theta=0=\lambda$, and therefore $r=(\frac{\theta}{\eta}=)\,0$: whence you conclude that the Area lying along the Abſciſs infinitely produced beyond the Ordinate, is infinitely great. Whereas, ſince in the former Caſe it is $\theta=1=\eta$, hence $r=(\frac{\theta}{\eta}=)$ 1, and therefore the Area adjacent to the Abſciſs and terminated at the Ordinate is finite, and ſo, according to this Prop. is thus expreſs'd $z\times\overline{1-\frac{1}{2}z+\frac{1}{3}z^2-\frac{1}{4}z^3+\text{\&c. in inf.}}$ or $z-\frac{1}{2}z^2+\frac{1}{3}z^3-\frac{1}{4}z^4$ &c. which converges, and approximates to the Value of the Area, if z do not exceed 1; otherwiſe not. But it muſt be obſerved that this *Exception* muſt be taken with this Limitation, *viz.* That the Term whoſe Denominator vaniſhes, have not it's Numerator equal to o at the ſame Time: for if that ſhould happen, the whole Term vaniſhes, and the Area is finite. There is another Exception mentioned by our Author againſt the general Rule about the Convergency of the Series by which the Area is expreſs'd: which is this, *if* $\frac{z^{\eta}}{e}=1$ *the Series does not converge.* In this Caſe the Powers of $\frac{z^{\eta}}{e}$ are all the ſame, *viz.* 1, and therefore by obſerving the Form and Progreſſion of the Series in Art. 125 *, where $\frac{z^{\eta}}{e}$ is the common Multiplier, in producing the laſt Factor of each

* Art. 26. of the Quadrature of Curves.

* Of this Explication.

each Term from that of the preceding, it appears that the Series cannot converge, unleſs it be by means of the Coefficients, which ſeldom happens: but that this ſometimes happens in the preſent Caſe, we ſhall demonſtrate juſt now: and therefore this Exception muſt be taken with ſome Limitation: as likewiſe muſt that which follows in this Art. *viz.* *That if* $\frac{z^\eta}{e}$ *be leſs than* 1, *the Series will converge in which the Index* η *is poſitive; but if* $\frac{z^\eta}{e}$ *be greater than* 1, *the other Series will converge.* For, that this will not always hold, will appear by the following Example. Let there be given this Equation $y = \frac{1}{e+fz} = z^{1-1} \times \overline{e+fz^1}|^{0-1}$: by comparing which with the general Expreſſion $z^{\theta-1} \times \overline{e+fz^\eta}|^{\lambda-1}$, you have $\theta = 1$, $\eta = 1$. $\lambda = 0$. $r = (\frac{\theta}{\eta} =) 1$, $s = 1$: whence the Series for the Area is $\frac{z}{e} \times \overline{1 - \frac{1}{2} \times \frac{fz}{e} + \frac{1}{3} \times \frac{f^2z^2}{e^2} - \frac{1}{4} \times \frac{f^3z^3}{e^3} + \frac{1}{5} \times \frac{f^4z^4}{e^4} - \&c.\ in\ inf.}$ where, if we ſuppoſe $\frac{fz}{e}$ to be any thing greater than Unity, the Series will not converge, whatever the Value of $\frac{z}{e}$ be. Thus ſuppoſe $f = 3$, $e = 2$, $z = 1$, and therefore $\frac{fz}{e} = \frac{3}{2}$: then the Series for the Area will be $\frac{1}{2} \times \overline{1 - \frac{1}{2} \times \frac{3}{2} + \frac{1}{3} \times \overline{\frac{3}{2}}|^2 - \frac{1}{4} \times \overline{\frac{3}{2}}|^3 + \frac{1}{5} \times \overline{\frac{3}{2}}|^4 - \&c.\ in\ inf.}$ where, it's evident, the Value of the Terms after the third, continues to increaſe ſtill more and more, and therefore the Series doth not converge although $\frac{z^\eta}{e} = \frac{z}{e} = \frac{1}{2}$ be leſs than 1. 2°. If it be $\frac{z}{e} = 1$, yet if f be any thing leſs than 1, the Series will converge by Art. 119, 121 *. for in that Caſe $\frac{fz}{e} < 1$. And ſo having ſhewn by this Example that the Exception, and Rules mentioned by our Author, with reſpect to the Value of $\frac{z^\eta}{e}$ in determining the Convergency or Non-convergency of the Series, muſt be taken with ſome Limitation, I ſhall here ſhew how you may determine when the Series for the Area of a binomial Curve will not converge; and when it will converge.

* Of this Explication.

133. In the Series for the Area of a binomial Curve, if $\frac{fz^\eta}{e}$ be greater than 1, the Series in which the Exponents of the Powers of z are poſitive cannot converge, whatever be the Value of $\frac{z^\eta}{e}$. For the Series for binomial Curves, putting A, B, C, D, &c. for the firſt, ſecond, third, fourth, &c. Terms (and not for their Coefficients) with their proper Signs, ſtands thus $z^\theta R^\lambda \times$

$\frac{\frac{1}{\eta}a}{re} - \frac{s}{r+1} \times \frac{fz^{\eta}}{e}A - \frac{s+1}{r+2} \times \frac{fz^{\eta}}{e}B - \frac{s+2}{r+3} \times \frac{fz^{\eta}}{e}C - \frac{s+3}{r+4} \times \frac{fz^{\eta}}{e}D -$ &c. *in inf.* Now, if you suppoſe s to be leſs than $r+1$, then it's evident, that, ſince the Cofficients $\frac{s}{r+1}$, $\frac{s+1}{r+2}$, $\frac{s+2}{r+3}$, &c. go on continually by the Addition of Units to the Numerator and Denominator, ſome Coefficient muſt at length become a proper poſitive Fraction, and always continue after that to increaſe, ſince s is leſs than $r+1$ (where a negative Quantity is conſidered as leſs than o, or any poſitive Quantity, or yet a leſs negative) becauſe the Difference continuing the ſame, the Ratio of the leſs to the greater will increaſe: therefore it's evident that if $\frac{fz^{\eta}}{e} > 1$, the Coefficient multiplied into $\frac{fz^{\eta}}{e}$, will at length exceed 1; and ſo the Terms, after that, muſt increaſe: but if s be greater than $r+1$, then it's evident that whether $\frac{fz^{\eta}}{e}$ be greater than 1; or equal to 1, the Value of the Terms muſt increaſe as the Series proceeds; and therefore whatever be the Values of r and s, the Series in which the Exponents of the Powers of z are poſitive cannot converge, if it be $\frac{fz^{\eta}}{e} > 1$.

134. In any binomial Curve thus generally expreſs'd $az^{\theta-1} \times \overline{e+fz^{\eta}}|^{\lambda-1} = y$, if the Quantity $\frac{fz^{\eta}}{e}$ be leſs than 1, the Series for the Area in which the Exponent η is affirmative, will converge: but if $\frac{fz^{\eta}}{e}$ be greater than 1, the other Series, in which η is negative. For the Series for the Areas of Curves, by applying it to the Caſe of binomial Curves, may ſtand thus $\frac{az^{\theta}}{\eta re}R^{\lambda} \times \overline{1 - \frac{s}{r+1} \times \frac{fz^{\eta}}{e} + \frac{s \times \overline{s+1}}{\overline{r+1} \times \overline{r+2}} \times \overline{\frac{fz^{\eta}}{e}}|^{2} - \frac{s \times \overline{s+1} \times \overline{s+2}}{\overline{r+1} \times \overline{r+2} \times \overline{r+3}} \times \overline{\frac{fz^{\eta}}{e}}|^{3} + \frac{s \times \overline{s+1} \times \overline{s+2} \times \overline{s+3}}{\overline{r+1} \times \overline{r+2} \times \overline{r+3} \times \overline{r+4}} \times \overline{\frac{fz^{\eta}}{e}}|^{4}}$ — &c. where, if it be $s < r+1$, in the Senſe already mentioned, the Coefficients, after a certain Number of Terms from the Beginning, muſt continually diminiſh (as is evident) and therefore the Series, in this Caſe, will converge by Lemmas 2. and 4. preceding. Again, if it be $s > r+1$, and you take n an integral Number not leſs than $\frac{sfz^{\eta} - \overline{r+1} \times e}{e - fz^{\eta}}$, the Terms of the Series after the $n+2$ Place ſhall continually diminiſh faſter and faſter, or in a greater Proportion continually; and therefore the Series ſhall converge. For calling P the Term which imme-

 diately

diately precedes the $n+2$ Term, and Q the next: from the Nature of the Series, you'll have $Q = \frac{s+n}{r+n+1} \times \frac{fz^n}{e}P$: where, by inserting for n, the Quantity $\frac{sfz^n - \overline{r+1} \times e}{e - fz^n}$, and reducing, you'll have $Q = \left(\frac{\overline{s-r-1} \times efz^n}{\overline{s-r-1} \times efz^n}P =\right) P$; consequently if n be taken greater, Q will be less than P: so that hence it appears that the Value of the Terms after this constantly diminishes: and it is moreover evident that they must diminish still in a greater Ratio, since $\frac{s+n}{r+n+1} > \frac{s+n+1}{r+n+2}$: wherefore, in this Case also, the Series will converge. But finally if it be $s = r+1$, then the Curve is quadrable: for $\lambda = 1$, since $s = r + \lambda$.

Again, if it be $\frac{fz^n}{e} > 1$, I say the Series will converge in which the Powers of z are negative: for since $\frac{fz^n}{e} > 1$, it must be $\frac{ez^{-n}}{f} < 1$, and therefore, since the Powers of $\frac{ez^{-n}}{f}$ which is less than 1, run through the Terms of the Series, it appears by Lemma 2, and 4. above, and what has been just now demonstrated, that the Series must converge. Therefore in binomial Curves, which are not quadrable, whether $\frac{fz^n}{e}$ be less than nothing or greater than nothing, one or other of the Series's for the Area will converge: and if it be $\frac{fz^n}{e} < 1$, the Convergence will be the quicker *(cæteris paribus)* the less $\frac{fz^n}{e}$ is; but if $\frac{fz^n}{e} > 1$, the quicker, the greater $\frac{fz^n}{e}$ is.

There is a third Case, *viz.* when $\frac{fz^n}{e} = 1$. In which Case the Convergency depends entirely upon the Coefficients: therefore if $s \geq r+1$, or, which is the same, if $\lambda > 1$, neither of the Series's for the Area will converge, because the Quantity $\frac{s+n}{r+n+1}$, by which the preceding Term of the Series is multiplied, becomes greater than Unity. But if $s < r+1$, or which is the same, $\lambda < 1$, both the Series's for the Area will converge; for the Quantity $\frac{s \times \overline{s+1} \times \overline{s+2} \times \overline{s+3}, \&c.}{\overline{r+1} \times \overline{r+2} \times \overline{r+3} \times \overline{r+4}, \&c.}$ must continually decrease within a finite Number of Terms, and become less than any given Quantity, and that in both Forms of the Series:

Series: and the Terms being affected with the Signs + and − alternately, each Term will be greater than all the subsequent Part of the Series. For let $a - b + c - d + e - f$, &c. represent a Series of an infinite Number of Terms, which continually diminish, and are alternate, the Amount of the Series is $a + c + e$, &c. $- b - d - f$, &c. now $a + c + e$, &c. $> b + d + f$, &c. but $b + d + f$, &c. $> c + e$, &c. therefore $a > a - b + c - d + e - f$, &c. *in inf.* Lastly, if $\lambda = 1$, the Curve is quadrable.

135. In Art. 29. our Author desires to remark, that, if the Ordinate of a Curve be contain'd under a rational Factor, which he denotes by Q, and another irreducible surd Factor, denoted by R^{π}; and the Quantity under the radical Sign, *viz.* R, do not divide the rational Factor Q, then the Expression of the Ordinate, remaining as it stands; $\lambda - 1$, the Index of R in the general Form of Ordinates, *viz.* $z^{\theta-1}R^{\lambda-1} \times \overline{a + bz^{n} + cz^{2n} + \&c.})$ must be put equal to π: so that $R^{\lambda-1} = R^{\pi}$. For in this Case, the Expression of the given Ordinate cannot be rendered more simple. And so in the Application of the general Theorem, we'll have $\lambda = \pi + 1$. But if Q can be divided by R, let it be divided, and then put $\lambda - 1 = \pi + 1$, or $R^{\lambda-1} = R^{\pi+1}$, so that $\lambda = \pi + 2$. Again, $Q \times R^{\pi}$ expressing any Ordinate of a Curve, if R divide Q twice, thrice, &c. after Q is divided twice, thrice, &c. respectively by R, you must put $\lambda - 1 = \pi + 2$, $\lambda - 1 = \pi + 3$, &c. respectively; or $R^{\lambda-1} = R^{\pi+2}$; $R^{\lambda-1} = R^{\pi+3}$, &c. This Operation reduces the Expression of the Ordinate to due Form; but it is evident it alters not it's Value: for $Q \times R^{\pi} = \frac{Q}{R} \times R^{\pi+1} = \frac{Q}{R^2} \times R^{\pi+2} = \frac{Q}{R^3} \times R^{\pi+3}$, &c.

Art. 29—31. of the Quadrature of Curves.

Thus, y and z standing for the Ordinate and Absciss, let it be $y = \overline{a^2 - z^2} \times \overline{a + z}|^{-\frac{1}{2}}$; here $Q = a^2 - z^2$, $R = a + z$, $\pi = -\frac{1}{2}$ and $R^{\pi} = \overline{a + z}|^{-\frac{1}{2}}$: and in regard $a^2 - z^2$ can be divided by $a + z$ once, I divide $a^2 - z^2$ by $a + z$, and it quotes $a - z$, but then to compensate that, I increase the Index of the Surd $\overline{a + z}|^{-\frac{1}{2}}$, by Unity: and so the Expression $y = \overline{a^2 - z^2} \times \overline{a + z}|^{-\frac{1}{2}}$ is reduced to this $\overline{a - z} \times \overline{a + z}|^{-\frac{1}{2} + 1 = \frac{1}{2}} = y$, where $R^{\lambda-1} = \overline{a + z}|^{\frac{1}{2}} = \overline{a + z}|^{\frac{3}{2} - 1}$, and so $\lambda = \frac{3}{2}$. Which Alteration being made upon the Form of the Ordinate, the Area is found by this Proposition. And so in other Cases.

* Of the Quadrature of Curves. 136. At Art. 30*. our Author puts another Case, *viz.* that the Ordinate of a Curve proposed, is a rational irreducible Fraction, having it's Denominator compounded of two or more Terms. In which Case, the Denominator of the Fraction must be resolved into it's prime or incomposite Divisors[1]: this done, either there must be some prime Divisor that has no Fellow, the same with it; or then each prime Divisor must have at least one Fellow equal to it. If the first be the Case, it's an Evidence the Curve cannot be squared (the Demonstration of which shall be given by and by *:) but when the last happens, you must reject one prime Divisor of each Set or Kind: this done, the Divisor, which remains; or the Product contained under all the remaining Divisors, when there are more left, is to be put for R, in the general Expression of the Ordinate, and -2 for $\lambda-1$ or R^{-2} for $R^{\lambda-1}$, and the Ordinate proposed, must be reduced to a Fraction having R^2 for it's Denominator: except it should happen, that the Product contained under the remaining prime Divisors, be a true Square, or Cube, or Biquadrate, &c. in which Case, the corresponding Square-root, or Cube-root, or Biquadratic-root, is to be put for R, and the Indexes made negative, *viz.* -2, or -3, or -4, &c. put for λ respectively; or $R^{-3} = R^{\lambda-1}$, $R^{-4} = R^{\lambda-1}$, $R^{-5} = R^{\lambda-1}$, &c. and then the Ordinate is to be reduced to a Fraction having R^3, or R^4, or R^5, &c. for it's Denominator. All which, with the Reason of it will be very plain by the following Examples.

*Art. 139. of this Explication.

137. Let $\frac{z^5 + z^4 - 8z^3}{z^5 + z^4 - 5z^3 - z^2 + 8z - 4}$ be an Ordinate of a Curve, which is a rational irreducible Fraction; and the Example proposed by the Author. The prime Divisors of the Denominator are $z-1$, $z-1$, $z-1$, one Set; $z+2$, $z+2$ another: here there is no prime Divisor wanting a Fellow. Now 'tis evident that the Ordinate proposed is equivalent to $\frac{1}{\overline{z-1}\times\overline{z+2}} \times \frac{z^5+z^4-8z^3}{\overline{z-1}\times\overline{z-1}\times\overline{z+2}}$, where the Denominator of the first Factor is made up of the several prime Divisors, which, according to the Rule, are to be rejected: and the Denominator of the second Factor, *viz.* $\overline{z-1}\times\overline{z-1}\times\overline{z+2}$, *i.e.* z^3-3z+2 is to be put for R, and $\frac{1}{\overline{\overline{z-1}\times\overline{z-1}\times\overline{z+2}}|^2} = \frac{1}{\overline{z^3-3z+2}|^2}$ or $\frac{1}{R^2}$ for $R^{\lambda-1}$, i.e. $\overline{z^3-3z+2}|^{-2}$ for $R^{\lambda-1}$, Wherefore, the proposed Ordinate must be reduced to the Denominator $\overline{\overline{z-1}\times\overline{z-1}\times\overline{z+2}}|^2$

[1] For doing this see *Arith. Universalis,* cap. *de Inventione Divisorum.*

$= \overline{z^3 - 3z + 2}|^2$. To do which, without altering the Value of the Expreſſion $\frac{z^5 + z^4 - 8z^3}{z^5 + z^4 - 5z^3 - z^2 + 8z - 4}$, you muſt multiply the Numerator $z^5 + z^4 - 8z^3$ by ſuch Diviſor (or the Product of ſuch Diviſors, if there were more than one) as is ſupernumerary in the Denominator $\overline{z-1} \times \overline{z-1} \times \overline{z+2}$ or $z^3 - 3z^2 + 2$ above thoſe in the Denominator $\overline{z-1} \times \overline{z+2}$, which were to be rejected: here there is only one ſuch ſupernumerary Diviſor, *viz.* $z - 1$: and ſo the Ordinate propoſed ſtands in this Form $\frac{z^6 - 9z^4 + 8z^3}{\overline{z^3 - 3z + 2}|^2}$, or $z^3 \times \overline{2 - 3z + z^3}|^{-2} \times \overline{8 - 9z + z^3}$.

For $\frac{z^5 + z^4 - 8z^3}{z^5 + z^4 - 5z^3 - z^2 + 8z - 4} = \frac{1}{\overline{z-1} \times \overline{z+2}} \times \frac{z^5 + z^4 - 8z^3}{\overline{z-1} \times \overline{z-1} \times \overline{z+2}} = \frac{z-1}{\overline{z-1} \times \overline{z-1} \times \overline{z+2}} \times \frac{z^5 + z^4 - 8z^3}{\overline{z-1} \times \overline{z-1} \times \overline{z+2}} = \frac{\overline{z-1} \times \overline{z^5 + z^4 - 8z^3}}{\overline{\overline{z-1} \times \overline{z-1} \times \overline{z+2}}|^2} = \frac{z^6 - 0z^4 + 8z^3}{\overline{z^3 - 3z + 2}|^2}$ $= \overline{z^6 - 9z^4 + 8z^3} \times \overline{z^3 - 3z + 2}|^{-2} = z^3 \times \overline{2 - 3z + z^3}|^{-2} \times \overline{8 - 9z + z^3}$, or, to bring it exactly to the general Form of Ordinates, $z^{4-1} \times \overline{2 - 3z + z^3}|^{-1-1} \times \overline{8 - 9z + z^3}$. And after the ſame Manner, may any other Ordinate, having the Properties mentioned, be reduced to due Form, and ſo be compared with the general Form of Ordinates, *viz.* $z^{\theta-1} R^{\lambda-1} \times \overline{a + bz^\eta + cz^{2\eta} + \&c.}$ or $z^{\theta-1} \times \overline{e + fz^\eta + gz^{2\eta} + \&c.}|^{\lambda-1} \times \overline{a + bz^\eta + cz^{2\eta} + \&c.}$ and by a due Subſtitution of Values, the Area found. Particularly, in the preſent Example, we will have $\theta = 4 . \eta = 1 . \lambda = -1 . r = (\frac{\theta}{\eta} =) 4 . s = (r + \lambda =) 3 . t = (s + \lambda =) 2 . v = (t + \lambda =) 1 . e = 2 . f = -3 . g = 0 . b = 1 . a = 8 . b = -9 . c = 0 . d = 1$. Which Values being inſerted in the Series for finding the Areas of Curves, *viz.*

$z^\theta R^\lambda \times \overline{\frac{\frac{1}{\eta}a}{re} + \frac{\frac{1}{\eta}b - sfA}{\overline{r+1} \times e} z^\eta + \frac{\frac{1}{\eta}c - \overline{s+1} \times fB - tgA}{\overline{r+2} \times e} z^{2\eta} + \&c}$, it comes out $z^4 \times \overline{2 - 3z + z^3}|^{-1} \times \overline{\frac{8}{8} + \frac{-9+9}{10} z + \frac{0}{12} z^2 + \frac{1-1}{14} z^3 + 0}$ $= \frac{z^4}{2 - 3z + z^3}$: for ſince the ſecond, third and fourth Terms are each equal to nothing, *i.e.* $B = 0 = C = D$, and the Factors $\overline{2 - 3z + z^3}|^{-1-1}$ and $\overline{8 - 9z + z^3}$ only Quadrinomials, it will be evident, by conſidering the Progreſſion of the Series, that all the ſubſequent Terms *in infinitum*, must vanish, and therefore the Area of the Curve is $\frac{[illegible]}{2-3z+z^3}$.

N. B. It is to be observed in this Example that the Factors $\overline{2-3z+z^3}|^{-2}$ and $\overline{8-9z+z^3}$, belonging to the Expression of the Ordinate, are to be considered as Quadrinomials, and not Trinomials: for the Name of the Quantity under the Vinculum, depends upon the Number of Terms, in which the Indexes of the Powers of z, constitute an arithmetical Progression: and thus these two Factors are to be considered as standing in this Form $\overline{2-3z+0z^2+z^3}|^{-2}$ and $\overline{8-9z+0z^2+z^3}$: for the Exponents of the Powers of z, *viz.* $0,\ \eta,\ 2\eta,\ 3\eta$, &c. in the general Form of the Ordinate $z^{\theta-1}\times\overline{e+fz^\eta+gz^{2\eta}+hz^{3\eta}+\&c.}|^{\lambda-1}\times\overline{a+bz^\eta+cz^{2\eta}+dz^{3\eta}+\&c.}$ constitute an arithmetical Progression.

138. But let us next suppose the Ordinate of a Curve to be $\frac{2+3x}{x^5+3x^6+3x^7+x^8}$, a rational irreducible Fraction: the prime Divisors of whose Denominator are $1+x.\ 1+x.\ 1+x$ one sort; and $x.\ x.\ x.\ x.\ x.$ another, and therefore no prime Divisor without a Fellow: reject one prime Divisor of each sort, and the Quotient is $\overline{1+x}\times\overline{1+x}\times x\times x\times x\times x=x^4+2x^5+x^6$ a true Square, having x^2+x^3 for it's Root; therefore, I put x^2+x^3 for R, $\overline{x^2+x^3}|^{-2}$ for R^λ; but $\overline{x^2+x^3}|^{-3}$ for $R^{\lambda-1}$; *viz.* by reducing the Ordinate to the Denominator $\overline{x^2+x^3}|^3$. The Manner of which Reduction, and bringing the proposed Ordinate to due Form, appears thus.

$$\frac{2+3x}{x^5+3x^6+3x^7+x^8}=\frac{2+3x}{\overline{1+x}\times\overline{1+x}\times\overline{1+x}\times x\times x\times x\times x\times x}=\frac{1}{\overline{1+x}\times x^2}\times$$

$$\frac{2+3x}{\overline{1+x}\times\overline{1+x}\times x\times x\times x\times x^{-1}}=\frac{1}{\overline{1+x}\times x^2}\times\frac{2+3x}{\overline{1+x}\times x^2\times\overline{1+x}\times x^2}=\frac{x^{-1}}{\overline{1+x}\times x^2}$$

$$\times\frac{2+3x}{\overline{1+x}\times x^2\times\overline{1+x}\times x^2}=\frac{x^{-1}\times\overline{2+3x}}{\overline{1+x\times x^2}|^3}=\frac{x\times\overline{2+3x}}{x^6\times\overline{1+x}|^3}=x^{-5}\times\overline{1+x}|^{-3}$$

$\times\,\overline{2+3x}$ or $x^{-4-1}\times\overline{1+x}|^{-2-1}\times\overline{2+3x}$, which is equal in Value to the proposed Ordinate $\frac{2+3x}{x^5+3x^6+3x^7+x^8}$; but brought to due Form, so as to be capable of being compared with the general Form of Ordinates, *viz.* $z^{\theta-1}R^{\lambda-1}\times\overline{a+bz^\eta+cz^{2\eta}+\&c.}$: by comparing of which, we shall have $\theta=-4.\ \eta=1.\ \lambda=-2.\ r=-4$, &c. Here we need proceed no further in the Comparison, since r is a negative

negative Integer: for this is an Evidence that the Area arising from this Form of the Ordinate is infinite *. Wherefore according to what was shewn at Art. 127 * I convert the Ordinate $x^{-5}\times\overline{1+x}|^{-3}\times\overline{2+3x}$ into the other Form, in which the Exponents of the Powers of x under the Vinculums, are negative: and it becomes $x^{-7}\times\overline{1+x^{-1}}|^{-3}\times\overline{3+2x^{-1}}$ or $x^{-6-1}\times\overline{1+x^{-1}}|^{-2-1}\times\overline{3+2x^{-1}}$: by comparing which with the general Form, we have $\theta=-6$. $\eta=-1$. $\lambda=-2$. $r=6$. $s=4$. $t=2$. $v=0$. $e=1$. $f=1$, $g=0$, &c. $a=3$. $b=2$. $c=0$, &c. Hence, by inserting these Values in the Series for *Areas*, we find $x^{-6}\times\overline{1+x^{-1}}|^{-2}\times-\frac{1}{2}=-\frac{1}{2x^4+4x^5+2x^6}$ for the Area of the Curve whose Ordinate was propos'd, all the Terms save the first vanishing. And you are to proceed in all other Cases of the like Nature.

139. But it was said above *, that, when the Denominator of a rational irreducible Fraction, expressing the Ordinate of a Curve, *viz.* in Terms made up of the Abscis and known Quantities, is resolved into it's constituent *prime* Divisors [1], if there be any of these prime Divisors without a Fellow equal to it, the Curve is not quadrable: the Demonstration of which I put off to this Place. And now in order to demonstrate it: I suppose it is easily understood, that, if any Fluent, be a rational Fraction, including one flowing Quantity only x, having it's Fluxion $\dot{x}=1$, the Fluxion will be a rational Fraction: and if any fluxionary Expression, including only one flowing Quantity x and it's Fluxion $\dot{x}=1$, be a rational Fraction, it's Fluent, *if finite*, must be a rational Fraction also; for the Fluxion of a Surd can never be a rational Quantity; nor the Fluxion of a rational Integer, be a Fraction. Which Things are easily deduced from what was shewn on Prop. 1.

*Art. 133. of this Explication.
* Of this Explication.
* Art. 30. of the Quadrature of Curves. Art. 136. of this Explication.

Wherefore let $\frac{N}{D}$ be a rational irreducible Fraction expressing the Area of a Curve, which therefore is quadrable: then $\frac{\dot{N}D-N\dot{D}}{D^2}$ is the Fluxion of the curvilinear Area $\frac{N}{D}$, and consequently the Value of the Ordinate when the Fluxion of the Abscis is Unity; and must be a rational fractional Quantity. Now in the Expression $\frac{\dot{N}D-N\dot{D}}{D^2}$, every prime Divisor of the Denominator D^2 must be capable of dividing it

[1] By prime Divisors here and in what follows in this Art. is meant such as include the Abscis.

some

some even Number of Times, since it is a Square, and of dividing D half that Number of Times, and $\dot{D}$ just once less than half the Number of Times, by Lemma third. Let P represent any such prime Divisor: then if P divide D^2 but twice, it won't divide the Numerator at all: because P divides $\dot{N}D$ the first Part of the Numerator, since it divides D once; but it won't divide the other Part of the Numerator, since it can't divide $\dot{D}$, by Lemma 3, nor N by Hyp.; N and D being prime to each other, and so not $N\dot{D}$; therefore it's plain, that P won't divide the Numerator $\dot{N}D - N\dot{D}$ at all, when it divides the Denominator D^2 but twice. Again, if P divide D^2 four Times, it can divide $\dot{N}D - N\dot{D}$ only once: for it divides $\dot{N}D$ twice and $-N\dot{D}$ but once, for the same Reason as before, therefore it can divide $\dot{N}D - N\dot{D}$ only once, when it divides D^2 four Times: and thus by the same way of reasoning, whatever Number of Times any prime Divisor divides the Denominator D^2, it will divide the Numerator $\dot{N}D - N\dot{D}$ just once less than half that Number of Times. Whence it plainly follows that when $\frac{\dot{N}D - N\dot{D}}{D^2}$ is in it's lowest Terms, there can be no prime Divisor in the Denominator without a Fellow equal to it. Therefore if the Ordinate of a Curve be a rational irreducible Fraction, which contains any prime Divisor in it's Denominator, without a Fellow equal to it, the Curve cannot be squared. Q. E. O.

140. If the Ordinate of a Curve be a rational irreducible Fraction, whose Denominator, consisting of several Terms, is not a true Square; and some prime Divisor of that Denominator, be not contain'd at least three Times in it, the Curve is not quadrable: as easily appears from what has been said. Which is still a further Limitation, beyond that mentioned by Sir *Isaac*. Moreover, let a, b, c, d, &c. be the prime Divisors contained in the Denominator thrice above; and let a^{α}, b^{β}, c^{γ}, d^{δ}, &c. be the highest Powers of a, b, c, d, &c. which divide the Denominator of the Fraction in it's lowest Terms, then the Curve is not quadrable, unless when the Denominator of the Fraction in it's lowest Terms being multiplied by $a^{\alpha-2}b^{\beta-2}c^{\gamma-2}d^{\delta-2}$, &c. makes a complete Square: for the Multiplication of the Numerator and Denominator by this last Quantity, must bring back the Expression of the Ordinate to it's original Form: as may easily appear from what has been said: and therefore if the Curve be quadrable, the Ordinate will be

be brought back to this Form $\frac{\dot{N}D-N\dot{D}}{D^2}$: in which the Denominator is a true Square.

141. At Art. 31 *, Sir *Isaac* directs how we are to manage, when the Ordinate is an irreducible Fraction, whose Denominator is made up of a rational Factor, which he calls Q, and an irreducible surd Factor, denoted by R^{λ}, that the proposed Ordinate may be brought to due Form. The Operation contains no Difficulty in it: and it amounts to this. After you have thrown by one prime Divisor of each Magnitude or Sort, contained in the Root R, see whether the Quantity Q, multiplied by the Divisors of R, which remain, either make up R, or any positive Power of R, whose Index is an Integer, denoted by *m*: then, if that be the Case, $R^{\lambda-1}$, in the general Expression of the Ordinate, becomes $R^{-\pi-m}$, *viz.* by reducing the proposed fractional Ordinate, to the Denominator $R^{\pi+m}$: and this Reduction will be always made thus: multiply the Numerator of the proposed Fraction, by the Divisors of the Root R, which remain, after one of each Magnitude or Kind is thrown by; and instead of the Denominator $Q \times R^{\pi}$, write $R^{\pi+m}$, and it is done. That the Thing may be perfectly evident, let $\frac{N}{Q \times R^{\pi}}$ represent the proposed fractional Ordinate, and suppose $a.b.c$ and $a.a.b.c.c.c$ to denote the component prime Divisors of Q and R respectively, so that the Ordinate may be represented thus $\frac{N}{abc \times \overline{aabccc}^{\,\pi}}$, where, after throwing by one prime Divisor of each Sort, of R, *viz.* $a.b.c$ and multiplying $Q = abc$ by all the remaining Divisors of R, *viz.* $a.c.c$, the Product is $aabccc = R$. Therefore, it's evident, if N be multiplied also by the remaining prime Divisors $a.c.c$, the Fraction $\frac{N}{abc \times \overline{aabccc}^{\,\pi}}$ will be converted into this equivalent one $\frac{acc \times N}{\overline{aabccc}^{\,1} \times \overline{aabccc}^{\,\pi}}$ or $\frac{acc \times N}{R^1 \times R^{\pi}} = \frac{acc \times N}{R^{\pi+1}}$. After the same manner, if we suppose $\frac{N}{Q \times R^{\pi}} = \frac{N}{a^3b^2c^4 \times \overline{a^2bc^3}^{\,\pi}}$, where, after throwing by $a.b.c$, one of each of the prime Divisors of R, and multiplying $Q = a^3b^2c^4$, by the Product of the remaining ones, *viz.* ac^2, $a^4b^2c^6$ arises, which is the Square of R; it will appear that $\frac{N}{Q \times R^{\pi}} = \left(\frac{N}{a^3b^2c^4 \times \overline{a^2bc^3}^{\,\pi}} = \frac{acc \times N}{a^4b^2c^6 \times \overline{a^2bc^3}^{\,\pi}} = \frac{acc \times N}{\overline{aabccc}^{\,2} \times \overline{aabccc}^{\,\pi}} =\right) \frac{acc \times N}{\overline{aabccc}^{\,\pi+2}}$. And so of others.

* Of the Quadrature of Curves.

Thus

Thus let $\frac{3q^5 - q^4x + 9q^3x^2 - q^2x^3 - 6qx^4}{q^2 - x^2 \times \overline{q^3 + q^2x - qx^2 - x^3}|^{\frac{1}{3}}}$ be the Ordinate of a Curve: the prime Divisors of $q^3 + q^2x - qx^2 - x^3$, are $q + x \cdot q + x \cdot q - x$: after throwing by $q + x$ and $q - x$ multiply the rational Factor $q^2 - x^2$, by the remaining prime Divisor $q + x$, the Product is $q^3 + q^2x - qx^2 - x^3$, therefore you transform the Ordinate by multiplying the Numerator by $q + x$ and making the Denominator $\overline{q^3 + q^2x - qx^2 - x^3}|^{\frac{1}{3}+1}$, whereby it stands thus

$$\frac{q + x \times 3q^5 - q^4x + 9q^3x^2 - q^2x^3 - 6qx^4}{\overline{q^3 + q^2x - qx^2 - x^3}|^{\frac{1}{3}+1}} = \frac{3q^6 + 2q^5x + 8q^4x^2 + 8q^3x^3 - 7q^2x^4 - 6qx^5}{\overline{q^3 + q^2x - qx^2 - x^3}|^{\frac{1}{3}+1}}$$

$$= x^{1-1} \times \overline{q^3 + q^2x - qx^2 - x^3}|^{\frac{1}{3}+1} \times \overline{3q^6 + 2q^5x + 8q^4x^2 + 8q^3x^3 - 7q^2x^4 - 6qx^5}:$$

by comparing which, with the general Ordinates $z^{\theta-1} \times R^{\lambda-1} \times \overline{a + bz^{\eta} + cz^{2\eta} + \&c}$; and then substituting the particular Values, in the Series for Areas, the Area of the Curve will be found to be $\frac{3q^3x + 3qx^3}{\overline{q^3 + q^2x - qx^2 - x^3}|^{\frac{1}{3}}}$.

142. Our Author, in this Proposition, exhibits a general Theorem or Canon, by which the Area of any Curve, whose Ordinate is of this Form $z^{\theta-1}R^{\lambda-1} \times \overline{a + bz^{\eta} + cz^{2\eta} + dz^{3\eta}, \&c.}$ may be determined; and that either perfectly true, *viz.* when the curvilinear Area admits of a geometrical Comparison with right-lined Figures; or else so near the Truth, that the Difference may still be made less and less without End, *viz.* when the Curve cannot be squared. But whereas, it may be of considerable Use, to know when a Curve may or may not be squared: *i. e.* when it's Area may be expressed by a finite Number of Terms, or cannot be so expressed: therefore I shall here lay down some general Rules respecting this Matter: and for that End shall distinguish some Classes of Curves that come under the general Form mentioned in this Proposition; or may be easily reduced to it. And they are these following:

1°. $az^{\theta-1} = y$. 2°. $az^{p} + bz^{q} + cz^{r} + dz^{s}$, &c. $= y$. 3°. $az^{\theta-1} \times \overline{e + fz^{\eta}}|^{\lambda-1} = y$. 4°. $az^{\theta-1} \times \overline{e + fz^{\eta} + gz^{2\eta}}|^{\lambda-1} = y$. 5°. $z^{\theta-1} \times \overline{e + fz^{\eta}}|^{\lambda-1} \times \overline{a + bz^{\eta}} = y$. 6°. $z^{\theta-1} \times \overline{e + fz^{\eta}}|^{\lambda-1} \times \overline{a + bz^{\eta} + cz^{2\eta}} = y$. 7°. $z^{\theta-1} \times \overline{e + fz^{\eta}}|^{\lambda-1} \times \overline{a + bz^{\eta} + cz^{2\eta}\ \&c.} = y$. 8. $z^{\theta-1} \times \overline{e + fz^{\eta} + gz^{2\eta}}|^{\lambda-1} \times \overline{a + bz^{\eta} + cz^{2\eta}\ \&c.} = y$. Where the Symbols are to be understood in the same Sense, they are taken by our Author, *viz.* z denotes the Abscis, and y the corresponding Ordinate:

nate: $a . b . c . d . e . f . g . \theta . \eta . \lambda$ alſo $p . q . r . s$ are conſtant Quantities; integral or fractional, poſitive or negative.

143. All Curves of this Form $az^{\theta - 1} = y$ may be ſquared, unleſs when it is $\theta = 0$: and the Area is $\frac{a}{\theta} z^{\theta}$.

For if the Ordinate $az^{\theta-1}$ or $z^{\theta-1} \times 1^{\lambda-1} \times a$ be compared with the general Form of Ordinates $z^{\theta-1}R^{\lambda-1} \times \overline{a + bz^{\eta}}$, &c. you find $f = 0 = g = h$, &c. $b = 0 = c$, &c. $R = (e =) \ 1 . a = a$, &c. Therefore inſert theſe Values into the general Expreſſion for the Area, belonging to this fifth Propoſition: and it will eaſily appear, that all the Terms of the Series after the firſt $\frac{\frac{1}{\eta}a}{re}$, vaniſhing, becauſe of the Letters $b, c, d,$ &c. $f, g, h,$ &c. being equal to nothing: the Area $z^{\theta}R^{\lambda} \times \frac{\frac{1}{\eta}a}{re}$ becomes $\frac{a}{\theta} z^{\theta}$. Which therefore always exhibits the Area exactly, and gives it finite, unleſs when $\theta = 0$: for then it becomes $\frac{a}{0}$ infinite. Q. E. O. See alſo Art. 52 *.

144. By this Rule all ſimple parabolical Curves, and hyperbolical Spaces lying betwixt the Curve and Aſſymptotes, unleſs in the Caſe of the *Apollonian* Hyperbola, are ſquared.

* Of this Explication.

Ex. 1. Suppoſe it be $pz = y^2$, or $p^{\frac{1}{2}} z^{\frac{1}{2}} = y$: here, becauſe $a = \sqrt{p}$. and $\theta - 1 = \frac{1}{2}$ or $\theta = \frac{3}{2}$, the Area is $\frac{2}{3} p^{\frac{1}{2}} z^{\frac{3}{2}}$: or by inſerting y for $\sqrt{pz}$, $\frac{2}{3} yz$: which gives the Area of the common Parabola: being $\frac{2}{3}$ of the Rectangle under the Abſciſs and Ordinate.

Ex. 2. Let it be $yz^2 - a = 0$ or $az^{-2} = y$: here $\theta - 1 = -2$ or $\theta = -1$, therefore the Area is $-az^{-1}$ or $-\frac{a}{z}$: which denotes an hyperbolical Space lying betwixt the Hyperbola and Aſſymptote infinitely produced below the Ordinate. But if it had been $zy = a$ or $y = az^{-1}$, where $\theta - 1 = -1$ or $\theta = 0$, the Area comes out $\frac{a}{0}$ infinitely great: which is the Caſe of the common Hyperbola: and what muſt be done here, ſhall be ſhewn afterwards *.

* Art. 190. of this Explication.

145. All Curves of this Form $az^p + bz^q + cz^r + dz^s$, &c. $= y$, when the Number of Terms of the Value of y, is finite, may be ſquared, unleſs ſome of the Exponents $p . q . r$, &c. be -1: and the Area is $\frac{a}{p+1} z^{p+1} + \frac{b}{q+1} z^{q+1} + \frac{c}{r+1} z^{r+1}$ &c.

For the complex Ordinate $az^p + bz^q$, &c. is to be considered as compounded of the several simple Ordinates az^p, bz^q, cz^r, &c: and therefore the Area of the Curve is compounded of the several Areas belonging to the simple Ordinates, which being assignable by the last Rule, the Area according thereto, will be $\frac{a}{p+1}z^{p+1} + \frac{b}{q+1}z^{q+1} + \frac{c}{r+1}z^{r+1}$ &c. Which is always finite, unless p or q, or r, &c. be equal to -1: for then the Area belonging to the Term, where that happens, becomes infinite by the last Rule. Therefore, *&c.* Q. E. O.

Ex. 1. Suppose $2 - 3z + 4z^2 - 5z^3 = y$ be an Equation to a Species of Parabola; then, by inserting $2, -3, 4, -5$ for a, b, c, d respectively; and $0, 1, 2, 3$ for p, q, r, s, the Area of the Curve is $2z - \frac{3}{2}z^2 + \frac{4}{3}z^3 - \frac{5}{4}z^4$.

Ex. 2. Let $z^2 + z^{-3} - z^{-4} = y$ express the Relation of the Abscis and Ordinate of a Curve: where $a = 1 . b = 1 . c = -1 . p = 2 . q = -3 . r = -4$: therefore by a proper Substitution of Values the Area will be $\frac{1}{3}z^3 - \frac{1}{2}z^{-2} + \frac{1}{3}z^{-3}$.

146. Cor. Hence it follows, if $a + bz^n + cz^{2n} + dz^{3n}$, &c. *in inf.* be a converging Series, approaching continually to the Value of the Ordinate of a Curve, then $az + \frac{b}{n+1}z^{n+1} + \frac{c}{2n+1}z^{2n+1} + \frac{d}{3n+1}z^{3n+1}$ &c. *in inf.* will be an infinite converging Series, approaching continually to the Value of the corresponding Area.

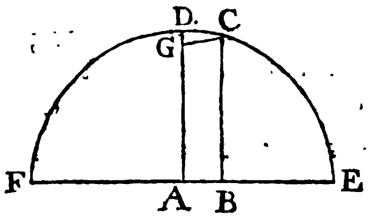
And by this means, the Areas of all Sorts of Curves, the Relation of whose Abscis and Ordinate is expressed by any algebraical Equation howsoever affected, may be determined by an infinite converging Series: when they cannot otherwise be squared, by the Methods shewn in this Treatise of Quadratures. For by means of such an Equation, the Value of the Ordinate may always be expressed by an infinite Series of simple Terms, containing the Powers of the Abscis z and known Quantities only, when it cannot be expressed in finite Terms: and that by the Methods taught by our Author elsewhere [1]: and thence by what has been just now said, the Area may be determined by an Approximation. Thus if $y^3 + a^2y + azy - 2a^3 - z^3 = 0$ be an Equation expressing the Relation of the Abscis and Ordinate of a Curve: by resolving the Equation according to his Method, by an infinite Series, it is $y = a - \frac{z}{4} + \frac{z^2}{64a} + \frac{131z^3}{512a^2}$, &c. *in inf.* whence,

[1] See his Treatise of *Analysis by Equations of an infinite Number of Terms*: likewise his *Method of Fluxions and infinite Series*, Chap. 1. published Anno 1736.

by

by what has been just now said, we shall have $az - \frac{1}{4}z^2 + \frac{z^3}{192a} + \frac{131z^4}{2048a^2}$, &c. an approximate Value of the Area.

Likewise if we have $\sqrt{rr - zz} = y$, an Equation to the Circle: by extracting the square Root of $rr - zz$, according to his binomial Theorem, it is $r - \frac{z^2}{2r} - \frac{z^4}{8r^3} - \frac{z^6}{16r^5} - \frac{5z^8}{128r^7}$ — &c. *in inf.* $= y$: wherefore the circular Area adjacent to the Abscisss z, which begins at the Center, is $rz - \frac{z^3}{6r} - \frac{z^5}{40r^3} - \frac{z^7}{112r^5} - \frac{5z^9}{1152r^7}$ — &c. r being the Radius. Thus let ADE be a Quadrant of a Circle (see Fig.) $AE = r$. $AB = z$, and let BC be perpendicular to AB. Put $r = 1$. $z = 0{,}1$. then the Area ABCD is $0{,}1 - \frac{0{,}001}{6} - \frac{0{,}00001}{40} - \frac{0{,}0000001}{112} - \frac{0{,}000000005}{1152}$ very nearly:



But if it be $AB = z = 1$, the Area of the Quadrant is $1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{5}{1152}$ nearly.

147. All Curves of this Form $az^{\theta-1} \times \overline{e + fz^{\eta}}|^{\lambda-1} = y$, may always be squared, when s, that is $\frac{\theta}{\eta} + \lambda$, is either equal to nothing, or a negative Integer, as -1, -2, -3, *&c.* unless when r, that is $\frac{\theta}{\eta}$, is either nothing, or a negative Interger, not greater than s, at the same Time: in which Case the Area is infinitely great. And the Area of the Curve is $\frac{az^{\theta}}{\eta} \times \overline{e + fz^{\eta}}|^{\lambda} \times$

$$\frac{1}{re} - \frac{sf}{r \times \overline{r+1} \times e^2} z^{\eta} + \frac{s \times \overline{s+1} \times f^2}{r \times \overline{r+1} \times \overline{r+2} \times e^3} z^{2\eta} - \frac{s \times \overline{s+1} \times \overline{s+2} \times f^3}{r \times \overline{r+1} \times \overline{r+2} \times \overline{r+3} \times e^4} z^{3\eta} + \&c.$$

Where the Number of Terms of the Series is one, when $s = 0$: 2, when $s = -1$: 3, when $s = -2$, *&c. i. e.* one more than the negative Units contain'd in s. In other Cases the Curve is not quadrable. But it must be observed, that both Expressions of the Ordinate must be tried by the Rule; as well that which has the Index of z under the Vinculum negative; as that, where it is positive.

For by considering the Series for Areas in general, exhibited by our Author in this Proposition *: and applying it to the Case of a simple binomial Curve, such as we are now speaking of: where it is $b = 0 = c = d$, &c. likewise $g = 0 = h = i$, &c. *in infinitum*,

* Art. 26. of the Quadrature of Curves.

tum, it will evidently be reduced to this Shape $\overline{z^{\theta} \times \overline{e + fz^{n}}|^{\lambda} \times}$

$$\overline{\frac{\frac{1}{n}a}{re} + \frac{-sfA}{r+1 \times e}z^{n} + \frac{-\overline{s+1} \times fB}{\overline{r+2} \times e}z^{2n} + \frac{-\overline{s+2} \times fC}{\overline{r+3} \times e}z^{3n}}, \text{ \&c.}:$$

Or, which comes to the same, $\overline{z^{\theta} \times \overline{e + fz^{n}}|^{\lambda} \times \frac{\frac{1}{n}a}{r} \times \frac{z^{0}}{e} -}$

$$\overline{\frac{sf \times \frac{1}{n}a}{r \times \overline{r+1}}\,\frac{z^{n}}{e^{2}} + \frac{s \times \overline{s+1} \times f^{2} \times \frac{1}{n}a}{r \times \overline{r+1} \times \overline{r+2}}\,\frac{z^{2n}}{e^{3}} - \frac{s \times \overline{s+1} \times \overline{s+2} \times f^{3} \times \frac{1}{n}a}{r \times \overline{r+1} \times \overline{r+2} \times \overline{r+3}} \times \frac{z^{3n}}{e^{4}}} + \text{\&c.}$$

as appears by Art. 125 *: for the whole Series, in this Case, is reduc'd to the first horizontal Row, as it is expressed towards the End of that Art. Which, by taking the common Factor $\frac{1}{n}a$, out of all the Terms, and prefixing it to the common Multiplier z^{θ}, makes the Series mentioned, *viz.* $\overline{\frac{az^{\theta}}{n} \times \overline{e + fz^{n}}|^{\lambda} \times}$

$$\overline{\frac{1}{re} - \frac{sf}{r \times \overline{r+1} \times e^{2}}z^{n} + \frac{s \times \overline{s+1} \times f^{2}}{r \times \overline{r+1} \times \overline{r+2} \times e^{3}}z^{2n} - \frac{s \times \overline{s+1} \times \overline{s+2} \times f^{3}}{r \times \overline{r+1} \times \overline{r+2} \times \overline{r+3} \times e^{4}}z^{3n}} \text{ \&c.}$$

* Of this Explication.

From which Series, expressing the Area of a simple binomial Curve in general, it appears that if it be $s = 0$, the second and all the subsequent Terms will vanish, because of the common Factor $s = 0$ found in them all: if it be $s + 1 = 0$ or $s = -1$, all the Terms after the second must vanish; because of the common Factor $s + 1 = 0$: if it be $s = -2$ or $s + 2 = 0$, all the Terms after the third must vanish, and so on continually: wherefore if s be nothing, or a negative Integer, the Series will terminate; and the Number of Terms, will be one more than the negative Units contain'd in $s = \frac{\theta}{n} + \lambda$: but if r be equal to nothing, or any negative Integer not greater than s, the Denominators of some one or more of the Terms preceding that Term, which should be the last, will vanish, and so make one or more Terms infinite: thus *e. g.* if it were $s = -2$ or $\overline{s+2} = 0$, there would be three Terms; but if, at the same Time, it be $r = -2$, or $r = -1$, or $r = 0$, i. e. $r + 2 = 0$, or $r + 1 = 0$, or $r = 0$, one or more of the preceding Denominators must contain a Factor equal to nothing, and thereby make the Term or Terms, to which it or they belong, infinite: and the Reasoning, from the Nature and Progression of the Series, is evidently the same, whenever r is nothing, or a negative Integer not greater than s.

Moreover, it is evident, that in no other Case, except when s is either nothing or a negative Integer, can the Series terminate: *i. e.* in no no other Caſe can the Curve be ſquared: theſe things being ſuppoſed to be applied to both Expreſſions of the Ordinate; as well that wherein the Index η is negative; as that wherein it is affirmative: which will appear by the Examples. Therefore, *&c.*

Ex. 1. Let $\frac{4z}{27 - 108z + 144z^2 - 04z^3} = y^2$ expreſs the Relation of the Abſciſs and Ordinate of a Curve: by extracting the ſquare Root, it is $\frac{2z^{\frac{1}{2}}}{\sqrt{27 - 108z + 144z^2 - 64z^3}} = y$: and becauſe the Quantity under the radical Sign in the Denominator, is a true Cube, whoſe cube Root is $3 - 4z$, therefore I reduce it to this Shape $\frac{2z^{\frac{1}{2}}}{\overline{3-4z}|^{\frac{3}{2}}} = y$, which is a ſimple binomial Curve. When it is brought to Form, it is either $2z^{\frac{3}{2}-1} \times \overline{3 - 4z}|^{-\frac{3}{2}-1}$ or $2z^{0-1} \times \overline{-4 + 3z^{-1}}|^{-\frac{3}{2}-1}$. In this laſt Caſe, by comparing with the general Form of Ordinates for binomial Curves $az^{\theta-1} \times \overline{e + fz^{\eta}}|^{\lambda-1}$, we have $\theta = 0 . \eta = -1 . r = (\frac{\theta}{\eta} =)$ 0, whence, without proceeding further, I conclude the Area, ariſing from the ſecond Form of the Ordinate, to be infinite: which is always the Caſe when $\theta = 0$. But I cannot thence conclude that the propos'd Curve cannot be ſquared, until I apply the Rule to the firſt Form of the Ordinate, *viz.* $2z^{\frac{3}{2}-1} \times \overline{3 - 4z}|^{-\frac{3}{2}-1}$: where we find $\theta = \frac{3}{2} . \eta = 1 . \lambda = -\frac{1}{2} . r = \frac{3}{2} . s = 2$, which being poſitive, ſhews that the Series for the Area, ariſing from this Form of the Ordinate, runs out into an infinite Series: and now I conclude that the propoſed Curve cannot be ſquared.

Ex. 2^d. Suppoſe it to be $2z^{-\frac{1}{2}} \times \overline{3 - 4z}|^{-\frac{3}{2}} = y$: this Ordinate may, when brought to due Form, ſtand thus $2z^{\frac{1}{2}-1} \times \overline{3-4z}|^{-\frac{1}{2}-1}$ or $2z^{-1-1} \times \overline{-4+3z^{-1}}|^{-\frac{1}{2}-1}$. From the firſt, we have $\theta = \frac{1}{2} . \eta = 1 . \lambda = -\frac{1}{2} . r = \frac{1}{2} . s = 0$: whence I conclude the Curve may be ſquared: and ſince moreover it is $e = 3 . f = -4 . a = 2$; the Area, which according to the Rule, is thus generally expreſſed $\frac{az^{\theta}}{\eta} \times \overline{e+fz^{\eta}}|^{\lambda} \times \frac{1}{re}$, will be, by Subſtitution of Values, $2z^{\frac{1}{2}} \times \overline{3-4z}|^{-\frac{1}{2}} \times \frac{1}{\frac{3}{2}} = \frac{4}{3}\sqrt{\frac{z}{3-4z}}$: although, if you try the other Form of the Ordinate $2z^{-1-1} \times \overline{-4+3z^{-1}}|^{-\frac{1}{2}-1}$, you would find the Expreſſion for the Area to run out into an infinite Series.

Ex.

Ex. 3^d. Let $c^4y - a^4z - 2c^2yz^2 + yz^4 = 0$ be an Equation, containing the Relation of the Abscifs and Ordinate of a Curve: by taking the Value of y, it is $y = \frac{a^4z}{c^4 - 2c^2z^2 + z^4}$: where the Denominator being a true Square, I make it ftand thus * $\frac{a^4z}{\overline{c^2 - z^2}|^2}$, or $a^4z \times \overline{c^2 - z^2}|^{-2}$: which, being reduced to due Form, is either $a^4z^{2-1} \times \overline{cc - z^2}|^{-1-1}$ or yet $a^4z^{-2-1} \times \overline{-1 + ccz^{-2}}|^{-1-1}$. From the firft Manner of expreffing, we have $\theta = 2 . \eta = 2 . \lambda = -1 . r = 1 . s = 0$: whence I conclude the Area arifing from this Form of the Ordinate to be finite and affignable: and fince it is moreover $e = cc . f = -1 . a = a^4$: the Area will be $\frac{a^4z^2}{2c^4 - 2ccz^2}$. Again, by comparing the fecond Form of the propos'd Ordinate, *viz.* $a^4z^{-2-1} \times \overline{-1 + ccz^{-2}}|^{-1-1}$, with the general Form, we have $\theta = -2 . \eta = -2 . \lambda = -1 . r = 1 . s = 0$: whence it appears that the Area, arifing from this other Form of the Ordinate, may be exactly determined likewife: and fince it is moreover $e = -1 . f = cc . a = a^4$; by a proper Subftitution and Reduction, the Area will come out $\frac{a^4}{2c^2 - 2z^2}$: By this means we obtain two Areas, perfectly affignable, and both belonging to the fame Curve: Of which we gave an Example formerly *.

* Art. 136, 137. of this Explication.

* Art. 128, 129. of this Explication.

Ex. 4^th. Let $az^{2\eta-1}\sqrt{e + fz^\eta}$ be the Ordinate of a Curve: it may ftand thus $az^{2\eta-1} \times \overline{e + fz^\eta}|^{\frac{3}{2}-1}$; or thus $az^{\frac{5}{2}\eta-1} \times \overline{f + ez^{-\eta}}|^{\frac{3}{2}-1}$. From the firft, it is $\theta = 2\eta . \eta = \eta . \lambda = \frac{1}{2} . r = 2 . s = \frac{7}{2}$: whence the Series for the Area, arifing from this Form of the Ordinate runs out infinitely. Wherefore I try the other Form $az^{\frac{5}{2}\eta-1} \times \overline{f + ez^{-\eta}}|^{\frac{3}{2}-1}$: here I find $\theta = \frac{5}{2}\eta . \eta = -\eta . \lambda = \frac{1}{2} . r = -\frac{5}{2} . s = -1$: whence I conclude that the Curve may be fquared: and fince it is moreover $e = f . f = e . a = a$, the Area by a proper Subftitution of Values in the Series for the Areas of binomial Curves, will be $\frac{az^{\frac{5}{2}\eta}}{-\eta} \times \overline{f + ez^{-\eta}}|^{\frac{3}{2}} \times \overline{\frac{1}{-\frac{5}{2}f} - \frac{-e}{\frac{15}{4}ff}z^{-\eta}}$ = (by a proper Reduction) $a \times \overline{e + fz^\eta}|^{\frac{3}{2}} \times \frac{-4e + 6fz^\eta}{15\eta f^2}$.

148. Cor. From what has been faid it appears, that, when we confider fimple binomial Curves that are capable of being fquared: fome are of that Nature that one Area only can be exactly found, as in the fecond and fourth Examples: which Area is fometimes deduced from

from that Form of the Ordinate, where η, the Index of z, is positive, as in the second Example; sometimes from the other Form of the Ordinate, where η is negative, as in the fourth Example. But there are other Curves of such a Nature that both Areas may be exactly found, as in Example third. This last Sort we may call *doubly* Quadrable. Therefore it may be of use to have a Rule ready at hand, whereby we may at first View determine, when any propos'd binomial Curve may, or may not be squared at all: and when it is doubly Quadrable.

Rule. To find when a simple binomial Curve may, or may not be squared at all: and when it is doubly Quadrable.

149. Having brought the Ordinate of the Curve proposed, to stand in due Form, *viz.* thus $az^{\theta-1} \times \overline{e+fz^{\eta}}|^{\lambda-1}$: being that Form, where η, the Index of z under the Vinculum, is suppos'd to be affirmative; θ and λ being either affirmative or negative.

If $\frac{\theta}{\eta}+\lambda$ be nothing, or a negative Integer; and at the same Time $\frac{\theta}{\eta}$ be nothing, nor a negative Integer as small or smaller than it; or if $-\frac{\theta}{\eta}+1$ be nothing, or a negative Integer; and at the same Time $-\frac{\theta}{\eta}+1-\lambda$ be not nothing, nor a negative Integer as small or smaller than it: then if either of these happen, the Curve may be squared: if neither of these happen, the Curve cannot be squared: if both of them happen, the Curve is doubly quadrable.

The Demonstration of this will be evident from what has been formerly said in Art. 147 *: if it be considered that the Ordinate $az^{\theta-1} \times \overline{e+fz^{\eta}}|^{\lambda-1}$ may be also expressed otherwise thus $az^{\theta+\lambda\eta-\eta-1} \times \overline{f+ez^{-\eta}}|^{\lambda-1}$: for from the first, we have $r=\frac{\theta}{\eta}$. $s=\frac{\theta}{\eta}+\lambda$: and from the second, $r=-\frac{\theta}{\eta}-\lambda+1$. $s=-\frac{\theta}{\eta}+1$.

* Of this Explication.

150. Hitherto we have been capable of laying down general Rules, by which, we may always determine, when any Curve, of the Kinds mentioned, is proposed; whether it may be squared, or not, and that immediately from the Properties of the Exponents. But with respect to trinomial Curves, *viz.* whose Ordinates come under this Form $az^{\theta-1} \times \overline{e+fz^{\eta}+gz^{2\eta}}|^{\lambda-1}$, all we can do by means of the Properties and Relations of the Symbols θ, η, λ, which belong to the Exponents, is to shew some Limitations, without which no such Curve can be squared. And whereas there are infinite different Relations, which

which ſubſiſting among all the Symbols $\theta . \eta . \lambda . e . f . g$ taken together, make the Curve quadrable, we can ſhew how ſome of theſe Relations may be inveſtigated. Finally, I ſhall lay down a Method or Rule, by which we can find, without much trouble, whether any propoſed ſimple trinomial Curve may, or may not be ſquared at all. In order to which it will eaſily appear, by conſidering the general Series for

* Art. 26. of the Quadrature of Curves. Art. 124. of this Explication.

Areas, exhibited by our Author in this Propoſition *, and applying it to the Caſe of a ſimple trinomial Curve, that it muſt put on this Shape

$$z^{\theta} \times \overline{e + fz^{\eta} + gz^{2\eta}}|^{\lambda} \times \overline{\frac{\frac{1}{\eta}a}{re} - \frac{sfA}{\overline{r+1} \times e} z^{\eta} - \frac{\overline{s+1} \times fB + tgA}{\overline{r+2} \times e} z^{2\eta} - \frac{\overline{s+2} \times fC + \overline{t+1} \times gB}{\overline{r+3} \times e} z^{3\eta}}$$

$$\overline{- \frac{\overline{s+3} \times fD + \overline{t+2} \times gC}{\overline{r+4} \times e} z^{3\eta} - \frac{\overline{s+4} \times fE + \overline{t+3} \times gD}{\overline{r+5} \times e}} - \&c.$$ the other Parts of the Numerators of each complex Term, vaniſhing, becauſe $b = o = c = d$, &c. and $b = o = i$, &c. *in infinitum.* From which we may deduce what follows.

151. Any Curve of this Form $az^{\theta-1} \times \overline{e + fz^{\eta} + gz^{2\eta}}|^{\lambda-1} = y$, being propoſed, ſet down the ſeveral Terms in order, of which it's Area is compoſed, by ſubſtituting the particular Values, in place of the Symbols, in the general Theorem or Series for the Area of trinomial Curves, expreſſed in the preceding Article: and if you come to two Terms [1], immediately ſucceeding one another, both ſeparately taken, equal to nothing, the Curve may be ſquared: And the Area is juſt equal to the preceding Part of the Series: provided always that none of the preceding Terms become infinitely great, by r becoming o or any negative Integer leſs than the Number of ſignificant Terms: which would make the Area infinitely great. If there be but two ſignificant Terms, it muſt neither be $r = 0$, nor $r = -1$: if there be three Terms preceding the two Terms which break off the Series, r muſt not be o, -1 nor -2, and ſo on: for there is no quadrable Curve of this Form, which can have but one Term, the Area being neceſſarily infinite.

The Demonſtration of this will be evident, if it be conſidered, that there are only two capital Letters, in each complex Term of the fore-

[1] By Terms here are meant Terms of the laſt Factor, conſtituting the Series $\frac{\frac{1}{\eta}a}{re} - \frac{sfA}{\overline{r+1} \times e} z^{\eta} - \frac{\overline{s+1} \times fB + tgA}{\overline{r+2} \times e}$, &c.

going

going Series for the Area of a trinomial Curve, after you paſs the ſecond Term, *in inf.* which two capital Letters always denote the Coefficients of the two immediately preceding Terms: and theſe Capitals are each a Factor of one of the two Parts of which the Numerator of the Term conſiſts: therefore, if any two Terms immediately ſucceeding one another, vaniſh; and conſequently their Coefficients, denoted by the two capital Letters, contain'd in the Term immediately following them, that Term muſt alſo vaniſh: and ſo alſo the next Term after it, for the like Reaſon: and therefore all the ſubſequent Terms *in inf.* by the ſame way of reaſoning. Whence if two immediately ſucceeding Terms vaniſh, all the ſubſequent Terms muſt alſo vaniſh: and therefore the Series muſt terminate with the Term immediately preceding them; and ſo exhibit the full Area of the Curve, and ſhew it to be quadrable: unleſs r by being equal to nothing; or to any negative Integer, leſs than the Number of ſignificant Terms do make the Denominator of ſome Term to vaniſh: it's Numerator being ſuppoſed not to vaniſh: for in ſuch a Caſe one or more of the preceding Terms muſt be infinitely great: as eaſily appears by conſidering the Series, and what was formerly ſaid.

Moreover, that there can be no quadrable Curve of this Form, whoſe Area conſiſts of no more but the firſt Term, *viz.* $\frac{\frac{1}{n}a}{re} \times z^{\theta} \times \overline{e + fz^{n} + gz^{2n}}|^{\lambda}$, will appear thus. When all the Terms, but the firſt vaniſh, we will have the third Term $- \frac{\overline{s+1} \times fB + tgA}{\overline{r+2} \times e} z^{2n} = 0$, that is, becauſe the ſecond Term alſo, and conſequently it's Coefficient denoted by B, vaniſhes, $tgA = 0$: which can only happen here, when it is $t = 0$: but ſeeing the ſecond Term $- \frac{sfA}{\overline{r+1} \times e} z^{n} = 0$: which cannot be in this Caſe, but when it is $s = 0$: therefore when all the Terms of the Series except the firſt vaniſh, it muſt be $t = (0 =) s$: but $t = s - \lambda$, therefore it is alſo $\lambda = 0$: again, $r = s - \lambda$, therefore it is $r = 0$: conſequently the firſt Term of the Series, and therefore the Area of the Curve is infinitely great. Which Things were to be ſhewn.

N. B. The Rule contain'd in this Article, may be applied to both Series's, which denote the Areas of the Curve; both to that, where the Index η is affirmative; and that where it is negative. And by ſo doing, if you come at two immediately ſucceeding Terms, equal each to nothing, in both Series's, the Curve is doubly quadrable. Moreover it muſt be remarked that r may poſſibly be equal to a negative Integer leſs than the Number of ſignificant Terms, and yet the Area be finite, and the Curve quadrable; if ſo be the Numerator of that Term vaniſh, whoſe Denominator vaniſhes. Thus if it be $r = -2$, the Denominator of the third Term, *viz.* $\overline{r+2} \times e$ vaniſhes, which makes that Term and conſequently the Area infinitely great, provided the Numerator $\overline{s+1} \times fB + tgA$ don't vaniſh at the ſame Time: but if that Numerator be nothing, then the third Term becomes nothing, and therefore the Area is finite, and the Curve will be quadrable upon the Conditions mentioned in this Art. And it muſt be obſerved, that the like Exception reſpecting the Infinity of Areas ariſing from the Value of r being nothing or a negative Integer, muſt be underſtood with the like Limitation in all the ſubſequent Rules contained in this Section.

152. In the preceding Article, I have demonſtrated that, when any two immediately ſucceeding Terms of either of the two Series's, for the Area of a ſimple trinomial Curve, vaniſh, and thereby break off the Series, the Curve is quadrable: and the Area made up of all the Terms preceding theſe. I ſhall next ſhew how one may inveſtigate certain Laws or Conditions, ariſing from the Properties and Relations of the Quantities, which enter the Expreſſion of the Ordinate: which Laws or Conditions obtaining, the Curve may be ſquared.

153. Put any two next adjoining Terms, beginning with the third and fourth *, of the Series expreſſing the Area of any ſimple trinomial Curve, each equal to nothing: and thence will ariſe a certain Relation, or certain Relations among the Quantities: which ſubſiſting, the Curve may ſquared.

*Art. 151. of this Explication.

Suppoſe the third and fourth Terms equal to nothing: *i. e.* $-\frac{\overline{s+1} \times fB + tgA}{\overline{r+2} \times e} z^{2n} = 0$ and $-\frac{\overline{s+2} \times fC + \overline{t+1} \times gB}{\overline{r+3} \times e} z^{3n} = 0$: then, becauſe $C = (-\frac{\overline{s+1} \times fB + tgA}{\overline{r+2} \times e} =)\ 0$, therefore, from the ſecond Equation, by inſerting 0 for C, and dividing by $\frac{-z^{3n}}{\overline{r+3} \times e}$ we will have $\overline{t+1} \times gB = 0$: *i. e.* becauſe neither g nor B is nothing, $t+1=0$ or $t = -1$: whence $s = (t - \lambda =) -1 - \lambda$ and $r = (s - \lambda =)$ $-1 - 2\lambda$: inſert theſe Values of s and t in place of them, in the firſt Equation $-\frac{\overline{s+1} \times fB + tgA}{\overline{r+2} \times e} z^{2n} = 0$, and divide it by $-\frac{z^{2n}}{\overline{r+2} \times e}$, and it becomes $-\lambda fB - gA = 0$: inſert for B it's Value, *viz.*

—

$-\frac{sfA}{\overline{r+1} \times e} = -\frac{\overline{\lambda+1} \times fA}{2\lambda \times e}$, and you will have $\frac{\overline{\lambda+1} \times f^2A}{2e} - gA =$ 0: *i.e.* by reducing, $\lambda = \frac{2eg}{f^2} - 1$.

Again $\frac{\theta}{\eta} = (r =) - 1 - 2\lambda$: therefore $\frac{\theta}{\eta} = 1 - \frac{4eg}{f^2}$.

Whence it follows, that, if it be $\lambda = \frac{2eg}{f^2} - 1$, and $\frac{\theta}{\eta} = 1 - \frac{4eg}{f^2}$; the third and fourth Terms of the Series, expreſſing the Value of the Area, both vaniſh: and therefore the Curve, that has ſuch an Ordinate, as that $\lambda = \frac{2eg}{f^2} - 1$, and $\frac{\theta}{\eta} = 1 - \frac{4eg}{f^2}$, may be ſquared: provided that $1 - \frac{4eg}{f^2}$, which is the Value of $\frac{\theta}{\eta}$ or r, be neither 0; nor -1: as appears by the laſt Article.

Example. Let the Ordinate of a Curve be $\frac{a}{\overline{2z^9 + 3z^{12} + 3z^{15}}|^{\frac{2}{3}}}$: reduce it to Form and it ſtands thus, $az^{-5-1} \times \overline{2 + 3z^3 + 3z^6}|^{\frac{1}{3}-1}$: where $\lambda = \frac{1}{3} . \theta = -5 . \eta = 3 . r\ (= \frac{\theta}{\eta}) = -\frac{5}{3} . e = 2 . f = 3 .$ $g = 3$: therefore $\lambda\ (= \frac{1}{3}) = \frac{2eg}{f^2} - 1$: again $r = (-\frac{5}{3} =)\ 1 - \frac{4eg}{f^2}$: wherefore the Curve may be ſquared, and accordingly, by inſerting the particular Values of the indefinite Symbols, in the Series which expreſſes the Value of the Area of a Trinomial *, you'll find the Area of the Curve propoſed, $\frac{0{,}3}{z^5} - \frac{0{,}1}{z^5} \times a\sqrt[3]{2 + 3z^3 + 3z^6}$: all the Terms after the firſt two, of the Series vaniſhing: for $t = (r + 2\lambda =) - 1$.

*Art. 150. of this Explication.

154. Now, by the very ſame way of reaſoning, you may inveſtigate and diſcover other Properties and Relations of the ſeveral Quantities r ($= \frac{\theta}{\eta}$) . λ . e . f . and g: which ſubſiſting, a ſimple trinomial Curve, whoſe Ordinate is of this Form $az^{\theta-1} \times \overline{e + fz^{\eta} + gz^{2\eta}}|^{\lambda-1}$, may be ſquared: *viz.* by putting the fourth and fifth Terms; the fifth and ſixth Terms; the ſixth and ſeventh Terms, and ſo on, equal to nothing: but the Inveſtigation becomes ſtill more and more tedious, and perplex'd: becauſe it involves you ſtill in higher Equations: and then after all, you ſhall not be able to diſcover a general Rule, by which you might always know when any ſimple trinomial Curve, may or may not be ſquared: for 'tis evident that all the Rules thus diſcovered, extend only to particular Caſes; by ſhewing ſome of the infinitely many Relations, which theſe Quantities bear, in order to make the Curve quadrable. And I ſuppoſe it may be eaſily gathered from what was ſaid

 formerly,

formerly, that this Method of Investigation may be applied to both Expressions of the Ordinate *.

*Art 147. of this Explication.

From what has been said in the four preceding Articles, we may easily deduce several things, serving as Limitations or Conditions, without which no trinomial Curve can be squared at all. Having therefore brought any propos'd Ordinate of a Curve of this Sort, to stand in this Form $az^{\theta-1}\times\overline{e+fz^{\eta}+gz^{2\eta}}|^{\lambda-1}$: where η, the Exponent of z under the Vinculum, must be positive: Then,

155. 1°. If both $\frac{\theta}{\eta}+2\lambda$ and $-\frac{\theta}{\eta}+2$ be nothing; Fractions of any Kind, or positive Integers, the Curve cannot be squared: for these are the two Values of the Symbol t, according to the two different Ways of expressing the Ordinate: and it is evident from what has been demonstrated, that the Series expressing the Area cannot terminate, unless t be a negative Integer: because no two Terms immediately succeeding one another can vanish, but when t is a negative Integer: as appears from Art. 153 *.

* Of this Explication.

156. 2°. If both $\frac{\theta}{\eta}$ and $-\frac{\theta}{\eta}-2\lambda+2$ be nothing or -1, the Curve cannot be squared: for these are the two Values of r, in the two different Expressions of the Ordinate: therefore, in this Case the Curve cannot be squared, the Area being infinitely great by Art. 151 *.

* Of this Explication.

157. 3°. If it be $\lambda=0$, the Curve cannot be squared: for in that Case $r=(t-2\lambda=)t$: but the Curve cannot be squared unless t be a negative Integer by Art. 155 *: now if t being a negative Integer, the Series terminate, the last significant Term of the Series becomes infinitely great: because it's Denominator vanishes, when $r=t$, and t a negative Integer: and therefore the Area is infinitely great: which will appear by considering the Series at Art. 150 *; and what has been demonstrating in Art. 151 *.

* Of this Explication.

* Of this Explication.

* Of this Explication.

And now, from the Consideration of all that has been said about trinomial Curves, we deduce this

RULE

158. To know when any simple trinomial Curve, whose Ordinate is of this Form $az^{\theta-1}\times\overline{e+fz^{\eta}+gz^{2\eta}}|^{\lambda-1}$, may, or may not be squared; and when it is doubly quadrable.

Any such Curve being proposed, set down it's Ordinate in that Form where η is affirmative: apply to it the Limitations mentioned in Art.

Art. 155. 156, 157 *: and if any of these Limitations shew that it cannot be squared, you proceed no further. But if none of them determine this: then set down likewise the other Form of the Ordinate, *viz.* that where η is negative: seek the two Values of t, belonging to the two different Expressions of the Ordinate; one or other, or both of them must be negative Integers, from the Limitations already mentioned: if only one of the Values of t be a negative Integer, you need only set down the Series for the Area, belonging to that Form of the Ordinate, according to which t is a negative Integer: but if both Values of t be negative Integers, you must set down both Series's for the Value of the Area: that where η is positive, and the other also where it is negative. Then, if one of the Values of t only be a negative Integer, it must be $t + 1 = 0$, or $t + 2 = 0$, or $t + 3 = 0$, &c. substitute the particular Values of the Letters, through all the different Terms of the corresponding Series for the Area, in order from the first, until you come to that Term of the Series, which immediately precedes the Term, where you have $t + 1 = 0$, or $t + 2 = 0$, or $\overline{t + 3} = 0$, &c. and if that Term be not nothing, the Curve cannot be squared at all: if it be nothing the Curve is quadrable: (provided still r be not equal to 0 nor any negative Integer so small or smaller than t) and the Area is equal to the preceding Terms.

* Of this Explication.

But if both Values of t be negative Integers, you must try both Series's for the Area, after the Manner just now mentioned: and if the Term of the Series's immediately preceding that Term, where you have $t + 1 = 0$, or $t + 2 = 0$, or $t + 3 = 0$, &c. be equal to nothing, in neither Series, the Curve is not quadrable: but if that Term be nothing, in either of the Series's, the Curve is quadrable, and you have the Area: if it be so in them both, the Curve is doubly quadrable, and you have both Areas: it being suppos'd still as before, that the Value or Values of r be not 0; nor any negative Integer or Integers so small or smaller than the Value or Values of t respectively. There needs nothing be said for demonstrating the Justness of this Rule or Solution, after what has been already shewn: unless perhaps it should be thought necessary to demonstrate, that if the Term of the Series for the Area, which immediately precedes that Term where it is $t + 1 = 0$, or $t + 2 = 0$, or $t + 3 = 0$, &c. be not nothing, the Terms of the Series will go on *in inf.* so as that it can never terminate: nor consequently the Area of the Curve, denoted by that Series, be exactly found. Which appears thus.

159. Suppose it were $t = -2$ or $t + 2 = 0$: (See the Series at Art. 150 *.) the Term where $t + 2$ is found, is the fifth. Now I say

* Of this Explication.

ſay if the fourth Term, *viz.* $-\frac{\overline{t+2}\times fC+\overline{t+1}\times gB}{\overline{r+3}\times e}z^{3n}$ be not equal to nothing, the Series will never terminate: for if it terminate the fourth Term, or ſome Term after the fourth muſt be the laſt ſignificant Term of the Series: it cannot be the fourth Term; for then, ſince the next two Terms vaniſh, *viz.* $-\frac{\overline{t+3}\times fD+\overline{t+2}\times gC}{\overline{r+4}\times e}z^{3n}$ and $-\frac{\overline{t+4}\times fE+\overline{t+3}\times gD}{\overline{r+5}\times e}z^{4n}$ we have $-\frac{\overline{t+4}\times fE+\overline{t+3}\times gD}{\overline{r+5}\times e}z^{4n}=0$: but it is alſo $E=0$: for E denotes the Coefficient of the preceding Term, which is nothing: therefore we will have $\frac{\overline{t+3}\times gD}{\overline{r+5}\times e}z^{4n}=0$, conſequently $\overline{t+3}\times gD=0$: but g is not nothing, from the Nature of the Trinomial; and D is not nothing by the Hyp., for it is the Coefficient of the fourth Term, therefore it muſt be $t+3=0$: but that is contrary to the Hyp., for it was ſuppoſ'd that $t+2=0$. Now by the very ſame way of reaſoning the fifth cannot be the laſt ſignificant Term, nor the ſixth, nor any ſubſequent Term whatſoever in the Series, ſtill for the ſame Reaſon, therefore the Series will never terminate. And it's evident, the Reaſoning is exactly the ſame if we ſuppoſe $t=-1$, or $t=-3$, or $t=-4$, *&c. i. e.* univerſally, whatever negative Integer be the Value of t. Therefore, *&c.* Q. E. D.

160. Cor. If the Series exhibiting the Value of the Area of a ſimple trinomial Curve, do terminate, the Area will at leaſt contain two Terms of the Series, multiplied into $z^{\theta}R^{\lambda}$, *viz.* when it is $t=-1$: the Area will be expreſſed by 3, 4, 5, *&c.* Terms multiplied into $z^{\theta}R^{\lambda}$, when it is $t=-2$, $t=-3$, $t=-4$, *&c.* reſpectively, *i. e.* the Number of Terms will be one more than the Number of negative Units contain'd in the Value of t.

Now I ſhall ſubjoin a few Examples of the Application of the preceding Rule.

Ex. 1. Let $\frac{d}{z}\sqrt{2-3z^2+4z^4}=y$ expreſs the Relation of the Abciſs and Ordinate of a Curve: when it is reduced to Form, according to the Rule, it ſtands thus $dz^{0-1}\times\overline{2-3z^2+4z^4}|^{\frac{1}{2}-1}$: here $\theta=0$. $\eta=2$. $\lambda=\frac{1}{2}$. therefore $\frac{\theta}{\eta}=0$; and $\frac{\theta}{\eta}+2\lambda=3$; and $-\frac{\theta}{\eta}+2=2$: therefore the Curve is of that Number that cannot be ſquared by Art. 155 *. The ſame may be concluded from Art. 156 *, ſince it is $\frac{\theta}{\eta}=0$ and $-\frac{\theta}{\eta}-2\lambda+2=-1$.

* Of this Explication.

Ex.

Ex. 2. Suppose $az^{-\frac{1}{2}} \times \overline{1 - 2z + 3z^2}|^{-\frac{7}{4}} = y$: reduce it to Form according to the Rule, and it ſtands thus $az^{\frac{1}{2}-1} \times \overline{1 - 2z + 3z^2}|^{-\frac{3}{4}-1}$: where we have $\frac{\theta}{\eta} + 2\lambda = -1$ and $-\frac{\theta}{\eta} + 2 = \frac{3}{2}$. further $\frac{\theta}{\eta} = \frac{1}{2}$ and $-\frac{\theta}{\eta} - 2\lambda + 2 = 0$: and ſince it is not $\lambda = 0$: hence it appears that this Curve comes not under the Limitations contain'd in Art. 155, 156, 157 *, ſo as that we can conclude it not to be quadrable. Therefore ſince it is $\frac{\theta}{\eta} + 2\lambda = -1$, and $\frac{\theta}{\eta} = \frac{1}{2}$: which are the Values of t and r belonging to that Form of the Ordinate, where η is poſitive: I ſet down the Series for the Area where the Value of η is poſitive: which by a proper Subſtitution of Values (ſee Art. 150 *) ſtands thus $z^{\frac{1}{2}} \times \overline{1 - 2z + 3z^2}|^{-\frac{3}{4}} \times \overline{2a - \frac{2a}{3}z + 2az^2}$, &c. Hence then I conclude the Curve cannot be ſquared: for ſince it is $t = -1$, the third Term muſt have been nothing, in order to make the Series terminate: and the other Value of t ariſing from the other Expreſſion of the Ordinate being $\frac{1}{2}$, the other Form of the Series for the Area, muſt run out infinitely.

* Of this Explication.

* Of this Explication.

N. B. The ſame thing might have been concluded from Art. 153 *.

* Of this Explication.

Ex. 3. Let it be $\frac{a}{\sqrt{z^6 - 2z^7 + 3z^8}} = y$: reduce it to the Form, where the Value of η is poſitive: and it ſtands thus $az^{-2-1} \times \overline{1 - 2z + 3z^2}|^{\frac{1}{2}-1}$: where it is $\theta = -2 . \eta = 1 . \lambda = \frac{1}{2}$: whence $\frac{\theta}{\eta} + 2\lambda = -1$; $-\frac{\theta}{\eta} + 2 = 4$, and $\frac{\theta}{\eta} = -2$: therefore I cannot conclude from Art. 155 — 157 *, that the Curve cannot be ſquared, only I know it is not doubly quadrable. Wherefore, agreeably to the Rule, ſince the Value of t, belonging to the Form of the Ordinate, having η poſitive, is -1, and that the correſponding Value of r is -2, a negative Integer not ſo ſmall as t; I ſet down the ſeveral Terms of the correſponding Series for the Area, according to the proper Values of the general Symbols: and thence I obtain $z^{-2} \times \overline{1 - 2z + 3z^2}|^{\frac{3}{2}} \times \overline{-\frac{1}{2}a - \frac{3a}{2}z - 0}$ where, becauſe the third Term, which immediately precedes that one, where we have $t + 1 = 0$, vaniſhes, the Curve is quadrable: and the full Area is, what I have juſt now ſet down; or $\overline{-\frac{3}{2z} + \frac{1}{2z^2}} \times a\sqrt{1 - 2z + 3z^2}$.

* Of this Explication.

161. Schol.

161. Schol. It appears from what has been ſaid about ſimple trinomial Curves, that Things of a like Nature might be demonſtrated with reſpect to ſimple Quadrinomials, *viz.* whoſe Ordinates are thus generally expreſſed $az^{\theta-1}\times\overline{e+fz^{\eta}+gz^{2\eta}+hz^{3\eta}}|^{\lambda-1}$; that is we might ſhew certain Conditions, without which no quadrinomial Curve can be ſquared at all, ariſing from the Relations and Properties of the Exponents $\theta.\eta.\lambda$, analogous to what has been laid down in Art. 155—157 * concerning Trinomials: likewiſe, how to inveſtigate certain Relations of all the Symbols $\theta.\eta.\lambda.e.f.g.h$: which obtaining, the Curve is quadrable: like to what was done in Art. 153 *: and finally, how a general Rule might be laid down; by which it might be known, when any ſimple quadrinomial Curve may or may not be ſquared at all: analogous to that laid down in Art. 158 *. And ſo likewiſe with reſpect to Curves, whoſe Ordinates contain ſtill more complicate radical Expreſſions. But this would lead us beyond our Deſign: it being ſufficient to have ſhewn theſe ſeveral things with reſpect to Trinomials: and thereby pointed out the Way for doing the ſame in theſe others, whoſe Ordinates are more complex.

* Of this Explication.

* Of this Explication.

* Of this Explication.

162. A Curve of this Form $z^{\theta-1}\times\overline{e+fz^{\eta}}|^{\lambda-1}\times\overline{a+bz^{\eta}}=y$ may be ſquared.

1°. When it is $\frac{\lambda\eta}{\theta}=\frac{be}{af}-1$; and the firſt Term of the Series for the Area multiplied into the common Factor $z^{\theta}R^{\lambda}$, *i. e.* $z^{\theta}\times\overline{e+fz^{\eta}}|^{\lambda}\times\frac{\frac{1}{\eta}a}{re}$ exhibits the full Area.

2°. When $s\ (=\frac{\theta}{\eta}+\lambda)$ is any negative Integer, as -1, -2, -3, &c. unleſs $r\ (=\frac{\theta}{\eta})$ be either 0 or a negative Integer not greater than s (in which Caſe the Area is infinitely great) and the Area of the Curve conſiſts of ſuch a Number of Terms of the Series, multiplied into the common Factor $z^{\theta}R^{\lambda}$, as exceeds the Number of negative Units contain'd in s, by one.

Which two Rules are to be applied to both Forms of the Ordinate: and if they agree to either, the Curve is quadrable. if to neither, the Curve cannot be ſquared: and if they agree to both, the Curve is doubly quadrable.

Theſe two Rules will eaſily be demonſtrated, from the Nature of the Series expreſſing the Area of ſuch a Curve: which Series, by reducing the general one, contain'd in our Author's fifth Propoſition,

to

to the present Case, will be thus expressed $z^{\theta} \times \overline{e + fz^{\eta}}|^{\lambda} \times$

$$\frac{\frac{1}{\eta}a}{re} + \frac{\frac{1}{\eta}b - sfA}{\overline{r+1} \times e}z^{\eta} - \frac{\overline{s+1} \times fB}{\overline{r+2} \times e}z^{2\eta} - \frac{\overline{s+2} \times fC}{\overline{s+3} \times e}z^{3\eta} - \frac{\overline{s+3} \times fD}{\overline{r+4} \times e}z^{3\eta}, \&c.$$

From the Nature of the Progression of this Series, it is evident, that, if any one Term, after you pass the first, vanish, all that follow it *in inf.* must vanish likewise: because the capital Letter, which is a Factor in each Term, denotes the Coefficient of the immediately preceding Term.

Now let us suppose the second Term $\frac{\frac{1}{\eta}b - sfA}{\overline{r+1} \times e}z^{\eta} = 0$: thence we have $\frac{1}{\eta}b - sfA = 0$: from which by inserting $\frac{\theta}{\eta} + \lambda$ for s, and $\frac{\frac{1}{\eta}a}{re}$ for A, and reducing, we obtain this Equation $\frac{\lambda\eta}{\theta} = \frac{be}{af} - 1$. Therefore conversely if it be $\frac{\lambda\eta}{\theta} = \frac{be}{af} - 1$, the second Term, and consequently all the subsequent ones, of the Series, vanish.

2°. If it be $s = -1$ or $\overline{s+1} = 0$, the third Term, and consequently all the subsequent Terms, vanish: if it be $s = -2$, or $s + 2 = 0$, the fourth Term, and all following it, vanish; and so on. Therefore since no Term after the second can vanish, but upon supposing $s + 1 = 0$ or $s + 2 = 0$, &c. and since the other things, with respect to the Values of r, which make the Area infinitely great, easily appear from what has been said formerly, the several things that were to be demonstrated, are evident.

Ex. 1. Suppose $\frac{2z - 3z^{\frac{3}{2}}}{\sqrt{3 - 4z^{\frac{1}{2}}}} = y$: reduce it to Form, and it may stand thus, $z^{2-1} \times \overline{3 - 4z^{\frac{1}{2}}}|^{\frac{1}{2}-1} \times \overline{2 - 3z^{\frac{1}{2}}}$: which Form falls under the Consideration of this Article: and by comparing it with the general Form of Ordinates of this Class, $z^{\theta-1} \times \overline{e + fz^{\eta}}|^{\lambda-1} \times \overline{a + bz^{\eta}}$, we find $\theta = 2 . \eta = \frac{1}{2} . \lambda = \frac{1}{2} . r = (\frac{\theta}{\eta} =) 4 . s = 4\frac{1}{2}$: whence it appears that we cannot determine the Curve to be quadrable, by the second Part of the Rule; from this Expression of the Ordinate, where the Exponent of z under the Vinculum is positive: for it ought to have been s equal to some negative Integer for that purpose. Therefore I try it by the first Part of the Rule, to see whether it be $\frac{\lambda\eta}{\theta} = \frac{be}{af} - 1$: and since it is $a = 2 . b = -3 . e = 3 . f = -4$,

 I find

I find $\frac{be}{af} - 1 = (\frac{9}{8} - 1 = \frac{1}{8} =) \frac{\lambda\eta}{\theta}$, therefore I conclude the Curve may be squared; all the Terms of the Series after the first vanishing. Accordingly, by a due Substitution, the Area will be found to be $\frac{x^2}{3}\sqrt{3-4x^2}$.

Ex. 2. Let it be $\frac{4b^3z + 2az^3}{\sqrt{4a^3 + 3az^2}} = y$, this Ordinate, reduced to Form, stands thus, $z^{2-1} \times \overline{4a^3 + 3az^2}|^{\frac{1}{2}-1} \times \overline{4b^3 + 2az^2}$; or thus, $z^{3-1} \times \overline{3a + 4a^3z^{-2}}|^{\frac{1}{2}-1} \times \overline{2a + 4b^3z^{-2}}$. In the first Form we have $\theta = 2$. $\eta = 2$. $\lambda = \frac{1}{2}$. $r = 1$. $s = \frac{1}{2}$. $e = 4a^3$. $f = 3a$. $a = 4b^3$. $b = 2a$: where since neither s is a negative Integer nor $\frac{\lambda\eta}{\theta} = \frac{be}{af} - 1$: I proceed to the other Form where η is negative, there I find $\theta = 3$. $\eta = -2$. $\lambda = \frac{1}{2}$. $r = -\frac{3}{2}$. $s = -1$, thence I conclude the Curve is quadrable, and the Area consists of two Terms of the Series multiplied into the common Factor $z^\theta R^\lambda$: and since in this Case it is $e = 3a$. $f = 4a^3$. $a = 2a$. $b = 4b^3$, by substituting these Values into the general Theorem for the Area, *viz.* $z^\theta \times \overline{e + fz^\eta}|^\lambda \times \frac{\frac{1}{\eta}a}{re} + \frac{\frac{1}{\eta}b - sfA}{\overline{r+1} \times e} z^\eta - \frac{\overline{s+1} \times fB}{\overline{r+2} \times e} z^{2\eta}$ &c. you'll obtain $\overline{\frac{4b^3}{3a} - \frac{16a^2}{27} + \frac{2z^2}{9}}\sqrt{4a^3 + 3az^2}$ for the full Area, after a proper Reduction.

163. A Curve of this Form $z^{\theta-1} \times \overline{e + fz^\eta}|^{\lambda-1} \times \overline{a + bz^\eta + cz^{2\eta}} = y$ may be squared.

1°. When it is $\frac{r^2 + r}{s+1} = \frac{rbef - saf^2}{ce^2}$: provided it be not $r = 0$, nor yet $r = -1$ (in which Case the Area will be infinitely great) and the Series for the Area terminates with the second Term.

2°. When s is equal to any negative Integer greater than -1: provided r be not 0 nor any negative Integer so small or smaller than s: (in which Cases the Area becomes infinitely great) and the Series expressing the Area, will consist of as many Terms and one more, multiplied into $z^\theta R^\lambda$, as s contains negative Units.

The Curve is quadrable, if these Conditions or Properties agree to the Ordinate in either of it's Forms: it is doubly quadrable, if they agree to both: but if none of them agree to either, the Curve cannot be squared at all.

These Things will easily appear, by considering the Series for the Areas of Curves, exhibited by our Author: when it is reduced to the present

present Caſe: for it will ſtand thus $z^{\theta} \times \overline{e + fz^{n}}|^{\lambda} \times \overline{\frac{\frac{1}{n}a}{re} + \frac{\frac{1}{n}b - sfA}{\overline{r+1} \times e} z^{n}}$

$$\overline{+ \frac{\frac{1}{n}c - \overline{s+1} \times fB}{\overline{r+2} \times e} z^{2n} - \frac{\overline{s+2} \times fC}{\overline{r+3} \times e} z^{3n} - \frac{\overline{s+3} \times fD}{\overline{r+4} \times e} z^{4n}}, \text{ \&c.}$$

where it is evident, if any Term after the ſecond, become nothing, all the ſubſequent Terms *in inf.* muſt vaniſh: as appears by what was ſaid in the preceding Art. Now if the third Term $\frac{\frac{1}{n}c - \overline{s+1} \times fB}{\overline{r+2} \times e} z^{2n}$ be put equal to nothing, you'll thence obtain $\frac{r^2 + r}{s+1} = \frac{rbef - saf^2}{ce^2}$: whence the firſt Part of the Rule appears.

Again, if it be $s = -2$ or $s + 2 = 0$, the fourth Term vaniſhes: if it be $s = -3$, or $s + 3 = 0$, the fifth Term vaniſhes, and ſo on.

But it is evident, from what has been juſt now ſaid, that the third Term of the Series cannot vaniſh, unleſs it be $\frac{r^2+r}{s+1} = \frac{rbef - saf^2}{ce^2}$: and if it don't vaniſh, none of the ſubſequent Terms can vaniſh, unleſs s be a negative Integer greater than -1: therefore if none of theſe happen, the Series will run out infinitely: conſequently, ſince the other things, reſpecting the Value of r, are plain by Inſpection of the Series: the ſeveral things belonging to this Rule, are abundantly evident.

164. When a Curve is of this Form $z^{\theta-1} \times \overline{e + fz^{n}}|^{\lambda-1} \times \overline{a + bz^{n} + cz^{2n} + dz^{3n} + \text{\&c.}} = y$, where the laſt Factor $a + bz^{n} + cz^{2n}$, &c. conſiſts of four, or five, or more Terms: it is quadrable

1°. When $\frac{1}{n}d = \overline{s+2} \times fC$; or $\frac{1}{n}e = \overline{s+3} \times fD$, &c. reſpectively.

2°. When s is equal to a negative Integer not leſs than 3 or 4, &c. reſpectively: provided ſtill that no Term become infinitely great by r being equal to nothing, or any negative Integer ſo ſmall or ſmaller than s. It being ſuppos'd, you apply theſe two Rules to both Forms of the Series.

And if neither of theſe happen the Curve is not quadrable. Which things are evident enough from what hath been already ſaid.

165. Finally, when a Curve of this Form $z^{\theta-1} \times \overline{e + fz^{n} + gz^{2n}}|^{\lambda-1} \times \overline{a + bz^{n} + cz^{2n} + \text{\&c.}} = y$ is propoſed, where the Quantity under the Vinculum of the Root is a Trinomial; and the laſt Factor $a + bz^{n}$, &c. conſiſts of two, or three, or more Terms: proceed thus.

Set down in order the several Terms of the Series expressing the Area, by substituting the particular Values answering to the general Symbols in the Theorem exhibited by our Author in this fifth Prop.; and after you have set down as many Terms of that Series as there are Terms in the last Factor of the Expression of the Ordinate, *viz.* $a + bz^n + cz^{2n}$, &c.

1°. If the last two of these Terms be each equal to nothing, the Curve is quadrable: and the Terms preceding these last two, multiplied into the common Factor $z^\theta R^\lambda$, make up the Value of the Area.

2°. If the last one of these Terms only be found equal to nothing, and the Symbol t be any negative Integer (among which o is included) equal to the Number of Terms contained in the Factor $a + bz^n + cz^{2n} +$ &c. diminished by 2, the Curve is quadrable; and the Area is equal to the preceding Terms multiplied into the common Factor $z^\theta R^\lambda$.

3°. If neither of the two preceding Cases happen, then see whether t be a negative Integer as great or greater than the Number of Terms less one, contain'd in the Factor $a + bz^n + cz^{2n}$, &c. for if it be not, the Curve cannot be squared: if it be, continue to write down the Terms of the Series for the Area, until you arrive at that Term which immediately precedes the one where it is $t + 1 = 0$, or $t + 2 = 0$, or $t + 3 = 0$, &c. and if such Term be not nothing the Curve is not quadrable: but if it be nothing the Curve is quadrable; and you have got the Value of the Area: for all the subsequent Terms *in inf.* vanish: it being still supposed in the Cases where I have determined the Curve to be quadrable, that the Symbol r be not nothing, nor any negative Integer so small as to render any Term from the first to the last mentioned, infinitely great, in the Manner formerly explained; which will appear as you proceed in writing down the Forms of the Series: and moreover, that the above Rules be applied to both Terms of the Series; both to that in which the Powers of z are positive; and also to that in which they are negative.

These things will easily appear to any one that considers the Progression of the Series for expressing the Areas of Curves; together with what hath been said already upon this Subject. And this is all I shall say with respect to such Curves, to determine when they are quadrable, and when they are not: because although there might be other particular Rules given for this Purpose, yet they are too particular and intricate to be of any considerable Use.

166. Cor. It easily follows from what has been said with respect to binomial Curves, thus universally expressed $z^{\theta-1} \times \overline{e + fz^n}|^{\lambda-1} \times$

5 $a +$

$\overline{a + bz^{\eta} + cz^{2\eta} + \&c.} = y$, that they may be squared when s ($= \frac{\theta}{\eta} + \lambda$) is equal to any negative Integer (among which o is included) not less than the greatest Repetition of η among the Exponents of z in the last Factor $a + bz^{\eta} + cz^{2\eta} + \&c.$ which is always one less than the Number of regular Terms contain'd in that Factor: *i. e.* if there be none but the first Term a; the Curve may always be squared, when s is o, or -1, or -2, *&c.*: when there are two Terms, *viz.* $a + bz^{\eta}$, it may be done if s be -1, or -2, or -3, *&c.*: if there are three Terms, as $a + bz^{\eta} + cz^{2\eta}$, or $a * + cz^{2\eta}$: it can be done when s is -2, or -3, or -4, *&c.* and so for others: provided still that r ($= \frac{\theta}{\eta}$) be not o, nor any negative Integer so small, or smaller than s.

In simple Binomials, where it is $z^{\theta-1} \times \overline{e + fz^{\eta}}|^{\lambda-1} \times az^{o\eta}$, or $az^{\theta-1} \times \overline{e + fz^{\eta}}|^{\lambda-1} = y$, the Curve can never be squared in any other Case *: *Art. 147. of this Explication.
and in others, it very seldom happens *: supposing still that both Forms or Expressions of the Ordinate be tried by this Rule. *Art. 163. of this Explication.

SECT. V.

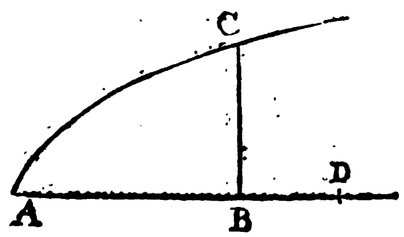
Of the Position and Limits of the Areas of Curves.

167. THE Areas of Curves, discovered by the Method of Quadratures, laid down by our Author in this Treatise, may have various Positions and Limits, according to the different Natures and Properties of the Curves, whose Areas are thus determined. Therefore that the young Geometrician may not be at a loss with respect to this Affair, I shall explain by some Examples, how this Variety happens: and then shew, by what means, the Area of a Curve, adjacent to any Portion of the Abscis, may always be found: when once the Area of the Curve in general is discovered, by the Methods taught by our Author.

168. In order to which, suppose $az^{\eta} = y$ be an Equation, expressing the Relation of the Abscis, and perpendicular Ordinate of a Curve. The Area, found by the Method of Quadratures, is $\frac{az^{\eta+1}}{\eta+1}$ *. *Art. 143. of this Explication.
And this Area, as likewise, every other curvilinear Area, found by the same Method, is bounded by the Ordinate drawn through the Extremity of the Abscis, and more or less of the Abscis and Curve-line, proceeding

proceeding from the Extremities of that Ordinate: but this Area, sometimes lyes upon the nearer Side of the Ordinate; (by which I mean the same Side the Beginning of the Abscifs lyes upon) at other times, it lyes upon the further Side of the Ordinate: it may happen likewise, that an Area may lye partly upon one Side, and partly upon the other, in some Curves, according to the different Terms, by which that Area is expressed. Now that you may know how far the Area extends, and upon what Side of the Ordinate it is situate, and consequently it's initial Limit, from whence it is to be computed, put the Expression of the Area equal to nothing, and the Value of z thence arising, or which agrees to that Supposition, will give you this initial Limit: *i. e.* an Ordinate drawn through the Extremity of z, taken of that Length; which is thus determin'd, will be the initial Limit of the Area: and so you'll have the Limits of the Area upon every Side.

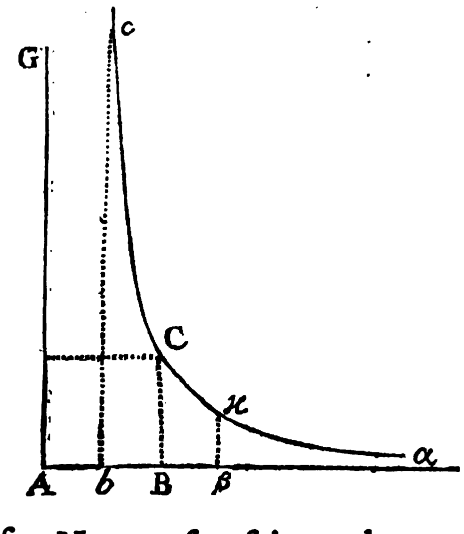
Thus in the Example just now mentioned, where the Area is $\frac{az^{\eta+1}}{\eta+1}$, put or suppose $\frac{az^{\eta+1}}{\eta+1}=0$: and if the Index $\eta+1$ be positive, it is evident, this will happen when $z=0$: and cannot happen in any other Case. But if the Index $\eta+1$ be negative, the Area denoted by the Expression above, *viz.* $\frac{az^{\eta+1}}{\eta+1}$, can never vanish, but when z is infinitely great. Hence you conclude, that in the former Case, the Area commences at the Beginning of the Abscifs, where also the Curve intersects it, and so reaches from that, along the Abscifs, all the way to the Ordinate. Whereas in the other Case, it reaches along the Abscifs infinitely produced beyond the Ordinate. Let AB be the Abscifs, and BC the perpendicular Ordinate, belonging to the Curve AC and take BD $=a$: suppose likewise, in the preceding Example, $\eta=\frac{1}{2}$: so that the Property of the Curve be expressed



by the Equation $az^{\frac{1}{2}}=y$; by inserting $\frac{1}{2}$ for η, the Area is $\frac{2az^{\frac{3}{2}}}{3}=$ $\frac{2}{3}$BD $\times\sqrt{}$AB$^{cub.}$ or $\frac{2}{3}$AB $\times$ BC, which is the Area ABC: since the Expression $\frac{2}{3}az^{\frac{3}{2}}$ can never be equal to nothing, but when $z=$ AB is nothing.

But, if the Relation of AB to BC be expressed by this Equation $x^{2}y=a$ or $y=ax^{-2}$ (see the Fig. in the next Page) the Area, by inserting -2 for η, becomes

becomes $-ax^{-1}$ or $-\frac{a}{x}$: which denotes the Area BCα lying upon the further Side of the Ordinate BC, and extending along the Line Bα infinitely produced: for if you put $-\frac{a}{z}=0$; z must be infinitely great, since a is a given Quantity: which shews that the initial Limit of the Area removes to an infinite Distance. This Area comes out negative, in regard it is situate upon the further Side of the Ordinate BC, and wholly above the Abscis AB produc'd: for since the Area ABC upon the nearer Side of the Ordinate, and above the Abscis is positive, the Area BCα ought to be represented negative, from the opposite Nature of positive and negative Quantities. Thus if we suppose $a=1$, and $z=AB=1$ also, we have $-\frac{a}{z}=-1$ for the Area BCα: whose positive Value therefore is $1=AB^2$ or the Square AC, for in this Case AB = BC.



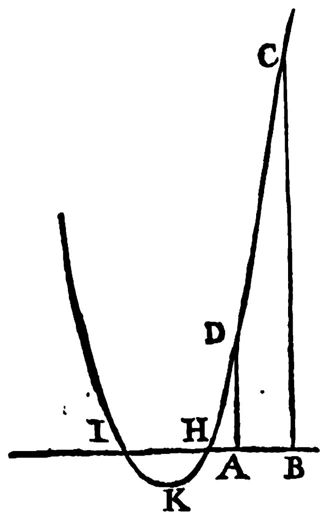
169. After the same Manner, and for the like Reason, it will be found, that the Area of every Curve, which consists of any Number of simple Terms, in every one of whose Numerators there is contain'd some positive Power of z, will have it's initial Limit, where the Abscis begins: And every Area whose Expression contains some positive Power of z in the Denominator of every Term, will be situate upon the further Side of the Ordinate, and extend itself along the Abscis infinitely produc'd beyond the Ordinate.

Let $a+bz+cz^2=y$ be an Equation, containing the Relation of the Abscis and Ordinate of a Curve. The Area, by Art. 145 *, is $az+\frac{1}{2}bz^2+\frac{1}{3}cz^3$: where the positive Powers of z run through the Numerators of all the Terms: therefore the initial Limit of that Area is at the Beginning of the Abscis: and the Area is bounded by an Ordinate drawn through the Beginning of the Abscis, the Ordinate belonging to the Abscis z, and the Abscis and Curve-line intercepted betwixt these two Ordinates.

* Of this Explication.

170. I have said, you must draw an Ordinate to the Curve at the Beginning of the Abscis in this Example, in order to have the initial Limit of the Area: because the Curve does not meet with the Abscis at the Beginning, as is evident: for since $y=a+bz+cz^2$; when the

the Abſciſs $z = 0$, we have $y = a$: which ſhews the Ordinate at the Beginning of the Abſciſs, to be equal to the given Quantity a, as is repreſented in the Fig. annex'd: where, if you draw AD perpendicular to the Abſciſs AB; meeting the Curve in the Point D, the Area ABCD is that denoted by the Expreſſion $az + \frac{1}{2}bz^2 + \frac{1}{3}cz^3$; and AD $= a$. And that you may know in every Caſe, where the Curve meets with the Baſe or Abſciſs produc'd, if need be, towards both Hands, ſuppoſe the Value of y equal to nothing, and the Value or Values of z, which ariſe from that Suppoſition, will ſhew you the Point or Points of the Abſciſs, in which the Curve-line meets with it. Thus in the Example juſt now mentioned, where $y = a + bz + cz^2$, put $cz^2 + bz + a = 0$: reſolve the Equation, and you'll find a double Value of z, *viz.* $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}$, both which Values are negative, and ſhew that if the Abſciſs AB be produced backwards, and the Points H and I be taken ſuch, that AH $= \frac{b - \sqrt{b^2 - 4ac}}{2c}$; and AI $= \frac{b + \sqrt{b^2 - 4ac}}{2c}$, the Curve CD continued, will meet with the Abſciſs produc'd, in the Points H and I: and in no other Points, in regard z has no other Value in the Equation $a + bz + cz^2 = 0$, *i. e.* when y is equal to nothing.



But if the given Quantities a, b and c be ſuch that it be $b^2 = 4ac$, then the two Points H and I fall into one, becauſe $\sqrt{b^2 - 4ac}$ becomes nothing: and ſo there will be but one Point of the Abſciſs, in which the Curve will meet with it, diſtant from the Beginning of the Abſciſs, produc'd backwards, by the given Quantity $\frac{b}{2c}$.

Again, if $4ac$ be greater than b^2, the Expreſſion $\sqrt{b^2 - 4ac}$ is impoſſible or imaginary, and ſo there will be no poſſible Value of z, or the Roots of the Equation $cz^2 + bz + a = 0$, are imaginary: and this ſhews that the Curve can no where meet with the Abſciſs, in this Caſe. And by the ſame Method, you may always find whether any Curve propoſed meets with it's Abſciſs or not; and if it do, in what Point or Points of the Abſciſs it meets with it. The Rule is this.

171. Put the Value of the Ordinate, expreſſed by given Quantities and the Abſciſs z, equal to nothing: and the Values of z thence ariſing, will be the Diſtances of theſe Points of Concurrence from the Beginning of the Abſciſs: and if all the Roots of the Equation be impoſſible, there is no Point of Concurrence.

172. But

172. But to return to the Limits of Areas. If the positive Powers of z run through the Denominators of all the Terms of the Expression of the Area, the initial Limit runs off to an infinite Distance; and so the Area extends itself from the Ordinate, on the further Side, all along the Abscis infinitely produc'd beyond the Ordinate. Thus in the Fig. p. 135, if the Relation of AB to BC were defin'd by this Equation $z^{-2}+z^{-\frac{3}{2}}=y$: the Area thence arising, is $-z^{-1}-2z^{-\frac{1}{2}}$, or $-\frac{1}{z}-\frac{2}{z^{\frac{1}{2}}}$ *; where the positive Powers of z are found in the Denominators of the several Terms; and both Terms are negative: therefore it is the Area BCα, infinitely extended upon the further Side of the Ordinate: for if you put $-\frac{1}{z}-\frac{2}{z^{\frac{1}{2}}}=0$, you find the Value of z, infinite: which shews that that Area can never be nothing, while the Abscis AB is of a finite Length: but it is evident from the very Nature of the Expression $-\frac{1}{z}-\frac{2}{z^{\frac{1}{2}}}$, that, as z or AB diminishes, it increases; and as AB increases, it continually diminishes: therefore, it must lye beyond the Ordinate BC: and it also lyes wholly above the Abscis, as will appear by the preceding Art. For if you put $\frac{1}{z^2}+\frac{1}{z^{\frac{3}{2}}}(=y)=0$, z can have no finite Value. From both which Considerations and Circumstances it appears, why it is represented negative. Let AB $(=z)=4$, then $-\frac{1}{z}-\frac{2}{z^{\frac{1}{2}}}=-\frac{1}{4}-\frac{2}{2}=-1\frac{1}{4}$, which being made positive, we have $1\frac{1}{4}$ for the Value of the Area BCα.

* Art. 145. of this Explication.

173. But sometimes the Area of a Curve will lye partly above, and partly below the Abscis or Base: which happens, when the Curve decussates or crosses the Base, and passes to the opposite Side, as is represented by the Figures in p. 136, 140 and 141. In order to discover whether there be any Points of Decussation; and if they be, where they are; do this. Find the Points of the Abscis, where the Curve meets with it, by Art. 171 *. And if there be no Point of Concurrence, it is evident, there can be no Point of Decussation: in which Case the Area must lye wholly upon one Side of the Abscis. But if you find Points of Concurrence, then you may know whether any of these be a Point of Decussation, thus: add to, and subtract from the given Length of the Abscis at the Point of Concurrence, a very small Quantity, which call p: and instead of z, in the Value of y, insert $z+p$, and then $z-p$: then neglecting all the Terms, where p rises higher than the Root, or other lowest Power; observe whether the Values of

* Of this Explication.

y, which remain, be, the one affirmative, the other negative: if they be, there is a Point of Decuſſation, at that Point of Concurrence: if not, but that both Values be affirmative, or both negative, there is no Point of Decuſſation, in that Place: and if one of the Values involve any impoſſible Suppoſition, the Curve is reflected back towards the Parts from whence it came.

The Reaſon of this Rule will eaſily appear, by conſidering, that, when there is a Point of Decuſſation, the Values of the Ordinate towards either hand of that Point, muſt be affected with oppoſite Signs: and the Powers of p above the Root, or above the loweſt Power, may be neglected, becauſe p is ſuppos'd to be diminiſhed infinitely. (See Art. 17*. and Note upon it.) The Thing will be further illuſtrated by Examples.

* Of the Quadrature of Curves.

Ex. 1. Let AB $= z$, and BC $= y$, be the Abſciſs and Ordinate of the Curve CDE (ſee the Fig. in p. 141.) and ſuppoſe $\frac{1}{z^2} - \frac{1}{z^{\frac{3}{2}}} = y$ expreſs the Relation of the Abſciſs and Ordinate: then putting $\frac{1}{z^2} - \frac{1}{z^{\frac{3}{2}}} = 0$, we have $z =$ infinite; and $z = 1$: ſo that at the Extremity of the Abſciſs 1, there is a Point of Concurrence, wherefore I take $z + p$, and then $z - p$, and inſert theſe for z, in the Value of y, *viz.* $\frac{1}{z^2} - \frac{1}{z^{\frac{3}{2}}}$: thence you obtain 1° $\frac{1}{\overline{z+p}|^2} - \frac{1}{\overline{z+p}|^{\frac{3}{2}}} = y$, i. e. by inſerting the Value of z at that Point, *viz.* 1, $\frac{1}{\overline{1+p}|^2} - \frac{1}{\overline{1+p}|^{\frac{3}{2}}} = y$, where $\frac{1}{\overline{1+p}|^2} - \frac{1}{\overline{1+p}|^{\frac{3}{2}}}$ is plainly negative: ſince $1 + p$ is greater than 1.

Again, by inſerting $1 - p$ for z you'll have 2° $y = \frac{1}{\overline{1-p}|^2} - \frac{1}{\overline{1-p}|^{\frac{3}{2}}}$ poſitive, becauſe $1 - p$ is leſs than 1: wherefore if you take AD $= 1$ the Point D is a Point of Decuſſation.

Ex. 2d. In the Curve CDHI, (ſee the Fig. in p. 136.) whoſe Abſciſs is AB $= z$, and Ordinate BC $= y$: let it be $y = a + bz + cz^2$, by putting $a + bz + cz^2 = 0$, you obtain $z = \frac{-b + \sqrt{b^2 - 4ac}}{2c}$ and $z = \frac{-b - \sqrt{b^2 - 4ac}}{2c}$, denoting two Points of Concurrence as before. For the firſt, take $\frac{-b + \sqrt{b^2 - 4ac}}{2c} + p$, and then $\frac{-b + \sqrt{b^2 - 4ac}}{2c} - p$, or by ſubſtituting q for $\frac{-b + \sqrt{b^2 - 4ac}}{2c}$, (for Brevity's ſake) $q + p$ and $q - p$: inſert

theſe

these for z in the Equation $y = a + bz + cz^2$: thence you have 1°

$$y = \begin{array}{l} cq^2 + 2cq \\ + bq \;\; + \;\; b \\ + a \end{array} p + cp^2$$

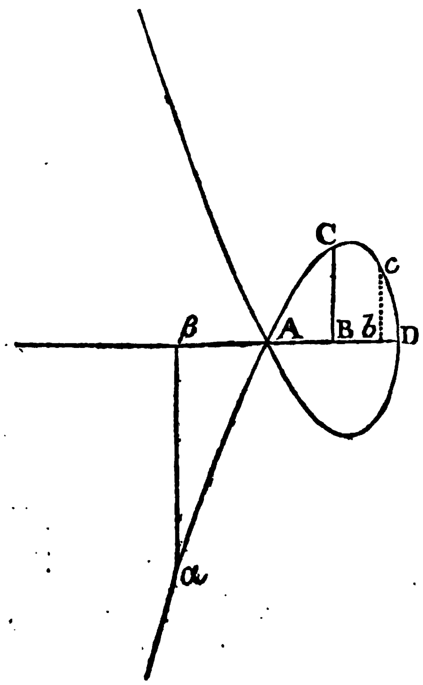
$$2^\circ\, y = \begin{array}{l} cq^2 - 2cq \\ + bq \;\; - \;\; b \\ + a \end{array} p + cp^2$$

Neglect or throw away what is common in these two Values of y; what remains is $2cqp + bp$ in the first; and $-2cqp - bp$ in the second: which are, the one affirmative; the other negative: therefore if you produce the Abscis AB backwards to the Point H, and take $AH = \frac{b - \sqrt{b^2 - 4ac}}{2c}$, the Point H will be a Point of Decussation. And after the same Manner, if you produce it still further, in the same Direction, to the Point I, so that it be $AI = \frac{b + \sqrt{b^2 - 4ac}}{2c}$, the Point I would be found to be another Point of Decussation.

Ex. 3. Suppose $4z^2 - z^3 = y^2$ express the Relation of the Abscis AB $= z$ and Ordinate BC $= y$, in the Curve ACD (see the Fig. in p. 140.) put the Value of y equal to nothing, or $4z^2 - z^3 = 0$: hence you obtain two Values of z, *viz.* $z = 0$; and $z = 4$: wherefore taking AD $= 4$, the Curve meets with the Abscis in the Points A and D. Therefore to find whether the Curve decussates the Base in the Point A, where $z = 0$, put $+p$, and then $-p$ instead of z in one and the same Value of y; suppose the positive Value of y, *viz.* $+\sqrt{4z^2 - z^3}$ or $z\sqrt{4 - z}$: (for it is $y = \pm\sqrt{4z^2 - z^3} = \pm z\sqrt{4 - z}$) hence you'll have $+p\sqrt{4 - p}$ and $-p\sqrt{4 + p}$ for the two Values of y; of which the first being positive, the other negative, shews the Curve decussates the Base at the Point A.

Again, for the other Point of Concurrence, *viz.* D; where you have $z = 4$, by inserting $\overline{4 + p}$ and then $\overline{4 - p}$ in place of z in the Equation $y = z\sqrt{4 - z}$, you have 1° $y = \overline{4 + p}\sqrt{-p}$, and 2° $y = \overline{4 - p}\sqrt{p}$: the first of which Values being impossible, hence you conclude that the Curve proceeds no further that way, but returns back towards the Beginning of the Abscis, making either a Cuspis, or continued Arch: which last it doth in this Case, forming another half upon the lower Side of the Base AB, exactly similar and equal to the half above: since the Property of the Curve is represented by the Equation $y^2 = 4z^2 - z^3$, and consequently $y = \pm\sqrt{4z^2 - z^3}$: which shews

ſhews that the Ordinate, at any Point of the Abſciſs, meets the Curve on both Sides, at the ſame Diſtance, as is repreſented by the Fig. annex'd, being the *nodated Parabola*, which conſtitutes the ſixty-eighth Species of Curves of the ſecond Order, according to our Author's Enumeration of Lines of the third Order.



And after the like Manner, in any other Caſe, it will appear whether the Curve proceed forward or not, at any Point of Concurrence with the Abſciſs: as for other Circumſtances, which may occur, with reſpect to the Points of Concurrence and Decuſſation, the Reflection and Continuation of Curves, they muſt be left to the young Geometrician's own Sagacity: becauſe the inſiſting upon theſe things in this Place would carry us too far away from our original Deſign [1].

174. If the Curve decuſſate the Baſe or Abſciſs betwixt the initial Limit of the Area and the Ordinate: whereby one Part of the curvilinear Area lyes above the Baſe, and another Part below, the Area which ariſes by the Method of Quadratures, will exhibit the Difference of the two Parts of the Area above and below the Baſe, lying betwixt the initial Limit and the Ordinate belonging to any given Abſciſs.

Thus in the Curve CDE (ſee the Fig. in the oppoſite Page) whoſe Abſciſs is AB$=z$, and perpendicular Ordinate BC$=y$, ſuppoſe it be $z^{-2} - z^{-\frac{3}{2}}$ $=y$: it was ſhewn above *, that the Curve will decuſſate it's Baſe AB, in one Point only, at the Diſtance 1 from A the Beginning of the Abſciſs: therefore, if we take AD$=1$, D is the Point, at which the Curve croſſes the Baſe, and afterwards extends itſelf infinitely below it: Now the Area of the Curve is $-\frac{1}{z}+\frac{2}{z^{\frac{1}{2}}}$ *: which Area extends itſelf, upon the further Side of the Ordinate BC, along the Abſciſs infinitely produc'd towards α; becauſe of the poſitive Powers of z, in the Denominators of the Terms *: but that Expreſſion denotes the Diffe-

* Art. 173. of this Explication.

* Art. 145. of this Explication.

* Art. 168, 169. of this Explication.

[1] Some Things of this ſort, reſpecting the Symptoms of Curves, will occur afterwards in the Notes upon the laſt Scholium.

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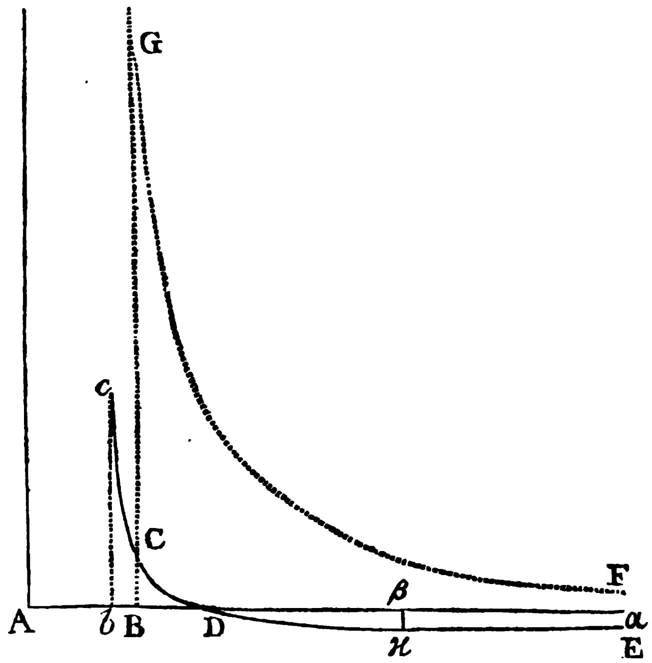
rence betwixt the Area BCD, above the Base, and the Area DEα, extended infinitely below the Base.

175. The thing is explain'd thus. Suppose the Curve GF to be describ'd by the Ordinate BG $= \frac{1}{z^2}$: then $- \frac{1}{z}$, which is the first Term of the Expression $- \frac{1}{z} + \frac{2}{z^{\frac{1}{2}}}$, denotes the Area GB$\alpha$, infinitely extended along Bα and lying wholly above it *. Again, let GC be always equal to $z^{-\frac{3}{2}}$ or $\frac{1}{z^{\frac{3}{2}}}$, the other



*Art. 168. of this Explication.

Part of the Expression of the proposed Ordinate, so that the Points G, C always lye in the Curve-lines GF and CDE: then, as easily appears from what has been formerly said, the Area belonging to the Ordinate $z^{-\frac{3}{2}} =$ CG, *viz.* $- \frac{2}{z^{\frac{1}{2}}}$ must be that which reaches from CG, all the way betwixt the two Curves CDE, GF infinitely produced: therefore, since the Ordinate BC $=$ BG $-$ BC, the Area of the Curve having BC for it's Ordinate, will be made up of the Difference of the former two Areas by subtracting the latter, *viz.* $- \frac{2}{z^{\frac{1}{2}}}$ from the former, *viz.* $- \frac{1}{z}$: wherefore $- \frac{1}{z} + \frac{2}{z^{\frac{1}{2}}}$ denotes the Area GBαF $-$ GCDEF, *i. e.* since GCDαF is common to both, CBD $-$ DEα. And since the Part CBD, lying beyond the Ordinate and above the Base, is always represented negative, in the algebraical Expression, for the Reason already assign'd *: the Part DEα lying below the Base and beyond the Ordinate will be positive, as being opposite in it's Nature to the other: therefore if the Expression $- \frac{1}{z} + \frac{2}{z^{\frac{1}{2}}}$ be in the whole negative, it gives the Excess of CBD above DEα; but if, on the whole, it

*Art. 168. of this Explication.

it be positive, it denotes the Excess of DEα above CBD; but if it amount to nothing, it denotes that the Areas CBD, DEα are equal.

Suppose AB $=\frac{1}{9}$, then $-\frac{1}{z}+\frac{2}{z^{\frac{1}{2}}}=-9+6=-3$, therefore in that Case CBD would exceed DEα by 3, but if it be AB $=\frac{4}{9}$, then $-\frac{1}{z}+\frac{2}{z^{\frac{1}{2}}}=-\frac{9}{4}+3=\frac{3}{4}$ positive, which shews that DEα exceeds CBD by so much. Lastly, if it were AB $=\frac{1}{4}$, then $-\frac{1}{z}+\frac{2}{z^{\frac{1}{2}}}=-4+4=0$: in which Case CBD and DEα are exactly equal.

176. But if you would have the Sum of both Areas CBD and DEα; find the Value of the Area DEα by itself, *viz.* by supposing the Point B the Foot of the Ordinate, to fall in with the Point of Decussation D: and so having obtain'd the Area DEα, lying all upon one Side of the Base: and having formerly found the Difference of the two Areas CBD and DEα, above and below the Base, you easily obtain the Area CBD also by itself, and consequently the Sum of both. Thus when B and D coincide AB $=z=1$ *: therefore the Area DEα $=(-\frac{1}{z}+\frac{2}{\sqrt{z}}=)\ 1$: but if AB $(=z)=\frac{1}{9}$ as formerly, then $-\frac{1}{z}+\frac{2}{z^{\frac{1}{2}}}=-3$, which is the Difference of the two Areas, the Difference then being made positive is 3: which is the Excess of the Part lying above the Base, above the Part lying below. Therefore the Part lying above is $3+1=4$, and the Sum of both is $4+1=5$: and so in other Cases.

* Art. 173. Ex. 1. of this Explication.

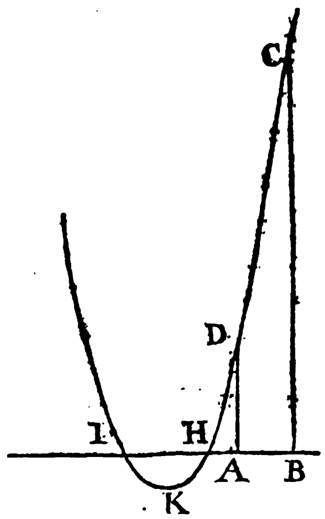
177. Cor. Hence it follows, that, if the Absciss AB, be of such a Length, as to make the Areas CBD and DEα above and below the Base, equal, and you take the Absciss A*b* less than AB, and draw the Ordinate *bc*: and then suppose A*b* $=z$, the Expression $-\frac{1}{z}+\frac{2}{z^{\frac{1}{2}}}$ will exhibit the Value of the Area BC*cb*, for that is the same with *c*D*b* — DEα. And thus the Ordinate BC, so drawn, will be an initial Limit, from whence the Area is computed, when the Abscissi A*b* is less than AB. And thus there are two initial Limits in this Curve: the one at an infinite Distance from A, the Beginning of the Abscissi, towards α; the other is an Ordinate drawn from that Point of the Abscissi; which is distant from A, by $\frac{1}{4}$AD $=\frac{1}{4}$. These initial Limits, both arise, by putting the Expression of the Area, *viz.* $-\frac{1}{z}+\frac{2}{z^{\frac{1}{2}}}=0$:

$=0$: for that will happen, both when z is infinitely great, and alſo when $z=\frac{1}{4}$: as will appear by reſolving the Equation.

178. By conſidering of this, you'll know how to apply any like Caſe, that may happen: where, by putting the Expreſſion for the Area equal to nothing, as directed above, in order to find it's initial Limit, you obtain different Values of z, or Roots of the Equation: for theſe different Values of z, give you ſo many different initial Limits, from any one of which, the Area may be computed. The Reaſon of which is evident: becauſe the curvilinear Area lying betwixt any two initial Limits amounts to nothing, in the algebraical Notation; the Parts above and below the Baſe, which are oppoſite Quantities, exactly compenſating one another.

But although the curvilinear Area may be thus computed from any one of the initial Limits mentioned: yet it is moſt ſimple and natural to compute the Area from the higheſt initial Limit. Thus when BC (ſee the Fig. in p. 141.) is an initial Limit: if you take $Ab=z$ leſs than AB, *e.g.* $\frac{1}{9}$, the Area thence ariſing, *viz.* $-\frac{1}{z}+\frac{2}{\sqrt{z}}=-3$, is the Area BC*cb*; as well as the Difference betwixt D*bc* and DE*a*.

179. Let it be $y=a+bz+cz^2$ (ſee the Fig. annexed) as before, the Area is $az+\frac{1}{2}bz^2+\frac{1}{3}cz^3$: put that Expreſſion of the Area $az+\frac{1}{2}bz^2+\frac{1}{3}cz^3=0$: thence you obtain, 1°. $z=0$. 2°. $z=\frac{-3b+\sqrt{9b^2-48ac}}{4c}$. 3°. $z=\frac{-3b-\sqrt{9b^2-48ac}}{4c}$. Whence you have three initial Limits: one at the Point A, where the Abſciſs begins: the other two beyond the Beginning of the Abſciſs AB, *viz.* produced towards H, as is denoted by the negative Values of z. The two laſt evidently fall into one, if ſo be $16ac=3b^2$: for then the common radical Part $\sqrt{9b^2-48ac}=0$, in the two negative Values of z: ſo that, in that Caſe, they both become $z=-\frac{3b}{4c}$. Let us ſuppoſe $a=1$. $b=4$. $c=3$: then $16ac=48$, $=3b^2$, therefore $\sqrt{9b^2-48ac}=0$: and ſo $z=\frac{-3b+\sqrt{9b^2-48ac}}{4c}=-1$; and $z=\frac{-3b-\sqrt{9b^2-48ac}}{4c}=-1$ likewiſe. In this Caſe the



the Points of Decussation H and I will be such, that AH $= -\frac{1}{3}$ and AI $= -1$: as will easily appear, by substituting these Values of the Symbols a, b, and c in place of them, in the Equations above, for determining these Points (see Ex. 2. Art. 173 *). Hence if you take AI $= -1$, *i. e.* produce the Abscis AB backwards to I, so that AI $= 1$, the Points A and I will be the only initial Limits: and the Area ADH = HKI *. To find the Area ADH, substitute $-\frac{1}{3}$ for z in the general Expression of the Area, *viz.* $az + \frac{1}{2}bz^2 + \frac{1}{3}cz^3$, and 1, 4, 3 for a, b, c; and it becomes $-\frac{1}{3} + \frac{2}{9} - \frac{1}{27} = -\frac{4}{27}$. Again, to find the Difference betwixt the two Areas ADH and HKI, substitute -1 for z in the Expression for the Area, *viz.* $az + \frac{1}{2}bz^2 + \frac{1}{3}cz^3$, and you'll have $-1 + 2 - 1 = 0$ for the Difference.

* Of this Explication.

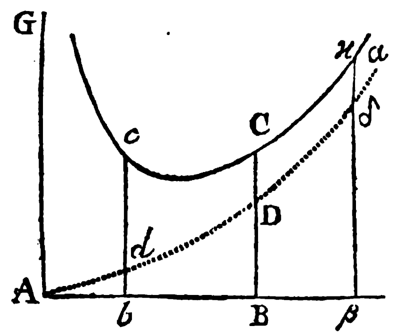
*Art. 178. of this Explication.

180. Resuming Ex. 3^d. of Art. 173 *, *viz.* $4z^2 - z^3 = y^2$, belonging to the Curve represented by the Fig. in p. 140, by extracting the square Root, it is $y = z\sqrt{4-z}$: therefore the Area is $\frac{-16-6z}{15} \times \overline{4-z}|^{\frac{3}{2}}$ *. To find the initial Limit, put $\frac{-16-6z}{15} \times \overline{4-z}|^{\frac{3}{2}} = 0$: and you find this to happen in two Cases, and no more, *viz.* 1°. when $z = 4$; 2°. when $z = -\frac{8}{3}$: therefore if you take in the Abscis AB, produc'd backwards, $A\beta = \frac{8}{3}$, and draw the perpendicular Ordinate $\alpha\beta$: the two initial Limits will be at $\alpha\beta$, and at the Point D, where the Curve meets with the Abscis: for it was demonstrated already that AD $= 4$ *. Wherefore take the Abscis AB $= 2 = z$: and the general Expression of the Area, *viz.* $\frac{-16-6z}{15} \times \overline{4-z}|^{\frac{3}{2}}$, becomes $\frac{-16-12}{15} \times 2^{\frac{3}{2}} = -\frac{28}{15}\sqrt{8}$: which being reckoned from the initial Limit D, denotes the Area DBC; or being reckoned from the initial Limit $\alpha\beta$ denotes the Difference of the two Areas ABC and $A\beta\alpha$ lying above and below the Base.

* Art. 147, 149. of this Explication.

*Art. 173. Ex. 3. of this Explication.

181. Sometimes the Area, which arises by the Method of Quadratures, is situate partly upon one Side, and partly upon the other Side of the same Ordinate: which happens when the Value of y is made up of simple Terms, whereof some have the positive Powers of z in the Numerators; and others have them in the Denominators.



Thus let the Relation of the Abscis AB $= z$, and Ordinate BC $= y$, be defin'd by this Equation $z^2 + \frac{1}{z^2} = y$: (see the Fig. annex'd.) The Area, by the Method of Quadratures, will be $\frac{1}{3}z^3 - \frac{1}{z}$ *: the first Part of which, $\frac{1}{3}z^3$ denotes an Area lying betwixt the Beginning

*Art. 145. of this Explication.

ginning of the Abscis and the Ordinate; the other Part of the Expression, *viz.* $-\frac{1}{z}$, denotes an Area situate upon the further Side of the Ordinate, and extends itself to an infinite Distance upon that Side *. *Art 169. of this Explication.

The Thing is explained after the same Manner as the Example in Art. 175 *. For let AdD be another Curve, whose Ordinate $BD = z^2$, the first Part of the Expression $z^2 + \frac{1}{z^2} = BC$; then, it is evident, the remaining Part of the Ordinate BC, *viz.* $DC = \frac{1}{z^2}$, the other Part of the Expression $z^2 + \frac{1}{z^2}$. Now the Area belonging to the Ordinate $BD = z^2$, *viz.* $\frac{1}{3}z^3$, is ABD; and the Area belonging to the other Ordinate $DC = \frac{1}{z^2}$, *viz.* $-\frac{1}{z}$, is $CD\alpha$, lying upon the further Side of the Ordinate BC, and extending itself infinitely betwixt the two Curve-lines $cC\alpha$ and $AdD\alpha$, after the Manner formerly explained in that Art. 175 *. * Of this Explication.

182. If we shall seek for the initial Limit of the Area in this Case, *viz.* by putting the Area $\frac{1}{3}z^3 - \frac{1}{z} = 0$, we find only one Value of z, *viz.* $z = \sqrt[4]{3}$: Take the Abscis $AB = \sqrt[4]{3}$, draw through the Point B, the Ordinate BDC, substitute $\sqrt[4]{3}$ for z in the Expression of the Area, and you'll have the Area $\frac{1}{3}z^3 - \frac{1}{z} = \frac{1}{3}\sqrt[4]{27} - \frac{1}{\sqrt[4]{3}} = \sqrt[4]{\frac{27}{81}} - \sqrt[4]{\frac{1}{3}} = 0$. This shews that when the Ordinate BDC is drawn at the Distance $\sqrt[4]{3}$ from A, the Areas ABD and $CD\alpha$ are equal.

183. If the given Abscis $A\beta$ be greater than the Abscis AB, which reaches to the Limit, the Expression $\frac{1}{3}z^3 - \frac{1}{z}$ is positive and denotes the Area $B\beta\kappa C$: but when the Abscis Ab is less than AB, the Expression $\frac{1}{3}z^3 - \frac{1}{z}$ is negative: but being changed into it's opposite, will denote the Area $BbcC$. For when $z = A\beta$, then $\frac{1}{3}z^3 - \frac{1}{z} = A\beta\delta - \alpha\delta\kappa = ABD + B\beta\delta D - CD\alpha + CD\delta\kappa = B\beta\delta D + CD\delta\kappa$ or $B\beta\kappa C$ for $ABD = CD\alpha$. Again, when $z = Ab$, then $\frac{1}{3}z^3 - \frac{1}{z} = Abd - \alpha dc$: but $Abd = ABD - bBDd$, and $\alpha dc = \alpha DC + DCcd$: subtract the latter from the former, and you have $\frac{1}{3}z^3 - \frac{1}{z} = ABD - bBDd - \alpha DC - DCcd = - bBDd - DCcd = - BbcC$.

Let it be $A\beta = 2$, then $\frac{1}{3}z^3 - \frac{1}{z} = \frac{8}{3} - \frac{1}{2} = 2\frac{1}{6}$, the Value of the Area $B\beta\kappa C$ lying betwixt the Ordinate $\beta\kappa$ and initial Limit BC. Again

 suppose

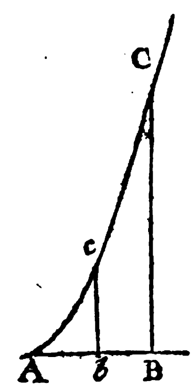
ſuppoſe A$b = \frac{2}{3}$, then $\frac{1}{3}z^3 - \frac{1}{z} = \frac{8}{81} - \frac{3}{2} = -1\frac{65}{162}$: which being made poſitive, is the Value of the Area BbcC.

184. Sometimes no initial Limit of the Area can be found at all. This will always happen, when, putting the Expreſſion for the Area equal to nothing, the Equation has no real Root or Value of z. Thus if the Curve be ſuch that $z^2 - \frac{1}{z^2} = y$: then the Area, by the Method of Quadratures, is $\frac{1}{3}z^3 + \frac{1}{z}$ *: if we put $\frac{1}{3}z^3 + \frac{1}{z} = 0$, thence we have $z^4 + 3 = 0$ or $z = \sqrt[4]{-3}$, an impoſſible Expreſſion: and thus, there is not any poſſible or real Value of z, in the Equation $\frac{1}{3}z^3 + \frac{1}{z} = 0$, or $z^4 + 3 = 0$. Therefore no initial Limit, from whence the curvilinear Area ought to be computed, can be found. Again, ſuppoſe the Curve to be defin'd by this Equation $z\sqrt{a^2 + z^2} = y$: The Area, by the Method of Quadratures, is $\frac{a^2+z^2}{3}\sqrt{a^2 + z^2}$ *. Put it equal to nothing, *i. e.* $\frac{a^2+z^2}{3}\sqrt{a^2 + z^2} = 0$, and you find this can never happen, unleſs it could be $a^2 + z^2 = 0$ or $z = \sqrt{-a^2}$, which is impoſſible. Therefore no initial Limit can be found: and ſo in other Caſes.

* Art. 145. of this Explication.

* Art. 147. of this Explication.

185. When this happens, you may, notwithſtanding, find the Area which is adjacent to any given Part of the Abſciſs, and lying betwixt the two Ordinates at the two Extremities of that given Part of the Abſciſs, *viz.* thus: ſubtract the Area belonging to the leſſer Abſciſs, ariſing by the Method of Quadratures, from the Area belonging to the greater Abſciſs, the Difference is the Value of the Area adjacent to the Difference of the Abſciſſes.



