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AN

ELEMENTARY TREATISE

ON

SOLID GEOMETRY

BY

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PREFACE TO THE FIRST EDITION.

THE present work is intended as an introductory text-boo for the use of Students reading for the Mathematica Tripos. Many of the higher applications of the subje are therefore either omitted entirely or treated very briefl At the same time the Author believes that the book in cludes as much as the great majority of Cambridge Studen have time to master thoroughly, while those who ar desirous of making farther acquaintance with the subje will perhaps find a work like the present not unsuitable a an introduction to the more complete treatises of Salmo and others.

The Author begs to thank those of his friends who hav kindly assisted him by revising the manuscript and proo sheets, and will feel obliged to any one who will offer corre tions or improvements.

Examples, selected chiefly from recent College and Uni-ersity Examination Papers, will be found at the end of each hapter.

CAMBRIDGE, *August*, 1865.

SECOND EDITION.

The present Edition has been revised and re-arranged ıd somewhat enlarged.

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THIRD EDITION.

The Third Edition has been revised and farther ılarged, chiefly by the addition of hints for the solution of ıe Examples.

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SOLID GEOMETRY.

CHAPTER I.

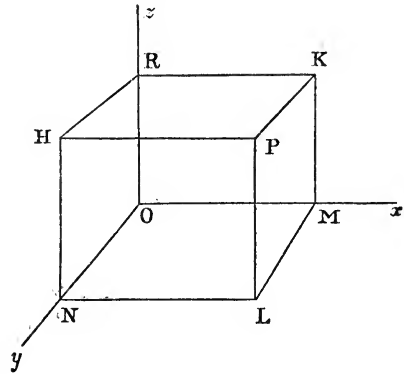
INTRODUCTORY THEOREMS.

1. THE position of a point in space is usually determined by referring it to three planes meeting in a point. This point is called the origin, the three planes the co-ordinate planes, and their three lines of intersection the co-ordinate axes. The point of intersection of the three planes is usually designated by the letter O, and their lines of intersection by the letters Ox, Oy, Oz. They are called the axes of x, y, and z respectively, and the planes yOz, zOx, xOy are called the planes of yz, zx, xy respectively. If the three planes of yz, zx, xy, and consequently the three lines Ox, Oy, Oz, are at right angles to each other, the co-ordinates are said to be rectangular, and in all other cases oblique. We shall generally make use of rectangular co-ordinates, but in some cases the proofs and the results obtained will hold good equally whether the axes be at right angles or not.

2. The position of any point P relatively to these three planes is known, if its distance from each, measured parallel to the intersection of the other two, be known.

For let PH, PK, PL be drawn through P parallel to Ox, Oy, Oz respectively to meet the planes of yz, zx, xy in H, K, L; and let a plane through PL, PK, which by Euclid, XI. 15, is parallel to the plane of yz, meet Ox in M. Let also a plane through PH, PL meet Oy in N, and a plane through PH, PK meet Oz in R. Then if KR, KM be joined, $KMOR$ is obviously a parallelogram, and KR therefore equal to OM. Similarly $RKPH$ is a parallelogram, and KR equal to PH.

Hence PH is equal to OM, and similarly PL to OR, PK to ON. If therefore we measure off from Ox, Oy, Oz, respectively, lengths OM, ON, OR equal to the given distances of P from the co-ordinate planes, and through M, N, R draw



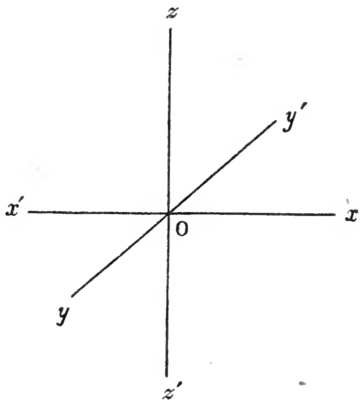
planes parallel to those of yz, zx, xy, these planes will intersect in P, the position of which is therefore determined. The lengths PH, PK, PL, or OM, ON, OR, which are equal to them, are called the co-ordinates of P, and are usually denoted by the letters x, y, z.

3. If the line xO be produced through O to x', and from Ox' we cut off a length OM' equal to OM, and repeat the preceding construction, we obtain a point P' whose absolute distances from the three co-ordinate planes are the same as those of P. We must therefore have some convention to enable us to distinguish between these two points. The following is usually adopted.

The co-ordinates are considered positive if measured in one direction along the axes from O, and negative if measured in the opposite.

The positive directions for the three axes are usually taken to be those represented in the figure by Ox, Oy, Oz, and the negative directions to be Ox', Oy', Oz'.

It will be seen that the whole of space is divided by the co-ordinate planes into eight compartments, and the signs of the co-ordinates of any point indicate in which of these compartments it is situated, while their absolute magnitudes indicate its position in that compartment. Thus the co-ordi-



nates of a point whose absolute distances from the co-ordinate planes are α, β, γ are represented by (α, β, γ), $(-\alpha, \beta, \gamma)$, $(\alpha, -\beta, \gamma)$, $(\alpha, \beta, -\gamma)$, $(\alpha, -\beta, -\gamma)$, $(-\alpha, \beta, -\gamma)$, $(-\alpha, -\beta, \gamma)$, $(-\alpha, -\beta, -\gamma)$, according as the point lies in the compartment $Oxyz$, $Ox'yz$, $Oxy'z$, $Oxyz'$, $Oxy'z'$, $Ox'yz'$, $Ox'y'z$, $Ox'y'z'$, respectively.

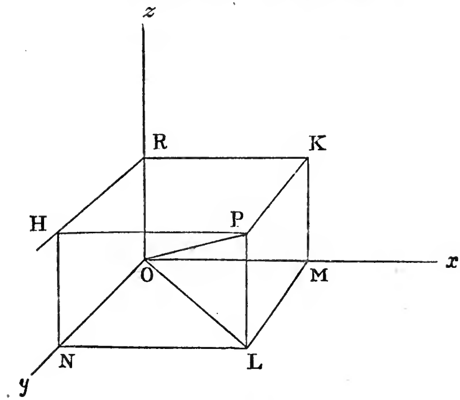
4. *To find the distance of a point from the origin in terms of its co-ordinates.*

In this and Articles 5, 6 and 8 the co-ordinates are supposed rectangular.

Let P be the point, x, y, z its co-ordinates. Through P draw planes parallel to the co-ordinate planes and forming with them a parallelepiped of which OP is the diagonal and PL the edge through P parallel to Oz.

Join OP and OL. Then since PL is parallel to Oz which is perpendicular to the plane of xy, PL is perpendicular to the plane of xy, and therefore to the line OL which lies in that plane. (Euclid, XI. Def. 3.)

Hence $$OP^2 = OL^2 + PL^2.$$



But $$OL^2 = OM^2 + ML^2;$$

$$\therefore\ OP^2 = OM^2 + ML^2 + PL^2 = x^2 + y^2 + z^2 \ldots\ldots(1).$$

5. Let α, β, γ be the angles between OP and the axes of x, y, z respectively. Join PM. Then since Ox is perpendicular to the plane PLM, it is perpendicular to PM.

Hence

$$\left.\begin{array}{l} OM = OP\cos POM = OP\cos\alpha;\ \text{or}\ x = r\cos\alpha \\ \text{Similarly,} \\ ON = OP\cos PON = OP\cos\beta;\ \text{or}\ y = r\cos\beta \\ OR = OP\cos POR = OP\cos\gamma;\ \text{or}\ z = r\cos\gamma \end{array}\right\}\ldots(2),$$

x, y, z being the co-ordinates of P, and OP being denoted by r.

Squaring and adding, we get

$$OM^2 + ON^2 + OR^2 = OP^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma),$$

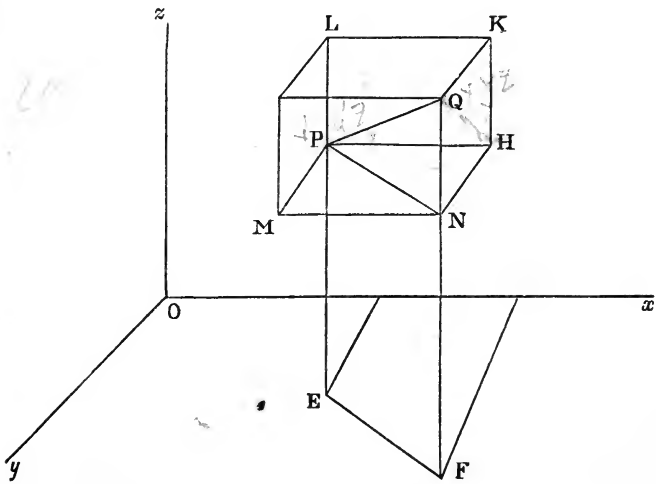
or taking account of (1),

$$1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma \ldots\ldots\ldots\ldots\ldots(3).$$

The letters l, m, n are frequently used to denote $\cos\alpha$, $\cos\beta$, $\cos\gamma$, which are called the direction-cosines of the line OP. It is usual to denote by α, β, γ the angles which OP makes with the *positive* directions of the axes, in which case the formulæ (2) hold for all positions of the point P.

6. *To find the distance between two points whose co-ordinates are given.*

Let P and Q be the two points; x_1, y_1, z_1; x_2, y_2, z_2 their co-ordinates. Join PQ, and through P and Q draw planes



parallel to each of the co-ordinate planes, thus forming a parallelepiped whose edges are parallel to the co-ordinate axes, and are equal in length to $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$, respectively.

As in Art. 4, we obtain

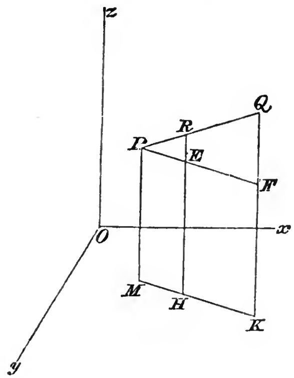
$$PQ^2 = PH^2 + HN^2 + NQ^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \quad \text{.........(4).}$$

We have also formulæ similar to those of equation (2), α, β, γ being the angles between PQ and the lines drawn through P parallel to the axes, viz.

$$\left.\begin{aligned} PH &= x_2 - x_1 = PQ \cos\alpha = lr \\ PM &= y_2 - y_1 = PQ \cos\beta = mr \\ PL &= z_2 - z_1 = PQ \cos\gamma = nr \end{aligned}\right\} \quad \text{...........(5),}$$

where r represents the length of PQ, and l, m, n are *the direction-cosines* of PQ.

7. *To find the co-ordinates of a point which divides the straight line joining two given points in a given ratio.*



Let P, Q be the two given points, and R the point in PQ which divides PQ in the given ratio of n_1 to n_2. Let x_1, y_1, z_1 be the co-ordinates of P, x_2, y_2, z_2 those of Q, x', y', z' those of R.

Draw PM, RH, QK parallel to the axis of z to meet the plane of xy in M, H, K. These points all lie in one straight line, namely that in which a plane through PQ parallel to the axis of z cuts the plane of xy. Draw PEF parallel to MHK to meet RH in E and QK in F.

Then $$PM = z_1, \quad RH = z', \quad QK = z_2.$$

Also $$\frac{RE}{QF} = \frac{PR}{PQ} = \frac{n_1}{n_1 + n_2},$$

or $$\frac{z' - z_1}{z_2 - z_1} = \frac{n_1}{n_1 + n_2},$$

whence $$z'(n_1 + n_2) = n_1 z_2 + n_2 z_1;$$

$$\therefore\ z' = \frac{n_1 z_2 + n_2 z_1}{n_1 + n_2}.$$

Similarly it may be shewn that

$$x' = \frac{n_1 x_2 + n_2 x_1}{n_1 + n_2}, \quad y' = \frac{n_1 y_2 + n_2 y_1}{n_1 + n_2}.$$

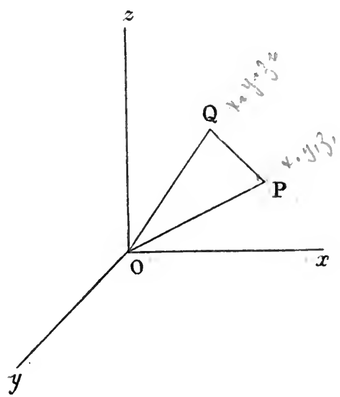
If R be the middle point of PQ, $n_1 = n_2$, and we have

$$x' = \frac{x_1 + x_2}{2}, \quad y' = \frac{y_1 + y_2}{2}, \quad z' = \frac{z_1 + z_2}{2}.$$

8. *To find the angle between two straight lines whose direction-cosines are given.*

Since by Euclid, XI. 10, the angle between any two straight lines is equal to that between any other two respectively parallel to them, we need only consider the case of two lines through the origin.

Let OP, OQ be the two lines; l, m, n the direction-cosines



of OP; l', m', n' those of OQ. Let x_1, y_1, z_1, be the co-ordinates of P any point in OP; x_2, y_2, z_2 those of Q any point in OQ.

Then by Art. (6)

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$
$$= x_2^2 + y_2^2 + z_2^2 + x_1^2 + y_1^2 + z_1^2 - 2(x_1x_2 + y_1y_2 + z_1z_2).$$

But by Art. (4)

$$x_1^2 + y_1^2 + z_1^2 = OP^2,$$
$$x_2^2 + y_2^2 + z_2^2 = OQ^2.$$

And by Art. (5)

$$x_1 = OP \, . \, l, \quad y_1 = OP \, . \, m, \quad z_1 = OP \, . \, n,$$
$$x_2 = OQ \, . \, l', \quad y_2 = OQ \, . \, m', \quad z_2 = OQ \, . \, n';$$

and $\therefore \; x_1x_2 + y_1y_2 + z_1z_2 = OP \, . \, OQ \, (ll' + mm' + nn')$.

Hence $PQ^2 = OP^2 + OQ^2 - 2OP \, . \, OQ \, (ll' + mm' + nn')$.

But by Trigonometry we have from the triangle OPQ

$$PQ^2 = OP^2 + OQ^2 - 2OP \, . \, OQ \, . \cos POQ.$$

Comparing these two expressions for PQ^2, we get

$$\cos POQ = ll' + mm' + nn' \quad \text{..............} (6).$$

The formulæ (1), (3), (4) and (6) are of very frequent use, and should be carefully remembered by the student.

From (6) we can deduce

$$\sin^2 POQ = 1 - (ll' + mm' + nn')^2$$
$$= (l^2 + m^2 + n^2)(l'^2 + m'^2 + n'^2) - (ll' + mm' + nn')^2$$
$$= (mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2.$$

9. If from the ends of a straight line PQ of limited length there be drawn perpendiculars on a fixed plane and the feet of these perpendiculars be joined by a straight line, the joining line is called the *projection of PQ on the plane.* Thus in the figure to Art. (6) if the edges LP, QN of the parallelepiped $PKQM$ be produced to meet the plane of xy in E and F, EF is the projection of PQ on the plane of xy, and is equal and parallel to PN. Also

$$PN = PQ \cos QPN.$$

But QPN is equal to the angle which PQ makes with the plane of xy. Hence we derive the theorem:

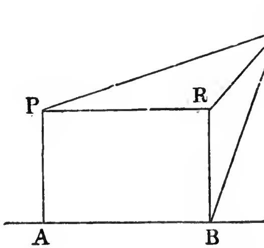
The projection of a straight line of limited length on a given plane is equal to the length of the line multiplied by the cosine of the angle between the line and plane.

10. If again from P and Q we draw perpendiculars on some fixed line, the portion of the second line intercepted between the feet of these perpendiculars is called the *projection of PQ on the fixed line,* and the following theorem holds:

The projection of a straight line of limited length on a second straight line, is equal to the length of the first line multiplied by the cosine of the angle between the two lines; understanding by the angle between two lines which do not meet, the angle between any two lines parallel to them which do meet.

This theorem is proved as follows:

Let PQ be the line of limited length, and AB the line on which it is to be projected. Through P draw PR parallel, and PA perpendicular to AB. Through Q draw a plane perpendicular to AB meeting AB in B, and PR in R. Join QR, RB, BQ. Then AB is the projection of PQ, for AB is perpendicular to QB which lies in the plane QBR. Then



since PR is parallel to AB, which is perpendicular to the plane RBQ, PR is also perpendicular to this plane and therefore perpendicular to QR and RB. Hence $PRBA$ is a parallelogram, and therefore $AB = PR$. But $PR = PQ \cos QPR$, since PRQ is a right angle.

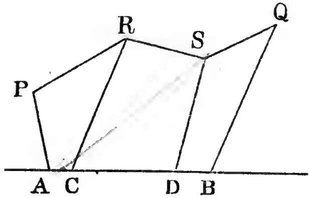
Therefore

$$AB = PQ \cos QPR,$$

the theorem required.

11. If we take any two points P, Q, and draw from P in any direction a straight line PR of any length, from R a straight line RS, and join SQ; and from P, R, S and Q draw perpendiculars PA, RC, SD, QB on AB; AC, CD and DB will be the projections of PR, RS and SQ on AB; and as long as A, C, D, B fall in the order represented in the figure, the arithmetic sum of these projections is equal to

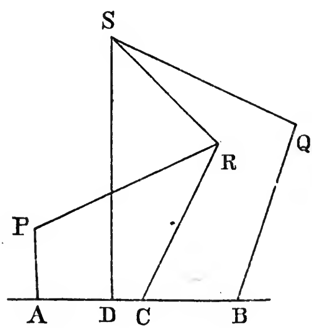
AB, the projection of PQ. The same would be true if we had taken any number of lines between P and Q. If however C fall to the right of D, or C or D fall to the right of



B or to the left of A, this will be no longer the case. We may agree to consider the projection of a line to be equal to its length multiplied by the cosine of the angle which it makes with the second line, those angles being always taken which are formed by the successive lines PR, RS, SQ with AB towards the same part. Thus if D come to the left of C, the angle between RS and AB will be obtuse, and the projection of RS will be negative. And since

$$AC - CD + DB = AB,$$

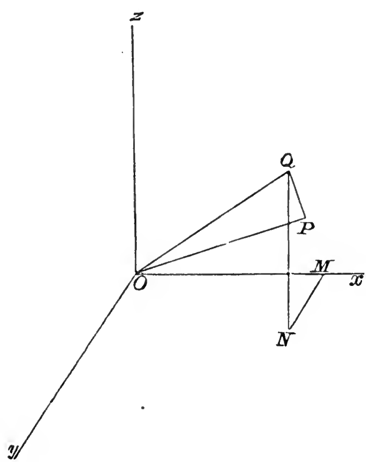
we still have the theorem that "*the algebraical sum of the projections on a given line, of a series of lines by which we*



pass from one point to a second, is equal to the projection on the same line, of the straight line joining the two points."

This statement may be illustrated thus. Suppose a point to move from P to Q along PR, RS, SQ, and from each of its successive positions imagine a perpendicular let fall on AB. As the point moves along PR, the foot of this perpendicular will move along AB from A towards B, or in the opposite direction, according as the angle between PR and AB is acute or obtuse, and the length traversed by it along AB is the projection of PR, and is positive if it travels from A towards B, and negative if in the opposite direction. It is clear that as the moving point passes from P to Q, the foot of the perpendicular will pass from A to B, and hence AB which is the projection of PQ will also be the algebraical sum of the distances travelled by the foot of the perpendicular, or of the projections of PR, RS, SQ. The same theorem will be obviously true if instead of three lines we have any number. By the angle between PR and AB is meant the angle which would be formed if from any point were drawn lines in the directions of PR and AB. Thus the angle between PR and AB is the supplement of that between RP and AB.

12. By means of the result of the last Article, another proof of the formula (6) of Art. 8 can be obtained.

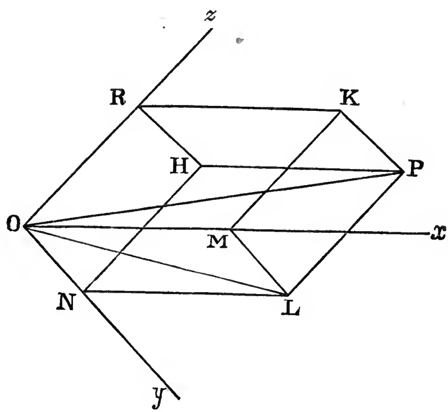


If, in the figure of that Article, QN be drawn parallel to the axis of z to meet the plane of xy in N, and NM drawn parallel to Oy to meet Ox in M, it follows that the projection of OQ on OP is equal to the sum of the projections of OM, MN and NQ on OP, that is, if θ be the angle POQ, and l, m, n; l', m', n' be the direction-cosines of OP and OQ respectively,

$$OQ \cos\theta = OM \,.\, l + MN \,.\, m + NQ \,.\, n$$
$$= OQ \,.\, l' \,.\, l + OQ \,.\, m' \,.\, m + OQ \,.\, n' \,.\, n;$$
$$\therefore \cos\theta = ll' + mm' + nn'.$$

13. *To find the distance of a point from the origin when the co-ordinates are oblique.*

The formulæ of Arts. 4, 5, 6 and 8 were obtained on the supposition of rectangular co-ordinates. Let Ox, Oy, Oz be oblique axes, and P any point. Through P draw planes parallel to the co-ordinate planes to meet the axes in M, N, R; and join OP. The ratios of OM, ON and OR to OP



will be clearly the same whatever be the position of P, provided it lie in the same straight line through O. These ratios are called the *direction-ratios* of the line OP, and are usually denoted by the letters l, m, n. We then get formulæ corresponding to those of Art. (5),

$$x = l \,.\, OP, \quad y = m \,.\, OP, \quad z = n \,.\, OP.$$

Again, let λ, μ, ν be the angles between $(Oy, Oz), (Oz, Ox), (Ox, Oy)$. Then we have, if PL be the edge of the parallelepiped through P parallel to Oz,

$$OL^2 = OM^2 + ML^2 - 2\,OM\,.\,ML \cos OML$$
$$= x^2 + y^2 + 2xy \cos \nu.$$

And $$OP^2 = OL^2 + PL^2 - 2OL\,.\,PL \cos OLP.$$

But the projection of OL on OR is equal to the sum of the projections of OM and ML on OR, or by Art. 9,

$$OL \cos ROL = OM \cos \mu + ML \cos \lambda = - OL \cos OLP;$$

and therefore

$$OP^2 = x^2 + y^2 + z^2 + 2yz \cos \lambda + 2zx \cos \mu + 2xy \cos \nu.$$

Combining this with the formulæ $x = l\,.\,OP$, $y = m\,.\,OP$, $z = n\,.\,OP$, we get

$$1 = l^2 + m^2 + n^2 + 2mn \cos \lambda + 2nl \cos \mu + 2lm \cos \nu \ldots (1),$$

the relation which holds between the direction-ratios of any straight line.

In the same manner we could shew that the distance between two points $x_1, y_1, z_1;\ x_2, y_2, z_2$ is

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + 2(y_1 - y_2)(z_1 - z_2) \cos \lambda$$
$$+ 2(z_1 - z_2)(x_1 - x_2) \cos \mu + 2(x_1 - x_2)(y_1 - y_2) \cos \nu:$$

And as in (8) that the cosine of the angle between two lines whose direction-ratios are $l, m, n;\ l', m', n'$ is

$$ll' + mm' + nn' + (mn' + m'n) \cos \lambda$$
$$+ (nl' + n'l) \cos \mu + (lm' + l'm) \cos \nu \ldots (2).$$

14. The volume of the parallelepiped of which OP is the diagonal is evidently equal to the product of the area of the parallelogram $OMLN$ into the perpendicular from R on the plane of xy. If θ be the angle between OR and a line perpendicular to the plane of xy, this volume would equal

$$OM\,.\,ON \sin \nu \times OR \cos \theta$$
$$= xyz\,.\,\sin \nu\,.\,\cos \theta.$$

But if l', m', n' be the direction-ratios of the line through O perpendicular to the plane of xy, since it is perpendicular to Ox and Oy whose direction-ratios are (1, 0, 0), (0, 1, 0) respectively, we have, by formula (2) of the last Article,

$$l' + m' \cos \nu + n' \cos \mu = 0 \quad \text{.............. (1)},$$

$$l' \cos \nu + m' + n' \cos \lambda = 0 \quad \text{.............. (2)}.$$

And since it makes an angle θ with Oz whose direction-ratios are (0, 0, 1) we have

$$n' + l' \cos \mu + m' \cos \lambda = \cos \theta \quad \text{........... (3)}.$$

From these, since by formula (1) of the last Article

$$l' (l' + m' \cos \nu + n' \cos \mu) + m' (m' + n' \cos \lambda + l' \cos \nu)$$
$$+ n' (n' + l' \cos \mu + m' \cos \lambda) = l'^2 + m'^2 + n'^2$$
$$+ 2m'n' \cos \lambda + 2n'l' \cos \mu + 2l'm' \cos \nu = 1,$$

we have $$n' \cos \theta = 1 \quad \text{...................... (4)}.$$

And from (1) and (2) we have

$$\frac{l'}{\cos \mu - \cos \lambda \cos \nu} = \frac{m'}{\cos \lambda - \cos \mu \cos \nu} = \frac{n'}{\cos^2 \nu - 1}$$

$$= \frac{\cos \theta}{\cos^2 \lambda + \cos^2 \mu + \cos^2 \nu - 2 \cos \lambda \cos \mu \cos \nu - 1} \text{ by (3)},$$

whence we get

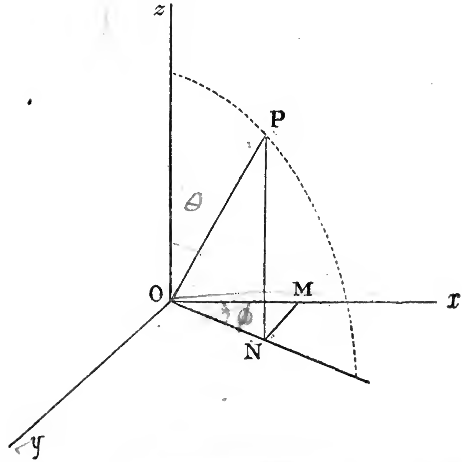
$$\cos^2 \theta \sin^2 \nu = 1 - \cos^2 \lambda - \cos^2 \mu - \cos^2 \nu + 2 \cos \lambda \cos \mu \cos \nu.$$

And the volume of the parallelepiped becomes

$$xyz \sqrt{1 - \cos^2 \lambda - \cos^2 \mu - \cos^2 \nu + 2 \cos \lambda \,.\, \cos \mu \,.\, \cos \nu}.$$

The volume of the tetrahedron cut off from the co-ordinate axes by a plane through R, M, N, is evidently one-sixth of the above expression.

15. The position of a point in space is sometimes determined by means of *polar co-ordinates*. Thus if Ox, Oy, Oz be rectangular axes and P any point, the position of P is clearly determined if we know OP the distance of P from the origin; the angle POz which OP makes with a fixed line the axis of z; and thirdly, the angle between the plane through OP and Oz and some fixed plane through Oz, as the



plane of zx. These are called the polar co-ordinates of P and are usually denoted by the letters r, θ, ϕ. They are connected with the rectangular co-ordinates of P referred to the axes Ox, Oy, Oz by very simple relations which can be obtained thus. Draw PN parallel to Oz to meet the plane of xy in N, and NM parallel to Oy to meet Ox in M. Join ON.

Then

$$x = OM = ON \cos\phi = OP \sin\theta \cos\phi = r \sin\theta \cos\phi,$$

$$y = MN = ON \sin\phi = OP \sin\theta \sin\phi = r \sin\theta \sin\phi,$$

$$z = PN = OP \cos\theta = r \cos\theta,$$

from which we can obtain the equivalent system

$$r^2 = x^2 + y^2 + z^2,$$

$$\tan\theta = \frac{\sqrt{x^2 + y^2}}{z},$$

$$\tan\phi = \frac{y}{x};$$

which give r, θ, ϕ in terms of x, y, z.

EXAMPLES. CHAPTER I.

1. Find the distances between each pair of the points whose co-ordinates are (1, 2, 3), (2, 3, 4), (3, 4, 5) respectively.

2. Prove that the triangle formed by joining the three points whose co-ordinates are (1, 2, 3), (2, 3, 1), (3, 1, 2) respectively is an equilateral triangle.

3. The direction-cosines of a straight line are proportional to 1, 2, 3; find their values.

4. The direction-cosines of a straight line are proportional to 2, 3 and 6; find their values. Find also the angle between this line and that in question (3).

5. Find the angle between two straight lines whose direction-cosines are proportional to 1, 2, 3 and (5, -4, 1) respectively.

6. A, B, C are three points on the axes of x, y, z respectively; if $OA=a$, $OB=b$, $OC=c$, find the co-ordinates of the middle points of AB, BC and CA respectively.

7. In the last question find the co-ordinates of the centre of gravity of the triangle ABC and the distances of this point from A, B, C respectively.

8. Shew that if D, E be the middle points of BC, CA in the last question, $DE=\frac{1}{2}BC$.

9. Find the distance between two points in terms of their polar co-ordinates.

10. The co-ordinates of a point are $(\sqrt{3}, 1, 2\sqrt{3})$; find its polar co-ordinates.

11. The polar co-ordinates of a point are $\left(4, \frac{\pi}{6}, \frac{\pi}{3}\right)$; find its rectangular co-ordinates.

CHAPTER II.

THE STRAIGHT LINE AND PLANE.

16. BEFORE proceeding to find the equations of the straight line and plane, we must examine the nature of the locus represented by an equation of the form

$$F(x, y, z) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Solving with respect to z we obtain

$$z = f(x, y),$$

where z may have one or more values for each set of values of x and y. Hence if we take any point in the plane of xy whose co-ordinates are a, b we get one or more values of z, that is, the straight line drawn through the point (a, b) parallel to the axis of z will meet the locus in one or more definite points. Hence the equation (1) must represent a surface and not a solid figure.

Two equations

$$F_1(x, y, z) = 0,$$
$$F_2(x, y, z) = 0,$$

considered as simultaneous will be satisfied by the co-ordinates of all the points of intersection of the two surfaces

$$F_1(x, y, z) = 0,$$
$$F_2(x, y, z) = 0,$$

that is, will represent a line.

The simplest line with which we are acquainted is the straight line, and the simplest surface the plane. It would perhaps be more logical to find the equation of the plane first, and then, since any two planes intersect in a straight line, the equations of two planes considered as simultaneous would represent a straight line. The equations of a straight line can however be obtained most simply without reference to that of a plane, and we shall therefore invert the apparently natural order.

17. *To find the equations of a straight line.*

Let l, m, n be the direction-cosines of the straight line, α, β, γ the co-ordinates of some fixed point in it, and x, y, z those of any other point in it. Also let r be the distance between these points. Then by Art. (6) we have

$$x-\alpha=lr,\quad y-\beta=mr,\quad z-\gamma=nr,$$

or

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r\text{.................}(1).$$

These are the symmetrical equations of a straight line. If A, B, C be any quantities which are proportional to l, m, n, we can replace these equations by

$$\frac{x-\alpha}{A}=\frac{y-\beta}{B}=\frac{z-\gamma}{C}\text{ .... }(2),$$

but these fractions are no longer equal to r. Conversely any equations of the form (2) represent a straight line whose direction-cosines are proportional to A, B, C. The values of these direction-cosines can be found; for supposing them to be l, m, n, we have

$$\frac{l}{A}=\frac{m}{B}=\frac{n}{C}=\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{A^2+B^2+C^2}}=\frac{1}{\sqrt{A^2+B^2+C^2}}.$$

The equations (2) can be also written thus:

$$y=\frac{B}{A}x+\left(\beta-\frac{B}{A}\alpha\right),$$

$$z=\frac{C}{A}x+\left(\gamma-\frac{C}{A}\alpha\right).$$

Or writing

$$\frac{B}{A}=m,\ \beta-\frac{B}{A}\alpha=p,\ \frac{C}{A}=n,\ \gamma-\frac{C}{A}\alpha=q,$$

$$\left.\begin{aligned} y &= mx + p \\ z &= nx + q \end{aligned}\right\} \quad\text{.........................(3)},$$

which are the simplest forms of the equations of a straight line, and useful in many cases. The student is however advised especially to attend to the forms (1) and (2).

The equations in (3) are those of planes drawn through the line parallel to the axes of z and y respectively, the intersections of which with the planes of xy and zx are the projections of the given line on those planes. (Art. 19.)

18. *To find the equations of a straight line passing through two given points.*

Let α, β, γ; α', β', γ' be the co-ordinates of the two given points.

By the last article the equations of any straight line through (α, β, γ) can be written in the form

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad\text{..................(1)}.$$

But if the line also pass through the point $(\alpha', \beta', \gamma')$ we must have

$$\frac{\alpha'-\alpha}{l} = \frac{\beta'-\beta}{m} = \frac{\gamma'-\gamma}{n} \quad\text{..................(2)}.$$

Dividing each member of (1) by the corresponding member of (2), we get as the equations required

$$\frac{x-\alpha}{\alpha'-\alpha} = \frac{y-\beta}{\beta'-\beta} = \frac{z-\gamma}{\gamma'-\gamma}.$$

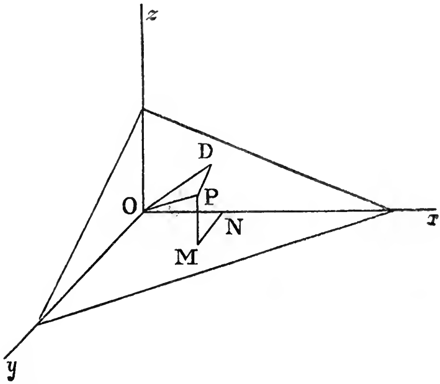
19. *To find the equation of a plane.*

Let OD be drawn perpendicular on the plane from the origin, and let the length of OD be p, and l, m, n its direction-cosines. Let P be any point in the plane. Then since OD is perpendicular to the plane it is perpendicular to PD. Hence OD is the projection of OP on OD.

Draw PM parallel to Oz to meet the plane of xy in M, and MN parallel to Oy to meet Ox in N. Then the projection of OP on OD is the sum of the projections of ON, NM

and MP on OD. But these are lx, my, nz, respectively, and the projection of OP on OD is p. Hence

$$lx + my + nz = p \text{.....................}(1);$$



a relation which is satisfied by the co-ordinates of any point in the plane, and therefore the equation of the plane.

If the plane is perpendicular to one of the co-ordinate planes, as for instance that of xy, OD will lie in that plane, and we have $n = 0$. Hence the equation in that case becomes

$$lx + my = p \text{.......................}(2),$$

and does not contain the variable z.

If the plane is perpendicular to two of the co-ordinate planes, as those of xy and zx, $l = 1$, $m = 0$, $n = 0$, and the equation becomes

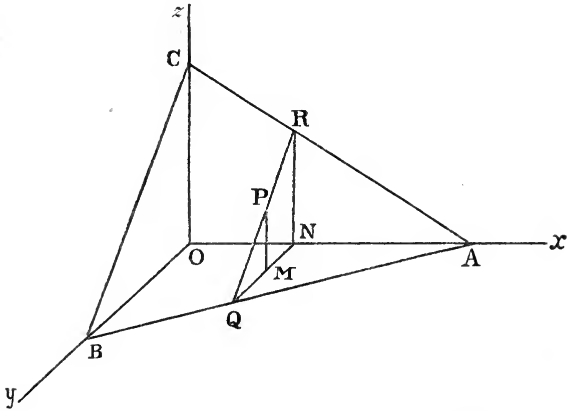
$$x = p \text{..........................}(3).$$

These results are geometrically evident.

20. *To find the equation of the plane in terms of its intercepts on the axes.*

This can be deduced from the equation (1) in the last article, but may also be obtained independently thus.

Let the plane cut the axes in A, B, C; and let any plane parallel to that of yz cut the co-ordinate planes of zx, xy in



the lines RN, NQ, and the given plane in RQ. Let P be any point in RQ and therefore *any point* in the plane. Then by Euclid, XI. 16, the lines RN, NQ, QR are parallel to the lines CO, OB and BC, respectively. Draw PM parallel to RN to meet QN in M.

Let $ON = x$, $NM = y$, $MP = z$, $OA = a$, $OB = b$, $OC = c$.

Then by similar triangles

$$\frac{PM}{RN} = \frac{MQ}{NQ} = 1 - \frac{NM}{NQ}.$$

Also
$$\frac{RN}{CO} = \frac{AN}{AO} = \frac{NQ}{BO}.$$

Hence
$$\frac{PM}{RN} \times \frac{RN}{CO} = \frac{AN}{AO} - \frac{NM}{NQ} \times \frac{NQ}{BO},$$

$$\therefore \frac{PM}{CO} + \frac{MN}{BO} = \frac{AN}{AO} = 1 - \frac{ON}{AO};$$

or
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \text{.....................} (4).$$

21. All these forms of the equation of the plane are included in the form

$$Ax + By + Cz = D \text{ .... } (5).$$

Conversely we can shew that any equation of the form (5) represents a plane.

For let α, β, γ; α', β', γ' be the co-ordinates of any two points in the locus represented by (5). The equations of the straight line joining these two points are

$$\frac{x-\alpha}{\alpha'-\alpha}=\frac{y-\beta}{\beta'-\beta}=\frac{z-\gamma}{\gamma'-\gamma}\quad\text{..................(6).}$$

But since (α, β, γ), $(\alpha', \beta', \gamma')$ lie on (5) we have

$$A\alpha + B\beta + C\gamma = D,$$
$$A\alpha' + B\beta' + C\gamma' = D.$$

Subtracting, $A(\alpha-\alpha')+B(\beta-\beta')+C(\gamma-\gamma')=0.$

And therefore by (6)

$$A(x-\alpha)+B(y-\beta)+C(z-\gamma)=0,$$

where x, y, z are the co-ordinates of any point in the line (6),

or $$Ax+By+Cz=A\alpha+B\beta+C\gamma=D.$$

Hence x, y, z, the co-ordinates of any point in the line (6), satisfy the equation of the locus. That is, if any two points be taken in the locus of (5) and be joined by a straight line, this straight line lies wholly in that locus. Therefore the surface represented by (5) is a plane.

An equation of the form

$$Ax+By=D$$

represents a plane perpendicular to the plane of xy, and an equation of the form

$$Ax=D$$

represents a plane perpendicular to the axis of x, (Art. 19). These are particular cases of (5), and may be obtained from it by making first C to vanish, and secondly both B and C to vanish.

22. To find the distance from the origin of the point at which the plane (5) cuts the axis of x we must put $y=0$ and $z=0$. We thus obtain $Ax=D$ or $x=\frac{D}{A}$; or if this distance be called a, $\frac{D}{A}=a$. Similarly $\frac{D}{B}=b$, $\frac{D}{C}=c$; and substituting for A, B, C in (5) we get

$$\frac{Dx}{a} + \frac{Dy}{b} + \frac{Dz}{c} = D,$$

$$\text{or} \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

the equation found in Art. 20.

23. By Art. 19 it appears that every plane can be represented by an equation of the form

$$lx + my + nz = p,$$

where l, m, n are the direction-cosines, and p the length, of the perpendicular from the origin on the plane. But

$$Ax + By + Cz = D$$

represents a plane. Hence if these represent the same plane, we have

$$\frac{l}{A} = \frac{m}{B} = \frac{n}{C} = \frac{p}{D}.$$

$$\text{Also} \quad l^2 + m^2 + n^2 = 1;$$

$$\therefore l = \frac{A}{\sqrt{A^2 + B^2 + C^2}},$$

$$m = \frac{B}{\sqrt{A^2 + B^2 + C^2}},$$

$$n = \frac{C}{\sqrt{A^2 + B^2 + C^2}},$$

$$\text{and} \quad p = \frac{D}{\sqrt{A^2 + B^2 + C^2}}.$$

Thus the direction-cosines of the perpendicular from the origin on the plane

$$Ax + By + Cz = D$$

are proportional to A, B, C, and the length of the perpendicular is $\dfrac{D}{\sqrt{A^2 + B^2 + C^2}}$.

24. The angle between any two planes whose equations are

$$Ax + By + Cz = D,$$

$$A'x + B'y + C'z = D',$$

is the same as the angle between the perpendiculars on them from the origin. But the direction-cosines of these perpendiculars are (Art. 23)

$$\frac{A}{\sqrt{A^2+B^2+C^2}}, \quad \frac{B}{\sqrt{A^2+B^2+C^2}}, \quad \frac{C}{\sqrt{A^2+B^2+C^2}},$$

$$\frac{A'}{\sqrt{A'^2+B'^2+C'^2}}, \quad \frac{B'}{\sqrt{A'^2+B'^2+C'^2}}, \quad \frac{C'}{\sqrt{A'^2+B'^2+C'^2}},$$

and the cosine of the angle between the planes is therefore equal to

$$\frac{AA'+BB'+CC'}{\sqrt{A^2+B^2+C^2}\sqrt{A'^2+B'^2+C'^2}}.$$

The condition that the two planes should be at right angles is therefore

$$AA'+BB'+CC'=0.$$

The conditions that they should be parallel may be obtained by equating the cosine of the angle between them to unity. It will be found that this leads to the conditions

$$\frac{A}{A'}=\frac{B}{B'}=\frac{C}{C'}.$$

These may be also obtained independently from the consideration that the direction-cosines of the perpendicular on the one plane are proportional to A, B, C, and those of the perpendicular on the other to A', B', C'; and if the planes be parallel, and consequently the perpendiculars from the origin on them coincident, we must have A, B, C proportional to A', B', C', or

$$\frac{A}{A'}=\frac{B}{B'}=\frac{C}{C'}.$$

25. The equation of a plane through a point (α, β, γ) parallel to the plane

$$Ax+By+Cz=D \quad \text{.....................(1)}$$

is easily seen to be

$$A(x-\alpha)+B(y-\beta)+C(z-\gamma)=0,$$

$$\text{or} \quad Ax+By+Cz=A\alpha+B\beta+C\gamma \quad \text{......... (2).}$$

For this equation does represent a plane parallel to (1) by the last article, and it is satisfied by the values

$$x=\alpha,\ y=\beta,\ z=\gamma.$$

Now the length of the perpendicular from the origin on the plane (1)

$$=\frac{D}{\sqrt{A^2+B^2+C^2}},$$

and the length of the perpendicular from the origin on the plane (2) is similarly

$$\frac{A\alpha+B\beta+C\gamma}{\sqrt{A^2+B^2+C^2}}.$$

The difference of these, or

$$\frac{(A\alpha+B\beta+C\gamma)\sim D}{\sqrt{A^2+B^2+C^2}},$$

is the length of the perpendicular from the point (α, β, γ) on the plane (1).

If we take the equation of the plane in the form

$$lx+my+nz-p=0,$$

the numerical value of the length of the perpendicular from any point (x, y, z) on this plane is

$$\pm(lx+my+nz-p).$$

It is easily seen that the expression

$$lx+my+nz-p$$

is positive if the point (x, y, z) is on the opposite side of the plane from the origin, and negative when the point (x, y, z) is on the same side of the plane as the origin. If the expression be denoted by α, the length of the perpendicular from any point on the plane

$$\alpha=0$$

is $+\alpha$ or $-\alpha$, according as the point and the origin are on the same or opposite sides of the plane.

26. If we take four planes forming a tetrahedron, whose equations are

$$\alpha=0,\ \beta=0,\ \gamma=0,\ \delta=0,$$

all expressed in the form

$$lx+my+nz-p=0,$$

any other plane may be represented by the equation

$$l\alpha+m\beta+n\gamma+q\delta=0.$$

For this represents some plane, being of the first degree in x, y, z, and since it contains three arbitrary constants, namely, the ratios of three of the quantities l, m, n, q to the fourth, it may be made to satisfy three conditions, and may therefore be made to represent any plane.

This method of representing planes may be developed in a similar manner to that used for straight lines in Plane Coordinate Geometry (Todhunter's *Conic Sections*, Chap. IV.). Thus the equations of the two planes bisecting the angles between the planes $\alpha=0$, $\beta=0$, will be

$$\alpha-\beta=0 \text{ and } \alpha+\beta=0,$$

the former bisecting that angle within which the origin lies, and the latter the supplementary angle.

Any equation which is not homogeneous in α, β, γ, δ, can be rendered so by means of the relation

$$A\alpha+B\beta+C\gamma+D\delta=-3V,$$

where V is the volume of the tetrahedron, and A, B, C, D the areas of its faces. This equation merely states that the algebraic sum of the four tetrahedra whose vertices are at the point $(\alpha, \beta, \gamma, \delta)$ is equal to the fundamental tetrahedron.

27. If a straight line

$$\frac{x-\alpha}{A}=\frac{y-\beta}{B}=\frac{z-\gamma}{C} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

is parallel or perpendicular to a plane

$$A'x+B'y+C'z=D \quad\ldots\ldots\ldots\ldots\ldots\ldots\ (2),$$

it is perpendicular or parallel respectively to the perpendicular on that plane, whose direction-cosines are proportional to A', B', C'.

The condition that (1) may be parallel to (2) is therefore

$$AA'+BB'+CC'=0,$$

and the conditions that (1) may be perpendicular to (2) are

$$\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'}.$$

28. It is often requisite to know the length of the perpendicular on a given straight line from a given point.

Let the equations of the straight line be

$$\frac{x-\alpha}{A} = \frac{y-\beta}{B} = \frac{z-\gamma}{C} \quad \text{.................. (1)},$$

and let α', β', γ' be the co-ordinates of the given point.

The equation of any plane through $(\alpha', \beta', \gamma')$ is

$$\lambda(x-\alpha') + \mu(y-\beta') + \nu(z-\gamma') = 0 \quad \text{.......... (2)}.$$

If this plane be perpendicular to (1) we have

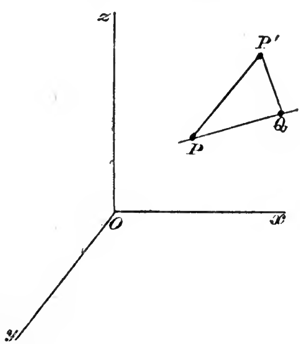
$$\frac{\lambda}{A} = \frac{\mu}{B} = \frac{\nu}{C},$$

and its equation becomes

$$A(x-\alpha') + B(y-\beta') + C(z-\gamma') = 0 \quad \text{......... (3)}.$$

The point where this plane meets the line (1) is evidently the foot of the perpendicular from $(\alpha', \beta', \gamma')$ on (1).

Let then P be the point (α, β, γ), P' the point $(\alpha', \beta', \gamma')$,



and Q the foot of the perpendicular from P' on the line (1); therefore PQ is the perpendicular from P on the plane (3), and we have $$PQ = \frac{A(\alpha' - \alpha) + B(\beta' - \beta) + C(\gamma' - \gamma)}{\sqrt{A^2 + B^2 + C^2}},$$

and $P'Q^2 = PP'^2 - PQ^2$ by the right-angled triangle $P'QP$;

$$\therefore P'Q^2 = (\alpha - \alpha')^2 + (\beta - \beta')^2 + (\gamma - \gamma')^2 - \frac{\{A(\alpha' - \alpha) + B(\beta' - \beta) + C(\gamma' - \gamma)\}^2}{A^2 + B^2 + C^2}.$$

29. *To find the conditions that a straight line may lie wholly in a given plane.*

Let $$\frac{x - \alpha}{A} = \frac{y - \beta}{B} = \frac{z - \gamma}{C} \quad \text{.................. (1)}$$

be the equations of the line,

$$A'x + B'y + C'z = D \quad \text{.................. (2)}$$

the equation of the plane.

Put each of the fractions in (1) equal to k.

Therefore

$$x = \alpha + Ak, \; y = \beta + Bk, \; z = \gamma + Ck,$$

and if the line (1) lies wholly in (2), these values of x, y, z must satisfy (2) whatever be the value of k. Hence the equation

$$A'\alpha + B'\beta + C'\gamma - D + (AA' + BB' + CC')k = 0,$$

must be satisfied independently of k. This gives us the two conditions

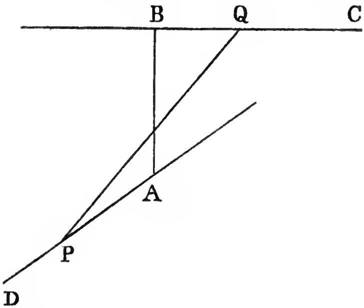
$$A'\alpha + B'\beta + C'\gamma - D = 0,$$
$$AA' + BB' + CC' = 0.$$

The first of these equations denotes that the point (α, β, γ) lies in the plane (2), and the second that the angle between the line (1) and the perpendicular on the plane (2) is a right angle. These are evidently necessary and sufficient conditions.

30. *To find the shortest distance between two straight lines whose equations are given.*

We must first prove that the shortest line between two given straight lines is perpendicular to each of them.

Let BC, AD be the two straight lines, and AB a line perpendicular to each of them. Then AB is clearly shorter than the line joining A with any other point of BC, and also than the line joining B with any other point of AD. Let P be any point in AD, and Q any point in BC. Then



PA and QB are both perpendicular to AB, and therefore AB is the projection of PQ on AB, and is equal to the length of PQ multiplied by the cosine of the angle between them, and is therefore less than PQ, since the cosine of any angle is less than unity. if ∠ is not 0° or multiple of π

Let
$$\frac{x-\alpha}{A}=\frac{y-\beta}{B}=\frac{z-\gamma}{C} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\frac{x-\alpha'}{A'}=\frac{y-\beta'}{B'}=\frac{z-\gamma'}{C'} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

be the equations of the two straight lines. Let the equation of any plane through (1) be

$$P(x-\alpha)+Q(y-\beta)+R(z-\gamma)=0 \ldots\ldots\ldots (3).$$

Then we have, since (3) contains (1),

$$PA+QB+RC=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

And if we take the plane through (1) to be also parallel to (2), we have

$$PA'+QB'+RC'=0 \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

From (4) and (5) we have

$$\frac{P}{BC'-B'C}=\frac{Q}{CA'-C'A}=\frac{R}{AB'-A'B}.$$

The equation of a plane through (1) parallel to (2) is therefore

$$(BC'-B'C)(x-\alpha)+(CA'-C'A)(y-\beta)+(AB'-A'B)(z-\gamma)=0 \quad (6).$$

Similarly the equation of a plane through (2) parallel to (1) is

$$(BC'-B'C)(x-\alpha')+(CA'-C'A)(y-\beta')+(AB'-A'B)(z-\gamma')=0 \quad (7).$$

The length of the perpendicular from the origin on (6) is

$$\frac{(BC'-B'C)\,\alpha+(CA'-C'A)\,\beta+(AB'-A'B)\,\gamma}{\sqrt{(BC'-B'C)^2+(CA'-C'A)^2+(AB'-A'B)^2}},$$

and the length of the perpendicular on (7) is

$$\frac{(BC'-B'C)\,\alpha'+(CA'-C'A)\,\beta'+(AB'-A'B)\,\gamma'}{\sqrt{(BC'-B'C)^2+(CA'-C'A)^2+(AB'-A'B)^2}}.$$

The difference of these, or

$$\frac{(BC'-B'C)(\alpha-\alpha')+(CA'-C'A)(\beta-\beta')+(AB'-A'B)(\gamma-\gamma')}{\sqrt{(BC'-B'C)^2+(CA'-C'A)^2+(AB'-A'B)^2}}$$

is clearly the perpendicular distance between the two given lines.

The equations of the line AB can be obtained by finding the equations of two planes, one of which contains the straight line BC and is perpendicular to the plane (6), and the other contains the line AD and is perpendicular to the same plane. Each of these planes evidently contains the straight line AB, and their equations considered as simultaneous determine the line. The requisite conditions for the two planes will be found in Articles 24 and 29.

31. *To find the condition that two straight lines whose equations are given may intersect.*

Let the equations of the straight lines be

$$\frac{x-\alpha}{A}=\frac{y-\beta}{B}=\frac{z-\gamma}{C} \quad \text{.....................} (1),$$

$$\frac{x-\alpha'}{A'}=\frac{y-\beta'}{B'}=\frac{z-\gamma'}{C'} \quad \text{.....................} (2).$$

Then if they intersect, a plane can be made to pass through both of them. Let this plane be

$$Px + Qy + Rz = D.$$

Since this contains the line (1) we have, by Art. 29,

$$P\alpha + Q\beta + R\gamma = D \quad\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$PA + QB + RC = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

And since it contains the line (2) we have

$$P\alpha' + Q\beta' + R\gamma' = D \quad\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

$$PA' + QB' + RC' = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

From (3) and (5) we have

$$P(\alpha - \alpha') + Q(\beta - \beta') + R(\gamma - \gamma') = 0 \quad\ldots\ldots\ldots (7).$$

And eliminating P, Q, R from (4), (6) and (7) we get with the usual notation of determinants,

$$\begin{vmatrix} A & B & C \\ A' & B' & C' \\ \alpha - \alpha' & \beta - \beta' & \gamma - \gamma' \end{vmatrix} = 0,$$

or

$$(\alpha-\alpha')(BC'-B'C)+(\beta-\beta')(CA'-C'A)+(\gamma-\gamma')(AB'-A'B)=0.$$

A result which might have been obtained from the last article by the consideration that if two straight lines intersect their shortest distance vanishes.

If the two straight lines be given by the equations

$$\left.\begin{array}{r} Ax + By + Cz = D \\ A'x + B'y + C'z = D' \end{array}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

$$\left.\begin{array}{r} Px + Qy + Rz = S \\ P'x + Q'y + R'z = S' \end{array}\right\} \quad\ldots\ldots\ldots\ldots\ldots (9),$$

the condition of intersection is obtained from the consideration that these four equations must be able to be satisfied by the same values of x, y, z. The condition for this is

$$\begin{vmatrix} A & B & C & D \\ A' & B' & C' & D' \\ P & Q & R & S \\ P' & Q' & R' & S' \end{vmatrix} = 0.$$

EXAMPLES. CHAPTER II.

1. Find the equations of a straight line passing through the point (1, 2, 3) and whose direction-cosines are proportional to $\sqrt{3}$, 1 and $2\sqrt{3}$.

2. Find the equations of the straight line joining the two points whose co-ordinates are (1, 2, 3) and (3, 2, 1) respectively.

3. Find the equations of the sides of the triangle formed by joining the points (1, 2, 3), (3, 2, 1), (2, 3, 1). Deduce the values of the angles of the triangle.

4. Find the equation of the plane which passes through the three points in the last question, and the length of the perpendicular on it from the origin.

5. Find the equations of a straight line which passes through the point (1, 2, 3), and is perpendicular to the plane

$$x + 2y + 3z = 6.$$

6. Find the equations of a straight line which passes through the point (1, 2, 3), and is perpendicular to the two straight lines in questions (1) and (2).

