SOLID GEOMETRY

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By

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CHAPTER IV. **Areas and Volumes**

CHAPTER V. **Similarity and Symmetry**

THE PREFACE

The *Solid Geometry* is prepared for the same purpose and with the same general features as is the *Plane Geometry.* In both books the two main characteristics are *analysis* and *emphasis.*

One of the great objectives in education is to train young people to attack difficulties through an analysis of the problems presented. It is because of this fact that both the *Plane* and the *Solid Geometry* are prepared as suggestive method texts with the suggestions in the form of a logical analysis.

Moreover, if the mind is to retain any lasting impression of the work covered, distinctions in emphasis are necessary. The material presented in both the *Plane* and the *Solid Geometry* has been arranged with this fact in mind. Attention is called particularly to chapters ii, iv, and v. Chapter ii discusses the nature and properties of the various surfaces and solids ordinarily studied in solid geometry with the exception of the sphere, which is studied in chapter iii. All areas and volumes are considered in chapter iv. Similarity is considered in chapter v.

There are several advantages gained from this arrangement. It enables the pupil to take up the study of areas and volumes with a clear idea of the solids considered. It makes possible a more logical arrangement of material. Cylinders are compared with prisms and cones with pyramids when the properties of these solids are studied; but as the volume of the pyramid is obtained from that of the prism, a different order is used when volumes are studied. Moreover, the theorems concerning areas and volumes may be worked into a logical whole when considered together, which is not possible in the traditional arrangement. For example, Theorems **121**, and **122** serve not only as a necessary preparation for the measurement of the sphere, but also as a fitting climax to the work on cylinders and cones.

Both the *Plane* and the *Solid Geometry* are written with the firm conviction that if geometry is taught by analysis and if at the same time proper distinctions in emphasis are made, pupils will reach the end of their course with more real education and with a much clearer and more lasting impression of the meaning of the great concepts of geometry than can possibly be the case under traditional methods.

Attention is called also to certain minor features:

The approach to the early theorems through the introductory material is natural and easy.

As the first difficulty, perhaps the only real difficulty, in solid geometry, is the inability of pupils to visualize figures in space, the use of models made by pupils is strongly recommended. These models should be used for demonstration work and should precede the use of blackboard figures until the pupil is clearly out of "flatland." If possible, spherical triangles should be studied from a slated globe.

The treatment of loci in §§ 41–47 and the treatment of similarity in chapter v deserve attention.

The formal study of the theory of limits is omitted. The treatment of areas and volumes of round bodies by this method is given in outline only. It is intended merely to make the results appear reasonable to the pupil.

By proper choice of material the study of volumes may be made to depend upon Cavalieri's theorem. See §§ 173–179 and § 154.

The "Notes on Arithmetic and Algebra" and the "Outline of Plane Geometry" given will be found convenient for reference. The "Topics and References for Mathematic Clubs" are intended as suggestions only.

M. S.
C. E. C.

CHICAGO, ILLINOIS
March, 1922

SOLID GEOMETRY

CHAPTER I

Lines and Planes

INTRODUCTION

SUBJECT MATTER OF GEOMETRY

1. In plane geometry our study was confined to figures that could be drawn on a plane and drawn with ruler and compass only. Such figures are constructed of points and lines.

In solid geometry we extend our study to figures in space. These figures are constructed of points, lines, surfaces, and solids.

2. The space occupied by a ball is a geometrical solid. The outside of the ball is a surface; it is the boundary of the solid and separates the space within from the space without (Fig. 1).

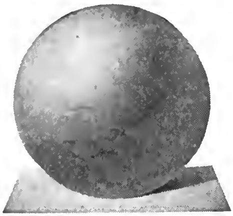


Fig. 1

In general we may say that—

The space occupied by any object is called a **geometrical solid.**

The boundaries of a solid are called surfaces.

The surfaces of a solid separate the solid from the remainder of space.

We may define a **surface** as any boundary between two parts of space.

The space occupied by a tomato can is a geometrical solid. The surface of the can separates the space within from the space without. We may consider the total surface of the can as composed of three parts separated by the edges of the can (Fig. 2).

Fig. 2

The edges of a solid are called lines. We may consider the edges of a solid as separating the parts of its surface.

We may define a **line** as the boundary between two parts of any surface.

The equator is a line separating the northern and southern hemispheres, two parts of the surface of the earth.

The space occupied by a rectangular block is a geometrical solid. The lateral surface of the block (Fig. 3) consists of four parts which we may think of as separated by the lateral edges, or lines. The edge about the upper base may be considered as composed of four parts separated by the corners, or points.

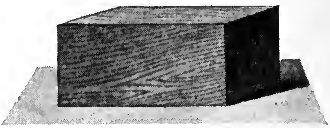


FIG. 3

The corners of a solid are called points. We may consider the corners of a solid as separating the parts of its edges.

We may define a **point** as the boundary between two parts of any line.

A point on a straight line divides the line into two parts called **rays.**

DETERMINATION OF PLANES

3. The following **assumptions** concerning planes are important:

As. 1. A straight line joining any two points in a plane lies wholly in the plane.

This is the **fundamental characteristic** of planes.

Exercise. Is it possible to find two points on the curved surface of a tomato can, such that the line joining them lies wholly on the surface? Why is this surface not a plane?

As. 2. A plane is unlimited in extent.

As. 3. Through a given straight line an unlimited number of planes may be passed.

If a number of planes pass through a given line, they are said to form a **pencil of planes,** and the line is said to be the **axis** of the pencil.

In Fig. 4, AB is said to be the axis of the pencil of planes.

As. 4. If a plane passes through a given straight line, it may be revolved so as to contain any given point in space.

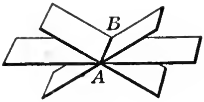


FIG. 4

As. 5. Through a given straight line and a given point without the line only one plane can be passed.

COR. I. **A plane is located definitely if it contains three given points not in the same straight line.**

Suggestion. Show that this is equivalent to locating the plane by a straight line and a point without the line.

COR. II. **A plane is located definitely by two given intersecting straight lines.**

Suggestion. Show that this is equivalent to locating the plane by one of the lines and any point in the other. See As. 1.

COR. III. **A plane is located definitely by two parallel straight lines.**

Suggestion. By definition two parallel lines lie in the same plane. Only one plane can contain the two parallel lines. If there could be two planes each containing the two parallels, there could be two planes each containing a line and the same point without the line. This is impossible.

RELATIVE POSITIONS OF TWO PLANES

4. Two planes either intersect or do not intersect.

Two planes that do not intersect are said to be **parallel.**

This definition is the **fundamental test for parallel planes.**

If two planes intersect, the **intersection** of the two planes is the locus of the points common to the two planes.

As. 6. **Two intersecting planes have at least two common points.**

5. THEOREM 1. **The intersection of two planes is a straight line.**

Analysis: To prove that the intersection is a straight line, join two points in the intersection by a straight line and prove that

I. Every point in this line lies in each plane.

II. No point outside this line lies in both planes.

Theorem 1 may be stated thus: **The intersection of two given planes determines a straight line.**

RELATIVE POSITIONS OF A STRAIGHT LINE AND A PLANE

6. If a straight line lies in a plane, the plane is said to **contain the line.**

If the plane does not contain the line, the line may intersect the plane, or it may not intersect the plane.

If the line and the plane do not intersect, they are said to be **parallel.** This definition is the **fundamental test for lines parallel to planes.**

When a given straight line intersects a given plane, one point and only one point is determined; or

As. 7. **A straight line can intersect a plane in but one point.**

RELATIVE POSITIONS OF STRAIGHT LINES IN SPACE

7. Two straight lines in a plane either intersect or are parallel. Two straight lines in space either intersect or are parallel or skew.

Two straight lines are said to be **parallel** if they are in the same plane and do not meet.

Two straight lines in space are said to be **skew** if they neither intersect nor are parallel.

Exercise. Hold two pencils to represent two skew lines. Point out two skew lines in the room in which you are sitting.

EXERCISES INVOLVING THE DETERMINATION OF LINES AND PLANES

8. 1. Will an object mounted on three legs always stand firm? Why? Will one mounted on four legs always stand firm? Why?

2. What practical uses are made of the fact referred to in the previous exercise?

3. Will three concurrent straight lines always lie in the same plane? Illustrate your answer. How many planes may be determined by three concurrent straight lines?

Tell how many planes would be determined in each of the following cases (Ex. 4–10) and discuss all possibilities. Illustrate your answers with pencils or toothpicks and give reasons.

4. Two intersecting straight lines and a point not in the plane of the lines.

5. Four concurrent straight lines.

6. Three concurrent straight lines and one point.

7. Four points not all in the same plane. How many lines are determined in this case?

8. Five points no four of which lie in the same plane. How many lines are determined in this case?

9. Three parallel lines that are not all in the same plane.

10. Four parallel lines no three of which lie in the same plane.

11. Must three intersecting straight lines lie in the same plane? Why?

12. Hold three pencils to illustrate the various relative positions of three straight lines in space. How many planes are determined in each case?

13. Answer Ex. 12 for four straight lines.

14. What is a skew quadrilateral? Can a skew quadrilateral be a parallelogram? How many planes are determined by the sides of a skew quadrilateral?

15. Hold two cards to illustrate the possible relative positions of two planes in space. How many lines are determined in each case?

16. Answer Ex. 15 for three planes.

17. Can skew straight lines be parallel to the same plane? Illustrate your answer with two pencils and a card.

18. What are the least number of planes that can completely inclose a space? How many lines are determined in this case? Illustrate your answer.

19. If five planes intersect so as completely to inclose a space, how many lines are determined?

PARALLEL LINES AND PLANES

TEST FOR LINES PARALLEL TO PLANES

9. THEOREM 2. **If a plane contains one and only one of two parallel straight lines, it is parallel to the other.**

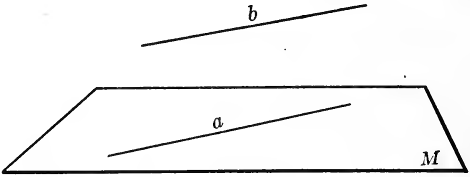


FIG. 5

Hypothesis: Lines a and b are parallel and plane M contains line a but not line b.

Conclusion: Line b is parallel to plane M.

Analysis:

I. To prove line $b \parallel$ plane M, prove that line b cannot meet plane M.

II. $\therefore$ show that if line b met plane M, it would also meet line a.

Proof:

STATEMENTS	REASONS
I. *a.* Lines a and b lie in the same plane.	I. *a.* Two parallel lines determine a plane.
b. Line a is the intersection of plane M and plane ab.	*b.* Line a lies in both planes.

c. If line b meets plane M, it will meet line a.

c. Line a is the intersection of plane M and plane ab.

d. But line b cannot meet line a.

d. Why?

II. $\therefore$ line b cannot meet plane M and is $\parallel$ it.

II. The supposition that line b would meet plane M is eliminated.

Ex. 1. Show that one plane and only one may be passed through one of two skew lines and parallel to the other.

Suggestion. Let a and b represent the two given skew lines. Through any point in a draw b' parallel to b. Lines a and b' determine a plane parallel to b. Why?

Ex. 2. Show that one plane and only one may be passed through a given point and parallel to two given skew lines in space.

Suggestion. Let a and b represent the two given skew lines and O the given point. Draw through O, $a' \parallel a$ and $b' \parallel b$. Lines a' and b' determine a plane parallel to line a and to line b. Why?