Let the Curve AC (ſee Fig. annexed) whoſe Baſe is AB, be defin'd by the Equation $z\sqrt{a^2 + z^2} = y$, mentioned juſt now: the Area, by the Doctrine of Quadratures is $\frac{a^2+z^2}{3}\sqrt{a^2 + z^2}$. Wherefore, to find the curvilinear Area adjacent to any given Part of the Abſciſs as Bb: let AB $= 2$, A$b = 1$: ſubſtitute $2 =$ AB for z in the general Expreſſion of the Area $\frac{a^2+z^2}{3}\sqrt{a^2 + z^2}$, and it becomes $\frac{a^2+4}{3}\sqrt{a^2 + 4}$: again ſubſtitute $1 =$ Ab for z, and you'll have $\frac{a^2+1}{3}\sqrt{a^2 + 1}$: ſubtract this

this last from the other, and it gives $\frac{a^2+4}{3}\sqrt{a^2+4} - \frac{a^2+1}{3}\sqrt{a^2+1}$, for the just Value of the Area B*bc*C adjacent to the given Part of the Absciss B*b*.

186. Universally: call the greater Abscis AB $= z$; the lesser A*b* $= \zeta$: and then the Area B*bc*C is $\frac{a^2+z^2}{3}\sqrt{a^2+z^2} - \frac{a^2+\zeta^2}{3}\sqrt{a^2+\zeta^2}$. If ζ (= A*b*) = o, then $\frac{a^2+z^2}{3}\sqrt{a^2+z^2} - \frac{a^2+\zeta^2}{3}\sqrt{a^2-\zeta^2} = \frac{a^2+z^2}{3}\sqrt{a^2+z^2} - \frac{a^3}{3}$, for the Value of the Area ABC, reckoning from the Beginning of the Abscis. It comes to the same Purpose to suppose $z = 0$ in the general Expression $\frac{a^2+z^2}{3}\sqrt{a^2+z^2}$, and then subtract what arises, *viz.* $\frac{a^2}{3}\sqrt{a^2} = \frac{a^3}{3}$, from that general Expression, giving $\frac{a^2+z^2}{3}\sqrt{a^2+z^2} - \frac{a^3}{3}$ for the Value of the Area ABC adjacent to the whole Abscis AB.

Let it be $a = 1 =$ AB; then the Area ABC $= (\frac{a^2+z^2}{3}\sqrt{a^2+z^2} - \frac{a^3}{3} =)\ \frac{2}{3}\sqrt{2} - \frac{1}{3}$.

187. The same Method may be used in the Case of any other Curve, whether it admit of an initial Limit for the Area, according to the Rule delivered above, or not. Yea in regard in some Curves, one may be perplex'd, and ready to mistake, you do best to follow this same Method, in the Case of any Curve proposed.

Thus in the Curve AC*c*D, represented by the Fig. p. 140. where we have $y = z\sqrt{4-z}$: let it be required to find the Area B*bc*C adjacent to B*b* a given Part of the Abscis. The general Expression of the Area is $\frac{-16-6z}{15} \times \overline{4-z}|^{\frac{3}{2}}$. Calling therefore the lesser Abscis AB $= z$; and the greater Abscis A*b* $= \zeta$, subtract the Area belonging to z from the Area belonging to ζ, and you'll have $\frac{-16-6\zeta}{15} \times \overline{4-\zeta}|^{\frac{3}{2}} + \frac{16+6z}{15} \times \overline{4-z}|^{\frac{3}{2}}$, for the Value of the Area B*bc*C. And if z (= AB) = o you'll have $\frac{-16-6\zeta}{15} \times \overline{4-\zeta}|^{\frac{3}{2}} + \frac{16}{15} \times 8 = \frac{128 - \overline{16+6\zeta} \times \overline{4-\zeta}|^{\frac{3}{2}}}{15}$, for the true Value of the Area A*bc*. Let it be AB $= z = 2$, and A*b* $= \zeta = 3$, then $\frac{-16-6\zeta}{15} \times \overline{4-\zeta}|^{\frac{3}{2}} + \frac{16+6z}{15} \times \overline{4-z}|^{\frac{3}{2}} = -\frac{34}{15} + \frac{28}{15}\sqrt{8} =$ Area B*bc*C. Again, putting $z =$ AB $= 0$, you'll have (by retaining the same Value of A*b* $= \zeta$) $\frac{128 - \overline{16+6\zeta} \times \overline{4-\zeta}|^{\frac{3}{2}}}{15} = \frac{128-34\times 1}{15} = \frac{94}{15}$ for

 the

the Value of the Area Abc, reckoning from the Beginning of the Abſciſs at A.

188. Thus likewiſe, when the Curve is doubly quadrable: *e. g.* let it be $y = \frac{a}{\overline{1+z}|^2}$ (ſee the Fig. at Art. 128 *.) It was there demonſtrated that the Area of the Curve was either $\frac{az}{1+z}$; or yet $\frac{-a}{1+z}$: either of which Expreſſions will ſerve to find the Area $BbcC$ adjacent to any given Part of the Abſciſs as Bb. Calling the leſſer Abſciſs $Ab = \zeta$, the firſt Form will give you $\frac{az}{1+z} - \frac{a\zeta}{1+\zeta}$: the ſecond Form will give you $\frac{-a}{1+z} + \frac{a}{1+\zeta}$: either of which exhibits the Value of the Area $BbcC$. Let it be $AB = z = 2$; $Ab = \zeta = 1$, then $\frac{az}{1+z} - \frac{a\zeta}{1+\zeta} = \frac{2a}{3} - \frac{a}{2}$ $= \frac{1}{6}a$ for the Area $BbcC$. Again $\frac{-a}{1+z} + \frac{a}{1+\zeta} = \frac{-a}{3} + \frac{a}{2} = \frac{1}{6}a$, the ſame as before. In the firſt Expreſſion, if you put $\zeta = Ab = 0$, in order to find the Area ABC adjacent to the whole Abſciſs $AB = 2$, then $\frac{az}{1+z} - \frac{a\zeta}{1+\zeta}$ becomes $\frac{az}{1+z} = \frac{2a}{3} =$ ABC. Again, the ſecond Expreſſion, *viz.* $\frac{-a}{1+z} + \frac{a}{1+\zeta}$ becomes $\frac{-a}{1+z} + a = \frac{-a}{3} + a = \frac{2a}{3}$, the ſame.

* Of this Explication.

189. Let the Nature of the Curve be defin'd by this Equation $z^2 + \frac{1}{z^2} = y$, ſee the Fig. p. 144: The general Expreſſion of the Area was found to be $\frac{1}{3}z^3 - \frac{1}{z}$ *. Call $A\beta = z$ and $Ab = \zeta$, then the Area $b\beta xc =$ $\frac{1}{3}z^3 - \frac{1}{z} - \frac{1}{3}\zeta^3 + \frac{1}{\zeta}$: ſuppoſe $A\beta = z = 2$. $Ab = \zeta = \frac{2}{3}$: then $\frac{1}{3}z^3 - \frac{1}{z} - \frac{1}{3}\zeta^3 + \frac{1}{\zeta} = \frac{8}{3} - \frac{1}{2} - \frac{8}{81} + \frac{3}{2} = 3\frac{46}{81}$ the Value of the Area $b\beta xc$. Again, ſuppoſe $\zeta = \sqrt[4]{3}$, and $z = 2$, as before: then $\frac{1}{3}z^3 - \frac{1}{z} - \frac{1}{3}\zeta^3 + \frac{1}{\zeta} = \frac{8}{3} - \frac{1}{2} - \frac{\sqrt[4]{27}}{3} + \frac{1}{\sqrt[4]{3}} = \frac{8}{3} - \frac{1}{2} = 2\frac{1}{6}$: as was found by means of the initial Limit before *.

*Art. 181. of this Explication.

*Art. 183. of this Explication.

But, if you either ſuppoſe $Ab = \zeta = 0$, or yet $A\beta = z =$ infinite: then the Expreſſion $\frac{1}{3}z^3 - \frac{1}{z} - \frac{1}{3}\zeta^3 + \frac{1}{\zeta}$, in both Caſes, becomes infinitely great: for when $\zeta = 0$, the Term $\frac{1}{\zeta}$ is infinite: again when $z =$ infinite, the Term $\frac{1}{3}z^3$ is infinite. The firſt ſhews that the Area upon the nearer Side of any Ordinate, comprehended betwixt it and AG perpendicular

pendicular to AB, is infinitely great: the other signifies that the Area situate upon the further Side of the Ordinate, and extending along the Abscifs infinitely produc'd, is also infinitely great. Therefore in this and the like Cases, we must be contented to find the curvilinear Area, adjacent to any middle Part of the Abscifs, as has been just now shewn.

190. And this naturally leads us to speak of the hyperbolical Space contain'd betwixt the Curve and Assymptote in the Case of the *Apollonian* Hyperbola, whose Assymptotes are at right Angles to each other: the Consideration of which was put off to this Place.

Suppose $cC\kappa\alpha$ to be such an Hyperbola; (see the Fig. p. 135.) AB, AG it's Assymptotes: the Equation to it is $zy=a$, or $y=\frac{a}{z}$, a denoting a given superficial Space. The Area, by the Method of Quadratures, is $\frac{-az^0}{0}$ * *Art. 144. of this Explication.
$=$ infinite: whence it is evident that we can neither find the Area on the nearer Side of any Ordinate BC, lying between it and the Assymptote AG; nor the Area $CB\alpha$, lying on the further Side of the Ordinate BC, along $B\alpha$ infinitely produced; these Spaces, in both Cases being infinite: nor yet can we, by the foregoing Method, find the Value of any intermediate Area, as $BbcC$, adjacent to any middle Part of the Abscifs Bb, as long as the Point A continues to be the Beginning of the Abscifs. Therefore, to remedy this, you must alter the Beginning of the Abscifs: and suppose it to be at some Point in the Assymptote, as at the Point B, at a given Distance from A: which Distance, call d: draw the Ordinate BC; likewise the Ordinates $\beta\kappa$, bc, on each Side of BC. Then you'll have $B\beta=z$, $Bb=-z$, $A\beta=d+z$, and $Ab=d-z$: and so the Ordinate $\beta\kappa=\frac{a}{d+z}$, and $bc=\frac{a}{d-z}$. Reduce the Ordinate $\frac{a}{d+z}$ to Form, and it stands thus $z^{1-1}\times\overline{d+z}|^{0-1}\times a$: which belongs to a Curve that is not quadrable, as appears by Art. 149 *. Wherefore we must content ourselves with an Approximation. Accordingly, by substituting 1 for θ, η, r, s and f; and d, a for e, a respectively, in the general Series for the Area of a simple Binomial, *viz.* $z^\theta\times\overline{e+fz^\eta}|^\lambda\times$ * Of this Explication.

$$\overline{\frac{\frac{1}{\eta}a}{re}+\frac{-sfA}{\overline{r+1}\times e}z^\eta+\frac{-\overline{s+1}\times fB}{\overline{r+2}\times e}z^{2\eta}+\frac{-\overline{s+2}\times fC}{\overline{r+3}\times e}z^{3\eta}+\frac{-\overline{s+3}\times fD}{\overline{r+4}\times e}z^{4\eta}\&c.}$$

you'll have $z\times\overline{d+z}|^0\times\overline{\frac{a}{d}-\frac{a}{2d^2}z+\frac{a}{3d^3}z^2-\frac{a}{4d^4}z^3+\frac{a}{5d^5}z^4-\&c.}$

$=\frac{az}{d}\times\overline{1-\frac{z}{2d}+\frac{z^2}{3d^2}-\frac{z^3}{4d^3}+\frac{z^4}{5d^4}-\&c.}$ *in inf.* which denotes the

Area

Area $B\beta xC$, becauſe of the poſitive Powers of z contain'd in the Numerators of the ſeveral Terms *.

* Art. 169. of this Explication.

After the ſame manner, if you take $y = \frac{a}{d-z}$, you'll find, by making the like Subſtitution in the general Theorem, expreſſing the Areas of binomial Curves, $\frac{az}{d} \times \overline{1 + \frac{z}{2d} + \frac{z^2}{3d^2} + \frac{z^3}{4d^3} + \frac{z^4}{5d^4} + \&c.\ in\ inf.}$ for the Value of the Area $BbcC$, for the like Reaſon.

191. Whence it follows, that, if $B\beta = Bb$, then by adding both Series's together, you'll have $\frac{2az}{d} \times \overline{1 + \frac{z^2}{3d^2} + \frac{z^4}{5d^4} + \frac{z^6}{7d^6} + \&c.\ in\ inf.}$ for the Value of the Area $b\beta xc$, made up of both the preceding Areas: which Series converges double as faſt as the preceding ones. And thus you may, moſt conveniently, find an approximate Value of any hyperbolical Space, ſuch as $b\beta xc$, adjoining to any intermediate Part $b\beta$ of the Aſſymptote, *viz.* by ſuppoſing the Beginning of the Abſciſs z to be at the Point B, exactly in the Middle betwixt b and β: for by this means the Series made up of the two others, as above, converges faſter, than by ſuppoſing the Beginning of the Abſciſs to be at one of the Extremities of the given Part of the Aſſymptote; or by throwing the Beginning of the Abſciſs without the Part $b\beta$, and working as in Art. 186 *: which may likewiſe be done here.

* Of this Explication.

Ex. Let $Ab = 0{,}9$, $A\beta = 1{,}1$: and you would have the Area $b\beta xc$. Take $bB = 0{,}1 = B\beta = z$, and conſequently $AB = d = 1$: then ſubſtitute 0,1 for z and 1 for d, in the preceding Series, and you'll have $0 . 2a \times \overline{1 + \frac{0.01}{3} + \frac{0.0001}{5} + \frac{0.000001}{7} + \&c.} = a \times \overline{0.2 + 0.000666666, \&c. + 0.000004 + 0.000000028, \&c. + \&c.} = 0.20067069, \&c. \times BC$ for an approximate Value of the Area $b\beta xc$: and ſince the Progreſſion of the Series is manifeſt, the Value of the Area, may be carried to any Degree of Exactneſs you pleaſe. Now although the Series's above, would converge very ſlowly, if it were $z = d$; and not at all, if it were z greater than d *, yet it is evident, by the above expedient, z muſt always be leſs than d: for $Bb = B\beta = z$ muſt always be leſs than $AB = d$.

* Art. 28. of the Quadrature of Curves.

And thus we have ſhewn at full Length, what is to be done in order to find the curvilinear Area adjacent to any middle Part of the Aſſymptote, in the Caſe of the *Apollonian* Hyperbola, as we promis'd above *.

* Art. 144. of this Explication.

192. And as it has been ſhewn, in the Caſe of the Hyperbola, how you may change the Beginning of the Abſciſs; it may not be improper

per here to obſerve, that the ſame Expedient may be uſed, in the Caſe of any other Curve whatſoever: which may be of conſiderable Uſe, upon Occaſions, for extricating one out of ſeveral Difficulties that may occur in this Doctrine of Curve-lines: particularly, by this means, you may expreſs one and the ſame curvilinear Area, by infinite different Ways: ſince every different Beginning of an Abſciſs, which may be infinitely varied, will afford you a different Expreſſion for the Area. And thus likewiſe you may get free of an hyperbolical Term, which may enter at any Time into the Expreſſion of the Ordinate of a Curve. I call an hyperbolical Term, that which contains the firſt Power or Root of the Abſciſs z in it's Denominator: which always denotes an hyperbolical Ordinate, applied to the Aſſymptote. Let $z + \frac{2}{z} = y$ be an Equation to a Curve: then becauſe of the hyperbolical Term $\frac{2}{z}$ the Area becomes infinite, *viz.* $\frac{1}{2}z - \frac{2z^0}{0}$: but if you ſuppoſe the Abſciſs z to be increas'd or diminiſh'd by any given Length, *e.g.* to be diminiſh'd by 1, by taking a new Abſciſs $\zeta = z - 1$ or $1 + \zeta = z$: and ſubſtituting $1 + \zeta$ for z in the Equation to the Curve: hence you obtain $y = 1 + \zeta + \frac{2}{1+\zeta}$: and therefore ſince the Area correſponding to $1 + \zeta$ is $\zeta + \frac{1}{2}\zeta^2$, and that correſponding to the Part $\frac{2}{1+\zeta}$ is $2\zeta - \frac{2\zeta^2}{2} + \frac{2\zeta^3}{3} - \frac{2\zeta^4}{4} +$ &c. *in inf.* after the Manner already demonſtrated: hence by joining theſe Areas together, you'll have $3\zeta - \frac{\zeta^2}{2} + \frac{2}{3}\zeta^3 - \frac{1}{2}\zeta^4 + \frac{2}{5}\zeta^5 -$ &c. *in inf.* for an approximate Value of the Area of the Curve propoſed.

193. And ſo I ſhall finiſh this Section, reſpecting the Poſition and Limits of the Areas of Curves, being afraid I have inſiſted too long upon it already, when I ſhall have remarked that, if the Curve decuſſate it's Baſe, betwixt the two Extremities of that Part of it, to which the Area to be found, adjoins, then, by obſerving the Rule delivered above, *viz.* by ſubtracting the Area belonging to the ſhorter Abſciſs from the Area belonging to the longer Abſciſs, found by the Method of Quadratures, the thing which ariſes is the Difference between that Part of the intervening Area, which lyes above the Baſe; and that Part, which lyes below, the latter being ſubtracted from the former. Thus if it be AB $= z$ and A$\beta = \zeta$ (ſee the Fig. p. 141.): the Area belonging to the leſſer Abſciſs AB $= z$, is $-\frac{1}{z} + \frac{2}{z^{\frac{1}{2}}}$: and that belonging to the

larger Abſciſs $A\beta = \zeta$, is $-\frac{1}{\zeta} + \frac{2}{\zeta^{\frac{1}{2}}}$ therefore $-\frac{1}{\zeta} + \frac{2}{\zeta^{\frac{1}{2}}} + \frac{1}{z} - \frac{2}{z^{\frac{1}{2}}}$ denotes the Area DBC — $D\beta x$.

Let $AB = z = \frac{4}{9}$ and $A\beta = \zeta = \frac{9}{4}$, then $-\frac{1}{\zeta} + \frac{2}{\zeta^{\frac{1}{2}}} + \frac{1}{z} - \frac{2}{z^{\frac{1}{2}}} = -\frac{4}{9} + \frac{4}{3} + \frac{9}{4} - 3 = \frac{5}{36}$ the Difference ariſing by ſubtracting the Area $D\beta x$ from D*bc*.

Again, let it be $AB = z = \frac{81}{100}$, and $A\beta = \zeta = 4$: then $-\frac{1}{\zeta} + \frac{2}{\zeta^{\frac{1}{2}}} + \frac{1}{z} - \frac{2}{z^{\frac{1}{2}}} = -\frac{1}{4} + 1 + \frac{100}{81} - \frac{20}{9} = -\frac{77}{324}$ the Difference ariſing by ſubtracting the Area $D\beta x$ below the Baſe from the Area DBC above the Baſe: which comes out negative, in regard, $D\beta x$ is, in this Caſe, greater than DBC.

Finally, let it be $AB = z = \frac{4}{9}$, and $A\beta = 4$: then $-\frac{1}{\zeta} + \frac{2}{\zeta^{\frac{1}{2}}} + \frac{1}{z} - \frac{2}{z^{\frac{1}{2}}} = -\frac{1}{4} + 1 + \frac{9}{4} - 3 = 0$: which ſhews that the Parts of the Area above and below the Baſe are equal.

SECT. VI.

Containing Notes on Art. 32.

DEMONSTRATION of PROP. VI.

THE Demonſtration of this Prop. which our Author has omitted, is as follows.

194. By Prop. 4. and Art. III. the following Ordinates and Areas of Curves correſpond.

$$
\begin{array}{l|l}
\quad\quad\quad\text{Ordinates.} & \text{Areas.} \\
\left.\begin{array}{llll}
1^{\circ}\ \theta ckA & \begin{smallmatrix}+\theta\\+\lambda\eta\end{smallmatrix} fkAz^{\eta} & \begin{smallmatrix}+\theta\\+2\lambda\eta\end{smallmatrix} gkAz^{2\eta} & \begin{smallmatrix}+\theta\\+3\lambda\eta\end{smallmatrix} hkAz^{3\eta} \\
 & \begin{smallmatrix}+\theta\\+\mu\eta\end{smallmatrix} dlAz^{\eta} & \begin{smallmatrix}+\theta\\+\lambda\eta\\+\mu\eta\end{smallmatrix} flAz^{2\eta} & \begin{smallmatrix}+\theta\\+2\lambda\eta\\+\mu\eta\end{smallmatrix} glAz^{3\eta}\ \&c. \\
 & & \begin{smallmatrix}+\theta\\+2\mu\eta\end{smallmatrix} emAz^{2\eta} & \begin{smallmatrix}+\theta\\+\lambda\eta\\+2\mu\eta\end{smallmatrix} fmAz^{3\eta} \\
 & & & \begin{smallmatrix}+\theta\\+3\mu\eta\end{smallmatrix} enAz^{3\eta}
\end{array}\right\} \times z^{\theta-1}R^{\lambda-1}S^{\mu-1} & Az^{\theta}R^{\lambda}S^{\mu}
\end{array}
$$

- -

2°.

	Ordinates.		Areas.
2°.	$\overline{\theta+\eta}\,ekBz^{\eta} + \overline{\theta+\eta+\lambda\eta}\,fkBz^{2\eta} + \overline{\theta+\eta+2\lambda\eta}\,gkBz^{3\eta} + \overline{\theta+\eta+\mu\eta}\,elBz^{2\eta} + \overline{\theta+\eta+\lambda\eta+\mu\eta}\,flBz^{3\eta} + \overline{\theta+\eta+2\mu\eta}\,emBz^{3\eta}$ &c.	$\times z^{\theta-1}R^{\lambda-1}S^{\mu-1}$	$Bz^{\theta+\eta}R^{\lambda}S^{\mu}$
3°.	$\overline{\theta+2\eta}\,ekCz^{2\eta} + \overline{\theta+2\eta+\lambda\eta}\,fkCz^{3\eta} + \overline{\theta+2\eta+2\mu\eta}\,elCz^{3\eta}$ &c.	$\times z^{\theta-1}R^{\lambda-1}S^{\mu-1}$	$Cz^{\theta+2\eta}R^{\lambda}S^{\mu}$
4°.	$\overline{\theta+3\eta}\,ekDz^{3\eta}$ &c.	$\times z^{\theta-1}R^{\lambda-1}S^{\mu-1}$	$Dz^{\theta+3\eta}R^{\lambda}S^{\mu}$

Now if the Sum of theſe Ordinates be ſuppos'd equal to the Ordinate $\overline{a+bz^{\eta}+cz^{2\eta}+dz^{3\eta}+\&c.}\times z^{\theta-1}R^{\lambda-1}S^{\mu-1}$ propoſed, then the Sum of all the Areas, correſponding to the ſeveral Ordinates, *i. e.* $Az^{\theta}R^{\lambda}S^{\mu}+Bz^{\theta+\eta}R^{\lambda}S^{\mu}+Cz^{\theta+2\eta}R^{\lambda}S^{\mu}+Dz^{\theta+3\eta}R^{\lambda}S^{\mu}+\&c.$ or $\overline{A+Bz^{\eta}+Cz^{2\eta}+Dz^{3\eta}+\&c.}\times z^{\theta}R^{\lambda}S^{\mu}$, ſhall be equal to the Area of the Curve, whoſe Ordinate is $\overline{a+bz^{\eta}+cz^{2\eta}+dz^{3\eta}+\&c.}\times z^{\theta-1}R^{\lambda-1}S^{\mu-1}$.

Wherefore, let the correſpondent Terms of the Ordinates be put equal, *i. e.* ſuch Terms as have the ſame Power of z in them: that by that means, we may diſcover the Values of A, B, C, D, &c. which are of certain Values, although not yet determined. Hence we have theſe Equations.

1°. $a=\theta ekA$.

2°. $b=\overline{\theta+\lambda\eta}\times fkA+\overline{\theta+\mu\eta}\times elA+\overline{\theta+\eta}\times ekB$.

3°. $c=\overline{\theta+2\lambda\eta}\times gkA+\overline{\theta+\lambda\eta+\mu\eta}\times flA+\overline{\theta+2\mu\eta}\times emA+\overline{\theta+\eta+\lambda\eta}\times fkB+\overline{\theta+\eta+\mu\eta}\times elB+\overline{\theta+2\eta}\times ekC$.

4°. $d=\overline{\theta+3\lambda\eta}\times hkA+\overline{\theta+2\lambda\eta+\mu\eta}\times glA+\overline{\theta+\lambda\eta+2\mu\eta}\times fmA+\overline{\theta+3\mu\eta}\times enA+\overline{\theta+\eta+2\lambda\eta}\times gkB+\overline{\theta+\eta+\lambda\eta+\mu\eta}\times flB+\overline{\theta+\eta+2\mu\eta}\times emB+\overline{\theta+2\eta+\lambda\eta}\times fkC+\overline{\theta+2\eta+\mu\eta}\times elC+\overline{\theta+3\eta}\times ekD$.

&c.

In which Equations, if you put or suppose $\frac{\theta}{\eta} = r \,.\, r + \lambda = s \,.\, s + \lambda = t \,.\, t + \lambda = v$. &c. And again $r + \mu = s' \,.\, s + \mu = t'$ &c. as they are expressed by the Author himself: they will appear thus.

1°. $a = r\eta ek\mathrm{A}$.

2°. $b = s\eta fk\mathrm{A} + s'\eta el\mathrm{A} + \overline{r+1} \times ek\mathrm{B}$.

3°. $c = t\eta gk\mathrm{A} + t'\eta fl\mathrm{A} + t''\eta em\mathrm{A} + \overline{s+1} \times \eta fk\mathrm{B} + \overline{s'+1} \times \eta el\mathrm{B} + \overline{r+2} \times \eta ek\mathrm{C}$.

4°. $d = v\eta hk\mathrm{A} + v'\eta gl\mathrm{A} + v''\eta fm\mathrm{A} + v'''\eta en\mathrm{A} + \overline{t+1} \times \eta gk\mathrm{B} + \overline{t'+1} \times \eta fl\mathrm{B} + \overline{t''+1} \times \eta em\mathrm{B} + \overline{s+2} \times \eta fk\mathrm{C} + \overline{s'+2} \times \eta el\mathrm{C} + \overline{r+3} \times \eta ek\mathrm{D}$.

&c.

From which Equations, by a due Reduction, you may find the Values of A, B, C, D, *&c.* which are the following, *viz.*

$$\mathrm{A} = \frac{a}{r\eta ek} = \frac{\frac{1}{\eta}a}{rek}$$

$$\mathrm{B} = \frac{\frac{1}{\eta}b - sfk\mathrm{A} - s'el\mathrm{A}}{\overline{r+1} \times ek}$$

$$\mathrm{C} = \frac{\frac{1}{\eta}c - \overline{s+1} \times fk\mathrm{B} - \overline{s'+1} \times el\mathrm{B} - tgk\mathrm{A} - t'fl\mathrm{A} - t''em\mathrm{A}}{\overline{r+2} \times ek}$$

$$\mathrm{D} = \frac{\frac{1}{\eta}d - \overline{s+2} \times fk\mathrm{C} - \overline{s'+2} \times el\mathrm{C} - \overline{t+1} \times gk\mathrm{B} - \overline{t'+1} \times fl\mathrm{B} - \overline{t''+1} \times em\mathrm{B} - vhk\mathrm{A} - v'gl\mathrm{A} - v''fm\mathrm{A} - v'''en\mathrm{A}}{\overline{r+3} \times ek}$$

Wherefore, since, by putting the Sum of the Ordinates above mentioned, equal to the Ordinate $\overline{a + bz^{\eta} + cz^{2\eta} + dz^{3\eta} + \&c.} \times z^{\theta-1} \mathrm{R}^{\lambda-1}\mathrm{S}^{\mu-1}$; and the corresponding Terms of the one and of the other equal likewise, these Values of the Coefficients A, B, C, D, *&c.* do arise, it follows that, if you insert these Values of A, B, C, D, *&c.* so determin'd, in place of them, in the Sum of the Areas, *viz.* $\overline{\mathrm{A} + \mathrm{B}z^{\eta} + \mathrm{C}z^{2\eta} + \mathrm{D}z^{3\eta} + \&c.} \times z^{\theta}\mathrm{R}^{\lambda}\mathrm{S}^{\mu}$; what results will be the Area of the Curve, whose Ordinate was propos'd, which will be thereby determined: and accordingly it will be expressed by the following Series.

$$z^{\theta}R^{\lambda}S^{\mu}\times+\frac{\frac{1}{n}a}{rek}$$

$$+\frac{\left(\frac{1}{n}b-sfk-s'el\right)A}{\overline{r+1}\times ek}z^{n}$$

$$+\frac{\left(\frac{1}{n}c-\overline{s+1}\times fk-\overline{s+1}'\times el\right)B\left(-tgk-t'fl-t''em\right)A}{\overline{r+2}\times ek}z^{2n}$$

$$+\frac{\left(\frac{1}{n}d-\overline{s+2}\times fk-\overline{s+2}'\times el\right)C\left(-\overline{t+1}\times gk-\overline{t+1}'\times fl-\overline{t+1}''\times em\right)B\left(-vhk-v'gl-v''fm-v'''en\right)A}{\overline{r+3}\times ek}z^{3n}$$

$$+\ \&c.$$

In which Series, although the Capitals A, B, C, *&c.* are retain'd, yet it is evident, they are now fully known: for $A=\frac{\frac{1}{n}a}{rek}$ the first Term, which is wholly known: consequently the Coefficient of the second Term is wholly known, *i. e.* B is wholly known: and so on for all the rest. And as A, B, C, *&c.* are the Values of the Coefficients of the first, second, third, *&c.* Terms; so, when any of these Coefficients is negative, the capital Letter, which stands for it, must be negative likewise. And therefore if $\overline{a+bz^{n}+cz^{2n}+dz^{3n}+\&c.}\times z^{\theta-1}R^{\lambda-1}S^{\mu-1}$ be the Ordinate of a Curve, the Series expressed above, will exhibit it's Area: and that either perfectly, *viz.* when the Series terminates; or else by Approximation, *viz.* when the Series runs out infinitely. But it must still be remember'd, that every Ordinate of a Curve may be expressed in two different Forms: for either the Exponents of the Powers of z, contain'd in R, S and $\overline{a+bz^{n}+cz^{2n}+\&c.}$ may be positive, or negative: which was sufficiently explain'd upon the foregoing Proposition: and accordingly both Forms of the Ordinate must be tried, in the Application of this Theorem: and if the Curve be quadrable, the Series will terminate in one or other of the Cases: but if otherwise, the Series will converge, either when n is positive, or else when it is negative; and so it will exhibit an approximate Value of the Area.

X 2 195. What-

195. Whatever Observations or Remarks were made upon the foregoing Prop. the like may be made upon this, with respect to the Laws or Conditions of the Termination, Continuation and Convergency of the Series: and the bringing Ordinates of Curves propos'd, which fall under this Prop. into due Form, as likewise with respect to the Position and Limits of Areas, and other things relating thereto. But that would be too large a Subject to enter upon: and therefore I shall let it alone.

The Series exhibiting the Areas of Curves falling under this Prop. may be easily continued, by considering the Law of the Progression. It is evident likewise that the Series of such Propositions, as the fifth and sixth, is infinite; as well as that of the third and fourth, which was mentioned already *, *i. e.* we may suppose T to stand for $\alpha + \beta z^{\eta} + \gamma z^{2\eta} + \&c.$ and ν for any numeral Index: and then if the Ordinate of a Curve be of this Form $z^{\theta-1}R^{\lambda-1}S^{\mu-1}T^{\nu-1} \times \overline{a + bz^{\eta} + cz^{2\eta} + dz^{3\eta} + \&c.}$ you may exhibit a Series for it's Area, analogous to these in Prop. fifth and sixth, by considering what was said in Art. 117 *, and applying that to this Case, as Prop. third and fourth have been applied for demonstrating Prop. fifth and sixth: and so likewise for other Ordinates still more complex.

*Art. 117. of this Explication.

* Of this Explication.

SECT. VII.

Containing Notes on Art. 33——39.

EXPLICATION of PROP. VII.

CASE I.

196. SUpposing the same things mentioned by the Author in this Prop. Let $R = e + fz^{\eta}$ a Binomial: I say if the Area of any one of the infinitely many Curves, whose Ordinates are thus generally express'd $z^{\theta \pm \sigma\eta}R^{\lambda \pm \tau}$, be given, the Areas of all these Curves will be given.

The Demonstration of this Prop. as far as it respects binomial Curves, is left by our Author to be gathered from his Demonstration of it in the Case of Trinomials: which therefore I shall here supply.

The first of the four Cases into which this Prop. is divided by him, respects the Increase or Diminution of the Index of z, *viz.* $\theta \pm \sigma\eta$, by any Repetition of η.

197. Let

197. Let the Area of the Curve, whose Ordinate is $z^{\theta-1}R^{\lambda-1}$, be given: and call it A, *i. e.* multiplying both by the Quantity p, whose Value is afterwards to be determined, we shall have $pz^{\theta-1}R^{\lambda-1}$ the Ordinate of a Curve, whose Area is pA. But, by Prop. 3, $\overline{\theta e+\overline{\theta+\lambda\eta}\times fz^{\eta}}\times z^{\theta-1}R^{\lambda-1}$, is the Ordinate of a Curve, whose Area is $z^{\theta}R^{\lambda}$: subtract the former Ordinate and Area from the latter Ordinate and Area respectively; and you'll have $\overline{\theta e-p+\overline{\theta+\lambda\eta}\times fz^{\eta}}\times z^{\theta-1}R^{\lambda-1}$, and $z^{\theta}R^{\lambda}-pA$, the Remainders; the first the Ordinate of a Curve, whereof the last is the Area: and that whatever be the Value of p, which is the Quantity by which the given Ordinate and Area, *viz.* $z^{\theta-1}R^{\lambda-1}$ and A, were multiplied. Now put $p=\theta e$ a known Quantity, and substitute the one for the other in the last mentioned Area, and thence you'll have $\overline{\theta+\lambda\eta}\times fz^{\eta}\times z^{\theta-1}R^{\lambda-1}$ and $z^{\theta}R^{\lambda}-\theta eA$, for correspondent Ordinate and Area. Again, dividing both by the given Quantity $\overline{\theta+\lambda\eta}\times f$, we have $z^{\theta+\eta-1}R^{\lambda-1}$ the Ordinate of a Curve, whose Area is $\dfrac{z^{\theta}R^{\lambda}-\theta eA}{\overline{\theta+\lambda\eta}\times f}$, which is wholly known.

That the Process may appear at one View, I shall set down the several Steps in Order.

	Ordinates.	Areas.
1°.	$p \quad * \quad \times z^{\theta-1}R^{\lambda-1}$	pA
2°.	$\overline{\theta e+\overline{\theta+\lambda\eta}\times fz^{\eta}}\times z^{\theta-1}R^{\lambda-1}$	$z^{\theta}R^{\lambda}$
3°.	$\overline{\theta e-p+\overline{\theta+\lambda\eta}\times fz^{\eta}}\times z^{\theta-1}R^{\lambda-1}$	$z^{\theta}R^{\lambda}-pA$.
4°.	$\overline{\theta+\lambda\eta}\times fz^{\eta}\times z^{\theta-1}R^{\lambda-1}$	$z^{\theta}R^{\lambda}-\theta eA$.
5°.	$z^{\theta+\eta-1}R^{\lambda-1}$	$\dfrac{z^{\theta}R^{\lambda}-\theta eA}{\overline{\theta+\lambda\eta}\times f}$.

198. By the same way of reasoning, if you call this Area $\dfrac{z^{\theta}R^{\lambda}-\theta eA}{\overline{\theta+\lambda\eta}\times f}=B$, whose Ordinate is $z^{\theta+\eta-1}R^{\lambda-1}$, you may find the Area of another Curve, whose Ordinate is $z^{\theta+2\eta-1}R^{\lambda-1}$: by considering what is expressed in Art. 114 *. Which is done by subtracting the Ordinate $qz^{\theta+\eta-1}R^{\lambda-1}$ or $qz^{\eta}\times z^{\theta-1}R^{\lambda-1}$, and correspondent Area qB from the Ordinate $\overline{\overline{\theta+\eta}\times ez^{\eta}+\overline{\theta+\eta+\lambda\eta}\times fz^{2\eta}}\times z^{\theta-1}R^{\lambda-1}$, and correspondent Area $z^{\theta+\eta}R^{\lambda}$ respectively: which last Ordinate and Area

* Of this Explication.

Area belonging to the same Curve by that Art. and then putting $qz^{\eta} = \overline{\theta+\eta}\times ez^{\eta}$; or $q=\overline{\theta+\eta}\times e$: and substituting this Value in place of it: for thence you'll obtain $\overline{\theta+\eta+\lambda\eta}\times fz^{2\eta}\times z^{\theta-1}R^{\lambda-1}$ and $z^{\theta+\eta}R^{\lambda} - \overline{\theta+\eta}\times eB$; or, by dividing both by $\overline{\theta+\eta+\lambda\eta}\times f$, $z^{\theta+2\eta-1}R^{\lambda-1}$ and $\frac{z^{\theta+\eta}R^{\lambda}-\overline{\theta+\eta}\times eB}{\overline{\theta+\eta+\lambda\eta}\times f}$, for Ordinate and Area belonging to one and the same Curve: which Ordinate has the Exponent of z, *viz.* $\theta+2\eta-1$ greater than the Exponent of z in the Ordinate $z^{\theta-1}R^{\lambda-1}$ belonging to the given Area A, by 2η: and since B is known, being formerly found, the Area $\frac{z^{\theta+\eta}R^{\lambda}-\overline{\theta+\eta}\times eB}{\overline{\theta+\eta+\lambda\eta}\times f}$ is also wholly known.

And thus it is evident, that, by the like way of proceeding, you may find the Areas of all the Curves, whose Ordinates are thus generally expressed $z^{\theta+\sigma\eta-1}R^{\lambda-1}$, σ being a positive Integer. For if $z^{\theta+\sigma\eta-1}R^{\lambda-1}$ be the Ordinate of a Curve, and A denote the Area of that Curve, then $\frac{z^{\theta+\sigma\eta}R^{\lambda}-\overline{\theta+\sigma\eta}\times eA}{\overline{\theta+\sigma\eta+\lambda\eta}\times f}$ will be the Area of another Curve, belonging to that same Progression of Curves, which has the Index of z in it's Ordinate greater by once η than it is in the Ordinate of the other: as appears from Art. 114 *, compared with what hath been already said.

* Of this Explication.

199. Again, by the like way of reasoning, it will appear, that if the Area of the Curve be given, whose Ordinate is $z^{\theta-1}R^{\lambda-1}$ as formerly, which call A, then the Area of the Curve, whose Ordinate is $z^{\theta-\eta-1}R^{\lambda-1}$, may be thence found, and will be $\frac{z^{\theta-\eta}R^{\lambda}-\overline{\theta-\eta+\lambda\eta}\times fA}{\overline{\theta-\eta}\times e}$. For you'll have Ordinates and Areas corresponding to each other as follows.

	Ordinates.	Areas.
1°.	$p\times z^{\theta-1}R^{\lambda-1}$	pA.
2°.	$\overline{\theta-\eta}\times ez^{-\eta}+\overline{\theta-\eta+\lambda\eta}\times f\times z^{\theta-1}R^{\lambda-1}$	$z^{\theta-\eta}R^{\lambda}$ *.

*Art. 114. of this Explication.

Subtract first Step from second: put $p=\overline{\theta-\eta+\lambda\eta}\times f$: and divide by $\overline{\theta-\eta}\times e$.

3°.	$z^{\theta-\eta-1}R^{\lambda-1}$	$\frac{z^{\theta-\eta}R^{\lambda}-\overline{\theta-\eta+\lambda\eta}\times fA}{\overline{\theta-\eta}\times e}$

This last Area being found, call it P, and then from the known Area P, you proceed to find the Area of the Curve next in order in this

this Progression of Curves, *viz.* the Area of the Curve, whose Ordinate is $z^{\theta-2\eta-1}R^{\lambda-1}$: which by the same way of reasoning, will be found to be $\frac{z^{\theta-2\eta}R^{\lambda}-\overline{\theta-2\eta+\lambda\eta}\times fP}{\overline{\theta-2\eta}\times e}$: and so you may proceed from this last Area, now given, to determine the Area of that Curve which comes next in the Order of the Progression of Curves, *viz.* that whose Ordinate is $z^{\theta-3\eta-1}R^{\lambda-1}$: and so on continually. So that universally if you put P for the Area of any Curve whose Ordinate is $z^{\theta-\sigma\eta-1}R^{\lambda-1}$, then $\frac{z^{\theta-\sigma\eta-\eta}R^{\lambda}-\overline{\theta-\sigma\eta-\eta+\lambda\eta}\times fP}{\overline{\theta-\sigma\eta-\eta}\times e}$ shall be the Area of a Curve whose Ordinate is $z^{\theta-\sigma\eta-\eta-1}R^{\lambda-1}$: as easily appears by substituting $\theta-\sigma\eta$ for θ in the first Part of this Art.

200. And thus it appears that the Area of a Curve, whose Ordinate is $z^{\theta-1}R^{\lambda-1}$, being given, the Area of every other Curve, whose Ordinate is thus generally expressed $z^{\theta\pm\sigma\eta-1}R^{\lambda-1}$, is likewise given. Wherefore by a contrary Analysis, you may come back from the Area of any Curve, whose Ordinate is $z^{\theta\pm\sigma\eta-1}R^{\lambda-1}$, to the Area of the Curve, whose Ordinate is $z^{\theta-1}R^{\lambda-1}$. For, if you put A for the Area of any Curve, whose Ordinate is $z^{\theta+\sigma\eta-1}R^{\lambda-1}$: and B for the Area of the Curve whose Ordinate is $z^{\theta+\sigma\eta+\eta-1}R^{\lambda-1}$: then, by what has been demonstrated, $B=\frac{z^{\theta+\sigma\eta}R^{\lambda}-\overline{\theta+\sigma\eta}\times eA}{\overline{\theta+\sigma\eta+\lambda\eta}\times f}$: therefore it follows contrarily by resolving the Equation, that if B be given and A sought, it will be $A=\frac{z^{\theta+\sigma\eta}R^{\lambda}-\overline{\theta+\sigma\eta+\lambda\eta}\times fB}{\overline{\theta+\sigma\eta}\times e}$. And again, in the other Progression of Areas proceeding the contrary way, if P stand for the Area of a Curve, whose Ordinate is $z^{\theta-\sigma\eta-1}R^{\lambda-1}$; and Q for the Area of a Curve, whose Ordinate is $z^{\theta-\sigma\eta-\eta-1}R^{\lambda-1}$: it has been demonstrated that $Q=\frac{z^{\theta-\sigma\eta-\eta}R^{\lambda}-\overline{\theta-\sigma\eta-\eta+\lambda\eta}\times fP}{\overline{\theta-\sigma\eta-\eta}\times e}$: Wherefore, supposing Q given, and P sought, by resolving the Equation, we have $P=\frac{z^{\theta-\sigma\eta-\eta}R^{\lambda}-\overline{\theta-\sigma\eta-\eta}\times eQ}{\overline{\theta-\sigma\eta-\eta+\lambda\eta}\times f}$. From both which it plainly follows, that, if the Area of any one of the infinitely many Curves, whose Ordinates are generally expressed thus $z^{\theta\pm\sigma\eta-1}R^{\lambda-1}$, be given, the Areas of all the rest are also given, *viz.* when $R=e+fz^{\eta}$ a Binomial.

N. B. It's the same thing whether the general Form of the Ordinates be expressed thus $z^{\theta\pm\sigma\eta-1}R^{\lambda-1}$; or thus $z^{\theta\pm\sigma\eta}R^{\lambda\pm\tau}$, as it is by the Author himself in the Prop. you need only suppose that θ in the last is equivalent to $\theta-1$ in the first: and that $\lambda\pm\tau$ in the last is equivalent

equivalent to $\lambda - 1$ in the first; or $\lambda \pm \tau + 1$ in the last, the same with λ in the first.

201. It appears from what has been shewn in the three preceding Art. that if A and B be made to stand for the Areas of any two Curves next to one another, whether they be in the Progession running towards the one hand, or towards the other: so that the Exponent of z in the Expression of the Ordinate answering to the Area B, exceed the Exponent of z in the Ordinate corresponding to A, by $+\eta$: then, we may make $z^{\theta-1}R^{\lambda-1}$ and $z^{\theta+\eta-1}R^{\lambda-1}$ to represent any two such Ordinates universally, since θ may represent any Index positive or negative, integral or fractional. From whence, by considering what hath been said, we shall have the two following Theorems for determining the Areas of all binomial Curves, from any one Area being given.

202. Th. 1. $B = \dfrac{z^{\theta}R^{\lambda} - \theta e A}{\overline{\theta + \lambda\eta} \times f}$.

203. Th. 2. $A = \dfrac{z^{\theta}R^{\lambda} - \overline{\theta + \lambda\eta} \times fB}{\theta e}$.

204. And thus having demonstrated the first Case of the Prop. when $R = e + fz^{\eta}$, a Binomial; the Demonstration of the first Case of the Prop. accommodated to the Supposition of $R = e + fz^{\eta} + gz^{2\eta}$, a Trinomial, set down by the Author himself, includes no new Difficulty. The way of finding the Area of the Curve whose Ordinate is $z^{\theta+2\eta-1}R^{\lambda-1}$, from the Areas of the two Curves, whose Ordinates are $z^{\theta-1}R^{\lambda-1}$ and $z^{\theta+\eta-1}R^{\lambda-1}$, being given, he shews himself at length: and so you have the Method of determining the Area of any Curve whose Ordinate is $z^{\theta+\sigma\eta-1}R^{\lambda-1}$, when the Areas of the two Curves are given, whose Ordinates are $z^{\theta-1}R^{\lambda-1}$ and $z^{\theta+\eta-1}R^{\lambda-1}$. And he tells you that the Method of proceeding, in order to find the Areas of Curves, in the Progression of Curves, which goes the contrary way, *viz.* those Curves whose Ordinates are thus generally expressed $z^{\theta-\sigma\eta-1}R^{\lambda-1}$, is like to that same, which he shews for the Areas of Curves in the other Progression. That the Truth of this may appear I shall set down the Process of the Steps in order.

	Ordinates.	Areas.
1º.	$p \times z^{\theta-1}R^{\lambda-1}$	pA.
2º.	$qz^{\eta} \times z^{\theta-1}R^{\lambda-1}$	qB.
3º.	$\overline{\theta-\eta} \times e z^{-\eta} + \overline{\theta-\eta+\lambda\eta} \times f z^{0} + \overline{\theta-\eta+2\lambda\eta} \times g z^{\eta} \times z^{\theta-1}R^{\lambda-1}$	$z^{\theta-\eta}R^{\lambda}$ *

* Art. 114. of this Explication.

Subtract the first and second Steps from the third: put $p = \overline{\theta - \eta + \lambda\eta} \times f$ and $q = \overline{\theta - \eta + 2\lambda\eta} \times g$: and divide by $\overline{\theta - \eta} \times e$.

	Ordinates.	Areas.
4º.	$z^{\theta-\eta-1}R^{\lambda-1}$	$\dfrac{z^{\theta-\eta}R^{\lambda} - \overline{\theta-\eta+\lambda\eta} \times fA - \overline{\theta-\eta+2\lambda\eta} \times gB}{\overline{\theta-\eta} \times e}$

Which

Which laſt Area is that which was to be found, and is now wholly known, ſince A and B were given. And after the ſame Manner that our Author ſhews how to find the Area of the Curve, whoſe Ordinate is $z^{\theta+2\eta-1}R^{\lambda-1}$, from the Areas of two Curves being given, whoſe Ordinates are $z^{\theta-1}R^{\lambda-1}$ and $z^{\theta+\eta-1}R^{\lambda-1}$: and that I have juſt now ſhewn, how from theſe ſame Areas given, to find the Area of the Curve, whoſe Ordinate is $z^{\theta-\eta-1}R^{\lambda-1}$, the Areas of all the Curves in both Progreſſions may be found. For, univerſally if A and B ſtand for the Areas of two Curves next one another in the firſt Progreſſion, any where, *viz.* ſuch whoſe Ordinates are $z^{\theta+\sigma\eta-1}R^{\lambda-1}$ and $z^{\theta+\sigma\eta+\eta-1}R^{\lambda-1}$, then $\frac{z^{\theta+\sigma\eta}R^{\lambda} - \overline{\theta+\sigma\eta} \times eA - \overline{\theta+\sigma\eta+\lambda\eta} \times fB}{\overline{\theta+\sigma\eta+2\lambda\eta} \times g}$ will be the Area of the Curve following next in that Progreſſion, *viz.* whoſe Ordinate is $z^{\theta+\sigma\eta+2\eta-1}R^{\lambda-1}$: as eaſily appears by inſerting $\theta+\sigma\eta$ inſtead of θ in our Author's Demonſtration, *i. e.* by ſuppoſing $\theta+\sigma\eta$ here, to be equivalent to θ there.

Again, if P and Q ſtand for the Areas of any two Curves next to each other in the Progreſſion of Curves going the contrary way, *i. e.* if they be the Areas of any two Curves, whoſe Ordinates are $z^{\theta-\sigma\eta-1}R^{\lambda-1}$ and $z^{\theta-\sigma\eta-\eta-1}R^{\lambda-1}$, then $\frac{z^{\theta-\sigma\eta-2\eta}R^{\lambda} - \overline{\theta-\sigma\eta-2\eta+\lambda\eta} \times fQ - \overline{\theta-\sigma\eta-2\eta+2\lambda\eta} \times gP}{\overline{\theta-\sigma\eta-2\eta} \times e}$ muſt be the Area of the Curve following next in order in this Progreſſion, *viz.* the Curve whoſe Ordinate is $z^{\theta-\sigma\eta-2\eta}R^{\lambda-1}$: which eaſily appears by inſerting $\theta-\sigma\eta-\eta$ inſtead of θ in the Demonſtration juſt now given; *i. e.* by ſuppoſing $\theta-\sigma\eta-\eta$ here, to be equivalent to θ there. And therefore it hence appears that, if the Areas of the two Curves, whoſe Ordinates are $z^{\theta-1}R^{\lambda-1}$ and $z^{\theta+\eta-1}R^{\lambda-1}$, be given, the Areas of all the Curves in both Progreſſions, are thence alſo given. And therefore by a contrary Analyſis, if the Areas of any two Curves, whether both in the Progreſſion of Curves proceeding the ſame way, or one in the Progreſſion proceeding the one way, and the other in the Progreſſion that proceeds the contrary way, be given, the Areas of the two Curves whoſe Ordinates are $z^{\theta-1}R^{\lambda-1}$ and $z^{\theta+\eta-1}R^{\lambda-1}$; and alſo of all the others whoſe Ordinates are thus generally expreſſed, $z^{\theta\pm\sigma\eta-1}R^{\lambda-1}$, may thence be found. For let A, B, C, D, &c. denote the Areas of the Curves belonging to the Progreſſion, whoſe Ordinates are thus expreſſed $z^{\theta+\sigma\eta-1}R^{\lambda-1}$, where $\sigma\eta$ is poſitive, and A, the Area of the Curve whoſe Ordinate is $z^{\theta-1}R^{\lambda-1}$: and again let A, P, Q, R, &c. denote the Areas of the Curves belonging to the Progreſſion proceeding the contrary way, *viz.* thoſe whoſe Ordinates are thus expreſſed $z^{\theta-\sigma\eta-1}R^{\lambda-1}$. A being as before the Area belonging to the

Curve, whose Ordinate is $z^{\theta-1}R^{\lambda-1}$: which Curve stands in the Middle betwixt the two Progressions, or rather is the first of both. Then by what has been shewn, we have an Equation defining the Relation of A, B and C; another defining the Relation of B, C and D; a third defining the Relation of C, D and E, and so on: whereby we have a new Equation for every new Area. Again, we have an Equation defining the Relation of B, A and P, when we proceed now the contrary way: we have another, exhibiting the Relation of A, P and Q: another that of P, Q and R, and so on: therefore, it is evident, there will be as many distinct Equations except two, as there are Areas: consequently if any two Areas be given, all the rest may be found.

205. Although in the preceding Art. we have considered the Curves whose Ordinates are generally expressed thus $z^{\theta+\sigma\eta-1}R^{\lambda-1}$, and those whose Ordinates are expressed thus $z^{\theta-\sigma\eta-1}R^{\lambda-1}$, distinctly: and accordingly exhibited two distinct Equations, for determining the Areas of the Curves belonging to the one, and the other: yet they are both in effect one and the same, and both virtually included in that one Equation or Theorem mentioned by the Author himself. For if A, B and C represent the Areas of three different Curves next to one another, so that A be the Area of a Curve any where belonging to the Class of Curves, whose Ordinates are thus generally expressed $z^{\theta\pm\sigma\eta-1}R^{\lambda-1}$; B the Area of a Curve, in whose Ordinate the Index of z is greater by $+\eta$; and C the Area of a third, in whose Ordinate the Index of z is greater by $+2\eta$, than it is in the Ordinate corresponding to the Area A; then we may make $z^{\theta-1}R^{\lambda-1}$ to represent universally the Ordinate of the first; $z^{\theta+\eta-1}R^{\lambda-1}$ the Ordinate of the second; and $z^{\theta+2\eta-1}R^{\lambda-1}$ to represent the Ordinate of the third, corresponding to the three Areas A, B, C: so that the Equation $z^{\theta}R^{\lambda} = \theta eA - \overline{\theta+\lambda\eta} \times fB = \overline{\theta+2\lambda\eta} \times gC$ mentioned by him, may serve for finding all the Areas in this whole Class of Curves, whose Ordinates are thus generally expressed $z^{\theta\pm\sigma\eta-1}R^{\lambda-1}$, any two of them being given: which contains these three Theorems following.

206. Th. 1. $C = \dfrac{z^{\theta}R^{\lambda} - \theta eA - \overline{\theta+\lambda\eta} \times fB}{\overline{\theta+2\lambda\eta} \times g}$.

207. Th. 2. $B = \dfrac{z^{\theta}R^{\lambda} - \theta eA - \overline{\theta+2\lambda\eta} \times gC}{\overline{\theta+\lambda\eta} \times f}$.

208. Th. 3. $A = \dfrac{z^{\theta}R^{\lambda} - \overline{\theta+\lambda\eta} \times fB - \overline{\theta+2\lambda\eta} \times gC}{\theta e}$.

209. But, says our Author, if any of the Terms θ, $\theta+\lambda\eta$ and $\theta+2\lambda\eta$ be wanting or vanish, or break off the Series: *let the Area*

*p*A

pA be assum'd in the Beginning of the one Progression, and the Area qB in the Beginning of the other, and from these two Areas, will be given all the Areas in both Progressions.

From the preceding Theorems it is evident, that if θ or $\theta + \lambda\eta$ or $\theta + 2\lambda\eta$ be equal to nothing, the Series of Curves is broke off or interrupted: thus in the general Equation or Theorem $z^{\theta}R^{\lambda} - \theta eA - \overline{\theta + \lambda\eta} \times fB = \overline{\theta + 2\lambda\eta} \times gC$, if it be $\theta = 0$, A disappears; if it be $\theta + \lambda\eta = 0$, B disappears; and if it be $\theta + 2\lambda\eta = 0$, C disappears. But the Remedy our Author proposes in this Case, I acknowledge I don't sufficiently understand, according as he expresses it above. For suppose we call A, B, C, D, *&c.* the ascending Progression; and A, P, Q, R, *&c.* the descending Progression, signifying as above: the Meaning of what he says, would seem to be this, that when $\theta = 0$, from the Assumption of B and P, all the Areas in both Progressions on both Sides of A may be found: that if it were $\theta + \lambda\eta = 0$, all the Areas on both Sides of B may be found from the Assumption of A and C; and that if it be $\theta + 2\lambda\eta = 0$, all the Areas on both Sides of C may be found from the Assumption of B and D. But this is not true: for when $\theta + \lambda\eta = 0$, from the Assumption of A and C, you cannot find the Area of any other Curve in either Progression. For in this Case A and C are not sufficient for determining B: since the Equation above $z^{\theta}R^{\lambda} - \theta eA - \overline{\theta + \lambda\eta} \times fB = \overline{\theta + 2\lambda\eta} \times gC$ becomes $z^{\theta}R^{\lambda} - \theta eA = \lambda\eta \times gC$, where B disappears: but without B and C, neither D, nor any of the Areas above C can be found; and without B and A both, neither P, nor any of the Areas in the descending Progression can be found: as may easily appear from the Consideration of the three preceding Theorems, and what has been said formerly. However in the Case proposed, *viz.* when any of the Terms θ, $\theta + \lambda\eta$, $\theta + 2\lambda\eta$ is wanting, 1°, If it be $\theta = 0$, from the Assumption of A and B, all the other Areas in both Progressions may be found. 2°. If it be $\theta + \lambda\eta = 0$, from the Assumption of the same two Areas, all the others may be found. 3°. If it be $\theta + 2\lambda\eta = 0$, from the Assumption of A and B, the Areas of the Curves in the ascending Progression cannot be found: but if you assume A and C, all the Areas in both Progressions will be found, as appears from the foregoing Theorems. Yet it will not thence follow universally, that any two Areas being given, all the Areas can thence be found. For, 1°. when it is $\theta = 0$, if the two given Areas be both in the ascending Progression beyond A, none of the Areas in the descending Progression can be thence found: since the Equation that includes B and C, cannot include A, the Term θeA vanishing: and this evidently prevents the passing from the ascending to the descending Progression. 2°. If it be $\theta + \lambda\eta = 0$, from A and C given, none of the other Areas can be found, as was shewn already. 3°. If it be $\theta + 2\lambda\eta = 0$, from the Areas A and B given, or any two Areas in the descending Progression, none of the Areas in the ascending Progression beyond B, can be thence found: for the Equation including A and B, doth not include C, and therefore this interrupts the Passage from A and B to the other Areas in the ascending Progression. But if you except these Cases, any two Areas being given, either both in the ascending, or both in the descending; or one in the one, and another in the other, all the other Areas in both Progressions may be found by help of the preceding Theorems.

Schol. 1. In the preceding Cases, from one Area only given, in the Case of Trinomials, other Areas are found. Thus 1°. when it is $\theta = 0$, one Area above A in the ascending Progression, such as B, C, *&c.* being given, all the others in that Progression, except A, may be found. 2°. If it be $\theta + \lambda\eta = 0$, the Area C may be found from A, or A from C. 3°. If it be $\theta + 2\lambda\eta = 0$, from B or A, or any one Area in the descending Progression given, all the other Areas in that Progression (including also B) may be found: as may easily appear from the preceding Theorems, and what has been said.

Schol. 2. There are analogous Observations which hold with respect to other more complex Curves, such as Quadrinomials, Quinquinomials, *&c.* And particularly with respect to binomial Curves, it

* Of this Explication.

appears from Art. 202, 203 *, that when $\theta = 0$, from B, or any one of the Areas in the ascending Progression above B, given, neither A, nor any of the Areas in the descending Progression can be found. 2°. When $\theta + \lambda\eta = 0$, from the Area A, or any one in the descending Progression, given, neither B, nor any other Area in the ascending Progression can be thence found. But that in the first Case, the Curve whose Ordinate is $az^{\eta-1} \times \overline{e + fz^{\eta}}|^{\lambda-1}$, is quadrable, it's Area being $\frac{a}{\lambda\eta f} \times \overline{e + fz^{\eta}}|^{\lambda}$: and in the second Case, the Curve whose Ordinate is $az^{\theta-1} \times \overline{e + fz^{\eta}}|^{\lambda-1}$, is quadrable; and it's Area $\frac{az^{\theta} \times \overline{e + fz^{\eta}}|^{\lambda}}{\lambda\eta f}$: as appears also by Art. 149 *.

* Of this Explication.

CASE II.

210. The second Case of the Prop. is where λ, the Exponent of R, in the Expression of the Ordinate, is to be increased, or diminished,

ed, by the continual Addition, or Subtraction of Units. This Caſe, as far as it reſpects binomial Curves, *viz.* when $R = e + fz^{\eta}$, is left out by our Author: and is ſupplied in what follows.

Suppoſe the Area of one ſuch binomial Curve to be given, whoſe Ordinate is $z^{\theta-1}R^{\lambda}$, and that Area to be called A. Then $pz^{\theta-1}R^{\lambda}$ and pA are correſpondent Ordinate and Area, p being any conſtant Quantity *. Further, divide, and then multiply the Ordinate $pz^{\theta-1}R^{\lambda}$ by R or $e + fz^{\eta}$, and it appears thus $\overline{pe + pfz^{\eta}} \times z^{\theta-1}R^{\lambda-1}$ different in Appearance, but the ſame in Effect, with what it was before, ſince the Multiplication and Diviſion by the ſame thing, mutually compenſate each other. Then, ſince the Ordinate $\overline{\theta e + \overline{\theta + \lambda\eta} \times fz^{\eta}} \times z^{\theta-1}R^{\lambda-1}$ and Area $z^{\theta}R^{\lambda}$ anſwer to each other; by Prop. 3, add the former Ordinate and Area to the latter Ordinate and Area, reſpectively: thence you'll have $\overline{\overline{p+\theta} \times e + \overline{p + \theta + \lambda\eta} \times fz^{\eta}} \times z^{\theta-1}R^{\lambda-1}$ for the Ordinate belonging to the Area $pA + z^{\theta}R^{\lambda}$: ſuppoſe the ſecond Term in the Value of the Ordinate equal to nothing, that is $\overline{p + \theta + \lambda\eta} \times fz^{\eta} = 0$, or $p + \theta + \lambda\eta = 0$: thence $p = -\overline{\theta + \lambda\eta}$: inſert this Value of p in place of it, in the firſt Term of the Ordinate, and in the correſponding Area, and you'll have $-\lambda\eta e \times z^{\theta-1}R^{\lambda-1}$ and $-\overline{\theta + \lambda\eta} \times A + z^{\theta}R^{\lambda}$, or, by dividing both by $-\lambda\eta e$, $z^{\theta-1}R^{\lambda-1}$ and $\frac{\overline{\theta+\lambda\eta} \times A - z^{\theta}R^{\lambda}}{\lambda\eta e}$, an Ordinate and Area belonging to the ſame Curve: which Area is wholly known, ſince A is given; and the Ordinate has the Index of R leſs by Unity, than the Index of R in the Ordinate of the given Curve. The Steps of the Proceſs, ſet down diſtinctly, follow.

* Art. 116. of this Explication.

	Ordinates.		Areas.
1°.	$\overline{pe + pfz^{\eta}}$	$\times z^{\theta-1}R^{\lambda-1}$ - - - - - -	pA.
2°.	$\overline{\theta e + \overline{\theta+\lambda\eta} \times fz^{\eta}}$	$\times z^{\theta-1}R^{\lambda-1}$ - - - - - - -	$z^{\theta}R^{\lambda}$ *.

* Prop. 3.

Put the Sum of the ſecond Terms equal to nothing, or $p = -\overline{\theta + \lambda\eta}$.

3°. - - - - $-\lambda\eta e \times z^{\theta-1}R^{\lambda-1}$ - - - - - - $\overline{\theta + \lambda\eta} \times A + z^{\theta}R^{\lambda}$.

Divide both by $-\lambda\eta e$.

4°. - - - - - - - - $z^{\theta-1}R^{\lambda-1}$ - - - - - - $\frac{\overline{\theta+\lambda\eta} \times A - z^{\theta}R^{\lambda}}{\lambda\eta e} = B$.

Now,

Now, after the same Manner, that, from the Area A belonging to the Ordinate $z^{\theta-1}R^{\lambda}$, given, the Area belonging to the Ordinate $z^{\theta-1}R^{\lambda-1}$, was just now found, the Area belonging to this last Ordinate, being given, you may find the Area belonging to another Curve, whose Ordinate is $z^{\theta-1}R^{\lambda-2}$: the Method of doing it and the Demonstration differs in nothing, but inserting $\lambda-1$, every where, for λ in the foregoing: so that if the Area just now found be called B; and the Area belonging to the Ordinate $z^{\theta-1}R^{\lambda-2}$, be denoted by C, then $C=\frac{\overline{\theta+\lambda\eta+\eta}\times B-z^{\theta}R^{\lambda-1}}{\overline{\lambda\eta+\eta}\times e}$. And so you may proceed to find the Area belonging to the Ordinate $z^{\theta-1}R^{\lambda-3}$, by a like Process, *viz.* by inserting $\lambda-2$ for λ, and so on continually. For in effect, the Theorem, which makes the Conclusion of the first Demonstration, *viz.* $B=\frac{\overline{\theta+\lambda\eta}\times A-z^{\theta}R^{\lambda}}{\lambda\eta e}$, includes all possible Cases of binomial Curves, in which the Area of any one being given, the Area of another is sought, in whose Ordinate, the Index of R, is lessened by Unity: because whatever be the Index of R, in the Ordinate belonging to the given Area; whether it be affirmative, or negative, you may suppose it to be represented by λ: and again, whatever be the Index of z without the Vinculum, it may be always represented by $\theta-1$. Therefore also conversely, if A and B represent the Areas of any two binomial Curves, such, that the Index of R be greater by a positive Unit, in the Ordinate belonging to A than in the Ordinate belonging to B, by resolving the Equation, and considering B as given, and A as sought, you'll have $A=\frac{\lambda\eta eB+z^{\theta}R^{\lambda}}{\overline{\theta+\lambda\eta}\times e}$. Which furnishes us with the two following Theorems, serving universally for carrying on the Progression of binomial Curves, from the Area of any one being given, in whose Ordinates the Index of R is to be diminish'd or increas'd by the continual Subtraction or Addition of Units.

211. Th. 1. $B=\frac{\overline{\theta+\lambda\eta}\times A-z^{\theta}R^{\lambda}}{\lambda\eta e}$.

212. Th. 2. $A=\frac{\lambda\eta eB+z^{\theta}R^{\lambda}}{\theta+\lambda\eta}$.

213. I shall illustrate what concerns the Increase and Diminution of the Index of R by Units; and of the Index of z without the Vinculum by η: for which I gave two Theorems formerly * by an Example, in the Case of a binomial Curve, which will make all evident.

* Art. 202, 203. of this Explication.

Let

Let $\frac{z^2}{\sqrt{a^2 - z^2}} = y$, define the Relation of the Abſciſs and Ordinate of a Curve: and ſuppoſe I would compare the Area of this Curve, with the Area of another Curve, whoſe Ordinate is $\sqrt{a^2 - z^2}$, z being the common Abſciſs to both: which laſt Ordinate plainly belongs to a Circle, in which a is the Radius, and the Abſciſs z reaching from the Center to the Ordinate: and therefore the circular Area, with which the Area of the Curve propoſed, is to be compared, is that lying betwixt the Ordinate and the Radius of the Circle parallel to it, as appears by Art. 146 *: which Area is repreſented by ABCD in the Fig. belonging to that Art. when AB $= z$. Now, to make this Compariſon, let the given Ordinate, *viz.* $\frac{z^2}{\sqrt{a^2 - z^2}}$ be brought to the Form $z^{3-1} \times \overline{a^2 + z^2}|^{-\frac{1}{2}}$: call the Area β, and from the Aſſumption of β, let us firſt find the Area belonging to an Ordinate expreſſed thus $z^{3-1} \times \overline{a^2 - z^2}|^{\frac{1}{2}}$ where the Exponent of the Quantity under the Vinculum, *viz.* $a^2 - z^2$, anſwering to R, is greater by Unity. Call the Area belonging to this laſt Ordinate, α: then, by comparing this Example with the two general Theorems juſt now laid down, you'll find $A = \alpha \,.\, B = \beta \,.\, \theta = 3 \,.\, \eta = 2 \,.\, \lambda = \frac{1}{2} \,.\, e = aa$. Therefore by the ſecond Theorem you have $\alpha = \frac{a^2\beta + z^1 \times \overline{a^2 - z^2}|^{\frac{3}{2}}}{4}$, for the Relation of the Areas α and β.

* Of this Explication.

Again from the Area α, belonging to the Ordinate $z^{3-1} \times \overline{a^2 - z^2}|^{\frac{1}{2}}$, I find the Area belonging to the Ordinate $z^{1-1} \times \overline{a^2 - z^2}|^{\frac{1}{2}}$ or $\sqrt{a^2 - z^2}$: which is the former Ordinate, having the Index of z without the Vinculum, diminiſh'd by $2 = \eta$. This is done by Caſe 1. of this Prop. For calling the Area correſponding to this laſt circular Ordinate, *viz.* $\sqrt{a^2 - z^2}$, γ; and reducing the two laſt Ordinates to due Form, you have $z^{3-1} \times \overline{a^2 - z^2}|^{\frac{3}{2}-1}$, anſwering to the Area α; and $z^{1-1} \times \overline{a^2 - z^2}|^{\frac{3}{2}-1}$ anſwering to the Area γ: wherefore, by comparing this Example with Art. 201, 202, 203 *, you'll find $B = \alpha$ and $A = \gamma \,.\, \theta = 1 \,.\, \eta = 2 \,.\, \lambda = \frac{3}{2} \,.\, e = aa \,.\, f = -1$. And therefore by Art. 203 * you'll have $\gamma = \frac{z \times \overline{a^2 - z^2}|^{\frac{3}{2}} + 4\alpha}{aa} =$ (by inſerting the Value of α ſet down above) $\frac{z \times \overline{a^2 - z^2}|^{\frac{3}{2}} + a^2\beta + z^3 \times \overline{a^2 - z^2}|^{\frac{1}{2}}}{a^2} =$) $z\sqrt{a^2 - z^2} + \beta$. Whence $\beta = \gamma - z\sqrt{a^2 - z^2}$, *i. e.* the Area of the

* Of this Explication.

* Of this Explication.

the Curve propofed is equal to the circular Area, lefs the Rectangle contain'd under the common Abfcifs z and the Ordinate of the Circle, *i. e.* if you draw CG parallel to AB in the Fig. Art. 146 *, the Area CDG.

* Of this Explication.

Art. 36. of the Quadrature of Curves.

214. Our Author fhews at full Length, how, from the two Areas given, belonging to the two trinomial Curves, whofe Ordinates are $z^{\theta-1}R^{\lambda}$ and $z^{\theta+\eta-1}R^{\lambda}$, you may find the Area correfponding to the Ordinate $z^{\theta+\eta-1}R^{\lambda}$: and then he fays by the like Means you may find the Area belonging to the Curve, whofe Ordinate is $z^{\theta-1}R^{\lambda-1}$. The Manner of doing it is thus.

Let the four Ordinates, and four correfponding Areas be fet down, as he hath done:

$$\overline{\begin{array}{llll} \theta ae & +\overline{\theta+\lambda\eta}\times af\times z^{\eta} & +\overline{\theta+2\lambda\eta}\times ag\times z^{2\eta} & +\overline{\theta+\eta+2\lambda\eta}\times bg\times z^{3\eta} \\ +pe & +\overline{\theta+\eta}\times be & +\overline{\theta+\eta+\lambda\eta}\times bf & +qg \\ & +pf & +pg & \\ & +qe & +qf & \end{array}}\times z^{\theta-1}R^{\lambda-1} \quad \text{Sum of the Ord}^{s}.$$

$$pA+qB+az^{\theta}R^{\lambda}+bz^{\theta+\eta}R^{\lambda} \quad \text{Sum of the Areas.}$$

Then fuppofe the Coefficients of all the Terms, fave the firft, of the complex Ordinate, equal, each to nothing: by which you'll have three diftinct Equations. From the laft of which, *viz.* $\overline{\theta+\eta+2\lambda\eta}\times bg+qg=0$, find a Value of q, which is $q=-\overline{\theta+\eta+2\lambda\eta}\times b$. Then infert this Value of q in place of it, in the fecond Equation, which arifes from the third Term, *viz.* $\overline{\theta+2\lambda\eta}\times ag+\overline{\theta+\eta+\lambda\eta}\times bf+pg+qf=0$, thence you have a Value of p, *viz.* $p=\lambda\eta\times\frac{bf}{g}-\overline{\theta+2\lambda\eta}\times a$. Again infert thefe Values of p and q now found, in the firft Equation, arifing from the fecond Term, which is $\overline{\theta+\lambda\eta}\times af+\overline{\theta+\eta}\times be+pf+qe=0$: and fo you'll have an Equation free of p and q, *viz.* $-\lambda\eta\times af-2\lambda\eta\times be+\lambda\eta\times\frac{bf^{2}}{g}=0$: which by a proper Reduction, gives you the Value of b, *viz.* $b=\frac{afg}{f^{2}-2ge}$. Next, fubftitute this Value of b in place of it, in the preceding Equations, expreffing the Values of q and p, *viz.* $q=-\overline{\theta+\eta+2\lambda\eta}\times b$: and $p=\lambda\eta\times\frac{bf}{g}-\overline{\theta+2\lambda\eta}\times a$: and fo you'll have $q=\frac{\overline{\theta+\eta+2\lambda\eta}\times afg}{2eg-f^{2}}$. $p=\frac{\lambda\eta af^{2}}{f^{2}-2eg}-\overline{\theta+2\lambda\eta}\times a$. And fo having found thefe Values of

q,

q, p and b, ariſing from the Suppoſition of the Coefficients of the three laſt Terms of the complex Ordinate being equal to nothing: put theſe Values of q, p and b in place of them in the Sum of the Ordinates and Areas, and you'll obtain $\theta ae + \overline{\frac{\lambda\eta af^2 e}{f^2 - 2eg} - \overline{\theta + 2\lambda\eta} \times ae} \times z^{\theta-1}R^{\lambda-1}$ for the Sum of the Ordinates. And $\overline{\frac{\lambda\eta af^2}{f^2 - 2eg} - \theta + 2\lambda\eta \times a} \times A - \frac{\overline{\theta + \eta + 2\lambda\eta} \times afg}{f^2 - 2eg}B + az^{\theta}R^{\lambda} + \frac{afg}{f^2 - 2eg}z^{\theta+\eta}R^{\lambda}$ Sum of the Areas.

Multiply both by the given Quantity $f^2 - 2eg$, and then divide both by the given Quantity $4\lambda\eta aeeg - \lambda\eta aef^2$ or $-\lambda\eta ea \times \overline{f^2 - 4eg}$, and you'll obtain the Ordinate $z^{\theta-1}R^{\lambda-1}$, and correſpondent Area

$$\frac{\overline{2\theta + 4\lambda\eta} \times eg - \overline{\theta + \lambda\eta} \times f^2 \times A - \overline{\theta + \eta + 2\lambda\eta} \times fq \times B + fgz^{\theta+\eta}R^{\lambda} + \overline{f^2 - 2eg} \times z^{\theta}R^{\lambda}}{-\lambda\eta e \times \overline{f^2 - 4eg}}.$$

Which therefore is the Area ſought, and is wholly known, ſince A and B were given. Wherefore, if you call this Area C, and call A, B, and D the Areas correſponding to the Ordinates $z^{\theta-1}R^{\lambda}$, $z^{\theta+\eta-1}R^{\lambda}$ and $z^{\theta+\eta-1}R^{\lambda-1}$ reſpectively, as they are uſed by the Author: by conſidering what he hath here demonſtrated in this 36th Art *. and what is demonſtrated above, you'll have the two following general Theorems ſerving to find the Areas of all trinomial Curves whoſe Ordinates are thus generally expreſſed $z^{\theta\pm\sigma\eta}R^{\lambda\pm\tau}$, when the Areas of any two of them are given, where the Index of z without the Vinculum is greater by $+\eta$ in the Ordinates of the Curves whoſe Areas are repreſented by B and D, than in the Ordinates of thoſe repreſented by A and C: but the Index of R in the Ordinates correſponding to the Areas A and B greater by Unity than the Index of R in the Ordinates of thoſe whoſe Areas are C and D.

* Of the Quadrature of Curves.

215. Th. 1. $D = \frac{fz^{\theta}R^{\lambda} + 2g \times z^{\theta+\eta}R^{\lambda} - \theta fA - \overline{\theta + \eta + 2\lambda\eta} \times 2gB}{\lambda\eta \times \overline{f^2 - 4eg}}$.

216. Th. 2. $C = \frac{\overline{f^2 - 2eg} \times z^{\theta}R^{\lambda} + fg \times z^{\theta+\eta}R^{\lambda} + \overline{\overline{\theta + 2\lambda\eta} \times 2eg - \overline{\theta + \lambda\eta} \times f^2} \times A - \overline{\theta + \eta + 2\lambda\eta} \times fg \times B}{-\lambda\eta e \times \overline{f^2 - 4eg}}$.

217. By theſe two Theorems, if what was ſaid formerly *, be duly conſidered, and the Areas of two of the moſt ſimple trinomial Curves, be given, all others, in whoſe Ordinates, the Index of R under the Vinculum, is increas'd or diminiſh'd by any Number of Units, may be found.

*Art. 210. of this Explication.

CASES 3, 4.

218. And now it appears, by what means, if both the Index of z without the Vinculum is increas'd or diminiſh'd by the continual Ad- Art 37, 38 of the Quadrature of Curves.

dition or Subtraction of η; and the Index of R under the Vinculum by the continual Addition or Subtraction of Unity: the Areas answering to all the Ordinates will be given, when the Area of any one of the binomial Curves is given: or the Areas of two of the trinomial Curves are given. For this is evidently done by the first and second Cases conjoin'd. An Example of this when the Curves are Binomials, was given already *. And the like may be done in the Case of Trinomials, when the Areas of any two of them are given. For the Ordinates of any two trinomial Curves may be represented thus, $z^{\theta}R^{\lambda}$ and $z^{\theta\pm\sigma\eta}R^{\lambda\pm\tau}$: supposing then the Areas of the two Curves, to which these Ordinates belong, to be given: assume the Area belonging to the Ordinate $z^{\theta+\eta}R^{\lambda}$, as if it were given: then from the two Areas corresponding to the two Ordinates $z^{\theta}R^{\lambda}$ and $z^{\theta+\eta}R^{\lambda}$, you can, by Case first, find the Areas corresponding to the Ordinates $z^{\theta\pm\sigma\eta}R^{\lambda}$ and $z^{\theta\pm\sigma\eta+\eta}R^{\lambda}$ from the two last, you may find the Area belonging to the Ordinate $z^{\theta\pm\sigma\eta}R^{\lambda\pm\tau}$ by Case second: and this Area being the other one which was given, it follows conversely that from the two Areas, belonging to the Ordinates $z^{\theta}R^{\lambda}$ and $z^{\theta\pm\sigma\eta}R^{\lambda\pm\tau}$, given, you may find the Area belonging to the Ordinate $z^{\theta+\eta}R^{\lambda}$, which was formerly assumed. And therefore the two Areas, corresponding to the two Ordinates $z^{\theta}R^{\lambda}$ and $z^{\theta+\eta}R^{\lambda}$, being now both given, it is evident you can thence find the Area of any other trinomial Curve, whose Ordinate is thus indefinitely expressed $z^{\theta\pm\sigma\eta}R^{\lambda\pm\tau}$.

* Art. 213. of this Explication.

As to the fourth Case of the Prop. when R is a Quadrinomial, or any other Multinomial: the Demonstration of it is made out after the like Manner as when R was a Trinomial.

DEMONSTRATION of PROP. VIII.

CASE I.

The first Case of the Prop. is when the Index of z without the Vinculum, is increas'd or diminish'd by the continual Addition or Subtraction of η.

219. Let the Ordinates of two Curves be $pz^{\theta-1}R^{\lambda-1}S^{\mu-1}$ and $qz^{\theta+\eta-1}R^{\lambda-1}S^{\mu-1}$, and the corresponding Areas pA and qB, when $R = e + fz^{\eta}$ and $S = k + lz^{\eta}$, both Binomials: then you'll have as follows.

Ordinates.

	Ordinates.	Areas.
1°.	p - - * - - - - - - - - * - - - - - - - $\times z^{\theta-1}R^{\lambda-1}S^{\mu-1}$	- - pA.
2°.	* - - qz^{η} - - - - - - * - - - - - - - $\times z^{\theta-1}R^{\lambda-1}S^{\mu-1}$	- - qB.
3°.	$\overline{\theta ek + \overline{\theta+\lambda\eta}\times fkz^{\eta} + \overline{\theta+\lambda\eta+\mu\eta}\times flz^{2\eta}} \times z^{\theta-1}R^{\lambda-1}S^{\mu-1}$ $+\overline{\theta+\mu\eta}\times el$	- - $z^{\theta}R^{\lambda}S^{\mu}$ *

* Prop. 4.

Suppose $p+\theta ek=0$, and $q+\overline{\theta+\lambda\eta}\times fk+\overline{\theta+\mu\eta}\times el=0$, or $p=-\theta ek$, and $q=-\overline{\theta+\lambda\eta}\times fk-\overline{\theta+\mu\eta}\times el$: and then take the Sum of the Ordinates and Areas.

4°. - - - - - - - - - $\overline{\theta+\lambda\eta+\mu\eta}\times flz^{2\eta}\times z^{\theta-1}R^{\lambda-1}S^{\mu-1}$ - - $z^{\theta}R^{\lambda}S^{\mu}-\theta ekA-\overline{\theta+\lambda\eta}\times fkB$
$-\overline{\theta+\mu\eta}\times el$

Divide both by the given Quantity $\overline{\theta+\lambda\eta+\mu\eta}\times fl$.

5°. - - - - - - - - - - - - - - - $z^{\theta+2\eta-1}R^{\lambda-1}S^{\mu-1}$ - - $\dfrac{z^{\theta}R^{\lambda}S^{\mu}-\theta ekA-\overline{\theta+\lambda\eta}\times fkB-\overline{\theta+\mu\eta}\times el}{\overline{\theta+\lambda\eta+\mu\eta}\times fl}=C$

Which last Area belongs to the Ordinate $z^{\theta+2\eta-1}R^{\lambda-1}S^{\mu-1}$, and is wholly known when A and B are known. And by the like way of proceeding, if you call this last Area C; from the Areas B and C you may find a fourth Area corresponding to the Ordinate $z^{\theta+3\eta-1}R^{\lambda-1}S^{\mu-1}$: and so on. And the Method of proceeding in the Progression going the contrary way, *viz.* from A and B to find the Areas corresponding to the Ordinates $z^{\theta-\eta-1}R^{\lambda-1}S^{\mu-1}$, $z^{\theta-2\eta-1}R^{\lambda-1}S^{\mu-1}$, *&c. in inf.* is after the same Sort *. And contrarily, from any other two Areas, you may, by Analysis, return back to the Areas A and B: as will easily appear by considering what was said upon the foregoing Prop. so that from any two of these Areas given, all the others will also be given.

* Art. 115. of this Explication.

220. And when R and S taken together consist of five Terms or Members, then there must be three Areas given: if R and S taken together consist of six Members, the Areas of four Curves must be given, *&c.* and then the Areas of all the rest may be found, after the like Manner, as hath been done, when R and S are each Binomials, and the Areas of two Curves given.

CASE 2.

The second Case of the Prop. is when λ, the Index of R, is increas'd or diminish'd by the continual Addition or Subtraction of Unity.

221. Let the Ordinates $pz^{\theta-1}R^{\lambda}S^{\mu}$ and $qz^{\theta+\eta-1}R^{\lambda}S^{\mu}$ have pA and qB for their corresponding Areas: and let the Ordinates be first multiplied, and then divided by $R=e+fz^{\eta}$, as in Case second of the last Prop. by which they stand thus $\overline{pe+pfz^{\eta}}\times z^{\theta-1}R^{\lambda-1}S^{\mu}$ and $\overline{qez^{\eta}+qfz^{2\eta}}\times z^{\theta-1}R^{\lambda-1}S^{\mu}$: again, by Prop. 4. the Ordinate $\overline{\theta ek+\overline{\theta+\lambda\eta}\times fk+\overline{\theta+\mu\eta+\eta}\times el\times z^{\eta}+\overline{\theta+\lambda\eta+\mu\eta+\eta}\times flz^{2\eta}}$ $\times z^{\theta-1}R^{\lambda-1}S^{\mu}$ and Area $z^{\theta}R^{\lambda}S^{\mu+1}$ belong to one and the same Curve. Therefore take the Sum of the three Ordinates and the Sum of the three Areas, which will make an Ordinate and Area corresponding to each other: then supposing the Coefficients of the first and third Terms of the complex Ordinate, each equal to nothing, *viz.* $pe+\theta ek=0$ and $\overline{\theta+\lambda\eta+\mu\eta+\eta}\times fl+qf=0$; or, by reducing, $p=-\theta k$ and $q=-\overline{\theta+\lambda\eta+\mu\eta+\eta}\times l$: which Values substitute in place of p and q: divide by $\lambda\eta\times\overline{fk-el}$: and thence you'll obtain the Area $\frac{z^{\theta}R^{\lambda}S^{\mu+1}-\theta kA-\overline{\theta+\lambda\eta+\mu\eta+\eta}\times lB}{\lambda\eta\times\overline{fk-el}}$, corresponding to the Ordinate $z^{\theta+\eta-1}R^{\lambda-1}S^{\mu}$, as follows.

	Ordinates.		Areas.
1°.	$\overline{pe+pfz^{\eta}}$ - - - - - - - - - - - - - - -	$\times z^{\theta-1}R^{\lambda-1}S^{\mu}$ - -	pA
2°.	- - - $\overline{qez^{\eta}+qfz^{2\eta}}$	$\times z^{\theta-1}R^{\lambda-1}S^{\mu}$ - -	qB
3°.	$\overline{\theta ek+\overline{\theta+\lambda\eta}\times fkz^{\eta}+\overline{\theta+\lambda\eta+\mu\eta+\eta}\times flz^{2\eta}}$ $+\overline{\theta+\mu\eta+\eta}\times el$	$\times z^{\theta-1}R^{\lambda-1}S^{\mu}$ - -	$z^{\theta}R^{\lambda}S^{\mu+1}$
4°.	- - - - - - - - - - - - - - - -	$z^{\theta+\eta-1}R^{\lambda-1}S^{\mu}$ - -	$\frac{z^{\theta}R^{\lambda}S^{\mu+1}-\theta kA-\overline{\theta+\lambda\eta+\mu\eta+\eta}\times lB}{\lambda\eta\times\overline{fk-el}}$

If, instead of putting the Coefficients of the first and third Terms equal to nothing, in the complex Ordinate, you had put the Coefficients of the second and third Terms equal to nothing, you would have thence obtained the Area corresponding to the Ordinate $z^{\theta-1}R^{\lambda-1}S^{\mu}$: and calling these two Areas C and D, after the same Manner that from A and B given, the Areas C and D are found, you may from the Areas C and D, proceed to find the Areas corresponding to the Ordinates $z^{\theta-1}R^{\lambda-2}S^{\mu}$ and $z^{\theta+\eta-1}R^{\lambda-2}S^{\mu}$, and so on continually. And conversely if C and D be given, by Analysis, you find A and B; and thence the Areas of the Curves belonging to the Progression going

going the contrary way, *viz.* where λ is continually increaſed by Unity: and the reſt follows as formerly.

Moreover the ſame may be done, by the like Means, when R or S, or both of them are more than Binomials: provided ſtill the Number of given Areas be but two leſs than the Number of Members or Terms of which R and S, taken together, do conſiſt.

CASE 3.

222. The third Caſe of the Prop. is when the Index of S is increas'd or diminiſhed by the continual Addition or Subtraction of Unity. Let pA and qB be the Areas to which the Ordinates $pz^{\theta-1}R^{\lambda}S^{\mu}$ and $qz^{\theta+\eta-1}R^{\lambda}S^{\mu}$ correſpond: when R and S are Binomials. Then from theſe two Areas pA and qB given, you may find other two Areas belonging to the Curves whoſe Ordinates are $z^{\theta-1}R^{\lambda}S^{\mu-1}$ and $z^{\theta+\eta-1}R^{\lambda}S^{\mu-1}$, after the like Manner that, by Caſe ſecond, you found the Areas correſponding to the Ordinates $z^{\theta-1}R^{\lambda-1}S^{\mu}$ and $z^{\theta+\eta-1}R^{\lambda-1}S^{\mu}$: all the Difference is, that the Ordinates of the two Curves whoſe Areas are given, are firſt to be multiplied, and then divided by $S = k + lz^{\eta}$, inſtead of being multiplied and divided by $R = e + fz^{\eta}$: and then all the reſt goes on as formerly.

CASE 4.

223. The fourth Caſe of the Prop. is when the Index of z is increas'd, or diminiſh'd by the continual Addition or Subtraction of η, and likewiſe the Exponents of R and S by the continual Addition or Subtraction of Unity. In which Caſe the Areas anſwering to each of the Ordinates ſhall be given, upon the Conditions mentioned in the Prop. and it is done by the three former Caſes jointly.

Let $z^{\theta-1}R^{\lambda}S^{\mu}$ and $z^{\theta\pm\sigma\eta-1}R^{\lambda\pm\tau}S^{\mu\pm\upsilon}$ repreſent the Ordinates of any two Curves whatſoever, whoſe Areas are given, R and S being Binomials: then aſſume, or ſuppoſe to be given, the Area anſwering to the Ordinate $z^{\theta+\eta-1}R^{\lambda}S^{\mu}$. From the Areas correſponding to the Ordinates $z^{\theta-1}R^{\lambda}S^{\mu}$ and $z^{\theta+\eta-1}R^{\lambda}S^{\mu}$, you may, by Caſe firſt, find the Areas anſwering to the Ordinates $z^{\theta\pm\sigma\eta-1}R^{\lambda}S^{\mu}$ and $z^{\theta\pm\sigma\eta-\eta}R^{\lambda}S^{\mu}$: from theſe laſt two Areas you can find, by Caſe ſecond, other two Areas, correſponding to the Ordinates $z^{\theta\pm\sigma\eta-1}R^{\lambda\pm\tau}S^{\mu}$ and $z^{\theta\pm\sigma\eta+\eta-1}R^{\lambda\pm\tau}S^{\mu}$: again from theſe laſt two Areas, you may, by Caſe third, find the Area anſwering to the Ordinate $z^{\theta\pm\sigma\eta-1}R^{\lambda\pm\tau}S^{\mu\pm\upsilon}$, which was one of the Areas originally given: therefore converſely, by

Analyſis,

Analyſis, when the Areas belonging to the Ordinates $z^{\theta-1}R^{\lambda}S^{\mu}$ and $z^{\theta\pm\sigma\eta-1}R^{\lambda\pm\tau}S^{\mu\pm\upsilon}$ are given, as at firſt, thence may be found the Area which anſwers to the Ordinate $z^{\theta+\eta-1}R^{\lambda}S^{\mu}$; which was at firſt aſſumed: and conſequently it being now given, together with the Area correſponding to the Ordinate $z^{\theta-1}R^{\lambda}S^{\mu}$, from theſe two Areas, now given, the Area of any other Curve in general belonging to the Claſs of Curves, mentioned in this Prop. may be found, when R and S are Binomials. And the like is the Reaſoning in all other Caſes, provided the Number of given Areas be ſuch, as is mentioned in the Prop.

SECT. VIII.

Containing Notes on Art. 40——51.

PROP. IX.

Art. 40. of the Quadrature of Curves.

224. THE Truth contained in this Prop. *viz.* That the Areas of theſe Curves are equal whoſe Ordinates are reciprocally proportional to the Fluxions of their Abſciſſes, was at large inſiſted upon towards the Beginning, and was there demonſtrated *.

* Art. 38. of this Explication.

COR. 1.

225. By means of this Corollary, we may find innumerable Curves, whoſe Areas ſhall all be equal: of which take the following Examples.

Ex. 1. Let $dx - x^2 = v^2$ be an Equation given, belonging to a Circle: and it is required to find other Curves, whoſe Areas may be equal to that of the Circle.

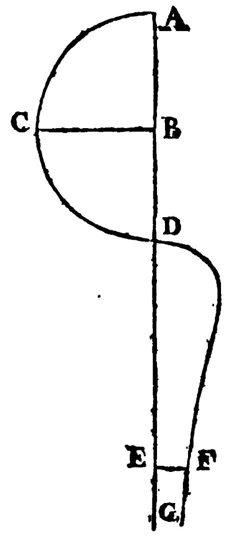
Let z and y be the Abſciſs and Ordinate of another Curve, having it's Area equal to that of the Circle: and ſuppoſe $dx = z^2$ expreſs the Relation of the Abſciſſes.

From this Equation $dx = z^2$, find the Relation of their Fluxions, by Prop. 1, which is $d\dot{x} = 2z\dot{z}$, that is by ſuppoſing $\dot{x} = 1$, $d = 2z\dot{z}$ or $\dot{z} = \frac{d}{2z}$: wherefore by this Prop. $\dot{z} = \frac{d}{2z} : \dot{x} = 1 :: v = \sqrt{dx - x^2} : y$: whence $y = \frac{2z}{d}\sqrt{dx - x^2}$, but $dx = z^2$, by Hyp. or $x = \frac{z^2}{d}$: which Value being inſerted for x, you have $y = \frac{2z^2}{d^2}\sqrt{d^2 - z^2}$ for the Property of the Curve ſought, whoſe Area is equal to that of the Circle.

* Prop. 1.

Again let it be $d^2 = xz$: then $\dot{x}z + x\dot{z} = 0$ *, or $\dot{z} = -\frac{z}{x}$: wherefore ſince the Ordinates and Fluxions of the Abſciſſes are reciprocally pro-

proportional, you'll have $-\frac{\dot{z}}{x} : 1 :: \sqrt{dx - x^2} : y$, that is $y = -\frac{x}{\dot{z}}\sqrt{dx - x^2}$; or (by inserting $\frac{d^2}{z}$ for x, from the given Equation $dd = xz$) $y = -\frac{d^3}{z^3}\sqrt{-d^2 + dz}$, the Equation to the Curve sought: here the curvilinear Area is situate beyond the Ordinate, extending itself along the Abscis infinitely produc'd: for $\dot{z}$ is negative, z diminishing as x increases, and contrarily: and when $x = 0$, z is infinite, but when $x = d . z = d$: which things easily appear from the Equations above.



Thus let ABCD be a Semicircle, the Diameter AD $= d$, the Abscis AB $= x$, and Ordinate BC $= v$: produce AD indefinitely towards G, take AE a third proportional to AB and AD, which call z: whence it is $z = \frac{dd}{x}$: draw EF perpendicular to AE, and take it equal to $\frac{d^3}{z^3}\sqrt{-d^2 + dz} = y$. Then the Ordinate EF shall describe a curvilinear Area extending itself infinitely along the right Line DG, which is the Assymptote to the Curve: so that the Area EFG adjacent to the Abscis infinitely produced beyond the Ordinate, shall be equal to the circular Area ABC adjacent to the Abscis AB; and the remaining Area DEF, lying above the Ordinate EF, and adjacent to the Part of the Abscis DE, is equal to the remaining circular Area BCD: and the whole curvilinear Area DGFD = the Semicircle ACDA. And if you take DE = AD, you shall have DEFD equal to the Quadrant of the Circle.

And thus you may, by assuming new Relations of x and z, find infinite other Curves, whose Areas shall be equal to the Area of the Circle.

Ex. 2. Let $\frac{aa}{d+x} = v$ be an Equation to the Hyperbola: and you would find another Curve, whose Area is equal to the hyperbolical Area: call it's Abscis and Ordinate z and y, as formerly: then take any Relation of the Abscisses, as thus, $bx = z^2$: thence by Prop. 1, $b = 2z\dot{z}$ or $\dot{z} = \frac{b}{2z}$: therefore $\frac{b}{2z} : 1 :: \frac{a^2}{d+x} : y$, whence $y = \frac{2a^2z}{bd+bx}$ = (by inserting the Value of x, *viz.* $\frac{z^2}{b}$) $\frac{2a^2z}{bd+z^2}$, which shews the Property of the Curve, whose Area is equal to the hyperbolical Area. After the like manner you may find infinite other Curves, having their Areas

Areas equal to the hyperbolical Area. And likewise any known Curve being assign'd, you may, by the like means, find as many Curves as you please, having their Areas, each, equal to the Area of the known Curve.

N. B. This Operation may be called *Transformation of Curves:* because by it you are taught a general Method of transforming any given Curve into as many other equivalent Curves, as you please: which may be of great Use in the Solution of Problems in Geometry.

226. It would be much the same, if it were required to find any Number of Curves, each of whose Areas bears any assignable Relation to the Area of any known Curve.

Thus let x, v; z, y represent the Abscisses and Ordinates, of the given Curve, and of the Curve sought, as before: moreover let s denote the circular, elliptical, hyperbolical, or any other Area of a known Curve; and t the Area of the other Curve. And suppose the given Curve be defined by the Equation $a^2p - px^2 = av^2$ or $\sqrt{\frac{a^2p - px^2}{a}} = v$: which is to the Ellipse, having a the half Axe, and p it's half Parameter: let $2s + t - xv = 0$, be the assigned Relation of the Areas. Then, you assume any Relation of the Abscisses x and z, as $px^2 + az^2x^2 = pa^2$. And by means of these three Equations you may find the Value of y, *i. e.* the Property of the Curve, whose Area bears the Relation assigned to the elliptical Area.

The three given Equations are, 1°. $a^2p - px^2 = av^2$, being to the given Curve: 2°. $px^2 + az^2x^2 = pa^2$, defining the Relation of the Abscisses. 3°. $2s + t - xv = 0$, containing the given Relation of the Areas, from which y is sought.

*Art. 108. of this Explication. Now since $\dot{t} = y\dot{z}$ or $y = \frac{\dot{t}}{\dot{z}}$ *, you must find the Ratio of $\dot{t}$ to $\dot{z}$ or $\dot{z} \times 1$ in Terms made up of z and known Quantities, which can always be done by means of the three given Equations. In order to which, from the second Equation $px^2 + az^2x^2 = paa$, take the Relation of the Fluxions, *viz.* $2px\dot{x} + 2az\dot{z}x^2 + 2az^2x\dot{x} = 0$, or $\dot{z} = \frac{-az^2\dot{x} - p\dot{x}}{azx}$, *i. e.* by supposing $\dot{x} = 1$ (for we may always suppose one of the flowing Quantities to flow uniformly) $\dot{z} = \frac{-az^2 - p}{azx}$: which Value of $\dot{z}$ you substitute for it in the Equation $y = \frac{\dot{t}}{\dot{z}}$, and it becomes $y = \frac{-azx\dot{t}}{az^2 + p}$, having one fluxionary Letter only: that you may get free of it likewise, from the third given Equation $2s + t - xv = 0$, take the Relation of the Fluxions, *viz.* $2\dot{s} + \dot{t} - \dot{x}v - x\dot{v} = 0$, whence

whence you have $\dot{t} = (\dot{x}v + x\dot{v} - 2\dot{s} = v + x\dot{v} - 2\dot{s} =$ (becaufe $\dot{s} = \dot{x}v = v$) $x\dot{v} - v$: but from the firft of the given Equations, *viz.* $a^2p - px^2 = av^2$, we have $-2px\dot{x} = 2av\dot{v}$ or $\dot{v} = \frac{-px}{av}$: fubftitute this for $\dot{v}$ in the Equation $\dot{t} = x\dot{v} - v$, and you have $\dot{t} = \frac{-px^2}{av} - v$, *i. e.* by inferting $\sqrt{\frac{a^2p - px^2}{a}}$ for v, from the firft of the three given Equations, $\dot{t} = \left(\frac{-px^2}{a\sqrt{\frac{a^2p - px^2}{a}}} - \sqrt{\frac{a^2p - px^2}{a}} = \right) - \frac{a^2p}{\sqrt{a^3p - apx^2}}$: wherefore, fince it was $y = \frac{-axx\dot{t}}{ax^2 + p}$, we fhall now have, by putting in $-\frac{a^2p}{\sqrt{a^3p - apx^2}}$ for $\dot{t}$, $y = \frac{a^3pxx}{\overline{p + ax^2}\sqrt{a^3p - apx^2}}$. Finally from the fecond of the three given Equations, *viz.* $px^2 + az^2x^2 = a^2p$, find the Value of x, which is $\sqrt{\frac{a^2p}{p + az^2}}$, infert it for x in the laft Equation, and reduce, and you'll obtain, at length, $y = \frac{a^2p}{p + az^2}$ the Equation defining the Property of the Curve fought: whofe Area is to the Area of the Ellipfe in the Relation propofed. And after the fame manner, if other Relations of the Abfciffes of x and z, different from the foregoing, were affumed, the Relation of the Areas continuing the fame, you may find other Curves, each of whofe Areas fhall be in the fame given Relation to the Area of the given Ellipfe. And by the fame way of proceeding, when any known Curve is affigned, you may find infinite other Curves, whofe Areas fhall have any given Relation to the Area of that Curve.

COR. 2.

227. The Truth of this Corollary appears thus. Since by the Hyp. $z^s = x$, thence by taking the Fluxions, you have $sz^{s-1}\dot{z} = \dot{x}$, *i. e.* by fubftituting $\frac{\eta}{\nu}$ for s, x for z^s and $x^{-\frac{\nu}{\eta}}$ for z^{-1} or $x^{\frac{-\nu+\eta}{\eta}}$ for z^{s-1}, which is deduced from the given Equation $z^s = x$) $\frac{\eta}{\nu}x^{\frac{-\nu+\eta}{\eta}}\dot{z} = \dot{x}$: hence you obtain the Ratio of the Fluxions of x and z, *viz.* $\dot{x} : \dot{z} :: \frac{\eta}{\nu}x^{\frac{-\nu+\eta}{\eta}} : 1$. But, by this Prop. thefe Curves have equal Areas, whofe Ordinates are reciprocally proportional to the Fluxions of their Abfciffes: wherefore, if to the three Quantities $\frac{\eta}{\nu}x^{\frac{-\nu+\eta}{\eta}}$, 1 and

$z^{\theta-1}\times\overline{e+fz^{\eta}+gz^{2\eta}+\&c.}|^{\lambda}$, you find a fourth Proportional, that shall be the Ordinate of a Curve, whose Area is equal to the Area of the Curve proposed, it's Abscifs being x: but a fourth Proportional to these three Quantities is $\frac{z^{\theta-1}\times\overline{e+fz^{\eta}+gz^{2\eta}+\&c.}|^{\lambda}}{\frac{\eta}{\nu}x^{\frac{-\nu+\eta}{\eta}}}$, where if you substitute x^{ν} for z^{η}, and $x^{\frac{\nu\theta-\nu}{\eta}}$ for $z^{\theta-1}$, from the Equation $z^{s}=x$ given at the Beginning, you shall have $\frac{\nu}{\eta}x^{\frac{\nu\theta-\eta}{\eta}}\times\overline{e+fx^{\nu}+gx^{2\nu}+\&c.}|^{\lambda}$, for the Ordinate of a Curve, which having x for it's Abscifs, has it's Area equal to the Area of the propos'd Curve, whose Abscifs being z has it's Ordinate $z^{\theta-1}\times\overline{e+fz^{\eta}+gz^{2\eta}+\&c.}|^{\lambda}$.

COR. 3.

Art. 43. of the Quadrature of Curves.

228. This Corollary appears thus: since by the Hyp. it is $x=z^{s}=z^{\frac{\eta}{\nu}}$, thence by Prop. 1. $\dot{x}=\frac{\eta}{\nu}z^{\frac{\eta-\nu}{\nu}}\dot{z}$: wherefore if you put $\dot{z}=1$, you'll have $\dot{x}=\frac{\eta}{\nu}z^{\frac{\eta-\nu}{\nu}}$: consequently, if to the three Terms $\frac{\eta}{\nu}z^{\frac{\eta-\nu}{\nu}}$, 1, and $z^{\theta-1}\times\overline{a+bz^{\eta}+cz^{2\eta}+\&c.}\times\overline{e+fz^{\eta}+gz^{2\eta}+\&c.}|^{\lambda}$, you find a fourth Proportional, that shall be the Ordinate of the Curve whose Abscifs is x, and it's Area equal to the Area of the Curve proposed: the fourth Proportional is $\frac{z^{\theta-1}\times\overline{a+bz^{\eta}+cz^{2\eta}+\&c.}\times\overline{e+fz^{\eta}+gz^{2\eta}+\&c.}|^{\lambda}}{\frac{\eta}{\nu}z^{\frac{\eta-\nu}{\nu}}}$, *i.e.* (by substituting x^{ν} for z^{η}, $x^{\frac{\nu\theta-\nu}{\eta}}$ for $z^{\theta-1}$ and $x^{\frac{\eta-\nu}{\eta}}$ for $z^{\frac{\eta-\nu}{\nu}}$, from the Equation $x=z^{\frac{\eta}{\nu}}$, and reducing) $\frac{\nu}{\eta}x^{\frac{\nu\theta-\eta}{\eta}}\times\overline{a+bx^{\nu}+cx^{2\nu}+\&c.}\times\overline{e+fx^{\nu}+gx^{2\nu}+\&c.}|^{\lambda}$.

COR. 4.

Art. 44. of the Quadrature of Curves.

224. Since the Relation of the Abscisses, and consequently of their Fluxions is the same as before; to the three Quantities $\frac{\eta}{\nu}z^{\frac{\eta-\nu}{\nu}}$, 1, and $z^{\theta-1}\times\overline{a+bz^{\eta}+cz^{2\eta}+\&c.}\times\overline{e+fz^{\eta}+gz^{2\eta}+\&c.}|^{\lambda}\times\overline{k+lz^{\eta}+mz^{2\eta}+\&c.}|^{\mu}$, take a fourth Proportional, substitute x^{ν} for z^{η}, &c. as in the last, and you'll obtain $\frac{\nu}{\eta}x^{\frac{\nu\theta-\eta}{\eta}}\times\overline{a+bx^{\nu}+cx^{2\nu}+\&c.}\times\overline{e+fx^{\nu}+gx^{2\nu}+\&c.}|^{\lambda}$

×

$\times \overline{k + lz^{\nu} + mz^{2\nu} + \&c.}|^{\mu}$ for the Ordinate of a Curve, whose Abscifs being $x = z^{\frac{n}{\nu}}$, shall have it's Area equal to that of the Curve proposed.

COR. 5.

230. Since $x = \frac{1}{z}$, thence $\dot{x} = -z^{-2}\dot{z}$: therefore, calling the Ordinate of the Curve, whose Abscifs is x, v, you'll have this Prop. $-z^{-2} : 1 :: z^{\theta-1} \times \overline{e + fz^{n} + gz^{2n} + \&c.}|^{\lambda} : v$, *i. e.* $v = -z^{\theta+1} \times \overline{e + fz^{n} + gz^{2n} + \&c.}|^{\lambda}$, or, by substituting x^{-n} for z^{n} and $x^{-\theta-1}$ for $z^{\theta+1}$ from the given Equation $x = z^{-1}$, $v = -\frac{1}{x^{\theta+1}} \times \overline{e + fx^{-n} + gx^{-2n} + \&c.}|^{\lambda}$: *i. e.* when the Quantity under the Vinculum is a Binomial, by dividing by $x^{n\lambda}$, and then multiplying by it, $v = -\frac{1}{x^{\theta+1+n\lambda}} \times \overline{f + ex^{n}}|^{\lambda}$: but if the Quantity under the Vinculum be a Trinomial, then by first dividing, and then multiplying by $x^{2n\lambda}$, it is $v = -\frac{1}{x^{\theta+1+2n\lambda}} \times \overline{g + fx + ex^{2n}}|^{\lambda}$: which are the Ordinates of Curves, having $x = \frac{1}{z}$ for their Abscifs, and their Areas equal to the Areas of the Curves whose Abscisses being z have their Ordinates $z^{\theta-1} \times \overline{e + fz^{n}}|^{\lambda}$, and $z^{\theta-1} \times \overline{e + fz^{n} + gz^{2n}}|^{\lambda}$ respectively. Art. 45. of the Quadrature of Curves.

COR. 6.

231. The Truth of this Corollary appears the same way: for since $\dot{z} : \dot{x} :: 1 : -z^{-2}$, you'll find that $z^{\theta-1} \times \overline{e + fz^{n} + gz^{2n}\ \&c.}|^{\lambda} \times \overline{k + lz^{n} + mz^{2n} + \&c.}|^{\mu} : \frac{1}{x^{\theta+1}} \times \overline{e + fx^{-n} + gx^{-2n} + \&c.}|^{\lambda} \times \overline{k + lx^{-n} + mx^{-2n} + \&c.}|^{\mu} :: -z^{-2} : 1$, by considering that $x = \frac{1}{z}$. And when the Quantities under the Vinculums are Binomials, the general Expression of the Ordinate $-\frac{1}{x^{\theta+1}} \times \overline{e + fx^{-n} + gx^{-2n} + \&c.}|^{\lambda} \times \overline{k + lx^{-n} + mx^{-2n} + \&c.}|^{\mu}$, by dividing the first Factor by $x^{n\lambda+n\mu}$; and multiplying the second Factor by $x^{n\lambda}$ and the last Factor by $x^{n\mu}$, which Operations compensate each other, becomes $-\frac{1}{x^{\theta+1+n\lambda+n\mu}} \times \overline{f + ex^{n}}|^{\lambda} \times \overline{l + kx^{n}}|^{\mu}$: and if there be three Members Art. 46. of the Quadrature of Curves.

 under

under the former Vinculum and two under the latter; by dividing the first Factor by $x^{2\eta\lambda+\eta\mu}$; and multiplying the second Factor by $x^{2\eta\lambda}$, and the last by $x^{\eta\mu}$, the Ordinate appears thus $-\frac{1}{x^{\theta+1+2\eta\lambda+\eta\mu}}\times \overline{g+fz^{\eta}+ex^{2\eta}}|^{\lambda}\times\overline{l+kz^{\eta}}|^{\mu}$. And so in other Cases.

Art. 47. of the Quadrature of Curves. 232. The Ordinates belonging to the Curves whose Abscis is x, in the two last Corollaries, come out negative in the Operation, whereby they are discovered: although the negative Sign be not prefixt by the Author: which arises from this, that the Fluxion of x is negative; or x diminishes as z increases, and contrarily: whence it happens, that, if the Area of the Curve having z for it's Abscis, lye upon the nearer Side of the Ordinate, and adjacent to the Abscis, the Area of the other Curve, having x for it's Abscis, and whose Area is equal to that of the former, will be situate along the Abscis produc'd upon the further Side of it's Ordinate, and contrarily, as is observed by him in this Art. For if the Area of the Curve, having z for it's Abscis, be adjacent to the Abscis, and so increase as the Abscis z increases; the Area of the other Curve, that it may increase at the same Time, must be situate beyond it's Ordinate, for the Increase of z is the Diminution of x. But if the Area of the Curve, whose Abscis is z, lye upon the further Side of the Ordinate, the equal Area of the other will lye upon the nearer Side of the Ordinate; for the same Reason. See Art. 225 *.

* Of this Explication.

COR. 7.

Art. 48. of the Quadrature of Curves. 233. The Truth of this Corollary appears thus. It is supposed that 1°. $s=\frac{\eta-\delta}{\eta}$. 2°. $x=\frac{1}{s}z^{s}$. 3°. $\lambda=\frac{\eta-\delta}{\alpha\delta+\beta\eta}$.

Find the Relation of the Fluxions x and z, from the second of these Equations, *viz.* $\dot{z}:\dot{x}::(1:z^{s-1}::z:z^{s}::)\ s^{\frac{1}{s}}x^{\frac{1}{s}}:sx$. Therefore, since the Areas of the Curves are equal by the Hyp. it is $s^{\frac{1}{s}}x^{\frac{1}{s}}:sx::v:y$; whence you have $y=s^{\frac{s-1}{s}}x^{\frac{s-1}{s}}v$: from thence $y^{\alpha}=s^{\frac{s\alpha-\alpha}{s}}x^{\frac{s\alpha-\alpha}{s}}v^{\alpha}$, and $y^{\eta}=s^{\frac{s\eta-\eta}{s}}x^{\frac{s\eta-\eta}{s}}v^{\eta}$; &c. Again $z^{\beta}=s^{\frac{\beta}{s}}x^{\frac{\beta}{s}}$, and $z^{\delta}=s^{\frac{\delta}{s}}x^{\frac{\delta}{s}}$, &c. Wherefore in the Equation to the given Curve, *viz.* $y^{\alpha}\times\overline{e+fy^{\eta}z^{\delta}+gy^{2\eta}z^{2\delta}+hy^{3\eta}z^{3\delta}+\&c.}=z^{\beta}\times\overline{k+ly^{\eta}z^{\delta}+my^{2\eta}z^{2\delta}+\&c.}$ by inserting in place of y^{α}, y^{η}; z^{β}, z^{δ}, these Values just now found, it will stand thus

$$s^{\frac{s\alpha-\alpha}{s}}x^{\frac{s\alpha-\alpha}{s}}v^{\alpha}\times\overline{e+fs^{\frac{s\eta-\eta}{s}}x^{\frac{s\eta-\eta}{s}}v^{\eta}s^{\frac{\delta}{s}}x^{\frac{\delta}{s}}+gs^{\frac{2s\eta-2\eta}{s}}x^{\frac{2s\eta-2\eta}{s}}v^{2\eta}s^{\frac{2\delta}{s}}x^{\frac{2\delta}{s}}+\&c.}$$

$=$

$= s^{\frac{\beta}{s}} x^{\frac{\beta}{s}} \times \overline{k + l s^{\frac{sn-n}{s}} x^{\frac{sn-n}{s}} v^n s^{\frac{\delta}{s}} x^{\frac{\delta}{s}} + m s^{\frac{2sn-2n}{s}} x^{\frac{2sn-2n}{s}} v^{2n} s^{\frac{2\delta}{s}} x^{\frac{2\delta}{s}} + \&c.}$

But $\frac{sn-n}{s} = -\frac{\delta}{s}$, from the firſt of the original Equations, wherefore, by throwing out repugnant Expreſſions, the Equation is $s^{\frac{s\alpha-\alpha}{s}} x^{\frac{s\alpha-\alpha}{s}} v^\alpha \times \overline{e + fv^n + gv^{2n} + \&c.} = s^{\frac{\beta}{s}} x^{\frac{\beta}{s}} \times \overline{k + lv^n + mv^{2n} + \&c.}$ Divide both Sides of the Equation by $s^{\frac{\beta}{s}} \times \overline{k + lv^n + mv^{2n} + \&c.}$ and moreover by $x^{\frac{s\alpha-\alpha}{s}}$, and you'll have it ſtanding thus $s^{\frac{s\alpha-\alpha-\beta}{s}} v^\alpha \times \overline{e + fv^n + gv^{2n} + \&c.} \times \overline{k + lv^n + mv^{2n} + \&c.}^{-1} = x^{\frac{\beta - s\alpha + \alpha}{s}}$. But $\frac{s\alpha-\alpha-\beta}{s} = -\frac{\alpha\delta+\beta n}{n-\delta} = -\frac{1}{\lambda}$, from the firſt and third of the original Equations. Wherefore by inſerting theſe Values, and raiſing both Sides of the Equation to the Power whoſe Exponent is λ, we ſhall have the Equation fully reduced, ſtanding thus $\frac{1}{s} v^{\alpha\lambda} \times \overline{e + fv^n + gv^{2n} + hv^{3n} + \&c.}|^{\lambda} \times \overline{k + lv^n + mv^{2n} + \&c.}^{-\lambda} = x$.

Which expreſſes the Relation betwixt the Abſciſs and Ordinate, in the Curve having x and v for it's Abſciſs and Ordinate, and that by an Equation not affected: and whoſe Area is equal to the Area of the Curve propos'd, whoſe Abſciſs and Ordinate are z and y. By means of which the Abſciſs x is determin'd, when the Ordinate v is given.

COR. 8.

234. In this Corollary it is ſuppoſed that, 1°. $s = \frac{n-\delta}{n}$. 2°. $x = \frac{1}{s} z^s$. 3°. $\mu = \frac{\alpha\delta+\beta n}{n-\delta}$. 4°. $\nu = \frac{\alpha\delta+\gamma n}{n-\delta}$. Art. 49. of the Quadrature of Curves.

Whence the Relation of the Ordinates y and v, is the ſame as in the preceding Corollary: ſince, by the ſecond Equation, the Relation of the Abſciſſes, and conſequently of their Fluxions, is the ſame: ſo that the Values of y^α, y^n; z^β, z^δ may be thence transferred. We have moreover in this Corollary $z^\gamma = s^{\frac{\gamma}{s}} x^{\frac{\gamma}{s}}$. Wherefore, by inſerting theſe Values in the given Equation $y^\alpha \times \overline{e + fy^n z^\delta + gy^{2n} z^{2\delta} + \&c.} = z^\beta \times \overline{k + ly^n z^\delta + my^{2n} z^{2\delta} + \&c.} + z^\gamma \times \overline{p + qy^n z^\delta + ry^{2n} z^{2\delta} + \&c.}$ we ſhall have $s^{\frac{s\alpha-\alpha}{s}} x^{\frac{s\alpha-\alpha}{s}} v^\alpha \times \overline{e + f s^{\frac{sn-n}{s}} x^{\frac{sn-n}{s}} v^n s^{\frac{\delta}{s}} x^{\frac{\delta}{s}} + \&c.} = s^{\frac{\beta}{s}} x^{\frac{\beta}{s}} \times \overline{k + l s^{\frac{sn-n}{s}} x^{\frac{sn-n}{s}} v^n s^{\frac{\delta}{s}} x^{\frac{\delta}{s}} + \&c.} + s^{\frac{\gamma}{s}} x^{\frac{\gamma}{s}} \times \overline{p + q s^{\frac{sn-n}{s}} x^{\frac{sn-n}{s}} v^n s^{\frac{\delta}{s}} x^{\frac{\delta}{s}} + \&c.}$

But,

But, as in the last Corollary, $\frac{s\eta - \eta + \delta}{s} = 0$: and moreover, in this, $\frac{\beta - s\alpha + \alpha}{s} = \mu$, and $\frac{\gamma - s\alpha + \alpha}{s} = \nu$: as appears from the first and fourth original Equations compared, therefore, by throwing out Repugnancies, dividing both Sides of the Equation by $s^{\frac{s\alpha - \alpha}{s}} x^{\frac{s\alpha - \alpha}{s}}$, and substituting μ and ν for their Values, we shall have $v^{z} \times \overline{e + fv^{\eta} + gv^{2\eta} + \&c.} = s^{\mu}x^{\mu} \times \overline{k + lv^{\eta} + mv^{2\eta} + \&c.} + s^{\nu}x^{\nu} \times \overline{p + qv^{\eta} + rv^{2\eta} + \&c.}$ The Equation sought, expressing the Relation of the Abscis and Ordinate x and v, belonging to a Curve whose Area is equal to the Area of the other Curve proposed, whose Abscis and Ordinate are z and y: and that less affected.

COR. 9.

Art. 50. of the Quadrature of Curves.

235. This Corollary is demonstrated after the same Manner as Corollaries fifth and sixth. Here it is supposed that, 1°. $\theta = \lambda\nu$. 2°. $x = \overline{ez^{\nu} + fz^{\nu+\eta} + gz^{\nu+2\eta} + \&c.}|^{\pi}$. 3°. $\sigma = \frac{\tau}{\pi}$. 4°. $\vartheta = \frac{\lambda - \pi}{\pi}$. From the second, by Prop. 1, we have the Relation of the Fluxions of x and z, which is expressed by this Proposition $\dot{x} : \dot{z} :: \pi \times \overline{ez^{\nu} + fz^{\nu+\eta} + gz^{\nu+2\eta} + \&c}|^{\pi-1} \times \overline{\nu e z^{\nu-1} + \overline{\nu+\eta} \times fz^{\nu+\eta-1} + \overline{\nu+2\eta} \times gz^{\nu+2\eta-1} + \&c}$: 1 : but $\dot{x} : \dot{z} :: y : v$, by this Prop. Therefore the two last Ratios of these two Proportions are equal: consequently, if you reduce them into an Equation, and in place of y, insert it's Value from the Corollary itself, you'll have

$$v = \frac{\pi z^{\theta-1} \times \overline{\nu e + \overline{\nu+\eta} \times fz^{\eta} + \overline{\nu+2\eta} \times gz^{2\eta} + \&c.} \times \overline{e + fz^{\eta} + gz^{2\eta} + \&c.}|^{\lambda-1} \times \overline{a + b \times \overline{ez^{\nu} + fz^{\nu+\eta} + gz^{\nu+2\eta} + \&c}|^{\tau}}|^{\omega}}{\pi \times \overline{ez^{\nu} + fz^{\nu+\eta} + gz^{\nu+2\eta} + \&c.}|^{\pi-1} \times \overline{\nu e z^{\nu-1} + \overline{\nu+\eta} \times fz^{\nu+\eta-1} + \overline{\nu+2\eta} \times gz^{\nu+2\eta-1} + \&c.}}$$

Which, by dividing the first Factor of the Numerator by the first Factor of the Denominator, the second by the third, and the third by the second, will stand thus $v = z^{\theta-1} \times z^{-\nu+1} \times z^{\nu-\pi\nu} \times \overline{e + fz^{\eta} + gz^{2\eta}}|^{\lambda-\pi} \times \overline{a + b \times \overline{ez^{\nu} + fz^{\nu+\eta} + gz^{\nu+2\eta} + \&c.}|^{\tau}}|^{\omega}$.

That is by multiplying the Factors, and inserting x for $\overline{ez^{\nu} + fz^{\nu+\eta} + gz^{\nu+2\eta} + \&c.}|^{\pi}$ $v = z^{\theta-\pi\nu} \times z^{\pi\nu-\lambda\nu} x^{\frac{\lambda-\pi}{\pi}} \times \overline{a + bx^{\frac{\tau}{\pi}}}|^{\omega}$.

That is, (because $\lambda\nu = \theta$ and $\frac{\tau}{\pi} = \sigma$, and $\frac{\lambda - \pi}{\pi} = \vartheta$ from the original Equations) $v = x^{\vartheta} \times \overline{a + bx^{\sigma}}|^{\omega}$.

Which

Which therefore exhibits the Property of a Curve having x for it's Abſciſs, and equal in Area to the Curve propoſed, whoſe Abſciſs is z: the Relation of x and z being that mentioned above.

236. Sir *Iſaac* deſires to remark three different Caſes, or Suppoſitions, upon which the former of the two Ordinates in this Corollary becomes more ſimple. The firſt is when $\lambda = 1$: for then the Ordinate is $\overline{\pi z^{\theta-1} \times e + \overline{\nu + \eta} \times fz^{\eta} + \overline{\nu + 2\eta} \times gz^{2\eta} + \&c.} \times \overline{a + b \times \overline{ez^{\nu} + fz^{\nu+\eta} + \&c.}\rceil^{\tau}}\rceil^{\omega}$.

The ſecond Suppoſition is, if $\tau = 1$, and the Root denoted by the Index ω be capable of being extracted: which would evidently make the Expreſſion more ſimple.

The third Caſe mentioned, is when $\omega = -1$ and $\lambda = 1 = \tau = \sigma = \pi$: for then the former Ordinate will put on this Form $\frac{\overline{z^{\theta-1} \times e + \overline{\theta+\eta} \times fz^{\eta} + \overline{\theta+2\eta} \times gz^{2\eta} + \&c.}}{a + b \times \overline{ez^{\theta} + fz^{\theta+\eta} + gz^{\theta+2\eta} + \&c.}}$: and the latter this $\frac{1}{a + bx}$.

And beſides theſe three Caſes, there are a Variety of others, upon Suppoſition of which, the propoſed Ordinate will become more ſimple.

COR. 10.

237. By ſome means or another an Error has crept into this Corollary, both in the Edition publiſhed by the Author himſelf, at the End of his Optics; and that publiſhed by Mr. *Jones*, with other original Pieces of Sir *Iſaac*'s.

The moſt ſimple Correction is by altering the Sign of the Index υ, belonging to S in the laſt Factor of the Expreſſion of the Ordinate propoſed: ſo that, inſtead of $\overline{\pi S^{r} + \varphi R^{s}} \times R^{\lambda-1} S^{\mu-1} \times \overline{aS^{\upsilon} + bR^{\tau}}\rceil^{\omega}$, it be made $\overline{\pi S^{r} + \varphi R^{s}} \times R^{\lambda-1} S^{\mu-1} \times \overline{aS^{-\nu} R^{\tau}}\rceil^{\omega}$, and the other things remain the ſame. Or the Correction may be made thus: ſuppoſing the Expreſſion of the given Ordinate to be made the ſame, inſtead of $\frac{\mu - \upsilon\omega}{\lambda} = \frac{\upsilon}{\tau} = \frac{\varphi}{\pi}$, put $\frac{\mu + \upsilon\omega}{\lambda} = -\frac{\upsilon}{\tau} = \frac{\varphi}{\pi}$, that is, you change the Sign of ν in theſe Equations, into it's oppoſite. But without one or other of theſe Corrections, the Curve whoſe Ordinate is propos'd, cannot be transform'd into the other that is mentioned. Therefore I ſhall firſt demonſtrate the Truth of the Corollary ſuppoſing the Corrections, which I have mentioned to be made: and then ſhew that the Corollary will not hold as the things are expreſſed in the printed Copies.

238. For $ez^{\nu} + fz^{\nu+\eta} + gz^{\nu+2\eta} +$ &c. $\nu ez^{\nu-1} + \overline{\nu+\eta} \times fz^{\nu+\eta-1} + \overline{\nu+2\eta} \times gz^{\nu+2\eta-1} +$ &c. $k + lz^{\eta} + mz^{2\eta} +$ &c. and $\eta lz^{\eta-1} + 2\eta mz^{2\eta-1} +$ &c. write R, $\dot{r}$, S, and $\dot{s}$ respectively, and every Curve whose Ordinate is $\overline{\pi S\dot{r} + \varphi R\dot{s}} \times R^{\lambda-1}S^{\mu-1} \times \overline{aS^{-\nu} + bR^{\tau}}^{\omega}$, if it be $\frac{\mu - \nu\omega}{\lambda} = \frac{\nu}{\tau} = \frac{\varphi}{\pi}$, $\frac{\tau}{\pi} = \sigma$, $\frac{\lambda - \pi}{\pi} = \vartheta$, and $R^{\pi}S^{\varphi} = x$, passes into another one equal to it in Area, whose Ordinate is $x^{\vartheta} \times \overline{a + bx^{\sigma}}^{\omega}$.

For since $x = R^{\pi}S^{\varphi}$, we have by Prop. 1, $\dot{x} = \pi R^{\pi-1}\dot{R}S^{\varphi} + \varphi R^{\pi}S^{\varphi-1}\dot{S}$: but it's evident that $\dot{r} = \frac{\dot{R}}{\dot{z}}$ and $\dot{s} = \frac{\dot{S}}{\dot{z}}$; or $\dot{R} = \dot{r}\dot{z}$ and $\dot{S} = \dot{s}\dot{z}$: by inserting of which Values of $\dot{R}$ and $\dot{S}$ in the preceding Equation, you'll have $\dot{x} = \pi R^{\pi-1}\dot{r}\dot{z}S^{\varphi} + \varphi R^{\pi}S^{\varphi-1}\dot{s}\dot{z}$, whence $\dot{x} : \dot{z} : \pi R^{\pi-1}\dot{r}S^{\varphi} + \varphi R^{\pi}S^{\varphi-1}\dot{s} : 1$. Wherefore by this Prop. if the given Ordinate $\overline{\pi S\dot{r} + \varphi R\dot{s}} \times R^{\lambda-1}S^{\mu-1} \times \overline{aS^{-\nu} + bR^{\tau}}^{\omega}$ be divided by $\pi R^{\pi-1}\dot{r}S^{\varphi} + \varphi R^{\pi}S^{\varphi-1}\dot{s}$, we shall have $R^{\lambda-\pi}S^{\mu-\varphi} \times \overline{aS^{-\nu} + bR^{\tau}}^{\omega}$, for the Ordinate of another Curve, equal in Area to the former, and having x for it's Abscis. But, because $x = R^{\pi}S^{\varphi}$ or $R^{\tau} = x^{\frac{\tau}{\pi}}S^{-\frac{\varphi\tau}{\pi}}$, hence the Ordinate may stand thus, $R^{\lambda-\pi}S^{\mu-\varphi} \times \overline{aS^{-\nu} + bx^{\frac{\tau}{\pi}}S^{-\frac{\varphi\tau}{\pi}}}^{\omega}$, or, because $\frac{\nu}{\tau} = \frac{\varphi}{\pi}$, *i. e.* $S^{-\frac{\varphi\tau}{\pi}} = S^{-\nu}$, it will stand thus $R^{\lambda-\pi}S^{\mu-\varphi-\nu\omega} \times \overline{a + bx^{\frac{\tau}{\pi}}}^{\omega}$. Where if, instead of $\mu - \nu\omega$ you put in $\frac{\lambda\varphi}{\pi}$; and instead of $\frac{\tau}{\pi}$, σ, deduced from the original Equations, you'll have $R^{\lambda-\pi}S^{\frac{\lambda\varphi - \pi\varphi}{\pi}} \times \overline{a + bx^{\sigma}}^{\omega}$, that is, because $R^{\pi}S^{\varphi} = x$, and $\frac{\lambda - \pi}{\pi} = \vartheta$, $x^{\vartheta} \times \overline{a + bx^{\sigma}}^{\omega}$: which is that mentioned by our Author, and is the Ordinate of a Curve equal in Area to the Curve proposed; and hav- x for it's Abscis.

239. Again, if the Values of R, $\dot{r}$, S and $\dot{s}$ be the same as formerly, and the Ordinate of a Curve be $\overline{\pi S\dot{r} + \varphi R\dot{s}} \times R^{\lambda-1}S^{\mu-1} \times \overline{aS + bR}^{\omega}$; and moreover $\frac{\mu + \nu\omega}{\lambda} = \frac{-\nu}{\tau} = \frac{\varphi}{\pi}$, $\frac{\tau}{\pi} = \sigma$, $\frac{\lambda - \pi}{\pi} = \vartheta$ $R^{\pi}S^{\varphi} = x$, the Curve will pass into another equal to it, whose Ordinate is $x^{\vartheta} \times \overline{a + bx^{\sigma}}^{\omega}$.

For

For we prove, as formerly, that the Ordinate of the new Curve is $R^{\lambda-\pi}S^{\mu-\varphi} \times \overline{aS^{v}+bx^{\frac{\tau}{\pi}}S^{-\frac{\varphi\tau}{\pi}}}|^{\omega}$, *i. e.* becauſe $-\frac{v}{\tau}=\frac{\varphi}{\pi}$, or $-\frac{\varphi\tau}{\pi}=v$, $R^{\lambda-\pi}S^{\mu-\varphi+v\omega} \times \overline{a+bx^{\frac{\tau}{\pi}}}|^{\omega}$; which, by inſerting $\frac{\lambda\varphi}{\pi}$ for $\mu+v\omega$ and σ for $\frac{\tau}{\pi}$, from the original Equations, ſtands thus, $R^{\lambda-\pi}S^{\frac{\lambda\varphi-\pi\varphi}{\pi}} \times \overline{a+bx^{\sigma}}|^{\omega}$, that is $x^{\vartheta} \times \overline{a+bx^{\sigma}}|^{\omega}$ as formerly.

240. But if the Signs of all the Quantities be retained the ſame as in the printed Copies, then it may be demonſtrated, as above, that the Ordinate of the Curve will be $R^{\lambda-\pi}S^{\mu-\varphi} \times \overline{aS^{v}+bR^{\tau}}|^{\omega}$ or $R^{\lambda-\pi}S^{\mu-\varphi} \times \overline{aS^{v}+bx^{\frac{\tau}{\pi}}S^{-\frac{\varphi\tau}{\pi}}}|^{\omega}$, *i. e.* $R^{\lambda-\pi}S^{\mu-\varphi} \times \overline{aS^{v}+bx^{\sigma}S^{-v}}|^{\omega}$ or $R^{\lambda-\pi}S^{\frac{\lambda\varphi-\pi\varphi}{\pi}} \times \overline{aS^{2v}+bx^{\sigma}}|^{\omega} = x^{\vartheta} \times \overline{aS^{2v}+bx^{\sigma}}|^{\omega}$: whereas it ought to be $x^{\vartheta} \times \overline{a+bx^{\sigma}}|^{\omega}$.

241. Our Author takes notice of certain Caſes, among many others, in which the former of the two Ordinates becomes more ſimple.

SECT. IX.

Containing Notes on Art. 52——64.

PROP. 10. and it's Corollaries explained and illuſtrated.

242. THE Deſign of this Propoſition is to diſcover the moſt ſimple Figures, with which any propoſed Curve may be geometrically compared: which is done by the Help of the foregoing Propoſitions. When any propoſed Curve is capable of being compared with a rectilinear Figure, ſo as the Relation betwixt them may be exactly determined, then ſuch a Curve is ſtrictly *quadrable:* becauſe we can find a Square equal to it by Prop. 14. B. 2d. Elem. When this can't be done, the Curve, properly ſpeaking, is not quadrable: yet a Curve may be ſuch, that, though it's Area be not capable of a geometrical Compariſon with a right-lined Figure, it may be compared with the Area of another Curve, or Curves more ſimple than it: by which means we may have a more clear and diſtinct Conception of it. Thus if a Curve be ſuch, that it is capable of being compared geometrically with a Circle, one ſhall have a more diſtinct and ſatisfactory Notion of it upon that very account: becauſe the Circle is a Figure ſo fami-

liar and well underſtood in Geometry, and ſo eaſily deſcribed. So likewiſe with reſpect to ſuch Figures as may be compared with the Ellipſe and Hyperbola, they will be better conceived of, upon that very account, than if they were not capable of ſuch Compariſon, and their Areas more eaſily exhibited. So that, conſidering how well the Conic Sections are known, any Curve that admits of a geometrical Compariſon with them, may, in ſome Senſe, be accounted *quadrable.* Accordingly our Author in the ſecond of the two following Tables, has given us a Catalogue of the more ſimple Kind of Curves that admit of a geometrical Compariſon with the Ellipſe (including the Circle) and Hyperbola: even as in the firſt, we have a Catalogue of the more ſimple Curves, that are ſtrictly quadrable.

CASE 1.

243. This Caſe is plain: every Curve of this Sort being quadrable. See Art. 52. and 143 *.

* Of this Explication.

CASE 2.

244. If $az^{\theta-1} \times \overline{e + fz^{\eta} + gz^{2\eta} + \&c.}|^{\lambda-1}$ be the Ordinate of a Curve, the Curve may admit of a Compariſon with a rectilinear Figure or not.

If it do, you'll have it's Area exactly by Prop. 5. and that you may know when this will happen, in the Caſe of the more ſimple Curves, belonging to this general Claſs, and have the Canons at hand for determining theſe Areas, conſult Art. 147—161 *.

* Of this Explication.

245. But if the propoſed Curve cannot be compared with a right-lin'd Figure, the next thing is to find out the moſt ſimple Curve or Curves, with which it may be compared, whether it be Conic Section or any others. To which purpoſe, you firſt make uſe of Cor. 2[d]. Prop. 9. which ſee. By it, if you aſſume ν any Quantity you pleaſe, and put $x = z^{\frac{\eta}{\nu}}$, the Curve whoſe Ordinate is $az^{\theta-1} \times \overline{e + fz^{\eta} + gz^{2\eta} + \&c.}|^{\lambda-1}$ may be changed into another of an equal Area, whoſe Ordinate is $\frac{\nu}{\eta} ax^{\frac{\nu\theta-\eta}{\eta}} \times \overline{e + fx^{\nu} + gx^{2\nu} + \&c.}|^{\lambda-1}$: and therefore if you put $\nu = 1$, the Curve propoſed will be changed into one more ſimple whoſe Ordinate is $\frac{a}{\eta} x^{\frac{\theta}{\eta}-1} \times \overline{e + fx + gx^{2} + \&c.}|^{\lambda-1}$, where the Indexes of x under the Vinculum, go on in the natural Order of the Numbers.

Ex.

Ex. Let $\frac{4z^2}{2+3z^3} = 4z^{3-1} \times \overline{2+3z^3}|^{0-1}$ be the Ordinate of a Curve: I perceive it cannot be compared with a rectilinear Figure, by Art. 139, or 149*. Wherefore, by comparing this Ordinate with the general Ordinate $az^{\theta-1} \times \overline{e+fz^\eta + \&c.}|^{\lambda-1}$, I find $\theta = 3 . \eta = 3 . \lambda = 0 . a = 4 . e = 2 . f = 3$: therefore the Curve proposed is converted into another equal to it, whose Ordinate is $\frac{4}{3} \times \overline{2+3x}|^{-1}$ or $\frac{4}{6+9x}$, whose Abscis is $x = z^3$: and which Ordinate belongs to the rectangular Hyperbola, as it is applied to the Assymptote: and therefore the Area of the Curve propos'd, whose Ordinate is $\frac{4z^2}{2+3z^3}$, is reducible to an hyperbolical Area. * Of this Explication.

The Comparison is thus. Reduce the Ordinate $\frac{4}{6+9x}$ to $\frac{\frac{4}{9}}{\frac{2}{3}+x}$, where the numeral Coefficient of x is Unity: wherefore, having drawn the right Line ABN for one of the Assymptotes, (see Fig. 1. Tab. 2. p. 25.) in it take $A\alpha = \frac{2}{3}$, (linear Unity being suppos'd to be already determined) and $\alpha B = x$: likewise having drawn αG perpendicular to AB, and equal to $\frac{2}{3}$: if with the Center A and Assymptote ABN you describe a rectangular Hyperbola GDS through the Point G; and then draw the perpendicular Ordinate BD, the Area αBDG shall be equal to the Area of the Curve proposed. For from the Property of the Hyperbola $AB = \frac{2}{3} + x : A\alpha = \frac{2}{3} :: \alpha G = \frac{2}{3} : BD$, which therefore is equal to $\frac{\frac{4}{9}}{\frac{2}{3}+x}$ or $\frac{4}{6+9x}$: so that αBDG is equal to the Area of the Curve proposed: for that is the Area belonging to the Ordinate BD and Abscis $\alpha B = x$ *. *Art. 190. of this Explication.

N. B. It would have come to the same, if you had expressed the Ordinate $\frac{4}{6+9x}$ thus $\frac{4}{9} \times \frac{1}{\frac{2}{3}+x}$; only in that Case you must take $\alpha G = \frac{3}{2}$, and then the Area of the Curve will be to the hyperbolical Area αBDG as 4 to 9.

246. But if the given Ordinate $az^{\theta-1} \times \overline{e+fz^\eta + gz^{2\eta} + \&c.}|^{\lambda-1}$ being reduced to the Ordinate $\frac{a}{\eta}x^{\frac{\theta}{\eta}-1} \times \overline{e+fx+gx^2+\&c.}|^{\lambda-1}$, this last one is not yet so simple as may be, you must then diminish the Indexes $\frac{\theta}{\eta} - 1$ and $\lambda - 1$ by Unity always, by Prop. 7. until they be as low as possible, and so you shall have the most simple Figures, which can be discovered by this Method, with which the Curve proposed may be geometrically compared.