7. Find the equation of a plane passing through two given points and perpendicular to a given plane.

8. Find the equations of a straight line passing through the point (1, 2, 3) and parallel to the plane in question (4) and to the plane of xy.

9. Find the equation of a plane passing through the point (2, 3, 4) and the straight line in question (1).

10. Find the equations of a straight line drawn from the origin of co-ordinates at right angles to one given straight line, and making a given angle with another. If the given straight lines be at right angles to each other and the given angle be $\frac{\pi}{4}$, shew that there are two solutions, and that the two straight lines so found are at right angles to each other.

11. Find the equation of a plane which passes through a given point, and is perpendicular to each of two given planes.

12. Shew that the equation of a plane in oblique co-ordinates can be put in the form

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p,$$

where p is the length of the perpendicular on the plane from the origin, and α, β, γ the angles which it makes with the axes.

13. Shew that if α, β, γ be the angles between any straight line and the axes of co-ordinates, l, m, n the direction-ratios of the line, and λ, μ, ν have the meanings given in Art. 13,

$$\cos \alpha = l + m \cos \nu + n \cos \mu,$$

$$\cos \beta = m + n \cos \lambda + l \cos \nu,$$

$$\cos \gamma = n + l \cos \mu + m \cos \lambda.$$

14. Deduce the conditions that in oblique co-ordinates the straight line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

may be perpendicular to the plane

$$Ax + By + Cz = D.$$

15. Shew that the locus of a point which moves so as always to be equidistant from two given points, is a plane which bisects at right angles the straight line joining the two points.

16. What loci are represented by each of the equations

$$f(x) = 0;\ f(r) = 0;\ f(\theta) = 0;\ f(\phi) = 0;$$

where r, θ, ϕ are the usual polar co-ordinates?

17. Interpret the equations:

(1) $\theta = 0$; (2) $\begin{cases} \theta = \alpha, \\ \phi = \beta; \end{cases}$ (3) $\begin{cases} \phi = 0, \\ r = a. \end{cases}$

18. Find the polar equation of a plane.

19. Find the angle between the two lines given by

$$\left.\begin{array}{r} x+y+z=3 \\ 3x+4y+5z=12 \end{array}\right\} (1), \quad \text{and } x=y=z \quad (2).$$

20. Three planes are at perpendicular distances p_1, p_2, p_3 from the origin; three planes are drawn through the lines of intersection of any two perpendicular to the third; shew that the last three planes will intersect in a straight line passing through the origin if

$$p_1 \cos A = p_2 \cos B = p_3 \cos C,$$

where A, B, C are the angles between the first three planes.

21. Shew that through two given points (a, b, c), (a', b', c'), two planes may be drawn cutting off from the axes intercepts whose sum is zero; and these two planes will be at right angles to each other if

$$\frac{1}{a-a'} + \frac{1}{b-b'} + \frac{1}{c-c'} = 0.$$

22. Find the cosine of the angle between the two straight lines represented by

$$x+y+z=0,$$

$$\frac{3}{y-z} + \frac{5}{z-x} - \frac{8}{x-y} = 0.$$

23. Find the condition that the two straight lines whose direction-cosines are given by the equations

$$Al + Bm + Cn = 0,$$

$$Pl^2 + Qm^2 + Rn^2 = 0,$$

may be at right angles to each other.

24. If the co-ordinates of four points be $a-b$, $a-c$, $a-d$; $b-c$, $b-d$, $b-a$; $c-d$, $c-a$, $c-b$; $d-a$, $d-b$, $d-c$, respectively, prove that the straight line joining the middle points of any two opposite edges of the tetrahedron formed by joining the points, will pass through the origin.

25. Shew analytically that the least distance between two straight lines is perpendicular to each of them.

26. The shortest distance between the lines

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{l} \text{ and } \frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'},$$

intersects the latter in the point whose co-ordinates are

$$\alpha' + l' \operatorname{cosec}^2 \theta \,(u' + u \cos \theta),$$

and two similar expressions where θ is the angle between the lines and

$$u = l(\alpha' - \alpha) + m(\beta' - \beta) + n(\gamma' - \gamma),$$
$$u' = l'(\alpha - \alpha') + m'(\beta - \beta') + n'(\gamma - \gamma').$$

27. Prove that the straight lines joining the middle points of opposite edges of a tetrahedron all meet in a point and bisect one another.

28. If x, y be the lengths of two of the straight lines joining the middle points of opposite edges of a tetrahedron, ω the angle between these lines, and a, a' those edges of the tetrahedron which are not met by either of the lines, prove that

$$\cos \omega = \frac{a^2 \sim a'^2}{4xy}.$$

29. Find the shortest distance between the diagonal of a cube and any edge which it does not meet.

30. Find the area of the triangle formed by joining the three points where the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

cuts the axes.

31. From the origin are drawn three equal straight lines of length p, such that the inclinations of the first to the axes of x, y, z respectively, are the same as those of the second to y, z, x, and of the third to z, x, y. A plane is drawn perpendicular to each of them through its extremity. Find the co-

ordinates of the point of intersection of these three planes and the equations of the line joining it with the origin.

32. A straight line is drawn from the origin to meet the straight line

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$$

at right angles. Shew that its equations are

$$\frac{x}{a-lt}=\frac{y}{b-mt}=\frac{z}{c-nt},$$

where
$$t=\frac{al+bm+cn}{l^2+m^2+n^2}.$$

33. Shew that by a proper choice of axes the equations of any two straight lines can be put in the forms

$$z=c,\ y=mx;\ z=-c,\ y=-mx.$$

CHAPTER III.

ON CERTAIN SURFACES OF THE SECOND ORDER.

32. We have shewn that the general equation of the first degree represents a plane. Before proceeding to the discussion of the general equation of the second degree, we shall find the equations of certain special surfaces included in the class represented by the equation of the second degree.

33. *The Sphere.*

A sphere is a surface every point of which is at a constant distance from a fixed point called the centre. The constant distance is called the radius.

Let a, b, c be the co-ordinates of the centre, r the radius, x, y, z the co-ordinates of any point on the surface. Then the distance of the point (x, y, z) from the centre is equal to

$$\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}.$$

But this distance must equal the radius r. Hence for all points on the surface

$$\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}=r,$$

or

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2 \qquad\qquad (1),$$

which is the equation required.

Conversely any equation of the form

$$x^2+y^2+z^2+Ax+By+Cz+D=0$$

represents a sphere. For it can be put into the form

$$\left(x+\frac{A}{2}\right)^2+\left(y+\frac{B}{2}\right)^2+\left(z+\frac{C}{2}\right)^2=\frac{A^2+B^2+C^2}{4}-D,$$

and, comparing this with (1), we see that it represents a sphere whose centre is at a point $\left(-\frac{A}{2}, -\frac{B}{2}, -\frac{C}{2}\right)$ and whose radius is

$$\sqrt{\frac{A^2+B^2+C^2}{4}-D}.$$

34. *The Cone.*

A cone is a surface generated by a straight line which always passes through a fixed point called the vertex, and through a fixed curve.

We shall only discuss in this and the next Article the case when the fixed curve is a plane curve of the second degree.

Take the plane of the curve as the plane of xy, and let the equation of the curve be

$$Ax^2 + Cy^2 + Ex = 0 \qquad\qquad (1),$$

to which form the equation of any conic section can be reduced; and let α, β, γ be the co-ordinates of the vertex.

The equations of any straight line through the vertex are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad\qquad (2);$$

when this meets the plane of xy we have $z = 0$, and therefore

$$x = \alpha - \frac{l}{n}\gamma, \quad y = \beta - \frac{m}{n}\gamma.$$

These values of x and y must satisfy the equation (1), since the line always passes through some point in the curve represented by (1). Hence we have

$$A\left(\alpha - \frac{l}{n}\gamma\right)^2 + C\left(\beta - \frac{m}{n}\gamma\right)^2 + E\left(\alpha - \frac{l}{n}\gamma\right) = 0;$$

or, multiplying by n^2,

$$A(n\alpha - l\gamma)^2 + C(n\beta - m\gamma)^2 + En(n\alpha - l\gamma) = 0.$$

This is a relation which must be satisfied by l, m, n if the straight line (2) meet the curve (1). But if (x, y, z) be any point in (2) we have

$$\frac{l}{x-\alpha}=\frac{m}{y-\beta}=\frac{n}{z-\gamma}.$$

Consequently, if (x, y, z) be any point in any straight line joining (α, β, γ) with some point of the curve (1), we must have

$$A\{\alpha(z-\gamma)-\gamma(x-\alpha)\}^2+C\{\beta(z-\gamma)-\gamma(y-\beta)\}^2$$
$$+E(z-\gamma)\{\alpha(z-\gamma)-\gamma(x-\alpha)\}=0;$$

or reducing,

$$A(\alpha z-\gamma x)^2+C(\beta z-\gamma y)^2+E(z-\gamma)(\alpha z-\gamma x)=0\ldots(3),$$

which is therefore the equation of the cone.

If we transfer the origin to the point (α, β, γ) we must put

$$x=x'+\alpha,\ y=y'+\beta,\ z=z'+\gamma,$$

and the equation becomes

$$A(\alpha z'-\gamma x')^2+C(\beta z'-\gamma y')^2+Ez'(\alpha z'-\gamma x')=0,$$

of which every term is of the second degree in x', y', z'. The equation of a cone of the second degree whose vertex is at the origin is therefore homogeneous. Conversely every homogeneous equation of the second degree represents a cone whose vertex is at the origin. For let

$$Px^2+Qy^2+Rz^2+P'yz+Q'zx+R'xy=0\ldots\ldots(4),$$

be the equation. And let x_1, y_1, z_1 be the co-ordinates of any point on the locus. Then the equations of the straight line joining (x_1, y_1, z_1) with the origin are

$$\frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

But, since (x_1, y_1, z_1) is a point in (4),

$$Px_1^2+Qy_1^2+Rz_1^2+P'y_1z_1+Q'z_1x_1+R'x_1y_1=0,$$

and therefore by (5), if (x, y, z) be any point in (5),

$$Px^2+Qy^2+Rz^2+P'yz+Q'zx+R'xy=0.$$

Hence every point on the straight line joining the origin with (x_1, y_1, z_1) lies on the surface. Thus, the surface is generated by a straight line which always passes through the origin, and is therefore a cone.

35. *The Cylinder.*

A cylinder is a surface generated by a straight line which always passes through a fixed curve and remains parallel to itself.

Let the plane of the curve be taken as the plane of xy, and let its equation be

$$Ax^2 + Cy^2 + Ex = 0 \qquad (1).$$

Also let l, m, n be the direction-cosines of the straight line to which the generating line always continues parallel. Let α, β, 0 be the co-ordinates of the point in the curve (1) through which any generating line passes. The equations of this line will therefore be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z}{n} \qquad (2);$$

$$\therefore \alpha = x - \frac{lz}{n}, \quad \beta = y - \frac{mz}{n}.$$

But α, β are the co-ordinates of some point in (1), and therefore we have by substitution

$$A\left(x - \frac{lz}{n}\right)^2 + C\left(y - \frac{mz}{n}\right)^2 + E\left(x - \frac{lz}{n}\right) = 0,$$

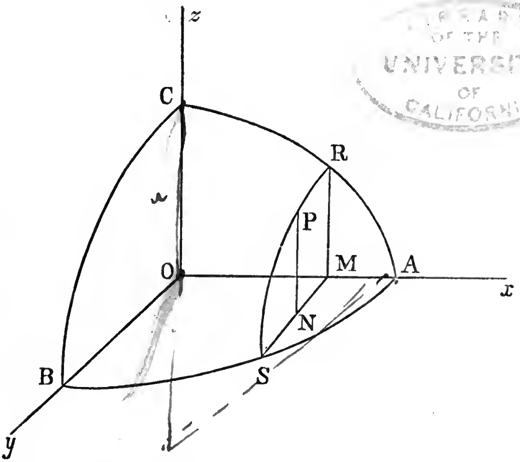
or $$A(nx - lz)^2 + C(ny - mz)^2 + nE(nx - lz) = 0 \qquad (3),$$

which, being a relation satisfied by the co-ordinates of any point in any one of the generating lines, is the equation of the surface.

36. *The Ellipsoid.*

The ellipsoid is a surface generated by a variable ellipse which always moves parallel to itself, and has its vertices on two ellipses whose planes are perpendicular to each other and to the plane of the moving ellipse, and which have one axis common.

Let the planes of the fixed ellipses be taken as the planes of zx and xy, and the direction of their common axis as the



axis of x. The plane of the moving ellipse will be parallel to the plane of yz.

Let COA, AOB be the fixed ellipses, $OA = a$, $OB = b$, $OC = c$. And let RPS be any position of the moving ellipse, MR, MS its semi-axes, P any point in it.

Draw PN parallel to Oz to meet MS in N.

Let $OM = x$, $MN = y$, $NP = z$.

From the ellipse RPS,

$$\frac{z^2}{RM^2} + \frac{y^2}{MS^2} = 1 \quad\text{.....................(1).}$$

From the ellipse COA,

$$\frac{RM^2}{c^2} = 1 - \frac{x^2}{a^2} \quad\text{.......................(2).}$$

From the ellipse AOB,

$$\frac{MS^2}{b^2} = 1 - \frac{x^2}{a^2} \quad\text{.......................(3).}$$

Whence substituting in (1)

$$\frac{z^2}{c^2}+\frac{y^2}{b^2}=1-\frac{x^2}{a^2};$$

$$\text{or}\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

If the two semi-axes OC and OB be equal, it can be seen from (2) and (3) that MR and MS are also equal. Now an ellipse whose axes are equal is a circle. Hence the surface in this case would be generated by the revolution of the ellipse BOA round OA, and its equation becomes

$$\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1.$$

The surface is called an oblate or prolate spheroid according as the semi-axis a is less or greater than b. If all the three semi-axes OA, OB, OC be equal, the equation becomes

$$x^2+y^2+z^2=a^2,$$

which shews that the surface in that case becomes a sphere whose centre is at O.

37. *The Hyperboloid of one Sheet.*

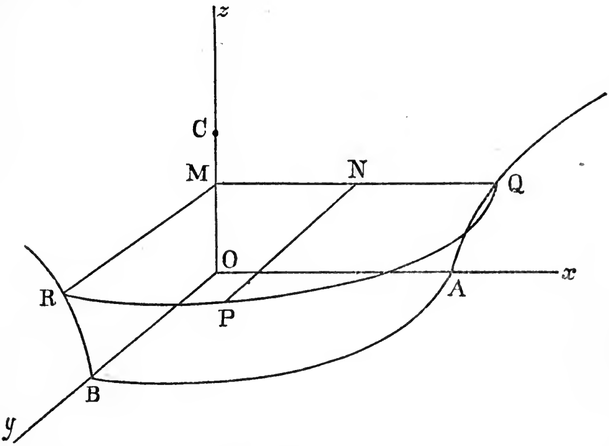
The hyperboloid of one sheet is generated by a variable ellipse which moves parallel to itself, and has its vertices on two hyperbolas whose planes are perpendicular to each other and to the plane of the moving ellipse, and which have a common conjugate axis.

Let AQ be one hyperbola in the plane of zx, BR the other in the plane of yz, and RPQ any position of the moving ellipse, RM and QM its semi-axes, and P any point on it. Let $OA=a$, $OB=b$, and OC, the common conjugate semi-axis, $=c$. Draw PN parallel to MR to meet MQ in N. Let $OM=z$, $MN=x$, $NP=y$. Then from the ellipse RPQ,

$$\frac{y^2}{MR^2}+\frac{x^2}{MQ^2}=1,$$

from the hyperbola AQ, $\frac{MQ^2}{a^2} = 1 + \frac{z^2}{c^2}$,

from the hyperbola BR, $\frac{MR^2}{b^2} = 1 + \frac{z^2}{c^2}$;



$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{z^2}{c^2},$$

or
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

the equation required.

38. *The Hyperboloid of two Sheets.*

This is generated as the last surface except that the hyperbolas have a common *transverse axis.*

Take the direction of the common axis as axis of x, the planes of the hyperbolas as the planes of zx, xy, and the plane of yz parallel to that of the moving ellipse. Let $OA = a$ be the common transverse semi-axis, and $OB = b$, $OC = c$, the two conjugate semi-axes. Let QPR be any position of the moving ellipse, MQ, MR its semi-axes, and P any point in it. Draw PN parallel to QM to meet RM in N.

Let $OM=x$, $MN=y$, $NP=z$.

From the ellipse QPR, $\frac{y^2}{MR^2}+\frac{z^2}{MQ^2}=1$,

from the hyperbola AQ, $\frac{MQ^2}{c^2}=\frac{x^2}{a^2}-1$,

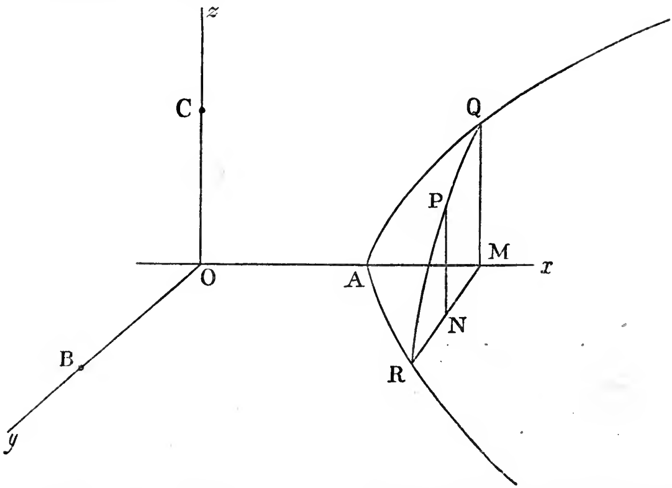
from the hyperbola AR, $\frac{MR^2}{b^2}=\frac{x^2}{a^2}-1$;

$$\therefore \frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{x^2}{a^2}-1,$$

or

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

the equation required.



These three surfaces, the ellipsoid, the hyperboloid of one sheet, and the hyperboloid of two sheets, are all included in the equation

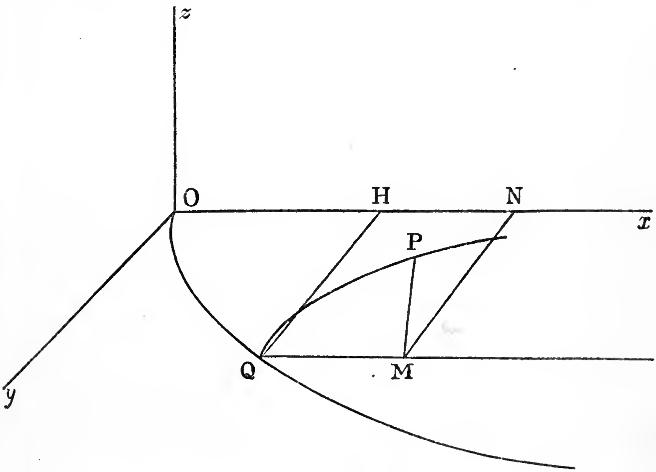
$$Ax^2+By^2+Cz^2=1.$$

39. *The Elliptic Paraboloid.*

The elliptic paraboloid is generated by a parabola which moves with its vertex in a fixed parabola, the planes of the two *parabolas being at right angles, their axes parallel, and their concavities turned in the same direction.*

Take the plane of the fixed parabola as plane of xy, its vertex as origin, and its axis as axis of x. Then the plane of the moving parabola is parallel to that of zx.

Let PQ be any position of the moving parabola, P any point in it, l' its latus rectum, and let l be the latus rectum of the fixed parabola. Draw PM parallel to Oz to meet the



axis of the moving parabola in M, and draw QH and MN parallel to the axis of y.

Then from the parabola PQ,

$$PM^2 = z^2 = l' \, . \, QM,$$

and from the parabola QO,

$$QH^2 = y^2 = l \, . \, OH = lx - l \, . \, QM$$

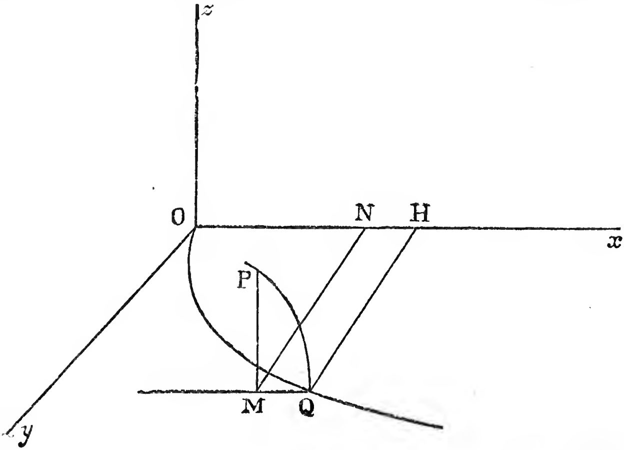
$$= lx - \frac{lz^2}{l'};$$

$$\therefore \; \frac{y^2}{l} + \frac{z^2}{l'} = x.$$

40. *The Hyperbolic Paraboloid.*

This is generated in the same manner as the last surface except that the concavities are turned in opposite directions.

Let OQ be the fixed parabola in the plane of xy, PQ any position of the moving parabola parallel to the plane of zx,



P any point in it. Draw PM parallel to Oz, MN and QH parallel to Oy. Let l and l' be the latera recta of the two parabolas OQ, PQ.

From the parabola PQ,

$$PM^2 = z^2 = l' \,.\, QM,$$

from the parabola OQ,

$$\begin{aligned} QH^2 = y^2 &= l \,.\, OH \\ &= l \,.\, (x + QM) \\ &= lx + \frac{lz^2}{l'}\,; \end{aligned}$$

$$\therefore \frac{y^2}{l} - \frac{z^2}{l'} = x.$$

The two paraboloids are both included in the equation

$$By^2 + Cz^2 = x.$$

We shall shew hereafter that any equation of the second degree in x, y, z can be reduced to that of one of the surfaces whose equations we have considered in this chapter.

41. *Asymptotic surfaces.*

The equation of the hyperboloid of one sheet is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \text{ ....} (1),$$

which can be put into the form

$$\frac{z}{c}=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{1}{2}}\left(1-\frac{a^2b^2}{a^2y^2+b^2x^2}\right)^{\frac{1}{2}}$$

$$=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{1}{2}}-\frac{ab}{2\,(a^2y^2+b^2x^2)^{\frac{1}{2}}}+\dots,$$

where the remaining terms contain higher powers of $a^2y^2+b^2x^2$ in the denominator.

Hence, if we increase x or y, or both, indefinitely, the value of z approaches indefinitely near to

$$c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}.$$

And if we construct the surface

$$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2} \text{ ....} (2),$$

(which by Art. 34 represents a cone whose vertex is the origin), the ordinate of this surface parallel to Oz, corresponding to any given values of x and y, approaches indefinitely near to equality with the ordinate of the hyperboloid corresponding to the same values of x and y, when these values are increased indefinitely; that is, the cone (2) is asymptotic to the hyperboloid.

Similarly the cone whose equation is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=0,$$

is asymptotic to the surface

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

42. The equation of the hyperbolic paraboloid is

$$\frac{y^2}{l} - \frac{z^2}{l'} = x \text{ .... } (1);$$

$$\therefore y = \pm \sqrt{\frac{l}{l'}} z \left(1 + \frac{l'x}{z^2}\right)^{\frac{1}{2}}$$

$$= \pm \sqrt{\frac{l}{l'}} z \left(1 + \frac{l'x}{2z^2} + \ldots\right)$$

$$= \pm \sqrt{\frac{l}{l'}} \left(z + \frac{l'x}{2z} + \ldots\right).$$

Now if z be increased indefinitely and x be not very large, the second and all the succeeding terms of the series on the right will diminish indefinitely. Hence the equations

$$y = \pm \sqrt{\frac{l}{l'}} z \text{ ....} (2),$$

represent two planes which are asymptotic to the surface (1) at points for which y and z are increased indefinitely while x remains finite.

EXAMPLES. CHAPTER III.

1. Find the polar equation of a sphere, any point not the centre being the pole. Shew that if through a fixed point O any chord OPQ be drawn meeting a sphere in P and Q, the rectangle $OP . OQ$ is invariable.

2. From any point O a straight line is drawn to meet a given plane in P. In OP a point Q is taken so that the rectangle $OP . OQ$ is equal to a given constant k^2. Find the locus of Q.

3. From any point O a straight line is drawn to meet a given sphere in P. In OP a point Q is taken so that the rectangle $OP . OQ$ is equal to a given constant k^2. Find the locus of Q.

4. Shew that if through any point of a sphere a plane be drawn perpendicular to the straight line joining the centre with that point, the plane will only meet the sphere in that one point.

5. A and B are two fixed points, P a point which moves so that PA is to PB in a constant ratio. Find the locus of P.

6. A and B are two fixed points, P a point which moves so that the angle APB is a right angle. Find the locus of P.

7. Find the surface generated by the line of intersection of two planes which pass each through a fixed straight line and are at right angles to each other.

8. Shew that all the points of intersection of two spheres lie on a circle whose plane is perpendicular to the straight line joining the centres of the spheres.

9. About three fixed points as centres, spheres are described having variable radii which are always in the same ratio to each other. Shew that they always intersect two and two on three fixed spheres, and that these three spheres have one circle common.

10. Prove that the planes of the three circles in which three spheres intersect each other two and two, all intersect in a straight line which is perpendicular to the plane containing the centres of the three spheres.

11. Prove that the six planes of intersection of four spheres two and two have one point common to them all.

12. Shew that if each of six equal spheres intersects all the rest but one, so that the radii at the line of intersection are inclined at 60°, the portion of space common to all will have eight solid angles coinciding with those of a cube whose side is $\frac{1}{\sqrt{18}}$ of the diameter of the sphere.

13. A straight line moves so that three given points of it lie respectively in three planes at right angles to each other.

Shew that a fourth point in the straight line, whose distances from the other three are respectively a, b, c, traces out an ellipsoid.

14. The two straight lines

$$\frac{x \pm a}{0} = \frac{\pm y}{\cos \alpha} = \frac{z}{\sin \alpha}$$

meet the axis of x in O, O', and P, P' are points on the two lines such that $OP . O'P' = c^2$; shew that the surface traced out by the straight line PP' is the hyperboloid

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 \cos^2 \alpha} + \frac{z^2}{c^2 \sin^2 \alpha} = 1,$$

P, P' being taken on the same side of the plane xy.

15. Find the surface generated by a straight line which revolves round a fixed straight line which it does not meet.

16. Find the surface which is the locus of the family of curves defined by the equations

$$x^2 + y^2 + z^2 = a^2 \text{ and } y^2 + z^2 = n^2a^2 - c^2,$$

where a is a variable parameter and c an absolute constant; and discuss its form for different values of n.

17. A perpendicular PN is let fall from a point P in a right cone on a plane through the vertex perpendicular to the axis, and a point P' is taken in PN or PN produced such that $PN . P'N$ is constant. Find the locus of P'.

CHAPTER IV.

TRANSFORMATION OF CO-ORDINATES.

43. MANY of the equations which we shall have occasion to employ will be much simplified by a proper choice of axes. It is necessary therefore to investigate the relations which hold between the co-ordinates of any point when referred to two different sets of axes.

The simplest case is that in which the directions of the two sets of axes are identical, the origin only being different.

Let x, y, z be the co-ordinates of P referred to the old set of axes; x', y', z', the co-ordinates of the same point referred to the new set. Let α, β, γ be the co-ordinates of the new origin referred to the old axes. Then the distance of P from the old plane of yz is equal to the distance of P from the new plane of yz together with the distance between these two planes, or

$$x = x' + \alpha.$$

Similarly

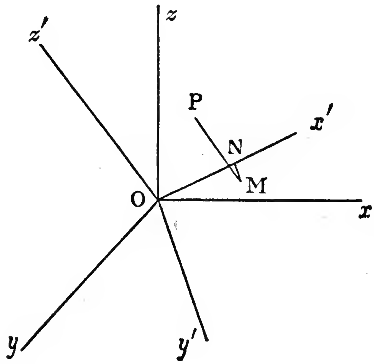
$$y = y' + \beta,$$
$$z = z' + \gamma.$$

These results will hold whether the axes be oblique or rectangular.

44. *To find the co-ordinates of a point* P *referred to one set of rectangular axes, in terms of the co-ordinates of the same point referred to another set of axes, also rectangular, with the same origin.*

Let Ox, Oy, Oz be the old axes; Ox', Oy', Oz' the new. Let x, y, z be the co-ordinates of P referred to the old axes;

x', y', z' the co-ordinates of the same point referred to the new axes. Let l_1, m_1, n_1 be the direction-cosines of Ox' re-



ferred to Ox, Oy, Oz; l_2, m_2, n_2 those of Oy', and l_3, m_3, n_3 those of Oz'.

Through P draw PM parallel to Oz' to meet the plane $Ox'y'$ in M, and through M draw MN parallel to Oy' to meet Ox' in N. Then $ON = x'$, $NM = y'$, $MP = z'$.

Also the projection of OP on Ox is x. And the projections of ON, NM, MP on Ox are l_1x', l_2y', l_3z', respectively, since l_1, l_2, l_3 are the cosines of the angles between Ox and ON, NM and MP, respectively. But the projection of OP on any straight line is equal to the sum of the projections of ON, NM and MP on the same line. Hence

$$x = l_1x' + l_2y' + l_3z'.$$

Similarly by projecting on the lines Oy and Oz we get

$$y = m_1x' + m_2y' + m_3z',$$
$$z = n_1x' + n_2y' + n_3z'.$$

The nine quantities l_1, m_1, n_1, l_2, m_2, n_2, l_3, m_3, n_3 are not independent, but are connected by six relations. For since l_1, m_1, n_1 are the *direction-cosines* of Ox', we have

$$l_1^2 + m_1^2 + n_1^2 = 1.$$

Similarly

$$l_2^2 + m_2^2 + n_2^2 = 1,$$
$$l_3^2 + m_3^2 + n_3^2 = 1.$$

Also the cosine of the angle between Oy' and Oz' is equal to $l_2l_3+m_2m_3+n_2n_3$; but this angle being a right angle, its cosine is equal to zero;

$$\therefore\ l_2l_3+m_2m_3+n_2n_3=0.$$

Similarly
$$l_3l_1+m_3m_1+n_3n_1=0,$$
$$l_1l_2+m_1m_2+n_1n_2=0.$$

These relations may be replaced by the six equations

$$\begin{aligned}
l_1^2+l_2^2+l_3^2&=1,\\
m_1^2+m_2^2+m_3^2&=1,\\
n_1^2+n_2^2+n_3^2&=1,\\
m_1n_1+m_2n_2+m_3n_3&=0,\\
n_1l_1+n_2l_2+n_3l_3&=0,\\
l_1m_1+l_2m_2+l_3m_3&=0.
\end{aligned}$$

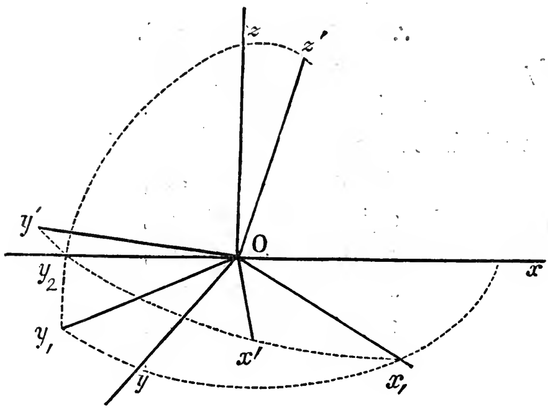
These equations can be algebraically deduced from the previous set, but they can be more easily proved independently thus:

l_1, m_1, n_1 are the cosines of the angles between Ox' and Ox, Oy, Oz; l_2, m_2, n_2 those of the angles between Oy' and Ox, Oy, Oz; and l_3, m_3, n_3 of the angles between Oz' and Ox, Oy, Oz. Consequently l_1, l_2, l_3 are the cosines of the angles between Ox and Ox', Oy', Oz'; m_1, m_2, m_3 those of the angles between Oy and Ox', Oy', Oz'; and n_1, n_2, n_3 those of the angles between Oz and Ox', Oy', Oz'. Considering Ox', Oy', Oz' as axes, and remembering that Ox, Oy, Oz are mutually at right angles, we obtain the above formulæ at once.

45. The formulæ given in the last Article are extremely useful, and from their symmetrical character are easy to remember. They are liable to the objection that nine constants are introduced of which six are superfluous, and other formulæ have been proposed which employ only three constants.

Let Ox, Oy, Oz be the old axes; Ox', Oy', Oz' the new ones. Let the plane of $x'y'$ cut the plane of xy in Ox_1, and let a plane through Oz and Oz', which is therefore by Euclid, XI. 18, perpendicular to the planes of xy and $x'y'$, cut these planes in Oy_1, Oy_2, respectively.

Then since Oz is perpendicular to the plane of xy it is perpendicular to Ox_1, and since Oz' is perpendicular to the



plane of $x'y'$, it also is perpendicular to Ox_1. Hence Ox_1 is perpendicular to the lines Oz and Oz', and is therefore perpendicular to the plane in which they lie, and therefore perpendicular to Oy_1, Oy_2. Hence by Euclid, XI. *Def.* 6, the angle y_1Oy_2 is the angle between the planes of xy and $x'y'$. Let this angle be called θ, and let the angle between Ox and Ox_1 be called ϕ, and the angle between Ox_1 and Ox' be called ψ.

Let x, y, z be the co-ordinates of any point P referred to the axes Ox, Oy, Oz. Then if we take Ox_1, Oy_1, and Oz as axes, the ordinate z will be unaltered, and if x_1, y_1 be the new co-ordinates parallel to Ox_1, Oy_1, we have by the ordinary formulæ of transformation in plane co-ordinates,

$$x = x_1 \cos\phi - y_1 \sin\phi,$$
$$y = x_1 \sin\phi + y_1 \cos\phi.$$

Again, if we take Ox_1, Oy_2, Oz' as axes, the x_1 will be unaltered, and if y_2, z' be the new co-ordinates parallel to Oy_2, Oz', we have

$$y_1 = y_2 \cos\theta - z' \sin\theta,$$
$$z = y_2 \sin\theta + z' \cos\theta.$$

And lastly, taking Ox', Oy', Oz' as axes, the z' will be unaltered, and we get

$$x_1 = x' \cos \psi - y' \sin \psi,$$
$$y_2 = x' \sin \psi + y' \cos \psi.$$

And, making the substitutions for x_1, y_1, y_2, we get finally

$$x = x' (\cos \phi \cos \psi - \sin \phi \sin \psi \cos \theta) - y' (\sin \psi \cos \phi + \cos \theta \cos \psi \sin \phi) + z' \sin \phi \sin \theta,$$
$$y = x' (\sin \phi \cos \psi + \cos \phi \sin \psi \cos \theta) - y' (\sin \phi \sin \psi - \cos \phi \cos \psi \cos \theta) - z' \sin \theta \cos \phi,$$
$$z = x' \sin \psi \sin \theta + y' \cos \psi \sin \theta + z' \cos \theta.$$

These are called Euler's Formulæ. They are useful in discussing the nature of the sections of surfaces, but their unsymmetrical character renders them difficult to remember.

46. If we wish to change both the origin and the direction of the axes we have only to combine the formulæ of Arts. 43 and 44. For changing the origin to a point whose co-ordinates are α, β, γ, and keeping the direction of the axes unchanged, we get $x = x_1 + \alpha$, $y = y_1 + \beta$, $z = z_1 + \gamma$. And then changing the directions of the axes we get

$$x_1 = l_1 x' + l_2 y' + l_3 z',$$
$$\text{or } x = l_1 x' + l_2 y' + l_3 z' + \alpha.$$

Similarly
$$y = m_1 x' + m_2 y' + m_3 z' + \beta,$$
$$z = n_1 x' + n_2 y' + n_3 z' + \gamma.$$

47. The formulæ for transformation of co-ordinates in Art. 44 hold also when the axes are oblique if l_1, m_1, n_1 denote the *direction-ratios* of the new axis of x with respect to the old axes. The six relations which hold between the nine constants involved, which can be obtained from Art. 13, are in general very cumbrous.

48. A proof exactly similar to that given in Todhunter's *Conic Sections*, Art. 87, will shew that the degree of any expression involving x, y, z is unaltered by transformation of co-ordinates.

49. The following proposition is useful in many questions of transformation of co-ordinates.

The condition that the expression

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy \ldots\ldots (1)$$

should be the product of two linear expressions in x, y, z, is

$$ABC + 2A'B'C' - AA'^2 - BB'^2 - CC'^2 = 0.$$

For if one of the factors be

$$\lambda x + \mu y + \nu z \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

it is evident, by considering the coefficients of x^2, y^2 and z^2 in (1), that the other factor must be

$$\frac{A}{\lambda} x + \frac{B}{\mu} y + \frac{C}{\nu} z \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Multiplying (2) by (3) and equating the coefficients of yz, zx and xy in the product, to those of the same terms in (1) we have

$$B\frac{\nu}{\mu} + C\frac{\mu}{\nu} = 2A',$$

$$C\frac{\lambda}{\nu} + A\frac{\nu}{\lambda} = 2B',$$

$$A\frac{\mu}{\lambda} + B\frac{\lambda}{\mu} = 2C',$$

whence by multiplication we get

$$8A'B'C' = 2ABC + A\left(B^2\frac{\nu^2}{\mu^2} + C^2\frac{\mu^2}{\nu^2}\right) + B\left(C^2\frac{\lambda^2}{\nu^2} + A^2\frac{\nu^2}{\lambda^2}\right)$$
$$+ C\left(A^2\frac{\mu^2}{\lambda^2} + B^2\frac{\lambda^2}{\mu^2}\right)$$

$$= 2ABC + A(4A'^2 - 2BC) + B(4B'^2 - 2CA) + C(4C'^2 - 2AB),$$

or transposing and dividing by 4,

$$2A'B'C' + ABC - AA'^2 - BB'^2 - CC'^2 = 0.$$

The expression $2A'B'C' + ABC - AA'^2 - BB'^2 - CC'^2$ is called *the discriminant* of the expression (1).

50. It is evident that in any transformation of co-ordinates from one set of axes to another, the origin being unchanged, the expression $x^2+y^2+z^2$ will be transformed into $x'^2+y'^2+z'^2$ if both sets of axes be rectangular; or the expression

$$x^2+y^2+z^2+2yz\cos\lambda+2zx\cos\mu+2xy\cos\nu$$

will be transformed into

$$x'^2+y'^2+z'^2+2y'z'\cos\lambda'+2z'x'\cos\mu'+2x'y'\cos\nu'$$

if the axes are oblique, the expressions in each case representing the square of the distance of the point whose co-ordinates are considered, from the common origin.

Thus if the axes are rectangular, and the expression

$$Ax^2+By^2+Cz^2+2A'yz+2B'zx+2C'xy \ldots\ldots (1)$$

become by transformation

$$Px'^2+Qy'^2+Rz'^2+2P'y'z'+2Q'z'x'+2R'x'y' \ldots (2);$$

we shall have also the expression

$$Ax^2+By^2+Cz^2+2A'yz+2B'zx+2C'xy-\lambda(x^2+y^2+z^2) \ldots (3),$$

where λ is any constant, transformed into

$$Px'^2+Qy'^2+Rz'^2+2P'y'z'+2Q'z'x'+2R'x'y'-\lambda(x'^2+y'^2+z'^2) \ldots (4).$$

But if, for any values of λ the expression (3) be the product of two linear expressions in x, y, z, the expression (4) must, for the *same values* of λ, be the product of the two expressions in x', y', z' into which the former two would be reduced by the transformation. Hence the discriminant of (3) is identical with that of (4), or the two equations

$$(A-\lambda)(B-\lambda)(C-\lambda)-A'^2(A-\lambda)-B'^2(B-\lambda)-C'^2(C-\lambda) + 2A'B'C'=0 \ldots\ldots (5),$$

$$(P-\lambda)(Q-\lambda)(R-\lambda)-P'^2(P-\lambda)-Q'^2(Q-\lambda)-R'^2(R-\lambda) + 2P'Q'R'=0 \ldots\ldots (6),$$

are identical, and satisfied by the same values of λ. Thus the coefficients of the different powers of λ in these equations must be equal, and we have

$$\left.\begin{aligned} A+B+C&=P+Q+R,\\ BC+CA+AB-A'^2-B'^2-C'^2&\\ =QR+RP+PQ&-P'^2-Q'^2-R'^2,\\ ABC+2A'B'C'-AA'^2-BB'^2-CC'^2&\\ =PQR+2P'Q'R'&-PP'^2-QQ'^2-RR'^2 \end{aligned}\right\} \quad (7).$$

The expressions on the left-hand side of the equations (7) are called *invariants* of the expression (1).

51. As a particular case of the foregoing, let us suppose if possible, as it will be proved to be hereafter, that the expression (1) is transformed into an expression of the form

$$Px'^2+Qy'^2+Rz'^2.$$

The equation (6) then becomes

$$(P-\lambda)(Q-\lambda)(R-\lambda)=0,$$

and the roots of this equation are P, Q, R, the coefficients of x'^2, y'^2, z'^2 in the transformed expression. These coefficients are therefore the roots of the equation (5) with which (6) is identical, namely,

$$(A-\lambda)(B-\lambda)(C-\lambda)-A'^2(A-\lambda)-B'^2(B-\lambda)-C'^2(C-\lambda)$$
$$+2A'B'C'=0.$$

Another proof of this result will be given hereafter (Art. 86).

EXAMPLES. CHAPTER IV.

1. The co-ordinates of a point are $(1, 2, 3)$. Find its co-ordinates relative to new axes whose equations are $x=y=z$; $2x=-y=2z$; $x=-z$, $y=0$.

2. Transform the expression $xy+yz+zx$ to the new axes in the last question.

3. Shew that $x^2+y^2+z^2+yz+zx+xy$ can be reduced by transformation of co-ordinates to the form

$$A(x'^2+y'^2)+Bz'^2.$$

4. From the formulæ in Art. 44 prove that

$$l_1=m_2n_3-m_3n_2,\quad l_2=m_3n_1-m_1n_3,\quad l_3=m_1n_2-m_2n_1.$$

5. Find the values of P, Q, R when the expression

$$x^2+y^2+z^2-4xy-4yz-4zx$$

is transformed into the form $Px'^2+Qy'^2+Rz'^2$.

6. Shew that if the expression

$$Ax^2+By^2+Cz^2+2A'yz+2B'zx+2C'xy$$

be transformed into $Px'^2+Qy'^2+Rz'^2$ where the first axes are inclined at angles λ, μ, ν, and the new axes are rectangular, P, Q, R will be the values of k given by the cubic equation

$$\begin{aligned}(A-k)(B-k)(C-k)&-(A'-k\cos\lambda)^2(A-k)\\&-(B'-k\cos\mu)^2(B-k)-(C'-k\cos\nu)^2(C-k)\\&+2(A'-k\cos\lambda)(B'-k\cos\mu)(C'-k\cos\nu)=0.\end{aligned}$$

7. Prove that the equation

$$\sqrt{x}+\sqrt{y}+\sqrt{z}=0$$

represents a cone of revolution round the line

$$x=y=z,$$

whose semi-vertical angle is $\cot^{-1}\sqrt{2}$.

CHAPTER V.

ON GENERATING LINES AND SECTIONS OF QUADRICS.

52. We have seen (Arts. 34, 35) that the cone and cylinder admit of being generated by the motion of a straight line. This is also the case with the hyperboloid of one sheet and with the hyperbolic paraboloid, but not with any other surfaces whose equations are of the second degree in x, y, z.

Surfaces whose equations are of the second degree in (x, y, z) are called *Quadrics*, or, following the analogy of the terms ellipsoid, &c., *Conicoids*.

53. *On the generating lines of the hyperboloid of one sheet.*

The equation of the hyperboloid of one sheet is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

$$\text{or } \frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{y^2}{b^2} \quad\text{.................... (1).}$$

This equation is satisfied by all values of x, y, z which satisfy either of the pairs of equations

$$\left.\begin{aligned}\frac{x}{a}-\frac{z}{c}&=\mu\left(1-\frac{y}{b}\right)\\ \frac{x}{a}+\frac{z}{c}&=\frac{1}{\mu}\left(1+\frac{y}{b}\right)\end{aligned}\right\}\quad\text{.................. (2),}$$

$$\text{or } \left.\begin{aligned}\frac{x}{a}-\frac{z}{c}&=\mu\left(1+\frac{y}{b}\right)\\ \frac{x}{a}+\frac{z}{c}&=\frac{1}{\mu}\left(1-\frac{y}{b}\right)\end{aligned}\right\}\quad\text{.................. (3),}$$

whatever be the value of μ. Each of these pairs represents a straight line. There are thus two systems of straight lines lying wholly on the surface. We shall first prove that all the straight lines of one system intersect all the straight lines of the other; and secondly, that no two lines of the same system intersect one another.

54. The equations of any two straight lines of opposite systems are

$$\left.\begin{aligned} \frac{x}{a}-\frac{z}{c}&=\mu\left(1-\frac{y}{b}\right)\\ \frac{x}{a}+\frac{z}{c}&=\frac{1}{\mu}\left(1+\frac{y}{b}\right)\end{aligned}\right\}\text{..................} (1),$$

$$\left.\begin{aligned} \frac{x}{a}-\frac{z}{c}&=\mu'\left(1+\frac{y}{b}\right)\\ \frac{x}{a}+\frac{z}{c}&=\frac{1}{\mu'}\left(1-\frac{y}{b}\right)\end{aligned}\right\}\text{..................} (2).$$

And if the straight lines represented by these equations meet, these four equations must be satisfied by the same values of x, y, z. But the four equations are all satisfied if we take

$$\left.\begin{aligned} \frac{x}{a}-\frac{z}{c}&=\mu\left(1-\frac{y}{b}\right)\\ \frac{x}{a}+\frac{z}{c}&=\frac{1}{\mu}\left(1+\frac{y}{b}\right)\\ \text{and }\ \mu'\left(1+\frac{y}{b}\right)&=\mu\left(1-\frac{y}{b}\right)\end{aligned}\right\}\text{..................} (3).$$

From which we obtain

$$\frac{y}{b}=\frac{\mu-\mu'}{\mu+\mu'},\quad \frac{x}{a}=\frac{1+\mu\mu'}{\mu+\mu'},\quad \frac{z}{c}=\frac{1-\mu\mu'}{\mu+\mu'}\text{.........}(4).$$

Hence any two generating lines of opposite systems meet in a point.

Conversely, through any point of a hyperboloid of one sheet two straight lines can be drawn lying wholly on the surface. For if we assume the co-ordinates of the point to be x, y, z; from equations (3) we can determine μ and μ', and therefore the equations of the two generating lines through the point in question.

55. Secondly, no two lines of the same system intersect. For let their equations be

$$\left.\begin{aligned} \frac{x}{a}-\frac{z}{c} &= \mu\left(1-\frac{y}{b}\right) \\ \frac{x}{a}+\frac{z}{c} &= \frac{1}{\mu}\left(1+\frac{y}{b}\right) \end{aligned}\right\},$$

$$\left.\begin{aligned} \frac{x}{a}-\frac{z}{c} &= \mu'\left(1-\frac{y}{b}\right) \\ \frac{x}{a}+\frac{z}{c} &= \frac{1}{\mu'}\left(1+\frac{y}{b}\right) \end{aligned}\right\}.$$

From the first and third we get by subtraction,

$$(\mu-\mu')\left(1-\frac{y}{b}\right)=0;$$

therefore $\mu=\mu'$, or $y=b$.

From the second and fourth we get by subtraction,

$$\left(\frac{1}{\mu}-\frac{1}{\mu'}\right)\left(1+\frac{y}{b}\right)=0;$$

$$\therefore \ \mu=\mu', \text{ or } y=-b.$$

Hence since we cannot have y equal both to b and $-b$ we must have $\mu=\mu'$, or the lines must coincide. Therefore no two lines of the same system intersect.

56. The equation of the Hyperbolic paraboloid is

$$\frac{y^2}{l}-\frac{z^2}{l'}=x,$$

which will be satisfied by all values of x, y, z, which satisfy either of the pairs of equations

$$\left.\begin{aligned} \frac{y}{\sqrt{l}}+\frac{z}{\sqrt{l'}} &= \frac{x}{\mu} \\ \frac{y}{\sqrt{l}}-\frac{z}{\sqrt{l'}} &= \mu \end{aligned}\right\},$$

$$\text{or } \left.\begin{aligned} \frac{y}{\sqrt{l}} - \frac{z}{\sqrt{l'}} = \frac{x}{\mu} \\ \frac{y}{\sqrt{l}} + \frac{z}{\sqrt{l'}} = \mu \end{aligned}\right\}.$$

Hence in this case also there are two systems of straight lines lying wholly on the surface. A proof similar to that of the last two articles will shew that all the straight lines of one system intersect all those of the other, and that no two straight lines of the same system intersect one another.

It may be noticed that from the form of the second equation in each set, it follows that all the lines of each system are parallel to a fixed plane.

57. We have shewn in the preceding articles that the hyperboloid of one sheet and the hyperbolic paraboloid admit of rectilinear generators; we shall now shew that these are the only surfaces among those which we have considered, besides the cone and cylinder, with which this is the case.

Let us first take the equation

$$Ax^2 + By^2 + Cz^2 = 1 \quad \text{.................(1)},$$

which includes the ellipsoid and the two hyperboloids; and if possible let the line whose equations are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \quad \text{.............. (2)},$$

lie wholly on the surface (1).

From (2),

$$x = \alpha + lr, \quad y = \beta + mr, \quad z = \gamma + nr;$$

and if the straight line (2) lies wholly on (1) the equation

$$A(\alpha + lr)^2 + B(\beta + mr)^2 + C(\gamma + nr)^2 = 1$$

must be satisfied for all values of r.

The conditions for this are

$$A\alpha^2 + B\beta^2 + C\gamma^2 = 1 \quad \text{..................... (3)},$$
$$Al\alpha + Bm\beta + Cn\gamma = 0 \quad \text{..................... (4)},$$
$$Al^2 + Bm^2 + Cn^2 = 0 \quad \text{..................... (5)}.$$

The first of these equations merely expresses the condition that the point (α, β, γ) may lie on the surface. The second and third are the conditions which l, m, n must satisfy. They will in general give two values for the ratios $l : m : n$. It remains to examine whether these values are real or not.

From (4) we have

$$Cn = -\frac{Al\alpha + Bm\beta}{\gamma}.$$

Substituting in (5) we get

$$CAl^2\gamma^2 + CBm^2\gamma^2 + (Al\alpha + Bm\beta)^2 = 0,$$

which is a quadratic in $\frac{l}{m}$.

The roots of this quadratic will be possible or impossible according as

$$(AC\gamma^2 + A^2\alpha^2)(BC\gamma^2 + B^2\beta^2) < \text{ or } > A^2B^2\alpha^2\beta^2,$$

or as $$ABC^2\gamma^4 + A^2BC\alpha^2\gamma^2 + B^2AC\beta^2\gamma^2 < \text{ or } > 0,$$

or as $$ABC\gamma^2(A\alpha^2 + B\beta^2 + C\gamma^2) < \text{ or } > 0,$$

or as $$ABC < \text{ or } > 0.$$

Hence that the generating lines may be real we must have ABC a negative quantity; thus one or three of the quantities A, B, C must be negative. If they are all three negative, the surface is impossible, so that the only *possible* surface is the hyperboloid of one sheet in which one is negative. In this case we may take

$$A = \frac{1}{a^2}, \quad B = \frac{1}{b^2}, \quad C = -\frac{1}{c^2},$$

and the equations which determine the directions of the generating lines are

$$\frac{l^2}{a^2} + \frac{m^2}{b^2} - \frac{n^2}{c^2} = 0,$$

$$\frac{l\alpha}{a^2} + \frac{m\beta}{b^2} - \frac{n\gamma}{c^2} = 0.$$

58. It may be noticed that since for either of the generating lines we have

$$Al\alpha + Bm\beta + Cn\gamma = 0,$$

and for any point in either line we have

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

we must also have the equation

$$A\alpha(x-\alpha)+B\beta(y-\beta)+C\gamma(z-\gamma)=0,$$

satisfied for any point in either of the straight lines through the point (α, β, γ). But this is the equation of a plane: it is therefore the equation of the plane containing the two straight lines.

The equation can be written

$$A\alpha x+B\beta y+C\gamma z=A\alpha^2+B\beta^2+C\gamma^2=1,$$

and it may be noticed that whether the lines themselves be real or not, this plane is a real plane. We shall prove hereafter that it is the *tangent* plane to the surface at any point (α, β, γ).

59. The equation of the projection of either line on the plane of xy is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m},$$

$$\text{or}\quad y=\frac{m}{l}x+\beta-\frac{m}{l}\alpha \text{ .... (1)},$$

the values of $\frac{m}{l}$ being deduced from the quadratic equation given in Art. 57.

$$Al^2(C\gamma^2+A\alpha^2)+2AB\alpha\beta\, lm+Bm^2(C\gamma^2+B\beta^2)=0,$$

$$\text{or}\quad Al^2(1-B\beta^2)+2AB\alpha\beta\, lm+Bm^2(1-A\alpha^2)=0;$$

$$\therefore Al^2+Bm^2=AB(l\beta-m\alpha)^2.$$

Hence the equation (1) can be written

$$y=\frac{m}{l}x\pm\sqrt{\frac{1}{B}+\frac{1}{A}\cdot\frac{m^2}{l^2}},$$

which is a well-known form of the equation of the tangent to the curve

$$Ax^2+By^2=1.$$

But this curve is the ellipse in which the given surface is cut by the plane of xy. Hence the projections of the generating lines on the plane of xy are tangents to the curve in which the surface is cut by that plane.

The same is true for the planes of yz and zx.

60. The equations of the two paraboloids are both included in the equation

$$By^2 + Cz^2 = x \quad\text{.......................(1).}$$

The conditions that a straight line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad\text{..................(2),}$$

should lie wholly on the surface (1) are found by a process similar to that of Art. 57 to be

$$B\beta^2 + C\gamma^2 = \alpha \quad\text{....................(3),}$$

$$Bm^2 + Cn^2 = 0 \quad\text{....................(4),}$$

$$2Bm\beta + 2Cn\gamma - l = 0 \quad\text{....................(5).}$$

The first equation indicates that the point (α, β, γ) lies on the surface (1). The second and third give the values of the ratios $l : m : n$. These values will be real if B and C have opposite signs, so that the surface must be the hyperbolic paraboloid.