TESTS FOR PARALLEL LINES

10. THEOREM 3. **If a straight line is parallel to a plane, it is parallel to the intersection of the given plane with any intersecting plane containing the line.**

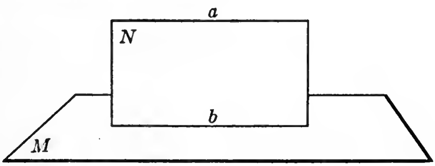


FIG. 6

Analysis: To prove $a \parallel b$, prove that they are in the same plane and cannot meet.

COR. **If a straight line and a plane are parallel, a straight line through any point of the given plane parallel to the given line lies wholly in the given plane.**

Analysis:

I. To prove b lies in plane M, prove that b coincides with a line that does lie in plane M.

II. $\therefore$ pass a plane through a and point P, intersecting M in line x, and prove that x coincides with b.

III. To prove that x coincides with b, show that x and b are both parallel to a through P (Fig. 7).

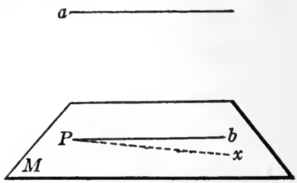


FIG. 7

11. THEOREM 4. **If two parallel planes are cut by a third plane, the intersections are parallel.**

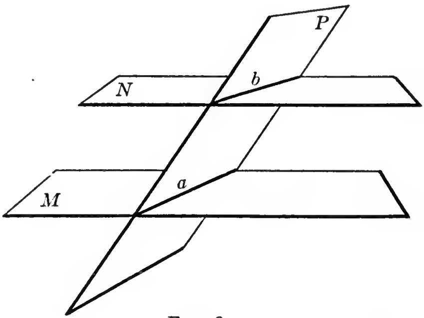


FIG. 8

Hypothesis: M and N are two parallel planes cut by plane P in lines a and b respectively.

Conclusion: Lines a and b are parallel.

Analysis: To prove lines a and b parallel, prove that they are in the same plane and do not meet.

Exercise. Illustrate Ths. 1–4 by planes from the room in which you are sitting and by pencils and pieces of cardboard.

TEST FOR PARALLEL PLANES

12. THEOREM 5. **If two intersecting straight lines are parallel respectively to two other intersecting straight lines, the plane of the first pair is parallel to the plane of the second pair.**

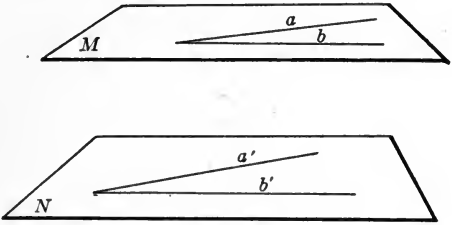


FIG. 9

Hypothesis: M and N are two planes with lines a and b in plane M parallel respectively to lines a' and b' in plane N.

Conclusion: Planes M and N are parallel.

Analysis:

I. Show that plane M cannot meet plane N.

II. $\therefore$ show that if plane M met plane N in any line, this line would be parallel to each of the intersecting lines a and b (Th. 3).

III. $\therefore$ prove that a and b are each parallel to plane N.

Proof:

STATEMENTS	REASONS
I. *a.* Line $a \parallel$ line a'.	
b. Line $a \parallel$ plane N.	Th. 2
c. Similarly line $b \parallel$ plane N.	
II. *a.* Suppose M and N meet in some line. Call this intersection line p.	
b. $\therefore a$ would be $\parallel p$.	Th. 3
c. Similarly b would be $\parallel p$.	
III. But a and b cannot both be $\parallel$ same line.	Why?
IV. $\therefore M$ cannot meet N and is $\parallel N$.	

The pupil should quote in full all theorems referred to by number.

TEST FOR PARALLEL LINES

13. THEOREM 6. **Two straight lines parallel to a third straight line are parallel to each other.**

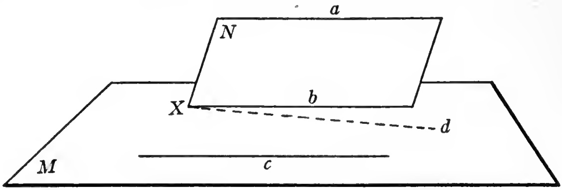


FIG. 10

Hypothesis: Lines a and b are each parallel to c.

Conclusion: Line a is parallel to line b.

Analysis and construction:

I. To prove $a \parallel b$, prove that b coincides with a line that is $\parallel a$.

II. Let M be the plane of b and c, and N be the plane of a and X, any point in b. Let N intersect M in line d. Prove that d is $\parallel a$ and that d coincides with b.

III. To prove that d and b coincide, prove that they are both parallel to c through X.

Proof:

STATEMENTS	REASONS
I. *a.* Line $c \parallel N$.	I. *a.* Th. 2
b. Line $d \parallel$ line c, through X.	*b.* Th. 3
c. Line $b \parallel$ line c, through X.	*c.* Hypothesis
d. $\therefore$ lines d and b coincide.	*d.* As. 30, *Plane Geometry*
II. but line $d \parallel a$.	II. Ths. 2 and 3
$\therefore$ line $b \parallel a$.	Why?

NOTE. Th. 6 is proved here for the sake of the exercises in §14. Another proof is given on page 19.

EXERCISES INVOLVING PARALLELS

14. To prove **two planes parallel,** prove that

1. They cannot meet, or
2. Two intersecting lines in one are parallel respectively to two intersecting lines in the other.

To prove **a line and a plane parallel,** prove that

1. They cannot meet, or
2. The plane contains a line parallel to the given line.

To prove **two lines parallel,** prove that

1. They are in the same plane and do not meet, or
2. One is parallel to a given plane and the other is the intersection of the given plane and any plane containing the line, or
3. They are the intersection of two parallel planes cut by a third, or
4. They are parallel to the same line.

1. Are two lines parallel to the same plane parallel to each other? Hold a card and two pencils so as to illustrate your answer.

2. Are two planes parallel to the same line parallel to each other? Illustrate your answer.

3. Through a given point how many lines can be drawn parallel to a given plane? Why?

4. Parallel segments between parallel planes are equal and cut off on the given planes segments that are equal and parallel.

5. If a line is parallel to a given plane, what is its relation to the lines of the plane? Illustrate your answer. To what lines of the plane is it parallel?

6. If a straight line and a plane are parallel, a pencil of planes through the given line intersects the given plane in parallel lines.

7. If a line is parallel to one of two parallel planes, it is parallel to the other or lies in the other.

Suggestion. Pass a plane through the given line so as to cut the two parallel planes.

8. If one of two parallel lines is parallel to a given plane, the other is also parallel to the plane or lies in the plane.

PERPENDICULAR LINES AND PLANES

PRELIMINARY THEOREMS

15. THEOREM 7. **In space but one line can be drawn perpendicular to a given line from a given point without the line.**

THEOREM 8. **In space any number of lines can be drawn perpendicular to a given line through a given point on the line.**

The proofs to Ths. 7 and 8 are left to the pupil.

FUNDAMENTAL TEST FOR PERPENDICULARS TO PLANES

16. THEOREM 9. **If a line is perpendicular to each of two lines at their intersection, it is perpendicular to all lines that are in their plane and are drawn through their intersection.**

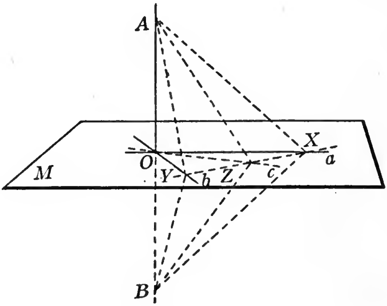


FIG. 11

Hypothesis: Line $AO \perp$ lines a and b at their intersection O. M is the plane of a and b.

Conclusion: $AO \perp$ all lines in M through O.

Analysis and construction:

I. To prove $AO \perp$ all lines in M through O, draw c any line in M through O and prove $AO \perp c$.

II. Draw any line in M intersecting a, b, and c in X, Y, and Z, respectively. Extend AO, making $AO = OB$. Join AX, AZ, AY, BX, BZ, and BY.

III. To prove $AO \perp c$, prove $ZA = ZB$ (*Plane Geometry*, Th. 86, Cor.).

IV. $\therefore$ prove $\triangle AXZ \cong \triangle BXZ$.

V. $\therefore$ " $AX = XB$ and $\angle AXZ = \angle BXZ$.

VI. To prove $\angle AXZ = \angle BXZ$, prove $\triangle AXY \cong \triangle BXY$.

VII. $\therefore$ prove $AX = BX$ and $AY = BY$.

In the same way AO may be proved perpendicular to all lines in M through O.

Exercise. Make a model of toothpicks, string, and cardboard to illustrate Fig. 11 and give the analysis above from your model.

The intersection of a line and a plane is called the **foot** of the line.

A line that is perpendicular to all lines in a plane passing through its foot is said to be **perpendicular** to the **plane.**

Theorem 9 may be stated: **If a line is perpendicular to each of two lines at their point of intersection, it is perpendicular to their plane.**

COR. **If a line is perpendicular to a plane, it is perpendicular to all lines in the plane passing through its foot.**

17. THEOREM 10. **If from a point in a perpendicular to a plane equal oblique segments are drawn to the plane, they cut off equal distances on the plane from the foot of the perpendicular and conversely** (Fig. 12).

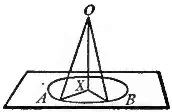


FIG. 12

Exercise. If from a point in a perpendicular to a plane unequal oblique segments are drawn to the plane, the longer segment cuts off the greater distance from the foot of the perpendicular and conversely (Fig. 13).

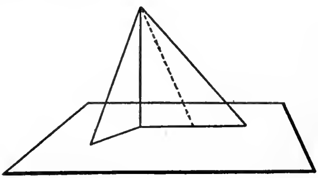


FIG. 13

DETERMINATION OF PLANES

18. THEOREM 11. **Not more than one plane can be drawn containing a given point and perpendicular to a given line.**

Case A. When the given point is on the given line.

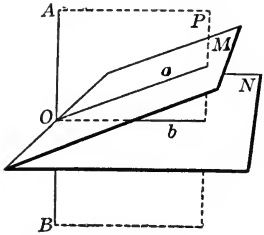


FIG. 14

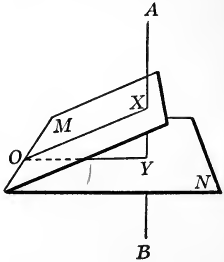


FIG. 15

Analysis and construction (Fig. 14):

I. Suppose that two planes M and N are both $\perp$ AB at O.

II. Show that this supposition would give two lines in the same plane $\perp$ the same line at the same point.

III. $\therefore$ pass any plane through AB intersecting M and N in lines a and b respectively.

Outline of proof:

I. *a.* Line $a \perp AB$ at O in plane P.
b. Line $b \perp AB$ at O in plane P.
c. This is impossible. (*Plane Geometry*, As. 7.)

II. $\therefore$ M and N are not both $\perp$ AB at O.

Case B. When the given point is without the given line.

Analysis and construction (Fig. 15):

I. Suppose that two planes M and N are both $\perp$ AB from O.

II. Show that this supposition would give two lines $\perp$ AB from a point without AB.

III. $\therefore$ join O with X and Y, the intersection of AB with M and N respectively. Prove OX and $OY \perp AB$.

We will assume from Th. 9 that **one plane can be drawn perpendicular to a given line and contain a given point** (see §57, Ex. 1 and 2).