Ex. 1. Let the Ordinate $\frac{dz^{3\eta-1}}{e+fz^\eta}$ or $dz^{3\eta-1}\times\overline{e+fz^\eta}|^{0-1}$ be proposed. By putting $x=z^\eta$, I reduce it, by the Method already shewn, to this other Ordinate $\frac{d}{\eta}x^2\times\overline{e+fx}|^{-1}$ belonging to a Curve of an equal Area. Then, by rejecting the Units from the Index of x without the Vinculum, by Prop. 7. Case 1, I reduce the Ordinate further, *viz.* thus: call the Area belonging to the Curve proposed, t, and consequently the Area belonging to the Ordinate $x^2\times\overline{e+fx}|^{-1}$, $\frac{\eta t}{d}$, you'll find, by applying the general Theorem at Art. 203 *, *viz.* $A=\frac{z^\theta R^\lambda-\overline{\theta+\lambda\eta}\times fB}{\theta e}$ that $\frac{dx^2-2\eta ft}{2ed}$ is the Area of a Curve whose Ordinate is $x\times\overline{e+fx}|^{-1}$. Again, by repeating the Application of the same Theorem, you'll find $\frac{2edx-dfx^2+2\eta fft}{2ee}$ the Area belonging to the Ordinate $\frac{d}{e+fx}$, which is to the Hyperbola as before. Therefore if we call this hyperbolical Area, s, we will have $\frac{2dex-dfx^2+2\eta fft}{2e^2}=s$; whence, by proper Reduction, $t=\left(\frac{2e^2s-2edx+dfx^2}{2\eta f^2}=\right)\frac{e^2s}{\eta f^2}-\frac{dex^\eta}{\eta f^2}+\frac{dx^{2\eta}}{2\eta f}$, and therefore, if the hyperbolical Area s be given, the Area of the Curve proposed is given likewise.

* Of this Explication.

Ex. 2. Let the Ordinate $3z^3\sqrt{1+4z^2+3z^4}$ be proposed: which, reduced to Form, is $3z^{4-1}\times\overline{1+4z^2+3z^4}|^{\frac{3}{2}-1}$. By Cor. 2. Prop. 9, if you put $z^2=x$, you'll have $\frac{3}{2}x\sqrt{1+4x+3x^2}$ for the Ordinate of another more simple but equal Curve.

This done, I next reject Unity from the Index of x without the Vinculum, twice. To do which, bring it to this Form $\frac{3}{2}x^{2-1}\times\overline{1+4x+3x^2}|^{\frac{3}{2}-1}$, and call the Area belonging to it, t; that is the Area belonging to the Ordinate $x^{2-1}\times\overline{1+4x+3x^2}|^{\frac{3}{2}-1}$ is $\frac{2t}{3}$. Now suppose s and σ denote the Areas belonging to the Ordinates $\sqrt{1+4x+3x^2}$ and $x^{-1}\sqrt{1+4x+3x^2}$: then, here are three Ordinates $x^{0-1}\times\overline{1+4x+3x^2}|^{\frac{3}{2}-1}$, and $x^{1-1}\times\overline{1+4x+3x^2}|^{\frac{3}{2}-1}$ and $x^{2-1}\times\overline{1+4x+3x^2}|^{\frac{3}{2}-1}$, whose Areas are denoted by σ, s and $\frac{2t}{3}$ respectively. Therefore, by applying the Theor. in Art. 206 *, *viz.* $C=\frac{z^\theta R^\lambda-\theta eA-\overline{\theta+\lambda\eta}\times fB}{\overline{\theta+2\lambda\eta}\times g}$, you'll have $\theta=0$. $\eta=1$. $\lambda=\frac{3}{2}$. $e=1$. $f=4$

* Of this Explication.

$f=4 \cdot g=3 \cdot A=\sigma \cdot B=s \cdot C=\frac{2t}{3} \cdot z=x$. Whence $\frac{2t}{3}=\frac{\overline{1+4x+3x^2}|^{\frac{3}{2}}-6s}{9}$; for since $\theta=0$, σ goes out. Wherefore if you multiply by $\frac{1}{2}$ you have $t=\frac{\overline{1+4x+3x^2}|^{\frac{3}{2}}}{6}-s$ the Area of the Curve proposed: so that when the Area s belonging to the Ordinate $\sqrt{1+4x+3x^2}$ is given, the Area t will likewise be given. Now the Ordinate $\sqrt{1+4x+3x^2}$ is an Ordinate to the Hyperbola: which is thus constructed. (See Fig. 3^d. Tab. 2^d. p. 25.) With the determinate Axis $Ka=\frac{2}{3}$ and Latus Rectum 2, describe the Hyperbola aGDS, having A for it's Center: take $a\alpha=Aa$, $\alpha B=x=z^2$, and draw the Ordinates αG, BD: then you'll have $BD=\sqrt{1+4x+3x^2}$ and $\alpha BDG=s$, and consequently $\frac{1}{6}BD^3-\alpha BDG=t$ the Area of the Curve whose Ordinate was proposed, *viz.* $3z^3\sqrt{1+4z^2+3z^4}$: for, from the Property of the Hyperbola, putting L for the Latus Rectum, it is $\overline{Ka+aB}\times aB : BD^q :: Ka : L$, that is in the present Case (where we have $Ka=\frac{2}{3}$, $aB=\frac{1}{3}+x$ and $L=2$) $\overline{1+x}\times\overline{\frac{1}{3}+x} : BD^q :: \frac{2}{3} : 2$: whence $BD=\sqrt{1+4x+3x^2}$: and the initial Limit of the Area being αG, as may be collected from Sect. 5. concerning the Position and Limits of curvilinear Areas compared with Prop. 5. it must be $\alpha BDG=s$: and therefore $\frac{1}{6}BD^3-\alpha BDG$ is the Area of the Curve proposed.

247. But although in the last Example, the Area of a trinomial Curve is compared geometrically with the Area of one Curve only, which is more simple than itself: yet it most frequently happens, that the Areas of two Curves must be supposed to be given, that you may have the Area of a trinomial Curve. See Art. 209 *. Schol. 1.

* Of this Explication.

Thus if $\frac{\sqrt{1+4z^2+3z^4}}{3}$ was the Ordinate of a Curve proposed, in order to find the most simple Curves with which it may be compared, you will find that it requires two Conic Sections to it's Quadrature. The Method of proceeding is thus. The proposed Ordinate $z^{0-1}\times\overline{1+4z^2+3z^4}|^{\frac{1}{2}}$ by putting $z^2=x$, according to Cor. 2. Prop. 9, is changed into the Ordinate $\frac{1}{2}x^{-1}\times\overline{1+4x+3x^2}|^{\frac{1}{2}}$: where $\overline{1+4x+3x^2}|^{\frac{1}{2}}$ is the Ordinate of an Hyperbola as was shewn formerly: so that if we should call the Area of the Curve proposed, t; and the Area of the Hyperbola, s, we would have $2t$ and s, for the Areas of two Curves

Curves whose Ordinates are $x^{-1} \times \overline{1+4x+3x^2}|^{\frac{1}{2}}$ and $x^0 \times \overline{1+4x+3x^2}|^{\frac{1}{2}}$: from one of which being given, the other cannot be found: as you will find by trial, by applying any of the Canons contain'd in Art. 206, 207, or 208 *. Wherefore, as our Author next directs, I transform the Ordinate $x^{0-1} \times \overline{1+4x+3x^2}|^{\frac{1}{2}}$ into this other $-\frac{2}{\xi^2} \times \overline{3+4\xi+\xi^2}|^{\frac{1}{2}}$, which belongs to an equal Curve by Cor. 5. Prop. 9, by putting $\frac{1}{x} = \xi$: where the radical Part $\overline{3+4\xi+\xi^2}|^{\frac{1}{2}}$ is the Ordinate of an Hyperbola. Suppose you call the hyperbolical Area corresponding to the Ordinate $\overline{3+4\xi+\xi^2}|^{\frac{1}{2}}$, σ: then by means of the two Areas s and σ you may find t the Area sought. For by Cor. 5. Prop. 9, the Ordinate $\overline{1+4x+3x^2}|^{\frac{1}{2}}$ is transform'd into the Ordinate $-\frac{1}{\xi^3} \times \overline{3+4\xi+\xi^2}|^{\frac{1}{2}}$, whose corresponding Area is s: wherefore there are three Ordinates, *viz.*

* Of this Explication.

1°. $\xi^{-2-1} \times \overline{3+4\xi+\xi^2}|^{\frac{3}{2}-1}$. 2°. $\xi^{-1-1} \times \overline{3+4\xi+\xi^2}|^{\frac{3}{2}-1}$. 3°. $\xi^{1-1} \times \overline{3+4\xi+\xi^2}|^{\frac{3}{2}-1}$,

all belonging to the same Form of Curves; whose corresponding Areas are $-s$, $-2t$ and σ, two whereof being given the other is given likewise by Prop. 7. For by assuming the Areas $-s$ and $-2t$ belonging to the Ordinates $\xi^{-2-1} \times \overline{3+4\xi+\xi^2}|^{\frac{3}{2}-1}$ and $\xi^{-1-1} \times \overline{3+4\xi+\xi^2}|^{\frac{3}{2}-1}$, you'll find, by applying the Canon Art. 206 *, *viz.* $C = \frac{x^{\theta}R^{\lambda} - \theta bA - \overline{\theta+\lambda\eta} \times fB}{\overline{\theta+2\lambda\eta} \times g}$, that $\xi^{-2} \times \overline{3+4\xi+\xi^2}|^{\frac{3}{2}} - 6s - 4t$, is the Area of the Curve next in order, *viz.* that whose Ordinate is $\xi^{0-1} \times \overline{3+4\xi+\xi^2}|^{\frac{3}{2}-1}$. Renew the Application of the same Canon to find the Relation of the Areas belonging to the three Ordinates $\xi^{-1-1} \times \overline{3+4\xi+\xi^2}|^{\frac{3}{2}-1}$, $\xi^{0-1} \times \overline{3+4\xi+\xi^2}|^{\frac{3}{2}-1}$ and $\xi^{1-1} \times \overline{3+4\xi+\xi^2}|^{\frac{3}{2}-1}$: which Areas are $-2t$, $\xi^{-2} \times \overline{3+4\xi+\xi^2}|^{\frac{3}{2}} - 6s - 4t$, and σ, and you will obtain this Equation $\sigma = \frac{\xi^{-1} \times \overline{3+4\xi+\xi^2}|^{\frac{3}{2}} - 2\xi^{-2} \times \overline{3+4\xi+\xi^2}|^{\frac{3}{2}} + 2t + 12s}{2}$, that is, by reducing, $t = \sigma - 6s + \overline{3\xi^{-2} + \frac{5}{2}\xi^{-1} - 1 - \frac{1}{2}\xi} \times \overline{3+4\xi+\xi^2}|^{\frac{1}{2}}$, which, by substituting Υ for $\overline{3+4\xi+\xi^2}|^{\frac{1}{2}}$, and v for $\xi^{-1} \times \overline{3+4\xi+\xi^2}|^{\frac{1}{2}} = \overline{1+4x+3x^2}|^{\frac{1}{2}}$, will give $t = \sigma - 6s + 3xv + \frac{5}{2}v - \Upsilon - \frac{1}{2}\xi\Upsilon$, the Area sought. And the Description of the Conic Sections may be easily gathered from what was said formerly.

* Of this Explication.

248. Now,

248. Now, after you have found out the most simple Curves with which any proposed Curve may be compared, by the preceding Methods: you may sometimes find Curves yet more simple, provided the former ones can be so compared and joined together, by the Help of Prop. 3, as to constitute a new Curve, either of the Form mentioned in Cor. 9; or of that mentioned Cor. 10. Prop. 9: which will sometimes happen.

CASE 3.

249. When any proposed Ordinate is of this Form $z^{\theta-1} \times \overline{a + bz^{n} + cz^{2n} + \&c.} \times \overline{e + fz^{n} + gz^{2n} + \&c.}|^{\lambda-1}$: and the Curve cannot be squared by Prop. 5, then you are to distinguish it's Ordinate into it's constituent Parts, which are to be considered as so many distinct Ordinates: every one of which must be handled, as the Ordinate in Case 2: and having by that means discovered the Areas belonging to these several Ordinates, join them all together by their proper Signs, and you'll have the Area required. See Art. 165 *. Yet it may be proper here to observe, in order to prevent Mistakes, that such a Curve may be quadrable, although all the Parts of which the Ordinate is compos'd are not quadrable, as appears by comparing Art. 147 with Art. 162 and 163 *: such is the Ordinate $\frac{2z-aa}{\sqrt{z^2-aaz}}$.

* Of this Explication.

* Of this Explication.

CASE 4.

250. This Case respects Curves of the Form mentioned in Prop. 4th and 6th: which are to be treated much after the same Manner as those mentioned in Case 2d: according as our Author directs.

CASE 5.

251. The Meaning of what is affirmed by our Author in this Case, together with the Reason of it, appears from what has been said formerly in Case 3d, and Art. 164 and 165 *: and will be further illustrated by what is said in the Cor. succeeding.

* Of this Explication.

COR. 1.

252. It is evident from what has been said, that, if the Relation of the Abscis and Ordinate be expressed by an Equation in which the Ordinate *y* ascends not above the Square, such Curve may be compared with the most simple Figures with which it admits of a geometrical Comparison, whether rectilinear or curvilinear. Thus in the Example adduced

adduced by our Author: from the Equation $a^2y^2 + z^2y^2 = 2a^3y + 2z^3y - z^4$, is deduced $y = \frac{a^3 + z^3 \pm a\sqrt{a^4 + 2az^3 - z^4}}{a^2 + z^2}$: where the Value of y is made up of two Parts, which are to be considered as two distinct Ordinates, *viz.* $\frac{a^3 + z^3}{a^2 + z^2}$ and $\frac{\pm a\sqrt{a^4 + 2az^3 - z^4}}{a^2 + z^2}$: which must belong to Curves that may either be squared, or else compared with the most simple Curves, by the Methods already delivered.

253. It may be proper here to observe, that, when the Ordinate y has different Values (as in the above Example there are two different Values, according as the irrational Part is affected with the positive or negative Sign) there will be as many distinct Values of the Area, as there are Values of y: for in so many different Points will the Ordinate meet with the Curve: and so each particular Area must be considered as extending along that Part of the Curve, to which the corresponding Value of the Ordinate belongs: by observing of which, the several Areas may easily be distinguished from each other.

COR. 2.

254. By means of this Corollary, any Curve whose Ordinate is defin'd by an affected Equation of that Sort which is mentioned in Cor. 7. Prop. 9. *viz.* such a Curve as may be made to pass into another equal to itself, whose Abscis may be determined from the Ordinate given, by an Equation not affected, may either be squared, or else compared with the most simple Curves it can be compared with.

For the general Equation to that Sort of Curves is

$$y^\alpha \times \overline{e + fy^\eta z^\delta + gy^{2\eta}z^{2\delta} + \&c.} = z^\beta \times \overline{k + ly^\eta z^\delta + my^{2\eta}z^{2\delta} + \&c.}$$

which by putting $s = \frac{\eta - \delta}{\eta}$. $x = \frac{1}{s}z^s$ and $\lambda = \frac{\eta - \delta}{\alpha\delta + \beta\eta}$. passes into another equal Curve, in which the Relation of the Abscis and Ordinate is defin'd by the Equation $\frac{1}{s}v^{\alpha\lambda} \times \overline{e + fv^\eta + gv^{2\eta} + \&c.}|^{\lambda} \times \overline{k + lv^\eta + mv^{2\eta} + \&c.}|^{-\lambda} = x$; by which, from the Ordinate v given, the Abscis x is determined, and that by an Equation not affected. Therefore, by taking the Fluxions, according to Prop. 1, we have $\dot{x} = \frac{\alpha\lambda}{s}v^{\alpha\lambda - 1}\dot{v} \times$

$$\overline{e + fv^\eta + \&c.}|^{\lambda} \times \overline{k + lv^\eta + \&c.}|^{-\lambda} + \lambda \times \overline{\eta fv^{\eta-1}\dot{v} + 2\eta gv^{2\eta-1}\dot{v} + \&c.}$$
$$\times \overline{e + fv^\eta + \&c.}|^{\lambda-1} \times \overline{k + lv^\eta + \&c.}|^{-\lambda} \times \frac{1}{s}v^{\alpha\lambda} - \lambda \times \overline{\eta lv^{\eta-1}\dot{v} + 2\eta mv^{2\eta-1}\dot{v} + \&c.}$$
$$\times \overline{k + lv^\eta + \&c.}|^{-\lambda-1} \times \overline{e + fv^\eta + \&c.}|^{\lambda} \times \frac{1}{s}v^{\alpha\lambda}.$$

And

And if you multiply both Sides of the Equation by v, it becomes $\dot{x}v = \dot{v} \times \frac{\alpha\lambda}{s} v^{\alpha\lambda} \times \overline{e + fv^{\eta} + \&c.}|^{\lambda} \times \overline{k + lv^{\eta} + \&c.}|^{-\lambda} + \dot{v} \times \frac{\lambda}{s} v^{\alpha\lambda} \times \overline{\eta fv^{\eta} + 2\eta gv^{2\eta} + \&c.} \times \overline{e + fv^{\eta} + \&c.}|^{\lambda - 1} \times \overline{k + lv^{\eta} + \&c.}|^{-\lambda} - \dot{v} \times \frac{\lambda}{s} v^{\alpha\lambda} \times \overline{\eta lv^{\eta} + 2\eta mv^{2\eta} + \&c.} \times \overline{k + lv^{\eta} + \&c.}|^{-\lambda - 1} \times \overline{e + fv^{\eta} + \&c.}|^{\lambda} = \dot{z}y$, because $\dot{x}v$ and $\dot{z}y$ are the Fluxions of equal Curves: consequently, by Prop. 10, a Curve having v for it's Abscifs, and $\frac{\alpha\lambda}{s} v^{\alpha\lambda} \times \overline{e + fv^{\eta} + \&c.}|^{\lambda} \times \overline{k + lv^{\eta} + \&c.}|^{-\lambda} + \frac{\lambda}{s} v^{\alpha\lambda} \times \overline{\eta fv^{\eta} + 2\eta gv^{2\eta} + \&c.} \times \overline{e + fv^{\eta} + \&c.}|^{\lambda - 1} \times \overline{k + lv^{\eta} + \&c.}|^{-\lambda} - \frac{\lambda}{s} v^{\alpha\lambda} \times \overline{\eta lv^{\eta} + 2\eta mv^{2\eta} + \&c.} \times \overline{k + lv^{\eta} + \&c.}|^{-\lambda - 1} \times \overline{e + fv^{\eta} + \&c.}|^{\lambda}$ for it's Ordinate, is equal to the Curve proposed, having z and y for it's Abscifs and Ordinate.

Now the Ordinate of the Curve having v for it's Abscifs, is made up of these Parts.

1°. $\frac{\alpha\lambda}{s} v^{\alpha\lambda} \times \overline{e + fv^{\eta} + \&c.}|^{\lambda} \times \overline{k + lv^{\eta} + \&c.}|^{-\lambda}$.

2°. $\frac{\eta\lambda}{s} v^{\alpha\lambda + \eta} \times \overline{f + 2gv^{\eta} + \&c.} \times \overline{e + fv^{\eta} + \&c.}|^{\lambda - 1} \times \overline{k + lv^{\eta} + \&c.}|^{-\lambda}$.

3°. $-\frac{\eta\lambda}{s} v^{\alpha\lambda + \eta} \times \overline{l + 2mv^{\eta} + \&c.} \times \overline{e + fv^{\eta} + \&c.}|^{\lambda} \times \overline{k + lv^{\eta} + \&c.}|^{-\lambda - 1}$.

And the Curves having these several Parts of their Ordinates, and v for their common Abscifs, may either be squared by Prop. fifth and sixth, or compared with the most simple Curves, they can be compared with by Cases third and fourth of this Prop. and consequently the Curve proposed is either squared; or compared with the most simple Curves, it admits of a geometrical Comparison with.

255. Accordingly, as our Author affirms, every Curve whose Property is defin'd by any Equation consisting of three Terms only, however affected, may either be squared by this Corollary; or compared with the most simple Curves possible.

Ex. Let $y^3 - azy + z^3 = 0$ be such an Equation: by reducing it to the Form in Cor. 7. Prop. 9, it may stand thus $y^3 \times 1 = z^3 \times \overline{-1 + az^{-2}y}$: which compared with the general Form, gives $\alpha = 3 = \beta$, $\delta = -2$, $\eta = 1 . e = 1 . k = -1 . l = a . s = (\frac{\eta - \delta}{\eta} =) 3$, $x = \frac{1}{3}z^3$, $\lambda = -1$: whence by a proper Substitution you'll obtain $-\frac{1}{3v^3} + \frac{a}{3v^2} = x$ for the Relation of the Abscifs and Ordinate of an equal Curve: and by taking the Fluxions; $\frac{\dot{v}}{v^4} - \frac{2a\dot{v}}{3v^3} = \dot{x}$; and fur-

 ther,

ther, by multiplying both Sides of the Equation by v, it is $\frac{\dot{v}}{v^3} - \frac{2a\dot{v}}{3v^2} = \dot{x}v = \dot{z}y$: wherefore the Area of the Curve, which has v for it's Abſciſs, and $\frac{1}{v^3} - \frac{2a}{3v^2}$ for it's Ordinate, is equal to the Area of the Curve propoſed: which therefore, by the Method of Quadratures, is $-\frac{1}{2v^2} + \frac{2a}{3v}$ *. Now to reduce this to an equivalent Expreſſion in Terms of z and y: you have $\dot{x} : \dot{z} :: y : v$, by what has been ſhewn, and again from the Equation $\frac{1}{3}z^3 = x$, by taking the Fluxions, you have $\dot{x} : \dot{z} :: z^2 : 1$: therefore $y : v :: z^2 : 1$; whence $v = \frac{y}{z^2}$. Subſtitute this Value of v in the Expreſſion $-\frac{1}{2v^2} + \frac{2a}{3v}$, and you'll obtain $\frac{2az^2}{3y} - \frac{z^4}{2y^2}$ for the Value of the Area required in Terms of z and y.

* Art. 145. of this Explication.

Therefore, if you aſſume z or y of any determined Value, and ſubſtitute that Value of z or y, in the Equation to the Curve, you'll, by the Reſolution of the Equation, find the correſponding Value or Values of the other (for there may be as many Values of the unknown Quantity, retain'd in the Equation, as it has Dimenſions) and ſo by ſubſtituting theſe Values of z and y in the Expreſſion of the Area, you'll obtain ſo many Values of the Area, as the Nature of the Curve requires. Thus in the foregoing Example, if you take the Abſciſs $z = \frac{1}{2}a$, then by ſubſtituting this Value for z in the Equation to the Curve, *viz.* $z^3 - axy + y^3 = 0$, it becomes $\frac{1}{8}a^3 - \frac{1}{2}aay + y^3 = 0$; whence you'll obtain three Values of y correſponding to that Value of z, *viz.* 1°. $y = \frac{1}{2}a$: 2°. $y = \frac{\sqrt{5}-1}{4}a$. 3°. $y = -\frac{\sqrt{5}+1}{4}a$: the firſt two poſitive; the other negative. Wherefore if you ſubſtitute theſe Values of y for it, and $\frac{1}{2}a$ for z in the Expreſſion $\frac{2az^2}{3y} - \frac{z^4}{2y^2}$, you'll have 1°. $\frac{5}{24}aa$. 2°. $\frac{27-11\sqrt{5}}{-96+48\sqrt{5}}a^2$. 3°. $-\frac{27+11\sqrt{5}}{90+48\sqrt{5}}aa$ for the three correſponding Values of the Area, according to the different Parts of the Curve, to which the different Values of y belong: which Areas are to be computed from the Beginning of the Abſciſs *; the firſt two upon the one Side of the Abſciſs; and the laſt, which is negative, upon the other Side of the Abſciſs, becauſe of the negative Value of the Ordinate.

* Art. 169. of this Explication.

256. The Reſult will be the ſame, if we proceed as follows. After having found, as above, that $x = \frac{a}{3v^2} - \frac{1}{3v^3}$, inſtead of conſidering x as the Abſciſs and v as it's correſponding Ordinate, you may, on the contrary, conſider v as the Abſciſs, and x as it's Ordinate: and then multiply-

multiplying both by $\dot{v}$, it is $x\dot{v} = \frac{a\dot{v}}{3v^2} - \frac{\dot{v}}{3v^3}$, which is the Fluxion of the curvilinear Space, that, together with the curvilinear Area, belonging to the Abſciſs x and Ordinate v, makes up the circumſcribed Rectangle xv: wherefore find the Area of the Curve having v for it's Abſciſs and $\frac{a}{3v^2} - \frac{1}{3v^3}$ for it's Ordinate: which is $-\frac{a}{3v} + \frac{1}{6v^2}$; ſubtract it from $xv = \frac{a}{3v} - \frac{1}{3v^2}$, and there remains $\frac{2a}{3v} - \frac{1}{2v^2}$, for the Area required, to be converted into Terms of z and y as formerly.

COR. 3.

257. What our Author lays down in this Corollary is evident, *viz.* that any Curve whoſe Ordinate is defin'd by any affected Equation whatſoever, that by the Application of Cor. 8. Prop. 9. can be made to paſs into an affected quadratic Equation, may either be ſquared; or elſe compared with the moſt ſimple Figures, that it can be compared with. Which appears ſufficiently from what has been ſaid.

258. Schol. But if the Equation defining the Relation of the Abſciſs and Ordinate of any Curve propoſed, be ſuch, that none of the preceding Methods are ſufficient for finding it's Area: then we muſt have recourſe to the Method of infinite Series, laid down and explain'd by our Author in his Analyſis by Equations of an infinite Number of Terms: by means of which, any Equation how high or affected ſoever it be, that defines the Relation of the Abſciſs and Ordinate of a Curve, may be reduced into an infinite converging Series, which exhibits the Value of the Ordinate, by an infinite Number of ſimple Terms, made up of the Abſciſs and known Quantities: each of which Terms is quadrable by Art. 145 *: and therefore we thence find an approximate Value of the curvilinear Area. An Example of this we gave already Art. 146 *: more of which, and the Method of Reſolution, may be ſeen in the Treatiſe itſelf: which, becauſe of the Affinity of the Subject, I have here annexed.

* Of this Explication.
* Of this Explication.

259. And thus far I have endeavoured to explain the Principles upon which the Quadrature of Curves, and the Doctrine of Fluxions in general, are founded; and to illuſtrate the foregoing Propoſitions contained in this Treatiſe, in which, the general Methods for finding the Areas of Curves are laid down. Theſe the ſkilful and cautious Geometrician may have recourſe to and conſult, as often as Occaſion requires: but in regard it would be a troubleſom and tedious Affair to be obliged always to have recourſe to thoſe general Methods, when the Area of any Curve is required: therefore our Author has furniſhed

us with the two following Tables. The firſt of which exhibits immediately, the Areas of the more ſimple Kinds of Curves, that are capable of being ſquared geometrically. The other contains the more ſimple Kinds of Curves, that are capable of a geometrical Compariſon with the Ellipſe and Hyperbola, under the firſt of which the Circle is comprehended: in which the Relation of the curvilinear Areas, to the Areas of the Conic Sections, is ſet before our View, ſo as that it may be ſeen at once. They are both divided into certain Forms, Orders, or Claſſes, which contain certain Species's of Curves under them: which Tables are a ſort of *Speculum*, exhibiting the Areas of infinite Numbers and Varieties of Curves.

EXPLICATION of TABLE I.

260. This Table contains four different Forms or Orders of Curves: the firſt two Orders contain only one Species each; the other two Forms contain under them innumerable Species's of Curves, although the firſt four Species's, which are the moſt ſimple, be only inſerted in the Table.

Form 1ſt. 261. The Curves of the firſt Form are quadrable: and the Area is that ſet down in the Table as appears by Art. 143 *. But if it be $\eta = 0$, appearing thus $dz^{-1} = y$, it belongs to the Hyperbola: for which ſee Art. 190, 191 *.

* Of this Explication.

* Of this Explication.

Form 2d. 262. All Curves belonging to the ſecond Form are doubly quadrable, as appears by Art. 149 *: and the Area is to be found as our Author directs Art. 30 †: that is to ſay, you muſt bring the Expreſſion of the Ordinate, *viz.* $\frac{dz^{\eta-1}}{e^2+2efz^{\eta}+f^2z^{2\eta}}$ to ſtand thus, $\frac{dz^{\eta-1}}{\overline{e+fz^{\eta}}|^2}$, which is expreſſed thus $dz^{\eta-1} \times \overline{e+fz^{\eta}}|^{-2}$; or thus $dz^{-\eta-1} \times \overline{f+ez^{-\eta}}|^{-2}$ *: to the firſt Form of the Ordinate belongs the Area $\frac{dz^{\eta}}{\eta e^2+\eta efz^{\eta}}$; to the ſecond the Area $\frac{-d}{\eta ef+\eta f^2z^{\eta}}$, as may be eaſily collected from Art. 147 *; in which the third Example belongs to this Form. The firſt Area commences at the Beginning of the Abſciſs; the other lyes along the Abſciſs infinitely produced beyond the Ordinate. See Sect. 5. concerning the Poſition and Limits of Areas.

* Of this Explication.

† Of the Quadrature of Curves.

*Art. 127. of this Explication.

* Of this Explication.

263. The other Species of Curves under this Form, whoſe Ordinates are denoted thus $\frac{dz^{p\eta-1}}{\overline{c+fz^{\eta}}|}$, where p is any Integer, poſitive or negative, are not quadrable, as appears by Art. 147 and 149 *.

* Of this Explication.

264. The

264. The Curves belonging to Form third are quadrable by Art. 147 and 149 *: which may be thus generally expressed $dz^{pn-1}\sqrt{e+fz^{n}}=y$: p being a positive Integer: but by them it appears that the Area, arising from the Ordinate retained in this Form, runs out into an infinite Series: and therefore it must be converted into the other Form, *viz.* $dz^{\frac{2pn+n}{2}-1}\times\sqrt{f+ez^{-n}}$: and thence you'll find the Area by Prop. 5. according to Art. 147 *. The fourth Example in that Art. is the same with the Curve of Species second of this Order: which you may see. Form 3d. * Of this Explication. * Of this Explication.

265. Now that we may have a general Theorem for the Areas of all Curves belonging to this Form; and know how these Areas may be continued through all the different Species's *in inf.* let the general Ordinate $dz^{pn-1}\sqrt{e+fz^{n}}$, turned into this Shape $dz^{\frac{2pn+n}{2}-1}\times\overline{f+ez^{-n}}|^{\frac{3}{2}-1}$, be compared with the Ordinate for binomial Curves Art. 147 *, and you'll find by substituting $\frac{2pn+n}{2}$ for θ, $-\eta$ for η, $\frac{3}{2}$ for λ, $-\frac{2p+1}{2}$ for r, and $-p+1$ for s, that the Series for the Area will be $dz^{\frac{2pn+n}{2}}\times\overline{f+ez^{-n}}|^{\frac{3}{2}}$ into $\frac{2}{\overline{2p+1}\times nf}-\frac{2\times2\times\overline{p-1}\times e}{\overline{2p+1}\times\overline{2p-1}\times nf^{2}}z^{-n}$ $+\frac{2\times2\times2\times\overline{p-1}\times\overline{p-2}\times e^{2}}{\overline{2p+1}\times\overline{2p-1}\times\overline{2p-3}\times nf^{3}}z^{-2n}-\frac{2\times2\times2\times2\times\overline{p-1}\times\overline{p-2}\times\overline{p-3}\times e^{3}}{\overline{2p+1}\times\overline{2p-1}\times\overline{2p-3}\times\overline{2p-5}\times nf^{4}}z^{-3n}+$ $\frac{2\times2\times2\times2\times2\times\overline{p-1}\times\overline{p-2}\times\overline{p-3}\times\overline{p-4}\times e^{4}}{\overline{2p+1}\times\overline{2p-1}\times\overline{2p-3}\times\overline{2p-5}\times\overline{2p-7}\times nf^{5}}z^{-4n}-$ &c. where you may observe, * Of this Explication.

1°. That the Number of Terms must be equal to p.

2°. Supposing p to be 1 . 2 . 3 . 4 . 5, &c. successively, the numeral Coefficient of the Numerator of the last Term will be 2, -4, $+16$, -96, $+868$, &c. respectively: which are produced by multiplying by -2, -4, -6, -8, &c. continually, *i. e.* multiply the first, *viz.* 2 by -2 that produces the 2ᵈ, -4; multiply this by -4, it produces $+16$; this by -6, produces -96, &c. as easily appears from the Progression of the Terms.

3°. p standing for any positive Integer whatsoever, it will easily appear, by considering the Progression of the Series, that the numeral Coefficients of the Numerators of the several Terms, beginning with the last and going backwards, are formed by dividing by the Terms of this Series, 2 . 4 . 6 . 8, &c. and the Denominators of the Terms, in in the ſame Order, are formed by dividing by the Terms of this Series $3 \cdot 5 \cdot 7 \cdot 9$, *&c.* Wherefore, the numeral Coefficients of the ſeveral Terms, beginning with the laſt and going backwards, if you reſpect the Signs, are formed by multiplying by the Terms of this Series $-\frac{3}{2}, -\frac{5}{4}, -\frac{7}{6}, -\frac{9}{8}$, *&c.*

Therefore, if the foregoing Expreſſion of the Area be duly reduced, by bringing all the Terms under one common Denominator; and the Order of the Terms be inverted, the Relation of the numeral Coefficients of the ſeveral Terms will remain the ſame as before, the firſt Term now coming in place of the laſt formerly; the ſecond in place of the laſt ſave one, *&c.* and ſince that which becomes the common Denominator, is the ſame with the Denominator of the laſt Term formerly, as in the Series above: hence it appears, that, after the Expreſſions of the Areas are reduced to the ſame Denominator, if p be $1 \cdot 2 \cdot 3 \cdot 4 \cdot$ *&c.* ſucceſſively, the Denominator will be $3 \cdot 15 \cdot 105 \cdot 945$, *&c.* reſpectively, formed by multiplying the Numbers $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9$, *&c.* into one another continually.

And ſo you have the Demonſtration of what Sir *Iſaac Newton* ſays at the End of the ſecond Table, with reſpect to the Continuation of the Series of Curves belonging to this third Form. Only you may obſerve here, that the firſt Factor, *viz.* $dz^{\frac{2pn+n}{2}} \times \overline{f+ez^{-n}}|^{\frac{3}{2}}$, being multiplied by the negative Power of z contain'd in the laſt Term of the Series for the Area above, *viz.* $-\overline{p-1} \times n$, will always become $d \times \overline{e+fz^n}|^{\frac{3}{2}}$ or dR^3: to compenſate which Multiplication, all the other Terms are divided by the ſame negative Power of z; and ſo the Expreſſion for the Area ſhall put on the ſame Form as in the Table.

Form 4th. 266. The Species's of Curves belonging to Form fourth are likewiſe infinite, and may be thus generally expreſſed $\frac{dz^{pn-1}}{\sqrt{e+fz^n}}$. The Area ariſing from the Form of the Ordinate $dz^{pn-1} \times \overline{e+fz^n}|^{-\frac{1}{2}}$, runs out into an infinite Series, as appears from Art. 147 and 149 *: but, when the Ordinate is turned into this other Form, *viz.* $dz^{\frac{2pn-n}{2}-1} \times \overline{f+ez^{-n}}|^{-\frac{1}{2}}$, you'll find, by theſe ſame Art. that the Expreſſion for the Area terminates: and therefore all the Curves belonging to this Form *in inf.* are quadrable.

* Of this Explication.

267. Let the Curve of the firſt Species be propoſed, *viz.* $dz^{n-1} \times \overline{e+fz^n}|^{-\frac{1}{2}} = y$: I turn the Ordinate into this other Form, *viz.* $dz^{\frac{1}{2}n-1} \times \overline{f+ez^{-n}}|^{-\frac{1}{2}}$: then by comparing it with the general Form of Ordi-

5 nates

nates of binomial Curves Art. 147 *, I find $\theta = \frac{1}{2}\eta \,.\, \eta = -\eta \,.\, \lambda = \frac{1}{2} \,.$ $r = -\frac{1}{2} \,.\, s = 0 \,.\, a = d \,.\, e = f \,.\, f = e.$ Therefore by a due Substitution, the Area comes out $\frac{d}{-\eta} z^{\frac{1}{2}\eta} \times \overline{f + ez^{-\eta}}|^{\frac{1}{2}} \times \frac{1}{-\frac{1}{2}f} = \frac{2d}{\eta f}\sqrt{e + fz^{\eta}}$; as in the Table.

* Of this Explication.

And after the same manner, the Quadratures of all the other Species's of Curves, belonging to this Form, may be demonstrated, *viz.* by the direct and immediate Application of Art. 147 *.

* Of this Explication.

268. But the Areas of all the Curves in this Species, and the preceding *in inf.* may be found, and the Series of Areas continued after the Area of the first is known, by the Help of the Theorem Art. 202 *, *viz.* $B = \frac{z^{\theta}R^{\lambda} - \theta e A}{\overline{\theta + \lambda\eta} \times f}$: thus, to find the Area of the second Species of this Form, whose Ordinate is $dz^{2\eta - 1} \times \overline{e + fz^{\eta}}|^{-\frac{1}{2}}$, you'll have $\theta = \eta \,.$ $\lambda = \frac{1}{2} \,.\, A = \frac{2}{\eta f}\sqrt{e + fz^{\eta}}$: therefore, by a due Substitution of Values,

it is $B = \frac{z^{\eta}\sqrt{e + fz^{\eta}} - \frac{2e}{f}\sqrt{e + fz^{\eta}}}{\frac{3}{2}\eta f} = \frac{2fz^{\eta} - 4e}{3\eta f^{2}}\sqrt{e + fz^{\eta}}$: which being multiplied by d, gives $\frac{-4e + 2fz^{\eta}}{3\eta f^{2}} d\sqrt{e + fz^{\eta}}$ for the Area of the Curve constituting the second Species of this Order: and so for the others. But the Species's of Curves belonging to the Progression, which runs backward above the first Species, both in this and the preceding Form, expressed thus $dz^{-p\eta - 1}\sqrt{e + fz^{\eta}}$ and $\frac{dz^{-p\eta - 1}}{\sqrt{e + fz^{\eta}}} = y$, cannot be found by the Help of the Theor. Art. 203 *: See Schol. 2. Art. 209 †: neither are they quadrable, as appears by Art. 149 ‡; and they constitute Forms third and fourth, Tab. 2.

* Of this Explication.

† Of this Explication.

‡ Of this Explication.

269. You may likewise, by Prop. 5, and Art. 147 *, find a general Area corresponding to the general Ordinate $dz^{p\eta - 1} \times \overline{e + fz^{\eta}}|^{-\frac{1}{2}}$, as was shewn in the Curves belonging to the third Order: which general Expression will stand thus $dz^{\frac{2p\eta - \eta}{2}} \times \overline{f + ez^{-\eta}}|^{-\frac{1}{2}}$ into

* Of this Explication.

$$\frac{2}{\overline{2p-1} \times \eta f} - \frac{4 \times \overline{p-1} \times e}{\overline{2p-1} \times \overline{2p-3} \times \eta f^{2}} z^{-\eta} + \frac{8 \times \overline{p-1} \times \overline{p-2} \times e^{2}}{\overline{2p-1} \times \overline{2p-3} \times \overline{2p-5} \times \eta f^{3}} z^{-2\eta} - \frac{16 \times \overline{p-1} \times \overline{p-2} \times \overline{p-3} \times e^{3}}{\overline{2p-1} \times \overline{2p-3} \times \overline{2p-5} \times \overline{2p-7} \times \eta f^{4}} z^{-3\eta} + \&c.$$

Which Series will serve to find the Area of any Curve belonging to this Order, since the Progression of the Series is evident. And by considering

ſidering it in the ſame Light, in which the general Series for the Areas of the Curves of the preceding Order, was conſidered above, you'll eaſily ſee the Demonſtration of what Sir *Iſaac* ſays, concerning the Continuation of the Series of Curves of this Order *in inf.* at the End of the ſecond Tab. which it would be needleſs to inſiſt upon particularly, after what has been ſhewn at length, with reſpect to this Matter, upon Order third.

270. To ſhew the Application of the preceding general Canon, put $p=4$, then you'll have $dz^{\frac{7}{2}n} \times \overline{f+ez^{-n}}|^{\frac{1}{2}}$ into $\frac{2}{7nf} - \frac{4\times 3\times e}{7\times 5\times nf^2}z^{-n} + \frac{8\times 3\times 2\times e^2}{7\times 5\times 3\times nf^3}z^{-2n} - \frac{16\times 3\times 2\times 1\times e^3}{7\times 5\times 3\times 1\times nf^4}z^{-3n}$, which, by reducing all to the Denominator of the laſt Term, and freeing the Expreſſion of the negative Powers of z, becomes $d\sqrt{e+fz^n} \times \frac{30f^3z^{3n} - 36ef^2z^{2n} + 48e^2fz^n - 96e^3}{105nf^4}$ the ſame as in the Tab.

271. If in the Ordinates of Curves belonging to the laſt two general Forms, the Index of the Radical was $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, &c. in the third Form; or $-\frac{3}{2}$, $-\frac{5}{2}$, $-\frac{7}{2}$, &c. in the fourth, they may eaſily be reduced to the Curves in the Tab. by Caſe ſecond of Prop. 7, *viz.* by diminiſhing or increaſing that Index by Unity, till it become $\frac{1}{2}$ or $-\frac{1}{2}$, according to the Canons in Art. 211 and 212 *.

* Of this Explication.

EXPLICATION of TABLE II.

272. When we enquire into the Area of any Curve propoſed, that Area may be exhibited either *arithmetically:* which is done by Prop^s. fifth and ſixth; or *geometrically* by finding and deſcribing other more ſimple Curves, with which it may be compared. Now by the Help of Prop^s. ninth and tenth, our Author has conſtructed this Table or Catalogue of Curves, that are capable of being compared geometrically with the Ellipſe and Hyperbola; ſo that their Areas may be exhibited by the Deſcription of theſe Figures: and conſequently given, when theſe Figures are given.

273. It is not my Deſign here to go through all the Orders of Curves contained in this Tab. and demonſtrate the Quadratures of the ſeveral Species's of Curves, belonging to each Order. All I ſhall do, is to ſhew by a few Examples, the Manner of demonſtrating all the reſt: then make ſome general Remarks upon the Conſtruction and Uſe of this Table: and finally ſhew the Application of it in ſome Examples moſtly taken from the Author himſelf.

274. The Curve of the first Species of Form first, is expressed thus $\frac{dz^{\eta-1}}{e+fz^{\eta}}=y$: which is not capable of a geometrical Comparison with a right-lined Figure, as appears by Art. 147 and 149 *. Wherefore, putting $z^{\eta}=x$, (by Cor. 4. Prop. 9.) it passes into another equal Curve, whose Ordinate is $\frac{1}{\eta}\times\frac{d}{e+fx}$: x being the Abscis. Therefore, if the Area of the Curve which hath $\frac{d}{e+fx}$ for it's Ordinate, be called s, the Area of the Curve proposed shall be $\frac{1}{\eta}s$, since η is a given Quantity. Now $\frac{d}{e+fx}=v$ is an Equation to the rectangular Hyperbola, as was shewn formerly *. * Of this Explication. *Art. 245. of this Explication.

The Construction of the Hyperbola, which serves equally for all the Species's of Curves under this general Form, is thus. See Fig. 1. belonging to this Table, p. 25. Draw the right Line ABN for one of the Assymptotes, make f linear Unity, take $A\alpha=e$, and $\alpha B=x$: draw αG perpendicular to ABN, and equal to $\frac{d}{e}$, or a third Proportional to the three Lines e, f and d: through G, with the Center A and Assymptote ABN, describe the rectangular Hyperbola GDS, draw the Ordinate BD: then you have $v=BD=\frac{d}{e+fx}$: and $s=\alpha GDB$: therefore it is $\frac{\alpha GDB}{\eta}=t$ the Area required *. *Art. 190. of this Explication.

275. The Quadratures of the other Species's of this Order, are demonstrated by the Help of the Canon in Art. 211 *: the Area of the first Species being once found. * Of this Explication.

The Series of Curves belonging to this Order, may be carried on likewise *in inf.* by Division. Thus let the fourth Species be proposed $\frac{dz^{4\eta-1}}{e+fz^{\eta}}=y$: divide $dz^{4\eta-1}$ by $fz^{\eta}+e$, and the Quote is $\frac{d}{f}z^{3\eta-1}-\frac{de}{ff}z^{2\eta-1}+\frac{de^2}{f^3}z^{\eta-1}-\frac{\frac{de^3}{f^3}z^{\eta-1}}{e+fz^{\eta}}$, which Division is to be carried on, till you have as many Terms in the Quote, as the Number prefixt to η in the Index of z without the Vinculum, less one: to which annex the Remainder with it's proper Sign, and $e+fz^{\eta}$ under it for it's Denominator: and then all the Terms save the last, denote Ordinates of Curves belonging to Order first, Table first, which are thence quadrable; and the last Term is the Ordinate of a Curve, belonging to Species first of this Order: therefore all the Areas, joined with their proper Signs, make the Area required, *viz.* $\frac{d}{3\eta f}z^{3\eta} - \frac{de}{2\eta f^2}z^{2\eta} + \frac{de^2}{\eta f^3}z^{\eta} - \frac{e^3}{\eta f^3}s$. And so of others.

If we put $\frac{dz^{p\eta-1}}{e+fz^{\eta}} = y$, it is easy to see what is the Progression of the general Series for the Area, *viz.*

$$\frac{d}{\overline{p-1}\times\eta f}z^{p\eta-\eta} - \frac{de}{\overline{p-2}\times\eta f^2}z^{p\eta-2\eta} + \frac{de^2}{\overline{p-3}\times\eta f^3}z^{p\eta-3\eta} - - - - - \pm\frac{e^{p-1}}{\eta f^{p-1}}s.$$

Form 3d. 276. The Quadrature at Species first, Order third, is thus demonstrated. Since the Property of the Curve is expressed thus $\frac{d}{z}\sqrt{e+fz^{\eta}} = y$: by putting $\frac{1}{z^{\eta}} = x^2$ *, an equal Curve, whose Abscis is x, will have it's Ordinate $v = -\frac{2d}{\eta}x^{-2}\sqrt{f+ex^2}$. Wherefore, from the Assumption of the Area t, belonging to the Ordinate $-\frac{2d}{\eta}x^{-2}\sqrt{f+ex^2}$, or of the Area $-\frac{\eta t}{2d}$ belonging to the Ordinate $x^{-2}\sqrt{f+ex^2}$, find (by Prop. 7.) the Area belonging to the Ordinate $\sqrt{f+ex^2}$: which you do by the Theorem in Art. 202 *, *viz.* $B = \frac{x^{\theta}R^{\lambda} - \theta e A}{\overline{\theta+\lambda\eta}\times f}$. For by assuming the Area $-\frac{\eta t}{2d}$ agreeing to the Ordinate $x^{-1-1}\times\overline{f+ex^2}|^{\frac{1}{2}-1}$, I find the Area belonging to the Ordinate $x^{1-1}\times\overline{f+ex^2}|^{\frac{3}{2}-1}$, by putting $\theta = -1$. $\eta = 2$. $\lambda = \frac{1}{2}$. $e = f$. $f = e$ and $A = -\frac{\eta t}{2d}$: which accordingly is $\frac{x^{-1}\times\overline{f+ex^2}|^{\frac{3}{2}} - \frac{\eta f t}{2d}}{2e}$: call this Area s, then you have this Equation $s = \frac{\overline{f+ex^2}|^{\frac{3}{2}}}{2ex} - \frac{\eta f t}{4de}$: by reducing of which, you obtain $t = \frac{4de}{\eta f}\times\overline{\frac{\overline{f+ex^2}|^{\frac{3}{2}}}{2ex} - s}$ the Area of the Curve proposed: which therefore is given, when s is given.

Art. 42. of the Quadrature of Curves.

* Of this Explication.

Now s denotes the Area of a Conic Section, for it belongs to the Curve whose Ordinate is $\sqrt{f+ex^2}$, which, when f and e are both positive, is an Hyperbola, *viz.* that represented at Fig. 2. p. 25. but if f be negative, it belongs to the Hyperbola at Fig. 3. p. 25. lastly, if e be negative, it belongs to the Ellipse at Fig. 4. p. 25.

The

The Construction for the first Case is thus: with the Center A, the half transverse Axis AP $= \sqrt{f}$ (that is a mean Proportional betwixt Unity and f) and Parameter of the Axis $\frac{2\sqrt{f}}{e}$ (that is a fourth Proportional to the Lines e, 2 and AP) describe the Hyperbola PGD: draw the indeterminate Axis AN, in which take AB $= x = \frac{1}{z^{\frac{1}{2}n}}$: to which draw the Ordinate BD: so shall it be BD $= v = \sqrt{f + ex^2}$, and s $=$ APDB. For from the Property of the Hyperbola, calling the Latus Transversum t, and it's Parameter p, it is $\overline{BD + AP} \times \overline{BD - AP}$: $AB^q :: t : p$ or $BD^q - AP^q : AB^q :: t : p$, that is, by inserting the Values according to the Construction, $BD^q - f : x^2 :: 2\sqrt{f} : \frac{2}{e}\sqrt{f}$: whence BD $= \sqrt{f + ex^2} = v$, as it ought to be; and so you'll have $s =$ ABDP.

Now in order to find what the Area of the Curve proposed is equal to, in the geometrical Description; we must find the Value of the Expression $\frac{\overline{f + ex^2}|^{\frac{3}{2}}}{2ex}$, or which is the same, $\frac{v^3}{2ex}$. For this end, draw the Tangent DT meeting AB in T: then from the Property of the Tangent, you'll have AT $= \frac{AB \times AP^q}{AO^q - AP^q}$, that is, by substituting the Symbols, AT $= \frac{\frac{1}{4}tt}{ex}$: add AB $= x$ to each, and it is BT $= \frac{ex^2 + \frac{1}{4}tt}{ex}$, that is, because $\frac{1}{4}tt = f$, BT $= (\frac{ex^2 + f}{ex} =) \frac{v^2}{ex}$: therefore, multiplying by $\frac{BD}{2} = \frac{v}{2}$, you obtain $\frac{BT \times BD}{2}$, *i. e.* TDB $= \frac{v^3}{2ex}$.

Hence the Area of the Curve proposed, whose Ordinate is $\frac{d}{z}\sqrt{e + fz^n}$, which was shewn to be $\frac{4de}{nf} \times \overline{\frac{v^3}{2ex} - s}$, is $\frac{4de}{nf}$ into TDB $-$ APDB: which is set down in the Table thus $\frac{4de}{nf}$ into APDB $\div$ TDB, denoting that it is ambiguous, whether the second is to be subtracted from the first, or the first from the second: for sometimes TDB may exceed APDB, sometimes be less: but the Area of the Curve proposed is equal to that Difference: if TDB $-$ APDB be positive, the Area required being positive, lyes above the Ordinate; if TDB $-$ APDB be negative, the Area is situate upon the other Side of the Ordinate; and adjacent to the Abscis produced beyond the Ordinate.

277. If the Quantity f be negative in the Value of y, so that it be $y = \frac{d}{z}\sqrt{e - fz^n}$, then the Area of the Curve is found by the Description

tion of the Hyperbola likewise, but according to a different Construction, *viz.* at Fig. 3. p. 25. For now the Curve proposed, will be equal to that whose Ordinate is $-\frac{2d}{n}x^{-2}\sqrt{-f+ex^2}$: and so the Conic Section must be such, as to have it's Ordinate $\sqrt{-f+ex^2}$. The Construction is thus: with the Center A, transverse Axe $Ka = 2\sqrt{\frac{f}{e}}$, and Parameter $2\sqrt{ef}$, describe the Hyperbola *a*GDS: take AB $= x$, draw the Ordinate BD and the Tangent DT: and the Area, which now by changing the Sign of f, becomes $-\frac{4de}{n\sqrt{f}}\times\overline{\frac{v^3}{2ex}-s}$ or $\frac{4de}{n\sqrt{f}}\times\overline{s-\frac{v^3}{2ex}}$, will be, according to this Construction, $\frac{4de}{n\sqrt{f}}$ into *a*GDB $-$ TDB or $\frac{4de}{n\sqrt{f}}\times -$ *a*GDT: which shews the Area to be on the further Side of the Ordinate.

For from the Property of the Hyperbola, (using the Symbols t and p in the same Sense as formerly) $KB \times Ba : BD^q :: t : p$; or $x^2 - \frac{1}{4}tt : v^2 :: t : p$; that is, by inserting the Values of t and p from the Construction, $x^2 - \frac{f}{e} : v^2 :: 2\sqrt{\frac{f}{e}} : 2\sqrt{ef}$: whence we have $v = \sqrt{-f+ex^2}$, as it ought to be. Again, from the Property of the Tangent, the Subtangent $BT = x - \frac{\frac{1}{4}tt}{x} = \frac{x^2 - \frac{1}{4}tt}{x} =$ (by inserting $\frac{tv^2}{p}$ for $x^2 - \frac{1}{4}tt$, as above) $\frac{tv^2}{px}$ or $\frac{2\sqrt{\frac{f}{e}} \times v^2}{2\sqrt{fe} \times x} = \frac{v^2}{ex} =$ BT: therefore, multiplying by $\frac{v}{2} = \frac{BD}{2}$, it is $\frac{v^3}{2ex} = \frac{BT \times BD}{2} = \Delta$ TDB. Wherefore, since $s =$ *a*GDB, you'll have $\frac{4de}{n\sqrt{f}} \times \overline{s - \frac{v^3}{2ex}} = \frac{4de}{n\sqrt{f}}$ into *a*GDB $-$ TDB or $\frac{4de}{n\sqrt{f}}$ into $-$ *a*GDT. Which negative Value shews that the Area of the Curve is adjacent to the Abscis produced beyond the Ordinate.

278. If e be negative, the Construction is by means of the Ellipse at Fig. 4. p. 25. for now it is $v = \sqrt{f - ex^2}$. The Construction is thus: upon the Center A, with the transverse Axis $Ka = 2\sqrt{\frac{f}{e}}$, and half conjugate Axis $AP = \sqrt{f}$, describe the Ellipse *a*DPSK: take AB $= x$, draw the Ordinate DB, then it will be DB $= (v =) \sqrt{f - ex^2}$, and APDB $= s$: consequently the Area, which now, by changing the Sign of e, becomes $-\frac{4de}{n\sqrt{f}} \times \overline{\frac{v^3}{-2ex} - s}$, or $\frac{4de}{n\sqrt{f}} \times \frac{v^3}{2ex} + s$, will be $\frac{4de}{n\sqrt{f}}$ into TDB$+$APDB; or else $\frac{4de}{n\sqrt{f}}$ into TDB$-$*a*GDB, that is $\frac{4de}{n\sqrt{f}} \times$ *a*GDT.

For

For from the Property of the Ellipse, KB × Ba : BD^q :: KA^q : AP^q, that is by inserting the Symbols and Values from the Construction, $\frac{1}{4}tt - x^2 : v^2 :: \frac{1}{4}tt : f$ or $\frac{f}{e} - x^2 : v^2 :: \frac{f}{e} : f$; whence $v = \sqrt{f - ex^2}$, $=$ BD as it ought to be.

279. As to the two-fold Value of the Area, *viz.* $\frac{4de}{nf} \times \overline{TDB + APDB}$ and $\frac{4de}{nf} \times \overline{TDB - aGDB}$, it is to be observed, that $-s$, in the Expression denoting the Area of the Curve, may signify, either that the Area of the Conic Section, which is adjacent to the Abscis, is subtracted from the other Quantities contain'd in the Expression; or that the Area of the Conic Section, lying upon the further Side of the Ordinate and along the Abscis produced, is added to the other Quantities: even so, $+s$ denotes indifferently either that the first is to be added, or the second subtracted, as it may seem convenient; or the Case require. Wherefore in the Expression $\frac{4de}{nf} \times \overline{\frac{v^3}{2ex} + s}$, the Term $+s$ denotes either that the Area APDB is to be added to $\frac{v^3}{2ex} =$ TDB; or that aGDB is to be subtracted from TDB. And this last Value is that which is exhibited in the Table by Sir *Isaac*: because it is the proper Value in this Case. For if you express the Area of the Curve, whose Ordinate is $\frac{d}{z}\sqrt{-e + fz^n} = dz^{\frac{1}{2}n-1}\sqrt{f - ez^{-n}}$, analytically by a converging Series, according to Prop. 5, you'll find by the Method of determining the Limits of Areas (Sect. 5.) that the Area vanishes when it is $z^n = \frac{e}{f}$, that is when $x = \sqrt{\frac{f}{e}}$: because $z^n = \frac{1}{x^2}$: but according to the Construction $\sqrt{\frac{f}{e}} = $ Aa; therefore the Area of the Curve vanishes, when AB ($= x$) becomes equal to Aa: which cannot happen to the Expression $\frac{4de}{nf} \times \overline{\frac{v^3}{2ex} + s}$, but when $+s$ signifies that aGDB is to be subtracted from $\frac{v^3}{2ex} =$ TDB.

And thus I have shewn at full length, what way the Conic Sections are to be constructed, with which the Curves belonging to Species first, Order third, are geometrically compared, in all possible Cases: and what the Value of the Area in each Case will be.

How-

Fig. 1.

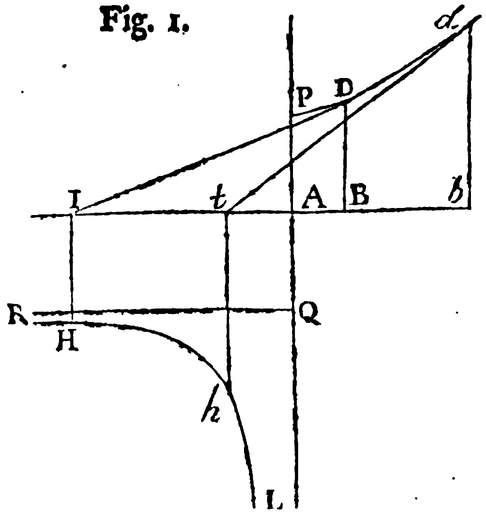


Fig. 2.

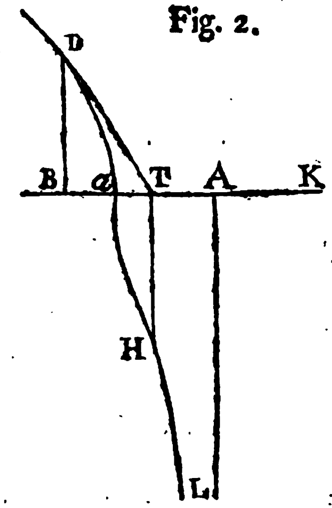
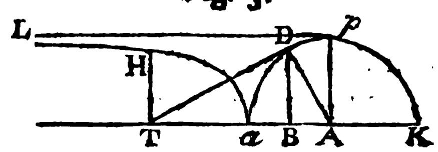


Fig. 3.



However, to render this Affair as plain as possible, and assist the young Geometrician how to proceed in like Cases, I shall illustrate what hath been said in these last three Articles by an Example.

Let HL be a Curve having the Abscis AT $= z$, and Ordinate TH $= y = \frac{\sqrt{1+z^2}}{z}$ Fig. 1. or $\frac{\sqrt{1-z^2}}{z}$, Fig. 2. or $\frac{\sqrt{-1+z^2}}{z}$, Fig. 3. where $e = 1 = f = d$, $\eta = 2$, $x^2 = \frac{1}{z^\eta} = \frac{1}{z^2}$ or $x = \frac{1}{z}$. and $v = (\sqrt{f + ex^\eta}$ $=)$ $\sqrt{1+x^2}$, Fig. 1; or $\sqrt{-1+x^2}$, Fig. 2. or $\sqrt{1-x^2}$, Fig. 3. so that the Axes and Parameters of the Conic Sections being equal they become equilateral Hyperbolas in the first two Cases, and the Circle in the other Case: in all which the Semiaxes is 1.

Wherefore, when it is $y = \frac{\sqrt{1+z^2}}{z}$; upon the Center A with the semitransverse Axis AP $= 1$, and indeterminate Axis AT (Fig. 1.) describe the equilateral Hyperbola PD*d*, to which draw the Tangent TD through the Point T, and through D the Ordinate DB: then from the Property of the Tangent (as in Art. 276 *.) $z = (AT = \frac{AB \times AP^q}{BD^q - AP^q} = \frac{AP^q}{AB} =) \frac{1}{AB}$, whence AB $= x$ and BD $= v = \sqrt{1+x^2}$ =

* Of this Explication.

$= \sqrt{f + ex^2}$; BT $= \frac{1+x^2}{x} = \frac{f+ex^2}{x} = \frac{v^2}{ex}$ and ΔTDB $= \frac{v^3}{2ex}$; likewise APDB $= s$: therefore the Area of the Curve, *viz.* $\frac{4de}{\sqrt{f}} \times \overline{\frac{v^3}{2ex} - s}$, is in this Case twice ΔTDB — APDB. Whence (by Sect. 5.) if the Point t be such, that Δtdb — AP$db = 0$, the curvilinear Area is nothing, and the Ordinate tb is the initial Limit, from whence the Area must be reckoned: so that if TDB — APDB be positive, the curvilinear Area THbt lyes upon the same Side of the Ordinate TH with the Point A the Beginning of the Absciss: but if TDB—APDB be negative, it lyes upon the opposite Side of the Ordinate. Now although this initial Limit cannot be found in this Case, without drawing td so as to make tdb = APDB, which is equivalent to the Quadrature of the Hyperbola, yet nevertheless the curvilinear Area THbt lying betwixt any two Ordinates TH, tb, is equal to the hyperbolical Area TDdt. For the Area corresponding to the Ordinate TH having the greatest Absciss AT, is TDB — APDB; and that corresponding to the Ordinate tb, having the lesser Absciss At, is tdb — APdb; therefore it will appear from the Consideration of what hath been said, that, whatever way the Areas lye, the Area THbt inclosed betwixt the two Ordinates TH, tb, is TDB — APDB — tdb + APdb = TDdt, the Difference betwixt the other two: agreeably to what was shewn Art. 185—187 *.

* Of this Explication.

If through the opposite Vertex of the Hyperbola PDd, *viz.* Q, you draw QR perpendicular to PQ; then PQ and QR are two Assymptotes to the Curve HbL.

But, 2°. If it be $y = \frac{\sqrt{1-z^2}}{z}$ (see Fig. 2. p. 206.) with the Center A, and semitransverse Axis Aa = 1, describe the equilateral Hyperbola aD: from the Point T draw the Line TD touching the Hyperbola in D; and from D, the Ordinate DB: then it is AB $= x$, since AT $= z$, Aa = 1 and AB are proportional, from the Property of the Tangent. Moreover the Curve HL passes through the Point a: for since TH $= y = \frac{\sqrt{1-z^2}}{z}$, put $\frac{\sqrt{1-z^2}}{z} = 0$, and thence you have $z = 1 =$ * Aa. Now it may be easily shewn here, as in the preceding Case, that ΔTDB $= \frac{v^2}{2x} = \frac{v^3}{2ex}$; and aDB $= s$: therefore since the curvilinear Area is $\frac{4de}{\sqrt{f}} \times \overline{s - \frac{v^3}{2ex}}$, as was shewn above, it becomes in this Case $2 \times \overline{a\text{DB} - \text{TDB}} = -2a$DT negative: wherefore, since

* Art. 171. of this Explication.

aDT

aDT $= 0$ when AT $=$ Aa, the curvilinear Area commences at a, being aTH, which lyes on that Side of the Ordinate, which is opposite to A. The indeterminate Axis of the Hyperbola, *viz.* AL is an Assymptote to the Curve AHL: and if AT be taken $= 0{,}816$, *&c.* of Aa, the Ordinate TH will pass through a Point of contrary Flexure.

Again, 3°. If it be $y = \frac{\sqrt{-1+z^2}}{z}$ (see Fig. 3.) with the Radius Aa $= 1$, describe the Semicircle ADPK; through T draw the Tangent TD, and through D the Ordinate DB: then since AT$=z$, A$a = 1$ and AB are proportional, you have AB $= x$: and as formerly ΔTDB $= \frac{v^3}{2ex}$; and now it is APDB $= s$ and aDB $= -s$: wherefore the curvilinear Area, as was shewn above, must either be $2\times\overline{\text{TDB}+\text{APDB}}$ or $2\times\overline{\text{TDB}-\text{ADB}} = 2a$DT: the last of which is equal to the curvilinear Area aTH: for since the Curve HL passes through the Point a, the Areas aTH and aDT both commence at the Point a: and the Fluxion of ATH is double the Fluxion of aDT, since HT $=$ ($y = \frac{\sqrt{-1+z^2}}{z} = \sqrt{1-x^2} = v =$) DB, and the Fluxion of AHT is to the Fluxion of aDT, as HT $\times$ Fluxion of aT to $\frac{1}{2}$DB $\times$ Fluxion of aT (by Art. 48 *.) that is as 2 to 1: therefore (by Art. 40 *.) aTH $=$ $2a$DT. Draw AP perpendicular to Aa; and the Tangent PL is an Assymptote to the Curve HL.

* Of this Explication.

* Of this Explication.

280. But by supposing a different Relation of the Abscisses z and x, *viz.* that $\frac{1}{z^\eta} = x$, as in the Table, you may obtain the Area of the Curve after a different Manner. For upon that Supposition, you'll find the Curve proposed to pass into another equal Curve having $-\frac{d}{\eta}x^{-2}\sqrt{fx+ex^2}$ it's Ordinate: so that, by Prop. 7, you may find the Relation of the Curve proposed, to the Curve whose Ordinate is $\sqrt{fx+ex^2}$: which you do by turning the Ordinate $-\frac{d}{\eta}x^{-2}\sqrt{fx+ex^2}$ into $-\frac{d}{\eta}x^{-\frac{3}{2}}\sqrt{f+ex}$, then increasing the Exponent of x, *viz.* $-\frac{3}{2}$ by 2: and so you may construct the Conic Section, with which the Curve may be compared. The other Species's of Curves of this Form are deduced from the first by Prop. 7.

Form 5th. 281. I shall next shew the Demonstration of the Quadrature of the Curve, constituting Species second of Form fifth, which requires two Conic Sections.

The

The Curve is expressed thus $\frac{dz^{2\eta-1}}{e+fz^\eta+gz^{2\eta}}=y$: and the Relation of the Abscis z to the Abscisses x and ξ, which are these of the two Conic Sections, is expressed thus $\sqrt{\frac{d}{e+fz^\eta+gz^{2\eta}}}=x$, and $fz^\eta+gz^{2\eta}=\xi$. Wherefore, I first seek for the Ordinate of a Curve, which having it's Abscis x, shall be equal to the Curve proposed, *viz.* thus. The Ordinate sought (by Prop. 9.) is equal to $\frac{y\dot{z}}{\dot{x}}$, that is, by supposing $\dot{z}=1$, $\frac{y}{\dot{x}}$: or by putting in the Value of y, $\frac{dz^{2\eta-1}}{\dot{x}\times\overline{e+fz^\eta+gz^{2\eta}}}$: wherefore from the Equation $\sqrt{\frac{d}{e+fz^\eta+gz^{2\eta}}}=x$, find the Relation of the Fluxions, which is $\dot{x}=-\frac{\eta fz^{\eta-1}x+2\eta gz^{2\eta-1}x}{2e+2fz^\eta+2gz^{2\eta}}$: insert this in place of $\dot{x}$, and the Ordinate sought will be $\frac{-2dz^\eta}{\eta fx-2\eta gz^\eta x}$: where, if you put for z^η it's Value, $\sqrt{\frac{ff-4eg}{4gg}+\frac{d}{gx^2}}-\frac{f}{2g}$, from the preceding Equation, and then reduce, you'll obtain $\frac{df}{2\eta g^2\sqrt{\frac{d}{g}+\frac{ff-4eg}{4gg}x^2}}-\frac{d}{\eta gx^2}$, for the Value of the Ordinate of a Curve, which has x for it's Abscis, and is equal to the Curve proposed at first: which Ordinate is made up of two distinct Parts. The Area belonging to the first Part of the Ordinate, may be compared with the Area of the Conic Section whose Ordinate is $v=\sqrt{\frac{d}{g}+\frac{ff-4eg}{4gg}x^2}$, by help of the Theorem in Art. 211 *, *viz.* $B=\frac{\overline{\theta+\lambda\eta}\times A-z^\theta R^\lambda}{\lambda\eta e}$: by applying of which, you'll obtain $\frac{2fs-fxv}{2\eta g}$ for the Area belonging to the Ordinate $\frac{df}{2\eta g^2\sqrt{\frac{d}{g}+\frac{ff-4eg}{4gg}x^2}}$: which might also be deduced from the Quadrature of the Curve of the first Species of this Order.

* Of this Explication.

The other Part of the Ordinate is $-\frac{d}{\eta gx^2}$: and in order to find the Area belonging to this Part of the Ordinate, take $x=\sqrt{\frac{d}{e+\xi}}$ for the Relation of the Abscisses of two Conic Sections, as it is in the

Table, whose Areas are equal; the Ordinate of the first being $-\frac{d}{ngx}$: from whence find the Ordinate of the other. It will be, by Prop. 9, $\frac{\dot{x} \times -\frac{d}{ngx}}{\dot{\xi}}$, or $-\frac{d}{ngx\dot{\xi}}$, by putting $\dot{x} = 1$: instead of x insert it's Value $\sqrt{\frac{d}{e+\xi}}$; and instead of $\dot{\xi}$ it's Value $\frac{2e+2\xi}{-\sqrt{\frac{d}{e+\xi}}}$, deduced from the Equation $x = \sqrt{\frac{d}{e+\xi}}$, and so you'll obtain $\frac{d}{2ng} \times \frac{1}{e+\xi}$ for the Ordinate of a Curve, which having ξ for it's Absciss, is equal to the Curve which has x for it's Absciss and $-\frac{d}{ngx}$ for it's Ordinate.

Now, according to the Table, $\Upsilon = \frac{1}{e+\xi}$: which denotes the Ordinate to the Assymptote in a rectangular Hyperbola *, and to which the correspondent Area is σ: that is the Area belonging to the Ordinate $\frac{d}{2ng} \times \frac{1}{e+\xi}$ is $\frac{d\sigma}{2ng}$: which therefore is the Area belonging to the Ordinate $-\frac{d}{ngx}$. See Art. 190—192 *.

* See Form 1st.

* Of this Explication.

Whence by putting the two Areas together, you'll have $\frac{d\sigma + 2fs - fxv}{2ng} = t$ the Area of the Curve proposed, expressible by the Description of two Conic Sections.

282. The other Species's of Curves of this Order *in inf.* may be found by Division, the Areas of the two first being once found. Thus for the third Species, where it is $\frac{dz^{3\eta-1}}{e+fz^{\eta}+gz^{2\eta}} = y$, divide $dz^{3\eta-1}$ by $gz^{2\eta} + fz^{\eta} + e$, and the Quotient is $\frac{d}{g}z^{\eta-1} - \frac{\frac{df}{g}z^{2\eta-1}}{e+fz^{\eta}+gz^{2\eta}} - \frac{\frac{de}{g}z^{\eta-1}}{e+fz^{\eta}+gz^{2\eta}}$, which is the Value of the Ordinate of the Curve constituting the third Species of this Form: the first Part of which is the Ordinate of a Curve quadrable by Theorem first, Table first; the other two belong to the first and second Species of this Form fifth: whence the Area may be found.

Forms 7th and 11th.

283. I have said as much already upon Example 2[d], Case 2[d], Prop. 10, as may serve to illustrate the Quadratures of the Curves belonging to Form 7. Therefore I shall only take one Instance more: which shall be the Curve constituting the first Species of the last Form, whose Property

Property is thus expressed $dz^{-1}\sqrt{\frac{e+fz^n}{g+hz^n}}=y$: and two Conic Sections are required to it's Quadrature. In order to which, I find the Ordinate of a Curve, which having x for it's Abscis, shall be equal to the Curve proposed, supposing $x=\sqrt{g+hz^n}$: the Ordinate sought is $\frac{\dot{z}y}{\dot{x}}$ or $\frac{y}{\dot{x}}$, putting $\dot{z}=1$. Wherefore, if you put in the Value of y, *viz.* $dz^{-1}\sqrt{\frac{e+fz^n}{g+hz^n}}$, and the Value of $\dot{x}$, *viz.* $\frac{nhz^{n-1}}{2x}$, deduced from the Equation $x=\sqrt{g+hz^n}$, the Ordinate sought will be $\frac{2dx\sqrt{e+fz^n}}{nhz^n\sqrt{g+hz^n}}$: which, by inserting $\frac{x^2-g}{h}$ for z^n and x for $\sqrt{g+hz^n}$, and bringing to Order, gives $\frac{2d}{n}\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^2}\times\overline{-g+x^2}|^{-1}$ for the Ordinate of a Curve equal to the Curve at first proposed.

I next find the Ordinate of a Curve, whose Abscis, as before, being x, shall have it's Area equal to σ, which is the Area of the second Conic Section, whose Abscis being ξ, has it's Ordinate $\Upsilon=\sqrt{\frac{fg-eh}{g}+\frac{e}{g}\xi^2}$, the Relation of the Abscisses being $\frac{x^2-g}{h}=\frac{g}{\xi^2-h}$: which last is easily deduced from the Values of x and ξ in the Tab. And by proceeding after the same manner, as in the preceding Part of this Art. you'll find that Ordinate to be $-gh\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^2}\times\overline{-g+x^2}|^{-2}$. So that now you have three Ordinates, *viz.* 1°. $\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^2}\times\overline{-g+x^2}|^0$. 2°. $\sqrt{\frac{eh-fg}{h}+\frac{f}{g}x^2}\times\overline{-g+x^2}|^{-1}$. 3°. $\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^2}\times\overline{-g+x^2}|^{-2}$, to which the three corresponding Areas are s, $\frac{nt}{2d}$ and $-\frac{\sigma}{gh}$: so that from any two of them given, the third may be found by Prop. 8: by applying of which to the present Case, you'll obtain $t=2fs-hx\times\overline{-g+x^2}|^{-1}\times\overline{\frac{eh-fg}{h}+\frac{f}{h}x^2}|^{\frac{3}{2}}+2e\sigma\times\frac{2d}{neh-nfg}$; or by a due Substitution of xz^{-n} for $hx\times\overline{-g+x^2}|^{-1}$, and v^3 for $\overline{\frac{eh-fg}{h}+\frac{f}{h}x^2}|^{\frac{3}{2}}$, $t=\frac{2dxv^3z^{-n}-4dfs-4de\sigma}{nfg-neh}$, the Area required.

284. And thus I have shewn in a Variety of Cases the Demonstration of the Quadratures of Curves belonging to several different

Forms in this Table: by obſerving of which carefully, all the others may be demonſtrated in like manner: wherefore, I now proceed to make ſome general Obſervations upon this Affair, taken moſtly from our Author.

285. 1°. The Series of Curves in each Form may be continued *in inf.* towards both Hands, by the Help of Propoſition third, fourth, ſeventh and eighth: that is, the Progreſſion of Curves may be carried backwards by diminiſhing the Index of z continually by η; as well as forwards according to the Order in the Table, by increaſing the Index of z continually by η. For which purpoſe you may conſult the

* Of this Explication. Theorems contained in Art. 202, 203, 206, 207, 208, 209 *: the Application of which Theorems has been exemplified in ſeveral Inſtances, in the preceding Work. And the Series of Curves belonging to the firſt, ſecond, fifth, ſixth, ninth and tenth Forms may be carried forward by Diviſion only, after the Manner ſhewn above in Or-

*Art. 282. of this Explication. der fifth *. But whereas theſe Theorems will not ſerve for finding the Curves belonging to the backward Progreſſion in ſome Forms.

* Of this Explication. See Art. 209 *, therefore we ſhall ſpeak of every Form particularly afterwards *.

*Art. 318. of this Explication. 286. 2°. Although the Signs of the given Quantities d, e, f, g, h and i are all affirmative in the Equations to the Curves, contained in this Table: yet one or more of theſe may be negative: and when that happens, ſuch Quantity or Quantities muſt be made negative in the Values of the Abſciſs and Ordinate of the Conic Section or Sections, with which any ſuch Curve is compared; and likewiſe in the Expreſſion of the Area. This I illuſtrated in the Curves belonging to Spe-

* Art. 276 —279. of this Explication. cies firſt Form third *. Where it appears, which I would have to be obſerved, that the changing the Signs of theſe Quantities, alters the Conſtruction of the Conic Section or Sections with which the Curve is compared. Thus I ſhew'd that when the Equation belonging to the Curve, is $\frac{d}{z}\sqrt{e+fz^{\eta}}=y$; and conſequently that to the Conic Section, $\sqrt{f+ex^2}=v$, ſuch Conic Section, with which the given Curve is compared, is at Fig. 2. p. 25: if it be $\frac{d}{z}\sqrt{e-fz^{\eta}}=y$, and conſequently $\sqrt{-f+ex^2}=v$; the Conic Section is at Fig. 3. p. 25: but if it be $\frac{d}{z}\sqrt{-e+fz^{\eta}}=y$, and $\sqrt{f-ex^2}=v$, in that Caſe, the Area of the Curve depends upon the Conſtruction of the Ellipſe at Fig. 4. p. 25. And the like happens in other Curves, upon like Suppoſitions.

287. 3°. These Quantities d, e, f, g, &c. may be any Quantities whatsoever, *i. e.* bear any possible Relations to one another; and be affected with any Signs; excepting only these Cases. 1°. That they be not such as to make the Sum of the Quantities under any of the radical Signs, negative: for that would make an impossible or imaginary Expression. And therefore also, 2°. That in the Curves belonging to Form sixth, when the Quantities e and g are affected with the same Sign, $4eg$ do not exceed ff: for then $p = \sqrt{ff - 4eg}$, would be impossible, and so render the Ordinate of the Conic Section, and consequently the Expression of the Area in the Table impossible.

288. 4°. In order to construct the Conic Sections, which belong to the several Orders, and Species's of these Orders; and every particular Case of any Curve, coming under any Species, according to any Supposition, either with respect to the Signs, or Proportions of the given Quantities d, e, f, g, &c. You must consider the four Figures annexed to the Table, p. 25; and investigate the Equations expressing the Relations of the Abscisses and Ordinates in each Figure, from the primary Properties of these Conic Sections, and that, according as the Abscis x is either AB, aB, or αB, the Ordinate v being always BD: and then compare these Equations with the Equations to the Conic Sections contained in the Table. For which purpose I shall run through the several Figures.

For Fig. 1.

289. Let it be $\alpha B = x$. $BD = v$. $A\alpha = a$ a given Line; $\alpha G = b$ a given Line likewise. Then, when the Point B is further distant from the Center than α is, we have from the Property of the rectangular Hyperbola $AB \times BD = A\alpha \times \alpha G$, that is in Symbols $\overline{a+x} \times v = ab$, whence - - - - - - - - - - - - - 1°. $v = \frac{ab}{a+x}$.

290. But if B fall betwixt A and α, in which Case $\alpha B = x$ is negative, the Equation puts on this Form - - 2°. $v = \frac{ab}{a-x}$.

291. And these two Forms, or Values of the Ordinate, will serve for all the Varieties of Curves belonging to Order first. The first Form, *viz.* $\frac{ab}{a+x} = v$, answers exactly to that in the Table $\frac{d}{e+fx} = v$: which is the proper Form of the Ordinate, when it is $\frac{d}{e+fz^n} = y$. If it be $\frac{d}{e-fz^n} = y$, and consequently $\frac{d}{e-fx} = v$; the second Form, *viz.* $\frac{ab}{a-x} = v$, answers to that Case. If it be $\frac{-d}{e+fx^n} = y$ that only denotes the Area

Area belonging to the first Form taken negatively: and so it will be evident that all the Varieties that can happen, by varying the Signs of the Quantities d, e, f, after any manner whatsoever, may be reduced to these two Forms expressed above.

292. As to the Comparison of the corresponding Parts of the two different Expressions of the hyperbolical Ordinate, it is made thus: since $\frac{d}{e+fx} = v = \frac{ab}{a+x}$ you'll have $a = e \,.\, 1 = f$ and $ab = d$ or $b = \frac{d}{e}$. Therefore when the Conic Section is to be constructed, you use f for linear Unity, take $A\alpha = e$, $\alpha G = \frac{d}{e}$ or $\frac{df}{e}$ that is a fourth Proportional to the three Lines e, d and f; or $A\alpha$, d, and Unity: and therefore, if through the Point G you describe a rectangular Hyperbola, with the Center A and Assymptote ABN, that shall be it you want: so that taking $\alpha B = x$, and drawing the Ordinate BD, you'll have

*Art. 190. of this Explication. $BD = v = \frac{d}{e+fx}$; and $\alpha GDB = s$ *.

And after the like manner, you make the Construction upon other Suppositions.

For Fig. 2.

293. Let it be $\alpha B = x \,.\, BD = v \,.\, A\alpha = a$ a given Line; the transverse Axe t; it's Parameter p. Then from the Property of the Figure $\overline{BD + AP} \times \overline{BD - AP} : AB^q :: t : p$, that is in Symbols, $v^2 - \frac{1}{4}tt : a^2 + 2ax + x^2 :: t : p$. Whence

$$1^\circ.\ v = \sqrt{\frac{a^2t + \frac{1}{4}ptt}{p} + \frac{2at}{p}x + \frac{t}{p}x^2}.$$

294. Which Form of the Ordinate, when B falls betwixt A and α, *i. e.* when x becomes negative, falls into this other.

$$2^\circ.\ v = \sqrt{\frac{a^2t + \frac{1}{4}ptt}{p} - \frac{2at}{p}x + \frac{t}{p}x^2}.$$

295. Again, the other Things remaining the same as in the first Supposition: only now let $A\alpha = a$ vanish, or the Points A and α coincide, by which it becomes $AB = x$: in that Case, the Equation will appear in this Form.

$$3^\circ.\ v = \sqrt{\tfrac{1}{4}tt + \tfrac{t}{p}x^2}.$$

For Fig. 3.

296. Let it be $\alpha B = x \,.\, BD = v \,.\, a\alpha = a \,.\, t$ and p as formerly: and let α fall betwixt a and B. Then from the Property of the Hyperbola,

perbola, KB×αB : BD² :: $t : p$, that is $\overline{t+a+x}\times\overline{a+x} : v^2 :: t : p$. Whence we deduce,

$$1^\circ.\ v=\sqrt{\frac{apt+a^2p}{t}+\frac{pt+2ap}{t}x+\frac{p}{t}x^2}.$$

297. Which by supposing B to fall betwixt a and α, and by that means x to become negative, falls into this other.

$$2^\circ.\ v=\sqrt{\frac{apt+aap}{t}-\frac{pt+2ap}{t}x+\frac{p}{t}x^2}.$$

298. If in the first Form, you make $a\alpha=a$ to vanish, so that it be aB $=x$, then you'll have

$$3^\circ.\ v=\sqrt{px+\frac{p}{t}x^2}.$$

299. But further, if you suppose that the Point α falls beyond the Vertex a without the Hyperbola, αB being equal to x, as in the first Form: then it will turn into this

$$4^\circ.\ v=\sqrt{\frac{-apt+a^2p}{t}+\frac{pt-2ap}{t}x+\frac{p}{t}x^2}.$$

Which includes under it three different Forms, according to the various Positions of α.

For, 1°. If α fall betwixt a and A, so that t be greater than $2a$, the Order of the Signs will be - - - - - - - $-++$.

But if α fall betwixt A and K, so that t be greater than a; but less than $2a$, you'll have this Order of Signs - - $--+$.

Again, if α fall beyond K, so that a be greater than t, this Order of the Signs will arise, *viz.* - - - - - $+-+$.

In the other two Positions of α, *viz.* when it falls in either with A or K, the Radical is reduced to two Terms.

300. Thus if α fall in with the Center A, so that it be $2a=t$ and AB $=x$, then it is

$$5^\circ.\ v=\sqrt{-\tfrac{1}{4}pt+\frac{p}{t}x^2}.$$

301. Lastly, if α fall in with K, so that K be the Beginning of the Abscis; and $t=a$, then it is

$$6^\circ.\ v=\sqrt{-px+\frac{p}{t}x^2}.$$

So that in all, there are seven different Forms under which the Value of the Ordinate BD $=v$ appears: and one of these admits of a two-fold Construction, *viz.* Form 2ᵈ and Case 3ᵈ of Form 4ᵗʰ.

For Fig. 4.

302. Put αB $=x$. BD $=v$. $a\alpha=a$. t the transverse Axe and p it's Parameter: and let α fall betwixt a and B. Then from the Pro-

 perty

perty of the Ellipse $KB \times Ba : BD^q :: t : p$, that is in Symbols $\overline{t-a-x} \times \overline{a+x} : v^2 :: t : p$, or $\overline{at-aa} + \overline{t-2a} \times x - x^2 : v^2 :: t : p$. Whence we have

$$1^\circ.\ v = \sqrt{\frac{apt-aap}{t} + \frac{pt-2ap}{t}x - \frac{p}{t}x^2}.$$

Which according to the different Values of $a\alpha = a$, will appear differently: for if a be less than $\frac{1}{2}t$, the Order of the Signs will be

$$+ + -.$$

If a be greater than $\frac{1}{2}t$, the Order is - - $+ - -$.

303. But if α fall in with the Center so that $\alpha B = x = AB$, and $a = \frac{1}{2}t$, then the Ordinate appears in this Form

$$2^\circ.\ v = \sqrt{\frac{1}{4}t - \frac{p}{t}x^2}.$$

304. Again, if you suppose $a = a\alpha$ to vanish, by which means $\alpha B = x = aB$, then the Ordinate appears thus

$$3^\circ.\ v = \sqrt{px - \frac{p}{t}x^2}.$$

305. Now, if instead of supposing the Point α the Beginning of the Abscis, to fall betwixt a and B, you should suppose B to fall in betwixt the Points a and α, so that $x = \alpha B$ become negative, then the first Form of the Ordinate will turn into this other

$$4^\circ.\ v = \sqrt{\frac{apt-aap}{t} - \frac{pt-2ap}{t}x - \frac{p}{t}x^2}.$$

Where you'll have different Orders of the Signs according as a is less or greater than $\frac{1}{2}t$. For if it be $a = a\alpha$ less than $\frac{1}{2}t$, the Order of the Signs will be this

$$+ - -.$$

But if a be greater than $\frac{1}{2}t$, but so as that the Point α still fall within the Ellipse, and therefore that a be less than t, then the Order of the Signs will be

$$+ + -.$$

306. Again, supposing the same things: only that now the Point α fall in with the Center so that $a = (a\alpha =)\ \frac{1}{2}t$, then the 4th Form appears thus

$$5^\circ.\ v = \sqrt{\frac{1}{4}pt - \frac{p}{t}x^2}.$$

The very same with the second.

307. And by supposing the Point α to fall in with the Point K, so that $a = (a\alpha =)\ t$ you'll obtain

$$6^\circ.\ v = \sqrt{px - \frac{p}{t}x^2}.$$

The

The very same with the third.

308. But now the same things remaining as in Form fourth, only suppose $a = \alpha a$ to be greater than t, so that the Beginning of the Abscirs lye beyond K, then you'll have that Form always putting on this Order of Signs

$$= + -.$$

309. Finally, the same things remaining as mentioned in Form first: only now suppose the Point α to fall without the Ellipse, beyond the Vertex a, then the first Form will appear thus

$$7^{\circ}.\ v = \sqrt{-\frac{apt + a^2p}{t} + \frac{pt + 2ap}{t}x - \frac{p}{t}x^2}.$$

Which always gives this Order of Signs, *viz.* $- + -$.

310. By considering the several Forms under which the Ordinate of the Conic Section appears in the foregoing Articles, it will be found, that, whatever be the Signs, or Proportions of the given Quantities d, e, f, g, &c. in the Equations expressing the Nature of the Curves, contained in the several Orders succeeding the first (of which we spoke already: *) and whatever be the Value of v and Υ in the Table, which denote the Ordinates of the Conic Sections, by the Construction of which, these Curves are determined, you'll always find some of these Forms, that may be compared with the Value of v and Υ in the Table. For there are three general Forms, under which the Value of v appears in the Orders succeeding the first in the Table: which may be represented without the Signs thus. $1^{\circ}.\ v = \sqrt{.\ \alpha\ .\ \beta x^2}$. $2^{\circ}.\ v = \sqrt{.\ \alpha x\ .\ \beta x^2}$. $3^{\circ}.\ v = \sqrt{.\ \alpha\ .\ \beta x\ .\ \gamma x^2}$. The first two admit of three Varieties of the Signs, each: the last admits of seven Varieties, all possible. If it be $v = \sqrt{\alpha + \beta x^2}$, you have it Art. 295 *: if it be $v = \sqrt{\alpha - \beta x^2}$, you have it Art. 303 and 306 †: if $v = \sqrt{-\alpha + \beta x^2}$, you see it in Art, 300 ‡: The Varieties of the second are contained in Art. 298, 301, 304, 307 ‖. And the seven possible Cases of the third are contained in Art. 293, 294, 296, 297, 299, 302, 305, 308, 309 *. By which it appears, that there is no Case can happen, with respect to the various Dispositions of the Signs, in any possible Value of the Ordinate of the Conic Sections in the Table, but you'll find a corresponding Form of the Ordinate in these Articles; and in several Cases, different corresponding Forms; so that in all possible Cases, you may make Constructions of Conic Sections; with which, the several Species's of Curves in all the different Orders, according to all possible Varieties in the Signs, and Proportions of the Quantities, d, e, f, &c. may be *geometrically* compared. In order to which Construction, you need only to compare the homologous or

* Art. 289—292. of this Explication.

* Of this Explication.

† Of this Explication.

‡ Of this Explication.

‖ Of this Explication.

* Of this Explication.

correſpondent Terms: in doing of which if any impoſſible Root or Value ariſe, by comparing with one Form, you muſt make the Compariſon with another.

311. Thus if it were required to conſtruct the Conic Section neceſſary to the Quadrature of the Curve belonging to Species firſt Order fifth expreſſed by this Equation $\frac{dz^{\eta-1}}{-e+fz^{\eta}+gz^{2\eta}}=y$: I find the Ordinate of the Conic Section in the Table, agreeing to this to be $\sqrt{\frac{d}{g}+\frac{f^2+4eg}{4g^2}x^2}$ * $=v$: therefore I compare this with the Form of the Ordinate contained in Art. 295 *: which ſtands thus $v=\sqrt{\frac{1}{4}tt+\frac{t}{p}x^2}$ and belongs to Fig. 2. p. 25: wherefore I compare the correſponding Terms or Parts of theſe two Expreſſions of the Value of v: which gives theſe two Equations.

*Art. 286. of this Explication.

* Of this Explication.

$$1^{\circ}.\ \tfrac{1}{4}tt=\frac{d}{g}.\quad 2^{\circ}.\ \frac{t}{p}=\frac{f^2+4eg}{4g^2}.$$

By the firſt you find $t=2\sqrt{\frac{d}{g}}$: inſert this Value of t in the ſecond, and it becomes $\frac{2}{p}\sqrt{\frac{d}{g}}=\frac{ff+4eg}{4gg}$, whence you obtain $p=\frac{8g^2}{ff+4eg}\sqrt{\frac{d}{g}}$: ſo that the tranſverſe Axis and Parameter of the Hyperbola being known, the Hyperbola may be conſtructed: with which the propoſed Curve may be geometrically compared.

312. Again, ſuppoſe the Curve was propoſed, whoſe Property is thus expreſſed $dz^{\eta-1}\sqrt{e+fz^{\eta}+gz^{2\eta}}=y$: which belongs to Species ſecond Order ſeventh, you find the Ordinate of the Conic Section, with which it is compared, in the Table to be $v=\sqrt{e+fx+gx^2}$. Where, ſince all the Quantities e, f and g are poſitive, it may be compared with the Form contained in Art. 293 *; which ſtands thus $v=\sqrt{\frac{a^2t+\frac{1}{4}ptt}{p}+\frac{2at}{p}x+\frac{t}{p}x^2}$, belonging to Fig. 2. p. 25: or, it may be compared with the Form contained in Art. 296 *, *viz.* $v=\sqrt{\frac{apt+aap}{t}+\frac{pt+2ap}{t}x+\frac{p}{t}x^2}$, which belongs to Fig. 3. p. 25. By the firſt you have theſe three Equations.

* Of this Explication.

* Of this Explication.

$$1^{\circ}.\ \frac{a^2t+\frac{1}{4}ptt}{p}=e.\quad 2^{\circ}.\ \frac{2at}{p}=f.\quad 3^{\circ}.\ \frac{t}{p}=g.$$

From which, ſuppoſing e, f, g given, you'll obtain the Values of t, p and a, *viz.*

$t=$

$t = \sqrt{\frac{4eg-ff}{g}}.\ p = \frac{1}{g}\sqrt{\frac{4eg-ff}{g}}.\ a = \frac{f}{2g}.$

By the ſecond Compariſon you have theſe three Equations, *viz.*

$1°.\ \frac{apt + a^2p}{t} = e.\quad 2°.\ \frac{pt + 2ap}{t} = f.\quad 3°.\ \frac{p}{t} = g.$

From whence you obtain the Values of t, p and a, *viz.*

$t = \sqrt{\frac{ff-4eg}{g}}.\ p = \sqrt{ff-4eg}.\ a = \frac{f-\sqrt{ff-4eg}}{2g}.$

So that the Conſtructions of the Hyperbolas are manifeſt: only it muſt be obſerved that if $4eg$ exceed ff, you muſt uſe the Conſtruction, ariſing from the firſt Compariſon, repreſented at Fig. 2. p. 25: but if f^2 exceed $4eg$, you muſt make the Conſtruction that ariſes from the ſecond Compariſon, and is repreſented at Fig. 3. p. 25: the Reaſon of which is manifeſt, from the Conſideration of the radical Expreſſions.

313. And if the Signs of one or two of the Quantities e, f, g be negative, you'll always find one or more Forms of Ordinates in the preceding Articles, with which the Ordinate of the Conic Section in the Table may be compared; and one or more Conſtructions made accordingly: by means of which the Area of any ſuch Curve may be found. And ſo in other Caſes.

314. 5°. It muſt likewiſe be obſerved that when the Signs of the numeral Indexes η and θ are negative in the Equation expreſſing the Nature of the Curve, they muſt be changed in the Values of the Areas. Thus in the Curves belonging to Species third Order fourth: which are thus expreſſed $\frac{d}{z^{2\eta+1}\sqrt{e+fz^\eta}} = y$; if η be made negative, ſo that it ſtands thus $\frac{d}{z^{-2\eta+1}\sqrt{e+fz^{-\eta}}} = y$, by which both the Sign of η and $\theta = 2\eta$ are changed, you muſt change the Sign of η, both in the Equation expreſſing the Relation of the Abſciſſes, and in the Value of the Area: by which means you'll have $z^\eta = x$, and $t = -\frac{d}{\eta e} \times \overline{3s-2xv}$: that is, if the Value of the Area was formerly $\frac{d}{\eta e} \times \overline{3s-2xv}$, it will now, by the Change of the Sign of η, become $\frac{d}{\eta e} \times \overline{2xv-3s}$: and ſo of others. You may likewiſe obſerve that, when this Change is made upon the Sign of η, the Species of the Curve may belong to a new Order. Thus in the preſent Caſe, when you have $\frac{d}{z^{-2\eta+1}\sqrt{e+fz^{-\eta}}} = y$, or $dz^{2\eta-1} \times \overline{e+fz^{-\eta}}|^{-\frac{1}{2}} = y$, it is the ſame

 thing

correspondent Terms: in doing of which if any impossible Root or Value arise, by comparing with one Form, you must make the Comparison with another.

311. Thus if it were required to construct the Conic Section necessary to the Quadrature of the Curve belonging to Species first Order fifth expressed by this Equation $\frac{dz^{\eta-1}}{-e+fz^{\eta}+gz^{2\eta}}=y$: I find the Ordinate of the Conic Section in the Table, agreeing to this to be $\sqrt{\frac{d}{g}+\frac{f^2+4eg}{4g^2}x^2}$ * $=v$: therefore I compare this with the Form of the Ordinate contained in Art. 295 *: which stands thus $v=\sqrt{\frac{1}{4}tt+\frac{t}{p}x^2}$ and belongs to Fig. 2. p. 25: wherefore I compare the corresponding Terms or Parts of these two Expressions of the Value of v: which gives these two Equations.

* Art. 286. of this Explication.
* Of this Explication.

$$1°.\ \tfrac{1}{4}tt=\frac{d}{g}.\quad 2°.\ \frac{t}{p}=\frac{f^2+4eg}{4g^2}.$$

By the first you find $t=2\sqrt{\frac{d}{g}}$: insert this Value of t in the second, and it becomes $\frac{2}{p}\sqrt{\frac{d}{g}}=\frac{ff+4eg}{4gg}$, whence you obtain $p=\frac{8g^2}{ff+4eg}\sqrt{\frac{d}{g}}$: so that the transverse Axis and Parameter of the Hyperbola being known, the Hyperbola may be constructed: with which the proposed Curve may be geometrically compared.

312. Again, suppose the Curve was proposed, whose Property is thus expressed $dz^{\eta-1}\sqrt{e+fz^{\eta}+gz^{2\eta}}=y$: which belongs to Species second Order seventh, you find the Ordinate of the Conic Section, with which it is compared, in the Table to be $v=\sqrt{e+fx+gx^2}$. Where, since all the Quantities e, f and g are positive, it may be compared with the Form contained in Art. 293 *; which stands thus $v=\sqrt{\frac{a^2t+\frac{1}{4}ptt}{p}+\frac{2at}{p}x+\frac{t}{p}x^2}$, belonging to Fig. 2. p. 25: or, it may be compared with the Form contained in Art. 296 *, *viz.* $v=\sqrt{\frac{apt+aap}{t}+\frac{pt+2ap}{t}x+\frac{p}{t}x^2}$, which belongs to Fig. 3. p. 25. By the first you have these three Equations.

* Of this Explication.
* Of this Explication.

$$1°.\ \frac{a^2t+\frac{1}{4}ptt}{p}=e.\quad 2°.\ \frac{2at}{p}=f.\quad 3°.\ \frac{t}{p}=g.$$

From which, supposing e, f, g given, you'll obtain the Values of t, p and a, *viz.*

$t=$

$$t = \sqrt{\frac{4eg - ff}{g}}.\ p = \frac{1}{g}\sqrt{\frac{4eg - ff}{g}}.\ a = \frac{f}{2g}.$$

By the ſecond Compariſon you have theſe three Equations, *viz.*

$$1^\circ.\ \frac{apt + a^2p}{t} = e. \quad 2^\circ.\ \frac{pt + 2ap}{t} = f. \quad 3^\circ.\ \frac{p}{t} = g.$$

From whence you obtain the Values of t, p and a, *viz.*

$$t = \sqrt{\frac{ff - 4eg}{g}}.\ p = \sqrt{ff - 4eg}.\ a = \frac{f - \sqrt{ff - 4eg}}{2g}.$$

So that the Conſtructions of the Hyperbolas are manifeſt: only it muſt be obſerved that if $4eg$ exceed ff, you muſt uſe the Conſtruction, ariſing from the firſt Compariſon, repreſented at Fig. 2. p. 25: but if f^2 exceed $4eg$, you muſt make the Conſtruction that ariſes from the ſecond Compariſon, and is repreſented at Fig. 3. p. 25: the Reaſon of which is manifeſt, from the Conſideration of the radical Expreſſions.

313. And if the Signs of one or two of the Quantities e, f, g be negative, you'll always find one or more Forms of Ordinates in the preceding Articles, with which the Ordinate of the Conic Section in the Table may be compared; and one or more Conſtructions made accordingly: by means of which the Area of any ſuch Curve may be found. And ſo in other Caſes.

314. 5°. It muſt likewiſe be obſerved that when the Signs of the numeral Indexes η and θ are negative in the Equation expreſſing the Nature of the Curve, they muſt be changed in the Values of the Areas. Thus in the Curves belonging to Species third Order fourth: which are thus expreſſed $\frac{d}{z^{2\eta+1}\sqrt{e + fz^{\eta}}} = y$; if η be made negative, ſo that it ſtands thus $\frac{d}{z^{-2\eta+1}\sqrt{e + fz^{-\eta}}} = y$, by which both the Sign of η and $\theta = 2\eta$ are changed, you muſt change the Sign of η, both in the Equation expreſſing the Relation of the Abſciſſes, and in the Value of the Area: by which means you'll have $z^{\eta} = x$, and $t = -\frac{d}{\eta e} \times \overline{3s \div 2xv}$: that is, if the Value of the Area was formerly $\frac{d}{\eta e} \times \overline{3s - 2xv}$, it will now, by the Change of the Sign of η, become $\frac{d}{\eta e} \times \overline{2xv - 3s}$: and ſo of others. You may likewiſe obſerve that, when this Change is made upon the Sign of η, the Species of the Curve may belong to a new Order. Thus in the preſent Caſe, when you have $\frac{d}{z^{-2\eta+1}\sqrt{e + fz^{-\eta}}} = y$, or $dz^{2\eta-1} \times \overline{e + fz^{-\eta}}|^{-\frac{1}{2}} = y$, it is the ſame

 thing

thing with $dz^{3\eta-1} \times \overline{fz^{\eta} + ez^{2\eta}}|^{-\frac{1}{2}} = y$, which falls in with Species third of Order eight, by supposing the Term *e* in the Table to be nothing, and substituting *e here* instead of *g there*. By observing which, you'll have $z^{\eta} = x$: $\sqrt{fx + ex^2} = v$, and $t = \left(\frac{3dffs - 2dffxv}{-\eta ffe} =\right) \frac{d}{\eta} \times 2xv - 3s$ as formerly. Thus likewise the Curve defin'd by the Equation $\frac{d}{z}\sqrt{e + fz^{\eta}} = y$, which belongs to Species first Order third, by changing the Sign of η, may be turned into this Shape $\frac{d}{z^{\frac{1}{2}\eta+1}}\sqrt{f + ez^{\eta}} = y$: which belongs to a new Order of Curves, as our Author observes at Art. 64 *.

* Of the Quadrature of Curves.

315. 6°. When the defining Equation of any Curve is compounded of several Equations of different Orders; or of different Species's of the same Order, it's Area must be compounded of the corresponding Areas: taking care however that they may be rightly connected with their proper Signs: for we must not always add or subtract Ordinates to or from Ordinates; and at the same time add or subtract corresponding Areas to or from corresponding Areas: but sometimes the Sum of the Ordinates, and the Difference of the Areas are to be taken to constitute a new Ordinate and corresponding Area. Which must be done, when the constituent Areas are posited on the opposite Sides of the Ordinate. For understanding of which, see Sect. 5. on the Position and Limits of Areas: which Matter must be determined from the Nature and Circumstances of the Curve. And that the cautious Geometrician may not fall into any Inconveniency upon this Account, our Author tells us, he has prefixt their proper Signs to the Values of the Areas, though sometimes negative, as in the fifth and seventh Orders. And as to the Value of *t*, if it comes out affirmative, it denotes the Area of the Curve proposed, adjoining to its Abscifs: and contrarywise, if it be negative, it represents the Area on the other Side of the Ordinate. And as to the Meaning of $+ s$ and $- s$, it has been explain'd formerly *; and will further appear by the subsequent Examples.

*Art 279. of this Explication.

316. 7°. As to the initial Limits of the Areas, in every Case; they are to be determined, as directed in Sect. 5. of the Position and Limits of Areas. From whence also you gather that if any Part of the Area is posited below the Abscifs, *t* will denote the Difference betwixt that and the Part above the Abscifs.

317. 8°. When any Curve is proposed to find it's Area, you must endeavour to bring the Value of it's Ordinate into such a Form, as that you may be able to compare it with some Species of Curves belonging

longing to some of the Orders or Forms contained in the Tables. If this cannot be done, you must transform the Equation, defining the Nature of the Curve, into other Equations of Curves, bearing any assumed Relations to it, after the Manner shewn in Art. 225, 226 *, till you find a Curve, whose Area may be known by the Tables. And when all Endeavours are used, and yet no such Curve can be found: our Author tells us, it may be certainly concluded, that the Curve proposed cannot be compared either with rectilinear Figures; or with the Conic Sections. It may be observed likewise, that, when the Dimensions of the Terms, in the Values of x, v and t shall ascend too high or descend too low, in comparing any proposed Curve with the Forms of Curves in the Table, you may reduce them to a just Degree, by dividing or multiplying so often by any given Quantity, which may be supposed to perform the Office of Unity, as shall be necessary to bring these Terms to the just Number of Dimensions: as will appear in the subsequent Examples.

318. 9°. With respect to the Continuation of the several Forms, both forwards and backwards, let the following things be observed. 1°. The Series's belonging to Forms first, second, fifth, sixth, ninth and tenth may be carried forwards according to the Order of Progression in the Table by Division; as was said in the first general Remark *. 2°. Those belonging to Forms third, fourth, seventh and eighth are continued forwards by the two Theorems in Art. 203 and 206 *: and the Series's of Form eleventh by the Theorem Step fifth, Art. 219 *. But with respect to the Continuation of the Series's in the several Forms, the contrary way, by going backwards from the first Species in each Order, this cannot be done in every one of them, by Propositions seventh and eighth. Wherefore, 3°. It may be done in Forms second, sixth, seventh and eleventh, *viz.* in Form second, by Art. 203 *: in the sixth and seventh, by Art. 208 †: and in Form eleventh, by Art. 219 ‡. In the other Forms we cannot pass from the first Species's in the Table to the Species's immediately before them, by going backwards, by means of Propositions seventh and eighth, as will appear by consulting what we have said in Art. 209 *. Notwithstanding, these other Forms may be continued backwards, in the following Manner. 4°. The Series's in Form first, fifth and eighth thus, let p denote any positive Integer, so that $\frac{dz^{-pn-1}}{e+fz^{n}}=y$ may denote any Species in the backward Progression of Form first: divide the Numerator and Denominator by z^{n}, and it stands thus $\frac{dz^{-pn-n-1}}{f+ez^{-n}}=y$, which

* Of this Explication.

* Art. 285. of this Explication.

* Of this Explication.

* Of this Explication.

* Of this Explication.

† Of this Explication.

‡ Of this Explication.

* Of this Explication.

which belongs to some Species in the other Progression, by only changing η into $-\eta$. See Remark fifth *. The same way you proceed in Form fifth and eighth: by converting $\frac{dz^{-p\eta-1}}{e+fz^{\eta}+gz^{2\eta}}$, into $\frac{dz^{-p\eta-2\eta-1}}{g+fz^{-\eta}+ez^{-2\eta}}$ in Form fifth; and $\frac{dz^{-p\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}}$, into $\frac{dz^{-p\eta-\eta-1}}{\sqrt{g+fz^{-\eta}+ez^{-2\eta}}}$, in Form 8th. The same might be done likewise in Forms second and sixth. 5°. As to Forms third and fourth, the Curves belonging to the backward Progression are quadrable, and constitute the Curves of Forms third and fourth in Table 1. See Schol. 2. Art. 209 *. Wherefore, all that remains is to shew after what manner the Series of Forms ninth and tenth are to be carried backwards. 6°. Now in order to find the Area of the Curve whose Ordinate is $\frac{dz^{-1}\sqrt{e+fz^{\eta}}}{g+hz^{\eta}}$, let A and B denote the Areas of two Curves whose Ordinates are $\frac{z^{\eta-1}\sqrt{e+fz^{\eta}}}{g+hz^{\eta}}$, and $\frac{z^{-1}\sqrt{e+fz^{\eta}}}{\overline{g+hz^{\eta}}|^{0}} = z^{-1}\sqrt{e+fz^{\eta}}$: the first of which belongs to Species first of this ninth Form; and the other to Species first Form third, therefore A and B are given. Now if you take the indefinite Quantities p and q, and multiply these Ordinates and Areas by them, and proceed as in Art. 219 *, you'll have, by putting $R=e+fz^{\eta}$ and $S=g+hz^{\eta}$, $pz^{\eta}\times z^{-1}R^{\frac{1}{2}}S^{-1}$, and $q\times\overline{g+hz^{\eta}}\times z^{-1}R^{\frac{1}{2}}S^{-1}$, Ordinates to the Curves whose Areas are pA, and qB; and by taking the Sum of the Ordinates and Areas, it is $qg+\overline{qh+p}\times z^{\eta}\times z^{-1}R^{\frac{1}{2}}S^{-1}$, and $pA+qB$, for corresponding Ordinate and Area: whence, by putting $qh+p=0$ or $p=-qh$; and reducing, you'll obtain $\frac{B-hA}{g}$ the Area belonging to the Ordinate $\frac{z^{-1}\sqrt{e+fz^{\eta}}}{g+hz^{\eta}}$, or $\frac{dB-dhA}{g}$ belonging to the Ordinate $\frac{dz^{-1}\sqrt{e+fz^{\eta}}}{g+hz^{\eta}}$: which was that sought. After the same manner, from the Areas of the Curves whose Ordinates are $\frac{z^{\eta-\theta}}{g+hz^{\eta}\sqrt{e+fz^{\eta}}}$ and $\frac{z^{-1}}{\sqrt{e+fz^{\eta}}}$, belonging to Species's first Forms tenth and fourth, you may find the Area belonging to the Ordinate

*Art. 314. of this Explication.

* Of this Explication.

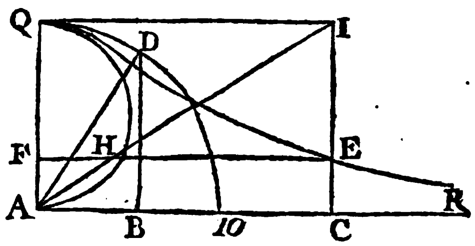
* Of this Explication.

nate $\frac{dz^{-1}}{g+hz^{\eta}\sqrt{e+fz^{\eta}}}$: which is that immediately preceding Species first of Form tenth: and so these Curves being found which immediately precede the first Species's in Forms ninth and tenth, the rest of the backward Progression *in inf.* may be found by Prop. 8, according to the Theor. Step fifth Art. 219 *. And thus I have shewn particularly, how the Series's of Curves belonging to each Form, may be continued both forwards and backwards *in inf.*

* Of this Explication.

And now I proceed to shew the Use of the preceding Tables of Quadratures in some Examples.

319. Example 1. Let QER be a Conchoidal of such a kind, that the Semicircle QHA being described, and AC being erected perpendicular to the Diameter AQ: if the Parallelogram QACI be completed, the Diagonal AI be drawn, meeting the Semicircle in H; and the Perpendicular HE be let fall to IC, then the Point E will describe a Curve, whose Area ACEQ is sought.



Therefore making $AQ = a$, $AC = z$, $CE = y$: because of the continual Proportionals AI, AQ, AH, EC *, it will be EC or $y = \frac{a^3}{a^2+z^2}$.

* Elem. Prop. 4. B. 6.

Now that this may acquire the Form of the Equations in the Tables, make $\eta = 2$, and for z^2 in the Denominator write z^{η}, and $a^3 z^{\frac{1}{2}\eta - 1}$ for a^3 or $a^3 z^{1-1}$ in the Numerator, and there will arise $y = \frac{a^3 z^{\frac{1}{2}\eta - 1}}{a^2 + z^{\eta}}$, an Equation of the first Species of the second Order of Table second, which stands thus $\frac{dz^{\frac{1}{2}\eta - 1}}{e+fz^{\eta}} = y$: so that the Terms being compared it will be $d = a^3$, $e = a^2$, and $f = 1$; and further $\sqrt{\frac{a^3}{a^2+z^2}} = x$; $\sqrt{a^3 - a^2x^2} = v$ and $xv - 2s = t$.

Now that the Values found of x and v may be reduced to a just Number of Dimensions, chuse any given Quantity as a, by which, as Unity, a^3 may be multiplied once, in the Value of x; and in the Value of v, a^3 may be divided once, and a^2x^2 twice. And by this means you'll obtain $\sqrt{\frac{a^4}{a^2+z^2}} = x$, $\sqrt{a^2 - x^2} = v$ and $xv - 2s = t$.

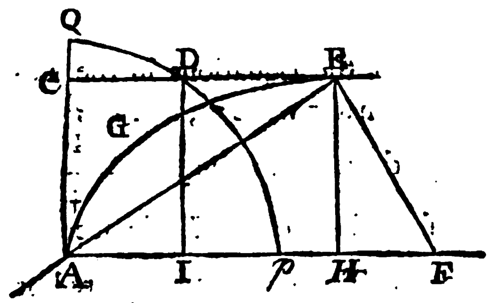
Where

Where it is evident, that, ſince we have $\sqrt{a^2 - x^2} = v$ for the Ordinate of the Conic Section, with which the propoſed Curve is compared, that Conic Section becomes a Circle of which $a = AQ$ is the Radius: for $\sqrt{a^2 - x^2} = v$ is the Equation to ſuch a Circle. Which likewiſe appears by comparing this Value of v, with that contained in Art. 303 *, *viz.* $\sqrt{\frac{1}{4}pt - \frac{p}{t}x^2} = v$: which gives $\frac{p}{t} = 1$ and $\frac{1}{2}t = a$: that is the tranſverſe Axe and it's Parameter are equal, and either of them equal to $2a$. Further, ſince it is $x = \sqrt{\frac{a^4}{a^2 + z^2}} = \sqrt{\frac{AQ^{qq}}{AQ^q + AC^q}} = \frac{AQ^q}{AI} = AH$, the Conſtruction will be manifeſt: which is this.

* Of this Explication.

From the Center A with the Radius AQ deſcribe the Quadrant QDP, take in the Radius AP, $AB = x = AH$, draw BD perpendicular to AB, ſo is $BD = v$, $ABDQ = s$ and $ABD = \frac{1}{2}xv$: and therefore $xv - 2s = 2ABD - 2ABDQ = -2AQD$, or $xv - 2s = 2ABD + 2BDP$ * $= 2DAP$: the laſt of theſe Values is affirmative, and therefore denotes the Area ACEQ adjacent to the Abſciſs AC, and lying on the nearer Side of the Ordinate; the negative Value $-2AQD$ belongs to the Area RECR upon the further Side of the Ordinate, adjacent to the Abſciſs produced infinitely.

*Art. 279. of this Explication.



320. Ex. 2. Let AGE be a Curve, which is deſcribed by the angular Point E of the Norma AEF, while one of the Legs AE, being indefinite towards A, paſſes continually through that given Point; and the other EF of a given Length, ſlides upon the right Line AF given in Poſition. Let fall EH perpendicular to AF, and complete the Parallelogram AHEC; and calling $AC = z$, $CE = y$ and $EF = a$: becauſe HF, HE, HA are continual Proportionals *; and $HF = \sqrt{EF^q - EH^q} = \sqrt{a^2 - z^2}$, you'll have HA or $y = \frac{z^2}{\sqrt{a^2 - z^2}}$.

* Prop 8. B. 6. Elem.

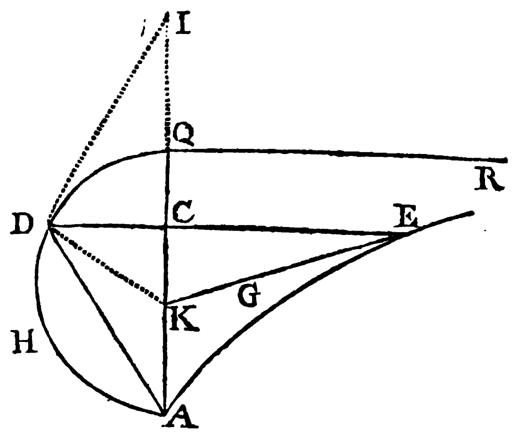
Now that the Area AGEC may be known, by comparing the Equation to the Curve with the Equation to ſome Species of Curves in the Table, ſuppoſe $z^2 = z^\eta$ or $2 = \eta$, then you'll have $\frac{z^{\frac{3}{2}\eta - 1}}{\sqrt{a^2 - z^\eta}} = y$.

Where

Where, becauſe z in the Numerator is of a fracted Dimenſion, and no Equation in the Table can be found, with which it may be immediately compared, I divide the Numerator and Denominator by $z^{\frac{1}{2}\eta}$ or $\sqrt{z^\eta}$: by which means, it will be turned into this Shape $\frac{z^{\eta-1}}{\sqrt{-1+aaz^{-\eta}}}$ or $\frac{1}{z^{-\eta+1}\sqrt{-1+a^2z^{-\eta}}}=y$: which I find to be an Equation of the ſecond Species of Order fourth Table ſecond, which ſtand thus $\frac{d}{z^{\eta+1}\sqrt{e+fz^\eta}}=y$. By comparing of which Values, you have $d=1$, $e=-1$, and $f=a^2$: and becauſe η is negative, and equal to 2, it is $z^2=(\frac{1}{z^{-\eta}}=)\,x^2$ *; $\sqrt{a^2-x^2}=v$; and $s-xv=t$. *Art. 314. of this Explication.

Now ſince it is $v=\sqrt{a^2-x^2}$, which is an Equation to a Circle having $a=EF$ for it's Radius, hence it appears that the Curve propoſed is compared with a Circle whoſe Radius is equal to EF: and ſince moreover it is $x=z=AC$, the Conſtruction will be thus. Upon the given Point A as a Center with the Radius $AQ=EF$, deſcribe the Quadrant QDP, cutting CE in the Point D: complete the Parallelogram ACDI, then is $AC=z=x$, $CD=v$, $ACDP=s$, $ACDI=xv$: and the Area ſought $AGEC=(s-xv=)\,ACDP-ACDI=IDP$.

321. Ex. 3. Let AGE be the Ciſſoid, belonging to the Circle ADQ, deſcribed with the Diameter AQ. Let DCE be drawn perpendicular to the Diameter, and meeting the Curves in D and E. And calling $AC=z$, $CE=y$ and $AQ=a$: becauſe of CD, CA, CE continual Proportionals, from the Nature of the Ciſſoid, it will be CE or $y=\frac{z^2}{\sqrt{az-zz}}$, that is by dividing by z, $y=\frac{z}{\sqrt{az^{-1}-1}}=\frac{1}{z^{-2+1}\sqrt{-1+az^{-1}}}$: where becauſe it is $\eta=-1$ or $z^\eta=z^{-1}$, it will be an Equation of the third Species of Order fourth Table ſecond, which ſtands thus $\frac{d}{z^{2\eta+1}\sqrt{e+fz^\eta}}=y$. So that by com-



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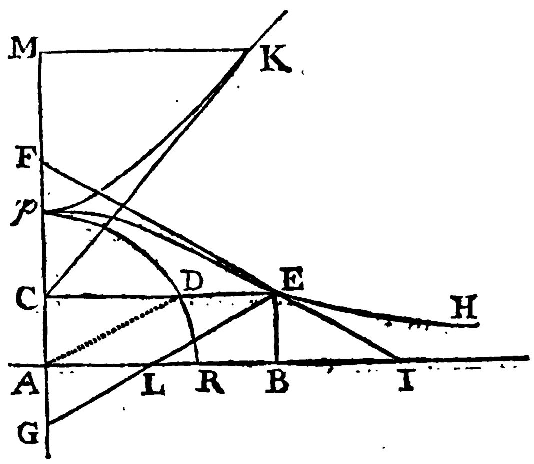
paring the Terms it will be $d = 1$, $e = -1$. $f = a$. $z = (\frac{1}{z^n} =) x$, $\sqrt{ax - x^2} = v$: and $3s - 2xv = t$. Now since it is $v = \sqrt{ax - x^2}$, it is evident as formerly that the Cissoid is compared with the Circle whose Diameter is $a =$ AQ, that is ADQ. Therefore since $z =$ AC $= x$, you'll have DC $= v$, ACDH $= s$, and (drawing AD) ΔACD $= \frac{1}{2}xv$: consequently 3ACDH $-$ 4ACD $= 3s - 2xv = t =$ Area of the Cissoid ACEGA, or by adding ACD to both; 3 Segments ADHA $=$ Area ADEGA: or 4 Segments ADHA $=$ Area AHDEGA.



322. Ex. 4. Let PE be the first Conchoid of the Ancients, described from the Center G, with the Assymptote AL, and Distance LE. Draw it's Axis GAP, and let fall the Ordinate EC. Then calling AC $= z$, CE $= y$, GA $= b$ and AP $= c$, because it is AC : CE—AL :: GC : CE, you'll thence obtain CE or $y = \frac{b+z}{z}\sqrt{c^2 - z^2}$: *viz.* thus: with the Radius AP describe the quadrantal Arch PDR cutting CE in the Point D, join AD: then because of parallel Lines CA : EL $=$ PA $=$ AD :: AG : GL; therefore the two Triangles ACD, GAL are similar *, and therefore AL $=$ DE * and CE $-$ AL $=$ CD $= \sqrt{cc - z^2}$: by substituting which Value for CE $-$ AL, and the Symbols for the other Quantities, in the preceding Proportion, *viz.* AC : CE $-$ AL :: GC : CE, it is $z : \sqrt{c^2 - z^2} :: b + z : y$; whence $y = \frac{b+z}{z}\sqrt{cc - zz}$ defines the Nature of the Curve: which being a compounded Ordinate, it's two Parts must be considered separately; which two Parts are $\frac{b}{z}\sqrt{cc - zz}$, and $\sqrt{cc - zz}$: the last of which is the Ordinate of a Circle having $c =$ AP for it's Radius, and $z =$ AC for it's Absciss; which therefore is the Line CD. Therefore, when the Area PCE is required, the Area PCD belonging to the Circle PDR, makes one Part of it, and is therefore known. And the other Part DPED, described by the Ordinate DE $= \frac{b}{z}\sqrt{cc - z^2}$, is to be found. Where, if you suppose $z = \eta$, it becomes $\frac{b}{z}\sqrt{cc - z^{\eta}} =$ DE, which belongs to

* Prop. 7. B. 6. Elem.

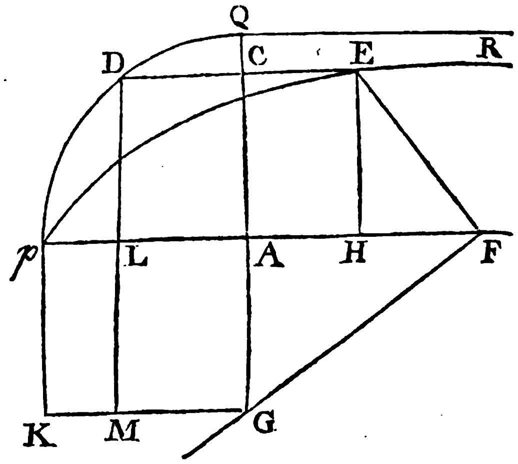
* Prop. 33. B. 1. ibid.

to Species first, Order third of Table second, which stands thus $\frac{d}{z}\sqrt{e+fz^{\eta}}=y$: so that, by comparing the Parts, you'll have $d=b$, $e=c^2$, $f=-1$, $\frac{1}{z}=\sqrt{\frac{1}{z^{\eta}}}=x$, $\sqrt{-1+c^2x^2}=v$. In order to complete the Dimensions, by some given Quantity considered as Unity; which let be c, there will arise $\frac{c^2}{z}=x$ and $\sqrt{-c^2+x^2}=v$: by comparing which with Art. 300 *, you'll find it to belong to an Hyperbola, having it's transverse Axe $2c=2AP$ and it's Latus Rectum the same: and the tabular Area, *viz.* $\frac{4de}{\eta f}\times\overline{\frac{v^3}{2ex}-s}$, becomes $-2bc^2\times\overline{-s+\frac{v^3}{2c^2x}}=2bc^2s-\frac{bv^3}{x}$; or, by completing the Dimensions $\frac{2bs}{c}-\frac{bv^3}{cx}=\frac{2b}{c}\times\overline{s-\frac{v^3}{2x}}=t$. * Of this Explication.

By considering of which things, the Construction is thus. With the Center A and Vertex Principalis P, describe the rectangular Hyperbola PK: take $AM=x$*, a third Proportional to AC and AP, for $x=\frac{c^2}{z}$: draw the Ordinate MK which is equal to v: join KC, which will touch the Hyperbola in the Point K from the Property of the Tangent. Then $PMKP=s$, and $\Delta CMK=\frac{v^3}{2x}$ (for which see Art. 276 *.) Whence $-CKPC=s-\frac{v^3}{2x}$, and $\frac{2GA}{AP}\times -CKPC=\frac{2b}{c}\times\overline{s-\frac{v^3}{2x}}=$ Area PDEP: which comes out negative; because that Area lyes upon the further Side of the Ordinate *. Therefore it is AP : 2AG :: CKPC : DPED the Area required. *Art. 300. of this Explication. * Of this Explication. *Art. 316. of this Explication.

323. Ex. 5. Let the Norma GFE so revolve about the Pole G, as that it's angular Point F may continually slide upon the right Line AF, given in Position: then conceive the Curve PE to be described by any Point E, in the Leg EF, while the other Leg FG always passes through the given Point G. Now that the Area of this Curve may be found, let fall GA and EH perpendicular to the right Line AF, and completing the Parallelogram AHEC, call $AC=z$, $CE=y$, $AG=b$ and $EF=c$; and because of the Proportionals HF : EH :: AG : AF, *i. e.* $\sqrt{c^2-z^2} : z :: b$: AF, you'll have $AF=\frac{bz}{\sqrt{cc-zz}}$: therefore CE or $y=(AF-HF=)\ \frac{bz}{\sqrt{cc-zz}}-\sqrt{cc-zz}$. But whereas $\sqrt{cc-zz}$ is the Ordinate of a Circle, described with the Semi-diameter c

$= EF = AP$: about the Center A, let ſuch a Circle PDQ be deſcribed, which CE produced meets in D; then it will be $DE = (DC + CE =) \frac{bz}{\sqrt{cc - zz}}$: by the Help of which Equation, the Area PDEP on the nearer Side of the Ordinate, or DERQ on the further Side of the Ordinate, remains to be determined. Suppoſe therefore $\eta = 2$, and it will be $\frac{bz^{\eta-1}}{\sqrt{cc - z^{\eta}}}$, an Equation of the 1ſt Species Order 4th Tab.



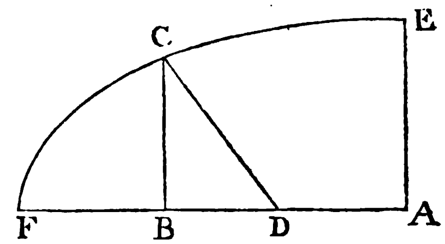
firſt, which ſtands thus $\frac{dz^{\eta-1}}{\sqrt{e + fz^{\eta}}} = y$: ſo that the Terms being compared, it will be $d = b$, $e = cc$, and $f = -1$: ſo that $t = (\frac{2d}{\eta f}R =) - b\sqrt{cc - zz}$ the Value of the Area ſought: which being negative, deſigns the Area lying beyond the Line DE.

In order therefore to know it's initial Limit upon that further Side, put $-b\sqrt{cc - zz} = 0$; thence ariſes $z = c = AQ$: wherefore, drawing the Ordinate QR, you have that for the Limit, and conſequently the Area $DQRED = b\sqrt{cc - zz} =$ Rectangle contain'd under AG and CD: ſo that if you complete the Parallelogram PAGK, and draw DLM perpendicular to AP, you'll have the Area DQRED = Parallelogram AM.

If you would know the Quantity of the Area PDE, poſited at the Abſciſs AC, and co-extended with it, without inquiring into the Limit QR, ſubtract the Value of t at the Beginning of the Abſciſs, *i. e.* when $z = 0$, which therefore is $-bc$, from it's Value, when $z = AC$, *viz.* $-b\sqrt{cc - zz}$; and the Difference $bc - b\sqrt{cc - zz} = AG \times AP - AG \times AL$ or the Rectangle PKML, is the Value of the Area PDE ſought.

324. Ex. 6. Let AE and AF be two given right Lines at right Angles to each other: and let the right Line CD = AE, ſlide along AF with it's Extremity D, and at the ſame time, turn about D as a Pole or Center, in ſuch manner that, if from it's Extremity C you draw

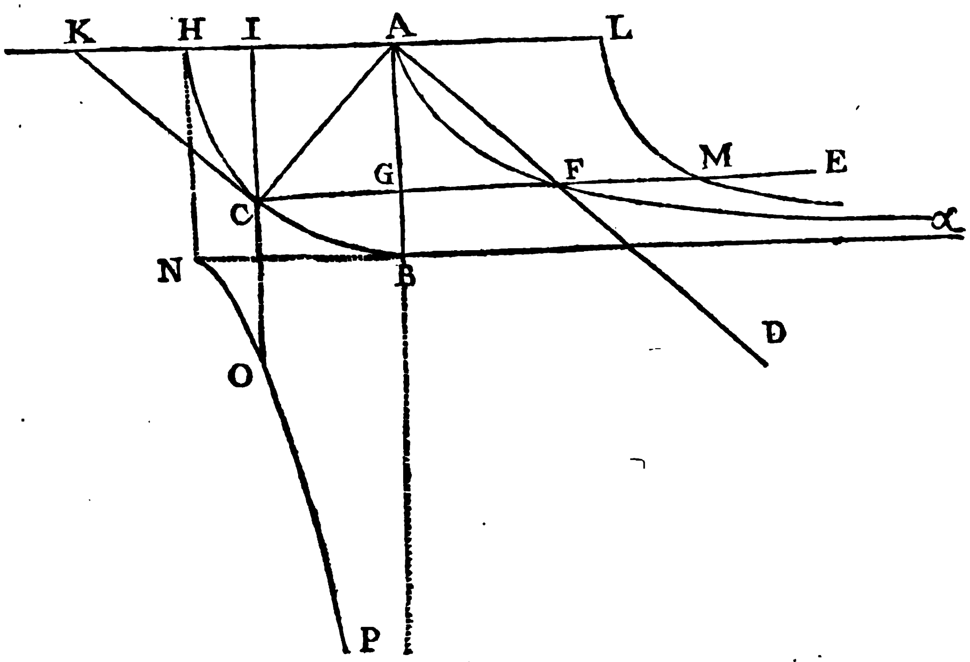
draw CB perpendicular to AF, it ſhall always be AB : BD :: AF : AE, the Point C touches a Curve, whoſe Nature and Area are required.



Call AB $= z$, BC $= y$, AE $= a$, AF $= b$: then becauſe AF $= b$: AE $= a$ :: AB $= z$: BD, hence we have BD $= \frac{az}{b}$: and becauſe of the right-angled Triangle BCD, BC $= \sqrt{\overline{CD^q - BD^q}}$, that is $y = \sqrt{a^2 - \frac{a^2z^2}{b^2}}$: By comparing this with Art. 306 *, where you have $v = \sqrt{\frac{1}{4}pt - \frac{p}{t}x^2}$, it appears to belong to an Ellipſe: and by comparing the homologous Terms, you have $\frac{1}{4}pt = aa$ and $\frac{p}{t} = \frac{a^2}{b^2}$; whence ariſes $t = 2b = 2$AF, and $p = \frac{4a^2}{2b} = \frac{4AE^q}{2AF}$, that is the tranſverſe Axis is 2AF and it's Parameter a third Proportional to 2AF and 2AE; ſo that it's conjugate Axis is 2AE, and A the Center.

* Of this Explication.

325. Ex. 7. Let AB be a given right Line, CAD a rectangular Norma, revolving upon the Point A as a Pole, of which one Leg AC



$=$ AB

$=AB$; the other AD indefinite: and while the Norma revolves, let the indefinite right Line CE drawn through C and cutting AB at right Angles in G, always intersect AD in the Point F: the Point F describes a Curve, whose Area is sought. Call $AG=z$, $GF=y$, $AB=a=AC$: then $CG=(\sqrt{AC^q-AG^q}=)\ \sqrt{aa-z^2}$: and because $CG=\sqrt{a^2-z^2}$, $AG=z$, and $GF=y$ are proportional, hence it is $y=\frac{z^2}{\sqrt{a^2-z^2}}$, the Equation to the Curve whose Area is sought. By comparing which with Ex. 2. Art. 320 *, it appears to be the same Curve: and therefore it's Area is found, as is there shewn, by comparing it with Species second, Form fourth, Table second. But that you may see how one and the same Curve may belong to different Orders and Species's in the Table, you may compare it with Species third, Form eight, Table second, *viz.* $\frac{dz^{3\eta-1}}{\sqrt{e+fz^\eta+gz^{2\eta}}}=y$; by putting $\eta=1=d$ and $f=0$: and moreover $e=aa$, $g=-1$, $z=x$, $\sqrt{a^2-x^2}=v$, and $s-xv=t$, as formerly. Now since it is $x=z$, and $v=\sqrt{a^2-x^2}=\sqrt{a^2-z^2}$, hence it appears that the Conic Section with which the Curve is compared is a Circle, whereof the Radius is $a=AB$, and the Abscis x, beginning at the Center, equal to $z=AG$. Wherefore, if with the Center A and Radius AB, the Quadrant BCH be described, and CI drawn parallel to AB, then $GC=v$, the Rectangle $GI=xv$, and the Area $AGCH=s$: consequently $AGCH-GI=CHI=s-xv=t=$ curvilinear Area AGF required. And if Bα be drawn perpendicular to AB, it is evident it will be Assymptote to the Curve; and that the remaining Part of the curvilinear Area, lying below GF, are infinitely extended along the Assymptote, *viz.* BGFα = Area ABCI: and therefore the whole curvilinear Area ABα = the Quadrant ABCH.

* Of this Explication.

326. Schol. 1. If you draw the Tangent CK, meeting AH produced in the Point K, it appears that the two Triangles AGF, CIK are equal and similar: and therefore the Ordinate $GF=IK$ the Subtangent of the Arch CH; and $AF=CK$ it's Tangent: so that ABαFA is a Figure of Subtangents in the Circle, ordinately applied to the right Sines of the Arches as their Abscisses: any Part of which Figure of Subtangents, as AGF is equal to the correspondent circular Area CHI, and the whole equal to the Quadrant. And universally, since in any Curve whatsoever, if you call z the Abscis, y the Ordinate, and s the Subtangent, you have $\dot{y}:\dot{z}::y:s$ *, or $\dot{y}s=\dot{z}y$, the first of which is

* Art. 42. of this Explication.

is the Fluxion of the Figure of Subtangents applied to the Ordinates or right Sines as their Abſciſſes; and the other, the Fluxion of the curvilinear Area *: [1] hence the Fluents are equal, ſince they both begin to be generated at the ſame time *, *i. e.* the Figure of Subtangents applied to the Ordinates of any Curve for their Abſciſſes, is equal to the Area of the Curve.

* Art. 35. of this Explication.
* Art. 38. of this Explication.

327. Schol. 2. Moreover, it appears, that the Line CGF, in the preceding Figure, is equal to the Secant of the Arch CH: and therefore AHBαFA is equal to a Figure of Secants ordinately applied to the right Sines as their Abſciſſes: which Figure of Secants, therefore, adjacent to the whole Radius, is equal to the Semicircle. It's Deſcription may be thus: produce HA to L, ſo that AL $=$ AH, and take every where upon GF produced, FM $=$ CG, then the Curve LM, which M touches, belongs to a Figure of Secants, ſuch as has been mentioned; having AG $=$ IC for the Abſciſs, and GM $=$ AK for it's Ordinate. And ſo the Area AGML $=$ (AGCHA $+$ CHI $=$) twice Sector ACH: and the whole Area ABαML infinitely extended along the Aſſymptote Bα, is equal to twice the Quadrant AHB: and when the Arch HC $= 45°$, the Ordinate GM biſects the curvilinear Area, making each half equal to the Quadrant. Accordingly GM $=$ $\left(GF + GC = \frac{z^2}{\sqrt{a^2 - z^2}} + \sqrt{a^2 - z^2} =\right) \frac{a^2}{\sqrt{a^2 - z^2}}$ or $\frac{a^2}{z\sqrt{-1 + aaz^{-2}}}$: which belongs to Species 1. Order 4. Table 2, *viz.* $\frac{d}{z\sqrt{e + fz^{\eta}}} = y$; as appears by putting $d = a^2$, $e = -1$, $f = a^2$, $\eta = -2$: ſo that it is $x^2 = \left(\frac{1}{z^{\eta}} =\right) z^2$, or $x = z =$ AG, $v = (\sqrt{a^2 - x^2} = \sqrt{aa - z^2} =)$ CG, and $\frac{4d}{\eta f} \times \overline{\frac{1}{2}xv \div s} = 2 \times \overline{s - \frac{1}{2}xv} =$ 2ACH $=$ AGML required. The ſame may be found by comparing the given Ordinate $\frac{a^2}{\sqrt{a^2 - z^2}}$ with Species 1. Order 8. as in Art. 325 *.

* Of this Explication.

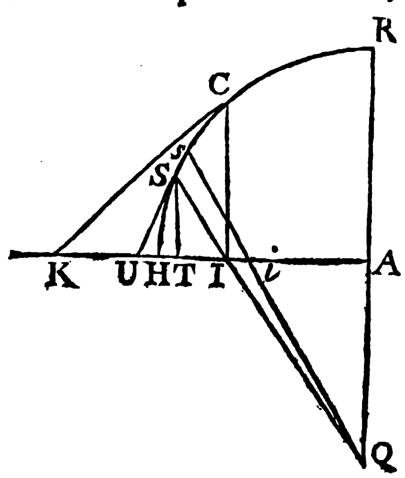
328. Schol. 3. But if the Secants be ordinately applied to the verſed Sines of their Arches, that is, if the rectangular Ordinate IO be always equal to the Secant AK, then AI, AH, AO are proportional: therefore if you complete the Square AN, it is evident that the Curve

1 When I ſay here that $\dot{y}s$ and $\dot{z}y$ are the Fluxions of the Figure of Subtangents and Area of the Curve, I hope the Reader won't miſtake me, or alledge that I have departed from the true Notion of Fluxions laid down in the Beginning of this Work: as if I meant that theſe Expreſſions were in ſtrict Propriety Fluxions or Velocities of flowing: it is enough that they are Repreſentations or the Exponents of the Fluxions and proportional to them, which is all that the Reaſoning requires. This I thought proper to obſerve in this Place once for all: that ſo the Reader may not miſtake me in any Place, where he meets with the like way of ſpeaking.

NOP,

NOP, described by the Point O, is the rectangular Hyperbola having A for it's Center and N for it's Vertex Principalis. Which is the orthographical Projection of the Figure of Secants applied ordinately to the Arches as their Abscisses, these Secants, made Ordinates, being supposed to be perpendicular to the Plane of the Circle, which is itself perpendicular to the Plane of Projection. Thus likewise in the Ellipse and Hyperbola, if you put a for the half Axis and z for the Abscis commencing at the Center, you'll have $\frac{a^2}{z}$ for the Secant: which shews that in these Curves likewise the Secants applied as Ordinates to the Axis, will touch with their Extremities rectangular Hyperbolas, whose Assymptotes are the transverse and conjugate Axes of the Conic Sections, and the Axes of those are to the Axes of these in the subduplicate Ratio of 2 to 1: which rectangular Hyperbolas are likewise the orthographic Projections of the Figures of Secants applied as Ordinates to the Arches for their Abscisses in the Conic Sections, in the same Sense as in the Case of the Circle already mentioned. And the Hyperbola, which is formed by the Secants, in the Case of the Ellipse, ordinately applied to the versed Sines or Sagittas as their Abscisses, is the same with the Hyperbola which is formed by the Secants in the Circle similarly applied, when the Diameter of the Circle is the same with the Axis of the Ellipse: because when a Circle is circumscribed about an Ellipse, or inscribed in it, the Secants of the Arches, whose versed Sines are equal, are themselves equal: as is evident from the common Property of the Tangent in both.