61. The equation of the projection of one of the generating lines on the plane of xy is

$$y = \frac{m}{l}x + \left(\beta - \frac{m}{l}\alpha\right) \quad\text{...............(6).}$$

But from (5)

$$(2Bm\beta - l)^2 = 4C^2n^2\gamma^2$$
$$= -4BC\gamma^2m^2 \text{ from (4)};$$
$$\therefore\ 4Bm^2(B\beta^2 + C\gamma^2) - 4Blm\beta + l^2 = 0;$$
$$\therefore\ 4Bm^2\alpha - 4Blm\beta + l^2 = 0 \text{ from (3)};$$
$$\text{or } l\beta - m\alpha = \frac{1}{4B}\cdot\frac{l^2}{m}.$$

And the equation (6) becomes

$$y = \frac{m}{l} x + \frac{1}{4B} \cdot \frac{l}{m},$$

a well-known form of the equation of the tangent to the curve $By^2 = x$.

Hence the projection of the generating line on the plane of xy is a tangent to the curve in which that plane is cut by the surface. A similar proof holds for the projection on the plane of zx.

The equation of the projection on the plane of yz is

$$\frac{y-\beta}{m} = \frac{z-\gamma}{n},$$

or

$$y = \frac{m}{n} z + \beta - \frac{m}{n} \gamma \quad \text{.................. (7).}$$

But $$Bm^2 + Cn^2 = 0\,;\ \therefore \frac{m}{n} = \pm\sqrt{-\frac{C}{B}},$$

and the equation (7) becomes

$$y = \pm\sqrt{-\frac{C}{B}} \cdot z + \left(\beta \mp \sqrt{-\frac{C}{B}} \cdot \gamma\right).$$

Hence the projections of the generating lines on the plane of yz are parallel to the two straight lines in which the surface is cut by that plane.

62. The sections of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{.................... (1),}$$

made by planes parallel to either of the co-ordinate planes are ellipses. For taking the equation of a plane parallel to that of xy to be

$$z = \gamma \quad \text{.......................... (2),}$$

we get for the points where this meets (1)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{\gamma^2}{c^2}.$$

This is the equation of the projection of the curve of section on the plane of xy. But since the cutting plane is parallel to the plane of xy, the projection of the curve of section on that plane is equal and similar to the curve itself. Hence this curve is an ellipse. And it may be noticed that this ellipse is always similar to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, in which the surface is cut by the plane of xy.

In a similar manner the sections by planes parallel to the other co-ordinate planes may be shewn to be ellipses.

The sections of the hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

by planes parallel to that of xy are ellipses, and those by planes parallel to the planes of yz or zx are hyperbolas.

The sections of the hyperboloid of two sheets

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

by planes parallel to those of zx or xy are hyperbolas, and by planes parallel to that of yz are ellipses, which are impossible if the value of x for points in the cutting plane is numerically less than a.

The sections of the two paraboloids

$$\frac{y^2}{l}+\frac{z^2}{l'}=x,$$

$$\frac{y^2}{l}-\frac{z^2}{l'}=x,$$

by planes parallel to those of zx or xy are parabolas whose latera recta are l' and l respectively.

Their sections by planes parallel to that of yz are respectively ellipses and hyperbolas, the former being impossible when the cutting plane is to the left of the origin.

To find the nature of the sections of these surfaces by planes not parallel to the co-ordinate planes it is no longer sufficient to find the equations of the projections of the curve of section on the co-ordinate planes, since the projection will not in general be similar to the curve itself. The simplest method is to transform the co-ordinates so that the plane of xy shall be parallel to the cutting plane, and then the nature of the section will be given as above by its projection on the plane of xy. For this transformation the formulæ of Art. 45 are very useful. We may in general avoid the third substitution, and since we wish to find merely the nature of the sections by planes parallel to that of $x'y'$; which we shall prove in the next article to be always similar to the section by the plane of $x'y'$ itself, we may before substitution put $z'=0$. The required substitutions will then be derived from the formulæ in Art. 45 by putting $\psi=0$ and $z'=0$. We thus get

$$x = x'\cos\phi - y'\cos\theta\sin\phi,$$
$$y = x'\sin\phi + y'\cos\theta\cos\phi,$$
$$z = y'\sin\theta.$$

If the equation of the cutting plane be given in the form $lx+my+nz=p$, we have $\tan\phi = -\dfrac{l}{m}$, and $\cos\theta = n$. The above substitutions then become

$$x = \frac{mx' + lny'}{\sqrt{l^2+m^2}},\quad y = \frac{mny' - lx'}{\sqrt{l^2+m^2}},\quad z = y'\sqrt{l^2+m^2},$$

where we assume that $l^2+m^2+n^2=1$.

63. We shall first prove the following general proposition.

All sections of surfaces of the second order made by parallel planes are similar and similarly situated.

Take the plane of xy parallel to the system of cutting planes. The equation of the surface can be put into the form

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy + 2A''x + 2B''y + 2C''z + F = 0 \quad\ldots\ldots(1).$$

The curve in which this is cut by the plane

$$z = \gamma \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

is given by the equation

$$Ax^2 + By^2 + 2C'xy + (2B'\gamma + 2A'')\,x + (2A'\gamma + 2B')\,y + C\gamma^2 + 2C''\gamma + F = 0.$$

And whatever be the value of γ this curve is always similar and similarly situated to the curve

$$Ax^2 + By^2 + 2C'xy + 2A''x + 2B''y + F = 0,$$

in which the surface is cut by the plane of xy.

Hence in discussing the form of the sections of surfaces by a series of planes, we need only consider planes through the origin.

This method will not fail even if the curve of section by a plane through the origin become impossible, since the terms of the second degree in the equation of this curve are the same as in the equations of the possible curves formed by the intersection of parallel planes with the surface.

64. We shall consider first the equation

$$Ax^2 + By^2 + Cz^2 = 1,$$

which includes the three central surfaces.

Making the substitutions suggested in Art. 62, we get as the equation of the curve of section

$$x'^2(A\cos^2\phi + B\sin^2\phi) + 2x'y'(B - A)\cos\phi\sin\phi\cos\theta + y'^2(A\cos^2\theta\sin^2\phi + B\cos^2\theta\cos^2\phi + C\sin^2\theta) = 1.$$

And the section will therefore be an ellipse or hyperbola according as

$$(B - A)^2\cos^2\theta\cos^2\phi\sin^2\phi - (A\cos^2\phi + B\sin^2\phi)(A\cos^2\theta\sin^2\phi + B\cos^2\theta\cos^2\phi + C\sin^2\theta)$$

is negative or positive. This expression can be reduced to the form

$$-\{BC\sin^2\theta\sin^2\phi + CA\sin^2\theta\cos^2\phi + AB\cos^2\theta\}.$$

In the case of the ellipsoid A, B and C are all positive, and this expression is therefore always negative. All sections of the ellipsoid are therefore ellipses. The investigation of the nature of the sections in the other surfaces is long and the results uninteresting, except in the particular case in which the section becomes a circle.

The conditions that this may be the case are, that the coefficient of $x'y'$ should vanish and the coefficients of x'^2 and y'^2 should be equal. We have therefore

$$(B-A)\cos\theta\sin\phi\cos\phi=0,$$

$$A\cos^2\phi+B\sin^2\phi=A\cos^2\theta\sin^2\phi+B\cos^2\theta\cos^2\phi+C\sin^2\theta.$$

From the first equation we must have either $B=A$, in which case it is already obvious that all sections parallel to the plane of xy are circles, or

$$\cos\theta\,.\,\sin\phi\,.\,\cos\phi=0.$$

If $\cos\theta=0$, we have $\theta=90^\circ$, and the second equation gives

$$A\cos^2\phi+B\sin^2\phi=C=C(\cos^2\phi+\sin^2\phi);$$

$$\therefore\ \tan^2\phi=\frac{C-A}{B-C},$$

and if the values of $\tan\phi$ be real, we get circular sections by two planes through the axis of z.

If we take $\cos\phi=0$; we have $\phi=90^\circ$, or the plane passes through the axis of y, and the second condition gives

$$B=A\cos^2\theta+C\sin^2\theta;$$

and therefore $$\tan^2\theta=\frac{A-B}{B-C},$$

and if the values of $\tan\theta$ be real, we get circular sections by planes through the axis of y.

Similarly from the condition $\sin\phi=0$, we get circular sections by planes through the axis of x inclined to the plane of xy at angles given by the equation

$$\tan^2\theta=\frac{A-B}{C-A}.$$

In all cases the circular sections are made by planes passing through one of the axes. It only remains to examine in what cases they are real.

Only one of the three quantities

$$\frac{C-A}{B-C},\ \frac{A-B}{B-C},\ \frac{A-B}{C-A}$$

can be positive, consequently there are only two real central circular sections, and they pass through the axis of z, y or x, according as the first, second, or third of these expressions is positive.

(1) In the ellipsoid A, B, C are all positive, and if we take them in order of magnitude, the second of the above expressions is positive. Consequently the central circular sections of an ellipsoid are made by planes through the *mean* axis.

(2) In the hyperboloid of one sheet C is negative, and if we suppose $A > B$, it is again the second of the above expressions that is positive, and the circular section is made by a plane through the greater real axis, since

$$A = \frac{1}{a^2},\ B = \frac{1}{b^2},$$

and A being $> B$, $a < b$.

(3) In the hyperboloid of two sheets, B and C are negative, and if we suppose B numerically greater than C, or $b < c$, $B - C$ will be negative, the first of the above expressions is positive, and the circular section is made by a plane through the greater impossible axis.

65. We have shewn in the last article that the *only* planes which give circular sections of central quadrics are certain planes through one of the axes. It is easy to shew without transformation that these planes do give circular sections.

Thus the equation of the ellipsoid can be written in the form

$$\frac{x^2 + y^2 + z^2}{b^2} + x^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + z^2\left(\frac{1}{c^2} - \frac{1}{b^2}\right) = 1,$$

or

$$x^2 + y^2 + z^2 + \left\{\frac{z}{c}\sqrt{b^2 - c^2} - \frac{x}{a}\sqrt{a^2 - b^2}\right\}\left\{\frac{z}{c}\sqrt{b^2 - c^2} + \frac{x}{a}\sqrt{a^2 - b^2}\right\} = b^2,$$

which shews that either of the planes

$$\frac{z}{c}\sqrt{b^2-c^2}-\frac{x}{a}\sqrt{a^2-b^2}=0 \text{ ....} (1),$$

or

$$\frac{z}{c}\sqrt{b^2-c^2}+\frac{x}{a}\sqrt{a^2-b^2}=0 \text{ ....} (2),$$

cuts the ellipsoid in the same points in which it cuts the sphere

$$x^2+y^2+z^2=b^2.$$

But every plane section of a sphere is a circle. Hence the planes (1) and (2) and consequently by Art. 63 all planes parallel to them cut the ellipsoid in circles.

The circular sections of the hyperboloids of one and two sheets can be deduced in a similar manner.

66. The two paraboloids are included in the equation

$$By^2+Cz^2=x.$$

Making the same substitutions as in Art. 64 we obtain for the equation of the curve of intersection,

$$B\sin^2\phi x'^2+2B\sin\phi\cos\phi\cos\theta\, x'y'$$
$$+y'^2(B\cos^2\theta\cos^2\phi+C\sin^2\theta)=x'\cos\phi-y'\cos\theta\sin\phi,$$

which will represent an ellipse, parabola, or hyperbola, according as

$$B^2\sin^2\phi\cos^2\phi\cos^2\theta-B\sin^2\phi\,(B\cos^2\theta\cos^2\phi+C\sin^2\theta)$$

is negative, zero, or positive. That is, according as

$$BC\sin^2\phi\sin^2\theta$$

is positive, zero, or negative.

The sections of both paraboloids are therefore parabolas if ϕ or θ vanish, that is, if the cutting plane pass through the axis of x or coincide with the plane of xy. In all other cases the sections of the elliptic paraboloid are ellipses, and of the hyperbolic paraboloid, hyperbolas.

The conditions that the section may be a circle are

$$B \sin\phi \cos\phi \cos\theta = 0,$$

$$B \sin^2\phi = B \cos^2\phi \cos^2\theta + C \sin^2\theta.$$

From the first equation

$$\sin\phi = 0, \ \cos\phi = 0, \ \text{or} \ \cos\theta = 0.$$

If $\sin\phi = 0$, the coefficient of x'^2 vanishes, and the section reduces to a straight line or parabola.

If $\cos\phi = 0$, we have from the second equation $B = C\sin^2\theta$, and if B and C are of the same sign and $B < C$ this gives two possible values of θ. If $\cos\theta = 0$, we get $\sin^2\phi = \frac{C}{B}$, and this gives two possible values of ϕ if $C < B$, and B and C have the same sign. Thus we get real circular sections of the elliptic paraboloid passing through the axis of y or z, according as $B <$ or $> C$, that is as $l >$ or $< l'$.

If B and C have opposite signs, there are no real circular sections.

67. The equation of the elliptic paraboloid can be put into the form

$$\frac{x^2 + y^2 + z^2}{l'} - \frac{x^2}{l'} + \frac{y^2}{l} - \frac{y^2}{l'} = x,$$

or $$\frac{x^2 + y^2 + z^2}{l'} + \left(y\sqrt{\frac{1}{l} - \frac{1}{l'}} - \frac{x}{\sqrt{l'}}\right)\left(y\sqrt{\frac{1}{l} - \frac{1}{l'}} + \frac{x}{\sqrt{l'}}\right) = x.$$

Thus each of the planes

$$y\sqrt{\frac{1}{l} - \frac{1}{l'}} - x\sqrt{\frac{1}{l'}} = 0,$$

$$y\sqrt{\frac{1}{l} - \frac{1}{l'}} + x\sqrt{\frac{1}{l'}} = 0,$$

and therefore all planes parallel to them will cut the surface in circles. These planes are real if $l' > l$. If $l' < l$ we can shew similarly that the planes

$$z\sqrt{\frac{1}{l'}-\frac{1}{l}}-x\sqrt{\frac{1}{l}}=0,$$

$$z\sqrt{\frac{1}{l'}-\frac{1}{l}}+x\sqrt{\frac{1}{l}}=0,$$

cut the surface in circles.

68. We shall conclude this chapter with the investigation of the position and magnitude of the axes of the section of an ellipsoid by a plane through its centre.

Let $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad\ldots\ldots\ldots\ldots (1)$$

be the equation of the ellipsoid,

$$lx+my+nz=0 \quad\ldots\ldots\ldots\ldots (2)$$

the equation of the cutting plane.

Let $$\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}=r \quad\ldots\ldots\ldots\ldots (3)$$

be the equations of any straight line in the plane (2), and let r be the distance from the origin of the point where it meets the ellipsoid; therefore

$$\frac{1}{r^2}=\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}+\frac{\nu^2}{c^2} \quad\ldots\ldots\ldots\ldots (4),$$

and $$l\lambda+m\mu+n\nu=0 \quad\ldots\ldots\ldots\ldots (5),$$

since the line (3) lies in the plane (2).

Also if r be the length of one of the semiaxes of the section of (1) by (2), we must have r a maximum or minimum by the variation of λ, μ, ν, which are connected by the relation (5) and also by the relation

$$\lambda^2+\mu^2+\nu^2=1 \quad\ldots\ldots\ldots\ldots (6).$$

Differentiating (4) we get when r is a maximum or minimum

$$0=\frac{\lambda d\lambda}{a^2}+\frac{\mu d\mu}{b^2}+\frac{\nu d\nu}{c^2}.$$

And from (5) and (6) respectively,

$$0 = l d\lambda + m d\mu + n d\nu,$$

$$0 = \lambda d\lambda + \mu d\mu + \nu d\nu.$$

Whence by indeterminate multipliers,

$$\frac{\lambda}{a^2} + kl + k'\lambda = 0 \quad \dots\dots\dots\dots\dots\dots (7),$$

$$\frac{\mu}{b^2} + km + k'\mu = 0 \quad \dots\dots\dots\dots\dots\dots (8),$$

$$\frac{\nu}{c^2} + kn + k'\nu = 0 \quad \dots\dots\dots\dots\dots\dots (9).$$

Multiplying (7) by λ, (8) by μ, (9) by ν, and adding, we get

$$\frac{1}{r^2} + k' = 0,$$

and therefore

$$\lambda\left(\frac{1}{a^2} - \frac{1}{r^2}\right) = -kl, \qquad \therefore \lambda = \frac{klr^2a^2}{a^2 - r^2},$$

$$\mu\left(\frac{1}{b^2} - \frac{1}{r^2}\right) = -km, \qquad \mu = \frac{kmr^2b^2}{b^2 - r^2},$$

$$\nu\left(\frac{1}{c^2} - \frac{1}{r^2}\right) = -kn, \qquad \nu = \frac{knr^2c^2}{c^2 - r^2}.$$

And therefore from (5),

$$\frac{l^2a^2}{a^2 - r^2} + \frac{m^2b^2}{b^2 - r^2} + \frac{n^2c^2}{c^2 - r^2} = 0 \quad \dots\dots\dots\dots (10),$$

which is a quadratic equation and gives two values of r^2.

The product of these two values

$$= \frac{a^2b^2c^2}{l^2a^2 + m^2b^2 + n^2c^2},$$

and the area of the section is therefore

$$= \frac{\pi abc}{\sqrt{l^2a^2 + m^2b^2 + n^2c^2}} \quad \dots\dots\dots\dots\dots\dots (11).$$

The directions of the two axes may be obtained by eliminating k and k' from equations (7), (8) and (9); we then get

$$\begin{vmatrix} \frac{\lambda}{a^2} & l & \lambda \\ \frac{\mu}{b^2} & m & \mu \\ \frac{\nu}{c^2} & n & \nu \end{vmatrix} = 0,$$

or $l\mu\nu\left(\frac{1}{b^2}-\frac{1}{c^2}\right)+m\nu\lambda\left(\frac{1}{c^2}-\frac{1}{a^2}\right)+n\lambda\mu\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=0$ ... (12),

which united with (5) and (6) gives two sets of values of λ, μ, ν.

The expression for the area of a section of an ellipsoid by a plane not passing through the centre will be given in a future article. (Art. 79.)

EXAMPLES. CHAPTER V.

1. Shew that the two generating lines of the surface

$$c^2(x^2+y^2)-a^2z^2=a^2c^2$$

drawn through a point for which $z=\pm c\sqrt{\frac{a^2-c^2}{a^2+c^2}}$ are at right angles to each other.

2. Shew that all the points on the surface

$$\frac{x^2+y^2}{a^2}-\frac{z^2}{c^2}=1,$$

for which the generating lines are inclined at an angle α, lie in one or other of two fixed planes.

3. Find the angle between the two generating lines of the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

at the point α, β, γ.

4. If the surface

$$(x^2+y^2+z^2)^2 = a^2x^2+b^2y^2+c^2z^2$$

be cut by a central circular section of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

the sum of the squares on any two perpendicular radii vectores of the curve of section is constant.

5. The equation of a surface can be put into the form

$$x^2+y^2+z^2+(lx+my+nz-p)\,(l'x+m'y+n'z-p')=0,$$

find the planes which give circular sections.

6. Prove that the sections of the surface

$$xy+yz+zx=1,$$

by planes parallel to $x+y+z=0$, are circles.

7. If the two generators drawn from a point O on the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

intersect the principal ellipse in points P, P' at the ends of conjugate diameters, then will

$$OP^2+OP'^2=a^2+b^2+2c^2.$$

8. Find the circular sections of the surface

$$\frac{yz}{a^2}+\frac{zx}{b^2}+\frac{xy}{c^2}=1.$$

9. Prove that if the section of the surface

$$\frac{yz}{a^2}+\frac{zx}{b^2}+\frac{xy}{c^2}=1$$

by the plane $lx+my+nz=0$ be a rectangular hyperbola,

$$\frac{1}{la^2}+\frac{1}{mb^2}+\frac{1}{nc^2}=0.$$

10. The angle between the generating lines of

$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1$ at the point (x, y, z) is $\cos^{-1}\frac{\lambda_1+\lambda_2}{\lambda_1 \sim \lambda_2}$,

where λ_1 and λ_2 are the two roots of

$$\frac{x^2}{a(a+\lambda)}+\frac{y^2}{b(b+\lambda)}+\frac{z^2}{c(c+\lambda)}=0.$$

11. Prove that the foci of all centric sections of the surface

$$ax^2+by^2+cz^2=1$$

lie on the surface

$$(x^2+y^2+z^2)(1-ax^2-by^2-cz^2)\{a(c-b)^2y^2z^2+b(a-c)^2z^2x^2+c(b-a)^2x^2y^2\}$$
$$=(ax^2+by^2+cz^2)\{(c-b)^2y^2z^2+(a-c)^2z^2x^2+(b-a)^2x^2y^2\}.$$

12. Find the equation of a right circular cylinder whose axis is the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n},$$

and whose radius is a.

13. Find the condition that the cone

$$Ax^2+By^2+Cz^2+2A'yz+2B'zx+2C'xy=0$$

may have three generating lines mutually at right angles.

14. Find the equation of the right cone which has a centric circular section of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

for its base and its altitude equal to b.

15. Find the equation of a right circular cone referred to rectangular axes, having its vertex at the origin, and meeting each of the co-ordinate planes in one line only.

16. Find the equation of a right circular cone whose axis is the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$, and semi-vertical angle α.

17. Find the equation of a right circular cone which contains three given straight lines passing through the origin.

18. Find the locus of the points at which the two generating lines of the surface

$$Ax^2 + By^2 + Cz^2 = 1$$

are at right angles.

19. If a plane be drawn through the straight line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n},$$

the two other straight lines in which it cuts the cone

$$(B-C)yz(mz-ny)+(C-A)zx(nx-lz)+(A-B)xy(ly-mx)=0$$

will be at right angles to each other.

20. Shew that any point on the hyperboloid of one sheet may be represented by the equations

$$x = a \cos\phi \sec\theta,$$
$$y = b \sin\phi \sec\theta,$$
$$z = c \tan\theta;$$

and find the equations of the generating lines through that point.

21. Shew that if the two generating lines at any point o the surface

$$\frac{x^2+y^2}{a^2} - \frac{z^2}{c^2} = 1$$

be at right angles respectively to those of opposite system through a second point, the two points are either in a plane through the axis of z or equally distant from the plane of xy

22. If two planes be drawn passing respectively through two generating lines of the same system at the extremities o the major axis of the principal elliptic section of a hyperboloi of one sheet and intersecting in any third generating line, th traces of these planes on either of two fixed planes will be a right angles.

23. If $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$ be the equations of the fou faces of a tetrahedron expressed as in Art. 26, the equatio of a hyperboloid of one sheet passing through two opposit edges is

$$P\alpha\beta + Q\gamma\delta + R\delta\alpha + S\beta\gamma = 0.$$

CHAPTER VI.

DIAMETRAL PLANES.

69. It will be useful to commence the chapter with the following definitions.

1. *The centre of a surface is a point such that all chords passing through it are bisected by it.*

2. *The locus of the middle points of a system of parallel chords of a surface is called the diametral surface of the system.*

We shall shew that if the original surface be a quadric, the diametral surface of any system of parallel chords is a plane. In this case we shall require the following definition.

3. *A principal plane of a quadric is a plane perpendicular to the chords which it bisects.*

We shall shew hereafter that such a plane can always be found.

70. *If a quadric have a centre and be referred to a system of axes with the centre as origin, the equation will not contain any terms of the first degree.*

For the general equation of the second degree is

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy + 2A''x + 2B''y + 2C''z + F = 0 \quad \ldots\ldots (1).$$

Then if x_1, y_1, z_1 be the co-ordinates of any point on the surface, $-x_1, -y_1, -z_1$ must also satisfy the equation (1), since the origin is the centre. Hence we have

$$Ax_1^2+By_1^2+Cz_1^2+2A'y_1z_1+2B'z_1x_1+2C'x_1y_1 \\ +2A''x_1+2B''y_1+2C''z_1+F=0,$$

$$Ax_1^2+By_1^2+Cz_1^2+2A'y_1z_1+2B'z_1x_1+2C'x_1y_1 \\ -2A''x_1-2B''y_1-2C''z_1+F=0.$$

Subtracting we obtain

$$4(A''x_1+B''y_1+C''z_1)=0 \quad\ldots\ldots\ldots\ldots\ldots (2).$$

This equation must be satisfied for all values of x_1, y_1, z_1 consistent with (1). But unless $A''=0$, $B''=0$, $C''=0$, equation (2) can only be satisfied by the co-ordinates of points lying in the plane

$$A''x+B''y+C''z=0.$$

Consequently we must have

$$A''=0,\ B''=0,\ C''=0,$$

or the equation (1) does not involve the first powers of x, y, z.

Conversely, if the equation of a quadric do not involve the first powers of x, y, z, the origin is the centre of the surface. Moreover, if the equation can be put in the form

$$Ax^2+By^2+Cz^2=F \quad\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

the axes being rectangular, the co-ordinate planes will be principal planes. For if x_1, y_1, z_1 satisfy the equation (3), so do $-x_1$, y_1, z_1. Hence the plane of yz bisects all ordinates parallel to the axis of x, and similarly for the other co-ordinate planes.

Conversely, if each co-ordinate plane bisect all chords parallel to the corresponding axis the equation must assume the above form.

71. *To find the locus of the middle points of a system of parallel chords drawn in an ellipsoid.*

Let the equation of the ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and let the equations of any one of the system of parallel chords be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \quad\text{.............. (2)},$$

where l, m, n are direction-cosines.

To find the points where (2) meets (1) we have

$$\frac{(\alpha+lr)^2}{a^2}+\frac{(\beta+mr)^2}{b^2}+\frac{(\gamma+nr)^2}{c^2}=1,$$

or $r^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)+2r\left(\frac{l\alpha}{a^2}+\frac{m\beta}{b^2}+\frac{n\gamma}{c^2}\right)+\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1=0 \ldots (3).$

This equation gives two values of r which are the distances from the point (α, β, γ) of the two points where the straight line (2) cuts the ellipsoid. If (α, β, γ) be the middle point of the chord these two values must be equal, and opposite in sign; the coefficient of r in the equation (3) must therefore vanish, or

$$\frac{l\alpha}{a^2}+\frac{m\beta}{b^2}+\frac{n\gamma}{c^2}=0.$$

Hence (α, β, γ) always lies in the plane

$$\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}=0 \quad\text{.................. (4)},$$

which is therefore the equation of the locus of the middle points of the system of chords.

72. If x_1, y_1, z_1 be the co-ordinates of the point in which the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ meets the ellipsoid, that is, the co-ordinates of the extremity of the diameter drawn parallel to the system of parallel chords, we have

$$\frac{x_1}{l}=\frac{y_1}{m}=\frac{z_1}{n},$$

and the equation (4) of the last article may be written

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}+\frac{z_1z}{c^2}=0 \quad\text{.................. (1)}.$$

Also if x_2, y_2, z_2 be the co-ordinates of any point in th curve in which this plane cuts the ellipsoid, we have

$$\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} + \frac{z_1z_2}{c^2} = 0,$$

which shews that the point (x_1, y_1, z_1) lies in the plan which bisects all chords parallel to the diameter throug (x_2, y_2, z_2).

The planes which bisect chords parallel to the two diame ters through (x_1, y_1, z_1), (x_2, y_2, z_2) will intersect in a straigh line. Let the co-ordinates of the point where this line meet the ellipsoid be x_3, y_3, z_3. Then since (x_3, y_3, z_3) lies in th plane which bisects chords parallel to the diameter throug (x_1, y_1, z_1) we have

$$\frac{x_3x_1}{a^2} + \frac{y_3y_1}{b^2} + \frac{z_3z_1}{c^2} = 0,$$

and since it lies in the plane which bisects chords parallel t the diameter through (x_2, y_2, z_2), we have

$$\frac{x_2x_3}{a^2} + \frac{y_2y_3}{b^2} + \frac{z_2z_3}{c^2} = 0.$$

These last equations shew that (x_1, y_1, z_1), (x_2, y_2, z_2) bot lie in the plane which bisects all chords parallel to the diame ter through (x_3, y_3, z_3).

Hence the three diameters have this property, that th plane through any two of them bisects chords parallel to th third.

The three diameters are called conjugate diameters.

73. *The equation of the ellipsoid when referred to a sys tem of three conjugate diameters as axes assumes the form*

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1,$$

where a′, b′, c′ *are the lengths of the conjugate semi-diameters*

For the equation must be of the second degree by Art. 48 and since each co-ordinate plane bisects chords parallel t the corresponding axis, by Art. 70 the equation must assum the form

$$Ax^2 + By^2 + Cz^2 = F.$$

When the axis of x meets the surface we have

$$x = a', \quad y = 0, \quad z = 0,$$

and therefore
$$a'^2 = \frac{F}{A}.$$

Similarly
$$b'^2 = \frac{F}{B},$$

$$c'^2 = \frac{F}{C}.$$

And the equation becomes

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1.$$

74. The co-ordinates of the extremities of three conjugate diameters are connected by the relations

$$\left.\begin{array}{l} \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} + \dfrac{z_1^2}{c^2} - 1 = 0 \\ \dfrac{x_2^2}{a^2} + \dfrac{y_2^2}{b^2} + \dfrac{z_2^2}{c^2} - 1 = 0 \\ \dfrac{x_3^2}{a^2} + \dfrac{y_3^2}{b^2} + \dfrac{z_3^2}{c^2} - 1 = 0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\left.\begin{array}{l} \dfrac{x_2x_3}{a^2} + \dfrac{y_2y_3}{b^2} + \dfrac{z_2z_3}{c^2} = 0 \\ \dfrac{x_3x_1}{a^2} + \dfrac{y_3y_1}{b^2} + \dfrac{z_3z_1}{c^2} = 0 \\ \dfrac{x_1x_2}{a^2} + \dfrac{y_1y_2}{b^2} + \dfrac{z_1z_2}{c^2} = 0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Squaring all these equations, and adding twice the squares of the second three to the squares of the first three, we get

$$\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right)^2 + \left(\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} + \frac{z_2^2}{c^2} - 1\right)^2 + \left(\frac{x_3^2}{a^2} + \frac{y_3^2}{b^2} + \frac{z_3^2}{c^2} - 1\right)^2$$

$$2\left(\frac{x_2x_3}{a^2} + \frac{y_2y_3}{b^2} + \frac{z_2z_3}{c^2}\right)^2 + 2\left(\frac{x_3x_1}{a^2} + \frac{y_3y_1}{b^2} + \frac{z_3z_1}{c^2}\right)^2 + 2\left(\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} + \frac{z_1z_2}{c^2}\right)^2 = 0.$$

Expanding, and rearranging the terms we get

$$\left(\frac{x_1^2}{a^2}+\frac{x_2^2}{a^2}+\frac{x_3^2}{a^2}-1\right)^2+\left(\frac{y_1^2}{b^2}+\frac{y_2^2}{b^2}+\frac{y_3^2}{b^2}-1\right)^2+\left(\frac{z_1^2}{c^2}+\frac{z_2^2}{c^2}+\frac{z_3^2}{c^2}-1\right)^2$$

$$+2\left(\frac{y_1z_1}{bc}+\frac{y_2z_2}{bc}+\frac{y_3z_3}{bc}\right)^2+2\left(\frac{z_1x_1}{ca}+\frac{z_2x_2}{ca}+\frac{z_3x_3}{ca}\right)^2+2\left(\frac{x_1y_1}{ab}+\frac{x_2y_2}{ab}+\frac{x_3y_3}{ab}\right)^2=$$

Whence

$$\left.\begin{aligned} x_1^2+x_2^2+x_3^2&=a^2\\ y_1^2+y_2^2+y_3^2&=b^2\\ z_1^2+z_2^2+z_3^2&=c^2\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$\left.\begin{aligned} y_1z_1+y_2z_2+y_3z_3&=0\\ z_1x_1+z_2x_2+z_3x_3&=0\\ x_1y_1+x_2y_2+x_3y_3&=0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

This transformation can be easily seen to be equivalent to that effected in Art. 44, using $\frac{x_1}{a}$ for l_1, and so on. And the method of that article may be employed to deduce (3) and (4) from (1) and (2).

Similar relations exist between the direction-cosines of the normals to the three planes, each of which bisects chords parallel to the intersection of the other two. For if l_1, m_1, n_1 be the direction-cosines of the normal to the plane bisecting chords parallel to the line

$$\frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1},$$

we have

$$\frac{l_1}{\frac{x_1}{a^2}}=\frac{m_1}{\frac{y_1}{b^2}}=\frac{n_1}{\frac{z_1}{c^2}},$$

or

$$\frac{al_1}{\frac{x_1}{a}}=\frac{bm_1}{\frac{y_1}{b}}=\frac{cn_1}{\frac{z_1}{c}},$$

and similar relations for l_2, m_2, n_2. Whence equations (2) easily give

$$\left.\begin{aligned} a^2l_1l_2+b^2m_1m_2+c^2n_1n_2&=0\\ a^2l_2l_3+b^2m_2m_3+c^2n_2n_3&=0\\ a^2l_3l_1+b^2m_3m_1+c^2n_3n_1&=0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(5),$$

and obviously also

$$\left.\begin{aligned} l_1^2+m_1^2+n_1^2=1 \\ l_2^2+m_2^2+n_2^2=1 \\ l_3^2+m_3^2+n_3^2=1 \end{aligned}\right\}\text{.....................(6).}$$

75. From equations (3) of the last article we obtain by addition

$$a'^2+b'^2+c'^2=a^2+b^2+c^2 \text{...............(1),}$$

where a', b', c' are the lengths of the conjugate semi-diameters.

Let λ, μ, ν be the angles between (b', c'), (c', a') and (a', b'), respectively.

Then since the direction-cosines of a' referred to the principal axes of the ellipsoid are $\frac{x_1}{a'}, \frac{y_1}{a'}, \frac{z_1}{a'}$, and similarly for those of b', c', we have, by Art. (8),

$$\sin^2\lambda = \frac{(y_2z_3-y_3z_2)^2+(z_2x_3-z_3x_2)^2+(x_2y_3-x_3y_2)^2}{b'^2c'^2};$$

$$\therefore\ b'^2c'^2\sin^2\lambda = (y_2z_3-y_3z_2)^2+(z_2x_3-z_3x_2)^2+(x_2y_3-x_3y_2)^2.$$

But we have $\frac{x_1}{a}\cdot\frac{x_2}{a}+\frac{y_1}{b}\cdot\frac{y_2}{b}+\frac{z_1}{c}\cdot\frac{z_2}{c}=0,$

$$\frac{x_1}{a}\cdot\frac{x_3}{a}+\frac{y_1}{b}\cdot\frac{y_3}{b}+\frac{z_1}{c}\cdot\frac{z_3}{c}=0;$$

$$\therefore\ \frac{\frac{x_1}{a}}{\frac{y_2z_3-y_3z_2}{bc}}=\frac{\frac{y_1}{b}}{\frac{z_2x_3-z_3x_2}{ca}}=\frac{\frac{z_1}{c}}{\frac{x_2y_3-x_3y_2}{ab}}$$

$$=\sqrt{\frac{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}}{\left(\frac{y_2z_3-y_3z_2}{bc}\right)^2+\left(\frac{z_2x_3-z_3x_2}{ca}\right)^2+\left(\frac{x_2y_3-x_3y_2}{ab}\right)^2}}$$

$$=\sqrt{\frac{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}}{\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}\right)\left(\frac{x_3^2}{a^2}+\frac{y_3^2}{b^2}+\frac{z_3^2}{c^2}\right)-\left(\frac{x_2x_3}{a^2}+\frac{y_2y_3}{b^2}+\frac{z_2z_3}{c^2}\right)^2}}=\pm 1$$

by equations (1) and (2) of the last article.

Hence $b'^2c'^2\sin^2\lambda = b^2c^2\dfrac{x_1^{\ 2}}{a^2} + c^2a^2\dfrac{y_1^{\ 2}}{b^2} + a^2b^2\dfrac{z_1^{\ 2}}{c^2}$.

Similarly $c'^2a'^2\sin^2\mu = b^2c^2\dfrac{x_2^{\ 2}}{a^2} + c^2a^2\dfrac{y_2^{\ 2}}{b^2} + a^2b^2\dfrac{z_2^{\ 2}}{c^2}$,

$$a'^2b'^2\sin^2\nu = b^2c^2\frac{x_3^{\ 2}}{a^2} + c^2a^2\frac{y_3^{\ 2}}{b^2} + a^2b^2\frac{z_3^{\ 2}}{c^2}.$$

Adding, we get

$$(b'c'\sin\lambda)^2 + (c'a'\sin\mu)^2 + (a'b'\sin\nu)^2 = b^2c^2 + c^2a^2 + a^2b^2 \ldots (2).$$

Again, if p be the perpendicular from the point (x_3, y_3, z_3) on the plane which contains a' and b', whose equation is

$$\frac{xx_3}{a^2} + \frac{yy_3}{b^2} + \frac{zz_3}{c^2} = 0,$$

we have $p = \dfrac{\dfrac{x_3^{\ 2}}{a^2} + \dfrac{y_3^{\ 2}}{b^2} + \dfrac{z_3^{\ 2}}{c^2}}{\sqrt{\left(\dfrac{x_3^{\ 2}}{a^4} + \dfrac{y_3^{\ 2}}{b^4} + \dfrac{z_3^{\ 2}}{c^4}\right)}}$

$$= \frac{1}{\sqrt{\left(\dfrac{x_3^{\ 2}}{a^4} + \dfrac{y_3^{\ 2}}{b^4} + \dfrac{z_3^{\ 2}}{c^4}\right)}}.$$

Hence squaring and multiplying by the value previously obtained for $a'^2b'^2\sin^2\nu$ we get

$$p^2a'^2b'^2\sin^2\nu = a^2b^2c^2 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

But $a'b'\sin\nu$ is the area of the parallelogram whose edges are a' and b', and $pa'b'\sin\nu$ is the volume of the parallelepiped whose base is this parallelogram and whose altitude is p, that is, the volume of the parallelepiped whose three edges are a', b', c'.

By Art. 14 this volume can be expressed in the form

$$a'b'c'\sqrt{1 - \cos^2\lambda - \cos^2\mu - \cos^2\nu + 2\cos\lambda\cos\mu\cos\nu}.$$

Hence this expression is equal to abc.

76. Another method of obtaining these relations is afforded by the consideration that the expression

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + k(x^2 + y^2 + z^2)$$

is transformed by taking three conjugate diameters as axes to the expression

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}+k(x^2+y^2+z^2+2yz\cos\lambda+2zx\cos\mu+2xy\cos\nu).$$

Consequently, if for any value of k the first expression split up into two linear factors, the second expression will do so likewise for the *same value* of k.

By Art. 49 the requisite values of k for the two expressions are given respectively by the equations

$$\left(k+\frac{1}{a^2}\right)\left(k+\frac{1}{b^2}\right)\left(k+\frac{1}{c^2}\right)=0,$$

and

$$\left(k+\frac{1}{a'^2}\right)\left(k+\frac{1}{b'^2}\right)\left(k+\frac{1}{c'^2}\right)$$
$$-k^2\cos^2\lambda\left(k+\frac{1}{a'^2}\right)-k^2\cos^2\mu\left(k+\frac{1}{b'^2}\right)-k^2\cos^2\nu\left(k+\frac{1}{c'^2}\right)$$
$$+2k^3\cos\lambda\cos\mu\cos\nu=0,$$

which when cleared of fractions and expanded become respectively,

$$a^2b^2c^2k^3+(a^2b^2+b^2c^2+c^2a^2)k^2+(a^2+b^2+c^2)k+1=0,$$

and

$$a'^2b'^2c'^2(1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu)k^3$$
$$+(b'^2c'^2\sin^2\lambda+c'^2a'^2\sin^2\mu+a'^2b'^2\sin^2\nu)k^2$$
$$+(a'^2+b'^2+c'^2)k+1=0.$$

And since these equations are identical we get the relations (1), (2) and (3).

They can also be obtained geometrically by a series of transformations; or by finding the values of the maximum radius vector of the surface when referred to three conjugate diameters as axes. The result will be a cubic equation in r^2, and the three values of r^2 will be a^2, b^2, c^2; whence the values of

$$a^2b^2c^2,\quad a^2b^2+c^2a^2+b^2c^2,\quad a^2+b^2+c^2$$

are known in terms of a', b', c'.

The formulæ obtained in Arts. 71—76 hold for the other central surfaces if the proper changes be made in the signs of a^2, b^2 and c^2.

77. The equation of the plane which bisects all chords of the ellipsoid parallel to the line

$$\frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1}\text{.......................}(1)$$

is

$$x\,.\,\frac{x_1}{a^2}+y\,.\,\frac{y_1}{b^2}+z\,.\,\frac{z_1}{c^2}=0\text{..............}(2).$$

Conversely the chords which are bisected by the plane

$$lx+my+nz=0\text{.....................}(3)$$

are parallel to the line

$$\frac{x}{a^2l}=\frac{y}{b^2m}=\frac{z}{c^2n}\text{.......................}(4).$$

The line (4) is said to be conjugate to the plane (3).

By Art. 72 every system of chords parallel to any line which lies in the plane (3) is bisected by some plane passing through (4).

Hence the plane passing through the origin which bisects any system of parallel chords of the section of the ellipsoid by a plane

$$lx+my+nz-p=0\text{.................}(5)$$

parallel to (3), must contain the straight line (4). Whence it easily follows that the point where (4) meets (5) is the centre of the section of the ellipsoid made by (5). The co-ordinates of this centre are therefore given by

$$\frac{x}{a^2l}=\frac{y}{b^2m}=\frac{z}{c^2n}=\frac{lx+my+nz}{a^2l^2+b^2m^2+c^2n^2}=\frac{p}{a^2l^2+b^2m^2+c^2n^2}\text{....}(6).$$

78. The co-ordinates of the centre of the section of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\text{......................}(1)$$

by the plane

$$lx+my+nz=p\text{....................}(2)$$

can also be obtained in the following manner.

Let α, β, γ be the co-ordinates required, and let

$$\frac{x-\alpha}{\lambda}=\frac{y-\beta}{\mu}=\frac{z-\gamma}{\nu}=r \ldots\ldots\ldots\ldots\ldots(3)$$

be the equations of any straight line drawn in the plane (2) to meet the ellipsoid, r being the length of the radius vector. Then if x, y, z be the co-ordinates of the point where (3) meets (1), we have from (3)

$$x=\alpha+\lambda r,\quad y=\beta+\mu r,\quad z=\gamma+\nu r,$$

and therefore from (1) by substitution

$$r^2\left(\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}+\frac{\nu^2}{c^2}\right)+2r\left(\frac{\lambda\alpha}{a^2}+\frac{\mu\beta}{b^2}+\frac{\nu\gamma}{c^2}\right)+\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1=0\ldots(4).$$

But if α, β, γ be the co-ordinates of the centre of the section of (1) by (2), the two values of r given by (4) must be equal in magnitude and of opposite sign for all straight lines lying in (1); that is, we must have

$$\frac{\lambda\alpha}{a^2}+\frac{\mu\beta}{b^2}+\frac{\nu\gamma}{c^2}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

for all values of λ, μ, ν consistent with the equation

$$\lambda l+\mu m+\nu n=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

which is the condition that (3) may lie in (2).

Hence the equations (5) and (6) must be identical, or we have

$$\frac{\alpha}{la^2}=\frac{\beta}{mb^2}=\frac{\gamma}{nc^2},$$

and as in the last article each of these fractions

$$=\frac{p}{a^2l^2+b^2m^2+c^2n^2}.$$

79. The equation (4) of the last article, when the values of α, β, γ are substituted in it, becomes

$$r^2\left(\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}+\frac{\nu^2}{c^2}\right)=1-\frac{p^2}{a^2l^2+b^2m^2+c^2n^2}.$$

Comparing this with equation (4) of Art. 68 we see that if r_1 be the central radius vector which is parallel to r, we have

$$r^2 = r_1^{\,2}\left(1 - \frac{p^2}{a^2l^2 + b^2m^2 + c^2n^2}\right).$$

Consequently, since the areas of similar figures are proportional to the squares of any corresponding lines in the figures, if A be the area of the section of (1) by (2), and A_1 the area of the parallel central section,

$$A = A_1\left\{1 - \frac{p^2}{a^2l^2 + b^2m^2 + c^2n^2}\right\}$$

$$= \frac{\pi abc}{\sqrt{a^2l^2 + b^2m^2 + c^2n^2}}\left\{1 - \frac{p^2}{a^2l^2 + b^2m^2 + c^2n^2}\right\}.$$

80. The result of the last article can also be obtained in the following manner.

Let α, β, γ be the co-ordinates of the centre of the section. Then the equation

$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} + \frac{(z-\gamma)^2}{c^2} = k^2 \qquad \text{..........(1)}$$

represents an ellipsoid whose centre is at (α, β, γ), and whose semi-axes are ka, kb, kc.

At the points where this cuts the given ellipsoid we have by subtraction

$$\frac{2\alpha x}{a^2} + \frac{2\beta y}{b^2} + \frac{2\gamma z}{c^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} + 1 - k^2.$$

Or, putting for α, β, γ their values from equation (6) of Art. 77,

$$2(lx + my + nz) = \frac{a^2l^2 + b^2m^2 + c^2n^2}{p}\left\{\frac{p^2}{a^2l^2 + b^2m^2 + c^2n^2} + 1 - k^2\right\},$$

and if this equation be identical with

$$lx + my + nz = p \qquad \text{.....................(2)},$$

the sections of the two ellipsoids by this latter plane will coincide.

The condition for this is

$$1 - k^2 = \frac{p^2}{a^2l^2 + b^2m^2 + c^2n^2};$$

$$\therefore\ k^2 = 1 - \frac{p^2}{a^2l^2 + b^2m^2 + c^2n^2}.$$

But the area of the section of (1) by the plane (2) which passes through its centre, by Art. (68)

$$= \frac{\pi k^3 abc}{\sqrt{a^2k^2l^2 + b^2k^2m^2 + c^2k^2n^2}} = \frac{\pi abck^2}{\sqrt{a^2l^2 + b^2m^2 + c^2n^2}}$$

$$= \frac{\pi abc}{\sqrt{a^2l^2 + b^2m^2 + c^2n^2}}\left\{1 - \frac{p^2}{a^2l^2 + b^2m^2 + c^2n^2}\right\},$$

which is therefore the area required.

81. It can be shewn by an investigation similar to that in Art. 71, that the locus of the middle points of a system of parallel chords of the surface

$$By^2 + Cz^2 = x,$$

whose direction-cosines are l, m, n, is

$$2Bmy + 2Cnz = l.$$

Also the equation of the surface, when two diametral planes and a plane through the point where their line of intersection cuts the surface, parallel to the two systems of chords bisected by them, are taken as planes of zx, xy and yz respectively, will assume the form

$$B'y^2 + C'z^2 = x,$$

where B' and C' have the same or opposite signs according as B and C have.

We shall however at once proceed to the more general problem.

82. *To find the locus of the middle points of a system of parallel chords in any quadric.*

Let the equation of the surface be

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy$$
$$+ 2A''x + 2B''y + 2C''z + F = 0 \ \ldots\ldots(1),$$

which we will denote by $F(x, y, z) = 0$.

And let $$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \;\ldots\ldots\ldots\ldots\ldots (2),$$

be the equations of any one of the system of parallel chords.

To find the points where (2) meets (1) we must substitute $\alpha+lr$, $\beta+mr$, $\gamma+nr$ for x, y, z in (1). We thus get

$$F(\alpha+lr,\ \beta+mr,\ \gamma+nr)=0,$$

or $$F(\alpha,\beta,\gamma)+\left\{l\frac{dF}{d\alpha}+m\frac{dF}{d\beta}+n\frac{dF}{d\gamma}\right\}r+Pr^2=0 \ldots (3),$$

where $\frac{dF}{d\alpha}$, $\frac{dF}{d\beta}$, $\frac{dF}{d\gamma}$ are the partial differential coefficients of $F(\alpha,\beta,\gamma)$ with respect to α, β, γ respectively and P is some function of l, m, n.

The equation (3) gives two values of r, which are the distances from (α,β,γ) of the two points where the line (2) cuts the surface (1). If (α,β,γ) be the middle point of the chord these two values must be equal and opposite in sign, and the coefficient of r in the above quadratic must vanish;

$$\therefore\ l\frac{dF}{d\alpha}+m\frac{dF}{d\beta}+n\frac{dF}{d\gamma}=0,$$

or writing out the values of $\frac{dF}{d\alpha}$, $\frac{dF}{d\beta}$ and $\frac{dF}{d\gamma}$, and rearranging,

$$\alpha(Al+C'm+B'n)+\beta(C'l+Bm+A'n)+\gamma(B'l+A'm+Cn)$$
$$+A''l+B''m+C''n=0,$$

which shews that the locus required is a plane.

83. The diametral plane will not in general be perpendicular to the chords which it bisects. There are however certain directions of the chords for which this is the case. Let us suppose l, m, n to be the direction-cosines of any chord of the system.

The equation of the diametral plane is therefore by the last article,

$$x(Al+C'm+B'n)+y(C'l+Bm+A'n)+z(B'l+A'm+Cn)$$
$$+A''l+B''m+C''n=0.$$

If this plane be perpendicular to the system of chords we must have, by Art. 23,

$$\frac{Al + C'm + B'n}{l} = \frac{C'l + Bm + A'n}{m} = \frac{B'l + A'm + Cn}{n}.$$

Let each of these fractions be put equal to some quantity s. We have then

$$\left.\begin{array}{l}(A-s)\,l + C'm + B'n = 0\\ C'l + (B-s)\,m + A'n = 0\\ B'l + A'm + (C-s)\,n = 0\end{array}\right\} \ldots\ldots\ldots\ldots\ldots (1).$$

Whence eliminating l, m, n, we get

$$\begin{vmatrix} (A-s), & C', & B' \\ C', & (B-s), & A' \\ B', & A', & (C-s) \end{vmatrix} = 0,$$

or $(A-s)(B-s)(C-s) - A'^2(A-s) - B'^2(B-s) - C'^2(C-s)$
$$+ 2A'B'C' = 0 \ldots\ldots (2).$$

This cubic equation will certainly give *one* real value of s, and the corresponding values of l, m, n are known from any two of the three equations (1). From the second and third we get

$$\frac{m}{A'B' - C'(C-s)} = \frac{n}{A'C' - B'(B-s)},$$

or $m\,\{A'C' - B'(B-s)\} = n\,\{A'B' - C'(C-s)\}$
$$= l\,\{B'C' - A'(A-s)\} \ldots\ldots (3),$$
by symmetry.

And when the value of s is known, equations (3) give the corresponding values of l, m, n.

In Todhunter's *Theory of Equations*, Art. 176, it is shewn that all three roots of the cubic are real.

The equation (2) is frequently called the discriminating cubic of the quadric (1).

EXAMPLES. CHAPTER VI.

1. If A_1, A_2, A_3 be the areas of the sections of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

made by planes perpendicular to any three generators of the cone

$$x^2(a^2-d^2)+y^2(b^2-d^2)+z^2(c^2-d^2)=0,$$

and if p_1, p_2, p_3 be the perpendiculars on the planes from the origin, then

$$A_1(p_2^{\,2}-p_3^{\,2})+A_2(p_3^{\,2}-p_1^{\,2})+A_3(p_1^{\,2}-p_2^{\,2})=0.$$

2. Find the locus of the centres of sections of an ellipsoid, the areas of which are always in a constant ratio to the areas of the parallel central sections.

3. OL, OM, ON are conjugate semi-diameters of an ellipsoid; x_1, y_1, z_1 the co-ordinates of L; x_2, y_2, z_2 and x_3, y_3, z_3 those of M and N respectively. Prove that the equation of the plane LMN is

$$\frac{x}{a^2}(x_1+x_2+x_3)+\frac{y}{b^2}(y_1+y_2+y_3)+\frac{z}{c^2}(z_1+z_2+z_3)=1.$$

4. Find the area of the section of the ellipsoid by the plane LMN in the last example.

5. OL, OM, ON are conjugate semi-diameters of an ellipsoid; a perpendicular is drawn from O on the plane LMN meeting it at Q; and a diametral plane is drawn parallel to the plane LMN. Shew that the cone which has its vertex at Q and for its base the section of the ellipsoid by the diametral plane, is of constant volume.

6. Find the locus of the directrices of all sections of an ellipsoid made by planes passing through the least axis.

7. Shew that a straight line parallel to the least axis of an ellipsoid will be the directrix of two plane sections of the ellipsoid, provided the straight line be situated between two definite cylindrical surfaces.

8. Find the locus of the centres of sections of an ellipsoid made by planes at a constant distance from the origin.

9. If A, B, C be the areas of any three conjugate diametral sections of an ellipsoid; X, Y, Z those of the sections made by planes respectively parallel to them and intersecting in a point on the surface, prove that

$$\frac{X}{A} + \frac{Y}{B} + \frac{Z}{C} = 2.$$

10. Any generating line of the cone

$$Px^2 + Qy^2 + Rz^2 = 0$$

being taken, a plane is drawn diametral to it with respect to the surface

$$Ax^2 + By^2 + Cz^2 = 1.$$

Shew that the principal axes of the sections of the latter surface by such planes all lie on the surface

$$\frac{Px^2}{A^2}\{(A-B)y^2 + (A-C)z^2\}^2 + \frac{Qy^2}{B^2}\{(B-C)z^2 + (B-A)x^2\}^2 + \frac{Rz^2}{C^2}\{(C-A)x^2 + (C-B)y^2\}^2 = 0.$$

11. Find the co-ordinates of the centre of the section of the surface

$$By^2 + Cz^2 = x$$

made by the plane $lx + my + nz = p$.

Find the locus of the centres of all sections made by planes passing through a fixed point.

12. If in question 3, the point L remain fixed, shew that the perpendicular from the origin on the plane LMN describes the cone

$$a^2x^2 + b^2y^2 + c^2z^2 = 3(xx_1 + yy_1 + zz_1)^2.$$

13. If the plane $lx + my + nz = p$ cut the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

in a parabola, prove that

$$a^2l^2 + b^2m^2 - c^2n^2 = 0.$$

CHAPTER VII.

THE GENERAL EQUATION OF THE SECOND DEGREE.

84. THE general equation of the second degree can be written

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy$$
$$+ 2A''x + 2B''y + 2C''z + F = 0 \ldots\ldots (1),$$

which we will denote by $F(x, y, z) = 0$.

The object of the present chapter is to examine the nature of the different surfaces represented by (1), and the conditions that it may represent any particular kind of surface.

We shall first examine whether the locus represented by (1) has a centre.

If it has a centre and this point be taken for origin we know, by Art. (70), that the terms of the first degree must disappear.