19. THEOREM 12. **All perpendiculars to a given straight line at a given point lie in a plane that is perpendicular to the line at the given point.**

Suggestion. Prove that the plane determined by OX and OY coincides with the plane determined by OY and OZ. $\therefore$ prove planes XOY and YOZ each $\perp$ AB at O.

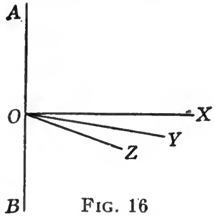


FIG. 16

DETERMINATION OF LINES

20. THEOREM 13. **Not more than one perpendicular can be drawn to a given plane from a given point.**

Case A. When the point is in the plane.

Suggestion. Suppose a and b (Fig. 17) are two perpendiculars to M from O. Let the plane of a and b intersect M in line RS. Show that there would be at point O two lines in the same plane perpendicular to line RS.

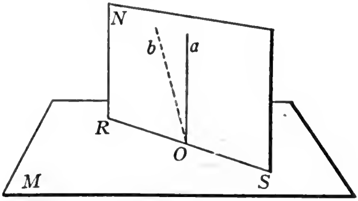


FIG. 17

Case B. When the point is without the plane.

Suggestion. How is line RS (Fig. 18) determined? Show that there would be from point O two lines in the same plane perpendicular to the line RS.

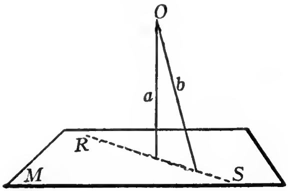


FIG. 18

We will assume that **one line can be drawn perpendicular to a given plane from a given point** (see §57, Ex. 3 and 4).

COMPARATIVE LENGTHS OF SEGMENTS

21. THEOREM 14. **The perpendicular from a point to a plane is the shortest distance from the point to the plane.**

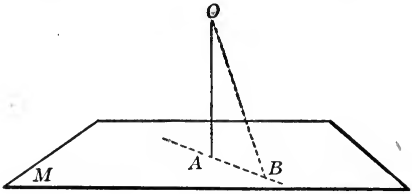


FIG. 19

Suggestion. If OA is not the shortest distance from O to plane M, suppose OB to be the shortest distance. Find the intersection of plane AOB and M. Show that this contradicts Th. 56, *Plane Geometry*, the perpendicular is the shortest segment from a point to a straight line.

EXERCISES INVOLVING PERPENDICULARS

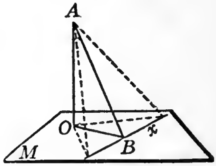


FIG. 20

22. 1. In Fig. 20, AO is perpendicular to plane M. OB is perpendicular to line x, any line in M. Prove that AB is perpendicular to x and that x is perpendicular to plane AOB.

2. In Fig. 20, suppose that AO is perpendicular to plane M and that AB is drawn from point A perpendicular to line x. Line x is any line in M. Prove that BO is perpendicular to line x and that x is perpendicular to plane AOB.

3. In Fig. 21, AB is perpendicular to plane M and BC is oblique to plane M. CD and CE are two lines in plane M making equal angles with CB. Prove that CD and CE make equal angles with CA.

4. In Fig. 21, suppose that AB is drawn perpendicular to plane M and BC is oblique to plane M. If CD and CE are in plane M and make equal angles with AC, prove that they make equal angles with line BC.

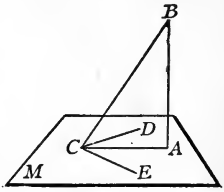


FIG. 21

5. If CA bisects $\angle DCE$ and AB is perpendicular to the plane of $\angle DCE$, prove that any point in AB is equally distant from CD and CE. (Use Ex. 2, p. 16.)

6. If three segments are equal and parallel and are not all in the same plane, the segments joining corresponding extremities form congruent triangles.

EQUAL ANGLES IN SPACE

23. THEOREM 15. **If two angles lying in different planes have their sides respectively parallel and lie on the same side of the line joining their vertices, the angles are equal.**

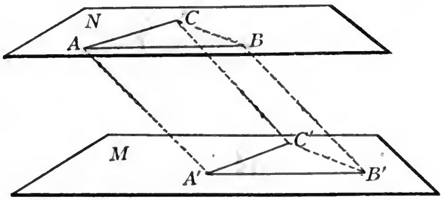


FIG. 22

Hypothesis: $\angle s$ BAC and $B'A'C'$, lying in planes N and M, have $AB \parallel A'B'$, $AC \parallel A'C'$, and AB, $A'B'$, AC, and $A'C'$ lying on the same side of the line AA' which joins their vertices.

Conclusion: $\angle BAC = \angle B'A'C'$.

Analysis and construction:

I. To prove $\angle A = \angle A'$, prove them corresponding angles of congruent triangles.

II. $\therefore$ make $AB = A'B'$, $AC = A'C'$, join BC and $B'C'$, and prove $BC = B'C'$.

III. To prove $BC = B'C'$, join BB' and CC' and prove $BB'C'C$ a $\square$.

IV. $\therefore$ prove $BB' \parallel$ and $= CC'$.

V. $\therefore$ prove BB' and CC' each $\parallel$ and equal to AA'.

VI. $\therefore$ prove $AA'B'B$ and $AA'C'C$ $\boxed{s}$.

The proof is left to the pupil.

PERPENDICULARS AND PARALLELS

TEST FOR PERPENDICULARS TO PLANES

24. THEOREM 16. **If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.**

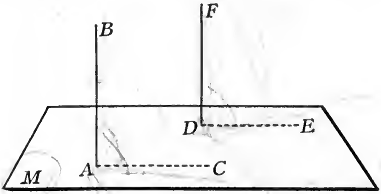


FIG. 23

Hypothesis: AB and DF are two parallel straight lines cutting plane M at A and D respectively. $AB \perp$ plane M.

Conclusion: $DF \perp$ plane M.

Analysis and construction.

I. To prove $DF \perp M$, prove $DF \perp$ to two lines in M.

II. To prove $DF \perp$ one line, as DE, in M, draw AC in M from A and $\parallel DE$ and prove that
(1) $\angle FDE = \angle BAC$ and (2) $\angle BAC$ is a rt. $\angle$.

III. In the same way DF can be proved $\perp$ any other line in M through D.

TEST FOR PARALLEL LINES

25. THEOREM 17. **Two lines perpendicular to the same plane are parallel to each other.**

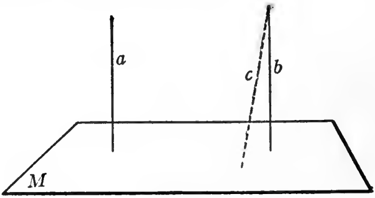


FIG. 24

Hypothesis: Lines a and b are each $\perp$ plane M.
Conclusion: $a \parallel b$.

Analysis and construction:

I. To prove $b \parallel a$, prove that b coincides with a line that is $\parallel a$.

II. $\therefore$ construct c from a point in $b \parallel a$ and prove that c coincides with b.

III. To prove that c and b coincide, show that c and b are both $\perp M$ from the same point.

COR. **Two straight lines parallel to a third are parallel to each other.**

Suggestion. If lines a and b are both parallel to line c, pass a plane $\perp$ line c and prove that a and b are both $\perp$ this plane.

TEST FOR PERPENDICULARS TO PLANES

26. THEOREM 18. **If a straight line is perpendicular to one of two parallel planes, it is perpendicular to the other.**

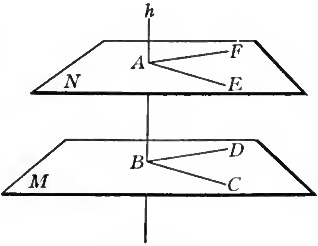


FIG. 25

Hypothesis: Plane $M \parallel$ plane N. Line $h \perp$ plane N.

Conclusion: Line $h \perp$ plane M.

Analysis and construction:

I. To prove line $h \perp M$, prove $h \perp$ two lines in M.

II. $\therefore$ construct BC and BD any two lines in M through B and prove $h \perp BC$ and BD.

III. To prove $h \perp BC$ and BD, prove $h \perp$ lines that are $\parallel BC$ and BD.

IV. $\therefore$ let the plane of h and BC cut N in line AE and the plane of h and BD cut N in line AF and prove $AE \parallel BC$ and $AF \parallel BD$.

TEST FOR PARALLEL PLANES

27. THEOREM 19. **Two planes perpendicular to the same straight line are parallel.**

Outline of proof:

I. If the two planes were not parallel, they would intersect.

II. Suppose the two planes intersect. Let P represent some point in the intersection.

III. We should then have two planes containing a given point and perpendicular to a given line.

IV. This is impossible.

V. $\therefore$ the planes are parallel.

MISCELLANEOUS EXERCISES

28. 1. If two angles lying in different planes have their sides parallel and lying on opposite sides of the line joining their vertices, the angles are equal.

2. When are two angles lying in different planes supplementary? Give proof.

3. Two planes parallel to a third are parallel to each other.

Analysis:

I. To prove plane $M \parallel$ plane N, prove them $\perp$ the same line.

II. $\therefore$ construct a line $\perp$ the third plane and prove this line $\perp$ M and N.

4. Are two lines perpendicular to the same plane coplanar? Are three lines perpendicular to the same plane coplanar?

5. How many planes are determined by three lines perpendicular to the same plane? Show that each of these planes is parallel to one of the given lines.

6. Show that it is not always true in space that a line perpendicular to one of two parallels is perpendicular to the other. State this theorem so that it is always true in space.

7. Do you know any other theorems that may be true on a plane that are not true in space?

8. If a line is parallel to a plane, it is everywhere equally distant from the plane.

9. Two points on the same side of a plane and equally distant from it determine a line parallel to the given plane.

10. Two parallel planes are everywhere equally distant.

11. If lines a and b are parallel and planes M and N are perpendicular respectively to lines a and b, prove that plane M is either parallel to or coincides with plane N.

12. If planes M and N are parallel and lines a and b are perpendicular respectively to planes M and N, then line a is either parallel to or coincides with line b.

13. A line and a plane perpendicular to the same line are parallel unless the line lies in the plane.

14. If a line and a plane are parallel, is any line perpendicular to the given line perpendicular also to the given plane? Illustrate your answer by using pencils and pieces of cardboard.

15. Given line a parallel to plane M. From any point in a draw line b perpendicular to M. Prove that b is perpendicular to a.

16. If each of two intersecting planes contains one of two parallel lines, the intersection of the planes is parallel to each of the lines and to their plane.

17. AB, CD, and EF are three equal segments perpendicular to plane M at the points A, C, and E, respectively. Prove that the plane determined by points B, D, and F is parallel to plane M.

DIHEDRAL ANGLES

29. The two parts into which a straight line divides a plane are called **half-planes.** The line is called the **edge** of the half-plane.

Two half-planes with a common edge form a **dihedral angle.** The common edge is the **edge** of the dihedral angle. The half-planes are the **faces** of the dihedral angle.

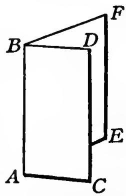


FIG. 26

In Fig. 26, the half-planes $ABDC$ and $ABFE$ have a common edge AB. AB is the edge and $ABDC$ and $ABFE$ are the faces of the dihedral angle. The angle may be read $D\text{–}AB\text{–}E$, or, if there is no ambiguity, AB.