329. Schol. 4. Let the Hyperbola NOP remain as before, *viz.* a Figure of Secants ordinately applied to the versed Sines of their Arches as the Abscisses: and suppose now that the Quadrant ABH is the fourth Part of the primitive Circle, upon whose Plan the Sphere is *stereographically* projected, that primitive Circle being the Equator: in which A is the Representation of the Pole, and AH that of the Half of a meridian Circle.



According to the known Law of this Sort of Projection, the right Line AI is the Tangent of one half of the meridional Arch of which it is the Representation. For in the Figure adjoined, let the Eye be supposed to be at Q, one of the Poles; R being the other, QR the Axis, AH the Line of Measures, lying in the Plane of the Equator: and let

let AH in this Figure, be equal to AH in the preceding: then the Quadrant HR being described is a fourth Part of a meridian Circle: draw QIS, cutting AH in I, and HR in S, then it is evident that AI is the Tangent of the Half of RS. Now suppose the Line QIS to revolve upon Q as a Pole, while IO in the preceding Figure moves parallel to itself, so that AI and HI in both Figures be always of the same Length: and let QIS move into the new Place Q*is*, the two Triangles QS*s*, QI*i*, in their nascent State, are similar: whence S*s* : I*i* :: QS : Q*i* or QI; but, if you draw ST perpendicular to AH at the Point T, from the similar Triangles AIQ, IST, it is QS : QI :: AT : AI, therefore S*s* : I*i* :: AT : AI; but S*s* : I*i* :: Fluxion of HS : Fluxion of HI *, that is, calling HS$=z$ and HI$=x$, $\dot{z} : \dot{x}$ :: (AT : AI ::) Cos. HS : T. $\frac{1}{2}$ comp. HS. Again, in the preceding Figure, F.IN : F.IOHN (:: HN : IO :: AI : AH) :: T. $\frac{1}{2}$ comp. z : R; but F.IN : F.IOHN :: $\frac{\text{F.IN}}{\text{HN}} = \dot{x} : \frac{\text{F.IOHN}}{\text{HN}}$, therefore $\dot{x} : \frac{\text{F.IOHN}}{\text{HN}}$ :: T. $\frac{1}{2}$ comp. z : R. But it was just now demonstrated that $\dot{z} : \dot{x}$ :: Cos. z : T. $\frac{1}{2}$ comp. z; therefore by Equality $\dot{z} : \frac{\text{F.IOHN}}{\text{HN}}$ :: Cos. z : R :: R : Sec. z. But supposing p to express the Length of the meridional Arch HS or z, as protracted in the nautical Chart, according to *Mercator*'s Projection, from the known Property of that Projection, it is $\dot{z} : \dot{p}$ :: R : Sec. z; wherefore, by taking equal Ratios, it is $\dot{z} : \frac{\text{F.IONH}}{\text{HN}}$ :: $\dot{z} : \dot{p}$; whence $\frac{\text{F.IONH}}{\text{HN}} = \dot{p}$, and therefore $\frac{\text{IONH}}{\text{HN}} = p$ * the protracted meridional Arch HS. Wherefore the hyperbolical Area IONH is equal to the protracted meridional Arch multiplied into the Radius. Now from the known Property of the Hyperbola, the Area IONH is the Logarithm or Measure of the Ratio of AH to AI, that is, calling the Rad. AH, 1, the Logarithm of $\frac{1}{1-x} = \frac{\text{R}}{\text{T.}\ \frac{1}{2}\ \text{comp.}\ z}$: and whereas the hyperbolical Logarithm of any Ratio, is to the tabular Logarithm of that Ratio, taken from *Briggs*'s or the common Tables as 2.302585, *&c.* to 1: therefore the Logarithm of $\frac{\text{R}}{\text{T.}\ \frac{1}{2}\ \text{comp. lat.}}$ taken from the common Tables, being multiplied by 2.302585, *&c.* will give the hyperbolical Area IONH, or the Length of the protracted meridional Arch, the Radius being Unity: so that the Rule is this; subtract logarithmical Tangent of $\frac{1}{2}$ comp. lat. from logarithmical Rad. and multiply the Difference by 2.302585, *&c.* Moreover, if you put A and a for a greater and lesser Arch of the Meridian, reckoning from the Equator, and l for an hyperbolical Logarithm: then

* Art. 28. of this Explication.

* Art. 40. of this Explication.

it appears from what has been said, that the protracted Length of the meridional Arch A — a is $l.\frac{R}{T.\frac{1}{2}\text{comp. A}} - l.\frac{R}{T.\frac{1}{2}\text{comp. }a} = l.T.\frac{1}{2}$ comp. $a - l.T.\frac{1}{2}$ comp. A, that is, subtract the Log. of the Tangent of the half Complement of the greater Arch from the Log. of the Tangent of the half Complement of the least Arch, and the Difference is the protracted Length of the Arch A — a: meaning by Logarithm the hyperbolical Logarithm. Wherefore when you take the logarithmical or artificial Tangents from the common trigonometrical Tables, the Difference mentioned above must be multiplied by 2.302585, *&c.* to have the protracted meridional Arch lying betwixt the two Latitudes A and a, the Rad. of the Globe being 1. But if you would have this meridional Arch expressed in the common way by Miles or Minutes, so that each Minute of Latitude be called 1, then the Expression already found must be multiplied by 3437,74677, *&c.* because so many times is one Minute of a great Circle contained in the Radius. Or, by one Operation, multiply the Difference of the artificial Tangents of the half Complements of the Latitudes taken from the common Tables, by 7915,70446789, *&c.* = 2.30258509, *&c.* × 3437,74677078, *&c. e. g.* Let it be required to find the Length of the meridional Arch in *Mercator*'s Chart, lying betwixt the Tropic in Lat. 23° 30′, and the polar Circle in 66° 30′: which being Complements to each other, therefore the Halves of their Complements are 33° 15′, and 11° 45′, whose artificial Tangents are 9.8166580 and 9.3180640, whose Difference is 0.498594, which multiplied by 7915,704, *&c.* gives 3946,722, *&c.* Miles or Minutes of a great Circle, contained in that meridional Arch protracted.

330. Schol. 4. Because this Affair is under Consideration, and the Figures suited to the Purpose, I shall shew another Method of investigating and demonstrating this Property of the meridian Line in *Mercator*'s Projection. In the last Figure draw IC perpendicular to AH, meeting the Circumference of the Circle in C: at C and S draw the Tangents CK, SV, meeting AH produced in K and V. Then, using the Symbols x, z and p as formerly, and supposing the same things as before, from the Property of the Tangent, you have AV : AH : AT ∺ and AK : AH : AI ∺, whence AV : AK :: (AI : AT :: QI : QS :: Ii : Ss ::) $\dot{x} : \dot{z}$, that is AV × $\dot{z}$ = AK × $\dot{x}$: but the first is the Fluxion of the Figure of Secants applied to their Arches as Abscisses; the other is the Fluxion of a Figure of Secants applied to the versed Sines as their Abscisses; the corresponding or synchronal Abscisses being HS and HI: therefore the Areas of these Figures are equal *, that is

* Art. 38. of this Explication.

is the hyperbolical Area HION (in the preceding Figure) is equal to the Area of the Figure of Secants ordinately applied to the Arches, having HS for the Abſciſs, in this laſt Figure. But the Area laſt mentioned is to a Rectangle having it's Baſe equal to HS or x, and Altitude, the Radius, as the protracted Length of the Arch HS, to it's natural Length (for the Fluxions of theſe Figures are as the Secant of HS to Radius) therefore HION : $z \times R :: p : z$; that is HION $\times z = z \times R \times p$, or $\frac{HION}{R} = p$: ſo that when $R = 1$, HION $= p$ the protracted meridional Arch HS: and therefore the reſt follows as before.

And here it may be obſerved that a Method is ſuggeſted of finding a geometrical Curve having it's Area equal to that of a mechanical Curve: ſince it has been demonſtrated that the hyperbolical Area HION is equal to the Area of the Curve which has an Arch of a Circle, *viz.* HS for it's Abſciſs, and the Secant AV for it's Ordinate: of which more immediately.

PROBLEM.

331. *To find the Areas of* mechanical *Curves.*

A Curve, whoſe Nature or Property may be expreſſed by an algebraical Equation of any finite Order or Degree, which defines the Relation of the Abſciſs and Ordinate to each other, is commonly called an *algebraical* or *geometrical* Curve: ſo that, according to the Number of Dimenſions of the Equation by which ſuch Relation is defined, Curves are diſtinguiſhed into different Orders: when the Equation is of two Dimenſions, the Curve is of the firſt Order, the Circle and Conic Sections are the only Curves of this Order: when the Equation is of three Dimenſions, the Curves are of the ſecond Order, ſuch are the cubical Parabola, the Ciſſoid of *Diocles*; and others, an Enumeration of which our learned Author hath made in a ſeparate Treatiſe by itſelf, in which he reduces them all to ſeventy-two Kinds. And after the like manner, Curves of the third, fourth and higher Orders are ſuch, the Relation of whoſe Ordinates to their Abſciſſes, is defined by Equations of four, five or more Dimenſions reſpectively. It amounts to the ſame to ſay that Curves of the firſt, ſecond, third, *&c.* Orders are ſuch as may be cut by a right Line in two, three, four, *&c.* Points reſpectively. And a Curve of the firſt Order is eſteemed a Line of the ſecond Order; a Curve of the ſecond Order, a Line of the third, *&c.* the right Line being the firſt Order of Lines; which can be interſected by another, only in one Point. Accordingly a Curve which may be interſected by a right Line in an infinite Number of Points, is ſaid to be of an infiniteſimal Order: theſe Curves, or ſuch, the Relation of whoſe Abſciſs to the Ordinate cannot be expreſſed by an algebraical Equation of any *finite* Order or Dimenſions, are termed *mechanical* or *tranſcendental*, ſuch as the common Cycloid, Quadratrix, Spiral and infinite others [1].

Theſe mechanical Curves (at leaſt the Order of them) are ſuppoſed to have the Abſciſs or Ordinate equal to ſome known Curve, as a Circle or Conic Section; or even one of ſome higher Order; or elſe to bear ſome known Relation to ſuch Curve-line; or curvilinear Area applied to Unity. Now although the Relation of the Abſciſs and Ordinate in theſe mechanical or tranſcendental Curves, cannot be expreſſed by any algebraical Equation of finite Dimenſions: yet the Relation of the Fluxions of the Abſciſs and Ordinate, may be expreſſed by an algebraical Equation; by means of which we may find a geometrical Curve which has the ſame Fluxion: and thence the Area of the mechanical Curve may be found, by the Quadrature of Curves: of which take the following Examples.

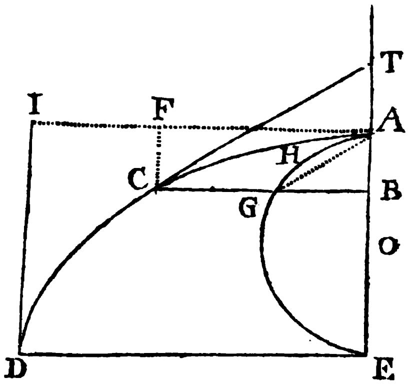
332. Ex. 1. Let the common Cycloid or Trochoid ABCD be propoſed, generated by the Rolling of the Circle AHGE upon the Baſe

[1] *Des Cartes* calls thoſe Curves, in which the Relation of the Abſciſs and Ordinate is expreſſed by an Equation of two Dimenſions, Curves of the firſt Order or Kind, *viz.* the Circle, Ellipſe, Parabola and Hyperbola: but under Curves of the ſecond Order, he reckons both ſuch, the Relation of whoſe Abſciſs and Ordinate is expreſſed by an Equation of three Dimenſions; and likewiſe thoſe, in which it is expreſſed by an Equation of four Dimenſions. And under Curves of the third Kind or Order, he reckons all thoſe in which the Relation of the Abſciſs and Ordinate is defined either by an Equation of four or of five Dimenſions, and ſo on. See his *Geometry*, Book 2. But the Diviſion above, which is Sir *Iſaac Newton's*, is the moſt ſimple and natural. See his *Enumeratio Linearum tertii Ordinis* at the Beginning. And whereas there have been ſome Differences among learned Men, with reſpect to what Curves ought to be called geometrical; and as to their Preference in geometrical Conſtructions of Problems. See our Author's Opinion of this Matter in the Appendix to his *Arith. Univ. de Conſtructione Lineari.* Where he ſhews that the Rule by which to determine the Preference of one Curve to another in Geometry, is the Eaſineſs of the Deſcription; and not the greater Simplicity of the Equation by which it is defined: which laſt is a Conſideration entirely algebraical, otherwiſe the Parabola ought to be preferred to the Circle in geometrical Conſtructions: which no one will affirm.

The general Diviſion of Curves by our Author is into *geometrically rational, and geometrically irrational.* " *Curvas geometricè rationales appello, quarum puncta omnia per longitudines* " *æquationibus definitas,* i. e. *per longitudinum rationes complicatas, determinari poſſunt; cæteraſque (ut Spirales, Quadratrices, Trochoides) geometricè irrationales. Nam longitudines quæ* " *ſunt vel non ſunt ut Numerus ad Numerum (quemadmodum in decimo Elementorum) ſunt arithmeticè rationales vel irrationales.*" Phil. Nat. Prin. Lib. 1. Cor. ad Lem. 28. By which it appears that theſe Curves which he calls geometrically rational, are the ſame with ſuch as we (after others) have called *geometrical* or *algebraical*: and the geometrically irrational, the ſame, we have termed *mechanical* or *tranſcendental*: which include thoſe called *exponential* by Mr. *Leibnitz*, *viz.* ſuch whoſe Equations involve ſome Power of the Abſciſs or Ordinate, that has a variable Quantity for it's Exponent as x^y, x^x &c.

ED.

ED, of which the Diameter AE is the Axis of the Cycloid: from any Point of which as B, having drawn the rectangular Ordinate BGC, meeting the Circumference of the Circle in G, and the Curve in C: complete the Parallelograms ABCF, AEDI; draw the Chord AG; and ſuppoſe CT a Tangent to the Curve at C. The Area ABC is required.



Call AB $= z$, BC $= y$, BG $= v$, AE $= d$: then from the Nature of the Curve (as ſhall likewiſe be ſhewn afterwards) CT is parallel to AG, ſo that the two Triangles BCT and BGA are ſimilar, whence $\dot{z} : \dot{y} ::$ (TB : BC* ::) $z : v$, that is $\dot{y} = \frac{v\dot{z}}{z} =$ (by inſerting $\sqrt{dz - z^2}$ inſtead of y, from the Property of the Circle) $\frac{\dot{z}\sqrt{dz - zz}}{z}$: whence you have the Relation of the Fluxions of the Abſciſs and Ordinate. Therefore, while AB flows with the Fluxion $\dot{z}$, BC or AF flows with the Fluxion $\frac{\dot{z}\sqrt{dz - zz}}{z}$: conſequently, if you multiply this laſt Fluxion into CF $= z$, you'll have $\dot{z}\sqrt{dz - zz}$, for the Fluxion of the external Area AFC, which, with ABC, completes the Parallelogram ABCF. Wherefore (by Prop. 9.) the Area AFC is equal to the Area of a geometrical Curve having z for it's Abſciſs, and $\sqrt{dz - zz}$ for it's Ordinate: which is the circular Area ABG deſcribed by the Ordinate BG $= \sqrt{dz - zz}$ and Abſciſs AB $= z$: ſo that the complemental Area AFC is equal to the circular Area AHGB; and the whole Area AIDCA equal to the Semicircle AGEA: therefore the Area ABC $=$ ABCF $-$ AHGB; and the Semicycloid AED $=$ AEDI $-$ AHGE, *i. e.* (becauſe the Baſe ED is equal to the Semicircumference AHGE, and therefore the circumſcribed Rectangle AD $=$ AE $\times$ ED $=$ AE $\times$ AHGE $=$ 4AHGEA) the Semicycloid AED $=$ 3 Semicircles AHGEA; and ſo the whole Cycloid is equal to three times the generating Circle, and ACDEGHA equal to that Circle.

* Art. 41. of this Explication.

333. Ex. 2. After the ſame manner, if ACDE was a Figure of Arches of any geometrical Curve, as the Circle, Ellipſe or any other known Curve: ſo that the Ordinate BC applied to the verſed Sine AB, were equal to the Arch AHG: you may find the complemental Area AFC, and thence the Area AHGB as formerly. For, ſince BC $=$ AHG,

* Art. 40. of this Explication. AHG, their Fluxions are equal*: therefore, from the known Property of the Curve ABG, you may find the Relation of the Fluxions of AB and AHG or BC, that is AF; and then you proceed as in the last Article.

Thus if AGE be a Circle, using the same Symbols as before, the Fluxion of BG or AHG, *viz.* $\dot{y} = \sqrt{\dot{z}^2 + \dot{v}^2}$*, that is, (by substituting for $\dot{v}^2$ it's Value, *viz.* $\dot{z}^2 \times \frac{d^2 - 4dz + 4z^2}{4dz - 4z^2}$, deduced from the Equation to the Circle, $v = \sqrt{dz - z^2}$, and reducing) $\dot{y} = \frac{\frac{1}{2}d\dot{z}}{\sqrt{dz - z^2}}$: multiply this into FC $= z$, and you have $\dot{y}z = \frac{\frac{1}{2}dz\dot{z}}{\sqrt{dz - zz}}$: wherefore a geometrical Curve, whose Absciss is $z =$ AB, and Ordinate $\frac{\frac{1}{2}dz}{\sqrt{dz - zz}}$, is equal to the complemental Area AFC*: which may be reduced to Species second, Order eight, Table second, expressed thus $\frac{dz^{2\eta - 1}}{\sqrt{e + fz^\eta + gz^{2\eta}}} = y$, by putting $\eta = 1$, $d = \frac{1}{2}d$, $e = 0$, $f = d$, $g = -1$, $x = (z^\eta =)\ z =$ AB, $\sqrt{e + fx + gx^2} = v = \sqrt{dz - z^2} =$ BG, and $t = 2s - xv = 2$ABGHA $- 2\triangle$ABG $= 2$AGHA: which therefore is equal to the complemental Area AFC; and so the whole Area AID is equal to twice the Semicircle. Whence, by subtracting twice the Segment AGHA from the given Rectangle AC, there remains the Area ABC required: and since ED = AHGE, and consequently the Rectangle AD $= 4$AHGEA, and AIDCA $= 2$AHGEA, therefore the Figure of Arches AEDCA is just equal to the Circle whose Diameter is AE: so that the Rectangle AD is bissected by the Curve-line ACD, and the Area AED by the Semicircumference AHGE.

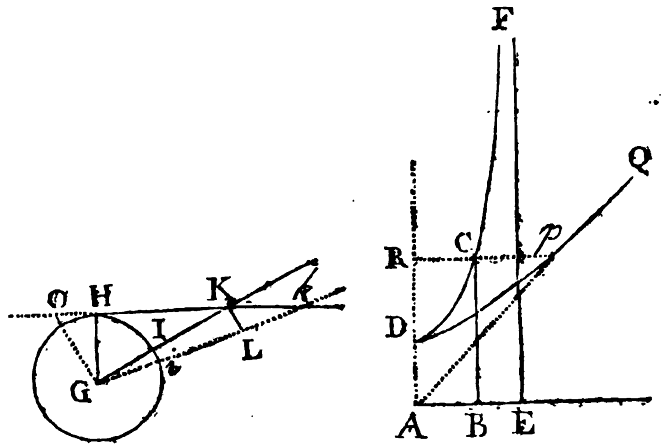
* Art. 42. of this Explication.

* Prop 9.

The same things may be found by comparing the Ordinate, after it is reduced to this Form $y = \frac{\frac{1}{2}d}{\sqrt{-1 + dz^{-1}}}$, with Case second, Species second, Order fourth, Table second, by putting $\eta = -1$, *&c.* And thus you might find the Areas of Figures of Arches in the Ellipse and Hyperbola, and compare them with geometrical Curves: only in this Case, it will be proper to compute the Beginning of the Absciss from the Centers of the Conic Sections, rather than from the Extremities of the Diameters: as we have done in the Case of the Circle.

334. Ex. 3. Let GHI be a Circle, which the right Line HK touches in the Point H, and suppose the right Line GK to revolve about the Center G as a Pole, and always intersect the Tangent HK in the Point K,

K, and the Circumference of the Circle in I: and let FCD be a Curve of such a Nature that it's Abscis A B be equal to the Arch HI, and it's rectangular Ordinate BC equal to the Secant GK, so as to constitute such a Figure of Secants as was mentioned formerly: the Area ABCD is required.



Join GH, and let GIK move into the new Place G*ik*; draw KL perpendicular to G*k*: call HI or AB $= x$, and GK or BC $= y$: then $\dot{x}$ is to $\dot{y}$ in the prime Ratio of the nascent Augments I*i* and L*k* *, that is, (because the Triangles GI*i*, GKL, and LK*k*, GHK are in that Case similar) $\dot{x} : \dot{y} :: Ii : KL + KL : Lk$ or $\dot{x} : \dot{y} :: GI : GK + GH : HK$, that is, if you call the Radius r; $\dot{x} : \dot{y} :: r^2 : y\sqrt{y^2 - rr}$, or $\dot{x}y\sqrt{y^2 - rr} = \dot{y}rr$, which defines the Relation of the Fluxions of AB and BC. But whereas the Value of $\dot{y}$ is mixt, having y in it's Composition, otherwise than happened in the preceding Examples, I can't find the complemental Area as in them; but I take the Value of $\dot{x}$, *viz.* $\dot{x} = \frac{\dot{y}rr}{y\sqrt{y^2 - rr}}$, and multiply it into BC $= y$, *Art. 3, 12 of the Quadrature of Curves.

which produces $\frac{\dot{y}rr}{\sqrt{y^2 - rr}} = \dot{x}y$ the Fluxion of the curvilinear Area ABCD: and therefore that curvilinear Area is equal to the Area of a geometrical Curve having $y =$ BC or GK for it's Abscis, and $\frac{rr}{\sqrt{y^2 - rr}}$ for it's Ordinate *, that is, if you draw GO perpendicular to GK, meeting the Tangent KH produced, in O, the Line HO, which is the Cotangent of HI. * Prop. 9.

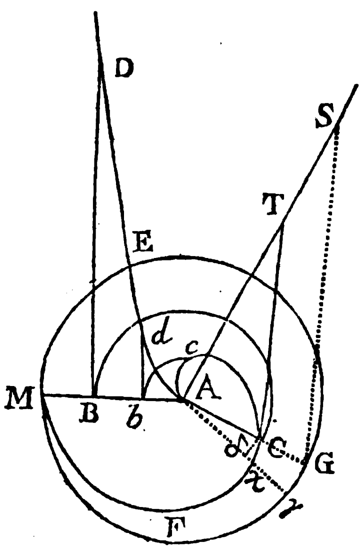
Now this Ordinate $\frac{rr}{\sqrt{y^2 - rr}}$, being made to stand thus $\frac{r^2}{y\sqrt{1 - \frac{r^2}{y^2}}}$, belongs to Species 1. Order 4. Table 2, expressed thus $\frac{d}{z\sqrt{e + fz^{\eta}}} = y$, by putting $d = r^2$, $e = 1$, $f = - r^2$, $z = y$, $\eta = - 2$: in which Species,

Species, becauſe $\frac{1}{z^{\eta}} = x^2$, $\sqrt{f + ex^2} = v$, and $\frac{Ad}{rf} \times \overline{\frac{1}{2}xv - s}$: by making a proper Subſtitution of Values you'll find that the Conic Section, with which the Curve is compared, has it's Abſciſs $x = y =$ BC, or GK, and it's Ordinate $v = \sqrt{-r^2 + x^2} = \sqrt{\overline{BC}^2 - \overline{AD}^2} =$ HK; and that the Area of the Curve ſought is $2 \times \overline{\frac{1}{2}xv - s}$.

Which points out this Conſtruction: with the Center A and half tranſverſe Axis AD, deſcribe the rectangular Hyperbola DPQ, through C draw PCR parallel to AB meeting the Hyperbola in P, and the tranſverſe Axis in R; join AP, then the curvilinear Area ABCD $=$ 2APDA. For AR $= x$, RP $= \sqrt{\overline{AR}^2 - \overline{AD}^2} =$ HK $= \sqrt{x^2 - r^2} = v$. ARP $= \frac{1}{2}xv$ and DRP $= s$. And this hyperbolical Sector APDA is equal to one half of the hyperbolical Area IONH in the Figure Art. 325 *, according to the Meaning of Art. 329 †, when the Arch HS in the Figure Art. 329 ‡, and the Arch HI in this Figure are equal, the Radius's of both Circles being ſuppoſed equal: as might be eaſily demonſtrated.

* Of this Explication.
† Of this Explication.
‡ Of this Explication.

335. The Areas of Spirals may be determined in the following manner. Let FC be a Spiral of any Kind deſcribed from the Point A as a Pole; BC an Arch of a Circle deſcribed from the ſame Point A as a Center, moving uniformly along the indefinite right Line AB given in Poſition, as a Baſe or Abſciſs: and let the Arch BC, while it moves, always touch the Spiral with it's Extremity C: then it is evident that the ſpiral Space terminated at BC, is deſcribed with the ſame Fluxion as a Curve having AB for the Abſciſs, and a right Line equal to BC for the rectangular Ordinate; and therefore the ſpiral Area and curvilinear Area adjacent to the ſame Part of the Abſciſs, muſt be equal *. Thus, if at the Point B of the Baſe AB, you draw the right Line BD perpendicular to AB, and equal to the Arch BC, ſo that while BC moves along AB, BD may move together with it, and be always equal to it, ſo as to deſcribe the Curve *d*ED, the Fluxions of the ſpiral and curvilinear Areas are equal: therefore if from the Point *b* you draw the Perpendicular *bd* $=$ *bc*, the Areas *b*BD*d*



* Art. 40. of this Explication.

*b*BD*d* and *b*BC*cb* are equal *. Wherefore calling AB $= z$, BC or BD $= y$, the Relation of z and y are defined by an Equation expressing the Nature of the Spiral; and thence the Area may be found, as formerly, from the Quadrature of Curves. * Art. 40. of this Explication.

Thus let $y^m = a^p z^n$ be an Equation defining the Relation of z and y; or AB and BC or BD; which may denote an infinite Number of different Kinds of Spirals: thence we have $y = a^{\frac{p}{m}} z^{\frac{n}{m}}$ for the Value of the Ordinate BD of the Curve *d*ED: therefore the Area is $\frac{m}{n+m} a^{\frac{p}{m}} z^{\frac{n+m}{m}}$ * denoting the curvilinear Area ABD or spiral Space ABCA, beginning at the Pole A, when the Exponent $\frac{n+m}{m}$ is positive; and adjacent to the Abscis AB and bounded by BD and DA, or BC and C*c*A *: but if $\frac{n+m}{m}$ be negative, the Area lyes upon the other Side of BD or BC †: and in every Case, the Area adjacent to any Part of the Abscis B*b*, *viz.* B*bc*CB is found, by subtracting the Area belonging to the lesser Abscis from that belonging to the greater *. * By Form 1. Tab. 1. * Art. 168. of this Explication. † Ibid. * Art. 187. of this Explication.

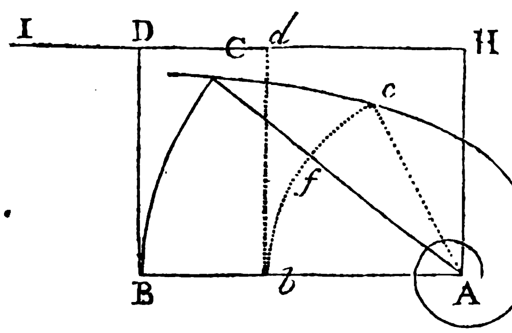
Ex. 1. Let the Equation be $y = a^{-1} z^2$, where we have $m = 1$, $p = -1$ and $n = 2$: therefore the Area is $\frac{1}{3}\frac{z^3}{a} =$ (by substituting y for $\frac{z^2}{a}$) $\frac{1}{3}zy =$ ABCA $=$ ABDA. In which Case it appears that the Curve AED is the common Parabola having it's principal Vertex in A, where AB touches it, and BD a Diameter; and a the Latus Rectum to the Axis: for it is $a \times \text{BD} = \text{AB}^2$. And the Spiral is that of *Archimedes.* For suppose ACFM to be his first Spiral, AM the Line of the first Rotation; draw ACG cutting the Circle MEGM, which bounds the Spiral, in the Point G: call AM $= r =$ AG, MEGM $= c$, MEG $= v$, and retain z and y as formerly. Then from the Description of *Archimedes*'s Spiral $z : r :: v : c$, but $z : r :: y : v$, therefore, by equating the two Values of v, deduced from these two Proportions, you have $\frac{cz}{r} = \frac{ry}{z}$, or $\frac{cz^2}{r^2} = y$: which is the same with the assumed Equation $\frac{z^2}{a} = y$, by supposing $a = \frac{r^2}{c}$, that is a third Proportional to the Circumference and Radius of the Circle MEGM bounding the first Spiral: which therefore is the Latus Rectum of the Parabola AED, whose external Area ABD is equal to the external spiral Space ABCA.

Moreover, if you suppose the right Line AD drawn, the parabolical Segment AEDA is equal to the internal spiral Area A*c*CA: for

 the

the whole circular Sector ABCA $= \frac{AB \times AC}{2} = \frac{AB \times BD}{2} = \Delta ABD$: but the external ſpiral Space AB*CcA* = ABDEA, therefore the internal Space A*c*CA = AEDA. From which, as likewiſe from $\frac{1}{3}zy$ denoting the ſpiral Space ABC*c*A, as was demonſtrated, it follows that ABC*c*A = $\frac{2}{3}$ Sector ABCA, and A*c*CA = $\frac{1}{3}$ of the ſame: that the whole internal ſpiral Area A*c*CMA = $\frac{1}{3}$ Circle MEGM, and the external Part lying without that, and within the Circumference of the Circle MEGM is $\frac{2}{3}$ of the ſame Circle. And by a like way of reaſoning it may be ſhewn, that if the right Line AM be ſuppoſed to be produced indefinitely, and by revolving to deſcribe Spirals of the ſecond, third, fourth, *&c.* Revolutions the Proportions of the ſeveral ſpiral Spaces, belonging to the ſeveral Revolutions, may be ſhewn to be what *Archimedes* has demonſtrated in his Treatiſe *de Lineis Spiralibus.* Moreover, the ſpiral Space A*c*CA is to the whole ſpiral Space A*c*CMA, as the circular Sector ABCA, to the whole Circle MEGM, that is in a Ratio compounded of the Sector ABCA to the Sector AMEGA, and the Sector AMEGA to the whole Circle MEGM; or of AC^q to AG^q and MEG to MEGM, or AC to AG, that is the Ratio of AC^{cub} to AM^{cub}, as *Pappus* has demonſtrated [1].

Ex. 2. Let the Nature of the Spiral be expreſſed by this Equation $\frac{z^3}{a} = y^2$, or $\frac{z^{\frac{3}{2}}}{\sqrt{a}} = y$: whence the ſpiral Area A*c*CBA $= \frac{2}{5}\frac{z^{\frac{5}{2}}}{\sqrt{a}} = \frac{2}{5}zy$ = ABDEA: where the Curve AED to whoſe Area the ſpiral Space is equal, is a Parabola of the ſecond Kind: and the external ſpiral Space A*c*CBA is $\frac{4}{5}$ of the circular Sector ABCA; and therefore the internal A*c*CA is $\frac{1}{5}$ thereof: and the whole A*c*CMA is $\frac{1}{5}$ of the whole Circle MEGM.



Ex. 3. Let A*c*C be a Spiral deſcribed from the Pole A, of ſuch a Nature that if the right Line AB (called the firſt Radius) be drawn, and from the Center A, with any Diſtance AB, the Arch BC of a given Length, be deſcribed, it's Extremity C may always touch the Spiral C*c*A. In this Caſe calling AB $= z$ and BC $= y$ as before; the general Equation $y^m = a^p z^n$, becomes $y = az^0$ or $y = a$, and therefore the Area is

[1] Pappi Collect. Mathem. Prop. 22. Lib. 4.

$az = yz.$

$az = yz$. Wherefore if the right Line HA be drawn perpendicular to AB, so that AH be equal to BC; and through H, HI parallel to AB; HI shall be an Assymptote to the Spiral: and if through B, *b*, you draw BD, *bd*, perpendicular to AB, meeting HI in B, *b*, the spiral Space BC*cb* = Rectangle BD*db*. Moreover let AC intersect *bc* in the Point *f*, then, if from $BCcb = Bb \times bc$, you take away $BCfb = Bb \times \frac{bc + bf}{2}$, there remains $Cfc = Bb \times \frac{fc}{2}$, to which add $Afc = Ab \times \frac{fc}{2}$, and you have the spiral Sector $ACc = \frac{AB \times fc}{2}$: therefore when the Abscisses AB, A*b* are given, the spiral Sector AC*c* is found. This Spiral is called the reciprocal Spiral, with respect to the common Spiral of *Archimedes:* because, as in that of *Archimedes*, the Radius BC is proportional to the Angle BAC, which it makes with the first Radius, called the Beginning of the Circulation; so in this other, BC is reciprocally proportional to the Angle BAC, as easily appears.

Ex. 4. But if the Spiral C*c*A be of such a Nature, that the Arch BC be reciprocally proportional to AB, or BC : *bc* :: A*b* : AB: then the Equation to it is $y = az^{-1}$, which plainly appears to be an Equation to the Hyperbola, reckoning the Absciss *z* from the Center of the Hyperbola upon one of the Assymptotes, and the Ordinate drawn from the Hyperbola upon the Assymptote, parallel to the other Assymptote. Therefore, suppose the Point *b* such, that *bc* = A*b*; complete the Square A*d*; and with the Center A and principal Vertex *d*, describe an equilateral Hyperbola, to which AB, AH are the Assymptotes: draw the Ordinates BD, *bd*, then it is evident, the spiral Area BC*cb* is equal to the hyperbolical Area BD*db*, the right Line *d*DI in the former Example, becoming an Hyperbola in this. This Spiral continually approaches to AB as an Assymptote. Moreover, since it is AB : A*b* :: *bc* : BC, the Sectors ABC, A*bc* are always equal, hence it appears that the spiral Sector AC*c* = BC*cb*. Whence it follows that BC*cb* and AC*c* are as the Logarithm of the Ratio of AC to A*c*.

And these Examples shall suffice to shew after what manner *mechanical* Curves may be reduced to geometrical Figures, and thereby their Areas determined: which was what we proposed to do in this Problem. I shall next shew after what manner the Areas of Curves may be determined, when they are supposed to be generated by the Revolution of a Radius round some immoveable Point as a Pole or Center.

PROBLEM.

336. *To find the Areas of Curves supposed to be generated by the Revolution of a Radius about a fixt Point.*

Although in this Treatise of the Quadrature of Curves, the *illustrious* Author considers curvilinear Areas as generated by the Motion of a rectangular Ordinate along a Base or Absciss: which is certainly the most natural and convenient Conception of the Production of geometrical Figures; and likewise the most Part of mechanical Ones; in order to determine their Areas by his Method of Fluxions: yet the Method of Fluxions, as applied to the Quadrature of Curves, is not limited to this manner of conceiving the Production of curvilinear Areas; but may be applied to other manners of conceiving their Production or Generation, by Motion variously modified: particularly when they are produced, or supposed to be produced, by the Revolution of a right Line about an immoveable Point. In which Case, even as in the other, to which our Author confines himself, there is some certain Line supposed to flow with an uniform or constant Fluxion; and some Area likewise to flow with a constant Fluxion, which arises from the Multiplication of the Fluxion of the uniformly flowing Line into a given Line: then, the Relation of the Fluxion of the curvilinear Area, described by the Revolution of a Radius, to the Fluxion of the uniformly flowing Area, is investigated, and thence by means of the Equation to the Curve, the Relation of the Fluents is found as in the other Method. But whereas this way of conceiving the Description of Areas, is of very little use, unless in the Case of Spirals, I shall just shew in an Example or two, the Application of it to this sort of Curves.

Ex. 1. Let ACM be the Spiral of *Archimedes*, (see the Fig. at Art. 335 *. p. 240.) the Lines AM, ACG as before: so that the Radius ACG revolving about the Center A with an uniform or equable Motion, may with the Point G describe the Circumference of the Circle MEGM, and with the Point C the spiral Line A*c*CFM. In this Case, it is evident, that both the Circumference and Area of the circular Sector AMEGM flow with a constant Fluxion: and moreover that the Fluxion of the spiral Sector A*c*CA is to the Fluxion of the circular Sector AMEGM as AC^q to AG^q; for this is the prime Ratio of the nascent Augments of these Sectors. Wherefore, call AG or AM $=a$, MEGM$=c$, MEG $=z$, AC$=y$, the circular Sector$=$S, the spiral

* Of this Explication.

ſpiral Sector $= s$, and you'll have $\dot{S} : \dot{s} :: a^2 : y^2$, or $\dot{s} = \frac{y^2}{a^2}\dot{S}$. But ſince the Fluxion of MEG or z is $\dot{z}$; or of $1 \times z$ is $1 \times \dot{z}$, thence the Fluxion of the circular Sector is $\frac{a\dot{z}}{2}$ *: ſubſtitute this for $\dot{S}$ in the preceding Equation $\dot{s} = \frac{y^2\dot{S}}{a^2}$, and you'll have $\dot{s} = \frac{y^2\dot{z}}{2a}$, which is the general Formula for the Fluxion of all Spirals. But from the Property of the Spiral, it is $y : a :: z : c$, or $y = \frac{az}{c}$; by ſubſtituting of which Value of y in place of it, you have $\dot{s} = \frac{az^2\dot{z}}{2cc}$: wherefore it is $s = \frac{az^3}{6cc}$ *, the Value of the ſpiral Sector A*c*CA: and when $z = c$, it is $\frac{ac}{6} = \frac{\text{AM} \times \text{MEGM}}{6} = \frac{1}{3}$ of the Circle MEGM $=$ A*c*CFMA, as before. Since it is $s = \frac{az^3}{6cc}$, if you ſubſtitute for z it's Value, *viz.* $\frac{cy}{a}$, the ſpiral Sector A*c*CA will be otherwiſe expreſſed thus $\frac{cy^3}{6a^2}$: which includes the ſame things that were demonſtrated above.

*Art. 3. 12 of the Quadrature of Curves.

* Art. 42. of this Explication, and Form 1. Tab. 1.

Ex. 2. Let MFC (ſee the ſame Fig. p. 240.) be ſuppoſed a Spiral of ſuch a ſort, that if from it's Pole A with the given Diſtance AM, the Circle MGEM be deſcribed, and any Radius ACG be drawn, the Sum of the Squares of AC and MG be equal to a given Area, *e. g.* to the Square of AM. Then putting AM $= a$, MG $= z$, and AC $= y$, it is $y^2 + z^2 = a^2$, or $y^2 = a^2 - z^2$: but the Fluxion of the ſpiral Sector is $\frac{yy\dot{z}}{2a}$, as was ſhewn in the preceding Example, in which fluxionary Expreſſion inſert $a^2 - z^2$ for y^2, and it becomes $\frac{a\dot{z}}{2} - \frac{z^2\dot{z}}{2a}$; the Fluent of which, *viz.* $\frac{az}{2} - \frac{z^3}{6a}$ is the ſpiral Area ſought, *viz.* AMFCA: which Expreſſion conſiſts of two Parts; the firſt $\frac{az}{2}$ denotes the circular Sector AMGA; the other $\frac{z^3}{6a}$, the external ſpiral Space MFCGM. And when $z = a$ or MG $=$ AM, the Point C falls into A; and the whole internal ſpiral Space is $\frac{1}{2}a^2 - \frac{1}{6}a^2 = \frac{1}{3}a^2 = \frac{1}{3}\text{AM}^2$.

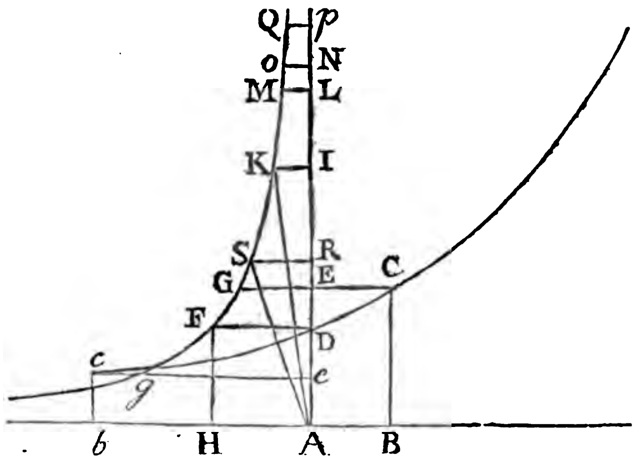
And thus you may feign any Relations of z and y, you pleaſe, by which an infinite Variety of Spirals will ariſe, whoſe Areas may be determined after the like manner as in theſe two Examples; or elſe may be compared with the Areas of Conic Sections, or other the moſt ſimple geometrical Curves, they can be compared with.

SECT.

SECT. X.

Of the Reduction of the Hyperbola and Ellipse; together with all curvilinear Areas, and Fluents, which depend upon these Conic Sections, to the Measures of Ratios, and Angles; or Logarithms and circular Arches.

337. DEF. Let the right Line BC move along the right Line AB in a given Angle, so that while AB increases or decreases with an equable or uniform Velocity, the Ordinate BC increases or decreases with a Velocity always proportional to itself: then AB is called the *Logarithm* of the Ratio of BC to AD; AD being the same with BC when the Points B and A coincide: and the Curve DC which the Point C describes is called the *logarithmical* or *logistical* Curve: because, if the Abscis AB be divided into very small and equal Parts, and Ordinates drawn thro' all the Points of Division, they will constitute a Series of continual geometrical Proportionals, falling in betwixt BC and AD; so that if the Ratio of BC to AD be considered as compounded or made up of the equal Ratios of all the adjoining Ordinates beginning with BC and ending with AD, the Abscis AB will express the Number of the equal compounding Ratios, which make up the Ratio of BC to AD: which is the true and original Notion of the Word *Logarithm*. The right Line AB is likewise called the *Measure* of the Ratio of BC to AD. According to which Meaning of the Word Ratio, the Measure of the Duplicate, Triplicate, Quadruplicate, &c. of any Ratio is double, triple, quadruple, &c. and the Measure of the Subduplicate, Subtriplicate, &c. is $\frac{1}{2}$, $\frac{1}{3}$, &c. of the Measure of the simple Ratio: the Measure of the Ratio of Equality is 0; because the Duplicate-triplicate, &c. or Subduplicate-



duplicate-ſubtriplicate, &c. of a Ratio of Equality is ſtill the ſame with itſelf, and therefore according to the former Notion of Ratio, it's Meaſure muſt be 0: accordingly when BC coincides with AD, the Abſciſs AB vaniſhes. And when the Meaſure of the Ratio of greater to leſs is conſidered as poſitive; upon the contrary the Meaſure of the Ratio of leſs to greater, is negative: for when BC by moving towards AD, and getting upon the oppoſite Side of it, comes into the Poſition *bc*, the Abſciſs AB by diminiſhing firſt becomes nothing, from being poſitive; and afterwards becomes negative, repreſented by A*b*, which is negative with reſpect to AB, and denotes the Meaſure of the Ratio of *bc* to AD, or it's Logarithm.

338. Let the Parallelogram ADFH be completed; and with the Aſſymptotes AD, AH, and principal Vertex F, deſcribe the Hyperbola FG: through C draw CEG parallel to AB, cutting AD in E, and the Hyperbola in G, the hyperbolical Area DEGF is proportional to the right Line AB. For, from the known Property of the Hyperbola, EG is reciprocally proportional to AE, that is to the Velocity or Fluxion of AE = BC, ſince BC was ſuppoſed to increaſe, or decreaſe with a Velocity proportional to itſelf *: in which Caſe the Fluxion of the hyperbolical Area DEGF is conſtant and invariable (as eaſily follows from Art. 37 *.) but ſo is the Fluxion of AB alſo; therefore the Fluents being as the Time in which they are deſcribed, theſe Fluents, *viz.* the right Line AB and hyperbolical Area DEGF are always proportional. Whence it follows that the Area DEGF is the Logarithm, or Meaſure, of the Ratio of AE to AD, or DF to EG. And in this Caſe likewiſe, as in the preceding, the Meaſure of the Ratio of Equality is 0, for DEGF vaniſhes, when the Point E falls in with D; and the Logarithm or Meaſure of the Ratio of leſs to greater is negative, for D*eg*F, the Meaſure of the Ratio of A*e* to AD, is negative, *viz.* when the Point E falls in *e* betwixt A and D.

* Art. 337. of this Explication.

* Of this Explication.

339. If through any two Points K and M of the Hyperbola, the Ordinates KI, ML be drawn, parallel to the Aſſymptote AH, the Area IKML, will be the Logarithm or Meaſure of the Ratio of AL to AI or IK to LM: for if you take AE : AD :: AL : AI, and ſuppoſe the Fluxions of AE, AL, as formerly to be proportional to theſe Lines, the Lines DE, IL; and conſequently the Areas DFGE, IKML, are generated in equal Times; and therefore, ſince the Fluxions with which the Augments DFGE, IKML are deſcribed, are likewiſe conſtantly the ſame * theſe Augments are equal: but DFGE is the Logarithm or Meaſure of the Ratio of AE to AD, therefore IKLM is likewiſe the Meaſure of that Ratio, *i. e.* of AL to AI or IK to LM.

* Art. 338. above.

340. Hence

340. Hence follows the fundamental Property upon which logarithmical Arithmetic is founded, *viz.* that the Logarithm of any Ratio made up or compounded of other Ratios is equal to the Sum of the Logarithms of the compounding Ratios: and the Logarithm of a Ratio, which arises by subducting [1] one Ratio from another is equal to the Difference of their Logarithms. For let the Ratio of AR to AD be compounded of the Ratios of AP to AN and AL to AI; through R, P, N draw the Ordinates RS, PQ, NO. Take the Point E such that it may be AP : AN :: AR : AE; then must it be AL : AI :: AE : AD, otherwise the Ratio of AR to AD would not be compounded of the Ratios of AP to AN, and AL to AI. Wherefore it appears, by the last Article, that the Sum of the two Areas NOPQ, IKML is equal to the Area DFSR: the first of which is the Sum of the Logarithms of the compounding Ratios, and the last is the Logarithm of the compounded Ratio, and the rest easily follows. Whence it follows that the Sum of the Logarithms of two Numbers is equal to the Logarithm of their Product, and the Difference of the Logarithms of two Numbers is the Logarithm of their Quotient, *&c.* For what is ordinarily called the Logarithm of any Number, is, properly speaking, the Logarithm of the Ratio of the Number to Unity.

341. If BC, and consequently AD, be perpendicular to AB, the Hyperbola FG is rectangular: in which Case, if you call $AD = a$, a constant Quantity, $DE = x$, and $EG = y$, two variable Quantities, then, from the Property of the Hyperbola $\overline{a + x} \times y = a^2$, or $y = \frac{a^2}{a+x}$: but the Fluxion of DFGE, that is of the Logarithm, or Measure of the Ratio of AE to AD, or $a + x$ to a, is $\dot{x}y = \frac{a^2\dot{x}}{a+x}$, so that if $AD = (a =)\ 1$, the Logarithm or Measure of the Ratio of $1 + x$ to 1 is $\frac{\dot{x}}{1+x} =$ (by Division) $\dot{x} - \dot{x}x + \dot{x}x^2 - \dot{x}x^3 + \dot{x}x^4$ — *&c. in inf.* all the Terms of which being squared, you'll have $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5$ — *&c. in inf.* for the Value of the hyperbolical Area DFGE; and consequently of the Logarithm, or

[1] By subducting of one Ratio from another, I mean the Reverse of that which is called compounding of Ratios, agreeable to the Way in which we consider the Doctrine of Ratios in this Place. Thus, if the Ratio of A to B were compounded or made up of the Ratio of C to D, and of E to F; which is when A : B :: C × E : D × F; so contrarily the Ratio of E to F subducted from the Ratio of A to B, leaves the Ratio of C to D, *viz.* when $\frac{A}{E} : \frac{B}{F}$ :: C : D. This last Operation might be called Resolution of Ratios, being the opposite of Composition of Ratios. But it is all one how it be termed, if the thing itself be understood.

Measure

Meaſure of the Ratio of $1 + x$ to 1: which is commonly called it's hyperbolical Logarithm, to diſtinguiſh it from the common or tabular Logarithm of that Ratio, commonly called ſimply the Logarithm of $1 + x$ [1]. By the ſame way of reaſoning, the Area DF*ge* will be found to be $- x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 -$ &c. *in inf.* by ſuppoſing D*e* $= - x$ *: which therefore is the Logarithm of the Ratio of $1 - x$ to 1: or, which is the ſame, $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 +$ &c. *in inf.* is the Logarithm of the Ratio of AD to A*e*, or 1 to $1 - x$. Conſequently if it be D*e* = DE, the Logarithm of the Ratio of AE to A*e* or $1 + x$ to $\overline{1 - x}$, *viz.* EG*ge*, is equal to the Sum of $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 +$ &c. and $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 +$ * &c. that is $2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7$, &c. *in inf.* $= 2 \times \overline{x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \&c.}$ *in inf.* which Series converges twice as faſt as the former ones: and affords one of the moſt convenient and expeditious Methods of conſtructing and computing a Table of Logarithms. If the Difference betwixt the two Areas DFGE and DF*ge* be ſought, ſubtract the leſſer DFGE $= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 +$ &c. from the greater DF*ge* $= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 +$ &c. and there remains $x^2 + \frac{1}{2}x^4 + \frac{1}{3}x^6 + \frac{1}{4}x^8 +$ &c. = DF*ge* — DFGE.

* Art. 190, 191. of this Explication.

* Art. 340. of this Explication.

Now if you ſuppoſe DG = D*g* = 0.1, that is AG = 1.1 and A*g* = 0.9: by ſubſtituting 0.1 for x in the Series $2 \times \overline{x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \&c.}$ you'll find the Area EG*ge* = 0.200670695462, &c *: which is therefore the hyperbolical Logarithm, or Meaſure of the Ratio of 1.1 to 0.9 or 11 to 9, in this particular Syſtem. And by ſubſtituting 0.1 for x in the other Series $x^2 + \frac{1}{2}x^4 + \frac{1}{3}x^6 + \frac{1}{4}x^8 +$ &c. you'll obtain 0.0100503358, &c. for the Value of DF*ge* — DFGE: wherefore by adding this Number to and ſubtracting it from the former, the Half of the Sum, *viz.* 0.1053605156 = DF*ge*; and the Half of the Difference, *viz.* 0.0953101798, &c. = DFGE; the firſt of which is the Logarithm of the Ratio of 1.1 to 1; and the other the Logarithm of the Ratio of 1 to 0.9. After the ſame manner by ſuppoſing DG = 0.2 = D*g*, you'll find the Logarithm of the Ratio of 1.2 to 0.8, or 12 to 8, or 3 to 2 to be 0.4054651081, &c. of the Ratio 1.2 to 1 to be 0.2231435513, &c. and of 1 to 0.8 to be 0.1823215567, &c. And thus by ſuppoſing DE = D*e* = 0.01, 0.02, &c. 0.001, 0.002, &c. you may inveſtigate the Logarithms of other Ratios, in the ſame

* Art. 191. of this Explication.

[1] The Tables of Logarithms, which are in common uſe, are called *Briggs*'s Logarithms, to diſtinguiſh them from that Form of Logarithms firſt invented by Lord *Neper*: which laſt are the ſame with thoſe which I have mentioned under the Name of hyperbolical Logarithms. The Difference betwixt the two Forms will appear in what follows.

manner: by the Addition of which Logarithms, their Subtraction, Multiplication and Division, all other Logarithms belonging to this System may be found most conveniently and expeditiously: particularly since $\frac{2}{0.8}\times\frac{2}{0.8}=\frac{2}{1}$ or 2, and $\frac{2}{1}\times\frac{1.2}{0.8}=\frac{3}{1}=3$, and $\frac{2}{1}\times\frac{2}{0.8}=\frac{5}{1}=5$, and $\frac{5}{1}\times\frac{2}{1}=\frac{10}{1}=10$: hence you'll find by Addition of the Logarithms formerly found, agreeably to what was said in the preceding Article, the Logarithms of the Numbers 2, 3, 5, 10; or rather of their Ratios to Unity, to be 0.6931471805, *&c.* 1.0986122886, *&c.* 1.6094379124, *&c.* and 2.302585092999, *&c.* And since the Logarithm of 10, or of the decuple Ratio in the common or *Briggs's* Tables is 1, hence you see what Proportion the Logarithm or Measure of the decuple Ratio in the hyperbolical Logarithms, bears to the Logarithm of that same Ratio in the common System of Logarithms, *viz.* that of 2.302585092999, *&c.* to 1: and the same is the Proportion that the hyperbolical Logarithm of any other Ratio bears to the tabular Logarithm of that same Ratio: whence the hyperbolical Logarithms being found, in the Manner just now shewn, the tabular Logarithms may thence be found, and contrarily: more of which by and by.

342. Schol. As there may be an infinite Variety of different Hyperbolas, hence it appears that there may be an infinite Variety of different Systems or Scales of Logarithms, or Measures of Ratios: and what Proportion the Logarithms or Measures of the same Ratio, in the different Systems, bear to each other, will appear by what follows.

PROP.

343. If from any two Points K and S of an Hyperbola FSK (see the foregoing Fig. p. 246.) the Semi-diameters KA, SA be drawn, and the two Ordinates KI, SR, as before, the hyperbolical Sector KAS is equal to the Area KIRS; and consequently, is the Logarithm or Measure of the Ratio of AI to AR, or RS to IK, in that particular System.

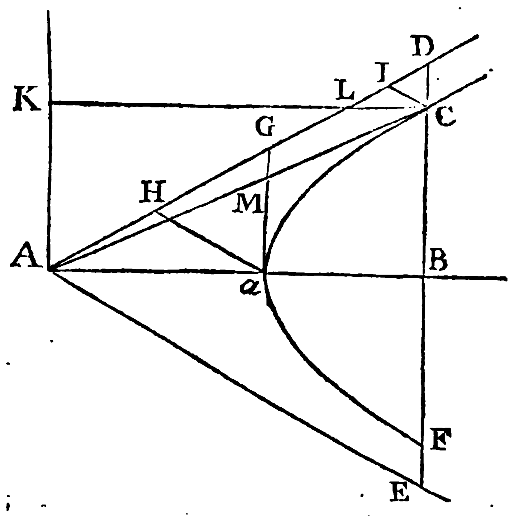
For the two Triangles ARS, AIK are equal, since the Angles at R and I are equal, and the Sides about these equal Angles reciprocally proportional: wherefore, if from the Area ASKIA, these Triangles be subtracted separately, there remains KIRS = KAS. Wherefore also KAS is the Logarithm or Measure of the Ratio of AI to AR, or RS to IK *.

* Art. 338. of this Explication.

PROP.

344. Let aC be an Hyperbola describ'd with the Centre A, semi-transverse Aa, and Assymptotes AD, AE: through any Point of the Hyperbola

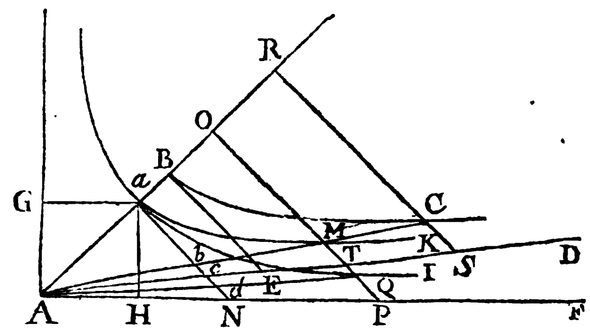
Hyperbola C draw the Ordinate CBF to the transverse Axis, cutting it in B, the Curve in C and F, and the Assymptotes in D and E; draw likewise CLK an Ordinate to the conjugate AK, meeting the Assymptote AD in L; draw also the Semi-diameter AC: let *a*MG be a Tangent at *a* meeting AC, AD in M and G: and through *a* and C draw *a*H, CI parallel to the Assymptote AE, and meeting the other Assymptote AD in H and I: then I say the Area *a*HIC, or Sector AC*a*, is the Measure of the Ratio of FD to *a*G or *a*G to CD; or KC + KL to A*a* or A*a* to KC — KL; or BD + BC to $\sqrt{BD^q - BC^q}$ or *a*G + *a*M to $\sqrt{aG^q - aM^q}$; or KC + KL to $\sqrt{KC^q - KL^q}$. For from parallel Lines, *a*H : CI : : *a*G : CD. Again, from the Property of the Hyperbola, *a*G : CD :: CE = DF : *a*G. Likewise, from similar Triangles A*a*H, LCI, it is *a*H : CI :: A*a* : LC; and from the Property of the Hyperbola A*a* : LC :: KC + KL : A*a*. Since then all these Ratios are equal to the Ratio of *a*H to CI, which the Area *a*HIC or Sector AC*a* is the Measure of: hence it appears that *a*HIC or AC*a* is the Measure of any of these Ratios, which have been mentioned, in the System belonging to the given Hyperbola. And as to the other Ratios mentioned in the Prop. *viz.* these of BD + BC to $\sqrt{BD^q - BC^q}$ &c. since it is BD + BC, *a*G, and BD — BC, proportional; hence BD + BC = FD : *a*G :: $\sqrt{BD + BC}$: $\sqrt{BD - BC}$ :: BD + BC : $\sqrt{BD^q - BC^q}$: hence it appears that the Ratio of BD + BC to $\sqrt{BD^q - BC^q}$ is the same with the Ratio of FD to *a*G. And by the same way it may be shewn that the Ratio of KC + KL to $\sqrt{KC^q - KL^q}$ is the same with that of KC + KL to A*a*. Moreover it is BD : BC :: *a*G : *a*M, therefore it follows that BD + BC : $\sqrt{BD^q - BC^q}$ :: *a*G + *a*M : $\sqrt{aG^q - aM^q}$. Whence it follows, that *a*HIC or A*a*C is likewise the Measure of the Ratio of BD + BC to



$\sqrt{BD^q - BC^q}$ or $aG + aM$ to $\sqrt{aG^q - aM^q}$, or of KC + KL to $\sqrt{KC^q - KL^q}$ in the same System.

345. Schol. It would be the same thing if A*a* and AK were any two conjugate Diameters, for the Demonstration is the same, as when they are the two Axes.

PROP.



346. Let *a*I and BC be any two Hyperbolas described from a common Centre A, with the semi-transverse Axes A*a*, AB; and Assymptotes AF, AD respectively: draw *a*N, BE Tangents to the Curves at *a* and B, meeting their Assymptotes in N and E: then I say the Logarithm, or Measure of any Ratio, in the System belonging to the Hyperbola *a*I; is to the Logarithm, or Measure of the same Ratio, in the System belonging to the Hyperbola BC, as the Triangle A*a*N to the Triangle ABE.

For with the Center A, principal Vertex *a*, and Assymptote AD, describe another Hyperbola *a*K, which therefore is similar to the Hyperbola BC. From any Point O in the Axis produced, draw the Ordinate OP, cutting the Assymptotes AF, AD, in P and T, and the Hyperbola's *a*I, *a*K in Q and M: join AQ, AM, intersecting *a*N in *d* and *b*: and let AM, produced if need be, meet the Hyperbola BC in the Point C: through C, draw the Ordinate RCS to the Axis, meeting it in R, and the Assymptote AD in S: moreover let the Assymptote AD and Tangent *a*N intersect each other in the Point *c*.

Then because the Hyperbolas BC and *a*K are similar, their similar Sectors ABC, A*a*M are to one another, as the Triangle ABE to the Triangle A*ac* (as is demonstrated in the Lemma annexed.) Again, Sector A*a*M : Sector A*a*Q :: ΔA*ac* : ΔA*a*N (as is there demonstrated also) therefore *ex æquo* Sector ABC : Sector A*a*Q :: ΔABE : ΔA*a*N. Now it appears, by the Lemma, that AC : AM :: AB : A*a* :: BE : *ac*; but AC : AM :: CS : MT, therefore CS : MT :: BE : *ac*; or by Alternation, CS : BE :: MT : *ac*.

Again,

Again, in comparing the two Hyperbolas aK, aI together, you'll have, from the Property of the Hyperbola, $OT^q - OM^q : OP^q - OQ^q :: (ac^q : aN^q ::) OT^q : OP^q$; and by Alternation, Inversion, and Division, $OT^q : OM^q :: OP^q : OQ^q$; or $OT : OM :: OP : OQ$; by Division, $OT : OP :: MT : QP$; but $OT : OP :: ac : aN$, whence $MT : QP :: ac : aN$, or $MT : ac :: QP : aN$; but it was shewn at the End of the last Paragraph, that $MT : ac :: CS : BE$, therefore $QP : aN :: CS : BE$: wherefore, by the last Prop. the Sectors AaQ and ABC are Logarithms or Measures of the same Ratio, in their respective Systems, belonging to the Hyperbolas aI and BC: but, as has been said, and is demonstrated in the Lemma subjoined, these Sectors are as the Triangles AaN, Aac: therefore the Logarithms or Measures of the same Ratio, in the different Systems, are as the Triangles AaN, Aac. Q. E. D.

LEMMA.

347. It was said that Sector ABC : Sector $AaM :: \Delta ABE : \Delta Aac$; and that Sector AaM : Sector $AaQ :: \Delta Aac : \Delta AaN$; and consequently, by Equality, that Sector ABC : Sector $AaQ :: \Delta ABE : \Delta AaN$. Which things appear thus.

Because the transverse and conjugate Axes of the two Hyperbolas BC and aK are in the same Ratio, therefore, from the Property of the Ordinates, $AR^q - AB^q : AO^q : - Aa^q :: (RC^q : OM^q ::) AR^q : AO^q$; or by Division, $AB^q : Aa^q :: AR^q : AO^q :: AC^q : AM^q$; but $F.ABC : F.AaM :: AC^q : AM^q$, since this is the Ratio of their nascent Augments, and $AC^q : AM^q :: AB^q : Aa^q :: \Delta ABE : \Delta Aac$; therefore $F.ABC : F.AaM :: \Delta ABE : \Delta Aac$, which is a given or constant Ratio, hence it follows that Sector ABC : Sector AaM * $:: \Delta BE : \Delta Aac$: which was the first thing to be shewn. * Art. 40. of this Explication.

Again it is demonstrated, as in the Prop. itself, that $OM : OQ :: ac : aN :: \Delta Aac : \Delta AaN$, which is a given Ratio; and that being the Ratio of the Fluxions of the hyperbolical Areas, or half Segments, aOM, aOQ, hence these Areas themselves are in the same Ratio: but the Triangles AOM, AOQ are in that Ratio, consequently the hyperbolical Sectors AaM, AaQ, which are their Differences, are in the same Ratio, *viz.* of the Triangles Aac, AaN. By comparing of which with the Proportion at the End of the last Paragraph, *viz.* Sector ABC : Sector $AaM :: \Delta ABE : \Delta Aac$, it is, by Equality, Sector ABC : Sector $AaQ :: \Delta ABE : \Delta Aac$. Q. E. D.

348. Def. The Triangle AaN is called by the learned and ingenious Mr. *Cotes*, the *Modulus* of the System, belonging to the Hyper-

bola aI; and the like in other Hyperbolas. It is the ſame thing to take the Parallelogram $AHaG$, inſcribed betwixt the Hyperbola, and it's Aſſymptotes AF, AG; for $AN = 2AH$: yea, and although Aa be not the Axis, but any other Diameter, the Triangle AaN and Parallelogram $AHaG$ are always equal to one another, and to the Parallelogram $AHaG$ inſcribed at the Vertex Principalis: therefore any of them will be the Modulus. Whence it follows that the Logarithm or Meaſure of any Ratio, in one Syſtem, or Hyperbola, is to the Logarithm or Meaſure of the ſame Ratio in another Syſtem; as the Modulus of the firſt, to the Modulus of the other.

349. Wherefore, by the Help of what was demonſtrated Art. 341 *, the Modulus of the common Syſtem of Logarithms, is eaſily found: for ſuppoſing AaI to be an equilateral Hyperbola, in which $AH = 1$ or $Aa = \sqrt{2} = aN$, the Modulus is 1, and the Meaſure of the decuple Ratio, as was there ſhewn, is 2.302585092994, *&c.* and the Logarithm of the Number 10, *i. e.* of the decuple Ratio in *Briggs*'s Logarithms, is 1: therefore a third Porportional to 2.30258, *&c.* and 1, that is $\frac{1}{2.30258509, \&c.} = 0.434294481903$, *&c.* is the Modulus of the common Syſtem of Logarithms.

* Of this Explication.

350. From what has been ſaid, it likewiſe appears, that the hyperbolical Logarithm of any Ratio being found by Art. 341 *, in which the Modulus is 1, that Logarithm multiplied by the Modulus of any other Syſtem, will give the Logarithm of that ſame Ratio, in that other Syſtem. Moreover, that, if the tabular Logarithm of any given Ratio be called l; and M denote the Modulus of any other Syſtem, and in the Meaſure of the given Ratio in that Syſtem, then it will be $M = \frac{m \times 0.43429, \&c.}{l}$: and $m = \frac{M \times l}{0.43429, \&c.}$ or $M \times l \times 2.30258509$, *&c.* ſince the Meaſure of any given Ratio is always as the Modulus of the Syſtem to which it belongs.

* Of this Explication.

351. Def. The Ratio, of which the Modulus in any Syſtem, is the Meaſure, is called, by the ſame ingenious Author, the *Ratio Modularis.* And it is evident, that this Ratio is the ſame in every Syſtem: for the Meaſure of any given Ratio is as the Modulus of the Syſtem *. Therefore, if in the forementioned equilateral Hyperbola, you find the Ratio whoſe Meaſure is 1, that ſhall be the Ratio Modularis. In order to which (ſee Fig. Art. 337 *. p. 246.) if you put $AD = 1$, and $DE = x = De$, then by Art. 341 † you have the Logarithm or Meaſure of the Ratio of AE to $Ae = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7$, *&c.* which Series expreſſes the Value of the Area $EGge$.

*Art. 348. of this Explication.
* Of this Explication.
† Of this Explication.

Where-

Wherefore, if you put that Series equal to Unity, and by the Method of Reversion of Series (taught by our Author [1]) find the Value of x $=$ DE or De, by a converging Series, and add that Value of x to AD $=1$, and subtract it from AD; the first will give AE the Antecedent, and the second Ae the Consequent of the Ratio Modularis, it's Measure being 1, which is here the Modulus. You might find the same Ratio by putting $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$, &c. $= 1$; or $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4$, &c. $= 1$; and the Value of x, resulting from the first added to AD $= 1$, will give the Antecedent of the Ratio Modularis to AD or 1, the Consequent; and the Value of x resulting from the other, subtracted from 1, gives the Consequent of that Ratio, to the Antecedent 1: as appears from Art. 341 *, and what hath been said: but the former Series converging twice as fast as these two last, is to be preferred. By which means it will be found, that the Ratio Modularis is that of 2.71828181828459, &c. to 1, or 1 to 0.367879441171, &c. the first of which Numbers has 0.43429448, &c. it's Log. in the common Tables. * Of this Explication.

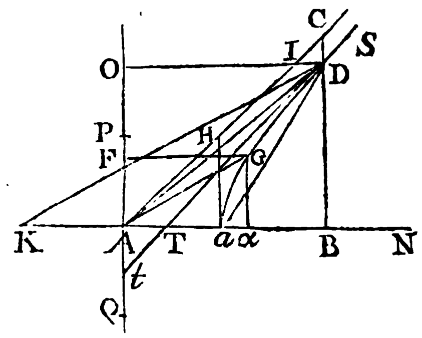
352. And here by the by you see the Solution of this Problem, *From the Logarithm of a Number or Ratio given, to find the Number or Ratio*: which is the Converse of the Prob. treated of Art. 341 *. * Of this Explication.

353. According to the Notation made use of by Mr. *Cotes*, let $M\big|\frac{A}{C}$ denote the Measure of the Ratio of A to C, in the System whose Modulus is M; A being the Antecedent, and C the Consequent: then if n denote any Number, integral or fractional, you shall have $M\big|\frac{A}{C} = nM\Big|\frac{A^{\frac{1}{n}}}{C^{\frac{1}{n}}}$, and $M\big|\frac{A}{C} = \frac{M}{n}\Big|\frac{A^n}{C^n}$. For $M\big|\frac{A}{C} : nM\big|\frac{A}{C} :: 1 : n$ *; and $nM\big|\frac{A}{C} : nM\Big|\frac{A^{\frac{1}{n}}}{C^{\frac{1}{n}}} :: n : 1$ *. Therefore *ex æquo* $M\big|\frac{A}{C} : nM\Big|\frac{A^{\frac{1}{n}}}{C^{\frac{1}{n}}} :: 1 : 1$. And the same is the Reasoning in the other Case. *Art. 348. of this Explication. *Art. 337. of this Explication.

354. Hence it follows (see Fig. Art. 344*. p. 251.) that the Area aHICa, or Sector AaC $= \frac{1}{2}\Delta$AaG $\big|\frac{EC}{CD}$ or $\frac{1}{2}\Delta$AaG $\big|\frac{BD+BC}{BD-BC}$: likewise aHICa, or AaC $= \frac{1}{2}\Delta$AaG $\big|\frac{KC+KL}{KC-KL}$. For EC : aG : CD $\div\!\!\div$, and also KC + KL : Aa : KC − KL $\div\!\!\div$, from the Property of the Hyperbola: hence the thing is evident by comparing the last Article with Art. 348 *. * Of this Explication. * Of this Explication.

[1] See Analysis by means of Equations infinite in Number of Terms, Sect. Invention of the Base from the Area given, likewise our Author's Letter to Mr. *Oldenburg*, 24 *Octob.* 1676.

355. Schol. What is principally to be remarked here is, that by means of a logarithmical Canon, any hyperbolical Space may be found. For let *a*GDS be any conical Hyperbola, defcribed with the Center A, tranfverfe and conjugate Axes K*a*, PQ; and Affymptote AC. Draw the Ordinates DB, DO, meeting the Axes in B and O, and the Affymptote in C and I: likewife from any other Point G the Ordinates G*α*, GF, meeting the tranfverfe and conjugate Axes in the Points *α* and F: at *a* draw the Tangent *a*H, meeting the Affymptote in H: join AG, AD, KD; and let DT*t* be a Tangent at D, meeting the tranfverfe Axis in T, and Conjugate in *t*. Then it appears from what has been faid, that the hyperbolical Spaces ADG*a*, *a*GDB, *α*GDB, *a*GD*a*, *a*GDT, *a*GDK, AODG*a*, FODG, &c. may be found by logarithmical Canon. For the Sector ADG*a* is the Meafure of the Ratio of *a*H to DC, as it is varioufly expreffed Art. 344 *, in the Syftem whofe Modulus is the Triangle A*a*H, and therefore if the Logarithm of that Ratio be taken from the logarithmical Canon, the Modulus of the logarithmical Canon is the Logarithm of the Ratio of *a*H to DC; as the Triangle A*a*H is to the Sector ADG*a* *, which therefore is found. The fame way is the Sector AG*a* found. And therefore, fince all the other hyperbolical Areas are made up of thefe and known rightlined Figures, they are found.



* Of this Explication.

* Art. 350. of this Explication.

356. Wherefore, if the Lines belonging to the hyperbolical Areas, be expreffed by Numbers, thefe Areas are moft conveniently found by a common Table of Logarithms: which is a Syftem of the Meafures of Ratios expreffed by Numbers (for there may be a great Variety of Syftems, in which thefe Meafures may be expreffed by any other Kind of Quantities). Thus in this Figure, fuppofe $Ka = 1$. $aH = 0{,}4$. $aB = 0{,}5$: fo that the Triangle $AaH = 0.1$, the Modulus of the Syftem. And becaufe $Aa = 0.5 : aH = 0.4 :: AB = 1 : BC$, therefore $BC = 0.8$: again, fince $aH^q = BC^q - BD^q$, therefore $BD = (\sqrt{BC^q - aH^q} = \sqrt{0.64 - 0.16} =)\ 0.6928203$, &c. and therefore $DC = (BC - BD = 0.8 - 0.6928203 =)\ 0.1071796$, &c. Wherefore the Sector $AaGD = 0.1 \times 2.30258509 \times$ Log. Ratio of 0.4 to 0.1071796. Now the Logarithm of any Ratio taken from a Table of Logarithms, is found by fubtracting the Logarithm of the Confequent from the Logarithm of the Antecedent: for if N and *n* denote

denote any two Numbers, then N : *n* :: N : 1 + 1 : *n*; but the Logarithm of N to 1, is that which is called the Logarithm of N; and the Logarithm of 1 to *n* is the opposite of the Logarithm of *n* to 1, that is — Log. *n*: so that the Logarithm of N to *n*, is Log. N — Log. *n* *. Therefore Sector A*a*GD = 0.1 × 2.302585o9 × Log. 0.4 — Log. 0.1071796 = 0.2302585, &c. × $\overline{1}$.6020600 — $\overline{1}$.0301121 = 0.2302585 × 0.5719479 = 0.131696, &c. = Sector A*a*GD nearly. Subtract this from ΔABD = $\frac{AB \times BD}{2}$ = 0.346410, &c. remains 0.214714 — = *a*GDB. Subtract the same Sector from ΔAD*a* = $\frac{Aa \times DB}{2}$ = 0.1732050, remains 0.041509 — = Segment *a*GD*a*. Again, to the Sector A*a*GD add that ΔAD*a* = ΔADK, the Sum is 0.304901 = *a*GDK. Then for the Space *a*GDT, from the Property of the Tangent, you have AB = 1 : A*a* = 0.5 :: A*a* : AT = 0.25. Therefore ΔADT = $\frac{AT \times BD}{2}$ = 0.0866020; which subtracting from the Sector A*a*GD = 0.131696, remains 0.045094 = *a*GDT. Further to the Sector A*a*GD add ΔAOD = $\frac{AB \times BD}{2}$ = 0.346410, the Sum is 0.478106 = Area AODG*a*. And it is abundantly evident, how, by the same Method, the Sector *a*GA may be found, and from thence the Areas *a*G*α* and AFG*α*: which being subtracted from *a*GDB and AODG*a*, you shall thereby find *α*GDB and FGDO. Which were the things I proposed to shew.

*Art. 340. of this Explication.

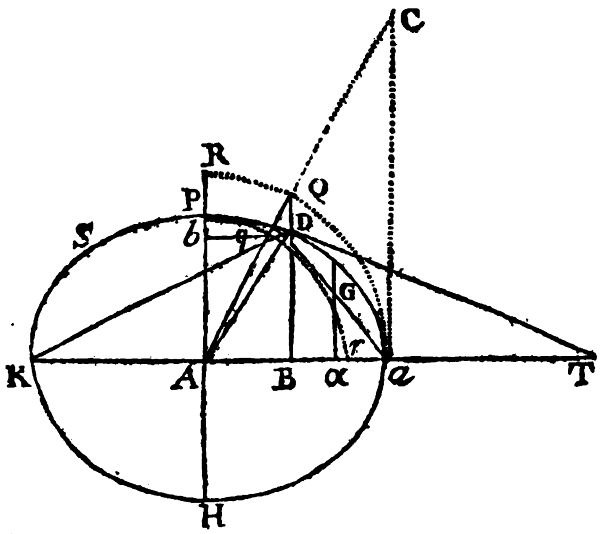
357. Hence it appears that the Areas of all those Curves, which are capable of being compared with the Hyperbola, may be found, by the Measures of Ratios, or by the Help of a Table of Logarithms: and therefore also, all such Fluents of any kind, whose Fluxions are analogous to the Fluxions of these Curves. More of which afterwards.

PROP.

358. Every elliptical Area may be reduced to a circular Area: and so be found by means of the trigonometrical Canon, and the known Proportion of the Radius of a Circle to the Circumference.

Let *a*GS be any given Ellipse, described with the Center A, transverse and conjugate Axes K*a*, PH: let AD be any Radius of the Ellipse and *a*GDA a Sector adjacent to the transverse Axe, and DPA a Sector adjacent to the shorter Axe: with the Radius A*a*, describe the quadrantal Arch *a*QR, meeting AP produced, in R: through D draw the Ordinate DB, meeting the transverse Axis in B, and produce it till it cut the Circle in Q, join AQ. I say the elliptical Sector *a*GDA :

circular Sector aQA :: BD : BQ :: AP : Aa; and therefore Sector $aGDA = \frac{1}{2}AP \times aQ$. And Sector DPA $= \frac{1}{2}AP \times QR$. For from the Property of the Ordinates BD : BQ :: AP : AR, therefore $aGDB$: aQB :: AP : AR, since their Fluxions are in that given Ratio: but ΔABD : ΔABQ :: BD : BQ :: AP : AR; therefore $aGDA$: aQA :: AP : AR :: BD : BQ; wherefore it is $aGDA = \frac{aQA \times AP}{AR} = \frac{1}{2}AP \times aQ$.



And thence it follows that Sector DPA $= \frac{1}{2}AP \times QR$.

PROP.

259. If with the Radius AP (see the preceding Figure) you describe the quadrantal Arch Pqr, meeting the longer Axe in r, and the Ordinate Db, belonging to the shorter Axe, in the Point q, the Sector DPA $= \frac{1}{2}Aa \times Pq$, and the Sector $aGDA = \frac{1}{2}Aa \times qr$. It is demonstrated the same way as the preceding.

360. Since it is $bD : bq$ :: ($Aa : Ar$ ::) BQ : BD, and by Division $bD - AB$: qD :: DB : QD, hence it appears that the Points A, q, Q lye in the same right Line: so that the circular Sectors RQA and PqA; aQA and rqA; and Arches RQ and Pq, aQ and rq, in this and the preceding Proposition, are similar. Moreover, if the Tangent aC be drawn at the Point a, and AqQ be produced till it meet it in C, the Arch aQ is such an Arch as has Aa for it's Radius, aC for it's Tangent and AC for it's Secant: which are proportional to AB, BQ and AQ or Aa. And it is evident that the like Construction may be made for the Circle Pqr; and that the Arches RQ and Pq, are such that Ab, bq and Aq or AP are as the Radius, Tangent and Secant of these Arches.

361. Hence it appears that as the hyperbolical Sectors are Measures of Ratios, so the elliptical Sectors are Measures of Angles: thus the elliptical Sector $aGDA$ is the Measure of the Angle aAQ: for since $aGDA = \frac{1}{2}AP \times aQ$, it appears that $aGDA$ increases and diminishes

minishes in the same Ratio the Arch aQ doth, that is in the same Ratio the Angle aAQ doth. But as the Measure of any given Ratio is not sufficiently determined, unless you know to what System it belongs: which may be known from the Modulus (as was explained at full Length) so neither can the Measure of a given Angle be known, unless you know to what System of Measures of Angles it belongs. Therefore it is necessary to consider the Measure of some certain determinate Angle, as a Modulus, to which the Measures of all other Angles in the System, may be referred and compared, and thereby determined. Now there may be conceived various Kinds of Measures of Angles: Arches of Circles, the angular Point being at the Center, make one System of Measures: circular Sectors make another.

The Measures in the first Case being Lines, in the other plane Surfaces; but both increase and decrease in the same Ratio with the Angles *. There may be different Modulus's: but the most convenient in the Systems of Arches, upon several Accounts, particularly because of the *Analogy* betwixt the Measures of Ratios and Angles, and the *natural* Passage from the one to the other, is the Radius. Which being established for the Modulus in this Kind, the Half of the Square of the Radius will be the Modulus in the Systems of circular Sectors: for the Radius bears the same Ratio to an Arch, which subtends any given Angle, as half the Square of the Radius bears to the circular Sector, which has the same Arch for it's Base; and is the Measure of the same given Angle in the System of circular Sectors: which is the Property and Notion of a Modulus.

* Eucl. Elem. lib. 6. Prop. 33.

362. Hence it follows that in a System of elliptical Sectors, considered as the Measures of Angles, the Modulus is half the Rectangle contained under the semitransverse and semiconjugate Axes, *i. e.* (in the foregoing Fig.) if you suppose the right Line Pa joined, the Triangle AaP: which is the same with the Modulus of the System of Measures of Ratios, belonging to an Hyperbola, having the same Axes with the Ellipse. For $\frac{AP \times Aa}{2} : \frac{AR \times Aa}{2} :: aGDA : aQA$; which two last are the Measures of the same Angle aAQ in the respective Systems: and therefore since $\frac{AR \times Aa}{2}$ is the Modulus of the one System, $\frac{AP \times Aa}{2}$ will be the Modulus of the other System.

363. From what has been said it appears, that, in a given Ellipse aP, whose transverse and conjugate Axes are known, if moreover the Ratio of Aa to aC or to AC, which are the Tangent and Secant of the Angle aAQ, in the Circle whose Radius is Aa, be given, the Arch

 aQ,

*a*Q, and elliptical Sector *a*GDA may be found. Now the Ratio of A*a* to *a*C, or, which is the same, of AB to BQ is given, when AB is given, for $BQ = \sqrt{Aa^q - AB^q}$: so that the Sector *a*GDA is the Measure of an Angle whereof the Radius, Tangent and Secant [1] are A*a*, *a*C and AC; or AB, BQ and A*a*, the Modulus being $\frac{AP \times Aa}{2}$. And it is evident from A*b* given, by the like means the other elliptical Sector PDA is found, by supposing a Tangent to the Circle P*qr*, drawn at the Point P, and produced till it meet A*q*C. For it is the Measure of an Angle, whose Radius, Tangent, and Secant are as A*b*, *bq* and AP, the Modulus being the same as before, *viz.* $\frac{AP \times Aa}{2}$.

364. Hence it appears, that, when the Lengths of the given Lines are expressed by Numbers, the Number of Degrees, Minutes, *&c.* contained in the Arches *a*Q and P*q* may be found, with great ease, by means of the trigonometrical Canon: and consequently the elliptical Sectors will be found, from the known Ratio of the Radius of a Circle to it's Circumference, whereby the Length of an Arch of a known Number of Degrees, Minutes, *&c.* with respect to the Length of the Radius is determin'd: thus, in the preceding Fig. p. 258 *. $aGDA = \frac{1}{2}AP \times aQ$ *, and $PDA = \frac{1}{2}AP \times RQ$ *. And therefore, in this Case, the Areas of any other elliptical Spaces may be found. That is, if you join KD, draw the Tangent DT meeting the Axis in T, and the Ordinate G*a*, you may find the Areas of the elliptical Spaces *a*GDB, *a*DG*a*, *a*GDT, *a*GDK, A*b*DG*a*, P*b*D, ABDP, *a*GDB, *&c.* after the same Manner as was shewn formerly, in the Case of the Hyperbola: since all these elliptical Spaces are made up of elliptical Sectors and known right-lined Figures, by Addition or Subtraction.

* Of this Explication.

*Art. 258. of this Explication.

Ex. Let us suppose K*a* = 1, HP = 0.8, AB = 0.25. Then AB = 0.25 : AQ = 0.5 :: Rad. : Sec. ∠ *a*AQ; whence Sec. ∠ *a*AQ = 2 × Rad. wherefore add the Log. of 2 to the logarithmical Rad. *i. e.* 0.3010300 to 10.0000000, and the Sum 10.3010300 will be found in the Table of artificial Secants, opposite to an Angle of 60°: therefore the Arch *a*Q is an Arch of 60°, *i. e.* $\frac{1}{6}$ of the whole Circumference. But the Circumference of a Circle is to the Rad. as 6.283185307179, *&c.* to 1: therefore that Number divided by 6, quotes 1.047197551196, *&c.* the Length of $\frac{1}{6}$ of the Periphery of a Circle having it's Radius 1: and since it is $Aa = \frac{1}{2}$, if you divide the former Number by 2, you'll

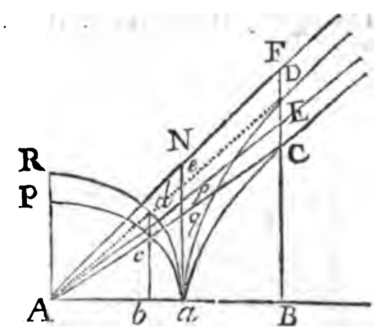
[1] When in this Place, or any other, I mention the Radius of an Angle, in conjunction with the Tangent and Secant, I mean, as is evident, the Radius, Tangent and Secant of a circular Arch which measures that Angle.

have

have 0.523598775598 the Length of aQ: which multiplied by $\frac{1}{2}$AP $=0.2$, gives $0.1047197551196 = a$GDA nearly.

Again $aK^q = 1 : HP^q = 0.64 :: KB \times Ba = 0.1875 : BD^q$; therefore BD $= \sqrt{0.64 \times 0.1875} = 0.346410161513$, &c. and therefore the Triangle ABD $= \frac{AB \times BD}{2}$ 0.043301270189, which subtracted from aGDA $= 0.104719755119$, leaves $0.061418484493 = a$GDB nearly. And we need not insist particularly, in shewing how all the other elliptical Areas mentioned, are deduced after the same Manner: after what has been done already, in the like Case of the hyperbolical Areas.

365. And from what has been said, with respect to the Ellipse, it appears, that the Areas of all such Curves as are capable of being compared geometrically with the Ellipse: and universally, all Fluents, whose Fluxions are analogous to the Fluxions of such curvilinear Areas, or are expressed by the same algebraical Quantities, may be found, and exhibited, by the Measures of Angles, *i.e.* by circular Arches, or Sectors, in a linear Description: or, by means of the trigonometrical Canon, (when the Values of the Quantities are given in Numbers) in an arithmetical Expression.

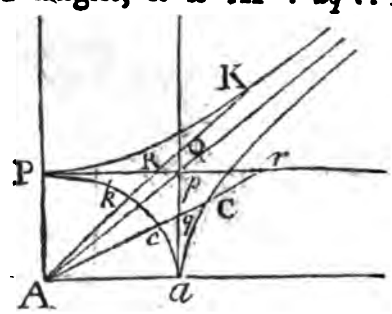


366. And now, it will be proper to shew the Analogy there is betwixt the Measures of Ratios, and the Measures of Angles, as they belong to the Hyperbola and Ellipse. In order to which let aC and aP be an Hyperbola and Ellipse, described with the Center A, and the same semitransverse and semiconjugate Axes, Aa, AP; and let AE be an Assymptote to the Hyperbola aC: draw the right Line AcC, and from C, c, the Lines CB, cb, perpendicular to the Axis Aa, at the Points B, b: at the Vertex a, draw the Tangent apq, cutting AC, AE in q and p. I say the hyperbolical Sector AaC $= \frac{Aa \times AP}{2} \Big| \frac{AP + aq}{\sqrt{AP^q - aq^q}}$: and the elliptical Sector A$ac = \frac{Aa \times AP}{2}$ $\Big| \frac{AP \,.\, aq}{\sqrt{AP^q + aq^q}}$: by the last of which two Notations, I mean the Measure of an Angle, whose Radius, Tangent, and Secant are AP, aq, and $\sqrt{AP^q + aq^q}$, to the Modulus $\frac{Aa \times AP}{2}$: where it must be observed that of the two Quantities as AP . aq . the first always designs the Radius and

and the second the Tangent. For if with the Center A, and Semiaxis Aa, the equilateral Hyperbola aD, and quadrantal Arch aR be described; the Assymptote ANF to the Hyperbola aD be drawn: and if BC be produced till it meet the Hyperbola aD in D, and the Assymptotes AE, AF, in E and F; and bc be produced till it meet the Circle in the Point d; and moreover aqp, produc'd, if need be, till it meet the Assymptote AF in N: then, because it is $bc : bd ::$ (AP $=$ *Art. 346. of this Explication. ap : AR $= a$N :: BE : BF ::) BC : BD *, therefore the Points A, d, D, lye in the same right Line. Let that right Line AdD be joined, and intersect aN, in the Point o.

*Art. 344. of this Explication. Now the hyperbolical Sector AaC $= \frac{Aa \times AP}{2} \left| \frac{AP + aq}{\sqrt{AP^2 - aq^2}} \right.$ *: for AP *Art. 363. of this Explication. $= ap$. Again the elliptical Sector A$ac = \frac{Aa \times AP}{2} \left| \frac{Aa \, . \, ao}{Ao} \right.$ * $= \frac{Aa \times AP}{2} \left| \frac{aN \, . \, ao}{\sqrt{aN^2 + ao^2}} \right.$ $=$ (because aN : ao :: BF : BD :: BE : BC :: ap=AP : aq) $\frac{Aa \times AP}{2} \left| \frac{AP \, . \, aq}{\sqrt{AP^2 + aq^2}} \right.$, as was to be shewn.

367. Cor. 1. Let Pp be joined and produced indefinitely towards r; and AcC produc'd, if need be, meet it in r: then, from similar Triangles, it is AP : aq :: Pr : Aa; therefore it follows that the



same hyperbolical Sector AaC $= \frac{Aa \times AP}{2} \left| \frac{Pr + Aa}{\sqrt{Pr^2 - Aa^2}} \right.$: and the elliptical Sector A$ac = \frac{Aa \times AP}{2} \left| \frac{Pr \, . \, Aa}{\sqrt{Pr^2 + Aa^2}} \right.$. And when the Points q and r by approaching to the Point p, coincide with it, the hyperbolical Sector AaC becomes infinite, and so doth the Measure of the Ratio; for the Consequent of the Ratio vanishes. But the elliptical Sector Aac becomes the Measure of an Angle of 45°, because AP and aq or Pr and Aa, that is the Radius and Tangent became equal. And if q and r appear upon the opposite Side of p, *viz.* in Q and R, the hyperbolical Sector AaC becomes $\frac{Aa \times AP}{2} \left| \frac{AP + aQ}{\sqrt{AP^2 - aQ^2}} \right.$ or $\frac{Aa \times AP}{2} \left| \frac{PR + Aa}{\sqrt{PR^2 - Aa^2}} \right.$ both impossible, as including negative Squares. But the elliptical Sector, now A$ak = \frac{Aa \times AP}{2} \left| \frac{AP \, . \, aQ}{\sqrt{AP^2 + aQ^2}} \right.$, or $\frac{Aa \times AP}{2} \left| \frac{PR \, . \, Aa}{\sqrt{PR^2 + Aa^2}} \right.$.

368. Cor.

368. Cor. 2. In this last Case, *viz.* when the Points q and r fall in Q and R, with the semitransverse and semiconjugate Axes AP, Aa, describe PK a conjugate Hyperbola to the former aC, then, by the same way of reasoning as before, the hyperbolical Sector APK $= \frac{Aa\times AP}{2}\Big|\frac{Aa+PR}{\sqrt{Aa^2-PR^2}}$; or $\frac{Aa\times AP}{2}\Big|\frac{aQ+AP}{\sqrt{aQ^2-AP^2}}$: and the elliptical Sector APk $= \frac{Aa\times AP}{2}\Big|\frac{Aa\,.\,PR}{\sqrt{Aa^2+PR^2}}$ or $\frac{Aa\times AP}{2}\Big|\frac{aQ\,.\,AP}{\sqrt{aQ^2+AP^2}}$.

369. Cor. 3. By the same way of reasoning it appears, that the hyperbolical Sector AaC $= \frac{Aa\times AP}{2}\Big|\frac{Ar+Aq}{\sqrt{Ar^2-Aq^2}}$: and the elliptical Sector Aac $= \frac{Aa\times AP}{2}\Big|\frac{Ar\,.\,Aq}{\sqrt{Ar^2+Aq^2}}$. And APK $= \frac{Aa\times AP}{2}\Big|\frac{AQ+AR}{\sqrt{AQ^2-AR^2}}$: and AP$k$ $= \frac{Aa\times AP}{2}\Big|\frac{AQ\,.\,AR}{\sqrt{AQ^2+AR^2}}$.

370. Schol. Because the Measure of any Ratio or Angle is increased and diminished in the same Proportion as the Modulus is *, hence it appears that the Expressions $\frac{Aa\times AP}{2}\Big|\frac{AP+aq}{\sqrt{AP^2-aq^2}}$, and $\frac{Aa\times AP}{2}\Big|\frac{AP\,.\,aq}{\sqrt{AP^2+aq^2}}$, are the same thing as $\frac{1}{2}Aa\times AP\Big|\frac{AP+aq}{\sqrt{AP^2-aq^2}}$, and $\frac{1}{2}Aa\times AP\Big|\frac{AP\,.\,aq}{\sqrt{AP^2+aq^2}}$: in which last two Expressions, AP only is considered as the Modulus, and the Measure of the Ratio, or Angle, to that Modulus, is multiplied by $\frac{1}{2}$Aa; and so in other like Cases. Moreover it may be observed, that the Terms AP . aq . $\sqrt{AP^2 \mp aq^2}$ may be multiplied or divided by any Quantity, without altering the Measure of the Ratio or Angle.

* Art. 350, 361. of this Explication.

371. I have insisted the longer upon this Affair of the Measures of Ratios and Angles, as it is connected with hyperbolical and elliptical Areas, that I might thereby shew the Connection there is betwixt our Author's Method of reducing curvilinear Areas, and consequently other analogous Fluents, to hyperbolical, elliptical and circular Areas; and the Method of the ingenious Mr. *Cotes*, by which he reduces such Fluents, (with many other besides) to the Measures of Ratios and Angles, in his *Harmonia Mensurarum*, published and enlarged by the learned Dr. *Smith*. So that what I have said, may serve as an Introduction to the young Geometrician, to the reading and understanding Mr. *Cotes*'s Method, in the Book just now mentioned. For a further Commentary upon which, the Reader may consult a Performance entitled *Epistola ad Amicum de Cotesii Inventis*,

372. And

372. And now I shall illustrate this Connection of the two different Methods by an Example. Take the Curve belonging to Form third Species first of Sir *Isaac Newton's* second Table, defined by this Equation $\frac{d}{z}\sqrt{e+fz^{\eta}}=y$: the Quadrature of which was demonstrated, and explained at full Length already, according to our Author's Method *. The analogous Fluxion and Fluent are found in Form third of Mr. *Cotes's* first Table, *viz.* $d\dot{z}z^{-1}\sqrt{e+fz^{\eta}}$, and $\frac{2}{\eta}dP-\frac{2}{\eta}dR\left|\frac{R+T}{S}\right.$: where the Values of P. R. T. S. are $P=\sqrt{e+fz^{\eta}}$. $R=\sqrt{e}$. $T=\sqrt{e+fz^{\eta}}$. $S=\sqrt{fz^{\eta}}$.

* Art. 276—279. of this Explication.

Now by what was demonstrated Art. 276 *, if (see Fig. 2. Tab. 2. p. 25.) with the Center A, and half transverse Axis $AP=\sqrt{f}$ and half Parameter $\frac{\sqrt{f}}{e}$; you describe the Hyperbola PGDS; and, upon the conjugate Axis AN, you take $AB=(x=)\frac{1}{\sqrt{z^{\eta}}}$, draw the Ordinate BD, and the Tangent DT meeting AB in T: then the Area of the Curve sought is $\frac{4d}{\eta f}\times\overline{TDB-APDB}$. Join AD, and $\frac{1}{2}$ the conjugate Axis is $\sqrt{\frac{f}{e}}$; therefore if to $AO=\sqrt{f+ex^2}$, $OD=x$, and $AP=\sqrt{f}$, you find a fourth Proportional, *viz.* $\frac{x\sqrt{f}}{\sqrt{f+ex^2}}=$ (by inserting $\frac{1}{\sqrt{z^{\eta}}}$ for x) $\sqrt{\frac{f}{e+fz^{\eta}}}$: then you'll have, by what has been demonstrated *, the Sector $APGD=\frac{f}{2\sqrt{e}}\left|\frac{\sqrt{\frac{f}{e}}+\sqrt{\frac{f}{e+fz^{\eta}}}}{\sqrt{\frac{f}{e}-\frac{f}{e+fz^{\eta}}}}\right.=$ (by reducing) $\frac{f}{2\sqrt{e}}\left|\frac{\sqrt{e}+\sqrt{e+fz^{\eta}}}{\sqrt{fz^{\eta}}}\right.$: to which if you add the Triangle $ADB=\frac{x\sqrt{f+ex^2}}{2}=\frac{z^{-\eta}\sqrt{e+fz^{\eta}}}{2}$, the Sum $\frac{z^{-\eta}\sqrt{e+fz^{\eta}}}{2}+\frac{f}{2\sqrt{e}}\left|\frac{\sqrt{e}+\sqrt{e+fz^{\eta}}}{\sqrt{fz^{\eta}}}\right.=APDB$. But the Triangle TDB was shewn to be equal to $\frac{v^3}{2ex}=\frac{\overline{f+ex^2}^{\frac{3}{2}}}{2ex}=\frac{\overline{f+ex^2}\sqrt{f+ex^2}}{2ex}=\frac{\overline{f+ex}^{-1}}{2e}\sqrt{e+fz^{\eta}}$: therefore TDB—APDB $=\frac{\overline{f+ex}^{-\eta}}{2e}\sqrt{e+fz^{\eta}}-\frac{z^{-\eta}\sqrt{e+fz^{\eta}}}{2}-\frac{f}{2\sqrt{e}}\left|\frac{\sqrt{e}+\sqrt{e+fz^{\eta}}}{\sqrt{fz^{\eta}}}\right.=\frac{f}{2e}\sqrt{e+fz^{\eta}}-\frac{f}{2\sqrt{e}}\left|\frac{\sqrt{e}+\sqrt{e+fz^{\eta}}}{\sqrt{fz^{\eta}}}\right.=\frac{f}{2e}\sqrt{e+fz^{\eta}}-\frac{f}{2e}\sqrt{e}\left|\frac{\sqrt{e}+\sqrt{e+fz^{\eta}}}{\sqrt{fz^{\eta}}}\right.=\frac{f}{2e}P-\frac{f}{2e}R\left|\frac{R+T}{S}\right.$:

* Of this Explication.

* Art. 366. of this Explication.

$\left|\frac{R+T}{S}\right.$: wherefore $\frac{4de}{nf}\times\overline{TDB-APDB}=\frac{2d}{n}P-\frac{2d}{n}R\left|\frac{R+T}{S}\right.$: as it is in Mr. *Cotes*'s Table.

373. But if f be negative, so that it be $\frac{d}{z}\sqrt{e-fz^n}=y$: then by what was shewn at Art. 277 *, (see Fig. 3. Tab. 2. p. 25.) if, with the Center A, half Transverse $Aa=\sqrt{\frac{f}{e}}$ and half Conjugate $=\sqrt{f}$, you describe the Hyperbola aGDS; take AB $=(x=)\frac{1}{\sqrt{z^n}}$, draw the Ordinate BD, the Line AD, and Tangent DT, then the Area of the Curve sought is equal to $-\frac{4de}{nf}\times a$GDT. But, by what has been demonstrated Art. 366 *, if to AB $=x$, BD $=v=\sqrt{-f+ex^2}$, and $Aa=\sqrt{\frac{f}{e}}$, you find a fourth Proportional, *viz.* $\frac{\sqrt{-\frac{ff}{e}+fx^2}}{x}=\sqrt{\frac{ef-ffz^n}{e}}$, then you shall have the hyperbolical Sector AaGD $=$ $\frac{f}{2\sqrt{e}}\left|\frac{\sqrt{f}+\sqrt{\frac{ef-ffz^n}{e}}}{\sqrt{\frac{f^2z^n}{e}}}\right.=\frac{f}{2e}\sqrt{e}\left|\frac{\sqrt{e}+\sqrt{e-fz^n}}{\sqrt{fz^n}}\right.$: from which subtract the Triangle ADT $=\left(\frac{AD\times BD}{2}=\frac{\overline{BA-BT}\times BD}{2}=\frac{\overline{x-\frac{v^2}{ex}}\times\sqrt{-f+ex^2}}{2}=\overline{x+\frac{f-ex^2}{ex}}\times\frac{\sqrt{-f+ex^2}}{2}=\right)\frac{f}{2e}\sqrt{e+fz^n}$, there remains aGDT $=$ $\frac{f}{2e}\sqrt{e}\left|\frac{\sqrt{e}+\sqrt{e-fz^n}}{\sqrt{fz^n}}\right.-\frac{f}{2e}\sqrt{e+fz^n}=\frac{f}{2e}R\left|\frac{R+T}{S}\right.-\frac{f}{2e}P$; wherefore $-\frac{4de}{nf}\times a$GDT $=\frac{2}{n}dP-\frac{2}{n}dR\left|\frac{R+T}{S}\right.$, the same as in Mr. *Cotes*'s Table.

374. Finally, if it be $\frac{d}{z}\sqrt{-e+fz^n}=y$: then, by what was demonstrated Art. 278 * (see Fig. 4. Tab. 2. p. 25.) if with the Center A, Semitransverse $Aa=\sqrt{\frac{f}{e}}$, and Semiconjugate AP $=\sqrt{f}$ (the same as in the preceding Figure) the Ellipse aGDS be described; the Radius AD, and Tangent DT be drawn; and you take AB $=(x=)$ $\frac{1}{\sqrt{z^n}}$, the Area of the Curve sought is $\frac{4de}{nf}\times a$GDT. But (by what has been demonstrated Art. 366 *.) if to AB $=x$, BD $=\sqrt{f-ex^2}$, and

* Of this Explication.

* Of this Explication.

* Of this Explication.

* Of this Explication.

and $Aa = \sqrt{\frac{f}{e}}$, you find a fourth Proportional, *viz.* $\frac{\sqrt{\frac{f^2}{e} - fx^2}}{x} = \sqrt{\frac{-ef + ffz^n}{e}}$, the elliptical Sector $aGDA = \frac{f}{2\sqrt{e}} \Big| \frac{\sqrt{f} \cdot \sqrt{\frac{-ef+ffz^n}{e}}}{\sqrt{\frac{f^2 z^n}{e}}} = \frac{f}{2e}\sqrt{e} \Big| \frac{\sqrt{e} \cdot \sqrt{-e+fz^n}}{\sqrt{fz^n}}$: which being subtracted from the Triangle ADT $= \left(\frac{AT \times BD}{2} = \frac{\frac{f}{ex} \times \sqrt{f - ex^2}}{2} = \right) \frac{f}{2e}\sqrt{-e+fz^n}$, there remains $\frac{f}{2e}\sqrt{-e+fz^n} - \frac{f}{2e}\sqrt{e} \Big| \frac{\sqrt{e} \cdot \sqrt{-e+fz^n}}{\sqrt{fz^n}} = \frac{f}{2e}P - \frac{f}{2e}R \Big| \frac{R.T}{S}$: wherefore $\frac{4de}{nf} \times aGDT = \frac{2}{n}dP - \frac{2}{n}dR \Big| \frac{R.T}{S}$: the same as in Mr. *Cotes*'s Table.

375. Schol. It may be observed that the same Notation, making allowance for the Difference of the Signs, serves all the Cases, in this Example: and the same happens in all others: when the Fluents are expressed by the Measures of Ratios, and Angles. In the last Case, where the Area is expressed by the elliptical Area; and the Fluent by the Measure of an Angle, we might have taken AB for the Radius, and $\sqrt{Aa^q - AB^q}$, for the Tangent of the Angle to the same Modulus as before: and so it would have been $\frac{f}{2\sqrt{e}} \Big| \frac{x \cdot \sqrt{\frac{f}{e} - x^2}}{\sqrt{\frac{f}{e}}} = \frac{f}{2\sqrt{e}} \Big| \frac{\sqrt{e} \cdot \sqrt{-e+fz^n}}{\sqrt{fz^n}}$, for the Value of the Sector $aGDA$, as before. It may likewise be observed that the two Hyperbolas are conjugate to one another, the transverse Axis of the one being the conjugate Axis of the other; and the conjugate Axis the same with the transverse: and the Ellipse has the same Axes with the Hyperbolas; so as to be that which is inscribed within the conjugate Hyperbolas, as represented by the Fig. at Art. 367 *. Moreover the Expressions for the Measures of the Ratios and Angles might have been deduced otherwise, by considering what was shewn Art. 367, 368, 369 *.

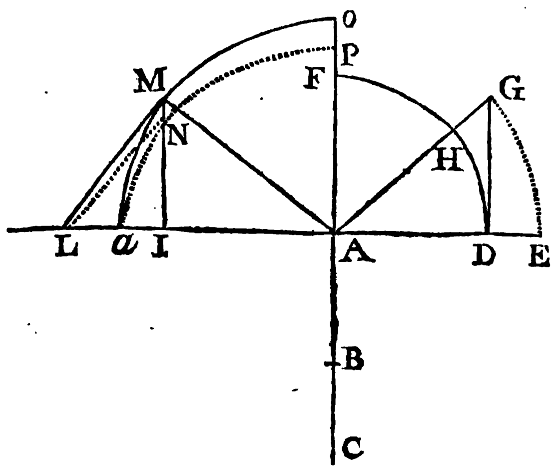
* Of this Explication.

* Of this Explication.

376. If the Area of the Curve whose Equation is $\frac{d}{z}\sqrt{-e+fz^n} = y$; or the analogous Fluent of the Fluxion $d\dot{z}z^{-1}\sqrt{-e+fz^n}$, be required in a linear Description by means of a Circle, it may be thus. Take some Quantity for linear Unity in the Construction, by which

let

let the Dimensions of the several Terms be reduced to the just Number, by Multiplication and Division thereby: suppose it be f. Then take AP $= f$, and upon AP produced beyond A, take AB $= e$, and AC $= z^n$; and having described upon PB and PC as Diameters, two Semicircles; through A draw ADE perpendicular to AP, meeting the Semicircles in D and E: with the Radius AD describe the Quadrant DHF, draw the Tangent DG; and with the Radius AE describe the Arch EG cutting the Tangent in G, join AG: and the Area, or Fluent required is equal to $\frac{2d}{n} \times \overline{DG - DH}$.



For it is evident by the Construction, that AD $= \sqrt{e}$ and AE $= \sqrt{fz^n} =$ AG, and therefore DG $= \sqrt{-e + fz^n}$: whence it follows that the Arch DH $= \sqrt{e} \left| \frac{\sqrt{e} \cdot \sqrt{-e+fz^n}}{\sqrt{fz^n}} \right.$; and DG $-$ DH $\sqrt{-e+fz^n} - \sqrt{e} \left| \frac{\sqrt{e} \cdot \sqrt{-e+fz^n}}{\sqrt{fz^n}} \right. = P - R \left| \frac{R \cdot T}{S} \right.$; therefore $\frac{2d}{n} \times \overline{DG - DH} = \frac{2}{n} dP - \frac{2}{n} dR \left| \frac{R \cdot T}{S} \right.$, as it ought to be *.

377. The Construction may be made otherwise thus. Supposing as before AP $= f = 1$, AC $= z^n$, AB $= e$, AE $= \sqrt{fz^n} = \sqrt{z^n}$, and AD $= \sqrt{e}$: produce EA towards L, upon which take AI a third Proportional to AE $= \sqrt{z^n}$ and AP $= 1$, so is AI $= \left(\frac{1}{z^n} =\right) x$ (see Table 2. Form 3. Quad. p. 25. and take Aa a third Proportional to AD $= \sqrt{e}$ and AP $= 1$, so is A$a = \frac{1}{\sqrt{e}}$. With the Radius Aa describe the Quadrant aMO: through I draw IM perpendicular to Aa, cutting the Quadrant in M: draw the Radius AM, and the Tangent ML meeting Aa produced in L, then the Area of the Curve whose Ordinate is $\frac{d}{z}\sqrt{-e + fz^n}$, or the analogous Fluent whose Fluxion is $\frac{d\dot{z}}{z}\sqrt{-e + fz^n}$ is $\frac{2d}{n} \times \overline{LM \times AB - aM \times AB}$. For with the Semi-

*Art. 374. of this Explication.

 axes

axes A*a*, AP, describe the Quadrant of the Ellipse *a*NP, cutting IM in the Point N; join NL, which is a Tangent to the Ellipse at the Point N, then by what was formerly demonstrated *, the Area or Fluent is equal to $\frac{4de}{nf} \times a\text{NL}$; but it is $a\text{NL} : a\text{ML} :: (\text{IN} : \text{IM} ::) \text{AP} = 1 : \text{A}a = \frac{1}{\sqrt{e}}$, therefore $a\text{NL} = a\text{ML} \times \sqrt{e} = \overline{\text{LM} - a\text{M}} \times \frac{\text{A}a}{2} \times \sqrt{e} = \overline{\text{LM} - a\text{M}} \times \frac{1}{2\sqrt{e}} \times \sqrt{e} = \overline{\text{LM} - a\text{M}} \times \frac{1}{2}$: substitute this for *a*NL in the foregoing Expression $\frac{4de}{nf} \times a\text{NL}$, 1 for *f*, and AB for *e*, and it becomes $\frac{2d}{n} \times \overline{\text{LM} \times \text{AB} - a\text{M} \times \text{AB}}$ for the Area or Fluent required.

*Art. 278. of this Explication.

SECT. XI.

Containing the Demonstration of Prop. 11.

378. SUpposing the same Things as are supposed by our Author in this Proposition, (which see with the Figure belonging to it) only let us further suppose that the Areas of the Curves ADB, AEB, AFB, *&c.* are represented by α, β, γ, *&c.* and consequently that the Ordinates BE, BF, BG, *&c.* are $\frac{\alpha}{1}$, $\frac{\beta}{1}$, $\frac{\gamma}{1}$, *&c.* Then I say, if the Areas α, β, γ, *&c.* be terminated at the whole given Abscis $\text{AC} = t$, and at the Ordinate CI given in Position and infinitely produced, *i. e.* when they are the Areas ADIC, AEKC, AFLC, *&c.* and the Areas A, B, C, *&c.* be terminated at the same Abscis and Ordinate, then it shall be

1°. $\text{ADIC} = \text{A}$.
2°. $\text{AEKC} = t\text{A} - \text{B}$.
3°. $\text{AFLC} = \frac{t^2\text{A} - 2t\text{B} + \text{C}}{2}$.
4°. $\text{AGMC} = \frac{t^3\text{A} - 3t^2\text{B} + 3t\text{C} - \text{D}}{6}$.
5°. $\text{AHNC} = \frac{t^4\text{A} - 4t^3\text{B} + 6t^2\text{C} - 4t\text{D} + \text{E}}{24}$.
&c.

That is to say universally, if ε represent the Area of any Curve in the Series α, β, γ, &c. at any Distance of Place from the first α; and such indefinite Distance be called *d*, then, when ε and A, B, C, &c. are terminated at the whole given Abscis AC, and infinite Ordinate CI, it shall be

$\varepsilon =$

$$s = \frac{t^d A - dt^{d-1} B + d \times \frac{d-1}{2} t^{d-2} C - d \times \frac{d-1}{2} \times \frac{d-2}{3} t^{d-3} D + \&c.}{d \times \overline{d-1} \times d - 2 \ \&c.}$$

Where the numeral Coefficients of the Terms in the Numerator, *viz.* 1, $-d$, $d \times \frac{d-1}{2}$ &c. are the same with those which belong to the Terms of a Residual, such as $a - b$, raised to the Power whose Exponent is d, *i. e.* of $\overline{a-b}|^d$ when actually involved: the Exponent of the Power of t in the first Term is d, in the second, $d - 1$, in the third, $d - 2$, &c. the same as the Exponents of the Powers of a the first Member of the Residual, and A, B, C, &c. are Factors of the first, second, third, &c. Terms respectively. And the Factors by whose Multiplication the Denominator is formed, diminish each by Unity from the first d, and are continued till the last be equal to Unity; or which is the same, till their Number be equal to d.

379. To demonstrate which, I shall prove that, when the Areas A, B, C, D, &c. and s, have any common Abscis $z = AB$, the Relation of the Areas shall be expressed after the same manner, *viz.* thus

$$s = \frac{z^d A - dz^{d-1} B + d \times \frac{d-1}{2} z^{d-2} C - d \times \frac{d-1}{2} \times \frac{d-2}{3} z^{d-3} D + \&c.}{d \times \overline{d-1} \times \overline{d-2} \ \&c.}$$

For by Proposition 1, the Fluxion of

$$\frac{z^d A - dz^{d-1} B + d \times \frac{d-1}{2} z^{d-2} C - d \times \frac{d-1}{2} \times \frac{d-2}{3} z^{d-3} D + \&c.}{d \times \overline{d-1} \times \overline{d-2} \ \&c.}$$

is $\dfrac{z^{d-1} \dot{z} A}{\overline{d-1} \times \overline{d-2} \ \&c.} + \dfrac{z^d \dot{A}}{d \times \overline{d-1} \times \overline{d-2} \ \&c.}$

$- \dfrac{\overline{d-1} \times z^{d-2} \dot{z} B}{\overline{d-1} \times \overline{d-2} \ \&c.} - \dfrac{dz^{d-1} \dot{B}}{d \times \overline{d-1} \times \overline{d-2} \ \&c.}$

$+ \dfrac{\overline{d-1} \times \frac{d-2}{2} z^{d-3} \dot{z} C}{\overline{d-1} \times \overline{d-2} \ \&c.} + \dfrac{d \times \frac{d-1}{2} z^{d-2} \dot{C}}{d \times \overline{d-1} \times \overline{d-2} \ \&c.}$

$- \dfrac{\overline{d-1} \times \frac{d-2}{2} \times \frac{d-3}{3} z^{d-4} \dot{z} D}{\overline{d-1} \times \overline{d-2} \ \&c.} - \dfrac{d \times \frac{d-1}{2} \times \frac{d-2}{3} z^{d-3} \dot{D}}{d \times \overline{d-1} \times \overline{d-2} \ \&c.}$

&c. &c.

But $\dot{A} = \dot{z}y$. $\dot{B} = (\dot{z}zy =) \ z\dot{A}$. $\dot{C} = (\dot{z}z^2y =) \ z^2\dot{A}$. $\dot{D} = (\dot{z}z^3y =) \ z^3\dot{A}$, &c. which Values of $\dot{B}$, $\dot{C}$, $\dot{D}$, &c. being put for them, in the in the Terms which constitute the second Column, it becomes

$$\frac{z^d\dot{A} - dz^d\dot{A} + d\times\frac{d-1}{2}z^d\dot{A} - d\times\frac{d-1}{2}\times\frac{d-2}{3}z^d\dot{A} + \&c.}{d\times\overline{d-1}\times\overline{d-2}\ \&c.} \text{ or } \frac{z^d\dot{A}}{d\times\overline{d-1}\times\overline{d-2}\ \&c.}$$

$\times\overline{1-d+d\times\frac{d-1}{2}-d\times\frac{d-1}{2}\times\frac{d-2}{3}+\&c.}$ where the last Factor $1-d+d\times\frac{d-1}{2}-\&c.$ being the numeral Coefficients of a Residual raised to the Power whose Exponent is d, is therefore equal to nothing: consequently all the Terms constituting the second Column, taken together are equal to nothing. Wherefore the Terms constituting the first Column, make up the Fluxion of the Quantity

$$\frac{z^dA - dz^{d-1}B + d\times\frac{d-1}{2}z^{d-2}C - d\times\frac{d-1}{2}\times\frac{d-2}{3}z^{d-3}D + \&c.}{d\times\overline{d-1}\times\overline{d-2}\ \&c.}$$

Which therefore is

$$\dot{z}\times\frac{z^{d-1}A - \overline{d-1}\times z^{d-2}B + \overline{d-1}\times\frac{d-2}{2}z^{d-3}C - \&c.}{d-1\times\overline{d-2}\ \&c.}$$

Wherefore, if the first of these express the Area of a Curve whose Abscis is z, the second will express the Fluxion of that Area: and therefore, if it be divided by $\dot{z}$, it will express the Value of the Ordinate of that Curve. In which last Expression, the numeral Coefficients of the Terms in the Numerator are the same with those of the Power of a Residual, whose Exponent is $d-1$: the Indexes of the Powers of z are $d-1$, $d-2$, $d-3$, &c. and A, B, C, &c. are Factors of the first, second, third, &c. Terms respectively. Whence it appears that the Number of Terms in this Expression of the Ordinate, is one less than in the Expression of the corresponding Area: and further, the Denominator of the last Expression is equal to the Denominator of the former divided by d.

Moreover if d be 1; so that the Area of the Curve be $zA - B$, the Fluxion is $\dot{z}A + \dot{A}z - \dot{B} = (\dot{z}A + \dot{z}zy - \dot{z}zy =)\ \dot{z}A$, and consequently the Ordinate is A, by considering of which things it appears that in the Series

$$A\ .\ zA - B\ .\ \frac{z^2A - 2zB + C}{2}\ .\ \frac{z^3A - 3z^2B + 3zC - D}{6}\ .\ \frac{z^4A - 4z^3B + 6z^2C - 4zD + E}{24}$$

&c. in inf.

If you take any two next adjoining Terms, the first of the two being the Ordinate of a Curve whose Abscis is z, the last of the two will

will be the Area. Or, which is the same, if you consider all the Terms as Areas of Curves, having the common Abscifs z; and suppose $\dot{z}$ or $\dot{z} \times 1 = 1$, then each Area will be the Fluxion, or, if you please, the Exponent of the Fluxion of the one immediately following. But $\alpha . \beta . \gamma . \delta$ &c. is a Series of Areas having the same Abscifs, and Relation as the former: and the first Area A is equal to the first Area α, since they have the same Abscifs z and Ordinate y: therefore all the succeeding Terms of the one Series are equal to all the succeeding Terms of the other, each to each. Now suppose the Abscifs z to become equal to AC or t, then α, β, γ, δ, &c. become ADIC, AEKC, AFLC, &c. and therefore it will be

1°. $ADIC = A$.
2°. $AEKC = tA - B$.
3°. $AFLC = \frac{t^2A - 2tB + C}{2}$.
4°. $AGMC = \frac{t^3A - 3t^2B + 3tC - D}{6}$.
5°. $AHNC = \frac{t^4A - 4t^3B + 6t^2C - 4tD + E}{24}$.
&c.

This was the first thing to be demonstrated.

380. The second thing to be demonstrated is, that, if $x = t - z$, and P . Q . R . S, &c. be a Series of Areas having x for their common Abscifs, and y, xy, x^2y, x^3y, &c. for their Ordinates respectively, when the Areas P . Q . R . S &c. as well as A . B . C . D &c. are terminated at the whole given Abscifs $AC = t$, and infinite Ordinate CI, it shall be

$P = A$.
$Q = tA - B$.
$R = t^2A - 2tB + C$.
$S = t^3A - 3t^2B + 3tC - D$.
&c. *in inf.*

Or universally, if d denote the Distance of Place of any Area from the first P, in the Series of Areas P . Q . R . S . &c. that Area shall be equal to

$$t^dA - dt^{d-1}B + d \times \frac{d-1}{2}t^{d-2}C - d \times \frac{d-1}{2} \times \frac{d-2}{3}t^{d-3}D, \text{ \&c.}$$

the same Series as that mentioned in the preceding Part of the Proposition, only wanting the Denominator.

For the Curve belonging to the Series P . Q . R . S &c. which is at the Distance d from the first, has x^dy for it's Ordinate: but $x = t - z$,

or

or $x^d = (\overline{t-z}|^d =)\ t^d - dt^{d-1}z + d \times \frac{d-1}{2} t^{d-2}z^2 - d \times \frac{d-1}{2} \times \frac{d-2}{3} t^{d-3}z^3 +$ &c. therefore $x^dy = t^dy - dt^{d-1}zy + d \times \frac{d-1}{2} t^{d-2}z^2y - d \times \frac{d-1}{2} \times \frac{d-2}{3} t^{d-3}z^3y +$ &c.

Now the Ordinates y, zy, z^2y, &c. belong to the Areas A, B, C, &c. consequently, since t is a given Quantity, the Area belonging to the compounded Ordinate $t^dy - dt^{d-1}zy + d \times \frac{d-1}{2} t^{d-2}z^2y - d \times \frac{d-1}{2} \times \frac{d-2}{3} t^{d-3}z^3y +$ &c. will be $t^dA - dt^{d-1}B + d \times \frac{d-1}{2} t^{d-2}C - d \times \frac{d-1}{2} \times \frac{d-2}{3} t^{d-3}D +$ &c. the Abscis being z: and the same is true, when the Abscis z grows into the whole given Abscis t. Therefore, since the Ordinate x^dy at every Point of the given Abscis AC $= t$, is equal to the compounded Ordinate, the Area described by the Ordinate x^dy drawn along the whole Abscis AC, is equal to the former compounded Area. Hence the second Part of the Proposition appears.

And therefore by considering what has been demonstrated in this, and the preceding Article, the Truth of what our Author asserts in this Proposition is manifest; and the Corollary, thence deduced, plainly appears from the Proposition itself.

SECT. XII.

Containing an Explication of our Author's last Scholium; with the Application of the preceding Doctrine to the Solution of some Problems.

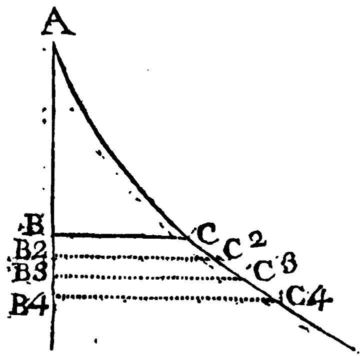
381. THE Affair of the different Orders of Fluxions was fully explained at the Beginning of this Work in Sect. 2; and the Justness of the Conception vindicated from the Objections that have been raised against it by a late Author. In the same Place I likewise shewed how these different Orders of Fluxions may be expounded by the Ordinates of Curves, after such manner as is mentioned by our Author here in Art. 71—74 *: so that it would be to no Purpose to spend more Time in explaining that Matter. But what is contained in Art. 70 † must be illustrated a little: and so much the rather, that there seems to be some Mistake in the Text.

* Of the Quadrature of Curves.

† Of the Quadrature of Curves.

382. Let

382. Let ABC be a Curve described by the Ordinate BC moving along the Abscis AB. Call AB $=z$, BC $=y$: and let the Curve be of that Nature that $y=z^n$. Let BC, B2C2, B3C3, B4C4, &c. be successive Positions of the Ordinate, so that the Distances B B2, B2 B3, B3 B4, &c. be equal, any of which Distances call o. Then the successive Values of the Abscis are z, $z+o$, $z+2o$, $z+3o$, &c: to which the successive corresponding Values of the Ordinate are z^n, $\overline{z+o}|^n$, $\overline{z+2o}|^n$, $\overline{z+3o}|^n$. By throwing these Values of y into Series's, you'll have them expressed thus. Art. 70. of the Quadrature of Curves.



1°. z^n.

2°. $z^n + nz^{n-1}o + n\times\frac{n-1}{2}z^{n-2}oo + n\times\frac{n-1}{2}\times\frac{n-2}{3}z^{n-3}o^3 + n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}z^{n-4}o^4$, &c.

3°. $z^n + 2nz^{n-1}o + 4n\times\frac{n-1}{2}z^{n-2}oo + 8n\times\frac{n-1}{2}\times\frac{n-2}{3}z^{n-3}o^3 + 16n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}z^{n-4}o^4$, &c.

4°. $z^n + 3nz^{n-1}o + 9n\times\frac{n-1}{2}z^{n-2}oo + 27n\times\frac{n-1}{2}\times\frac{n-2}{3}z^{n-3}o^3 + 81n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}z^{n-4}o^4$, &c.

&c. *in inf.*

Where the first Term of each Series is the flowing Quantity z^n itself: the numeral Coefficients of the second Terms are 1 . 2 . 3 . 4, &c; of the third Terms, 1 . 4 . 9 . 16, &c; of the fourth, 1 . 8 . 27 . 64, &c; of the fifth, 1 . 16 . 81 . 256, &c. that is, the numeral Coefficients of the second, third, fourth, &c. Terms are Series's of the first, second, third, fourth, &c. Powers of the natural Numbers.

If you subtract the first Value of the Ordinate from the second, the second from the third, the third from the fourth, &c. you'll have the first Differences of the Ordinates BC, B2C2, B3C3, B4C4, &c. the first Term of which in them all is $nz^{n-1}o$. Again, by subtracting the first Differences from each other, you have the Differences of the Differences, or second Differences of the Ordinates, the first Term of which is $2n\times\frac{n-1}{2}z^{n-2}oo = n\times\overline{n-1}z^{n-2}oo$. After the same manner, by subtracting the second Differences from each other, you'll find the third Differences of the Ordinates, the first Term of which in all is $6n \times \frac{n-1}{2} \times \frac{n-2}{3} z^{n-3} o^3 = n \times \overline{n-1} \times \overline{n-2} \times z^{n-3} o^3$. So likewise the first Term of the fourth Differences will be $24n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-1}{4} z^{n-4} o^4 = n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times z^{n-4} o^4$: and so on through the subsequent Differences *in inf.* as is deduced from the Differences of the Series's of Powers of the natural Numbers, to which the numeral Coefficients of the Terms in the successive Values of the Ordinate z^n correspond. Now by considering these several Orders of Differences, it will appear that there are infinite Orders of them in every Case, unless when n is a positive Integer: in which last Case the Number of Orders is the same with n. And the first of the first Order is called the first Difference z^n; the first of the second Order is called the second Difference of z^n; the first of the third Order, the third Difference of z^n, and so on. Only it may be observed that these various Orders of Differences of z^n, are generally understood by Writers upon this Subject, as meant of them when infinitely diminished: in which Case any one of the same Order of Differences of the Ordinates is the same with the first of that Order. And so it appears that the first Terms of the different Orders of Differences of z^n, are equal to the second Term; twice the third Term; six times the fourth Term; twenty-four times the fifth Term, &c. respectively, of the converging Series $z^n + nz^{n-1}o + n \times \frac{n-1}{2} z^{n-2} o^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} z^{n-3} o^3 + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} z^{n-4} o^4$, &c. $= \overline{z+o}|^n$.

Now when the Increment of z, *viz.* o, which denotes the Distances of the Ordinates, is diminished more and more *in inf.* the prime or ultimate Ratios of the first Terms of the Differences, to the whole Differences respectively, will be a Ratio of Equality, by reasoning after the same Manner as in Art. 11 *, and in the Notes upon it: wherefore, hence it appears, that, in the converging Series $z^n + nz^{n-1}o + n \times \frac{n-1}{2} z^{n-2} o^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} z^{n-3} o^3 +$ &c. the first Term z^n being the Fluent, the second, third, fourth, &c. Terms will be proportional to the first, second, third, &c. Differences of z^n, it being understood that the Quantities are taken in their prime or ultimate Ratio, *viz.* when these Differences are in their *nascent* or *evanescent* State. But from what was demonstrated in the Beginning of this Work, the Fluxion of z being uniform and constant, and put equal to

* Of the Quadrature of Curves.

to Unity, the first, second, third, *&c.* Fluxions of z^n will be nz^{n-1}, $n \times \overline{n-1} \times z^{n-2}$, $n \times \overline{n-1} \times \overline{n-2} \times z^{n-3}$, *&c.* Hence it appears, that these first, second, third, *&c.* Fluxions are proportional to the second, third, fourth, *&c.* Terms of the converging Series; and consequently to the first, second, third, fourth, *&c.* Differences of z^n, when these Differences are in their nascent or evanescent State.

But here you must observe, that I don't mean that there is any Ratio subsists betwixt Fluxions, represented here by finite assignable Quantities, and evanescent Differences: but only that a Fluxion of any Order is to a Fluxion of that same Order, as a Difference of the like Order is to another Difference of that Order, taken in their prime Ratio. And this I take to be the Meaning of what our Author here advances, which seems not to be expressed with that Clearness and Accuracy, which he was Master of upon other Occasions. For the second, third, fourth, *&c.* Terms of the converging Series are said to be *equal* to the first, second, third, *&c.* Increments or Differences; which certainly is not true. Therefore Dr. *Keil* (see *Commercium Epist. pag.* 41.) supposes that the Particle *ut* has stood originally betwixt the Word *erit* and Incrementum, so as to make the Text run, *tertius* $\frac{n^2-n}{2}ooz^{n-2}$ *erit* ut *ejus Incrementum secundum:* and again, *quartus* $\frac{n^3-3n^2+n}{6}o^3z^{n-3}$ *erit* ut *ejus Incrementum tertium*, &c. and that this Particle *ut* by some Accident or another hath been left out. But even when this Correction is made, I don't think the Text clear'd: because the plain and natural Meaning would still appear to be that the second Term of the converging Series will be the first Increment or Difference *absolutely*; and the third Term of the Series will be *as* the second Increment or Difference *absolutely*, &c. contrary to what has been shewn. But if we shall suppose that by first, second, third, *&c.* Increments or Differences of z^n, Sir *Isaac Newton* meant the second, third, fourth, *&c.* Terms of the converging Series $z^n + noz^{n-1} + \frac{n^2-n}{2}ooz^{n-2} + \frac{n^3-3n^2+2n}{6}o^3z^{n-3}$, *&c.* (as he might have done) the Addition of *cui nascenti* is superfluous. However I suppose I have explained the Author's Meaning: and the *Omission*, *Inaccuracy*, or *Mistake*, no way affects the Subject. There was no less than twenty-eight or thirty Years interven'd betwixt Sir *Isaac*'s writing the former Part of the Treatise, and this Scholium, which was annex'd A°. 1704.

383. And hence by the by it appears how the *differential Method* of Mr. *Leibnitz*, and the foreign Mathematicians, will agree with the

Method of Fluxions in it's Conclusions; because the Ratios of the Differences are the same with the Ratios of the Fluxions, although the Principles they reason upon are not so just and *geometrical.*

In order to understand the subsequent Part of this Scholium: the Application of which is of very great Extent, there are several things to be taken Notice of, and a little explain'd, with respect to the Nature of *Fluents* and *Fluxions*, and the Constitution of *Equations* defining the *Relations* of the one or the other:

384. It was observed in Section second, that every fluxional Equation must contain the Fluxions of two Quantities at least, either expressed or understood *: because Fluxion or Velocity being a relative Term, there must always be the Fluxions of two flowing Quantities at least, in every such Equation; even as there must be at least two flowing Quantities in every fluential Equation. Wherefore if any fluxional Equation be proposed, in which the Fluxion of one Quantity only is expressed, the Fluxion of some other flowing Quantity is understood, although it don't appear; by which all the Terms, that don't include the Fluxion of the first mentioned Quantity, must be supposed to be multiplied. Which other flowing Quantity is supposed to flow uniformly, and to have it's Fluxion equal to Unity, by which means it disappears in these Terms, and must be supplied, after the Manner taught in that Place *.

* Art. 95. of this Explication.

* Art. 98, 99. of this Explication.

385. In the Doctrine of Fluxions, there are two different ways of expressing each of the two general Problems, that contain the *direct*, and *inverse* Method of Fluxions: which, although they may be reduced to one manner of Expression, yet that the young Geometrician may still preserve clear and distinct Ideas, I shall explain a little in this Place, to make the Way plainer to what follows.

In the direct Method, an Equation is sometimes given, which must contain at least two variable or flowing Quantities, and the Problem is expressed thus: *From the given Equation expressing the Relation of the Fluents, to find an Equation defining the Relation of the Fluxions.* Thus our Author expresses it in Prop. 1: and the Meaning of it was explained at full length, in the Notes upon that Prop. which see. Again, the same Problem is expressed otherwise thus: *Any Fluent being given, to find it's Fluxion.* Thus the flowing Quantity x^n being given, to find it's Fluxion: the Fluent $ax^3 - 4abx^2 + 8bx^2 + ab$ being given, to find it's Fluxion: likewise, the flowing Quantity $x^3 - 3x^2y + 3xy^2 - y^3 + ab$ being proposed, to find it's Fluxion: and so for other Fluents, including more variable Quantities than two. The Meaning of all which must be conceived to be this. The Fluxions

ions or Velocities of flowing of the variable Quantities, which compose these Fluents, being supposed to be known, to find what are the Fluxions or Velocities with which the whole Fluents flow: or what Relation they bear to the Fluxion, or Fluxions of the flowing Quantity, or Quantities of which they are compounded. Thus the Fluxion of x^n is $nx^{n-1}\dot{x}$; the Fluxion of $ax^3 - 4abx^2 + 8bx^2 - ab$, is $3ax^2\dot{x} - 8abx\dot{x} + 16bx\dot{x}$; of $x^3 - 3x^2y + 3xy^2 - y^3 + ab$ is $3x^2\dot{x} - 6x\dot{x}y - 3x^2\dot{y} + 3\dot{x}y^2 + 6xy\dot{y} - 3y^2\dot{y}$: the Meaning of all which is, that, when Fluxions of x and y are $\dot{x}$ and $\dot{y}$, the Fluxions of the Fluents mentioned, are such, or bear such Relation to them, as the fluxional Expressions set down, import. Whence the two different Ways of expressing the second general Problem, containing the inverse Method of Fluxions, may be easily understood. One of which is this: *Any Equation being given, defining the Relation of the Fluxions of any flowing Quantities; to find an Equation, defining the Relation of the Fluents.* The other runs thus: *To find the Fluent of any fluxionary Expression proposed.* Which are the Converse of the former two Problems.

386. Now although these be different Ways of expressing the two general Problems, yet in effect they mean the same thing. For any Fluent being proposed, to find it's Fluxion, and conversely, you may put the flowing Quantity in the first Case, equal to some flowing Quantity expressed by a Letter or Symbol: and the fluxionary Expression in the second Case, to the Fluxion of some flowing Quantity expressed by the proper Notation of a fluxionary Symbol. Thus when it is proposed, as above, to find the Fluxion of $ax^3 - 4abx^2 + 8bx^2 - ab$; suppose the Fluent equal to z, so that it stand thus $ax^3 - 4abx^2 + 8bx^2 - ab = z$. And then from this Equation expressing the Relation of the Fluents x and z, you find an Equation expressing the Relation of their Fluxions. Again, when the fluxionary Expression $3x^2\dot{x} - 6x\dot{x}y - 3x^2\dot{y} + 3\dot{x}y^2 + 6xy\dot{y} - 3y^2\dot{y}$ is proposed, to find it's Fluent: you suppose it equal to the Fluxion of z, so as to stand thus, $3x^2\dot{x} - 6x\dot{x}y - 3x^2\dot{y} + 3\dot{x}y^2 + 6xy\dot{y} - 3y^2\dot{y} = \dot{z}$: and so it will amount to this, from the Equation just now mentioned, expressing the Relation of the Fluxions $\dot{x}$, $\dot{y}$ and $\dot{z}$, to find an Equation expressing the Relation of the Fluents: for the Import of which see Art. 89, 90 *, so that the two different Ways of expressing the two general Problems, will in effect amount to one and the same thing.

* Of this Explication.

387. But here something occurs that must be carefully observed, *viz.* that the Fluent belonging to any Fluxion, is either *pure* or *mixt*. Thus in one of the Examples above, it was shewn that $3ax^2\dot{x} -$

8abxẋ

$8abx\dot{x} + 16bx\dot{x}$ was the Fluxion of $ax^3 - 4abx^2 + 8bx^2 - ab$: but it is alſo the Fluxion of the Expreſſion $ax^3 - 4abx^2 + 8bx^2$, becauſe $-ab$ is a conſtant Quantity, producing no Fluxion. The laſt, I call a *pure* Fluent; becauſe it contains no Part but what flows, and leaves ſome Veſtige of it in the Fluxion: whereas the other is called a *mixt* Fluent, as being made up of Parts, which don't all flow; there being a Part $-ab$ which is conſtant, producing no Fluxion, and therefore leaving no Veſtige of itſelf in the fluxionary Expreſſion, by which one might know that it was in the Fluent.

Now the finding the pure Fluents from the Fluxions given, is a determinate Problem, which in many Caſes may be ſolved by the Quadrature of Curves, or the Method of finding their Areas taught by our Author in this Treatiſe: whereas, to find a mixt Fluent belonging to any propoſed Fluxion is indeterminate, ſince there may be an infinite Number of ſuch. However, there may be ſuch Limitations either aſſigned, or ariſing from the Conditions and Circumſtances of the Caſe, as may make this laſt Problem determinate; although otherwiſe it would not. Thus if you know that the Fluent ought to be equal to a certain known Quantity in certain Circumſtances, you may thence find the *mixt* Fluent you want; provided you can firſt find the *pure* Fluent.

Thus if the Fluxion $\frac{ax\dot{x}}{\overline{b+cx^2}|^2}$ was propoſed, in order to find a mixt Fluent, to which that Fluxion belongs, ſo circumſtantiate or limited, as that, when $x = b$, the Fluent may be 0, or b; or $\frac{bc}{a}$ or $b - a$; or any other known Quantity whatſoever: then I firſt ſeek a pure Fluent, which will be the ſame Expreſſion with that, which exhibits the Area of a Curve, whoſe Abſciſs is x, and Ordinate $\frac{ax}{\overline{b+cxx}|^2}$; as will be ſhewn more clearly below: this Area or pure Fluent you'll

* Of this Explication. find (by Prop. 5. or Art. 147 *; or, more ſhortly, by comparing $\frac{ax}{\overline{b+cxx}|^2}$ with the Ordinate of the Curve belonging to Form ſecond, Table firſt) to be $\frac{ax^2}{2b^2+2bcx^2}$. Which found, ſuppoſe you take q for any given Quantity, to be juſt now determined: put $\frac{ax^2}{2b^2+2bcx^2} + q = 0$, or b, or $\frac{bc}{a}$, or $b - a$, or any other given Quantity, to which the mix'd Fluent muſt be equal, when $x = b$, that is, (by putting

putting $x = b$) $\frac{a}{2+2bc} + q = 0$, or b, or $\frac{bc}{a}$; or $b - a$, &c. Whence you find $q = -\frac{a}{2+2bc}$; or $q = b - \frac{a}{2+2bc}$; or $q = \frac{bc}{a} - \frac{a}{2+2bc}$ or $q = b - a - \frac{a}{2+2bc}$, &c. Therefore, in order to find the true Fluents anſwering to theſe ſeveral Caſes or Conditions: to the pure Fluent formerly found, you muſt annex the correſponding Values of q, which will give $\frac{ax^2}{2b^2+2bcx^2} - \frac{a}{2+2bc}$; $\frac{ax^2}{2b^2+2bcx^2} + b - \frac{a}{2+2bc}$; $\frac{ax^2}{2b^2+2bcxx} + \frac{bc}{a} - \frac{a}{2+2bc}$; $\frac{ax^2}{2b^2+2bcx^2} + b - a - \frac{a}{2+2bc}$, &c. for the true Fluents required, correſponding to the ſeveral Caſes, or Suppoſitions put. And the ſame may be done in other Caſes, when like Limitations, or Conditions of the Fluent are aſſigned or known.