Assume α, β, γ as the co-ordinates of the centre. The equation when the origin is transferred to this point is obtained by substituting in (1) $x' + \alpha$, $y' + \beta$, $z' + \gamma$ for x, y, z, respectively (Art. 43), and is therefore

$$F(x' + \alpha,\ y' + \beta,\ z' + \gamma) = 0,$$

which can be written

$$F(\alpha, \beta, \gamma) + x' \frac{dF}{d\alpha} + y' \frac{dF}{d\beta} + z' \frac{dF}{d\gamma} + \ldots = 0,$$

the remaining terms being of the second order in x', y', z', and $\frac{dF}{d\alpha}$, $\frac{dF}{d\beta}$, $\frac{dF}{d\gamma}$ having the same meaning as in Art. 82.

If the coefficients of x', y', z' vanish, we have

$$\frac{dF}{d\alpha}=0,\ \frac{dF}{d\beta}=0,\ \frac{dF}{d\gamma}=0,$$

or writing them out at length,

$$\left.\begin{aligned} A\alpha+C'\beta+B'\gamma+A''&=0\\ C'\alpha+B\beta+A'\gamma+B''&=0\\ B'\alpha+A'\beta+C\gamma+C''&=0\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (2).$$

These equations determine α, β, γ. We get from them

$$-\frac{\begin{vmatrix} A'' & C' & B' \\ B'' & B & A' \\ C'' & A' & C\end{vmatrix}}{\begin{vmatrix} A & C' & B' \\ C' & B & A' \\ B' & A' & C\end{vmatrix}}=\frac{A''(A'^2-BC)+B''(CC'-A'B')+C''(BB'-C'A')}{ABC+2A'B'C'-AA'^2-BB'^2-CC'^2}$$

ilarly

$$\left.\begin{aligned}\beta&=\frac{A''(CC'-A'B')+B''(B'^2-CA)+C''(AA'-B'C')}{ABC+2A'B'C'-AA'^2-BB'^2-CC'^2}\\ \gamma&=\frac{A''(BB'-C'A')+B''(AA'-B'C')+C''(C'^2-AB)}{ABC+2A'B'C'-AA'^2-BB'^2-CC'^2}\end{aligned}\right\}\ (3).$$

We can therefore always obtain finite values of α, β, γ except when

$$ABC+2A'B'C'-AA'^2-BB'^2-CC'^2=0,$$

in which case the surface has not a centre unless the numerators of the above three fractions vanish, when the values of α, β, γ become indeterminate; the reason of such indeterminateness being that the three equations (2) are not all independent. (Todhunter's *Algebra*, Arts. 214, 215.)

If the denominator do not vanish the surface has a centre whose co-ordinates are given by (3).

It may be noticed that the equations (2) are the conditions that the point (α, β, γ) shall lie in the diametral plane to all systems of chords. (Art. 82.)

85. We see from the last article that it is not always possible to get rid of the terms involving x, y, z. We shall now shew that it is *always* possible to simplify the equation by transformation so as to get rid of the terms involving yz, zx and xy.

By Art. 83 we know that there is at least one system of parallel chords which is perpendicular to its diametral plane.

Let a straight line parallel to these chords be taken as the axis of z and let the transformed equation be

$$Px^2 + Qy^2 + Rz^2 + 2P'yz + 2Q'zx + 2R'xy + 2P''x + 2Q''y + 2R''z + F = 0.$$

The direction-cosines of the chords which are perpendicular to their diametral plane are given by the equations

$$Pl + R'm + Q'n = sl,$$
$$R'l + Qm + P'n = sm,$$
$$Q'l + P'm + Rn = sn.$$

But since these chords are parallel to the axis of z, these equations must be satisfied by

$$l = 0, \quad m = 0, \quad n = 1.$$

Whence we get $Q' = 0$, $P' = 0$; and the equation of the surface is

$$Px^2 + Qy^2 + Rz^2 + 2R'xy + 2P''x + 2Q''y + 2R''z + F = 0.$$

Turning the axes of x and y in their own plane through an angle θ given by the equation

$$\tan 2\theta = \frac{2R'}{P - Q} \text{ (Todhunter's } \textit{Conic Sections}\text{, Art. 271)},$$

the term involving xy disappears, and the equation assumes the form

$$Px^2 + Qy^2 + Rz^2 + 2P''x + 2Q''y + 2R''z + F = 0.$$

The equations which determine the directions of the principal diametral planes are now satisfied by $l = 1$, $m = 0$, $n = 0$, or by $l = 0$, $m = 1$, $n = 0$. Consequently each of the axes of x and y as well as that of z is parallel to one of the three lines determined by equations (1) of Art. 83.

We thus have an independent proof that these three directions are all real and at right angles to each other.

86. We have now shewn that by a proper choice of axes the terms involving yz, zx and xy can be made to disappear. It remains to explain how the coefficients of the different terms in the resulting equation can be determined.

Let l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 be the direction-cosines of the new axes. These values all satisfy the equations (1) of Art. 83. Let s_1, s_2, s_3 be the corresponding values of s.

By Art. 44 the required transformation will be effected by substituting for x, y, z the expressions

$$l_1x' + l_2y' + l_3z', \quad m_1x' + m_2y' + m_3z', \quad n_1x' + n_2y' + n_3z',$$

respectively. If therefore the original equation be

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy \\ + 2A''x + 2B''y + 2C''z + F = 0,$$

the coefficient of x'^2 in the result will be

$$Al_1^2 + Bm_1^2 + Cn_1^2 + 2A'm_1n_1 + 2B'n_1l_1 + 2C'l_1m_1.$$

But from Art. 83 we have

$$\begin{aligned} Al_1 + C'm_1 + B'n_1 &= s_1l_1, \\ C'l_1 + Bm_1 + A'n_1 &= s_1m_1, \\ B'l_1 + A'm_1 + Cn_1 &= s_1n_1. \end{aligned}$$

Multiplying these equations by l_1, m_1, n_1, respectively, and adding, we get

$$Al_1^2 + Bm_1^2 + Cn_1^2 + 2A'm_1n_1 + 2B'n_1l_1 + 2C'l_1m_1 = s_1.$$

Hence P the coefficient of x'^2 is s_1. Similarly $Q = s_2$, $R = s_3$, or P, Q, R are the three roots of the discriminating cubic.

It follows from this that the coefficients of the discriminating cubic remain unaltered in value however the axes may be turned about the origin.

The results of this article have been already obtained by a different method in Art. 51.

87. It is easy to verify that the coefficients of $y'z'$, $z'x'$ and $x'y'$ disappear; since l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the direction-cosines of lines such that any one is parallel to each of the planes which bisect chords parallel to either of the others, and thus $l_1, m_1, n_1, l_2, m_2, n_2$, satisfy the relation

$$Al_1l_2 + Bm_1m_2 + Cn_1n_2 + A'(m_1n_2 + m_2n_1) + B'(n_1l_2 + n_2l_1) + C'(l_1m_2 + l_2m_1) = 0,$$

and the expression on the left-hand side of this equation is the coefficient of $x'y'$ in the transformed equation.

The coefficients of x', y' and z' in the transformed equation will be

$$2(A''l_1 + B''m_1 + C''n_1),\quad 2(A''l_2 + B''m_2 + C''n_2)$$

and $$2(A''l_3 + B''m_3 + C''n_3),$$

respectively, and the constant term remains unchanged.

88. The equation when transformed to

$$Px^2 + Qy^2 + Rz^2 + 2P''x + 2Q''y + 2R''z + F = 0$$

can be farther simplified by a change of origin.

Suppose first that none of the quantities P, Q, R vanish, that is, that none of the roots of the discriminating cubic vanish, which will be the case if the constant term of the cubic, or

$$ABC + 2A'B'C' - AA'^2 - BB'^2 - CC'^2,$$

be different from zero.

In this case the equation can be written

$$P\left(x + \frac{P''}{P}\right)^2 + Q\left(y + \frac{Q''}{Q}\right)^2 + R\left(z + \frac{R''}{R}\right)^2 = \frac{P''^2}{P} + \frac{Q''^2}{Q} + \frac{R''^2}{R} - F = F',$$

and transferring the origin to the point whose co-ordinates are

$$\left(-\frac{P''}{P},\ -\frac{Q''}{Q},\ -\frac{R''}{R}\right),$$

this becomes

$$Px^2 + Qy^2 + Rz^2 = F'.$$

This represents an ellipsoid, a hyperboloid of one or two sheets, or an impossible locus, respectively, according as the quantities $\frac{F'}{P}$, $\frac{F'}{Q}$, $\frac{F'}{R}$ are all positive, two positive and one negative, one positive and two negative, or all negative.

Thus unless

$$ABC + 2A'B'C' - AA'^2 - BB'^2 - CC'^2$$

vanish, the surface has a centre and is one of the surfaces whose equations we have already investigated.

Now if we had first changed the origin to be the centre, we should have got rid of the terms of the first degree, and the equation would have been

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy = F' \ldots\ldots (1),$$

which by turning round the axes would become

$$Px^2 + Qy^2 + Rz^2 = F',$$

and consequently, if F' be positive the surface (1) will represent an ellipsoid, a hyperboloid of one or two sheets, or an impossible locus according as the roots of the discriminating cubic are all positive, two positive and one negative, one positive and two negative, or all negative. If F' be negative the order of the statement must be reversed.

89. If F' vanish the surface is a cone. Now returning to Art. 84 we see that $F' = -F(\alpha, \beta, \gamma)$, where α, β, γ are determined from the equations

$$\left.\begin{aligned} A\alpha + C'\beta + B'\gamma + A'' &= 0 \\ C'\alpha + B\beta + A'\gamma + B'' &= 0 \\ B'\alpha + A'\beta + C\gamma + C'' &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (2).$$

Multiplying the first of these by α, the second by β, the third by γ and adding, we get

$$A\alpha^2 + B\beta^2 + C\gamma^2 + 2A'\beta\gamma + 2B'\gamma\alpha + 2C'\alpha\beta + A''\alpha + B''\beta + C''\gamma = 0.$$

But

$$A\alpha^2 + B\beta^2 + C\gamma^2 + 2A'\beta\gamma + 2B'\gamma\alpha + 2C'\alpha\beta + 2A''\alpha + 2B''\beta + 2C''\gamma + F = F(\alpha, \beta, \gamma) = -F'.$$

Subtracting the first of these from the second, we get

$$-F' = A''\alpha + B''\beta + C''\gamma + F.$$

Hence if the surface be a cone

$$A''\alpha + B''\beta + C''\gamma + F = 0.$$

And eliminating α, β, γ between this equation and the three equations (2), we get as the condition that the surface represents a cone

$$\begin{vmatrix} A & C' & B' & A'' \\ C' & B & A' & B'' \\ B' & A' & C & C'' \\ A'' & B'' & C'' & F \end{vmatrix} = 0.$$

90. Suppose, secondly, that one of the quantities P, Q, R vanishes, as P. From this it follows that the constant term of the cubic in s must vanish, or

$$ABC + 2A'B'C' - AA'^2 - BB'^2 - CC'^2 = 0,$$

which we saw in Art. 84 indicated that there was not a definite centre.

The equation becomes

$$Qy^2 + Rz^2 + 2P''x + 2Q''y + 2R''z + F = 0,$$

and by changing the origin we can get rid of the terms in y and z, and the constant term; the equation thus becomes

$$Qy^2 + Rz^2 + 2P''x = 0,$$

which represents an elliptic or hyperbolic paraboloid according as Q and R have the same or opposite signs, or according as

$$BC+CA+AB-A'^2-B'^2-C'^2,$$

which is the coefficient of s in the cubic, and therefore equal to the product of the two finite roots, is positive or negative.

91. Thirdly, let two of the quantities P, Q, R vanish, which necessitates the two conditions,

$$ABC+2A'B'C'-AA'^2-BB'^2-CC'^2=0,$$
$$BC+CA+AB-A'^2-B'^2-C'^2=0.$$

The equation now becomes

$$Rz^2+2P''x+2Q''y+2R''z+F=0.$$

And by changing the origin, the term involving z and the constant term may be removed, and we get

$$Rz^2+2P''x+2Q''y=0.$$

By turning the axes of x and y round in their own plane, the equation can be reduced to the form

$$Rz^2+2P'''x=0,$$

which represents a parabolic cylinder whose generating lines are parallel to the axis of y.

The two conditions

$$ABC+2A'B'C'-AA'^2-BB'^2-CC'^2=0,$$
$$BC+CA+AB-A'^2-B'^2-C'^2=0,$$

can be replaced by simpler ones. For the first equation is equivalent to either of the forms

$$(CA-B'^2)(AB-C'^2)=(B'C'-AA')^2,$$
$$(AB-C'^2)(BC-A'^2)=(C'A'-BB')^2,$$
$$(BC-A'^2)(CA-B'^2)=(A'B'-CC')^2,$$

whence it follows that the three quantities $AB-C'^2$, $CA-B'^2$, $BC-A'^2$ have all the same sign, and therefore if their sum vanishes they must vanish separately, and we must have

$$BC-A'^2=0,\quad CA-B'^2=0,\quad AB-C'^2=0.$$

We must also have

$$B'C' - AA' = 0, \quad C'A' - BB' = 0, \quad A'B' - CC' = 0,$$

but these are included in the former.

92. If only one of the quantities P, Q, R, as P, vanish, and P'' also vanish, the equation becomes

$$Qy^2 + Rz^2 + 2Q''y + 2R''z + F = 0,$$

which can be reduced to the form

$$Qy^2 + Rz^2 + F' = 0,$$

and therefore represents an elliptic or hyperbolic cylinder according as Q and R have the same or opposite signs, that is, according as

$$BC - A'^2 + CA - B'^2 + AB - C'^2$$

is positive or negative.

If Q, R and F' have all the same sign the locus is an impossible one.

The condition that P'' may vanish is, that

$$A''l_1 + B''m_1 + C''n_1$$

should vanish, where l_1, m_1, n_1 are the values of l, m, n derived from equations (1) of Art. 83 by putting $s = 0$. But these values are proportional to

$$\frac{1}{B'C' - AA'}, \quad \frac{1}{C'A' - BB'}, \quad \frac{1}{A'B' - CC'},$$

so that we get

$$\frac{A''}{B'C' - AA'} + \frac{B''}{C'A' - BB'} + \frac{C''}{A'B' - CC'} = 0.$$

This condition may be obtained in another form from the consideration that the equations

$$\left.\begin{aligned} A''l_1 + B''m_1 + C''n_1 &= 0 \\ Al_1 + C'm_1 + B'n_1 &= 0 \\ C'l_1 + Bm_1 + A'n_1 &= 0 \\ B'l_1 + A'm_1 + Cn_1 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

must be all satisfied by the same values of l_1, m_1, n_1, and the requisite conditions that this may be the case are

$$ABC + 2A'B'C' - AA'^2 - BB'^2 - CC'^2 = 0,$$

united with any one of the set,

$$\left.\begin{aligned} A''(CC' - A'B') + B''(B'^2 - CA) + C''(AA' - B'C') = 0, \\ A''(A'^2 - BC) + B''(CC' - A'B') + C''(BB' - C'A') = 0, \\ A''(BB' - C'A') + B''(AA' - B'C') + C''(C'^2 - AB) = 0. \end{aligned}\right\}$$

The equations (1) are evidently the conditions that the three equations (2) of Art. 84 should not be *independent*, and consequently there is a line of centres.

93. If two of the roots of the discriminating cubic as P and Q vanish, and P'', Q'' also vanish, the locus reduces to

$$Rz^2 + 2R''z + F = 0,$$

which represents two parallel planes. The conditions for the two roots vanishing are

$$BC - A'^2 = 0, \quad CA - B'^2 = 0, \quad AB - C'^2 = 0 \ldots\ldots(2),$$

and l_1, m_1, n_1 are only restricted by the equation

$$Al_1 + C'm_1 + B'n_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

with which the other two equations in (1) Art. 83 become identical.

If we have also $A''l_1 + B''m_1 + C''n_1 = 0$, for all values of l_1, m_1, n_1 consistent with (3) we must have

$$\frac{A''}{A} = \frac{B''}{C'} = \frac{C''}{B'},$$

or from (2) $$\frac{A''}{\sqrt{A}} = \frac{B''}{\sqrt{B}} = \frac{C''}{\sqrt{C}}.$$

94. On the whole then we have the following results.

I. If $ABC + 2A'B'C' - AA'^2 - BB'^2 - CC'^2$ be not zero, the equation represents an ellipsoid, a hyperboloid, or an impossible locus, with the cone as a variety of the hyperboloids.

II. If $ABC + 2A'B'C' - AA'^2 - BB'^2 - CC'^2$ vanishes, the equation in general represents an elliptic or hyperbolic paraboloid according as

$$BC + CA + AB - A'^2 - B'^2 - C'^2$$

is positive or negative; which may degenerate into an elliptic or hyperbolic cylinder, with an impossible locus, a straight line or two intersecting planes, as particular cases.

III. If $BC - A'^2$, $CA - B'^2$, $AB - C'^2$ all vanish, the equation represents a parabolic cylinder which may degenerate into two parallel or coincident planes.

The conditions that the equation may represent a surface of revolution may be obtained from the consideration that two roots of the cubic in s are equal. This is discussed in Todhunter's *Theory of Equations*, Art. 179, to which the reader is referred.

The reduction of the equation in the particular case when

$$ABC + 2A'B'C' - AA'^2 - BB'^2 - CC'^2 = 0$$

may be effected by writing it in the form

$$(Ax + C'y + B'z)^2 + (AB - C'^2)y^2 + 2(AA' - B'C')yz$$
$$+ (CA - B'^2)z^2 + A(2A''x + 2B''y + 2C''z + F) = 0,$$

or putting $$\frac{AA' - B'C'}{AB - C'^2} = p = \frac{CA - B'^2}{AA' - B'C'},$$

$$(Ax + C'y + B'z)^2 + (AB - C'^2)(y + pz)^2$$
$$+ A(2A''x + 2B''y + 2C''z + F) = 0.$$

And if we take as co-ordinate planes the planes

$$Ax + C'y + B'z = 0,$$
$$y + pz = 0,$$
$$2A''x + 2B''y + 2C''z + F = 0,$$

this equation will in general assume the form

$$Py^2 + Qz^2 + Rx = 0,$$

which represents one of the paraboloids. The axes are not however rectangular. The exceptional cases can be deduced from the consideration that the reduction fails when any two of the three planes are parallel, or when one of them is parallel to the intersection of the other two.

We shall conclude this chapter with the following general proposition.

95. *If two surfaces of the second degree intersect in one plane curve, all their other points of intersection lie in another plane curve.*

For let $S=0$ and $S'=0$ be the equations of the two surfaces, and $lx+my+nz-p=0$, or $\alpha=0$, the equation of the plane of intersection. Then the curve in which $\alpha=0$ cuts the surface $S=0$ coincides with the curve in which it cuts the surface $S'=0$. So that the three equations $S=0$, $S'=0$, $\alpha=0$ are satisfied by an indefinite number of values of x, y and z.

Consequently the expression S must be identical with $kS'+\alpha\beta$, where k is a constant and β a linear function of x, y, z.

Hence when $S=0$ and $S'=0$, we have $\alpha=0$ or $\beta=0$, that is, all the points of intersection lie in one of the two planes $\alpha=0$, or $\beta=0$.

EXAMPLES. CHAPTER VII.

1. Investigate the nature of the surfaces,

(1) $2x^2+5y^2+3z^2+2yz-8zx-2xy-1=0$.

(2) $x^2+4y^2-z^2-2yz-zx+4xy+2z=0$.

2. Interpret the equations:

(1) $yz+zx+xy-x-2y-3z+2+a=0$.

(2) $x^2+2y^2-3z^2+2yz-4zx-2xy+3x=0$.

(3) $x^2+9y^2-6xy+2y-4z=0$.

(4) $x^2+y^2-z^2+2yz+2zx-2xy+2x+2y+2z=a^2$.

3. Shew that the two surfaces whose equations are

$$(h^2+b^2+c^2)x^2+(h^2+c^2+a^2)\,y^2+(h^2+a^2+b^2)\,z^2 - 2bcyz - 2cazx - 2abxy = 1,$$

and $$(cy-bz)^2+(az-cx)^2+(bx-ay)^2=1,$$

have their axes coincident in direction. What kind of surface are they respectively?

4. Discuss the surfaces obtained by giving different values to μ in the equation

$$x^2+2y^2+2z^2-(2-2\mu)\,yz-2zx=c^2.$$

5. Find the nature of the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+\frac{yz}{bc}+\frac{zx}{ca}+\frac{xy}{ab}-\frac{3x}{a}-\frac{3y}{b}-\frac{3z}{c}+3=0.$$

and shew that it *touches* the co-ordinate planes.

6. If one of the angles between the co-ordinate axes be a right angle and the other two be supplementary, prove that the sum of the squares of the axes of the surface

$$xy+yz+zx+d^2=0$$

is $12d^2$ (Ex. 7, Chap. IV.).

7. Shew that if two generators of a hyperboloid of one sheet be taken as two of the axes of co-ordinates, the equation is of the form

$$z^2+az=lyz+mzx+nxy.$$

8. Find by the method of Art. 68 the position and magnitude of the axes of the section of the surface

$$Ax^2+By^2+Cz^2+2A'yz+2B'zx+2C'xy=1$$

by the plane

$$lx+my+nz=0.$$

9. Find by the method of Art. 78 the axes of the section of the surface

$$\sqrt{x}+\sqrt{y}+\sqrt{z}=0$$

by the plane

$$lx+my+nz=1.$$

10. If the equation

$$ax^2+by^2+cz^2+2b'zx+2c'xy+2a''x+2b''y+2c''z+d=0$$

represent a paraboloid of revolution, prove that $c=b\pm a$. If the upper sign be taken, prove that the equations to the axis are

$$cz + c'' = 0, \quad (cx + a'') \sqrt{a} + (cy + b'') \sqrt{b} = 0,$$

and find the condition that the paraboloid may reduce to a circular cylinder.

11. Find the equation of a surface of the second degree which contains two given straight lines at right angles, and the condition that it may be a hyperboloid of one sheet.

Take the shortest distance between the lines as axis of z, the middle point of it as origin, and the axes of x and y parallel to the two lines.

12. Find the equation of the surface generated by a straight line which meets three straight lines which are mutually at right angles, but which do not intersect.

13. Shew that the section of the surface

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy = 1,$$

by the plane $lx + my + nz = 0$, will be a circle if

$$\frac{Bn^2 + Cm^2 - 2A'mn}{m^2 + n^2} = \frac{Cl^2 + An^2 - 2B'nl}{n^2 + l^2} = \frac{Am^2 + Bl^2 - 2C'lm}{l^2 + m^2}.$$

14. Shew that the axes of the surface

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy = 1$$

lie on the two cones

$$C'(x^2 - y^2) - B'yz + A'zx - (A - B)\, xy = 0,$$
$$A'(y^2 - z^2) - (B - C)\, yz - C'zx + B'xy = 0.$$

15. A cone whose equation referred to its principal axes is

$$\alpha^2 x^2 + \beta^2 y^2 = (\alpha^2 + \beta^2)\, z^2,$$

is thrust into an elliptical hole whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Shew that when the cone fits the hole its vertex must lie on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + z^2 \left(\frac{1}{a^2} + \frac{1}{b^2}\right) = 1.$$

CHAPTER VIII.

ON TANGENT LINES AND PLANES.

96. THE straight line joining any point P on a surface to another point Q on the surface, is called a chord. If the point Q be made to approach indefinitely near to P, the limiting position of the chord PQ is said to be a tangent line to the surface at the point P.

In general all the tangent lines at the point P lie in a plane, which is called the tangent plane at P. This we will now prove.

Let x, y, z be the co-ordinates of any point P on a surface whose equation is

$$F(x, y, z) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

And let the equations of any straight line through P be

$$\frac{x' - x}{l} = \frac{y' - y}{m} = \frac{z' - z}{n} = r \quad \ldots\ldots\ldots\ldots (2),$$

where x', y', z' are current co-ordinates.

To find the points where (2) meets (1) we must substitute

$$x + lr,\ y + mr,\ z + nr \text{ for } x, y, z \text{ in } (1);$$

we thus get the equation

$$F(x + lr, y + mr, z + nr) = 0;$$

$$\text{or } F(x, y, z) + r\left(l\frac{dF}{dx} + m\frac{dF}{dy} + n\frac{dF}{dz}\right)$$
$$+ \frac{r^2}{\underline{|2}}\left\{l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right\}^2 F(x, y, z)$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ \frac{r^p}{\underline{|p}}\left\{l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right\}^p F(x, y, z) = 0 \ldots (3),$$

supposing $F(x, y, z)$ to be of the p^{th} degree in x, y, z.

This equation gives the distances from P of the different points in which (2) cuts (1), and since (x, y, z) is a point on the surface (1), $F(x, y, z)$ vanishes and the equation (3) is satisfied by one value of r equal to zero.

If l, m, n be such as to satisfy the equation

$$l\frac{dF}{dx} + m\frac{dF}{dy} + n\frac{dF}{dz} = 0 \ldots\ldots\ldots\ldots\ldots (4),$$

two values of r are zero, and the line (2) meets the surface in two coincident points, and is therefore a tangent line to the surface at (x, y, z). Equation (4) is therefore a condition which must be satisfied by the direction-cosines of all tangent lines at the point P.

But for all points in any such tangent line we have

$$\frac{x' - x}{l} = \frac{y' - y}{m} = \frac{z' - z}{n}.$$

Consequently for all points in any such tangent line we have

$$(x' - x)\frac{dF}{dx} + (y' - y)\frac{dF}{dy} + (z' - z)\frac{dF}{dz} = 0 \ldots (5),$$

whence it follows that all the tangent lines in general lie in a plane whose equation is (5).

97. It may happen that at a given point of a surface the three quantities $\frac{dF}{dx}$, $\frac{dF}{dy}$ and $\frac{dF}{dz}$ all vanish.

If this be the case, the equation (3) of the last article always gives two values of r equal to zero, and all lines through the point P meet the surface in two coincident points. The vertex of a cone is such a point. If we take l, m, n such as to satisfy the condition

$$l^2 \frac{d^2F}{dx^2} + m^2 \frac{d^2F}{dy^2} + n^2 \frac{d^2F}{dz^2} + 2mn \frac{d^2F}{dy\,dz} + 2nl \frac{d^2F}{dz\,dx} + 2lm \frac{d^2F}{dx\,dy} = 0 \ldots (1),$$

three values of r will be zero, and the straight lines whose direction-cosines satisfy this equation meet the surface in three coincident points; eliminating l, m, n, we have as the equation of the locus of all such straight lines

$$(x'-x)^2 \frac{d^2F}{dx^2} + (y'-y)^2 \frac{d^2F}{dy^2} + (z'-z)^2 \frac{d^2F}{dz^2} + 2(y'-y)(z'-z) \frac{d^2F}{dy\,dz} + 2(z'-z)(x'-x) \frac{d^2F}{dz\,dx} + 2(x'-x)(y'-y) \frac{d^2F}{dx\,dy} = 0 \ldots\ldots (2),$$

which is the equation of a cone of the second degree whose vertex is at the point (x, y, z). See Art. 34.

A point at which $\frac{dF}{dx}$, $\frac{dF}{dy}$ and $\frac{dF}{dz}$ all vanish is called a singular point on the surface, and the cone (2) is called the tangent cone at that point.

98. In the case of Art. 96 we see that all straight lines whose direction-cosines satisfy (4) meet the surface in two coincident points. If we take l, m, n such as to satisfy both the conditions

$$\left.\begin{aligned} & l \frac{dF}{dx} + m \frac{dF}{dy} + n \frac{dF}{dz} = 0, \\ & l^2 \frac{d^2F}{dx^2} + m^2 \frac{d^2F}{dy^2} + n^2 \frac{d^2F}{dz^2} \\ & \quad + 2mn \frac{d^2F}{dy\,dz} + 2nl \frac{d^2F}{dz\,dx} + 2lm \frac{d^2F}{dx\,dy} = 0 \end{aligned}\right\} \ldots (1),$$

the straight lines whose direction-cosines are obtained from these equations meet the surface in three coincident points. They are therefore tangents to the curve in which the tangent plane meets the surface. This curve, therefore, has a double point at the point of contact, since the above equations in general give two values of the ratios $l:m:n$, which values may be possible or impossible.

If the surface be of the second degree, the two straight lines given by (1) lie wholly on the surface, and are possible if the surface be a hyperboloid of one sheet or a hyperbolic paraboloid, and impossible in other cases.

99. The equation of a surface is often given in the form

$$z = f(x, y), \text{ or } z - f(x, y) = 0.$$

In this case $\dfrac{dF}{dx}$ becomes $-\dfrac{df}{dx}$ or $-\dfrac{dz}{dx}$, $\dfrac{dF}{dy}$ becomes $-\dfrac{dz}{dy}$, and $\dfrac{dF}{dz}$ becomes unity. The equation of the tangent plane becomes therefore

$$z' - z = \frac{dz}{dx}(x' - x) + \frac{dz}{dy}(y' - y).$$

It is usual to denote the quantities $\dfrac{dz}{dx}$ and $\dfrac{dz}{dy}$ by the letters p, q, and the quantities $\dfrac{d^2z}{dx^2}$, $\dfrac{d^2z}{dy^2}$, $\dfrac{d^2z}{dx\,dy}$ by the letters r, t, s, respectively.

100. The equation of the tangent plane being

$$(x' - x)\frac{dF}{dx} + (y' - y)\frac{dF}{dy} + (z' - z)\frac{dF}{dz} = 0,$$

the length of the perpendicular on it from the origin is

$$\frac{x\dfrac{dF}{dx} + y\dfrac{dF}{dy} + z\dfrac{dF}{dz}}{\sqrt{\left(\dfrac{dF}{dx}\right)^2 + \left(\dfrac{dF}{dy}\right)^2 + \left(\dfrac{dF}{dz}\right)^2}} \quad \text{..............(1)}.$$

The letters U, V, W are frequently used to denote

$$\frac{dF}{dx},\ \frac{dF}{dy},\ \frac{dF}{dz},$$

and the letters u, v, w, u', v', w' to denote

$$\frac{d^2F}{dx^2},\ \frac{d^2F}{dy^2},\ \frac{d^2F}{dz^2},\ \frac{d^2F}{dy\,dz},\ \frac{d^2F}{dz\,dx},\ \frac{d^2F}{dx\,dy},$$

respectively. With this notation the above expression becomes

$$\frac{Ux+Vy+Wz}{\sqrt{U^2+V^2+W^2}} \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If we take the form of the equation in Art. 99, the length of the perpendicular is

$$\frac{z-px-qy}{\sqrt{1+p^2+q^2}} \quad\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

101. As an example take the tangent plane at any point (x, y, z) of an ellipsoid whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Here $$U=\frac{2x}{a^2},\ V=\frac{2y}{b^2},\ W=\frac{2z}{c^2};$$

and the equation of the tangent plane is

$$(x'-x)\frac{x}{a^2}+(y'-y)\frac{y}{b^2}+(z'-z)\frac{z}{c^2}=0,$$

$$\text{or } \frac{x'x}{a^2}+\frac{y'y}{b^2}+\frac{z'z}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad\ldots\ldots (2).$$

The equation of every plane can be expressed in the form

$$\lambda x'+\mu y'+\nu z'=p \quad\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where p is the length, and λ, μ, ν are the direction-cosines, of the perpendicular on it from the origin.

If we suppose (2) identical with (3), we get

$$\frac{\lambda}{\frac{x}{a^2}} = \frac{\mu}{\frac{y}{b^2}} = \frac{\nu}{\frac{z}{c^2}} = \frac{p}{1} \quad\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

or $p = \dfrac{a\lambda}{\frac{x}{a}} = \dfrac{b\mu}{\frac{y}{b}} = \dfrac{c\nu}{\frac{z}{c}} = \dfrac{\sqrt{a^2\lambda^2 + b^2\mu^2 + c^2\nu^2}}{\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}} = \sqrt{a^2\lambda^2 + b^2\mu^2 + c^2\nu^2}.$

And the equation of the tangent plane becomes

$$\lambda x' + \mu y' + \nu z' = \sqrt{a^2\lambda^2 + b^2\mu^2 + c^2\nu^2} \quad\ldots\ldots\ldots\ldots (5),$$

a form which is often useful.

The length of the perpendicular on (2) from the origin

$$= \frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}.$$

The values of λ, μ, ν the direction-cosines of this perpendicular are $\dfrac{px}{a^2}$, $\dfrac{py}{b^2}$, $\dfrac{pz}{c^2}$ by (4), and the co-ordinates of the foot of this perpendicular are consequently $\dfrac{p^2x}{a^2}$, $\dfrac{p^2y}{b^2}$, $\dfrac{p^2z}{c^2}$.

102. The equation of a paraboloid being

$$\frac{y^2}{l} + \frac{z^2}{l'} = x \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

the equation of the tangent plane at (x, y, z) becomes

$$(x' - x) - \frac{2y}{l}(y' - y) - \frac{2z}{l'}(z' - z) = 0,$$

or
$$x' - \frac{2y}{l} \cdot y' - \frac{2z}{l'} \cdot z' = x - \frac{2y^2}{l} - \frac{2z^2}{l'} = -x,$$

or
$$\frac{2y}{l} \cdot y' + \frac{2z}{l'} \cdot z' = x' + x \quad\ldots\ldots\ldots\ldots\ldots (2).$$

This can be put into another form, for comparing it with

$$\lambda x' + \mu y' + \nu z' = p,$$

we get
$$\frac{\mu}{\frac{2y}{l}} = \frac{\nu}{\frac{2z}{l'}} = \frac{\lambda}{-1} = \frac{p}{x},$$

or
$$x = -\frac{p}{\lambda}, \quad y = -\frac{l\mu}{2\lambda}, \quad z = -\frac{l'\nu}{2\lambda},$$

and therefore from (1),

$$\frac{l\mu^2 + l'\nu^2}{4\lambda^2} = -\frac{p}{\lambda}, \quad \therefore p = -\frac{l\mu^2 + l'\nu^2}{4\lambda};$$

and the equation of the tangent plane becomes

$$\lambda x' + \mu y' + \nu z' = -\frac{l\mu^2 + l'\nu^2}{4\lambda} \quad \text{..........} \quad (3).$$

103. *The normal to a surface at any point is the straight line drawn through that point perpendicular to the tangent plane.*

The equation of the tangent plane at (x, y, z) is

$$(x' - x)\frac{dF}{dx} + (y' - y)\frac{dF}{dy} + (z' - z)\frac{dF}{dz} = 0,$$

and the equations of a straight line through the point (x, y, z) perpendicular to this plane are

$$\frac{x' - x}{\frac{dF}{dx}} = \frac{y' - y}{\frac{dF}{dy}} = \frac{z' - z}{\frac{dF}{dz}} \quad \text{..................} \quad (1).$$

These are therefore the equations of the normal.

The equations of the normal to an ellipsoid at the point (x, y, z) are

$$\frac{a^2(x' - x)}{x} = \frac{b^2(y' - y)}{y} = \frac{c^2(z' - z)}{z}.$$

If we take the equation of the surface to be

$$z = f(x, y),$$

the equation of the tangent plane is

$$z' - z - p(x' - x) - q(y' - y) = 0,$$

and the equations of the normal are therefore

$$\left.\begin{aligned} x' - x + p(z' - z) = 0 \\ y' - y + q(z' - z) = 0 \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (2).$$

104. The equation of the tangent plane to a surface

$$F(x, y, z) = 0 \dots\dots\dots\dots\dots\dots\dots (1)$$

at the point (x, y, z) is

$$(x' - x)\frac{dF}{dx} + (y' - y)\frac{dF}{dy} + (z' - z)\frac{dF}{dz} = 0.$$

If this plane pass through a point whose co-ordinates are α, β, γ, we have

$$(\alpha - x)\frac{dF}{dx} + (\beta - y)\frac{dF}{dy} + (\gamma - z)\frac{dF}{dz} = 0 \dots\dots\dots (2).$$

This relation is satisfied by the co-ordinates of all points, the tangent planes at which pass through a given point (α, β, γ). It is the equation of a surface which by its intersection with (1) determines the points of contact of tangent planes to (1) drawn through (α, β, γ).

105. We can shew that all these points of contact lie on a surface of the degree next below that of the original surface. For let $F(x, y, z)$ be of the p^{th} degree, and let us assume

$$F(x, y, z) = u_p + u_{p-1} + u_{p-2} + \dots + u_2 + u_1 + u_0,$$

where $u_p, u_{p-1}\dots$ denote the terms of the p^{th}, $(p-1)^{\text{th}}\dots$ degrees respectively.

Then the points of contact are determined by (1) and (2), and the latter may be written

$$\alpha\frac{dF}{dx} + \beta\frac{dF}{dy} + \gamma\frac{dF}{dz} = x\frac{dF}{dx} + y\frac{dF}{dy} + z\frac{dF}{dz}.$$

But by a well-known theorem (see Todhunter's *Diff. Calc.* Chapter VIII. Ex. 3),

$$x\frac{du_p}{dx}+y\frac{du_p}{dy}+z\frac{du_p}{dz}=pu_p$$

$$x\frac{du_{p-1}}{dx}+y\frac{du_{p-1}}{dy}+z\frac{du_{p-1}}{dz}=(p-1)\,u_{p-1}$$

....

$$\therefore x\frac{dF}{dx}+y\frac{dF}{dy}+z\frac{dF}{dz}=pu_p+(p-1)u_{p-1}+\ldots+2u_2+u_1 \ldots (3).$$

But for all the points of contact we have

$$F(x, y, z)=0;$$

therefore $\quad 0=pu_p+pu_{p-1}+\ldots+pu_2+pu_1+pu_0 \ldots\ldots\ldots (4).$

Subtracting (4) from (3) we get

$$x\frac{dF}{dx}+y\frac{dF}{dy}+z\frac{dF}{dz}=-u_{p-1}-2u_{p-2}-\ldots-(p-2)\,u_2-(p-1)u_1-pu_0,$$

and equation (2) becomes

$$\alpha\frac{dF}{dx}+\beta\frac{dF}{dy}+\gamma\frac{dF}{dz}+u_{p-1}+2u_{p-2}+\ldots+(p-1)\,u_1+pu_0=0 \ldots (5).$$

Now $\frac{dF}{dx}$, $\frac{dF}{dy}$, $\frac{dF}{dz}$ are of the $(p-1)^{\text{th}}$ degree, consequently (5) represents a surface of the $(p-1)^{\text{th}}$ degree.

If the original surface be of the second degree, all the points of contact lie in a plane.

106. The equation of the tangent plane to an ellipsoid at the point (x, y, z) is

$$\frac{x'x}{a^2}+\frac{y'y}{b^2}+\frac{z'z}{c^2}=1.$$

If this pass through a point (α, β, γ), we must have

$$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

a relation which is satisfied by the co-ordinates of all the points of contact, and which is therefore the equation of the plane of contact.

The plane (1) is called the polar plane of the point (α, β, γ) with respect to the ellipsoid; and (α, β, γ) is called the pole of the plane (1).

If all the points in which (1) cuts the ellipsoid be joined with (α, β, γ) the joining lines will form a cone, and will all touch the ellipsoid, since each of them lies in the tangent plane at the point where it meets the surface. This cone is called an enveloping cone.

Conversely, if at all points at which any plane cuts an ellipsoid, tangent planes be drawn, these planes will all meet in one point, which is the pole of the cutting plane.

If a series of planes be drawn passing through a fixed point and cutting an ellipsoid, the poles of these planes will all lie in a fixed plane which is the polar of the fixed point. Let (α, β, γ) be the fixed point, and (x, y, z) the pole of any plane through (α, β, γ).

The equation of the polar of (x, y, z) is

$$\frac{x'x}{a^2} + \frac{y'y}{b^2} + \frac{z'z}{c^2} = 1.$$

If this plane pass through (α, β, γ) we must have

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = 1,$$

which shews that (x, y, z) lies on the polar of (α, β, γ).

If a series of planes be drawn passing through two fixed points and therefore through a fixed straight line, the poles of these planes will all lie in each of two fixed planes which are the polar planes of the two fixed points, that is, they will all lie in a fixed straight line.

Similar results hold for all the surfaces of the second degree.

107. The equation of the enveloping cone can be found by a process similar to that adopted in Art. 34. The equations of any generating line can be written

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \qquad \text{..............(1)},$$

and the equations of the curve of contact are

$$\left.\begin{aligned} F(x, y, z) &= 0 \\ (\alpha-x)\frac{dF}{dx}+(\beta-y)\frac{dF}{dy}+(\gamma-z)\frac{dF}{dz} &= 0 \end{aligned}\right\} \text{.......(2)}.$$

By substituting for x, y, z from (1) in the equations (2) their values $\alpha+lr$, $\beta+mr$, $\gamma+nr$ and eliminating r, we obtain a relation which l, m, n must satisfy in order that the line (1) may pass through some point of the curve (2).

The equations (2) can be reduced to one equation of the p^{th} degree, and one of the $(p-1)^{\text{th}}$, and the result of substituting for x, y, z from (1) will therefore be

$$\left.\begin{aligned} A_p r^p + A_{p-1}r^{p-1}+\ldots+A_1 r+A_0 &= 0 \\ B_{p-1}r^{p-1}+B_{p-2}r^{p-2}+\ldots+B_1 r+B_0 &= 0 \end{aligned}\right\} \text{.........(3)},$$

where A_p is a homogeneous function in l, m, n of the p^{th} degree, A_{p-1} and B_{p-1} are homogeneous functions of the $(p-1)^{\text{th}}$ degree, and so on.

The equations (3) can therefore be expressed in the form

$$A_p'(nr)^p + A'_{p-1}(nr)^{p-1}+\ldots+A_1'nr+A_0=0,$$
$$B'_{p-1}(nr)^{p-1}+B'_{p-2}(nr)^{p-2}+\ldots+B_1'nr+B_0=0,$$

where A_p', A'_{p-1}, $\ldots A_1'$, A_0, B'_{p-1}, $\ldots B_1'$, B_0 are functions of $\frac{l}{n}$, $\frac{m}{n}$, and the result of eliminating nr between them will be of the form

$$\phi\left(\frac{l}{n}, \frac{m}{n}\right)=0,$$

and the equation of the cone is therefore

$$\phi\left(\frac{x-\alpha}{z-\gamma}, \frac{y-\beta}{z-\gamma}\right)=0.$$

108. In the case of an ellipsoid the equation of the plane of contact is

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} - 1 = 0 \quad\text{.....................(1)},$$

and we have to substitute $\alpha + lr$, $\beta + mr$, $\gamma + nr$, for x, y, z in (1), and in the equation of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad\text{.....................(2)}.$$

We thus get

$$\left(\frac{\alpha l}{a^2} + \frac{\beta m}{b^2} + \frac{\gamma n}{c^2}\right) r + \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1 = 0 \quad\text{..........(3)},$$

and $$\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) r^2 + 2\left(\frac{\alpha l}{a^2} + \frac{\beta m}{b^2} + \frac{\gamma n}{c^2}\right) r$$

$$+ \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1 = 0 \quad\text{.........(4)};$$

and substituting for r from (3) in (4) we obtain

$$\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1\right) = \left(\frac{\alpha l}{a^2} + \frac{\beta m}{b^2} + \frac{\gamma n}{c^2}\right)^2 \quad\text{...(5)}.$$

This is the relation which l, m, n must satisfy in order that the straight line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$$

may pass through some point in the curve of intersection of (1) and (2).

The equation of the enveloping cone is obtained by substituting $x - \alpha$, $y - \beta$, $z - \gamma$ for l, m, n, and is therefore

$$\left\{\left(\frac{x-\alpha}{a}\right)^2 + \left(\frac{y-\beta}{b}\right)^2 + \left(\frac{z-\gamma}{c}\right)^2\right\}\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1\right)$$

$$= \left\{\frac{(x-\alpha)\,\alpha}{a^2} + \frac{(y-\beta)\,\beta}{b^2} + \frac{(z-\gamma)\,\gamma}{c^2}\right\}^2 \quad\text{......(6)}.$$

109. This equation can be obtained in another form by the aid of the following proposition.

Let $S=0$ be the equation of any surface of the second degree, and let $u=0$, $v=0$ be the equations of two planes. Then the equation

$$S+\lambda uv=0 \quad\text{.......................(1)},$$

where λ is some constant, will represent any surface of the second degree passing through the curves of intersection of $S=0$ with $u=0$ and $v=0$. For if $S'=0$ be the equation of any such surface, it is evident that S' cannot assume any other form than $k(S+\lambda uv)$ consistently with the suppositions that it is of the second degree, and is satisfied by all values of x, y, z which make S and u vanish simultaneously, and also by all values which make S and v vanish.

Again, if we suppose the plane $u=0$ to change its position so as to coincide with $v=0$, the equation (1) represents any surface *touching* $S=0$ along the curve in which the latter is cut by $v=0$, and becomes

$$S+\lambda v^2=0.$$

Hence the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1+\lambda\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}-1\right)^2=0 \quad\text{......(2)},$$

represents any surface of the second degree touching the ellipsoid at all the points of contact of tangent planes through (α, β, γ). If we take λ such that (2) shall pass through (α, β, γ) it must represent the enveloping cone. Substituting α, β, γ for x, y, z, we get

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1+\lambda\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)^2=0.$$

Whence the equation of the enveloping cone becomes

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)=\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}-1\right)^2 \quad\text{...(3)}.$$

This equation can of course be deduced from that of the last article.

110. If we suppose the point (α, β, γ) to recede from the origin to an infinite distance, the cone will ultimately become a cylinder whose generating lines are parallel to the line joining (α, β, γ) with the origin. This is called an enveloping cylinder, and the equation of any such cylinder can be found from that of the cone, by putting $\alpha = \lambda k$, $\beta = \mu k$, $\gamma = \nu k$, where λ, μ, ν are the direction-cosines of the generating lines, dividing by the highest power of k, and then making k infinite. The equation of the enveloping cylinder of an ellipsoid deduced in this manner from either of the equations in Arts. 108, 109 is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}+\frac{\nu^2}{c^2}\right)=\left(\frac{\lambda x}{a^2}+\frac{\mu y}{b^2}+\frac{\nu z}{c^2}\right)^2.$$

111. The equation of the cylinder which envelopes a given surface

$$F(x, y, z) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

can however be obtained independently of the enveloping cone.

For let λ, μ, ν be the direction-cosines of one of the generating lines; x, y, z the co-ordinates of the point where it touches (1). Then since this generating line of the cylinder is a tangent line to (1) at (x, y, z), we must have

$$\lambda\frac{dF}{dx}+\mu\frac{dF}{dy}+\nu\frac{dF}{dz}=0 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

This equation combined with (1) gives the locus of the points at which the enveloping cylinder touches the surface, and we have only to find the equation of a cylinder with its generating lines in a given direction, and passing through the curve given by (1) and (2), which can be done as in Art. 35.

If x', y', z' be the co-ordinates of any point in the generating line which touches (1) at the point (x, y, z), we have

$$\frac{x'-x}{\lambda}=\frac{y'-y}{\mu}=\frac{z'-z}{\nu}=-k \text{ suppose},$$

or $$x = x' + \lambda k, \quad y = y' + \mu k, \quad z = z' + \nu k.$$

Substituting these values of x, y, z in the equations (1) and (2), and eliminating k between the two equations, we get a relation between x', y', z' which is the equation of the enveloping cylinder.

112. In the case of the ellipsoid, the curve of contact is determined by the equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

$$\frac{\lambda x}{a^2}+\frac{\mu y}{b^2}+\frac{\nu z}{c^2}=0.$$

Putting $x'+\lambda k$, $y'+\mu k$, $z'+\nu k$ for x, y, z we get

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}-1+2\left(\frac{\lambda x'}{a^2}+\frac{\mu y'}{b^2}+\frac{\nu z'}{c^2}\right)k+\left(\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}+\frac{\nu^2}{c^2}\right)k^2=0,$$

$$\left(\frac{\lambda x'}{a^2}+\frac{\mu y'}{b^2}+\frac{\nu z'}{c^2}\right)+\left(\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}+\frac{\nu^2}{c^2}\right)k=0.$$

Substituting for k from the second in the first we get

$$\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}-1\right)\left(\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}+\frac{\nu^2}{c^2}\right)=\left(\frac{\lambda x'}{a^2}+\frac{\mu y'}{b^2}+\frac{\nu z'}{c^2}\right)^2,$$

the same equation as we obtained in Article 110.

113. Let the equation of a surface be given in the form

$$\phi(\alpha, \beta, \gamma, \delta)=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where α, β, γ, δ are the lengths of the perpendiculars from any point on the four faces of a tetrahedron, and let any straight line be drawn through the point $(\alpha, \beta, \gamma, \delta)$. Then if α', β', γ', δ' be the values of α, β, γ, δ for any other point in the line we shall have by obvious geometry

$$\frac{\alpha'-\alpha}{l}=\frac{\beta'-\beta}{m}=\frac{\gamma'-\gamma}{n}=\frac{\delta'-\delta}{q}=k \quad\ldots\ldots\ldots(2),$$

where l, m, n, q are the cosines of the angles between the line and the perpendiculars on the four faces of the tetrahedron, and k is the distance between the two points.

We obtain the value of k for the points where the line (2) meets (1) from the equation

$$\phi(\alpha+lk,\ \beta+mk,\ \gamma+nk,\ \delta+qk)=0,$$

or $$\phi(\alpha, \beta, \gamma, \delta) + k\left(l\frac{d\phi}{d\alpha} + m\frac{d\phi}{d\beta} + n\frac{d\phi}{d\gamma} + q\frac{d\phi}{d\delta}\right) + Ak^2 + Bk^3 + \ldots = 0 \ldots\ldots (3).$$

This equation gives as many values of k as the degree of the equation (1).

Since $(\alpha, \beta, \gamma, \delta)$ is a point on (1), $\phi(\alpha, \beta, \gamma, \delta)$ vanishes, and one value of k is zero. If l, m, n, q be restricted by the relation

$$l\frac{d\phi}{d\alpha} + m\frac{d\phi}{d\beta} + n\frac{d\phi}{d\gamma} + q\frac{d\phi}{d\delta} = 0,$$

two values of k vanish, and the line (2) is a tangent line to (1) at $(\alpha, \beta, \gamma, \delta)$. Hence eliminating l, m, n, q by means of (2) the equation of the locus of the tangent lines at $(\alpha, \beta, \gamma, \delta)$ is

$$(\alpha' - \alpha)\frac{d\phi}{d\alpha} + (\beta' - \beta)\frac{d\phi}{d\beta} + (\gamma' - \gamma)\frac{d\phi}{d\gamma} + (\delta' - \delta)\frac{d\phi}{d\delta} = 0,$$

or

$$\alpha'\frac{d\phi}{d\alpha} + \beta'\frac{d\phi}{d\beta} + \gamma'\frac{d\phi}{d\gamma} + \delta'\frac{d\phi}{d\delta} = \alpha\frac{d\phi}{d\alpha} + \beta\frac{d\phi}{d\beta} + \gamma\frac{d\phi}{d\gamma} + \delta\frac{d\phi}{d\delta}.$$

But the expression $\phi(\alpha, \beta, \gamma, \delta)$ may be supposed homogeneous, since if it be not, it can be made so by means of the relation given in Art. 26; and if it be of the p^{th} degree, we have by a well-known formula

$$\alpha\frac{d\phi}{d\alpha} + \beta\frac{d\phi}{d\beta} + \gamma\frac{d\phi}{d\gamma} + \delta\frac{d\phi}{d\delta} = p\phi(\alpha, \beta, \gamma, \delta) = 0,$$

since the point $(\alpha, \beta, \gamma, \delta)$ is on the surface (1). Hence the equation of the tangent plane at $(\alpha, \beta, \gamma, \delta)$ becomes

$$\alpha'\frac{d\phi}{d\alpha} + \beta'\frac{d\phi}{d\beta} + \gamma'\frac{d\phi}{d\gamma} + \delta'\frac{d\phi}{d\delta} = 0 \ldots\ldots\ldots (4).$$

EXAMPLES. CHAPTER VIII.

1. Find the locus of the point of intersection of three tangent planes to an ellipsoid which are mutually at right angles.

2. Find the locus of a point which moves so that the locus of the centre of the section of an ellipsoid by its polar plane with respect to that ellipsoid is a similar and similarly situated ellipsoid whose axes are each half of the corresponding axis of the original ellipsoid.

3. Shew that the polar equation of the locus of the foot of the perpendicular from the origin on the tangent plane to an ellipsoid is

$$r^2 = a^2 \sin^2\theta \cos^2\phi + b^2 \sin^2\theta \sin^2\phi + c^2 \cos^2\theta.$$

4. Find the equation of the locus of the foot of the perpendicular from a point (α, β, γ) on the tangent planes of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

5. Find the equation of the locus of the poles of all tangent planes of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

with respect to a sphere whose centre is at the point (α, β, γ) and whose radius is k.

6. Shew that in general six normals can be drawn through a given point to an ellipsoid, and that these six all lie on a cone of the second degree, three of whose generating lines are parallel to the axes of the ellipsoid.

7. If normals be drawn to an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at the points where it is cut by the cone

$$\frac{l}{x}+\frac{m}{y}+\frac{n}{z}=0,$$

prove that these normals all pass through a diameter of the ellipsoid.

8. In an ellipsoid whose semi-axes are a, b, c, plane sections are drawn so as always to touch a confocal ellipsoid (see Art. 160). Shew that the centres of these sections always lie on a surface of the fourth degree which intersects the ellipsoid in the cone

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}=0.$$

9. Prove that through any central radius of an ellipsoid one plane can be drawn cutting the ellipsoid in a curve of which that radius is a semi-axis. Shew that if it be so for more than one section it must be so for all such sections.

10. On a plane section of a given ellipsoid as base two cones are constructed of which the vertices are the centre of the surface and the pole of the section. If the ratio of the volumes of these cones is constant, prove that each of them is constant; and find the volume when the ratio is one of equality.

11. Find the locus of a luminous point, in order that the boundary of the shadow of an ellipsoid cast by it upon a given principal plane may be circular.

12. Prove that the right circular cylinders described about the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

b being the mean semi-axis, are represented by the equation

$$(b^2-c^2)x^2-(c^2-a^2)y^2+(a^2-b^2)z^2\pm 2(a^2-b^2)^{\frac{1}{2}}(b^2-c^2)^{\frac{1}{2}}zx=(a^2-c^2)b^2.$$

13. The shadow of a ball is cast by a candle on an inclined plane in contact with the ball; prove that as the candle burns down, the locus of the centre of the shadow is a straight line.

14. Find the equation of the tangent plane to the surface

$$xyz = a^3,$$

and the volume cut off by this plane from the axes.