MEASUREMENT OF DIHEDRAL ANGLES

30. An angle formed by two straight lines, one in each face of the dihedral angle, perpendicular to the edge at the same point is called a **plane angle** of the dihedral angle. In Fig. 27, $\angle XOY$ is a plane angle of the dihedral angle $D\text{-}AB\text{-}E$ if OX is in face $ABFE$ and OY is in face $ABDC$ and OX and OY are each perpendicular to AB at the same point O.

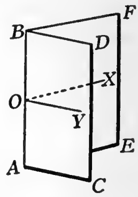


FIG. 27

THEOREM 20. **All plane angles of the same dihedral angles are equal.**

The proof is left to the pupil.

THEOREM 21. **If a plane is perpendicular to the edge of a dihedral angle, its intersections with the faces of the dihedral angle form a plane angle of the dihedral angle.**

31. Two dihedral angles are said to be **congruent** if they can be made to coincide. Two dihedral angles are said to be **equal** if they have the same measure number.

The **measure of a dihedral angle** will be defined as the measure of its plane angle. From this definition we have

THEOREM 22. **If two dihedral angles are equal, their plane angles are equal.**

THEOREM 23. **If the plane angles of two dihedral angles are equal, the dihedral angles are equal.**

Exercise. Cut and fold a card as shown in Fig. 28 so as to illustrate the definition of the measure of a dihedral angle.

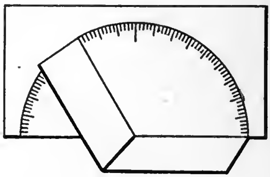


FIG. 28

As. 8. **Two dihedral angles that are congruent are equal, and, conversely, two dihedral angles that are equal are congruent.**

RELATED DIHEDRAL ANGLES

32. Two dihedral angles are said to be **complementary** or **supplementary** according as their plane angles are complementary or supplementary.

Two dihedral angles are said to be **adjacent** if they have a common edge and a common face separating the angles.

Two dihedral angles are said to be **vertical** if the faces of one are prolongations of the faces of the other.

33. THEOREM 24. **Two vertical dihedral angles are equal.**

Suggestion. In this theorem and in the following exercises pass a plane perpendicular to the edge of one of the dihedral angles and show that this will determine the plane angles of all the dihedral angles.

Ex. 1. If one plane meets another, the adjacent dihedral angles formed are supplementary.

Suggestion. Pass a plane perpendicular to the common edge and reduce to a plane geometry theorem.

Ex. 2. If two parallel planes are cut by a third plane, the alternate interior dihedral angles are equal.

Ex. 3. Prove Ex. 2 for the corresponding dihedral angles.

Ex. 4. Can you state any other cases of equal or supplementary dihedral angles? Give proof.

PERPENDICULAR PLANES

FUNDAMENTAL TEST

34. If the plane angle of a dihedral angle is a right angle, the **dihedral angle is said to be a right dihedral angle** and the faces of the dihedral angle are said to be perpendicular to each other.

If two planes are perpendicular to each other, the dihedral angles formed are equal and the plane angles of the dihedral angles are right angles.

These definitions follow at once from the definition of the measure of a dihedral angle.

TEST FOR PERPENDICULAR PLANES

35. THEOREM 25. **If a straight line is perpendicular to a given plane, every plane passed through the line is perpendicular to the given plane.**

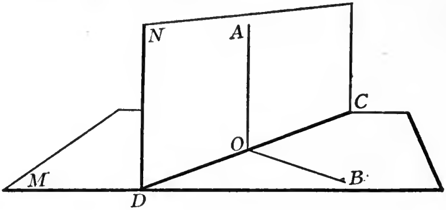


FIG. 29

Hypothesis: Line $AO \perp$ plane M at point O. N is any plane passed through line AO intersecting plane M in line CD.

Conclusion: Plane $N \perp$ plane M.

Analysis and construction:

I. To prove plane $N \perp$ plane M, prove dihedral angle A–CD–B a right dihedral angle.

II. $\therefore$ prove a plane angle of dihedral angle A–CD–B a right angle.

III. $\therefore$ draw OB in plane $M \perp CD$ at O and prove
  (1) $\angle AOB$ the plane angle of dihedral angle A–CD–B.
  (2) $\angle AOB$ a right angle.

Ex. 1. If a plane is perpendicular to the edge of a dihedral angle, it is perpendicular to each of its faces.

Ex. 2. The rays that form a plane angle of a dihedral angle lie in a plane which is perpendicular to the faces of the dihedral angle.

Ex. 3. If three or more planes intersect in parallel lines, a plane perpendicular to one of the lines is perpendicular to all of the planes.

Ex. 4. A given line is oblique to a given plane. Show how to pass a plane through the given line so that it will be perpendicular to the given plane.

TEST FOR PERPENDICULARS TO PLANES

36. THEOREM 26. **If two planes are perpendicular to each other, a line drawn in one of them perpendicular to the intersection is perpendicular to the other.**

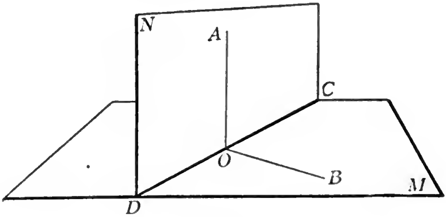


FIG. 30

Hypothesis: Plane $N \perp$ plane M, intersecting plane M in line CD. AO is a line in plane $N \perp CD$ at O.

Conclusion: Line $AO \perp$ plane M.

Analysis and construction:

I. To prove $AO \perp$ plane M, prove $AO \perp$ two lines in plane M through point O.

II. Since $AO \perp CD$, construct OB in plane $M \perp CD$ at O and prove $AO \perp OB$.

III. To prove $AO \perp OB$, prove $\angle AOB$ a plane angle of dihedral angle $A\text{–}CD\text{–}B$.

The proof is left to the pupil.

Ex. 1. If a line and a plane not containing the line are perpendicular to the same plane, they are parallel.

Suggestion. Line a and plane M are both $\perp$ plane N. Draw a line in $M \perp$ the intersection of M and N. Prove the line $\parallel a$.

Ex. 2. If a line is parallel to a given plane, any plane perpendicular to the line is perpendicular also to the given plane.

Suggestion. Line a is $\parallel$ plane M and $\perp$ plane N. Pass any plane through a intersecting M in line b. Prove $b \perp N$.

Ex. 3. A plane perpendicular to one of two parallel planes is perpendicular to the other.

Suggestion. Use a construction line.

DETERMINATION OF LINES

37. THEOREM 27. **If two planes are perpendicular, a line drawn perpendicular to the first from any point in the second lies wholly in the second.**

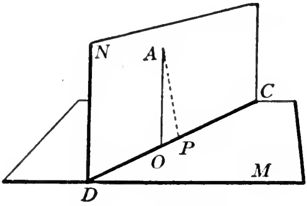


FIG. 31

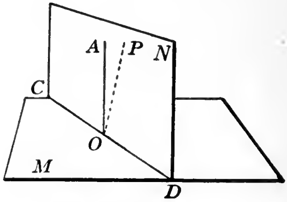


FIG. 32

Hypothesis: Planes M and $N \perp$ each other and intersect in line CD. Line AO is drawn from a point in plane $N \perp$ plane M.

Conclusion: AO lies wholly in plane N.

Case I. AO is drawn from point A, a point not in CD.

Analysis and construction (FIG. 31):

I. To prove that AO lies wholly in plane N, prove that AO coincides with a line that does lie in plane N.

II. $\therefore$ construct AP from $A \perp CD$ and prove that AO coincides with AP.

III. To prove that AO coincides with AP, show that AO and AP are both $\perp$ plane M from A.

Case II. AO is drawn from point O in line CD (Fig. 32).

Analysis and proof are left to the pupil.

Ex. 1. The plane perpendicular to the edge of a dihedral angle from any point P contains the perpendiculars from P to the faces of the dihedral angle.

Ex. 2. Prove that the perpendiculars from P referred to in Ex. 1 form an angle which is the supplement of a plane of the dihedral angle.

TEST FOR PERPENDICULARS TO PLANES

38. THEOREM 28. **If two intersecting planes are perpendicular to a third plane, their intersection is perpendicular to that plane.**

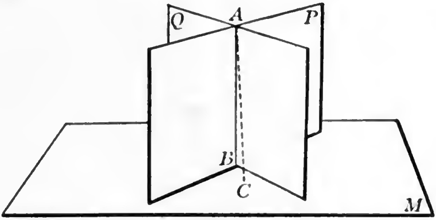


FIG. 33

Hypothesis: Planes P and $Q \perp$ plane M. Planes P and Q intersect in line AB.

Conclusion: $AB \perp$ plane M.

Analysis and construction:

I. To prove $AB \perp$ plane M, prove that AB coincides with a line that $\perp$ plane M.

II. $\therefore$ construct AC from any point in $AB \perp$ plane M and prove that AC coincides with AB.

III. To prove that AC coincides with AB, prove that AC lies in both plane P and plane Q.

The proof is left to the pupil.

Ex. 1. Prove Th. 28 by drawing the construction line referred to in step I of the analysis from B perpendicular to plane M.

Ex. 2. If three non-parallel planes are each perpendicular to a third plane, their intersections are parallel.

Ex. 3. If two intersecting planes are perpendicular respectively to two intersecting lines, the line determined by the planes is perpendicular to the plane determined by the lines.

Ex. 4. Two dihedral angles with their edges parallel and their faces perpendicular to each other are either equal or supplementary.

Suggestion. Pass a plane $\perp$ the two parallel edges.

DETERMINATION OF PLANES

39. Theorem 29. **Through a straight line oblique to a plane one plane can be passed perpendicular to the plane and only one.**

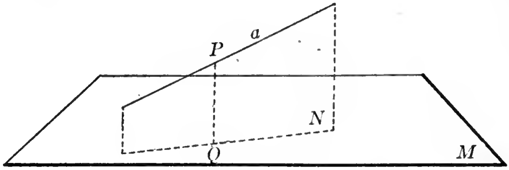


Fig. 34

Hypothesis: Line a is oblique to plane M.

Conclusion: (1) One plane can be passed through $a \perp M$. (2) Only one plane can be passed through $a \perp M$.

Analysis and construction (1):

I. A plane $\perp M$ must contain a line $\perp M$.

II. $\therefore$ from any point in a draw $PQ \perp M$ and prove the plane of a and $PQ \perp M$.

Analysis (2): To prove plane N the only plane through $a \perp M$, prove that there is

a. Only one $\perp$ from P to M.

b. Only one plane containing a and PQ.

c. Perpendiculars to M from all points in a lie wholly in N.

Let the pupil give the proof.

Ex. 1. In Fig. 34 how many planes could be drawn perpendicular to plane M and containing line a if line a were perpendicular to plane M?

Ex. 2. How many planes can be drawn to contain a given point and perpendicular to a given plane? How would these planes be obtained?

Ex. 3. Can a plane be drawn through a given point perpendicular to each of two given planes, (1) when the planes intersect; (2) when the planes do not interesect? Show how this plane is obtained in each case.