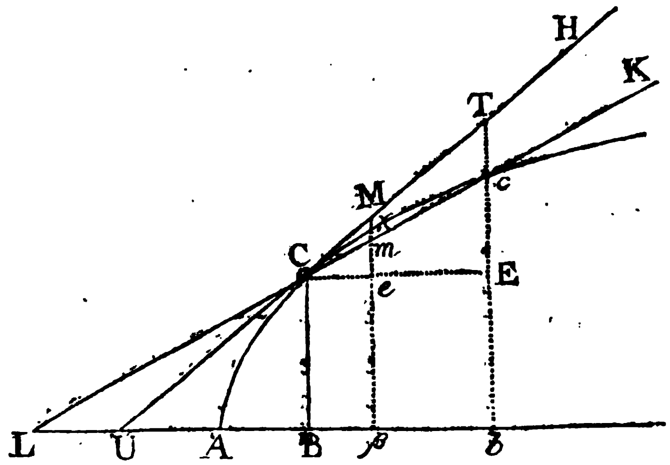
388. It may likewiſe be obſerved, that there are ſometimes two pure Fluents belonging to one and the ſame Fluxion: thus the fluxionary Expreſſion $\frac{ax\dot{x}}{\overline{b+cx^2}|^2}$, mentioned above, has another pure Fluent, *viz.* $\frac{-a}{2bc+2ccx^2}$; which is found by changing the Form of the given Fluxion $\frac{ax\dot{x}}{\overline{b+cx^2}|^2}$, into this other $\frac{ax^{-3}\dot{x}}{\overline{bx^{-2}+c}|^2}$; and finding the Area of a Curve correſponding to the Ordinate $\frac{ax^{-3}}{\overline{c+bx^{-2}}|^2}$: which Area is the laſt mentioned Fluent, $\frac{-a}{2bc+2ccx^2}$; as you find by Prop. 5, or by conſulting Form ſecond of Table firſt. So that the ſame Rules, which were formerly given in Art. 147—176 *, for determining when a propoſed Curve is quadrable, or not; and when it is doubly quadrable, equally ſerve to find whether a propoſed fluxionary Expreſſion, has any *finite* Fluent, *i. e.* ſuch as can be fully expreſſed; or not: and when there are two Fluents, and likewiſe to find what that Fluent or theſe Fluents are.

* Of this Explication.

The two pure Fluents of any fluxionary Expreſſion, including only one unknown or variable Quantity, with it's Fluxion multiplied into each Term, (when there are two ſuch Fluents) correſpond to the two Areas of a Curve, which lye upon the oppoſite Sides of the Ordinate: hence it follows that theſe two Fluents will be affected with oppoſite Signs.

389. Fur-

389. Further it is to be observed, that when any Equation, exhibiting the Relation of the Fluxions of any two flowing Quantities, is given, the Relation of the Fluents defin'd by the fluential Equation thence deriv'd, is the same, whether you suppose the one, or the other; or neither of the Quantities to flow with an uniform and constant Fluxion. For as these different Suppositions don't alter the Relation of the Fluxions *; so neither will they alter the Relation of the Fluents, which depends upon the Relation of the Fluxions. Thus in the curvilinear Figure ABC, whose Abscifs is AB and Ordinate BC; and CV a Tangent meeting AB in the Point V; calling AB $= z$, BC $= y$, the Relation of $\dot{z}$ to $\dot{y}$, is the same whether z flow uniformly, or y flow uniformly; or though neither of them do so, *viz.* that of VB to BC: and the Relation of the Fluents AB and BC in every Case is the same.



* Art. 95. of this Explication.

390. The finding the Fluents from the Fluxions given, by means of the Quadrature of Curves, depends upon this Principle or Position, *That any variable or flowing Quantity, which flows at the same Rate, or according to the same Law, that any curvilinear Area flows, may be represented or expounded by such curvilinear Area.* For whatever Relation a superficial Unit bears to the curvilinear Area, that same Relation will an Unit of any other Kind bear to a variable or flowing Quantity of that Kind, which flows in an analogous Manner. Hence it is, that by the Quadrature of Curves, we may determine the *Lengths* of Curve-lines; the *superficial* and *solid* Contents of Bodies; the Centers of *Gravity*, *Percussion* or *Oscillation* of all Kinds of Magnitudes: and in general, Quantities of *any Kind*, considered as flowing at the same Rate, or according to the same Law, that any curvilinear Area flows: just after the same Manner that in the fifth Book of *Euclid*'s Elements, Quantities of all Kinds are represented and expounded by right Lines.

Thus

Thus if the Fluxion $\sqrt{\dot{x}^2 + \frac{9x\dot{x}^2}{4a}}$ or $\dot{x}\sqrt{1 + \frac{9x}{4a}}$ was propofed, to find the Fluent: I confider $\dot{x}\sqrt{1 + \frac{9x}{4a}}$ as the Fluxion of a curvilinear Area, whofe Abfcifs is x and Ordinate $\sqrt{1 + \frac{9x}{4a}}$: whatever be the Nature of the flowing Quantity, which flows with the propofed Fluxion. Then by the Quadrature of Curves, I find the Area of the Curve, *i.e.* I put $\sqrt{1 + \frac{9x}{4a}} = y$: and thence I find the Area by Prop. 5. Or thus: I confult the Rules laid down in Art. 147, 149 *; by which I find the Curve is quadrable, and that it's Area is $\frac{8a + 18x}{27}\sqrt{1 + \frac{9x}{4a}}$. The fame thing you find more expeditioufly, by comparing the given Equation $\sqrt{1 + \frac{9x}{4a}} = y$, with the Equation belonging to the Curves of Species 1. Order 3. Table 1. Therefore $\frac{8a + 18x}{27}\sqrt{1 + \frac{9x}{4a}}$ is the Fluent belonging to the given Fluxion, by confidering that Expreffion as no longer reprefenting or denoting an Area of a Curve, but a Quantity of any other Kind, whether Line, Body, or any other, according to the Nature of the Fluent, whofe Fluxion was propofed. More Examples of which may be feen below.

* Of this Explication.

Wherefore I come now, as I promifed at the Beginning, to fhew how this Treatife of Quadratures is to be applied to the Solution of fome of thofe Problems mentioned Art. 100 *, for I cannot here undertake to go through them all: it being fufficient to my Purpofe to fhew this in fome Inftances, by which the Reader may be enabled to form a Judgment concerning the reft.

* Of this Explication.

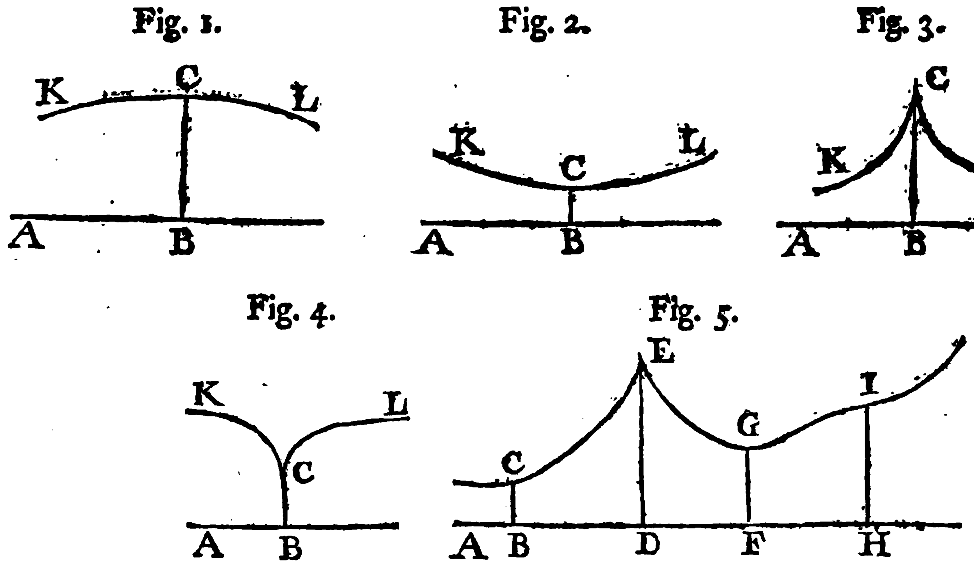
PROB.

To determine the Maxima *and* Minima *of Quantities.*

391. A variable or flowing Quantity is often of fuch a Nature, that it paffes through a State where it neither increafes nor decreafes, but paffes from increafing to decreafing, or from decreafing to increafing: in either of which Cafes it arrives at an extreme Value, fo as to be called a ***Maximum***, in the firft Cafe, and a ***Minimum***, in the other. Now fince the variable Quantity, at this Inftant, neither flows backwards, nor forwards, therefore any Fluent being propofed, or an Equation defining the Relation of flowing Quantities, find the Fluxions,

by our Author's first Proposition, and put the Fluxion of the Quantity which is to be determined to an extreme Value equal to nothing; and the Equation resulting, together with the Equation proposed, will help to discover and determine where and when the Quantity arrives at a *Maximum* or *Minimum*. The same thing is likewise done sometimes by putting the Reciprocal of the Fluxion of the Quantity equal to nothing: because in passing through a *Maximum* or *Minimum*, the Fluxion of the Quantity becomes infinitely great, in some Cases.

392. The Matter may be explained thus: let KCL (Fig. 1, 2, 3, 4.) be a Curve described with the Base AB and Ordinate BC: while the Abscis AB increases (Fig. 1, and 3.) the Ordinate BC increases together with it, until it arrive at the Position represented in these Figures, afterwards it decreases: therefore at that Instant of Time when BC is betwixt the increasing and decreasing, it attains a greater Length than in the Times or Positions preceding and succeeding, for some Space,



and therefore it is said to be a *Maximum:* for which end the Fluxion or Velocity with which BC flows must from being positive become negative: which it must do either by vanishing, as at Fig. 1, or by becoming infinitely great in comparison of the Fluxion of AB, as at Fig. 3. Again (at Fig. 2, and 4.) while AB increases BC diminishes until it arrive at the Position represented in these Figures, afterwards it increases, therefore in the intermediate Instant it is called a *Minimum:* at which Time the Fluxion of BC from being negative be-

2 comes

comes positive: to do which it must either vanish (as at Fig. 2.), or become infinitely great in comparison of the Fluxion of AB (as at Fig. 4.)

393. Sometimes in Curves of the higher Orders the Ordinate may arrive at a *Maximum* or *Minimum*, in different Places of the same Curve, as in Fig. 5. where the Ordinate is represented as a *Minimum* or *Maximum* in the several Positions BC, DE, FG. Whence it appears that by a *Maximum* or *Minimum* in this Doctrine, Geometricians don't mean a variable Quantity's attaining it's greatest or least Value it can possibly at any Time arrive at; but only such a Value as is greater or less than the Values for some Time or Space before or after that Instant at which it is said to be a *Maximum* or *Minimum*.

394. It is evident, by considering what has been said, that, if you call x the Abscis, and y the Ordinate of a Curve, when the Fluxion of y vanishes, the Tangent becomes parallel to the Abscis, as in Fig. 1, and 3; and when the Fluxion of y is infinitely great in comparison of the Fluxion of x, the Tangent falls in with the Ordinate, as in Fig. 2, and 4: but yet the Fluxion of y may vanish, or be infinite, and consequently the Tangent parallel to the Abscis, or coincident with the Ordinate, when the Ordinate is neither *Maximum* nor *Minimum*. For the Fluxion of y may vanish and the Tangent become parallel to the Abscis, by means of a Point of contrary Flexure, where the Ordinate doth not attain an extreme Value at that Place, as you may see with respect to the Ordinate HI at Fig. 5: such a Point of contrary Flexure may likewise make the Fluxion of y to be infinitely great in respect of the Fluxion of x, *viz.* when the right Line touching the Curve at the Point of contrary Flexure coincides with the Ordinate. The Fluxion of y may vanish, or be infinitely great likewise in another Case, when the Ordinate is no *Maximum* nor *Minimum*, *viz.* when at such Point where the Tangent becomes parallel to the Abscis; or coincident with the Ordinate, the Curve is reflected back again from the Ordinate, having no Branch upon the other Side of the Ordinate, at the Point where it meets with it: but either forming a Cuspis or turning back by a continued Curvature. But the Places where the Ordinate arrives at a *Maximum* or *Minimum* may be distinguished from these others, by considering that in the Case of a Point of contrary Flexure, the Ordinate upon the one Side of the Point of contrary Flexure, is greater; and that upon the other always less than the intermediate Ordinate which passes through the Point of contrary Flexure, or Inflection. Again when there is a Point of Reflec-

 tion

tion or Retrogression, the Ordinate upon the one Side of that Point, for some Space at least, is not to be found.

395. Wherefore an Equation being given including two unknown Quantities x and y, which you may conceive as the Abscis and Ordinate of a Curve, find the Relation of their Fluxions by the Author's first Proposition: then put $\dot{y}$ or $\frac{\dot{y}}{\dot{x}}=0$, and the Equation thence resulting together with the Equation proposed will serve to exterminate one of the unknown Quantities as y, and give a new Equation including only the other x, whose Root or Roots will shew the Point or Points of the Abscis at which an Ordinate or Ordinates being applied will pass through a Point, or Points of the Curve, where the Tangent is parallel to the Abscis: and again by putting $\frac{\dot{x}}{\dot{y}}=0$, and proceeding the same way you may discover the Points of the Curve where the Tangent coincides with the Ordinate, having found these Values of x, insert $x+p$; and then $x-p$ instead of x in a Value of y arising from the Equation to the Curve, which will give two new Values of y, one on each Side of the Ordinate whose Fluxion is nothing or infinite: subtract the Value of y deduced from the given Equation, from these two new Values, which will give two Differences: in which Differences insert for x it's corresponding Value, then suppose p to be diminished infinitely, so as that all the Terms where p is found of any higher Dimensions, than is necessary for this Effect, be neglected: Which done, if both the resulting Differences be positive, y is there a *Minimum*; if both be negative, it is a *Maximum*: but if the one be positive and the other negative, there is a Point of contrary Flexure; and if one of the Differences include any possible or imaginary Expression, the Curve is reflected from the Ordinate. The Reason of all which will easily appear to the attentive Reader *: who must likewise know that in this Rule for determining where the Ordinate is a *Maximum* or *Minimum* by the Signs of the Differences, I suppose the Value of the Ordinate to be positive: for if it be negative (which may be found from the Expression of it's Value) two positive Differences denote a *Maximum*, and two negative Differences a *Minimum*. Moreover I only account that Ordinate a *Maximum* or *Minimum*, which has the Curve continued on both Sides of it. I shall now give some Examples of this Method.

*Art. 173. of this Explication.

Ex. 1. Let $y+x^2-2ax-b^2=0$, be an Equation propos'd, and it is required to determine y to an extreme Value. The Relation of the Fluxions is $\dot{y}+2x\dot{x}-2a\dot{x}=0$: put $\frac{\dot{y}}{\dot{x}}=0=2a-2x$,

thence

thence you have $x = a$. To find whether y be at an extreme Value, and if so whether it be a *Maximum* or *Minimum*, put $\overline{x+p}$, and then $\overline{x-p}$ instead of x, in the Equation $y = b^2 + 2ax - x^2$; whence you have 1^mo^ $y = b^2 + 2ax + 2ap - x^2 - 2px - p^2$. 2^do^ $y = b^2 + 2ax - 2ap - x^2 + 2px - p^2$. From which two last Values of y subtract $b^2 + 2ax - x^2$, and the Differences are 1^mo^ $-2ap + 2px - p^2$. 2^do^ $2ap - 2px - p^2$: where if you insert for x, it's corresponding Value a, and throw out repugnant Terms, the resulting Differences are both $-p^2$, therefore y is a *Maximum*, since both Differences are negative.

Again, if you put $\dot{y}$ infinite in respect of $\dot{x}$, or $\frac{\dot{x}}{\dot{y}} = 0$, thence you have $\frac{1}{2a-2x} = (\frac{\dot{x}}{\dot{y}} =)\ 0$, which can only happen when x is infinitely great, where no *Maximum* nor *Minimum* can take place.

Ex. 2. Let $abx - bx^2 - ay^2 = 0$, be an Equation defining the Relation of x the Abscis and y the Ordinate of a Curve, and it's required to determine y to an extreme Value. By proceeding as before you have $ab\dot{x} - 2bx\dot{x} - 2ay\dot{y} = 0$: whence by putting $\dot{y} = 0$, you will find, by reducing, $x = \frac{1}{2}a$, therefore when $x = \frac{1}{2}a$, the Tangent is parallel to the Abscis. But to find whether y be in that Case a *Maximum* or *Minimum* put $\overline{x+p}$, and then $\overline{x-p}$ in place of x in a Value of y deduced from the given Equation, *viz.* $y = \sqrt{\frac{abx - bx^2}{a}}$, and you have 1^mo^ $y = \sqrt{\frac{abx + abp - bx^2 - 2bxp - bp^2}{a}}$. 2° $y = \sqrt{\frac{abx - abp - bx^2 + 2bxp - bp^2}{a}}$: subtract $\sqrt{\frac{abx - bx^2}{a}}$, from these two last Values of y, insert for x it's Value $\frac{1}{2}a$, and throw out repugnant Terms, hence you'll have 1^mo^ $\sqrt{\frac{ab}{4} - \frac{bp^2}{a}} - \sqrt{\frac{ab}{4}}$. 2° $\sqrt{\frac{ab}{4} - \frac{bp^2}{a}} - \sqrt{\frac{ab}{4}}$, for the Differences, both the same, and negative, whatever be the Value of p, as is evident, therefore y is a *Maximum.*

Again, if you put $\dot{y} = inf.$ or $\frac{\dot{x}}{\dot{y}} = 0$, thence you find $\frac{2ay}{ab - 2bx} = (\frac{\dot{x}}{\dot{y}} =)\ 0$: which can only happen when it is $y = 0$; that is (by putting 0 for y in the Equation to the Curve $abx - bx^2 - ay^2 = 0$) when $abx - bx^2 = 0$: in which case you have $x = 0$, and $x = a$: therefore when the Abscis vanishes, and when it is equal to a, the Tangent is coincident with the

the Ordinate. Now if you proceed as before, to insert $x+p$ and $x-p$ instead of x, in the Value of y, and then put o for x which is the one Value, and suppose p infinitely diminished, the resulting Values of y are $\sqrt{bp}$ and $\sqrt{-bp}$, which last being an imaginary Quantity, shews that the Curve is not found above the Beginning of the Abscis, but is reflected back and there touches the Ordinate. After the same manner, by inserting a for x, you would find that the Curve is reflected back from the Ordinate by touching it, when the Abscis is equal to a.

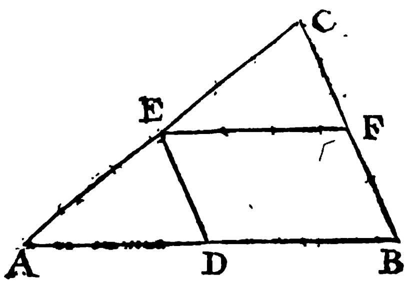
N. B. It may be remarked that when $x+p$, and $x-p$ are substituted for x in the Value of y, and the two Differences found, there is no Odds betwixt these two Differences but that the Terms where p is of an odd Number of Dimensions, are affected with opposite Signs, as is evident, which may help to abridge the Work.

Ex. 3. Let it be $x^3 - ax^2 + \frac{1}{3}a^2x - \frac{1}{27}a^3 = y$; and it is required to determine when y is a Maximum or Mininum, if ever it can become such: wherefore putting $\frac{\dot{y}}{\dot{x}} = 0$, you find $3x^2 - 2ax + \frac{1}{3}a^2 = (\frac{\dot{y}}{\dot{x}} =) 0$: whence, by reducing the Equation, you obtain one Value of x only, *viz.* $x = \frac{1}{3}a$. Wherefore in the given Equation insert $x+p$ and $x-p$ instead of x, and you obtain two new Values of y, *viz.* $x^3 \pm 3x^2p + 3xp^2 \pm p^3 - ax^2 \mp 2axp - ap^2 + \frac{1}{3}a^2x \pm \frac{1}{3}a^2p - \frac{1}{27}a^3$: from which subtracting $x^3 - ax^2 + \frac{1}{3}a^2x - \frac{1}{27}a^3$, and inserting the corresponding Value of x, *viz.* $\frac{1}{3}a$, you'll find, by throwing out repugnant Terms, the resulting Differences to be $\pm p^3$: which being affected with opposite Signs, shews that the Ordinate passes through a Point of contrary Flexure, and therefore is not a Maximum nor Minimum.

396. It may be remarked, that when y is of more Dimensions than one in the Equation proposed, the several Values of y must be found by resolving the Equation, in order to determine whether all or any of these Values of y become a Maximum or Minimum, as easily appears from what has been said. However it may be proper to observe likewise that, if the Equation by which the Value of x is determined, from putting $\dot{y} = 0$, or infinite, be only a simple Equation, y must be a Maximum or Minimum. But I must not insist upon these things. Whoever inclines to see more of this Subject, may consult the learned Mr. *Maclaurin*'s Treatise of Fluxions, where he has explained this Affair more fully and accurately than any other Writer upon the Doctrine of Fluxions: by which, and by the Whole of that excellent Performance, he has discovered how just and comprehensive Views he

he has of these Matters. Wherefore all that I propose further to do upon this Subject, is to shew the Application of the general Doctrine concerning the Maxima and Minima to the Solution of a Problem or two.

397. *Let it be required to inscribe in any given Triangle* ABC *the greatest possible Parallelogram* DF. Call AB $= a$, BC $= b$, AD $= x$, therefore DB $= a - x$, and because AB, BC, AD and DE are proportional, hence DE $= \frac{bx}{a}$. Now since the Angles of the Parallelogram are given, hence it is a Maximum when DB × DE, *i. e.* $\frac{abx - bx^2}{a}$ is a Maximum; or (because $\frac{b}{a}$ is a given Quantity) when $ax - x^2$ is a Maximum: wherefore suppose $ax - x^2 = y$, where you may imagine x and y to be related to one another as the Abscis and Ordinate of a Curve, in which you are to determine y to an extreme Value. Therefore you'll have $\frac{\dot{y}}{\dot{x}} = 0 = a - 2x$, whence arises $x = \frac{1}{2}a$, *i. e.* AD $= \frac{1}{2}$AB, in order to make the incribed Parallelogram a Maximum. And this coincides with Prop. 27. Lib. 6. Elem.

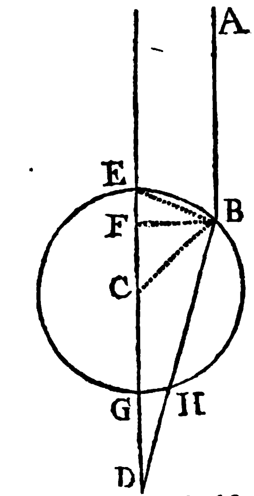


And after the same manner you might determine the inscrib'd Parallelogram to a Maximum, if ABC had been a Segment of any known Curve: thus if it had been a Portion of a Parabola, having AB for the Diameter and BC an Ordinate bounding the Figure, then because AB (a), AD (x), BCq (b^2), DEq are proportional, hence DE $= b\sqrt{\frac{x}{a}}$: therefore put the Fluxion of DE × DB, *i. e.* $\overline{a - x} \times b\sqrt{\frac{x}{a}}$ or, which is the same (by dividing by the given Quantity $\frac{b}{\sqrt{a}}$) of $ax^{\frac{1}{2}} - x^{\frac{3}{2}}$ equal to nothing, that is $\frac{1}{2}ax^{-\frac{1}{2}}\dot{x} - \frac{3}{2}x^{\frac{1}{2}}\dot{x} = 0$: whence, by reducing you find $x = \frac{1}{3}a$: which shews that the inscribed Parallelogram is in this case a Maximum, when AD is $\frac{1}{3}$ of AB. And so you proceed in other like Cases.

398. *Let it be proposed to find such a part of a spherical Superficies, as can be illuminated in it's farther Part, by Light coming from a great Distance, and refracted by the nearer Hemisphere.*

To do which, let AB be a Ray of Light coming from a great Distance, falling upon the nearer Hemisphere at B: let BEGH be a Section

Section of the Sphere through the Center C: draw the Diameter ECG parallel to AB: let the Ray be refracted at B into BHD, meeting the Circumference of the Circle in H, and the Diameter EG produced, if need be, in D. From B draw BF perpendicular to EG at F, join BE, BC, and GH.



Then because the Angles BCE and CBD are equal to the Angles of Incidence and Refraction, therefore in the Triangle BCD, the Ratio of BD to DC is a given invariable Ratio, being that of the Sine of the Angle of Incidence to the Sine of the Angle of Refraction, wheresoever the Point B falls, for the Ray AB is always parallel to EG, since the Light is suppos'd to come from a great Distance. But it is evident that the right Line GH is the Chord of half the spherical Segment, the Circumference of whose Base is enlightened by all the Rays which fall upon the nearer Hemisphere at the same Distance from the Point E as AB falls: when the Point B is very near to E, GH is very small; when EB increases, GH increases with it for some time, but afterwards decreases: therefore the Problem will come to this *to determine when* GH *is a Maximum?* To do which call the Radius EC $= 1$, EF $= x$, CD $= z$: whence because GE, EB, EF are proportional, you have EB $= \sqrt{2x}$: moreover, if a express the Ratio of the Sine of the Angle of Incidence to the Sine of the Angle of Refraction, then BD $= az$: likewise from the similar Triangles DBE, DGH, you have DB (az) : BE $(\sqrt{2x})$ :: DG $(z - 1)$: GH, whence GH $= \frac{z-1}{az}\sqrt{2x}$: therefore $\frac{z-1}{az}\sqrt{2x}$, or, which is the same (by dividing by the given Quantity $\frac{\sqrt{2}}{a}$) $x^{\frac{1}{2}} - \frac{x^{\frac{1}{2}}}{z}$, must be determined to a Maximum: wherefore by putting it's Fluxion equal to nothing, you have $\frac{\dot{x}}{2\sqrt{x}} - \frac{\dot{x}}{2z\sqrt{x}} + \frac{\dot{z}\sqrt{x}}{z^2} = 0$, that is (by multiplying by $2z^2\sqrt{x}$) $z^2\dot{x} - z\dot{x} + 2x\dot{z} = 0$.

Now in order to eliminate one of the unknown Quantities, and it's Fluxion, you must find an Equation expressing the Relation of x and z, which you may deduce from the right-angled Triangle DBF where DB^q $(a^2z^2) =$ DF^q $(z^2 + 2z - 2zx - 2x + x^2 + 1) +$ BF^q $(2x - x^2)$ that is $a^2z^2 = z^2 + 2z - 2xz + 1$: from which you have this fluxional Equation $2a^2z\dot{z} = 2z\dot{z} + 2\dot{z} - 2\dot{x}z - 2x\dot{z}$: whence

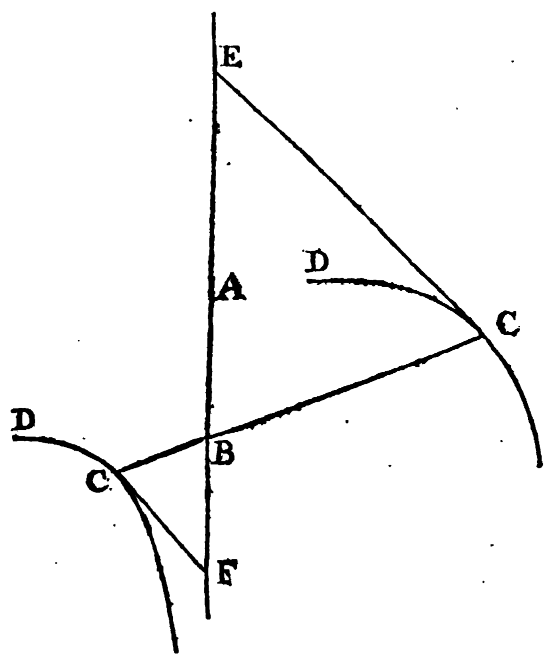
whence $\dot{z} = \frac{z\dot{x}}{z - a^2z - x + 1}$: which Value of $\dot{z}$ being inserted for it in the preceding fluxional Equation, *viz.* $z^2\dot{x} - z\dot{x} + 2x\dot{z} = 0$, and the Equation reduced, you'll have $z^2 - a^2z^2 - xz + a^2z + 3x - 1 = 0$: by means of which, and the preceding Equation, *viz.* $a^2z^2 = z^2 + 2z - 2xz + 1$, if you throw out x, you'll obtain this cubical Equation $z^3 - a^2z^3 - a^2z^2 + z^2 + 3z + 3$, whose Roots are $z = -1 . z = -\sqrt{\frac{3}{a^2-1}} . z = \sqrt{\frac{3}{a^2-1}}$: of which the first two being negative, can be of no use in this Affair, since it is evident the Point D must fall upon the same Side of the Center with G: therefore GH is a Maximum when CD $= \sqrt{\frac{3}{a^2-1}}$. by inserting which Value of z for it in the preceding Equation $a^2z^2 = z^2 + 2z - 2xz + 1$, and reducing, you'll get $x = \frac{1}{2\sqrt{\frac{3}{a^2-1}}} + 1 - \frac{1}{2}aa - \frac{1}{2}\sqrt{\frac{3}{a^2-1}} =$ EF. Thus, for Example, suppose the Globe was Glass, in which Case the Sine of the Angle of Incidence to the Sine of the Angle of Refraction is nearly as 3 to 2, therefore by inserting $\frac{3}{2}$ for a, you'll have CD $= \sqrt{\frac{3}{\frac{9}{4}-1}} = \sqrt{2.4} = 1,5492$, &c. of CG; and EF $= 0.3545$, &c. of CE. And since it was GH $= \frac{x-1}{ax}\sqrt{2x}$: hence by substituting for z and x their Values, it is GH $= 0.198$, &c. which is nearly the Subtense of an Arch 11°, $22'$, in a Circle whose Rad. is 1: therefore the Part of the farther Hemisphere which is enlightened, is a Segment, the Diameter of whose Base subtends an Arch of 22°, $44'$, nearly: and in this Case EB is nearly an Arch of 49°, $48'$.

399. Schol. By means of the above Equation $a^2z^2 - z^2 - 2z + 2xz - 1 = 0$, defining the Relation of $x =$ EF and $z =$ CD, you may determine several things in this Affair: *e. g.* by putting $z = 1$, you may find the Distance of those Rays from E, which after Refraction will pass through C. By putting $x = 1 =$ Rad. you may find where those Rays that touch the Globe after Refraction at their Entrance would cut CG. By putting $x = 0$ the Value of z will give the Focus of parallel Rays after Refraction at the first Surface, &c. And if any two corresponding Values of x and z are given, you may thence find a, that is the Law of Refraction.

PROB.

To draw Tangents to all sorts of Curves.

400. The Principle upon which the Solution of this Problem depends is laid down by our Author Art. 4 *; and was explain'd at full Length Art. 41, 42 †. which amounts to this: if the Curve DC be described by the Ordinate moving along the Abscifs AB; and CE be a Tangent to the Curve at the Extremity of the Ordinate, meeting the Abscifs in E, the Sides BE, BC and CE of the Triangle BCE are proportional to the Fluxions of AB, BC and DC, the Abscifs, Ordinate and Curve-line: and it is the same thing whether BC be a rectangular Ordinate; or an Ordinate applied in any other given Angle. Wherefore calling $AB = x$, $BC = y$, and $BE = s$ (which last is termed the Subtangent) you have this Proportion $\dot{y} : \dot{x} :: y : s$, or $y \times \frac{\dot{x}}{\dot{y}} = s$: which is the general Formula for finding the Subtangent BE. Therefore if an Equation be given defining the Relation of x and y, we may thence find the Relation of their Fluxions, and consequently from BC or AB given you find the Point E, so that EC being join'd may be a Tangent at the Point C.



401. Hence it is evident, that if, while the Abscifs increases, the Ordinate increase at the same time, then $\frac{\dot{x}}{\dot{y}}$ is positive, and consequently the Subtangent is positive: in which Case BE lyes upon the same Side of the Ordinate that the Point A the Beginning of the Abscifs does: if $\dot{y}$ become infinitely great in comparison of $\dot{x}$, then $\frac{\dot{x}}{\dot{y}}$ vanishes, and consequently EB or s, if y be finite, and the Tangent falls in with the Ordinate: if $\dot{y}$ vanish, $\frac{\dot{x}}{\dot{y}}$ becomes infinitely great, consequently EB is infinite; and the Tangent becomes parallel to AB: but

* Of the Quadrature of Curves.
† Of this Explication.

but if BC diminish while AB increases, and consequently have a negative Fluxion, then $\frac{\dot{x}}{\dot{y}}$, and consequently $y \times \frac{\dot{x}}{\dot{y}} = s$, is negative, in which Case BE lyes upon that Side of the Ordinate which is opposite to A.

Ex. 1. Let $y^n = a^{n-1}x$ define the Nature of the Curve: then, by Prop. 1, $ny^{n-1}\dot{y} = \overline{a^{n-1}\dot{x}}$, or $\frac{\dot{x}}{\dot{y}} = \frac{ny^{n-1}}{a^{n-1}}$, therefore $\frac{ny^n}{a^{n-1}} = (y \times \frac{\dot{x}}{\dot{y}} =)$ s the Length of the Subtangent: or if you would have the Subtangent expressed in Terms of the Abscis, insert for y^n it's Value $a^{n-1}x$ from the Nature of the Curve, and you have $s = nx$: wherefore if you take BE : AB :: n : 1, and join EC it is the Tangent required, Thus if it be $n = 2$, the Equation belongs to the common Parabola, and so EB $= 2$AB. And if n be 3, 4, 5, *&c.* which gives a Series of Parabolas of different Orders, you have EB equal 3AB, 4AB, 5AB, *&c.* respectively. Again, if it be $n = -1$, then the Equation becomes $y^{-1} = a^{-2}x$ or $yx = x^2$, which is to the common Hyperbola, having the Center A, one of it's Assymptotes AB, and the other parallel to BC: here you have $s = -x$, therefore you must take EB $=$ AB upon the other Side of the Ordinate. After the same manner if you take $n = -2$, -3, -4, *&c.* you'll have different Orders of Hyperbolas, whose Subtangents are $-2x$, $-3x$, $-4x$, *&c.* *i. e.* 2AB, 3AB, 4AB, *&c.* taken upon the Abscis produced beyond the Ordinate.

Ex. 2. Let $abx - bx^2 - ay^2 = 0$ express the Relation of AB and BC, in which case the Curve DC is an Ellipse, A the Vertex of the Diameter, a the Diameter, b, its Parameter: then by taking the Fluxions, it is $ab\dot{x} - 2bx\dot{x} - 2ay\dot{y} = 0$; whence $\frac{\dot{x}}{\dot{y}} = \frac{2ay}{ab - 2bx}$, therefore $s = \frac{2ay^2}{ab - 2bx} =$ (by substituting $abx - bx^2$ for ay^2 from the Equation to the Curve) $\frac{2abx - 2bx^2}{ab - 2bx} = \frac{ax - xx}{\frac{1}{2}a - x} =$ BE: whence BE is a fourth Proportional to $\frac{1}{2}a - x$, $a - x$, and x, as is known from other Principles. Hence it appears that, since the Value of BE the Subtangent, doth not include the Parameter b, any two Diameters of any two Ellipses, and their Abscisses, being equal, the Subtangents are equal: and therefore if there be any Number of Ellipses having a common Axis, including also the Circle, the Tangents drawn from the Extremities of the Ordinates applied to the same Point of the common Axis, will meet that Axis in the same Point. If the Equation had been abx

$+ bx^2 - ay^2 = 0$, it would have been to the Hyperbola, and $\frac{ax+x^2}{\frac{1}{2}a+x} = s = BE$.

Ex. 3. Let AGE be the Ciſſoid of *Diocles* (ſee Fig. Art. 321 *) belonging to the Circle ADQ, the ſame things being ſuppoſed as in that Article, call AQ $= a$, AC $= x$, CE $= y$: and it is required to draw a Tangent to the Curve at the Point E.

* Of this Explication.

The Equation to the Curve is $y = \frac{x^2}{\sqrt{ax - x^2}}$; or, by ſquaring, $ay^2 - xy^2 = x^3$, therefore, by taking the Fluxions, $2ay\dot{y} - 2xy\dot{y} - \dot{x}y^2 = 3x^2\dot{x}$, whence $\frac{\dot{x}}{\dot{y}} = \frac{2ay - 2xy}{y^2 + 3x^2}$: wherefore $\frac{\dot{x}}{\dot{y}} \times y = \frac{\overline{2a - 2x} \times y^2}{y^2 + 3x^2} =$ (by inſerting $\frac{x^3}{a-x}$ for y^2, and reducing) $\frac{2ax - 2x^2}{3a - 2x} = \frac{CD^q}{\frac{1}{2}AQ + QC}$, which points out this Conſtruction: produce AQ to I, ſo that QI be equal to the Radius of the Circle ADQ, join ID, draw DK perpendicular to it, cutting the Abſciſs AC in K, join KE, and it is the Tangent required.

Ex. 4. Let PEH be the Conchoid of *Nicomedes* (ſee the Fig. Art. 322 *) and ſuppoſing the ſame things as there mentioned, only conſider AB as the Abſciſs, and BE as the rectangular Ordinate, which call x and y, the other Symbols remaining the ſame; it is required to draw a Tangent to the Curve at the Point E.

* Of this Explication.

The Equation to the Curve is $xy = \overline{b + y}\sqrt{cc - yy}$: therefore by taking the Fluxions, you have $\dot{x}y + x\dot{y} = \dot{y}\sqrt{cc - yy} + \overline{b + y} \times -\frac{y\dot{y}}{\sqrt{cc - yy}}$ *, whence $\frac{\dot{x}}{\dot{y}} = \frac{\sqrt{cc - yy} - \frac{by + y^2}{\sqrt{cc - yy}} - x}{y}$: therefore $\frac{y\dot{x}}{\dot{y}} = \sqrt{cc - yy} - \frac{by + y^2}{\sqrt{cc - yy}} - x = CD - \frac{GC \times AC}{CD} - AB = s$ the Subtangent: which Value being always negative, in regard AB always exceeds CD, ſhews that it is to be taken upon the Abſciſs AB produced beyond the Ordinate: wherefore if you take the Point T in the Aſſymptote AB, upon the further Side of the Ordinate equal to $AL + \frac{GC \times AC}{CD}$, and join TE, it ſhall be the Tangent required.

* Art. 87. of this Explication.

402. This Curve affords us an Example of a Point of contrary Flexure, which ſeparates betwixt the Part of the Curve that is concave to the Aſſymptote AB and the Part that is convex; which may be thus determined from the Conſideration of the Tangent. When AB is nothing, and the Point E coincident with P the Vertex of the Axis, the

Tangent

Tangent ET is parallel to the Aſſymptote AT, in which caſe the Subtangent BT is infinite; and if you ſuppoſe ET produced till it meet the Axis AP in F, the Part PF vaniſhes, when AB and conſequently PC is of any finite Magnitude BC and PF are finite Magnitudes: ſo that, ſuppoſing the Abſciſs AB ariſing from nothing to increaſe, the Line AT will decreaſe and AF increaſe, until ſuch time, as the Ordinate BE fall upon the Point of contrary Flexure: after which, AB ſtill increaſing, AT will begin to increaſe and AF to diminiſh. Hence it is evident that AT and AF attain an extreme Value, the firſt becoming a Minimum, the other a Maximum, when the Ordinate falls upon the Point of contrary Flexure. Therefore if you take the Value of any of theſe Lines, and put it's Fluxion equal to nothing, the reſulting Equation will determine the Point of contrary Flexure.

Now ſince we found $-BT = \sqrt{cc-yy} - \frac{by+y^2}{\sqrt{cc-yy}} - x$, hence calling $AT = t$, we ſhall have $-\sqrt{cc-yy} + \frac{by+y^2}{\sqrt{cc-y^2}} + 2x = t$. From which find an Equation expreſſing the Relation of the Fluxions, in which ſuppoſe $\dot{t} = o$ *; for x and $\dot{x}$ inſert their Values $\frac{b+y}{y}\sqrt{cc-y^2}$, and $\frac{\dot{y}}{y}\sqrt{cc-y^2} - \frac{b\dot{y}+y\dot{y}}{\sqrt{cc-yy}} - \frac{x\dot{y}}{y}$, from the Equation to the Curve, and reduce the reſulting Equation, and it will give you $y^3 - 3by^2 - 2bc^2 = o$. Therefore if this Equation be conſtructed, the Length of BE will be thereby found: to which if you take AC equal, and draw CE parallel to AB, the Point E ſhall be the Point of contrary Flexure required. And after the ſame manner the Points of contrary Flexures, in other Curves where there are ſuch Points, may be inveſtigated.

* Art. 391. of this Explication.

403. Although the Curves which are propoſed, be *mechanical*, Tangents may be drawn to them by the like means as if they were *geometrical*. All that is neceſſary is to find the Relation of the Fluxions of the Abſciſs and Ordinate, expreſſed by finite Lines: which may be done although there be no finite Equation that can define the Relation of the Abſciſs and Ordinate.

Ex. Take the common Cycloid or Trochoid ACD (ſee the Fig. Art. 332 *) and ſuppoſe the ſame things as are there mentioned, only call $AB = x$ and $AHG = z$; and let it be required to draw a Tangent to the Curve at the Point C.

* Of this Explication.

To do which, from the Property of the Curve, you have $BC = BG + AHG$, that is $y = v + z$: therefore, by taking the Fluxions, you

* Of this Explication. you have $\dot{y} = \dot{v} + \dot{z} =$ (becauſe $\dot{z} = \sqrt{\dot{x}^2 + \dot{v}^2}$ by Art. 42 *.) $\dot{v} + \sqrt{\dot{x}^2 + \dot{v}^2}$: but from the Property of the Circle, *viz.* $v^2 = dx - x^2$, by taking the Fluxions, you get $\dot{v} = \frac{\frac{1}{2}d - x}{\sqrt{dx - xx}}\dot{x}$: which Value of $\dot{v}$ being ſubſtituted in the preceding Equation $\dot{y} = \dot{v} + \sqrt{\dot{x}^2 + \dot{v}^2}$, and it duly reduced, you have $\dot{y} = \dot{x}\sqrt{\frac{d-x}{x}}$, whence $\frac{\dot{x}}{\dot{y}} = \sqrt{\frac{x}{d-x}}$, and $\frac{\dot{x}}{\dot{y}}y = \frac{y\sqrt{x}}{\sqrt{d-x}} = BC \times \frac{AB}{BG} = s = BT$. Whence it appears that the Subtangent BT is a fourth Proportional to BG, AB and BC; ſo that the Tangent CT is parallel to the Chord AG: which Property we aſſumed at Art. 326 *, in order to abridge the Demonſtration.

* Of this Explication.

404. And by the like Methods may Tangents be drawn to other mechanical Curves, which are related after any other manner to geometrical Curves: but whereas it is not ſo evident how to proceed in the Caſe of Spirals, I ſhall ſhew the young Geometrician how Tangents may be drawn to them.

* See Fig. Art. 335. of this Explication. Let ACM be a Spiral *, MEGM a Circle deſcrib'd from the Center A with a given Radius AM: and let the Radius ACG beginning at AM revolve upon A as a Pole, and in the mean time the Point C, moving along AC, deſcribe the Spiral ACFM: draw ATS perpendicular to ACG, and let CT be a Tangent to the Spiral at C meeting ATS in T: through G draw GS parallel to CT, meeting ATS in S. Suppoſe ACG to move forward into the new Poſition Acγ, with the Center A and Diſtance AC draw the ſmall Arch Cδ meeting Acγ in δ: then by ſuppoſing the Angle GAγ to be juſt vaniſhing, the Triangle Cδc is ultimately ſimilar to the Triangle ACT *, ſo that it is δc : δC :: AC : AT. Again from ſimilar Arches it is δC : γG :: (AC : AG ::) AT : AS, therefore *ex æquo* δc : γG :: AC : AS; but δc is to γG as the Fluxion of AC to the Fluxion of MEG *. Therefore an Equation being given from the Nature of the Spiral, defining the Relation of AC and MEG, thence find the Ratio of their Fluxions as formerly: and take AS to AC in the Ratio of the Fluxion of MEG to the Fluxion of AC, ſo ſhall SG be parallel to the Tangent at C: therefore SE being join'd, through C draw CT parallel to it, and it ſhall be a Tangent to the Spiral at C.

* Art. 41. of this Explication.

* Art. 28. of this Explication.

Ex. 1. Call MEG $= x$, and AC $= y$, which you may conſider as Abſciſs and Ordinate; and let it be $x : y :: a : b$, a given Ratio, ſo that the Spiral ſhall be that of *Archimedes*: then $bx = ay$, whence $b\dot{x} = a\dot{y}$, ſo that $\dot{y} : \dot{x} :: b : a :: y : \frac{ay}{b} =$ AS: therefore take AS a fourth Proportional

tional

tional to b, a, and AC, join GS, and CT drawn parallel to it touches the Spiral at C. If the given Circle MEGM be that which bounds the firſt Spiral, then a is to b as the Circumference of a Circle to it's Radius: which therefore will be the Ratio of AS to AC: conſequently when the Points C and G coaleſce at M, and conſequently T with S, the Subtangent AT is equal to the Circumference of the Circle MEGM, as *Archimedes* has demonſtrated.

Ex. 2. Let the Equation to the Spiral be generally expreſſed thus, $ay^m = bx^n$: then by taking the Fluxions it is $may^{m-1}\dot{y} = nbx^{n-1}\dot{x}$: whence $\dot{y} : \dot{x} :: nbx^{n-1} : may^{m-1} :: y : \text{AS}$, therefore $\text{AS} = \frac{may^m}{nbx^{n-1}}$ $=$ (by inſerting $\frac{ab^{\frac{1}{n}}y^m}{a^{\frac{1}{n}}by^{\frac{m}{n}}}$ for x^{n-1}, deduced from the given Equation and reducing) $\frac{ma^{\frac{1}{n}}y^{\frac{m}{n}}}{nb^{\frac{1}{n}}}$.

Ex. 3. Let the Equation $x^2 - 2ax + by^2 = 0$ expreſs the Relation of x and y: then by taking the Fluxions, you have $2x\dot{x} - 2a\dot{x} + 2by\dot{y} = 0$, whence $2a - 2x : 2by :: (\dot{y} : \dot{x} ::)\ y : \text{AS} = \frac{2by^2}{2a - 2x}$ $=$ (by ſubſtituting $a - \sqrt{a^2 - by^2}$ for x, deduced from the given Equation) $\frac{by^2}{\sqrt{a^2 - by^2}}$: therefore if you call $\text{AM} = r$, then ſince $r : y :: \text{AS} : \text{AT}$, it will be $\frac{by^3}{r\sqrt{a^2 - by^2}} = \text{AT}$, the Subtangent. And ſo you may proceed in all other like Caſes.

405. Theſe two general Problems, which we have been explaining, are an Application of our Author's firſt Propoſition, and belong to the direct Method of Fluxions, becauſe they only require the finding the Fluxions from the Fluents given. But the far greater Number of Problems to which the Doctrine of Fluxions is applied, require the finding the Fluents from the Fluxions given, in order to their Solution. This cannot always be done by the Quadrature of Curves; but may be done by it in many Caſes, of which theſe that follow are Examples.

PROB.

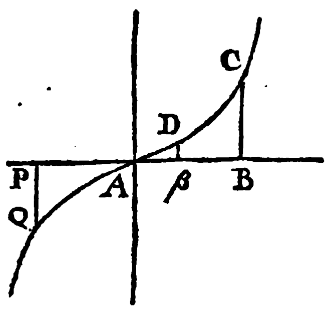
PROB.

To find the Length of Curve-lines.

406. The Principle, upon which the Solution of this Problem depends, was explained and demonstrated in Art. 41 and 42 * (see the Fig. belonging to that Place) which is this: if you put the Abscis $AB = x$, the rectangular Ordinate $BC = y$, and the Curve-line terminated at the Extremity of the Ordinate as $AC = z$, then it is $\dot z = \sqrt{\dot x^2 + \dot y^2}$: where you must observe that it is not necessary to suppose the Curve-line to begin at A or even to pass through A, although it do so in this Figure: but the initial Limit must be determined from the Nature of the Curve, as will appear afterwards. Therefore, if, from the Nature of the Curve, an Equation be given, expressing the Relation of x and y, thence by Prop. 1. you may find the Relation of their Fluxions in another Equation: from which two Equations you may find the Value of the Fluxion of y, in Terms made up of the Fluxion of x, x itself, and known Quantities: which Value of the Fluxion of y being substituted for it in the Equation $\dot z = \sqrt{\dot x^2 + \dot y^2}$, the second Member shall contain no variable Quantity but x, and no Fluxion but that of x; and so the Fluent z, that is the Length of the Curve-line, may be found by the Quadrature of Curves, as will appear by what follows.

* Of this Explication.

407. Let $x^n = ay^{n-1}$ be an Equation to a Parabola of any Order, as AC, where n represents any positive Integer: and it is required to find the Length of the Curve, which call z.



Then you have $\dot z = \sqrt{\dot x^2 + \dot y^2}$: but from the Equation to the Curve, *viz.* $x^n = ay^{n-1}$, by taking the Fluxions, it is $nx^{n-1}\dot x = \overline{n-1} \times ay^{n-2}\dot y$, whence $\dot y = \frac{nx^{n-1}\dot x}{\overline{n-1} \times ay^{n-2}}$, that is (by inserting $\frac{x^{\frac{n^2-2n}{n-1}}}{a^{\frac{n-2}{n-1}}}$ for y^{n-2}, deduced from the given Equation) $\dot y = \frac{nx^{\frac{1}{n-1}}\dot x}{\overline{n-1} \times a^{\frac{1}{n-1}}}$, or by squaring, $\dot y^2 = \frac{n^2x^{\frac{2}{n-1}}\dot x^2}{\overline{n^2 - 2n + 1} \times a^{\frac{2}{n-1}}}$: now if this Value of $\dot y^2$ be put in place of of it in the fluxional Equation $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$, hence will arise $\dot{z} =$

$\dot{x}\sqrt{1 + \frac{n^2 \times x^{\frac{2}{n-1}}}{\overline{n^2 - 2n + 1} \times a^{\frac{2}{n-1}}}}$. But it appears that this Expression denotes the Fluxion of a curvilinear Area, whose Abscis is x, and rectangular Ordinate the radical $\sqrt{1 + \frac{n^2}{\overline{n^2 - 2n + 1} \times a^{\frac{2}{n-1}}} x^{\frac{2}{n-1}}}$: therefore the Curve-line z, and such curvilinear Area, flow according to the same Law; the one as a Line, the other as a plane Superficies, consequently the two Fluents will be analogous, supposing them to begin at the same time: wherefore the curviliner Area having such Abscis and Ordinate, being found by the Quadrature of Curves, apply it to linear Unity, and that shall exhibit the Length of z required; or the curvilinear Area being found in an algebraical Expression, consider such Expression no longer as denoting a Superficies, but a Length, the same algebraical Expression serving for both, since the Division by Unity makes no Alteration upon it. Wherefore if the Value of n be such, that the Ordinate $\sqrt{1 + \frac{n^2}{\overline{n^2 - 2n + 1} \times a^{\frac{2}{n-1}}} x^{\frac{2}{n-1}}}$ belong to a Curve that is quadrable, the Length of the Curve-line is found exactly; if not, the Length of the Curve-line is found approximately, by Propositions fifth and sixth: or in this last Case, may be exhibited by the Area of a Conic Section; or perhaps of some other Curve applied to Unity. So that our Author's Tables of Quadratures serve for this Purpose, as well as for discovering the Areas of Curves: these Lengths likewise may sometimes be found by Logarithms and circular Arches, according to Sect. 10. For finding when these Curve-lines can be rectified exactly, or not consult Art. 147—149 *. * Of this Explication.

Thus if it be $n = 3$, so that the Equation to the Parabola ABC be $x^3 = ay^2$, which is a Parabola of the second Kind: then $\dot{z} = \dot{x}\sqrt{1 + \frac{9x}{4a}}$: which is analogous to the Fluxion of the Area of the Curve whose Ordinate is $\sqrt{1 + \frac{9x}{4a}}$, which may be squared by Art. 147—149 *, and belongs to Species 1. Form 3. Table 1. which stands thus $dz^{\eta-1}\sqrt{e + fz^{\eta}} = y$, by putting $\eta = 1$, $d = 1$, $e = 1$, * Of this Explication.

 $f =$

$f=\frac{9}{4a}$: wherefore the Fluent $\frac{2d}{3\eta f}R^3$ in the Table, is by a proper Substitution of Values, $\frac{8a}{27}\times\overline{1+\frac{9x}{4a}}|^{\frac{3}{2}}$ or $\frac{8a+18x}{27}\sqrt{1+\frac{9x}{4a}}$: which therefore denotes the Length of this Parabola. And after the same manner, the Lengths of all these Parabolas may be found where n is any positive odd Number greater than 3, as 5, 7, 9, &c. by help of the second, third, fourth, &c. Species's of Form third, Table first. The same thing might be done by having recourse to Prop. 5. See Art. 390 *.

* Of this Explication.

408. But whereas it is not yet determined from what initial Limit, the Length of the Curve so found is to be computed, this may be done by putting the Quantity $\frac{8a+18x}{27}\sqrt{1+\frac{9x}{4a}}$ equal to nothing, and the Value of x thence arising will determine the initial Limit; as was shewn formerly with respect to curvilinear Areas in Sect. 5. Thus if we put $\frac{8a+18x}{27}\sqrt{1+\frac{9x}{4a}}=0$, thence arises $\frac{8a+18x}{27}=0$, or $1+\frac{9x}{4a}=0$, either of which gives $x=-\frac{4}{9}a$: which Value being negative, shews that if you take upon the Abscis AB produced beyond A, the Point P such that it be $AP=\frac{4}{9}a$, and draw the Ordinate PQ, the Part $QAC=\frac{8a+18x}{27}\sqrt{1+\frac{9x}{4a}}$. If you would compute the Length of the Curve from the Beginning of the Abscis at A, then here is a known Condition or Limitation of the Fluent, *viz.* that it is nothing when $z=0$, therefore by Art. 387 *, $z=\frac{8a+18x}{27}\sqrt{1+\frac{9x}{4a}}-\frac{8a}{27}$. More universally: if you would have any Part of the Curve-line as γC, lying betwixt any two given Ordinates $\beta\gamma$, BC, subtract the Value of z arising from putting $x=A\beta$ the lesser Ordinate, from the Value of z arising by putting $x=AB$ the greater Ordinate, the Difference is the Length of γC *: thus if you call $AB=\xi$ and $AB=x$, then you have $\gamma C=\frac{8a+18x}{27}\sqrt{1+\frac{9x}{4a}}-\frac{8a+18\xi}{27}\sqrt{1+\frac{9\xi}{4a}}$: where if ξ vanish, the Computation is from the Beginning of the Abscis, or rather the Ordinate passing through it.

* Of this Explication.

*Art. 187. of this Explication.

409. Suppose now that in the preceding general Equation $x^n=ay^{n-1}$, n is 4, so that the Property of the Parabola AC be expressed thus $x^4=ay^3$; then it will be $\dot{z}=\dot{x}\sqrt{1+\frac{16x^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}$, by comparing this

with

with Art. 149 *, you'll find the radical Expreſſion denotes the Ordinate of a Curve that cannot be ſquared: therefore I next conſider whether it may be reduced to any Ordinate found in our Author's ſecond Table of Quadratures; which as it ſtands, it cannot; but if you reduce it to this other Form $x^{\frac{1}{3}}\sqrt{\frac{16}{9a^{\frac{2}{3}}}+x^{-\frac{2}{3}}}$ *, you find it to belong to Species third, Order third of Table ſecond, *viz.* $y = \frac{d}{z^{2\eta+1}}\sqrt{e+fz^{\eta}}$, by putting $\eta = -\frac{2}{3}$, $d = 1$, $e = \frac{16}{9a^{\frac{2}{3}}}$, $f = 1$: by comparing which with Art. 298 *, you'll find the Conic Section, by means of which the Area of the Curve, whoſe Ordinate is $\sqrt{1+\frac{16x^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}$, may be compared to be an Hyperbola belonging to Fig. third Table ſecond, whoſe Latus Rectum is 1 and Lat. Tranſv. is $\frac{9a^{\frac{2}{3}}}{16}$. And if you put $a = 1$, or $x^4 = 1y^3$, for the Equation to the Curve propoſed, you'll have the following Conſtruction. (See Fig. third, Table ſecond, Quad. p. 25.) With the Parameter a, and tranſverſe Axe $Ka = \frac{9}{16}a$ deſcribe the Hyperbola aGDS, take $a\text{B} = \text{AB}^{\frac{2}{3}}$, that is, take aB the ſecond of two middle Proportionals betwixt a and AB the Abſciſs of the given Parabola AC, draw the Ordinate BD, then the Area of the Curve whoſe Ordinate is $\sqrt{1+\frac{16x^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}$, is $\frac{3}{2}a$GDB: conſequently, the Area aGDB being applied to the Lat. Rectum a, $1\frac{1}{2}$ of the Line ariſing ſhall be equal to the Curve-line of the propoſed Parabola AC, of the third Kind.

* Of this Explication.

*Art. 127, and 317. of this Explication.

* Of this Explication.

Now if we would know from whence the Beginning of the Curve-line is to be computed, it is evident, that, ſince $z = \frac{\frac{3}{2}a\text{GDB}}{a}$, when $a\text{GDB} = 0$, then $z = 0$: but aGDB is nothing when the Abſciſs aB is nothing, and $a\text{B} = \text{AB}^{\frac{2}{3}}$, therefore z is nothing when AB the Abſciſs of the propoſed Parabola AC is nothing: therefore the Length of the Curve-line found is to be computed from A, being AC.

410. If we would have the Length of the Curve-line expreſſed by Logarithms, according to what has been delivered in Sect. 10: then ſee Fig. Art. 355 *, which ſubſtitute in place of Fig. 3. Table 2. Quad. ſuppoſing it deſcribed with the ſame Latus Tranſverſum, and Rectum, as before: then $z = \frac{\Delta\text{ABD}}{a} - \frac{\frac{1}{2}\text{A}a \times a\text{H}}{a}\Big|\frac{a\text{H}}{\text{DC}}$ *, *e.g.* ſuppoſe the

* Of this Explication.

* Art. 355, 370. of this Explication.

Length of the Abſciſs AB $= x = a = 1$, then in the Hyperbola by making $aB = (AB^{\frac{2}{3}} =)\ 1$, you'll find $aH = \frac{2}{3} = 0.6666$, *&c. in inf.* $BC = \frac{17}{12} = 1.41666$, *&c. in inf.* $BD = \frac{5}{4} = 1.25$, whence $DC = 0.16666$, *in inf.* therefore it is $z = 1.80569$, *&c.* — 0.296296, *&c.* $\frac{0.6666 \textit{ in inf.}}{0.1666 \textit{ in inf.}}$: wherefore, since the Modulus of the tabular Logarithm is 0.434294, *&c* *. hence it follows that it is $z = 1.80569$, *&c.* — $\frac{0.296296, \&c.}{0.434294, \&c.} \times \overline{\text{Log. } 0.6666 \textit{ in inf.} - \text{Log. } 0.1666 \textit{ in inf.}}$* $= 1.80569$, *&c.* — 0.682247, *&c.* $\times$ Log. $4 = 1.80569$, *&c.* — 0.682247, *&c.* $\times 0.6020600 = 1.80569$, *&c.* — 0.41075, *&c.* $= 1.4949$ nearly the Length of the Curve-line AC required.

*Art. 349. of this Explication.
*Art. 350. of this Explication.

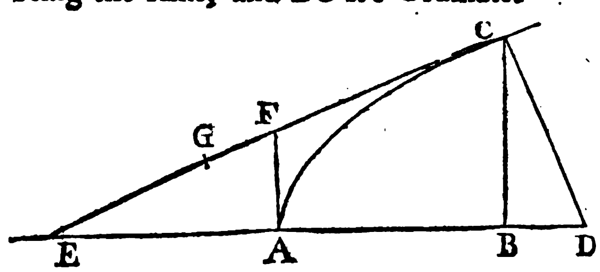
The ſame thing may be done by reducing the Value of the Area into a converging Series by Prop. 5. And after the like manner you may find the Lengths of the Curve-lines belonging to the other Orders of Parabolas, when n is 6, 8, 10 or any other even Number.

411. Ex. 2. Let it be propoſed to find the Length of the Cycloid (ſee the Fig. at Art. 332 *). Uſe the ſame Symbols as in that Article, only call the Abſciſs AB $= x$, and the Curve-line z.

* Of this Explication.

Then by what is there demonſtrated, you have $\dot{y} = \frac{\dot{x}\sqrt{dx - xx}}{x}$, or $\dot{y}^2 = \dot{x}^2 \times \frac{d-x}{x}$: inſert this Value of $\dot{y}^2$ in the general Formula $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$, and it ſtands thus $\dot{z} = \sqrt{\dot{x}^2 + \frac{d-x}{x}\dot{x}^2} = \dot{x}\sqrt{\frac{d}{x}}$ or $d^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}$: which belongs to Form firſt, Table firſt, Quad. *viz.* $dz^{\eta-1} = y$; ſo that if you ſubſtitute $d^{\frac{1}{2}}$ for d, $\frac{1}{2}$ for η, and x for z, the Fluent is $2\sqrt{dx} = 2$ Chord AG: which therefore denotes the Length of the Arch of the Cycloid AC: for it's initial Limit is at A, ſince $2\sqrt{dx}$ is nothing when $x =$ AB is nothing. Hence it follows that ACD $= 2$AE, twice the Diameter of the generating Circle.

412. Ex. 3. Let the common Parabola ABC be propoſed, and it is required to find the Length of the Part AC of the Curve-line, AB being the Axis, and BC it's Ordinate.



To do which, uſing the ſame Symbols for Abſciſs AB, Ordinate BC and Curve-line as formerly; call the Parameter a: then ſince it is $ax = y^2$; hence by taking

taking the Fluxions you $a\dot{x} = 2y\dot{y}$, that is $\dot{y} = (\frac{a\dot{x}}{2y} =) \frac{a^{\frac{1}{2}}\dot{x}}{2x^{\frac{1}{2}}}$, or $\dot{y}^2 = \frac{a\dot{x}^2}{4x}$ insert this Value of $\dot{y}^2$ in the general Formula $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$, and it becomes, by a due Reduction, $\dot{z} = \dot{x} \times \frac{1}{2}\sqrt{4 + ax^{-1}}$: which being the same with the Fluxion of a Curve that is not quadrable (by Art. 147, 149 *.) I find it belongs to Species second, Order third, Table second, *viz.* $\frac{d}{z^{n+1}}\sqrt{e + fz^n} = y$, by putting $n = -1$, &c. By which means it may be found, by applying an hyperbolical Space to linear Unity, as was shewn before in the Case of a Parabola of the third Order.

* Of this Explication.

413. Moreover it may be observed, that as we arrive at the Lengths of Curves by substituting the Value of $\dot{y}^2$ in Terms made up of x and it's Fluxions, in the general Formula $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; so you may substitute $\dot{x}^2$ it's Value made up of y and it's Fluxion, and by that means do the same: thus because it is in this Example, $a\dot{x} = 2y\dot{y}$, substitute $\frac{2y\dot{y}}{a}$ for $\dot{x}$, or $\frac{4y^2\dot{y}^2}{a^2}$ for $\dot{x}^2$, thence it will be $\dot{z} = \sqrt{\frac{4y^2\dot{y}^2}{a^2} + \dot{y}^2} = \dot{y}\sqrt{1 + \frac{4y^2}{a^2}}$: which agrees with the general Expression for all Parabolas, as in Art. 407 *, *viz.* $\dot{z} = \dot{x}\sqrt{1 + \frac{n^2 x^{\frac{2}{n-1}}}{n^2 - 2n + 1 \times a^{\frac{2}{n-1}}}}$, with this only Difference, that you have y here for x there, arising from this Circumstance, that the Abscis in the former Case was taken upon a Line perpendicular to the Axis upon which the Abscis is taken in this Example, as is evident. Therefore now that it is $\dot{z} = \dot{y}\sqrt{1 + \frac{4y^2}{aa}}$ or $\dot{z} = y\dot{y}\sqrt{\frac{4}{a^2} + y^{-2}}$, which belongs to Species second, Order third, Table second, *viz.* $\frac{d}{z^{n+1}}\sqrt{e + fz^n} = y$, by putting $n = -2$, $z = y$, &c. it will appear by consulting Art. 295 *, and comparing it with the Expressions $\frac{1}{z^n} = x^2$, $\sqrt{f + ex^2} = v$, and $-\frac{2d}{n}s = t$, in the Table, that, assuming any Line for linear Unity, if you take (see Fig. 2. Table 2. p. 25.) AP = 1, and with it for the half transverse Axis, and $\frac{1}{4}\frac{aa}{AP}$ for the Latus Rectum, describe the Hyperbola PGDS, take upon AN the indeterminate Axis, AB equal to BC the Ordinate of the pro-

* Of this Explication.

* Of this Explication.

posed Parabola ABC, and draw the Ordinate BD, then you shall have $\frac{ABDP}{1} = z =$ AC the parabolical Arch required.

414. Whence we learn to express the Length of the parabolical Arch AC by the Measures of Ratios thus. Draw the Tangent CE meeting the Axis in E, through A draw AF perpendicular to AB, meeting the Tangent in F, likewise CD perpendicular to the Tangent meeting the Axis in G: then take FG the Measure of the Ratio of AE + EF to AF to the Modulus $\frac{1}{2}$BD, so shall it be CG = Arch AC: which according to the Notation in Sect. 10. is expressed thus; $AC = CF + \frac{1}{2}BD \left| \frac{AE+EF}{AF} \right.$.

For since it has been shewn that $AC = \frac{ABDP}{1}$ in the preceding Art. and ABDP = ΔABD + Sector APGD; and moreover that in the Hyperbola you have AP = 1, Lat. Rect. = $\frac{1}{2}aa$, AB = y and BD = $\sqrt{1 + \frac{4}{a^2}y^2}$, it will follow by considering Art. 344 *, that the hyperbolical Sector APGD $= \frac{1}{4}a \left| \frac{\sqrt{\frac{1}{4}aa + yy} + y}{\frac{1}{2}a} \right. =$ (by the Construction above in the Parabola) $\frac{1}{2}BD \left| \frac{CD+BC}{BD} \right.$, or (from similar Triangles) $\frac{1}{2}BD \left| \frac{FE+EA}{AF} \right.$ and the Triangle ABD in the Hyperbola is equal to $\frac{y\sqrt{1+\frac{4}{aa}y^2}}{2} = \frac{1}{2} \times \frac{y\sqrt{\frac{1}{4}aa+yy}}{\frac{1}{2}a} =$ (in the Parabola) $\frac{1}{2} \times \frac{BC \times CD}{BD} =$ CF. Therefore the parabolical Arch $AC = (CF + \frac{1}{2}BD \left| \frac{FE+EA}{AF} \right. = CF + FG) = CG$. As was to be shewn. And this shall suffice for explaining and solving this general Problem: *to find the Length of Curve-lines.*

* Of this Explication.

415. Schol. Here it is proper to observe that you may find as many Curves as you please, whose Lengths are equal to known right Lines; or may be compared to the Areas of known Curves applied to Unity, like to what was shewn formerly with respect to the Areas of Curves. See Art. 100, 225, 226 *; *e. g.* let it be required to find the Curve whose Length $z = \frac{2}{3}x^{\frac{3}{2}}$.

* Of this Explication.

Therefore by taking the Fluxions, it is $\dot{z} = x^{\frac{1}{2}}\dot{x}$ and by squaring, $\dot{z}^2 = x\dot{x}^2$: but it is $\dot{z}^2 = \overline{\dot{x}^2 + \dot{y}^2}$, therefore by Substitution, $x\dot{x}^2 = \dot{x}^2 + \dot{y}^2$, or $\overline{x-1} \times \dot{x}^2 = \dot{y}^2$, that is by extracting the Square Root

$\dot{x}\sqrt{x-1}$

$\dot{x}\sqrt{x-1}=\dot{y}$; therefore it appears from what has been said, that the Area of a Curve whose Abscis is x and Ordinate $\sqrt{x-1}$, applied to Unity, will give the Length of y the Ordinate of the Curve required: which you'll find, by comparing the Expression $\sqrt{x-1}$ with Species 1. Order 3. Table 1. to be $\frac{2}{3}\times\overline{x-1}|^{\frac{3}{2}}$: therefore $y=\frac{2}{3}\times\overline{x-1}|^{\frac{3}{2}}$ defines the Property of the Curve sought.

And thus you may find as many other Curves as you please, which may be rectified; or compared with the Areas of known Curves aplied to Unity, by assuming or feigning any Relations of x and z you please.

PROB.

To find the Areas of Curve Surfaces.

416. The Principle upon which the Solution of this Problem depends, was explained and demonstrated Art. 50 *, which amounts to this: if any plane Figure ABC (see the Fig. belonging to that Art.) revolve upon AB as an Axis of Rotation, and thereby generate a Solid, call AB $=z$, it's rectangular Ordinate BC $=y$, the Surface generated by the revolving of the Curve-line AC $=s$; then supposing AB, BC and AC to flow, it is $\dot{s}=\frac{c}{r}\sqrt{\dot{z}^2+\dot{y}^2}$: where c denotes the Circumference of any Circle whose Radius is r. Therefore if from the Equation to the generating Curve ABC, defining the Relation of z and y, you find a new fluxional Equation containing the Relation of their Fluxions, by means of this last Equation, and the Equation to the Curve, you may always find such a Value of $\dot{s}$, as shall only include one of the unknown Quantities z and y, and it's Fluxion, with known Quantities: so that by the Quadrature of Curves the Value of s may be found, after the same Manner as in the Solution of the preceding Problem, as will appear by the following Examples.

* Of this Explication.

417. Ex. 1. Let it be proposed to find the superficial Content of a Solid generated by the Rotation of a Segment or Frustum of the common Parabola round it's Axis.

Then from the Property of the Parabola (if you put a for the Parameter) it is $az=y^2$, whence $a\dot{z}=2y\dot{y}$: but by the preceding Article $\dot{s}=\frac{c}{r}\sqrt{\dot{z}^2+\dot{y}^2}$: wherefore, by inserting $\frac{4y^2\dot{y}^2}{aa}$ for $\dot{z}^2$, deduc'd from the preceding fluxional Equation, it becomes $\dot{s}=(\frac{c}{r}\sqrt{\frac{4y^2\dot{y}^2}{aa}+\dot{y}^2}$

=)

$=)\ \dot{y}\times\frac{c}{r}\sqrt{1+\frac{4}{aa}y^2}$. Now the Expression $\frac{c}{r}\sqrt{1+\frac{4}{aa}y^2}$ belongs to a Curve that may be squared (as appears by Art. 147, 149 *.) and is the same with the Ordinate of a Curve belonging to Species 1. Order 3. Table 1. *viz.* $dz^{\eta-1}\sqrt{e+fz^{\eta}}$, whose Area is $\frac{2d}{3\eta f}\times\overline{e+fz^{\eta}}|^{\frac{3}{2}}$ so that by substituting y for z, 2 for η, $\frac{c}{r}$ for d, 1 for e, and $\frac{4}{aa}$ for f, the Fluent analogous to the Area of the Curve, is $\frac{a^2c}{12r}\times\overline{1+\frac{4}{aa}y^2}|^{\frac{3}{2}}$. But to ascertain the true Value of the Fluent we want, it must be computed from the Vertex of the Paraboloid as it's initial Limit, which corresponds to the Computation of the curvilinear Area from the Beginning of the Abscis, where $y=0$. Here then is a Condition limiting the Fluent, *viz.* that it must be nothing when y is nothing: therefore by considering Art. 387 *, the true Fluent is $\frac{a^2c}{12r}\times\overline{1+\frac{4}{aa}y^2}|^{\frac{3}{2}}-\frac{a^2c}{12r}$: for this only is the Fluent that vanishes when y vanishes: and therefore $\frac{a^2c}{12r}\times\overline{1+\frac{4}{aa}y^2}|^{\frac{3}{2}}-\frac{a^2c}{12r}=s$ the superficial Content of the parabolical Conoeid required.

* Of this Explication.

* Of this Explication.

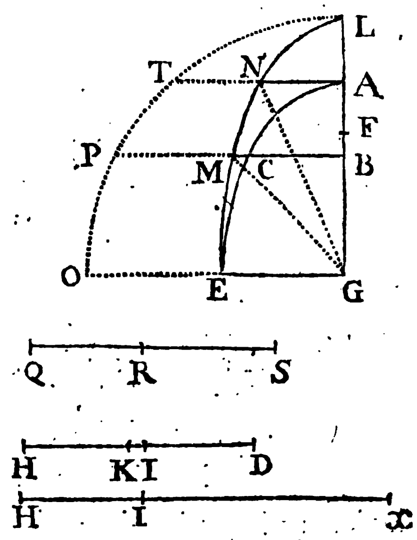
418. Ex. 2. To find the Surface of a spheroidical Segment, the Spheroid being that which is formed by the Rotation of an Ellipse round it's longer Axis, called an oblong Spheroid.

Let c, r, z, y, s denote the same things as formerly: moreover let k and l denote the Latus Transversum and Latus Rectum of the Ellipse. Then from the Property of the Ellipse, you have $y^2=\frac{klz-lz^2}{k}$: whence, by taking the Fluxions, $2y\dot{y}=\frac{kl\dot{z}-2lz\dot{z}}{k}$, that is $\dot{y}^2=\dot{z}^2\times\frac{k^2l-4klz+4lz^2}{4k^2z-4kz^2}$; by inserting of which Value of $\dot{y}^2$ instead of it in the general Formula $\dot{s}=\frac{c}{r}y\sqrt{\dot{z}^2+\dot{y}^2}$, and $\sqrt{\frac{klz-lz^2}{k}}$ for y, and making the proper Reduction, you shall obtain $\frac{c}{r}\dot{z}\sqrt{\frac{1}{4}ll+\frac{kl-ll}{k}z-\frac{kl-ll}{k^2}z^2}=\dot{s}$. Now by comparing this radical Expression with the Form of the Ordinate Art. 302 *, *viz.* $v=\sqrt{\frac{apt-a^2p}{t}+\frac{pt-2ap}{t}x+\frac{p}{t}x^2}$, you will find that it denotes the Ordinate of an Ellipse: in which, by comparing the homologous Terms

* Of this Explication.

Terms by the Rules of Algebra, you'll get 1mo. $t = k\sqrt{\frac{k}{k-l}}$ 2°. $p = l\sqrt{\frac{k}{k-l}}$ 3°. $a = \frac{1}{2}k\sqrt{\frac{k}{k-l}} - \frac{1}{2}k$ or $\frac{1}{2}t - \frac{1}{2}k$. By conſidering which you may deduce the following Conſtruction.

Let AGE be a Quadrant of the generating Ellipſe, AG half the longer and GE half the ſhorter Axis, BC an Ordinate to the longer Axis, F the Focus. Then upon GA produc'd take GL a third Proportional to GF and GA: with the ſemitranſverſe GL and ſemiconjugate GE, deſcribe the Ellipſis LE, draw AN perpendicular to AG, meeting the Ellipſe LE in the Point N, and produce BC till it meet it in M: then the Area ABMN is to the Surface of the ſpheroidical Segment generated by the Rotation of ABC, as r to c, that is as Radius of a Circle to the Circumference.



For ſince it is $GA = \frac{1}{2}k$, and half Parameter $= \frac{1}{2}l$, hence it is $GE = \frac{1}{2}\sqrt{kl}$; therefore $GF^q = (GA^q - GE^q =)\ \frac{1}{4}k^2 - \frac{1}{4}kl$, or $GF = \sqrt{\frac{1}{4}k^2 - \frac{1}{4}kl}$, conſequently $GL = \left(\frac{GA^q}{GF} =\right) \frac{1}{2}k\sqrt{\frac{k}{k-l}}$: but from the Property of the Ellipſis LGE, $BM^q : GE^q\ (\frac{1}{2}\sqrt{kl}) :: GL^q - GB^q\ \left(\frac{\frac{1}{4}k^3}{k-l} - \overline{\tfrac{1}{2}k - z}^2\right) : GL^q\ \left(\frac{\frac{1}{4}k^3}{k-l}\right)$: from whence, by a proper Reduction, you have $BM = \sqrt{\frac{1}{4}ll + \frac{kl-ll}{k}z - \frac{kl-ll}{k^2}z^2}$. Wherefore the Fluxion of the elliptical Space ABMN is $\dot{z}\sqrt{\frac{1}{4}ll + \frac{kl-ll}{k}z - \frac{kl-ll}{k^2}z^2}$: but it was ſhewn above that $\dot{s} = \frac{c\dot{z}}{r}\sqrt{\frac{1}{4}ll + \frac{kl-ll}{k}z - \frac{kl-ll}{k^2}z^2}$, therefore the Fluxions are to one another as r to c, a given Ratio, and the Fluents have both the ſame initial Limit at AN, therefore ABMN : the Surface of the ſpheroidical Segment :: $r : c$.

419. Cor. 1. Hence it appears that the whole ſpheroidical Surface is to 4AGEN, as the Periphery of a Circle to it's Radius.

420. Cor. 2. If GE remaining the ſame, GA grow ſhorter and ſhorter, GL will thereby grow longer and longer, until GA become

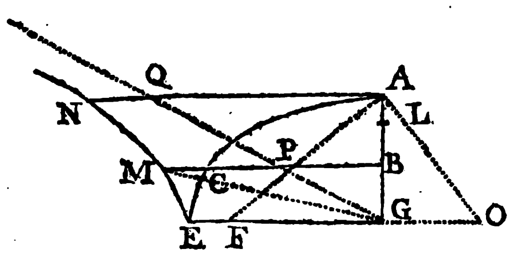
 equal

equal to GE, when the revolving Ellipse becomes a Circle, and the Spheroid, a Sphere, and the Focus F falls in with G; and then GL which is a third Proportional to GF and GA, turning infinite, EMN becomes a strait Line parallel to GA, and so AGEN and ABMN are Rectangles: whence it follows that the Surface of the spherical Segment is equal to the corresponding convex Surface of a right Cylinder having GE for the Radius of its Base: for the Rectangle AGEN, by it's Rotation about AG, generates a right Cylinder, and EMN the convex cylindrical Surface: where it is evident the Part of the cylindrical Surface generated by MN is to the Rectangle ABMN as c to r. Whence appears the Proportion betwixt the Surface of the Sphere and that of the circumscribed Cylinder; and that the Surface of the Sphere is equal to four times the Area of a great Circle.

421. Cor. 3. But upon the contrary if the Ellipse AGE be supposed to grow more and more excentric, by the Center G removing, the Points A and F remaining, until G be infinitely distant, the Ellipse becomes a Parabola, having A for it's principal Vertex and F for it's Focus. Now if the Surface of any Segment of this Paraboloid was required, you need only make such a Change upon the former fluxionary Equation $\dot{s} = \frac{c}{r}\dot{z}\sqrt{\frac{1}{4}ll + \frac{kl-ll}{k}z - \frac{kl-ll}{k^2}z^2}$ as the Supposition of k being infinite requires: which will turn it into this $\dot{s} = \frac{c}{r}\dot{z}\sqrt{\frac{1}{4}ll + lz}$: which being compared with Species 1. Form 3. Tab. 1. is found to denote the Fluxion of a quadrable Curve whose Area is, by a proper Substitution, $\frac{2c}{3r} \times \overline{\frac{1}{4}l + z}\sqrt{\frac{1}{4}ll + lz}$, which will be found to be the same with the Expression formerly found Art. 417 *. but to have the true Fluent, it must be made $\frac{2c}{3r} \times \overline{\frac{1}{4}l + z}\sqrt{\frac{1}{4}ll + lz} - \frac{cll}{12r}$: for which a geometrical Construction may be easily made.

* Of this Explication.

422. Cor. 4. If the revolving Axis AG be supposed to grow shorter than the Axis GE of the Ellipsis, as in the Figure adjoin'd: then, supposing k to denote the shorter Axis, and l it's Latus Rectum, the Expression for the Fluxion of the Surface of the Segment of the Spheroid, now become oblate, will be the same as before, only l being now greater than k, there will be a Difference in the Signs of the Terms of the radical Expression so as to give $\dot{s} = \frac{c\dot{z}}{r}\sqrt{\frac{1}{4}ll - \frac{ll-kl}{k}z + \frac{ll-kl}{k^2}z^2}$: by



by comparing of which with Art. 294 *, the radical Expreſſion will be found to denote an Ordinate to the ſecond or conjugate Axis of an Hyperbola. And if the Terms of the radical Expreſſion $\sqrt{\frac{1}{4}ll - \frac{ll-kl}{k}z + \frac{ll-kl}{k^2}z^2}$ be compared with the homologous Terms of the hyperbolical Ordinate in that Art. *viz.* $\sqrt{\frac{a^2t + \frac{1}{4}ptt}{p} - \frac{2at}{p}x + \frac{t}{p}x^2}$, as formerly, you will obtain theſe Values of t, p and a, *viz.* $t = \sqrt{kl}$, $p = \frac{k^2}{l-k}\sqrt{\frac{k}{l}}$ and $a = \frac{1}{2}k$. Whence ariſes this Conſtruction.

* Of this Explication.

F being the Focus of the Ellipſe, join AF, draw AO perpendicular to AF, meeting EG produced in the Point O: with half the tranſverſe Axis GE, and half the conjugate GL = GO, and the principal Vertex E, deſcribe the Hyperbola EMN; draw AN perpendicular to AG meeting the Hyperbola in N; produce BC till it meet the Hyperbola in M: then the hyperbolical Area ABMN is to the Surface of the Segment of the oblate Spheroid generated by the Revolution of ABC as r to c.

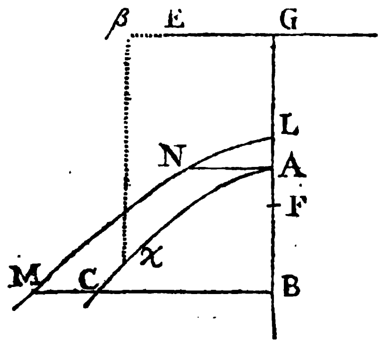
For $GF = (\sqrt{AF^q - AG^q} =) \sqrt{\frac{1}{4}kl - \frac{1}{4}k^2}$, and $GL = \left(\frac{AG^q}{FG} =\right) \frac{\frac{1}{4}k^2}{\sqrt{\frac{1}{4}kl - \frac{1}{4}k^2}}$: but from the Property of the Hyperbola it is $BM^q - GE^q : BG^q :: GE^q : GL^q$, or by Alternation and Compoſition, $BM^q : GE^q \left(\frac{1}{4}kl\right) :: BG^q + GL^q \left(\overline{\frac{1}{2}k - z}^2 + \frac{\frac{1}{4}k^3}{l-k}\right) : GL^q \left(\frac{\frac{1}{4}k^3}{l-k}\right)$, whence $BM = \left(\sqrt{\frac{GE^q \times \overline{BG^q + GL^q}}{GL^q}} =\right)$ by a due Reduction $\sqrt{\frac{1}{4}ll - \frac{ll-kl}{k}z + \frac{ll-kl}{k^2}z^2}$: whence it follows as in the preceding Conſtruction, that the hyperbolical Space ABMN is to the Surface of the Segment of the oblate Spheroid, as Radius of a Circle to it's Circumference.

423. Schol. By comparing theſe two Caſes of the oblong and oblate Spheroid together, and by ſuppoſing the one to paſs through the Sphere into the other, you ſee the gradual Progreſſion of the Values of the Surfaces, by the Line EMN paſſing from an Ellipſe, concave to AB, into a right Line parallel to it, and then into an Hyperbola convex to it: and that the revolving Axis GE is always one Axis of the Conic Section EMN, and the other GL a third Proportional to GF and GA.

In the foregoing Inveſtigations of the Surfaces of the oblong and oblate Spheroid, if you had ſuppoſed GB = z, ſo that the Abſciſs had

had begun at the Center, the Solution would have been more eaſy and ſimple: for in that caſe you would have had $\dot{s} = \frac{c}{r}\dot{z}\sqrt{\frac{1}{4}kl - \frac{kl-ll}{k^2}z^2}$.

424. Suppoſe now the revolving Conic Section to be an Hyperbola ABC, G the Center, GA half the tranſverſe, GE half the conjugate Axis, and F the Focus as before; and imagine it to revolve about the tranſverſe Axis GAB; BC being an Ordinate thereto as formerly: and the Surface of the Segment of the hyperbolical Conoeid is required. Suppoſe AB $= z$, BC $= y$, and k and l to denote the tranſverſe Axis and it's Parameter as before; then from the Property of this Conic Section it is $y^2 = \frac{lkz + lz^2}{k}$, and by taking the Fluxions $2y\dot{y} = \frac{lk\dot{z} + 2lz\dot{z}}{k}$: whence you obtain $\dot{y}^2 = \dot{z}^2 \times \frac{\frac{1}{4}lk^2 + lkz + lz^2}{k^2z + kz^2}$, by inſerting which in the general Formula $\dot{s} = \frac{y}{r}\sqrt{\dot{z}^2 + \dot{y}^2}$, and bringing all to order as before, you'll thereby obtain this Equation $\dot{s} = \frac{c\dot{z}}{r}\sqrt{\frac{1}{4}ll + \frac{ll+lk}{k}z + \frac{ll+lk}{k^2}z^2}$: by comparing this with Article 296 *, where it is $v = \sqrt{\frac{apt + a^2p}{t} + \frac{pt + 2ap}{t}x + \frac{p}{t}x^2}$, you may ſee that it belongs to an Hyperbola; ſo that by comparing the homologous Terms as before, it will give $t = k\sqrt{\frac{k}{l+k}} \cdot p = l\sqrt{\frac{l+k}{k}} \cdot a = \frac{1}{2}k - \frac{1}{2}k\sqrt{\frac{k}{l+k}}$ or $\frac{1}{2}k - \frac{1}{2}t$. Whence ariſes the following Conſtruction.



* Of this Explication.

Take GL a third Proportional to GF and GA, then with the Center G, half tranſverſe GL, and half conjugate GE, deſcribe the Hyperbola LNM, and finiſh the reſt of the Conſtruction as before: then the hyperbolical Area ABMN is to the Surface of the Segment of the hyperbolical Conoeid formed by the Revolution of ABC and AB, as Radius of a Circle to it's Circumference. The Demonſtration of which is the ſame as before. And as GL was a third Proportional to GF and GA, and 2GE was a common Axis to both Conic Sections in the preceding Caſes; it is the ſame in this Caſe.

425. Schol.

425. Schol. If the Points A and F continuing, the Center G be ſuppoſed to remove to an infinite Diſtance, AC becomes a Parabola, and LNM another, having both the ſame Latus Rectum, and AL = AF: for by ſuppoſing k infinite, the fluxional Equation $\dot{s} = \frac{c\dot{z}}{r}\sqrt{\frac{1}{4}ll + \frac{ll+lk}{k}z + \frac{ll+lk}{k^2}z^2}$ becomes $\dot{s} = \frac{c\dot{z}}{r}\sqrt{\frac{1}{4}ll + lz}$: where l denotes the Latus Rectum of the Axis of the revolving Parabola. Wherefore ſince it is AL = (AF =) $\frac{1}{4}l$, you have LB = $\frac{1}{4}l + z$, multiply this by l, and it produces $\frac{1}{4}ll + lz$, which is the Square of the Ordinate BM, therefore LNM is a Parabola having the ſame Latus Rectum with the revolving Parameter AC: ſo that the Surface of the Segment of the Paraboloid, generated by the Rotation of the Parabola ABC, is to the parabolical Space ABMN, as c to r.

426. Finally, if the Hyperbola AC be ſuppoſed to turn round upon the conjugate Axis GE, and thereby generate a Surface convex towards GE; take upon GE, produced if need be, any Abſciſs Gβ, and draw an Ordinate βx: then calling GB = z, and $\beta x = y$, you may find as before another Hyperbola giving an hyperbolical Space, which ſhall be to the Surface generated by the Rotation of Ax, as r to c. This other Hyperbola has GA for it's half Tranſverſe, and a third Proportional to GF and GE for it's half Conjugate: the ſame Property as the Conic Sections formerly found.

427. If the Area ABMN in all the preceding Caſes be applied to the Abſciſs AB, and the Line reſulting be made the Altitude of a Rectangle whereof AB is the Baſe; and ſuch Rectangle, by revolving round AB as an Axis, generate a right Cylinder, the convex Surface of the Cylinder is equal to the Curve-Surface of the Segment of the Spheroid or Conoeid generated by the Rotation of ABC about AB. For ſuch cylindrical Surface is to the Area ABMN as c to r: as is evident. Underſtand the ſame of the Surface formed by the Rotation of Ax about Aβ, when the hyperbolical Area ariſing is applied to Gβ.

428. By conſidering what has been demonſtrated, and comparing it with what was ſhewn formerly in Sect. 10. you may know how to expreſs theſe ſeveral things by the Meaſures of Angles and Ratios. Thus in the Caſe of the oblong Spheroid (ſee Fig. Art. 418.*) if you imagine the two right Lines or Radius's GM, GN to be drawn, the Area ABMN = LBMNL − LAN = Sector LGM − ΔBGM − Sector LGN + ΔAGN: but the Sector LGM = $\frac{1}{2}$GE × $\overline{GL}\left|\frac{GB.\sqrt{GL^q - GB^q}}{GL}\right.$, and the Sector LGN = $\frac{1}{2}$GE × $\overline{GL}\left|\frac{GA.\sqrt{GL^q - GA^q}}{GL}\right.$*; the

* Of this Explication.

* Art. 363, 370. of this Explication.

the $\triangle BGM = \frac{BG \times BM}{2}$, and $\triangle AGN = \frac{AG \times AN}{2}$: therefore the Area $ABMN = \frac{1}{2}GE \times GL \left| \frac{GB . \sqrt{GL^q - GB^q}}{GL} \right. - GL \left| \frac{GA . \sqrt{GL^q - GA^q}}{GL} \right. + \frac{AG \times AN - BG \times BM}{2}$: wherefore if this Quantity be multiplied by $\frac{c}{r} = 6.2831853$, *&c.* the Quantity arising will denote the Value of the Surface generated by the Rotation of AC about AB.

429. If the Surface generated by the Rotation of the Arch EC were required, then it is equal to $\frac{c}{r} \times BGEM$: but $BGEM = ($Sector $EGM + \triangle BGM =) \frac{1}{2}GE \times GL \left| \frac{\sqrt{GL^q - GB^q} . GB}{GL} \right.$* $+ \frac{BG \times BM}{2}$: therefore if the Arch which measures the Angle whose Radius, Tangent and Secant are to one another as $\sqrt{GL^q - GB^q}$, GB, and GL, or (which is the same) whose Radius is to it's Sine as GL to GB, be multiplied into $\frac{1}{2}$GE, and the Triangle BGM added thereto, then that Sum increased in the Ratio of c to r shall be equal to the Curve-Surface required. Which may be constructed thus.

* Art. 363, 370. of this Explication.

With the Center G and Radius GL describe the Quadrant LPO, produce BCM till it meet it in P: take QR : BG :: BM : GE; to QR add RS = OP, then the Surface generated by the Rotation of the Arch CE is to the Circle described with the Radius GE as QS to GE.

For the Surface generated by the Rotation of BC is $\frac{1}{2}GE \times GL \left| \frac{\sqrt{GL^q - GB^q} . GB}{GL} \right. + \frac{BG \times BM}{2} \times \frac{c}{r} = \frac{1}{2}GE \times OP + \frac{GE \times QR}{2} \times \frac{c}{r} = \frac{1}{2}GE \times QS \times \frac{c}{r}$: but the Circle described with the Radius GE is $\frac{1}{2}GE^q \times \frac{c}{r}$: therefore, dividing both by $\frac{\frac{1}{2}GE \times c}{r}$, they are to one another as QS to GE. And this is the Construction given by *Cotes*, *Harmonia Mensurarum*, *p.* 28.

And after the same manner, if you produce AN till it meet the Circle in T, take HI : GA :: AN : GE and HK : GB :: BM : GE; and upon HI produced take ID = PT, then the Surface produced by the Rotation of the Arch AC, shall be to the Circle described with the Radius GE, as KD to GE.

Moreover, if you take HI the same as before, and add to it Ix = OT, then the Surface generated by the Rotation of ACE, that is the half

half of the Surface of the oblong Spheroid, is to the Circle described with the Radius GE, as HX to GE.

All which things may be easily found by the trigonometrical Canon, and the known Ratio of the Radius of a Circle to it's Circumference.

430. With respect to the oblate Spheroid (see Fig. Art. 422 *.) the Surface generated by the Rotation of CE about BG is equal to $\frac{c}{r}$ $\times$ BGEM: now if you draw the Radius GM to the Hyperbola NEM, it appears that the Area BGEM=Sector GEM+$\triangle$BGM: but if you draw the Assymptote GPQ, cutting BM and AN in P and Q, the Sector GEM may be variously expressed by the Measure of a Ratio, according to Art. 344, 366, 367 *, as thus $\frac{1}{2}LG \times GE \left|\frac{BM+BP}{GE}\right.$ or $\frac{1}{2}LG \times GE \left|\frac{GE}{MP}\right.$ or $\frac{1}{2}LG \times GE \left|\frac{BM+BP}{\sqrt{BM^q - BP^q}}\right.$ &c. or $\frac{1}{2}LG \times GE \left|\frac{BM+\sqrt{BM^q - GE^q}}{GE}\right.$: wherefore BGEM = $\triangle$BGM, together with the Measure of any of these Ratios to the Modulus GE multiplied into $\frac{1}{2}$LG: for which it might be easy to make a Construction, similar to that for the Surface of the oblong Spheroid, and like what Mr. *Cotes* has given in the Place already mentioned. And if the Lengths of the Lines GE, GA, GB be expressed in Numbers, you may easily compute the Surface of any Part or the Whole of the oblate Spheroid by means of a Table of Logarithms. The same things are to be understood of the other Solids which have been mentioned. Which I shall leave to the young Geometrician for his own Exercise: and pass on to the next Problem that I may hasten to a Conclusion.

* Of this Explication.

* Of this Explication.

PROB.

To find the Contents of Solids generated by the Rotation of plane Figures.

431. The Principle upon which the Solution of this Problem depends, was fully explained Art. 43, 44 *, which amounts to this: if $\dot{z}$ denote the Fluxion of a Rectangular parallelopiped, whose Breadth and Thickness is 1, or having a square Base whose Side is 1; and the variable Length of the Parallelopiped, z, then the Fluxion of any Body, generated by the Rotation of any plane Figure, having z for it's Abscissa and y for it's perpendicular Ordinate, will be equal to $\dot{z}$ multiplied into the Circle whose Radius is y. Now if r and c signify the same thing as before, then the Area of a Circle whose Radius is y, will be designed by $\frac{cy^2}{2r}$: therefore if the solid Content of any such

* Of this Explication.

ſuch Body be called s, then you have this general Formula $\dot{s} = \frac{cy^2\dot{z}}{2r}$. Wherefore an Equation being given containing the Relation of z and y, you may by help of it throw y out of the Equation $\dot{s} = \frac{cy^2\dot{z}}{2r}$; or, if you find it more convenient, you may throw out $\dot{z}$, and retain a Value of $\dot{s}$ including only y and it's Fluxion: by either of which means you may find the Fluent by the Quadrature of Curves, by finding the Area of a Curve having an analogous Fluxion; which Area being found, and ſuppoſed to be multiplied by 1, gives the Content of the Solid. See Art. 390, 407 *.

* Of this Explication.

432. Ex. 1. Let it be propoſed to find the Solidity of a right Cone; which is generated by the Revolution of a right-angled Triangle round one of the Sides including the right Angle. Wherefore calling the Side about which the Triangle revolves a, and the other Side, which generates the circular Baſe, b; let z and y denote an Abſciſs and Ordinate in that Triangle: then $z : y :: a : b$, or $y = \frac{bz}{a}$: wherefore in the general Formula, *viz.* $\dot{s} = \frac{cy^2\dot{z}}{2r}$, inſert this Value of y, and it becomes $\dot{s} = \frac{cb^2z^2\dot{z}}{2ra^2}$, which may denote the Fluxion of a quadrable Curve belonging to Form 1. Table 1. or the Fluents may be found by the Converſe of our Author's firſt Propoſition; by either of which you'll have $s = \frac{cb^2z^3}{6ra^2} =$ (by inſerting y^2 for $\frac{b^2z^2}{a^2}$) $\frac{cy^2z}{6r}$, the Value of the Cone having z for it's Altitude: which when z and y become a and b, is $\frac{cb^2a}{6r}$, *i. e.* a third Part of the Cylinder having the ſame Baſe and Altitude, which is $\frac{cb^2a}{2r}$.

433. Ex. 2. Let it be required to find the Solidity of any Segment of an oblong Spheroid, as that generated by the Rotation of ABC (ſee Fig. Art. 418 *.) about the Axis AG. Let a and b denote the Halves of the longer and ſhorter Axis: call $AB = z$, $BC = y$, then the Equation to the Curve is $y^2 = \frac{2b^2z}{a} - \frac{b^2z^2}{a^2}$, therefore by inſerting this Value of y^2 into the general Formula $\dot{s} = \frac{cy^2\dot{z}}{2r}$, it becomes $\dot{s} = \dot{z} \times \overline{\frac{cb^2z}{ra} - \frac{cb^2z^2}{2ra^2}}$: wherefore if you find (by Art. 145 *, or otherwiſe) the Area of a Curve whoſe Abſciſs is z, and Ordinate $\frac{cb^2z}{ra} - \frac{cb^2z^2}{2ra^2}$; it will give the Value of the Segment required, *viz.* $\frac{cb^2z^2}{2ra} - \frac{cb^2z^3}{6ra^2} = \frac{cb^2}{2r}$ $\times$

* Of this Explication.

* Of this Explication.

$\times \overline{\frac{z^2}{a} - \frac{z^3}{3a^2}}$ or $\frac{cz^2}{2r} \times \overline{\frac{b^2}{a} - \frac{b^2 z}{3a^2}}$: the first of which Expressions denotes a Cylinder having b for the Radius of it's Base, and a third Proportional to a and z for it's Altitude, lessened by a Cylinder standing on the same Base whose Altitude is to that of the former as z to $3a$; or by a Cone standing upon the same Base, whose Altitude is to that of the Cylinder as z to a: and after the like manner may you resolve the other Expression.

433. Whence it follows that if z become equal to a, the former Expressions become $\frac{cb^2}{2r} \times \frac{2}{3}a$, and $\frac{ca^2}{2r} \times \frac{2}{3}\frac{b^2}{a}$: which denote $\frac{2}{3}$ of a Cylinder having the same Base and Altitude with the Half of the Spheroid; the other a Cone having half the longer Axe for the Radius of it's Base, and the Latus Rectum for it's Altitude. When a becomes equal to b, so that the Spheroid become a Sphere, it is easy to apply the Expressions to that Case: and likewise when a becomes shorter than b, so that the Spheroid become oblate: also to the Cases of the hyperbolical and parabolical Conoeids: particularly you'll find the parabolical Conoeid to be $\frac{1}{2}$ of the Cylinder having the same Base and Altitude.

434. Ex. 2. Let it be required to find the solid Content of a Body formed by the Rotation of the Cissoid ACE (see Fig. Art. 321 *.) round it's Axis ACQ.

* Of this Explication.

Call AC $= z$, CE $= y$, then the Equation to the Curve $y^2 = \frac{z^3}{a-z}$: insert this Value of y^2 in place of it in the general Formula $\dot{s} = \frac{cy^2\dot{z}}{2r}$, and you have $\dot{s} = \frac{cz^3\dot{z}}{2r \times \overline{a-z}}$, which I find to belong to Form 1. Tab. 2. *viz.* the fourth Species, by which means therefore it may be found, by help of the equilateral Hyperbola; and consequently by Logarithms or Measures of Ratios: which I shall omit, and shew how to express it by applying directly to our Author's fifth Proposition. The Ordinate then of the Curve which flows according to the same Law with this Solid, is $\frac{cz^3}{2r \times \overline{a-z}}$, which being compared with the general Form for Binomials, *viz.* $y = az^{\theta-1} \times \overline{e + fz^{\eta}}|^{\lambda-1}$, will give by Art. 147 *, for the Area $\frac{c}{2r} \times \overline{\frac{z^4}{4a} + \frac{z^5}{5a^2} + \frac{z^6}{6a^3} + \frac{z^7}{7a^4} + \frac{z^8}{8a^5} + \&c.\ in\ inf.}$ which therefore denotes the Value of the Solid generated by the Rotation of the Cissoid.

* Of this Explication.

435. And now I have ſhewn what way this Treatiſe of the Quadrature of Curves is applied to the Solution of ſome Problems: by which the young Geometrician may ſee how extenſive this Doctrine is; and underſtand how it is to be applied in other Caſes. In general it will ſerve for finding the Fluent of any given Fluxion (of whatever Kind the Fluent be) where the given Fluxion can be reduced to ſuch an Expreſſion as denotes the Fluxion of any curvilinear Area. And I ſhall now proceed to explain the ſeveral Caſes of fluxional Equations, mentioned by our Author, from whence the Fluents can be found by the Quadrature of Curves.

436. The firſt Caſe is contained in Art. 75 *, and it is explained and illuſtrated thus. Let $ax - e^2\dot{z} = 2efx^2\dot{z} + f^2x^4\dot{z}$: where x is ſuppoſed to flow uniformly, and $\dot{z}$ is the firſt Fluxion of the other flowing Quantity, *viz.* z. I ſay the Relation of x and z may be found, ſo that x being given, z ſhall be given. In order to which, multiply the Term ax by $\dot{x} = 1$, according to Art. 335 *, by which the Equation ſtands thus $ax\dot{x} - e^2\dot{z} = 2efx^2\dot{z} + f^2x^4\dot{z}$: thence reduce the Equation ſo, as that $\dot{z}$ ſtand upon one Side alone, which gives this $\frac{ax\dot{x}}{e^2 + 2efx^2 + f^2x^4} = \dot{z}$: where if x be ſuppoſed the Abſciſs of a Curve, and $\frac{ax}{e^2 + 2efx^2 + f^2x^4}$ the correſponding Ordinate: and the Area of that Curve be found, it ſhall be equal to z; whence z ſhall be determined. Now, in the preſent Example, if you compare the Ordinate $\frac{ax}{e^2 + 2efx^2 + f^2x^4}$ with the Ordinate belonging to Order 2. Table 1. you'll obtain, by a proper Subſtitution of Values, $\frac{ax^2}{2ee + 2efxx} = z$; or $\frac{-a}{2ef + 2f^2x^2} = z$, either of which gives the Value of z, and expreſſes the Relation of x and z. Which you may try by Prob. 1, to ſee whether both theſe fluential Equations, will produce the given Relation of the Fluxions. The Relation of the Fluxions ariſing from the firſt, *viz.* $\frac{ax^2}{2e^2 + 2efx^2} = z$ is $\frac{2ax\dot{x} \times \overline{2e^2 + 2efx^2} - 4efx\dot{x} \times ax^2}{4e^4 + 8e^3fx^2 + 4e^2f^2x^4}$ *, or by reducing $\frac{ax\dot{x}}{e^2 + 2efx^2 + f^2x^4} = \dot{z}$, that ariſing from the ſecond Equation, *viz.* $\frac{-a}{2ef + 2f^2x^2} = z$, is $\frac{4af^2x\dot{x}}{4e^2f^2 + 8ef^3x^2 + 4f^4x^4}$ or $\frac{ax\dot{x}}{e^2 + 2efx^2 + f^2x^4} = \dot{z}$; both the ſame with the fluxional Equation at firſt propoſed, ſuppoſing $\dot{x} = 1$.

* Of the Quadrature of Curves.

* Of this Explication.

* Art. 84. of this Explication.

437. So likewise, if the Equation had been $ax - e^2\ddot{z} = 2efx^2\ddot{z} + f^2x^4\ddot{z}$: where you have the second Fluxion of z, and any other flowing Quantity x, whose Fluxion is equal to 1, you may find the Relation of the Fluents by the Quadrature of Curves. For supposing as before $x = AB$ the Absciss [see our Author's Figure] and now $\ddot{z}$ or $\frac{\ddot{z}}{\dot{x}} = BD$ the Ordinate, and consequently $\dot{z} = BE$ or $\frac{ABD}{1}$, and $z = BF$ or $\frac{ABE}{1}$: from the given Equation reduced, suppose, to this Form $\ddot{z} = \frac{ax}{e^2 + 2efx + x^2}$ or $\ddot{z} = \frac{ax\dot{x}^2}{e^2 + 2efx + x^2}$, find the Relation of $\dot{z}$ and x that is of BE and AB, by finding the Area of the Curve ABD, whose Absciss being x has it's Ordinate $BD = \frac{ax}{e^2 + 2efx + x^2} = \ddot{z}$: which is done as before. This gives $\dot{z}$, or the Relation of x and $\dot{z}$; the same as of x and z, in the preceding Case, *viz.* $\frac{ax^2}{2e^2 + 2efx^2} = \dot{z} = BE$. From which given Relation of AB and BE or x and $\dot{z}$, you find $z = BF$ or $\frac{ABE}{1}$, by finding the Area of the Curve ABE, whose Absciss is $AB = x$, and Ordinate $BE = (\dot{z} =) \frac{ax^2}{2e^2 + 2efx^2}$: which therefore exhibits the Relation of z and x sought. But whereas the Curve which has x for it's Absciss, and $\frac{ax^2}{2e^2 + 2efx^2}$ for it's Ordinate, cannot be squared *, therefore the Expression of the Area, found by Prop. 5, runs out into an infinite Series: so that the Relation of z and x cannot be expressed by any other than infinite Equation, *algebraically*. Yet it may be exhibited *geometrically* by the Description of a Conic Section, since the Curve comes under Species second, Order second, Table second. And thus you proceed in all like Cases, where the given Equation is of that Sort mentioned by our Author in this Article.

* Art. 139. of this Explication.

438. Hence it appears that if the Curve ADB be quadrable, the Relation of the Fluents may be expressed by a finite Equation, when it is a first Fluxion that enters into the given Equation. If the Curves ADB and AEB be quadrable, the Relation of the Fluents may be expressed by a finite Equation, when it is a second Fluxion that is contained in the given Equation. If the Curves ADB, AEB and AFB can be squared, then the same thing is true with respect to the Fluents, when it is a third Fluxion that enters the given Equation, *&c.* but if these Curves cannot be squared, the Relations of the Fluents cannot be exhibited by finite Equations, in the Cases mentioned. And you know when these Curves may be squared by Prop. 2, *viz.*

 then,

then, when, calling $AB = z$ and $BD = y$, the Curves having the Abſciſs z and Ordinates y, zy, z^2y, &c. can be ſquared: from which Propoſition you likewiſe diſcover what the Areas of the Curves ADB, AEB, AFB, *&c.* in that Caſe are. Again if theſe Curves ADB, AEB, AFB, *&c.* are capable of a geometrical Compariſon with the Conic Sections; then the Relations of the Fluents may be exhibited *geometrically*, by the Deſcription of the Conic Sections.

Art 76, 77 of the Quadrature of Curves. 439. What our Author ſays with reſpect to Equations that involve only one flowing Quantity and it's firſt Fluxion; or only any two next adjoining Orders of Fluxions of the ſame flowing Quantity, without the other flowing Quantity to which it is related, *viz.* that, in ſuch Caſes, the Fluents may be found by means of the Quadrature of Curves, may be further explained and illuſtrated, in the following manner.

Let the Equation $aa\dot{v} = av + v^2$ be propoſed: then ſuppoſe any other flowing Quantity of the ſame ſort with the Quantity v: call it z, and let it's Fluxion $\dot{z} = 1$. Wherefore, by completing the Dimenſions *, the given Equation is $aa\dot{v} = av\dot{z} + v^2\dot{z}$. Let v, $\dot{v}$ and z be repreſented by BE or $\frac{ABD}{1}$, BD and AB reſpectively. Now ſince the Equation $aa\dot{v} = av\dot{z} + v^2\dot{z}$ exhibits the Relation of the Fluxions $\dot{v}$ and $\dot{z}$, when $\dot{z} = 1$; the Relation of the Fluents will be the ſame, if you change the Suppoſition, and imagine that v flows uniformly, making $\dot{v} = 1$ *. And therefore, bringing the Equation to this Form $\frac{aa\dot{v}}{va + vv} = \dot{z}$, it appears, that, if you find the Area of a Curve whoſe Abſciſs is v, and Ordinate $\frac{aa}{av + vv}$, that Area is equal to z: ſo that hence you have the Relation of v and z, or BE and AB, as formerly: and therefore z being aſſumed, v is thence determined, either *algebraically* or *geometrically*.

*Art. 384. of this Explication.

*Art. 389. of this Explication.

440. Moreover, if the Equation $aa\ddot{v} = a\dot{v} + \dot{v}\dot{v}$ was propoſed: ſuppoſe z another variable Quantity, whoſe Fluxion $\dot{z} = 1$; then by the like Operation as in the preceding Article, ſuppoſing v, $\dot{v}$, $\ddot{v}$ and z to be expounded by BF, BE, BD and AB reſpectively; by ſquaring the Curve, whoſe Abſciſs is $\dot{v}$ and Ordinate $\frac{aa}{a\dot{v} + \dot{v}\dot{v}}$, you find the Area equal to z; ſo that you obtain the Relation betwixt $\dot{v}$ and z, or BE and AB. Again from the given Relation of $\dot{v}$ and z or BE and AB, you find by the Quadrature of Curves, the Area of the Curve AEB, whence you have the Relation of AB and BF, that is z and v, ſince $BF = \frac{AEB}{1}$. And the Method of proceeding is the

ſame

ſame, when the given Equation contains any two Fluxions of any the ſame flowing Quantity, of any two Orders next to each other, without including the other Fluent to which it is related.

441. Our Author ſhews in this Article how a fluxional Equation including three unknown Quantities may ſometimes be reduced to another including only two unknown Quantities: in which Caſe, the Relation of the Fluents may be found by the Quadrature of Curves as before. Thus let $a - bx^m = cxy^n\dot{y} + dy^{2n}\dot{y}\dot{y}$ be propoſed, including three unknown Quantities, *viz.* x, y and $\dot{y}$, and the Relation of the Fluents x and y is ſought. Put $y^n\dot{y} = \dot{v}$, then the Equation ſtands thus $a - bx^m = cx\dot{v} + d\dot{v}\dot{v}$, which includes only two unknown Quantities, *viz.* x and $\dot{v}$: from whence as in Art. 75*, you may find the Relation of x and v. For by reducing the Quadratic Equation, you get $\dot{v} = \sqrt{\frac{a}{d} + \frac{c^2x^2}{4d^2} + \frac{b}{d}x^m} - \frac{cx}{2d}$, or by multiplying by $\dot{x} = 1$, $\dot{v} = \dot{x} \times \sqrt{\frac{a}{d} + \frac{c^2x^2}{4d^2} - \frac{b}{d}x^m} - \frac{cx}{2d}$: whence it appears, that the Area of a Curve whoſe Abſciſs is x and Ordinate $\sqrt{\frac{a}{d} + \frac{c^2x^2}{4d^2} - \frac{b}{d}x^m} - \frac{cx}{2d} = \dot{v}$, is equal to v: whence, by the Quadrature of Curves, you obtain the Relation betwixt x and v. Again from the other Equation, *viz.* $y^n\dot{y} = \dot{v}$, we find the Relation of v and y by the Converſe of Prop. 1, which is $\frac{1}{n+1}y^{n+1} = v$: but the Relation of x and v was found formerly: therefore you have the Relation of x and y, ſo that y is found when x is given. Art. 78. of the Quadrature of Curves.

* Of the Quadrature of Curves.

442. Our Author having ſhewn in the preceding Article that the Fluents may be found from fluxional Equations including three unknown Quantities, when they are capable of being reduced to ſuch as involve only two unknown Quantities, proceeds, in this Article, to ſhew that ſometimes the Fluents may be obtained from ſuch Equations, even when they cannot be reduced to Equations that have but two unknown Quantities. Thus let the Equation $\overline{ax^m + bx^n}|^p = rex^{r-1}y^s + sex^r\dot{y}y^{r-1} - f\dot{y}y^t$ including the Fluents x and y together with the Fluxion of y, be propoſed: and y is ſought for x given. Multiply the Terms where $\dot{y}$ is not found by $\dot{x} = 1$, and you have $\dot{x} \times \overline{ax^m + bx^n}|^p = re\dot{x}x^{r-1}y^s + sex^r\dot{y}y^{r-1} - f\dot{y}y^t$: of which Equation, the Fluent of the laſt Part, *viz.* $re\dot{x}x^{r-1}y^s + sex^r\dot{y}y^{s-1} - f\dot{y}y^t$, may be had by going back from the Fluxion to the Fluent, by a reverſe Art. 79. of the Quadrature of Curves.

verse Operation to that by which the Fluxion of a given Fluent is found, *viz.* by increasing the Indexes of the flowing Quantities, in the Terms where their Fluxions are, by Unity, and dividing the Terms by the Indexes so increased, and dashing out the fluxionary Expressions: after which, if there result any Quantities that are equal, dash all of them out but one: and the Terms remaining make the Fluent (if it can be found by the Converse of Prop. 1.) And to ascertain the Truth, try whether the Quantity so found produce the same Fluxion as that at first: which if it do, you have the right Fluent, otherwise not. By proceeding thus, you find from the last Side of the Equation, $ex^r y^s + ex^r y^s - \frac{1}{t+1} y^{t+1}$: where the Term $ex^r y^s$ being twice found, dash out one of them, and so you'll have $ex^r y^s - \frac{1}{t+1} y^{t+1}$ for the true Fluent of that Side of the Equation. And the other Side of the Equation, *viz.* $\dot{x} \times \overline{ax^m + bx^n}|^p$ expresses a Fluxion analogous to that of a Curve whose Absciss is x, and Ordinate $\overline{ax^m + bx^n}|^p$: wherefore, if you find the Area of that Curve, the Expression of the Value of that Area will be equal to $ex^r y^s - \frac{1}{t+1} y^{t+1}$: and thence therefore you obtain the Relation of the Fluents x and y, so that y is given from x given.

Art. 80. of the Quadrature of Curves.

443. Suppose you have such an Equation as this $\dot{x} \times \overline{ax^m + bx^n}|^p = \frac{d\dot{y}y^{\eta-1}}{\sqrt{e+fy^\eta}}$: which involves both the Fluents x and y, and their Fluxions (where, if one of the Fluxions had been wanting it had been the same thing, because it behov'd to have been supplied). Here, the two Sides of the Equation may be considered as the Fluxions of Curves, whereof the one has x for it's Absciss, and $\overline{ax^m + bx^n}|^p$ for it's Ordinate; and the other y for it's Absciss, and $\frac{dy^{\eta-1}}{\sqrt{e+fy^\eta}}$ for it's Ordinate: for these two Areas would be equal, by Prop. 9. Wherefore, if these Areas be found, and equated, you'll thence obtain the Relation of x and y. Now the Area of the Curve whose Ordinate is $\frac{dy^{\eta-1}}{\sqrt{e+fy^\eta}}$ by Species 1. Order 4. Table 1, is $\frac{2d}{\eta f}\sqrt{e+fy^\eta}$: therefore if you put this equal to the Area of the Curve whose Absciss is x and Ordinate $\overline{ax^m + bx^n}|^p$ (which may be found by the Quadrature of Curves) thence the

the Fluent y will be known, when x is given. Thus ſuppoſe $m = 2$, $n = 4$, $p = \frac{1}{2}$: then the Ordinate of the Curve having x for it's Abſciſs, is $\sqrt{ax^2 + bx^4} = x\sqrt{a + bx^2}$, which belongs to the Curves of the firſt Species of Form third, Table 1: according to which, it's Area would be $\frac{a + bx^2}{3b}\sqrt{a + bx^2}$. Wherefore $\frac{a + bx^2}{3b}\sqrt{a + bx^2} = \frac{2d}{nf}\sqrt{e + fy^n}$ gives the Relation of the Fluents ſought.

444. And thus our Author has ſhewn in what Caſes of fluxional Equations, the Fluents may be found, by means of the Quadrature of Curves: in all which, there are always the Fluxions of two flowing Quantities either expreſſed or underſtood. One of theſe flowing Quantities, which our Author in this Schol. calls x or z is ſuppoſed to flow after an equable and uniform manner, whoſe Fluxion is deſigned by Unity: the other flowing Quantity, he calls y or v, which is ſuppos'd to flow according to any Law whatſoever. The firſt of theſe Quantities he conſiders as a Standard, by which the other is to be meaſured or eſtimated; even as the Abſciſs is with reſpect to the Ordinate and Area of a Curve. And as in this Caſe, we reckon the Ordinate and Area known, when their Relation to the Abſciſs is expreſſed by any algebraical Equation: ſo in the Caſe of the Fluents x and y, y is conſidered as the Fluent ſought, and therefore when it's Relation to x is given, it is ſaid to be given. See Art. 75. and Art. 78, 79 and 80 * at the End. But now, after the Fluent y is thus found by the Quadrature of Curves; it muſt be conſidered that this is what we formerly called the *pure* Fluent, ſince it has no Part but what flows: and therefore the Fluent ſo found, may be increaſed or diminiſhed by any *conſtant* Quantity, and yet ſtill be the Fluent, *i. e.* a flowing Quantity, which has the ſame Fluxion as formerly, ſince the conſtant Quantity added to, or ſubtracted from the pure Fluent, having itſelf no Fluxion, can make no change in the Fluxion of the former Fluent. And thus you ſee the Reaſon of what our Author advances upon this Head in Art. 81 *. And therefore, as was obſerved before †, when the Fluents are found from the Fluxions, theſe Fluents, in the Application to any particular Caſe, muſt be ſo ordered and limited, as the Nature and Circumſtances of the Caſe or Queſtion do neceſſarily require. It would be the ſame thing to ſay, that, after an Equation is found expreſſing the Relation of the Fluents, all the Parts of which are flowing, you may add to either or both the Sides of it, what conſtant Quantities you pleaſe, and it ſhall yet be an Equation exhibiting the

Art. 81. of the Quadrature of Curves.

* Of the Quadrature of Curves.

* Of the Quadrature of Curves.

† Art. 338, 339. of this Explication.

the Relation of the Fluents: for it produces the ſame Relation of Fluxions as before.

Art. 82. of the Quadrature of Curves. 445. Our Author tells us that where there ariſes any Doubt about the Truth of the Concluſion, in finding the Fluents from the Fluxions, you may eaſily aſcertain it, by collecting the Fluxions of the Fluents ſo found, by Prop. 1: for if they be the ſame with the Fluxions that were at firſt propoſed, the Operation is juſt, and the Fluents found are true Fluents, otherwiſe not. In which laſt Caſe they muſt be corrected in ſuch manner, as to make their Fluxions agree with the Fluxions at firſt propoſed. For, ſays he, a Fluent may be aſſumed at pleaſure, and the Aſſumption corrected, by putting the Fluxion of the aſſumed Fluent equal to the propos'd Fluxion, and comparing the *homologous* Terms with one another. But after what manner the Fluent of any Fluxion is to be found by ſuch an Aſſumption of the Fluent, and Compariſon of the homologous Terms of the Fluxions: as likewiſe how this general Problem, including the whole of the *inverſe* Method of Fluxions, *viz. An Equation being given involving any Number of Fluxions, to find the Fluents,* i. e. an Equation defining their Relation, which is the Converſe of the firſt Propoſition of this Treatiſe: I ſay, how theſe things are to be performed, as it don't properly belong to the Doctrine of Quadratures, I have no Deſign nor Inclination to ſhew in this Place: ſince it would open too large a Field, and carry us beyond the original Deſign of this Work, which was to explain and vindicate the firſt Principles upon which the Doctrine of Fluxions is founded, and to explain and illuſtrate this Treatiſe of the Quadrature of Curves.

446. Our celebrated Author crowns all with a Saying, that expreſſes the *Greatneſs* of his Modeſty, and *Vaſtneſs* of his Underſtanding. What Man but Sir *Iſaac Newton* himſelf, could call the *noble* and *ſublime* Diſcoveries contained in this ſhort Treatiſe, by the Name of *Principles* or *Beginnings* of Knowledge, the clear apprehending and full underſtanding of which, after they are diſcovered and laid before us, requires *Pains* and *Application* to Underſtandings of the *common* Size. Notwithſtanding, it is true, what he affirms, that theſe pave the Way to innumerable ſtill more valuable and ſublime Diſcoveries in Mathematics and *Natural Philoſophy*, as he himſelf hath partly ſhewn in his other Performances, and eſpecially in that *invaluable* and *inimitable* one his mathematical Principles of natural Philoſophy.

The End of the Quadrature of Curves.

OF

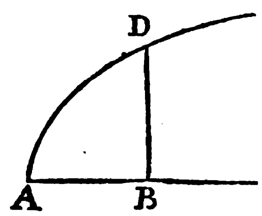
ANALYSIS

BY

Equations of an infinite Number of Terms.

1. *THE General Method, which I had devised some considerable Time ago, for measuring the Quantity of Curves, by Means of Series, infinite in the Number of Terms, is rather shortly explained, than accurately demonstrated in what follows.*

2. Let the Base AB of any Curve AD have BD for it's perpendicular Ordinate; and call AB=x, BD=y, and let a, b, c, &c. be given Quantities, and m and n whole Numbers. Then



The Quadrature of Simple Curves,

RULE I.

3. If $ax^{\frac{m}{n}}=y$; it shall be $\frac{an}{m+n}x^{\frac{m+n}{n}}$ = Area ABD.

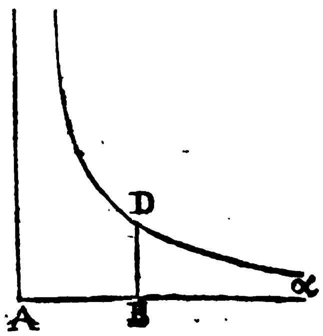
The thing will be evident by an Example.

1. If x^2 $(=1x^{\frac{2}{1}})=y$, that is $a=1=n$, and $m=2$; it shall be $\frac{1}{3}x^3$ = ABD.

2. Suppose $4\sqrt{x}$ $(=4x^{\frac{1}{2}})=y$; it will be $\frac{8}{3}x^{\frac{3}{2}}$ $(=\frac{8}{3}\sqrt{x^3})=$ ABD.

3. If $\sqrt[3]{x^5}$ $(=x^{\frac{5}{3}})=y$; it will be $\frac{3}{8}x^{\frac{8}{3}}$ $(=\frac{3}{8}\sqrt[3]{x^8})=$ ABD.

4. If $\frac{1}{x^2}=x^{-2})=y$, that is if $a=1=n$, and $m=-2$;



It will be $\frac{1}{-1}x^{\frac{-1}{1}}=)-x^{-1}$ $(=\frac{-1}{x})=\alpha$BD, infinitely extended towards α, which the Calculation places negative, because it lyes upon the other side of the Line BD.*

5. If $\frac{1}{\sqrt{x^3}}$ $(x^{-\frac{3}{2}})=y$; it will be $(\frac{2}{-1}x^{-\frac{1}{2}}=)$ $\frac{2}{-\sqrt{x}}=$ BDα.

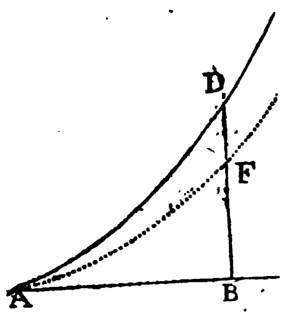
6. If $\frac{1}{x}$ $(=x^{-1})=y$; it will be $\frac{1}{0}x^{\frac{0}{1}}=\frac{1}{0}x^0=\frac{1}{0}\times 1=\frac{1}{0}=$ an infinite Quantity; such as is the Area of the Hyperbola upon both Sides of the Line BD.

The Quadrature of Curves compounded of simple ones.

RULE II.

4. If the Value of y be made up of several such Terms, the Area likewise shall be made up of the Areas which result from every one of the Terms.

The first Examples.



5. If it be $x^2+x^{\frac{3}{2}}=y$; it will be $\frac{1}{3}x^3+\frac{2}{5}x^{\frac{5}{2}}=$ ABD.

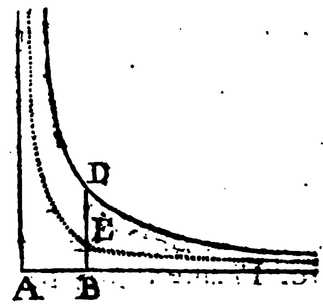
For if it be always $x^2=$ BF and $x^{\frac{3}{2}}=$ FD, you will have by the preceding Rule $\frac{1}{3}x^3=$ Superficies AFB; described by the Line BF; and $\frac{2}{5}x^{\frac{5}{2}}=$ AFD described by DF; wherefore $\frac{1}{3}x^3+\frac{2}{5}x^{\frac{5}{2}}=$ the whole Area ABD.

Thus if it be $x^2-x^{\frac{3}{2}}=y$; it will be $\frac{1}{3}x^3-\frac{2}{5}x^{\frac{5}{2}}=$ ABD. And if it be $3x-2x^2+x^3-5x^4=y$; it will be $\frac{3}{2}x^2-\frac{2}{3}x^3+\frac{1}{4}x^4-x^5=$ ABD.

* Whatever is laid down by our Author with respect to the Position of Areas of Curves in this and the following Rules, is explained at full length in Sect. 5 of the preceding Treatise. Which see.

The

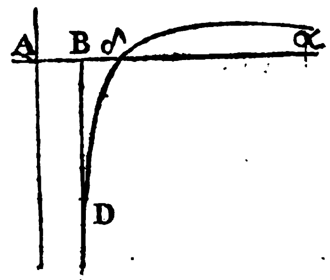
The second Examples.



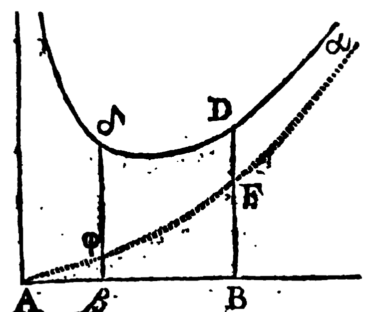
6. If $x^{-2} + x^{-\frac{3}{2}} = y$; it will be $- x^{-1} - 2x^{-\frac{1}{2}} = \alpha$BD. Or if it be $x^{-2} - x^{-\frac{3}{2}} = y$; it will be $- x^{-1} + 2x^{-\frac{1}{2}} = \alpha$BD.

And if you change the Signs of the Quantities, you will have the affirmative Value ($x^{-1} + 2x^{-\frac{1}{2}}$, or $x^{-1} - 2x^{-\frac{1}{2}}$) of the Superficies αBD, provided the whole of it fall above the Base ABα.

7. But if any Part fall below (which happens when the Curve decussates or crosses it's Base betwixt B and α, as you see here in δ) you are to subtract that Part from the Part above the Base; and so you shall have the Value of the Difference: but if you would have their Sum, seek both the Superficies's separately, and add them. And the same thing I would have observed in the other Examples belonging to this Rule.



The third Examples.



8. If $x^2 + x^{-2} = y$; it will be $\frac{1}{3}x^3 - x^{-1} =$ the Superficies described.

But here it must be remarked that the Parts of the said Superficies so found, lye upon opposite Sides of the Line BD.

That is, putting $x^2 =$ BF, and $x^{-2} =$ FD; it shall be $\frac{1}{3}x^3 =$ ABF the Superficies described by BF, and $- x^{-1} =$ DFα the Superficies described by DF.

9. And this always happens when the Indexes $\left(\frac{m+n}{n}\right)$ of the Ratios of the Base x in the Value of the Superficies sought, are affected with different Signs. In such Cases any middle part BD$\delta\beta$ of the Superficies (which only can be given, when the Superficies is infinite upon both Sides) is thus found.

Subtract the Superficies belonging to the lesser Base Aβ from the Superficies belonging to the greater Base AB, and you shall have βBDδ the Superficies insisting upon the difference of the Bases. Thus in this Example (see the preceding Fig.)

If AB $=2$, and A$\beta=1$; it will be βBD$\delta=\frac{17}{6}$:

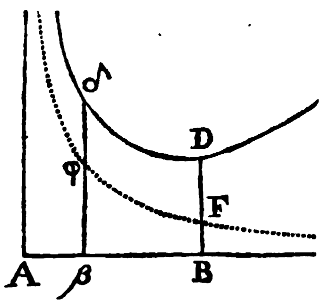
For the Superficies belonging to AB (*viz.* ABF — DFα) will be $\frac{8}{3}-\frac{1}{2}$ or $\frac{13}{6}$; and the Superficies belonging to Aβ (*viz.* A$\phi\beta$ — $\delta\phi\alpha$) will be $\frac{1}{3}-1$, or $-\frac{2}{3}$: and their Difference (*viz.* ABF — DFα — A$\phi\beta$ + $\delta\phi\alpha$ — βBDδ) will be $\frac{13}{6}+\frac{2}{3}$ or $\frac{17}{6}$.

After the same manner, if A$\beta=1$, and AB $=x$; it will be βBDδ $=\frac{2}{3}+\frac{1}{3}x^3-x^{-1}$.

Thus if $2x^3-3x^5-\frac{2}{3}x^{-4}+x^{-\frac{3}{5}}=y$, and A$\beta=1$;

It will be βBD$\delta=\frac{1}{2}x^4-\frac{1}{2}x^6+\frac{2}{9}x^{-3}+\frac{5}{2}x^{\frac{2}{5}}-\frac{49}{18}$.

10. Finally it may be observed, that if the Quantity x^{-1} be found in the Value of y, that Term (since it generates an hyperbolical Surface) is to be considered apart from the rest.



As if it were $x^2+x^{-3}+x^{-1}=y$: let it be $x^{-1}=$ BF, and $x^2+x^{-3}=$ FD; and A$\beta=1$; and it will be $\delta\phi$FD $=\frac{1}{6}+\frac{1}{3}x^3-\frac{1}{2}x^{-2}$, as being that which is generated by the Terms x^2+x^{-3}.

Whererefore if the remaining Superficies $\beta\phi$FB, which is hyperbolical, be given by any Method of Computation, the whole βBDd will be given.

The Quadrature of all other Curves.

RULE III.

11. But if the Value of y, or any of it's Terms be more compounded than the foregoing, it must be reduced into more simple Terms; by performing the Operation in Letters, after the same Manner as Arithmeticians divide in Decimal Numbers, extract the Square Root, or resolve affected Equations; and afterwards by the preceding Rules you will discover the Superficies of the Curve sought.

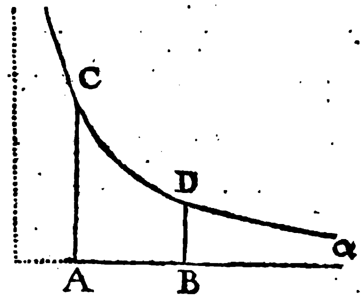
Examples, where you divide.

12. Let $\frac{aa}{b+x}=y$; *Viz.* where the Curve is an Hyperbola.

Now that that Equation may be freed from it's Denominator, I make the Division thus.

$b+x)$

$$
\begin{array}{l}
b+x)\,aa+0\left(\frac{aa}{b}-\frac{aax}{b^2}+\frac{aax^2}{b^3}-\frac{aax^3}{b^4}\ \&c.\right.\\
\underline{aa+\frac{aax}{b}}\\
\quad 0-\frac{aax}{b}+0\\
\quad \underline{-\frac{aax}{b}-\frac{aax^2}{b^2}}\\
\qquad 0+\frac{aax^2}{b^2}+0\\
\qquad \underline{+\frac{aax^2}{b^2}+\frac{aax^3}{b^3}}\\
\qquad\quad 0-\frac{aax^3}{b^3}+0\\
\qquad\quad \underline{-\frac{aax^3}{b^3}-\frac{aax^4}{b^4}}\\
\qquad\qquad 0+\frac{aax^4}{b^4}\\
\qquad\qquad \&c.
\end{array}
$$



And thus in Place of this $y=\frac{aa}{b+x}$, a new Equation arises, *viz.* $y=\frac{a^2}{b}-\frac{a^2x}{b^2}+\frac{a^2x^2}{b^3}-\frac{a^2x^3}{b^4}$ &c. this Series being continued infinitely; and therefore (by the second Rule)

The Area sought ABDC will be equal to $\frac{a^2x}{b}-\frac{a^2x^2}{2b^2}+\frac{a^2x^3}{3b^3}-\frac{a^2x^4}{4b^4}$ &c. an infinite Series likewise, but yet such, that a few of the initial Terms are exact enough for any Use, provided that b be equal to x repeated some few times.

13. After the same Manner if it be $\frac{1}{1+xx}=y$, by dividing there arises

$y=1-xx+x^4-x^6+x^8$ &c. Whence (by the second Rule)

You will have $ABDC=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\frac{1}{9}x^9$ &c.

Or if x^2 be made the first Term in the Divisor, *viz.* thus: $x^2+1)$ there will arise $x^{-2}-x^{-4}+x^{-6}-x^{-8}$ &c. for the Value of y; whence (by the second Rule)

It will be $BD\alpha=-x^{-1}+\frac{1}{3}x^{-3}-\frac{1}{5}x^{-5}+\frac{1}{7}x^{-7}$ &c. You must proceed in the first Way when x is small enough, but the second Way, when it is supposed great enough.

14. Finally, if it be $\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1 + x^{\frac{1}{2}} - 3x} = y$; by dividing there arises $2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^2 + 34x^{\frac{5}{2}}$ &c. whence it will be ABDC $= \frac{4}{3}x^{\frac{3}{2}} - x^2 + \frac{14}{5}x^{\frac{5}{2}} - \frac{13}{3}x^3$ &c.

Examples, where the Square Root must be extracted.

15. If it be $\sqrt{aa + xx} = y$, I extract the Root thus:

$$aa + xx \left(a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} \text{ \&c.} \right.$$

$$aa$$

$$0 + x^2$$

$$x^2 + \frac{x^4}{4a^2}$$

$$0 - \frac{x^4}{4a^2}$$

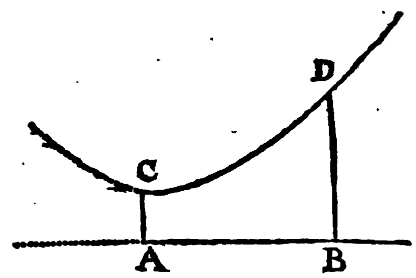
$$-\frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6}$$

$$0 + \frac{x^6}{8a^4} - \frac{x^8}{64a^6}$$

$$+\frac{x^6}{8a^4} + \frac{x^8}{16a^6} - \frac{x^{10}}{64a^8} + \frac{x^{12}}{256a^{10}}$$

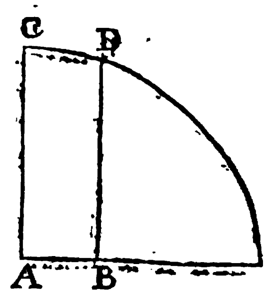
$$0 - \frac{5x^8}{64a^6} + \frac{x^{10}}{64a^8} - \frac{x^{12}}{256a^{10}}$$

&c.



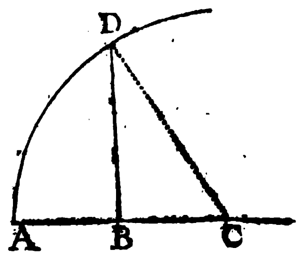
Whence for the Equation $\sqrt{aa + xx} = y$, a new one is produced, viz. $y = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$ &c. And (by the second Rule) You will have the Area sought ABDC $= ax + \frac{x^3}{6a} - \frac{x^5}{40a^3} + \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7}$ &c.

And this is the Quadrature of the Hyperbola.



16. After the same Manner if it be $\sqrt{aa - xx} = y$, it's Root will be $a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$ &c. and therefore the Area sought ABDC will be equal to $ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7}$ &c. And this is the Quadrature of the Circle.

17.



17. Or if you suppose $\sqrt{x - xx} = y$, the Root will be equal to this infinite Series $x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}} - \frac{5}{128}x^{\frac{9}{2}}$ &c.

And the Area sought ABD will be $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}} - \frac{5}{704}x^{\frac{11}{2}}$ &c.

Or $x^{\frac{1}{2}} \times \overline{\frac{2}{3}x - \frac{1}{5}x^2 - \frac{1}{28}x^3 - \frac{1}{72}x^4 - \frac{5}{704}x^5}$ &c.

And this is the Quadrature of the Circle's Area.

18. If $\frac{\sqrt{1 + ax^2}}{\sqrt{1 - bx^2}} = y$ (whose Quadrature gives the Length of the Curve of the Ellipse [1]) by extracting both Roots, there arises

$$\frac{1 + \frac{1}{2}ax^2 - \frac{1}{8}a^2x^4 + \frac{1}{16}a^3x^6 - \frac{5}{128}a^4x^8}{1 - \frac{1}{2}bx^2 - \frac{1}{8}b^2x^4 - \frac{1}{16}b^3x^6 - \frac{5}{128}b^4x^8} \text{ \&c.}$$

And by dividing as is done in the Case of Decimal Fractions, you will have

$$\begin{array}{lllll} 1 & + \frac{1}{2}b\,x^2 & + \frac{3}{8}b^2\,x^4 & + \frac{5}{16}b^3\,x^6 & + \frac{35}{128}b^4\,x^8 \text{ \&c.} \\ & + \frac{1}{2}a & + \frac{1}{4}ab & + \frac{3}{16}ab^2 & + \frac{5}{32}ab^3 \\ & & - \frac{1}{8}a^2 & - \frac{1}{16}a^2b & - \frac{3}{64}a^2b^2 \\ & & & + \frac{1}{16}a^3 & + \frac{1}{32}a^3b \\ & & & & - \frac{5}{128}a^4 \end{array}$$

And therefore the Area required

$$\begin{array}{llll} x & + \frac{1}{6}b\,x^3 & + \frac{3}{40}b^2\,x^5 \text{ \&c.} \\ & + \frac{1}{6}a & + \frac{1}{20}ab \\ & & - \frac{1}{40}a^2 \end{array}$$

19. But it is to be observed that the Operation may be often abbreviated by a due Preparation of the Equation, as in the Example just now adduced $\frac{\sqrt{1 + ax^2}}{\sqrt{1 - bx^2}} = y$. If you multiply both Parts of the Fraction by $\sqrt{1 - bx^2}$, there will arise $\frac{\sqrt{1 \begin{smallmatrix} +a \\ -b \end{smallmatrix} x^2 - abx^4}}{1 - bx^2} = y$; and the rest of the work is performed by extracting the Root of the Numerator only, and then dividing by the Denominator.

20. From these Examples, I suppose, it will be sufficiently evident after what Manner any Value of y may be reduced (with whatever Roots or Denominators it may be involved, as you may see here $x^3 + \frac{\sqrt{x - \sqrt{1 - xx}}}{\sqrt[3]{axx + x^3}} - \frac{\sqrt[5]{x^3 + 2x^5 - x^{\frac{3}{2}}}}{\sqrt[3]{x + x^2} - \sqrt{2x - x^{\frac{2}{3}}}} = y$) into infinite Series of simple Terms, from which, by the second Rule, the Superficies required may be known.