15. Find the equation of the tangent plane at any point of the surface

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

Find also the length of the perpendicular on it from the origin, and the area of the triangle intercepted on the tangent plane by the co-ordinate planes. Shew that the sum of the squares of the intercepts on the axes of co-ordinates is constant.

16. Find the equation of the enveloping cone of the surface $By^2 + Cz^2 = x$, whose vertex is at a point (α, β, γ).

17. Find the length of the normal at any point of an ellipsoid cut off by the plane of xy. Find also the co-ordinates of its point of intersection with the plane of xy.

18. Find the equations of the normal at any point of the surface

$$By^2 + Cz^2 = x.$$

Find the locus of the points in which the normals to the surface drawn at all points of its intersection with the plane $x = \alpha$ cut the plane of yz.

19. Shew that the points on the surface

$$xyz = c^3$$

at which the normals intersect a fixed line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

all lie on the surface

$$\alpha x (my - nz) + \beta y (nz - lx) + \gamma z (lx - my) = x^2 (my - nz) + y^2 (nz - lx) + z^2 (lx - my).$$

20. Find the locus of the point of intersection of three tangent planes to a paraboloid which are mutually at right angles.

21. Find the equation of a surface of the second degree which passes through all the points of contact of tangent planes drawn through an external point (α, β, γ) to the surface.

$$x^3 + y^3 + z^3 - 3xyz = c^3,$$

and discuss its nature for different positions of (α, β, γ).

22. Find the equation of a surface of the second degree which passes through all the points of contact of tangent planes drawn through an external point (α, β, γ) to the surface

$$xyz = a^3,$$

and discuss its nature for different positions of (α, β, γ).

23. Find the equation of the locus of the foot of the perpendicular from the origin on the tangent planes of the surface

$$By^2 + Cz^2 = x.$$

24. Shew that the plane

$$lx + my + nz = 0$$

will touch the cone

$$Ax^2 + By^2 + Cz^2 = 0$$

if l, m, n satisfy the condition

$$\frac{l^2}{A} + \frac{m^2}{B} + \frac{n^2}{C} = 0.$$

25. Shew that the axes of a central section of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by a plane parallel to the tangent plane at (α, β, γ) are given by the equation

$$r^4 - (a^2 + b^2 + c^2 - \alpha^2 - \beta^2 - \gamma^2)\, r^2 + \frac{a^2b^2c^2}{p^2} = 0,$$

where p is the perpendicular from the centre on the tangent plane.

CHAPTER IX.

ON CURVES IN SPACE.

114. We have seen (Art. 16) that any two equations

$$\left.\begin{aligned} F_1(x, y, z) = 0 \\ F_2(x, y, z) = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

since they are satisfied by the co-ordinates of all the points of intersection of the surfaces represented by each equation, will in general represent a curve.

These equations can be reduced to the form

$$\left.\begin{aligned} z = f_1(x) \\ y = f_2(x) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

by eliminating y and z in turn between the two equations (1). It may be noticed that the two equations (2) will in some cases represent a curve not included in (1). For instance, if the two equations (1) were of the first and second degrees respectively, by eliminating y and z in turn we should get two equations of the second degree, and the first two equations would represent one plane curve, while the second pair would represent the original curve, and another plane curve besides. (See Art. 95.)

Assuming x to be any arbitrary function of a new variable t, the equations (2) can be replaced by the three

$$x = \phi(t), \; y = \psi(t), \; z = \chi(t) \ldots\ldots\ldots\ldots (3).$$

This third form possesses many advantages from its symmetrical character, and we shall in general use it.

115. As an example the pair of equations

$$\left.\begin{aligned} Ax + By + Cz = D \\ A'x + B'y + C'z = D' \end{aligned}\right\} \quad\text{.................. (1)}$$

represent a straight line.

Eliminating y and z in turn we get the two equations

$$\left.\begin{aligned} z = \frac{A'B - AB'}{B'C - BC'} x + \frac{B'D - BD'}{B'C - BC'} \\ y = \frac{C'A - CA'}{B'C - BC'} x + \frac{CD' - C'D}{B'C - BC'} \end{aligned}\right\} \quad\text{.............. (2),}$$

which correspond to the form (2) in the last Article.

Lastly, assuming $x = (B'C - BC')\,t$, we get

$$\left.\begin{aligned} x = (B'C - BC')\,t,\ y = (C'A - CA')\,t + \frac{CD' - C'D}{B'C - BC'} \\ z = (A'B - AB')\,t + \frac{B'D - BD'}{B'C - BC'} \end{aligned}\right\} \text{...(3),}$$

which correspond to the form (3) in the last Article.

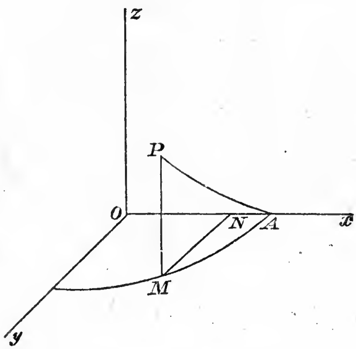
116. The curves of the most frequent occurrence and greatest importance are plane curves, the discussion of which properly belongs to plane geometry. As an instance of a curve not plane we may take the helix.

This is the curve formed by the thread of a screw. It may be produced by wrapping a right-angled triangle round a circular cylinder, the base of the triangle being at right angles to the axis of the cylinder.

Take the axis of the cylinder as axis of z, a plane through the base of the triangle as plane of xy, and a line through the acute angle at the base of the triangle as axis of x.

Let O be the origin; x, y, z the co-ordinates of any point P in the curve, a the radius of the cylinder, θ the angle AOM between the axis of x and OM the projection of OP on the plane of xy, and α the acute angle at the base of the triangle. We obtain without difficulty,

$$x = ON = OM \cos\theta = a\cos\theta,$$
$$y = MN = OM \sin\theta = a\sin\theta,$$
$$z = PM = \text{arc } AM \times \tan\alpha = a\theta\tan\alpha,$$



or if $a\tan\alpha = c$,

$$x = a\cos\theta,\ y = a\sin\theta,\ z = c\theta \ldots\ldots\ldots\ldots (1).$$

Whence $$x = a\cos\frac{z}{c},\ y = a\sin\frac{z}{c} \ldots\ldots\ldots\ldots\ldots (2).$$

Either (1) or (2) may be considered as the equations of the helix.

117. The limiting position of the straight line joining two points of a curve when the second point moves up indefinitely near to the first, is called the *tangent* to the curve at that point.

Let the equations of the curve be

$$x = \phi(t),\ y = \psi(t),\ z = \chi(t) \ldots\ldots\ldots\ldots\ldots (1),$$

and let t and $t + \tau$ be the values of t for two points on the curve. The equations of the straight line joining these are

$$\frac{x' - \phi(t)}{\phi(t+\tau) - \phi(t)} = \frac{y' - \psi(t)}{\psi(t+\tau) - \psi(t)} = \frac{z' - \chi(t)}{\chi(t+\tau) - \chi(t)},$$

x', y', z' being current co-ordinates;

or $$\frac{x' - \phi(t)}{\dfrac{\phi(t+\tau) - \phi(t)}{\tau}} = \frac{y' - \psi(t)}{\dfrac{\psi(t+\tau) - \psi(t)}{\tau}} = \frac{z' - \chi(t)}{\dfrac{\chi(t+\tau) - \chi(t)}{\tau}}.$$

But when τ is diminished indefinitely the two points coincide and the straight line joining them becomes the tangent at (x, y, z). Also the limit of $\dfrac{\phi(t+\tau) - \phi(t)}{\tau}$ is $\phi'(t)$ or $\dfrac{dx}{dt}$, and similarly for the other denominators.

Hence the equations of the tangent at (x, y, z) are

$$\frac{x' - x}{\dfrac{dx}{dt}} = \frac{y' - y}{\dfrac{dy}{dt}} = \frac{z' - z}{\dfrac{dz}{dt}} \quad \text{................ (2).}$$

118. The length of the chord joining two points (x, y, z) and (x_1, y_1, z_1) is

$$\sqrt{(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2}.$$

But by Newton (Section I. Lemma VII.) when the two points approach indefinitely near to each other, the ratio of the arc to the chord becomes ultimately a ratio of equality. Hence if s and $s + \delta s$ be the lengths of the arcs measured from some fixed point up to the points (x, y, z), (x_1, y_1, z_1) respectively, the fraction

$$\frac{\delta s}{\sqrt{(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2}}$$

becomes ultimately equal to unity, or

$$\frac{\delta s}{\tau} = \sqrt{\left(\frac{x_1 - x}{\tau}\right)^2 + \left(\frac{y_1 - y}{\tau}\right)^2 + \left(\frac{z_1 - z}{\tau}\right)^2} \text{ ultimately;}$$

$$\therefore \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \quad \text{.......... (1).}$$

From this result we see that the cosine of the angle which the tangent at (x, y, z) makes with the axis of x, which by Art. 17 is

$$\frac{\frac{dx}{dt}}{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}},$$

is equal to $$\frac{\frac{dx}{dt}}{\frac{ds}{dt}} \text{ or } \frac{dx}{ds}.$$

And similarly, the cosines of the angles which the tangent makes with the axes of y and z are $\frac{dy}{ds}$ and $\frac{dz}{ds}$ respectively.

Dividing by $\left(\frac{ds}{dt}\right)^2$ the equation (1) reduces to the form

$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1 \quad \text{..............} (2).$$

119. Any straight line through the point (x, y, z) perpendicular to the tangent is called a *normal line.* All such lines lie in a plane through (x, y, z) perpendicular to the tangent, which is called the *normal plane.* Its equation is at once seen to be

$$(x'-x)\frac{dx}{dt}+(y'-y)\frac{dy}{dt}+(z'-z)\frac{dz}{dt}=0.$$

120. It is always possible to draw a plane through any three points of a curve. The limiting position of this plane when two of the points move up indefinitely near to the third is called the *osculating plane* at that point.

Let the equations of the curve be

$$x=\phi(t),\ \ y=\psi(t),\ \ z=\chi(t) \quad \text{...........}(1),$$

and let t, $t+\tau$, $t+2\tau$ be the values of t corresponding to three points on the curve. Let the equation of any plane be

$$Ax' + By' + Cz' = D \quad \text{..................(2).}$$

If this plane pass through the three points t, $t+\tau$, $t+2\tau$, we have

$$A\phi(t) + B\psi(t) + C\chi(t) = D \quad \text{..............(3),}$$

$$A\phi(t+\tau) + B\psi(t+\tau) + C\chi(t+\tau) = D \quad \text{......(4),}$$

$$A\phi(t+2\tau) + B\psi(t+2\tau) + C\chi(t+2\tau) = D \quad \text{......(5).}$$

Subtracting the first of these equations from the second we have

$$A\{\phi(t+\tau) - \phi(t)\} + B\{\psi(t+\tau) - \psi(t)\} + C\{\chi(t+\tau) - \chi(t)\} = 0.$$

Or, dividing by τ,

$$A\frac{\phi(t+\tau) - \phi(t)}{\tau} + B\frac{\psi(t+\tau) - \psi(t)}{\tau} + C\frac{\chi(t+\tau) - \chi(t)}{\tau} = 0.$$

Subtracting twice the second from the sum of the first and third and dividing by τ^2, we get

$$A\frac{\phi(t+2\tau) - 2\phi(t+\tau) + \phi(t)}{\tau^2} + B\frac{\psi(t+2\tau) - 2\psi(t+\tau) + \psi(t)}{\tau^2} + C\frac{\chi(t+2\tau) - 2\chi(t+\tau) + \chi(t)}{\tau^2} = 0.$$

But if we make the three points coincide, τ vanishes, and these two equations become (Todhunter's *Diff. Calc.* Art. 127)

$$A\frac{dx}{dt} + B\frac{dy}{dt} + C\frac{dz}{dt} = 0,$$

$$A\frac{d^2x}{dt^2} + B\frac{d^2y}{dt^2} + C\frac{d^2z}{dt^2} = 0;$$

$$\therefore \frac{A}{\dfrac{dy}{dt}\dfrac{d^2z}{dt^2} - \dfrac{dz}{dt}\dfrac{d^2y}{dt^2}} = \frac{B}{\dfrac{dz}{dt}\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt}\dfrac{d^2z}{dt^2}} = \frac{C}{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}.$$

And subtracting (3) from (2) we have

$$A(x'-x)+B(y'-y)+C(z'-z)=0.$$

Whence the equation of the osculating plane at the point (x, y, z) becomes

$$(x'-x)\left\{\frac{dy}{dt}\frac{d^2z}{dt^2}-\frac{dz}{dt}\frac{d^2y}{dt^2}\right\}+(y'-y)\left\{\frac{dz}{dt}\frac{d^2x}{dt^2}-\frac{dx}{dt}\frac{d^2z}{dt^2}\right\}$$
$$+(z'-z)\left\{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}\right\}=0.$$

121. The osculating plane is sometimes defined as the plane which lies closer to a curve at a given point than any other plane, and its equation is obtained in the following manner.

Let $\quad A(x'-x)+B(y'-y)+C(z'-z)=0\text{.........}(1)$

be the equation of any plane through (x, y, z). The perpendicular on this from a point (x_1, y_1, z_1) is

$$\frac{A(x_1-x)+B(y_1-y)+C(z_1-z)}{\sqrt{A^2+B^2+C^2}}.$$

But if (x_1, y_1, z_1) be a point on the curve corresponding to a value $t+\tau$ of t,

$$x_1=x+\tau\frac{dx}{dt}+\frac{\tau^2}{\underline{|2}}\frac{d^2x}{dt^2}+\ldots\ldots$$

$$y_1=y+\tau\frac{dy}{dt}+\frac{\tau^2}{\underline{|2}}\frac{d^2y}{dt^2}+\ldots\ldots$$

$$z_1=z+\tau\frac{dz}{dt}+\frac{\tau^2}{\underline{|2}}\frac{d^2z}{dt^2}+\ldots\ldots$$

Hence the length of the perpendicular becomes

$$\frac{\tau\left(A\frac{dx}{dt}+B\frac{dy}{dt}+C\frac{dz}{dt}\right)+\frac{\tau^2}{\underline{|2}}\left(A\frac{d^2x}{dt^2}+B\frac{d^2y}{dt^2}+C\frac{d^2z}{dt^2}\right)+\text{terms involving }\tau^3.}{\sqrt{A^2+B^2+C^2}}$$

And when τ is diminished indefinitely, the succeeding terms are very small compared with the first and second, and the smallest value which this fraction can assume will be when A, B, C are determined by the equations

$$A\frac{dx}{dt} + B\frac{dy}{dt} + C\frac{dz}{dt} = 0,$$

$$A\frac{d^2x}{dt^2} + B\frac{d^2y}{dt^2} + C\frac{d^2z}{dt^2} = 0,$$

whence we obtain the same result as in the last Article.

122. All straight lines drawn through the point (x, y, z) perpendicular to the tangent at that point are normals. That normal which lies in the osculating plane may be considered as the normal drawn in the plane of the curve, and is called the principal normal. The equations of the normal plane and the osculating plane considered as simultaneous are the equations of this line. These are

$$(x'-x)\frac{dx}{dt} + (y'-y)\frac{dy}{dt} + (z'-z)\frac{dz}{dt} = 0,$$

$$(x'-x)\left\{\frac{dy}{dt}\frac{d^2z}{dt^2} - \frac{dz}{dt}\frac{d^2y}{dt^2}\right\} + (y'-y)\left\{\frac{dz}{dt}\frac{d^2x}{dt^2} - \frac{dx}{dt}\frac{d^2z}{dt^2}\right\}$$
$$+ (z'-z)\left\{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}\right\} = 0.$$

If we put these equations in the form

$$\frac{x'-x}{P} = \frac{y'-y}{Q} = \frac{z'-z}{R},$$

the value of P is

$$\frac{dy}{dt}\left\{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}\right\} - \frac{dz}{dt}\left\{\frac{dz}{dt}\frac{d^2x}{dt^2} - \frac{dx}{dt}\frac{d^2z}{dt^2}\right\}$$
$$= \frac{dx}{dt}\left\{\frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2}\right\} - \frac{d^2x}{dt^2}\left\{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}$$

But by Art. 118

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2;$$

therefore differentiating,

$$\frac{ds}{dt}\frac{d^2s}{dt^2} = \frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2}.$$

$$\text{Hence } P = \frac{dx}{dt}\left\{\frac{ds}{dt}\frac{d^2s}{dt^2} - \frac{dx}{dt}\frac{d^2x}{dt^2}\right\} - \frac{d^2x}{dt^2}\left\{\left(\frac{ds}{dt}\right)^2 - \left(\frac{dx}{dt}\right)^2\right\}$$

$$= \frac{ds}{dt}\left\{\frac{dx}{dt}\frac{d^2s}{dt^2} - \frac{ds}{dt}\frac{d^2x}{dt^2}\right\};$$

and similar values may be found for Q and R. Hence the equations of the principal normal are

$$\frac{x' - x}{\dfrac{ds}{dt}\dfrac{d^2x}{dt^2} - \dfrac{d^2s}{dt^2}\dfrac{dx}{dt}} = \frac{y' - y}{\dfrac{ds}{dt}\dfrac{d^2y}{dt^2} - \dfrac{d^2s}{dt^2}\dfrac{dy}{dt}} = \frac{z' - z}{\dfrac{ds}{dt}\dfrac{d^2z}{dt^2} - \dfrac{d^2s}{dt^2}\dfrac{dz}{dt}},$$

which may be written in either of the forms

$$\frac{x' - x}{\dfrac{d}{dt}\left(\dfrac{dx}{ds}\right)} = \frac{y' - y}{\dfrac{d}{dt}\left(\dfrac{dy}{ds}\right)} = \frac{z' - z}{\dfrac{d}{dt}\left(\dfrac{dz}{ds}\right)} \quad \ldots\ldots\ldots\ldots\ldots (1),$$

or

$$\frac{x' - x}{\dfrac{d^2x}{ds^2}} = \frac{y' - y}{\dfrac{d^2y}{ds^2}} = \frac{z' - z}{\dfrac{d^2z}{ds^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

123. The equations (2) of the last article can also be obtained as follows.

If $\lambda, \mu, \nu; \lambda', \mu', \nu'$ be the direction-cosines of two straight lines, the direction-cosines of the two straight lines which bisect the angles between them are proportional to $\lambda + \lambda'$, $\mu + \mu'$, $\nu + \nu'$ and $\lambda - \lambda'$, $\mu - \mu'$, $\nu - \nu'$.

For planes through the origin perpendicular to the two given straight lines have their equations

$$\lambda x + \mu y + \nu z = 0 \ldots\ldots\ldots (1)$$

and

$$\lambda' x + \mu' y + \nu' z = 0 \ldots\ldots\ldots (2) \text{ respectively.}$$

By Art. 26 the equations of two planes which bisect the angles between (1) and (2) are

$$(\lambda + \lambda') x + (\mu + \mu') y + (\nu + \nu') z = 0,$$
$$(\lambda - \lambda') x + (\mu - \mu') y + (\nu - \nu') z = 0.$$

And the direction-cosines of the normals to these planes, which are evidently parallel to the bisectors of the angles between the two original straight lines, are proportional to $\lambda + \lambda'$, $\mu + \mu'$, $\nu + \nu'$ and $\lambda - \lambda'$, $\mu - \mu'$, $\nu - \nu'$ respectively.

If l, m, n be the actual values of the direction-cosines of the latter line, we have

$$l = \frac{\lambda - \lambda'}{\sqrt{(\lambda - \lambda')^2 + (\mu - \mu')^2 + (\nu - \nu')^2}}$$
$$= \frac{\lambda - \lambda'}{\sqrt{2 - 2\cos\theta}} = \frac{1}{2}(\lambda - \lambda') \operatorname{cosec}\frac{\theta}{2},$$

if θ be the angle between the two straight lines.

124. Let now λ, μ, ν be the direction-cosines of the tangent to a curve at the point (x, y, z), and λ', μ', ν' their values at an adjacent point on the curve distant δs from the former. Then ultimately if the two points be made to approach indefinitely near to each other and coincide, of the two bisectors considered in the last article, the one will coincide with either tangent, and the other will be the principal normal. The former will evidently have its direction-cosines proportional to $\lambda + \lambda'$, $\mu + \mu'$, $\nu + \nu'$, and the latter must have its direction-cosines proportional to $\lambda - \lambda'$, $\mu - \mu'$, $\nu - \nu'$.

But $$\lambda' = \lambda + \frac{d\lambda}{ds}\delta s + \ldots\ldots \text{ terms involving } (\delta s)^2$$

$$\mu' = \mu + \frac{d\mu}{ds}\delta s + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\nu' = \nu + \frac{d\nu}{ds}\delta s + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence the direction-cosines of the principal normal are proportional to $\frac{d\lambda}{ds}\delta s$, $\frac{d\mu}{ds}\delta s$, $\frac{d\nu}{ds}\delta s$, or to $\frac{d\lambda}{ds}$, $\frac{d\mu}{ds}$, $\frac{d\nu}{ds}$, and putting for λ, μ, ν their values $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ the equations of the principal normal become as before

$$\frac{x'-x}{\frac{d^2x}{ds^2}} = \frac{y'-y}{\frac{d^2y}{ds^2}} = \frac{z'-z}{\frac{d^2z}{ds^2}}.$$

125. If the curve be a plane curve, the equation of the osculating plane must reduce to the equation of the plane in which the curve lies. Hence the ratios

$$\left(\frac{dy}{dt}\frac{d^2z}{dt^2} - \frac{dz}{dt}\frac{d^2y}{dt^2}\right) : \left(\frac{dz}{dt}\frac{d^2x}{dt^2} - \frac{dx}{dt}\frac{d^2z}{dt^2}\right) : \left(\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}\right)$$

must be constant for all points on the curve.

We may therefore assume

$$\frac{dy}{dt}\frac{d^2z}{dt^2} - \frac{dz}{dt}\frac{d^2y}{dt^2} = \lambda v \quad\quad (1),$$

$$\frac{dz}{dt}\frac{d^2x}{dt^2} - \frac{dx}{dt}\frac{d^2z}{dt^2} = \mu v \quad\quad (2),$$

$$\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2} = \nu v \quad\quad (3),$$

where λ, μ, ν are constants, and v some function of t.

Eliminating λ, μ and v from (1) and (2) by differentiating, we get

$$\left(\frac{dz}{dt}\frac{d^2x}{dt^2} - \frac{dx}{dt}\frac{d^2z}{dt^2}\right)\left(\frac{dy}{dt}\frac{d^3z}{dt^3} - \frac{dz}{dt}\frac{d^3y}{dt^3}\right)$$

$$-\left(\frac{dy}{dt}\frac{d^2z}{dt^2} - \frac{dz}{dt}\frac{d^2y}{dt^2}\right)\left(\frac{dz}{dt}\frac{d^3x}{dt^3} - \frac{dx}{dt}\frac{d^3z}{dt^3}\right) = 0,$$

or reducing and dividing by $\frac{dz}{dt}$,

$$\frac{d^3x}{dt^3}\left(\frac{dy}{dt}\frac{d^2z}{dt^2}-\frac{dz}{dt}\frac{d^2y}{dt^2}\right)+\frac{d^3y}{dt^3}\left(\frac{dz}{dt}\frac{d^2x}{dt^2}-\frac{dx}{dt}\frac{d^2z}{dt^2}\right)$$
$$+\frac{d^3z}{dt^3}\left(\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}\right)=0,$$

which may be written

$$\begin{vmatrix} \frac{dx}{dt}, & \frac{dy}{dt}, & \frac{dz}{dt} \\ \frac{d^2x}{dt^2}, & \frac{d^2y}{dt^2}, & \frac{d^2z}{dt^2} \\ \frac{d^3x}{dt^3}, & \frac{d^3y}{dt^3}, & \frac{d^3z}{dt^3} \end{vmatrix}=0 \quad\ldots\ldots\ldots\ldots (4).$$

The symmetry of this relation shews that we should get the same result by eliminating μ, ν and v from equations (2) and (3).

This relation may be also obtained from Art. 121, since if the curve lie in the plane (1), the perpendicular on this plane from any point in the curve must vanish. We must therefore have

$$A\frac{dx}{dt}+B\frac{dy}{dt}+C\frac{dz}{dt}=0,$$

$$A\frac{d^2x}{dt^2}+B\frac{d^2y}{dt^2}+C\frac{d^2z}{dt^2}=0,$$

$$A\frac{d^3x}{dt^3}+B\frac{d^3y}{dt^3}+C\frac{d^3z}{dt^3}=0,$$

whence the relation (4) follows. We must also have

$$A\frac{d^nx}{dt^n}+B\frac{d^ny}{dt^n}+C\frac{d^nz}{dt^n}=0,$$

for all values of n. But this will be the case if equation (4) is satisfied for all points in the curve, as may be seen by differentiating.

126. *If a curve be drawn on a given surface such that the inclination of its tangent to a given fixed plane is always greater than that of any other tangent line to the surface at the same point, the curve is called a line of greatest slope to the given plane.*

Let $$F(x, y, z) = 0 \quad\text{.....................}\quad (1)$$

be the equation of the given surface, and let

$$Ax + By + Cz = D \quad\text{.................}\quad (2)$$

be the equation of the given plane.

The direction-cosines of the tangent line to the curve at any point (x, y, z) are $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$.

The equation of the tangent plane to (1) at (x, y, z) is

$$(x' - x)\frac{dF}{dx} + (y' - y)\frac{dF}{dy} + (z' - z)\frac{dF}{dz} = 0,$$

and the direction-cosines of the line of intersection of this plane with the plane (2) are proportional to

$$B\frac{dF}{dz} - C\frac{dF}{dy},\quad C\frac{dF}{dx} - A\frac{dF}{dz},\quad A\frac{dF}{dy} - B\frac{dF}{dx},$$

and it is evident that the tangent line to the curve of greatest slope must be perpendicular to the intersection of the tangent plane with the plane (2), whence we get

$$\frac{dx}{ds}\left(B\frac{dF}{dz} - C\frac{dF}{dy}\right) + \frac{dy}{ds}\left(C\frac{dF}{dx} - A\frac{dF}{dz}\right) + \frac{dz}{ds}\left(A\frac{dF}{dy} - B\frac{dF}{dx}\right) = 0 \ldots (3)$$

The integral of this equation united with (1) gives the curves required. The integration will introduce one arbitrary constant which is determined if one point on the curve be known. Hence, a line of greatest slope can be drawn through any point on the surface.

If the given plane be the plane of xy, $A = 0$, $B = 0$, and the equation (3) becomes

$$\frac{dF}{dx}\frac{dy}{ds} - \frac{dF}{dy}\frac{dx}{ds} = 0,$$

or $$\frac{dF}{dx}\frac{dy}{dx} - \frac{dF}{dy} = 0 \quad\text{.......................}\quad (4).$$

As an example of the last case take the equation of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad\text{.......................}\quad (5).$$

Equation (4) becomes

$$\frac{x}{a^2}\frac{dy}{dx}-\frac{y}{b^2}=0;$$

$$\therefore \frac{1}{a^2}\log y-\frac{1}{b^2}\log x=\text{constant};$$

$$\therefore y=mx^{\frac{a^2}{b^2}} \quad\text{.......................}\quad (6).$$

This equation united with (5) gives the lines of greatest slope. If $a=b$, (6) becomes

$$y=mx,$$

so that in the case of a spheroid the meridians are the lines of greatest slope to the plane of circular section.

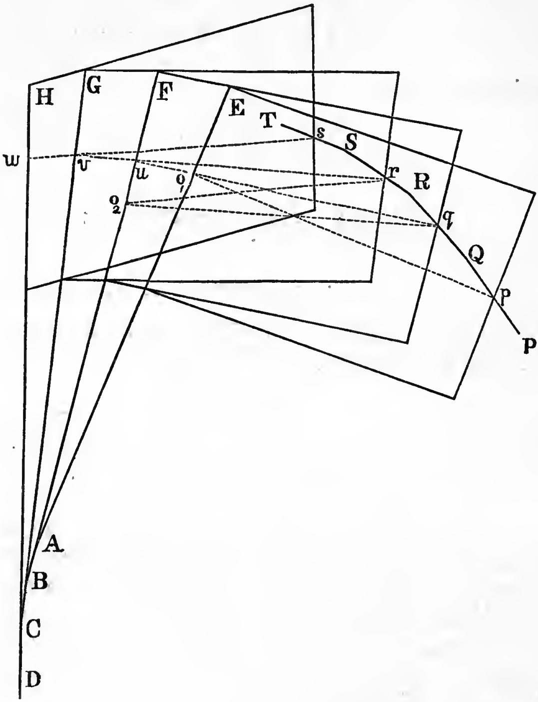
127. We shall devote the remainder of this Chapter to the discussion of the curvature of curves in space. This is of two kinds, the first being the curvature of the curve considered as lying in its osculating plane, and the second, the curvature by which it leaves the osculating plane, which is called the curvature of torsion. On this account curves in space are called curves of double curvature.

Before proceeding to the formulæ relating to the two kinds of curvature at any point of a curve some geometrical explanations and definitions must be given.

Let PQ, QR, RS, ST,... be a series of lines of equal length, which when their length is diminished indefinitely become ultimately small portions of a continuous curve. Let p, q, r, s... be their middle points.

Through p let a plane be drawn perpendicular to PQ and through q, r, s... planes perpendicular to QR, RS, ST,... respectively. These will ultimately be normal planes to the curve at consecutive points. Let the planes through p, q

intersect in a line AE, and the planes through q, r in a line BF which cuts AE in some point A, and so on.



Let the plane which passes through P, Q, R meet AE in O_1, and the plane through Q, R, S meet BF in O_2. It is evident that the point O_1 is equidistant from P, Q and R, and a circle with centre O_1 and radius O_1P will pass through Q and R. This circle will ultimately pass through three consecutive points of the curve, and lies in the plane $PQRO_1$, which is ultimately the osculating plane at Q. Hence it is the circle of curvature of the curve considered as a plane curve lying in the osculating plane. It is called the *circle of absolute or circular curvature*, and the point O_1 is called the *centre of absolute or circular curvature.*

Again, all points in the straight line AE are equidistant from the three points P, Q and R. All points in the straight line BF are equidistant from Q, R and S. Hence the point A, where AE and BF meet, is equidistant from the four points P, Q, R, and S, and a sphere with centre A and radius AP will ultimately pass through four consecutive points of the curve. The point A is called the *centre of spherical curvature*, and the length AP the *radius of spherical curvature*.

The lines AE, BF, CG ... ultimately generate a surface which is touched by the normal planes of the curve, and the ultimate intersections of these lines produce a curve which is called the edge of regression of this surface.

128. The locus of the centres of absolute curvature is not an evolute, but an infinite number of evolutes can be drawn on the surface generated by the lines AE, BF,... For let O_1 be any point in AE, and let pO_1, qO_1 be joined and qO_1 be produced to meet BF in u; join ru and produce it to meet CG in v; join sv and produce it to meet DH in w, and so on. We have

$$O_1p = O_1q;$$

$$\therefore uO_1 + O_1p = uq,$$

$$vu + uO_1 + O_1p = vu + uq = vu + ur = vr,$$

....

Hence if a string be laid along the curve $wvuO_1$ and its end be at p, as it is unwrapped this extremity will pass through $qrst$... and describe ultimately the original curve.

An evolute can thus be found passing through any point of any one of the lines AE, BF...

129. The centre of absolute curvature may be defined as the point where the line of intersection of two consecutive normal planes meets the osculating plane.

Let the equation of the normal plane at a point (x, y, z) be denoted by

$$F(t) = 0 \qquad \text{.......................... (1).}$$

Any other normal plane can be represented by

$$F(t_1) = 0 \qquad \text{.......................... (2),}$$

where t_1 is the corresponding value of t.

At the points where (1) and (2) meet, we have

$$F(t_1) - F(t) = 0,$$

$$\text{or} \quad \frac{F(t_1) - F(t)}{t_1 - t} = 0.$$

And this latter equation when $t_1 - t$ is indefinitely diminished becomes

$$\frac{dF}{dt} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Hence the line of intersection of two consecutive normal planes is given by the two equations

$$F(t) = 0, \quad \frac{dF}{dt} = 0.$$

But $\quad F(t) = (x' - x)\frac{dx}{dt} + (y' - y)\frac{dy}{dt} + (z' - z)\frac{dz}{dt},$

$$\therefore \frac{dF}{dt} = (x' - x)\frac{d^2x}{dt^2} + (y' - y)\frac{d^2y}{dt^2} + (z' - z)\frac{d^2z}{dt^2} - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \Big(\frac{d}{d}$$

$$= (x' - x)\frac{d^2x}{dt^2} + (y' - y)\frac{d^2y}{dt^2} + (z' - z)\frac{d^2z}{dt^2} - \left(\frac{ds}{dt}\right)^2.$$

Hence the line of intersection of two consecutive normal planes is given by

$$(x' - x)\frac{dx}{dt} + (y' - y)\frac{dy}{dt} + (z' - z)\frac{dz}{dt} = 0 \ldots\ldots\ldots\ldots (4),$$

$$\text{and} \quad (x' - x)\frac{d^2x}{dt^2} + (y' - y)\frac{d^2y}{dt^2} + (z' - z)\frac{d^2z}{dt^2} = \left(\frac{ds}{dt}\right)^2 \ldots\ldots\ldots (5).$$

The point where this line meets the osculating plane is given by (4) and (5) united with

$$(x' - x)\left(\frac{dy}{dt}\frac{d^2z}{dt^2} - \frac{dz}{dt}\frac{d^2y}{dt^2}\right)$$

$$+ (y' - y)\left(\frac{dz}{dt}\frac{d^2x}{dt^2} - \frac{dx}{dt}\frac{d^2z}{dt^2}\right)$$

$$+ (z' - z)\left(\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}\right) = 0 \ldots\ldots\ldots\ldots (6).$$

From (4) and (6) we obtain, as in Art. 122,

$$\frac{x'-x}{\frac{d^2x}{dt^2}\frac{ds}{dt}-\frac{dx}{dt}\frac{d^2s}{dt^2}}=\frac{y'-y}{\frac{d^2y}{dt^2}\frac{ds}{dt}-\frac{dy}{dt}\frac{d^2s}{dt^2}}=\frac{z'-z}{\frac{d^2z}{dt^2}\frac{ds}{dt}-\frac{dz}{dt}\frac{d^2s}{dt^2}}\text{.....}(7),$$

and by equation (5) each of these fractions is equal to

$$\frac{\left(\frac{ds}{dt}\right)^2}{\left\{\left(\frac{d^2x}{dt^2}\right)^2+\left(\frac{d^2y}{dt^2}\right)^2+\left(\frac{d^2z}{dt^2}\right)^2\right\}\frac{ds}{dt}-\frac{ds}{dt}\left(\frac{d^2s}{dt^2}\right)^2}$$

$$=\frac{\frac{ds}{dt}}{\left(\frac{d^2x}{dt^2}\right)^2+\left(\frac{d^2y}{dt^2}\right)^2+\left(\frac{d^2z}{dt^2}\right)^2-\left(\frac{d^2s}{dt^2}\right)^2}\text{...}(8).$$

Also each of them is equal to

$$\sqrt{\frac{(x'-x)^2+(y'-y)^2+(z'-z)^2}{\left(\frac{d^2x}{dt^2}\frac{ds}{dt}-\frac{dx}{dt}\frac{d^2s}{dt^2}\right)^2+\left(\frac{d^2y}{dt^2}\frac{ds}{dt}-\frac{dy}{dt}\frac{d^2s}{dt^2}\right)^2+\left(\frac{d^2z}{dt^2}\frac{ds}{dt}-\frac{dz}{dt}\frac{d^2s}{dt^2}\right)^2}}$$

$$=\frac{\rho}{\left(\frac{ds}{dt}\right)\sqrt{\left(\frac{d^2x}{dt^2}\right)^2+\left(\frac{d^2y}{dt^2}\right)^2+\left(\frac{d^2z}{dt^2}\right)^2-\left(\frac{d^2s}{dt^2}\right)^2}},$$

where ρ is the radius of absolute curvature.

Hence

$$\frac{\left(\frac{ds}{dt}\right)^4}{\rho^2}=\left(\frac{d^2x}{dt^2}\right)^2+\left(\frac{d^2y}{dt^2}\right)^2+\left(\frac{d^2z}{dt^2}\right)^2-\left(\frac{d^2s}{dt^2}\right)^2\text{........}(9),$$

or if s be taken as independent variable,

$$\frac{1}{\rho^2}=\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2\text{............}(10).$$

Equation (9) or (10) gives ρ, and equations (7) and (8) give x', y', z' the co-ordinates of the centre of absolute curvature.

130. The results of the last article can be obtained by means of the formulæ proved in Article 123.

For if δs be the arc between two points of a curve, θ the angle between the tangents at those points, and λ, μ, ν; λ', μ', ν' the direction-cosines of those tangents, we have

$$\rho = \text{limit} \frac{\delta s}{\theta},$$

when δs is indefinitely diminished.

Also if l, m, n be the direction-cosines of the principal normal, we have

$$l = \text{limit of } \frac{1}{2}(\lambda \sim \lambda') \operatorname{cosec} \frac{\theta}{2}$$

$$= \text{limit of } \frac{\delta\lambda}{\delta s} \cdot \frac{\delta s}{\theta} \cdot \frac{\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \rho \frac{d\lambda}{ds} = \rho \frac{d^2x}{ds^2}.$$

Similarly $\quad m \quad = \rho \dfrac{d\mu}{ds} = \rho \dfrac{d^2y}{ds^2}$

$$n = \rho \frac{d\nu}{ds} = \rho \frac{d^2z}{ds^2}.$$

But also by Art. 6 if x', y', z' be the co-ordinates of the centre of absolute curvature,

$$x' - x = l\rho = \rho^2 \frac{d^2x}{ds^2},$$

$$y' - y = m\rho = \rho^2 \frac{d^2y}{ds^2},$$

$$z' - z = n\rho = \rho^2 \frac{d^2z}{ds^2},$$

whence squaring and adding and dividing by ρ^4, we get

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2.$$

131. *To find the centre and radius of spherical curvature.*

The centre of spherical curvature is the point in which three consecutive normal planes intersect.

If $F(t)=0$ be the equation of any normal plane, the line of intersection of this with the consecutive plane is given by

$$F(t)=0, \text{ and } F'(t)=0 \quad\text{.................. (1).}$$

And if $F(t_1)=0$, $F'(t_1)=0$ be the equations of any other line of intersection of consecutive normal planes, at the point of intersection of these two lines,

$$\frac{F'(t_1)-F'(t)}{t_1-t}=0 \quad\text{.................. (2).}$$

And ultimately when $t=t_1$ the four equations give

$$F(t)=0,\ F'(t)=0,$$

and from (2)

$$F''(t)=0 \quad\text{.......................... (3).}$$

But

$$\left.\begin{aligned} F(t)&=(X-x)\frac{dx}{dt}+(Y-y)\frac{dy}{dt}+(Z-z)\frac{dz}{dt}=0\\ F'(t)&=(X-x)\frac{d^2x}{dt^2}+(Y-y)\frac{d^2y}{dt^2}+(Z-z)\frac{d^2z}{dt^2}-\left(\frac{ds}{dt}\right)^2=0\\ F''(t)&=(X-x)\frac{d^3x}{dt^3}+(Y-y)\frac{d^3y}{dt^3}+(Z-z)\frac{d^3z}{dt^3}-3\frac{ds}{dt}\frac{d^2s}{dt^2}=0\end{aligned}\right\}\text{...(4).}$$

These equations determine $X-x$, $Y-y$, $Z-z$, where X, Y, Z are the co-ordinates of the centre of spherical curvature. We get from them

$$X-x=\frac{\left(\frac{ds}{dt}\right)^2\left(\frac{dz}{dt}\frac{d^3y}{dt^3}-\frac{dy}{dt}\frac{d^3z}{dt^3}\right)+3\frac{ds}{dt}\frac{d^2s}{dt^2}\left(\frac{dy}{dt}\frac{d^2z}{dt^2}-\frac{dz}{dt}\frac{d^2y}{dt^2}\right)}{\begin{vmatrix}\frac{dx}{dt}, & \frac{dy}{dt}, & \frac{dz}{dt}\\ \frac{d^2x}{dt^2}, & \frac{d^2y}{dt^2}, & \frac{d^2z}{dt^2}\\ \frac{d^3x}{dt^3}, & \frac{d^3y}{dt^3}, & \frac{d^3z}{dt^3}\end{vmatrix}}\text{......(4)}$$

and similar values for $Y-y$, $Z-z$. The radius of spherical curvature R, which

$$=\sqrt{(X-x)^2+(Y-y)^2+(Z-z)^2},$$

is then known.

132. *To find the angle and radius of torsion at any point of a curve.*

The angle of torsion ($\delta\epsilon$) is the angle between two consecutive osculating planes.

Let λ, μ, ν be the direction-cosines of the normal to one osculating plane; then those of the normal to the osculating plane corresponding to the value $t+\tau$ of the independent variable will be

$$\lambda+\frac{d\lambda}{dt}.\tau+\ldots,\quad \mu+\frac{d\mu}{dt}.\tau+\ldots,\quad \nu+\frac{d\nu}{dt}.\tau+\ldots$$

And the sine of the angle between these lines is (Art. 8)

$$\tau\sqrt{\left(\mu\frac{d\nu}{dt}-\nu\frac{d\mu}{dt}\right)^2+\left(\nu\frac{d\lambda}{dt}-\lambda\frac{d\nu}{dt}\right)^2+\left(\lambda\frac{d\mu}{dt}-\mu\frac{d\lambda}{dt}\right)^2}.$$

But

$$\mu=k\left(\frac{dz}{dt}\frac{d^2x}{dt^2}-\frac{dx}{dt}\frac{d^2z}{dt^2}\right),\quad \nu=k\left(\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}\right),$$

where $$\frac{1}{k^2}=\left(\frac{dy}{dt}\frac{d^2z}{dt^2}-\frac{dz}{dt}\frac{d^2y}{dt^2}\right)^2+\left(\frac{dz}{dt}\frac{d^2x}{dt^2}-\frac{dx}{dt}\frac{d^2z}{dt^2}\right)^2$$
$$+\left(\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}\right)^2;$$

$$\therefore\ \mu\frac{d\nu}{dt}-\nu\frac{d\mu}{dt}=k^2\left(\frac{dz}{dt}\frac{d^2x}{dt^2}-\frac{dx}{dt}\frac{d^2z}{dt^2}\right)\left(\frac{dx}{dt}\frac{d^3y}{dt^3}-\frac{dy}{dt}\frac{d^3x}{dt^3}\right)$$
$$-k^2\left(\frac{dz}{dt}\frac{d^3x}{dt^3}-\frac{dx}{dt}\frac{d^3z}{dt^3}\right)\left(\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}\right),$$

$$= k^2 \frac{dx}{dt} \cdot \begin{vmatrix} \frac{dx}{dt}, & \frac{dy}{dt}, & \frac{dz}{dt} \\ \frac{d^2x}{dt^2}, & \frac{d^2y}{dt^2}, & \frac{d^2z}{dt^2} \\ \frac{d^3x}{dt^3}, & \frac{d^3y}{dt^3}, & \frac{d^3z}{dt^3} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Whence the sine of the angle of torsion becomes equal to

$$k^2\tau \frac{ds}{dt} \cdot \begin{vmatrix} \frac{dx}{dt}, & \frac{dy}{dt}, & \frac{dz}{dt} \\ \frac{d^2x}{dt^2}, & \frac{d^2y}{dt^2}, & \frac{d^2z}{dt^2} \\ \frac{d^3x}{dt^3}, & \frac{d^3y}{dt^3}, & \frac{d^3z}{dt^3} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and the radius of torsion ρ_1 which is equal to the limit of the small element of arc δs divided by the angle of torsion is given by the equation

$$\frac{1}{\rho_1} = k^2 \begin{vmatrix} \frac{dx}{dt}, & \frac{dy}{dt}, & \frac{dz}{dt} \\ \frac{d^2x}{dt^2}, & \frac{d^2y}{dt^2}, & \frac{d^2z}{dt^2} \\ \frac{d^3x}{dt^3}, & \frac{d^3y}{dt^3}, & \frac{d^3z}{dt^3} \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

133. It may be noticed that both the radius of spherical curvature and the radius of torsion become infinite if

$$\begin{vmatrix} \frac{dx}{dt}, & \frac{dy}{dt}, & \frac{dz}{dt} \\ \frac{d^2x}{dt^2}, & \frac{d^2y}{dt^2}, & \frac{d^2z}{dt^2} \\ \frac{d^3x}{dt^3}, & \frac{d^3y}{dt^3}, & \frac{d^3z}{dt^3} \end{vmatrix} = 0.$$

This is in fact the condition that four consecutive points of the curve should lie in one plane (see Art. 125).

We have also

$$\frac{1}{k^2} = \left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}\left\{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2\right\}$$
$$- \left(\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2}\right)^2$$
$$= \left(\frac{ds}{dt}\right)^2 \left\{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2 - \left(\frac{d^2s}{dt^2}\right)^2\right\}$$
$$= \frac{\left(\frac{ds}{dt}\right)^6}{\rho^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where ρ is the radius of absolute curvature.

Hence if we denote the above determinant by the symbol Δ we have

$$\frac{1}{\rho_1} = \frac{\rho^2 \Delta}{\left(\frac{ds}{dt}\right)^6} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

EXAMPLES. CHAPTER IX.

1. Find the equations of the osculating plane at any point of the curves

(1) $x = a\cos\theta, \quad y = a\sin\theta, \quad z = c\theta.$

(2) $x^2 + y^2 - ry = 0, \quad z^2 + ry = r^2.$

Find also the length of the arc of (1) between two points whose co-ordinates are given.

2. The equations of a curve of double curvature being given, find the equation of the surface formed by making it revolve round the axis of z.

Ex. $x = a\cos\theta, \quad y = a\sin\theta, \quad z = c\theta.$

3. A helix joins two points, the distance between which is b, the angle between the tangent lines and the axis of the generating cylinder being a given angle α; prove that if the length of the helix is a maximum, the helix has a constant length, and that the radius of the generating circle is $\dfrac{b \tan \alpha}{2n\pi}$, where n is a positive integer.

4. A curve is traced on a right circular cylinder of radius a, such that if the cylinder were unrolled into a plane the curve would become a catenary whose axis formed one of the generating lines, and directrix the base, of the cylinder. Shew that

$$\rho = \frac{az^2}{c\sqrt{c^2+a^2}}, \quad \rho_1 = \frac{az^2}{cs},$$

ρ, ρ_1 being the radii of absolute curvature and torsion, z the ordinate, s the arc measured from the vertex, and c the constant of the catenary.

5. Find the equation of the osculating plane at any point of the curve given by the equations

$$x + y + z = 1,$$
$$ax^2 + by^2 + cz^2 = 1.$$

6. Find the equations of a curve traced on a sphere so as to cut all the great circles passing through a fixed point at the same angle.

7. Find the equations of the lines of greatest slope to the plane of xy on the surfaces

(1) $xyz = a^3$,

(2) $z = x + \dfrac{a}{2} \log \dfrac{x^2+y^2}{a^2}$.

8. Shew geometrically and analytically that if a sphere be described concentric with a given ellipsoid, the tangent line to the curve of intersection of the sphere and ellipsoid is parallel to one of the principal axes of the central section of the ellipsoid which passes through that tangent line.

9. Find the equations of a curve traced on a sphere, such that the sum of the arcs of great circles joining any point on it with two fixed points on the sphere, the arc joining which is a quadrant, is constant.

10. Find the equations of a curve traced on a sphere by a point which moves with constant velocity along the arc of a great circle while the great circle revolves with constant velocity round a fixed diameter.

11. A point moves on an ellipsoid so that its direction of motion always passes through the perpendicular from the origin on the tangent plane to the ellipsoid at that point. Shew that the curve traced out by the point is given by the intersection of the ellipsoid with the surface

$$x^{m-n}\, y^{n-l}\, z^{l-m} = \text{constant},$$

l, m, n being inversely proportional to the squares of the axes of the ellipsoid.

12. Find the equation of a curve traced on a right cone which cuts all the generating lines at a constant angle.

Find the length of the curve measured from the vertex.

13. A straight line is drawn on a plane which is then wrapped on a cone. Shew that if ρ be the radius of absolute curvature of the curve on the cone at a distance r from the vertex

$$r^3 = a^2 \rho,$$

where a is a constant.

14. Find the values of the radii of absolute and spherical curvature at any point of a helix.

15. Find the locus of the centres of spherical curvature of a helix.

16. If, at any point of a curve, equal lengths δs be measured along the curve and its circle of curvature, the dis-

tance between the extremities of these lengths is ultimately equal to

$$\frac{\delta s^3}{6\rho}\sqrt{\frac{1}{\sigma^2}+\frac{1}{\rho^2}\left(\frac{d\rho}{ds}\right)^2},$$

ρ being the radius of curvature and σ the radius of torsion at the point.

17. Shew that the normal plane at any point to the locus of the centres of circular curvature of any curve bisects the radius of spherical curvature at the corresponding point of the original curve.

18. If a curve be drawn on a right circular cone cutting all the generating lines at a constant angle β, shew that the radius of absolute curvature at any point is to the corresponding radius of curvature when the cone is developed in the ratio of $\sin\alpha$ to $\sqrt{\sin^2\alpha\cos^2\beta+\sin^2\beta}$.

CHAPTER X.

ON ENVELOPES.

134. Let the equation of any surface be

$$F(x, y, z, a) = 0 \qquad (1),$$

where a is a constant. If a be changed to a' we obtain the equation of another surface

$$F(x, y, z, a') = 0 \qquad (2),$$

differing from (1) in magnitude or position or both, but of the same general nature.

These two equations will both be satisfied by the coordinates of all points in the curve of intersection of the two surfaces, and if we suppose the value of a' to approach indefinitely near to that of a, this curve of intersection approaches some limiting position. The locus of all such limiting positions for different values of a is a surface which is called the *envelope* of the surface (1). Its equation can be found in the following manner.

At all points for which (1) and (2) are satisfied, we have

$$F(x, y, z, a) = 0,$$

$$\frac{F(x, y, z, a') - F(x, y, z, a)}{a' - a} = 0.$$

But ultimately when a' becomes equal to a these equations reduce to

$$F(x, y, z, a) = 0 \qquad (3),$$

$$\frac{d}{da} F(x, y, z, a) = 0 \qquad (4),$$

which are therefore the equations of the ultimate position of the curve of intersection of (1) and (2). Eliminating a between (3) and (4) we obtain the equation of the locus of such curves, or the envelope of the surface (1).

135. The curve given by the two equations (3) and (4) of the last article is called *the characteristic* of the envelope.

If we take the equations of two consecutive characteristics and treat them as in Art. 131 we get, to determine their point of intersection, the three equations

$$F(x, y, z, a) = 0,$$
$$F'(x, y, z, a) = 0,$$
$$F''(x, y, z, a) = 0.$$

If between these three equations we eliminate a we shall get two relations between x, y, z which are the equations of the locus of ultimate intersections of two consecutive characteristics. The curve so obtained is called the *edge of regression* of the envelope, or sometimes simply the edge of the envelope.

Thus the line given by equations (4) and (5) of Art. (129) is the characteristic of the envelope of the normal planes to the curve, while the locus of the centres of spherical curvature is the edge of regression of the same envelope.

136. We will now shew that the *envelope* obtained in Article 134 touches each of the series of intersecting surfaces.

For suppose from equation (4) of that article we obtain a value of a,

$$a = \phi(x, y, z).$$

Substituting in (3), the equation of the envelope becomes

$$F\{x, y, z, \phi(x, y, z)\} = 0 \qquad\text{..................(1).}$$

The values of $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at any point of this surface are given by the equations

$$\left.\begin{aligned}\frac{dF}{dx} + \frac{dF}{dz}\frac{dz}{dx} + \frac{dF}{d\phi}\left(\frac{d\phi}{dx} + \frac{d\phi}{dz}\frac{dz}{dx}\right) = 0\\ \frac{dF}{dy} + \frac{dF}{dz}\frac{dz}{dy} + \frac{dF}{d\phi}\left(\frac{d\phi}{dy} + \frac{d\phi}{dz}\frac{dz}{dy}\right) = 0\end{aligned}\right\} \qquad\text{..........(2).}$$

At any point of the surface $F(x, y, z, a) = 0$ the values of $\frac{dz}{dx}$ and $\frac{dz}{dy}$ are given by the equations

$$\left.\begin{aligned} \frac{dF}{dx} + \frac{dF}{dz}\frac{dz}{dx} = 0 \\ \frac{dF}{dy} + \frac{dF}{dz}\frac{dz}{dy} = 0 \end{aligned}\right\} \quad \text{.....................(3)}.$$

But at the points where the envelope meets the surface

$$F(x, y, z, a) = 0,$$

we have

$$a = \phi(x, y, z) \text{ and } \frac{dF}{da} = 0.$$

Now $\frac{dF}{d\phi}$ only differs from $\frac{dF}{da}$ in having $\phi(x, y, z)$ instead of a, consequently at all the points of intersection of the surface

$$F(x, y, z, a) = 0$$

with the envelope, $\frac{dF}{d\phi} = 0$, and the equations (2) become identical with equations (3). The values of $\frac{dz}{dx}$ and $\frac{dz}{dy}$ being the same for the surface and its envelope, the two surfaces touch.

137. If the equation of a surface be

$$F(x, y, z, a, b) = 0 \text{.....................(1)},$$

when a and b are constants, any two other surfaces formed by giving new values to a and b will intersect (1) in a point or points, which assume a limiting position when the new values of a and b approach indefinitely near to their first values. The locus of such limiting positions is called the envelope of the surface (1).

Let a and b become $a + h$, $b + k$ respectively. The equation of the corresponding surface is

$$F(x, y, z, a + h, b + k) = 0,$$

$$\text{or } F(x, y, z, a, b) + hF'(a + \theta h) + kF'(b + \theta k) = 0 \text{...(2)},$$

where θ is a proper fraction and $F'(a+\theta h)$ means that $F(x, y, z, a, b)$ has been differentiated with respect to a, and $a+\theta h$ put in the result for a.

At the points of intersection of (1) and (2) we have

$$\left.\begin{aligned} F(x, y, z, a, b) = 0 \\ hF'(a+\theta h) + kF'(b+\theta k) = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (3),$$

and whatever be the ratio of h to k, when h and k are diminished indefinitely all the curves of intersection given by (3) pass through the points given by

$$F(x, y, z, a, b) = 0, \quad F'(a) = 0, \quad F'(b) = 0.$$

By eliminating a and b between these equations we obtain the equation of the envelope.