PROPORTIONAL SEGMENTS

40. THEOREM 30. **If two straight lines are cut by three parallel planes, the corresponding segments are proportional.**

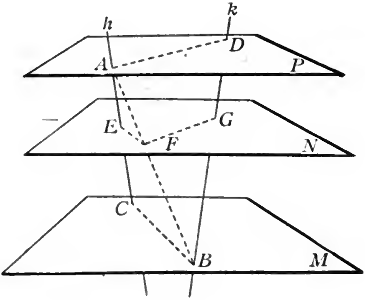


FIG. 35

Hypothesis: The parallel planes M, N, and P cut the lines h and k at points A, E, C, and D, G, B respectively.

Conclusion: $\frac{AE}{EC}=\frac{DG}{GB}$.

Analysis and construction:

I. To prove $\frac{AE}{EC}=\frac{DG}{GB}$, prove each ratio equal to a third ratio.

II. $\therefore$ join AB and prove $\frac{AE}{EC}=\frac{AF}{FB}$ and $\frac{DG}{GB}=\frac{AF}{FB}$.

LOCI IN SPACE

41. As in plane geometry, so in solid geometry the locus of points that obey one or more requirements consists of all points that satisfy the requirements and of no other points. For a complete proof of a locus theorem it is necessary to prove that

(1) All points on the locus obey the requirements.

(2) All points that obey the requirements are on the locus.

IMPORTANT SPECIAL CASES

42. THEOREM 31. **The locus of points equally distant from the faces of a dihedral angle is the half-plane bisecting the dihedral angle.**

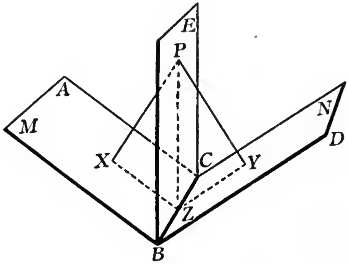


FIG. 36

Hypothesis: $A\text{–}BC\text{–}D$ is a dihedral angle bisected by the plane E.

Conclusion: Plane E is the locus of points equally distant from the faces M and N; that is,

a. Every point in E is equally distant from M and N.

b. Every point equally distant from M and N lies in E.

Analysis and construction for a:

I. Let P be any point in plane E.

II. $\therefore$ draw $PX \perp M$ and $PY \perp N$ and prove $PX = PY$

III. To prove $PX = PY$, let the plane of PX and PY intersect M in XZ, and N in YZ, and E in PZ and prove $\triangle PXZ \cong \triangle PYZ$.

IV. To prove $\triangle PXZ \cong \triangle PYZ$, prove $\angle PZX = \angle PZY$.

V. $\therefore$ prove $\angle PZX$ and $\angle PZY$ plane angles of the dihedral angles $A\text{–}BC\text{–}E$ and $D\text{–}BC\text{–}E$.

VI. $\therefore$ prove plane $PXY \perp BC$.

Analysis and construction for b:

I. Let P be any point equally distant from M and N. $\therefore$ let $PX = PY$.

II. To prove that P lies in plane E, draw plane PBC and prove that plane PBC bisects the dihedral angle $A-BC-D$.

III. $\therefore$ let the plane of PX and PY intersect M in XZ, N in ZY, and PBC in PZ and prove that

(1) $\angle PZX = \angle PZY$. (Prove $\triangle PZX \cong \triangle PZY$.)

(2) $\measuredangle PZX$ and PZY are plane angles of the dihedral angles.

43. THEOREM 32. **The locus of points equally distant from two given points is a plane bisecting at right angles the segment joining the given points.**

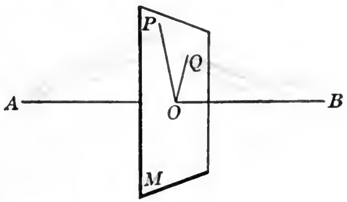


FIG. 37

Hypothesis: M is a plane bisecting at right angles the segment joining A and B.

Conclusion: M is the locus of points equally distant from A and B; that is,

a. Every point in M is equally distant from A and B.

b. Every point equally distant from A and B lies in M.

Analysis for a:

I. Let P be any point in M.

II. To prove P equally distant from A and B, join PO and prove $PO \perp AB$ (*Plane Geometry*, Th. 86).

Analysis for b:

I. Let Q be any point equally distant from A and B; that is, let $QA = QB$.

II. To prove Q lies in M, prove $QO \perp AB$ (see Th. 12).

GENERAL DISCUSSION

44. In solid geometry two kinds of loci of points are to be considered:

I. Loci of points fulfilling one requirement. In this case the locus is, in general, a surface or a number of surfaces.

II. Loci of points fulfilling two requirements. In this case the locus is, in general, a line or a number of lines.

If three requirements are given, one or more points are, in general, determined.

LOCUS OF POINTS FULFILLING ONE REQUIREMENT

45. The definitions and exercises in this article give examples of loci of points fulfilling one condition.

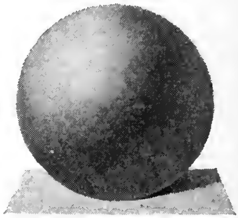


FIG. 38

The surface formed when a circle is revolved about a diameter is called a spherical surface. A **spherical surface** may be defined as the locus of points in space at a given distance from a given point (Fig. 38).

Ex. 1. From what plane locus is a spherical surface obtained? How?

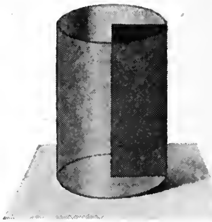


FIG. 39

The surface formed when one of two parallel lines revolves about the other is called a **cylindrical surface of revolution.** It is the locus of points in space at a given distance from a given line (Fig. 39).

Ex. 2. From what plane locus may a cylindrical surface be obtained. How?

Ex. 3. Find the locus of points in space that are equally distant from two parallel planes.

Ex. 4. Find the locus of points in space that are at a given distance from a given plane.

LOCI OF POINTS FULFILLING TWO REQUIREMENTS

46. In the following exercises the loci required are obtained by the intersection of two surfaces. The method of treatment is indicated in the suggestions that follow Ex. 1.

Ex. 1. Find the locus of points in space that are equally distant from two given parallel planes and also equally distant from two given points.

Solution:

I. All points equally distant from two given parallel planes lie in what plane? Call this plane locus I.

II. All points equally distant from two given points lie in what plane? Call this plane locus II.

III. Let locus I intersect locus II in line a. ∴ all points that fulfill both requirements lie in line a.

Discussion:

I. Ordinarily the required locus consists of one line, the intersection of loci I and II.

II. If loci I and II coincide, the required locus is a plane.

III. If loci I and II are parallel, the problem has no solution.

Find the locus of points in a given plane which are also

Ex. 2. Equally distant from two given parallel planes.

Ex. 3. Equally distant from two given points not in the given plane.

Ex. 4. At a given distance from a second given plane.

Ex. 5. Equally distant from two other given intersecting planes.

Find the locus of points in space which fullfiill the requirements given in Ex. 6–9.

Ex. 6. At a given distance from a given plane and also equally distant from two given points not in the given plane.

Ex. 7. Equally distant from two given points A and B and also equally distant from two other given points C and D.

Ex. 8. Equally distant from three given points.

Ex. 9. Equally distant from three planes that meet in a point.

Ex. 10. Equally distant from two given points A and B and also equally distant from two given planes, (1) when the planes are parallel, (2) when the planes intersect.

EXERCISES INVOLVING MISCELLANEOUS LOCI

47. 1. Find the locus of points in space equally distant from the vertices of a triangle. Prove.

2. Find the locus of points in space equally distant from every point on a circle. Prove.

3. Determine a point in a given plane which is equally distant from the vertices of a triangle. Discuss all possibilities.

4. Determine a point in a given plane that is equally distant from every point in a circle. Discuss all possibilities.

5. Why are there two parts to Fig. 40?

We have in solid geometry not only loci of points, but also loci of lines and even of surfaces. The following definitions and exercises illustrate some of the possibilities.

The surface formed when one of two intersecting lines that form an acute angle with each other revolves about the other as an axis is called a **conical surface of revolution.** It is the locus of lines that make a given acute angle with a given line at a given point in the line (Fig. 40).

The line used as an axis about which the other line revolves is not shown in Fig. 40. What is the position of this axis? What does the figure show?

6. What is the locus generated when the lines are at right angles?

7. What is the locus of lines that are parallel to a given line and at a given distance from it?

8. What is the locus of lines that make a given angle with a given plane at a given point in the plane? Discuss various possibilities.

9. What is the locus of lines that pass through a fixed point and are parallel to a fixed plane?

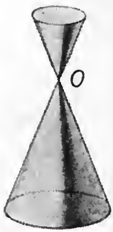


FIG. 40

10. Cut out a rectangle and imagine it to move in a direction perpendicular to its surface. What is the path of the surface of the moving rectangle?

11. Answer the preceding question using a circle instead of a rectangle.

12. Can you move the rectangle referred to in Ex. 10 so that the path of its surface will be different from that obtained in Ex. 10?

13. Answer questions 10 and 12 for various kinds of triangles instead of for a rectangle.

14. Can you make any general statement concerning the path of (1) a moving point; (2) a moving line; (3) a moving surface? Suppose in each case that the moving point, line, or surface obeys one or more requirements.

15. What is the path of a moving solid?

PROJECTION

48. Objects are represented or pictured on surfaces by means of **projection.** Projections are of two kinds, **parallel** and **central.**

Fig. 41 illustrates parallel projection. Parallel rays through the points A, B, C, D intersect the plane M in the points A', B', C', D'. $A'B'C'D'$ is the projection of $ABCD$ on the plane M. The projecting lines may make any angle with the plane. The plane M may be replaced by any surface whatsoever. A shadow made by the sun is an example of parallel projection.

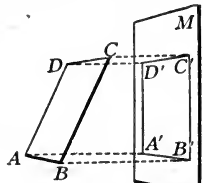


FIG. 41

By the **projection of a point on a plane** is meant the intersection of the projecting ray and the plane.

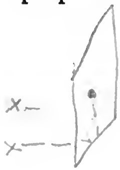
By the **projection of a line on a plane** is meant the locus of the projections of all of its points.

If the projecting rays are perpendicular to the plane, the projection is said to be **orthogonal.**

If the rays start from a common origin, instead of being parallel, the projection is said to be **central.** A shadow made by a candle is an example of central projection.

Hereafter when the word "projection" is used orthogonal projection is intended.

49. THEOREM 33. **The projection of a straight line not perpendicular to a plane upon that plane is a straight line.**



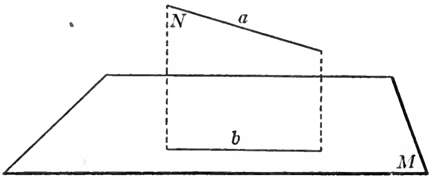


FIG. 42

Analysis and construction:

I. Prove that the projection of a on M coincides with a line that is straight.

II. ∴ construct plane N through $a \perp M$, intersecting M in b. Prove that b is the projection of a on M.

III. ∴ prove that b contains the projections of all points in a.

IV. ∴ prove that $\perp$s to M from all points in a lie in N.

COR. **If a straight line is oblique to a given plane, its projection upon that plane is the intersection of the given plane with a plane through the line perpendicular to the plane.**

The **projection of a segment on a plane** is the segment between the feet of the perpendiculars drawn to the plane from the extremities of the given segment.

Ex. 1. If a segment is parallel to a plane, it is parallel and equal to its projection on the plane.

Ex. 2. Can the projection of a curve upon a plane be a straight line? How?

Ex. 3. Can the projection of a curve upon each of two intersecting planes be straight lines? How?