[1] For the Explication of this see Art. 341—343 of the preceding Treatise.

Examples

Examples by the Resolution of EQUATIONS.

The Numeral Resolution of Affected Equations.

21. Because the whole Difficulty lyes in the Resolution, I shall first illustrate the Method I use in a numeral Equation.

Let the Equation $y^3 - 2y - 5 = 0$ be proposed to be resolved: and let 2 be a number which differs from the Root sought, by less than a tenth Part of itself. Then I put $2 + p = y$, and I substitute this Value in Place of it in the Equation; and thence a new Equation arises, *viz.* $p^3 + 6p^2 + 10p - 1 = 0$, whose Root p is to be sought for, that it may be added to the Quotient: *viz.* thus (neglecting $p^3 + 6p^2$ upon the Account of their smallness) $10p - 1 = 0$, or $p = 0,1$ is near the Truth; therefore I write 0,1 in the Quotient, and then suppose $0,1 + q = p$, and this it's value I substitute, as formerly, whence results $q^3 + 6,3q^2 + 11,23q + 0,061 = 0$.

22. And since $11,23q + 0,061 = 0$ comes near to the Truth, or since q is almost equal to $-0,0054$ (*viz.* by dividing until as many Figures arise as there are places betwixt the first Figures of this and the principal Quotient) I write $-0,0054$ in the lower Part of the Quotient, since it is negative.

23. And then supposing $-0,0054 + r = q$, I substitute this as formerly, and thus the Operation is continued as far as you please. But if I desire to continue the Work only to twice as many Figures as there are in the Quotient except one, instead of q I substitute $-0,0054 + r$ into this $6,3q^3 + 11,23q + 0,061$, *viz.* neglecting it's first Term (q^3) upon the Account of it's Smallness, and there arises $6,3r^2 + 11,16196r + 0,000541708 = 0$ almost, or (rejecting $6,3r^2$) $r = \frac{-0,000541708}{11,16196} = -0,00004853$ almost, which I write in the negative Part of the Quotient. Finally, subducting the negative Part of the Quotient from the affirmative, I have 2,09455147 the Quotient sought.

$y^3 -$

$y^3-2y-5=0$		$+2,10000000$ $-0,00544853$ $+2,09455147=y$
$2+p=y$	$+y^3$ $+2y$ -5	$+8+12p+6p^2+p^3$ $-4-2p$ -5
	Sum	$-1+10p+6p^2+p^3$
$0,1+q=p$	$+p^3$ $+6p^2$ $+10p$ -1	$+0,001+0,03q+0,3q^2+q^3$ $+0,06+1,2+6,0$ $+1,+10$ $-1,$
	Sum	$+0,061+11,23q+6,3q^2+q^3$
$-0,0054+r=q$	$+6,3q^2$ $+11,23q$ $+0,061$	$+0,000183708-0,06804r+6,3r^2$ $-0,060642+11,23$ $+0,061$
	Sum	$+0,000541708+11,16196r+6,3r^2$
$-0,00004854+s=r$		

24. Equations of more Dimensions are resolved the same way, and you'll save a good deal of Trouble, towards the End, as was done here, if you omit the first Terms gradually.

25. Moreover it is to be observed that in this Example, if I had any doubt whether $0,1=p$ approached near enough to the Truth, instead of $10p-1=0$, I had feigned $6p^2+10p-1=0$, and writ the first Figure of it's Root in the Quotient; and indeed it's convenient to search out the second and third Figure of the Quotient after this Manner, when in the last resulting Equation, the Square of the Coefficient of the penult Term is not ten times greater than the Product of the last Term and Coefficient of the antepenult Term.

26. Yea you will for the most part save Labour, especially in Equations of many Dimensions, if you find out all the Figures to be adjoined to the Quotient by the same Means, *i. e.* by extracting the lesser of the Roots out of the three last Terms of the last resulting Equation: For in that way you will gain double as many Figures, at each Time, to be annexed to the Quotient.

27. Whether this Method of resolving Equations be vulgarly practised I cannot tell, but surely to me it appears simple in Comparison of others, and more accommodated to Practice. The Demonstration of it appears from the very Manner of Operation, whence, as Occasion requires, you may easily call it into your Mind.

28. Equations wanting any of the Terms, or not, are managed almost with the same Ease: And an Equation is always left, whose Root, together with the Quotient already found, is equal to the Root of the Equation at first proposed. Whence you may examine or prove the Work here as well as in other Parts of Arithmetic, *viz.* by taking away the Quotient from the Root of the first Equation (as is known to Analysts) that so the last Equation; or two or three of it's last Terms may be thence produced. Any Labour there is in this Affair will be found to arise from the substituting the one Kind of Quantities for the other: Which you may do different Ways, but I think the following the most expeditious, especially when the numeral Coefficients consist of many Figures.

29. Let $p+3$ be to be substituted for y in this Equation $y^4-4y^3+5y^2-12y+17=0$: And since it may be resolved into this Form, $\overline{\overline{\overline{y-4}\times y+5}\times y-12}\times y+17=0$, the new Equation will be generated thus; $\overline{p-1}\times\overline{p+3}=p^2+2p-3$, and $\overline{p^2+2p+2}\times\overline{p+3}=p^3+5p^2+8p+6$, and $\overline{p^3+5p^2+8p-6}\times\overline{p+3}=p^4+8p^3+23p^2+18p-18$, and $p^4+8p^3+23p^2+18p-1=0$, which was sought.

The literal Resolution of affected Equations.

30. These things being thus shewn in Numbers: Let $y^3+a^2y-2a^3+axy-x^3=0$ be proposed to be resolved.

I first search for the Value of y when x is nothing, that is, I seek the Root of this Equation $y^3+a^2y-2a^3=0$, and I find it to be $+a$. Therefore I write $+a$ in the Quotient; and supposing $+a+p=y$, I substitute for y it's Value, and put the Terms thence resulting ($p^3+3ap^2+4a^2p$ &c.) in the Margin; out of which I assume $+4a^2p+a^2x$, *viz.* the Terms where p and x separately are of the least Dimensions, and I suppose them equal almost to nothing, or $p=-\frac{1}{4}x$ almost, or $p=-\frac{1}{4}x+q$. And writing $-\frac{1}{4}x$ in the Quotient, I substitute $-\frac{1}{4}x+q$ for p; and the Terms thence resulting I write again in the Margin, as you may see in the Scheme annexed; and

and thence I assume the Quantities $4a^2e - \frac{1}{16}ax^2$, *viz.* those in which q and x separately are of the least Dimensions, and I feign $q = \frac{x^2}{64a}$ almost, or $q = \frac{x^2}{64a} + r$; and subjoining $+\frac{x^2}{64a}$ to the Quotient, I substitute $\frac{xx}{64a}$ for q; and thus I proceed as far as you please.

$$y^3 + a^2y - 2a^3 + axy - x^3 = 0$$
$$y = a - \frac{x}{4} + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3} \text{ \&c.}$$

$+a+p=y$	$+y^3$ $+a^2y$ $+axy$ $-2a^3$ $-x^3$	$+a^3 + 3a^2p + 3ap^2 + p^3$ $+a^3 + a^2p$ $+a^2x + axp$ $-2a^3$ $-x^3$
$-\frac{1}{4}x+q=p$	$+p^3$ $+3ap^2$ $+4a^2p$ $+axp$ $+a^2x$ $-x^3$	$-\frac{1}{64}x^3 + \frac{3}{16}x^2q - \frac{3}{4}xq^2 + q^3$ $+\frac{3}{16}ax^2 - \frac{3}{2}axq + 3aq^2$ $-a^2x + 4a^2q$ $-\frac{1}{4}ax^2 + axq$ $+a^4x$ $-x^3$
$+\frac{x^2}{64a}+r=q$	$+3aq^2$ $+4a^2q$ $-\frac{1}{2}axq$ $+\frac{3}{16}x^2q$ $-\frac{1}{16}ax^2$ $-\frac{65}{64}x^3$	$+\frac{3x^4}{4096a} + \frac{3}{32}x^2r + 3ar^2$ $+\frac{1}{16}ax^2 + 4a^2r$ $-\frac{1}{128}x^3 - \frac{1}{2}dxr$ $+\frac{3x^4}{1024a} + \frac{3}{16}x^2r$ $-\frac{2}{16}ax^2$ $-\frac{65}{64}x^3$
$+4a^2 - \frac{1}{2}ax + \frac{9}{32}x^2) + \frac{131}{128}x^3 - \frac{15x^4}{4096a} . (+ \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$		

31. But if I desire only double as many Terms save one to be further adjoined to the Quotient: I omit the first Term (q^3) of the last resulting Equation, and likewise that Part ($-\frac{3}{4}xq^2$) of the second Term, where x is of as many Dimensions as in the penult Term of the Quotient; and substitute $\frac{x^2}{64a} + r$ for q into the other Terms ($3aq^2 + 4a^2q$ &c.) placed in the Margin as you see; and then from the last two Terms

 $(\frac{15x^4}{4096a}$

$\left(\frac{15x^4}{4096a} - \frac{131}{128}x^3 + \frac{9}{32}x^2r - \frac{1}{2}axr + 4a^2r\right)$ of the Equation thence resulting, by dividing thus:

$4a^2 - \frac{1}{2}ax + \frac{9}{32}x^2) + \frac{131}{128}x^3 - \frac{15x^4}{4096a}$ (I bring out $+\frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$ to be adjoined to the Quotient.

32. Finally that Quotient $(a - \frac{x}{4} + \frac{x^2}{64a}$ &c.) by the second Rule, will give $ax - \frac{x^2}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048a^2} + \frac{509x^5}{81920a^3}$ &c. for the Area sought, which approaches so much nearer to the Truth, the less that x is.

Another Way of resolving the same Equations.

33. But if the Value of the Area ought to approach the nearer to the Truth the greater that x is; of which let this be an Example $y^3 + axy + x^2y - a^3 - 2x^3 = 0$. Therefore, being about to resolve this, I chuse out the Terms $y^3 + x^2y - 2x^3$, in which x and y either separately, or multiplied together are of the most and at the same Time of equal Dimensions every where; and from them as if equal to nothing I discover the Root. This I find to be x; which I write in the Quotient. Or, which comes to the same Purpose, I extract the Root out of $y^3 + y - 2 = 0$ (substituting Unity for x) which Root here comes out 1, and I multiply it by x, and write the Product (x) in the Quotient. Finally I put $x + p = y$, and so I proceed as in the former Example until I obtain the Quotient $x - \frac{a}{4} + \frac{a^2}{64x} + \frac{131a^3}{512x^2} + \frac{509a^4}{16384x^3}$ &c. and so the Area sought $\frac{x^2}{2} - \frac{ax}{4} + \boxed{\frac{a^2}{64x}}$[1] $- \frac{131a^3}{512x} - \frac{509a^4}{32768x^2}$; concerning which see the third Examples of the second Rule. For the sake of illustration I have given this Example the same in every Respect with the former, provided x and a be there substituted for one another, that there might not be any Necessity to adjoin here any other Example of this Resolution.

34. But the Area $\frac{x^2}{2} - \frac{ax}{4} + \boxed{\frac{a^2}{64x}}$ &c. is terminated at a Curve which extends itself infinitely along some Assymptote; and the initial

[1] Sir *Isaac* uses this Notation to denote the curvilinear Area belonging to the Ordinate $\frac{a^2}{64x}$, which belongs to the Hyperbola. For understanding of which consult Art. 190 — 192 of the preceding Treatise.

Terms

Terms $x - \frac{1}{4}a$) of the Value of y extracted, always terminate in that Assymptote; whence you may easily find the Portion of the Assymptote. The same thing is always to be observed when the Area is denoted by Terms divided more and more by x continually, only that in place of a right-lined Assymptote, sometimes you will have the conical Parabola, or perhaps one more compounded.

35. But passing this Manner of Resolution, as being particular, because not applicable to Curves which return into themselves like Ellipses; with respect to the other Manner of Resolution, which was shewn above in the Example $y^3 + a^2y + axy - 2a^3 - x^3 = 0$ (*viz.* that Method in which the Dimensions of x perpetually increase in the Numerators of the Quotient) the following Things may be remarked.

1. If it happen at any Time that the Value of y, when x is supposed to be nothing, be a surd Quantity; or one entirely unknown, yet you may design it by some Letter. Thus in the Example $y^3 + a^2y + axy - 2a^3 - x^3 = 0$, if the Root of this Equation $y^3 + a^2y - 2a^3 = 0$ had been a Surd, or unknown, I would have supposed any Quantity (b) put for it; and have performed the Operation as follows. Writing b in the Quotient, I suppose $b + p = y$, and substitute that for y, as you may see; whence a new Equation $p^3 + 3bp^2$ &c. $= 0$ arises, in which the Terms $b^3 + a^2b - 2a^3$ are to be rejected, as being equal to nothing, because b is supposed the Root of this Equation $y^3 + a^2y - 2a^3 = 0$. Then the Terms $3b^2p + a^2p + abx$ give $-\frac{abx}{3b^2 + a^2}$ to be adjoined to the Quotient; and $-\frac{abx}{3b^2 + a^2} + q$ to be substituted for p, &c.

$y^3 +$

$$y^3 + a^2y + axy - 2a^3 - x^3 = 0. \quad \text{Put } cc = 3b^2 + a^2.$$
$$y = b - \frac{abx}{cc} + \frac{a^4bx^2}{c^6} + \frac{x^3}{c^2} + \frac{a^3b^3x^3}{c^8} - \frac{a^6bx^3}{c^8} + \frac{a^8b^3x^3}{c^{10}} \text{ \&c.}$$

$b + p = y$	y^3 $+ axy$ $+ aay$ $- x^3$ $- 2a^3$	$+ b^3 + 3b^2p + 3bp^2 + p^3$ $+ abx + axp$ $+ aab + a^2p$ $- x^3$ $- 2a^3$
$\frac{-abx}{cc} + q = p$	p^3 $+ 3bp^2$ $+ axp$ $+ ccp$ $- x^3$ $+ abx$	$- \frac{a^3b^3x^3}{c^6}$ &c. $+ \frac{3a^2b^3x^2}{c^4} - \frac{6ab^2x}{cc}q$ &c. $- \frac{a^2bx^2}{cc} + axq$ $- abx + ccq$ $- x^3$ $+ abx.$

$$c^2 + ax - \frac{6ab^2x}{cc}\Big) \frac{a^4bx^2}{c^4} + x^3 + \frac{a^3b^3x^3}{c^5} \Big(\frac{a^4bx^2}{c^6} + \frac{x^3}{c^2} + \frac{a^3b^3x^3}{c^8} \text{ \&c.}$$

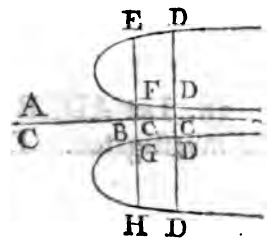
The Work being finiſhed, I take any Number for a, and I reſolve this Equation $y^3 + a^2y - 2a^3 = 0$, after the ſame Manner as was ſhewn above in the Caſe of Numeral Equations; and then I ſubſtitute it's Root in Place of b.

2°. If it ſhould happen that the ſaid Value be nothing, that is to ſay, if in the Equation to be reſolved, there be no Term but what is multiplied by x or y, as in this $y^3 - axy + x^3 = 0$; then I ſelect the Terms $(-axy + x^3)$ in which x ſeparately, and alſo y ſeparately, if that can be had, otherwiſe multiplied by x, is of the feweſt Dimenſions. And theſe Terms give $+\frac{xx}{a}$ for the firſt Term of the Quotient; and $\frac{xx}{a} + p$ to be ſubſtituted for y.

In this Equation $y^3 - a^2y + axy - x^3 = 0$, you may diſcover the firſt Term of the Quotient, either from $-a^2y - x^3$; or from $y^3 - a^2y$.

3°. If that Value of y be imaginary, as in this $y^4 + y^2 - 2y + 6 - x^2y^2 - 2x + x^2 + x^4 = 0$, I increaſe or diminiſh the Quantity x until the ſaid Value become real.

Thus



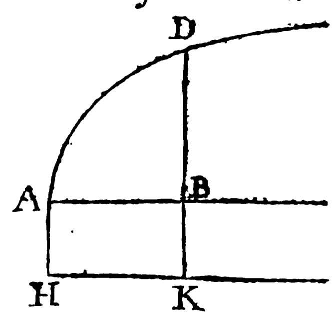
Thus in the Figure annexed, when AC (x) is nothing, then CD (y) is imaginary.

But if AC be diminished by the given Line AB, that so BC may become x; then supposing BC (x) to be nothing, CD (y) has a fourfold Value (CE, CF, CG or CH) each of them real; any of which Roots (CE, CF, CG or CH) may be the first Term of the Quotient, according as the Surface BEDC, BFDC, BGDC or BHDC is desired. In other Cases likewise, where you find any Stop, you may extricate yourself by the same Means.

Finally if the Index of the Power of x or y be a Fraction, I reduce it to an Integer; as in this Example $y^3 - xy^{\frac{1}{2}} + x^{\frac{4}{3}} = 0$; putting $y^{\frac{1}{2}} = v$, and $x^{\frac{1}{3}} = z$, there arises $v^6 - z^3v + z^4 = 0$, whose Root is $v = z + z^3$ &c. or (by restoring the Values) $y^{\frac{1}{2}} = x^{\frac{1}{3}} + x$ &c. and by squaring, $y = x^{\frac{2}{3}} + 2x^{\frac{4}{3}}$ &c.

36. **And this much shall suffice to have said touching the Investigation of the Areas of Curves.** Yea since all Problems concerning the Length of Curves; the Quantity and Surface of Solids; and the Centers of Gravity may at length be brought to this, *viz.* that the Quantity of some Plane Surface bounded by a Curve Line is sought, there is no Necessity to say any thing about them here. However I shall shew very briefly after what Manner I perform the Operation in these Things.

The Application of what has been said to other Problems of that Kind.



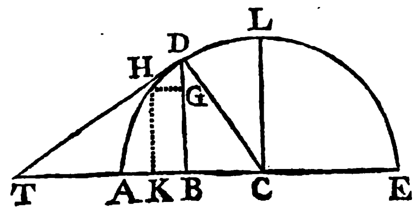
37. Let ABD be any Curve, and AHKB a Rectangle, whose Side AH or BK is Unity: And imagine the Right Line DBK to move uniformly from AH, so as to describe the Areas ABD and AK; and that BK (1) is the Moment with which AK (x), and BD (y) the Moment with which ABD is gradually encreased; and that from the Moment BD continually given, you can, by Means of the preceding Rules, investigate the Area ABD described by it, or compare it with AK (x), which is decribed with the Moment 1.

Now by the same Means that the Superficies ABD from it's Moment being at all Times given, is discovered by the foregoing Rules, by the like Means may any other Quantity be investigated from it's Moment given in like manner. The Thing will be clearer by an Example.

To

To find the Lengths of Curves.

38. Let ADLE be a Circle, the Length of whose Arch AD is to be investigated. Draw the Tangent DHT, and having compleated the indefinitely small Rectangle HGBK, and put $AE = 1 = 2AC$, it shall be as BK or GH the Moment of the Base AB (x) to HD the Moment of the Arch AD : : BT : DT : : BD ($\sqrt{x - xx}$) : DC ($\frac{1}{2}$) : : 1 (BK) : $\frac{1}{2\sqrt{x - xx}}$ (DH). And so



$\frac{1}{2\sqrt{x - xx}}$ or $\frac{\sqrt{x - xx}}{2x - 2xx}$ is the Moment of the Arch AD. Which being reduced makes $\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{4}x^{\frac{1}{2}} + \frac{3}{16}x^{\frac{3}{2}} + \frac{5}{32}x^{\frac{5}{2}} + \frac{35}{256}x^{\frac{7}{2}} + \frac{63}{512}x^{\frac{9}{2}}$ &c. Wherefore by Rule the second the Length of the Arch is $x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \frac{3}{40}x^{\frac{5}{2}} + \frac{5}{112}x^{\frac{7}{2}} + \frac{35}{1152}x^{\frac{9}{2}} + \frac{63}{2816}x^{\frac{11}{2}}$, &c. or $x^{\frac{1}{2}}$ into $1 + \frac{1}{6}x + \frac{3}{40}x^2 + \frac{5}{112}x^3 + \frac{35}{1152}x^4 + \frac{63}{2816}x^5$, &c.

39. After the same Manner by supposing CB to be x, the Radius CA to be 1, you will find the Arch LD to be $x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c.

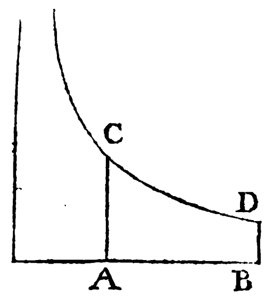
40. But it is to be remarked that that Unity which is put for the Moment, is a Superficies, when the Question is about Solids; and a Line when about Superficies; and a Point when it is about Lines (as in this Example.) Neither am I afraid to speak of Unity in Points, or Lines infinitely small, since Geometers are wont now to consider Proportions even in such a Case, when they make use of the Methods of Indivisibles.

41. From these Things one may guess how one ought to proceed in investigating the Superficies and Contents of Solids; and likewise the Centers of Gravity.

To find the Converse of these Things.

42. But if upon the contrary, from the Area, or Length, &c. of any Curve being given, the Length of the Base AB be required, then you must extract the Root x, out of the Equations which have been found by the preceding Rules.

To find the Base from the Area given.



43. Thus if from the Area ABDC of the Hyperbola ($\frac{1}{1+x} = y$) given I wanted to investigate the Base AB, calling the Area z, I extract the Root of this Equation z (ABCD) $= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$, *&c.* neglecting those Terms in which x is of more Dimensions than z is desired in the Quotient.

As if I would have z to rise to five Dimensions only in the Quotient, I neglect all the Terms $- \frac{1}{6}x^6 + \frac{1}{7}x^7 - \frac{1}{8}x^8$, *&c.* and extract the Root of this only $\frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - z = 0$.

$x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$ *&c.*		
$z + p = x$	$+ \frac{1}{5}x^5$	$+ \frac{1}{5}z^5$ *&c.*
	$- \frac{1}{4}x^4$	$- \frac{1}{4}z^4 - z^3p$ *&c.*
	$+ \frac{1}{3}x^3$	$+ \frac{1}{3}z^3 + z^2p + zp^2$ *&c.*
	$- \frac{1}{2}x^2$	$- \frac{1}{2}z^2 - zp - \frac{1}{2}p^2$
	$+ x$	$+ z + p$
	$- z$	$- z$
$\frac{1}{2}z^2 + q = p$	$+ zp^2$	$+ \frac{1}{4}z^5$ *&c.*
	$- \frac{1}{2}p^2$	$- \frac{1}{8}z^4 - \frac{1}{2}z^2q$
	$- z^3p$	$- \frac{1}{2}z^5$ *&c.*
	$+ z^2p$	$+ \frac{1}{2}z^4 + z^2q$
	$- zp$	$- \frac{1}{2}z^3 - zq$
	$+ p$	$+ \frac{1}{2}z^2 + q$
	$+ \frac{1}{5}z^5$	$+ \frac{1}{5}z^5$
	$- \frac{1}{4}z^4$	$- \frac{1}{4}z^4$
	$+ \frac{1}{3}z^3$	$+ \frac{1}{3}z^3$
	$- \frac{1}{2}z^2$	$- \frac{1}{2}z^2$
$1 - z + \frac{1}{2}z^2)\ \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5\ (\frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$		

44. I have laid the Steps of the Resolution before you, as you see, upon the Account of the two following Remarks.

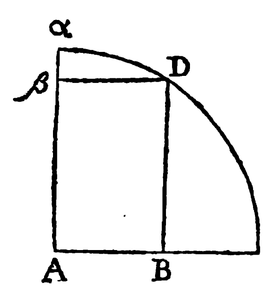
1. That in the Substitution, I always omit those Terms, which I foresee will be of no Use afterwards. Concerning which this is the

 Rule;

Rule; That after the first Term resulting from each Quantity that is collateral to it, I add no more Terms upon the right Hand than the Index of the Dimension of that first Term wants Units of the Index of the greatest Dimension. As in this Example, where the greatest Dimension is 5, I neglect all the Terms after z^5, I put one after z^4, and two only after z^3. When the Root (x) to be extracted, is every where of even or odd Dimensions, let this be the Rule: That after the first Term, resulting from each Quantity which is collateral to it, you add no more Terms towards the Right Hand, than what the Index of the Dimension of that first Term, wants Pairs of Units of the Index of the highest Dimension; or no more than what it wants Ternaries of Units, when the Indexes of the Dimensions of x differ by three Units; and so in others.

2. When I see that p, q, or r, *&c.* in the last resulting Equation, is found of one Dimension only, I seek it's Value, that is to say the remaining Terms, which are still to be added to the Quotient, by means of Division; as you see done here.

To find the Base from the Length of the Curve given.



45. If from the Arch αD given the Sine AB was required; I extract the Root of the Equation found above, *viz.* $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$ (it being supposed that AB $= x$, αD $= z$, and A$\alpha = 1$) by which I find $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$ *&c.*

46. And moreover if the Cosine Aβ were required from that Arch given, make Aβ $(= \sqrt{1 - xx}) = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 + \frac{1}{40320}z^8 - \frac{1}{3628800}z^{10}$, *&c.*

Concerning the Continuation of the Series of the Progressions.

47. Let it be observed here, by the bye, that when 5 or 6 Terms of those Roots are known, they may be continued at Pleasure for most Part, by observing the Analogy of the Progression.

Thus you may continue this $x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$, *&c.* by dividing the last Term by the following Numbers in Order, 2, 3, 4, 5, 6, 7, *&c.*

And this $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7$, *&c.* by dividing by these Numbers 2×3, 4×5, 6×7, 8×9, 10×11, *&c.*

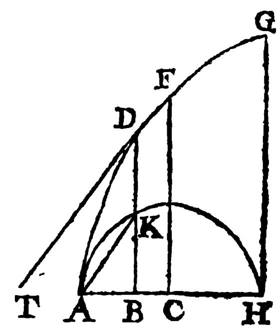
And

And this $x = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6$, &c. by these $1 \times 2, 3 \times 4, 5 \times 6, 7 \times 8, 9 \times 10$, &c.

And this $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c. by multiplying by these, *viz.* $\frac{1 \times 1}{2 \times 3}, \frac{3 \times 3}{4 \times 5}, \frac{5 \times 5}{6 \times 7}, \frac{7 \times 7}{8 \times 9}$, &c. And so in others.

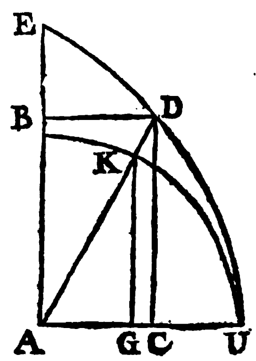
The Application of what has been said to Mechanical Curves.

48. And what has been said shall suffice with Respect to Geometrical Curves. But now, although the Curve be Mechanical, yet this Method of ours may be applied to it.



For Example, let ADFG be the Trochoid whose Vertex is A, and Axis AH, and AKH the Wheel or Circle with which it is described. And the Superficies ABD is required. In Order to which, putting AB $= x$ and BD $= y$ as formerly; and AH $= 1$, I first seek the Length of BD. Now from the Nature of the Trochoid, you have KD = Arch AK: Wherefore the whole Line BD = BD + Arch AK. But BK $(= \sqrt{x - xx}) = x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}$, &c. And (from what was formerly shewn) the Arch AK $= x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \frac{3}{40}x^{\frac{5}{2}} + \frac{5}{112}x^{\frac{7}{2}}$ &c. Therefore the whole BD $= 2x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} - \frac{1}{20}x^{\frac{5}{2}} - \frac{1}{56}x^{\frac{7}{2}}$ &c. And (by Rule the second) the Area ABD $= \frac{4}{3}x^{\frac{3}{2}} - \frac{2}{15}x^{\frac{5}{2}} - \frac{1}{70}x^{\frac{7}{2}} + \frac{1}{252}x^{\frac{9}{2}}$ &c.

Or more shortly thus: Since the Right Line AK is parallel to the Tangent TD, it will be AB to BK, as the Moment of the Line AB to the Moment of the Line BD, that is, $x : \sqrt{x - xx} :: 1 : \frac{1}{x}\sqrt{x - x^2} = x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}} - \frac{1}{8}x^{\frac{3}{2}} - \frac{1}{16}x^{\frac{5}{2}} - \frac{5}{128}x^{\frac{7}{2}}$ &c. Wherefore (by Rule the second) BD $= 2x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} - \frac{1}{20}x^{\frac{5}{2}} - \frac{1}{56}x^{\frac{7}{2}} - \frac{5}{576}x^{\frac{9}{2}}$ &c. And the Superficies ABD $= \frac{4}{3}x^{\frac{3}{2}} - \frac{2}{15}x^{\frac{5}{2}} - \frac{1}{70}x^{\frac{7}{2}} - \frac{1}{252}x^{\frac{9}{2}} - \frac{5}{3168}x^{\frac{11}{2}}$ &c.



49. Much after the same Manner (if C be the Center of the Circle and CB $= x$) you may find the Area CBDF &c,

50. Let it be required to find the Area ABDU of the Quadrature UDE (whose Vertex is U, and A the Center of the interior Circle UK to which it is fitted.

Having drawn any Line as AKD, I let fall the Perpendiculars DB, DC, KG. And so it shall be KG : AG :: AB (x) : BD (y), or or $\frac{x \times AG}{KG} = y$. But from the Nature of the Quadratrix, you have BA $(= DC) =$ Arch UK; or $UK = x$. Wherefore putting $AU = 1$, it will be $GK = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$, &c. from what was ſhewn above, and $GA = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$ &c.

And therefore $y \left(= \frac{x \times AG}{KG}\right) = \frac{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 \text{ \&c.}}{1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{1}{5040}x^6 \text{ \&c.}}$, or, by actual Diviſion $y = 1 - \frac{1}{3}x^2 - \frac{1}{45}x^4 - \frac{2}{945}x^6$ &c. and (by Rule the ſecond) the Area $AUDB = x - \frac{1}{9}x^3 - \frac{1}{225}x^5 - \frac{2}{6615}x^7$ &c.

51. Thus alſo the Length of the Quadratrix UD may be determined, although the Calculation be ſomething more difficult.

52. Neither do I know any Thing of this Kind to which this Method doth not extend; and that in various Ways. Yea Tangents may be drawn to Mechanical Curves by it, when it happens that it can be done by no other Means. And whatever the common Analyſis performs by Means of Equations of a finite Number of Terms (provided that can be done) this can always perform the ſame by Means of infinite Equations: So that I have not made any Queſtion of giving this the Name of *Analyſis* likewiſe. For the Reaſonings in this are no leſs certain than in the other; nor the Equations leſs exact; albeit we Mortals whoſe reaſoning Powers are confined within narrow Limits, can neither expreſs, nor ſo conceive all the Terms of theſe Equations, as to know exactly from thence the Quantities we want: Even as the ſurd Roots of finite Equations can neither be ſo expreſt by Numbers, nor any analytical Contrivance, that the Quantity of any one of them can be ſo diſtinguiſhed from all the reſt, as to be underſtood exactly.

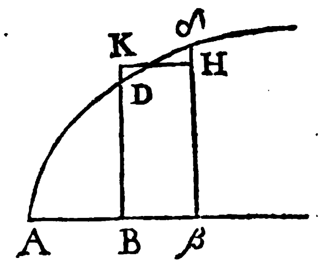
53. To conclude, we may juſtly reckon that to belong to the *Analytic* Art, by the Help of which the Areas and Lengths &c. of Curves may be exactly and geometrically determined (when ſuch a thing is poſſible). But this is not a Place for inſiſting upon theſe Things. There are two Things eſpecially which an attentive Reader will ſee need to be demonſtrated.

I. The Demonſtration of the Quadrature of Simple Curves belonging to Rule the firſt.

Preparation for demonſtrating the firſt Rule.

54. Let then $AD\delta$ be any Curve whoſe Baſe $AB = x$, the perpendicular Ordinate $BD = y$, and the Area $ABD = z$, as at the Beginning,

ning. Likewise put Bβ = o, BK = v; and the Rectangle BβHK (ov) equal to the Space B$\beta\delta$D.



Therefore it is Aβ = x + o, and A$\delta\beta$ = z + ov: Which Things being premised, assume any Relation betwixt x and z that you please, and seek for y in the following Manner.

Take at Pleasure $\frac{2}{3}x^{\frac{3}{2}} = z$; or $\frac{4}{9}x^3 = z^2$. Then x + o (Aβ) being substituted for x, and z + ov (A$\delta\beta$) for z, there arises $\frac{4}{9}$ into $x^3 + 3xo^2 + 3xo^2 + o^3$ = (from the Nature of the Curve) $z^2 + 2zov + o^2v^2$. And taking away Equals ($\frac{4}{9}x^3$ and z^2) and dividing the Remainders by o, there arises $\frac{4}{9}$ into $3x^2 + 3xo + oo = 2zv + ovv$. Now if we suppose B$\beta$ to be diminished infinitely and to vanish, or o to be nothing, v and y, in that Case will be equal, and the Terms which are multiplied by o will vanish: So that there will remain $\frac{4}{9} \times 3x^2 = 2zv$, or $\frac{2}{3}x^2$ ($= zy$) $= \frac{2}{3}x^{\frac{3}{2}}y$; or $x^{\frac{1}{2}}$ $\left(= \frac{x^2}{x^{\frac{3}{2}}}\right) = y$. Wherefore conversely if it be $x^{\frac{1}{2}} = y$, it shall be $\frac{2}{3}x^{\frac{3}{2}} = z$.

The Demonstration.

55. Or universally, if $\frac{n}{m+n} \times ax^{\frac{m+n}{n}} = z$; or, putting $\frac{na}{m+n} = c$, and $m + n = p$, if $cx^{\frac{p}{n}} = z$; or $c^nx^p = z^n$: Then by substituting x + o for x, and $z + ov$ (or which is the same $z + oy$) for z, there arises c^n into $x^p + pox^{p-1}$, &c. $= z^n + noyz^{n-1}$ &c. the other Terms, which would at length vanish being neglected. Now taking away c^nx^p and z^n which are equal, and dividing the Remainders by o, there remains $c^npx^{p-1} = nyz^{n-1}$ $\left(= \frac{nyz^n}{z}\right) = \frac{nyc^nx^p}{cx^{\frac{p}{n}}}$, or, by dividing by c^nx^p, it shall be $px^{-1} = \frac{ny}{cx^{\frac{p}{n}}}$; or $pcx^{\frac{p-n}{n}} = ny$; or by restoring $\frac{na}{m+n}$ for c, and $m + n$ for p, that is m for $p - n$, and na for pc, it becomes $ax^{\frac{m}{n}} = y$. Wherefore conversely, if $ax^{\frac{m}{n}} = y$, it shall be $\frac{n}{m+n}ax^{\frac{m+n}{n}} = z$. Q. E. D.

To

To find those Curves which can be squared.

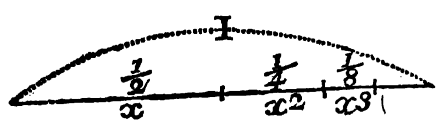
56. Hence by the Way you may obſerve after what Manner as many Curves as you pleaſe may be found, whoſe Areas are known; *viz.* aſſume any Equation you pleaſe for the Relation betwixt the Area z and Baſe x, and thence let the Ordinate y be ſought. Thus if you ſuppoſe $\sqrt{aa+xx} = z$, by performing the Calculation you will find $\frac{x}{\sqrt{aa+xx}} = y$. And ſo in other Caſes.

II. The Demonſtration of the Reſolution of Affected Equations.

57. The other Thing to be demonſtrated, is the literal Reſolution of affected Equations. *Viz.* That the Quotient, when x is ſufficiently ſmall, the further it is produced, approaches ſo much nearer to the Truth, ſo that the defect (p, q, or r &c.) by which it differs from the full Value of y, at Length becomes leſs than any given Quantity; and that Quotient being produced infinitely is exactly equal to y. Which will appear thus.

1°. Becauſe that Quantity in which x is of the loweſt Dimenſion (that is to ſay, more than the half of the laſt Term, provided you ſuppoſe x ſmall enough) in every Operation is perpetually taken out of the laſt Term of the Equations, of which p, q, r, &c. are the Roots: Therefore that laſt Term (by 1. 10. Elem.) at length becomes leſs than any given Quantity; and would entirely vaniſh, if the Operation were infinitely continued.

Thus if it be $x = \frac{1}{2}$, you have x the half of all theſe $x + x^2 + x^3 + x^4$, &c. and x^2 the half of all theſe $x^2 + x^3 + x^4 + x^5$, &c. Therefore if $x \sqsupset \frac{1}{2}$, x ſhall be greater than the half of all theſe $x + x^2 + x^3$, &c. and x^2 greater than the half of all theſe $x^2 + x^3 + x^4$, &c.



Thus if $\frac{x}{b} \sqsupset \frac{1}{2}$, you ſhall have x more than the half of all theſe $x + \frac{x^2}{b} + \frac{x^3}{b^2}$, &c. And the ſame Way of others. And as to the numeral Coefficients, for moſt part they perpetually decreaſe: Or if at any Time they increaſe, you need only ſuppoſe x ſome few Times leſs.

2°. If

2°. If the last Term of any Equation be continually diminished, till it at length vanish, one of it's Roots shall likewise be diminished, until the last Term vanishing, it vanish along with it.

3°. Wherefore one Value of the Quantities p, q, r, *&c.* continually decreases until at length, when the Operation is infinitely produced, it vanish entirely.

4°. But the Values of these Roots p, q, r, *&c.* together with the Quotient already extracted, are equal to the Roots of the proposed Equation (thus in the Resolution of the Equation $y^3 + a^2y + axy - 2a^3 - x^3 = 0$, shewn above, you will observe that $y = a + p = a - \frac{1}{4}x + q = a - \frac{1}{4}x + \frac{xx}{64a} + r$, *&c.*) Whence it is sufficiently evident that the Quotient infinitely produced, is one of the Values of y: Which was what I proposed to shew.

58. The same Thing will appear by substituting the Quotient instead of y in the Equation proposed. For you will perceive that those Terms perpetually destroy one another in which x is of the least Dimensions.

Sir

Sir *ISAAC NEWTON*'s

ANALYSIS

BY

Equations of an infinite Number of Terms, *explained.*

SECT. I.

INTRODUCTION.

1. HAVING endeavoured, in what goes before, to explain and illustrate Sir *Isaac Newton*'s Treatise of the Quadrature of Curves: In order to render this Doctrine, together with what depends upon it, as compleat as I can, for the Use of the young Geometrician, I now design to give him some Assistance, for his better understanding this other *twin* Performance of our celebrated Author, according as I promised. And therefore, as this Analysis treats of the Doctrine of Infinite Series and Equations, chiefly with a View to the Quadrature of Curves, and other Parts of the Geometry of Curve-Lines, I suppose it will neither be unsuitable to the Design, nor unacceptable to the Reader, to lay before him a brief Account of the Rise and Progress of the Contemplation of Infinites and the Quadrature of Curves, in Mathematics, from it's Infancy in the Method of *Exhaustions*, used by the Ancients, to it's present State, as it has been perfected by Sir *Isaac Newton* into the Method of *Infinite Series* and *Fluxions*.

2. When

2. When the Ancients began to indulge themſelves in the Study and Contemplation of the Properties and Relations of Magnitudes, they very ſoon found it neceſſary to conſider and inveſtigate the fundamental Principles of the Doctrine of Proportion: And accordingly we find it delivered by *Euclid* in the fifth Book of his Elements. In the third Definition, he defines Ratio or Proportion to be [1] *a certain Habitude of two Magnitudes of the* SAME KIND *to each other according to Quantity:* And in the next Definition, he defines Magnitudes capable of bearing Ratio or Proportion, to be ſuch, *as being multiplied or repeated ſome certain Number of Times, may* EXCEED *one another.* From whence it appears that Quantities that are of different Kinds, ſuch as Lines and Surfaces, Surfaces and Bodies; any of theſe and Time; Time and Velocity, &c. cannot be ſaid to have Ratio or Proportion to one another: Both becauſe they are of different Kinds; and likewiſe (which indeed follows from the other) becauſe no Multiplication or Repetition of the one can ever make it exceed the other; or even conſtitute any Quantity of that Kind: For no Multiplication of a Line can ever make any Surface; nor any Portion of Time, however multiplied, make any Degree of Velocity, &c. But it is evident, by the ſecond of the preceding Definitions, that *Euclid* deſigned by it to limit Magnitudes that bear Proportion to each other, to ſome other Idea than that of Homogeneity: Otherwiſe his fourth Definition had been ſuperfluous. And by what comes afterwards, in the ſucceeding Parts of theſe Elements, it appears, that he meant to lay a Foundation for the Doctrine of *Exhauſtions*, as it is wont to be called; as well as that of Incommenſurables. For being about to treat of the Doctrine of Incommenſurables in the tenth Book, he lays down and demonſtrates this Propoſition by way of a Lemma, *viz. Two unequal Magnitudes being propoſed, if from the greater you take away more than the half, and from the Remainder more than it's half; and ſo on continually, there will remain a Magnitude at length, by this continual Subduction, which is leſs than the leaſt of the two Magnitudes propoſed.* In demonſtrating which he aſſumes this Principle or Poſtulate, that *the leſſer of the two Magnitudes being multiplied, will at length exceed the greater:* For if not, they can bear no Ratio to one another by *Def.* 3. *B.* 5. which yet they are ſuppoſed to do, that they may be capable of any Compariſon or Relation to one another.

3. Now this Propoſition or Lemma is the Foundation upon which the Method of *Exhauſtions* is built. Which Method is made uſe of by *Euclid* in demonſtrating ſome Propoſitions, as the ſecond, tenth,

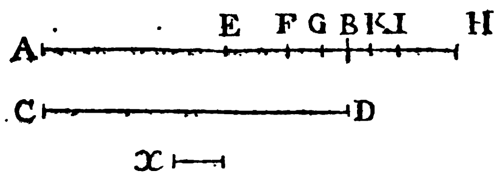
[1] Λόγος ἐςὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητα πρὸς ἄλληλα ποία σχέσις.

eleventh, twelfth, and eighteenth, B. 12. El. in which he proves that Circles are to one another as the Squares of their Diameters; that a Cone is a third part of a Cylinder of the ſame Baſe and Altitude; that Cones and Cylinders of the ſame Altitude are to one another as their Baſes; that ſimilar Cones and Cylinders are to one another in a triplicate Proportion of the Diameters of their Baſes; and that Spheres are to one another in a triplicate Proportion of their Diameters. And *Archimedes* makes conſtant Uſe of it throughout all his Writings. For he, beginning where *Euclid* and other Geometricians before himſelf left off, found that this, or ſome Principle equivalent to it, was abſolutely neceſſary to make any Progreſs in the Contemplation of Plain Curvilinear Figures; and Solid Figures generated by their Rotation; which were not capable of being divided into Parts bounded by Right Lines, or Plain Surfaces, and therefore we find him, in ſeveral different Places, aſſuming this Poſtulate, that *of two unequal Magnitudes of the ſame Kind, the Exceſs of the greater above the leſſer cannot be ſo ſmall, but that it may be ſo multiplied as to exceed either of them; or any other Magnitude of the ſame Kind.* Accordingly, by the Help of the Principle, he has demonſtrated, by the Method of Exhauſtion or continual Approach, the Quadrature of the Parabola; having ſhewn that any Segment of a Parabola cut off by a Right Line is to the inſcribed Triangle of the ſame Baſe and Altitude as 4 to 3; that a Circle is equal to a right-angled Triangle whereof the two Sides including the Right-angle, are equal the one to the Semi-diameter, the other to the Periphery of the Circle; and has ſhewn how to approach to the Quadrature of the Circle and Ellipſe. He has demonſtrated the Ratio of the Cylinder, Hemiſphere and Cone having the ſame Baſe and Height, both with reſpect to their ſolid Contents, and Surfaces; and ſhewn in general how to compare Spheres, Spheroids, Parabolical, and Hyperbolical Conoeids, or any Portions of them, with Cones or Cylinders: And with Reſpect to the Surface of the Sphere, he has demonſtrated that it is equal to 4 times the Area of one of it's great Circles. And beſides many noble and uſeful Diſcoveries with Reſpect to theſe Figures mentioned, he wrote a moſt elaborate Treatiſe upon the Spiral Line, which commonly goes under the Name of *Archimedes*'s Spiral, in which he demonſtrates the Proportion of the Spiral Spaces to the Areas of the circumſcribing Circles; and that of the Subtangents in the Spiral, to the Perimeters of the Circles. All theſe Things are demonſtrated moſtly by the Method of Exhauſtions. Which may be conſidered as the firſt Step towards the general Method of Quadratures, and infinite converging Series lately introduced. This hath got the Name of the

Method

Method of *Exhaustions*, because, in comparing two unequal Quantities with one another, by taking more than half the Difference, and adding it to the least, or subducting it from the greatest; and again taking more than the half of the remaining Difference, and adding it to the least, or subducting it from the greatest; and so on continually, the Difference betwixt the two Quantities is gradually *exhausted*, so as to become less than any given Quantity, in virtue of what is demonstrated by *Euclid* in that Lemma, which hath been mentioned: And it would be the same thing, if, instead of taking more than half the Difference, you take half the Difference, or even any given Part of the Difference, and add it continually to the least; or subduct it continually from the greatest of the two Quantities: For this likewise will gradually *exhaust* the Difference, so as to make it less than any given Quantity.

4. That the Way of proceeding in this Method may be the better understood, I shall shew how *Archimedes* demonstrates by it, that a Circle is equal to a right-angled Triangle, whereof one of the Sides including the Right Angle, is equal to the Radius of the Circle, and the other to it's Circumference. In Order to which, let the Circle and Triangle be represented by the two Right Lines AB and CD, as Symbols. Moreover let AE, AF, AG, &c. represent a Square, an eight sided, a sixteen sided Figure, &c. regularly inscribed in the Circle: The Number of Sides being always doubled. Again let AH, AI, AK, &c. represent a Square, a regular eight sided, sixteen sided, &c. Figure, circumscribed about the Circle. Then I say AB = CD. For if not, then AB is either greater than CD; or less. If AB be greater than CD, let the Excess be x: Then since EB, FB, GB, &c. are the Differences betwixt the inscribed Figures and the Circle, EF is greater than the half of EB; FG greater than the half of FB, &c. (as may easily be demonstrated) consequently (by Prop. 1. Elem. 10.) you must come at length to some Figure inscribed as AG, whose Difference from the Circle AB, *viz.* GB is less than x: Wherefore AG must be greater than the Triangle CD, since AB was equal to CD + x: But AG, being an inscribed Figure, is only equal to a right-angled Triangle having one of the sides equal to the Perpendicular drawn from the Center upon one of the Sides of the Polygon; and the other to the Sum of the Sides of the Polygon; the first of which is less than the Radius of the Circle; and the other less than the Circumference;



to which the Sides of the Triangle CD are equal: Therefore AG is less than CD, contrary to what has been demonstrated. Wherefore it is not true that AB is greater than CD. I say neither is AB less than CD. For if AB be less than CD; let the Defect be x, so that it be $AB + x = CD$; then it is easy to shew that HI is greater than $\frac{1}{2}$ HB; that IK is greater than $\frac{1}{2}$ IB, &c. and that therefore you must (by the Proposition already mentioned) at length arrive at some Difference as KB less than x: Wherefore AK is less than CD, since $AB + x = CD$. But AK, being a regular Polygon circumscribed about the Circle, is equal to a right-angled Triangle having one of it's Sides equal to the Radius, and the other equal to the Sum of all the Sides of the Polygon, *i. e.* greater than the Circumference of the Circle: Which Triangle therefore is greater than the Triangle CD: Wherefore AK is also greater than CD; contrary to what has been demonstrated: For it was shewn to be less. Therefore the Circle AB is not less than the Triangle CD; and it was demonstrated that it is not greater. Therefore they are equal. Q. E. D.

5. By this example it may appear in what Manner the ancient Geometricians proceeded, when they demonstrated any Proposition by the Method of Exhaustions. The *Characteristic* of which is, that one Quantity makes a *continual Approach* to another, so as to *exhaust* the Difference by degrees, until it become less than any given Quantity. And it is the same in Ratio's: When a Ratio, which is variable, continually approaches to another constant Ratio, so as to differ from it at length by less than any given Ratio.

Where they could discover any two Quantities, or two Ratios, which were both Limits to which the same variable Quantity, or Ratio, continually approached, so as at length to come nearer them than by any given Difference, but never to go beyond them, they concluded these Quantities, or Ratios, which were the Limits, to be equal. And this they demonstrated by that kind of Demonstration which is called *ducens ad absurdum*: Which Demonstrations were sometimes long; but always strict and exact. To avoid the Tediousness of such Demonstrations, it was that our Author invented his Method of *prime and ultimate Ratios* [1]. Which is founded upon a like Principle, as the Method of Exhaustions; and the same Objections lye against the one as against the other, as we have shewn towards the Beginning of our Notes upon the Quadrature of Curves.

6. As, by what has been said, it appears in what Manner the Ancients entered into the Contemplation of Curvilinear Areas, and Solids

[1] Phil. Nat. Prin. Sect. 1. Schol.

generated

generated by their Revolution; so likewise we may observe the Doctrine of *infinite converging Series*, beginning to appear in Embryo, as it were, in their continual Approaches, by exhausting the Difference. For the Steps of the Approach, may be supposed to be continued indefinitely: For although a finite number, or even a very few of them, is sufficient to make out their Demonstrations; yet it is evident that no finite Number of them can ever *entirely* exhaust the Difference. Thus in the preceding Example taken from *Archimedes*, AE + EF + FG, *&c. in inf.* represents an infinite Series of simple Terms, according to Sir *Isaac Newton*'s Method; which is that generally used; where the Aggregate of all the Terms, infinite in Number, is equal to AB; and the greater the Number of them, taken from the Beginning, is, the nearer doth the Aggregate of them approach to AB: So that a finite Number of Terms differs from AB by less than any given Difference. Again AH . AE, AI . AF, AK . AG, *&c.* represents Mr. *James Gregory*'s Method of converging Series in his excellent Treatise *Vera Circuli & Hyperbolæ Quadratura*. In which he supposes the Terms of the Series to proceed by Pairs, as A and B, C and D, E and F, *&c.* such, that the Difference betwixt A and B is greater than the Difference betwixt C and D; and that after the same Manner that C and D are formed by an analytical Operation from A and B; after the like Manner, or by the like analytical Operation, E and F are formed from C and D; and so on continually: Every new Pair of Terms being produced always in the same Manner from the Pair of Terms immediately preceding. By which the Difference continually lessening, becomes less than any given Quantity; and the Series being supposed to be continued *in infinitum*, that Difference quite vanishes, and the two Terms become equal; either of which is the Quantity sought; and the other Terms approach so much nearer to it, in order, the further they are distant from the Beginning of the Series. He mentions another Form of converging Series consisting of simple Terms, to which the foregoing may be reduced; which he defines thus: Let A, B, C, D, E, *&c.* be a Series of simple Terms, and let them be of such a Nature that the third Term C is compounded after the same Manner of A and B, the first and second, that the fourth D is of B and C; and the fifth E is of C and D, and so on *in infinitum*: And let the Difference of the antecedent Terms A and B, be always greater than the Difference of the immediately subsequent B and C. This Series being supposed to be continued infinitely, each new Term makes a nearer Approach to the Quantity required, and you may continue it so far as to arrive at a Term, which differs from it by less than any given Difference: So that what

what he calls the last Term (for want of another Word to express it) is the very Quantity required. Such Series appears in Embryo, likewise, in the preceding Proposition from *Archimedes*; where AB being the Quantity sought, AE, AF, AG, &c. may represent the Terms of one such Series; and AH, AI, AK, &c. the Terms of another.

7. But notwithstanding there is some Analogy in all the three Methods, *viz.* that of Exhaustions; that of Mr. *Gregory*'s Forms of Series; and that of Sir *Isaac*'s; as they represent a continual Approach to some determined Limit; yet there is a manifest *Gradation* as to their *Extent* and *Universality*. For the Approach used by the Ancients proceeded by taking away or adding the half Difference, or more than the half Difference, continually. In Mr. *Gregory*'s Form of converging Series, it is sufficient if the Terms by which the continual Approach is made be so related, that the subsequent ones be always compounded of the preceding ones after the same Manner, by any analytical Operation whatsoever, including Addition, Subtraction, Multiplication, Division, and Extraction of Roots. Sir *Isaac Newton*'s Form of Infinite Series doth not even require this; it being sufficient that the Sum of the Terms from the Beginning, continually approach to the Quantity sought; so as a finite Number of them may differ from the thing sought by less than any given Difference, without any further Limitation; although the Law according to which the Approximation is made, may often, if not always, be discovered. The Method of investigating these Series's being much more general, as well as the Relation of the Terms of which they are composed, Mr. *Gregory* afterwards fell into the same Method; when he saw it to have so much Advantage over the Method he had formerly used in his Quadrature of the Circle, Ellipse and Hyperbola; as we shall see afterwards.

8. After *Archimedes* there was little or nothing done in the Geometry of Curve Lines, till towards the Middle of the last Century. In the Year 1635 *Bonaventura Cavallerius* introduced his Method of *Indivisibles*: By which he endeavoured to render the Method of *Exhaustions* of the Ancients, somewhat easier and shorter. In this Method Figures are conceived as made up of infinitely small or *indivisible* Parts or Elements; a Line of an infinite Number of Points; a Surface of innumerable Lines, right or curve, as is most convenient; or perhaps of other Elements, as infinitely small Triangles, Sectors, &c. a Solid, of an infinite Number of Surfaces, plain or curve; or perhaps of other Elements; as a Sphere of innumerable small Pyramids, all terminating in the Center, and constituting the spherical Surface with their Bases, &c. In which Method, Points, Lines, and Surfaces, considered

ſidered as the Elements of which Magnitudes are compoſed, are not to be conceived as *geometrical* Points, Lines and Surfaces; but as geometrical Lines, Surfaces and Solids, of infinitely ſmall Length, Breadth and Thickneſs. By this Method many uſeful Truths were diſcovered; and demonſtrated in a much more conciſe Way than by the Method of Exhauſtions: Yet one had need proceed with great Caution, otherwiſe one may be very readily led, by it, into Errors, and falſe Concluſions. It was found of very great Uſe for inveſtigating thoſe Properties and Relations of Magnitudes; which could not be eaſily diſcovered by the former Methods: Which being once diſcovered might always be aſcertained, by reducing the Demonſtrations to the Apagogical Form of the Ancients; where there was any Scruple left as to their Truth; which indeed was frequently the Caſe. For the Foundation, upon which the Geometry of Indiviſibles is built, ſeems not to be quite ſo ſatisfying, or ſtrictly *geometrical*; in Regard the Suppoſition of finite Magnitudes being made up of infinitely ſmall and indiviſible Parts, is a Notion hard to digeſt; and, to ſay the beſt of it, is not capable of being altogether freed from ſome *Confuſion*, which ſtill remains in the Mind, after all the Pains we can be at to comprehend it.

9. After *Cavallerius* and his Scholar *Torricellius*, there aroſe a great Number of famous Men in the Mathematical Way, who have conſiderably enlarged and promoted this Science, in it's higher Parts; ſuch were M. *Des Cartes*, *Fermat*, *Huddenius*, *Gregory* of St. *Vincent*, Mr. *Huygens*, Dr. *Barrow*, and many others: But the moſt conſiderable Step by far that was made in the Geometry of Curve Lines, and the Doctrine of Quadratures, was by Dr. *Wallis* in his *Arithmetic of Infinites*, publiſhed in the Year 1656. The Doctrine of infinite Series had been very little conſidered before this Time: But he perceiving the Analogy betwixt the Terms of certain Series's and the Ordinates of certain Curves, ſought out Rules for finding the amount of all the Terms of theſe Series's: By Means of which he was capable of ſquaring all thoſe Curves, whoſe Ordinates were proportional to the Terms of ſuch Series's. He began by conſidering an arithmetical Progreſſion: From that he proceeded to Progreſſions of Powers, that is whoſe Terms were as the Squares, Cubes, Biquadrates, &c. of the Terms of an Arithmetical Progreſſion as 0, 1, 2, 3, 4, &c. then to Progreſſions of Roots, that is, whoſe Terms are as the Square Roots, Cube Roots, Biquadratic Roots, &c. of the Terms of an arithmetical Progreſſion. Afterwards he was led to conſider Progreſſions whoſe Terms are any Dimenſion whatſoever of the Terms of the Arithmetical Progreſſion, *i. e.* the Indexes or Exponents of whoſe Dimenſions are any Numbers integral, fractional,

fractional, or surd, whether positive, or negative [1]. He considered these Progressions as consisting of an infinite Number of Terms, the last Term, which represented the lowest Ordinate of the Curve, being still finite, and the intermediate Terms from o to the last, being infinite in Number, represented Ordinates applied to the Axis at infinitely small and equal Distances, betwixt the Vertex and lowest Ordinate: Or perhaps these Terms represented any other Lines Right or Curve; or any Plain or Curve Surfaces, in the Case of Solids, which were proportional to them. And he found, *in a Way of Induction*, this most general and comprehensive Proposition, *That the Sum of all the Terms of any such Series is to so many times the greatest, as Unity is to the Index or Exponent of the Power, Root, or Dimension whatsoever, encreased by* 1. Which amounts to this: Supposing 0, 1, 2, 3, 4, &c. x to be an Arithmetical Progression consisting of an infinite Number of Terms, in the natural Order of the Numbers, having the last Term x; and let 0^m, 1^m, 2^m, 3^m, &c. x^m be a Progression of Terms, which are any the same Power, Root, or Dimension whatsoever, of the former Terms, whose Exponent is denoted by m, then shall the Sum of this last Series or Progression be equal to $\frac{x^{m+1}}{m+1}$. And this is in Substance the same with the first Rule of this Analysis of Sir *Isaac Newton*'s, which was discovered by an Induction of Particulars by Dr. *Wallis*; but is demonstrated universally, by Means of an indefinite Index, by Sir *Isaac*.

10. From thence Dr. *Wallis* proceeded to the Consideration of such Progressions as have their Terms made up of Combinations of the simple Terms of the forementioned Series's, by Addition and Subtraction; and likewise of the Powers of such Sums and Differences: And thence he investigated what is in Substance the same with our Author's second Rule of his Analysis, for the Quadrature of compound Curves. By this Means he was enabled to give the Quadrature of all sorts of Parabolas, of spiral spaces, and of many plain Figures which extend themselves infinitely along an assymptote: Likewise the Contents of such solids as are generated by the Revolution of these Parabolical and Hyperbolical Curves; and thence discovered other Things, which depended upon these Quadratures, such as the rectifying of several Curve Lines, and plaining of Curve Surfaces; as he has shewn in his Arithmetic of Infinites, and other mathematical Works.

11. But he found his Method fail him in the Case of those Progressions, whose Terms were the Roots of the Sums or Differences of simple

[1] Dr. *Wallis* is reckoned the first Person who brought in the Use of fractional and negative Indexes; and likewise of indefinite and surd ones.

Terms,

Terms, called Roots universal, such as $\sqrt{r^2 - 0a^2}$, $\sqrt{r^2 - 1a^2}$, $\sqrt{r^2 - 4a^2}$, $\sqrt{r^2 - 9a^2}$, $\sqrt{r^2 - 16a^2}$, &c. $\sqrt{r^2 - r^2}$: Which he calls a Series of Terms in the subduplicate Ratio of a Series of Equals lessened by a Series of Secundans or Squares. Where if r stand for the Radius of a Circle; and a for any of the infinitely small and equal Distances of the Ordinates in the Quadrant of the Circle, beginning at it's Center, such Quadrant is equal to the Sum of all the Terms of this Progression; as he shews in Prop. 121. Arith. Inf. The same Series with the Sine of the second Term under the Vinculum, changed into it's Opposite, that is $\sqrt{r^2 + 0a^2}$, $\sqrt{r^2 + 1a^2}$, $\sqrt{r^2 + 4a^2}$, &c. $\sqrt{r^2 + r^2}$, being summed up, would give the Quadrature of the equilateral Hyperbola. He likewise shews other two Series's, by the suming up of whose Terms, the Quadrature of the Circle and Hyperbola would be found, but labouring under the same Inconveniency. Here Dr. *Wallis* stuck: But this Difficulty was removed afterwards by our incomparable Author, by his third Rule in this Analysis, as we shall see anon. However Dr. *Wallis* being very solicitous to do something towards the Quadrature of the Circle; which was his principal View when he engaged in the Prosecution of these Enquiries (as he tells us in the Preface to that Work) he thought upon another Method, which he calls *Interpolation* of Series. By which he means a Method of discovering certain intermediate Terms of a regular Series or Progression, by considering the Properties of the Progression, and the Relations of the Terms to each other. Of this he gives several Instances for finding the Area of the Circle. Of which this is one: In the Progression $1, \frac{3}{2}, \frac{15}{8}, \frac{105}{48}$, &c. whose Terms are produced by the continual Multiplication of $1 \times \frac{3}{2} \times \frac{5}{4} \times \frac{7}{6}$, &c. to find the intermediate Term betwixt 1 and $\frac{3}{2}$. Such intermediate Term he expresses by this Symbol □, even as the Square Root of 2 is expressed thus, $\sqrt{2}$; and he shews that the Circle is to the Square of it's Diameter as 1 to □. But the Result of his Enquiry was, that the Value of □ cannot be adequately expressed by any received Way of Notation; which is nothing more strange than that $\sqrt{2}$ or any other surd Root is not capable of being adequately expressed, in that Way: But whereas the Value of $\sqrt{2}$, or any other Surd, may be expressed approximately by the common Notation, so likewise he found that the approximate Value of □, was $1 \times \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \text{ \&c. in inf.}}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \text{ \&c. in inf.}}$ or $1 + \frac{1}{8}A + \frac{1}{24}B + \frac{1}{48}C + \frac{1}{80}D$ &c. *in inf.* the Capitals A, B, C, &c. denoting the first, second, third, &c. Terms.

12. When these Discoveries, which were indeed very noble and useful, and in Point of Generality and Extent, far exceeding any thing

that had been formerly done by others in the Geometry of Curve Lines, were made public, it was objected to him by Monsieur *Fermat*[a], and others, that however valuable his Discoveries were, and true in themselves, yet the Demonstrations of them in the Way of Induction by Dr. *Wallis*, did not come up to that Accuracy which a Geometrical Subject required, and the ancient Geometricians had all along observed in all their Performances. To which Dr. *Wallis* made Answer (even as he had remarked in the Arithmetic of Infinites itself) that he did not so much design to demonstrate his Discoveries, as to lay open to others the Method he used in making them; which the Ancients purposely concealed; that, notwithstanding, he thought the Proof by way of Induction was satisfying and convincing; that it would be an easy Matter for any Person moderately skilled in Geometry to demonstrate these Things with all the Pomp and Apparatus made Use of by the Ancients: But that was a Labour he never designed to undertake. However, to give some Satisfaction in this Matter, he shews by some Examples in the 78th Chap. of his Algebra, how the Propositions he had discovered, might be demonstrated after the Manner of the Ancients, in imitation of what had been done by *Archimedes* in the 10th and 11th Propositions of his Treatise of Spiral Lines: In which *Archimedes* demonstrates, what is the Sum of all the Terms of a Progression of Squares, whose Sides constitute an arithmetical Progression of Lines, having the common Difference equal to the first Term; when compared with so many times the greatest Square: And the Limits betwixt which the Sum of the Terms of such a Progression is contained, although the common Difference of their Sides be not the same with the first of them. Which he applies to the finding the Relation of the Spiral Spaces to the Circular Sectors; even as Dr. *Wallis*, by prosecuting this Affair to a much greater Length, shews how to find not only the Sum of all the Squares; but the Sum of any Powers, or Roots whatsoever, of an arithmetical Progression; and thereby to compare infinite Numbers of curvilinear Areas with right-lined Figures, and with one another. And truly, when one attentively considers this elaborate Treatise of *Archimedes*, and the other Works of that subtile and penetrating Genius, one cannot help seeing Dr. *Wallis*'s Arithmetic of Infinites, Mr. *Gregory*'s Method of Inscription and Circumscription of Polygons; and even our second *Archimedes*'s Method of prime and ultimate Ratios, beginning to discover themselves, as it were in Embryo, in order to be brought forth afterwards to open Light.

[a]. See Dr. *Wallis*'s Algebra, Chap. 79.

13. And

13. And now we are arrived at the last Step of this Progress, as it was perfected into the Method of infinite Series and Fluxions by our celebrated Author. By what Means this was done, I cannot better represent to the Reader than in the Account he gives of it himself in his Letter to Mr. *Oldenburg*, *Octob.* 24, 1676.[a]

14. " About the Beginning of my mathematical Studies, as soon " as the Works of our celebrated Countryman Dr. *Wallis* fell into my " Hands, by considering the Series's, by the Intercalation of which, " he exhibits the Area of the Circle and Hyperbola; *viz.* that in a " Series of Curves, whose common Base or Axis is x, and the Ordinates " $\overline{1-xx}|^{\frac{0}{2}}$, $\overline{1-xx}|^{\frac{1}{2}}$, $\overline{1-xx}|^{\frac{2}{2}}$, $\overline{1-xx}|^{\frac{3}{2}}$, $\overline{1-xx}|^{\frac{4}{2}}$, $\overline{1-xx}|^{\frac{5}{2}}$, *&c.* " if the Areas of every other of them, that is x, $x-\frac{1}{3}x^3$, $x-\frac{2}{3}x^3$ " $+\frac{1}{5}x^5$, $x-\frac{3}{3}x^3+\frac{3}{5}x^5-\frac{1}{7}x^7$, *&c.* could be interpolated, we " would have the Areas of the intermediate ones, of which the first " $\overline{1-xx}|^{\frac{1}{2}}$ is the Circle: To interpolate these, I observed that the " first Term in all of them was x, and that the second Terms $\frac{0}{3}x^3$, " $\frac{1}{3}x^3$, $\frac{2}{3}x^3$, $\frac{3}{3}x^3$, *&c.* were in arithmetical Progression; and that there" fore the two first Terms of the Series's to be intercalated, should be " $x-\frac{\frac{1}{2}x^3}{3}$, $x-\frac{\frac{3}{2}x^3}{3}$, $x-\frac{\frac{5}{2}x^3}{3}$, *&c.*

" For intercalating the rest, I considered that the Denominators 1, " 3, 5, 7, *&c.* were in arithmetical Progression, and that nothing " remained but to investigate the numeral Coefficients of the Numera" tors. Now these I saw, in the alternate Areas that were known, " to be the Figures which express the Powers of the Number 11, *viz.* " 11^0, 11^1, 11^2, 11^3, 11^4, *&c.* that is the first was 1; the second " 1, 1; the third 1, 2, 1; the fourth 1, 3, 3, 1; the fifth 1, 4, 6, " 4, 1, *&c.*

" Wherefore I enquired how, from the two first Figures given in " these Series's the rest might be derived; and I found that if the " second Figure be called m, the rest might be produced by a conti" nual Multiplication of the Terms of this Series $\frac{m-0}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times$ " $\frac{m-3}{4}\times\frac{m-4}{5}$, *&c.*

" For Example, let the second Term $m=4$, then $4\times\frac{m-1}{2}$, that " is 6, will be the third Term; and $6\times\frac{m-2}{3}$, that is 4, the fourth

[a] See this Letter, which contains many valuable Discoveries of Sir *Isaac* in the Com. Epist. N° 55.

" Term; and $4\times\frac{m-3}{4}$, that is 1, the fifth; and $1\times\frac{m-4}{5}$, that is 0, " the sixth Term: With which the Series, in this Case, terminates.

" Therefore I applied this Rule for intercalating the Series's. And " since, for the Circle, the second Term was $\frac{\frac{1}{2}x^3}{3}$, I put $m=\frac{1}{2}$, and " the Terms arising were $\frac{1}{2}\times\frac{\frac{1}{2}-1}{2}$, or $-\frac{1}{8}$; $-\frac{1}{8}\times\frac{\frac{1}{2}-2}{3}$, or $+\frac{1}{16}$; " $+\frac{1}{16}\times\frac{\frac{1}{2}-3}{4}$, or $-\frac{5}{128}$; and so on *in inf.* Whence I came to " understand that the Area of the circular Segment, which I wanted, " was $x-\frac{\frac{1}{2}x^3}{3}-\frac{\frac{1}{8}x^5}{5}-\frac{\frac{1}{16}x^7}{7}-\frac{\frac{5}{128}x^9}{9}$, &c.

" And the like Method may be made use of to interpolate other " Series's, and that although they be deficient by the Intervals of two " or more Terms together. Such was my first Entry upon these Studies; " which had certainly escaped me, had I not accidentally cast my Eye " upon some loose Papers that came in my Way a few Weeks ago. " But having arrived this Length, I immediately began to consider that " the Terms $\overline{1-xx}|^{\frac{0}{2}}$, $\overline{1-xx}|^{\frac{2}{2}}$, $\overline{1-xx}|^{\frac{4}{2}}$, $\overline{1-xx}|^{\frac{6}{2}}$, &c. that is " to say 1, $1-x^2$, $1-2x^2+x^4$, $1-3x^2+3x^4-x^6$, &c. might " be intercalated after the same Manner as the Areas generated by " them; and that nothing was required for this Purpose but to omit " the Denominators 1, 3, 5, 7, &c. which are in the Terms expressing " the Areas: That is to say that the Coefficients of the Terms of the " Quantity to be intercalated $\overline{1-xx}|^{\frac{1}{2}}$, or $\overline{1-xx}|^{\frac{3}{2}}$; or in general of " $\overline{1-xx}|^{m}$, arise by the continual Multiplication of the Terms of " this Series $m\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}$, &c. And thus I perceived *e. g.* " that $\overline{1-xx}|^{\frac{1}{2}}$ was equivalent to $1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6$; and $\overline{1-xx}|^{\frac{3}{2}}$, " to $1-\frac{3}{2}x^2+\frac{3}{8}x^4+\frac{1}{16}x^6$, &c. and $\overline{1-xx}|^{\frac{1}{3}}$, to $1-\frac{1}{3}x^2-\frac{1}{9}x^4$ " $-\frac{5}{81}x^6$, &c.

" And thus I was possessed of a general Method of reducing Radicals " to infinite Series, by the Rule I mentioned towards the Beginning of " my former Letter [1], and that before I was acquainted with the

" Extractions

[1] This refers to his Letter wrote to Mr. *Oldenburg* 13 *June* 1676, at the Beginning of which he had set down his famous Binomial Theorem thus, $\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ$, &c. which is what he understands by the Rule he mentioned. See

Com.

" Extractions of Roots. But once this was known, I could not remain " long ignorant of that other. For in order to prove these Operations, " I multiplied $1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6$, &c. into itself, and I found " the Product $1 - xx$, the rest of the Terms *in inf.* vanishing, by the " Continuation of the Series. And thus likewise the Series $1 - \frac{1}{3}x^2 -$ " $\frac{1}{9}x^4 - \frac{5}{81}x^5$, &c. multiplied twice into itself produced $1 - x^2$. " Which, as it afforded a certain Proof of the Truth of these Con-" clusions; so it led me to try whether, conversely, these Series's, " which thus appeared to be the Roots of the Quantity $1 - x^2$, " might not in an arithmetical Manner, be extracted out of it. " The Matter succeeded to my Mind. The Form of the Operation " in the Extraction was this. *(See it under the Examples for extracting " Roots in this Analysis. Art.* 15. *and following.)*

" Having discovered this, I entirely neglected the Interpolation of " Series's, and made use of these Operations only, as a more genuine " Foundation. And as to Reduction into infinite Series by Division, " it did not lye long concealed, being a much easier Affair. Yea I " immediately attempted the Resolution of affected Equations, and " obtained it likewise. Whence the Ordinates, the Segments of the " Axes, and any other Right Lines, were at once known, from the " Areas or Arches of the Curves being given.

" About that Time the Plague breaking out [a], obliged me to get " hence (from *Cambridge)* and think of something else. Notwith-" standing, I added to my other Discoveries, at that Time, a certain " Construction of Logarithms deduced from the Area of the Hyperbola." And after shewing how this Construction was affected, he proceeds thus:

" I am ashamed to tell to how many Places of Figures I carried " these Computations, having no other Business at that Time: For I " pleased myself too much with these Discoveries. But as soon as " that ingenious Performance of *Nicolaus Mercator* appeared, I mean " his *Logarithmotechnia* (who I doubt not, first discovered the Things " he published) I began to cool a little upon these Things, suspecting " that either he understood the Extraction of Roots; as well as Division " of Fractions; or at least that others upon the Discovery of Division, " would find out the rest, before I could be of a sufficient Age for " writing.

Com. Epist. N° 48. These two Letters of Sir *Isaac Newton* to Mr. *Oldenburg* give an Account of many of his admirable Discoveries in the Business of the Quadrature of Curves, and infinite Series; and upon that Account deserve the mathematical Reader's careful Perusal.

[a] This happened in the Years 1665 and 1666.

" Yet

" Yet about the ſame Time that Performance appeared, I commu-
" nicated to Mr. *Collins* by the Intervention of our Friend Dr. *Barrow*
" (then Profeſſor of Mathematics at *Cambridge*) a certain Compend
" of theſe Series's [3]. In which I had ſignified that the Areas and
" Lengths of all Curves; and the Surfaces and Contents of Solids,
" might be determined from Right Lines given; and converſely
" from theſe given, the others might be found: And I illuſtrated
" the Method I pointed out, by ſeveral Series's.

" After this a literary Correſpondence having ariſen betwixt us,
" Mr. *Collins*, a Perſon born for the Advancement of mathematical
" Learning, inſiſted ſtrongly upon my publiſhing theſe other Things.
" Accordingly five Years ago (1671) when at the Perſuaſion of my
" Friends, I had reſolved to publiſh a Treatiſe of the Refraction of
" Light; and of Colours, which I had then ready by me, I began
" to turn my Thoughts again to theſe Series's, and wrote likewiſe
" a Treatiſe upon them [1], that I might publiſh both together.

" But having wrote you a Letter upon the Subject of the reflecting
" Teleſcope, in which I explained briefly my Notions of the Nature
" of Light, ſomething unexpected fell out, which made me imagine
" it concerned me to write to you ſpeedily about printing that Letter.
" Immediately upon the back of this I met with frequent Interrup-
" tions, from ſeveral Perſons letters, taken up with Objections, and
" other Things, which frighted me entirely from executing my De-
" ſign; and made me find Fault with myſelf for my Imprudence,
" that by catching at a Shadow, I had been ſo far deprived of the Peace
" and Quiet of my own Mind, which is a thing of more ſubſtantial
" Worth.

" About that Time, *James Gregory*, from one certain Series of
" mine, which Mr. *Collins* had ſent him, after much Thought and Ap-
" plication (as he himſelf wrote back to *Collins*) lighted upon the
" ſame Method, and hath left a Treatiſe upon it, which we hope
" will be publiſhed by his Friends. Since, conſidering his extraordina-
" ry Genius, he muſt have added many new Diſcoveries of his own,
" which would be a great Loſs to mathematical Learning, ſhould they
" be loſt [2]. " But

[3] By which he means this Analyſis by infinite Equations.

[1] This Treatiſe is mentioned by Mr. *Collins* in two Letters publiſhed in the Com. Epiſt. N° 22, 23; and it appears, by the Account given of it by him, and by what follows in this Letter of the Author, to be the ſame with that publiſhed by the ingenious Mr. *John Colſon*, now Profeſſor of Mathematics in the Univerſity of *Cambridge*, A° 1736, under the Title of *The Method of Fluxions and infinite Series; with it's Application to the Geometry of Curve Lines.*

[2] Here we have Sir *Iſaac Newton*'s own Teſtimony that Mr. *James Gregory* diſcovered a General Method of Quadratures and Series's, of the ſame Sort with his own. This is ſufficient-ly

" But as for my Treatise I mentioned, I had not quite finished it, " when I gave over my Design: And I have had no Inclination to " this very Day, to add the other Things to it. For it wanted that " Part, in which I had designed to explain the Method of solving " such Problems as cannot be reduced to Quadratures: Although I had " laid down something of the Foundation of this also. However the " Doctrine of Infinite Series made but a small Part of that Treatise.

" I put a good many Things together, among which was the Me- " thod of drawing Tangents; which the sagacious *Slusius* communi- " cated to you some two or three Years ago; concerning which (at

ly confirmed by Mr. *Gregory*'s Letters to Mr. *Collins*, published in the Com. Epist. N° 16, 17, 18, 20, wrote in the Years 1670 and 1671; by Mr. *Oldenburg*'s Letter to Mr. *Leibnitz*, *Ap.* 15. A° 1675. N° 36. and by Mr. *Collins*'s Letter to *David Gregory*, Brother to *James*, *Aug.* 11. A° 1676. N° 47; which was the Year after *James Gregory* died. But although this be evident from these and other incontestable Proofs, yet by any thing that ever I could learn, Mr. *Gregory* left no such Treatise behind him, at his Death, as Sir *Isaac Newton* speaks of in this Place: And had there been any such, it is highly probable it would have been made public by his Nephew, Dr. *David Gregory*, late Professor of Astronomy at *Oxford*, with whom all his Uncle's Papers were left. These Papers, which are now in the Custody of Dr. *David Gregory*, Canon of *Christ*'s Church in that University, according to his known Humanity, were allowed me and another Gentleman to peruse in the Year 1736, with the Hopes of finding some such Treatise as that mentioned by Sir *Isaac Newton*; but we could not discover any such Thing among these Papers, although we saw several curious ones upon particular Subjects, which are not in Print. On the contrary, by some Letters we saw, it appears that *James Gregory* had not compiled any Treatise containing the Foundations of this General Method, a very short Time before his Death. So that all that can be known about his Method, can only be collected from his Letters published in the Com. Epist. and the short History of his Mathematical Discoveries compiled by Mr. *Collins* in twelve Sheets after his Death, at the Desire of the Gentlemen of the Royal Academy at *Paris*: Which was reposited among the Archives of the Royal Society at *London*. See Com. Epist. N° 47. The Principles of his Method seem to have been few and simple; which he could easily retain in his Mind, and apply as Occasion offered. And I am much mistaken, if they have not been the very same, or nearly the same, with those of Sir *Isaac Newton*'s Method; as may be collected from his Letters to Mr. *Collins*, mentioned already. In one of which he says, *I had not observed that Mr.* Newton's *Series for the Zone of a Circle (which you did me the Pleasure to transmit some Time ago) together with an infinite Number of such Series's, are easy Consequences from that one which I sent you about Logarithms*; viz. *a Logarithm being given, to find it's Number*; *or to change the Root of any pure Power into an infinite Series*. And in another he says: *As to Mr.* Newton's *General Method, I am possessed of it in some Measure, I think, as well with respect to mechanical as geometrical Curves*. And, after giving him a considerable Number of different Series's for the Circle, Ellipse, and Hyperbola, he adds: *I would not incline to give you so much Trouble*; *nor have I a Mind to print any thing, unless it be to give a new Edition of my Quadrature of the Circle, with some trifling Additions. As to my Method of finding the Roots of all Equations*; *one Series discovers one Root only*; *but there are infinite such for every Root: Although it requires some Pains to begin the Series right, and to distinguish the Root to which it belongs. But I shall perhaps write you more of this afterwards. You need be in no Strait to communicate whatever I send you, to any Person: For I am not at all anxious, whether it be published under my Name, or that of any other*. This last Sentence shews a worthy Disposition of Mind, which regarded the Discovery of Truth only, without affecting the Honour of being the Discoverer. From these and some other Passages in other Letters in the Com. Epist. it appears that Mr. *Gregory*'s Method he made use of after the Year 1670, and Sir *Isaac Newton*'s were in Effect the same.

" the

" the Desire of Mr. *Collins*) you writ back that I was in the Know-" ledge of the same. We fell upon it, it is true, in different Ways. " When one knows the Foundation I proceed upon, he cannot draw " Tangents, after any other Manner, unless he designedly go out of " the right Way.

" Moreover there is no Stop, whatever Way the Equations be af-" fected with Radicals, affecting one or both the indefinite Quantities; " but without any Reduction of such Equations (which would render " the Work frequently insupportable) the Tangent is directly drawn. " And the Thing is much the same in the Questions *de Maximis* & " *Minimis*; and some others, which I shall not here mention.

" The Foundation of these Operations, which is obvious enough " in itself (because I can't here pursue the Explication of it) I chuse " rather to conceal in Cypher thus *6accdæ13eff7i3l9n4o4qrr4s9t12vx*[1].

" Upon this Foundation I have endeavoured likewise to render the " Speculation concerning the Quadrature of Curves, more simple; " and have arrived at some more general Theorems."

Here he gives an Account of these Theorems, and some Examples of their Application: By which it appears, that, by applying the Foundation just now mentioned to render the Speculations about the Quadrature of Curves more simple, he means the Substance of the Doctrine contained in his Treatise of the Quadrature of Curves; which likewise appears from that Part of the *Method of Fluxions and Infinite Series* itself, published by Mr. *Colson*, which treats of this Affair. (*See Prob.* 9. *throughout, especially Art.* 63.) And after he has given some Account of these Theorems; and shewn how, at length, they become very complex, when the Progressions of the Curves are continued; he adds, " So that I am of Opinion that they could scarce be found " out by the Transmutation of Figures, which *James Gregory*[2] and " some others have made use of, without some other Foundation.

" For my own Part, I could not arrive at any thing that was *general* " in these Matters, until I abstracted from the Contemplation of Fi-" gures, and reduced the whole Affair to the simple Consideration of " the Ordinates only.

[1] This being decyphered, is, *Datâ Equatione quotcunque Fluentes Quantitates involvente, Fluctiones invenire; & vice versa. i. e.* An Equation being given, involving any Number of Flowing Quantities, to find the Fluxions; and contrarily.

[2] This is the Method of Quadratures which Mr. *James Gregory* had formerly made use of in his Book published at *Padua* in 1668, under the Title of *Geometriæ pars universalis inserviens Quantitatum Curvarum Transmutationi & Mensuræ*.

15. And

15. And thus I have given some Account [3] of the Rise and Progress of the Contemplation of Curves and Infinite Series: And therefore come now, as I proposed, to give some Account of the Contents of this Treatise of *Analysis by Equations of an infinite Number of Terms*; and to shew wherein it agrees with the other Treatise of the Quadrature of Curves; and wherein they differ.

16. As to the Analysis by Equations of an infinite Number of Terms; we may take it up as containing the following Things.

1°. The Quadrature of simple Curves; where the Ordinate is equal to one Term only, including some Power of the Base or Absciss x, any Way involved with known Quantities; thus generally expressed, $y = ax^{\frac{m}{n}}$.

2°. The Quadrature of compound Curves: *i. e.* where the Ordinate is equal to an Aggregate of such simple Terms.

3°. The Quadrature of all other Curves, where the Value of the Ordinate y, or some Part of it, is more complex than in the two preceding Cases. And here, such complex Value, or Part of the Value of y, may be of three Kinds. 1°. A Fraction, which, after it is reduced as far as may be, by the common Rules of Algebra, still retains the Denominator a compound Quantity, including the Absciss x in one or more of it's Terms, as $\frac{aa}{b+x}$. 2°. Some complex Radical, involving the Absciss x in one or more Terms below the Vinculum, as $y = \sqrt{aa-xx}$. 3°. The Root of an affected Equation, as $y^3 + a^2y + axy - 2a^3 - x^3 = 0$. In all these Cases, we are here taught how to reduce the Value of y into an infinite Series of simple Terms. For doing of which we have three different Methods, corresponding to these three different Cases; which are Division, Extraction of the Roots of compound Quantities, and Extraction of the Roots of affected Equations; all explained here in Order. And under the Head of the Extraction of Roots of affected Equations, our Author shews the Method of doing it in *numeral* Equations, as well as *specious*.

[3] In this short Account of the Doctrine of Curves and Infinite Series, I have omitted making Mention of a great Number of ingenious and learned Men, both of our own Nation and Foreigners, who by their Industry and Labours, have deserved well of Mathematicks in these Particulars. For, an exact Account of their Discoveries and Improvements, both before and since our Author's Time, would take up a large Volume of itself; which I think might be a very useful Work. Neither have I entered into the Dispute betwixt the *English* and Foreigners about the first Inventor of the Method of Fluxions. With respect to which I am of Opinion, that no disinterested Person, who will take the Trouble to read the *Commercium Epistolicum*, published by Order of the Royal Society, can be in any Doubt.

4°. He likewise shews how the foregoing Doctrine of the Quadrature of Curves, may be applied to other Subjects; and gives an Example in finding the Lengths of Curves.

5°. He lets us see how we may return again from the Area, or Length, &c. given, to the Base or Abscifs of the Curve, by Means of the Resolution of affected Equations.

6°. He shews how this Doctrine may be applied to mechanical Curves.

7°. He gives a Demonstration of the Quadrature of simple Curves; which in Effect contains a Demonstration of the Quadrature of all others. And here, by the Way, he shews what Way you may find as many Curves as you please, that can be squared.

Finally, you have a Demonstration of the preceding Method of resolving affected Equations.

17. Wherefore by what is delivered in this short Compend of Sir *Isaac Newton*'s first Discoveries, we are taught the Method of finding the Areas of all Sorts of Curves, either exactly or approximately. But this wonderful Person, not content with this noble Discovery, which infinitely surpassed all that had been done before in this Affair, still proceeded further, and made considerable Improvements in it; which are delivered in the other twin Performance, his Treatise of *the Quadrature of Curves.* For it often happens that those Curves, whose Areas can only be exhibited by *infinite* Equations, according to the Methods laid down in this Analysis, are capable of being expressed by finite Equations, and so may be geometrically squared; or they may be compared with other more simple Curves, whose Areas may in some Respect be considered as known; such as the Conic Sections. Accordingly he has taught us the Way of doing this in his Quadrature of Curves: and thereby rendered this Speculation more simple, and advanced that Doctrine to a greater Degree of Perfection. For as the Solution of a Problem by an Equation of an higher Order, when it may be solved by one of a lower, is reckoned ungeometrical, because not so simple as the Nature of the Problem will admit; so, much rather a Solution by means of an infinite Equation is not to be admitted, when the same Thing can be done by means of a finite Equation. For if this can be done, the Curve is geometrically squared, *i. e.* compared with rectilinear Figures. And when this cannot be done, yet it is of considerable Use to reduce the Areas of Curves, to the most simple Curves, with which they admit of a geometrical Comparison. The fifth and sixth Propositions of the Quadrature of Curves present us with two General Theorems for the Areas of all Curves which are thus

thus generally expressed $y = z^{\theta-1} \times \overline{e + fz^{\eta} + gz^{2\eta} + hz^{3\eta} \&c.}|^{\lambda-1} \times \overline{a + bz^{\eta} + cz^{2\eta} + dz^{3\eta} \&c.}$ or $y = z^{\theta-1} \times \overline{e + fz^{\eta} + gz^{2\eta} \&c.}|^{\lambda-1} \times \overline{k + lz^{\eta} + mz^{2\eta} \&c.}|^{\mu-1} \times \overline{a + bz^{\eta} + cz^{2\eta} \&c}$: y being the Ordinate; z the Abscifs; $\theta, \eta, \lambda, \mu$, any Numbers whatsoever, integral or fractional, positive or negative; the other Letters representing any known Quantities. Where the Series's expressing the Areas often terminate; and thereby shew the Curve to be geometrically quadrable; when at the same Time the Area exhibited by the Method in the Analysis, can be exprest no other Way but by an infinite Series.

For understanding of which the Reader may consult these two Propositions, and our Explication of them.

18. There are other Cases of Curves likewise, which may be geometrically squared by the ninth and tenth Propositions: whose Areas can only be exprest by infinite Equations, according to the Method of the Analysis. And by the tenth Prop. of the Treatise of Quadratures, you can always discover what are the most simple Figures, whether rectilinear, or curvilinear, with which any Curve can be geometrically compared, whose Ordinate y is determined from the Abscifs z given, by an Equation not affected. By Cor. 1. of that Prop. you can find the most simple Figures with which any Curve can be compared, whose Ordinate y is the Root of an affected quadratic Equation: such as $a^2y^2 + z^2y^2 = 2a^3y + 2z^3y - z^4$. And therefore also, every Curve, whose Ordinate is defined by *any* affected Equation, which can be made to pass into another Curve whose Ordinate is defined by a *quadratic* affected Equation, by the Method taught in Cor. 8. Prop. 9. may be geometrically squared; or compared with the most simple Figures with which it admits of a geometrical Comparison. Moreover by Cor. 2. of that Prop. every Curve whose Ordinate is defined by such an affected Equation, as may be made to pass into an Equation not affected, by the Method taught in Cor. 7. Prop. 9. can be geometrically squared; or compared with other the most simple Curves, it admits of being compared with. Under this last head are included all such Curves, whose Equation consists of three Terms, whatever Way affected: thus generally exprest $az^l + bz^m y^s + cy^t = 0$.

19. These are the Cases of Curves whose Areas may be found by what is taught in our Author's Quadrature of Curves. By which also we find when they admit of a geometrical Quadrature, or Comparison with right-lined Figures: which often happens, when their Areas can only be exprest by an infinite Series, by the Methods taught

in this Analysis. For when the Ordinate, or any Part of it, is exprest by a complex Fraction or Radical, the Area which is exhibited by the Analysis is always exprest by an infinite Series: and it very seldom happens to be otherwise, when the Ordinate is the Root of an affected Equation. Wherefore it hence appears that our Author's Treatise of the Quadrature of Curves is a considerable Improvement of that Subject, and what is connected with it. Besides what has been said, it contains the fundamental Principles of the Method of Fluxions; many useful Things respecting the Comparison and Transmutation of Curves; and shews in the last Scholium, how far the Quadrature of Curves serves for finding the Fluents from the Fluxions given. Finally it presents us with two most useful Tables of Quadratures: the first of which exhibits the Areas of the more simple Kinds of Curves that can be squared; the other contains the more simple Kinds of Curves that may be compared with the Conic Sections. So that the Areas in both Cases may be found by simple Inspection.

20. But in regard there are Curves, whose Areas cannot be found at all by the Methods taught in the Quadrature of Curves, *viz.* those that are defined by Equations that come not under any of the Heads mentioned in Art. 17. and 18. above: therefore, in these Cases, we must have Recourse to the Method taught in this Analysis, by the Resolution of affected Equations into infinite Series. Which is one of the chief Reasons of annexing this Analysis with the Explication of it, to the foregoing Treatise of the Quadrature of Curves.

SECT. II.

Explication of the first Part of the Author's third Rule; which shews the Method of finding the Areas of those Curves, which require the Reduction of complicate Fractions and Radicals: contained in Art. 12 —— 20.

21. AS to the two first Rules for the Quadrature of *simple* and *compound* Curves, I think I have so fully explained the Doctrine contained in them, in the Notes upon the preceding Treatise, especially in Sect. 5. and the Explication of Prop. 5. that it would be a needless Repetition to say any thing upon that Head, in this Place.

22. I told

22. I told already that there are three different Ways of reducing Quantities into infinite Series. 1°. By *Division.* 2°. By extracting the Roots of *surd Quantities.* 3°. By extracting the Roots of *affected Equations.* The first two are to be explained in this Section.

23. Any Fraction, whose Denominator consists of more Terms than one after the Fraction is brought to it's most simple Expression, may be reduced to an infinite Series of simple Terms (*i. e.* which either have no Denominators; or else only simple ones) by dividing continually; as is done in the Case of Division in Decimal Arithmetic, where there is a Remainder: in which Case, as is commonly known, the Quotient may be continued in decimal Fractions: which very often run out infinitely. The Conveniency of thus reducing Fractions having compound Denominators; and the Use to be made of it in the Quadrature of Curves, appears from what our Author shews of it's Application to the equilateral Hyperbola; and some other Curves.

24. In the Application of this Operation to the squaring of Curves, where the Ordinate y, or any Part of it, is exprest by a Fraction; it is only then that the Denominator of the Fraction is reckoned compound, and the Fraction to be turned into an infinite Series, when, after it is reduced by the common Rules for reducing of Fractions in Algebra, the Denominator is still made up of several Terms, of which one or more include the Base or Abscis x: and no otherwise. Thus $\frac{aa}{b+x}$, $\frac{4a-4x}{2a^2-4ax+2x^2}$, are such complex Fractions; but $\frac{a^2+ax}{a-b}$, $\frac{a^2+2ax+x^2}{3a^2+3ax-ab-bx}$, are not reckoned such: since in the first of them, the Abscis x does not enter the Denominator; and the other, when reduced to it's most simple Expression, becomes $\frac{a+x}{3a-b}$, where the Denominator doth not include x neither. But in the other two Examples, the first, which is already in it's lowest Terms, includes x in one Term of it's Denominator; and the other, after Reduction, becomes $\frac{2}{a-x}$, which still includes x in one Part of it's Denominator.

25. The Author's first Example is for finding the Area of the external rectangular Hyperbola ABDC (*See the Fig. in the Analysis Art.* 12.) The Manner of Operation in dividing is clearly laid before us, and contains no Difficulty in it. In this Example, $AB = x$ the Abscis, which commences at the given Point A in one of the Assymptotes; $BD = y$ the perpendicular Ordinate; the Distance betwixt

twixt the Center and the Point A is denoted by b; and a denotes the Side of a Square inscribed betwixt the Hyperbola and it's Assymptotes; and AC is parallel to BD. We spoke of this Case at the End of Sect. 5. of the preceding Treatise: which the Reader may consult. To make the Series, which arises by this Operation, for the Value of the Area ABDC, to converge, it is necessary that b be greater than x. We may also suppose x to be negative, so that, the Point A remaining, the Point B fall betwixt it and the Center: then you have $y = (\text{BD} =) \frac{a^2}{b-x}$: and if you actually divide aa by $b-x$, the Quotient is $\frac{a^2}{b} + \frac{a^2x}{b^2} + \frac{a^2x^2}{b^3} + \frac{a^2x^3}{b^4}$, &c: therefore the Hyperbolical Area adjacent to AB, in this Case, by the Author's second Rule, is $\frac{a^2x}{b} + \frac{a^2x^2}{2b^2} + \frac{a^2x^3}{3b^3} + \frac{a^2x^4}{4b^4}$ &c: which differs from the other in no other Respect, but that all the Terms are positive, which in the other are alternately positive and negative: so that the other Series, by supposing x negative, becomes the same with this. Therefore if the Area adjacent to any Part of the Assymptote be required, let such Part be bissected: and calling b the Distance from the Center to the Point of Bissection (which is considered as a constant Quantity) and the variable Quantity x standing for the half of the foresaid Part of the Assymptote; and a being the same as before: the Area adjacent to the Part $2x$ will be $\frac{2a^2x}{b} + \frac{2a^2x^3}{3b^3} + \frac{2a^2x^5}{5b^5} + \frac{2a^2x^7}{7b^7}$ &c. made by adding together the Values of the two Areas lying on each Side of the Ordinate which bissects the Part of the Assymptote $2x$. Which Series will always converge, whatever be the Value of x, since, by the Hypothesis it is less than b. And the Convergency is much quicker than in the preceding Series's.

26. If we attempt the Division of aa by $x+b$, putting x first in the Divisor, it will give $y = \frac{a^2}{x} - \frac{a^2b}{x^2} + \frac{a^2b^2}{x^3} - \frac{a^2b^3}{x^4}$ &c. $= a^2x^{-1} - a^2bx^{-2} + a^2b^2x^{-3} - a^2b^3x^{-4}$: therefore, by the second Rule, the Area is $\frac{a^2x^0}{0} + a^2bx^{-1} - \frac{a^2b^2x^{-2}}{2} + \frac{a^2b^3x^{-3}}{3}$ &c. or $\frac{a^2}{0} + \frac{a^2b}{x} - \frac{a^2b^2}{2x^2} + \frac{a^2b^3}{3x^3}$ &c. which denotes the Area BDα upon the other Side of the Ordinate BD opposite to A, (*See Sect. 5. of the preceding Treatise at the beginning*) and infinitely extended along the Abscis produced. And this cannot be found, in regard the first Term $\frac{a^2}{0}$ is infinitely great.

27. But

27. But in our Author's second Example, *viz.* $y = \frac{1}{1+x^2}$, (Art. 13.) which belongs to an Hyperbola of a higher Order, if you divide 1 by $x^2 + 1$, putting x^2 foremost, the Quote is $x^{-2} - x^{-4} + x^{-6} - x^{-8}$ &c. $= y$: and therefore, by the second Rule, the Area is $- x^{-1} + \frac{x^{-3}}{3} - \frac{x^{-5}}{5} + \frac{x^{-7}}{7}$ &c. Which is the Value of the Area BDα, extending itself along the Abscifs infinitely produced beyond the Ordinate BD, the Signs of all the Terms being changed into their Oppofites. Which Value is finite, although the Area be infinitely extended: and the Series converges, if x be greater than 1. Whereas the other Series, *viz.* $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7$ &c. denotes the Area ABDC adjacent to the Abscifs AB: which converges when x is lefs than 1. Thefe two Expreffions of the Area are exactly the fame which the fifth Prop. of the Treatife of the Quadrature of Curves would be found to give, by putting $y = \overline{1 + x^2}|^{-1}$; or $x^{-2} \times \overline{1 + x^{-2}}|^{-1}$, and comparing them with the general Form in that Prop. *viz.* $y = z^{\theta - 1} \times \overline{e + fz^{\eta}}|^{\lambda - 1} \times \overline{a + bz^{\eta}}$ &c.

28. The Divifion in the Author's laft Example (Art. 14.) *viz.* $\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1 + x^{\frac{1}{2}} - 3x} = y$, is performed in the following Manner.

$$1 + x^{\frac{1}{2}} - 3x)\ 2x^{\frac{1}{2}} - x^{\frac{3}{2}}\ (2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^2 \ \&c. = y$$

$$\begin{array}{l}
2x^{\frac{1}{2}} + 2x - 6x^{\frac{3}{2}} \\
\hline
-2x + 5x^{\frac{3}{2}} \\
-2x - 2x^{\frac{3}{2}} + 6x^2 \\
\hline
+7x^{\frac{3}{2}} - 6x^2 \\
+7x^{\frac{3}{2}} + 7x^2 - 21x^{\frac{5}{2}} \\
\hline
-13x^2 + 21x^{\frac{5}{2}} \\
\&c.
\end{array}$$

And

And if you invert the Order of the Terms of the Divisor and Dividend, the Operation is as follows.

$$-3x+x^{\frac{1}{2}}+1\,\big)\,-x^{\frac{3}{2}}+2x^{\frac{1}{2}}\,\Big(\frac{x^{\frac{1}{2}}}{3}+\tfrac{1}{9}-\frac{14}{27x^{\frac{1}{2}}}-\frac{11}{81x}-\frac{53}{243x^{\frac{3}{2}}}\ \&c.=y$$

$$\underline{-x^{\frac{3}{2}}+\tfrac{1}{3}x+\tfrac{1}{3}x^{\frac{1}{2}}}$$

$$-\tfrac{1}{3}x+\frac{5x^{\frac{1}{2}}}{3}$$

$$\underline{-\tfrac{1}{3}x+\tfrac{1}{9}x^{\frac{1}{2}}+\tfrac{1}{9}}$$

$$+\frac{14x^{\frac{1}{2}}}{9}-\tfrac{1}{9}$$

$$\underline{+\frac{14x^{\frac{1}{2}}}{9}-\tfrac{14}{27}-\frac{14}{27x^{\frac{1}{2}}}}$$

$$+\tfrac{11}{27}+\frac{14}{27x^{\frac{1}{2}}}$$

$$\underline{+\tfrac{11}{27}-\frac{11}{81x^{\frac{1}{2}}}-\frac{11}{81x}}$$

$$+\frac{53}{81x^{\frac{1}{2}}}+\frac{11}{81x}$$

&c.

Therefore the Area in the former Case, which is adjacent to the Abscifs, is $\frac{4}{3}x^{\frac{3}{2}}-x^{2}+\frac{14}{5}x^{\frac{5}{2}}-\frac{13}{3}x^{3}$ &c. But in the other Value of y, there is included the Term $-\frac{11}{81x}$, where x is of one Dimenfion in the Denominator: which therefore denotes the Ordinate applied to the Affymptote in the Conical Hyperbola: the Quadrature of which Term produces an Area infinitely great. By fuch a Series you can neither find the Area adjacent to the whole Abfcifs x; nor adjacent to the Abfcifs produced infinitely beyond the Ordinate; but only the Area adjacent to fome intermediate Part of the Abfcifs: which would be the Cafe likewife here, although there were no fuch Hyperbolical Term, in regard the Powers of x would be found in the Numerators of fome of the Terms; and in the Denominators of other of the Terms of the Value of the Curvilinear Area. For underftanding of this confult Sect. 5. of the former Treatife.

From

From these Examples of the Quadrature of Curves, which require a Reduction of their Ordinates into infinite Series by Means of Division, you may understand how you are to proceed in any similar Case of Curves, where such Operation is required [1].

29. In the whole Business of Infinite Series, whether arising by Division; or otherwise, the Operation resembles those Operations in *Decimal* Arithmetic, where the Quotient runs out into a Number, which approximates to the Value of the Thing sought continually, but so as that it never terminates. Such is that Operation by which Vulgar Fractions are reduced to Decimal Fractions, when the Fractions cannot be wholly exprest in Decimals; and that whereby the Roots of such Numbers are extracted, which are not true Powers of their Kind. In which Cases you approximate to the Quotient, or Root, in Decimals; and may approach nearer to it than by any given Difference; but can never arrive at the full Value. The chief Difference betwixt these infinite Series's in Decimal Arithmetic, and those in the Literal or Specious, is, that, in the former, there is only one Scale or Progression of Terms, which decreases in a decuple Proportion; and the Coefficients

[1] Mr. *James Gregory*, in his *Vera Circuli & Hyperbolæ Quadratura*, published in the Year 1667, gave a converging Series; or at least shewed the Constitution of a converging Series, which proceeds by Pairs of Terms, of that Sort I have mentioned above, Art. 7. by which he shews how to approach to the Area of the Hyperbola, as well as that of the Circle, as near as you please. In *April* next Year, Lord Viscount *Brounker* published an infinite Series (of that kind, which I have called above, Art. 6. the *Newtonian*) for the same Purpose: which Series he expresses thus $\frac{1}{1\times 2}+\frac{1}{3\times 4}+\frac{1}{5\times 6}+\frac{1}{7\times 8}$, *&c. in inf.* That same Year Mr. *Nicholas Mercator*, an *Holsatian* by Birth, but who had spent a great Part of his Life in *England*, published his *Logarithmotechnia*, in which he shewed how Lord Viscount *Brounker*'s Series might be found, by reducing a complex Fraction to an infinite Series of simple Terms by Division. Which was but a small Improvement upon what Dr. *Wallis* had shewn in his *Math. Univ.* Chap. 33. with respect to Division (although the Doctor had not given any explicate Example, in which the Number of Terms in the Quotient was infinite, as some alledge) *Mercator* having reduced the complex Fraction to an infinite Series of simple Terms, had no more ado after that, but apply the same Person's Method of Quadratures in his *Arith. Inf.* for squaring the several Terms. Upon the Publication of the *Logarithmotechnia*, Dr. *Wallis* illustrated the Discovery; and gave another infinite Series for the same Purpose, in the *Philosoph. Trans.* for *Aug.* 1668. *Mercator* and *Wallis*'s Series's were in effect the same with those mentioned above, Art. 25. *viz.* $\frac{a^2x}{b}-\frac{a^2x^2}{2b^2}+\frac{a^2x^3}{3b^3}-\frac{a^2x^4}{4b^4}$, &c. and $\frac{a^2x}{b}+\frac{a^2x^2}{2b^2}+\frac{a^2x^3}{3b^3}+\frac{a^2x^4}{4b^4}$, &c. the former being *Mercator*'s; the latter Dr. *Wallis*'s. Towards the End of the same Year 1668, Mr. *James Gregory* published his *Exercitationes Geometricæ*: in which he promoted and enlarged *Mercator*'s Discovery, and gave a geometrical Demonstration of it, by means of summing up the Secants of a circular Arch. However it appears from what we have taken out of Sir *Isaac Newton*'s Letter to Mr. *Oldenburg*, recorded above, and other undoubted Evidences, that Sir *Isaac* was posseft of his *General Method of reducing Complicate Fractions, Radicals, and affected Equations, into infinite Series's*, above two Years before the *Logarithmotechnia* was published.

are all positive Integers below 10; whereas in the latter, which is of a more general and indefinite Nature, the Scales or Progressions may be infinitely varied, so as to decrease in a decuple, or any other Proportion whatsoever; and the numeral Coefficients may be any Numbers, integral or fractional; positive or negative. Thus we may conceive any integral Number as a Series of Powers with their proper Coefficients, *i. e.* the Number 86027 may be conceived as expressed thus; $8\times\overline{10}|^{4} + 6\times\overline{10}|^{3} + 0\times\overline{10}|^{2} + 2\times\overline{10}|^{1} + 7\times\overline{10}|^{0}$: and if the Number be continued into Places below the Place of Units, as thus 86027.4638, then it is equivalent to this $8\times\overline{10}|^{4} + 6\times\overline{10}|^{3} + 0\times\overline{10}|^{2} + 2\times\overline{10}|^{1} + 7\times\overline{10}|^{0} + 4\times\overline{10}|^{-1} + 6\times\overline{10}|^{-2} + 3\times\overline{10}|^{-3} + 8\times\overline{10}|^{-4}$: the fractional Part of which may be otherwise denoted thus $\frac{4}{\overline{10}|^{1}} + \frac{6}{\overline{10}|^{2}} + \frac{3}{\overline{10}|^{3}} + \frac{8}{\overline{10}|^{4}}$. We may likewise suppose 0.1 to be the Root instead of 10, and then the preceding Number 86027.4638 will stand thus $\frac{8}{\overline{0.1}|^{4}} + \frac{6}{\overline{0.1}|^{3}} + \frac{0}{\overline{0.1}|^{2}} + \frac{2}{\overline{0.1}|^{1}} + \frac{7}{\overline{0.1}|^{0}} + 4\times\overline{0.1}|^{1} + 6\times\overline{0.1}|^{2} + 3\times\overline{0.1}|^{3} + 8\times\overline{0.1}|^{4}$: the fractional Part of which, *viz.* .4638 may likewise stand thus $\frac{4}{\overline{0.1}|^{-1}} + \frac{6}{\overline{0.1}|^{-2}} + \frac{3}{\overline{0.1}|^{-3}} + \frac{8}{\overline{0.1}|^{-4}}$. All which Series's would be infinite if the Decimal Fraction were infinite. And after the same Manner a great Variety of other Scales, or Forms of Progression, might be devised, with suitable Roots for different Kinds of Arithmetic: such as the sexagenary used in Astronomy; in which the Root is 60 or it's Reciprocal. Further it may be observed, that the Operations in Decimal Arithmetic, by which the Quotient, or Quantity sought, is discovered, correspond to the Operations relating to infinite Series in specious Arithmetic; in this Respect that every new Step of the Operation, in both Cases, by which the constituent Parts of the Quotient are found, makes as great an Advance towards the Supply of what the Quotient is yet deficient, as the Nature of the Progression will admit.

30. In those Series's which belong to the literal Arithmetic, the Indexes or Exponents of the Powers or Dimensions, may be fractional as well as integral; and are sometimes left indefinite: but they are always supposed to proceed by equal Differences. The numeral Coefficients

cients are likewise sometimes left indefinite. And then these indefinite Exponents and Coefficients are exprest by Letters. Such a general and indefinite converging Series may be represented thus, $Ax^m + Bx^{m\pm n} + Cx^{m\pm 2n} + Dx^{m\pm 3n}$, *&c.* where $x^{\pm n}$ denotes the Root of the Scale or Progression; A, B, C, D, *&c.* any Coefficients, positive or negative; m and n any positive Numbers whatsoever: m may be nothing; and so may one or more of the Quantities, A, B, C, *&c.*

31. But every Series in specious Arithmetic, when it is to be applied in Practice, must have it's several Species and indefinite Characters, reduced to determinate and known Values, that so the Value of the several Terms, and consequently of the whole, may be collected. Thus suppose we are going to make use of the Series $\frac{2a^2x}{b} + \frac{2a^2x^3}{3b^3} + \frac{2a^2x^5}{5b^5} + \frac{2a^2x^7}{7b^7}$ *&c.* (See above Art. 25.) for finding the hyperbolical Area ABDC (See the Fig. referred to in that Place) where AB $= 2x$: a, and b, the same as there mentioned: suppose $a = 1 = b$; and $x = 0.1$ or AB $= 0.2$, then you'll have ABDC $(= \frac{2a^2x}{b} + \frac{2a^2x^3}{3b^3} + \frac{2a^2x^5}{5b^5} + \frac{2a^2x^7}{7b^7}$ *&c.*$) = 0.2 + \frac{0.002}{3} + \frac{0.00002}{5} + \frac{0.0000002}{7}$ *&c.* or $0.2 + 0.0006666$ *&c.* $+ 0.000004 + 0.000000028$ *&c.* $= 0.20067069$ *&c.*

I now proceed to explain the Reduction of complex Radicals into infinite Series: and the Quadratures depending thereon: of which our Author treats in Art. 15—20.

32. In the Account I have given above of the Rise and Progress of the Doctrine of Quadratures and Series's, I have shewn after what Manner, and by what Steps our Author discovered the Reduction of Radicals into infinite Series. As for the Manner of Operation, it resembles the like Operation in Decimal Arithmetic for extracting the Roots of Numbers which are not true Powers of their Kind: In which Case, the Root or Quotient runs out into an infinite Decimal. Wherefore this being understood; and the Reader being supposed to be acquainted with the Operations of literal Arithmetic; and the Foundation upon which the Extraction of Roots is built, I shall not detain him in explaining this Operation, when applied to the Reduction of complex Surds into infinite Series, further than to represent the Parts of the Operation in the Example Art. 15. a little more fully.

33. Let it be proposed then to extract the Square Root of $aa + xx$; or to find the Value of y in an infinite Series, supposing that $y = \sqrt{aa + xx}$. The Manner of proceeding is as follows.

$$aa+xx\left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}+\frac{7x^{10}}{256a^9}\right.\ \text{\&c. in inf.}$$

$$\begin{array}{ll}
 & aa \\
2a+\frac{x^2}{2a}\Big) & * + xx \\
\times\frac{x^2}{2a} & +xx+\frac{x^4}{4a^2} \\
\hline
2a+\frac{x^2}{a}-\frac{x^4}{8a^3}\Big) & * - \frac{x^4}{4a^2} \\
\times-\frac{x^4}{8a^3} & -\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6} \\
\hline
2a+\frac{x^2}{a}-\frac{x^4}{4a^3}\ *\ \Big) & * + \frac{x^6}{8a^4}-\frac{x^8}{64a^6} \\
\times+\frac{x^6}{16a^5} & +\frac{x^6}{8a^4}+\frac{x^8}{16a^6}-\frac{x^{10}}{64a^8}\ * \\
\hline
2a+\frac{x^2}{a}\ *\ *\ *\ \Big) & * - \frac{5x^8}{64a^6}+\frac{x^{10}}{64a^8}\ * \\
\times-\frac{5x^8}{128a^7} & -\frac{5x^8}{64a^6}-\frac{5x^{10}}{128a^8} \\
\hline
 & * + \frac{7x^{10}}{128a^8}\ \text{\&c.} \\
 & \text{\&c.}
\end{array}$$

Where, having set down the Quantity $aa+xx$, whose Square Root is to be extracted, you take the Square Root of a^2 the first Term, *viz.* a, which you place in the Quotient: and then subtracting a^2 from the proposed Quantity, the Remainder $+x^2$ is divided by $2a$, which gives $+\frac{x^2}{2a}$ to be subjoined to a in the Quotient: You annex it likewise to the Divisor $2a$, making $2a+\frac{x^2}{2a}$, and then multiplying this augmented Divisor by it, place the Product $+xx+\frac{x^4}{4a^2}$ under the Resolvend or Dividend $+xx$: subtract it from it, and the Remainder $-\frac{x^4}{4a^2}$ gives a new Resolvend or Dividend. For forming the Divisor to which, take double the Part of the Root or Quotient already found, *viz.* $2a+\frac{x^2}{a}$: which place for the Divisor as you see, leaving Room for another Term to be subjoined. Divide $-\frac{x^4}{4a^2}$ by $2a$, the first Term of this Divisor; which gives $-\frac{x^4}{8a^3}$ to be subjoined to the preceding Part

Part of the Quotient; and likewise to the Divisor: then multiply the augmented Divisor $2a + \frac{x^2}{a} - \frac{x^4}{8a^3}$ by it, forming thereby the Ablativum $-\frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6}$, to be subtracted from the Resolvend $-\frac{x^4}{4a^2}$: and so you proceed, repeating always the same Operation, until you obtain as many Terms in the Root as you desire. Where you may observe that those Terms may be neglected in the Operation, which have no Influence upon the Quotient within the proposed Limits. Thus you'll observe that if the Quotient is to be continued to the 10th Power of x only, those Terms are neglected, which can be of no Use within these Bounds: which I have accordingly marked with Asterisks.

34. The Operation being finished as far as you intend, the Truth of it may be proved by multiplying the Root into itself: for that ought to produce $a^2 + x^2$, the subsequent Terms *in inf.* destroying one another: which you will find to be so here, as far as the Product can be carried by the defective Root found, *viz.* to as many Terms as have been found in the Quotient. And after the same Manner you are to proceed in other Examples.

35. And thus having found $y = (\sqrt{a^2 + x^2} =)\ a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9}$ &c. the Area of the Curve whose Property is defined by $y^2 = a^2 + x^2$, or $y = \sqrt{a^2 + x^2}$, is (by the second Rule) $ax + \frac{x^3}{6a} - \frac{x^5}{40a^3} + \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7} + \frac{7x^{11}}{2816a^9}$ &c. Now if CD (*See the Fig. Art.* 15.) be a rectangular Hyperbola, A it's Center, AC $(= a)$ the half of the determinate Axis; and consequently of the Latus Rectum, AB $(= x)$ any Part of the second Axis, and BD $(= y)$ it's Ordinate: from the Property of the Hyperbola, you have $\overline{BD + AC} \times \overline{BD - AC} = ABq$, or $\overline{y + a} \times \overline{y - a} = x^2$, that is $y^2 - a^2 = x^2$ or $y^2 = a^2 + x^2$: which is the Property required. Therefore $ax + \frac{x^3}{6a} - \frac{x^5}{40a^3}$ &c. $=$ ABDC the Area required: for the Area begins at AC, since it vanishes, when $x =$ AB vanishes. And to make this Series converge, x should be less than a: and the less x is, a remaining, the quicker is the Convergency. The same hyperbolical Area would be found by Prop. 5. of the Treatise of Quadratures, to be $\overline{aa + xx}|^{\frac{3}{2}} \times \overline{\frac{x}{a^2} - \frac{4x^3}{3a^4} + \frac{8x^5}{5a^6} - \frac{64x^7}{35a^8} + \frac{128x^9}{63a^{10}}}$ &c. or $y \times \overline{x - \frac{x^3}{3a^2} + \frac{4x^5}{15a^4} - \frac{8x^7}{35a^6} + \frac{64x^9}{315a^8}}$ &c.

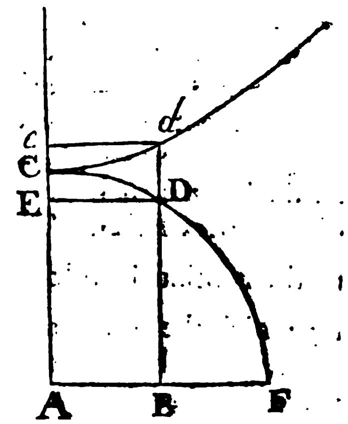
36. If

36. If it were $y = \sqrt{aa - xx}$, the Equation would be to the Circle (See the Fig. Art. 16.) ABCD: where the Radius AC $= a$, AB $= x$, BD $= y$: for $BD^2 = AC^2 - AB^2$, that is $y^2 = a^2 - x^2$. Now if the Square Root be extracted out of $a^2 - x^2$, after the Manner of the preceding Example, it will be found to be $a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} - \frac{7x^{10}}{256a^9}$ &c. which differs from the Series for the Ordinate to the second Axis of the Hyperbola, as above, in no other Respect but that all the Terms after the first are negative; which in the Hyperbola are positive and negative alternately. Whence the circular Area ABDC $= ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7} - \frac{7x^{11}}{2816a^9}$ &c. differing from the Series for the Hyperbolical Area in the same Respect those for the Ordinates differ. The double of this Series will give the Area of the Circular Zone adjacent to the Diameter, and having it's Right Sine or Altitude equal to x, *viz.* $2ax - \frac{x^3}{3a} - \frac{x^5}{20a^3} - \frac{x^7}{56a^5} - \frac{5x^9}{576a^7}$ &c. which by substituting R for a the Radius, and B for x the Sine, is $2RB - \frac{B^3}{3R} - \frac{B^5}{20R^3} - \frac{B^7}{56R^5}$ &c. the same with that which was sent by Mr. *Collins* in one of his Letters to Mr. *James Gregory*, as a Specimen of Sir *Isaac*'s Method of Series. See Com. Epist. N°. 18. The same circular Area ABDC would be found by Prop. 5. of the Quadratures to be $\overline{y \times x} + \frac{1}{3}\frac{x^3}{a^2} + \frac{4}{15}\frac{x^5}{a^4} + \frac{8}{35}\frac{x^7}{a^6} + \frac{64}{315}\frac{x^9}{a^8}$ &c. the same with that for the hyperbolical Area found by the same Prop. except only in the Signs. The former Series $ax \pm \frac{x^3}{6a} - \frac{x^5}{40a^3} \pm \frac{x^7}{112a^5}$ &c. and this one $\overline{y \times x} \mp \frac{1}{3}\frac{x^3}{a^2} + \frac{4}{15}\frac{x^5}{a^4}$ &c. will be found the same, if for y you insert it's Value, *viz.* $a \pm \frac{x^2}{2a} - \frac{x^4}{8a^3} \pm \frac{x^6}{16a^5}$ &c. and then multiply it by $x \mp \frac{x^3}{3a^2} + \frac{4x^5}{15a^4} \mp \frac{8x^7}{35a^6}$ &c.

37. When a few of the initial Terms of a Series are once found, it is of considerable Use to know the Law of the Progression by which the Series proceeds, that thereby it may be continued at Pleasure. This Law is found either by observing the Steps of the general Theorem from whence the Series is deduced; or by considering the Relations of those Terms which are already found in the beginning of the Series. Thus

Thus the Series $ax \pm \frac{x^3}{6a} - \frac{x^5}{40a^3} \pm \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7} \pm \frac{7x^{11}}{2816a^9}$ &c. for the hyperbolical; or circular Area: found as above by the Method of infinite Series, may be continued at Pleasure, by observing that the numeral Coefficients $\frac{1}{6} . \frac{1}{40} . \frac{1}{112} . \frac{5}{1152}$. &c. are formed by multiplying continually by $\frac{1\times1}{2\times3} . \frac{1\times3}{4\times5} . \frac{3\times5}{6\times7} . \frac{5\times7}{8\times9} . \frac{7\times9}{10\times11}$. &c. due Regard being had to the Signs. So likewise the other Series for the same Areas, derived from Prop. 5. of the Quadratures, *viz.* $y \times \overline{x \mp \frac{x^3}{3a^2} + \frac{4x^5}{15a^4} \mp \frac{8x^7}{35a^6} + \frac{64x^9}{315a^8}}$ &c. may be continued, by observing that the Numeral Coefficients $\frac{1}{3} . \frac{4}{15} . \frac{8}{35} . \frac{64}{315}$ &c. are formed by the continual Multiplication of the following Fractions $\frac{2}{3} . \frac{4}{5} . \frac{6}{7} . \frac{8}{9} . \frac{10}{11}$. &c.

38. Hence if CD, C*d*, be a Circle, and rectangular Hyberbola, described with the Center A, and Transverse Axis AC; and AB $= x$ the Base or Absciss, as before, upon the conjugate Axis; BD or B*d* $= y$ the Ordinate: and the Ordinates DE, *de*, to the Transverse Axis be drawn: then from ABDC $= ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7} - \frac{7x^{11}}{2816a^9}$ &c. subtracting the Rectangle ABDE $=$ AB$\times$BD $= x \times a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} - \frac{7x^{10}}{256a^9} =$



$ax - \frac{x^3}{2a} - \frac{x^5}{8a^3} - \frac{x^7}{16a^5} - \frac{5x^9}{128a^7} - \frac{7x^{11}}{256a^9}$ &c. there remains DCE $= \frac{x^3}{3a} + \frac{x^5}{10a^3} + \frac{3x^7}{56a^5} + \frac{5x^9}{144a^7} + \frac{35x^{11}}{1408a^9}$ &c. which serves for finding the Area or half Segment DCE from the Right Sine AB or DE given. And after the like manner, from AB or $de = x$ given in the Hyperbola, you'll find the Hyperbolical half Segment *d*C*e* equal to $\frac{x^3}{3a} - \frac{x^5}{10a^3} + \frac{3x^7}{56a^5} - \frac{5x^9}{144a^7} + \frac{35x^{11}}{1408a^9}$ &c. which two Series may be continued at Pleasure by forming the numeral Coefficients of each Term by the continual Multiplication of so many Terms of the following Series, as the Place of the Term denotes, *viz.* $\frac{1\times1}{1\times3} . \frac{1\times3}{2\times5} . \frac{3\times5}{4\times7} . \frac{5\times7}{6\times9} . \frac{7\times9}{8\times11}$. &c. By adding the two Series $\frac{x^3}{3a} + \frac{x^5}{10a^3} + \frac{3x^7}{56a^5}$ &c. and $\frac{x^3}{3a} - \frac{x^5}{10a^3}$

+

$+ \frac{3x^7}{56a^5}$ &c. together, you have $\frac{2x^3}{3a} + \frac{3x^7}{28a^5} + \frac{35x^{11}}{704a^9}$ &c. $=$ DCE $+ dCe$: and by subtracting the latter from the former you have $\frac{x^5}{5a^3} + \frac{5x^9}{72a^7}$ &c. $=$ DCE $- dCe$: the first is the Sum of the two half Segments; the last their Difference: both which converge very fast; so doth the Series $\frac{x^3}{3a} + \frac{x^7}{56a^5} + \frac{7x^{11}}{1408a^9}$ &c. $=$ the intermediate Area DC*d* the Difference betwixt ABDC and AB*d*C: and also $2ax - \frac{x^5}{20a^3} - \frac{5x^9}{576a^7}$ &c. equal to their Sum.

39. Our Author's next Example (Art. 17.) is $y = \sqrt{x - xx}$ or $\sqrt{1 \times x - xx}$: which is to the Circle ABD (*See the Fig. belonging to that Art.*) having it's Diameter equal to 1; and the Abscis AB $= x$, beginning at the extremity of the Diameter; in which Case you have yy ($= \text{BD}^2 = \overline{1-x} \times x$) $= x - x^2$; or $y = \sqrt{x - x^2}$. Now if the Square Root of $x - xx$ be extracted, you'll find it to be $x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}} - \frac{5}{128}x^{\frac{9}{2}} - \frac{7}{256}x^{\frac{11}{2}}$ &c. $= y$: and therefore by squaring all the Terms, it is $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}} - \frac{5}{704}x^{\frac{11}{2}} - \frac{7}{1664}x^{\frac{13}{2}}$ &c. $=$ Area ABD. The numeral Coefficients of which are produced by the continual Multiplication of $\frac{2}{3} \cdot \frac{-1\times3}{2\times5} \cdot \frac{1\times5}{4\times7} \cdot \frac{3\times7}{6\times9} \cdot \frac{5\times9}{8\times11} \cdot \frac{7\times11}{10\times13}$ &c. by which the Series may be continued as far as you please.

40. Now by Means of the foregoing Series's contained in the four preceding Articles, you may easily find any hyperbolical, or circular Area, in any Case proposed. For the Satisfaction of the young Geometrician, I shall lay before him the Computation of the circular Area from the last Series $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}}$ &c. $=$ ABD. Here the Diameter is 1: let us suppose AB ($= x$) to be any Part of the Diameter, the less Part it is taken, the Series will converge the more quickly: but because it is most convenient for the computing the Area of the whole Circle, let it be $x = \frac{1}{4}$ that is the versed Sine of 60 Degrees. By what was shewn in the last Article, the first Term being $\frac{2}{3}x^{\frac{3}{2}}$, the second Term will be found by multiplying the first by $-\frac{1\times3}{2\times5}x = -\frac{3}{10}x$; the third, by multiplying the second, *viz.* $\frac{-1}{5}x^{\frac{5}{2}}$ by $\frac{1\times5}{4\times7}x$ or $\frac{5}{28}x$: and so on. But in such numeral Calculations, you'll do best to reduce all to Decimals: and set down each Term, as you see in the Margin, after it is found. Now since $x = \frac{1}{4}$, it appears that $\frac{2}{3}x^{\frac{3}{2}} = \frac{2}{3} \times \frac{1}{8} = \frac{1}{12} = 0.083333$ &c. Which I set down in the Margin with it's

it's proper Sign. This multiplied by $-\frac{3}{10}x = -\frac{3}{40}$, makes -0.00625. Which being placed duly under the former with it's proper Sign, you multiply it by $\frac{5}{28}x = \frac{5}{112}$, and it produces -0.0002790178571429 &c. And so on for the rest; which will be such as you see expressed in the Margin. Where the first Line only is positive; and all the rest negative. Wherefore subtract the Sum of all the Negatives, *viz.* 0.0065602271702860&c. from the first Term 0.0833333333333333 &c. and there remains 0.0767731061630473 &c. = the half Segment ABD. The Radius CD being drawn, the Triangle $BCD = \frac{BC \times CD}{2} = \frac{\frac{1}{4} \times \sqrt{\frac{3}{16}}}{2} = \frac{\sqrt{3}}{32} =$ 0.0541265877365275 &c. which being added to ABD, makes 0.1308996938995 7 &c. = Sector ACD a sixth Part of the whole Circle: So that multiplying it by 6, you have 0.785398163339744 &c. for the Area of a Circle whose Diameter is 1: true to 14 Places of Decimals. And therefore the Diameter of any Circle being given, multiply this Number by the Square of that Diameter, and you have it's Area.

+ 0.0833333333333333
− 0.00625
− 0.0002790178571429
− 0.0000271267361111
− 0.0000034679066051
− 0.0000005135169396
− 0.0000000834465027
− 0.0000000144628917
− 0.0000000026285354
− 0.0000000004954581
− 0.0000000000961296
− 0.0000000000190948
− 0.0000000000038676
− 0.0000000000007963
− 0.0000000000001663
− 0.0000000000000352
− 0.0000000000000075
− 0.0000000000000016
− 0.0000000000000004
− 0.0000000000000001
− 0.0065602271702860
+ 0.0767731061630473
+ 0.0541265877365275
0.1308996938995 7

Moreover the Area being found divide by $\frac{1}{2}$ Radius, and you'll have the Circumference: that is in the present Case, divide the preceding Number by $\frac{1}{4}$ or multiply it by 4, and it gives 3.14159265358979 &c. for the Circumference of a Circle whose Diameter is 1.

41. After the like Manner you may find the Area of the Hyperbola from the Equation $y = \sqrt{x + xx}$: and both that of the Circle and that of the Hyperbola, by Means of the other Series's which have been mentioned. And it appears with what Ease and Expedition this is performed: So that for Ease and Expedition it vastly surpasses any former Method for approaching the Area of the Circle, made use of before our Author's Time: not excepting Mr. *James Gregory*'s, in his *Vera Circuli & Hyperbolæ Quadratura* (which was the best) as will easily appear by considering the Operation made use of for this Purpose in Prop. 29; and comparing it with the Method here shewn: which yet is capable of great Improvement.

 42. Our

42. Our Author's next Example (Art. 18.) is $\frac{\sqrt{1+ax^2}}{\sqrt{1-bx^2}}=y$: the Quadrature of which gives the Length of an Elliptical Arch. For suppose the Figure ACF, belonging to Art. 38 above, to be an Ellipse, having AF the half Transverse and AC the half conjugate Axis; and DB and DE Ordinates to them respectively: call $AB=x$, $BD=v$, $AF=t$, $AC=c$, and the Arch $CD=z$: then by what was demonstrated in the preceding Treatise (*See Notes upon Art. 5. of the Treatise of Quadratures*) it is $\dot{z}=\sqrt{\dot{x}^2+\dot{v}^2}$: that is (by inserting $\frac{t^2x^2\dot{x}^2}{c^4-c^2x^2}$ for $\dot{v}^2$, which is equal to it from the Property of the Curve, and reducing) $\dot{z}=\dot{x}\,\frac{\sqrt{c^4\begin{smallmatrix}+tt\\-cc\end{smallmatrix}x^2}}{\sqrt{c^4-ccx^2}}$. Wherefore a Curve which has x for it's Abscis and $\frac{\sqrt{c^4\begin{smallmatrix}+tt\\-cc\end{smallmatrix}x^2}}{\sqrt{c^4-ccx^2}}$ for it's Ordinate, flows with a Fluxion or Velocity analogous to that with which the Elliptical Arch CD flows: consequently the Area generated by that Ordinate is analogous to the Elliptical Arch CD: So that the same Algebraical Expression shall denote both agreeably to Art. 37. of this Analysis; and what was shewn with Respect to the Rectification of Curves in the preceding Treatise. Therefore if it be $y=\frac{\sqrt{1+ax^2}}{\sqrt{1-bx^2}}$, the Quadrature of it will give the Elliptical Arch, supposing $c=1$, $tt-cc=a$, and $cc=b$.

43. For this Purpose you may extract the Square Roots of $1+ax^2$ and $1-bx^2$, and divide the former by the latter: but, as our Author observes, you may abridge the Operation by multiplying the Numerator and Denominator by $\sqrt{1-bx^2}$: by which it becomes $\frac{\sqrt{1\begin{smallmatrix}+a\\-b\end{smallmatrix}x^2-abx^4}}{1-bx^2}$: and so you need only extract the Square Root of the Numerator, and divide by the Denominator, in order to have the Value of y. The Operation follows.

$1\begin{smallmatrix}+a\\-b\end{smallmatrix}x^2$

$$
\begin{array}{l}
1\begin{array}{l}+a\\-b\end{array}x^2 - abx^4\left(1\begin{array}{l}+\frac{1}{2}ax^2\\-\frac{1}{2}b\end{array}\begin{array}{l}-\frac{1}{8}a^2x^4\\-\frac{1}{4}ab\\-\frac{1}{8}bb\end{array}\begin{array}{l}+\frac{1}{16}a^3x^6\\+\frac{1}{16}a^2b\\-\frac{1}{16}ab^2\\-\frac{1}{16}bbb\end{array}\right. \&c. = \sqrt{1\begin{array}{l}+\frac{a}{b}\\-\end{array}x^2 - abx^4}\\
1\\
\hline
2\begin{array}{l}+\frac{1}{2}ax^2\\-\frac{1}{2}b\end{array}\Big)\ * \begin{array}{l}+ax^2\\-b\end{array} - abx^4\\
\qquad\begin{array}{l}+ax^2\\-b\end{array}\begin{array}{l}+\frac{1}{4}aax^4\\-\frac{1}{2}ab\\+\frac{1}{4}bb\end{array}\\
\hline
2\begin{array}{l}+ax^2\\-b\end{array}\begin{array}{l}-\frac{1}{8}aax^4\\-\frac{1}{4}ab\\-\frac{1}{8}bb\end{array}\Bigg)\ * \begin{array}{l}-\frac{1}{4}aax^4\\-\frac{1}{2}ab\\-\frac{1}{4}bb\end{array}\\
\qquad\begin{array}{l}-\frac{1}{4}aax^4\\-\frac{1}{2}ab\\-\frac{1}{4}bb\end{array}\begin{array}{l}-\frac{1}{8}a^3x^6\\-\frac{1}{8}a^2b\\+\frac{1}{8}ab^2\\+\frac{1}{8}b^3\end{array}\ *\\
\hline
\qquad *\ \begin{array}{l}+\frac{1}{8}a^3x^6\\+\frac{1}{8}a^2b\\-\frac{1}{8}ab^2\\-\frac{1}{8}b^3\\ \&c.\end{array}\ *
\end{array}
$$

Which Square Root, now reduced into a Series of ſimple Terms, may be eaſily divided, by $1 - bx^2$ after the Manner ſhewn above: by doing of which, you'll obtain this Quotient $1\begin{array}{l}+\frac{1}{2}ax^2\\+\frac{1}{2}b\end{array}\begin{array}{l}-\frac{1}{8}aax^4\\+\frac{1}{4}ab\\+\frac{3}{8}bb\end{array}\begin{array}{l}+\frac{1}{16}a^3x^6\\-\frac{1}{16}a^2b\\+\frac{3}{16}ab^2\\+\frac{5}{16}b^3\end{array}$ &c.

And therefore, by Rule the ſecond, the Terms being ſquared, will give

$$x\begin{array}{l}+\frac{1}{6}ax^3\\+\frac{1}{6}b\end{array}\begin{array}{l}-\frac{1}{40}a^2x^5\\+\frac{1}{20}ab\\+\frac{3}{40}bb\end{array}\begin{array}{l}+\frac{1}{112}a^3x^7\\-\frac{1}{112}a^2b\\+\frac{3}{112}ab^2\\+\frac{5}{112}b^3\end{array}\ \&c. = \text{Elliptical Arch CD}$$

for the initial Limit of the Arch is at the Point C: ſince the Series becomes nothing when $x = AB$ vaniſhes: that is when CD vaniſhes.

44. I ſhould next proceed to ſhew how the Cube Root; and other higher Roots, of complex Radicals may be extracted, and thereby reduced to infinite Series's: which Operations are performed after the ſame Manner as they are in ſpecious Arithmetic, by conſidering the Geneſis of the ſeveral Powers of a Binomial; and performing the Extraction or Reſolution in every Caſe, as the Compoſition of the Power, whoſe Root is extracted, may require; and continuing the Operation, after the Manner you do in common Arithmetic, in Decimal Fractions: as may be eaſily underſtood by the Method of reſolving quadratic Radicals ſhewn above. But I think it needleſs to detain the Reader with

 this:

this: since our Author's famous *Binomial Theorem*, mentioned already in the History of his Discoveries, serves more readily and generally, not only for extracting Roots properly so called; but for expressing by a Series of simple Terms, any Root, Power, or Dimension whatsoever of a compound Quantity, having any Index positive; negative; integral or fractional. Which, without all Peradventure, is one of the most general and useful Theorems, that ever were discovered.

45. Sir *Isaac* exhibits the Theorem thus: $\overline{P+PQ}^{\frac{m}{n}} + P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \&c.$[1] Where P + PQ represents any compound Quantity, whose Root or Dimension whatsoever is to be investigated: P, the first Term of that Quantity; Q, the rest of the Terms divided by the first; and consequently, PQ, all the rest of the Terms: $\frac{m}{n}$ the numeral Index of the Root, Power, or Dimension of P + PQ; whether that numeral Index be integral, or fractional; positive, or negative. Moreover the Capitals A, B, C, D, *&c.* stand for the first, second, third, fourth, *&c.* Terms: *i. e.* $A = P^{\frac{m}{n}}$; $B = \frac{m}{n}AQ$; $C = \frac{m-n}{2n}BQ$: *&c.* The same Theorem may stand otherwise thus: $\overline{P+PQ}^{\frac{m}{n}} = (P^{\frac{m}{n}} \times \overline{1+Q}^{\frac{m}{n}} =)\ P^{\frac{m}{n}} \times \overline{1 + \frac{m}{n}Q + \frac{m}{n} \times \frac{m-n}{2n}Q^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}Q^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}Q^4 + \&c.}$[2] Which is the same with the former, only differently exprest: sometimes the one Expression being more commodious; sometimes the other.

46. This celebrated Theorem was discovered at first by the Method of Trial and Induction, as may be seen in the foregoing Account of our Author's Discoveries, Art. 14. Since that Time several Demonstrations of it have been given; or alledged to be given. The Method of Fluxions is most frequently made use of for this Purpose: *viz.* this Prop. that the Fluxion of $ax^{\frac{m}{n}}$ is $\frac{m}{n}ax^{\frac{m}{n}-1}\dot{x}$. Which therefore ought to be proved independently of the Binomial Theorem: otherwise the Proof goes in a Circle. But we shall supersede the Demonstration of it, in this Place, reserving it till we have explained the

[1] See his Letter to Mr. *Oldenburg* *June* 13. 1676. Com. Epist. N° 48.
[2] See the Account of our Author's Discoveries above, in Art. 14.

Resolution

Resolution of affected Equations: and only shew, by some Examples, in what Manner it is applied.

47. Ex. 1. Let it be required to express $\sqrt{cc-xx}$ in an infinite Series; or to find the Value of y in such a Series, supposing $y=\overline{cc-xx}|^{\frac{1}{2}}$. By comparing this Example with the General Theorem $\overline{P+PQ}|^{\frac{m}{n}}=$ $=P^{\frac{m}{n}}+\frac{m}{n}AQ+\frac{m-n}{2n}BQ+\frac{m-2n}{3n}CQ$ &c. You have $P=cc$, $Q=\frac{-x^2}{c^2}$, $m=1$, $n=2$: therefore also $P^{\frac{m}{n}}=cc^{\frac{1}{2}}=c=A$, $\frac{m}{n}AQ=\frac{1}{2}c\times\frac{-x^2}{c^2}=-\frac{x^2}{2c}=B$, $\frac{m-n}{2n}BQ=\frac{-1}{4}\times-\frac{x^2}{2c}\times-\frac{x^2}{c^2}=-\frac{x^4}{8c^3}=C$, $\frac{m-2n}{3n}CQ=\frac{-3}{6}\times-\frac{x^4}{8c^3}\times-\frac{x^2}{c^2}=-\frac{x^6}{16c^5}=D$, $\frac{m-3n}{4n}DQ=\frac{-5}{8}\times-\frac{x^6}{16c^5}\times-\frac{x^2}{c^2}=-\frac{5x^8}{128c^7}=E$, &c. Therefore it is $y=(\sqrt{cc-xx}=)$ $c-\frac{x^2}{2c}-\frac{x^4}{8c^3}-\frac{x^6}{16c^5}\ \frac{5x^8}{128c^7}$ &c. the same as above, Art. 36, being the Sine of an Arch in a Circle, whose Radius is c, the Cosine being x.