138. The envelope in this case also touches each of the series of intersecting surfaces. For let the equation of one of the surfaces be

$$F(x, y, z, a, b) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The corresponding point on the envelope is given by (1) combined with

$$\frac{dF}{da} = 0, \quad \frac{dF}{db} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (2) we can obtain by solving for a and b

$$a = \phi_1(x, y, z), \quad b = \phi_2(x, y, z);$$

and the equation of the envelope becomes

$$F\{x, y, z, \phi_1(x, y, z), \phi_2(x, y, z)\} = 0.$$

The values of $\frac{dz}{dx}$ and $\frac{dz}{dy}$ for any point of the envelope are given by the equations

$$\frac{dF}{dx} + \frac{dF}{dz}\frac{dz}{dx} + \frac{dF}{d\phi_1}\left(\frac{d\phi_1}{dx} + \frac{d\phi_1}{dz}\frac{dz}{dx}\right) + \frac{dF}{d\phi_2}\left(\frac{d\phi_2}{dx} + \frac{d\phi_2}{dz}\frac{dz}{dx}\right) = 0,$$

$$\frac{dF}{dy} + \frac{dF}{dz}\frac{dz}{dy} + \frac{dF}{d\phi_1}\left(\frac{d\phi_1}{dy} + \frac{d\phi_1}{dz}\frac{dz}{dy}\right) + \frac{dF}{d\phi_2}\left(\frac{d\phi_2}{dy} + \frac{d\phi_2}{dz}\frac{dz}{dy}\right) = 0.$$

But at the points where (1) meets the envelope

$$a = \phi_1(x, y, z), \quad \frac{dF}{da} = 0,$$

$$b = \phi_2(x, y, z), \quad \frac{dF}{db} = 0;$$

consequently at those points $\frac{dF}{d\phi_1} = 0$, $\frac{dF}{d\phi_2} = 0$, and the above equations become

$$\frac{dF}{dx} + \frac{dF}{dz}\frac{dz}{dx} = 0,$$

$$\frac{dF}{dy} + \frac{dF}{dz}\frac{dz}{dy} = 0,$$

which are the same as the equations which give $\frac{dz}{dx}$ and $\frac{dz}{dy}$ for the point (x, y, z) of the surface (1). Hence at the points where (1) meets the envelope the values of $\frac{dz}{dx}$ and $\frac{dz}{dy}$ are the same for the surface and the envelope, which therefore touch one another at those points.

139. If the equation of a family of surfaces contains n arbitrary constants connected by $(n-1)$ equations there is really *one* independent constant, and the envelope can be found by substituting for $(n-1)$ of the constants their values in terms of the n^{th}. It is better in general to consider $(n-1)$ of the constants to be functions of the n^{th}, and differentiating all the equations to eliminate by undetermined multipliers.

If the n constants be connected by $(n-2)$ equations, two of the constants are arbitrary, and the envelope falls under the second class. The method of undetermined multipliers can be used in this case also.

For examples of the solution of problems the reader is referred to Todhunter's *Differential Calculus*, Chapter XXV., the methods employed there being equally applicable to the problems of Solid Geometry.

140. The polar plane of any point (α, β, γ) with respect to any quadric can be obtained as in Art. 106. If the point (α, β, γ) be constrained to lie on any given surface

$$f(x, y, z) = 0 \quad\text{.......................(1)},$$

the equation of its polar plane will contain three parameters α, β, γ connected by one relation

$$f(\alpha, \beta, \gamma) = 0.$$

The equation of its envelope can therefore be found by the methods of Art. 137.

Suppose this equation to be

$$\phi(x, y, z) = 0 \quad\text{.....................(2)}.$$

Then any point $(\alpha', \beta', \gamma')$ in (2) is the limiting position of the point of intersection of the polar plane of some point (α, β, γ) on (1) with the polar planes of points on (1) adjacent to (α, β, γ). Hence by Art. 106 the polar plane of $(\alpha', \beta', \gamma')$ with respect to the given quadric must pass through the point (α, β, γ) and other points on (1) contiguous to (α, β, γ), that is the polar plane of $(\alpha', \beta', \gamma')$ is a tangent plane to (1) at (α, β, γ). Thus the surface (1) is the envelope of the polar planes of all points on (2) with respect to the same quadric. The two surfaces are from this property called *reciprocal polars*.

Each surface may be also defined as the locus of the poles of the tangent planes to the other with respect to the given quadric.

141. Let the quadric with respect to which the polars are taken be the sphere,

$$x^2 + y^2 + z^2 = k^2 \quad\text{.....................(1)}.$$

The equation of the polar plane of any point (α, β, γ) with respect to this sphere is

$$\alpha x + \beta y + \gamma z = k^2 \quad\text{.....................(2)}.$$

Let the surface to be reciprocated be the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad\text{.................(3)}.$$

Hence we have to get the envelope of the plane (2), α, β, γ being parameters connected by the relation

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=1 \quad\text{.................. (4).}$$

Using the method of undetermined multipliers we get to determine the envelope,

$$\frac{\alpha}{a^2}+\lambda x=0,\quad \frac{\beta}{b^2}+\lambda y=0,\quad \frac{\gamma}{c^2}+\lambda z=0;$$

whence $1+\lambda k^2=0$;

$$\therefore\ \alpha=\frac{a^2x}{k^2},\quad \beta=\frac{b^2y}{k^2},\quad \gamma=\frac{c^2z}{k^2},$$

and substituting in (2) the envelope becomes

$$a^2x^2+b^2y^2+c^2z^2=k^4 \quad\text{.................. (5).}$$

The surface represented by (5) is often called the reciprocal ellipsoid of that represented by (1).

EXAMPLES. CHAPTER X.

1. FIND the envelope of the series of planes

$$\alpha x+\beta y+\gamma z=1,$$

where α, β, γ are parameters connected by the relations

$$\alpha^2+\beta^2+\gamma^2=1,$$
$$l\alpha+m\beta+n\gamma=0.$$

2. Find the envelope of a sphere of constant radius which moves with its centre on a fixed circle.

3. Find the envelope of central sections of an ellipsoid of which one axis is constant and equal to k.

4. Find the envelope of planes which are the polars of points on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

with respect to the ellipsoid

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1.$$

5. Find the envelope of a sphere of constant radius which moves with its centre in a fixed plane.

6. Find the envelope of an ellipsoid whose axes are given in direction and the product of whose axes is constant and equal to $8k^3$.

7. Find the envelope of the series of planes

$$lx+my+nz=v,$$

where l, m, n, v are parameters connected by the relations

$$l^2+m^2+n^2=1,$$

$$\frac{l^2}{v^2-a^2}+\frac{m^2}{v^2-b^2}+\frac{n^2}{v^2-c^2}=0.$$

8. Find the envelope of a sphere whose centre is at a point $(\alpha, \beta, 0)$, and radius is γ where α, β, γ are connected by the relation

$$\alpha^2+\beta^2+\gamma^2=k^2,$$

k being a constant.

9. Find the envelope of the surface

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1,$$

where α, β, γ are parameters connected by the relations

$$\frac{\alpha^2}{a^2}=\frac{\beta^2}{b^2}=\frac{c^2}{\gamma^2};$$

a, b, c being constants.

10. Find the envelope of all planes which cut off a constant volume from the co-ordinate axes.

11. Find the envelope of a series of planes which move so that the perpendicular on them from the origin is constant in length.

12. Find the envelope of a series of planes which move so that the area of the section of an ellipsoid made by any one is in a constant ratio to the area of the parallel section through the centre of the ellipsoid.

13. Find the envelope of a sphere of constant radius which moves with its centre on a fixed sphere.

14. Find the envelope of the plane

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = 1,$$

when α, β, γ are connected by the relations

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = 1,$$

$$l\alpha + m\beta + n\gamma = 1.$$

15. Through a given point (α, β, γ) a series of chords are drawn to an ellipsoid whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

in such directions that the line of intersection of the tangent planes at the extremities of each chord is perpendicular to that chord. Prove that the envelope of the lines of intersection of the tangent planes is a parabola which is the intersection of the polar plane of (α, β, γ) with the cone whose equation is

$$\frac{\sqrt{(b^2 - c^2)\,\alpha x}}{a} + \frac{\sqrt{(c^2 - a^2)\,\beta y}}{b} + \frac{\sqrt{(a^2 - b^2)\,\gamma z}}{c} = 0.$$

CHAPTER XI.

ON FUNCTIONAL AND DIFFERENTIAL EQUATIONS OF FAMILIES OF SURFACES.

142. *To find the general equation of conical surfaces.*

A conical surface is generated by a straight line which always passes through a fixed point and meets a fixed curve.

Let (α, β, γ) be the fixed point, and let the equations of any generating line be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{.................(1).}$$

Let the equations of the curve through which (1) always passes be

$$y=\phi(x), \quad z=\psi(x) \quad \text{.................(2).}$$

Since (1) always meets (2) we have

$$\beta+\frac{m}{l}(x-\alpha)=\phi(x),$$

$$\gamma+\frac{n}{l}(x-\alpha)=\psi(x).$$

And eliminating x between these equations, we shall get a relation between $\frac{n}{l}$ and $\frac{m}{l}$, which can be put into the form

$$\frac{n}{l}=F\left(\frac{m}{l}\right),$$

whence the equation of the cone becomes

$$\frac{z-\gamma}{x-\alpha}=F\left(\frac{y-\beta}{x-\alpha}\right) \qquad\qquad (3).$$

This is the functional equation of conical surfaces. In all cases it is clear that the equation is homogeneous in $x-\alpha$, $y-\beta$, $z-\gamma$; in fact the result we have obtained is the analytical statement of the fact that the equation of any conical surface whose vertex is at a point (α, β, γ) is homogeneous in $x-\alpha$, $y-\beta$, $z-\gamma$; an extension of the result of Art. 34.

A differential equation holding for all such surfaces can be deduced thus.

From (3) differentiating with respect to x,

$$\frac{dz}{dx}=F\left(\frac{y-\beta}{x-\alpha}\right)-\left(\frac{y-\beta}{x-\alpha}\right)F'\left(\frac{y-\beta}{x-\alpha}\right),$$

and with respect to y,

$$\frac{dz}{dy}=F'\left(\frac{y-\beta}{x-\alpha}\right);$$

$$\therefore\ \frac{dz}{dx}=\frac{z-\gamma}{x-\alpha}-\frac{y-\beta}{x-\alpha}\cdot\frac{dz}{dy},$$

$$\text{or}\ (x-\alpha)\frac{dz}{dx}+(y-\beta)\frac{dz}{dy}=z-\gamma \qquad (4).$$

143. *To find the general equation of cylindrical surfaces.*

A cylindrical surface is generated by a straight line which moves always parallel to itself and meets a fixed curve.

Let l, m, n be the direction-cosines of any one of the generating lines, and

$$\frac{X-x}{l}=\frac{Y-y}{m}=\frac{Z-z}{n}=r \qquad (1)$$

the equations of the line. Let the equations of the directing curve be

$$Y=\phi(X),\quad Z=\psi(X) \qquad (2).$$

Since (1) meets (2), we have

$$y+mr=\phi(x+lr),\quad z+nr=\psi(x+lr),$$

and by eliminating r between these two equations we get a relation between x, y, z, the co-ordinates of any point in any one of the generating lines, which is therefore the equation of the surface.

The general form of the result is obtained thus.

From (1) $$mX - lY = mx - ly,$$
$$nY - mZ = ny - mz.$$

But from (2) $mX - lY$ and $nY - mZ$ can ordinarily both be expressed as functions of X, and we can therefore deduce a relation of the form

$$mX - lY = F(nY - mZ);$$
$$\therefore\ mx - ly = F(ny - mz) \quad \text{.............. (3)},$$

which is the general functional equation of cylindrical surfaces.

The differential equation can be deduced. For from (3), differentiating with respect to x,

$$m = -mF'(ny - mz)\frac{dz}{dx},$$

and differentiating with respect to y

$$-l = \left(n - m\frac{dz}{dy}\right)F'(ny - mz),$$

whence $$l\frac{dz}{dx} = n - m\frac{dz}{dy},$$

or $$l\frac{dz}{dx} + m\frac{dz}{dy} = n. \quad \text{.................... (4)}.$$

If the direction of the generating line of the cylindrical surface be parallel to the axis of y we have $l = 0$, $m = 1$, $n = 0$, and equation (3) becomes

$$x = F(-z) \quad \text{or} \quad f(x, z) = 0 \quad \text{............. (5)}.$$

Any equation of this form represents therefore a cylindrical surface whose base is the curve of which (5) is the equation regarded as an equation restricted to the plane of zx.

Similarly the equations

$$f(x, y) = 0,$$
$$f(y, z) = 0,$$

represent cylindrical surfaces whose generating lines are parallel to the axes of z and x.

These results are obvious also from general considerations.

144. *To find the general equation of conoidal surfaces.*

A conoidal surface is a surface generated by a straight line which always meets a fixed straight line, is parallel to a fixed plane, and meets a fixed curve.

Let the equations of the fixed line be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \;\ldots\ldots\ldots\ldots\ldots(1),$$

and let the equation of the fixed plane be

$$l'x + m'y + n'z = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The co-ordinates of any point in (1) can be represented by $\alpha + lr$, $\beta + mr$, $\gamma + nr$, and the equations of any straight line through this point are

$$\frac{x-\alpha-lr}{\lambda} = \frac{y-\beta-mr}{\mu} = \frac{z-\gamma-nr}{\nu} \;\ldots\ldots\ldots(3).$$

If this be parallel to (2), we have

$$\lambda l' + \mu m' + \nu n' = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(4).$$

From (3) and (4)

$$l'(x-\alpha) + m'(y-\beta) + n'(z-\gamma) = (ll' + mm' + nn')\,r \ldots(5),$$

and from (3) eliminating r

$$\frac{n\lambda - l\nu}{n\mu - m\nu} = \frac{n(x-\alpha) - l(z-\gamma)}{n(y-\beta) - m(z-\gamma)} \;\ldots\ldots\ldots\ldots(6).$$

Now the condition that the straight line (3) may meet the fixed curve, combined with (4), will ordinarily enable us to express $\frac{\lambda}{\nu}$ and $\frac{\mu}{\nu}$ as functions of r, and consequently we can arrive at a result of the form

$$\frac{n\lambda - l\nu}{n\mu - m\nu} = F\{(ll' + mm' + nn')\,r\},$$

or

$$\frac{n(x-\alpha) - l(z-\gamma)}{n(y-\beta) - m(z-\gamma)} = F\{l'(x-\alpha) + m'(y-\beta) + n'(z-\gamma)\} \ldots (7),$$

which is the general functional equation of conoidal surfaces.

If the fixed plane be taken as the plane of xy, and the point where the fixed line meets it as the origin, we have

$$l' = 0, \quad m' = 0, \quad n' = 1, \quad \alpha = 0, \quad \beta = 0, \quad \gamma = 0,$$

and the equation (7) becomes

$$\frac{nx - lz}{ny - mz} = F(z) \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

If the fixed line be perpendicular to the fixed plane

$$l = 0, \quad m = 0, \quad n = 1,$$

and the equation of the surface becomes

$$\frac{x}{y} = F(z),$$

or

$$z = \psi\left(\frac{x}{y}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

In this case the surface is called a *right conoid.*

145. The differential equation of conoidal surfaces can be deduced from (7); for differentiating it with respect to x, we have

$$\frac{\left(n - l\frac{dz}{dx}\right)\{n(y-\beta) - m(z-\gamma)\} + m\frac{dz}{dx}\{n(x-\alpha) - l(z-\gamma)\}}{\{n(y-\beta) - m(z-\gamma)\}^2}$$

$$= \left(l' + n'\frac{dz}{dx}\right) F'\{l'(x-\alpha) + m'(y-\beta) + n'(z-\gamma)\};$$

and differentiating with respect to y, we have

$$-\frac{l\frac{dz}{dy}\{n(y-\beta)-m(z-\gamma)\}+\left(n-m\frac{dz}{dy}\right)\{n(x-\alpha)-l(z-\gamma)\}}{\{n(y-\beta)-m(z-\gamma)\}^2}$$

$$=\left(m'+n'\frac{dz}{dy}\right)F'\{l'(x-\alpha)+m'(y-\beta)+n'(z-\gamma)\},$$

and reducing and eliminating

$$F'\{l'(x-\alpha)+m'(y-\beta)+n'(z-\gamma)\}$$

we obtain

$$\left(m'+n'\frac{dz}{dy}\right)\left[n(y-\beta)-m(z-\gamma)+\frac{dz}{dx}\{m(x-\alpha)-l(y-\beta)\}\right]$$

$$+\left(l'+n'\frac{dz}{dx}\right)\left[n(x-\alpha)-l(z-\gamma)+\frac{dz}{dy}\{l(y-\beta)-m(x-\alpha)\}\right]=0,$$

or $m'\{n(y-\beta)-m(z-\gamma)\}+l'\{n(x-\alpha)-l(z-\gamma)\}$

$$+\frac{dz}{dx}\left[m'\{m(x-\alpha)-l(y-\beta)\}+n'\{n(x-\alpha)-l(z-\gamma)\}\right]$$

$$+\frac{dz}{dy}\left[n'\{n(y-\beta)-m(z-\gamma)\}+l'\{l(y-\beta)-m(x-\alpha)\}\right]=0\ldots(10).$$

The differential equation corresponding to equation (8) is obtained by putting

$$\alpha=0,\quad \beta=0,\quad \gamma=0,\quad l'=0,\quad m'=0,\quad n'=1,$$

and is therefore

$$(nx-lz)\frac{dz}{dx}+(ny-mz)\frac{dz}{dy}=0\ldots\ldots\ldots\ldots(11).$$

The differential equation of a right conoid is obtained from (11) by putting

$$l=0,\quad m=0,\quad n=1,$$

and is therefore

$$x\frac{dz}{dx}+y\frac{dz}{dy}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

The forms (11) and (12) can of course be obtained directly from (8) and (9) by differentiation.

For instance from (9), differentiating with respect to x, we have

$$\frac{dz}{dx} = \frac{1}{y}\psi'\left(\frac{x}{y}\right),$$

and differentiating with respect to y,

$$\frac{dz}{dy} = -\frac{x}{y^2}\psi'\left(\frac{x}{y}\right);$$

whence eliminating $\psi'\left(\frac{x}{y}\right)$, we have

$$x\frac{dz}{dx} + y\frac{dz}{dy} = 0.$$

146. The three classes of surfaces we have considered are all included in the general class of *ruled surfaces*, that is, surfaces which can be generated by the motion of a straight line. The first and second differ from the third in this, that any two consecutive generating lines in any surface of the first or second classes lie in one plane, whereas this is not in general the case with the third class. Ruled surfaces in which consecutive generating lines lie in one plane are called *developable surfaces*, while all other ruled surfaces are called *skew surfaces*. Thus the surface generated by the ultimate intersections of the normal planes to a given curve is developable.

Developable surfaces are so named for the following reason. Let a series of consecutive generating lines be drawn. The plane which passes through the first and second line intersects the plane which passes through the second and third line in the second line. The first plane may be turned round the second line till it coincides with the second plane, and thus three generating lines of the surface can be made to lie in one plane. Again, this plane can be turned round the third line till it coincides with the plane which passes through the third and fourth lines, and so four consecutive lines can be made to lie in one plane. In this manner the whole surface can be *developed* so as to lie in one plane without tearing.

Since any two consecutive generating lines of a developable surface lie in one plane, any such surface may be produced by the ultimate intersections of a series of planes, and since any two consecutive planes intersect in a *line* on the surface, the equation representing any one of the series can only involve *one* arbitrary constant (Art. 134).

147. Let the equation of one of the planes be

$$Ax + By + Cz - D = 0 \quad\text{..................(1)}.$$

Then since the equation only involves one arbitrary constant, A, B, C, D must be functions of one constant which we may call α. Thus equation (1) may be written

$$x\phi_1(\alpha) + y\phi_2(\alpha) + z\phi_3(\alpha) - \phi_4(\alpha) = 0 \quad\text{.........(2)},$$

and the envelope is found by eliminating α between (2) and the equation obtained by differentiating it with respect to α, viz.

$$x\phi_1'(\alpha) + y\phi_2'(\alpha) + z\phi_3'(\alpha) - \phi_4'(\alpha) = 0 \quad\text{.........(3)}.$$

To obtain the general differential equation of developable surfaces we must differentiate (2), considering α as a function of x, y, z determined from (3).

Differentiating with respect to x, we get

$$\phi_1(\alpha) + \frac{dz}{dx}\phi_3(\alpha)$$
$$+ \{x\phi_1'(\alpha) + y\phi_2'(\alpha) + z\phi_3'(\alpha) - \phi_4'(\alpha)\}\left\{\frac{d\alpha}{dx} + \frac{d\alpha}{dz}\frac{dz}{dx}\right\} = 0,$$

or by (3),
$$\phi_1(\alpha) + \frac{dz}{dx}\phi_3(\alpha) = 0 \quad\text{....................(4)}.$$

Similarly, differentiating with respect to y, we get

$$\phi_2(\alpha) + \frac{dz}{dy}\phi_3(\alpha) = 0 \quad\text{.................(5)}.$$

Eliminating α between (4) and (5), we get

$$\frac{dz}{dx} = f\left(\frac{dz}{dy}\right) \text{.........................} (6),$$

and differentiating again with respect to x and y in turn, we get

$$\frac{d^2z}{dx^2} = f'\left(\frac{dz}{dy}\right) \cdot \frac{d^2z}{dx\,dy},$$

$$\frac{d^2z}{dx\,dy} = f'\left(\frac{dz}{dy}\right) \cdot \frac{d^2z}{dy^2}.$$

And eliminating $f'\left(\frac{dz}{dy}\right)$, we get

$$\frac{d^2z}{dx^2} \cdot \frac{d^2z}{dy^2} - \left(\frac{d^2z}{dx\,dy}\right)^2 = 0,$$

which is the differential equation of developable surfaces.

148. *To find the general equation of surfaces of revolution.*

A surface of revolution is the surface produced by the revolution of a plane curve round a fixed straight line in its plane called the axis of revolution.

Let the equations of the axis of revolution be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \text{....................} (1).$$

And let $y = f(x)$ be the equation of the revolving curve when the axis of revolution is taken as the axis of x, and the point (α, β, γ) as origin. Let P be any point on the surface, PR perpendicular on the line (1), and Q the point (α, β, γ). Then from the definition of a surface of revolution,

$$PR = f(RQ) \text{.......................} (2).$$

But $\quad RQ = l(x-\alpha) + m(y-\beta) + n(z-\gamma),$

since it is the perpendicular from Q on a plane through P perpendicular to (1), and

$$PR^2 = (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 - \{l(x-\alpha) + m(y-\beta) + n(z-\gamma)\}^2.$$

Hence

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=\{l(x-\alpha)+m(y-\beta)+n(z-\gamma)\}^2$$
$$+[f\{l(x-\alpha)+m(y-\beta)+n(z-\gamma)\}]^2,$$

or $(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2$

$$=\phi\{l(x-\alpha)+m(y-\beta)+n(z-\gamma)\} \ldots\ldots (3),$$

which is the functional equation of surfaces of revolution.

The differential equation can be thus deduced.

Differentiating (3) with respect to x we get

$$2\left\{(x-\alpha)+(z-\gamma)\frac{dz}{dx}\right\}$$
$$=\left\{l+n\frac{dz}{dx}\right\}\phi'\{l(x-\alpha)+m(y-\beta)+n(z-\gamma)\},$$

and differentiating with respect to y

$$2\left\{(y-\beta)+(z-\gamma)\frac{dz}{dy}\right\}$$
$$=\left\{m+n\frac{dz}{dy}\right\}\phi'\{l(x-\alpha)+m(y-\beta)+n(z-\gamma)\}.$$

Eliminating ϕ' and reducing, we get

$$m(x-\alpha)-l(y-\beta)+\{m(z-\gamma)-n(y-\beta)\}\frac{dz}{dx}$$
$$+\{n(x-\alpha)-l(z-\gamma)\}\frac{dz}{dy}=0 \ldots\ldots\ldots\ldots (4),$$

which is the differential equation required.

149. The conditions that the general equation of the second degree should represent a surface of revolution, can be obtained either from the functional or differential equation of the last Article. We will obtain them from the functional equation.

Let the equation be

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy + 2A''x + 2B''y + 2C''z + F = 0 \ldots\ldots (1).$$

If this equation represents a surface of revolution it can be put into the form

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = P(lx+my+nz)^2 + Q(lx+my+nz) + R \ldots\ldots\ldots (2),$$

where P, Q, R are constants. This is evident from the considerations that the right-hand member must be some function of

$$l(x-\alpha) + m(y-\beta) + n(z-\gamma),$$

or of

$$lx + my + nz - (l\alpha + m\beta + n\gamma),$$

and that it cannot contain x, y, z to a higher degree than the second. Making the equations (1) and (2) identical, we obtain from the terms of the second degree

$$Pl^2 - 1 = kA \ldots\ldots (3), \qquad Pmn = kA' \ldots\ldots (6),$$
$$Pm^2 - 1 = kB \ldots\ldots (4), \qquad Pnl = kB' \ldots\ldots (7),$$
$$Pn^2 - 1 = kC \ldots\ldots (5), \qquad Plm = kC' \ldots\ldots (8),$$

where k is some constant.

Multiplying (7) by (8) and dividing the product by (6), we obtain

$$Pl^2 = k\frac{B'C'}{A'} = kA + 1;$$

$$\therefore \frac{B'C'}{A'} - A = \frac{1}{k} = \frac{C'A'}{B'} - B = \frac{A'B'}{C'} - C \ldots\ldots\ldots (9).$$

These are the conditions which must be satisfied by the coefficients of the equation.

The relations which must subsist between α, β, γ are obtained by equating the coefficients of the terms of the first degree in (1) and (2). We thus obtain

$$Ql + 2\alpha = 2kA'',$$
$$Qm + 2\beta = 2kB'',$$
$$Qn + 2\gamma = 2kC''.$$

Whence $$\frac{kA''-\alpha}{l}=\frac{kB''-\beta}{m}=\frac{kC''-\gamma}{n} \quad\ldots\ldots\ldots\ldots (10).$$

But $\dfrac{l^2}{\dfrac{B'C'}{A'}}=\dfrac{m^2}{\dfrac{C'A'}{B'}}=\dfrac{n^2}{\dfrac{A'B'}{C'}}$, and k is given by (9).

The three equations (10) being the relations which α, β, γ must satisfy are the equations of the axis of revolution.

150. The preceding investigation fails if the quantities

$$\frac{B'C'}{A'}-A,\ \frac{C'A'}{B'}-B,\ \frac{A'B'}{C'}-C \text{ vanish,}$$

for then k is required to be infinite.

We know that the equation (1) in this case represents a parabolic cylinder, or two parallel planes (Art. 91), consequently the surface cannot be a surface of revolution.

The investigation also fails if A', B', or C' vanish. Suppose $A'=0$. From equation (6), $mn=0$; $\therefore m=0$ or $n=0$, and therefore, B' or C' must vanish also. Suppose $n=0$, and therefore $B'=0$, we get then

$$Plm=kC',$$
$$kC=-1,$$
$$\text{and } (1+kA)(1+kB)=P^2l^2m^2=k^2C'^2;$$
$$\therefore (C-A)(C-B)=C'^2,$$

which with $B'=0$ is the condition required. The other exceptional cases can be treated in the same way.

151. The differential equations of the different classes of surfaces can be put into a more symmetrical form by the substitutions

$$\frac{dz}{dx}=-\frac{\dfrac{dF}{dx}}{\dfrac{dF}{dz}},\quad \frac{dz}{dy}=-\frac{\dfrac{dF}{dy}}{\dfrac{dF}{dz}},$$

and corresponding substitutions for the second differential coefficients of z, the equation of the surface being assumed to be

$$F(x, y, z) = 0.$$

Thus the differential equation of cylindrical surfaces becomes

$$l \frac{dF}{dx} + m \frac{dF}{dy} + n \frac{dF}{dz} = 0 \quad \ldots\ldots\ldots\ldots (1).$$

The equations can be more conveniently used in this form to discover whether a surface whose equation is given belongs to the peculiar class considered.

For instance, if the surface be cylindrical, there must be some values of l, m, n which shall make the expression

$$l \frac{dF}{dx} + m \frac{dF}{dy} + n \frac{dF}{dz} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

vanish identically for all values of x, y, z corresponding to any point on the surface.

The conditions that this may be possible will be that the coefficients of the several powers and products of x, y, z in (2) must vanish for the same values of l, m, n.

The differential equations can be found independently of the functional. For instance, equation (1) is the algebraical statement of the fact that at all points of the surface

$$F(x, y, z) = 0,$$

a straight line whose direction-cosines are l, m, n is a tangent line to the surface, a condition obviously satisfied by cylindrical surfaces only.

In the case of conical surfaces we at once obtain the differential equation

$$(x - \alpha) \frac{dF}{dx} + (y - \beta) \frac{dF}{dy} + (z - \gamma) \frac{dF}{dz} = 0,$$

from the consideration that the straight line joining any point (x, y, z) with the vertex is a tangent line to the surface at the point (x, y, z).

EXAMPLES. CHAPTER XI.

1. Shew how to find the functional and differential equations of a tubular surface, that is, a surface which is the envelope of a sphere of constant radius which moves with its centre on a fixed curve.

2. Prove that the surface

$$x^3 + y^3 + z^3 - 3xyz = a^3$$

is a surface of revolution round the line $x = y = z$. Find the equation of the generating curve.

3. Find the equation of a conoidal surface of which the generating lines pass through the axis of z and are parallel to the plane of xy, and whose directing curve is a circle with its centre in the axis of x and its plane parallel to that of yz. (*The Cono-Cuneus.*)

4. Find the equation of the surface generated by a straight line which passes through two fixed straight lines at right angles to each other, and also through a circle whose plane is parallel to each of the straight lines and whose centre is at the middle point of the shortest distance between them.

5. Find the equation of the surface generated by a straight line which always passes through the axis of z and some point of the curve

$$x = a \cos\theta, \quad y = a \sin\theta, \quad z = c\theta;$$

and is parallel to the plane of xy.

6. Find the equation of the surface generated by the tangent lines of the curve

$$x = a \cos\theta, \quad y = a \sin\theta, \quad z = c\theta.$$

7. Find the equation of a conical surface whose vertex is at any point on the surface of a sphere, and whose base is a small circle of the sphere.

Find also the curve in which the cone is cut by a plane through the centre of the sphere perpendicular to the diameter through the vertex.

8. Find the equation of the surface generated by the revolution of a circle round a straight line in its own plane which does not cut it.

9. Prove that all tangent planes to the surface in the last question which pass through its centre cut it in two circles.

10. A fixed straight line AB meets a fixed plane in A. A straight line AP moves so that the sine of the angle which it makes with AB bears a constant ratio to the sine of the angle which it makes with the fixed plane. Find the surface generated by AP.

11. Find the conditions that the surface

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy + 2A''x + 2B''y + 2C''z + F = 0$$

may be a cylindrical surface.

12. Shew that with the notation of Art. 100 the condition that the surface $F(x, y, z) = 0$ may be developable is

$$U^2(vw - u'^2) + V^2(wu - v'^2) + W^2(uv - w'^2) + 2VW(v'w' - uu') + 2WU(w'u' - vv') + 2UV(u'v' - ww') = 0.$$

Deduce the conditions that the surface in (11) may be developable.

13. Find the equation of the surface generated by all the normals drawn to an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

at the points where it is cut by the cone

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0.$$

14. A surface is generated by a straight line which passes through the axis of z, and the line $x=a$, $z=0$; remaining parallel to the plane $y=kz$. Shew that its equation is $x(y-kz)=ay$.

15. Describe the general nature of the surfaces represented by the several equations

(1) $f(r, \theta)=0$. (2) $f(r, \phi)=0$. (3) $f(\theta, \phi)=0$.

16. Examine the nature of the surfaces represented by

(1) $r^2=a^2 \cos 2\theta$. (2) $r^2=a^2 \cos 2\phi$.

17. Find the equation of a cylindrical surface having one central circular section of an ellipsoid for its guiding curve, and its axis perpendicular to the other circular section.

18. With the axis of z as axis a series of helices are described, all intersecting two given curves; prove that the functional equation of the surfaces generated is

$$\tan^{-1}\frac{y}{x}=z\,.\,F(x^2+y^2)+f(x^2+y^2),$$

and that the differential equation is

$$y^2\frac{d^2z}{dx^2}-2yx\frac{d^2z}{dx\,dy}+x^2\frac{d^2z}{dy^2}=x\frac{dz}{dx}+y\frac{dz}{dy}.$$

19. A candle is placed at a given distance in front of a plane vertical circular mirror on a line perpendicular to the plane of the mirror through the extremity of its horizontal diameter; shew that the boundary of the reflected light which falls on a wall of which the plane is perpendicular to that of the mirror is a parabola, and determine its latus rectum.

20. A straight line AB moves on two fixed straight lines not in the same plane so that the portion between the lines subtends a right angle at a fixed point O. Prove that the locus of this line is a skew surface of the second order.

21. Obtain the differential equation of surfaces of revolution from the consideration that at every point of such surfaces one tangent line is perpendicular to the plane containing that point and the axis of revolution.

22. Shew that if a section of a right conoid whose generating lines are parallel to the plane of xy be made by any plane parallel to that of xy, the normals at points in the lines of section will meet the plane of xy in concentric hyperbolas.

23. Prove that the general functional equation of the surfaces generated by a circle which always touches the axis of z at the origin may be written in the form

$$x^2 + y^2 + z^2 = 2cxf\left(\frac{y}{x}\right),$$

and that the differential equation is

$$2z\left\{x\frac{dz}{dx} + y\frac{dz}{dy}\right\} = z^2 - x^2 - y^2.$$

CHAPTER XII.

ON FOCI AND CONFOCAL QUADRICS.

152. A FOCUS of a conic section is a point such that the distance of any point on the curve from it can be expressed as a linear function of the co-ordinates of that point.

There are certain points which have analogous properties in reference to quadrics, and which may therefore be called foci of quadrics.

153. For instance the equation of the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad\text{......................}\quad (1),$$

where we will suppose a, b, c in descending order of magnitude. Also let e_1, e_2, e_3 be the excentricities of the sections of (1) by the planes of yz, zx, xy respectively.

The co-ordinates of the focus of the section by the plane of xy are ae_3, 0, 0. The square of the distance of any point (x, y, z) in (1) from this focus

$$= (x - ae_3)^2 + y^2 + z^2$$

$$= x^2 - 2ae_3x + a^2e_3^{\,2} + b^2\left(1 - \frac{x^2}{a^2} - \frac{z^2}{c^2}\right) + z^2$$

$$= x^2\left(1 - \frac{b^2}{a^2}\right) - 2ae_3x + a^2 - \frac{b^2 - c^2}{c^2}z^2.$$

$$= (e_3x - a)^2 - \frac{b^2e_1^{\,2}z^2}{c^2}$$

$$= (e_3x - e'z - a)(e_3x + e'z - a), \text{ if } e' = \frac{be_1}{c}.$$

Hence the square of the distance of any point on (1) from the focus of the section of (1) by the plane of xy is equal to the product of two linear functions of the co-ordinates of the point.

Or, geometrically, we may say that the square of the distance of any point on the quadric from the focus of the section of the quadric by the plane of xy, is proportional to the product of the distances of the point from two planes whose equations are

$$e_3x - e'z - a = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$e_3x + e'z - a = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

These two planes intersect in a line whose equations are $z = 0$, $e_3x - a = 0$, that is in the directrix of the section of the quadric by the plane of xy.

Similar properties hold for the foci of the sections of the quadric by the planes of yz and zx, but in these cases the two planes corresponding to (2) and (3) are impossible, though their line of intersection is real.

154. These points are not however the only points which have the same property. We will examine the conditions which must be satisfied by the co-ordinates of any point, in order that the square of its distance from any point on a given central quadric, may be proportional to the rectangle contained by the distances of the latter point from two planes, real or impossible.

If α, β, γ be the co-ordinates of such a point, we must have the expression $(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2$ identically equal to

$$\{l(x-\alpha')+m(y-\beta')+n(z-\gamma')\}\{l'(x-\alpha')+m'(y-\beta')+n'(z-\gamma')\},$$

for all values of x, y, z which satisfy the equation of the quadric; α', β', γ' being the co-ordinates of any point in the line of intersection of the planes.

Let the equation of the quadric be

$$Ax^2 + By^2 + Cz^2 = 1 \quad\text{.................. (1).}$$

Then the equation

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2$$
$$-\{l(x-\alpha')+m(y-\beta')+n(z-\gamma')\}\{l'(x-\alpha')+m'(y-\beta')+n'(z-\gamma')\}=0 \quad (2)$$

must be satisfied by all values of x, y, z which satisfy (1).

This can only be the case when the two equations are identical, and as first conditions for this the coefficients of yz, zx and xy in (2) must vanish. We thus get

$$mn' + n'm = 0, \quad nl' + n'l = 0, \quad lm' + l'm = 0,$$

which can only be satisfied by one of the sets of conditions

$$\left.\begin{aligned} l = 0, \quad l' = 0, \quad \frac{m'}{m} &= -\frac{n'}{n} \\ m = 0, \quad m' = 0, \quad \frac{n'}{n} &= -\frac{l'}{l} \\ \text{or} \qquad n = 0, \quad n' = 0, \quad \frac{l'}{l} &= -\frac{m'}{m} \end{aligned}\right\} \text{.................. (3).}$$

If we take the second set of these equations and put $\frac{l'}{l} = k$, the equation (2) becomes

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-kl^2(x-\alpha')^2+kn^2(z-\gamma')^2=0 \text{...... (4).}$$

Comparing the remaining terms of the second degree with those in (1) we obtain

$$\frac{1-kl^2}{A} = \frac{1}{B} = \frac{1+kn^2}{C},$$

or

$$kl^2 = 1 - \frac{A}{B}, \quad kn^2 = \frac{C}{B} - 1 \text{.................. (5).}$$

And by comparing the terms involving x, y, z, and the constant term in (4) with the corresponding terms in (1) we have

$$\alpha - kl^2\alpha' = 0,\ \ \beta = 0,\ \ \gamma + kn^2\gamma' = 0 \text{ ....(6)},$$

$$\alpha^2 + \beta^2 + \gamma^2 - kl^2\alpha'^2 + kn^2\gamma'^2 = -\frac{1}{B} \text{ ....(7)}.$$

And substituting for α', γ' from (6) in (7) we obtain by help of (5),

$$\frac{\alpha^2}{\frac{1}{A} - \frac{1}{B}} + \frac{\gamma^2}{\frac{1}{C} - \frac{1}{B}} = 1 \text{ .... (8)}.$$

The equation (8) combined with $\beta = 0$ gives a conic section in the plane of zx, all the points on which may be considered as foci of the quadric. This curve is called *a focal conic* of (1).

155. The equations (6) give values of α' and γ' corresponding to any particular focus (α, β, γ). These values determine the position of a straight line which we may call the directrix corresponding to that particular focus.

The directrices corresponding to the different foci lying on the conic (8) all lie on a cylinder whose equation will be found by eliminating α and γ between (6) and (8), to be

$$A^2\alpha'^2\left(\frac{1}{A} - \frac{1}{B}\right) + C^2\gamma'^2\left(\frac{1}{C} - \frac{1}{B}\right) = 1.$$

156. The other conditions in (3) will similarly give us two other focal conics in the planes of xy and yz whose equations are

$$\frac{\alpha^2}{\frac{1}{A} - \frac{1}{C}} + \frac{\beta^2}{\frac{1}{B} - \frac{1}{C}} = 1 \text{ ....(9)},$$

$$\frac{\beta^2}{\frac{1}{B} - \frac{1}{A}} + \frac{\gamma^2}{\frac{1}{C} - \frac{1}{A}} = 1 \text{ ....(10)};$$

and corresponding to any focus there will be a directrix perpendicular to the plane in which the focal conic lies.

Of these conics, whatever be the signs and relative magnitudes of A, B, C, one will be an ellipse, another an hyperbola, and the third an impossible locus.

157. For instance, in the ellipsoid whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

the equations of the focal conics will be

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1 \text{ in the plane of } xy,$$

$$\frac{x^2}{a^2-b^2}+\frac{z^2}{c^2-b^2}=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad zx,$$

$$\frac{y^2}{b^2-a^2}+\frac{z^2}{c^2-a^2}=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad yz.$$

And if we assume a, b, c to be in descending order of magnitude, the first of these is an ellipse the extremities of whose axes are the foci of the sections of the original ellipsoid by the planes of yz and zx: the second a hyperbola with its real axis in the axis of x, the extremities of this real axis being the foci of the section of the ellipsoid by the plane of xy: while the third is altogether an impossible locus.

Similar results may be obtained for the two hyperboloids.

158. The focal conics of a cone

$$Ax^2+By^2+Cz^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

can be deduced from those of a central quadric

$$Ax^2+By^2+Cz^2=\lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

by putting λ equal to zero.

The focal conics of (2) would be, writing $\frac{A}{\lambda}, \frac{B}{\lambda}, \frac{C}{\lambda}$ instead of A, B, C in the formulæ (8), (9), (10) of Articles 154 and 156,

$$\frac{\alpha^2}{\frac{\lambda}{A}-\frac{\lambda}{B}}+\frac{\gamma^2}{\frac{\lambda}{C}-\frac{\lambda}{B}}=1,$$

$$\frac{\alpha^2}{\frac{\lambda}{A}-\frac{\lambda}{C}}+\frac{\beta^2}{\frac{\lambda}{B}-\frac{\lambda}{C}}=1,$$

$$\frac{\beta^2}{\frac{\lambda}{B}-\frac{\lambda}{A}}+\frac{\gamma^2}{\frac{\lambda}{C}-\frac{\lambda}{A}}=1.$$

Or, multiplying these equations by λ and then making λ to vanish, the focal conics of the cone (1) become

$$\frac{\alpha^2}{\frac{1}{A}-\frac{1}{B}}+\frac{\gamma^2}{\frac{1}{C}-\frac{1}{B}}=0,$$

$$\frac{\alpha^2}{\frac{1}{A}-\frac{1}{C}}+\frac{\beta^2}{\frac{1}{B}-\frac{1}{C}}=0,$$

$$\frac{\beta^2}{\frac{1}{B}-\frac{1}{A}}+\frac{\gamma^2}{\frac{1}{C}-\frac{1}{A}}=0.$$

Of these, whatever be the signs of

$$\frac{1}{B}-\frac{1}{C},\quad \frac{1}{C}-\frac{1}{A},\quad \frac{1}{A}-\frac{1}{B},$$

one will give two straight lines, and the other two give a point, the vertex.

159. To find the focal conics of the paraboloid

$$By^2+Cz^2=x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

we must as in Art. 154 make the equation (1) identical with

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2$$
$$-\{l(x-\alpha')+m(y-\beta')+n(z-\gamma')\}\{l'(x-\alpha')+m'(y-\beta')+n'(z-\gamma')\}=0 \quad (2$$

The first conditions for this identity are the same as equations (3) of Art. 154, and if we take the second of those conditions and put $\frac{l'}{l}=k$, equation (2) becomes as in that Article

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-kl^2(x-\alpha')^2+kn^2(z-\gamma')^2=0.$$

And since (1) contains no term involving x^2 and no constant term, we get

$$1-kl^2=0, \qquad \alpha^2+\beta^2+\gamma^2-kl^2\alpha'^2+kn^2\gamma'^2=0;$$

and by comparing the remaining terms in the two equations, we have

$$\frac{1+kn^2}{C}=\frac{1}{B}=\frac{2(\alpha-kl^2\alpha')}{1},$$
$$\beta=0,\ 2(\gamma+kn^2\gamma')=0;$$

and thus we get for the locus of the foci the two equations

$$\beta=0 \text{ and } \alpha^2+\gamma^2-\left(\alpha-\frac{1}{2B}\right)^2+\frac{B\gamma^2}{C-B}=0,$$

or

$$\left.\begin{array}{r}\dfrac{C\gamma^2}{C-B}=-\dfrac{1}{B}\left(\alpha-\dfrac{1}{4B}\right)\\ \beta=0\end{array}\right\}.$$

and

By taking the third of the conditions (3) of Art. 154 we shall similarly get another focal conic in the plane of xy whose equations are

$$\gamma=0,$$
$$\frac{B\beta^2}{B-C}=-\frac{1}{C}\left(\alpha-\frac{1}{4C}\right).$$

The first of the conditions (3) of Art. 154 is in this case inadmissible inasmuch as (1) contains no term involving x^2.

Thus in this case the focal conics are two parabolas whose vertices are the foci of the sections of the surface (1) by the planes of xy and zx.

160. Two central quadrics

$$Ax^2 + By^2 + Cz^2 = 1,$$
$$A'x^2 + B'y^2 + C'z^2 = 1,$$

will have the same focal conics if

$$\frac{1}{B} - \frac{1}{C} = \frac{1}{B'} - \frac{1}{C'}, \quad \frac{1}{C} - \frac{1}{A} = \frac{1}{C'} - \frac{1}{A'}, \quad \frac{1}{A} - \frac{1}{B} = \frac{1}{A'} - \frac{1}{B'},$$

or as we may write the conditions, if

$$\frac{1}{A} - \frac{1}{A'} = \frac{1}{B} - \frac{1}{B'} = \frac{1}{C} - \frac{1}{C'}.$$

Two quadrics whose equations satisfy these conditions are called *confocal quadrics.*

Thus if the equation of an ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad\qquad (1),$$

all surfaces whose equations are of the form

$$\frac{x^2}{a^2+k} + \frac{y^2}{b^2+k} + \frac{z^2}{c^2+k} = 1 \qquad\qquad (2),$$

where k is any quantity positive or negative, are confocal with the ellipsoid.

161. If α, β, γ be the co-ordinates of any point, we can find the equation of a surface passing through (α, β, γ) and confocal with (1) by determining k from the equation

$$\frac{\alpha^2}{a^2+k} + \frac{\beta^2}{b^2+k} + \frac{\gamma}{c^2+k} - 1 = 0 \qquad\qquad (3),$$

which is the condition that (2) should pass through the point (α, β, γ). This equation is a cubic in k, of which it can be shewn that the roots are all real. There are therefore three quadrics confocal to (1) which pass through the point (α, β, γ), of which one can be shewn to be an ellipsoid, and the others to be hyperboloids of one and two sheets respectively.

162. Any two confocal quadrics intersect at right angles at all points where they meet.

For let x, y, z be the co-ordinates of any point common to the two quadrics

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\frac{x^2}{A+k} + \frac{y^2}{B+k} + \frac{z^2}{C+k} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The equation of the tangent plane to (1) at the point (x, y, z) is

$$\frac{x'x}{A} + \frac{y'y}{B} + \frac{z'z}{C} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

And the equation of the tangent plane to (2) at the same point is

$$\frac{x'x}{A+k} + \frac{y'y}{B+k} + \frac{z'z}{C+k} = 1 \ldots\ldots\ldots\ldots\ldots (4).$$

But from (1) and (2) by subtraction we obtain at all their points of intersection

$$\frac{x^2}{A(A+k)} + \frac{y^2}{B(B+k)} + \frac{z^2}{C(C+k)} = 0,$$

which is the condition that (3) and (4) should be at right angles to each other.

EXAMPLES. CHAPTER XII.

1. FIND the equations of the focal conics of the quadric

$$2x^2 + 3y^2 + 4z^2 = 9.$$

2. Find the equations of the quadrics confocal with the quadric

$$2x^2 + 3y^2 + 4z^2 = 9,$$

which pass through the point (1, 1, 1).

3. Find the locus of the points of contact of tangent planes drawn from a point in the axis of x to a series of confocal surfaces whose axes coincide with the axes of co-ordinates.

4. Shew that the surfaces

$$\frac{x}{\alpha} + \frac{y^2}{\alpha x - a^2} + \frac{z^2}{\alpha x - b^2} = 1,$$

$$\frac{x^2}{\beta y + a^2} + \frac{y}{\beta} + \frac{z^2}{\beta y + a^2 - b^2} = 1,$$

intersect everywhere at right angles.

5. Shew that if the foci of the principal sections of two paraboloids coincide, their focal conics will also coincide.

6. Extend the proposition of Art. 162 to the case of two confocal paraboloids.

CHAPTER XIII.

ON CURVATURE OF SURFACES.

163. Two surfaces are said to have contact of the first order at any point where they meet when they have a common tangent plane at that point. The necessary and sufficient conditions for this are that for the same values of x and y the values of z, $\frac{dz}{dx}$ and $\frac{dz}{dy}$ shall be the same for the two surfaces.

Two surfaces are said to have contact of the n^{th} order at a point where they meet when the sections of the two surfaces by every plane passing through that point have contact of the n^{th} order. This we will prove to be the case if the sections of the surfaces by all planes which contain any given straight line through the point of contact not lying in the tangent plane have contact of the n^{th} order.

For let the common point be taken as origin and the given line as axis of z. Let the equations of the two surfaces be

$$z = f(x, y) \dots\dots\dots\dots\dots\dots (1),$$

$$z = F(x, y) \dots\dots\dots\dots\dots\dots (2).$$

Expanding (1) and (2) we obtain

$$z_1 = \left(\frac{df}{dx}\right) x + \left(\frac{df}{dy}\right) y + \dots + \frac{1}{\lfloor n} \left(x \frac{d}{dx} + y \frac{d}{dy}\right)^n . f + \dots\dots (3),$$

$$z_2 = \left(\frac{dF}{dx}\right) x + \left(\frac{dF}{dy}\right) y + \dots + \frac{1}{\lfloor n} \left(x \frac{d}{dx} + y \frac{d}{dy}\right)^n . F + \dots . (4),$$

where z_1 and z_2 are the ordinates of the two surfaces corresponding to the same values of x and y; and in the quantities $\frac{df}{dx}, \frac{dF}{dx}, \ldots$ x and y are put equal to zero after the differentiations are performed.

Now since all sections of (1) and (2) by planes which contain the axis of z have contact of the n^{th} order, the difference of z_1 and z_2 must be of the $(n+1)^{\text{th}}$ degree in x and y. Hence we have

$$\left.\begin{array}{l} \dfrac{df}{dx}=\dfrac{dF}{dx},\ \dfrac{df}{dy}=\dfrac{dF}{dy},\ \dfrac{d^2f}{dx^2}=\dfrac{d^2F}{dx^2},\ \dfrac{d^2f}{dxdy}=\dfrac{d^2F}{dxdy},\ \ldots \\ \quad \dfrac{d^nf}{dx^n}=\dfrac{d^nF}{dx^n},\ \ldots \dfrac{d^nf}{dx^rdy^{n-r}}=\dfrac{d^nF}{dx^rdy^{n-r}},\ \ldots \end{array}\right\}\ldots(5).$$

If now the axes be changed in position, the origin remaining the same, since the new co-ordinates x', y', z' of any point are linear functions of the old co-ordinates, it is clear that any differential coefficient of the form $\frac{d^{r+s}z'}{dx'^r dy'^s}$ can be expressed in terms of the differential coefficients of z with respect to x and y of orders up to but not exceeding the $(r+s)^{\text{th}}$. Hence if the differential coefficients of z with respect to x and y for one surface, up to those of the n^{th} order inclusive, be respectively equal to the corresponding quantities for a second surface, the same will be true of the differential coefficients of z' with respect to x' and y'; that is, if conditions (5) be satisfied for two surfaces with any one set of axes, they will be also satisfied with any other set of axes.

Thus if the sections of the two surfaces (1) and (2) by all planes through the axis of z have contact of the n^{th} order, so will their sections by all planes through the common point.

The conditions that two surfaces should have contact of the n^{th} order at a given point are therefore that the values of

$$z,\ \frac{dz}{dx},\ \frac{dz}{dy},\ \ldots \frac{d^nz}{dx^n},\ \frac{d^nz}{dx^{n-1}dy},\ \ldots \frac{d^nz}{dy^n},$$

should be the same for the two surfaces for the given values of x and y.

164. *If two surfaces touch at a given point and the sections by a plane through the normal and any tangent line have contact of the second order, then all sections by planes through the same tangent line have contact of the second order.*

Take the common point as origin and the common normal as axis of z. Then, $z=f(x, y)$, $z=F(x, y)$ being the equations of the two surfaces, the values of $\frac{df}{dx}, \frac{df}{dy}, \frac{dF}{dx}, \frac{dF}{dy}$ vanish at the origin and the equations of the surfaces can be put in the form

$$z=ax^2+bxy+cy^2+\dots\dots\dots\dots\dots\dots\dots (1),$$

$$z=Ax^2+Bxy+Cy^2+\dots\dots\dots\dots\dots\dots\dots (2),$$

where a, b, c are the values of $\frac{1}{2}\frac{d^2f}{dx^2}, \frac{d^2f}{dx\,dy}, \frac{1}{2}\frac{d^2f}{dy^2}$ at the origin, and A, B, C those of $\frac{1}{2}\frac{d^2F}{dx^2}, \frac{d^2F}{dx\,dy}, \frac{1}{2}\frac{d^2F}{dy^2}$.

Also if the given tangent line be the axis of x, the sections by the plane of zx have contact of the second order, and we have $a=A$.

Consider now the sections by a plane through the axis of x whose equation is

$$y=mz\dots\dots\dots\dots\dots\dots\dots\dots (3),$$

we have for a given value x_1 of x, in the one surface

$$z_1=ax_1^2+bx_1y_1+cy_1^2+\dots,$$

and in the other

$$z_2=Ax_1^2+Bx_1y_2+Cy_2^2+\dots;$$

$$\therefore\ z_1-z_2=x_1(by_1-By_2)+cy_1^2-Cy_2^2+\dots$$

But z_1, z_2 being of the second degree in x_1, y_1 and y_2 are so also by (3), and therefore $x_1(by_1-By_2)$ is of the third degree, and therefore z_1-z_2 is of the third degree in x_1, and the sections of the two surfaces by (3) have contact of the second order.

Similarly if two surfaces have complete contact of the $(n-1)^{\text{th}}$ order at a given point, and the sections by any plane through the normal and a given tangent line have contact of the n^{th} order, then all sections by planes through this tangent line have contact of the n^{th} order.

165. From the proposition proved in the last article it follows that if R be the radius of curvature of any normal section of a surface, $R \cos \theta$ is the radius of curvature of an oblique section through the same tangent line inclined at an angle θ to the normal section. For if a sphere whose radius is R be described touching the surface at the given point, the normal sections of this sphere and the surface through the given tangent line have contact of the second order and therefore also any oblique sections.