Ex. 4. If two lines are parallel, their projections on the same plane coincide or are parallel.

Ex. 5. The projections of a segment upon two parallel planes are parallel and equal.

50. THEOREM 34. **The acute angle that a straight line makes with its own projection on a plane is the least angle that it makes with any line of the plane.**

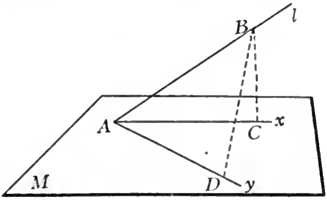


FIG. 43

Analysis and construction: To prove $\angle BAC < \angle BAD$, construct $BC \perp M$ from B, any point in l, and intersecting line x at C. Make AD on line y equal to AC. Join BD. Prove $BC < BD$ (see *Plane Geometry*, Th. 60).

An angle that a straight line makes with its own projection on a plane is called the **inclination of the line to the plane** or the **angle of the line and the plane.**

Ex. 1. If a line is oblique to a plane, what is the largest angle that it makes with any line of the plane? Why?

Ex. 2. If a straight line intersects two parallel planes, it makes equal angles with them.

51. THEOREM 35. **The length of the projection of a given segment on a plane is the length of the segment multiplied by the cosine of the angle of inclination of the line to the plane** (Fig. 44).

Suggestion. Show that $AB = AC \cos \angle BAC$. $\therefore XY = AC \cos \angle BAC$.

Exercise. Using the tables, find the lengths of the projections of the following segments. The lengths and the angles made with the planes are given:

a. 15 in., 42°.
b. 27 in., 38°.
c. 36 in., 67°.
d. 23 in., 52°.

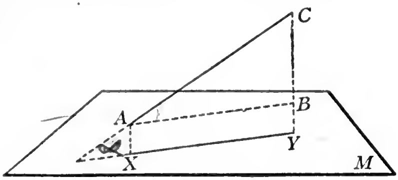


FIG. 44

POLYHEDRAL ANGLES

52. If a ray has a fixed origin and moves so as continually to intersect the perimeter of a fixed polygon not in the plane of its origin, the ray is said to generate a **polyhedral angle.**

In Fig. 45, O–$ABCD$ etc. is a polyhedral angle. O is the fixed origin of the ray and is called the **vertex** of the polyhedral angle. $ABCD$ etc. is the fixed polygon. OA, OB, etc., are called the **edges** of the polyhedral angle. The planes AOB, BOC, etc., are called its **faces.** The angles AOB, BOC, etc., made by two consecutive edges are called the **face angles** of the polyhedral angle. The dihedral angles made by two consecutive faces are called its **dihedral angles.**

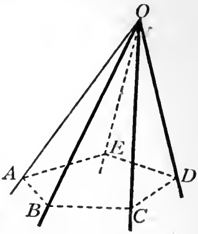


FIG. 45

The parts of a polyhedral angle are its face angles, its faces, and its dihedral angles.

A polyhedral angle is said to be **convex** or **concave** according as the fixed polygon is convex or concave. Only convex polyhedral angles will be considered.

A polyhedral angle with three faces is called a **trihedral angle.**

The **size** of a polyhedral angle depends upon the spread of the planes forming the faces of the polyhedral angle. The measure of the polyhedral angle is too difficult for this book.

A polyhedral angle is one kind of a **solid angle.** Another kind of a solid angle is the **conical angle.** A conical angle is generated by a ray that has a fixed origin and moves so as continually to intersect a fixed closed curve.

If the face angles and the dihedral angles of one polyhedral angle are equal respectively to the corresponding parts of the other, and arranged in the same order, the polyhedral angles are said to be **congruent.**

If the face angles and the dihedral angles of one polyhedral angle are equal respectively to the corresponding parts of the other, but arranged in the opposite order, the polyhedral angles are said to be **symmetric.**

Ex. 1. Construct out of cardboard two congruent trihedral angles; also two symmetric trihedral angles.

Ex. 2. If the edges of one polyhedral angle are extended through the vertex, a polyhedral angle is formed which is symmetric to the given polyhedral angle.

53. THEOREM 36. **Any face angle of a trihedral angle is less than the sum of the other two.**

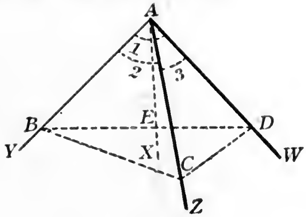


FIG. 46

Hypothesis: A–BCD is a trihedral angle with the face angles 1, 2, 3, and the edges AY, AZ, and AW.

Conclusion: $\angle 1 < \angle 2 + \angle 3$.

Case A. When $\angle 1 =$ either $\angle 2$ or $\angle 3$.

Case B. When $\angle 1 <$ either $\angle 2$ or $\angle 3$.

Case C. When $\angle 1 >$ either $\angle 2$ or $\angle 3$.

Proof is required for Case C only. Why?

Analysis and construction:

I. To prove $\angle 1 < \angle 2 + \angle 3$, construct $\angle XAW$ in $\angle 1$ so that $\angle XAW = \angle 3$ and prove $\angle YAX < \angle 2$.

II. $\therefore$ lay off equal segments AE and AC on AX and AZ respectively. Pass any plane through points E and C cutting AW and AY at D and B respectively. Prove $EB < BC$ (*Plane Geometry*, Th. 60).

III. $\therefore$ since $BD < BC + CD$, prove $DE = CD$.

54. THEOREM 37. **The sum of the face angles of any polyhedral angle is less than four right angles.**

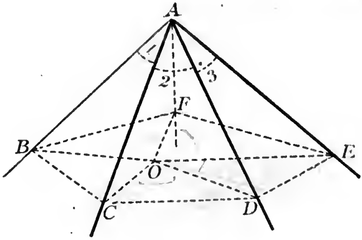


FIG. 47

Hypothesis: A–$BCDE$ etc. is any polyhedral angle with face angles 1, 2, 3, etc.

Conclusion: $\angle 1 + \angle 2 + \angle 3 +$ etc. < 4 rt. $\measuredangle$.

Analysis and construction:

I. To prove $\angle 1 + \angle 2 + \angle 3 +$ etc. < 4 rt. $\measuredangle$, compare them with angles that are equal to 4 rt. $\measuredangle$.

II. $\therefore$ construct plane $BCDE$ etc. cutting all the edges of A–$BCDE$ etc., but not containing point A. Take O any point in polygon $ABCD$ etc., join BO, CO, DO, etc., and prove the sum of the $\measuredangle$ about $A <$ the sum of the $\measuredangle$ about O.

III. $\therefore$ compare the sum of the $\measuredangle$ of the $A\triangle$ with the sum of the $\measuredangle$ of the $O\triangle$ and the sum of the base $\measuredangle$ of $A\triangle$ with the sum of the base $\measuredangle$ of $O\triangle$.

Outline of proof:

I. Sum of the $\measuredangle$ of $A\triangle =$ sum of the $\measuredangle$ of $O\triangle$.

II. $\angle BCA + \angle ACD > \angle BCD$.
$\angle CDA + \angle ADE > \angle CDE$, etc.
$\therefore$ sum of the base $\measuredangle$ of $A\triangle >$ sum of base $\measuredangle$ of $O\triangle$.

III. $\therefore$ sum of vertex $\measuredangle$ of $A\triangle <$ sum of vertex $\measuredangle$ of $O\triangle$. (See *Plane Geometry*, As. 34.)
$\therefore \angle 1 + \angle 2 + \angle 3 +$ etc. < 4 rt. $\measuredangle$.

Exercise. If a trihedral angle has two face angles right angles, two of its dihedral angles are right dihedral angles.

55. THEOREM 38. **If two trihedral angles have the three face angles of one equal to the three face angles of the other, the dihedral angles opposite equal face angles are equal.**

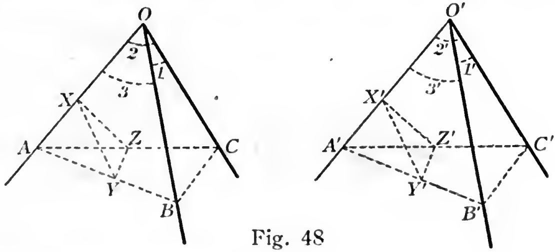


Fig. 48

Analysis and construction:

I. To prove dihedral angle OA equals dihedral angle $O'A'$, prove their plane angles equal.

II. ∴ take $O'A' = O'B' = O'C' = OA = OB = OC$. Join AB, BC, CA and $A'B'$, $B'C'$, $C'A'$. Take OX in OA equal to $O'X'$ in $O'A'$. Draw $XY \perp OA$ at X in AOB, $XZ \perp OA$ at X in AOC. Let XY and XZ meet AB and AC at Y and Z respectively. $X'Y'$ and $X'Z'$ are similarly drawn in the faces of angle O'. Join YZ and $Y'Z'$. Prove $\angle YXZ = \angle Y'X'Z'$.

III. ∴ prove $\triangle XYZ \cong \triangle X'Y'Z'$.

IV. ∴ prove $XY = X'Y'$, $XZ = X'Z'$, $YZ = Y'Z'$.

V. To prove $XY = X'Y'$, prove $\triangle AXY \cong \triangle A'X'Y'$.

VI. To prove $\triangle AXY \cong \triangle A'X'Y'$, prove . . . .

Let the pupil complete the analysis and give proof.

In the same way the other dihedral angles may be proved equal.

Question. It is necessary for the proof that XY does not meet AB on BA extended. How does the construction provide for this?

COR. **If the three face angles of one trihedral angle are equal to the three face angles of another, the trihedral angles are congruent if the parts are arranged in the same order and symmetric if arranged in the opposite order.**

SUMMARY AND SUPPLEMENTARY EXERCISES

56. SUMMARY OF IMPORTANT POINTS IN CHAPTER I

A. Determination of Lines and Planes:

I. **Only one plane can be drawn** containing
- *a.* Three given points (§3).
- *b.* One given line and a given point not in the line (§3).
- *c.* Two given intersecting lines (§3).
- *d.* Two given parallel lines (§3).
- *e.* One given point and be perpendicular to a given line (§18).
- *f.* One given line oblique to a given plane and be perpendicular to the given plane (§39).

II. **Only one line can be drawn** containing a given point and
- *a.* Be perpendicular to a given line if the point is not on the line (§15).
- *b.* Be perpendicular to a given plane (§20).
- *c.* Be parallel to a given line.
- *d.* Contain a second given point.

III. **A given line lies in a plane** if it contains
- *a.* A point in the plane and is parallel to a line that is parallel to the plane (§10).
- *b.* A point in the plane and is perpendicular to a line that is perpendicular to the plane (§19).
- *c.* A point in one of two perpendicular planes and is perpendicular to the other (§37).
- *d.* Two points in the plane (§3).

B. Tests:

I. To prove **planes parallel** to planes, prove that
- *a.* They cannot meet (§4).
- *b.* Two intersecting lines in one, etc. (§12).
- *c.* They are perpendicular to the same line (§27).

II. To prove **a line and a plane parallel,** prove that
a. They cannot meet (§6).
b. The plane contains a line that is parallel to the given line (§9).

III. To prove **two lines parallel,** prove that
a. They are in the same plane and do not meet (§7).
b. One is parallel to a given plane and the other is the intersection of the given plane with a plane containing the given line (§10).
c. They are the intersections of two parallel planes cut by a third (§11).
d. They are parallel to the same line (§§13, 25).
e. They are perpendicular to the same plane (§25).