48. Ex. 2. Let it be $y=\sqrt[3]{ax^2+x^3}$, y is sought. Here $P=ax^2$, $Q=\frac{x}{a}$, $m=1$, $n=3$: therefore also it is $P^{\frac{m}{n}}=a^{\frac{1}{3}}x^{\frac{2}{3}}=A$, $\frac{m}{n}AQ$ $=\frac{1}{3}a^{\frac{1}{3}}x^{\frac{2}{3}}\times\frac{x}{a}=\frac{x^{\frac{5}{3}}}{3a^{\frac{2}{3}}}=B$, $\frac{m-n}{2n}BQ=\frac{-2}{6}\times\frac{x^{\frac{5}{3}}}{3a^{\frac{2}{3}}}\times\frac{x}{a}=-\frac{x^{\frac{8}{3}}}{9a^{\frac{5}{3}}}=C$, $\frac{m-2n}{3n}CQ=\frac{-5}{9}\times-\frac{x^{\frac{8}{3}}}{9a^{\frac{5}{3}}}\times\frac{x}{a}=\frac{5x^{\frac{11}{3}}}{81a^{\frac{8}{3}}}=D$, $\frac{m-3n}{4n}DQ=\frac{-8}{12}\times\frac{5x^{\frac{11}{3}}}{81a^{\frac{8}{3}}}$ $\times\frac{x}{a}=-\frac{10x^{\frac{14}{3}}}{243a^{\frac{11}{3}}}=E$, &c. Wherefore it is $y=\overline{(ax^2+x^3}|^{\frac{1}{3}}=)$ $a^{\frac{1}{3}}x^{\frac{2}{3}}+\frac{x^{\frac{5}{3}}}{2a^{\frac{2}{3}}}-\frac{x^{\frac{8}{3}}}{9a^{\frac{5}{3}}}+\frac{5x^{\frac{11}{3}}}{81a^{\frac{8}{3}}}-\frac{10x^{\frac{14}{3}}}{243a^{\frac{11}{3}}}$ &c. In which Example you might have inverted the Order of the Terms of the Binomial, so as to make it stand thus $\overline{x^3+ax^2}|^{\frac{1}{3}}$: and in that Case it would have been $P=x^3$, and $Q=\frac{a}{x}$: and therefore $P^{\frac{m}{n}}=x=A$, $\frac{m}{n}AQ=\frac{a}{3}=B$, $\frac{m-n}{2n}BQ=-\frac{a^2}{9x}=C$, &c. and consequently $y=\overline{x^3+ax^2}|^{\frac{1}{3}}=x$ +

$+\frac{a}{3}-\frac{a^2}{9x}+\frac{5a^3}{81x^2}-\frac{10a^4}{243x^3}$ &c. The former is to be chosen when x is small; the latter, when it is great in Comparison of a.

49. Ex. 3. Let it be $y = \overline{c^5 + c^4x - x^5}|^{\frac{1}{5}}$: and suppose we take the second Form of the Theorem, *viz.* $\overline{P+PQ}|^{\frac{m}{n}} = (P^{\frac{m}{n}} \times \overline{1+Q}|^{\frac{m}{n}} =)\ P^{\frac{m}{n}} \times \overline{1+\frac{m}{n}Q+\frac{m}{n}\times\frac{m-n}{2n}Q^2 \text{ &c.}}$ Here we have $P = c^5$, $Q = \frac{c^4x - x^5}{c^5}$, $m = 1$, $n = 5$; $P^{\frac{m}{n}} = c$: and therefore it is $y = c \times$

$\overline{1+\frac{1}{5}\times\frac{c^4x-x^5}{c^5}+\frac{1}{5}\times\frac{-4}{10}\times\overline{\frac{c^4x-x^5}{c^5}}|^2+\frac{1}{5}\times\frac{-4}{10}\times\frac{-9}{15}\times\overline{\frac{c^4x-x^5}{c^5}}|^3+\frac{1}{5}\times\frac{-4}{10}\times\frac{-9}{15}\times\frac{-14}{20}\times\overline{\frac{c^4x-x^5}{c^5}}|^4 \text{ &c.}}$

Here it may be observed, when you make use of this Form of the General Theorem, the Series may be easily produced, by considering that the Factors both in the Numerators and Denominators of the Numeral Coefficients go on in Arithmetical Progressions. This Series by involving the Quantity $\frac{c^4x - x^5}{c^5}$ to the several Powers, and multiplying by c, is $y = c + \frac{c^4x - x^5}{5c^4} - \frac{2c^8x^2 - 4c^4x^6 + 2cx^{10}}{25c^9} + \frac{6c^{12}x^3 - 18c^8x^7 + 18c^4x^{11} - 6x^{15}}{125c^{14}}$ &c.

In this Example you might have inverted the Order of the Terms, making it stand thus $\sqrt[5]{-x^5 + c^4x + c^5}$: and then it would have been $P = -x^5$, $Q = \frac{c^4x + c^5}{-x^5}$; $P^{\frac{m}{n}} = -x$, &c. and so you would have had $y = -x + \frac{c^4x + c^5}{5x^4} + \frac{2c^8x^2 + 4c^9x + 2c^{10}}{25x^9}$ &c. The former Series converges when x is small; this when x is great.

50. Ex. 4. Suppose $y = \overline{a^2 + x^2}|^{\frac{4}{3}}$. Here you have $P = a^2$, $Q = \frac{x^2}{a^2}$, $m = 4$, $n = 3$: $P^{\frac{m}{n}} = a^{\frac{8}{3}}$, &c. Wherefore, by the second Form of the Theorem it is $y = (\overline{aa + xx}|^{\frac{4}{3}} =)\ a^{\frac{8}{3}} \times$

$\overline{1+\frac{4}{3}\frac{x^2}{a^2}+\frac{4\times1}{3\times6}\frac{x^4}{a^4}+\frac{4\times1\times-2}{3\times6\times9}\frac{x^6}{a^6}+\frac{4\times1\times-2\times-5}{3\times6\times9\times12}\frac{x^8}{a^8}+\frac{4\times1\times-2\times-5\times-8}{3\times6\times9\times12\times15}\frac{x^{10}}{a^{10}} \text{ &c.}}$

$= a^{\frac{8}{3}} + \frac{4a^{\frac{2}{3}}x^2}{3} + \frac{2x^4}{9a^{\frac{4}{3}}} - \frac{4x^6}{81a^{\frac{10}{3}}} + \frac{5x^8}{243a^{\frac{16}{3}}}$ &c.

You might also have made x^2 the first Member under the Vinculum of the Root: and the Series would come out the same, only placing a for x, and x for a in the several Terms.

51. Ex. 5.

51. Ex. 5. Let $y = \frac{\sqrt{c^4 \frac{-cc}{+rr} x^2}}{\sqrt{c^4 - ccx^2}}$ be propofed: and it is required to find the Value of y in an infinite Series, and then divide the one by the other; or you might firft multiply both by the Denominator, by way of Preparation, as was fhewn above, Art. 43: or laftly you may bring the Quantity $\sqrt{c^4 - ccx^2}$ from the Denominator to the Numerator, by changing the Sign of the Index $\frac{1}{2}$, into it's oppofite, fo that the Expreffion ftand thus $\overline{c^4 \frac{-cc}{+rr} x^2}|^{\frac{1}{2}} \times \overline{c^4 - ccx^2}|^{-\frac{1}{2}}$, and then reduce the two Factors into infinite Series by the Binomial Theorem; and afterwards multiply thefe two Series into one another. Suppofe we take this laft Method: then for Brevity fake, fubftitute a for cc and b for $-cc+rr$, fo that it be $y = \overline{a^2 + bx^2}|^{\frac{1}{2}} \times \overline{a^2 - ax^2}|^{-\frac{1}{2}}$. By comparing the firft of thefe with the general Theorem you have $P = aa$, $Q = \frac{bx^2}{a}$, $m = 1$, $n = 2$: by comparing the fecond, $P = aa$, $Q = \frac{-x^2}{a}$, $m = -1$, $n = 2$. And according to the firft Form of that Theorem, you'll have for the firft Series $P^{\frac{m}{n}} = a = A$, $\frac{m}{n}AQ = \frac{bx^2}{2a} = B$, $\frac{m-n}{2n}BQ = -\frac{b^2x^4}{8a^3} = C$, $\frac{m-2n}{3n}CQ = \frac{b^3x^6}{16a^5} = D$, $\frac{m-3n}{4n}DQ = -\frac{5b^4x^8}{128a^7}$ &c. and according to the fecond Form of the Theorem, you'll have for the fecond Series $P^{\frac{m}{n}} = aa^{-\frac{1}{2}} = \frac{1}{a}$: and therefore $\overline{a^2 - ax^2}|^{-\frac{1}{2}} = \frac{1}{a} \times 1 + \frac{x^2}{2a} + \frac{-1\times-3}{2\times4}\frac{x^4}{a^2} - \frac{-1\times-3\times-5}{2\times4\times6}\frac{x^6}{a^3} + \frac{-1\times-3\times-5\times-7}{2\times4\times6\times8}\frac{x^8}{a^4}$ &c. So that $\overline{a^2 + bx^2}|^{\frac{1}{2}} = a + \frac{bx^2}{2a} - \frac{b^2x^4}{8a^3} + \frac{b^3x^6}{16a^5} - \frac{5b^4x^8}{128a^7}$ &c. and $\overline{aa - ax^2}|^{-\frac{1}{2}} = \frac{1}{a} + \frac{x^2}{2a^2} + \frac{3x^4}{8a^3} + \frac{5x^6}{16a^4} + \frac{35x^8}{128a^5}$ &c. Multiply them as follows,

$$a + \frac{bx^2}{2a} - \frac{b^2x^4}{8a^3} + \frac{b^3x^6}{16a^5} - \frac{5b^4x^8}{128a^7} \text{ \&c.}$$
$$\frac{1}{a} + \frac{x^2}{2a^2} + \frac{3x^4}{8a^3} + \frac{5x^6}{16a^4} + \frac{35x^8}{128a^5} \text{ \&c.}$$
$$1 + \frac{bx^2}{2a^2} - \frac{b^2x^4}{8a^4} + \frac{b^3x^6}{16a^6} - \frac{5b^4x^8}{128a^8} \text{ \&c.}$$
$$+ \frac{x^2}{2a} + \frac{bx^4}{4a^3} - \frac{b^2x^6}{16a^5} + \frac{b^3x^8}{32a^7} \text{ \&c.}$$
$$+ \frac{3x^4}{8a^2} + \frac{3bx^6}{16a^4} - \frac{3b^2x^8}{64a^6} \text{ \&c.}$$
$$+ \frac{5x^6}{16a^3} + \frac{5bx^8}{32a^5} \text{ \&c.}$$
$$+ \frac{35x^8}{128a^4} \text{ \&c.}$$

and

and you obtain this Product $1+\frac{a+b}{2a^2}x^2+\frac{3a^2+2ab-b^2}{8a^4}x^4+\frac{5a^3+3a^2b-ab^2+b^3}{16a^6}x^6+\frac{35a^4+20a^3b-6a^2b^2+4ab^3-5b^4}{128a^8}x^8$ &c. Which therefore is the Value of $\frac{\sqrt{a^2+bx^2}}{\sqrt{a^2-ax^2}}$.

52. Hence it follows, by our Author's ſecond Rule, that a Curve having ſuch an Ordinate, has it's Area equal to $x+\frac{a+b}{6a^2}x^3+\frac{3a^2+2ab-b^2}{40a^4}x^5+\frac{5a^3+3a^2b-ab^2+b^3}{112a^6}x^7+\frac{35a^4+20a^3b-6a^2b^2+4ab^3-5b^4}{1152a^8}x^9$, &c. Which by replacing c^2 for a, and $rr-cc$ for b, makes $x+\frac{r^2}{6c^4}x^3+\frac{4r^2c^2-r^4}{40c^8}x^5+\frac{8c^4r^2-4c^2r^4+r^6}{112c^{12}}x^7+\frac{64c^6r^2-48c^4r^4+24c^2r^6-5r}{1152c^{16}}x^9$ &c. Which expreſſes the Length of an Elliptical Arch, beginning at the Extremity of either of the Axes whoſe Halves are r and c; and having x for it's right Sine (as appears from Art. 42.) And this is the Series which Mr. *James Gregory* ſent to Mr. *Collins*. See Com. Epiſt. N° 20. And the ſame Series will ſerve for the Length of an Arch of the Hyperbola, if all the particular Parts of each Term be made poſitive; and the whole of the 3d, 5th, 7th, &c. Terms be made negative: that is, if it ſtand thus $x+\frac{r^2}{6c^4}x^3-\frac{4r^2c^2+r^4}{40c^8}x^5+\frac{8c^4r^2+4c^2r^4+r^6}{112c^{12}}x^7-\frac{64c^6r^2+48c^4r^4+24c^2r^6+5r^8}{1152c^{16}}x^9$ &c.

And if you ſuppoſe p to be inſerted for $\frac{c^2}{r}$, that is half the Parameter of the Axis $2r$; or, which is the ſame, inſert pr for c^2 every where: the Series for the Arch of the Ellipſe will appear in this Form:

$$\begin{array}{llllllll} x+\frac{1}{6p^2}x^3 & +\frac{1}{10rp^3}x^5 & & +\frac{1}{14r^2p^4}x^7 & & +\frac{1}{18r^3p^5}x^9 & \&c. \\ & -\frac{1}{40p^4} & & -\frac{1}{28rp^5} & & -\frac{1}{24r^2p^6} & \\ & & & +\frac{1}{112p^6} & & +\frac{1}{48rp^7} & \\ & & & & & -\frac{1}{1152p^8} & \end{array}$$

Which is the Series given by Sir *Iſaac Newton* for the Elliptical Arch, in his firſt Letter to Mr. *Oldenburg*, Com. Epiſt. N° 48. And this Series may be continued by obſerving that the Numeral Coefficients of the uppermoſt Line, *viz.* $\frac{1}{6}.\frac{1}{10}.\frac{1}{14}$ &c. are Fractions whoſe Denominators go on by the common Difference 4, having Unity their common Numerator: and as to the Numeral Coefficients of all the inferior

Terms

Terms in every Column, that they arise by multiplying the numeral Coefficient of the uppermost Term continually, by the Terms of this Progression $\frac{\frac{1}{2}n-1}{2} \cdot \frac{\frac{1}{2}n-3}{4} \cdot \frac{\frac{1}{2}n-5}{6} \cdot \frac{\frac{1}{2}n-7}{8}$. &c. the Letter n denoting the Index of the Power of p, in the uppermost Term of the Column.

53. If in the preceding *Gregorian* Series, you suppose $r=c$; or in the *Newtonian* $r=p$; in which Case the Ellipse becomes a Circle: the Series becomes $x+\frac{x^3}{6r^2}+\frac{3x^5}{40r^4}+\frac{5x^7}{112r^6}+\frac{35x^9}{1152r^8}$ &c. Which therefore denotes the Arch of a Circle whose Right Sine is x, the Radius being r. Which Series may be produced to any Length by multiplying continually by the Numbers $\frac{1\times1}{2\times3} \cdot \frac{3\times3}{4\times5} \cdot \frac{5\times5}{6\times7} \cdot \frac{7\times7}{8\times9}$ &c. in forming the numeral Coefficients of the Terms. Which is the Series given by Sir *Isaac Newton* for this Purpose at Art. 38 of this Analysis, supposing $r=1$.

54. 'Tis evident from what has been said, that this Binomial Theorem will equally serve for reducing complicate Fractions to infinite Series; and also for involving any compound Quantity to any perfect Power: as it doth for reducing complicate Radicals.

Ex. 6. Let it be required to find the Value of y in simple Terms, supposing $y=\overline{a+x}|^5$. Here $P=a$, $Q=\frac{x}{a}$, $m=5$, $n=1$: consequently $P^{\frac{m}{n}}=a^5=A$, $\frac{m}{n}AQ=5a^4x=B$, $\frac{m-n}{2n}BQ=10a^3x^2=C$, $\frac{m-2n}{3n}CQ=10a^2x^3=D$, $\frac{m-3n}{4n}DQ=5ax^4=E$, $\frac{m-4n}{5n}EQ=x^5=F$, $\frac{m-5n}{6n}EQ=0=G$: and so the Series terminates, since G $(=0)$ enters into the Composition of every subsequent Term. So that you have $y=\overline{a+x}|^5=a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5$. And it's evident that in this Case of perfect Powers, the Series must always terminate; and that when the Number of Terms is one more than the Exponent of the Power.

55. And in order to reduce complicate Fractions, you have no more to do but bring the Denominator into the Numerator, with the Sign of the Index changed into it's opposite: and then apply the Theorem.

Ex. 7. Suppose $y=\frac{1}{a-x}$: I express it thus $y=\overline{a-x}|^{-1}$. By comparing which with the second Form of the general Theorem, you have $P=a$, $Q=-\frac{x}{a}$, $m=-1$, $n=1$. Therefore it is $y=$

$(\frac{1}{a} \times \overline{1 - \frac{1}{1} \times -\frac{x}{a} - \frac{1\times-2}{1\times2} \times \frac{x^2}{a^2} - \frac{1\times-2\times-3}{1\times2\times3} \times -\frac{x^3}{a^3} - \frac{1\times-2\times-3\times-4}{1\times2\times3\times4} \times \frac{x^4}{a^4} \&c.}$
$=)\ \frac{1}{a} + \frac{x}{a^2} + \frac{x^2}{a^3} + \frac{x^3}{a^4} + \frac{x^4}{a^5}$ &c.

Ex. 8. Let it be $y = \frac{a-x}{\overline{a^2+x^2}|^3}$: that is $y = \overline{a-x} \times \overline{a^2+x^2}|^{-3}$. By comparing $\overline{a^2+x^2}|^{-3}$ with the second Form of the Theorem, you have $P = a^2$, $Q = \frac{x^2}{a^2}$, $m = -3$, $n = 1$. Therefore by a due Substitution it is $\overline{a^2+x^2}|^{-3} = (a^{-6} \times \overline{1 - \frac{3x^2}{a^2} - \frac{3\times-4}{1\times2}\frac{x^4}{a^4} - \frac{3\times-4\times-5}{1\times2\times3}\frac{x^6}{a^6} - \frac{3\times-4\times-5\times-6}{1\times2\times3\times4}\frac{x^8}{a^8} \&c.}$ $=)\ \frac{1}{a^6} - \frac{3x^2}{a^8} + \frac{6x^4}{a^{10}} - \frac{10x^6}{a^{12}} + \frac{15x^8}{a^{14}}$ &c. Which being multiplied by $a - x$, produces $\frac{1}{a^5} - \frac{x}{a^6} - \frac{3x^2}{a^7} + \frac{3x^3}{a^8} + \frac{6x^4}{a^9} - \frac{6x^5}{a^{10}}$ &c. $= y$. And so the Area's of the Curves, having their Ordinates thus reduced into infinite Series, are easily found by the Author's second Rule.

56. And thus far I have shewn how the Binomial Theorem is applied, in a variety of Examples. But it may be proper in this Place to observe that, notwithstanding the Series for the Area; or Fluent, found by the Help of this Theorem, always becomes infinite, when the Index of the Power, Root, or Dimension of the Quantity expressing the Ordinate of the Curve; or Fluxion, is not a positive Integer: and consequently exhibits the Area or Fluent, only by an Approximation: yet it often happens that the Curve is *geometrically quadrable*; and so the Area or Fluent may be fully assigned. Which is done by Means of Propositions 5 and 6 of the Treatise of Quadratures. Which see, with our Notes upon them.

57. The same Thing may be found likewise by Means of the *Transmutation* of Curves, founded upon Prop. 9. of the Treatise of Quadratures. Thus let $y = dz^k \times \overline{e+fz^n}|^l$ define the Relation betwixt the Absciss z and Ordinate y of a Curve: and suppose that the Index l is such, that the Quantity $\overline{e+fz^n}|^l$, being reduced, runs out into an infinite Series: and you would know if the Curve be *geometrically quadrable*. Put $e+fz^n = x$: hence $z = \overline{\frac{x-e}{f}}|^{\frac{1}{n}}$, $z^k = \overline{\frac{x-e}{f}}|^{\frac{k}{n}}$, and $z^n = \frac{x-e}{f}$. Therefore, by taking the Fluxions of $e+fz^n = x$, you have $nfz^{n-1}\dot{z} = \dot{x}$, that is $\dot{x} : \dot{z} :: nfz^{n-1} : 1 :: nfz^n : z$. Now, suppose v to be the Ordinate of a Curve, whose Absciss is x, and it's Area

Area equal to the Area of the Curve proposed, having z for it's Abscis, and $y = dz^k \times \overline{e+fz^n}|^l$ for it's Ordinate: then, by that Prop. $y(= dz^k \times \overline{e+fz^n}|^l) : v :: (\dot{x} : \dot{z} ::) \; nfz^n : z$. Whence $v = \frac{d}{nf} z^{k-n+1} \times \overline{e+fz^n}|^l =$ (by substituting the preceding Values) $\frac{dx^l}{nf} \times \overline{\frac{-e+x}{f}}|^{\frac{k+1}{n}-1}$.

Therefore, if $\frac{k+1}{n} - 1$ be a positive Integer, the Quantity $\overline{\frac{-e}{f} + \frac{x}{f}}|^{\frac{k+1}{n}-1}$ is a perfect Power, which may be exprest by a finite Number of simple Terms (See Art. 54.) Let it be done, and each Term multiplied by $\frac{dx^l}{nf}$, and then squared by Rule 1st: So you shall have the Area required: unless perhaps the Values of k, l and n be such, as to give some Term with the Root x in it's Denominator: which would make the Area infinite (See Art. 28.) Therefore, when, in the Curve proposed $y = dz^k \times \overline{e+fz^n}|^l$, the Expression $\frac{k+1}{n}$, is an Integer, the Curve is quadrable, with the Limitation mentioned.

58. Moreover, since the Ordinate or Expression $dz^k \times \overline{e+fz^n}|^l$ may be converted into this other Form $dz^{k+ln} \times \overline{f+ez^{-n}}|^l$ (by Article 28. Treatise of Quad.) if you put $x = f + ez^{-n}$; and proceed as in the former Article, you'll find that the Curve whose Abscis is z, and Ordinate $y = dz^k \times \overline{e+fz^n}|^l$, is equal to another Curve, having the Abscis x, and Ordinate $- \frac{dx^l}{ne} \times \overline{-\frac{f}{e} + \frac{x}{e}}|^{\frac{k+ln+1}{-n}-1}$. Therefore if $\frac{k+ln+1}{-n}$ be a positive Integer, the Curve proposed will be quadrable, with the like Limitation as to an *hyperbolical* Term, as in the preceding Article: and the Area found the same Way. But these two Areas, found by this and the preceding Article, lye upon different Sides of the Ordinate (See Sect. 5 of the preceding Treatise.) And if they can both be exactly found, the Curve is of that Sort, which we call doubly Quadrable.

59. The 9th and 10th Corollaries of Prop. 9. of the Quadratures serve for the same Purpose. Thus, if in Cor. 9. you put $\nu = o = \theta$, $\pi = 1 = \tau = \sigma$, consequently $\vartheta = \lambda - 1$; and suppose it be $a = -\frac{e}{f}$, $b = \frac{1}{f}$, $g = o$ &c. then in the Curve whose Abscis is z and

 Ordinate

Ordinate $\eta f z^{\omega\eta+\eta-1} \times \overline{e+fz^{\eta}}|^{\lambda-1}$ paſſes into another which is equal to it, whoſe Ordinate is $x^{\lambda-1} \times \overline{-\frac{e}{f}+\frac{x}{f}}|^{\omega}$. Or (by multiplying both by the given Quantity $\frac{d}{\eta f}$) the Curve whoſe Ordinate is $dz^{\omega\eta+\eta-1} \times \overline{e+fz^{\eta}}|^{\lambda-1}$, paſſes into another equal Curve, whoſe Ordinate is $\frac{dn}{f} x^{\lambda-1} \times \overline{-\frac{e}{f}+\frac{x}{f}}|^{\omega}$. Compare the Ordinate $dz^{k} \times \overline{e+fz^{\eta}}|^{l}$, in Art. 57, with the former of theſe two, and you have $\eta = n$; $\omega\eta + \eta - 1 = k$, or $\omega = \frac{k-n+1}{n}$; $\lambda - 1 = l$: the reſt the ſame. Therefore, by a due Subſtitution, the given Curve, whoſe Ordinate is $dz^{k} \times \overline{e+fz^{n}}|^{l}$, paſſes into another of an equal Area, whoſe Ordinate is $\frac{dx^{l}}{nf} \times \overline{-\frac{e}{f}+\frac{x}{f}}|^{\frac{k+1}{n}-1}$. Whence, if $\frac{k+1}{n}$ be a poſitive Integer, the Curve is geometrically quadrable, as before. And the ſame way, it may be ſhewn that the Curve propoſed is likewiſe quadrable, if $\frac{k+ln+1}{-n}$ be a poſitive Integer.

But I treated more fully of this Affair; and in a different Manner, in the Explication of Prop. 5. Quad. Which the Reader may conſult.

SECT. III.

Explication of the Numeral Reſolution of affected Equations, contain'd in Art. 21——29.

60. IN order to underſtand our Author's Method of extracting the Roots of affected Numeral Equations, with the Abbreviations made uſe of therein, it will be proper to explain ſome Things which reſpect Decimal Fractions, and the Operations about them.

61. 1°. In Multiplication of Decimals, the Number of Places of Decimal Fractions in the Product, muſt be made equal to the Sum of the Places of Decimal Fractions in both Factors. And in Diviſion, the Number of ſuch Places in the Quotient, muſt be made equal to the Exceſs of thoſe in the Dividend above thoſe in the Diviſor. And if

if the Number of Figures in the Product be not so great as the foresaid sum, you must prefix Decimal Cyphers upon the left hand of the Product, until the Number of Places altogether be as great as that Sum. And in Division if there be a Defect in the Number of Places in the Quotient, it must be supplied the same way. But if the Places of Decimal Fractions in the Dividend, be fewer than those in the Divisor, you must add Decimal Cyphers upon the right hand of the Dividend, untill the Places of Decimal Fractions be as many as in the Divisor. Which Operations are commonly known; and the Reasons of them understood.

2°. When any two Decimal Fractions are multiplied, whereof one, or both, have any Number of Cyphers prefixt, the Number of Cyphers prefixt in the Product, must at least be as great as the Sum of those prefixt in both Factors. For, by the last, the Number of Decimal Places in the Product, is equal to the Sum of those in the two Factors: but the Number of significant Figures in the Product, towards the right hand, can never be greater than the Number of significant Figures in both Factors (which is evident from the Nature of Multiplication): therefore the Number of Cyphers prefixt in the Product is at least equal to those prefixt in both Factors. And oftentimes they are more. Thus $0.0024 \times 0.03 = 0.000072$.

3°. Hence it follows that if r denote any Decimal Fraction with n Number of Cyphers prefixt, then the Number of Cyphers prefixt in its several Powers, *viz.* r^2, r^3, r^4, r^5, *&c.* must be at least $2n$, $3n$, $4n$, $5n$, *&c. viz.* so many times n as the Exponent of the Power denotes.

4°. If two Numbers are proposed to be multiplied, whereof the one is either a Decimal Fraction or mixt Number; and the other a Decimal Fraction, with Cyphers prefixt; and the Product is only required to a certain Number of Decimal Places, you may cut off from the right hand of the first as many Decimal Places, as will make those which remain, together with the prefixed Cyphers in the other, equal to the Number of Decimal Places required in the Product: and neglect those which are cut off, in the Multiplication. Thus if it were required to find the Product of 123,4567 and 0.000089, only to 6 Places of Decimals, you may neglect 67 in the Multiplication: if the Product were required only to 5 Places, you may neglect 567: so as to multiply 0.000089 by 123.4 only: and if the Product be required only to 4 Places of Decimals, you need only multiply 0.000089 by the integral Part 123. But if the Number of Decimal Places, to which the Product is required, be less than the Number of Cyphers prefixt in the Decimal

Decimal Fraction, which is one of the Factors, *e.g.* 0.000089, then, you may neglect, not only all the Decimal Places of the other Factor 123,4567; but as many of the integral Places as the Decimal Places required in the Product, want of the Cyphers prefixt in the first Factor. Thus if the Product be required only to 3, 2, 1, Places of Decimals, you need only multiply 0.000089 by 12, by 1, by 0, respectively.

For since 0.000089 is less than 0.0001; and all after 123,45 less than 0.01: which two multiplied by one another produce only 0 000001 by Obs. 1. Therefore it is evident that 0.000089 multiplied by what follows 123,45, must produce something less than 0.000001: and therefore cannot affect the Product within the 6th Place of Decimals. And the reasoning is the same in every other Case, by considering the Decimal Progression of the Places.

62. These things with respect to Decimals being premised, our Author's Method of extracting the Roots of affected Numeral Equations, with the Abbreviations he makes use of therein, will be the better understood.

Let $ar + br^2 + cr^3 + dr^4$ &c. $= A$, represent any Numeral Equation, in which a, b, c, d, &c. and A are given Numbers, positive or negative, and r the Root whose Value is sought. Then by dividing by a, and transposing, you have $r = \frac{A}{a} - \frac{br^2}{a} - \frac{cr^3}{a} - \frac{dr^4}{a}$ &c. Now when one of the Values of r is small, so as it may be denoted by a Decimal Fraction with Cyphers prefixed, you may easily find the Limits within which the supplementary Parts $\frac{br^2}{a}$, $\frac{cr^3}{a}$, $\frac{dr^4}{a}$, &c. are contained: and consequently the Limits within which the Difference betwixt the true Value of r, and the Approximate Value of it, *viz.* $r = \frac{A}{a}$, must be contained. For since $\frac{br^2}{a}$ is an Expression, in which r^2 is the Square of r a Decimal Fraction having Cyphers prefixt; and therefore r^2 twice as many at least (by Obs. 3.) you may always find how many Cyphers will be prefixt in the Expression $\frac{br^2}{a}$, from the Nature of Decimal Arithmetic. And after the same Manner, how many such Cyphers will be prefixt in the Values of $\frac{cr^3}{a}$, $\frac{dr^4}{a}$, &c. For if $\frac{b}{a}$ be less than 1 (in which Case it's Value in Decimal Arithmetick is entirely below the Place of Unites) there must be as many Cyphers, at at least, prefixt in $\frac{br^2}{a}$, as in r^2 (by Obs. 2.) But if $\frac{b}{a}$ contain any Number of Integral Places, then the Cyphers prefixt in $\frac{br^2}{a}$ will, at least, be equal to the Excess of the Cyphers prefixt in r^2, above the Integral Places in $\frac{b}{a}$. Thus, if r had 4 Decimal Cyphers prefixt in it's Value; and suppose $a = 11, 16196$; $b = 6,3$; then, since $\frac{b}{a}$ $\left(= \frac{6.3}{11.16196}\right)$ is less than an Integer, and r^2 has 8 decimal Cyphers prefixt, $\frac{br^2}{a}$ must have as many prefixt at least. After the same Manner $\frac{br^3}{a}$ would have 12 at least: and universally $\frac{br^n}{a}$, as many, at least, as is denoted by $4n$. But if, the Value of r remaining, it were $a = 0.243$, $b = 6.3$; then since $\frac{b}{a}$ $\left(= \frac{6.3}{0.243}\right)$ is more than 1, and contains two integral Places, therefore $\frac{br^2}{a}$ may have only 6 $(= 8 - 2)$ Decimal Cyphers prefixt: although in some Cases, when the Figures are small, it may have more. And universally $\frac{br^n}{a}$ has $4n - 2$ at least. And so in other Cases. Which Things easily follow from what was shewn in the preceding Observations. And therefore you may proceed thus: 1°, Consider how many integral Places would be contained in $\frac{a}{b}$, (which may be easily found by Inspection) cut off as many of the Cyphers prefixt to the significant Figures in r^2, the remaining Cyphers are the fewest that can be prefixt to the significant Figures in $\frac{br^2}{a}$. 2°. If the Value of $\frac{b}{a}$ have no integral Places, but only decimal Places, then consider how many Decimal Cyphers would be prefixt to the significant Figures in the Value of $\frac{b}{a}$ (which you may easily discover by Inspection) and then the decimal Cyphers prefixt to the significant Figures in the Value of $\frac{br^2}{a}$ will be equal to the Sum of these prefixt in $\frac{b}{a}$ and r^2, both together.

63. From what has been said it appears, that, if the Number of Decimal Cyphers prefixt in the Value of r, be known, in the preceding Equation $r = \frac{A}{a} - \frac{br^2}{a} - \frac{cr^3}{a}$ &c. in which $\frac{A}{a}$ is an approximate

Value

Value of r, you may easily know to what Number of Decimal Places the approximate Value of r, *viz.* $\frac{A}{a}$, may be continued, before any of the supplementary Parts $\frac{br^2}{a}$, $\frac{cr^3}{a}$, &c. can affect the Root. Thus suppose you had this Equation $0.000541550536 + 11.16204748r + 6.2838r^2 + r^3 = 0$: Which is the third supplementary Equation in our Author's Example, supposing it to be fully drawn out: that is $r = -\frac{0.000541550536}{11.16204748} - \frac{6.2838r^2}{11.16204748} - \frac{r^3}{11.16204748}$, and it is known that the Value of r is a decimal Fraction having 4 Cyphers prefixt. Then because the Term $\frac{r^3}{11.16204748}$ is equal to $\frac{1}{11.162 \&c.} \times r^3$, of which the first Part is a Decimal Fraction with one Cypher prefixt; and the other a Decimal Fraction with 12 Cyphers prefixt, therefore the first significant Figure of it's Value will fall upon the 14th Place of Decimals (by Art. 62.) so that the Value of the Root r cannot be affected by it within the Limits of 13 Places. After the same Manner the Term $\frac{6.2838r^2}{11.16204748}$ will not affect the Root within the Limits of 8 Places of Decimals. Whence you conclude that the Value of r, deduced from supposing $r = -\frac{0.000541550536}{11.16204748}$ will be true to 8 Decimal Places, that is the Division of 0.000541550536 by 11.16204748 may be carried forward to so many Decimal Places: all which will be true unless perhaps the last Figure, which may sometimes be increased or diminished by 1, by the Addition or Subtraction of the supplementary Part. And since r^2 will, at least, have twice as many Decimal Cyphers prefixed in it's Value as r has, therefore, generally speaking, the Number of Decimal Places or Figures, to which the Quotient denoted by $\frac{A}{a}$, may be carried, will be twice as many as those to which the Root hath been found already, by the former Approximation: agreeably to what Sir *Isaac* remarks Art. 22. So that in this Method of resolving affected Numeral Equations, every new Operation, as it were, doubles the Number of Decimal Places formerly found in the Root: even supposing all the Terms of the supplementary Equations to be rejected, wherein the Supplements p, q, r, &c. rise above the 1st Power; unless perhaps when b exceeds a: in which Case the supplementary Part $\frac{b}{a}r^2$ may affect the last Figures.

64. By

64. By the Consideration of these Things you may abridge the Work considerably towards the latter End, by neglecting such Terms or Parts of Terms, as you foresee will be of no Use within the Limits to which the Root is proposed to be carried: as you see done by the Author in the Prosecution of the Root y, in the Example proposed by him. However it may be observed that the Quotient 0.00004853 &c. errs in the last figure 3, by his omitting the Term q^3 altogether, in transforming the Equation of q: for it ought to be 0.00004851 &c. only: as you may find by Art. 63.

65. *Schol.* Dr. *Halley's Rational* Formula for extracting the Roots of Equations is easily deduced from what hath been shewn above. Let $fz + gz^2 + hz^3 + iz^4$ &c. $= K$ be any Equation; z the unknown Quantity, the other Letters standing for known Quantities: and let l be a Number as near a Value of z as may be; found by Trial, or any other Way: substitute $l + r$ instead of z in the given Equation $fz + gz^2 + hz^3$ &c. $= K$: and let the transformed Equation be $ar + br^2 + cr^3 + dr^4$ &c. $= A$: where r is the Supplement to l; or the Difference betwixt l the approximate Value of z, and it's true Value: which therefore is supposed to be small. Wherefore $ar = A$ or $r = \frac{A}{a}$ nearly; or $ar + br^2 = A$ more nearly, that is $r = \frac{A}{a + br}$ *fere*, Substitute for r in the second Member of this last Equation, it's approximate Value deduced from the preceding $r = \frac{A}{a}$, and you'll have $r = \left(\frac{A}{a + \frac{bA}{a}} =\right) \frac{aA}{a^2 + bA}$ *quam proxime*. Wherefore you have $z = l + \frac{aA}{a^2 + bA}$ nearly. Which is Dr. *Halley's Rational* Formula: published in Phil. Transactions for *May* 1694, N° 210.

66. And if any one will be at Pains to consider the Business of the Resolution of Equations, there may be many Methods of approximating to their Roots discovered. Thus, retaining the same supplementary Equation $ar + br^2 + cr^3 + dr^4$ &c. $= A$, you have, as before $r = \frac{A}{a}$ nearly: or $r = \frac{A}{a} - \frac{br^2}{a}$ more nearly: insert in the latter Member of this last Equation, instead of r^2 it's approximate Value $\frac{A^2}{a^2}$, deduced from the preceding, and you'll obtain $r = \frac{A}{a} - \frac{b}{a} \times \frac{A^2}{a^2}$ for a nearer Approximation to the supplementary Root r. Which approximate Value of r being found, call it e: and in the Equation of r, insert

$e + s$ for r: from which you may find an approximate Value of s: and so on for the other subsequent Supplements.

67. Our Author tells us that when there is any Doubt whether the Value of p in the first supplementary Equation, approaches near enough the Truth, by supposing $10p - 1 = 0$, you may assume the three last Terms; thus, $6p^2 + 10p - 1 = 0$: and by the Solution of the Quadratic, find the first Figure of the Value of p. Now by the Solution of the Quadratic $6p^2 + 10p - 1 = 0$, you have $p = \frac{-5 \pm \sqrt{31}}{6}$: of which two-fold Value you take $\frac{-5 + \sqrt{31}}{6} = +0.09$, *quamproxime*; which is the least Root, and that which approaches nearest to the Value of p, by putting $10p - 1 = 0$. The Reason of which shall be shewn Art. 71. And thus you find 0.09 a nearer Approach to the Value of p than 0.1 is: and it gives you one Decimal Place more in the Quotient. Moreover, as he tells us, it is convenient to ascertain the second, or third Figure, of the Quotient, the same way, *when in the last resulting Equation, the Square of the Coefficient of the Penult Term is not ten times greater than the Product of the last Term multiplied into the Coefficient of the Antepenult Term.*

68. To understand the Reason of this: suppose $ar + br^2 = A$, to represent an Equation resulting, when all the Terms except the three last of any supplementary Equation, are rejected: and $ar = A$, that which arises by rejecting all save the two last Terms: then we may easily find the Difference betwixt the two corresponding Values of r arising from these two Equations. From the last, we have $r = \frac{A}{a}$; from the first, $r = \frac{-a \pm \sqrt{a^2 + 4bA}}{2b}$. That we may compare these two Values of r, suppose $a^2 = z \times bA$ (where z represents the Number which multiplying bA, makes the Product equal to a^2) that is $b = \frac{a^2}{zA}$: insert this Value of b instead of it, in the Equation $r = \frac{-a \pm \sqrt{a^2 + 4bA}}{2b}$, and you'll have $r = \overline{-\frac{1}{2}z \pm \sqrt{\frac{1}{4}z^2 + z}} \times \frac{A}{a}$: of which two-fold Value, $\overline{-\frac{1}{2}z + \sqrt{\frac{1}{4}z^2 + z}} \times \frac{A}{a}$ is to be taken, which is the least of the two Roots, and that which approaches nearest to the Root of the Equation $ar = A$, or $r = \frac{A}{a}$, betwixt which and it the Difference is sought. This Difference then is the same with the Difference betwixt $\frac{A}{a}$ and $\frac{A}{a} \times \overline{\sqrt{\frac{1}{4}z^2 + z} - \frac{1}{2}z}$: which will vary according

cording to the Value of z. If it be $z = 10$, then $\sqrt{\frac{1}{4}z^2 + z} - \frac{1}{2}z = 0.916$ &c. and therefore the Difference betwixt the two Roots, in this Case is $\overline{1 - 0.916} \times \frac{A}{a} = 0.083$ &c. $\times \frac{A}{a}$, which is less than $\frac{1}{10}$ of $\frac{A}{a}$: but if $z = 9$, then $\sqrt{\frac{1}{4}z^2 + z} - \frac{1}{2}z = 0.908$ &c. so that the Difference of the Roots would be nearly 0.091 &c. $\times \frac{A}{a}$: which is nigh to $\frac{1}{10}$ of $\frac{A}{a}$. And if z be 8 or any thing less, the Difference will be more than $\frac{1}{10}$ of $\frac{A}{a}$ (the Difference still increasing as z diminishes) Whence it appears, why, if z be less than 10, that is, if the Square of the Coefficient of the Penult Term be not ten times greater than the Product of the last Term and Coefficient of the Antepenult Term, it is proper to ascertain the first Figure of the Value of r, by assuming the three last Terms of the supplementary Equation; and taking the least Root of the Quadratic: because in this Case the approximate Root of r found by assuming only the two last Terms of the Equation, may differ from the true Root, by a Difference which may alter the first Figure of that approximate Value of r.

69. Nay, as Sir *Isaac* observes, if in the whole Course of the Operation, we assume the three last Terms of each supplementary Equation, instead of the two last only: and extract the least of the two Roots out of the Quadratic, we shall obtain about twice as many Figures in the Quotient the one way, as the other, at each Operation: as may be easily collected from what hath been said in Art. 63, 64: by which it appears that each new Operation; or supplementary Equation, will, as it were, triple the Number of Decimal Places already found in the Quotient. And it may be observed that by assuming the three last Terms of the supplementary Equation, as in the preceding Article, you have Dr. *Halley*'s *irrational* Formula, published in the Philosophical Transactions in the Place mentioned already *. And as to Mr. *Raphson*'s Method of Approximation in the extracting the Roots of Equations, published in his *Analysis Equationum universalis*, it is, *in Effect*, the very same with our Author's Method here laid down; which proceeds by assuming only the two last Terms of the supplementary Equations, at each new Operation.

* Art. 65.

70. This Method of extracting the Roots of affected Numeral Equations, may be successfully applied to all kinds of them, how high soever: whether the Terms be all compleat; or any Number of them wanting: and therefore also serves for extracting the Roots of pure

 Equations,

Equations, that is the Roots of Numbers, especially the higher Ones. Moreover, after you have found the Quotient to any Degree of Exactness you please, the Quotient so found together with the Root of the last resulting Equation, must be equal to the Root of the Equation at first proposed: as appears from the Process itself. And therefore, having continued the Work to any desired Length, in approximating to the Root, you may prove or ascertain the Truth of the Operation here, as well as in any other arithmetical Operation, which proceeds by Resolution. For if you substitute for y, in the Original Equation, the Quotient already found $+$ the Supplement q, r, or s, &c. the transformed Equation ought to be the same with the Supplementary Equation of q, r, or s, &c. Thus *e. g.* having found, in the Author's Example, $y = (2 + p =) 2.1 + q$: transform the original Equation $y^3 - 2y - 5 = 0$, by substituting $2.1 + q$ for y, and you have $0.061 + 11.23q + 6.3q^2 + q^3 = 0$, the same with the corresponding supplementary Equation: which ascertains the Truth of the Operation.

71. There seems to be no other Difficulty that can occur to the young Geometrician in this Matter: unless it be this. Every supplementary Equation is of the same Order or Rank with the original Equation proposed; and therefore the Roots or Values of the Supplement may be as many as the Dimensions of that Equation: how then doth it appear which Root of the supplementary Equation ought to be taken: and that the Value of the Supplement which arises by assuming the two last Terms of that Equation; or the least of the two Values of the Supplement, which arise by assuming the three last Terms, is an approximate Value of that which you want, to make up the true Root?

In order to clear up this Difficulty, let $ay + by^2 + cy^3$ &c. $= A$, represent any Numeral Equation, y being the unknown Quantity, and the other Letters standing for known Numbers; positive, or negative: and let the Roots of the Equation be called f, g, h, &c. positive, or negative. Then let n be a Number which approaches near to one of the Roots as f: which is found by Trial; or otherwise: and let it be $n + p = y$, where p is the Supplement. Substitute $n + p$ for y in the Equation $ay + by^2 + cy^3$ &c. $=$ A: and thence arises a new Equation, *viz.*

$$\left.\begin{array}{llll} -A & & & \\ +an & +ap & & \\ +bn^2 & +2bnp & +bp^2 & \\ +cn^3 & +3cn^2p & +3cnp^2 & +cp^3 \\ \&c. & \&c. & \&c. & \&c. \end{array}\right\} = 0,$$ which you may call the Equation of p;

and the other the Equation of y. In this Equation of p, the Roots or Values of p are just as many as the Roots or Values of y in the Equation

Values

of y, since $p = y - n$, that is $f - n$, $g - n$, $h - n$, &c. but of these Values of p, one, at least, is very small in Comparison of n, which is denoted by $f - n$, since n was supposed to differ but little from f; and this therefore is the Value of p, which is the Supplement sought. Wherefore in the Equation of p, the Terms which include the Powers of p, that is all before the two last Terms, will be small in Comparison of these two last Terms, if p be small enough in Comparison of n: which is the Case when the Root $f - n$ is sought: therefore these Terms may be neglected, when you are seeking a near Value of the Root $f - n$: so that you'll have $\left\{\begin{matrix} -A & \\ +an & +ap \\ +bn^2 & +2bnp \\ +cn^3 & +3cn^2p \\ \&c. & \&c. \end{matrix}\right\} = 0$ nearly, or $p = \frac{A - an - bn^2 - cn^3 \;\&c.}{a + 2bn + 3cn^2 \;\&c.}$ a near Value of the Supplement p, that is of $f - n$. And you may easily discover how near this Approach will come to the true Value of the Supplement p or $f - n$: which *(cæteris paribus)* will be different according to the different Proportion that n bears to p or $f - n$: suppose $p\,(= f - n) = \frac{n}{10}$, then in the preceding Equation of p, each Term, proceeding from the left to the right, abstracting from the numeral Coefficients, may be considered as ten times greater than the Term immediately succeeding it: in which Case the two last Terms of the Equation upon the left hand, are, abstracting from the Coefficients, more than $\frac{99}{100}$ of the whole; the three last Terms more than $\frac{999}{1000}$ of the whole, &c. And therefore the Value of the Supplement p deduced from the two last Terms of the Equation put equal to nothing, will not differ, generally speaking, from the true Value, but by a few hundred Parts. This will further appear if you compare the Value of p deduced from the abridg'd Equation, *viz.* $p = \frac{A - an - bn^2 - cn^3 \;\&c.}{a + 2bn + 3cn^2 \;\&c.}$, with it's Value derived from the entire Equation, which will be $p = \frac{A - an - bn^2 - cn^3 \;\&c.}{a + 2bn + 3cn^2 \;\&c.} - \frac{bp^2 + 3cnp^2 \;\&c.}{a + 2bn + 3cn^2 \;\&c.} - \frac{cp^3 \;\&c.}{a + 2bn + 3cn^2 \;\&c.}$; that is (by inserting $\frac{n}{10}$ for p) $p = \frac{A - an - bn^2 - cn^3 \;\&c.}{a + 2bn + 3cn^2 \;\&c.} - \frac{b \times 0.01n^2 + 3c \times 0.01n^3 \;\&c.}{a + 2bn + 3cn^2 \;\&c.} - \frac{c \times 0.001n^3 \;\&c.}{a + 2bn + 3cn^2 \;\&c.}$: from whence it appears that the approximate Value differs from the full Value, generally speaking, but by a few hundred Parts; since the Value of the Places in the Numerator of the second Term of the Value of p, are but hundred Parts of the Value of the Places in the Numerator of the first Term, the Denominators being the same. And after the same

Manner

Manner it will appear, that if $p = \frac{n}{100}$, the approximate Value will differ from the true Value of p, only by some thousand Parts, &c. but whatever be the Ratio of p to n, something depends upon the Coefficients: yet the exact Limits, in every particular Case, may easily be found, by applying what has been said in Art. 62, 63: by doing which, you may always discover to what Number of Places the Quotient may be continued, which arises from the Division of $A - an - bn^2 - cn^3$ &c. by $a + 2bn + 3cn^2$ &c. so as to give these Places all true; when you know the Ratio below which the Ratio of n to p doth not descend.

But if the Square of the Coefficient of the penult Term be not ten times greater than the Product of the last Term multiplied into the Coefficient of the antepenult Term, when n is not much more than $10p$, (of which Case I spoke Art. 68) then by assuming the three last Terms of the Equation of p, and putting them equal to nothing, the Solution of the Quadratic, gives a two-fold Value of p: now of these you take the least Value for the Supplement, because you still proceed upon the Supposition that the Value of p which you want is a small Value, *viz.* $f - v$, which may be supposed to be the least Value of p.

72. *Scholium.* There are some Things concerning our Author's Method of resolving numeral Equations, which it may be proper here to remark.

1°. He supposes that the approximate Value of y, which is assumed in the beginning, differs not a tenth part of itself from the true Value. Such an approximate Value may be found by the Methods for determining the Limits of the Roots of Equations, found in Algebraical Writers; or perhaps by repeated Trials.

2°. This Method of resolving numeral Equations, equally extends to the discovering of all the possible Roots of any Equation, whether they be positive or negative, commensurable or incommensurable: for whether the approximate Value of y in the proposed Equation be taken positive or negative; if it be such an approximation to any real Root as has been mentioned, the Operation goes on after the same Manner. However, if any one chuse, he may transform any proposed Equation having one or more of it's Roots negative, into another, in which all the Roots shall be positive, by increasing the Roots by a Number greater than any of the negative Roots, by *Transformation:* and so he shall only have positive Roots to extract. Moreover this Transformation is of Use for determining the Limits of the Roots: for when you have found the two immediate Limits betwixt which

any

any Root lyes, whereof one is greater and the other less than the Root sought, in the Manner taught by our Author in his *Arith. Univ. Cap, de Limitibus Equationum*, you may easily discover whether any of these Limits doth or doth not differ by $\frac{1}{10}$ of itself from the true Root: for if you substitute the Limits in Place of the unknown Quantity in the Equation: and then substitute the least Limit increased by $\frac{1}{10}$ of itself; or the greatest Limit diminished by $\frac{1}{10}$ of itself, in Place of the unknown Quantity; and observe the Signs with which the Aggregates of all the Terms of the Equation in every Case are affected, you'll thereby determine what you want: for if the Aggregate arising from the Substitution of the least Limit, be affected with a different Sign from that with which the Aggregate is affected by substituting that Limit increased by $\frac{1}{10}$ of itself, the least Limit doth not differ $\frac{1}{10}$ of itself from the true Root; otherwise if they be affected with the same Sign, it differs more than $\frac{1}{10}$: the same Rule is to be observed with respect to the greatest Limit: but here I suppose, that the Difference of the two Limits is greater than $\frac{1}{10}$ of the least Limit in the first Case; and greater than $\frac{1}{10}$ of the greatest Limit in the other Case: for if the Limits don't differ that much from one another; they will not differ from the true Root by $\frac{1}{10}$ of themselves respectively.

3°. This Method of resolving Equations is most properly applied for finding the incommensurable Roots: because the commensurable Roots are found more conveniently and directly by the Methods taught in Algebra.

4°. If the last Term of any supplementary Equation of p, q, r, &c. vanish, then the preceding Quotient already found is a true Root; or full Value of the unknown Quantity y: for in this Case the least Root of the Supplementary Equation is equal to nothing, that is the Supplement is equal to nothing.

SECT. IV.

Explication of the Resolution of affected specious or literal Equations, contain'd in Art. 30——34. 57.

73. THE specious Equations, whose Resolution our Author treats of in this Place, contain two unknown Quantities, as x and y: which are to be conceived as the Abscis, and rectangular Ordinate, of

of a Curve; whose Relation to one another is exprest by any such Equation: the other letters standing for definite known Quantities. And it is proposed to express the Root y in a Series of Terms including the Quantity x together with the other known Quantities. Thus in the Equation $y^3 + a^2y - 2a^3 + axy - x^3 = 0$, which he proposes, x is considered as the Abscifs, and y as the Ordinate, in a Curve, whose Nature is defined by that Equation. Now that the Value of y may be found in such a converging Series, it is necessary to consider x either, 1°. as very small; or, 2°. as very great; or then, 3°. as differing very little from some given Quantity. That so by supposing x to be very small, the Series may converge wherein the Root y is exprest by a Progression of Terms, in which the Dimensions of x increase in the Numerators: 2°. By supposing x to be very great, that Series may converge, in which the Dimensions of x continually increase in the Denominators of the Terms: Or that, 3°. By supposing x to differ but very little from some given Quantity, some Species or Letter, as z, being substituted for that Difference, it may come in Place of the Species x considered as very small. And all these three different Suppositions, may have each it's Conveniency, according to the Circumstances of the Case.

74. The first is the Case Sir *Isaac* chiefly insists upon: because most useful in the Affair of Quadratures: and to which the other Cases may be reduced.

In which he proceeds after this Manner. He supposes x entirely to vanish: and from the resulting Equation seeks a Value of y. This must be an approximate Value of y, because when x is very small, the Value of y cannot differ much from it's Value when x quite vanishes. But as this is but an approximate Value of y (suppose you call it A) he puts $y = A + p$: where p is the Supplement to the Root, as in the Resolution of Numeral Equations. Wherefore he substitutes $A + p$ instead of y, in the given Equation: whence arises a new Equation (which we shall call the Equation of p) from whence the Value of p is to be derived. Now although p may have several different Values, in this new Equation, yet it is the least Value that makes the Supplement (for which See Art. 71) Wherefore, since both x and p are very small, he rejects all the other Terms unless those where x and p are, separately, of the fewest Dimensions: and supposes them, as it were, equal to nothing from whence he finds an approximate Value of p: Which call Bx^n: and put $p = Bx^n + q$. Then he substitutes $Bx^n + q$ instead of p in the Equation of p; q being the Supplement to the approximate Value of p, *viz.* Bx^n; as p was formerly the

the Supplement to A, the approximate Value of y. And from hence you obtain a third Equation for finding an approximate Value of q, *viz.* by the like Means, that is by rejecting all the Terms of the Equation of q, save those wherein q and x are separately of the fewest Dimensions, the other Terms being comparatively of very little Significancy. Which near Value call Cx^m, or $q = Cx^m + r$: r being the Supplement. And so he proceeds. Whence he finds $y = A + p = A + Bx^n + q = A + Bx^n + Cx^m + r$, &c. By which you find a Series of Terms continually approaching to the Root y, that is to one of it's Values. The Process may be seen in the Treatise itself.

75. In which you may observe the following Things [1].

1°. That in the Equations of p, q, r, &c. upon the right Hand, those Parts of the last Term, in each Equation, will mutually destroy each other; in which x is of no higher Dimension than in the last, or immediately preceding Term of the Quotient. Thus in the Equation of p, the Parts $+ a^3 + a^3 - 2a^3$ of the ultimate Term; in the Equation of q, the Parts $- a^2x + a^2x$; in the Equation of r, the Parts $+ \frac{1}{16}ax^2 - \frac{1}{16}ax^2$ (and so for the rest) do destroy one another. The Reason of which is evident, by considering that those Parts which destroy one another in these last Terms, stand opposite to so many other Terms in the collateral Equation upon the left Hand, which were put equal to nothing. For since a, $\frac{1}{4}x$, $+ \frac{x^2}{64a}$, &c. are the Values of y, p, q, &c. in the Equations $y^3 + a^2y - 2a^3 = 0$, $4aap + a^2x = 0$, $4a^2q - \frac{1}{16}ax^2 = 0$, &c. Hence it follows that the Substitution of a, $- \frac{1}{4}x$, $+ \frac{x^2}{64a}$, &c. in place of y, p, q, &c. must make the Whole equal to nothing: So that those Parts of the last Terms of the Equations of p, q, r, &c upon the right Hand of the Diagram, which have been mentioned, must destroy each other. Whence it appears

2°. That the Powers of x in the last Term of the transformed Equations of p, q, r, &c. upon the right Hand, increase more and more continually; the lowest of which, after rejecting those Parts

[1] To prevent being misunderstood, let it be observed that the Series expressing the Value of y, is called the Quotient: and with respect to the Equations of p, q, r, &c. which stand upon the right-hand Column of the Diagram, I call that the *first Term*, in which p, q, r, &c. is of the highest Dimension: although it be placed last in the Diagram; and that the *last* Term of the Equation, which stands first in that Part of the Diagram: because the Order of the Terms ought to be computed by the Dimensions of p, q, r, &c. Moreover when any Term of an Equation is complex, I call the several simple Quantities, which make it up, *Parts* of that Term: thus $+ 3a^2p$, $+ a^2p$, $+ axp$, are the Parts of the penult Term of the Equation of p in our Author's Diagram.

which destroy one another (according to the last Remark) will be higher than the highest Dimension of x in the preceding part of the Quotient.

3°. Wherefore, when x is very small in Comparison of any other given Quantity, as $\overline{a}$, in the given Equation of y, the last Term of the supplementary Equation of p, q, r, &c. is continually diminishing, so as at length to vanish. And therefore one Value of the Quantities p, q, r, &c. continually diminishes, *viz.* the least Value, for
* Art. 71. that is the Supplement, as was observed already *: Consequently since
73. that Value of p, q, r, &c. together with the Quotient as far as formerly found, makes up the Value of y, it follows, that, by this Operation, you approach continually to the Root y, so as to differ from it at length by less than any given Difference: Because the last Term of the supplementary Equation, becoming at length less than any given Quantity, and vanishing, some Root of that Equation must become less than any given Quantity, and vanish; since the last Term of any Equation is the Product of all the Roots with contrary Signs: which diminishing and evanescent Root is the Supplement, as hath been just now said.

4° Whereas in every new Step, by which a new Term of the Value of y is found, our Author desires you to assume the Terms in which x and p; x and q; are separately of the fewest Dimensions, it may be observed that the Species p, q, r, &c. will always be found separately of one Dimension, without being affected with x: because in the first Supposition, you assume an Equation in which the Letter x is not found, and therefore the approximate Value of y thence deduced doth not include x; whence it is evident that the Parts of the Penult Term of the first transformed Equation, upon the right Hand Side of the Diagram, which stand collaterally to those Terms of the proposed Equation, wherein y is not affected with x, must contain p separately of one Dimension only, in the Equation of p: which in his Example are $3a^2p$ and a^2p: therefore in the next subsequent Transformation, *viz.* in the Equation of q, upon the right Hand Side of the Diagram, that part of the Penult Term which stands collateral to $4a^2p$ upon the left Hand, where p is separately but of one Dimension, must contain q separately of one Dimension only: and so the Reasoning is the same through all the subsequent Transformations *in inf.* So that in all Equations of this Kind, where you can deduce the first approximate Value of y, by supposing x to be nothing, the Parts of the transformed Equations to be selected, for deriving the Terms of the Quotient, in the Progress of the Operation, are manifest: and the Derivation of these

these Terms of the Quotient, requires no more but the dividing the Part of the last Term wherein x is of the least Dimension, with the Sign changed, by the Coefficient of such Part of the Penult Term, as includes but one Dimension of p, q, r, &c. separately.

76. And thus I have explained and demonstrated the general Process for deriving the several Terms of the converging Series for expressing the Value of y: but whereas in Art. 31, Sir *Isaac* mentions an Abridgment, which may serve to contract the Work, I must shew upon what Foundation it is built. In order to which, let it be observed that, when the Quotient is proposed to be continued only to a certain number of Terms; or (which is equivalent) to a given Dimension of x, all those Terms of any Equation upon the right Hand side of the Diagram, may be entirely omitted in the subsequent Transformation, in which, if, for the Supplement p, q, r, &c. that Power of x were substituted, which is found to be it's approximate Value, x would be of a higher Dimension than the proposed Limit [1]. The Reason of which is this: since the Species x is supposed indefinitely small, any Term, that, by the Substitution just now mentioned, would arise to a higher Dimension of x, than the Dimension to which the Quotient is proposed to be continued, is vastly less than any of the Terms in which the Dimension of x rises not above that Limit: and as such Term in the subsequent Transformation, must make that Part of the last Term of the Equation upon the right Hand, which stands collateral to it, of a higher Dimension of x than the proposed Dimension; and every subsequent Quantity upon that same Line of the Transformation, of still a higher Dimension of x, after a proper Substitution, as is evident, hence it appears that no such Term can any way affect the Quotient within the proposed Limits: since every new Quotient Term must have that Dimension of x, which is the lowest in the last Term of the supplemental Equation. Thus in the Operation in the Author, after you have arrived at the Equation of q; and found from it $q = \frac{x^2}{64a}$ nearly, the first Term q^3 is entirely neglected in the subsequent Transformation: because if $\frac{x^2}{64a} + r$ were substituted for

[1] The Reader must observe that this Rule which is given by Sir *Isaac*; and the Reason given for it here, can only be extended to those Cases in which the Supplements p, q, r, &c. are found in some Part of the penult Term of their respective Equations, without any Mixture of x: in other Cases the Rule will not hold. And therefore an universal Rule for all Cases shall be delivered afterwards.

q, it would give $q^3 = \frac{x^6}{262144a^3} + \frac{x^4r}{2048a^2} + \frac{x^2r^2}{32a} + r^3$, for it's collateral Value; in which the last Term $\frac{x^6}{262144a^3}$ rises above the fourth Power, which is the proposed Limit of the Dimension of x: and the other Terms would still rise to higher Dimensions of x, since the approximate Value of r must contain more Dimensions of x than q doth, which contains the Dimension x^2: whence it appears, that none of the Terms of the Value of q^3, in any subsequent Transformation, can give or produce any Quantity, but what shall contain either the 6th Dimension of x; or one higher in the ultimate Term: which therefore can have no Influence upon the Quotient within the proposed Limit. And the reasoning is exactly the same with respect to the Part $-\frac{3}{4}xq^2$ of the second Term of the Equation of q upon the right hand: which therefore is likewise entirely neglected in the subsequent Transformation.

77. But sometimes all the Terms of the transformed Value of a Quantity are not to be omitted, but only some of them, in order to carry the Quotient to a proposed Limit, or Dimension of x: for which you are to observe this Rule, *That after the first resulting Term of the transformed Value of any Quantity, standing collateral to it upon the Right-hand, so many more Terms are to be added, as the Index of the highest Power of* x, *in the Quotient, exceeds the Index of* x *in such first resulting Term of the transformed Value: provided the Dimensions of* x *ascend only by the Difference of one, as in the Example of the Author: where the Dimensions of* x *in the Terms of the Quote are* x^0, x^1, x^2, x^3, &c. Thus if the Quotient is to be continued only to the 4th Dimension of x, you omit all the Terms after x^4, and put one after x^3; and so in other Cases. Accordingly in the present Example, after you have arrived at the Equation of q, the transformed Value of the Quantity $3aq^2$ is $\frac{3x^4}{4096a} + \frac{3x^2r}{32} + 3ar^2$: of which the two Terms $\frac{3x^2}{r} + 3ar^2$ might be omitted, when the Quote is designed to be carried only to the 4th Power of x: for the same Reason, opposite to $\frac{3}{16}x^2q$, the Term $\frac{3}{16}x^2r$ might be omitted. The Reason of which may be easily deduced from what hath been said in the preceding Article. Which Rule includes the particular Rule given by Sir *Isaac*, Art. 31: as will be evident to one that considers it. But if the Dimensions of x in the Quote proceed by greater Differences, as x^0, x^2, x^4, x^6, &c. or x^0, x^3, x^6, x^9, &c. then, when

when the Differences are 2, after the first resulting Term of the transformed Value of any Quantity, you need only subjoin half the Number of Terms, which denotes the Excess of the Index of the highest Power of x, to which the Quotient is to be continued, above the Index of x in the first resulting Term of the Value of the Quantity, to which that Term stands collateral: And when the Differences of the Dimensions of x in the Terms of the Quote are 3, you need only adjoin a third of that Number of Terms, *&c.* because now the Differences of the Dimensions of x in the Terms of the transformed Value of any Quantity taken from the last resulting Equation, will be 2, 3, *&c.* after substituting for p, q, r, *&c.* their approximate Values. And after the Operation, in this abridged way, is carried so far, as that the Supplement q, r, *&c.* rises no higher than the Root, in the transformed Equation upon the Right-hand Side of the Diagram, you obtain the rest of the Terms by Division: which is to be continued untill the Quote be brought to the proposed Limit [1].

78. The Author's Example, with the Operation abridged, according to the preceding Rules, may stand thus: where the Mark * denotes the Parts of the Terms which are omitted: the Quotient or Value of y being proposed to be carried the Length of the 4th Power of x.

[1] These Things help to explain Art. 44. of the Author.

N. B.

N. B. The Quantities which are dashed thus $\cancel{a^3}$, are such as destroy one another.

$y^3 + aay - 2a^3 + axy - x^3 = 0.$			$y = a - \frac{x}{4} + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$ &c.
$y = +a + p.$	$+y^3 =$ $+a^2y =$ $-2a^3 =$ $+axy =$ $-x^3 =$	$+\cancel{a^3} + 3a^2p + 3ap^2 + p^3$ $+\cancel{a^3} + a^2p$ $-\cancel{2a^3}$ $+a^2x + axp$ $-x^3$	$+4a^2p + a^2x = 0$ *fere*, or $p = -\frac{x}{4}$ *fere.*
$p = -\frac{x}{4} + q.$	$+p^3 =$ $+3ap^2 =$ $+4a^2p =$ $+axp =$ $+a^2x =$ $-x^3 =$	$-\frac{1}{64}x^3 + \frac{3}{16}x^2q$ * * $+\frac{3}{16}ax^2 - \frac{3}{2}axq + 3aq^2$ $-\cancel{a^2x} + 4a^2q$ $-\frac{1}{4}ax^2 + axq$ $+\cancel{a^2x}$ $-x^3$	$+4a^2q - \frac{1}{16}ax^2 = 0$ *fere*, or $q = +\frac{x^2}{64a}$ *fere.*
$q = +\frac{x^2}{64a} + r.$	$+3aq^2 =$ $+4a^2q =$ $-\frac{1}{2}axq =$ $+\frac{3}{16}x^2q =$ $-\frac{1}{16}ax^2 =$ $-\frac{65}{64}x^3 =$	$+\frac{3x^4}{4096a}$ * * $+\cancel{\frac{1}{16}ax^2} + 4a^2r$ $-\frac{1}{128}x^3 - \frac{1}{2}axr$ $+\frac{3x^4}{1024a}$ * $-\cancel{\frac{1}{16}ax^2}$ $-\frac{65}{64}x^3$	$+4a^2r - \frac{1}{2}axr - \frac{131}{128}x^3 - \frac{15}{4096}\frac{x^4}{a} = 0$ *fere.* $r = \frac{+\frac{131}{128}x^3 + \frac{15}{4096}\frac{x^4}{a}}{+4a^2 - \frac{1}{2}ax}$ *fere.* $r = +\frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$ &c.

Now

Now because in the last Transformation, which gives the Equation of r, the Supplement r rises no higher than the Root, the rest of the Terms of the Quotient, as far as the 4th Power of x, which is the Limit proposed, are found by Division, *viz.* by dividing $\frac{131}{128}x^3 + \frac{15}{4096}\frac{x^4}{a}$ by $4a^2 - \frac{1}{2}ax$, which gives the Quote $\frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$ &c. which must not be carried beyond the 4th Power of x; in Regard the Terms affecting the 5th and higher Powers of x, were rejected.

79. Take another Example $x^2y^4 + c^2y^4 - x^4y^2 + c^4x^2 - c^6 = 0$, where the Dimensions of x in the Quotient, proceed by the Difference 2, and the highest Dimension of x to which the Quotient is proposed to be continued is the 6th: You assume the Terms $c^2y^4 - c^6 = 0$ *fere*, or $y = c + p$, and then proceed as you see:

	$x^2y^4 + c^2y^4 - x^4y^2 + c^4x^2 - c^6 = 0$		$y = c - \frac{x^2}{2c} + \frac{3x^4}{8c^3} - \frac{5x^6}{16c^5}$ &c.
$y = +c + p$	$+x^2y^4 =$	$+c^4x^2 + 4c^3x^2p + 6c^2x^2p^2 + 4cx^2p^3 + x^2p^4$	$4c^5p + 2c^4x^2 = 0$ *fere*
	$+c^2y^4 =$	$+\cancel{c^6} - 4c^5p + 6c^4p^2 + 4c^3p^3 + c^2p^4$	$p = -\frac{x^2}{2c}$ *fere*
	$-x^4y^2 =$	$-c^2x^4 - 2cx^4p - x^4p^2$	
	$+c^4x^2 =$	$+c^4x^2$	
	$-c^6 =$	$-\cancel{c^6}$	
$p = -\frac{x^2}{2c} + q.$	$+4c^3p^3 =$	$-\frac{1}{2}x^6$ * * *	$4c^5q - 2c^3x^2q - \frac{1}{2}c^2x^4 + 2x^6 = 0$ *fere*
	$+6c^4p^2 =$	$+\frac{3c^2x^4}{2} - 6c^3x^2q$ *	$q = \frac{+\frac{1}{2}c^2x^4 - 2x^6}{4c^5 - 2c^3x^2}$ *fere*
	$+6c^2x^2p^2 =$	$+\frac{3x^6}{2}$ * *	$4c^5 - 2c^3x^2)\frac{1}{2}c^2x^4 - 2x^6(+\frac{3x^4}{8c^3} - \frac{5x^6}{16c^5}$ &c. $= q$
	$-2cx^4p =$	$+x^6$ *	$\frac{1}{2}c^2x^4 - \frac{3}{4}x^6$
	$+4c^5p =$	$-\cancel{2c^4x^2} + 4c^5q$	* $-\frac{5}{4}x^6$
	$+4c^3x^2p =$	$-2c^2x^4 + 4c^3x^2q$	
	$+2c^4x^2 =$	$+\cancel{2c^4x^2}$	
	$-c^2x^4 =$	$-c^2x^4$	

Here

Here in this Example, becauſe the Differences of the Dimenſions of x in the Value of y are 2, and that Value or Quotient is to be carried forward only to the 6th Dimenſion of x, therefore in transforming the ſeveral Quantities of the Equation of p, you will obſerve that thoſe are entirely omitted in which if x^2, which is the Dimenſion of x in the approximate Value of p, were ſubſtituted for p, the Dimenſion would be higher than the 6th: no Term placed after the firſt reſulting Term, when the Index of x in it is 6: and only one Term after it, when the Exponent of x is 4. By which the Equation of q riſes not above the firſt Power of q: therefore you find other two Terms of the Quotient by Diviſion, as appears by the Operation itſelf. And ſo you proceed in other like Caſes, by which you ſhall obtain a Value of y in an infinite Series of ſimple Terms: and therefore if x and y denote the Abſciſs and perpendicular ordinate of a Curve, the correſpondent Curvilinear Area is found by our Author's ſecond Rule: which in this Caſe, where the Powers of x aſcend in the Numerators, approaches the nearer to the Truth the leſs that x is.

80. But if x be ſuppoſed to be very great in Compariſon of any other known Quantity in the given Equation (which is the ſecond Caſe we mentioned above *; and of which the Author treats in Art. 33, 34,) in this Caſe, the Quotient, or Value of y, muſt be ſuch, that the Dimenſions of x diminiſh in the Numerators; or increaſe in the Denominators of the Terms. Now in order to find ſuch a Value of y, our Author directs to proceed thus: let $y^3 + axy + x^2y - a^3 - 2x^3 = 0$ be propoſed, in which x is ſuppoſed very great in Compariſon of the given Quantity a, then he deſires you to ſelect thoſe Terms in which x and y either ſeparately or multiplied together are of the moſt and at the ſame Time equal Dimenſions, and put them as it were equal to nothing, and the Value of the Root y thence deduced, ſhall give the firſt Term of the Quotient: therefore in this Example you put $y^3 + x^2y - 2x^3 = 0$, whence you find $y = x$, an approximate Value of y, that is $y = x + p$, p being the Supplement. It will be the ſame Thing as Sir *Iſaac* obſerves if you turn the aſſumed Equation $y^3 + x^2y - 2x^3 = 0$, into this other $y^3 + y - 2 = 0$, and extract the Root y out of this laſt Equation; which will be found $y = 1$, and then multiply this Value of y by x: the Reaſon of which is this, that if the given Equation $y^3 + x^2y - 2x^3 = 0$, be transformed by putting $z = \frac{y}{x}$ or $zx = y$, it will give $x^3z^3 + x^3z - 2x^3 = 0$; or, by dividing by x^3, $z^3 + z - 2 = 0$, which is the ſame with $y^3 + y - 2 = 0$, whoſe Root 1 being multiplied by x, is the Root of the given Equation $y^3 + x^2y - 2x^3 = 0$, ſince $y = zx$. 81. Now

* Art. 72.

81. Now the first Term of the Quotient or Value of y being discovered in this Manner (the Reason of which shall be shewn afterwards) the subsequent Terms are found by substituting $x+p$ for y in the given Equation $y^3+axy+x^2y-a^3-2x^3=0$, and transforming it thereby into a new Equation; from which an approximate Value of p is to be found, by assuming those Terms of this new Equation, which are vastly greater than all the rest, and putting them, as it were, equal to nothing: and so carrying on the Operation for the subsequent Terms, after the like manner as has been shewn already, in the other Case, in which x was supposed to be very small. The Process of which is here represented to the Reader,

$$y=x+p.\quad \begin{array}{rcl} +y^3 &=& +\cancel{x^3}+3x^2p+3xp^2+p^3 \\ +axy &=& +ax^2+axp \\ +x^2y &=& +\cancel{x^3}+x^2p \\ -a^3 &=& -a^3 \\ -2x^3 &=& -\cancel{2x^3} \end{array} \qquad \begin{array}{l} 4x^2p+ax^2=0 \text{ fere} \\ p=\left(\frac{-ax^2}{4x^2}=\right)-\frac{1}{4}a \text{ fere} \end{array}$$

$$p=-\tfrac{1}{4}a+q.\quad \begin{array}{rcl} +p^3 &=& -\frac{1}{64}a^3+\frac{3}{16}a^2q-\frac{3}{4}aq^2+q^3 \\ +3xp^2 &=& +\frac{3}{16}a^2x-\frac{3}{2}axq+3xq^2 \\ +axp &=& -\frac{1}{4}a^2x+axq \\ +4x^2p &=& -\cancel{ax^2}+4x^2q \\ -a^3 &=& -a^3 \\ +ax^2 &=& +\cancel{ax^2} \end{array} \qquad \begin{array}{l} 4x^2q-\frac{1}{16}a^2x=0 \text{ fere} \\ q=+\frac{a^2}{64x} \text{ fere} \end{array}$$

$$q=+\frac{a^2}{64x}+r.\quad \begin{array}{rcl} +q^3 &=& +\frac{a^6}{262144x^3}+\frac{3a^4r}{4096x^2}+\frac{3a^2r^2}{64x}+r^3 \\ -\frac{3}{4}aq^2 &=& -\frac{3a^5}{16384x^2}-\frac{3a^3r}{128x}-\frac{3}{4}ar^2 \\ +3xq^2 &=& +\frac{3a^4}{4096x}+\frac{3a^2r}{32}+3xr^2 \\ +\frac{3}{16}a^2q &=& +\frac{3a^4}{1024x}+\frac{3}{16}a^2r \\ -\frac{1}{2}axq &=& -\frac{a^3}{128}-\frac{1}{2}axr \\ +4x^2q &=& +\cancel{\frac{a^2x}{16}}+4x^2r \\ -\frac{65a^3}{64} &=& -\frac{65a^3}{64} \\ -\frac{1}{16}a^2x &=& -\cancel{\frac{1}{16}a^2x} \end{array} \qquad \begin{array}{l} 4x^2r-\frac{131a^3}{128}=0 \text{ fere} \\ r=+\frac{131a^3}{512x^2} \text{ fere} \end{array}$$

$$4x^2-\tfrac{1}{2}ax \text{ \&c.}\Big)+\frac{131a^3}{128}-\frac{15a^4}{4096x} \text{ \&c.}\Big(+\frac{131a^3}{512x^2}+\frac{509a^4}{16384x^3} \text{ \&c.}=r$$

$$\begin{array}{l} +\frac{131a^3}{128}-\frac{131a^4}{1024x} \text{ \&c.} \\ \hline \qquad +\frac{509a^4}{4096x} \text{ \&c.} \end{array}$$

$$y=x-\tfrac{1}{4}a+\frac{a^2}{64x}+\frac{131a^3}{512x^2}+\frac{509a^4}{16384x^3} \text{ \&c. or } x-\tfrac{1}{4}a+\tfrac{1}{64}a^2x^{-1}+\tfrac{131}{512}a^3x^{-2}+\tfrac{509}{16384}a^4x^{-3} \text{ \&c.}$$

In

82. In which Operation, having obtained the Equation of p, I throw by the first and second Terms, as containing the Cube and Square of p: which being of small Value in Comparison of the other Terms, are to be neglected: likewise of the Penult Term, you only retain the Part $4x^2p$, in which x rises to the highest positive Dimension; or if there were no positive Dimension of x in it, you must take that Part only where x is of the lowest Negative Dimension [1], the rest you neglect, because by the Hypothesis x is supposed to be vastly great in Comparison of a; likewise of the ultimate Term I retain (after throwing by repugnant Parts) only the Part ax^2, *viz.* that in which x is of the highest positive Dimension; otherwise, if there be no positive Dimension, where it is of the lowest Negative Dimension: and so I put $4x^2p + ax^2 = 0$ almost, whence I deduce $p = -\frac{1}{4}a$ for an approximate Value of p; and consequently, for the second Term of the converging Series for the Value of y: and then putting $p = -\frac{1}{4}a + q$, you transform the Equation of p into a new Equation, as you see; which you manage after the same Manner as the preceding, by putting $4x^2q - \frac{1}{16}a^2x = 0$ almost, whence I deduce $q = +\frac{a^2}{64x}$ nearly, which is the third Term of the converging Series: and so you proceed to find the other Terms by the like Means. But here it may be observed, that when the Series for the Value of y is proposed to be continued only to a certain Limit, or Dimension of x, many unnecessary Terms, or Parts of Terms, in the Transformations of the Equations, may be omitted, *viz.* those that cannot affect the Quotient within the proposed Limit, agreeably to what was shewn formerly *, allowing for the Difference of the Cases: according to which those Quantities marked with an Asterisk above them, might have been omitted, if the Value of y was proposed to be carried no further than the Dimension $\frac{1}{x^3}$, or x^{-3}. Moreover it is to be observed, that, after you have obtained a certain Number of Terms of the Series, by a continued Transformation, you may obtain so many more by Division, as you see in the Operation. The Division is always performed by dividing the last Term with the Signs changed, by the Complex

* Art. 78. 79.

[1] I call a negative Dimension of x, a Power having a negative Index in the Numerator; or a positive Index in the Denominator as $x^{-3} = \frac{1}{x^3}$: accordingly I call $x^{-3} = \frac{1}{x^3}$ a lower negative Dimension of x than $x^{-4} = \frac{1}{x^4}$: likewise $x^0 = 1$ is to be accounted the lowest Dimension of x, whether positive or negative.

Coefficient

Coefficient of the Supplement q, r, &c. in the Penult Term of the last Equation. And in order to know to how many Terms the Quotient of the Division may be carried, observe this Rule. *In the last transformed Equation, suppose it be that of r, divide, or suppose to be divided, all the Terms of the Equation preceding the Penult, by* r, *some Power of which above the first, enters into them all: Then find the Power of x which enters into the first Term of the Value of r: which in the preceding Example I find to be* $\frac{1}{x^2}$ *or* x^{-2}, *by putting* $4x^2r - \frac{65a^3}{64} - \frac{a^3}{128} = 0$: *substitute that Dimension of x, viz.* $\frac{1}{x^2}$ *in Place of r, or suppose it to be substituted, in the Terms preceding the Penult, after they have been supposed to be divided by r as above; and let the lowest positive, or yet negative Dimension of x, thence arising, be* x^n *or* x^{-n} *respectively, according as x is supposed to be very small, or very great: then you may continue the Division untill you obtain that Term of the Quotient whose positive, or negative Dimension, is the next below* x^n *or* x^{-n} *respectively: only it must be observed, that, if there be any positive Dimension of x in any of the Terms preceding the Penult Term of the Equation, after the Division and Substitution mentioned above, and the positive Dimensions of x decrease in the Value of y, then the Division is to be continued only until you obtain that Term of the Quotient, whose positive Dimension is the very next* above *the highest in these Terms of the Equation. And you must take Care, in setting down the Parts of the Divisor and Dividend, that you make the lower Dimensions to precede the higher, when they are positive, and x very small; but the higher positive Dimensions to precede the lower, and the lower negative Dimensions to precede the higher, when x is very great.*

In this Rule it is supposed that all the Terms of the transformed Equations are compleat, although this be not always necessary [1]: and the Rule will serve in all Cases, whether the Dimensions of x be all positive and increase from the beginning of the Series; or 2°. the Dimensions of x be first negative and decrease, and afterwards become positive, and then increase: in which Cases x must be supposed to be very small; or, 3°. the Dimensions of x be negative and increase from

[1] There is not any Necessity for annexing any of the Terms, of the last transformed Equation, which precede the Antepenult: because the Dimensions of x contained in the Quantities, which lye upon the same horizontal Line, proceed always in an arithmetical Progression, after the Substitution mentioned in the Rule: nay and of the Antepenult Term, there is no Necessity for annexing any other Part or Quantity, but that one where x is of the least positive or least negative Dimension, according as x is very small or very great; since the Continuation of the Terms of the Quotient, arising by the Division, is regulated by it only.

 the

the beginning of the Series; or 4°. the Dimensions of x be first positive and decrease, and then become negative and afterwards increase: in which two last Cases, x must be supposed to be very great. And I need not take up Time in shewing the Foundation of this Rule: since the Reader will easily understand it from what was demonstrated formerly with respect to numeral Equations *, applied to this Case of literal Equations, which is very easy to do.

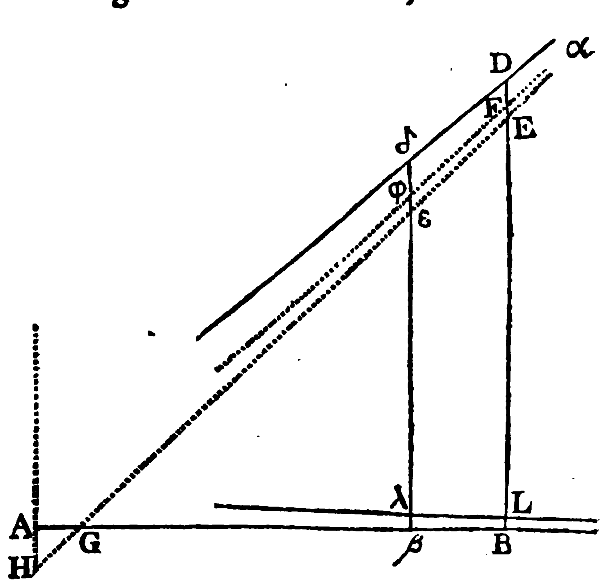
* Art. 63. 71.

83. From these Examples, I suppose, it will be manifest how you are to proceed in extracting the Roots of literal Equations including two unknown Quantities x and y; so as to express the Value of y by a Series of Terms including different Powers of x, and that whether x be supposed to be very small, or very great. But, notwithstanding that it has been shewn in the last Example, how to find the Terms of a converging Series expressing the Value of y, when x is very great in comparison of any other Quantity, as a, in the Equation, by a Method differing in some respect from that whereby the Value of y was found, when x was supposed to be very small; yet the same thing might be obtained, by the first Method, if you only suppose that the given Quantity a, in Comparison of which x is supposed to be very great, be considered as the indefinitely small Quantity coming in Place of x: and that you proposed to yourself to find a converging Series expressing the Value of y in Terms proceeding according to the positive Dimensions of a, instead of those of x: which is taken Notice of by our Author at Art. 33: The Reason of which is, that the Smalness or Greatness of x is comparative, *viz.* with respect to the Magnitude of the known Quantity a; so that the supposing of x to be very great is equivalent to the supposing of a to be very little.

84. And thus by either of these Methods the Value of y being extracted, and expressed by an infinite Series of simple Terms, including the Quantity x involved with known Quantities, the Area of the Curve, whose Absciss and Ordinate are x and y is found by the Author's second Rule. And the Area will approach the nearer to the Truth *(cæteris paribus)* the greater that x is, because this will evidently make both the Series for the Value of y; and that for the Value of the Area, thence deduced, to converge the faster: thus having found in the preceding Example $y = x - \frac{a}{4} + \frac{a^2}{64x} + \frac{131a^3}{512x^2} + \frac{509a^4}{16384x^3}$ &c. the Area belonging thereto is $\frac{x^2}{2} - \frac{ax}{4} + \boxed{\frac{a^2}{64x}} - \frac{131a^3}{512x} - \frac{509a^4}{32768x^2}$ &c. both of which converge the more quickly the greater

greater the Abſciſs x is. And as to the Curve, it approaches continually towards a rectilinear Aſſymptote, at which the Part of the Ordinate y denoted by the two firſt Terms of it's Value, *viz.* $x - \frac{1}{4}a$, always terminates; ſo that you may eaſily find the Poſition of that Aſſymptote, and any Portion of it, by aſſuming any two Points in the Abſciſs, and drawing two Perpendiculars at theſe two Points, equal to the Length of the Abſciſs leſs $\frac{1}{4}a$; the Line drawn betwixt the Extremities of the Perpendiculars, is the Portion of the rectilinear Aſſymptote, correſponding to the Part of the Abſciſs lying betwixt the two Points aſſumed therein. But that this, and likewiſe the way of computing the Area may appear the more clearly, I ſhall repreſent the thing to the Reader by an actual Deſcription, ſince the Author himſelf has juſt only mentioned it, Art. 34; and it may be of ſome Uſe to the young Geometer.

Draw the Right Line AB produced indefinitely towards B, upon which take $AG = \frac{1}{4}a$, and calling the Abſciſs $AB = x$, let the Curve $\delta D\alpha$ be deſcribed by the perpendicular Ordinate $BD = y$, ſo that it be $y = x - \frac{a}{4} + \frac{a^2}{64x} + \frac{131a^3}{512x^2}$ &c. upon BD take $BE = BG = x - \frac{a}{4}$, draw GE, and it is an Aſſymptote to the Curve $\delta D\alpha$, to which the Curve infinitely produced towards α, continually approaches. The Curvilinear Area was found to be $\frac{x^2}{2} - \frac{ax}{4} + \left|\frac{a^2}{64x}\right| - \frac{131a^3}{512x} - \frac{509a^4}{32768x^2}$, &c. of which that Part which is expreſt by the two firſt Terms $\frac{x^2}{2} - \frac{ax}{4}$ denotes a rectilinear Area adjacent to the Abſciſs AB: for if through A you draw AH perpendicular to AB, and meeting the Aſſymptote GE below the Abſciſs, in the point H, then the Triangle $GBE = \frac{1}{2}x^2 - \frac{1}{4}ax + \frac{1}{32}a^2$, and the Triangle $AGH = \frac{1}{32}a^2$, therefore $GBE - AGH = \frac{x^2}{2} - \frac{ax}{4}$. Again the Part $-\frac{131a^3}{512x}$



$-\frac{509a^4}{32768x^2}$ &c. denotes a curvilinear Area lying upon the other side of the Ordinate, terminated at the Curve infinitely produced upon that Side: because the positive Powers of x are found in the Denominators of all the Terms: the other Terminations of this Curvilinear Area may be determined by comparing the Ordinate $+\frac{131a^3}{512x^2}+\frac{509a^4}{16384x^3}$ &c. with which it is described, with the preceding Parts of the Value of y, viz. $x-\frac{a}{4}$, and $+\frac{a^2}{64x}$, of which the first Part, viz. $x-\frac{a}{4}$ is the right Line BE, which always touches the Assymptote GE; the other Part, *viz.* $\frac{a^2}{64x}$ will belong to an Hyperbola: for if in the Line ED you take $EF=\frac{a^2}{64x}$ the third Term of the Value of y, the Point F will touch an Hyperbola having HE and HA for it's Assymptotes; which let be described, and let it be Fϕ: from whence it appears that the Curvilinear Area denoted by the Terms $-\frac{131a^3}{512x}$ $-\frac{509a^4}{32768x^2}$ &c. is terminated by the Part of the Ordinate FD and the two Legs of the Curves δD and ϕF infinitely extended beyond the Ordinate BD towards α: but whereas the third part of the Curvilenear Area denoted by $\boxed{\frac{a^2}{64x}}$ is an infinite hyperbolical Space, lying betwixt the Hyperbola and it's Assymptote, infinitely produced, therefore we can only find the Curvilinear Area adjacent to any given Part of the Abscifs as Bβ, *viz.* the Area B$\beta\delta$D: which is done thus: subtract the Area belonging to the shorter Abscifs Aβ from the Area belonging to the longer Abscifs AB, without taking in the Hyperbolical Area $\boxed{\frac{a^2}{64x}}$ and the Difference shall give the Sum of the rectilinear Area B$\beta\varepsilon$E and the curvilinear Area F$\phi\delta$D: (for which see our Author's second Rule, Examp. 3. and Sect. 5. of the preceding Treatise) wherefore if to these you add the hyperbolical Area E$\varepsilon\phi$F, you shall have the entire Area B$\beta\delta$D. To find that hyperbolical Area E$\varepsilon\phi$F, you may proceed thus: Upon BD take BL = EF, and with the Center A and Assymptotes AB, AH describe the rectangular Hyperbola Lλ through the Point L, cutting $\beta\delta$ in the Point λ, and the Area B$\beta\lambda$L is equal to the Area E$\varepsilon\phi$F, as is evident; wherefore let the Area B$\beta\lambda$L belonging to the equilateral Hyperbola; and adjacent to the given part of the Assymptote Bβ be found by what hath been shewn

ſhewn formerly *: and this added to the Areas $B\beta\iota E$ and $F\varphi\delta D$ found already, will make up the whole Area $B\beta\delta D$ required. * Art. 25.

85. And thus I have endeavoured to illuſtrate the two different Caſes mentioned by Sir *Iſaac* in this Treatiſe, for diſcovering the Roots of affected literal Equations, and expreſſing them by infinite Series's, either when x is ſuppoſed to be very ſmall; or when it is ſuppoſed to be very great in Compariſon of any given Quantity in the propoſed Equation: but I mentioned a third Caſe, which is when x is ſuppoſed to differ very little from any given Quantity: in this Caſe you ſubſtitute ſome letter for that ſmall Difference in the propoſed Equation, as z; and ſo having the two unknown and variable Quantities y and z, of which z is conſidered as indefinitely ſmall in Compariſon of any given Quantity in the Equation, you may find a converging Series for the Value of y, in Terms made up of poſitive Powers of z, and known Quantities, as formerly: from which you may deduce a Series including x by replacing it in the Series for it's formerly ſubſtituted Value. Thus in the Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$, it being known or ſuppoſed that x is nearly of the ſame Quantity with a, you ſubſtitute z for their Difference, *i. e.* put $z = a - x$ or $x - a$, that is ſubſtitute $a - z$ or $a + z$ for x in the propoſed Equation: by ſubſtituting the firſt of which, there ariſes this new Equation $y^3 + 2a^2y - azy + 3a^2z - 3az^2 + z^3 - 3a^3 = 0$, from whence the Value of y may be found by the firſt Method, z being ſuppoſed very ſmall in Compariſon of a: which Value being found, you may replace x for $a - z$, that is $a - x$ for z; and ſo the Area may be found in Terms made up of a and x. And this third Method is proper to be taken when the Series for the Value of the Ordinate y, and conſequently that for the Value of the Area, will converge the more quickly the nearer that the Length of the Abſciſs x is to the Length of the given Line a.

86. Moreover, hence it appears that all the three different Methods for extracting the Roots of ſuch affected literal Equations, may be reduced to the firſt Method only: for this third Caſe mentioned in the preceding Art. falls in with it; and it was ſhewn in what Manner the ſecond Caſe, which ſuppoſes x to be very great, falls in with it alſo *. * Art. 83.
Which Caſe likewiſe you may reduce to the firſt, by ſubſtituting $\frac{1}{z}$ for x in the propoſed Equation, which muſt make z very ſmall when x is very great. Thus in the preceding Example $y^3 + axy + x^2y - a^3 - 2x^3 = 0$, in which x is ſuppoſed very great in Compariſon of a, ſubſtitute

substitute $\frac{1}{z}$ for x, and you will have $y^3 + \frac{ay}{z} + \frac{y}{z^2} - a^3 - \frac{2}{z^3} = 0$, that is, by multiplying by z^3, $z^3y^3 + az^2y + zy - a^3z^3 - 2 = 0$, in which z is very small: and from which the Root may be extracted upon that Supposition, by what will be said afterwards upon the second Remark under Art. 35 of the Author. Which Article I should now enter upon the Explication of directly; were it not that it will fall in to be explained afterwards to more Advantage in the Progress of the following Section.

SECT. V.

Of the various Methods of extracting and expressing the Roots of affected literal Equations, including two unknown Quantities, by Means of infinite Series's: in which the Author's 35th 42——46 Art. are explained.

87. ONE and the same Equation may afford different Series's for expressing the Root or Value of y, by *selecting* two or more Terms *differently*, out of the proposed Equation, and putting them equal to nothing, in order to discover from thence the first Term of a Series expressing the Value of y. In what goes before, it hath been shewn that one and the same Equation may afford two different Series's; one upon Supposition that x is very small, and another upon Supposition that x is very great, in Comparison of any known Quantity in the Equation: but it frequently happens that from the same Equation, there may be more Series's than one expressing the Value of y, when x is very small; and likewise more than one agreeing to the Supposition of x being very great: but then there is some Difficulty in finding out the first Terms of all those Series's: for understanding of which I shall demonstrate the following Lemmas.

LEMMA I.

88. In any Equation including two unknown Quantities x and y, from whence a converging Series is to be derived for expressing a Value of y in simple Terms involving different Powers of x; let Ax^n denote the first Term of such a Series, where n denotes any indefinite Exponent,

nent, poſitive or negative, integral or fractional; and A any indeterminate Coefficient: then Ax^n being ſubſtituted for y throughout the ſeveral Terms of the propoſed Equation, let the Dimenſions of x in two or more of the Terms be the ſame, and either leſs than the Dimenſions of x in all the other Terms; or greater than the Dimenſions of x in all the other Terms [1], I ſay theſe two or more Terms are vaſtly greater than all the reſt of the Terms of the Equation, according as x is ſuppoſed to be indefinitely leſs, or indefinitely greater, than any known Quantity in the Equation, *i.e.* if x be ſuppoſed to be indefinitely ſmall, theſe two or more Terms in which the Dimenſions of x are equal, and leſs than in all the reſt, are vaſtly greater than all the reſt: but if x be indefinitely great, the two or more Terms in which the Dimenſions of x are equal and greater than in all the reſt, are vaſtly greater than all the reſt of the Terms of the Equation.

Dem. For when x is ſuppoſed indefinitely little, the Dimenſions of x in the Terms of a converging Series, muſt be ſuppoſed to increaſe as the Series proceeds, ſo as to be leaſt of all in the firſt Term, *viz.* Ax^n: wherefore Ax^n will be vaſtly greater than all the remaining Terms, which will vaniſh in Compariſon of it, if you make x ſmall enough, ſo that Ax^n is, in that Caſe, nearly the Value of y: conſequently if Ax^n being ſubſtituted for y in the Terms of the Equation, raiſe two or more of them to equal and lower Dimenſions of x than in all the reſt, theſe two or more Terms muſt be vaſtly greater than all the reſt taken together. This is when x is ſuppoſed to be indefinitely ſmall.

And by the ſame Way of Reaſoning, if x be vaſtly great, it may be ſhewn, that, if there be two or more Terms in which the Dimenſions of x are equal and higher than the Dimenſions of x in all the other Terms of the Equation, after Ax^n is ſubſtituted for y, ſuch two or more Terms are vaſtly greater than all the reſt of the Terms taken together: by conſidering that in a converging Series anſwering to this Caſe, the Dimenſions of x muſt continually diminiſh; and that any higher Dimenſion of x is vaſtly greater in Value than all the lower Dimenſions of x taken together. Q. E. D.

[1] Here it muſt be obſerved that any negative Dimenſion of x is conſidered as lower than any poſitive Dimenſion; and of negative Dimenſions of x, the greater the negative Exponent is, the lower is the Dimenſion eſteemed to be: thus x^{-4} or $\frac{1}{x^4}$ is reckoned a lower Dimenſion of x than x^{-3} or $\frac{1}{x^3}$: ſo that $x^{-4}.x^{-3}.x^{-2}.x^{-1}.x^0.x^1.x^2.x^3$ &c. repreſents a Scale of Dimenſions of x aſcending.

LEMMA II.

89. If $x^p y^q \,.\, x^r y^s$ exhibit the Dimensions of x and y in any two Terms of such an Equation as has been mentioned, in which Terms Ax^n (signifying the same Thing as before) being substituted for y, makes the Dimensions of x equal, and less in them; or greater in them, than the Dimensions of x in all the rest of the Terms of the Equation, after Substitution of Ax^n for y in them likewise, these two Terms being put equal to nothing, will give an approximate Value of y, in which the Dimension of x shall be x^n, the same as in Ax^n the first Term of the converging Series [a].

Dem. For the two Terms including $x^p y^q$ and $x^r y^s$ being put equal to nothing, will give a Value of y, which contains this Dimension of x, viz. $x^{\frac{p-r}{s-q}}$, as is evident: now I say $\frac{p-r}{s-q} = n$. For, by inserting x^n for y in the two Terms of the Equation in which the Dimensions of x and y are $x^p y^q$ and $x^r y^s$, their Dimensions of x will be x^{p+qn} and x^{r+sn}, which by the Hypothesis are equal, that is $p + qn = r + sn$, whence you'll have $\frac{p-r}{s-q} = n$. Q. E. D.

90. Cor. 1. From the last Lemma it follows, that if there be any two Terms of such an Equation as has been mentioned, which being put equal to nothing, give a possible Value of y, in which Value the Dimension of x is such, that, being substituted for y in all the Terms of the Equation, it makes them all of higher, or all of lower Dimensions of x than in the two Terms mentioned, these two Terms, so assumed and put equal to nothing, will give the first Term of a converging Series for the Value of y, according as x is supposed to be indefinitely small; or indefinitely great.

Example I. Let the Equation $y^3 - axy - x^3 = 0$ be proposed, (which is that proposed by the Author in Art. 35. Rem. 2.) put the Terms $- axy - x^3 = 0$, thence arises $y = - \frac{x^2}{a}$, in which the Dimension of x is x^2: which being substituted for y in the first Term of the Equation, viz. y^3 would give x^6 for the Dimension of x; and that being higher than x^3; which is the Dimension of x in the two Terms assumed, hence $- \frac{x^2}{a}$ is the first Term of a converging Series for the Value of y, whence x is very little.

[a] The proposed Equation including x and y, is supposed here and all along to have all it's Terms brought to one Side, so that the whole Aggregate of the Terms may be equal to nothing.

After

After the ſame Manner, if you put $y^3 - x^3 = 0$, you have $y = x$, which will give the firſt Term of another converging Series for the Value of y, when x is very great: becauſe if x be ſubſtituted for y in the middle Term of the Equation, *viz.* $- axy$, it gives the Dimenſion x^2, lower than x^3, which is it's Dimenſion in the two Terms aſſumed.

Example II. Let the Equation $y^3 - a^2y + axy - x^3 = 0$ be propoſed: if you put $y^3 - a^2y = 0$, you have thence $y = + a$ or $- a$, where the Dimenſion of x is $x^0 = 1$, which being ſubſtituted for y through all the Terms, the Dimenſions of x will be x^0, x^0, x^1, x^3: which ſhews that $+ a$ or $- a$ may, either of them, be the firſt Term of a converging Series, when x is very little.

91. From which Example it is evident, that, in every ſuch Equation, if there be two or more Terms including no Dimenſion of x; from which put equal to nothing a Value of y may be deduced, ſuch Value will always give the firſt Term of a converging Series which ſhall be a Root of the Equation, when x is ſmall. And this coincides with our Author's Rule, Art. 30.

92. In the ſame Equation, aſſuming the two Terms $- a^2y - x^3 = 0$, you have $y = - \frac{x^3}{a^2}$: by inſerting of which Dimenſion of x for y through all the Terms, the Dimenſions of x ſtand thus x^9, x^3, x^4, x^3: whence it appears that $- \frac{x^3}{a^2}$ will be the firſt Term of another converging Series, when x is very little.

Again if you aſſume $y^3 - x^3 = 0$, you have $y = x$: by inſerting of which for y, the Dimenſions of x are x^3, x, x^2, x^3: whence it appears that x will be the firſt Term of a converging Series for the Value of y, when x is very great: becauſe the Dimenſions of x in the two aſſumed Terms are higher than in any of the reſt of the Terms of the Equation.

93. Cor. 2. But if any two Terms of ſuch an Equation as has been mentioned, be aſſumed, and put equal to nothing, and from thence no *poſſible* Value of y can be deduced; or if a real Value can be deduced, but ſuch a one that the Dimenſion of x it contains, being ſubſtituted in place of y through all the Terms, make the Dimenſions of x in ſome higher, and in ſome lower than in the two aſſumed Terms, in either Caſe the two aſſumed Terms are improper from whence to derive the firſt Term of a converging Series, either when x is ſmall or great. For in the firſt Caſe there is no real Value: and in the other Caſe where there is a real Value, it cannot be the firſt Term

of a converging Series, since the first Term of a converging Series must be deduced from those Terms only, which are much greater than all the rest: which the two Terms assumed would not be in this Case, whether you suppose x to be small or great: as easily appears from what hath been said above, especially in Lem. I. The same Thing is likewise evident of any two or more Terms in which the Dimension of y is the same: because the Value of y thence deduced is o.

Thus in the last Equation $y^3 - a^2y + axy - x^3 = 0$, if you put the two Terms $y^3 + axy = 0$, you have $y = \sqrt{-ax}$, an *imaginary* Expression: therefore no first Term of a Series consisting of *real* Terms, can be thence deduced.

Suppose you put $axy - x^3 = 0$, thence you have $y = \frac{x^2}{a}$ by substituting of which Dimension of x through the Terms of the Equation, the Dimensions of x are these, x^6, x^2, x^3, x^3: where it appears that there is one Dimension of x higher, and another lower, than in the two assumed Terms: and therefore these two Terms are improper for deriving the first Term of a converging Series, either when x is very small; or very great.

Finally, the two Terms $- a^2y + axy$ are improper: because the Value of y deduced from putting them equal to nothing is $y = 0$.

LEMMA III.

94. In any Equation including one *variable* or *flowing* Quantity only, as x, the rest being *constant* or *invariable* Quantities, all the Terms must include the same Dimension of x. [1]

Dem. For, if possible, let $ax^4 + bx^4 + cx^2 + cx^2 = 0$ be an Equation whose Terms include different Dimensions of x, viz. x^4 and x^2, and x at the same Time a variable or flowing Quantity: then divide all the Terms by the lowest Dimension of x, viz. x^2, and it is $ax^2 + bx^2 + c = 0$: whence $x^2 = \frac{-c}{a+b}$ or $x = \sqrt{\frac{-c}{a+b}}$: therefore x is a constant Quantity, contrary to Hyp.

Universally let $ax^p + bx^q + cx^r = 0$ be an Equation including one variable Quantity x, which hath different Dimensions, if possible, and let x^r be the lowest Dimension, whether the others be equal Dimensions of x or not: divide by x^r, and you have $ax^{p-r} + bx^{q-r} + c = 0$: which being an Equation of a *definite* or certain degree, x can

[1] In this Lemma it is supposed that the Coefficients a, b, c, &c. are not nothing.

only

only have a certain determinate Number of Values, which is contrary to the Nature of a *variable* Quantity; which admits of infinite different Values.

LEMMA IV.

95. Let ax^py^q, bx^ry^s, cx^ty^v, be three Terms of an Equation, which includes two unknown and *variable* Quantities x and y: and let these Terms be such, as that, when any Dimension of x, as x^n, is inserted for y in them, the Dimensions of x are equal: Moreover, when the three Terms $ax^py^q + bx^ry^s + cx^ty^v$ are put $= 0$, let a Value of y thence deduced be αx^ν, I say $n = \nu$.

Dem. For if you insert αx^ν for y in the Equation $ax^py^q + bx^ry^s + cx^ty^v = 0$, you shall have $a\alpha^q x^{p+q\nu} + b\alpha^s x^{r+s\nu} + c\alpha^v x^{t+v\nu} = 0$: where x being a variable Quantity, it's Dimensions through all the Terms must be equal *; therefore $p + q\nu = r + s\nu$; whence $\nu = \frac{p-r}{s-q}$: but by inserting x^n for y in the same three Terms, the Dimensions of x in them are $x^{p+qn} . x^{r+sn} . x^{t+vn}$: which by the Hyp. are equal, that is $p + qn = r + sn$ or $n = \frac{p-r}{s-q}$: therefore $n = \nu$. Q. E. D.

* Art. 94.

96. Cor. 1. The reasoning is the same if there be more than three Terms. And hence it follows, that, if there be three or more Terms of any proposed Equation, including two unknown Quantities x and y, which being put equal to nothing, give a possible Value of y, containing a certain Dimension of x [a] such, as that the Dimension of x, being substituted for y through all the Terms of the given Equation, makes the Dimensions of x greater in all the other Terms, or less in them all, than in the three or more assumed Terms, the Value of y thence deduced will give the first Term of a converging Series, for a Value of y in the proposed Equation: according as x is supposed to be very little, or very great. But if the Dimensions of x rise higher in some Terms, and lower in other Terms, of the proposed Equation, than they do in the three or more assumed Terms, these assumed Terms are improper for deriving the first Term of a converging Series, either when x is very small, or very great.

97. Cor. 2. Let $ax^py^q . bx^ry^s . cx^ty^v$ be three Terms selected out of the proposed Equation, which being put equal to nothing, give $y = Ax^n$, whether Ax^n be the first Term of a converging Series or not; then it shall be $p-r : s-q :: r-t : v-s$: or by Alternation, and

[a] Here $x^0 = 1$ is considered as a Dimension of x.

Composition

Composition $p-r : p-t :: s-q : v-q$: that is the Differences of the Indexes of x and y are proportional. For by inserting Ax^n for y, the Dimensions of x in the three Terms selected, are $x^{p+qn} . x^{r+sn}$.
* Art. 94. x^{t+vn} : which being equal *, you have 1°. $p+qn = r+sn$, 2°. $r+sn = t+vn$: from the first $n = \frac{p-r}{s-q}$; from the second $n = \frac{r-t}{v-s}$; whence $p-r : s-q :: r-t : v-s$. And the reasoning would be the same if the Terms were more than three. The Converse of which is likewise true, *viz.* If there be three or more Terms having the Indexes of x and y proportional, and if x^n be inserted for y in all of them, make the Dimension of x the same in any two of them, it will make it to be the same in all of them. For let $x^{p+qn} = x^{r+sn}$, then I say x^{t+vn} is equal to any of these: because since it is $p+qn = r+sn$, hence $n = \frac{p-r}{s-q}$: but $\frac{p-r}{s-q} = \frac{r-t}{v-s}$, whence $n = \frac{r-t}{v-s}$ or $vn - sn = r - t$, and by Transposition, $t + vn = r + sn$.

98. Cor. 3. If there be any Number of Terms more than two, of any proposed Equation, as the Terms $ax^p y^q . bx^r y^s . cx^t y^v$, from which put equal to nothing, the first Term of a converging Series is derived, *viz.* Ax^n : then any two of these Terms being put equal to nothing, will give a Value of y in which the Dimension of x is the same as in Ax^n. For if you put any two of the Terms assumed, as $ax^p y^q + bx^r y^s = 0$, you thence deduce $y = \frac{-a}{b} x^{\frac{p-r}{s-q}}$, where the Dimension of x is the same as in Ax^n by Lem. IV.

99. Cor. 4: From what has been said it appears, that, if any two Terms of a proposed Equation, being assumed, and put equal to nothing, a Value of y be thence deduced; and the Dimension of x contained in that Value, being substituted for y in all the Terms of the proposed Equation, make the Dimension of x the same in one or more Terms, as it is in the two assumed Terms; but higher in all the other Terms; or lower in all the other Terms; then such one or more Terms being joined to the two assumed Terms, and the Aggregate put equal to nothing, any real Value of y thence deduced, will make the first Term of a converging Series. Moreover any two Terms of the proposed Equation being taken, which belong to the Equation to be assumed for determining the first Term of a converging Series, you may always know if there are any more Terms to be joined with these two, from the Property of such Terms mentioned Art. 97. since the Differences of the Indexes of x and y must be proportional.

Examp.

Examp. Let the Equation $y^6 - 5xy^5 + \frac{x^3y^4}{a} - 7a^2x^2y^2 + 6a^3x^3 + b^2x^4 = 0$ be propoſed: aſſume the two Terms $y^6 - 7a^2x^2y^2 = 0$, thence you have $y + \sqrt[4]{7} \times \sqrt{ax}$, where the Dimenſion of x is $x^{\frac{1}{2}}$, which being ſubſtituted for y through the ſeveral Terms of the propoſed Equation, makes the Dimenſions of x theſe, *viz.* $x^3 . x^{\frac{7}{2}} . x^5 . x^3 . x^3 . x^4$: wherefore you put the three Terms $y^5 - 7a^2x^2y^2 + 6a^3x^3 = 0$, and thence ſeek the firſt Term of a converging Series, which you will find to be fourfold, *viz.* $+\sqrt{ax}$, $-\sqrt{ax}$, $+\sqrt{2ax}$, $-\sqrt{2ax}$, for all theſe are Roots of the Equation $y^6 - 7a^2x^2y^2 + 6a^3x^3$, as will appear by ſubſtituting them for y, one after another: therefore $+\sqrt{ax}$, or $-\sqrt{ax}$, or $+\sqrt{2ax}$, or $-\sqrt{2ax}$, may be the firſt Term of a converging Series, each of which Series's is a Root of the propoſed Equation, when x is very little. Moreover you ſee that the Differences of the Dimenſions of x and y are proportional in the three Terms $y^6 - 7a^2x^2y^2 + 6a^3x^3$ or $x^0y^6 - 7a^2x^2y^2 + 6a^3x^3y^0$, for $0-2 : 2-6 :: 2-3 : 0-2$ or $-2 : -4 :: -1 : -2$.

From the Symptoms or Properties of the Terms of any propoſed ſpecious Equation, including two unknown Quantities x and y, which are fit for deriving the firſt Term of a converging Series, which have been obſerved and demonſtrated in the preceding Lemmas and Corollaries, there may be various *Methods* deduced for diſcovering ſuch Terms.

METHOD I.

To be applied when there are two or more Terms of the propoſed Equation, which involve not the Quantity x; and one or more of which involve the Quantity y.

100. Put the Terms of the propoſed Equation which include not the Quantity x, equal to nothing, giving each Term it's proper Sign: the Value of y deduced from this fictitious Equation, will be the firſt Term of a converging Series for the Value of y; and if this fictitious Equation have more real Roots than one, each of theſe will give the firſt Term of a converging Series: when x is very ſmall.

This is the Rule propoſed by Sir *Iſaac* himſelf Art. 30. See the Demonſtration of it above, Art. 74. 91. and Examples in Art. 79. 80. and under Examp. 2. of Art 90.

But if y be of the ſame Dimenſion in all the Terms, which include not x, the firſt Term of a converging Series cannot be derived from them *.

* Art. 93. them *. And what may be done when all the Roots of the fictitious Equation are impossible shall be shewn afterwards.

METHOD II.

By which the first Terms of all the converging Series's may be discovered, which are the Roots of the proposed Equation, either when x is very small or very great.

101. This Method is by *Trial*, founded upon the Symptoms or Properties of the Terms to be selected for this Purpose, which have been explained and demonstrated above.

The Method of proceeding is this: Take the first Term of the proposed Equation, and join to it every one of the other Terms, each with it's proper Sign, one after another, forming so many fictitious Equations; from each of these seek the Value of y; and observe the Dimensions of x which each of these Values contains: substitute the Dimension of x contained in the Value of y derived from the first of those fictitious Equations, instead of y, through all the Terms of the proposed Equation, and observe the Dimensions of x through all the Terms; and if you find them either higher in all the other Terms, or lower in all the other Terms, than the Dimension of x in the two assumed Terms, of which the first fictitious Equation was formed, then the Value of y deduced from that fictitious Equation, is the first Term
* Art. 90. of a converging Series *; or if there be more Values of y than one deducible from that fictitious Equation, each of them will be the first Term of a converging Series. But if, besides the two Terms forming the first fictitious Equation, there be any other, in which the Dimension of x is the same as in these two Terms, and it's Dimensions in all the rest of the Terms, either all higher or all lower, in that Case, the other Term or Terms, in which the Dimension of x is the same as in the two first assumed Terms, must be joined with them, all with their proper Signs, from which you are to form a new Equation, by putting the Aggregate equal to nothing; and the Root or Roots of it will give the first Term or Terms, of one or more converging Series's,
* Art. 96. which will be a Root, or Roots, of the proposed Equation *.
98. But if the Dimensions of x, after Substitution through all the Terms of the proposed Equation as above, be higher in some Terms, and lower in other Terms, than what the Dimension of x is in the two Terms of the first fictitious Equation, the assumed Terms are improper to derive the first Term of a converging Series, whether x be very
* Art. 93. 96. little, or very great *.

Terms

This done, you must next join the second Term of the proposed Equation with all the other Terms except the first; or perhaps such other as it was joined with formerly, in deriving the first Term of a converging Series; and thereby form so many new fictitious Equations, as there are distinct Terms to join it with: which are to be managed after the same manner as the former fictitious Equations. Then proceed to the third Term of the proposed Equation, and join it with the other Terms that follow it, one after another, with the exception just now mentioned; and use these fictitious Equations as the former ones. And thus proceed to do with all the remaining Terms; and so you shall have discovered the first Terms of all the Converging Series's, which express the Roots of the proposed Equation, *i. e.* the Values of y, either when x is very little; or very great.

The Demonstration of this Method is contained in the preceding Lemmas and Corollaries, particularly in the Articles referred to: And the following Example will illustrate it.

Ex. Let the Equation $y^5 - by^2 + 9bx^2 - x^3 = 0$ be proposed: I form an Equation by joining the first two Terms together, which gives $y^5 - by^2 = 0$, from whence you have $y = \sqrt[3]{b}$ or $\sqrt[3]{b} \times x^0$: where the Dimension of x being $x^0 = 1$, I substitute it for y in the Terms of the given Equation; and they will contain these Dimensions of x, viz. $x^0 . x^0 . x^2 . x^3$: wherefore I conclude that $\sqrt[3]{b}$ is the first Term of a converging Series for y, when x is very small: since the Dimensions of x in the other Terms are greater than in the two first Terms, which were assumed.

But this I might have concluded immediately by the first Method.

Next I take the first and third Terms of the proposed Equation, and form this Equation $y^5 + 9bx^2 = 0$, from whence I have $y = \sqrt[5]{-9bx^2}$, in which the Dimension of x is $x^{\frac{2}{5}}$, which I substitute for y through all the Terms of the proposed Equation, from whence they have these Dimensions of x, viz. $x^2 . x^{\frac{4}{5}} . x^2 . x^3$. where I observe that the Dimension of x in the second Term is less, and in the fourth Term greater than the Dimension of x in the first and third Terms, which were those assumed: wherefore I know those two Terms are improper for deducing the first Term of a converging Series, either when x is very small, or very great.

3°. I form an Equation of the first and last Terms, *viz.* $y^5 - x^3 = 0$: whence I have $y = x^{\frac{3}{5}}$; by substituting which Dimension of x for y, you have x^3, $x^{\frac{6}{5}}$, x^2, x^3: from which I conclude

clude that $x^{\frac{1}{3}}$ is the first Term of a converging Series, when x is very great: because the Dimension of x in the two assumed Terms, is greater than in the other two.

And thus having joined the first Term of the proposed Equation with all the rest, I proceed to join the second Term with those that follow it; it having been joined with the first already. Wherefore by joining the second and third you form this Equation $-by^2+9bx^2=0$, whence I find $y=+3x$ or $-3x$; so that substituting this Dimension of x for y in the Terms of the proposed Equation, the Dimensions stand thus x^5, x^4, x^2, x^3: whence, as before, I conclude that $+3x$ and $-3x$ are the first Terms of two converging Series's, each of which will be a Value of y when x is very small, in Comparison of b.

And now I join the second Term of the proposed Equation with the last, and thereby form this Equation, viz. $-by^2-x^3=0$: in which I easily perceive that $y=\sqrt{-\frac{x^3}{b}}=\sqrt{-\frac{1}{b}}\times x^{\frac{3}{2}}$, which being the Square Root of a negative Quantity, supposing b to be positive, is impossible: from whence I conclude that these two Terms $-by^2-x^3$ are unfit for finding the first Term of a converging Series, by themselves: and I know they cannot be joined with any other Term, so as to form with it an Equation fit for that Purpose, because that would have appeared in the foregoing Part of the Operation, either when the first and second Terms were assumed; or when the second and third Terms were assumed, for forming these fictitious Equations *.

* Art. 95. 98.

102. But if this had not appeared from the foregoing Part of the Operation, then I would have proceeded to insert $x^{\frac{3}{2}}$ for y through the Terms of the proposed Equation, in order to find whether there might not be some other Term, which joined with these two, might have been proper to form an Equation, admitting of some real Root, for the first Term of a converging Series; for although two Terms don't yield a real Root when they are alone, yet I cannot from that conclude that they may not be joined with some other Term, which with them may form an Equation with one or more real Roots, which will be the first Term or Terms of one or more converging Series's. Thus if the proposed Equation had been $-by^2+2b^{\frac{1}{2}}x^{\frac{3}{2}}y+y+9bx^2-x^3=0$, then by joining the two extreme Terms, which are the same with those just now mentioned in the foregoing Equation, you have $-by^2-x^3=0$, whence $y=\sqrt{-\frac{1}{b}}\times x^{\frac{3}{2}}$, an impossible Value,

when

when b is positive; yet I substitute $x^{\frac{3}{2}}$ for y through the Terms of the Equation, and I find the Dimensions these $x^3 . x^3 . x^{\frac{3}{2}} . x^2 . x^3$: whence I am directed to frame this Equation $-by^2 + 2b^{\frac{1}{2}}x^{\frac{3}{2}}y - x^3 = 0$, or by changing all the Signs $by^2 - 2b^{\frac{1}{2}}x^{\frac{3}{2}}y + x^3 = 0$. where y has a real Value, *viz.* $+\sqrt{\frac{x^3}{b}}$; which will therefore make the first Term of a converging Series expressing one of the Roots of the Equation $-by^2 + 2b^{\frac{1}{2}}x^{\frac{3}{2}}y + y + 9bx^2 - x^3 = 0$, *viz.* when x is supposed much greater than b.

103. And now I have shewn what will be the Result of forming Equations by joining together every two Terms of the proposed Equation $y^5 - by^2 + 9bx^2 - x^3 = 0$, except the last two, from which it is plain no Value of y can be derived, since y doth not enter into any of them. And it appears that there may be three converging Series's when x is very little in Comparison of b, whose first Terms are $-b^{\frac{1}{3}}. +3x. -3x.$; and one converging Series when x is very great in Comparison of b, whose first Term is $+x^{\frac{3}{5}}$.

104. Schol. But here I suppose the Reader will easily understand that the whole of this Trial, to know what Terms of any proposed Equation are fit to derive converging Series's from, in the most Part of Cases, may be made in his own Mind, without the Necessity of always writing down all the Steps.

Thus if the Equation $by^3 - 2b^{\frac{1}{2}}x^{\frac{3}{2}}y - y - 9bx^2 + x^3 = 0$ be proposed, I easily perceive, by running over the Terms in my own Mind, so as to join them two by two, that the only fictitious Equations that can serve for finding the first Terms of converging Series's, are 1°. $by^3 - y = 0$ *. 2°. $by^3 + x^3 = 0$. 3°. $-y - 9bx^2 = 0$; the first and third when x is supposed to be very little; and the second, when it is supposed to be very great in Comparison of b; which first Terms therefore are $+\frac{1}{\sqrt{b}}$, $-\frac{1}{\sqrt{b}}$, $-\frac{x}{\sqrt[3]{b}}$, and $-9bx^2$.

* Art. 100.

METHOD III.

By which, for most part, the first Terms of two converging Series's may be discovered, one of them for the Case of x being very small; the other for the Case of x being very great.

105. Let x^p represent the Dimension of x in that Term of the proposed Equation in which x is separately (*i. e.* without any Mixture of y) of the lowest Dimension [1], and let $x^r y^s$ represent the Dimensions of x and y in every other Term of the Equation indifferently, one after another, (where r and s are supposed to put on as many different Values as the different Terms require) run over the several Terms, and observe where the Value of $\frac{p-r}{s}$ is greatest; then, that Term, or those Terms, if there be more of them, where this happens, being joined to the Term where you have the Dimension x^n, with their proper Signs, and put equal to nothing, the Root or Roots of this fictitious Equation will give the first Term or Terms of one or more converging Series's for the Value of y when x is very small.

Again if x^m represent the Dimension of x in that Term of the proposed Equation in which x is separately of the highest Dimension; and $x^r y^s$ represent the same thing as before, observe where the Expression $\frac{m-r}{s}$ is least of all, and such Term or Terms, being managed as before, will serve, in the same way, to discover the first Term or Terms of one or more converging Series's, when x is very great [2].

Dem. Let x^p denote the same thing as in this Art, formerly; and let $x^r y^s . x^{r'} y^{s'} . x^{r''} y^{s''}$. &c. represent the Dimensions of x and y in the other Terms of the proposed Equation, where I have given the Letters r and s Marks of Distinction, according as they are supposed to denote the Indexes of x and y in the different Terms: Now I say if $\frac{p-r}{s}$ be the greatest of all the corresponding Expressions, that is $\frac{p-r'}{s'}$, $\frac{p-r''}{s''}$ &c.

[1] Here it is supposed that the Exponents of the Powers of x and y are positive: and that x^0 therefore is the lowest Dimension of x in the proposed Equation.

[2] It must be observed that in computing the Value of the Expressions $\frac{p-r}{s}$ and $\frac{m-r}{s}$, a negative Value is less than any positive Value; and a greater negative is a less Value than a less Negative.

then

then the two Terms having x^p and $x^r y^s$ for their Dimensions, being put equal to nothing, are proper to derive the first Term of a converging Series from, when x is very small. For if these two Terms be put equal to nothing, the Value of y thence deduced, will contain this Dimension of x, viz. $x^{\frac{p-r}{s}}$: put $\frac{p-r}{s}=n$; insert x^n for y in the other Terms of the Equation, and the Dimensions of x in them will be $x^{r'+ns'}$ $x^{r''+ns''}$ &c. Now by Hyp. $n\ (=\frac{p-r}{s}) > \frac{p-r'}{s'}$ or $\frac{p-r''}{s''}$ &c. wherefore, if you multiply by s' or s'' &c. you'll have $ns' > p-r'$; $ns'' > p-r''$ &c. (and this is true whether $n=\frac{p-r}{s}$ be positive or negative) and by Transposition $r'-ns' > p$, $r''-ns'' > p$, &c. whence the thing is evident from Art. 90. And the Reader will easily understand, that, if there be any other Term, whose Dimensions of x and y are denoted by $x^t y^v$, such, that $\frac{p-t}{v}=\frac{p-r}{s}$, such Term must be joined with the other two, in forming the fictitious Equation for deriving the first Term of the converging Series *. * Art. 97.

Again if x^m denote the highest Dimension of x separately, and the other Symbols be used in the same Sense as before; by putting the Terms having the Dimensions x^m. $x^r y^s$, equal to nothing, you have a Value of y whose Dimension of x is $x^{\frac{m-r}{s}}$; put $\frac{m-r}{s}=n$, then by the Hyp. $n < \frac{m-r'}{s'}$; multiply both by s', and it is $ns' < m-r'$ (and that whether $n=\frac{m-r}{s}$ be positive or negative) wherefore, by Transposition, $r'+ns' < m$: and the rest of the Reasoning is as before, by Help of Art. 30. and 99. 99.

Ex. 1. Let the Equation $y^3 - axy + x^3 = 0$ be proposed, to find the first Term of a converging Series, when x is very small in Comparison of a, then $p=3$, and $m=3$ likewise, $r=1$. and $s=1$, when applied to the middle Term $-axy$, and $r=0$, $s=3$, when applied to the first Term y^3, so that the Values of $\frac{p-r}{s}$ or $\frac{m-r}{s}$ are $\frac{3-1}{1}=2$, and $\frac{3-0}{3}=1$; wherefore I put $-axy+x^3=0$ for finding the first Term of a converging Series, when x is very small; and $y^3+x^3=0$, when x is very great: which give $y=+\frac{x^2}{a}$; and $y=-x$ respectively.

Ex.

Ex. 2. Let $y^3 - a^2y + axy - x^3 = 0$ be proposed: here you find, by the same Method, the Terms $-a^2y - x^3$ proper for deriving the first Term of a converging Series, when x is very small; and $y^3 - x^3$ when x is very great: which therefore are $-\frac{x^3}{a^2}$, and x respectively.

Ex. 3. Let the Equation $by^3 - 2b^{\frac{1}{2}}x^{\frac{3}{2}}y - y - 9bx^2 + x^3 = 0$ be proposed: here you have $-9bx^2$ the Term wherein x is of the lowest Dimension separately, and therefore $p = 2$, and the Term $-y$ joined to it, is that which gives the Expression $\frac{p-r}{s}$ the greatest, which is $\frac{2-0}{1} = 2$, therefore the Equation $-y - 9bx^2 = 0$ gives the first Term of a converging Series, when x is very little, viz. $y = -9bx^2$. Again $+x^3$ is the Term where x is separately of the highest Dimension, therefore $m = 3$, and the Term by^3 is that which compared with $+x^3$, gives the Expression $\frac{m-r}{s}$ the least, wherefore I put $by^3 + x^3 = 0$, and thence deduce $y = -\frac{x}{\sqrt[3]{b}}$, for the first Term of a converging Series, when x is very great.

136. Schol. 1. This Method supposes that the Exponents of the Powers or Dimensions of x and y, through all the Terms of the proposed Equation, are positive; otherwise the Method will not answer, without other Limitations; as will appear, if you suppose that, in the first Part of the Demonstration, $n = \frac{p-r}{s}$ is positive; and $\frac{p-r'}{s'}$ and s' both negative.

107. Scol. 2. The Expression $\frac{p-r}{s}$, is always the greater the less that r is, if s be the same in different Terms of the proposed Equation: and if r be the same in different Terms of the Equation, $\frac{p-r}{s}$ is always the greater the less that s be, when $p-r$ is positive; but the greater that s be, if $p-r$ be negative: which is of Use in the Practice of selecting the Terms, to be used for discovering the first Term of a converging Series, when x is very small; and the like Observations may be made with Respect to the Expression $\frac{m-r}{s}$, to direct you in the selecting of the Terms proper for finding the first Term of such a Series, when x is very great. So that what has been said under the Head of this third Method, serves to illustrate our Author's second Remark under

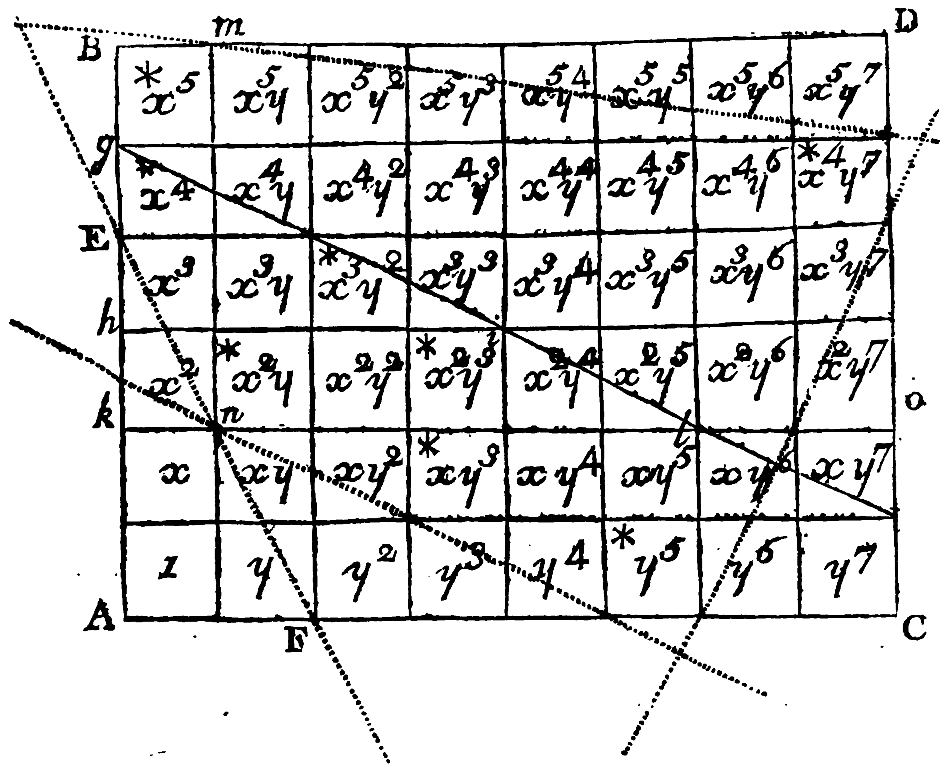
Art.

Art. 35. and, in general, his whole Method of proceeding in discovering the several Terms of the converging Series for the Value of *y*: which he calls the Quotient. And this leads us to

METHOD IV.

By Means of which the first Terms of all the converging Series's may be found which are Roots of the given Equation, whether x be very small, or very great.

108. This Method is by the help of a rectangular Parallelogram, ingeniously contrived by our celebrated Author.



Let AB, AC be two right Lines at right Angles to one another, divided each into some Number of equal Parts: and the Parallelogram AD being completed, through all the Points of Section of AB and AC draw Perpendiculars to these Lines, as you see, dividing the whole into equal Squares: which let be denominated from the Dimensions of of x and y; ſo that thoſe lying along AB may contain the Dimenſions of x ſeparately, *viz.* $1 = x^0 . x . x^2 . x^3 . x^4$ &c. thoſe along AC, the ſeparate Dimenſions of y, viz. $1 = y^0 . y . y^2 . y^3 . y^4$. &c. and every other Cell or Square, that Dimenſion of x that ſtands at the beginning of the ſame horizontal Row, and that Dimenſion of y that ſtands at the foot of the ſame perpendicular Row with itſelf, in the Manner the Figure repreſents. Then any Equation being propoſed, as the Equation $x^4y^7 - 4y^5 + 2axy^3 - 2bx^2y^3 + 3x^3y^2 - abx^2y + x^4 + 2ax^5 = 0$, mark ſuch of the ſmall Squares or Cells, as correſpond to the ſeveral Terms of the propoſed Equation, according to the Dimenſions of x and y they contain: which you ſee here done, by placing this Mark * in them: then let a Ruler as EF be applied to the loweſt of the cells that, being marked, lye along the line AB, (for in every Equation there muſt be at leaſt one ſuch Term) which in this Example is x^4, touching it in the Point E, as you ſee, then make the Ruler EF to turn from EA towards the Right Hand, until it touch, beſides the Cell x^4, one or perhaps more of the Cells that are marked, ſo as that all the reſt lye upon one Side of it: ſo here you find x^2y to be the only one it meets with at firſt. Wherefore you ſelect the two Terms of the propoſed Equation, which have the ſame Dimenſions of x and y, *viz.* $-abx^2y + x^4$, and form an Equation of them, *viz.* $-abx^2y + x^4 = 0$, from whence the Value of y being found, *viz.* $+\frac{x^2}{ab}$, gives the firſt Term of a converging Series, which converges the faſter the leſs that x is. Then I make the Ruler revolve the ſame Way about the Cell x^2y, until I find it touch, beſides it, one or more of the marked Cells, ſo as all the others lye upon one Side of the Ruler: here I find it to touch the three marked Cells $x^2y . xy^3 . y^5$: therefore I form another Equation of the three correſponding Terms of the given Equation, *viz.* $-4y^5 + 2axy^3 - abx^2y = 0$: from which you'll find a fourfold Root $+\sqrt{\frac{1}{4}a + \sqrt{-\frac{ab}{4} + \frac{1}{16}a^2}} \times x^{\frac{1}{2}}$, $+\sqrt{\frac{1}{4}a - \sqrt{-\frac{ab}{4} + \frac{1}{16}a^2}} \times x^{\frac{1}{2}}$, $-\sqrt{\frac{1}{4}a + \sqrt{-\frac{ab}{4} + \frac{1}{16}a^2}} \times x^{\frac{1}{2}}$, and $-\sqrt{\frac{1}{4}a - \sqrt{-\frac{ab}{4} + \frac{1}{16}a^2}} \times x^{\frac{1}{2}}$, any of which will be the firſt Term of a converging Series, when x is very ſmall in Compariſon of a and b.

Next I make the Ruler, according to the ſame Method, to touch the two marked Cells y^5 and x^4y^7, ſo as to leave all the reſt of the marked Cells

Cells upon the ſame Side of the Ruler, as you ſee by the Figure; and ſo I form an Equation of the two correſponding Terms, *viz.* $x^4y^7 - 4y^5 = 0$: whence you have $y = +\frac{2}{x^2}$ or $-\frac{2}{x^2}$, for the firſt Terms of other two Series's, which will alſo converge when x is very ſmall.

Hitherto the Ruler has been made to touch the outward marked Cells below, ſo as to have all the reſt ſtill lying as it were above it: and as no more of the extreme marked Cells can be touched by it in this Manner, you have got all the fictitious Equations from which the firſt Terms of Series's can be formed, which converge when x is very ſmall. And therefore you now begin to lay the Ruler to that marked Cell, which is the higheſt of ſuch that lye along the Line A B, which here is x^5, and making the Ruler to touch it above, let it revolve the contrary Way, and obſerve what extreme marked Cells can be touched by it at once, ſo as to leave all the others below it; and theſe will afford fictitious Equations for finding the firſt Terms of Series's, which will converge the faſter the greater that x is: and of ſuch I find only one, formed of the two Terms, which correſpond to the two Cells x^5 and x^4y^7: therefore I put $x^4y^7 + 2ax^5 = 0$, from whence you have $y = \sqrt[7]{2a} \times x^{\frac{1}{7}}$, which will be the firſt Term of a converging Series, wherein the Dimenſions of x ſhall continually diminiſh, correſponding to the Caſe of x being very great.

Here you may conceive the Ruler beginning to revolve from the Poſition E A towards E F, until it has made a half of an entire revolution, making it to touch the outward marked Cells below, as it revolves; and ſo you ſhall find the Terms of the propoſed Equation, which are proper for deriving the firſt Terms of all thoſe Series's, which converge the faſter the leſs that x is. (And again, if you conceive the Ruler, beginning at the point B, and lying in the direction of the Side A B of the Parallelogram, to turn about the contrary Way to what it did before, until it ſhall have turned about 180 Degrees, ſo as to make it touch the extreme marked Cells above, always leaving all the reſt entirely upon one Side of it, and as it were below it, you ſhall thereby obtain the Terms of the propoſed Equation, which are proper for forming fictitious Equations from which the firſt Terms of all ſuch Series's are to be derived, which converge when x is very great; that is in which the Dimenſions of x continually decreaſe. The ſame Thing may be conceived to be done by drawing a right lined Figure, every one of whoſe Sides ſhall touch two or more of the outward marked Cells, ſo as to leave the reſt entirely within the Figure, in the Manner repreſented in the Diagram itſelf.

 109. The

109. The Demonstration of these Things may be easily made out from a due Consideration of the Parallelogram, and what has been formerly said. In order to which, all that is necessary to be done, is to shew that, if the Cells which are touched by the lower Sides of the right-lined Figure, that is the Sides lying betwixt B A and C D upon the lower Sides of the Figure, be put equal to nothing; or (which, in this Case, is the same) the Terms of the proposed Equation, which correspond to these Cells, and the Dimension of x in the Value of y thence deduced, be incerted for y in all the rest of the marked Cells, they shall contain higher Dimensions of x than those Cells which the Line touches, after the Dimension of x is inserted in place of y in them also: and that these last mentioned shall contain the same Dimension of x.

And after the like Manner, that, if the Cells which are touched by the upper Sides of the right-lined Figure above, *i. e.* the Sides which lye betwixt B A and D C upon the upper Side of the Figure, be put equal to nothing, the Dimension of x contained in the Value of y thence deduced, being inserted instead of y in all the marked Cells, will produce lower Dimensions of x in them all, than in those which the Side of the Polygon touches: in which last the Dimensions of x shall be the same *.

* Art. 90, 96.

Now in order to shew this, let it be considered that if any Number of Cells are touched by any Line, as the Cells x^4, xy^2, and y^2 are by the Line E F; and as the Cells x^2y, xy^3 and y^5 are touched by the next Side of the Polygon, and so for any others, whatever way the Line touch them, whether above or below, and you take any Number of such Cells more than two, and that in any Order or at any Distances from each other, the Differences of the Indexes of x and y will be proportional, in the Sense of Art. 97. For let the right Line $g\,i\,l$ touch the Cells x^5, x^4y^2, x^3y^4, x^2y^6 upon the same Side: compare the Indexes of x and y in the three Cells x^5, x^3y^4, x^2y^6 together, and you find $5-3:4-0::5-2:6-0$. and so it is universally: for let the Cells x^5 x^3y^4, x^2y^6 be touched by the Line $g\,i\,l$ in the Points g, i, l, and let $i\,b$, $l\,k$, be the horizontal Lines drawn through i and l, meeting A B in the Points b and k, then the Triangles $g\,b\,i$, $g\,k\,l$ being similar, you have $g\,b : b\,i :: g\,k : k\,l$: but $g\,b$ and $g\,k$ measure the Differences of the Indexes of x, and $b\,i$ and $k\,l$ the Differences of the Indexes of y, as is evident by considering the Diagram: therefore the Differences of the Indexes of x and y are proportional, in the Terms x^5, x^3y^4, x^2y^6. And the Reasoning is the same, in every other Case.

110. From

110. From this it follows, that, if any Dimension of x, being inserted in place of y in all the Cells which are similarly touched by any right Line, make the Dimensions of x in two of these Cells equal, then the Dimensions of x in them all shall be equal *. * Art. 97.

111. Again, if a right Line touch any Number of Cells similarly below them (as the Line E F touches the Cells x^4, x^2y, y^2) then the Dimension of x, mentioned in the last Art. *viz.* that which being inserted for y in the touched Cells, makes them, or the Dimensions of x in them, equal; when it is inserted for y in the Cells that lye *entirely* above the Line E F, shall give higher Dimensions of x. For take any of the Cells which the Line E F touches below, as x^2y, which it touches in the Point n: let nm and no be the perpendicular and horizontal Lines upwards and towards the right Hand, then it will be evident, that all the Cells contained within the right Angle mno have either a greater Dimension of x, or a greater Dimension of y, than x^2y has; or both a greater Dimension of x and y: and the like must hold with respect to every other Cell that is similarly touched by E F, or by any other right Line which slopes downward from left to right: and these Cells which are contained within such angular Spaces, are all that lye *entirely* above the Line: but in this Case the Value of y that arises by putting any two of the Quantities contained in two of the touched Cells, equal to nothing, is a positive Dimension of x, as will be evident to any one that considers the Construction of the Parallelogram: therefore, when this positive Dimension of x is inserted for y in the other Cells which lye entirely above EF, the Dimension of x must be higher in every one of them.