But the radius of curvature of the oblique section of the sphere is obviously $R \cos \theta$; hence the radius of curvature of the oblique section of the given surface is also $R \cos \theta$. This proposition is called Meunier's Theorem.

166. If the tangent plane at any point be taken as the plane of xy and the point of contact as the origin, we have seen that the equation of the surface can be put into the form

$$z = ax^2 + bxy + cy^2 + \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where the remaining terms are of a higher degree than the second.

Consider the section of this surface by a plane through the axis of z whose equation is

$$y = x \tan \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The radius of curvature of this section is the limit of $\frac{x^2+y^2}{2z}$ when the values of x and y are diminished indefinitely. Hence if ρ be this radius, we have

$$\frac{1}{2\rho} = \text{lt.}\ \frac{ax^2 + bxy + cy^2 + Ax^3}{x^2+y^2}$$

$$= \text{lt.}\ \frac{a + b\tan\theta + c\tan^2\theta + Ax}{1+\tan^2\theta}$$

$$= a\cos^2\theta + b\sin\theta\cos\theta + c\sin^2\theta \ldots\ldots\ldots\ldots(3).$$

If we construct the conic section whose equation is

$$ax^2 + bxy + cy^2 = 1 \quad\text{.................. (4)},$$

it is evident from (3) that the square of any radius vector of this conic represents the diameter of curvature of the section of (1) by a normal plane passing through this radius vector. This conic section is called the *indicatrix* of the surface at the given point.

If in (1) we suppose x and y so small that the terms on the right hand after the third may be neglected, we get

$$z = ax^2 + bxy + cy^2 \quad\text{.................. (5)}.$$

The curve in which this surface is cut by a plane $z = k$ parallel to the plane of xy is similar and similarly situated to (4). Hence the indicatrix at any point of a surface may be defined as a curve similar and similarly situated to the limit of the curve in which the surface is cut by a plane indefinitely near to the tangent plane at the given point.

167. By choosing the axes of x and y so as to coincide with the principal axes of the indicatrix the equation (4) of the last article assumes the form

$$Ax^2 + Cy^2 = 1 \quad\text{..................... (1)}.$$

Also the radii vectores drawn in the directions of the principal axes are respectively the least and greatest radii of the curve. Hence the normal sections for which the radius of curvature is least and greatest respectively, pass through the principal axes of the indicatrix. The radii of curvature of these sections are called the principal radii of curvature at the given point, and the sections themselves, the principal sections.

Let R and R' be the principal radii of curvature, ρ and ρ' the radii of curvature of any other sections at right angles, which we may take to be the sections through the axes of x and y in equation (4) of the last article. Then

$$\frac{1}{2R} = A, \quad \frac{1}{2R'} = C,$$

$$\frac{1}{2\rho} = a, \quad \frac{1}{2\rho'} = c.$$

But $A + C = a + c$. (Todhunter's *Conic Sections*, Art. 274.)

And therefore

$$\frac{1}{R} + \frac{1}{R'} = \frac{1}{\rho} + \frac{1}{\rho'} \quad \text{.................(2).}$$

Also if the section whose radius of curvature is ρ be inclined at an angle θ to the principal section whose radius is R, we have from (1)

$$\frac{1}{2\rho} = A \cos^2\theta + C \sin^2\theta;$$

$$\therefore \frac{1}{\rho} = \frac{1}{R} \cos^2\theta + \frac{1}{R'} \sin^2\theta \quad \text{................ (3).}$$

We can thus obtain the radius of curvature of any normal section if we know those of the principal sections, and by Art. 165 we can deduce that of any oblique section. Hence, if we know the principal radii of curvature at any point of a surface, the curvatures of all sections of the surface at that point are known.

168. *To find the radius of curvature of any normal section of a surface at a given point.*

Let the equation of the surface be

$$F(x, y, z) = 0 \quad \text{..................... (1),}$$

and let x, y, z be the co-ordinates of the given point P. Let l, m, n be the direction-cosines of the tangent line at (x, y, z) through which the cutting plane passes. Also let $x + \alpha$, $y + \beta$, $z + \gamma$ be the co-ordinates of a point Q in the curve of section near to P. Let QR be drawn perpendicular on the tangent plane. Then, by Newton, the radius of curvature of the section is the limit of $\frac{PQ^2}{2QR}$ when Q is made to approach indefinitely near to P.

But the equation of the tangent plane is

$$(x' - x)\frac{dF}{dx} + (y' - y)\frac{dF}{dy} + (z' - z)\frac{dF}{dz} = 0 \quad \text{........(2).}$$

Hence

$$QR = \frac{\alpha \frac{dF}{dx} + \beta \frac{dF}{dy} + \gamma \frac{dF}{dz}}{\sqrt{\left\{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2\right\}}} = \frac{\alpha U + \beta V + \gamma W}{\sqrt{U^2 + V^2 + W^2}},$$

with the notation explained in Art. 100.

And $$PQ^2 = \alpha^2 + \beta^2 + \gamma^2.$$

The radius of curvature is therefore the limit of

$$\pm \frac{(\alpha^2 + \beta^2 + \gamma^2)\sqrt{U^2 + V^2 + W^2}}{2(\alpha U + \beta V + \gamma W)}.$$

But since the point $(x+\alpha, y+\beta, z+\gamma)$ is on the surface (1),

$$F(x + \alpha,\ y + \beta,\ z + \gamma) = 0,$$

or expanding, and remembering that $F(x, y, z) = 0$,

$$\alpha U + \beta V + \gamma W + \frac{1}{2}(\alpha^2 u + \beta^2 v + \gamma^2 w + 2\beta\gamma u' + 2\gamma\alpha v' + 2\alpha\beta w') + \ldots = 0.$$

Whence the above expression becomes

$$\mp \frac{(\alpha^2 + \beta^2 + \gamma^2)\sqrt{U^2 + V^2 + W^2}}{\alpha^2 u + \beta^2 v + \gamma^2 w + 2\beta\gamma u' + 2\gamma\alpha v' + 2\alpha\beta w' + \ldots}$$

where the remaining terms in the denominator are of higher dimensions than the second in α, β, γ.

Now, by Newton, the angle between PR and PQ diminishes indefinitely as Q approaches P. Hence we have ultimately

$$\frac{\alpha}{l} = \frac{\beta}{m} = \frac{\gamma}{n}.$$

And making these substitutions and diminishing α, β, γ indefinitely, we obtain for the radius of curvature

$$\rho = \frac{\mp\sqrt{U^2 + V^2 + W^2}}{ul^2 + vm^2 + wn^2 + 2u'mn + 2v'nl + 2w'lm} \quad \ldots\ldots (3).$$

169. The principal sections are those for which ρ is a maximum or minimum. Hence we have to make the expression

$$ul^2 + vm^2 + wn^2 + 2u'mn + 2v'nl + 2w'lm \ldots\ldots (1)$$

a maximum or minimum by the variation of l, m, n, which are connected by the relations

$$l^2 + m^2 + n^2 = 1 \ldots\ldots (2),$$
$$Ul + Vm + Wn = 0 \ldots\ldots (3),$$

the latter expressing the fact that the line whose direction-cosines are l, m, n lies in the tangent plane at the point (x, y, z). We shall denote the expression (1) by the symbol h.

Differentiating (1), (2) and (3) and using undetermined multipliers, we obtain

$$ul + w'm + v'n + kl + k'U = 0 \ldots\ldots (4),$$
$$w'l + vm + u'n + km + k'V = 0 \ldots\ldots (5),$$
$$v'l + u'm + wn + kn + k'W = 0 \ldots\ldots (6).$$

Multiplying (4) by l, (5) by m, and (6) by n, and adding, we get

$$h + k = 0 \ldots\ldots (7).$$

And the three equations (4), (5), and (6), become

$$(u-h)\,l + w'm + v'n = -k'U,$$
$$w'l + (v-h)\,m + u'n = -k'V,$$
$$v'l + u'm + (w-h)\,n = -k'W,$$

whence

$$\frac{l}{\begin{vmatrix} U, & w', & v' \\ V, & v-h, & u' \\ W, & u', & w-h \end{vmatrix}} = \frac{m}{\begin{vmatrix} U, & v', & u-h \\ V, & u', & w' \\ W, & w-h, & v' \end{vmatrix}} = \frac{n}{\begin{vmatrix} U, & u-h, & w' \\ V, & w', & v-h \\ W, & v', & u' \end{vmatrix}} \ldots (8).$$

Substituting in (3), and reducing

$$U^2\{(v-h)(w-h) - u'^2\} + V^2\{(w-h)(u-h) - v'^2\}$$
$$+ W^2\{(u-h)(v-h) - w'^2\}$$
$$+ 2VW\{v'w' - u'(u-h)\} + 2WU\{w'u' - v'(v-h)\}$$
$$+ 2UV\{u'v' - w'(w-h)\} = 0 \ldots (9).$$

From (9) we obtain two values of h and therefore of ρ, and from (8) we deduce the corresponding values of l, m, n.

170. The formulæ of the last two articles are somewhat simplified if we take the unsymmetrical form of the equation of a surface

$$z = f(x, y),$$

or

$$f(x, y) - z = 0.$$

The reductions may be effected by the substitutions

$$U = p, \qquad V = q, \qquad W = -1,$$

$$u = r, \qquad v = t, \qquad w = 0,$$

$$u' = 0, \qquad v' = 0, \qquad w' = s.$$

Moreover, instead of determining the tangent line through which the section is made by its direction-cosines, it is usual to determine it by its projection on the plane of xy, whose equation we may assume to be

$$y' - y = m(x' - x) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

The direction-cosines of the line of intersection of this plane with the tangent-plane at (x, y, z), whose equation is

$$p(x' - x) + q(y' - y) - (z' - z) = 0,$$

are proportional to $1, m, p + qm$, respectively.

The value of ρ becomes with these substitutions equal to

$$\frac{\sqrt{1 + p^2 + q^2}\,\{1 + p^2 + 2pqm + (1 + q^2)\,m^2\}}{r + 2sm + tm^2}.$$

171. The result of the last article can be obtained independently. Let a sphere be described having contact of the first order with the given surface at (x, y, z), and let the sections of the surface and the sphere by the plane (1) have contact of the second order. Then the sections of the surface and the sphere by a normal plane through the line in which (1) cuts the tangent plane will, by Meunier's Theorem, have contact of the second order with each other,

and the radius of the sphere is therefore the radius of curvature of the section required.

Let the equation of the sphere be

$$(X-a)^2+(Y-b)^2+(Z-c)^2=\rho^2 \quad\ldots\ldots\ldots(2);$$

$$\left.\begin{aligned} \therefore\ (X-a)+(Z-c)\frac{dZ}{dX}&=0 \\ (Y-b)+(Z-c)\frac{dZ}{dY}&=0 \end{aligned}\right\}\quad\ldots\ldots\ldots(3),$$

$$\left.\begin{aligned} 1+\left(\frac{dZ}{dX}\right)^2+(Z-c)\frac{d^2Z}{dX^2}&=0 \\ \frac{dZ}{dX}\cdot\frac{dZ}{dY}+(Z-c)\frac{d^2Z}{dX\,dY}&=0 \\ 1+\left(\frac{dZ}{dY}\right)^2+(Z-c)\frac{d^2Z}{dY^2}&=0 \end{aligned}\right\}\quad\ldots\ldots\ldots(4).$$

But at the point (x, y, z)

$$X=x,\quad Y=y,\quad Z=z,\quad \frac{dZ}{dX}=p,\quad \frac{dZ}{dY}=q,$$

since the sphere and surface have a common tangent plane. Also since their sections by the plane (1) have contact of the second order, the values of z in terms of $x'-x$, $y'-y$ for the sphere and surface must coincide as far as terms of the second degree in $x'-x$, $y'-y$ for points lying in the plane (1), whence we obtain

$$\frac{d^2Z}{dX^2}+2m\frac{d^2Z}{dX\,dY}+m^2\frac{d^2Z}{dY^2}=r+2ms+tm^2 \quad\ldots\ldots\ldots(5).$$

We deduce from (3)

$$\frac{x-a}{p}=\frac{y-b}{q}=\frac{z-c}{-1}=\frac{\rho}{\sqrt{1+p^2+q^2}},$$

and from (4)

$$\frac{d^2Z}{dX^2}=\frac{1+p^2}{c-z},\quad \frac{d^2Z}{dX\,dY}=\frac{pq}{c-z},\quad \frac{d^2Z}{dY^2}=\frac{1+q^2}{c-z}.$$

Whence from (5)

$$c - z = \frac{1 + p^2 + 2pqm + (1 + q^2)\, m^2}{r + 2sm + tm^2},$$

and

$$\therefore\ \rho = \sqrt{1 + p^2 + q^2} \cdot \frac{1 + p^2 + 2pqm + (1 + q^2)\, m^2}{r + 2sm + tm^2} \quad \text{.........(6).}$$

The equation which gives the sections of greatest and least curvature at any point is obtained by making this expression for ρ a maximum or minimum by the variation of m. Whence

$$\{pq + (1 + q^2)\, m\} \{r + 2sm + tm^2\}$$
$$- (s + tm) \{1 + p^2 + 2pqm + (1 + q^2)\, m^2\} = 0,$$

or $m^2 \{s (1 + q^2) - pqt\} + m \{r (1 + q^2) - t (1 + p^2)\}$
$$+ \{pqr - s (1 + p^2)\} = 0 \quad \text{.........(7).}$$

172. It may happen that at certain points of a surface the two principal radii of curvature become equal. It follows from Art. 167 that the radii of curvature of all normal sections at that point are equal, the indicatrix in this case being a circle. Such a point is called an *umbilicus*.

The conditions for the existence of an umbilicus can be deduced from the consideration that at such a point the expression

$$ul^2 + vm^2 + wn^2 + 2u'mn + 2v'nl + 2w'lm \quad \text{.........(1),}$$

must retain the same value for all values of l, m, n consistent with the conditions

$$Ul + Vm + Wn = 0 \quad \text{.....................(2),}$$

$$l^2 + m^2 + n^2 = 1 \quad \text{.....................(3).}$$

From (2) $\quad U^2l^2 + V^2m^2 + 2UVlm = W^2n^2;$

$$\therefore\ 2lm = \frac{W^2n^2 - U^2l^2 - V^2m^2}{UV}.$$

Similarly, $\quad 2nl = \dfrac{V^2m^2 - W^2n^2 - U^2l^2}{WU},$

$$2mn = \frac{U^2l^2 - V^2m^2 - W^2n^2}{VW}.$$

Whence, substituting in (1), the expression

$$l^2\left\{u+\frac{U^2u'}{VW}-\frac{Uv'}{W}-\frac{Uw'}{V}\right\}+m^2\left\{v+\frac{V^2v'}{WU}-\frac{Vw'}{U}-\frac{Vu'}{W}\right\}$$
$$+n^2\left\{w+\frac{W^2w'}{UV}-\frac{Wu'}{V}-\frac{Wv'}{U}\right\}$$

must have the same value for all values of l, m, n consistent with (3).

This gives the conditions

$$u+\frac{U}{VW}\{Uu'-Vv'-Ww'\}=v+\frac{V}{WU}\{Vv'-Ww'-Uu'\}$$
$$=w+\frac{W}{UV}\{Ww'-Uu'-Vv'\} \text{ ...... (4).}$$

If the equation of the surface be of the form

$$\phi_1(x)+\phi_2(y)+\phi_3(z)=0,$$

$u'=0$, $v'=0$, $w'=0$, and equations (4) become

$$u=v=w \text{ .... (5).}$$

If U, V or W vanish the investigation fails. Suppose $U=0$.

Then $$Vm+Wn=0, \text{ or } n=-\frac{V}{W}m,$$

and the expression (1) becomes

$$ul^2+vm^2+w\,.\,\frac{V^2}{W^2}\,.\,m^2-\frac{2u'V}{W}m^2+2lm\left(w'-\frac{Vv'}{W}\right),$$

which must remain constant for all values of m and n consistent with the relation

$$l^2+m^2\left(1+\frac{V^2}{W^2}\right)=1.$$

Hence $$Ww'-Vv'=0,$$

and $$u=\frac{v+\dfrac{V^2w}{W^2}-\dfrac{2u'V}{W}}{1+\dfrac{V^2}{W^2}}=\frac{V^2w+W^2v-2VWu'}{V^2+W^2} \text{ ...... (6).}$$

Similarly if $V=0$ or $W=0$ the requisite conditions may be deduced. In these cases, three conditions have to be satisfied by x, y, z besides the equation of the surface, which will not generally be consistent.

The conditions for an umbilicus when the unsymmetrical form of the equation of a surface is used may be deduced from the consideration that the value of ρ in Art. 171 must be independent of m. We thus get

$$\frac{1+p^2}{r}=\frac{pq}{s}=\frac{1+q^2}{t}.$$

173. The conditions for an umbilicus can be obtained in a slightly different form.

If h is the value of the expression (1) for all values of l, m, n consistent with (2), it is evident that the expression

$$ul^2+vm^2+wn^2+2u'mn+2v'nl+2w'lm \\ -h(l^2+m^2+n^2)\ldots\ldots\ldots(1)$$

must vanish for all values of l, m, n consistent with (2). Hence $Ul+Vm+Wn$ must be a factor of (6). The other factor must be

$$\frac{u-h}{U}l+\frac{v-h}{V}m+\frac{w-h}{W}n,$$

and multiplying these factors together and equating coefficients of mn, nl and lm as in Art. 49, we have

$$\frac{W}{V}(v-h)+\frac{V}{W}(w-h)=2u'$$

and two similar equations, whence

$$h=\frac{W^2v+V^2w-2u'VW}{V^2+W^2}$$

$$=\frac{V^2u+U^2v-2w'UV}{U^2+V^2}$$

$$=\frac{U^2w+W^2u-2v'WU}{W^2+U^2} \text{ by symmetry.}$$

174. *Lines of Curvature.*

A line of curvature on any surface is a curve such that the tangent line to it at any point coincides with the tangent line to one of the principal sections at that point.

The differential equation of such lines is obtained by substituting $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ for l, m, n respectively, in the equations which determine the directions of the principal sections in Art. 169. From the equations (4), (5) and (6) of that Article we have, eliminating k and k',

$$\begin{vmatrix} ul+w'm+v'n, & l, & U \\ w'l+vm+u'n, & m, & V \\ v'l+u'm+wn, & n, & W \end{vmatrix} = 0 \quad\ldots\ldots\ldots\ldots(1),$$

and replacing l, m, n by $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, respectively, we get the differential equation of the lines of curvature.

The differential equation of the projection of the lines of curvature on the plane of xy is obtained by writing $\frac{dy}{dx}$ for m in the equation (7) of Art. 171.

175. A line of curvature is sometimes defined as a curve such that the normals to the surface drawn at any two consecutive points of the curve intersect each other. This definition leads at once to the equation (1) of the last Article. For the equations of the normal at a point (x, y, z) are

$$\frac{x'-x}{U} = \frac{y'-y}{V} = \frac{z'-z}{W} \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The equations of the normal at a point $(x+\alpha, y+\beta, z+\gamma)$ are

$$\frac{x'-x-\alpha}{U+u\alpha+w'\beta+v'\gamma+\ldots} = \frac{y'-y-\beta}{V+w'\alpha+v\beta+u'\gamma+\ldots}$$

$$= \frac{z'-z-\gamma}{W+v'\alpha+u'\beta+w\gamma+\ldots} \quad\ldots\ldots\ldots\ldots\ldots(3),$$

where the remaining terms in the denominators contain higher powers of α, β, γ.

The condition that (2) and (3) should intersect is by Art. 31,

$$\begin{vmatrix} U + u\alpha + w'\beta + v'\gamma + \dots, & U, & \alpha \\ V + w'\alpha + v\beta + u'\gamma + \dots, & V, & \beta \\ W + v'\alpha + u'\beta + w\gamma + \dots, & W, & \gamma \end{vmatrix} = 0,$$

$$\text{whence} \quad \begin{vmatrix} u\alpha + w'\beta + v'\gamma, & U, & \alpha \\ w'\alpha + v\beta + u'\gamma, & V, & \beta \\ v'\alpha + u'\beta + w\gamma, & W, & \gamma \end{vmatrix} = 0 \qquad \text{(4)},$$

but ultimately α, β, γ are proportional to $\dfrac{dx}{ds}$, $\dfrac{dy}{ds}$, $\dfrac{dz}{ds}$, respectively, and the equation (4) reduces to the same as (1).

176. The radii of curvature at any point of a quadric can be obtained from the preceding formulæ. Some of the results are so simple and important that they deserve a separate consideration.

Since all parallel sections of a quadric are similar, it follows that the indicatrix at any point of such a surface is similar and similarly situated to the section of the quadric by a plane through the origin parallel to the tangent plane at the given point. Hence the tangents to the lines of curvature at any point are parallel to the axes of the section by this plane, and the umbilici are the points at which tangent planes can be drawn parallel to the planes which give circular sections.

The equation of the tangent plane at any point (α, β, γ) to an ellipsoid whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

$$\text{is} \quad \frac{x\alpha}{a^2} + \frac{y\beta}{b^2} + \frac{z\gamma}{c^2} = 1,$$

If this plane be parallel to either plane of circular section we have

$$\frac{\alpha}{a\sqrt{a^2-b^2}} = \frac{\beta}{0} = \frac{\gamma}{\pm c\sqrt{b^2-c^2}}, \text{ by Art. 65,}$$

and since (α, β, γ) is a point on the ellipsoid, each of these ratios $= \pm \dfrac{1}{\sqrt{a^2-c^2}}$.

Hence the ellipsoid has four umbilici whose co-ordinates are given by

$$\alpha = \pm a\sqrt{\frac{a^2-b^2}{a^2-c^2}}, \quad \beta = 0, \quad \gamma = \pm c\sqrt{\frac{b^2-c^2}{a^2-c^2}}.$$

177. If a tangent line be drawn to a surface of the second degree at the extremity of the axis of any plane section of that surface and lying in the cutting plane, the axis of the section and this tangent line are at right angles. This tangent line to the quadric is therefore also a tangent line to a sphere described with the origin as centre, and the length of the semi-axis of the section as radius.

Let the equation of an ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ .... (1),}$$

and let a sphere be described with the origin as centre and any radius k. The equation of this sphere is

$$x^2 + y^2 + z^2 = k^2 \text{........................(2).}$$

The equation of the cone formed by straight lines joining the origin with all the points of intersection of (1) and (2) is therefore

$$x^2\left(\frac{1}{a^2} - \frac{1}{k^2}\right) + y^2\left(\frac{1}{b^2} - \frac{1}{k^2}\right) + z^2\left(\frac{1}{c^2} - \frac{1}{k^2}\right) = 0 \text{...(3).}$$

For this equation does represent a cone whose vertex is the origin and being satisfied by all values of x, y, z which satisfy both (1) and (2) represents some surface passing through their intersection.

Now every plane which passes through the origin and any tangent line to the curve of intersection of (1) and (2) is evidently a tangent plane to the cone (3). Hence if we draw a tangent plane to (3) along any generating line OP_1, OP_1 is one axis of the section of (1) made by this plane. Let OR be the other axis and Q be the point of (1) at which a tangent plane can be drawn to (1) parallel to this section, then OQ is conjugate to the cutting plane and OP_1 is conjugate to the plane through OQ and OR.

The tangent to one line of curvature at Q is parallel to OR, and consequently lies in the plane QOR which is diametral to OP_1.

Let OP, OP_1, OP_2 be three consecutive generating lines of the cone (3); OQ, OQ_1 the lines conjugate to the planes POP_1, P_1OP_2 which are ultimately consecutive tangent planes to the cone (3). Then since OP_1 lies in a plane which is diametral to OQ, and also in a plane diametral to OQ_1, the plane QOQ_1 is diametral to OP_1 and therefore coincides with QOR, and the line joining QQ_1 is ultimately parallel to OR and therefore is the tangent line to one line of curvature which passes through Q. Hence one line of curvature through the point Q is the locus of the points at which tangent planes can be drawn to (1) parallel to the tangent planes to (3).

Hence if Q be any point on an ellipsoid, and r, k the semi-axes of the central section which is parallel to the tangent plane at Q, the axis k is constant for all points on the line of curvature whose tangent at Q is parallel to r. But if p be the perpendicular on the tangent plane at Q,

$$prk = abc \quad \text{Art. 75, equation (3),}$$

and therefore $$pr = \frac{abc}{k} = \text{constant.}$$

178. The equation of any tangent plane to (3) is

$$lx' + my' + nz' = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

where l, m, n are connected by the relation

$$\frac{l^2}{\frac{1}{a^2} - \frac{1}{k^2}} + \frac{m^2}{\frac{1}{b^2} - \frac{1}{k^2}} + \frac{n^2}{\frac{1}{c^2} - \frac{1}{k^2}} = 0 \ldots\ldots\ldots\ldots (5);$$

(See Chapter VIII. Ex. 24.)

nd the equation of a tangent plane to (1) at the point (x, y, z) s

$$\frac{x'x}{a^2}+\frac{y'y}{b^2}+\frac{z'z}{c^2}=1 \qquad (6).$$

Hence if (6) be parallel to (4)

$$\frac{x}{a^2l}=\frac{y}{b^2m}=\frac{z}{c^2n},$$

r from (5)

$$\frac{x^2}{a^2-\frac{a^4}{k^2}}+\frac{y^2}{b^2-\frac{b^4}{k^2}}+\frac{z^2}{c^2-\frac{c^4}{k^2}}=0,$$

nd subtracting this from the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

ve get

$$\frac{x^2}{a^2-k^2}+\frac{y^2}{b^2-k^2}+\frac{z^2}{c^2-k^2}=1,$$

which shews that the lines of curvature on an ellipsoid are its curves of intersection with confocal surfaces.

179. In the ellipsoid

$$U=\frac{2x}{a^2}, \qquad V=\frac{2y}{b^2}, \qquad W=\frac{2z}{c^2},$$

$$u=\frac{2}{a^2}, \quad v=\frac{2}{b^2}, \quad w=\frac{2}{c^2}, \quad u'=0, \quad v'=0, \quad w'=0.$$

Hence the differential equation of the lines of curvature is

$$\begin{vmatrix} \frac{1}{a^2}\frac{dx}{ds}, & \frac{x}{a^2}, & \frac{dx}{ds} \\ \frac{1}{b^2}\frac{dy}{ds}, & \frac{y}{b^2}, & \frac{dy}{ds} \\ \frac{1}{c^2}\frac{dz}{ds}, & \frac{z}{c^2}, & \frac{dz}{ds} \end{vmatrix}=0,$$

or $x\frac{dy}{ds}\frac{dz}{ds}(b^2-c^2)+y\frac{dz}{ds}\frac{dx}{ds}(c^2-a^2)+z\frac{dx}{ds}\frac{dy}{ds}(a^2-b^2)=0 \ldots (1).$

180. Taking the equation

$$\frac{x^2}{a^2+k}+\frac{y^2}{b^2+k}+\frac{z^2}{c^2+k}=1 \quad\text{..........(1)},$$

we have at the points where it meets the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad\text{..........(2)},$$

by subtraction

$$\frac{x^2}{a^2(a^2+k)}+\frac{y^2}{b^2(b^2+k)}+\frac{z^2}{c^2(c^2+k)}=0 \quad\text{..........(3)}.$$

Also by differentiating (1) and (3) we obtain

$$\frac{x\frac{dx}{ds}}{a^2+k}+\frac{y\frac{dy}{ds}}{b^2+k}+\frac{z\frac{dz}{ds}}{c^2+k}=0 \quad\text{..........(4)},$$

$$\frac{x\frac{dx}{ds}}{a^2(a^2+k)}+\frac{y\frac{dy}{ds}}{b^2(b^2+k)}+\frac{z\frac{dz}{ds}}{c^2(c^2+k)}=0 \quad\text{..........(5)}.$$

And from (3), (4) and (5) eliminating $\frac{1}{a^2+k}, \frac{1}{b^2+k}, \frac{1}{c^2+k}$, we obtain

$$\begin{vmatrix} \frac{x^2}{a^2}, & \frac{y^2}{b^2}, & \frac{z^2}{c^2} \\ x\frac{dx}{ds}, & y\frac{dy}{ds}, & z\frac{dz}{ds} \\ \frac{x}{a^2}\frac{dx}{ds}, & \frac{y}{b^2}\frac{dy}{ds}, & \frac{z}{c^2}\frac{dz}{ds} \end{vmatrix}=0 \quad\text{..........(6)},$$

which is the same as equation (1) of the last Article.

Thus we obtain an independent proof of the fact that the lines of curvature on an ellipsoid coincide with its curves of intersection with a series of confocal quadrics.

181. If we denote by l, m, n the direction-cosines of the tangent to either line of curvature at the point (x, y, z) on the ellipsoid they must satisfy the equations

$$\frac{x}{a^2} + \lambda l + \mu \frac{l}{a^2} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{y}{b^2} + \lambda m + \mu \frac{m}{b^2} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\frac{z}{c^2} + \lambda n + \mu \frac{n}{c^2} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

which are obtained from the equations (3), (4) and (5) of the last Article by the use of undetermined multipliers λ and μ.

But if r be the central radius vector of the ellipsoid parallel to the tangent line considered, and p the perpendicular from the centre on the tangent plane to the ellipsoid at the point (x, y, z), we have

$$\frac{1}{r^2} = \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Also from the equation of the ellipsoid, by differentiation

$$0 = \frac{lx}{a^2} + \frac{my}{b^2} + \frac{nz}{c^2} \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Differentiating (6), we have by means of (4)

$$\frac{1}{r^2} + \frac{x\dfrac{dl}{ds}}{a^2} + \frac{y\dfrac{dm}{ds}}{b^2} + \frac{z\dfrac{dn}{ds}}{c^2} = 0 \ldots\ldots\ldots\ldots(7).$$

Multiplying (1) by $\frac{x}{a^2}$, (2) by $\frac{y}{b^2}$ and (3) by $\frac{z}{c^2}$ and adding we have

$$\frac{1}{p^2} + \mu \cdot \left(\frac{lx}{a^4} + \frac{my}{b^4} + \frac{nz}{b^4}\right) = 0,$$

or using the result obtained by differentiating (5),

$$\frac{1}{p^2} - \frac{\mu}{p^3}\frac{dp}{ds} = 0.$$

Again, multiplying (1) by $\frac{dl}{ds}$, (2) by $\frac{dm}{ds}$, (3) by $\frac{dn}{ds}$ and adding, we have by (7) and (4) since also $l^2 + m^2 + n^2 = 1$,

$$-\frac{1}{r^2} - \frac{\mu}{r^3}\frac{dr}{ds} = 0.$$

Thus we obtain

$$\frac{1}{p}\frac{dp}{ds} = -\frac{1}{r}\frac{dr}{ds},$$

or

$$r\frac{dp}{ds} + p\frac{dr}{ds} = 0,$$

$$\therefore\ pr = \text{constant}.$$

182. A few propositions must be added concerning a class of lines of great importance, namely geodesic lines. These may be defined as follows :

A geodesic line on a surface is such that every small element PQ is the shortest line that can be drawn on the surface between P and Q.

The general equation of geodesic lines on a surface

$$F(x, y, z) = 0,$$

can be obtained by the help of Meunier's Theorem.

For if PQ be two points on a geodesic line, so near to one another that the arc between them may be considered as a plane curve, the length of PQ will be least when the curvature of the curve is least, or when the radius of curvature of the small arc PQ is greatest. But this will be the case when the section of the surface by a plane through the element PQ is a normal section. Hence the osculating plane at any point of the curve must contain the normal to the surface at that point. But the direction-cosines of that normal to the curve which lies in the osculating plane are proportional to

$$\frac{d^2x}{ds^2},\ \frac{d^2y}{ds^2},\ \frac{d^2z}{ds^2},$$

and the direction-cosines of the normal to the surface are proportional to

$$\frac{dF}{dx},\ \frac{dF}{dy},\ \frac{dF}{dz}.$$

Hence for all points in a geodesic line

$$\frac{\dfrac{d^2x}{ds^2}}{\dfrac{dF}{dx}} = \frac{\dfrac{d^2y}{ds^2}}{\dfrac{dF}{dy}} = \frac{\dfrac{d^2z}{ds^2}}{\dfrac{dF}{dz}} \quad \text{..........................(1).}$$

These equations can be also deduced by the Calculus of Variations. (Todhunter's *Int. Calc.* Art. 351.)

183. The equations of the last Article can be applied to discover the forms of the geodesic lines on any surface. In the case of developable surfaces, this object can often be more simply effected by the consideration that when the surface is developed, the geodesic must become a straight line. Thus the geodesic lines on a right circular cylinder are easily seen to be helices.

As an example in the case of a surface not developable, take the sphere

$$x^2 + y^2 + z^2 = a^2 \quad \text{........................(1).}$$

The differential equations of the geodesic lines become

$$\frac{\dfrac{d^2x}{ds^2}}{x} = \frac{\dfrac{d^2y}{ds^2}}{y} = \frac{\dfrac{d^2z}{ds^2}}{z};$$

$$\therefore z\frac{d^2y}{ds^2} - y\frac{d^2z}{ds^2} = 0;$$

$$\therefore z\frac{dy}{ds} - y\frac{dz}{ds} = \text{constant} = c_1 \quad \text{................(2).}$$

Similarly,

$$x\frac{dz}{ds} - z\frac{dx}{ds} = c_2 \quad \text{......................} \quad (3),$$

$$y\frac{dx}{ds} - x\frac{dy}{ds} = c_3 \quad \text{......................} \quad (4).$$

Multiplying (2) by x, (3) by y, (4) by z, and adding, we get

$$c_1x + c_2y + c_3z = 0 \quad \text{..................} \quad (5),$$

shewing that all geodesic lines are great circles.

184. As a second example take the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{..................} \quad (1).$$

The differential equations of the geodesic lines become

$$\frac{\frac{d^2x}{ds^2}}{\frac{x}{a^2}} = \frac{\frac{d^2y}{ds^2}}{\frac{y}{b^2}} = \frac{\frac{d^2z}{ds^2}}{\frac{z}{c^2}} \quad \text{..................} \quad (2).$$

Now let p be the perpendicular from the centre on the tangent plane to (1) at the point (x, y, z), and let r be the central radius of the ellipsoid drawn parallel to the tangent to the geodesic line at the point (x, y, z).

Then $$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4},$$

$$\frac{1}{r^2} = \frac{1}{a^2}\left(\frac{dx}{ds}\right)^2 + \frac{1}{b^2}\left(\frac{dy}{ds}\right)^2 + \frac{1}{c^2}\left(\frac{dz}{ds}\right)^2;$$

$$\therefore -\frac{1}{p^3}\frac{dp}{ds} = \frac{x}{a^4}\frac{dx}{ds} + \frac{y}{b^4}\frac{dy}{ds} + \frac{z}{c^4}\frac{dz}{ds} \quad \text{..................} \quad (3),$$

$$-\frac{1}{r^3}\frac{dr}{ds} = \frac{1}{a^2}\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{1}{b^2}\frac{dy}{ds}\frac{d^2y}{ds^2} + \frac{1}{c^2}\frac{dz}{ds}\frac{d^2z}{ds^2}$$

$$= \left(\frac{x}{a^4}\frac{dx}{ds} + \frac{y}{b^4}\frac{dy}{ds} + \frac{z}{c^4}\frac{dz}{ds}\right)k \quad \text{..............} \quad (4),$$

if k be put for each of the fractions in (2).

Now each of the fractions in (2)

$$= \frac{\frac{x}{a^2}\frac{d^2x}{ds^2} + \frac{y}{b^2}\frac{d^2y}{ds^2} + \frac{z}{c^2}\frac{d^2z}{ds^2}}{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$$

which by equation (7) of Article 181

$$= -\frac{\frac{1}{r^2}}{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}} = -\frac{p^2}{r^2}.$$

Hence from (3) and (4)

$$\frac{1}{p^3}\frac{dp}{ds} + \frac{r^2}{p^2} \cdot \frac{1}{r^3}\frac{dr}{ds} = 0,$$

$$\text{or} \quad r\frac{dp}{ds} + p\frac{dr}{ds} = 0;$$

whence

$$pr = \text{constant} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5).$$

This property is the same as that proved for lines of curvature, but the two systems of lines do not coincide.

Let ρ be the radius of absolute curvature of the geodesic at any point. Then each of the fractions in (2)

$$= \pm \sqrt{\frac{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}} = \pm \frac{\frac{1}{\rho}}{\frac{1}{p}} = \pm \frac{p}{\rho}.$$

Hence $$\pm \frac{p}{\rho} = -\frac{p^2}{r^2},$$

$$\text{or} \quad \rho = \pm \frac{r^2}{p} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6);$$

$$\therefore \rho = kr^3, \text{ where } k \text{ is some constant.}$$

185. We shall conclude this subject with the following proposition, known as Dupin's Theorem.

If there be three series of surfaces such that all the surfaces of each series intersect those of the other series at right angles, then the lines of intersection of the surfaces of different series are the lines of curvature on the surfaces.

Let O be the point of intersection of three surfaces, one of each system. Take O as origin, and the tangent planes of the three surfaces as co-ordinate planes. Let S_1, S_2, S_3 be the surfaces touched by the planes of yz, zx, xy, respectively, and let P, Q, R be points near O in the curves of intersection of S_2, S_3; S_3, S_1; S_1, S_2, respectively. Then since the surfaces S_2, S_3 cut at right angles, the normals at P to these surfaces are at right angles. Also since OP is ultimately a tangent line to both of them at P, the normals at P are both perpendicular to OP which is ultimately the axis of x. Let θ_3, θ_2 be the angles which the normals at P to S_3, S_2, respectively make with the planes of zx, xy, respectively; ϕ_1, ϕ_3 those which the normals at Q to S_1, S_3 make with the planes of xy, yz, respectively, and ψ_2, ψ_1 those which the normals at R to S_2, S_1 make with the planes of yz, zx, respectively. Let the lengths of OP, OQ, OR be α, β, γ, respectively.

Since the normal to S_2 lies in the tangent plane to S_3, the tangent of the angle which the normal to S_2 at O makes with the plane of xy is $\left(\frac{dz}{dy}\right)_3$, the suffix denoting the surface from which the differential coefficient is obtained. Hence the tangent of the angle which the normal to S_2 at P makes with the plane of xy is

$$\left(\frac{dz}{dy}\right)_3 + \alpha\frac{d}{dx}\left(\frac{dz}{dy}\right)_3 + \dots\dots$$

But $$\left(\frac{dz}{dy}\right)_3 = 0,$$

whence $$\theta_2 = \alpha\frac{d}{dx}\left(\frac{dz}{dy}\right)_3 \text{ ultimately.}$$

Similarly, $$\phi_1 = \beta \frac{d}{dy}\left(\frac{dz}{dx}\right)_3;$$

therefore $$\frac{\theta_2}{\alpha} = \frac{\phi_1}{\beta}.$$

Similarly, $$\frac{\theta_3}{\alpha} = \frac{\psi_2}{\gamma}; \quad \frac{\psi_1}{\gamma} = \frac{\phi_3}{\beta}.$$

But since the normals to S_2, S_3 at P are at right angles,

$$\theta_2 + \theta_3 = 0.$$

Similarly, $\phi_1 + \phi_3 = 0$, $\psi_2 + \psi_3 = 0$, whence $\theta_2 = 0$.

Hence the normals to S_2 at O and P both lie in the plane of xy and therefore intersect one another, and therefore OP is the tangent to the line of curvature on S_2 at O. Whence the theorem follows.

EXAMPLES. CHAPTER XIII.

1. Find the quadratic equation which gives the principal radii of curvature at any point of an ellipsoid.

Deduce the position of the umbilici.

2. Find the umbilici of the surfaces

(1) $xyz = a^3$,

(2) $$\frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}} + \frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}} + \frac{z^{\frac{2}{3}}}{c^{\frac{2}{3}}} = 1,$$

and find the value of the radius of curvature at the umbilicus in each case.

3. Find the equation of the projection of the lines of curvature of the surface $xyz = a^3$, on the plane of xy.

4. Deduce the formulæ for an umbilicus

$$\frac{1+p^2}{r}=\frac{pq}{s}=\frac{1+q^2}{t}:$$

first, from the consideration that the two principal radii of curvature are equal at an umbilicus; secondly, from the consideration that the directions of the lines of curvature at an umbilicus are indeterminate.

5. Find the condition that the two principal radii of curvature at any point of a surface may be equal in magnitude but opposite in sign.

Find the points on the surface

$$Ax^2+By^2+Cz^2=1$$

for which this is the case.

6. Shew that if the origin be at an umbilicus and the normal at that point the axis of z, the equation of an ellipsoid may be put into the form

$$x^2+y^2+kz(z-a)+byz+czx=0.$$

7. Any chord is drawn through an umbilicus of an ellipsoid, and its extremity is joined with the extremity of the normal at the umbilicus. Prove that the locus of the intersection of the joining line with the plane through the umbilicus perpendicular to the chord is a plane.

8. Prove that the lines of curvature of the surface

$$\frac{x}{a}+\frac{y^2}{ax-b^2}+\frac{z^2}{ax-c^2}=1$$

are circles, and that the plane of any one of them contains a fixed straight line lying wholly on the surface.

9. Shew that pr is constant for all lines of curvature which pass through the same umbilicus of an ellipsoid.

10. Shew that pr has the same value for all geodesic lines on an ellipsoid which touch the same line of curvature.

11. U and V are two adjacent umbilici of an ellipsoid, P is any point on the surface which is joined by geodesic arcs with U and V. Shew that the lines of curvature which pass through P bisect the interior and exterior angles between PU and PV.

12. If a point P move on an ellipsoid so that the sum or difference of the geodesic arcs PU, PV joining it with two adjacent umbilici of the ellipsoid is constant, shew that the locus of P is a line of curvature.

13. Shew that at every point of a geodesic circle round an umbilicus of an ellipsoid

$$\frac{a^2b^2c^2}{p^2d^2} = a^2 + c^2 - r^2,$$

where a, b, c are the semi-axes of the ellipsoid, r the central radius to the point, p the central perpendicular on the tangent plane, and d the semidiameter parallel to the tangent to the circle at that point.

14. The normal at each point of a principal section of an ellipsoid is intersected by the normal at a consecutive point not on the principal section; shew that the locus of the point of intersection is an ellipse having four real or imaginary contacts with the evolute of the principal section.

15. From the differential equation of geodesic lines investigate the nature of the geodesics on a right circular cylinder.

16. Find the equations of the geodesic lines on a right circular cone; first, from the differential equations, and secondly from the consideration that when the cone is developed the geodesics become straight lines.

17. Shew that the distance of any point of a geodesic traced on a surface of revolution from the axis varies inversely as the sine of the angle between the geodesic and the meridian of the surface which passes through that point.

18. Find expressions for the principal radii of curvature at any point of a surface of revolution round the axis of x.

19. Prove that the product of the principal radii of curvature at any point of a prolate spheroid varies as the product of the squares of the distances of the point from the foci of the generating ellipse.

20. Shew that the locus of the focus of an ellipse rolling along a straight line is a curve such that if it revolve about that line, the sum of the curvatures of any two normal sections at right angles is the same at every point of the surface generated.

21. If two surfaces cut each other at right angles, and R be the radius of curvature of the curve of intersection at any point, ρ_1, ρ_2 the radii of curvature of the normal sections of the two surfaces through the tangent line to the curve at that point, prove that

$$\frac{1}{R^2} = \frac{1}{\rho_1^2} + \frac{1}{\rho_2^2}.$$

22. If r, r' be the principal radii of curvature at any point of an ellipsoid on the line of intersection with a concentric sphere, shew that the expression $\dfrac{(rr')^{\frac{1}{4}}}{r+r'}$ is invariable.

23. If a geodesic line be drawn on a developable surface and cut any generating line of the surface at any angle ψ and at a distance t from the edge of regression measured along the generator, prove that

$$\frac{dt}{d\psi} + t\cot\psi = \rho,$$

where ρ is the radius of curvature of the edge of regression at the point where the generator touches it.

24. Prove that if r be the distance of any point of a geodesic from the origin, ρ the radius of absolute curvature, and p the perpendicular from the origin on the tangent plane to the surface,

$$p + \rho = \tfrac{1}{2}\rho \cdot \frac{d^2(r^2)}{ds^2}.$$

25. The centres of curvature of plane sections of a surface at any point lie on the surface

$$(x^2 + y^2 + z^2)\left(\frac{x^2}{\rho_1} + \frac{y^2}{\rho_2}\right) = z\,(x^2 + y^2),$$

the axes being the tangents to the lines of curvature at that point and the normal, and ρ_1, ρ_2 being the principal radii of curvature.

If these sections touch a right cone of semi-vertical angle α, about the axis of z, the centres lie on the elliptic paraboloid

$$\frac{x^2}{\rho_1} + \frac{y^2}{\rho_2} = z \sin^2 \alpha.$$

ANSWERS TO THE EXAMPLES.

CHAPTER I.

1. $\sqrt{3}$, $2\sqrt{3}$ and $\sqrt{3}$.

2. The length of each side is $\sqrt{6}$.

3. $\dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}$.   4. $\dfrac{2}{7}, \dfrac{3}{7}, \dfrac{6}{7}; \dfrac{26}{7\sqrt{14}}$.

5. $90°$.   6. $\dfrac{a}{2}, \dfrac{b}{2}, 0;\ 0, \dfrac{b}{2}, \dfrac{c}{2};\ \dfrac{a}{2}, 0, \dfrac{c}{2}$.

7. $\dfrac{a}{3}, \dfrac{b}{3}, \dfrac{c}{3}$;

$$\frac{1}{3}\sqrt{b^2+c^2+\frac{a^2}{4}},\quad \frac{1}{3}\sqrt{c^2+a^2+\frac{b^2}{4}},\quad \frac{1}{3}\sqrt{a^2+b^2+\frac{c^2}{4}}.$$

8. $DE^2 = \left(\dfrac{a}{2}-0\right)^2 + \left(0-\dfrac{b}{2}\right)^2 + \left(\dfrac{c}{2}-\dfrac{c}{2}\right)^2 = \dfrac{a^2+b^2}{4} = \tfrac{1}{4}AB^2.$

9. If $r_1, \theta_1, \phi_1, r_2, \theta_2, \phi_2$ be the polar co-ordinates of the points, the $(\text{dist.})^2$ between them by Arts. (6) and (15)

$$\begin{aligned}
&= (r_1 \sin\theta_1 \cos\phi_1 - r_2 \sin\theta_2 \cos\phi_2)^2 \\
&\qquad + (r_1 \sin\theta_1 \sin\phi_1 - r_2 \sin\theta_2 \sin\phi_2)^2 + (r_1\cos\theta_1 - r_2\cos\theta_2)^2 \\
&= r_1^2(\sin^2\theta_1\cos^2\phi_1 + \sin^2\theta_1\sin^2\phi_1 + \cos^2\theta_1) \\
&\qquad + r_2^2(\sin^2\theta_2\cos^2\phi_2 + \sin^2\theta_2\sin^2\phi_2 + \cos^2\theta_2) \\
&\qquad - 2r_1r_2\{\sin\theta_1\sin\theta_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2) \\
&\qquad\qquad + \cos\theta_1\cos\theta_2\} \\
&= r_1^2 + r_2^2 - 2r_1r_2\{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)\}.
\end{aligned}$$

10. $r=4, \quad \theta = \dfrac{\pi}{6}, \quad \phi = \dfrac{\pi}{6}$.

11. $x=1, \quad y=\sqrt{3}, \quad z = 2\sqrt{3}$.

CHAPTER II.

1. $\frac{x-1}{\sqrt{3}} = y - 2 = \frac{z-3}{2\sqrt{3}}$. 2. $x+z=4, \quad y=2$.

3. $\left.\begin{array}{r} x+z=4 \\ y=2 \end{array}\right\} \quad \left.\begin{array}{r} x+y=5 \\ z=1 \end{array}\right\} \quad \left. x-1=y-2=-\frac{z-3}{2} \right\}; \ \frac{\pi}{6}, \frac{\pi}{2}, \frac{\pi}{3}$.

4. $x+y+z=6$; $2\sqrt{3}$. 5. $x = \frac{y}{2} = \frac{z}{3}$.

6. $x - 1 = -\frac{y-2}{3\sqrt{3}} = z - 3$.

7. Let (α, β, γ); $(\alpha', \beta', \gamma')$, be the two points, $lx + my + nz = p$ the given plane. Then the required plane can have its equation in the form

$$A(x-\alpha) + B(y-\beta) + C(z-\gamma) = 0,$$

and A, B, C must satisfy the two conditions

$$A(\alpha'-\alpha) + B(\beta'-\beta) + C(\gamma'-\gamma) = 0, \quad Al + Bm + Cn = 0,$$

whence

$$A : B : C :: m(\gamma'-\gamma) - n(\beta'-\beta) : n(\alpha'-\alpha) - l(\gamma'-\gamma) : l(\beta'-\beta) - m(\alpha'-\alpha).$$

8. $z = 3, \quad x + y = 3$.

9. Let $A(x-2) + B(y-3) + C(z-4) = 0$ represent the plane required;

$$\therefore A(1-2) + B(2-3) + C(3-4) = 0, \quad \text{or } A + B + C = 0,$$

$$A . \sqrt{3} + B + C . 2\sqrt{3} = 0,$$

whence $A : B : C :: 2\sqrt{3} - 1 : -\sqrt{3} : 1 - \sqrt{3}$,

and the plane becomes

$$(2\sqrt{3}-1)(x-2) - \sqrt{3}(y-3) + (1-\sqrt{3})(z-4) = 0.$$

10. Let l, m, n; l', m', n' be the direction-cosines of the given lines; λ, μ, ν those of the required one;

$$\therefore \lambda l + \mu m + \nu n = 0, \quad \lambda l' + \mu m' + \nu n' = \cos\alpha.$$

The latter equation gives

$$(\lambda l + \mu m + \nu n)^2 = \cos^2\alpha\,(\lambda^2 + \mu^2 + \nu^2),$$

which combined with the former will give two values of the ratios $\lambda : \mu : \nu$, as in Art. 57. For the latter part put $\cos\alpha = \frac{1}{\sqrt{2}}$ and find the value of $\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2$; remembering that $ll' + mm' + nn'$ vanishes this will also be found to vanish.

11. Let (α, β, γ) be the given point, l, m, n; l', m', n' the direction-cosines of the perpendiculars on the two planes. The required plane is

$$(mn' - m'n)(x - \alpha) + (nl' - n'l)(y - \beta) + (lm' - l'm)(z - \gamma) = 0.$$

(See Art. 30.)

12. The proof of Art. 19 holds when the axes are not rectangular if l, m, n mean the cosines of the angles between OD and the axes.

13. Draw the oblique co-ordinates of the point D, and project OD on the axes in succession.

14. $$\frac{l + m\cos\nu + n\cos\mu}{A} = \frac{m + n\cos\lambda + l\cos\nu}{B}$$
$$= \frac{n + l\cos\mu + m\cos\lambda}{C}.$$

15. The condition is

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = (x - \alpha')^2 + (y - \beta')^2 + (z - \gamma')^2,$$

which reduces to

$$(\alpha' - \alpha)\left\{x - \frac{\alpha + \alpha'}{2}\right\} + (\beta' - \beta)\left\{y - \frac{\beta + \beta'}{2}\right\} + (\gamma' - \gamma)\left\{z - \frac{\gamma + \gamma'}{2}\right\} = 0.$$

16. (1) A series of planes parallel to that of yz; for $f(x) = 0$ gives a series of equations $x = a_1$, $x = a_2$, &c. (2) A series of spheres with the origin as centre. (3) A series of right circular cones with Oz as axis. (4) A series of planes passing through the line Oz.

17. (1) The axis of z. (2) A straight line OP through O inclined at an angle α to Oz, and such that the plane zOP makes an angle β with zOx. (3) A circle whose radius is a in the plane of zx and with its centre at the origin.

18. $A\cos\phi\sin\theta + B\sin\phi\sin\theta + C\cos\theta = \frac{D}{r}$. 19. $\frac{\pi}{2}$.

20. Let l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3; be the direction-cosines of the normals to the three planes. Then the equation of any plane through the line of intersection of the first and second is

$$(l_1 + \lambda l_2) x + (m_1 + \lambda m_2) y + (n_1 + \lambda n_2) z = p_1 + \lambda p_2,$$

where λ is a constant, and if this is perpendicular to the third,

$$l_3 (l_1 + \lambda l_2) + m_3 (m_1 + \lambda m_2) + n_3 (n_1 + \lambda n_2) = 0,$$

or
$$\cos B + \lambda \cos A = 0.$$

Also if the plane passes through the origin $p_1 + \lambda p_2 = 0$;

$$\therefore\ p_1 \cos A = p_2 \cos B,$$

and the plane becomes

$$(l_1 x + m_1 y + n_1 z) \cos A - (l_2 x + m_2 y + n_2 z) \cos B = 0.$$

If in addition $p_2 \cos B = p_3 \cos C$, the other two planes will have equations of a similar form and all three planes will intersect in one straight line through the origin.

21. Let $lx + my + nz = p$ be the equation of one of the planes;

$\therefore$ from the data $\dfrac{p}{l} + \dfrac{p}{m} + \dfrac{p}{n} = 0$, or $\dfrac{1}{l} + \dfrac{1}{m} + \dfrac{1}{n} = 0$ ....(1),

and $l(a' - a) + m(b' - b) + n(c' - c) = 0$............(2);

$\therefore$ substituting for n out of the second in the first

$$\frac{1}{l} + \frac{1}{m} - \frac{c' - c}{l(a' - a) + m(b' - b)} = 0;$$

$$\therefore\ l^2 (a' - a) + Plm + m^2 (b' - b) = 0,$$

which gives *two* values of $\dfrac{l}{m}$, and corresponding to each of these from (2) we can get one value of $\dfrac{n}{m}$. If $\dfrac{l_1}{m_1}$, $\dfrac{l_2}{m_2}$ be these two values, $\dfrac{l_1 l_2}{m_1 m_2} = \dfrac{b' - b}{a' - a}$. Similarly $\dfrac{n_1 n_2}{m_1 m_2} = \dfrac{b' - b}{c' - c}$.

Hence if the lines be at right angles

$$1 + \frac{l_1 l_2}{m_1 m_2} + \frac{n_1 n_2}{m_1 m_2} = 0;$$

$$\therefore\ 1 + \frac{b' - b}{a' - a} + \frac{b' - b}{c' - c} = 0; \quad \therefore\ \frac{1}{a' - a} + \frac{1}{b' - b} + \frac{1}{c' - c} = 0.$$

22. $\frac{\pi}{2}$. This and Example 23 are to be solved as the last example.