IV. To prove **a line perpendicular to a given plane,** prove that
a. It is perpendicular to two lines of the plane at their intersection (§16).
b. It is parallel to a line that is perpendicular to the plane (§24).
c. It is perpendicular to one of two parallel planes (§26).
d. It lies in one of two perpendicular planes and is perpendicular to their intersection (§36).
e. It is the intersection of two planes that are each perpendicular to the given plane (§38).

V. To prove **two lines perpendicular,** prove that one is perpendicular to a plane containing the other, at a point in the other (§16).

VI. To prove **two planes perpendicular,** prove that
a. One contains a line perpendicular to the other (§35).
b. The dihedral angle formed is a right dihedral angle (§34).

EXERCISES IN CONSTRUCTION

57. 1. Show how to construct a plane perpendicular to a given line at a given point in the line.

2. Show how to construct a plane perpendicular to a given line and containing a given point without the line.

3. From a point in a given plane construct a line perpendicular to the plane.

Analysis and construction (Fig. 49):

I. Draw OB any line in M through O. Construct plane $N \perp OB$ at O. Draw AO in $N \perp$ the intersection of M and N at O.

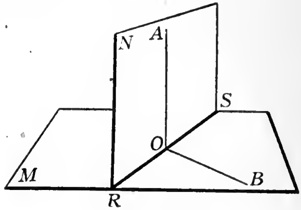


FIG. 49

II. Prove $N \perp M$. Use Ths. 25 and 26.

4. From a point without a given plane construct a line perpendicular to the given plane.

Analysis and construction (Fig. 50):

I. AO is to be in a plane that is $\perp M$ and $\perp$ the intersection of this plane and M.

II. Draw AB from $A \perp x$ any line in M. Draw line c from B, in M, $\perp x$. Draw $AO \perp$ line c from A.

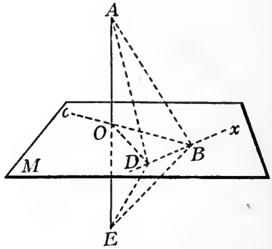


FIG. 50

5. Show that in Ex. 3 and 4 OA may be proved $\perp M$ by proving it $\perp$ to two lines in plane M.

6. Show that Ex. 4 may be solved as follows: Draw three equal segments from A to the plane, meeting the plane at B, C, and D. Find O, the center of the circle through B, C, and D. Join AO.

7. Through a given line pass a plane perpendicular to a given plane. How many such planes are possible? Discuss various cases.

8. Through a given point construct a plane that shall be perpendicular to each of two given planes: (1) when the planes intersect; (2) when the planes are parallel.

9. Construct a plane through a given point and parallel to a given plane. Is more that one such plane possible?

10. Through a given point in one of two planes draw a line parallel to the other plane: (1) when the planes intersect; (2) when the planes are parallel.

11. Through a given point in space draw a line parallel to each of two given planes (1) when the planes intersect; (2) when the planes are parallel.

12. Construct a common perpendicular to two skew lines (Fig. 51).

Analysis and construction:

I. If d is the required perpendicular, d must (1) be $\perp$ b, (2) be $\perp$ a line as c which is $\parallel$ a and intersects b, (3) lie in same plane as a and c.

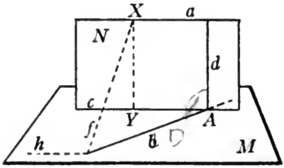


FIG. 51

II. Draw plane M through $b \parallel a$. Line a will be $\parallel$ any line in M that lies in the same plane as a.

III. $\therefore$ draw through a plane $N \perp M$, cutting M in line c. Let c cut b in point A.

IV. $\therefore$ draw $d \perp M$ at A.

13. Show that only one common perpendicular can be drawn to two skew lines.

Suggestion (Fig. 51). Show that if any other line (as f) is $\perp$ lines a and b, and if f intersects line a at point X, we could have two perpendiculars to M from X.

14. Show that the common perpendicular to two skew lines is the shortest distance between them.

Suggestion. Line $f > XY$ and $\therefore > d$.

15. Through a given point draw a straight line that will intersect each of two given skew lines.

Suggestion. Draw two planes each determined by the given point and one of the given lines.

16. Given three non-parallel non-intersecting straight lines. Show that any number of straight lines can be drawn that will intersect the three.

Suggestion. Construct a pencil of planes through one of the lines.

17. Construct a pair of parallel planes so that each shall contain one of two skew lines.

EXERCISES INVOLVING LOCI AND CONCURRENT LINES

58. 1. Prove that the planes bisecting the dihedral angles of a trihedral angle are concurrent.

Analysis:

I. Let a, b, and c be the faces of dihedral angle O.

II. To prove the bisectors of the dihedral angles concurrent, prove that the bisector of the angle made by a and b and that made by b and c meet in a line and that this line lies in the bisector of the angle made by a and c.

III. $\therefore$ prove that every point in this line is equally distant from a and c.

2. What locus is obtained by the intersection of three planes referred to in Ex. 1?

3. Find the locus of points in space that are equally distant from three given planes that intersect each other in three parallel lines.

4. Prove that the locus of points in space that are equally distant from two intersecting lines is the plane that is perpendicular to the plane of the lines and that contains the bisectors of the angles formed by the lines.

5. Prove that the planes perpendicular to the faces of a trihedral angle and passing through the bisectors of the face angles are concurrent. What locus is obtained?

6. Prove that the planes that pass through the edges of a trihedral angle and the bisectors of the opposite face angles are concurrent (Fig. 52).

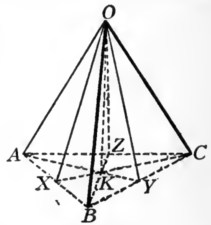


FIG. 52

Analysis and construction:

I. To prove OAY, OBZ, and OCX concurrent, prove that they have two points in common.

II. $\therefore$ make $OA = OB = OC$ and prove that OAY, OBZ, and OCX intersect plane ABC in concurrent lines.

III. $\therefore$ prove AY, BZ, and CX the medians of $\triangle ABC$.

7. Prove that the planes that are perpendicular to the chords of a circle at their midpoints are concurrent.

8. What locus is obtained by the intersection of the three planes referred to in Ex. 7? Give proof.

EXERCISES INVOLVING PROJECTIONS

59. 1. If two parallel lines are oblique to a plane, they make equal angles with the plane.

2. Two equal segments from a point to a plane have equal projections on the plane and make equal angles with the plane.

3. If a segment is oblique to a plane, it is longer than its projection on the plane.

4. If two unequal segments are drawn from a point to a plane, which has the longer projection on the plane? Give proof. Which makes the greater angle with the plane? Give proof.

5. If a straight line is perpendicular to one of two intersecting planes, its projection on the other is perpendicular to the intersection of the two planes.

6. If a right angle has one side parallel to a plane, its projection on the plane is a right angle. When is this not true?

7. When is the projection of a parallelogram on a given plane a parallelogram? Why?

8. Answer the question in the preceding exercise for a rectangle instead of a parallelogram.

9. The sides of an isosceles triangle make equal angles with any plane containing its base.

10. Parallel segments are proportional to their projections on the same plane.

11. If the projection of a given line upon each of two intersecting planes is a straight line, the given line is, in general, a straight line.

12. A point is 12 in. from a given plane. What is the length of a projection of a segment 13 in. long drawn from the point to the plane?

13. What is the length of the projection of a segment 5 in. long on a plane if the segment makes an angle of 30° with the plane? If it makes an angle of 60° with the plane? If it makes an angle of 45° with the plane? Trigonometry tables are not necessary.

CHAPTER II

PROPERTIES OF POLYHEDRONS, CYLINDERS, AND CONES

SOLIDS IN GENERAL

60. An inclosed portion of space is called a **solid.**

Solids are bounded or inclosed by **surfaces.**

If any solid is intersected by a plane, the figure formed by the intersections of the plane and the boundaries of the solid is called a **section** of the solid.

POLYHEDRONS IN GENERAL

61. In general, a solid all of whose bounding surfaces are planes is called a **polyhedron** (Fig. 53).

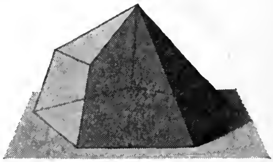


FIG. 53

The lines of intersection of the bounding planes are called the **edges** of the polyhedron.

The points of intersection of the edges are called the **vertices** of the polyhedron.

The polygons bounded by the edges of the polyhedron are called the **faces** of the polyhedron.

SOME ELEMENTARY SURFACES

62. We have seen that sometimes a surface is the locus of a point or of a line that obeys certain requirements. The surfaces studied in solid geometry are of this nature.

We shall study the following special surfaces:

If a straight line moves so as continually to intersect a given straight line, and at the same time to remain parallel to a third straight line not in the plane of the second, the surface generated is a **plane surface.**

A plane surface, or merely a **plane,** may be defined from its fundamental characteristic (§3) as a surface such that if any two points in it are chosen the straight line passing through these points will lie wholly in the surface.

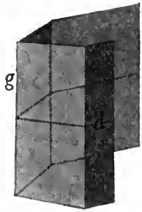


FIG. 54

If a straight line moves so as continually to intersect a chain of straight-line segments that lie in one plane, and at the same time to remain parallel to a fixed straight line not in this plane, the surface generated is called a **prismatic surface** (Fig. 54).

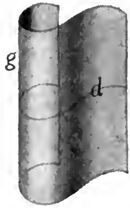


FIG. 55

If a straight line moves so as continually to intersect a fixed plane curve, and at the same time to remain parallel to a fixed straight line not in the plane of the curve, the surface generated is called a **cylindrical surface** (Fig. 55).

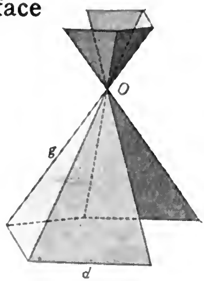


FIG. 56

If a straight line moves so as continually to intersect a chain of straight-line segments that lie in one plane, and at the same time to pass through a given point not in this plane, the surface generated is called a **pyramidal surface** (Fig. 56). The fixed point is called the **vertex** of the surface.

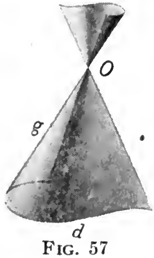


FIG. 57

If a straight line moves so as continually to intersect a fixed plane curve, and at the same time to pass through a given point not in the plane of the curve, the surface generated is called a **conical surface** (Fig. 57). The fixed point is called the **vertex** of the surface.

The prismatic and the pyramidal surfaces are examples of plane surfaces. The cylindrical and the conical surfaces are special kinds of **curved surfaces**. A curved surface is a surface no part of which is plane.

NOTE. There are many kinds of curved surfaces besides those mentioned above. If a circle is revolved about a diameter, a spherical surface is formed. We shall study spherical surfaces later. The reflector of a locomotive headlight is a surface formed by the revolution of a parabola.

63. In the surfaces defined above the moving line is called the **generator**. The series of straight-line segments or the curve is called the **director.** In Figs. 54–57, g is the generator and d is the director.

When the director is a closed polygon or a closed curve and the generator moves completely around the director, the surface is said to be a **closed surface** (Fig. 58).