But if the Line or Ruler, still touching below, slope downward from right to left, the Value of y which arises by putting any two of the Quantities, contained in two of the Cells which it touches similarly, equal to nothing, must contain a negative Dimension of x, as will be evident to one that considers it; and all the Cells contained within the right Angle which proceeds from any of the touched Cells towards the left Hand and upwards, contain either a higher Dimension of x with the same Dimension of y, or a lower Dimension of y with the same Dimension of x; or then, both a higher Dimension of x and a lower Dimension of y; in all which Cases, the negative Dimension of x inserted for y in such Cells, must make the Dimension of x higher in them than in the touched Cell: and this being true with respect to every one of the Cells, which are touched by a Line sloping downwards from right to left, hence the same Thing appears in this Case likewise, *viz.* that the negative Dimension of x, which inserted for y in the Cells which the Ruler touches similarly below, makes the Dimensions of x the same in them, will make the Dimensions of x higher in all the Cells, which lye entirely above the Line, when it is inserted for y in them.

112. And therefore, this being universally true, it plainly appears that the Cells which correspond to the Terms of any given Equation, which are touched by the lower Sides of the Polygon, must point out the Terms of the Equation, which are proper for deriving the first Terms of those Series's, which converge when x is very small. And by a like Way of Reasoning it may be demonstrated that the upper Sides of the Polygon placed as has been explained, must point out those Terms of the proposed Equation, which serve to discover the first Terms of those Series's which converge, when x is very great: which were the Things we proposed to shew.

113. Schol. 1. In the foregoing Demonstration I have not particularly mentioned the Cases in which the Ruler, touching any of the extreme marked Cells, either lies parallel to AB; or to AC: because in the first Case, it's plain no Value of y can be deduced from the fictitious Equation *; and the Reasoning in the other Case is most apparent.

* Art. 100.

114. Schol. 2. It may be observed likewise that Sir *I. Newton* makes use of the rectangular Parallelogram, only for discovering the Terms of any Equation, which are proper for finding the first Term of a converging Series, that will converge when x is very small; because he reduces all other Cases to this one *. And in particular, he uses it only for finding the first Term of such a converging Series, as may be deduced from the Supposition of the Ruler touching that Cell which contains the lowest Dimension of x separately, represented in the Diagram by the Line EF touching the Cells x^4 and x^2y [1]: which, I suppose, he doth, partly that he might not perplex his Readers with too great a Variety of Cases; and likewise because he uses the same uniform Method in finding the first, and all the other subsequent Terms of the Quotient or Value of y, as we shall see just now.

* Art. 86.

Of the different Methods of finding the subsequent Terms of the Quotient or converging Series, for the Value of y.

115. The first Term of the Quotient being found by any of the preceding Methods, the subsequent Terms may be discovered in diffe-

[1] See his Method of Fluxions and infinite Series, Introd. Art. 29, 30. and Com. Epist. No. 63.

rent

rent Ways: one of which is by Means of the Parallelogram already described. Thus, in any proposed Equation, having found the first Term of any Root or Value of y, when x is very small, to be Ax^n, you put $y = Ax^n + p$; and then transform the proposed Equation of y, by substituting $Ax^n + p$ for y, whence arises a new Equation, *viz.* that of p: which you now suppose to come in place of y: then mark those Cells of the Parallelogram, which correspond to the Terms of this secondary Equation: and laying a Ruler to the lowest of these Cells which lye adjacent to the Side A B, make it to revolve towards the right Hand and upwards, until it touch one or more of the marked Cells besides the first, in the Manner already explained; the Terms of the Equation of p, which correspond to the Cells which the Ruler touches, are to be feigned equal to nothing, and an approximate Value of p thence deduced, which will give the second Term of the Quotient: which let be denoted by Bx^{n+m}: then you suppose $p = Bx^{n+m} + q$, and transforming the former Equation of p into a new one, by substituting $Bx^{n+m} + q$, in place of p, you must find an approximate Value of q, by Means of the Parallelogram, after the same Manner as you find an approximate Value of p already, *i. e.* by applying the Ruler to the lowest marked Cell adjacent to A B. And so you proceed to find the subsequent Terms of the Quotient as far as you please to produce it. By proceeding thus, the Dimensions of x which are in the marked Cells upon the left Hand, adjacent to A B, will continually increase, as will be evident to one that considers it, and therefore you will continually approach to the true Root, by diminishing constantly the Supplement, p, q, &c *. * Art. 75.

116. Schol. In this Method of continuing the Terms of the Quotient, although you might find out the first Term variously, by different Applications of the Ruler; yet you find all the subsequent Terms in the same uniform Manner, *viz.* by applying the Ruler always to the lowest marked Cell adjacent to the Line A B: the Supplement p, q, r, &c. being supposed to be very small; as well as x. I have only considered the Case of a Root which converges when x is very small; because all other Cases may be reduced to this: and this Method of finding the subsequent Terms of the Quote cannot be so properly applied to the Case of x being very great: upon the account of the increasing negative Indexes of x.

117. But whereas the Parallelogram contains no negative nor fracted Powers of x and y, therefore it is proper to observe that, when any such occur, either in the Equation proposed at first, or in any of the secondary or supplementary Equations arising in the Progress of the

Operation,

Operation, they ought to be removed, before you proceed, by such Methods as are practised in Algebra: thus, if we had the Equation $x^3 + 3x^2y^{-1} - 2x^{-1} - 16y^{-3} = 0$, you free it from the negative Dimensions of x and y by multiplying by the positive Dimensions, which correspond to the highest negative Dimension, viz. x and y^3, by which Means it becomes $x^4y^3 + 3x^3y^2 - 2y^3 - 16x = 0$. After the same Manner the Equation $x = \frac{aa}{y} - \frac{2a^3}{y^2} + \frac{3a^4}{y^3}$ must be converted into $xy^3 = a^2y^2 - 2a^3y + 3a^4$ or $xy^3 - a^2y^2 + 2a^3y - 3a^4 = 0$. Which change being made, either upon the primary, or any of the secondary Equations, when necessary, the Operation will proceed.

118. Likewise in the Case of fractional Indexes, remove them by making a proper Substitution: thus if the Equation $y^3 - xy^{\frac{1}{2}} + x^{\frac{4}{3}} = 0$, were proposed, put $y^{\frac{1}{2}} = v$ and $x^{\frac{1}{3}} = z$, and by Substitution, you shall have $v^6 - z^3v + z^4 = 0$, out of which having extracted the Root v, you replace $y^{\frac{1}{2}}$ for v and $x^{\frac{1}{3}}$ for z: By extracting the Root v you would find $v = z + z^3 + 6z^5 + 51z^7$ &c. that is $y^{\frac{1}{2}} = x^{\frac{1}{3}} + x + 6x^{\frac{5}{3}} + 51x^{\frac{7}{3}}$ &c. and by squaring both Sides, $y = x^{\frac{2}{3}} + 2x^{\frac{4}{3}} + 13x^2 + 114x^{\frac{8}{3}}$ &c.

119. Moreover when any complicate Fractions, or surd Quantities, happen to occur, either in the proposed Equation; or to arise afterwards, they ought to be removed by the Methods used in common Algebra. Thus if the Equation $y^3 + \frac{a^2}{a-x}y^2 - x^3 = 0$, was proposed, you must free it of the complicate Fraction, by multiplying by $a - x$, whence arises $ay^3 - xy^3 + a^2y^2 - ax^3 + x^4 = 0$, out of which you extract the Root y: or you might substitute $\frac{v}{a-x}$ for y, and thence would arise $\frac{v^3}{\overline{a-x}|^3} + \frac{a^2v^2}{\overline{a-x}|^3} - x^3 = 0$, or $v^3 + a^2v^2 - a^3x^3 + 3a^2x^4 - 3ax^5 + x^6 = 0$: from whence having found the Value of v, divide it by $a - x$, and the Quotient gives the Root y. Likewise the Equation $y^3 - \sqrt{a-x} \times y^2 + x^3 = 0$ being given, to extract the Root y, you may substitute z for $\sqrt{a-x}$, whence you will have $y^3 - zy^2 + a^3 - 3a^2z^2 + 3az^4 + z^6 = 0$, out of which having extracted the Root y, replace $\sqrt{a-x}$ for z. Thus also the Equation $y^3 - \sqrt{a-x} \times y^2 - \sqrt{a-x} \times x^3 = 0$ being given, put $y = \frac{v}{\sqrt{a-x}}$, and thence arises $v^3 - av^2 + xv^2 - a^2x^3 + 2ax^4 - x^5$

$x^5 = 0$, the Root of which last Equation being extracted, divide it by $\sqrt{a-x} = a^{\frac{1}{2}} - \frac{1}{2}a^{-\frac{1}{2}}x - \frac{1}{8}a^{-\frac{3}{2}}x^2$ &c. and so you shall have a Value of y. Understand also that in the other Example which precedes, if the Area of the Curve be required, you should find the Expression of that Area in Terms of z before you replace $\sqrt{a-x}$ for z.

The Reduction and Preparation of Equations, mentioned in this Article, is always necessary in all the Methods of extracting the Roots of affected Equations, by Series; but those mentioned in the two preceding Articles are not always necessary in the other Methods; but are so in that which proceeds by Means of the Parallelogram.

120. Another Method for discovering the subsequent Terms of the Root, is that of our Author's, which he delivers in this Treatise, and which was formerly explained *. But because there are some Things in it about which the Reader may still require some further Satisfaction; and which may be of use in Practice, I shall illustrate it a little further. Wherefore let $ay^m + bx^qy^r + cx^sy^t$ &c. $+ bx^v = 0$ be a general Equation with indefinite Exponents, the Root y being proposed to be extracted, when x is supposed to be very small; to which all the other Cases may be reduced *: in which there will be one Term, at least, which do not include x, represented here by ay^m, and another one at least, that includes not y, which is here represented by $+ bx^v$: let the Terms $cx^sy^t + bx^v$ be those from which, put equal to nothing, the first Term of the Root may be deduced; found by any of the preceding Methods; which therefore is $\overline{\frac{-b}{c}}\Big|^{\frac{1}{t}} x^{\frac{v-s}{t}}$: put $\frac{v-s}{t} = n$, then $y = \overline{\frac{-b}{c}}\Big|^{\frac{1}{t}} x^n + p$: let the Equation be transformed, as follows, by the Help of the Binomial Theorem; omitting the numeral Coefficients.

* Sect. 4.

* Art. 86.

$$+ay^m = +a \times \overline{\frac{-b}{c}}\Big|^{\frac{m}{t}} x^{mn} + a \times \overline{\frac{-b}{c}}\Big|^{\frac{m}{t}-1} x^{mn-n}p + a \times \overline{\frac{-b}{c}}\Big|^{\frac{m}{t}-2} x^{mn-2n}p^2 \text{ \&c.}$$

$$+bx^qy^r = +b \times \overline{\frac{-b}{c}}\Big|^{\frac{r}{t}} x^{q+rn} + b \times \overline{\frac{-b}{c}}\Big|^{\frac{r}{t}-1} x^{q+rn-n}p + b \times \overline{\frac{-b}{c}}\Big|^{\frac{r}{t}-2} x^{q+rn-2n}p^2 \text{ \&c.}$$

$$+cx^sy^t = +c \times \overline{\frac{-b}{c}}\Big|^{\frac{t}{t}=1} x^{s+tn=v} + c \times \overline{\frac{-b}{c}}\Big|^{0} x^{v-n}p + c \times \overline{\frac{-b}{c}}\Big|^{-1} x^{v-2n}p^2 \text{ \&c.}$$

&c. &c.

$$+bx^v = +b \times x^v$$

By

By confidering of which the following Things will appear:

1°. That in the transformed Equation, *viz.* the Equation of p, thoſe Parts of the ultimate Term [1] which are collateral to the Terms $bx^v + cx^ty^s$ in the given Equation, muſt deſtroy one another. *

* Art. 75. Rem. 1.

2°. Theſe ſame Parts of the ultimate Term contain a lower Dimenſion of x, than any of the other Parts of that ultimate Term do: becauſe they are the Quantities which ariſe by inſerting the Dimenſion of x, which the Value of y contains, deduced from the fictitious Equation $bx^v + cx^ty^s = 0$, inſtead of y, in the ſeveral Terms of the propoſed Equation *.

* Art. 88.

3°. The Dimenſions of x in the ſeveral Quantities lying in the ſame horizontal Line of the Equation, conſtantly diminiſh by the ſame Difference; and thoſe of p increaſe by the ſame Difference; ſo that, abſtracting from the Coefficients, theſe Quantities go on in a continued Proportion, by the common Multiplier $x^{-n}p$: which is alſo the ſame in all the Lines. This appears from the Nature of Involution of Powers.

4°. From the laſt two Obſervations it follows, that the Part of the Penult Term, *viz.* $c \times \overline{\frac{-b}{c}}\Big|^{0} \times x^{v-n}p = cx^{v-n}p$, contains a lower Dimenſion of x, than any of the other Parts of that Penult Term: that the Part $c \times \overline{\frac{-b}{c}}\Big|^{-1} \times x^{v-2n}p^2$ of the antepenult Term upon that ſame Line contains a lower Dimenſion of x than any other Part or Quantity belonging to the Antepenult Term, &c. And moreover that, ſince the approximate Value of p; or the firſt Term of that Value, muſt contain a higher Dimenſion of x than x^n, which is it's Dimenſion in the firſt Term of the Quotient or Root, expreſſing the Value of y *, therefore, if the Dimenſion of x contained in that approximate Value of p, were ſubſtituted for p through all the Quantities, belonging to the Equation of p, the Dimenſions of x will increaſe, and that by the ſame common Difference in all the horizontal Lines, reckoning the Terms from the left Hand to the right: conſequently

* Art. 73.

5°. If $+ b \times \overline{\frac{-b}{c}}\Big|^{\frac{r}{s}} x^{q+m}$ be that Part of the Ultimate Term, which, after throwing by thoſe Parts which deſtroy each other, contains

[1] See Note 1. Art. 75.

the

the lowest Dimension of x, the approximate Value of the Supplement p must be found from the fictitious Equation $cx^{v-n}p + b \times \overline{\frac{-b}{c}}|^{\frac{r}{s}} x^{q+rn}$: and the same is to be understood of all the other secondary or supplementary Equations. Therefore you have this general Rule for finding all the subsequent Terms of the Value of y after the first, *viz. Let that Part of the ultimate Term of every supplementary Equation, which contains the lowest Dimension of* x, *having it's Sign changed, be divided by the Coefficient of* p *in that Part of the Penult Term, where* x *is of the lowest Dimension, and that shall be the next Term of the Quote to be annexed to what hath been formerly found.* The Observation of which, together with a Rule formerly given *, will mightily facilitate the Extraction of the Roots of such specious Equations. * Art. 82.

6°. The Coefficient of p, q, r, &c. in the Part of the Penult Term of any supplementary Equation, by which you divide, as in the last Remark, contains the same invariable Dimension of x throughout all these Equations: as will be evident if you consider, that, in the next supplementary Equation, *viz.* that of q, it will be found in the Transformation of the Equation of p, opposite to, or upon the same Line with, the Quantity $cx^{v-n}p$: so that if that Coefficient contain the Dimension x^{v-n} in the first supplementary Equation, it will contain the same in all the subsequent Ones.

7°. The Difference betwixt the Exponents of x in the two first Terms of the Series expressing the Root, is the same with the Difference of the Exponents of x, in the Parts of the ultimate Term of the Equation of p, in which x is of the lowest Dimension, and of the next higher, which are here bx^{v} and $b \times \overline{\frac{-b}{c}}|^{\frac{r}{s}} x^{q+rn}$: For the difference of the Dimensions of x in these Parts is $q + rn - v$: and the Value of the first Term of p, deduced from the fictitious Equation $+ c \times x^{v-n}p + b \times \overline{\frac{-b}{c}}|^{\frac{r}{s}} x^{q+rn} = 0$, contains this Power of x, viz. $x^{q+rn-v+n}$, whose Exponent exceeds the Exponent of x^{n}, which is the Power of x contained in the first Term of the Series, by $q + rn - v$, the same as the former: and the Reasoning is the same, although the Terms of the proposed Equation from which, the first Term of Series is derived, be any other than $cx^{q}y^{r} + bx^{v}$.

121. Although we have hitherto spoke of the converging Series, or Quotient, expressing the Value of y, as if the Exponents of the Powers of x in the Terms of the Series proceeded in an Arithmetical Progression; yet the Reader must know that, albeit this most frequently happens, yet it is not always so: for which Reason I shall resolve the following

PROBLEM,

To find whether the Exponents of the Powers of x *in the Terms of the converging Series go on in an Arithmetical Progression, or not: and if so, what the common Difference of that Progression is.*

122. Having found the Dimension or Power of x contained in the first Term of the Series, (*viz.* when x is very small) by any of the preceding Methods, which let be x^n, substitute x^n instead of y through all the Terms of the proposed Equation; and let the Exponents of the Powers of x after Substitution be A, A, B, C, E, *&c.* of which A is the least (and therefore found at least in two Terms) and the others B, C, E, *&c.* of which B is the least; C the next; E the next, *&c.* in an ascending Scale: then if B — A measure C — B, E — C, *&c.* throughout all the rest, the Exponents of x in the Terms of the Series proceed in an Arithmetical Progression, whose common Difference is B — A: but if B — A do not exactly measure all the Differences C — B, E — C, *&c.* the Dimensions of x in the Terms of the converging Series, will not constitute an Arithmetical Progression. Which will be plain by Examples.

Ex. 1. Take the Equation $y^3 + a^2y - 2a^3 + axy - x^3 = 0$, (which is the first of our Author's) by supposing $y^3 + a^2y - 2a^3 = 0$ *, we have a or ax^0 for the first Term of the Root; whose Dimension of x, viz. $x^0 = 1$, being put for y, in the Terms of the proposed Equation, the Exponents of x in the several Terms shall be 0, 0, 0, 1, 3, where the Difference betwixt 0 and 1, *viz.* 1. measures the Difference betwixt 1 and 3, *viz.* 2, therefore I conclude that the Exponents of the Powers of x in the Terms of the Quotient or Series, proceed in an Arithmetical Progression, whose Difference is 1.

* Art. 100.

Ex.

Ex. 2. Let the Equation $x^2y^4 + c^2y^4 - x^4y^2 + c^4x^2 - c^6 = 0$ be proposed: by feigning $c^2y^4 - c^6 = 0$ *, you find $+ c$ for the first Term of a converging Series; wherefore substitue x^0 for y, you have these Exponents of the Powers of x, viz. 2, 0, 4, 2, 0; which reduced to Order, and the coincident ones thrown into one, gives this Progression of distinct Terms 0, 2, 4, where $2 - 0\ (= 2)$ measures $4 - 2\ (= 2)$ therefore the Exponents of the Powers of x in the Terms of the Root, go on in an Arithmetical Progression, having 2 for the common Difference. *Art. 105.

Ex. 3. Let the Equation $y^5 - by^2 + 9bx^2 - x^3 = 0$ be proposed: then you may form the first Term of the Root either from the fictitious Equation $y^5 - by^2 = 0$; or $- by^2 + 9bx^2 = 0$ *: the first gives $y = \sqrt[3]{b}$ or $\sqrt[3]{b} \times x^0$; wherefore x^0 being inserted every where for y, the different Exponents of x in the several Terms, will be 0. 2. 3: here 2 being the Difference betwixt the least Exponents, measures not the Difference $3 - 2 = 1$, therefore the Exponents of x in the Terms of the Series will not proceed in an Arithmetical Progression. *Art. 100. 105.

But if the first Term of the Root be found from the other fictitious Equation $- by^2 + 9bx^2 = 0$; it is $+ 3x$ or $- 3x$: where the Dimension of x, viz. x^1, being inserted for y gives these Exponents of x in the Terms of the proposed Equation, *viz.* 5. 2. 3; which shews that $3 - 2 = 1$, measuring $5 - 3 = 2$, the Exponents of x in the Terms of this Root, will constitute an Arithmetical Progression, whose common Difference is 1.

Ex. 4. If $y^4 + x^3y - ax^3 - a^3x = 0$ be proposed, the first Term of the Root may be derived from putting $y^4 - a^3x = 0$ *, which will be $a^{\frac{3}{4}}x^{\frac{1}{4}}$; insert $x^{\frac{1}{4}}$ for y, and the Dimensions of x are 1. $\frac{13}{4}$. 3. where the Difference betwixt the two least 1 and 3, doth not measure the Difference betwixt 3 and $\frac{13}{4} = 3\frac{1}{4}$, therefore the Exponents of x in the Terms of the Quotient will not proceed in an Arithmetical Progression: for the Root is $a^{\frac{3}{4}}x^{\frac{1}{4}} + \frac{x^{\frac{9}{4}}}{4a^{\frac{5}{4}}} - \frac{x^{\frac{10}{4}}}{4a^{\frac{6}{4}}} - \frac{3x^{\frac{17}{4}}}{32a^{\frac{13}{4}}}$ &c. The Process of which, according to this Method, I shall here set down for the Reader's Satisfaction. Putting then $y = a^{\frac{3}{4}}x^{\frac{1}{4}} + p$, we have as follows: *Art. 105.

$$+y^4 = +\cancel{a^3x} + 4a^{\frac{9}{4}}x^{\frac{3}{4}}p + 6a^{\frac{3}{2}}x^{\frac{1}{2}}p^2 + \overset{*}{4a^{\frac{3}{4}}x^{\frac{1}{4}}p^3} + \overset{*}{p^4}$$

$$+x^3y = +a^{\frac{3}{4}}x^{\frac{13}{4}} + \overset{*}{x^3p}$$

$$-ax^3 = -ax^3$$

$$-a^3x = -\cancel{a^3x}$$

$4a^{\frac{9}{4}}x^{\frac{3}{4}}p - ax^3 = 0$ * *fere*

$p = +\dfrac{x^{\frac{9}{4}}}{4a^{\frac{5}{4}}} + q$ *Art. 120. Rem. 5.

$$+6a^{\frac{3}{2}}x^{\frac{1}{2}}p^2 = +\frac{3x^5}{8a} + 3a^{\frac{1}{4}}x^{\frac{11}{4}}q \quad *$$

$$+4a^{\frac{9}{4}}x^{\frac{3}{4}}p = +\cancel{ax^3} + 4a^{\frac{9}{4}}x^{\frac{3}{4}}q$$

$$-ax^3 = -\cancel{ax^3}$$

$$+a^{\frac{3}{4}}x^{\frac{13}{4}} = +a^{\frac{3}{4}}x^{\frac{13}{4}}$$

$$4a^{\frac{9}{4}}x^{\frac{3}{4}} + 3a^{\frac{1}{4}}x^{\frac{11}{4}}\Big) - a^{\frac{3}{4}}x^{\frac{13}{4}} - \frac{3x^5}{8a}\Big(-\frac{x^{\frac{10}{4}}}{4a^{\frac{6}{4}}} - \frac{3x^{\frac{17}{4}}}{32a^{\frac{13}{4}}} \text{ \&c.} = r.$$

$$y = a^{\frac{3}{4}}x^{\frac{1}{4}} + \frac{x^{\frac{9}{4}}}{4a^{\frac{5}{4}}} - \frac{x^{\frac{10}{4}}}{4a^{\frac{6}{4}}} - \frac{3x^{\frac{17}{4}}}{32a^{\frac{13}{4}}} \text{ \&c.}$$

In this Procefs the Reader will obferve, that I have omitted all thofe Quantities which cannot affect the Quotient within the Limit of that Power of x which is denoted by $x^{\frac{17}{4}}$ inclufive, which Quantities may be eafily known and diftinguifhed by what was formerly fhewn *, efpecially when joined with the Remarks in Art. 120.

* Art. 82.

123. Having illuftrated this Problem by Examples, I come now to give the Demonftration of it. Wherefore let $ay^m + bx^qy^r + cx^ty^s$ &c. $+ bx^v = 0$ reprefent any fpecious Equation as formerly: from which the firft Term of the Series being found by any of the preceding Methods, let the Equation be transformed into the firft fupplemental Equation, called that of p; in which Equation of p, let the Exponents of the Powers of x in the ultimate Term be A, A, B, C, E, &c. as in the Margin: of which two at leaft muft be equal, which are here reprefented by A and A: the Parts of the ultimate Term correfponding to which, do always deftroy one another *. Then fuppofe

Ult.	Penult.	Antepen.	&c.
~~A~~	*	*	*
~~A~~	A + D	A + 2D	A + 3D &c.
B	B + D	B + 2D	B + 3D &c.
C	C + D	C + 2D	C + 3D &c.
E	E + D	E + 2D	E + 3D &c.
&c.			

* Art. 75. Rem. 1.

pose the Exponent of the Power of x, which is contained in the first Term of the Value of p, found as above, to be D: so that after Substitution of such Power of x instead of p in the several Terms of the transformed Equation, *viz.* that of p, the Exponents of the Powers of x in the Penult Term may be $A+D$, $B+D$, $E+D$ *&c.* in the Antepenult $A+2D$, $B+2D$, *&c.* as in the Margin: for where there are Parts of these Terms of the Equation of p upon any horizontal Line, the Exponents of x in them, must be such as here represented, as appears by Remark 3 and 4, Art. 120. Wherefore it is evident, that, when the Equation of p is transformed into the second supplementary Equation, *viz.* that of q, the ultimate Term of this new Equation can contain no other Exponents of x but $A+D$, $A+2D$, $A+3D$, *&c.* B, $B+D$, $B+2D$ *&c.* C, $C+D$, *&c.* E, $E+D$ *&c.* But in the Equation of p, after throwing by A, the Exponent B is supposed to be the least of all the remaining ones B, C, E, *&c.* in the ultimate Term, therefore $A+D=B$ *; consequently $A+2D$ $=B+D$, $A+3D=B+2D$ *&c.* so that in the Equation of q, there shall be no other Exponents of x in the ultimate Term but what may be designed thus, B, $A+D$, $A+2D$, $A+3D$, *&c.* C, $C+D$, $C+2D$, *&c.* E, $E+D$, $E+2D$, *&c.* as in the Margin; of which B and $A+D$ being equal, the Parts of the ultimate Term corresponding to these must destroy one another, as before. Now after throwing by these, let $A+2D$ be the lowest Quantity in the ultimate Term, then it is evident that opposite to $A+D$, in this Equation of q, there must be a Part or Quantity belonging to the penult Term; but no Part belonging to any Term preceding the penult, because it is the Part of the Equation of q, which arises by the Transformation of the Part of the penult Term of the Equation of p, which corresponds to $A+D$, in which the Supplement p is in the first Power. And if you suppose that Power of x to be inserted for q, which is contained in the approximate Value of q found as above, the Exponent of x in this Part of the penult Term, must

* Art. 120. Rem. 4, 5.

Ult.	Penult.	Antepen.
~~B~~		
~~A+D~~	A + 2D	
A + 2D	A + 3D	A + 4D
A + 3D	A + 4D	A + 5D &c.
&c.	&c.	&c.
C		
C + D	C + 2D	
C + 2D	C + 3D	C + 4D
&c.	&c.	&c.
E		
E + D	E + 2D	
E + 2D	E + 3D	E + 4D
&c.	&c.	&c.

must be A + 2D, *viz.* the same as it is in that Part of the ultimate Term where it is lowest *: consequently the Exponents of x in the other Parts of the Penult Term, where there are such Parts, must exceed the Exponents of x in the corresponding Parts of the ultimate Term, by the same Difference D, after Substitution for q: and the Exponents of x in the Antepenult Term, must exceed those in the Penult by the same Difference still, *viz.* D; where there are such Parts belonging to the Antepenult Term: and so on for the rest of the Terms *, as you see represented in the Margin. And thus, by supposing the Equation of q to be transformed into the new supplemental Equation, *viz.* that of r; and so the Transformation to go on continually, it appears that the ultimate Term of every such Equation *in inf.* can contain no other Exponents of x but what fall in with some one or other of the Arithmetical Progressions A, A + D, A + 2D, A + 3D, &c. C, C + D, C + 2D, &c. E, E + D, &c. and that therefore if the first Progression A, A + D = B, &c. fall upon C, E, &c. the lowest Exponent of x in the ultimate Term of any supplemental Equation, will exceed the lowest Exponent of x in the ultimate Term of the immediately preceding supplemental Equation, by the constant Difference D = B — A: therefore the Exponents of the Powers of x in the converging Series for the Root y of the proposed Equation, ought likewise to proceed in an Arithmetical Progression (by Rem. 6. above) and that having the same Difference D or B — A (by Rem. 7.)

* Ibid.

* Art. 120. Rem. 3, 4.

But if the Arithmetical Progression A, A + D = B, A + 2 D, &c. do not fall upon the Terms C, E, &c. contained in the ultimate Term of the Equation of p, it appears from what has been said, that the Exponents of x, in the Terms of the Series which expresses the Root y, will not proceed in an Arithmetical Progression; but the Difference of the Exponents, after certain Intervals, become less.

And therefore, since the ultimate Term of the Equation of p, is the very same with what arises by substituting the first Term of the converging Series in place of y, throughout the Terms of the proposed Equation, the Thing we proposed is demonstrated.

123. Schol. It makes no Difference in the Demonstration, from whatever Terms selected out of the proposed Equation, the first Term of the Series be deduced, provided they be proper Terms for this Purpose: although the preceding Demonstration represents the first Term as deduced by the third Method preceding *.

* Art. 105.

124. Moreover it may be proper to advertise the Reader, that, although the Difference D = B — A do measure the other Differences

C — B,

C — B, E — C, *&c.* yet it may happen, that the known Coefficient of one or more Terms of the Series for the Root of the proposed Equation, may become equal to nothing: which may happen from certain Parts of the ultimate Term of any secondary or supplemental Equation destroying one another, sooner than they ought to do: which may make a Chasm in the Arithmetical Progression of the Exponents of x in the Series for the Root of the proposed Equation, when that happens; but the Exponents of the Powers of x through all the Terms of the Series, even in such a Case, must belong to the Arithmetical Progression mentioned in this Problem. And the Problem must therefore be understood to be taken with this Limitation.

125. And it may likewise happen that all the Parts, or Quantities, constituting the ultimate Term of any supplemental Equation, may totally destroy one another: in which Case, the Series for the Root of the proposed Equation terminates; and that Root is fully expressed by the Terms of the Quotient already found: since the remaining Supplement vanishes *.

* Art. 75. Rem. 3.

126. Besides the two Methods I have already mentioned, and explained, for extracting the Roots of affected specious Equations, containing two variable and unknown Quantities, by Means of infinite Series; there is a third, which I shall explain a little, lest the Reader should find Fault with the omitting of it entirely. It consists in the Assumption of a Series of Terms with indeterminate Coefficients, in which the Exponents of the Powers of x proceed in an Arithmetical Progression, either ascending or descending, according as x is supposed to be very small, or very great, in Comparison of the other known Quantities in the given Equation. Such a Series may be represented thus $Ax^n + Bx^{n+p} + Cx^{n+2p} + Dx^{n+3p}$, &c. which Series is supposed to express the Root of the proposed Equation; and therefore to be equal to y. And in order to determine the Values of the Coefficients A, B, C, D, *&c.* the proposed Equation is transformed into a new Equation, by substituting $Ax^n + Bx^{n+p} + Cx^{n+2p}$, &c. instead of y; and the Powers of it instead of the like Powers of y, in the Terms of the proposed Equation; whence there arises a new Equation, including only one variable Quantity, viz. x: of which Equation all the Terms together being equal to nothing; each Term by itself must be equal to nothing; that is the complex Coefficient of each Term must be equal to nothing: so that by putting the complex Coefficient of each Term equal to nothing, there will arise so many Equations, by the Resolution of which, the Values of the indeterminate Coefficients A, B, C, *&c.* may be found: and consequently the Root of the proposed

proposed Equation; provided the assumed Series be taken of a right *Form*.

127. To illustrate the Thing by an Example, let the Equation $y^3 + a^2y + axy - 2a^3 - x^3 = 0$ be proposed; to find y: then assume $y = A + Bx + Cx^2 + Dx^3$, &c. and let the given Equation be transformed as you see

$$\left.\begin{array}{llllll}
+y^3 = & +A^3 & +3A^2Bx & +3A^2Cx^2 & +3A^2Dx^3 & +3A^2Ex^4 \text{ \&c.} \\
 & & & +3AB^2 & +6ABC & +6ABD \\
 & & & & +BBB & +3ACC \\
 & & & & & +3BBC \\
+a^2y = & +a^2A & +a^2Bx & +a^2Cx^2 & +a^2Dx^3 & +a^2Ex^4 \text{ \&c.} \\
+axy = & * & +aAx & +aBx^2 & +aCx^3 & +aDx^4 \text{ \&c.} \\
-2a^3 = & -2a^3 & & & & \\
-x^3 = & * & * & * & -x^3 &
\end{array}\right\} = 0.$$

In which transformed Equation, the complex Coefficient of each Term must be equal to nothing, otherwise the Whole could not be equal to nothing, as we shall see by and by: therefore, by putting these Coefficients equal to nothing, there arises so many Equations, *viz.* 1°. $A^3 + a^2A - 2a^3 = 0$. 2°. $3A^2B + a^2B + aA = 0$. 3°. $3A^2C + a^2C + 3AB^2 + aB = 0$. 4°. $3A^2D + a^2D + 6ABC + B^3 + aC - 1 = 0$, &c. From the first of which find the Value of A: which Value of A being inserted for it in all the rest of the Equations; you may, from the second, find the Value of B: which Value of B being inserted for it in all the subsequent Equations, you may, from the third, find the Value of C; and so on *in inf.* From the first Equation $A^3 + a^2A - 2a^3 = 0$, you find $A = +a$: from the second you will find $B = -\frac{1}{4}$: from the third, $C = +\frac{1}{64a}$: from the fourth, $D = +\frac{131}{512a^2}$, &c. therefore, by substituting these Values of A, B, C, D, &c. instead of them in the assumed Series $y = A + Bx + Cx^2 + Dx^3$, &c. We have $y = a - \frac{x}{4} + \frac{x^2}{64a} + \frac{131x^3}{512a^2}$, &c. the same as formerly *.

* Art. 78.

128. The chief Difficulty in this Method is to find the right Form of a Series: for although the Series $A + Bx + Cx^2 + Dx^3$, &c. was that which we assumed in the preceding Example; and which indeed is the right Form for finding the Root of the Equation there proposed; yet in other Examples, the Series to be assumed, may be of

a different Form. And it's evident, that the Difference of the Forms depends upon theſe two Things, *viz.* 1°. The Dimenſion or Power of x in the firſt Term of the Series. 2°. The Difference betwixt the Exponents of the Powers of x in any two Terms of the Series that are next to another. The firſt of theſe Things may be found by any of the Methods that were formerly explained for diſcovering the firſt Term of a converging Series, which ſhall be the Root of the propoſed Equation.

129. The ſecond Thing, *viz.* the common Difference of the Exponents, may be found thus: inſert the Power or Dimenſion of x contained in the firſt Term of the Series, in place of y, in the ſeveral Terms of the propoſed Equation; then ſet down all the different Exponents of x (among which o is included) in order, which let be a, b, c, e, &c. find the greateſt Number, which will exactly meaſure all the Differences $b - a$, $c - b$, $e - c$, &c. call it p; then is p the Difference of the Indexes of x in the Terms of the converging Series: ſo that if n be the Exponent of x in the firſt Term of the Series, the right Form of the Series to be aſſumed ſhall be $Ax^n + Bx^{n+p} + Cx^{n+2p} + Dx^{n+3p}$, &c. The Thing will appear plainer by Examples.

Ex. 1. Let the Equation $y^3 + a^2y + axy - 2a^3 - x^3 = 0$, mentioned juſt now, be propoſed: then by feigning $y^3 + a^2y - 2a^3 = 0$, we have $y = +a$ or $+ax^0$; which is the firſt Term of the Root, when x is very ſmall *: next inſert x^0 inſtead of y in the Terms of *Art. 100.
the propoſed Equation, and the Indexes of the Powers of x are 0, 0, 1, 0, 3; ſo that the different ones, reduced into order, are 0, 1, 3, whoſe Differences 1, 2, have 1 for their greateſt common Meaſure: therefore $A + Bx + Cx^2 + Dx^3$, *&c.* is the proper Form of the Series to be aſſumed.

Ex. 2. Let the Equation $x^4y^7 - 4y^5 + 2axy^3 - 26x^2y^3 + 3x^3y^2 - abx^2y + x^4 + 2ax^5 = 0$, be given: by putting the Terms $-abx^2y + x^4 = 0$, you will have $+\frac{x^2}{ab}$ for the firſt Term of a converging Series, when x is very ſmall *: wherefore, inſert x^2 for y, through *Art. 105. 108.
the ſeveral Terms of the propoſed Equation; and the Exponents of x will be theſe, 18, 10, 7, 8, 7, 4, 4, 5, that is the different ones, reduced to order, are 4, 5, 7, 8, 10, 18, whereof the Differences are, 1, 2, 1, 2, 8, whoſe greateſt common Meaſure is 1: therefore the Form of the Series is $Ax^2 + Bx^3 + Cx^4$, *&c.*

But if the propoſed Equation remaining the ſame, you deduce the firſt Term of the Series from the fictitious Equation $-4y^5 + 2axy^3$

 $-abx^2y$

$-abx^2y=0$, you'll find it to be $+\sqrt{\frac{1}{4}a\pm\sqrt{-\frac{ab}{4}+\frac{aa}{16}}}\times x^{\frac{1}{2}}$ or

*Art.108. $-\sqrt{\frac{1}{4}a\pm\sqrt{-\frac{ab}{4}+\frac{aa}{16}}}\times x^{\frac{1}{2}}$ *, which gives the first Term of the Series, fourfold; but $n=\frac{1}{2}$ in all of them: and by inserting $x^{\frac{1}{2}}$ for y, the Exponents of the Powers of x will be $7\frac{1}{2}$. $2\frac{1}{2}$. $2\frac{1}{2}$. $3\frac{1}{2}$. 4. $2\frac{1}{2}$. 4. 5; that is the different ones when reduced to Order, are $2\frac{1}{2}$. $3\frac{1}{2}$. 4. 5. $7\frac{1}{2}$; whose Differences are 1. $\frac{1}{2}$. 1. $2\frac{1}{2}$ or $\frac{1}{2}$. 1. $2\frac{1}{2}$, whose greatest common Measure is $\frac{1}{2}$: Therefore the proper Form of a Series for any of these Roots will be $Ax^{\frac{1}{2}}+Bx+Cx^{\frac{3}{2}}+Dx^2$, &c.

Ex. 3. Let the Equation $y^3+5ay^2-34a^2y+x^2y-80a^3=0$ be proposed: by assuming the fictitious Equation $y^3+5ay^2-34a^2y-80a^3=0$, and resolving it, you'll find it to have these three Roots $+5a$, $-2a$, $-8a$; any of which will be the first Term of a con-

*Art.100. verging Series, when x is very small *: therefore insert x^0 for y, and the Indexes of the Powers of x in the proposed Equation, which are different, are none other but 0 and 2, whose Difference 2 has 2 for it's greatest measure: whence it follows that $A+Bx^2+Cx^4+Dx^6$ &c. is the true Form of the Series. Which Example, as to the Method of Operation, is here represented to the Reader.

$$\left.\begin{array}{llllll}
+y^3 & =+A^3 & +3A^2Bx^2 & +3A^2Cx^4 & +3A^2Dx^6 & +3A^2Ex^8 \text{ \&c.}\\
 & & & +3AB^2 & +6ABC & +6ABD\\
 & & & & +BBB & +3ACC\\
 & & & & & +3B^2C\\
+5ay^2 & =+5aA^2 & +10aABx^2 & +10aACx^4 & +10aADx^6 & +10aAEx^8 \text{ \&c.}\\
 & & & +5aBB & +10aBC & +10aBD\\
 & & & & & +5aCC\\
-34a^2y & =-34a^2A & -34a^2Bx^2 & -34a^2Cx^4 & -34a^2Dx^6 & -34a^2Ex^8 \text{ \&c.}\\
+x^2y & = \;\; * & + Ax^2 & + Bx^4 & + Cx^6 & + Dx^8 \text{ \&c.}\\
-80a^3 & =-80a^3 & & & &
\end{array}\right\}=0$$

Where, by putting the complex Coefficients of each Term of the transformed Equation, equal to nothing, you'll have 1°. $A^3+5aA^2-34a^2A-80a^3=0$. 2°. $3A^2B+10aAB-34a^2B+A=0$. 3°. $3A^2C+3AB^2+10aAC+5aB^2-34a^2C+B=0$: and so on, for the Coefficients of the subsequent Terms. The first of these Equations, *viz.* $A^3+5aA^2-34a^2A-80a^3=0$, is the same with

with the fictitious Equation at first assumed for finding the first Term of the Series, *viz.* $y^3 + 5ay^2 - 34a^2y - 80a^3$ (which will always be the Case) therefore A will have the same Values here, as y had there; that is $A = +5a$ or $-2a$ or $-8a$: any of which may be taken, according to the Root that is proposed to be extracted. Let the positive Value $+5a$ be taken: which therefore must be substituted in Place of A, in the subsequent Equations for discovering the Values of B, C, D, &c. by substituting it for A in the second Equation, *viz.* $3A^2B + 10aAB - 34a^2B + A = 0$, it becomes $75a^2B + 50a^2B - 34a^2B + 5a = 0$, whence $B = -\frac{5}{91a}$. Again, by substituting these Values of A and B, in Place of them, in the third Equation for determining C, *viz.* $3A^2C + 3AB^2 + 10aAC + 5aB^2 - 34a^2C + B = 0$, it becomes $75a^2C + \frac{375}{8281a} + 50a^2C + \frac{125}{8281a} - 34a^2C - \frac{5}{91a} = 0$; whence you have $C = -\frac{45}{753571a^3}$. After the same Manner, by substituting these known Values of A, B and C, in the fourth Equation, you may discover the Value of D, *viz.* $D = +\frac{6470}{\overline{91a}|^5}$: and by Means of the fifth Equation, in like Manner, you would find $E = +\frac{971855}{\overline{91a}|^7}$: and so on *in inf.* Therefore, by inserting these Values of A, B, C, &c. in the Equation $y = A + Bx^2 + Cx^4 + Dx^6 + Ex^8$ &c. $= 0$, we obtain $y = 5a - \frac{5x^2}{91a} - \frac{45x^4}{\overline{91a}|^3} + \frac{6470x^6}{\overline{91a}|^5} + \frac{971855x^8}{\overline{91a}|^7}$ &c. which is an Approximation to one of the Roots of the proposed Equation, when x is very small.

130. The Demonstration of this Practice may be easily derived from what was demonstrated with respect to the preceding Method*: For *Art. 122. 123. when any specious Equation, including two unknown Quantities x and y, is proposed, to find the Value of y in a converging Series; of which Series Ax^n is the first Term; and Ax^n is inserted for y through the Terms of the given Equation; if the assumed Series $Ax^n + Bx^{n+p} + Cx^{n+r}$ &c. be of such a Form, as that the Dimension of x in *every one* of the Terms of the proposed Equation (after substituting Ax^n for y in them) fall in with the Dimension of x in some Term of the assumed Series: I say if this be the Case, the young Geometrician will easily understand, from what hath been shewn in the foresaid Place, that there can be no Term of the Root of the proposed Equation, but such, whose Dimension of x must fall in with the Dimension of x in

in ſome Term or other of the aſſumed Series; ſince p, which is the common Difference of the Exponents of x in the aſſumed Series, is a common Meaſure of the Differences of the Dimenſions of x in the Terms of the propoſed Equation, after Subſtitution of Ax^n for y in them: and ſince p is the greateſt common Meaſure of theſe Differences, there cannot be any other Series aſſumed, having the Exponents of x in an arithmetical Progreſſion, whoſe common Difference is greater, which ſhall have theſe Properties. Conſequently, if the Values of the indeterminate Coefficients A, B, C, D, &c. in the aſſumed Series can be found, which we have ſhewn how to do, by the Examples we have adduced, then the ſeveral Terms of the Root of the propoſed Equation, will thereby be found. Which is all that is required.

131. All that the young Geometrician can require to know further, for his Satisfaction with reſpect to the Grounds of this Method of extracting the Roots of Equations, by the *Aſſumption* of a Series with indeterminate Coefficients is, 1°. Why the Exponents of x in the Terms of the transformed Equation, muſt conſtitute an arithmetical Progreſſion having the ſame common Difference as the Exponents of x in the Terms of the aſſumed Series: which is abſolutely neceſſary for finding the Values of A, B, C, D, &c.? This Queſtion will be eaſily ſatisfied, by conſidering the Nature of the Involution of Powers: by which it appears, that, if there be any Quantity conſiſting of any Number of Terms, finite or infinite, in which the Exponents of the Powers of any Letter as x, are in arithmetical Progreſſion, any *perfect* Power of that Quantity, as the Square, Cube, &c. will conſiſt of Terms in which the Exponents of x will conſtitute an arithmetical Progreſſion having that ſame Difference: ſo that any Power of $Ax^n + Bx^{n+p} + Cx^{n+2p}$ &c. or even any Power of it multiplied by any Power of x, will conſtitute a Series of Terms, in which the Exponents of x will make an arithmetical Progreſſion, having the ſame Difference as in $Ax^n + Bx^{n+p}$ &c. viz. p. From which it follows, that the Exponents of the Powers of x, in the ſeveral horizontal Lines of the transformed Equation, as in the preceding Examples, will proceed by the ſame common Difference in all, where there are more Terms than one upon any ſuch Line: and where there is but one, the Dimenſion of x in it, muſt fall in with the Dimenſion of x in ſome one, or more Quantities, of the other horizontal Lines; ſince the Progreſſion of the Exponents of x, in the Series $Ax^n + Bx^{n+p} + Cx^{n+2p}$ &c. falls upon the Exponent of x in ſuch Terms by the Hyp. Which Conſiderations, *by themſelves*, may furniſh a Proof of this Method.

132. The

132. The other Thing he may want to know (which I likewise promised to shew) is this: how it appears that, in any Equation such as $\overline{a-b}\times x^n + \overline{c-d}\times x^p + \overline{e-f}\times x^q$ &c. $=0$, in which x is a variable or flowing Quantity, and a, b, c, d, e, f, &c. constant Quantities, if the Terms contain different Powers of x, the Coefficients of every one of the Terms, that is $a-b$, $c-d$, $e-f$, &c. must be equal to nothing? The Thing appears thus: divide all the Terms of the Equation, by the lowest Power of x, which let be x^n, and you'll have $\overline{a-b} + \overline{c-d}\times x^{p-n} + \overline{e-f}\times x^{q-n} = 0$: which being true whatever be the Value of x, it is true when x vanishes: wherefore, let x vanish, and then it will become $a-b=0$: therefore $\overline{c-d}\times x^{p-n} + \overline{e-f}\times x^{q-n}$ &c. $=0$: consequently, by the same way of reasoning, we prove that $c-d=0$: and so on for all the rest: as was to be shewn.

133. The Reader will easily perceive that sometimes the Values of some of the indeterminate Coefficients B, C, D, &c. may be equal to nothing: which will always happen when the Exponents of x in the Terms of the Series for the Root, do not constitute an arithmetical Progression: which we shew'd how to discover formerly. In which Cases, there may perhaps Methods be fallen upon to abridge the Operation, by the Assumption of Series's with different Properties, when the Root is proposed to be carried only to a certain Limit: but upon this, and some other useful Things relating to this Method, I will not now insist, lest I should be thought to be too particular. Only the Reader may observe, that this Method of resolving specious Equations, serves for extracting the Roots, as well when the Species x is supposed very great, as when it is supposed very small: and the same Rules will serve for determining the Form of the Series to be assumed; only the Exponents of x in the Numerators of it's Terms must constitute a descending arithmetical Progression: thus $Ax^n + Bx^{n-p} + Cx^{n-2p}$ &c. where n may be positive or negative, and p always positive. The Operation, in one Example, I shall here set down. Let the Equation $y^3 + a^2y + axy - 2a^3 - x^3 = 0$, be proposed, to find the Value of y in a converging Series, when x is very great: the same Equation, which hath been proposed formerly, particularly under this third Method. I take the fictitious Equation $y^3 - x^3 = 0$, from whence I find $+x$ for the first Term of the Root, and likewise of the Series to be assumed: and by substituting x^2 for y in the proposed Equation, I find 1 for the greatest common Measure of the Exponents of x: whence I know it must be $y = Ax + Bx^{-1} + Cx^{-2}$ &c. Wherefore

$+y^3$

$$\left.\begin{array}{llllllll}
+y^3 = & +x^3 & +3Bx^2 & +3Cx & +3Dx^0 & +3Ex^{-1} & \&c. \\
 & & & +3B^2 & +6BC & +6BD & \\
 & & & & +BBB & +3CC & \\
 & & & & & +3B^2C & \\
+a^2y = & * & * & +aax & +a^2Bx^0 & +a^2Cx^{-1} & \&c. \\
+axy = & * & +ax^2 & +aBx & +aCx^0 & +aDx^{-1} & \&c. \\
-2a^3 = & * & * & * & -2a^3x^0 & & \\
-x^3 = & -x^3 & & & & &
\end{array}\right\} = 0.$$

In which transformed Equation, 1 is inserted instead of A, because it was found already that $Ax = 1x$: then for deriving the Values of B, C, D, E *&c.* we have these Equations, 1°. $3B + a = 0$. 2°. $3C + 3B^2 + aa + aB = 0$. 3°. $3D + 6BC + B^3 + aaB + aC - 2a^3 = 0$. 4°. $3E + 6BD + 3C^2 + 3B^2C + a^2C + aD = 0$. *&c.* by resolving of which, you'll find 1°. $B = -\frac{a}{3}$. 2°. $C = -\frac{a^2}{3}$. 3°. $D = +\frac{55a^3}{81}$. 4°. $E = +\frac{64a^4}{243}$, *&c.* whence you have $y =$ ($Ax + Bx^{-1} + Cx^{-2} + Dx^{-3} + Ex^{-4}$ *&c.* $=$) $x - \frac{a}{3x} - \frac{a^2}{3x^2} + \frac{55a^3}{81x^3} + \frac{64a^4}{243x^4}$ *&c.* the Root sought.

134. It may likewise be observed, that, in the foregoing Methods of resolving Equations into infinite Series, the Reduction of complex Fractions and Radicals into infinite Series, is *virtually* included. Thus if it were required to reduce the Fraction $\frac{aa}{b+x}$ into an infinite Series of simple Terms, you suppose $y = \frac{aa}{b+x}$, or (by multiplying by $b+x$) $by + xy = a^2$: from which you extract the Value of y. After the same Manner if it were proposed to reduce $\sqrt{aa - xx}$ into an infinite Series, you might put $y = \sqrt{a^2 - x^2}$, that is $y^2 = a^2 - x^2$: and if $\overline{a - x}|^{\frac{3}{4}}$ were proposed, you may put $y = \overline{a - x}|^{\frac{3}{4}}$, whence $y^4 = \overline{a - x}|^3$ or $y^4 = a^3 - 3a^2x + 3ax^2 - x^3$: and so the Values of y being found by the Resolution of Equations into Series, you'll obtain the Values of the Radicals in such a Series as was proposed.

Art. 43—46. 135. The preceding Doctrine of the Extraction of the Roots of specious Equations, may be conveniently applied to that Operation which goes under the Name of the *Reversion of Series*: which is nothing else but this: the Value of one of the indefinite Quantities, *e.g.* y, being exprest by a Series of simple Terms, including different Powers of the

the other indefinite Quantity x, thence, by a kind of *reverse* Operation, the value of x is found, exprest in a Series of simple Terms, including different Powers of the Quantity y. By Means of this Operation, the Areas or Lengths of Curves being given in a converging Series, the Bases or Abscisses, and consequently the Ordinates, of the Curves may be found: as our Author shews in Art. 43—46: likewise from the Logarithm given, we return to the Ratio of which it is the Logarithm.

Let CD (see the Fig. in the Author Art. 43) be an equilateral Hyperbola, AB $= x$, a Part of one of the Assymptotes, so that the Distance of the Point A from the Centre be called 1; AC and BD being two Ordinates; then the Area ABDC is the hyperbolical Logarithm of the Ratio of AC to BD, that is of $1 + x$ to 1: which Area, as has been shewn formerly, is $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5$ &c. so that, if you put l for the known Logarithm of the Ratio of $1 + x$ to x, that is, if you put $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$ &c. $- l = 0$, then by the preceding Methods, you may extract the Root x, after the Manner Sir *Isaac* teaches in this Art. which is nothing else but the Application of his Method of resolving specious Equations, applied to this particular Case: by which you find $x = l + \frac{1}{2}l^2 + \frac{1}{6}l^3 + \frac{1}{24}l^4 + \frac{1}{120}l^5$ &c. so that the Ratio, of which l was the given Logarithm is that of $1 + l + \frac{1}{2}l^2 + \frac{1}{6}l^3 + \frac{1}{24}l^4$ &c. to 1, that is according to the common way of speaking, the Logarithm of the Number $1 + l + \frac{1}{2}l^2 + \frac{1}{6}l^3$ &c. But if the Logarithm that is given be the common Tabular Logarithm, you must first reduce it to the hyperbolical Logarithm by multiplying by 2.30258 &c. which Doctrine we explained at fuller Length in our Notes upon the Quadrature of Curves.

Sir *Isaac* having set before the Reader the Reversion of this Series, by the Method he lays down for extracting the Roots of Equations in this Treatise; I shall shew how it is done by the Assumption of a Series with indeterminate Coefficients. Since x is supposed to be very small, you'll find by Art. 128, 129, that the Form of the Series must be $x = Al + Bl^2 + Cl^3 + Dl^4$ &c. wherefore transform thus;

$$
\left.\begin{array}{lcllllll}
-\,l & = & -\,l & & & & & \\
+\,x & = & +Al & +Bl^2 & +Cl^3 & +Dl^4 & +El^5 & \&c. \\
-\frac{1}{2}x^2 & = & * & -\frac{1}{2}A^2l^2 & -ABl^3 & -ACl^4 & -ADl^5 & \&c. \\
 & & & & & -\frac{1}{2}B^2 & -BC & \\
+\frac{1}{3}x^3 & = & * & * & +\frac{1}{3}A^3l^3 & +A^2Bl^4 & +A^2Cl^6 & \&c. \\
 & & & & & & +AB^2 & \\
-\frac{1}{4}x^4 & = & * & * & * & -\frac{1}{4}A^4l^4 & -A^3Bl^5 & \&c. \\
+\frac{1}{5}x^5 & = & * & * & * & * & +\frac{1}{5}A^5l^5 & \&c.
\end{array}\right\} = 0.
$$

Whence

Whence you obtain so many Equations, for determining the Values of the Coefficients A, B, C, &c. *viz.* 1°. $A - 1 = 0$. 2°. $B - \frac{1}{2}A^2 = 0$. 3°. $C - AB + \frac{1}{3}A^3 = 0$, &c. by reducing of which you'll find $A = 1$, $B = \frac{1}{2}$, $C = \frac{1}{6}$, $D = \frac{1}{24}$, $E = \frac{1}{120}$ &c. and therefore $x = l + \frac{1}{2}l^2 + \frac{1}{6}l^3 + \frac{1}{24}l^4 + \frac{1}{120}l^5$ &c. *in inf.*

136. There is, moreover, another ingenious Method, which our Author makes use of for the Reversion of Series: which is this [1]: let the Equation $z = t - \frac{1}{3}\frac{t^3}{r^2} + \frac{1}{5}\frac{t^5}{r^4} - \frac{1}{7}\frac{t^7}{r^6} + \frac{1}{9}\frac{t^9}{r^8}$ &c. be proposed, which was given by Mr. *James Gregory* [2]: by which the Length of the Arch of a Circle is found, from the Radius r and Tangent t being given, the Arch being called z: and it is required, conversely, to find the Tangent t from the Arch z given, in a converging Series extending to the 9th Power of z. In order to this, let the given Series $t - \frac{1}{3}\frac{t^3}{r^2} + \frac{1}{5}\frac{t^5}{r^4}$ &c. be involved to the 1st, 3d, 5th, 7th and 9th Powers, as follow:

$$z = + t - \frac{1}{3}\frac{t^3}{r^2} + \frac{1}{5}\frac{t^5}{r^4} - \frac{1}{7}\frac{t^7}{r^6} + \frac{1}{9}\frac{t^9}{r^8} \text{ \&c.}$$
$$z^3 = \quad * + t^3 - \frac{t^5}{r^2} + \frac{42}{45}\frac{t^7}{r^4} - \frac{818}{945}\frac{t^9}{r^6} \text{ \&c.}$$
$$z^5 = \quad * \quad * + t^5 - \frac{5}{3}\frac{t^7}{r^2} + \frac{19}{9}\frac{t^9}{r^4} \text{ \&c.}$$
$$z^7 = \quad * \quad * \quad * + t^7 - \frac{7}{3}\frac{t^9}{r^2} \text{ \&c.}$$
$$z^9 = \quad * \quad * \quad * \quad * + t^9$$

Here I observe, that, if the second Line were multiplied by $\frac{1}{3r^2}$, and then added to the first, it would make the second Term of the Series, *viz.* $-\frac{1}{3}\frac{t^3}{r^2}$ to disappear: by doing which you will obtain this Equation $z + \frac{z^3}{3r^2} = t * - \frac{2}{15}\frac{t^5}{r^4} + \frac{152}{945}\frac{t^7}{r^6} - \frac{503}{2835}\frac{t^9}{r^8}$ &c. then I observe, with respect to this last Equation, that the next Term $-\frac{2}{15}\frac{t^5}{r^4}$ would be destroyed, by multiplying both Sides of the Equation that comes next

[1] See Com. Epist. N°. 64.

[2] See Com. Epist. N°. 20.

next in the Example, *viz.* $z^5 = t^5 - \frac{5}{3}\frac{t^7}{r^2} + \frac{5.10}{3.9}\frac{t^9}{r^4}$ &c. by $\frac{2}{15r^4}$ and then adding the two Equations together; whence arises $z + \frac{z^3}{3r^2} + \frac{2z^5}{15r^4} = t * * - \frac{1}{3}\frac{7}{15}\frac{t^7}{r^6} + \frac{5}{5}\frac{9}{67}\frac{t^9}{r^8}$ &c. Next I perceive that the Term $-\frac{1}{3}\frac{7}{15}\frac{t^7}{r^6}$ in this last Equation may be destroyed by multiplying the fourth Equation in the Example, *viz.* $z^7 = t^7 - \frac{7}{3}\frac{t^9}{r^2}$ by $\frac{17}{315r^6}$, and then adding as before: for hence will arise $z + \frac{z^3}{3r^2} + \frac{2z^5}{15r^4} + \frac{17z^7}{315r^6} = t * * * - \frac{62}{2835}\frac{t^9}{r^8}$: finally, by multiplying $z^9 = t^9$ &c. by $\frac{62}{2835r^8}$, and adding as before, you'll destroy the Term $-\frac{62}{2835}\frac{t^9}{r^8}$: for there will arise $z + \frac{z^3}{3r^2} + \frac{2z^5}{15r^4} + \frac{17z^7}{315r^6} + \frac{62z^9}{2835r^8}$ &c. $= t$, the Equation sought; whereby the Tangent is found from the Arch: which is the same with Mr. *Gregory*'s [1]. And after the same Manner may any other Series be reverted.

137. By this Method of extracting the Roots of affected Equations, we are enabled to resolve Equations though infinitely affected: thus in the Equation:

$$\left.\begin{array}{l} -8 + z^2 - 4z^4 + 9z^6 - 16z^8 \text{ \&c.} \\ + y \times \overline{z^2 - 2z^4 + 3z^6 - 4z^8 \text{ \&c.}} \\ - y^2 \times \overline{z^2 - z^4 + z^6 - z^8 \text{ \&c.}} \\ + y^3 \times \overline{z^2 - \frac{1}{2}z^4 + \frac{1}{3}z^6 - \frac{1}{4}z^8 \text{ \&c.}} \end{array}\right\} = 0.$$

Let the Value of y be required as far as the 4th Dimension of z: then because z is supposed to be small, the first Term of the Root may be derived from this fictitious Equation $-8 + y^3z^2 = 0$ *; which *Art. 105.
therefore must be $+2z^{-\frac{2}{3}}$: therefore, I consider to what Length I propose to have the Root y extracted, that is, to what Dimension of z; that so the unnecessary Terms may be thrown by: and it being proposed to carry the Quote only to the 4th Dimension of z, all those Terms may be neglected from the beginning, in which the Dimension

[1] See Com. Epist. N°. 20.

of z, after Substitution of $z^{-\frac{2}{3}}$ for y, exceeds $z^{4+\frac{2}{3}}$, because they cannot affect the Quote within the proposed Limit, as appears from what
* Art. 82. 120. was shewn formerly * [a]: so that the Equation to be resolved, comes to this $\frac{1}{3}z^6y^3 - \frac{1}{2}z^4y^3 + z^2y^3 - z^6y^2 + z^4y^2 - z^2y^2 - 2z^4y + z^2y - 4z^4 + z^2 - 8 = 0$: out of which the Root y may be extracted by the preceding Methods.

138. Hence also the Reader may see, how the Root y may be extracted out of this infinite Equation $ay + by^2 + cy^3 + dy^4$ &c. $= gx + hx^2 + ix^3 + kx^4$ &c. the Resolution of which will give the Theorem for this Purpose, published by that famous Analyst Mr. *Abraham de Moivre*, in the Phil. Transf. N°. 249: and likewise, how still more general Theorems may be investigated, for the Solution of Equations in which the indefinite Species x and y are infinitely affected.

139. One of the chief Difficulties in the Practice of resolving affected Equations, by the Assumption of an infinite Series with indeterminate Coefficients, is, the raising of such an infinite Series to any Power required: and therefore, to serve this, as well as other Purposes, I shall here set down at length the distinct Terms, until the 7th, inclusive, of any indefinite Power of the infinite universal Series $Ax^n + Bx^{n+p} + Cx^{n+2p} + Dx^{n+3p} + Ex^{n+4p} + Fx^{n+5p} + Gx^{n+6p}$ &c. having m for it's Exponent, by Help of our Author's famous Binomial Theorem: by which duly applied we have

$$\overline{Ax^n + Bx^{n+p} + Cx^{n+2p} + Dx^{n+3p} + Ex^{n+4p} + Fx^{n+5p} + Gx^{n+6p} \&c.}|^m =$$

$$A^m \times x^{nm} + \frac{m}{1} \times A^{m-1} \times B \times x^{nm+p}$$

$$\left.\begin{array}{r} + \frac{m}{1} \times A^{m-1} \times C \\ + \frac{m}{1} \times \frac{m-1}{2} \times A^{m-2} \times B^2 \end{array}\right\} \times x^{nm+2p}$$

[a] When the proposed Equation, *viz.* that of y, is transformed (by putting $y = +2z^{-\frac{2}{3}} + p$) into the supplementary Equation of p, that Part of the penult Term of the Equation of p, which stands opposite to the Term $+y^3z^2$ in the Equation of y, is $+12z^{\frac{2}{3}}p$; in which the Dimension of z, viz: $z^{\frac{2}{3}}$, is lower than in any other Part of the penult Term of the Equation of p; hence it appears, that, in the Prosecution of the Root of the proposed Equation, the several Parts of the ultimate Terms of the supplementary Equations, must always be divided by that Power of z, that is, the Dimension of z in them must be lessened by $\frac{2}{3}$: whence it follows, that, when the Root of the proposed Equation is to be carried no further than z^4, all those Terms of that Equation, may be neglected from the beginning, where the Dimension of z rises above $z^{4+\frac{2}{3}}$, after substituting $+2z^{-\frac{2}{3}}$ for y, in those Terms. See Art. 120.

+

$$\left.\begin{array}{r} +\frac{m}{1}\times A^{m-1}\times D \\ +\frac{m}{1}\times\frac{m-1}{2}\times A^{m-2}\times 2BC \\ +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times A^{m-3}\times B^{3} \end{array}\right\}\times x^{nm+3p}$$

$$\left.\begin{array}{r} +\frac{m}{1}\times A^{m-1}\times E \\ +\frac{m}{1}\times\frac{m-1}{2}\times A^{m-2}\times \left\{\begin{array}{l}2BD\\CC\end{array}\right. \\ +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times A^{m-3}\times 3B^{2}C \\ +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times A^{m-4}\times B^{4} \end{array}\right\}\times x^{nm+4p}$$

$$\left.\begin{array}{r} +\frac{m}{1}\times A^{m-1}\times F \\ +\frac{m}{1}\times\frac{m-1}{2}\times A^{m-2}\times \left\{\begin{array}{l}2BE\\2CD\end{array}\right. \\ +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times A^{m-3}\times \left\{\begin{array}{l}3B^{2}D\\3BC^{2}\end{array}\right. \\ +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times A^{m-4}\times 4B^{3}C \\ +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}\times A^{m-5}\times B^{5} \end{array}\right\}\times x^{nm+5p}$$

$$\left.\begin{array}{r} +\frac{m}{1}\times A^{m-1}\times G \\ +\frac{m}{1}\times\frac{m-1}{2}\times A^{m-2}\times \left\{\begin{array}{l}2BF\\2CE\\D^{2}\end{array}\right. \\ +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times A^{m-3}\times \left\{\begin{array}{l}3B^{2}E\\6BCD\\C^{3}\end{array}\right. \\ +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times A^{m-4}\times \left\{\begin{array}{l}4B^{3}D\\6B^{2}C^{2}\end{array}\right. \\ +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}\times A^{m-5}\times 5B^{4}C \\ +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}\times\frac{m-5}{6}\times A^{m-6}\times B^{6} \end{array}\right\}\times x^{nm+6p}$$

&c. *&c.* *&c.*

By Help of which any Power of the universal infinite Series $Ax^{n} + Bx^{n+p} + Cx^{n+2p}$ &c. may be raised the Length of seven Terms, whatever be the particular Values of the Numbers n, p, m, which belong to the Exponents: *e. g.* if it was required to raise the infinite Series $A + Bx^{2} + Cx^{4} + Dx^{6}$ &c. to the third Power, you'll have $n = 0$, $p = 2$, $m = 3$: by inserting of which Values into the general Theorem, it becomes

$$\begin{array}{llll} A^{3} + 3A^{2}Bx^{2} & + 3A^{2}Cx^{4} & + 3A^{2}Dx^{6} & \&c. \\ & + 3AB^{2} & + 6ABC & \\ & & + BBB & \end{array}$$

The same Theorem will serve for extracting any Root of an infinite Series; and universally for exhibiting any Power, Root, Dimension, or Dignity whatsoever, of such a Series, whatever it's Exponent be, positive or negative, integral or fractional: *viz.* by inserting such it's Exponent in Place of m: and although the Exponents of x in the Terms of the Series, decrease instead of increasing, it will, notwithstanding, give you any desired Power, by only supposing p to be negative; and n also negative if necessary. Which therefore, because of it's great Excellency and Usefulness, I have laid before the Reader, to apply according to Occasion. The Demonstration of the Binomial Theorem shall conclude this Work, after we have explained a little the first and third Observations under our Author's 35th Art. and shewn by one Example, how this Doctrine of the Resolution of affected Equations is to be applied for finding the Areas of Curves.

Art. 35. 140. Sir *Isaac* takes Notice in the first of these Observations, that, although the Root of the fictitious Equation, by means of which the first Term of the converging Series for the Value of y in the proposed Equation is to be found, were a surd Quantity; or wholly unknown, as being perhaps the inextricable Root of an high affected Equation, you may notwithstanding, even in this Case, resolve the proposed Equation, provided the Root of the fictitious Equation be not impossible (of which immediately) by substituting some Letter for that unknown Root, and then proceeding with it as before, after the Manner he shews.

Thus if the Equation $y^{3} - 3a^{2}y + axy + a^{3} - x^{3} = 0$ were proposed, to find the Value of y in a converging Series of simple Terms, when x is very small: I find, by the Methods formerly taught, that there can be no other Equation assumed for finding the first Term of the Series, but $y^{3} - 3a^{2}y + a^{3} = 0$, whose Root is a Surd;

a Surd; therefore, instead of the Letter a, which denotes a known invariable Quantity, I substitute any Number I think proper, *e.g.* 1, by which it becomes $y^3 - 3y + 1 = 0$, which is to be resolved by our Author's Method of resolving affected numeral Equations: by which one of it's Roots will be found $y = + 1.53208$ &c. so that, having substituted any Letter, such as b, for the Root of that fictitious Equation $y^3 - 3a^2y + a^3 = 0$, and extracted the Root of the proposed Equation $y^3 - 3a^2y + axy + a^3 - x^3 = 0$, after the Manner our Author shews, to any desired Length, you substitute $+ 1.53208$ &c. every where for b; and 1 for a, which was the Value assumed for it; and so you shall have the required Number of Terms of a converging Series, for one of the Roots of the proposed Equation $y^3 - 3a^2y + axy + a^3 - x^3 = 0$. But, if you put $a = 1$, the proposed Equation may be first converted into this $y^3 - 3y + xy + 1 - x^3 = 0$: which being resolved by putting b for the first Term of the Series, as representing the Root of the fictitious Equation $y^3 - 3y + 1 = 0$, you'll obtain $y = b + \frac{b}{c}x + \frac{3b}{c^3}x^2 + \frac{18b^3 + cb^3 + 3cb - c^4}{c^5}x^3$ &c. where c stands for $3 - 3b^2$: and therefore, having found one of the Roots of $y^3 - 3y + 1 = 0$ to be $+ 1.53208$ &c. insert it instead of the fictitious Root b: and thus you shall have so many initial Terms of the converging Series for the Value of y: in which Value, Unity is equal to the given Quantity, or Line, a.

141. Yet perhaps in such Cases, some of the known Species or Letters belonging to the proposed Equation, may be retained in the Root: thus in the preceding Equation, having found that the first Term of the converging Series ought to be derived from $y^3 - 3a^2y + a^3 = 0$, you may divide the Root of this fictitious Equation by a, according to the known Rules of Algebra, by which it will become $y^3 - 3y + 1 = 0$, and having found any of the Values of y in this numeral Equation, multiply that Value by a, and you shall have an approximate Root of the Equation $y^3 - 3a^2y + a^3 = 0$, for the first Term of the converging Series.

Therefore, all literal Equations are to be freed as much as possible from the Letters they contain, by the Rules of Algebra: but if, after all, there remain one or more known Letters in the fictitious Equation, from which the first Term of the Series is to be derived, and the Root of that Equation be surd or wholly unknown, you may substitute Numbers for such Letters: which Numbers must bear the same Proportion to one another, that the Quantities represented by such Letters do bear, either exactly, if possible, or, if not, as near as may be; *viz.*

viz. when there are more than one: but if there be but one, such Letter remaining, you may put any Number for it, you think proper: and then you proceed with the Solution, as you have been directed.

142. I come now to speak a little of that Case, where the Equation to be assumed for deriving the first Term of the Series has no real or *possible* Root, mentioned by the Author in Obs. 3. Art. 35. In this Case he directs to increase, or diminish, the Quantity x, until you find such a fictitious Equation, as admits of one or more possible Roots.

Thus, if the Equation $y^4 - x^2y^2 + xy^2 + 2y^2 - 2y + 1 = 0$, were proposed; the Equation from which the first Term of a converging Series ought to be derived, when x is very small, is $y^4 + 2y^2 - 2y + 1 = 0$: whose Roots are all *imaginary*, therefore I suppose x to be increased, or diminished, by any given Quantity, which you may call a, and then substitute the Quantity x so increased, or diminished, in place of the former x: thus if you put $z = x - a$, or $z + a = x$, in the preceding Equation, it becomes $y^4 - z^2y^2 - 2azy^2 + zy^2 - a^2y^2 + ay^2 + 2y^2 - 2y + 1 = 0$: where, if you suppose z to vanish, this Equation results $y^4 - a^2y^2 + ay^2 + 2y^2 - 2y + 1 = 0$: in which Equation, if there be any Number, which being put in for a, will give an Equation with one or more real and commensurable Roots, I insert such Number, and then take the Root of that Equation for the first Term of the converging Series: thus I find that if a was supposed to be 2, the last fictitious Equation would become $y^4 - 2y + 1 = 0$, in which one of the Values of y is $+ 1$: therefore I suppose $z + 2 = x$, and so the transformed Equation $y^4 - z^2y^2 - 2azy^2 - a^2y^2 + zy^2 + ay^2 + 2y^2 - 2y + 1 = 0$, is converted into $y^4 - z^2y^2 - 3zy^2 - 2y + 1 = 0$, where the first Term of a converging Series for the Value of y, is $+ 1$; and if you prosecute the Extraction, you'll find $y = 1 + \frac{3z}{2} - 4z^2 + \frac{99z^3}{4}$ &c. which therefore is one of the Values of y, z being equal to $x - 2$; and x being supposed to differ very little from 2.

143. The Reader may conceive the Matter thus: the proposed Equation defines the Relation betwixt the Abscis x of some Curve, and it's Ordinate y: now if the Ordinate drawn through the beginning of the Abscis, cannot meet the Curve; it is evident y can have no possible Value, when x the Abscis is supposed to vanish: therefore the Author's Rule for finding the first Term of the Value of y cannot be applied (although perhaps some of the other Methods demonstrated above may) immediately and directly: yet if you suppose the

the beginning of the Abscifs to be changed, by removing it to a proper and sufficient Distance, the Ordinate passing through this new beginning of the Abscifs, will meet the Curve in one or more Points; and then his Method for finding the first Term of the Root of the proposed Equation, may be applied. Which may be done by inserting $a+z$, or $a-z$, instead of x in the proposed Equation; by Means of which, a new Equation arises, wherein y shall have one or more possible Values when z vanishes, if the Distance a betwixt the two beginnings of the Abscifs, be taken of a proper Length. And thus the Root of the proposed Equation may be found in a converging Series, which shall converge the faster the less that z is; or (which is the same Thing) the less that the Difference betwixt x and a is.

144. It is with a Design to explain this Affair that Sir *Isaac* introduces the Figure belonging to this Article, where it appears, that, when the Abscifs AC is supposed to begin at the given Point A, there is no possible Value of the Ordinate; yet if that Abscifs AC be lessened by the given Line AB; so that the Point B be conceived as the Commencement of the Abscifs BC, the Ordinate CD shall have a four fold Value, when BC vanishes: each of which Values may be the first Term of the Root or Quotient.

Suppose the Equation to the Curve to be $50y^4 + \frac{5}{\sqrt{51}}xy^3 - 2\sqrt{51}\times bxy^2 + x^2y^2 - \frac{10}{\sqrt{51}}b^2xy + \frac{5}{51}bx^2y + b^4 = 0$: here if you put $x=0$, you have $50y^4 + b^4 = 0$; where y has no possible Value: but if you suppose x to be diminished by the given Quantity $\sqrt{51}\times b$; or suppose $x = z + \sqrt{51}\times b$: by substituting this for that, the given Equation will become $50y^4 + 5by^3 + \frac{5}{\sqrt{51}}zy^3 - 51b^2y^2 + z^2y^2 - 5b^3y + \frac{5}{51}bz^2y + b^4 = 0$: where, when z vanishes, you have $50y^4 + 5by^3 - 51b^2y^2 - 5b^3y + b^4 = 0$, which has four real Roots, *viz.* 1°. $y = +b$. 2°. $y = +\frac{b}{10}$. 3°. $y = -\frac{b}{5}$. 4°. $y = -b$, corresponding to the four Values of the Ordinate, *viz.* BE, BF, BG, BH. Wherefore any of these may be made the first Term of a Series for expressing the Value of y, that is the Ordinate CD; and the other Terms must be found by the preceding Methods: And here it is evident, that, the nearer the Ordinate CD is to EBH, so much the faster must any of these Series's converge: for when the Points B and C coincide, the first Terms of the Series's already found, are the full Values of the Ordinate; and therefore, the less the Distance BC = z is,

z is, the nearer will the Lengths of the Ordinate CD approach to these: and accordingly the approximate Values of the Area's, thence deduced, *viz.* BCDE, BCDF, BCDG, BCDH, will approach the nearer to the Truth, in any given Number of Terms, the less that BC $= z$ is; or the less the Difference betwixt b and x is.

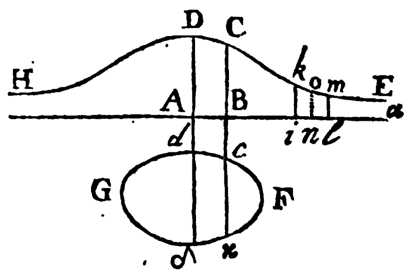
145. Schol. The Reader may observe from what hath been said, from first to last, upon this Subject, that the Roots of specious Equations, including two variable Quantities, may be extracted and expressed after a variety of different Ways. And if one would desire to find after how many Ways this may be done, you must try what are the Quantities, which being substituted for the indefinite Species x, in the proposed Equation, will make it divisible by $y +$ or $-$ any known Quantity; or perhaps by y alone: thus, in the Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$, if you substitute $+a$, $-a$, $-2a$, or $-\sqrt[3]{2} \times a$ for x, there will result these Equations $y^3 + 2a^2y - 3a^3 = 0$, $y^3 - 3a^3 = 0$, $y^3 - a^2y + 6a^3 = 0$, and $y^3 - \sqrt[3]{2} \times a^2y + a^2y = 0$, respectively; the first of which Equations is divisible by $y - a = 0$; the second by $y - \sqrt[3]{3} \times a = 0$; the third by $y + 2a = 0$; the fourth by $y = 0$, and besides by $y - \sqrt{-1 + \sqrt[3]{2}} \times a = 0$, and $y + \sqrt{-1 + \sqrt[3]{2}} \times a = 0$: whence it follows that the known Quantities joined to y, in the several lateral Equations, having their Signs changed, may be the first Terms of so many converging Series's for the Value of y in the proposed Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$; according as you suppose x to differ very little from $+a$ or $-a$ or $-2a$, or $-\sqrt[3]{2} \times a$, which were the Quantities substituted for x: the Reason of which is evident, since these will be Values of y when x is of the supposed Values: and therefore the several Roots may be prosecuted by inserting $z + a$, or $z - a$, or $z - 2a$, or $z - \sqrt[3]{2} \times a$ for x respectively, in the proposed Equation; as has been shewn already: after which, x, if you please may be restored.

And these you obtain by supposing the Differences betwixt x and $+a$, x and $-a$, x and $-2a$, x and $-\sqrt[3]{2} \times a$, that is z, to be very small; but you may obtain other Series's, which shall be Roots of the proposed Equation, by supposing these same Differences, that is z, to be indefinitely great, after the Manner shewn above. And all

all these different Series's you will obtain, besides those that are obtained by the Methods formerly taught. Nay moreover, you may substitute $z +$ or $-$ any known Quantity you please, for x, from whence an Equation will result, in which z being supposed to be very small, or very great, a converging Series may be derived by any of the Ways which have been explained. Finally, you may obtain the same Thing by substituting any fictitious Values at Pleasure for the indefinite Quantity x, as $az + bz^2$, $\frac{a}{b+z}$, $\frac{a+cz}{b+z}$, &c. and proceeding with the resulting Equations as before: so that there may be infinite different Ways of extracting and expressing the Roots of such specious Equations.

146. I come now to shew the Application of this Doctrine of the Extraction of the Roots of affected specious Equations, to the Quadrature of Curves, in one Example; by which the Reader may see how it is to be applied in other Examples of the Quadrature of Curves; for the sake of which indeed it was, that I was first induced to explain this Treatise of Analysis.

Let DEFGH be a Curve described with the Ordinate $CBc\varkappa$, moving perpendicularly along the Abscis or Base AB: call $AB = x$, BC, Bc, or $B\varkappa = y$; and a denoting a given Line, let the Nature of the Curve be defined by this Equation $y^3 + 5ay^2 - 34a^2y + x^2y - 80a^3 = 0$, the Area is required.



Draw through A the beginning of the Abscis, the Ordinate $DAd\delta$: then, if the Area ABCD, $ABcd$, or $AB\varkappa\delta$, be sought, whose initial Limit is $DAd\delta$; in the given Equation $y^3 + 5ay^2 - 34a^2y + x^2y - 80a^3 = 0$, suppose x to vanish, and you have $y^3 + 5ay^2 - 34a^2y - 80a^3 = 0$, whose Roots are AD, Ad, and $A\delta$. In order to find these Roots, you may suppose them divided by the Quantity a, by which the Equation will stand thus $y^3 + 5y^2 - 34y - 80 = 0$; in which the Values of y are $y = +5$. $y = -2$. $y = -8$; so that the Values of y in the Equation $y^3 + 5ay^2 - 34a^2y - 80a^3$ are $y = +5a = AD$, $y = -2a = Ad$ and $y = -8a = A\delta$: accordingly, any of these three will make the first Term of a Series which shall be a Root of the proposed Equation, according as BC, or Bc, or $B\varkappa$, is the Root required: and in order to discover the subsequent Terms of the Series, we must proceed by some of the Methods formerly explained. Thus by prosecuting the Extraction of the Root whose first Term is $+5a$, you would find $y = +5a - \frac{5x^2}{91a} - \frac{45x^4}{\overline{91a}|^3} + \frac{6470x^6}{\overline{91a}|^5} + \frac{971855x^8}{\overline{91a}|^7} + \frac{761845700x^{10}}{\overline{91a}|^9}$ &c. $= BC$. Wherefore by our Author's second Rule the Curvilinear Area $ABCD = 5ax - \frac{5x^3}{3\times 91a} - \frac{9x^5}{\overline{91a}|^3} + \frac{6470x^7}{7\times\overline{91a}|^5}$ &c. which converges the faster the less that x ($= AB$) is.

After the same Manner, by prosecuting the Extraction of the Roots whose first Terms are $-2a = Ad$, and $-8a = A\delta$, you would find $y = -2a - \frac{x^2}{21a} - \frac{11x^4}{\overline{21a}|^3}$ &c. $= Bc$, and $y = -8a + \frac{4x^2}{39a} + \frac{74x^4}{\overline{39a}|^3}$ &c. $= B\kappa$: and therefore the Area $ABcd = -2ax - \frac{x^3}{3\times 21a} - \frac{11x^5}{5\times\overline{21a}|^3}$ &c. and the Area $AB\kappa\delta = -8ax + \frac{4x^3}{3\times 39a} + \frac{74x^5}{5\times\overline{39a}|^3}$ &c. which Areas lying below the Base AB, are represented negative; and therefore their true Values are obtained by changing the Signs of all the Terms: which Expressions are so much the nearer the Truth, the less that x ($= AB$) is.

147. But besides the foregoing Value of y, exprest by Series's which converge when x is very small; there is another Value of y, the first Term of which is found by assuming the fictitious Equation $+x^2y - 80a^3 = 0$, *viz.* $+\frac{80a^3}{x^2}$, which agrees to the Supposition of x being very great: and which Value must denote the Ordinate BC lying above the Base, since the first Term is positive: this Root may be further prosecuted, and it will give $y = +\frac{80a^3}{x^2} + \frac{2720a^5}{x^4} - \frac{60480a^7}{x^6} + \frac{631680a^9}{x^8}$ &c. $= BC$: which converges only when x is great in Comparison of a; and, by the Dimensions of x increasing continually in the Denominators of the Terms, shews that the Line AB is an Assymptote to that Branch of the Curve, *viz.* DCE. And the Area derived from this Expression of the Ordinate (by the Author's second Rule) is $-\frac{80a^3}{x^2} - \frac{2720a^5}{3x^3} + \frac{60480a^7}{5x^5}$ &c. which denotes the Area $BC\alpha$ lying upon the other Side of the Ordinate, and adjacent to the Absciss infinitely extended with the Curve upon that Side. But upon the account of that Position, the Expression gives it negative; and therefore the Signs of the Terms must be changed: but it will only be of Use, when x (AB) is very great.

148. But if the curvilinear Area $ilmk$, adjacent to any given Part of the Base il, be required, then you may find the Areas $Almd$ and

and Aikd ſeparately, according to Art. 146, provided the Series's by which theſe Areas are expreſſed, converge ſufficiently; and ſubtract the latter from the former: or if theſe Series's do not converge ſufficiently, then find the Areas $ik\alpha$ and $lm\alpha$ according to Art. 147, if the Series's for theſe Areas converge ſufficiently, and ſubtract the one from the other. But if none of theſe two Methods will ſerve your Purpoſe, you muſt ſuppoſe the beginning of the Abſciſs to be changed, and brought to ſome other Point, ſuch as you find to be moſt convenient for the Purpoſe. In chuſing this new beginning of the Abſciſs, it is proper to enquire whether there be any known Length, which ſubſtituted for x in the given Equation $y^3 + 5ay^2 - 34a^2y + x^2y - 80a^3 = 0$, will produce an Equation having a commenſurable Root, at leaſt a Root which is not involved in a complex Radical Expreſſion, according to Art. 145, by which given Length if the Abſciſs x be ſhortned, by ſubſtituting $z +$ that Length for x, and the Root of the new transformed Equation extracted, by ſuppoſing z to be very little, the Root may converge ſufficiently for your Purpoſe: in which Caſe, you ſhould proceed by ſuppoſing the Abſciſs x to be ſhortened by that known Length. But if this cannot be obtained, you muſt ſubſtitute $z +$ any known Length for x, that will produce an Equation ſuch as is mentioned, and reſolved Art. 140, 141: which known Length ſhould be taken as convenient for the Purpoſe as poſſible.

149. That this may be explained by the Example before us; let it be A$i = 6a$, and A$l = 8a$: the Area $ilmk$ might be found by finding the Areas Aikd and Almd ſeparately, if the Series which ariſes by ſubtracting the two Series's denoting theſe two Areas, did ſufficiently converge: but you would find it by Trial to converge ſo ſlowly as to make this Method improper. Wherefore, you may next try to find the Area $ilmk$ by Means of the Areas $ik\alpha$ and $lm\alpha$: for finding the Area $ik\alpha$, ſubſtitute $6a =$ Ai for x in the general Series belonging to this Caſe, Article 147, *viz.* $-\frac{80a^3}{x} - \frac{2720a^5}{3x^3} + \frac{60480a^7}{5x^5} - \frac{631680a^9}{7x^7}$ &c. by which it becomes $-\frac{80a^2}{6} - \frac{2720a^2}{3\times216} + \frac{60480a^2}{5\times7776} - \frac{631680a^2}{7\times279936}$ &c. that is, when reduced, $\overline{-\frac{40}{3} - \frac{340}{81} + \frac{14}{9} - \frac{235}{729} \&c.} \times a^2$. Then you find the Series for the Value of the Area $lm\alpha$, by the like Means, by ſubſtituting $8a =$ Al for x in the ſame general Expreſſion; which, after due Reduction, becomes $\overline{-10 - \frac{85}{48} + \frac{189}{512} - \frac{705}{16384} \&c.} \times a^2$: both which Expreſſions are

 negative,

negative, and therefore if you subtract the first, which is greatest, from the last, you'll obtain $+\frac{10}{3}+\frac{3145}{1296}-\frac{5467}{4608}+\frac{3336259}{11943936}$ &c. $\times a^2 = 4.8$ &c. $\times a^2$ for an approximate Value of the Area *ilmk* required.

But if none of the preceding Methods would answer (or even allowing that any of them did answer in some Measure, but so as the Convergency of the Series were very slow) you must suppose the beginning of the Abscis to be changed, so as it may fall in with the Point *i* or *l*, or not be far distant from them, by the Method pointed out in Art. 145: but since I do not find any known Quantity such as $5a$, $6a$, $7a$, $8a$, or $\sqrt{35}\times a$ $\sqrt{48}\times a$ &c. which substituted for x in the given Equation $y^3+5ay^2-34a^2y+x^2y-80a^3=0$, would make it divisible by $y -$ any known Quantity, according to that Art. (which if there were and I could find that Quantity, and it were such as the Abscis being shortened by, would bring the beginning of it to be near to, or in the Line *il*, I would shorten the Abscis by that known Length) therefore you may alter the Beginning of the Abscis variously; but most conveniently by bringing it to the Point n, which bissects *il*: draw the Ordinate *no*; then is $An = 7a$: therefore, calling the new Abscis z, to be reckoned from the given Point n, substitute $7a+z$ for x in the proposed Equation, and manage it as directed in Art. 142. By such Substitution, the proposed Equation becomes $y^3+5ay^2+15a^2y+14azy+z^2y-80a^3=0$; from which assuming the fictitious Equation $y^3+5ay^2+15a^2y-80a^3=0$, I put any letter as b for it's Root; and so extract the Root of the Equation $y^3+5ay^2+15a^2y+14azy+z^2y-80a^3=0$, in the Manner our Author shews Art. 35. Rem. 1. by which you find $y=b-\frac{14ab}{c}z+\frac{2940a^4b+980a^3b^2-c^2}{c^3}z^2+\frac{42ab^2d-210a^3d+14abc^3}{c^5}z^3$ &c. where c stands for $15a^2+10ab+3b^2$, and d for $2940a^4b+980a^3b^2-c^2$. And therefore the Area $nlmo = bz+\frac{7ab}{c}z^2+\frac{2940a^4b+980a^3b^2-c^2}{3c^3}z^3+\frac{42ab^2d-210a^3d+14abc^3}{4c^5}z^4$ &c. Wherefore since $ni=nl$, but negative, as lying upon the other Side of the Point n, where the Abscis z begins, the Area *noki* will be denoted by the same Expression, only by making z negative, which is done by changing the Signs of the first, third, fifth, &c. Terms into their Opposites; that is the Area $noki = -bz+\frac{7ab}{c}z^2-\frac{2940a^4b+980a^3b^2-c^2}{3c^3}z^3+\frac{42ab^2d-210a^3d+14abc^3}{4c^5}z^4$ &c. Therefore, if you subtract this negative Value of *noki*; or add it's it's poſitive Value, which is the oppoſite of the former, you'll have the Area $ilmk = 2bz + \frac{5880a^4b + 1960a^3b - 2c^2}{3c^3}z^3$ &c. And in order to find the Value of b, you muſt extract the Root of the Equation $y^3 + 5y^2 + 15y - 80 = 0$, by our Author's Method of extracting the Roots of numeral Equations, and multiply it by a; by which you'll find $b = 2.42432070$ &c. $\times a$. So that if you ſubſtitute it for b, and a for $z = nl$, in the preceding Value, you ſhall have $ilmk = 4.848$ &c. $+ \frac{19006.674 \text{ \&c.} - 6469.576 \text{ \&c.}}{561937.669 \text{ \&c.}}$ &c. $\times a^2 = 4.86$ &c. $\times a^2$ the Area required.

150. Schol. 1. By thus making the new Beginning of the Abſciſs to fall exactly in the middle of the given Baſe il, the Series will be made to converge faſter than if you made it to fall in the Point i or l; and it is likewiſe more convenient for Calculation, than if it were brought to any other Point; ſince the ſame Series ſerves for the two different Parts $nlmo$ and $niko$, with the Change only of the Signs of ſome Terms: by the Addition of which Series to itſelf with ſuch Change of Signs, the reſulting Series, which expreſſes the Area $ilmk$ will converge the more quickly.

152. Schol. 2. The Reader may obſerve that this Curve belongs to the 40th Species of our Author's Enumeration of Lines of the third Order: and that the Areas belonging to the different Branches of the Curve, might much more eaſily be found by Prop. 5. of the Quad. of Curves, by taking the Abſciſs upon AD perpendicular to AB. It may be obſerved likewiſe that Cor. 7 and 8. Prop. 9. *ibid.* may be of conſiderable Uſe; and when Occaſion requires, ought to be applied, for reducing affected Equations, either to ſuch as are not affected; or ſuch as are leſs affected, before we attempt to find the Areas of thoſe Curves, the Relations of whoſe Ordinates and Abſciſſes are defined by ſuch Equations.

153. Schol. 3. With reſpect to the Doctrine of the Extraction of the Roots of affected ſpecious Equations, contained in this Section, it is to be obſerved, that although, in order to find an infinite converging Series, which approaches continually to the true Value of the Root y, the Quantity x muſt be ſuppoſed to be indefinitely ſmall; or indefinitely great, in Compariſon of any other known Quantity in the Equation, as appears from the Methods of extracting theſe Roots; yet after ſuch Series for the Root is thus inveſtigated, it will ſtill denote the Value of y, whatever be the Value of x: only the Series will not converge ſo faſt; or perhaps not at all. This appears from the

Analogy

Analogy of the Thing, eſpecially when x and y are conceived as denoting the Abſciſs and Ordinate of a Curve. Only, in theſe Equations, the Value of x may be ſo great, or ſo ſmall, as to render the Root impoſſible: which will happen when the Ordinate y cannot meet that Branch of the Curve, to which the infinite Equation belongs: and in this Caſe the Aggregate of all the Terms of the Series becomes infinitely great.

SECT. VI.

Containing the Inveſtigation and Demonſtration of the Binomial Theorem.

THE Demonſtration of this famous Theorem I promiſed to give Art. 45, having there illuſtrated it by ſeveral Examples.

154. I ſay then that $\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} \times$

$$\overline{1+\frac{m}{n}Q+\frac{m}{n}\times\frac{m-n}{2n}Q^2+\frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}Q^3+\frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}\times\frac{m-3n}{4n}Q^4 \ \&c.}$$

or $P^{\frac{m}{n}}+\frac{m}{n}AQ+\frac{m-n}{2n}BQ+\frac{m-2n}{3n}CQ+\frac{m-3n}{4n}DQ$ &c. where $P+PQ$ denotes the Quantity whoſe Power, Root, or Dimenſion whatſoever, is to be inveſtigated; P the firſt Term, Q the Quotient of all the other Terms divided by the firſt; $\frac{m}{n}$ the Index or Exponent of the Power, Root or Dimenſion whatſoever, which denotes any Number whatſoever, integral or fractional, poſitive or negative. Moreover, in the ſecond Form the Capitals A, B, C, &c. denote the firſt, ſecond, third, &c. Terms of the Series. It likewiſe appears evident that both the Forms amount to the ſame Thing: only when it is to be applied to particular Examples, the firſt Form may be more convenient for continuing the Terms of the Series, by ſhewing more evidently the Law of the Progreſſion; the other is more convenient for ſumming up the Terms of the Series, when the Letters have the Numbers which denote their particular Values, put in Place of them.

155. Caſe 1. When $n=1$ and m poſitive: then it ſhall appear in this Shape $\overline{P+PQ}|^{m} = P^{m} \times \overline{1+mQ+m\times\frac{m-1}{2}Q^2+m\times\frac{m-1}{2}\times\frac{m-2}{3}Q^3 \ \&c.}$

In

In order to demonstrate this first Case of the Prop. let $1+x$ denote a simple Binomial having Unity for it's first Term and x for the other; and let $p=m+1$: then if $\overline{1+x}|^m = 1 + mx + m\times\frac{m-1}{2}x^2 + m\times\frac{m-1}{2}\times\frac{m-2}{3}x^3$ &c. I say $\overline{1+x}|^p = 1 + px + p\times\frac{p-1}{2}x^2 + p\times\frac{p-1}{2}\times\frac{p-2}{3}x^3 + p\times\frac{p-1}{2}\times\frac{p-2}{3}\times\frac{p-3}{4}x^4$ &c.

For it is evident from the Involution of Powers by Multiplication, that the several Terms of the Power of $1+x$ whose Exponent is $p=m+1$, will be had by setting down in one Line the several Terms of the Power of $1+x$, whose Exponent is m, in Order, and then these several Terms, each multiplied by x, in the same Order in another Line, advancing each Term one Step, and so adding the two Lines together as follows.

$$1+mx+m\times\frac{m-1}{2}x^2+m\times\frac{m-1}{2}\times\frac{m-2}{3}x^3 \ \&c.$$
$$1x+mx^2 \qquad +m\times\frac{m-1}{2}x^3 \qquad +m\times\frac{m-1}{2}\times\frac{m-2}{3}x^4 \ \&c.$$

$$\text{Sum. } 1+\overline{1+m}x+\overline{m+m\times\frac{m-1}{2}}x^2+\overline{m\times\frac{m-1}{2}+m\times\frac{m-1}{2}\times\frac{m-2}{3}}x^3 \ \&c.$$

Here the Constitution of the Coefficient of any Term of the Sum is manifest; which may be generally expressed thus: $\frac{m\times\overline{m-1}\times\overline{m-2}\times\overline{m-3}\times\overline{m-4}\ \&c.}{1\times2\times3\times4\times5\ \&c.}+\frac{m\times\overline{m-1}\times\overline{m-2}\times\overline{m-3}\ \&c.}{1\times2\times3\times4\ \&c.}$, which I say is $=\frac{p\times\overline{p-1}\times\overline{p-2}\times\overline{p-3}\times\overline{p-4}\ \&c.}{1\times2\times3\times4\times5\ \&c.}$ where the Number of Factors in the Numerators and Denominators of the Terms must be the same; the Number of Factors in the first Term of the first Side of the Equation, the same with those in the Term standing upon the other Side of the Equation; but the Factors in the Numerator and Denominator of the second Term of the first Side of the Equation, one fewer.

For by inserting $p-1$ for m in the first Side of the Equation, that Side stands thus $\frac{\overline{p-1}\times\overline{p-2}\times\overline{p-3}\times\overline{p-4}\times\overline{p-5}\ \&c.}{1\times2\times3\times4\times5\ \&c.}+\frac{\overline{p-1}\times\overline{p-2}\times\overline{p-3}\times\overline{p-4}\ \&c.}{1\times2\times3\times4\ \&c.}$ $=\frac{\overline{p-1}\times\overline{p-2}\times\overline{p-3}\times\overline{p-4}}{1\times2\times3\times4}\times\overline{\frac{p-5}{5}+1}=\frac{\overline{p-1}\times\overline{p-2}\times\overline{p-3}\times\overline{p-4}}{1\times2\times3\times4}\times\frac{p}{5}=$ $\frac{p\times\overline{p-1}\times\overline{p-2}\times\overline{p-3}\times\overline{p-4}}{1\times2\times3\times4\times5}$. And it is evident that this will be true, whatever

whatever be the Number of Factors by whose Multiplication the Terms of the Equation are made up, according to the forementioned Conditions. Now let us suppose $m=2$, then it is true that $\overline{1+x}|^m = 1 + mx + m \times \frac{m-1}{2} x^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} x^3$ &c. $= 1+2x+x^2$, all the Terms after the third vanishing, because they involve the Factor $\frac{m-2}{3}=0$: therefore p $(=m+1)$ being 3, it follows that $\overline{1+x}|^p = 1+px + p\times\frac{p-1}{2}x^2 + p\times\frac{p-1}{2}\times\frac{p-2}{3}x^3$. And by the same way of reasoning, if $q=p+1$ then $\overline{1+x}|^q = 1+qx+q\times\frac{q-1}{2}x^2 + q\times\frac{q-1}{2} \times\frac{q-2}{3}x^3 + q\times\frac{q-1}{2}\times\frac{q-2}{3}\times\frac{q-3}{4}x^4$: and so on continually. Whence it appears that, if m be made to stand for any positive Integer whatsoever, it shall be true that $\overline{1+x}|^m = 1+mx+m\times\frac{m-1}{2}x^2+m\times \frac{m-1}{2}\times\frac{m-2}{3}x^3+m\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}x^4$ &c.

Now the Letter x may stand for any Quantity whatsoever, wherefore $\overline{1+Q}|^m = 1+mQ+m\times\frac{m-1}{2}Q^2+m\times\frac{m-1}{2}\times\frac{m-2}{3}Q^3+m \times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}Q^4$ &c. consequently $\overline{P+PQ}|^m = (P^m \times \overline{1+Q}|^m =)\ P^m \times \overline{1+mQ+m\times\frac{m-1}{2}Q^2+m\times\frac{m-1}{2}\times\frac{m-2}{3}Q^3+m\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}Q^4 \ \&c.}$ Which is the first Case of the Prop.

156. Schol. 1. Q may denote a Quantity consisting of any Number of Terms as $a+b+c+d$ &c. yet if we know how to raise the simple Binomial $1+a$ to any positive integral Power, in which Case Q is only one Term; the same Rule will serve to raise the Quantity $1+a+b+c+d$ &c. to any such Power, where Q consists of any Number of Terms whatsoever $a+b+c+d$ &c. For 1°. suppose $Q=a+b$ a Binomial, then $Q^2=a^2+2ab+b^2$; $Q^3=a^3+3a^2b+3ab^2+b^3$; $Q^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$: and so on, by what hath been demonstrated. Again 2°. suppose $Q=a+b+c$, a Trinomial, you may conceive it as a Binomial, whose two Members are a and $\overline{b+c}$: in which Case $Q^2=a^2+2a\times\overline{b+c}+\overline{b+c}|^2$; $Q^3=a^3+3a^2\times\overline{b+c}+3a\times\overline{b+c}|^2+\overline{b+c}|^3$: and so on for other higher Powers in this Case: and thus, by supposing the Number of Terms of which Q consists, to be increased continually, you may perceive by what Means the Quantity

P + PQ

$P+PQ$ or $P\times\overline{1+Q}$, may be raiſed to any poſitive integral Power, whatever Number of Terms Q or PQ conſiſt of; and that by the Help of what hath been demonſtrated: by which you'll obtain ſuch a Canon as we have exhibited in Art. 139.

157. Schol. 2. It is evident that, in this firſt Caſe, the Series for the Power muſt terminate, and the Number of Terms be one more than the Exponent m.

158. Caſe 2. When the Exponent $\frac{m}{n}$ is a poſitive Fraction, proper or improper, the Theorem will appear in this Form:

$$\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} \times \overline{1+\tfrac{m}{n}Q+\tfrac{m}{n}\times\tfrac{m-n}{2n}Q^2+\tfrac{m}{n}\times\tfrac{m-n}{2n}\times\tfrac{m-2n}{3n}Q^3 \text{ \&c.}}$$

For take the ſimple Binomial $1+x$ as before: then I ſay $\overline{1+x}|^{\frac{m}{n}}$ $=1+\frac{m}{n}x+\frac{m}{n}\times\frac{m-n}{2n}x^2+\frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}x^3$ *&c. in inf.*

To demonſtrate which ſuppoſe x to be a variable or flowing Quantity; and put $y=\overline{1+x}|^{\frac{m}{n}}$: raiſe both Sides of this laſt Equation to the Power whoſe Exponent is n, and it is $y^n=\overline{1+x}|^m=$ (by Caſe 1.) $1+mx+m\times\frac{m-1}{2}x^2+m\times\frac{m-1}{2}\times\frac{m-2}{3}x^3+m\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}x^4$ *&c.* wherefore if you extract the Root y of this Equation y^n-1-mx $-m\times\frac{m-1}{2}x^2-m\times\frac{m-1}{2}\times\frac{m-2}{3}x^3-m\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}x^4$ *&c.* $=$ o, by ſuppoſing x to be very ſmall: you ſhall have 1 for the firſt Term of a Series for that Root, by aſſuming the fictitious Equation $y^n-1=0$ *; and the Form of the Series for that Root ſhall be *Art. 128.
$y=1+Bx+Cx^2+Dx^3+Ex^4$ *&c.* * Wherefore, by Help of 129.
the firſt Caſe of this Prop. let the Equation $y^n-1-mx-m\times\frac{m-1}{2}x^2$ *&c.* $=0$, be transformed thus:

$$\left.\begin{array}{rlllll}
+y^n= & +1 & +nBx+ & nCx^2+ & & nDx^3 \text{ \&c.}\\
 & & & +n\times\frac{n-1}{2}B^2x^2+ & & n\times\frac{n-1}{2}\times 2BCx^3 \text{ \&c.}\\
 & & & & & +n\times\frac{n-1}{2}\times\frac{n-2}{3}B^3x^3 \text{ \&c.}\\
-1= & -1 & & & & \\
-mx= & * & -mx & & & \\
-m\times\frac{m-1}{2}x^2= & * & * & -m\times\frac{m-1}{2}x^2 & & \\
-m\times\frac{m-1}{2}\times\frac{m-2}{3}x^3= & * & * & * & & -m\times\frac{m-1}{2}\times\frac{m-2}{3}x^3\\
\text{\&c.} & & & & &
\end{array}\right\}=0$$

In which transformed Equation, if you put the complex Coefficients, each after the first, equal to nothing, you'll find the Values of the several indeterminate Coefficients B, C, D, &c. that of the first Term being already found to be $+1$.

The Equations are 1°. $nB - m = 0$, 2°. $nC + n \times \frac{n-1}{2}B^2 - m \times \frac{m-1}{2} = 0$. 3°. $nD + n \times \frac{n-1}{2} \times 2BC + n \times \frac{n-1}{2} \times \frac{n-2}{3}B^3 - m \times \frac{m-1}{2} \times \frac{m-2}{3} = 0$ &c.

From the first you find $B = + \frac{m}{n}$; by inserting which Value of B into the second, you have $nC + \frac{m^2n - m^2}{2n} - \frac{m^2 - m}{2} = 0$, that is $C = \left(\frac{m^2n - mn}{2n^2} - \frac{m^2n - m^2}{2n^2} = \frac{m^2 - mn}{2n^2} =\right) \frac{m}{n} \times \frac{m-n}{2n}$: and by inserting the Values of B and C in the third Equation, it becomes $nD + \frac{m^3n - m^3 - m^2n^2 + m^2n}{2n^2} + \frac{m^3n^2 - 3m^3n + 2m^3}{6n^2} - \frac{m^3 - 3m^2 + 2m}{6} = 0$, that is, by reducing, $D = \left(\frac{m^3 - 3m^2n + 2mn^2}{6n^3} =\right) \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}$: and thus you may proceed to find the Values of the other indeterminate Coefficients *in inf.* So that it appears that $\overline{1+x}|^{\frac{m}{n}}$ $(=y) = 1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}x^3$ &c.

Wherefore it follows, by what was shewn in Case 1, that $\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} \times \overline{1 + \frac{m}{n}Q + \frac{m}{n} \times \frac{m-n}{2n}Q^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}Q^3 \text{ \&c.}}$

159. Schol. It is evident that the Series in this second Case must be infinite; since no Repetition of n can be equal to m.

160. Case the third. When the Exponent $\frac{m}{n}$ denotes any negative Number integral or fractional, the Theorem will appear in this Form: $\overline{P+PQ}|^{\frac{-m}{n}} = P^{\frac{-m}{n}} \times \overline{1 - \frac{m}{n}Q - \frac{m}{n} \times \frac{-m-n}{2n}Q^2 - \frac{m}{n} \times \frac{-m-n}{2n} \times \frac{-m-2n}{3n}Q^3 \text{ \&c.}}$

For

For taking the Binomial $1+x$ as before, I say $\overline{1+x}|^{\frac{-m}{n}} = 1 - \frac{m}{n}x - \frac{m}{n} \times \frac{-m-n}{2n}x^2 - \frac{m}{n} \times \frac{-m-n}{2n} \times \frac{-m-2n}{3n}x^3$ &c. *in inf.*

To prove which, since $\overline{1+x}|^{\frac{-m}{n}} = \frac{1}{\overline{1+x}|^{\frac{m}{n}}} =$ (by Cases 1 and 2.)

$\frac{1}{1+\frac{m}{n}x+\frac{m}{n}\times\frac{m-n}{2n}x^2+\frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}x^3 \ \&c.}$, divide the Numerator of this Fraction, *viz.* 1, by the Denominator $1+\frac{m}{n}x+\frac{m}{n}\times\frac{m-n}{2n}x^2$ &c. and the Quotient will come out $1-\frac{m}{n}x-\frac{m}{n}\times\frac{-m-n}{2n}x^2-\frac{m}{n}\times\frac{-m-n}{2n}\times\frac{-m-2n}{3n}x^3$ &c. For these two Series multiplied into one another will produce 1, as they ought to do, the other Terms *in inf.* destroying one another. Whence $\overline{P+PQ}|^{\frac{-m}{n}} = P^{\frac{-m}{n}} \times \overline{1-\frac{m}{n}Q-\frac{m}{n}\times\frac{-m-n}{2n}Q^2-\frac{m}{n}\times\frac{-m-n}{2n}\times\frac{-m-2n}{3n}Q^3}$ &c. *in inf.* for it is evident that the Series cannot terminate.

From all which it appears, that, whatever be the Value of the numeral Exponent $\frac{m}{n}$, whether integral or fractional, positive or negative, it will still be true that $\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} \times \overline{1+\frac{m}{n}Q+\frac{m}{n}\times\frac{m-n}{2n}Q^2+\frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}Q^3}$ &c. or $P^{\frac{m}{n}}+\frac{m}{n}AQ+\frac{m-n}{2n}BQ+\frac{m-2n}{3n}CQ$ &c. Q. E. D.

161. This famous Theorem may be more expeditiously investigated and demonstrated, by the Help of the Doctrine of Fluxions; but it is necessary that the Rule for finding the Fluxion of any Power, perfect or imperfect, of any flowing Quantity, be demonstrated from other Principles, without making use of this Theorem itself for that Purpose: which our Author does *.

Now that Rule may be demonstrated independently of the Binomial Theorem thus: let n denote any positive whole Number, and x a flowing Quantity, then I say the Fluxion of x^n is $nx^{n-1}\dot{x}$.

* Quad. Curv. Art. 11.

In order to demonstrate which I proceed thus: let the Fluxion of x^n be supposed to be $nx^{n-1}\dot{x}$, then if $p=n+1$, the Fluxion of x^p is $px^{p-1}\dot{x}$. For $x^p=x^n\times x$: wherefore, if z denote that Term in the Geometrical Progression $1, x, x^2, x^3$, &c. whose Distance from the first is denoted by n, so that $z=x^n$, then it is $x^p=zx$; therefore their Fluxions are equal: now the Fluxion of zx is $\dot{z}x+z\dot{x}$ (*See Art.* 76 *of the Quadrature of Curves explained*) that is, by restoring x^n for z, and $nx^{n-1}\dot{x}$ for $\dot{z}$ (which are equal by the hyp. since $nx^{n-1}\dot{x}$ is supposed to be the Fluxion of x^n) $nx^n\dot{x}+x^n\dot{x}$ or $\overline{n+1}\times x^n\dot{x}$ that is $px^{p-1}\dot{x}$: which therefore is the Fluxion of x^p, if so be $nx^{n-1}\dot{x}$ be the Fluxion of x^n. Now, by the same way of reasoning, if $q=p+1$, it appears, that, if the Fluxion of $x^p=px^{p-1}\dot{x}$, the Fluxion of x^q shall be $qx^{q-1}\dot{x}$: and so on continually. But if $n=2$, it is demonstrated that the Fluxion of x^n $(=x^2)$ is $nx^{n-1}\dot{x}=2x\dot{x}$, in the forecited Place: therefore it follows, that, if n be 3, 4, 5, &c. that is any positive Integer whatsoever, the fluxion of x^n shall be $nx^{n-1}\dot{x}$. And from what is shewn in the same Place, it appears, that, if $nx^{n-1}\dot{x}$ be the Fluxion of x, when n is a positive Integer, it shall be so likewise when n denotes any other Number, whether it be a negative Integer, or positive or negative Fraction of any Kind: so that the Rule is demonstrated to hold universally.

162. Cor. Hence it appears that the Fluxion of $\overline{1+x}|^n$ is $n\times\overline{1+x}|^{n-1}\dot{x}$; n denoting any numeral Index as before.

163. Upon the preceding Foundation the Binomial Theorem is thus demonstrated: suppose $\overline{1+x}|^n=1+Bx+Cx^2+Dx^3+Ex^4$ &c. where n stands for any numeral Index, positive or negative, integral or fractional; and the Letters B, C, D, E, &c. stand for indeterminate Coefficients to be found: it being evident from the Nature of Powers and Roots, that the assumed Series ought to be of the Form here exprest, and have 1 for it's first Term. Now by taking the Fluxions of both Sides of the preceding Equation (by Art. 161, 162.) we have $n\times\overline{1+x}|^{n-1}\dot{x}=B\dot{x}+2Cx\dot{x}+3Dx^2\dot{x}+4Ex^3\dot{x}$, &c. and dividing by $\dot{x}$; $n\times\overline{1+x}|^{n-1}=B+2Cx+3Dx^2+4Ex^3$ &c. which being true whatever be the Value of the flowing Quantity x, it must be true likewise when x vanishes: wherefore, let x vanish, and you have $n\times\overline{1}|^{n-1}=B$, that is $n=B$. Again, assuming the last Equation, *viz.* $n\times\overline{1+x}|^{n-1}=B+2Cx+3Dx^2+4Ex^3$ &c. and putting it into Fluxion, you have $n\times\overline{n-1}\times\overline{1+x}|^{n-2}\dot{x}=2C\dot{x}+2\times$

$3D x\dot{x} + 3 \times 4 E x^2 \dot{x}$ &c. and if you divide both Sides by $\dot{x}$, and then suppose x to vanish as before, you find $n \times \frac{n-1}{2} = C$. Again by assuming the Equation $n \times \overline{n-1} \times \overline{1+x}|^{n-2} = 2C + 2 \times 3Dx + 3 \times 4Ex^2$ &c. (which arises by dividing the preceding by the Fluxion of x) and putting it into Fluxions, you have $n \times \overline{n-1} \times \overline{n-2} \times \overline{1+x}|^{n-3}\dot{x} = 2 \times 3D\dot{x} + 2 \times 3 \times 4Ex\dot{x}$, &c. which being reduced as before, gives $n \times \frac{n-1}{2} \times \frac{n-2}{3} = D$: and by proceeding after this Manner, it will be evident to any one that considers the Operation, that all the Coefficients of the subsequent Terms of the assumed Series $1 + Bx + Cx^2 + Dx^3 + Ex^4$ &c. will be found by the continual Multiplication of the Terms of this Series $1 \times n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$ *&c. in inf.* the Coefficient of each Term of the Series $1 + Bx + Cx^2 + Dx^3 + Ex^4$ &c. being the Product of as many Terms of the other Series, as it's Place denotes. So that by substituting the Values of B, C, D, *&c.* we find $\overline{1+x}|^n = 1 + nx + n \times \frac{n-1}{2}x^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3}x^3$ *&c.*

164. Whence it appears, from what hath been said formerly, that $\overline{P+PQ}|^n = P^n \times 1 + nQ + n \times \frac{n-1}{2}Q^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3}Q^3$ &c. or $P^n + nAQ + \frac{n-1}{2}BQ + \frac{n-2}{3}CQ$ *&c.* n denoting any numeral Index whatsoever, and A, B, C, *&c.* standing for the first, second, third, *&c.* Terms in the last Form.

165. What further remains of this Treatise of our celebrated Author with respect to the Application of the Doctrine of infinite Series and Quadratures to other Purposes in Geometry, being either sufficiently plain in itself, to one that understands what hath been said to explain the Foundations upon which those Things are built; or having been mostly explained in the Commentary upon the Quadrature of Curves, I thought it improper to trouble the Reader, now brought this Length, with Repetitions of Things to no Purpose.

GENERAL

GENERAL SCHOLIUM.

If Mankind be capable of ſuch Degrees of Knowledge, as the Inventions and Diſcoveries of a Sir *Iſaac Newton* prove they are; how much greater Degrees of Knowledge may we ſuppoſe *ſuperior Intelligences* capable of arriving at, whoſe better Faculties and more extended Views, furniſh them with a larger Stock of Ideas, the only *Materials* of Knowledge; and enable them to perceive and diſcern more *accurately* and quickly their Agreements and various Relations. And how much higher Degrees of Knowledge, ſhall we ourſelves arrive to, when we have got beyond the Confines of Mortality, and ſhaken off theſe Tabernacles of Clay, where our Souls being *confined* as in a Priſon, are kept from *extending* and *enlarging* their Views; and being *bore down* as by a heavy Weight, are greatly *retarded* in their Progreſs in the Ways of Knowledge?—Then ſhall we perceive more and *better* Objects, and judge and reaſon more accurately, quickly, and diſtinctly about them.—Happy Time, when the Diſcovery of Truth ſhall be attended with no ſuch *Toil* and *Labour*, as it often is here! In this *imperfect* State of Things, we are obliged to purſue long Trains of Reaſoning, in order to attain to the Knowledge of thoſe Truths; which, when our Faculties are enlarged and *perfected*, we ſhall perceive and diſcern as it were at one View; for as then our Knowledge will be more extenſive, ſo it will be more diſtinct and more *intuitive:* as a ſacred Writer expreſſes it, *Now we ſee through a Glaſs darkly, or in a Riddle; but then Face to Face*[1]: Happy Day! when

[1] 1 *Cor.* xiii. 12. Βλέπομεν γὰρ ἄρτι δι' ἐσόπτρου ἐν αἰνίγματι· τότε δὲ πρόσωπον πρὸς πρόσωπον. Here the Apoſtle *Paul* plainly inſinuates a remarkable Difference in our Knowledge in this, and the future State, as to the Diſtinctneſs of it, and the Manner of attaining it; as well as to the Extenſiveneſs of it. He deſcribes our Knowledge in this State, as being obſcure and dark, by ſaying we know only in an Enigma or Riddle, which is a dark and obſcure Repreſentation of ſome Truth, cloathed in Figure and Metaphor. And further he deſcribes it as acquired δι' ἐσόπτρου, by the intervention of a Speculum or Mirror; by which we behold the Images or Species's of things; and not the very things themſelves. Whereas τότε, in the future State, we ſhall ſee and know πρόσωπον πρὸς πρόσωπον, that is immediately, by a direct Intuition of things as they really are in themſelves. So that when this Manner of ſeeing and knowing takes Place, the former ſhall vaniſh or be done away: as the ſame Author aſſerts in the ſame Place. And by conſidering the whole Deſcription St. *Paul* gives in this Place, of the Knowledge of the future State, compared with that of the preſent, we ſhall find the Antitheſis to conſiſt in theſe Things. 1°. It will be vaſtly greater, more thorough and extenſive. 2°. More clear, diſtinct and ſatisfactory. 3°. More immediate and intuitive. 4°. Acquired with leſs Pain and Difficulty. Which Things he illuſtrates by a beautiful Variety of Expreſſions.

not only our intellectual, but our *moral* Faculties, ſhall be extended and enlarged far beyond their preſent Pitch: and that *bliſsful* State begin, where our whole reaſonable Nature, in all it's Powers and Faculties, ſhall make further and further Progreſs and Advancement, in a Likeneſs to God himſelf, the *beſt* and moſt *perfect* Being.—When we ſhall be admitted into the *immediate* Preſence and *beatifical* Viſion of the Divine Majeſty; and for ever employed in ſearching, contemplating and admiring his Perfections, and his Works: which will afford an inexhauſtible Treaſure of Knowledge and Delight, encreaſing to endleſs Ages.

The END.

ERRATA.

In the *Quadrature of Curves* and *Commentary.*

PAG. 7. Line 29. at the Beginning read 21.
p. 10. l. 5. r. $\frac{1}{n}c$
p. 16. l. 27. r. $z^{\theta + n}$
20. l. 4. r. $fy^{n}z^{\theta}$ Ibid. l. 21. Curves *r.* Caſes
25. to the Figures annex the N^os 1, 2, 3, 4.
Ibid. Tab. Form 1, throughout all the Species's in the Denom^r of the Value of *y*, r. $e + fz^{n}$; and in the Area of Species 1, r. $\frac{\alpha GDB}{n}$
26. Form 2. throughout all the Species's in the Ordinate of the the Conic Section r. $\sqrt{\frac{d}{f} - \frac{e}{f}x^{2}}$
Ibid. Form 4. Species 2. in the Area, r. αDGa
Ibid. Species 3. r. $3aDGa$
27. Form 6. Species 1. Ord. of the ſecond Conic Section r. $\sqrt{d + \frac{-f-p}{2g}z^{2}}$
Ibid Form 7. Species 1. in the Denominator of the Area r. $4neg$
29. in the Fig. betwixt K and M place L
37. l. 16. *r.* ſo as to come nearer
39. l. 23. *r.* wherefore
43. l. 17. *r.* which the point
44. l. 1. Terms *r.* Times
45. l. 18. r. BbC
48. l. 8. after Area, *r.* which we have to conſider, and this cannot be altered or influenced by any former or future Fluxion of the Abſciſs
52. l. 1. *r.* CEαC
55. l. 14. *r.* demonſtrated in the
56. l. 6. r. $\frac{n-2}{3}x^{n-3}o^{3}$
57. l. 31. 11^th r. n^th
61. l. 14. *r.* one uniformly flowing Quantity and the Fluxion of another flowing Quantity.
64. Margin 47 r. 46, and 46 r. 51
68. l. 32. r. 15—25
71. l. 3. *r.* $\frac{z}{\dot{x}}$ Ibid. l. 5. r. $a\dot{x}$
72. l. 13. r. $mx^{n}y^{m-1}\dot{y}z^{p}$
75. l. 11. r. $\frac{4a\dot{x}xy + ax^{2}\dot{y} + 6\dot{x}x^{3}}{2\sqrt{ay + xx}} = 0$
77. l. 35. *dele* &c.
78. l. 8. *r.* $+2c\dot{x}x = 0$ p. 79. l. 35. Times *r.* Terms
81. l. 38. or *r.* and
82. l. *penult* of the Notes, *r.* Mr. *David Gregory*
83. l. 28. BE *r.* BC
87. in the ſecond Term of both Forms of the Ordinate $2\lambda n$ r. λn

p. 87. in 2^d Term of 2^d Form $2\mu n$ r. μn
88. l. 22. r. $z^{\theta-1}R^{\lambda-1}$
89. l. 2. r. 26—31
91. Margin *r.* Art. 114. 117.
95. l. 8. ſetting *r.* ſet.
Ibid. l. 20. r. $\frac{\frac{1}{n}b}{r+1}$
99. l. 11. *dele* ſame
101. l. 15. but *r.* although
Ibid. l. 32. *r.* may be finite
103. l. 28. r. $e - fz^{n}$
109. l. 12. *r.* and ſo you
Ibid. l. 30. Fraction *r.* Fluxion
110. l. 28. *r.* above thrice
112. l. 1. r. $-x^{3}$ Ibid. l. 9. r. $-\frac{1}{3}-1$
117. l. 19. r. $s = 1$
119. l. 17. r. be not nothing
121. l. 29. r. $t = s + \lambda$
122. l. 26. *r.* may be ſquared
124. l. 29. *r.* demonſtrated
132. l. 29. Forms *r.* Terms, Terms *r.* Forms
145. l. 3. *r.* extending
146. l. 13. r. $\frac{a^{2} + x^{2}}{3}$
147. l. 13. r. $\frac{a^{3}}{3} =)\ \frac{1}{3}\sqrt{5} - \frac{1}{2}$
148. l. 14. ABC *r.* ABCD
150. l. 27. *r.* Now although the Series's above would converge very ſlowly, or not at all, if it were $z = d$, or $z > d$
151. l. 14. *r.* $\frac{1}{2}z^{2}$
153. in Term 2^d of 3^d Ord. $2\mu n$ r. μn
165. l. 30. r. $-\overline{\theta + \lambda n}$
168. l. 8. R^{λ} *r.* $R^{\lambda-1}$ Ibid. l. 15. r. pg
Ibid. l. penult & p. 169. l. 5. r. $\overline{-\theta + 2\lambda n} \times a$
Ibid. l. 10. fq r. fg
170. l. 14. Before *from* place a Colon
173. l. 32. r. $z^{\theta + \sigma n + n - 1}$
178. l. 5. r. $x^{\frac{r\theta - r}{n}}$
180. l. 4. r. $\overline{g + fx^{n}}^{\lambda} \times \overline{l + kx^{n}}^{\mu}$
181. l. 7. r. $\overline{k + lv^{n} + mv^{2n} + \&c.}^{-1}$
182. l. 2. *r.* firſt, third and fourth
183. l. 29. r r. v
184. l. 8. *r.* $\dot{R}$
186. l. 31. r. gx^{2r}
193. l. 16. of *r.* for
196. l. ult. after negative, *r.* (+1 only excepted)
199. l. 8. Species *r.* Form
201. on the Margin oppoſite to Art. 274. *r.* Form 1
204. l. penult *r.* $\frac{4de}{nf} \times \overline{\frac{v^{3}}{2ex} + s}$

ERRATA.

p. 208. l. 13. r. $-aDB$
209. l. 10. r. $+2ngz^n x$
216. l. 11. r. $\sqrt{\frac{1}{4}pt - \frac{p}{t}x^2}$
220. l. 4. r. $\frac{d}{ne} \times \overline{2xv - 3s}$
222. l. penult. r. $z^{n-\theta}$
230. l. 24. r. will be an Aſſymptote
Ibid. l. 25. are r. and
231. l. 30. AO r. IO
233. l. 14, 15, 17. JOHN r. JONH
Ibid. l. 21. r. $\dot{z} : \frac{F.JONH}{HN}$
Ibid. l. 22. r. $\frac{F.IONH}{HN} = \dot{p}$
234. l. 36. $\ddot{z}$ r. $\dot{z}$
235. l. 4. x r. z
236. l. 7. r. the loweſt Order
240. l. 1. r. $\overline{\frac{1}{2}xv - s} = t$
242. l. 1. r. $\frac{AB \times BC}{2}$
243. l. 11, 13. BC r. AC
244. l. 34, 36. r. AMEGA
247. l. 12. r. the equilateral Parallelogram
248. l. 47. r. the Fluxion of the Logarithm
249. l. 20. DG, Dg, AG, Ag, r. DE, De, AE, Ae
Ibid. l. 31. DG r. DE, and l. 32 Dg. r. De
250. l. 4. r. $\frac{1.2}{0.8} \times \frac{1.2}{0.9} = \frac{2}{1}$
253. l. 12, 14. Aac r. ABE
Ibid. l. 22. AO^q: r. AO^q
Ibid. l. 27. r. △ ABE
254. l. 26. r. and m the meaſure of
256. l. 17. r. by a logarithmical
Ibid. l. 22. r. is to the
258. l. 25. DB r. BQ
261. in the Fig. e r. o
Ibid. l. 33. r. $\sqrt{AP^q - aq^q}$
265. l. 13. r. $\frac{AT \times BD}{2}$
Ibid. l. 14, 15. r. $\sqrt{e - fz^n}$ Ibid. fx^n r. fz^n
267. l. 18. r. $\overline{DG - DH}$
Ibid. l. 20. r. $\sqrt{fz^n} = AG$.
Ibid. l. 21. r. $DH = \sqrt{-e + fz^n}$
Ibid. l. 27. $\frac{1}{z^n}$ r. $\frac{1}{\sqrt{z^n}}$
269. l. 1. in the Denominator r. $\overline{d-2}$
270. l. 11. in the Denominator r. $\overline{d-1}$
274. l. 5. r. $\frac{z-3}{4}$
Ibid. l. 13. r. Difference of z^n
Ibid. l. 18. r. In which Caſe a Difference of any Order is the ſame with the firſt Term of that Difference
277. l. 9. r. when the Fluxions

p. 280. l. 35. or r. and
283. l. 16. 3 r. 2 Ibid. l. 18. 2 r. 3
284. l. 26. r. any impoſſible
285. l. 6. r. $2ap - 2px$
Ibid. l. 7. r. $-2ap + 2px$
288. l. 16. r. the Chord of the Arch GH meaſuring the Diſtance of the Vertex of the Segment from the Circumference of its Baſe
289. l. 6. r. $3 = 0$ Ibid. l. 27. C r. G
290. l. 6. r. Ordinate BC
291. l. 15. x^2 r. a^2
292. l. 25. Point T. r. Line BT.
293. l. 4. BC r. BC r. BT
294. l. 32. SE r. SG
295. l. 9. r. $may^{m-1}\dot{y}$
296. l. 31. y^2 r. $\dot{y}^2$
Ibid. in the Fig. D r. γ
298. l. 20. z r. x Ibid. l. 26. r. $A\beta = \xi$
301. l. 1. after *you* r. *have*
304. l. penult r. $-\frac{p}{t}x^2$
305. l. 23. $\frac{1}{2}\sqrt{kl}$ r. $\frac{1}{4}\sqrt{kl}$
308. l. 29. and r. about
309. l. 10. r. revolving Parabola.
Ibid. l. 16. GB r. Gβ Ibid. l. 30 Aβ r. Gβ
314. l. 17. thence r. then
315. l. 20. r. than an
316. l. 24. r. $\frac{aav}{av + vv}$
317. l. 13. in this and the three next Pages the Numbers of the Articles of Quad. of Curves on the Margin to be encreaſed by 1.
Ibid. l. 31. for r. from.

In the *Analyſis, &c.* and *Commentary.*

322. l. 3. r. $\frac{1}{x^2} (= x^{-2})$
323. in the Fig. at the End of the Line AB place α
224. l. 5. r. $= \beta BD\delta$
327. l. 6. r. $-\frac{1}{28}x^3$
Ibid. in the Note r. 406 — 408
329. l. 5. $+2y$ r. $-2y$.
331. l. 1. $4a^2e$ r. $4a^2q$
339. l. 17. r. BD = BK
Ibid. l. 33. r. Quadratrix
341. l. 10. r. $3x^2o + 3xo^2$
346. l. 18. r. this Principle
357. l. 26. r. effected
377. l. 16. CD r. BD
378. l. 4. r. AC the half Tranſverſe and AF the half Conjugate
Ibid. l. 5. AF r. AC, and l. 6. AC r. AF.
Ibid. l. 16. r. Treatiſe Art. 406——415
379. l. 3. after the Extraction of the Root, $\frac{1}{8}ax^2$ r. $\frac{1}{2}ax^2$
380. l. 8. $+P^{\frac{m}{n}}$ r. $= P^{\frac{m}{n}}$

ERRATA.

p. 381. l. 11. *r.* $-\frac{5x^8}{128c^7}$

382. l. 7. r. $\overline{\frac{cx^4-x^5}{c^5}}\Big|^2$, $\overline{\frac{cx^4-x^5}{c^5}}\Big|^3$; $\overline{\frac{cx^4-x^5}{c^5}}\Big|^4$

383. l. 2. *r.* Series. In order to which you may extract the Square Root of the Numerator and Denominator, and then &c.

Ibid. l. 13. $\frac{bx^2}{a}$ r. $\frac{bx^2}{aa}$

384. l. 8. $-5r$ r. $-5r^8$

387. l. 12. *r.* a positive Integer

Ibid. l. ult. dele *in*

388. l. 4. $\frac{dn}{f}$ r. $\frac{d\eta}{f}$ Ibid. l. 5. fz^n r. fz^n

Ibid. l. 7. r. $\frac{k-\eta+1}{\eta}$ Ibid. l. 8. r. fz^n

396. at the Bottom *dele* Values

401. l. 22. a, $\frac{1}{4}x$, r. $+a$, $-\frac{1}{4}x$

402. l. 22. after q, r. &c.

404. l. 30. $\frac{3x^2}{r}$ r. $\frac{3x^2r}{32}$

406. l. antepen. *r.* $+\frac{15}{4096}\frac{x^4}{a}$

p. Ibid. l. penult and p. 407. l. 3. *r.* $-\frac{15}{4096}\frac{x^4}{a}$

407. l. 12. *r.* $+4c^5p$

418. l. ult. whence *r.* when

420. l. 25. dele cx^2

422. l. 8. Indexes *r.* Differences of the Indexes Ibid. l. 9. be *r.* being

423. l. 3. r. $y = +\sqrt[4]{7} \times \sqrt{ax}$

427. l. 15. *r.* $+b^{\frac{1}{3}}$

429. l. 10. r. $r' + ns'$ and $r'' + ns''$

434. l. 6. Sides *r.* Side

438. l. 32. *r.* $-z^6$

439. l. 19. *dele* which do not include x represented here by ay^m; and another one at least

441. l. ult. *r.* the Series

445. l. 2. *r.* to be such, that, after Substitution of that Power of x, &c.

449. l. 5. *r.* next to one another

466. l. 27. *r.* $-\frac{80a^3}{x}$

ADVERTISEMENT to the READER.

In revising the Sheets after they were printed off, for discovering the Errors of the Press, I found I had been guilty of some Mistakes, which I thought proper to point out to the Reader, in this Place, with the Corrections; and a few other Additions or Illustrations.

The Reader may be pleased to observe that by the Phrase *Instant of Time* made use of in the following Work, is not meant any Portion of Time, or Part of Duration, how small soever, which must be divisible into Parts, like all other kinds of Quantity; but an Instant of Time must be conceived as having no Parts, and to be with respect to any Portion of Time or Duration, what a Point is with Respect to a Line in Geometry.

Because Lemma 4. Page 41. may be thought not so clearly demonstrated, the following Demonstration may be substituted for the other.

Let V denote any given intermediate Velocity, betwixt M and N any two variable Velocities assumed, through which two Velocities, the variable Velocity, mentioned in this Lemma, passes; so that M be upon the one Side of V, and N upon the other. Imagine M and N to approach continually towards one another, and towards V, which always lies betwixt them: the Time which the variable Velocity takes in passing from M to N, may be so small, by the continual Approach of M and N, as to be less than any given Time: by which Means M and N shall approach so near to one another, as to differ from one another (and consequently from V) by less than any given Difference (as appears from the Nature of the variable Velocity mentioned in this Lemma.) Therefore since M and N approach continually to V, during a finite Time, so as before the End of that Time, to differ from it by less than any given Difference, they must at last be equal to it (by Art. 7.) But V was taken any intermediate Velocity whatsoever. Therefore, &c. as in the Lemma. Q. E. D.

Page 103. Art. 134. The Reader may observe that the first Paragragh of this Article respects the Case in which $\frac{fz^n}{e}$ is less than 1. In which Case there are two Sppositions made; either that s is greater than $r+1$; or less. Now because upon this second Supposition, the Demonstration for proving that the Series there mentioned, must converge, may not appear so plain; it may be more clearly demonstrated thus.

Again if it be $s > r+1$, the Quantity $\frac{s+n}{r+n+1}$ must, at a finite Distance from the Beginning of the Series, become a positive Quantity greater than 1; and after that Time continually diminish, since the Difference betwixt $s+n$ and $r+n+1$, is a given Number: so that the Ratio

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of $s+n$ to $r+n+1$ approaches nearer to the Ratio of Equality than any given Ratio of greater to less, that is than the Ratio of e to fz^n: therefore $\frac{s+n}{r+n+1} < \frac{e}{fz^n}$: whence $\frac{s+n}{r+n+1} \times \frac{fz^n}{e} < 1$: and therefore the Terms of the Series will continually diminish, and that still in a greater Ratio as the Terms advance, since $\frac{s+n}{r+n+1}$ continually diminishes: and therefore the Series must converge.

Page 104. last Paragraph Art 134. begins thus: "There is a 3[d] Case, &c. In which Case the "Convergency depends entirely upon the Coefficients:" whereas it should be said that it depends upon the Coefficients and the Signs of the Terms. Again where it is said, "If $s < r+1$, "both the Series's for the Area will converge," read, One, or both the Series's for the Area will converge, *viz* when the Signs of the Terms become alternate; otherwise not:

Page 299. Art. 410. An Overlight having been committed in the Operation in this Article, the Length of the Curve Line is not rightly expressed. Wherefore let that Article run thus.

If we would have the Length of the Curve Line expressed by Logarithms, agreeably to what hath been delivered in §. 10. then (see Fig. Art. 355) which substitute in Place of Fig. 3. Tab 2. Quad. of Curves, supposing the Hyperbola described with the same Latus Transversum and Rectum as before: then it will be $z = \frac{3}{2}\frac{\triangle ABD}{a} - \frac{\frac{1}{2}aA \times aH}{a} \Big| \frac{aH}{DC}$: for Example, suppose in the given Parabola, the Length of the Absciss AB (x) $= a = 1$, then in the Hyperbola, by taking $aB = (AB^{\frac{2}{3}} =)\ 1$, you'll find $AB = \frac{43}{32}$; and from the Property of the Hyperbola, and similar Triangles, $aH = \frac{3}{8}$, $BC = \frac{43}{24}$, $BD = \frac{5}{3}$, $DC = \frac{1}{8}$: therefore the Triangle $ABD = \left(\frac{AB \times BD}{2} =\right) \frac{215}{192}$; and the Triangle AaH (which is the Modulus) or $\frac{Aa \times aH}{2} = \frac{27}{512}$: whence you'll have $\triangle ABD - \frac{\frac{1}{2}Aa \times aH}{a} \Big| \frac{aH}{DC} = \frac{215}{192} - \frac{27}{512} \Big| \frac{3}{1}$, that is to say the Measure of the Ratio of 3 to 1 to the Modulus $\frac{27}{512}$ subtracted from $\frac{215}{192}$. But whereas the Modulus of the Tabular Logarithms is 0.434294 &c. (see Art. 349.) therefore the same Quantity expressed by Means of the Tab Log. will be (see Art. 350) $\frac{215}{192} - \frac{27}{512 \times 0.434294 \,\&c.} \times \text{Log.}\ \frac{3}{1} = \frac{215}{192} - \frac{27}{512 \times 0.434294 \,\&c.} \times 0.477122\ \&c.$, $=$ (by reducing all to Decimals) 1.0816293 nearly. Wherefore if you multiply this last Number by $\frac{3}{2}$, you shall have 1.622444 *p. min.* for the Length of the Curve Line AC required: which, by Means of a Table of Logarithms of more Places of Figures, may be carried to a proportionably greater Number of Figures.

Page 439. Art. 120. near the Beginning: it is said that in the general Equation there mentioned, there will be one Term at least which doth not include x; and *another* one at least that includes not y. Which is an Oversight: for it is true that there will be one Term at least that doth not include x: since if all the Terms of the proposed Equation do involve x, you may always free one Term of x by dividing by the lowest Dimension of x: and for the like Reason there will be one Term at least that doth not include y. But it will not always happen that these Terms are distinct: for the same Term that doth not include x, may be the only one likewise that doth not include y. However this Oversight makes no Alteration in the Reasoning there used. But it may be corrected by making it run thus: *in which there will be one Term at least which doth not include* y, *which is here represented by* hx^v, omitting what is in the middle.

Page 445. Art. 123. not far from the Beginning of the Article, there is a Mistake in supposing the Letter D to represent the Exponent of the Power of x contained in the first Term of the Value of p; and then, as a Consequence of that and what was supposed before that, making A+D, B+D, E+D, &c. to represent the Dimensions of x in the penult Term. Because the Dimensions of x in the penult Term will not exceed these in the ultimate by so great a Difference as is the Dimension of x in the first Term of the Value of p, as appears by Rem. 3. Art. 120. preceding. Yet this Mistake doth not affect the Reasoning: and it may be rectified thus.

Instead of saying, "Then suppose the Exponent of the Power of x, which is contained in "the first Term of the Value of p, found as above, to be D: so that after Substitution of such "Power of x, &c." read thus, *Then suppose the Exponent of the Power of* x, *which is contained in the first Term of the Value of* p, *found as above, to be such, that, after Substitution of that Power of* x &c. as it was formerly. And then the Reasoning proceeds and the Symbols answer.