23. $P(B^2+C^2)+Q(C^2+A^2)+R(A^2+B^2)=0$.

24. The co-ordinates of the middle points of the lines joining 1, 2 and 3, 4 are, Art. (7),

$$\tfrac{1}{2}(a-c),\quad \tfrac{1}{2}(a+b-c-d),\quad \tfrac{1}{2}(b-d),$$

and

$$\tfrac{1}{2}(c-a),\quad \tfrac{1}{2}(c+d-a-b),\quad \tfrac{1}{2}(d-b),$$

whence the result follows.

25. The co-ordinates of any point on one of the lines may be represented by $a+lt$, $b+mt$, $c+nt$; and those of any point on the other by $a'+l't'$, $b'+m't'$, $c'+n't'$. The square of the distance between these points is

$$(a-a'+lt-l't')^2+(b-b'+mt-m't')^2+(c-c'+nt-n't')^2.$$

The conditions that this may be a minimum by the variation of t and t' are

$$(a-a'+lt-l't')\,l+(b-b'+mt-m't')\,m+(c-c'+nt-n't')\,n=0,$$

and

$$(a-a'+lt-l't')\,l'+(b-b'+mt-m't')\,m'+(c-c'+nt-n't')\,n'=0,$$

which shew that the line joining the two points is perpendicular to both the given lines.

26. By the solution of the last question,

$$l(\alpha-\alpha')+m(\beta-\beta')+n(\gamma-\gamma')+t-t'\cos\theta=0,$$

$$l'(\alpha-\alpha')+m'(\beta-\beta')+n'(\gamma-\gamma')+t\cos\theta-t'=0,$$

whence

$$t'\sin^2\theta=u'+u\cos\theta.$$

27. Taking x_1, y_1, z_1, &c. as the co-ordinates of the angles of the tetrahedron it is easily shewn that the co-ordinates of the middle point of the line joining the middle points of two opposite edges are

$$\tfrac{1}{4}(x_1+x_2+x_3+x_4),\ \&c.$$

28. By the help of a figure and the last question it is easily seen that the two lines x, y are the diagonals of a parallelogram whose sides are $\frac{1}{2}a$ and $\frac{1}{2}a'$ and ω is the angle between the diagonals, whence by Trigonometry the result follows.

29. $\dfrac{c}{\sqrt{2}}$ if c is the edge of the cube.

30. $\frac{1}{2}\sqrt{b^2c^2 + c^2a^2 + a^2b^2}$.

31. The equations of the planes are

$$lx + my + nz = p, \quad mx + ny + lz = p, \quad nx + ly + mz = p;$$

$$\therefore\ x = y = z = \frac{p}{l + m + n}.$$

32. Any point on the given line can have its co-ordinates expressed by $a - lt$, $b - mt$, $c - nt$; the value of t is obtained from the condition of perpendicularity.

33. Take the shortest distance between the lines as axis of z, its middle point as origin, and the plane of zx to bisect the angle between the lines.

CHAPTER III.

1. $r^2 + r(A \sin\theta \cos\phi + B \sin\theta \sin\phi + C \cos\theta) + D = 0$.
This equation gives two values of r the product of which is D.

2. The polar equation of any plane is

$$A \sin\theta \cos\phi + B \sin\theta \sin\phi + C \cos\theta = \frac{D}{r}.$$

Hence if this be the equation of the locus of P, since $OP = \dfrac{k^2}{OQ}$, the equation of the locus of Q is

$$A \sin\theta \cos\phi + B \sin\theta \sin\phi + C \cos\theta = \frac{Dr}{k^2},$$

which is the polar equation of a sphere.

3. If the locus of P be

$$r^2 + r(A \sin\theta \cos\phi + B \sin\theta \sin\phi + C \cos\theta) + D = 0,$$

that of Q is

$$k^4 + k^2 r(A \sin\theta \cos\phi + B \sin\theta \sin\phi + C \cos\theta) + Dr^2 = 0,$$

which is another sphere.

4. The plane in question is

$$xx' + yy' + zz' = x'^2 + y'^2 + z'^2 = c^2,$$

also where it meets the sphere $x^2 + y^2 + z^2 = c^2$,

whence $x^2 + y^2 + z^2 - 2(xx' + yy' + zz') + x'^2 + y'^2 + z'^2 = 0$,

or

$$(x - x')^2 + (y - y')^2 + (z - z')^2 = 0;$$

$$\therefore\ x = x', \quad y = y', \quad z = z'.$$

5. Take A as origin and AB $(=a)$ as axis of x, the equation of the locus is

$$x^2 + y^2 + z^2 = m^2\{(x - a)^2 + y^2 + z^2\},$$

which reduces to the equation of a sphere.

6. With the same axes as in the last question the two lines whose direction-cosines are proportional to x, y, z and $x - a$, y, z must be at right angles. Hence $x(x - a) + y^2 + z^2 = 0$, a sphere, on AB as diameter.

7. Take for the equations of the fixed straight lines those given in Ex. 33, Chap. II. The equations of the two planes can then be written $y - mx + \lambda(z - c) = 0$ and $y + mx + \mu(z + c) = 0$ where λ and μ are constants. The condition of perpendicularity gives $1 - m^2 + \lambda\mu = 0$ and by substituting for λ and μ out of the first two in the third we get $(1 - m^2)(z^2 - c^2) + y^2 - m^2x^2 = 0$ as the locus.

8. If $S = 0$, $S' = 0$ be the equations of two spheres in their simplest form, the equation $S' - S = 0$ is easily seen to be a plane perpendicular to the line joining their centres, which must cut each sphere in a circle.

9. The equations of the spheres can be written

$$S - kr^2 = 0, \quad S' - k'r^2 = 0, \quad S'' - k''r^2 = 0,$$

where k, k', k'' are constants and r changes. The first and second intersect on the sphere $\dfrac{S}{k} - \dfrac{S'}{k'} = 0$, whence the rest will follow.

10 and 11. These follow easily from (8).

12. The six centres of the spheres must lie at the angular points of a regular octahedron the edge of which is the radius.

13. Take the three planes as co-ordinate planes, and let l, m, n be the direction-cosines of the straight line, x, y, z the co-ordinates of the point. Then by projecting on the axes in succession $x = la$, $y = mb$, $z = nc$;

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

14. We have

$$\frac{x' + a}{0} = \frac{y'}{\cos\alpha} = \frac{z'}{\sin\alpha} = OP,$$

and
$$\frac{x'' - a}{0} = \frac{-y''}{\cos\alpha} = \frac{z''}{\sin\alpha} = O'P',$$

if x', y', z' are the co-ordinates of P and x'', y'', z'' those of P'. Also if (x, y, z) be any point in PP'

$$\frac{x - x'}{x' - x''} = \frac{y - y'}{y' - y''} = \frac{z - z'}{z' - z''},$$

between these equations and $OP \,.\, O'P' = c^2$ we have to eliminate x', y', z', x'', y'', z'', OP, $O'P'$.

15. Take the line round which the line revolves as axis of z and the point where the shortest distance between the lines meets it as origin, and let c be the length of this shortest distance and θ the angle it makes with Ox. Let also α be the angle between the fixed and revolving lines. The equations of the revolving line are

$$\frac{x - c\cos\theta}{l} = \frac{y - c\sin\theta}{m} = \frac{z}{\cos\alpha}, \text{ where } l^2 + m^2 = \sin^2\alpha,$$

and since this is perpendicular to the line $\dfrac{x}{c\cos\theta} = \dfrac{y}{c\sin\theta} = \dfrac{z}{0}$, we have $l\cos\theta + m\sin\theta = 0$, whence eliminating l, m and θ we get $x^2 + y^2 = c^2 + \dfrac{z^2}{\cos^2\alpha}$.

16. $n^2x^2 + (n^2 - 1)(y^2 + z^2) = c^2$.

17. Let $y^2 + z^2 = x^2\tan^2\alpha$ be the right cone. Then the equation required is

$$x^2(y^2 + z^2) = k^4\tan^2\alpha.$$

CHAPTER IV.

1. $2\sqrt{3}$, 0, $-\sqrt{2}$. 2. $x'^2 - \dfrac{y'^2 + z'^2}{2}$.

3. Take $x = y = z$ and any two straight lines perpendicular to it as axes: the axes in the last question will do.

4. From the last two of the second set of relations the *ratios* of l_1, l_2, l_3 can be deduced, and their absolute values from the first, with the help of the other three.

5. 1, 1, 5, use Art. 51.

6. The proof is exactly similar to Art. 50 with the exception that

$$x^2 + y^2 + z^2 + 2yz \cos\lambda + 2zx \cos\mu + 2xy \cos\nu$$

is transformed into $x'^2 + y'^2 + z'^2$.

7. Transform so as to take the line $x = y = z$ as axis of x and any two lines perpendicular to it and each other as axes of y and z: as in Examples 1 and 2.

CHAPTER V.

1. The direction-cosines of the generating lines through any point (α, β, γ) are given by

$$\frac{l^2}{a^2} + \frac{m^2}{a^2} - \frac{n^2}{c^2} = 0, \qquad \frac{l\alpha}{a^2} + \frac{m\beta}{a^2} - \frac{n\gamma}{c^2} = 0.$$

The condition that these shall be at right angles is obtained as in Examples 21—23, Chapter II., and gives by the help of the relation $\dfrac{\alpha^2 + \beta^2}{a^2} - \dfrac{\gamma^2}{c^2} = 1$, a value of γ^2.

2 and 3. The direction-cosines of the generating lines are given by

$$\frac{l\alpha}{a^2} + \frac{m\beta}{b^2} - \frac{n\gamma}{c^2} = 0, \text{ and } \frac{l^2}{a^2} + \frac{m^2}{b^2} - \frac{n^2}{c^2} = 0.$$

From these we easily get, eliminating m,

$$\frac{l^2}{a^2}\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}\right) - \frac{2ln\alpha\gamma}{a^2c^2} + \frac{n^2}{c^2}\left(\frac{\gamma^2}{c^2} - \frac{\beta^2}{a^2}\right) = 0.$$

Whence

$$\frac{l_1 l_2}{n_1 n_2} = \frac{\frac{1}{c^2}\left(\frac{\gamma^2}{c^2} - \frac{\beta^2}{b^2}\right)}{\frac{1}{a^2}\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}\right)} = \frac{\frac{1}{c^2}\left(\frac{\alpha^2}{a^2} - 1\right)}{\frac{1}{a^2}\left(1 + \frac{\gamma^2}{c^2}\right)} = \frac{\alpha^2 - a^2}{\gamma^2 + c^2},$$

$$\frac{l_1}{n_1} + \frac{l_2}{n_2} = \frac{\frac{2\alpha\gamma}{a^2 c^2}}{\frac{1}{a^2}\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}\right)} = \frac{2\alpha\gamma}{\gamma^2 + c^2}.$$

Whence by symmetry we get

$$\frac{l_1 l_2}{\alpha^2 - a^2} = \frac{m_1 m_2}{\beta^2 - b^2} = \frac{n_1 n_2}{\gamma^2 + c^2} = \frac{m_1 n_2 + m_2 n_1}{2\beta\gamma} = \frac{n_1 l_2 + n_2 l_1}{2\gamma\alpha} = \frac{l_1 m_2 + l_2 m_1}{2\alpha\beta},$$

and if θ be the angle between the two straight lines, each of these ratios

$$= \frac{\cos\theta}{(\alpha^2 - a^2) + (\beta^2 - b^2) + (\gamma^2 + c^2)}$$

$$= \frac{\sin\theta}{\sqrt{4\{\beta^2\gamma^2 - (\beta^2 - b^2)(\gamma^2 + c^2)\} + \ldots \text{ similar terms}}};$$

$$\therefore \cot\theta = \frac{\alpha^2 + \beta^2 + \gamma^2 - a^2 - b^2 + c^2}{2\sqrt{\gamma^2(a^2 + b^2) + \beta^2(a^2 - c^2) + \alpha^2(b^2 - c^2) + b^2c^2 + a^2c^2 - a^2b^2}}.$$

The solution of (2) easily follows by putting $b = a$, and $\theta = \alpha$.

4. If $l, m, n; l', m', n'$ be the direction-cosines of the two radii vectores, these with $\frac{c\sqrt{a^2 - b^2}}{b\sqrt{a^2 - c^2}}$, 0, $\frac{a\sqrt{b^2 - c^2}}{b\sqrt{a^2 - c^2}}$ form a set of nine quantities satisfying the conditions of Art. 44. Also if r, r' be the two radii

$$r^2 = a^2 l^2 + b^2 m^2 + c^2 n^2, \quad r'^2 = a^2 l'^2 + b^2 m'^2 + c^2 n'^2;$$

$$\therefore r^2 + r'^2 = a^2(l^2 + l'^2) + b^2(m^2 + m'^2) + c^2(n^2 + n'^2)$$

$$= a^2\left\{1 - \frac{c^2(a^2 - b^2)}{b^2(a^2 - c^2)}\right\} + b^2 + c^2\left\{1 - \frac{a^2(b^2 - c^2)}{b^2(a^2 - c^2)}\right\}$$

$$= \frac{a^4(b^2 - c^2) + b^4(a^2 - c^2) + c^4(a^2 - b^2)}{b^2(a^2 - c^2)}.$$

5. Planes parallel to $lx+my+nz=0$ and $l'x+m'y+n'z=0$.

6. $xy+yz+zx \equiv \frac{(x+y+z)^2-x^2-y^2-z^2}{2}$, whence the result follows.

7. By Art. 59, the projections of OP, OP' on the plane of xy are tangents to the principal ellipse at the ends of conjugate diameters. The sum of the squares of these projections is therefore a^2+b^2. Also the height of O above the plane of xy can be easily shewn to be c, whence the result follows.

8 and 9. If λ, μ, ν be the direction-cosines and r the length of any radius vector in the plane $lx+my+nz=0$;

$$\frac{1}{r^2}=\frac{\mu\nu}{a^2}+\frac{\nu\lambda}{b^2}+\frac{\lambda\mu}{c^2} \ldots (1),$$

while $\lambda l+\mu m+\nu n=0 \ldots (2)$. If the section be a rectangular hyperbola two directions at right angles make $\frac{1}{r}$ vanish. By the methods of Ex. 21—23, Chap. II. the condition for this is found. The condition for a circular section is that $\frac{1}{r^2}$ shall be invariable for all values of λ, μ, ν consistent with (2); whence $\lambda l+\mu m+\nu n$ must be a factor of

$$\frac{\mu\nu}{a^2}+\frac{\nu\lambda}{b^2}+\frac{\lambda\mu}{c^2}-k(\lambda^2+\mu^2+\nu^2),$$

where k is the constant value of $\frac{1}{r^2}$. For the rest see Art. 49.

10. By Art. 63 the generating lines at any point (x, y, z) must be parallel to the asymptotes of a section by a plane through the centre parallel to that which cuts the surface in these two lines. The equation of such a plane is by Art. 58

$$\frac{\alpha x}{a}+\frac{\beta y}{b}+\frac{\gamma z}{c}=0,$$

if (α, β, γ) be the point: the semi-axes of the section by this plane are given {Art. 68 (10)} by the equation

$$\frac{\alpha^2}{a(a-r^2)}+\frac{\beta^2}{b(b-r^2)}+\frac{\gamma^2}{c(c-r)^2}=0.$$

Also if r_1^2, r_2^2 be the two values of r^2 in this equation and 2θ the angle between the asymptotes of the curve

$$\tan^2\theta = -\frac{r_1^2}{r_2^2};$$

$$\therefore \cos 2\theta = \frac{r_2^2 + r_1^2}{r_2^2 \sim r_1^2};$$

which is the required result.

11. The square of the distance of the focus of any section from the centre is the difference of the squares of the semi-axes of that section. Hence if ρ, λ, μ, ν be the radius vector and direction-cosines of any point in the locus, $\rho^2 = r_1^2 \sim r_2^2$, where r_1, r_2 are the two values of r in equation (10) of Art. 68, and λ, μ, ν are determined by equation (12) of that article; between these equations and (5) we have to eliminate l, m, n, and for $\lambda\rho$, $\mu\rho$, $\nu\rho$ to substitute x, y, z.

12. $x^2 + y^2 + z^2 - (lx + my + nz)^2$. See Art. (28).

13. $A + B + C = 0$. See Arts. 34 and 44.

14. If $lx + my + nz = 0$ ... (1) be the equation of the plane base, the co-ordinates of the vertex are given by $\frac{\alpha}{l} = \frac{\beta}{m} = \frac{\gamma}{n} = b$.

Let then $\frac{x - lb}{\lambda} = \frac{y - mb}{\mu} = \frac{z - nb}{\nu} = r$ ... (2) be the equations of the generating line; substitute for x, y, z from these equations in (1) and the equation of the ellipsoid, and eliminate r. Thus we get a relation between λ, μ, ν and then from (2) the equation of the cone as in Art. 34.

15. $x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0$.

16. $x^2 + y^2 + z^2 = (lx + my + nz)^2 \sec^2\alpha$. See Art. 28.

17. Determine l, m, n and α in the last question so as to make the cone contain the given lines.

18. Is solved in question (2).

19. Assume $\lambda x + \mu y + \nu z = 0$...(1) the plane; $\therefore \lambda l + \mu m + \nu n = 0$. Eliminate z between (1) and the given cone. We get a cubic equation in $\frac{y}{x}$ one value of which must be $\frac{m}{l}$; the product of the other values is easily obtained. See Ex. 23, Chapter II.

20. These values satisfy the equation of the hyperboloid whatever ϕ and θ may be. Substitute in the equations of Art. 57, and we shall get finally

$$\frac{x - a\cos\phi\sec\theta}{a\sin(\phi \pm \theta)} = \frac{y - b\sin\phi\sec\theta}{b\cos(\phi \pm \theta)} = \frac{z - c\tan\theta}{\pm c}.$$

21. Use the equations in the last question.

22. Any planes through the two generating lines in question may have their equations written

$$\frac{x}{a} - 1 + k\left(\frac{y}{b} - \frac{z}{c}\right) = 0, \quad \frac{x}{a} + 1 + k'\left(\frac{y}{b} + \frac{z}{c}\right) = 0.$$

The condition that the line of intersection of these should be a generating line is easily found to be $kk' = -1$.

It can be shewn that the intersections of these planes with either of the planes

$$\frac{y}{b}\sqrt{a^2 + c^2} \pm \frac{z}{c}\sqrt{a^2 - b^2} = 0$$

are always at right angles to each other. These are the planes which give circular sections.

23. Take the general homogeneous equation of the second degree in $\alpha, \beta, \gamma, \delta$. Find the conditions that this may be satisfied by either of the pairs $\alpha = 0$, $\delta = 0$, and $\beta = 0$, $\gamma = 0$.

CHAPTER VI.

1. If λ, μ, ν be the direction-cosines of any generator of the given cone $a^2\lambda^2 + b^2\mu^2 + c^2\nu^2 = d^2$, whence by Art. 79 the result follows.

2. Use equations (6) of Art. 77, and in the given case by Art. 78,

$$\frac{p^2}{a^2l^2 + b^2m^2 + c^2n^2} = k^2,$$

and the locus becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = k^2.$$

3. Use formulæ of Art. 74 to shew that the plane passes through the three given points.

4. $\dfrac{2\pi abc}{3p\sqrt{3}}$, where p is the perpendicular from the origin on the plane LMN.

5. volume $= \dfrac{\pi abc}{3\sqrt{3}}$.

6. A cylinder whose axis is parallel to Oz and whose trace on the plane of xy is given by

$$\frac{ab}{r} = \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta} \left\{1 - \frac{c^2 (a^2 \sin^2\theta + b^2 \cos^2\theta)}{a^2b^2}\right\}.$$

7. Let α, β be the co-ordinates of the point where the straight line cuts the plane of xy, and let a line be drawn inclined at an angle θ to Ox to cut the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ in two points. If r_1, r_2 be the distances of these two points from α, β, the square of the eccentricity of the vertical section through a straight line $x = \alpha$, $y = \beta$ supposed to be its directrix must $= \dfrac{r_1 \sim r_2}{r_1 + r_2}$, but it also equals $1 - \dfrac{c^2 (a^2 \sin^2\theta + b^2 \cos^2\theta)}{a^2b^2}$ by Art. 63, whence since r_1 and r_2 are expressed in terms of θ we can get a quadratic equation in $\tan^2\theta$ the roots of which must be real.

8. Use Art. 77, p being a constant:

$$p^2 \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)^2.$$

9. If a', b', c' be the conjugate semi-diameters, and x', y', z' the co-ordinates of the point in which the three planes meet

$$\frac{X}{A} = 1 - \frac{x'^2}{a'^2},$$

by similar triangles and Art. 79.

10. We have to find the directions of the axes of the section of $Ax^2 + By^2 + Cz^2 = 1$ by the plane $Alx + Bmy + Cnz = 0$, where $Pl^2 + Qm^2 + Rn^2 = 0$. See Art. 68, Equations 5 and 12 and eliminate l, m, n.

11. (1) $\alpha = \frac{p}{l} + \frac{m}{2l^2 B} + \frac{n}{2l^2 C}$, $\beta = -\frac{m}{2lB}$, $\gamma = -\frac{n}{2lC}$.

(2) $2B\beta(b-\beta) + 2C\gamma(c-\gamma) = a - \alpha$ if a, b, c be co-ordinates of the fixed point.

12. If x, y, z be the co-ordinates of any point on the perpendicular,

$$\frac{\dfrac{ax}{x_1+x_2+x_3}}{a} = \frac{\dfrac{by}{y_1+y_2+y_3}}{b} = \frac{\dfrac{cz}{z_1+z_2+z_3}}{c} = \frac{\sqrt{a^2x^2+b^2y^2+c^2z^2}}{\sqrt{3}}$$

by Art. 74,

$$= \frac{x}{\dfrac{x_1+x_2+x_3}{a^2}} = \frac{y}{\dfrac{y_1+y_2+y_3}{b^2}} = \frac{z}{\dfrac{z_1+z_2+z_3}{c^2}} = \frac{xx_1+yy_1+zz_1}{1} \ldots\ldots$$

whence the result follows.

13. If the curve be a parabola the line joining its centre to the origin must be parallel to the plane, whence the result follows.

CHAPTER VII.

1. (1) The discriminating cubic is $s^3 - 10s^2 + 13s + 55 = 0$. This has two positive roots and one negative root by Descartes' rule of signs, all the roots being real. Hence the equation represents a hyperboloid of one sheet.

(2) A hyperbolic cylinder.

2. (1) Hyperboloid of revolution whose centre is at the point (2, 1, 0); of one or two sheets according as $a >$ or < 2.

(2) Co-ordinates of centre $\frac{21}{8}$, $\frac{3}{8}$, $\frac{15}{8}$; hyperboloid of two sheets.

(3) A parabolic cylinder.

(4) A hyperboloid of one or two sheets as $a^2 >$ or < 3.

3. The two equations merely differ by $h^2(x^2+y^2+z^2)$ which remains unaltered by any transformation round the origin.

The second is a right circular cylinder, the first a spheroid.

4. An ellipsoid if $1-\mu<\sqrt{2}$, a hyperboloid of one sheet if $1-\mu>\sqrt{2}$.

5. An ellipsoid whose centre is at the point $\frac{3a}{4}, \frac{3b}{4}, \frac{3c}{4}$: the equation when $z=0$ can be put into the form

$$\left(\frac{x}{a}+\frac{y}{b}-2\right)^2+\left(\frac{x}{a}-1\right)^2+\left(\frac{y}{b}-1\right)^2=0.$$

6. See Example 6, Chapter IV. Wrong reference in question.

7. Take the general equation of the second degree and find the conditions that it may be satisfied when $x=0$ and $z=0$, and also when $y=0$ and $z=0$.

10. See Art. 150 for the condition that the equation represents a surface of revolution, and Art. 90. These conditions give if $c=a+b$, $b'=0$, $c'^2=ab$, and the equation can be written

$$(x\sqrt{a}+y\sqrt{b})^2+c\left(z+\frac{c''}{c}\right)^2+2a''x+2b''y+d-\frac{c''^2}{c}=0,$$

which can be again written

$$(x\sqrt{a}+y\sqrt{b}+k)^2+c\left(z+\frac{c''}{c}\right)^2+2(a''-k\sqrt{a})x$$
$$+2(b''-k\sqrt{b})y+d-\frac{c''^2}{c}-k^2=0.$$

And if k be so chosen that $x\sqrt{a}+y\sqrt{b}+k=0$, and the line

$$2x(a''-k\sqrt{a})+2y(b''-k\sqrt{b})=0,$$

are at right angles, the former united with $z+\frac{c''}{c}=0$ must give the axis.

11. $z^2+cxy=k^2$.

12. Take for the fixed straight lines $x=0$, $y=0$; $x=a$, $z=0$; $y=b$, $z=c$; and take the equations (3) of Art. 17 as the generating line: the equation becomes

$$ayz+bxz=cy(x-a).$$

13. The condition required is that

$$A\lambda^2+B\mu^2+C\nu^2+2A'\mu\nu+2B'\nu\lambda+2C'\lambda\mu$$

shall retain an invariable value for all values of λ, μ, ν consistent with $l\lambda+m\mu+n\nu=0$. See Art. 173.

14. Eliminate s between the equations (1) of Art. 83.

15. If x', y', z' be the co-ordinates of the vertex, the equation of the cone is

$$\frac{(x'z - xz')^2}{a^2} + \frac{(y'z - yz')^2}{b^2} = (z - z')^2.$$

And by Art. 50 equation (7) it follows that

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + z'^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) - 1 = \alpha^2 + \beta^2 - (\alpha^2 + \beta^2) = 0.$$

CHAPTER VIII.

1. $x^2 + y^2 + z^2 = a^2 + b^2 + c^2$. Use equation 5 of Art. 101.

2. A similar and similarly situated ellipsoid whose axes are double those of the first.

3. Use Art. 101.

4. $\{x(x - \alpha) + y(y - \beta) + z(z - \gamma)\}^2 = a^2x^2 + b^2y^2 + c^2z^2$.

5. $a^2x^2 + b^2y^2 + c^2z^2 = k^4$.

6. The conditions that the normal to the ellipsoid at (x, y, z) shall pass through (α, β, γ) are

$$\frac{a^2(x - \alpha)}{x} = \frac{b^2(y - \beta)}{y} = \frac{c^2(z - \gamma)}{z} = k,$$

and these combined with

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

give an equation of the sixth degree in k. All six lines lie on the cone

$$\frac{(b^2 - c^2)\alpha}{x - \alpha} + \frac{(c^2 - a^2)\beta}{y - \beta} + \frac{(a^2 - b^2)\gamma}{z - \gamma} = 0.$$

7. Obtain the condition that the normal at the point (x, y, z) may intersect a given diameter $\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu}$. By properly choosing λ, μ, ν this condition can be made identical with

$$\frac{l}{x} + \frac{m}{y} + \frac{n}{z} = 0.$$

8. The tangent plane to any such ellipsoid can have its equation written as

$$lx + my + nz = \sqrt{a^2l^2 + b^2m^2 + c^2n^2 - k^2},$$

whence by Art. 77 (6) the result can be obtained.

9. There will be one straight line in the tangent plane at the extremity of the radius, perpendicular to the radius.

10. If $lx + my + nz = p$ be the equation of the cutting plane, the first volume is given by multiplying the area of the section given in Art. 80 by $\frac{1}{3}p$. The co-ordinates of the pole of the section can be obtained from Art. 106, and the perpendicular on the plane from this point is found to be $\frac{a^2l^2 + b^2m^2 + c^2n^2 - p^2}{p}$: whence the ratio of the volumes is $\frac{a^2l^2 + b^2m^2 + c^2n^2 - p^2}{p^2}$, and if this be constant it easily follows that either volume is constant.

11. The shadow is the section by the plane, of the enveloping cone whose vertex is the luminous point.

12. Use Arts. 149, 150.

13. Take the centre of the ball as origin, a plane parallel to the inclined plane as plane of xy, and $\alpha - lt$, $\beta - mt$, $\gamma - nt$ as the co-ordinates of the luminous point at any time.

14. $yzx' + zxy' + xyz' = 3a^3$; $\frac{9a^3}{2}$.

15. $x'x^{-\frac{1}{3}} + y'y^{-\frac{1}{3}} + z'z^{-\frac{1}{3}} = a^{\frac{2}{3}}$.

16. $(B\beta^2 + C\gamma^2 - \alpha)(By^2 + Cz^2 - x) - \{B\beta y + C\gamma z - \frac{1}{2}(x + \alpha)\}^2 = 0$.

17. $c^2\sqrt{\frac{\alpha^2}{a^4} + \frac{\beta^2}{b^4} + \frac{\gamma^2}{c^4}}$; $\alpha\left(1 - \frac{c^2}{a^2}\right)$, $\beta\left(1 - \frac{c^2}{b^2}\right)$.

18. $\frac{x - \alpha}{-1} = \frac{y - \beta}{2B\beta} = \frac{z - \gamma}{2C\gamma}$; $\frac{By^2}{(1 + 2B\alpha)^2} + \frac{Cz^2}{(1 + 2C\alpha)^2} = \alpha$.

19. The equations of the normal at (x, y, z) are

$$x(x' - x) = y(y' - y) = z(z' - z).$$

Use the condition of Art. 31.

20. $4x + l' + l = 0$. Use equation (3) of Art. 102.

21. $a(x^2 - yz) + \beta(y^2 - zx) + \gamma(z^2 - xy) = c^3$: a hyperboloid of one or two sheets according as $a + \beta + \gamma$ is positive or negative.

22. $ayz + \beta zx + \gamma xy = 3a^3$: a hyperboloid of one or two sheets according as $a\beta\gamma$ is negative or positive.

23. $4x(x^2 + y^2 + z^2) + \dfrac{y^2}{B} + \dfrac{z^2}{C} = 0$.

24. The equation of any tangent plane to the cone can be put into the form $Axx' + Byy' + Czz' = 0$, where $Ax'^2 + By'^2 + Cz'^2 = 0$, and if

$$\frac{l}{Ax'} = \frac{m}{By'} = \frac{n}{Cz'},$$

we get the required result.

25. Use (10) of Art. 68 putting $\dfrac{a}{a^2}$, $\dfrac{\beta}{b^2}$, $\dfrac{\gamma}{c^2}$ for l, m, n and reducing. Or else use (1) and (3) of Art. 75.

CHAPTER IX.

1. (1) $x'y - y'x + \dfrac{a^2}{c}(z' - z) = 0$.

(2) If we assume $z = r\sin\phi$, the equation of the osculating plane can be written

$$2x'\cos^3\phi - y'\sin\phi(1 + 2\cos^2\phi) - 2z' + r\sin\phi(2 + \cos^2\phi) = 0.$$

2. Length of arc $= \sqrt{a^2 + c^2}\,.\,(\theta_1 - \theta_2)$. From the equations of the curve obtain $x^2 + y^2$ as a function of z: let $x^2 + y^2 = f(z)$. This is the equation required. Ex. $x^2 + y^2 = a^2$.

3. We easily get, if a be the radius of the generating cylinder,

$$b^2 = 4a^2\sin^2\frac{\theta_1 - \theta_2}{2} + a^2\cot^2 a(\theta_1 - \theta_2)^2 = 4a^2\sin^2\frac{\theta_1 - \theta_2}{2} + l^2\cos^2 a,$$

if l be the length. Hence, when l is a maximum, $\sin\dfrac{\theta_1 - \theta_2}{2} = 0$; and the maximum length $= \dfrac{b}{\cos a}$. But this maximum length $= a\operatorname{cosec} a(\theta_1 - \theta_2)$ and $\theta_1 - \theta_2 = 2n\pi$;

$$\therefore \frac{2n\pi a}{\sin a} = \frac{b}{\cos a}; \quad \therefore a = \frac{b\tan a}{2n\pi}.$$

4. The equations of the curve are

$$x = a\cos\theta,\ y = a\sin\theta,\ z = \frac{c}{2}\left(\epsilon^{\frac{a\theta}{c}} + \epsilon^{-\frac{a\theta}{c}}\right).$$

5. $x + y + z = 1$.

6. $r = a$, $\phi = \tan\beta \log\tan\frac{\theta}{2} + C$; r, θ, ϕ being polar co-ordinates.

7. (1) $y^2 - x^2 = c$. (2) $y + a\tan^{-1}\frac{y}{x} = c$.

8. Analytically. Differentiate the equations of the sphere and ellipsoid, and find the ratios $\frac{dx}{ds} : \frac{dy}{ds} : \frac{dz}{ds}$. The equation of the plane can then be found, and then the equation (12) of Art. 68 can be used.

9. (1) $\cos^{-1}(\cos\phi\sin\theta) + \cos^{-1}(\sin\phi\sin\theta) = \text{const.}$ which can be transformed into

(2) $\cos\phi\sin\theta\sqrt{1 - \sin^2\phi\sin^2\theta} + \sin\phi\sin\theta\sqrt{1 - \cos^2\phi\sin^2\theta} = \text{const.}$

or (3) $x\sqrt{a^2 - y^2} + y\sqrt{a^2 - x^2} = \text{const.}$

10. $\theta = a\phi$.

11. By Art. 101, $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ will be proportional to

$$\left(\frac{p^2}{a^2} - 1\right)x,\ \left(\frac{p^2}{b^2} - 1\right)y,\ \left(\frac{p^2}{c^2} - 1\right)z,$$

whence

$$\left(\frac{1}{b^2} - \frac{1}{c^2}\right)\frac{1}{x}\frac{dx}{ds} + \left(\frac{1}{c^2} - \frac{1}{a^2}\right)\frac{1}{y}\frac{dy}{ds} + \left(\frac{1}{a^2} - \frac{1}{b^2}\right)\frac{1}{z}\frac{dz}{ds} = 0.$$

12. $z = A\epsilon^{\theta\sin\alpha\cot\beta}$, where z is the distance of the point from the vertex, α the semivertical angle of the cone, β the fixed angle, and θ the angle made by the plane through the point on the curve and the axis of the cone, with some fixed plane. The length of curve between any two values of $\theta = \frac{A}{\sin\alpha\sin\beta}\left\{\epsilon^{\theta_2\sin\alpha\cot\beta} - \epsilon^{\theta_1\sin\alpha\cot\beta}\right\}$. At the vertex $\theta = -\infty$.

13. From the method of producing the curve we easily see that if s be the arc measured from the point nearest to the vertex,

$r^2 = c^2 + s^2$. Also if the axis of the cone be the axis of x, $x = r\cos\alpha$: whence $\frac{d^2x}{ds^2} = \frac{c^2\cos\alpha}{r^3}$. Also the principal normal to the curve is the normal to the cone at that point (Art. 182). Whence $\rho\frac{d^2x}{ds^2} = \pm\sin\alpha$, and $\therefore r^3 = a\rho$.

14. (1) $\frac{a^2+c^2}{a}$. (2) $\frac{a^2+c^2}{a}$.

15. $x = -\frac{c^2}{a}\cos\theta$, $y = -\frac{c^2}{a}\sin\theta$, $z = c\theta$.

16. Take the common tangent to the two curves as axis of x and the plane of the circle as plane of xy. Then if x_1, y_1, z_1 be the co-ordinates of a point on the circle at the end of the arc δs, and ρ the radius of the circle

$$x_1 = \rho\sin\frac{\delta s}{\rho} = \delta s - \tfrac{1}{6}\frac{\delta s^3}{\rho^2} + \dots;\quad y_1 = a\left(1-\cos\frac{\delta s}{\rho}\right) = \frac{\delta s^2}{2\rho} - \frac{\delta s^4}{\lfloor 4\,\rho^3} + \dots;\quad z_1 = 0$$

and if x_2, y_2, z_2 be the co-ordinates of the point on the curve we get

$$x_2 = \frac{dx}{ds}\delta s + \tfrac{1}{2}\frac{d^2x}{ds^2}\delta s^2 + \frac{1}{\lfloor 3}\frac{d^3x}{ds^3}\delta s^3 + \dots$$

and similar values for y_2, z_2. But it can easily be shewn by Arts. 119, 130 that

$$\frac{dx}{ds} = 1,\ \frac{dy}{ds} = 0,\ \frac{dz}{ds} = 0,\ \frac{d^2x}{ds^2} = 0,\ \frac{d^2y}{ds^2} = \frac{1}{\rho},\ \frac{d^2z}{ds^2} = 0;$$

whence the square of the distance required becomes

$$\left(\frac{\delta s^3}{\lfloor 3}\right)^2\left\{\left(\frac{d^3x}{ds^3} + \frac{1}{\rho^2}\right)^2 + \left(\frac{d^3y}{ds^3}\right)^2 + \left(\frac{d^3z}{ds^3}\right)^2\right\}.$$

And by differentiating the formula (10) of Art. 129, and (2) of Art. 118 the required result may be obtained.

17. Prove geometrically from the figure in Art. 127.

18. By Ex. 12 the equations of the curve may be written

$$x = A\tan\alpha\epsilon^{c\theta}\cos\theta,\ y = A\tan\alpha\epsilon^{c\theta}\sin\theta,\ z = A\epsilon^{c\theta},$$

where $c = \sin\alpha\cot\beta$; whence ρ can be obtained by (9) in Art. 129. When developed the curve is an equiangular spiral.

CHAPTER X.

1. $x^2 + y^2 + z^2 - (lx + my + nz)^2 = 1.$

2. $\sqrt{x^2 + y^2} + \sqrt{c^2 - z^2} = a.$

3. $x^2 + y^2 + z^2 - k^2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0.$

4. $\frac{a^2x^2}{\alpha^4} + \frac{b^2y^2}{\beta^4} + \frac{c^2z^2}{\gamma^4} = 1.$

5. $z^2 = c^2$, c being the radius, and the plane of xy the fixed plane.

6. $xyz = \frac{k^3}{3\sqrt{3}}.$

7. $\frac{a^2x^2}{a^2 - r^2} + \frac{b^2y^2}{b^2 - r^2} + \frac{c^2z^2}{c^2 - r^2} = 0$, where $r^2 = x^2 + y^2 + z^2$. The Wave Surface. See Chapter on Fresnel's Theory of Double Refraction.

8. The surface $x^2 + y^2 + 2z^2 = \frac{k^2}{2}$; and the curve $x^2 + y^2 = k^2$, $z = 0$.

9. $\frac{z^2}{c^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = \frac{1}{4}.$

10. $xyz = \frac{k^3}{27}$, where $\frac{1}{6}k^3$ is the given volume.

11. $x^2 + y^2 + z^2 = k^2.$ 12. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{1}{k^2}.$

13. $x^2 + y^2 + z^2 = (c \pm c')^2.$

14. $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)(a^2l^2 + b^2m^2 + c^2n^2 - 1) = (lx + my + nz - 1)^2.$

15. By Art. 106 all the lines in question lie in the polar plane of (α, β, γ). If α', β', γ' be the co-ordinates of one point in which any chord meets the ellipsoid, the line required will be given by the two equations

$$\frac{x\alpha}{a^2} + \frac{y\beta}{b^2} + \frac{z\gamma}{c^2} = 1, \text{ and } \frac{x\alpha'}{a^2} + \frac{y\beta'}{b^2} + \frac{z\gamma'}{c^2} = 1.$$

The condition that this line may be perpendicular to the line joining (α, β, γ) to $(\alpha', \beta', \gamma')$ can be reduced to the form

$$\frac{(b^2-c^2)\,\alpha}{\alpha-\alpha'}+\frac{(c^2-a^2)\,\beta}{\beta-\beta'}+\frac{(a^2-b^2)\,\gamma}{\gamma-\gamma'}=0.$$

Also the equation of a plane through the origin and the line required is

$$\frac{x\,(\alpha-\alpha')}{a^2}+\frac{y\,(\beta-\beta')}{b^2}+\frac{z\,(\gamma-\gamma')}{c^2}=0,$$

the envelope of which treating $\alpha-\alpha'$, $\beta-\beta'$, $\gamma-\gamma'$ as parameters gives us the cone required. That the curve is a parabola can be shewn because a plane through the origin parallel to the polar plane of (α, β, γ) can easily be shewn to touch the cone.

CHAPTER XI.

1. If $x=f_1(t)$, $y=f_2(t)$, $z=f_3(t)$ be the equations of the curve, we have to find the envelope of

$$\{x-f_1(t)\}^2+\{y-f_2(t)\}^2+\{z-f_3(t)\}^2=c^2,$$

where t is the parameter. The envelope is obtained from the intersection of the sphere with the normal plane to the curve at the point t.

2. The equation can be put in the form

$$\tfrac{3}{2}(x+y+z)\left\{(x^2+y^2+z^2)-\left(\frac{x+y+z}{\sqrt{3}}\right)^2\right\}=c^3,$$

and if the line $x=y=z$ be taken as axis of z' this becomes, by Arts. 25 and 28, $\dfrac{3\sqrt{3}}{2}z'(x'^2+y'^2)=c^3$, which is a surface formed by the revolution of the curve $z'x'^2=\dfrac{2c^3}{3\sqrt{3}}$ round the axis of z'. Or, apply Art. 148.

3. $a^2y^2+x^2z^2=c^2x^2$; $x=a$, $y^2+z^2=c^2$ being the equations of the circle.

4. See Ex. 11, Chap. VII. for choice of axes,

$$\frac{x^2}{(c+z)^2}+\frac{y^2}{(c-z)^2}=\frac{a^2}{c^2}.$$

5. $x \sin \frac{z}{c} = y \cos \frac{z}{c}$.

6. $x \cos \theta + y \sin \theta = a$, where $\theta = \frac{z}{c} - \frac{\sqrt{x^2 + y^2 - a^2}}{a}$.

7. (1) $(x^2+y^2)(k-na) + 2a(z-a)(lx+my) + (k+na)(z-a)^2$, the vertex being at the point $(0, 0, a)$ and the plane of the small circle being $lx + my + nz = k$.

(2) Put $z = 0$ in the above.

8. $\sqrt{x^2 + y^2} + \sqrt{c^2 - z^2} = a$, or

$$(x^2 + y^2 + z^2 + a^2 - c^2)^2 = 4a^2(x^2 + y^2) \quad \ldots\ldots\ldots\ldots\ldots (1).$$

9. The points at which the tangent plane passes through the origin are given by $z = \pm \frac{c}{a} \sqrt{a^2 - c^2}$, that is, they lie in two horizontal rings. Take one of these points in the plane of zx. The tangent plane at this point has for its equation

$$x = z \frac{\sqrt{a^2 - c^2}}{c} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Also the equation (1) can be put into the form

$$\{x^2 + y^2 + z^2 - (a^2 - c^2)\}^2 = 4c^2y^2 + 4c^2x^2 - 4(a^2 - c^2)z^2$$
$$= 4c^2y^2 + 4(cx - z\sqrt{a^2 - c^2})(cx + z\sqrt{a^2 - c^2}),$$

whence at the points of intersection of (2) with (1)

$$x^2 + y^2 + z^2 - (a^2 - c^2) = \pm 2cy.$$

Hence (2) cuts (1) in two circles. From the symmetry of the surface the same will be true for all the points.

10. The fixed plane being the plane of yz, and l, m, n the direction-cosines of AB, the equation of the surface is

$$(mz - ny)^2 + (nx - lz)^2 + (ly - mx)^2 = k^2(y^2 + z^2).$$

11. The conditions are given in Art. 92. See Art. 151.

13. $\frac{a^2x^2}{(a^2-k^2)^2} + \frac{b^2y^2}{(b^2-k^2)^2} + \frac{c^2z^2}{(c^2-k^2)^2} = 1,$

where $$k^2 = abc \cdot \frac{ayz + bzx + cxy}{bcyz + cazx + abxy}.$$

14. Take $y=px$ and $y=kz+q$ as equations of the generating line.

15. (1) A surface of revolution round Oz. (2) A surface such that all sections by planes through Oz are circles. (3) A cone whose vertex is O.

16. (1) A surface produced by the revolution of the lemniscate in the plane of zx round Oz. (2) A surface produced by the motion of a circle whose centre is O and radius is any radius of the same lemniscate placed in the plane of xy.

17. $$\left\{\frac{x}{c}\sqrt{b^2-c^2}-\frac{z}{a}\sqrt{a^2-b^2}\right\}^2\left(\frac{1}{c^2}-\frac{1}{a^2}\right)+b^2y^2\left(\frac{1}{c^2}+\frac{1}{a^2}-\frac{2}{b^2}\right)^2$$
$$=b^4\left(\frac{1}{c^2}+\frac{1}{a^2}-\frac{2}{b^2}\right)^2.$$

18. The equations of any helix can be written

$$x=a\cos\theta,\quad y=a\sin\theta,\quad z=c\theta+\gamma,$$

and by virtue of the given conditions γ and c must be expressible as functions of a. Hence since $a^2=x^2+y^2$ and $\theta=\tan^{-1}\frac{y}{x}$, and also $=\frac{z}{c}-\frac{\gamma}{c}$, we get

$$\tan^{-1}\frac{y}{x}=zF(x^2+y^2)+f(x^2+y^2).$$

The second part easily follows by differentiation.

19. The reflected light forms a cone of the second order, and the wall on which it falls is parallel to one of its generating lines.

20. If x_1, y_1, z_1; x_2, y_2, z_2 be the co-ordinates of the points A, B; O being the origin, the condition that AB subtends a right angle at O is $x_1x_2+y_1y_2+z_1z_2=0$. Also the equations of AB are

$$\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}=\frac{z-z_1}{z_1-z_2},$$

and from the equations of the straight lines x_2, y_2 can be expressed in terms of z_2 and x_1, y_1 in terms of z_1. Then eliminating z_1, z_2, between these equations we get a relation between x, y, z.

21. Equation (4) of Art. 148 is evidently the required condition.

22. If $\frac{x}{y}=f(z)$ be the equation of the surface, the locus required is

$$(x'-y'f)(y'+x'f)=\frac{z(1-f^4)}{f'},$$

where f, f' are the values of $f(z)$ and $f'(z)$ for the given value of z.

23. The equations of any such circle are $x^2+y^2+z^2=2ax$ and $y=mx$, also a must be expressible as a function of m, $=2cf(m)$ say. The differential equation can be easily deduced.

CHAPTER XII.

1. $6\alpha^2-12\gamma^2=1$, $\beta=0$; $4\alpha^2+12\beta^2=1$, $\gamma=0$; impossible locus.

2. If $\frac{2x^2}{2k+1}+\frac{3y^2}{3k+1}+\frac{4z^2}{4k+1}=9$ be either of the surfaces, the two values of k are the roots of the quadratic

$$k^2+\frac{3}{4}k+\frac{29}{216}=0.$$

3. Let a be the distance of the point along the axis of x, and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ one of the surfaces: the locus required is

$$\frac{x}{a}+\frac{y^2}{ax-(a^2-b^2)}+\frac{z^2}{ax-(a^2-c^2)}=1.$$

4. At the points of intersection we easily get $ax=\beta y+a^2$. Also the direction-cosines of the normal to the first surface at any such point are easily proved to be proportional to

$$\frac{1}{a}-\frac{a}{\beta^2}-\frac{az^2}{(ax-b^2)^2},\ \frac{2}{\beta},\ \frac{2z}{ax-b^2},$$

while those of the normal to the second are proportional to

$$\frac{2}{a},\ \frac{1}{\beta}-\frac{\beta}{a^2}-\frac{\beta z^2}{(ax-b^2)^2},\ \frac{2z}{ax-b^2},$$

and these lines are therefore perpendicular to each other since their direction-cosines satisfy the requisite condition.

5. If the two quadrics be $By^2 + Cz^2 = x$ and $B'y^2 + C'z^2 = x + h$, the coincidence of the foci involves

$$\frac{1}{4B} = \frac{1}{4B'} - h, \qquad \frac{1}{4C} = \frac{1}{4C'} - h,$$

whence also the focal conics will coincide, since

$$\frac{B - C}{BC} = \frac{B' - C'}{B'C'}.$$

6. At the points where the two quadrics in (5) cut, we have

$$(B - B')\,y^2 + (C - C')\,z^2 + h = 0,$$

or

$$4BB'y^2 + 4CC'z^2 + 1 = 0,$$

which is the condition that the tangent planes to the two quadrics at (x, y, z) should be at right angles.

CHAPTER XIII.

1. $$\frac{b^2c^2x^2}{p\rho - a^2} + \frac{c^2a^2y^2}{p\rho - b^2} + \frac{a^2b^2z^2}{p\rho - c^2} = 0$$

where p is the perpendicular from the centre on the tangent plane. This can be reduced to

$$p^2\rho^2 - (a^2 + b^2 + c^2 - r^2)\,p\rho + \frac{a^2b^2c^2}{p^2} = 0, \text{ where } r^2 = x^2 + y^2 + z^2.$$

For the umbilici the two roots will be equal. This will require one of the quantities x, y or z to vanish.

2. (1) $x = y = z = a$.

(2) When $xa^{\frac{1}{2}} = \pm\, yb^{\frac{1}{2}} = \pm\, zc^{\frac{1}{2}}$.

3. $$\left(\frac{dy}{dx}\right)^2 (x^2y^4 - a^6)\,x^2 + \frac{dy}{dx}\,x^3y^3\,(y^2 - x^2) - y^2\,(x^4y^2 - a^6) = 0.$$

4. (1) Eliminate m between equations (6) and (7) of Art. 171, writing $\rho = h\sqrt{1 + p^2 + q^2}$.

(2) The coefficients of the several powers of m in the equation (7) of Art. 171 must vanish.

5. The two values of h in (9) of Art. 169 must be equal and of opposite sign;

$$\therefore\ U^2(v+w)+V^2(w+u)+W^2(u+v)-2u'VW-2v'WU-2w'UV=0.$$

The points of intersection of the surface with the sphere

$$x^2+y^2+z^2=\frac{1}{A}+\frac{1}{B}+\frac{1}{C}.$$

6. Take the general equation of a quadric and determine the conditions that it may touch the plane of xy at the origin, and that sections by planes parallel to that plane may be circles.

7. Using the equation in the last question the locus required is

$$cx+by+k(z-a)-z=0.$$

8. See Ex. 4, Chap. XII. The surface in the question and the two surfaces

$$\frac{x^2}{\beta y+b^2}+\frac{y}{\beta}+\frac{z^2}{\beta y+b^2-c^2}=1,\quad \frac{x^2}{\gamma z+c^2}+\frac{y^2}{\gamma z+c^2-b^2}+\frac{z}{\gamma}=1,$$

can be shewn to cut always at right angles, where β and γ are any constants. Hence the intersections of these surfaces with the given one are its lines of curvature.

At the points of intersection of the first with the given surface we have $ax=\beta y+b^2$ a plane; and by combining this with the given equation, that can be written

$$\frac{x}{a}(ax-c^2)+\frac{y}{\beta}(\beta y+b^2-c^2)+z^2=ax-c^2,$$

which is the equation of a sphere. Hence the lines of curvature are circles: and the plane of any one of them being $ax=\beta y+b^2$ always contains the line $ax=b^2,\ y=0$.

9. The result follows from the fact that r has the same value for all tangent lines at the umbilicus.

10. At the points of contact pr has the same value for the geodesic and the line of curvature.

11. The value of pr is the same for the two geodesics through P since they each pass through an umbilicus. Hence the value of r is the same. The tangents to these two geodesics are there-

fore parallel to the equal radii of the indicatrix, and the tangents to the lines of curvature being parallel to the axes bisect the angles between these.

12. Can be proved from 11 by the method of infinitesimals.

13. The geodesic circle cuts all geodesics through the umbilicus at right angles. Hence if d, d' be the semidiameters parallel to the tangent to the geodesic circle and the line through the umbilicus, and ρ, ρ' be the semi-axes of the central section parallel to the tangent plane at the point

$$\frac{1}{d^2}+\frac{1}{d'^2}=\frac{1}{\rho^2}+\frac{1}{\rho'^2}=\frac{p^2(a^2+b^2+c^2-r^2)}{a^2b^2c^2}.$$

Ex. 25, Chap. VIII.

$$\therefore \frac{a^2b^2c^2}{p^2d^2}+\frac{a^2b^2c^2}{p^2d'^2}=a^2+b^2+c^2-r^2.$$

But $p^2d'^2=a^2c^2$ as can be ascertained from the known co-ordinates of the umbilici.

14. At any point in the principal section by the plane of yz the two roots of the equation in (1) can be shewn to be $\frac{a^2+b^2-r^2}{p}$ and $\frac{c^2}{p}$. The former root is the radius of curvature of the principal section: the latter gives the distance along the normal of the point whose locus is required which can then be worked out by plane geometry.

15. Taking $x^2+y^2=a^2$ as the equation of the cylinder we easily get for the geodesics $\frac{d^2z}{ds^2}=0$; therefore $\frac{dz}{ds}=c$, whence the curves are helices.

16. $s=\sqrt{z^2\sec^2\alpha-c^2}$, where α is the semi-vertical angle of the cone, and s the length of the arc from the nearest point to the vertex.

17. If $x^2+y^2=f(z)$ be the equation of the surface it easily follows from (1) of Art. 182 that for all points in any geodesic line

$$x\frac{dy}{ds}-y\frac{dx}{ds}=c.$$

And it can easily be proved that the sine of the angle required

$$=\frac{c}{\sqrt{x^2+y^2}}.$$

18. If $r=f(x)$ be the equation of the surface r^2 being y^2+z^2 the required expressions are

$$\frac{\left\{1+\left(\frac{dr}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2r}{dx^2}} \text{ and } r\sqrt{1+\left(\frac{dr}{dx}\right)^2}.$$

19. With the usual notation for an ellipse the product required is

$$\frac{CD^2}{PF}\times PG=\frac{CD^2}{PF}\cdot\frac{BC^2}{PF}=\frac{CD^4}{AC^2}\propto SP^2\,.\,S'P^2.$$

20. The radii of curvature of the principal sections are r and $\frac{r^2}{r-\rho\sin\phi}$, where r is the focal radius of the point on the ellipse which is in contact, ϕ the angle between that radius and the tangent, and ρ the radius of curvature of the ellipse (Besant on Glissettes, &c.). Hence the sum of the curvatures

$$=\frac{2}{r}-\frac{\rho\sin\phi}{r^2}=\frac{2}{r}-\frac{r(2a-r)}{ar^2}=\frac{1}{a}.$$

21. By Meunier's Theorem.

22. Use the quadratic equation in question (1) of this chapter, r being a constant.

23. Prove geometrically from the fact that when the surface is developed the geodesics become straight lines.

24. Differentiate $r^2=x^2+y^2+z^2$ twice and use the formulæ (1) of Art. 182, (10) of Art. 129, and (1) of Art. 100.

25. Use Meunier's Theorem, and (3) of Art. 167.

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