If the director is a convex polygon or curve, the surface generated is said to be a **convex surface.**

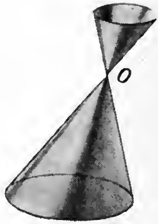


FIG. 58

The generator in any position is called an **element** of the surface.. The **edges** of a prismatic or pyramidal surface are the elements that pass through the points of intersection of the segments of the series that form the director.

Pyramidal and conical surfaces each consist of two parts called **nappes.** The generator of a pyramidal or conical surface is a line, not a ray. If, in Figs. 56, 57, and 58, O is the fixed point through which the generator passes, the part below point O generates one part or nappe of the surface formed, while the part above point O generates the other part or nappe.

The two nappes of a closed convex pyramidal surface contain polyhedral angles. The edges of one are prolongations of the edges of the other. Are the polyhedral angles equal or symmetric? Why?

If a plane cuts all the elements of a closed surface, the figure formed by the intersection of the plane and the surface is called a **transverse section** of the surface.

A transverse section formed by a plane perpendicular to the elements of a prismatic or cylindrical surface is called a **right section** of the surface ($VWXYZ$, Fig. 60).

PRISMS

DEFINITIONS

64. An unlimited closed prismatic surface is said to inclose a **prismatic space.**

A solid bounded by a closed prismatic surface and two parallel transverse sections is called a **prism** (Figs. 59 and 60). It is a portion of a prismatic space.

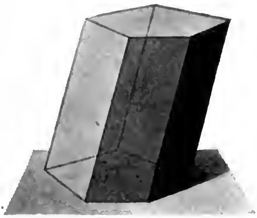


FIG. 59

The two transverse sections are called the **bases** of the prism. The prismatic surface and its edges are called respectively the **lateral surface** and **lateral edges** of the prism.

The perpendicular distance between the bases of a prism is called the **altitude** of the prism.

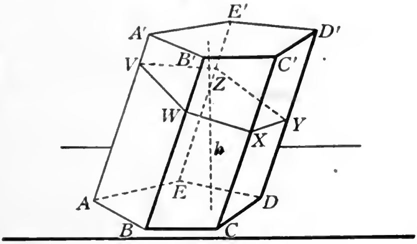


FIG. 60

Since the form of a prismatic space depends upon the form of its right section, prisms may be named from the form of their right sections, thus: A prism will be called a **square prism** if its right section is a square.

Exercise. Make of wood or cardboard a square prism in which the edges are not perpendicular to the base. Is the base a square?

GENERAL PROPERTIES OF PRISMS

65. THEOREM 39. **Two parallel transverse sections of a prismatic space are congruent.**

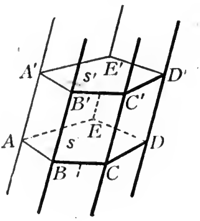


FIG. 61

Analysis:

I. To prove $s \cong s'$ prove the sides and angles equal in the same order.

II. To prove $AB = A'B'$, prove $ABB'A'$ a $\square$.

III. To prove $\angle ABC = \angle A'B'C'$, prove $AB \parallel A'B'$ and $BC \parallel B'C'$.

66. The following theorems are corollaries of Th. 39 and of the definition of a prism.

COR. I. **The bases of a prism are congruent polygons.**

COR. II. **The lateral edges of a prism are parallel and equal.**

COR. III. **The lateral faces of a prism are parallelograms.**

COR. IV. **All right sections of a prism are congruent.**

Ex. 1. Is it possible for only one lateral face of a prism to be a rectangle? Illustrate by a model.

Ex. 2. Is it possible for only two lateral faces of a prism to be rectangles? Prove.

Ex. 3. Every pair of lateral edges of a prism determines a plane parallel to each of the other lateral edges.

Ex. 4. If two intersecting planes each contain one and only one lateral edge of a prism, their intersection is parallel to the other lateral edges.

67. That part of a prism included between one base and a transverse section oblique to the base is called a **truncated prism** (Fig. 62).

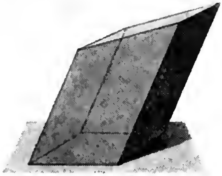


FIG. 62

SPECIAL PRISMS*

68. A prism is said to be a **right prism** if its base is a right section (Fig. 63). A prism that is not a right prism is called an **oblique prism** (see Fig. 59).

THEOREM 40. **The lateral faces of a right prism are rectangles.**

THEOREM 41. **The lateral faces of a right prism are perpendicular to the base.**

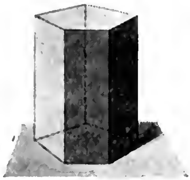


FIG. 63

69. A prism is said to be **regular** if it is a right prism whose base is a regular polygon.

70. A prism is called a **parallelepiped** if its bases are parallelograms (Fig. 64).

THEOREM 42. **Any two opposite faces of a parallelepiped are congruent and parallel.**

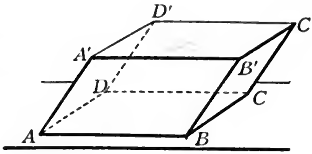


FIG. 64

Ex. 1. The edges of a parallelepiped may be divided into three groups of four parallel edges each.

Ex. 2. The diagonals of a parallelepiped meet in a point which is the midpoint of each diagonal.

NOTE. The intersection of the diagonals of a parallelepiped is called the **center** of the parallelepiped.

Ex. 3. Any line through the center of a parallelepiped and terminated by a pair of opposite faces is bisected by the center of the parallelepiped.

*Let the pupil construct a model for each of the special cases mentioned. These models may be constructed of wood, soap, or cardboard.

71. A parallelepiped is said to be a **right parallelepiped** if the base is a right section.

Ex. 1. What properties has a right parallelepiped by virtue of the fact that it is a parallelepiped and at the same time a right prism?

Ex. 2. Show by means of a model that a right parallelepiped is not right in all positions in which it may be placed.

72. A right parallelepiped is said to be a **rectangular parallelepiped** if its base is a rectangle (Fig. 65).

THEOREM 43. **If a parallelepiped is so constructed that each of its three edges that meet at a common vertex is perpendicular to the other two, the parallelepiped is rectangular.**

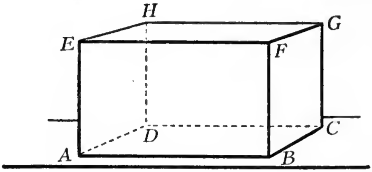


FIG. 65

Analysis:

I. Let each of the edges AB, AD, and AE be $\perp$ the other two.

II. To prove that the parallelepiped is rectangular, prove that
   (1) the base $ABCD$ is a rectangle.
   (2) the parallelepiped is a right parallelepiped.

III. To prove that it is a right parallelepiped, prove the edges AE, BF, CG, and DH each $\perp$ the base $ABCD$.

Ex. 1. Each face of a rectangular parallelepiped is a rectangle.

Ex. 2. Show that a rectangular parallelepiped is a right parallelepiped in all positions in which it may be placed.

Ex. 3. The diagonals of a rectangular parallelepiped are equal.

Ex. 4. Find the diagonal of a rectangular parallelepiped if three edges that meet in a point are respectively 4, 8, and 6 in. Find the diagonal if these edges are a, b, and c.

73. If the three edges of a rectangular parallelepiped that meet at one vertex are equal, the rectangular parallelepiped is called a **cube.**

THEOREM 44. **The faces of a cube are all squares.**

Ex. 1. Are the diagonals of a cube perpendicular to each other? Why?

Ex. 2. If the edge of a cube is E, the diagonal is $E\sqrt{3}$.

Ex. 3. One edge of a cube is 5 in. Find the area of a section made by passing a plane through two diagonally opposite edges.

CYLINDERS

DEFINITIONS

74. An unlimited closed cylindrical surface is said to inclose a **cylindrical space.**

A solid bounded by a closed cylindrical surface and two parallel transverse sections is called a **cylinder** (Figs. 66 and 67). It is a portion of a cylindrical space.

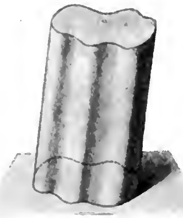


FIG. 66

The transverse sections are called the **bases** of the prism. The cylindrical surface and its elements are called respectively the **lateral surface** and the **elements** of the cylinder.

The perpendicular distance between the bases of the cylinder is called the **altitude** of the cylinder.

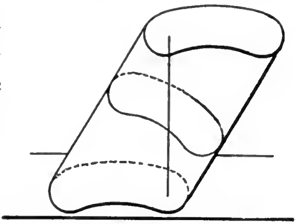


FIG. 67

A cylinder is said to be a **convex cylinder** if its cylindrical surface is convex.

That portion of a cylinder included between one base and a transverse section oblique to the base is called a **truncated cylinder** (Fig. 68).

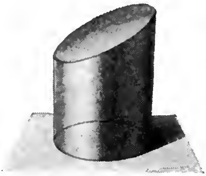


FIG. 68

GENERAL PROPERTIES OF CYLINDERS

75. THEOREM 45. **Two parallel transverse sections of a closed cylindrical space are congruent.**

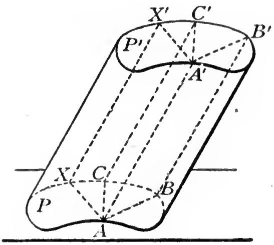


FIG. 69

Hypothesis: P and P' are two parallel transverse sections of a closed cylindrical space.

Conclusion: $P \cong P'$.

Analysis and construction:

I. To prove $P \cong P'$, prove that P will coincide with P' if superposed.

II. Take A, B, and C any three points in P. Draw the elements AA', BB', and CC' and show that P can be placed upon P' in such a way that A, B, and C fall respectively on A', B', and C' and at the same time any fourth point in P will fall on a point in P'.

III. To prove that A, B, and C can be made to fall on A', B', and C', prove $AB = A'B'$, $AC = A'C'$, and $\angle BAC = \angle B'A'C'$.

IV. To prove that a fourth point in P falls on a point in P', draw the element XX' and prove $AX = A'X'$ and $\angle BAX = \angle B'A'X'$.

Outline of proof:

I. $AA' \parallel BB'$.
$AB \parallel A'B'$.
$\therefore AA'BB'$ is a $\square$.
$\therefore AB = A'B'$.
Similarly $AC = A'C'$.

Also $\angle BAC = \angle B'A'C'$.

II. If P is placed on P' so that $\angle BAC$ falls on $\angle B'A'C'$, B will fall on B' and C on C'.

III. As in I, $AX = A'X'$ and $\angle BAX = \angle B'A'X'$.

IV. $\therefore$ if P is placed on P' so that $\angle BAC$ falls on $\angle B'A'C'$, point X will fall on point X'.

V. In the same way every point on P will fall on a point on P' if the three points A, B, and C fall on the three points A', B', and C'.

Let the pupil give the reasons.

COR. **The bases of a closed cylinder are congruent.**

76. THEOREM 46. **If a plane contains one element of a cylindrical surface and one other point of that surface, then it contains the element through that point.**

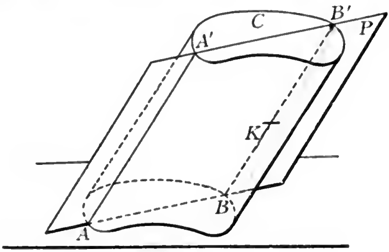


FIG. 70

Hypothesis: C is a cylindrical surface. P is a plane containing the element AA' and the point K of the cylindrical surface.

Conclusion: P contains the element BB' through K.

Analysis:

I. To prove that P contains the element BB', prove that BB' is parallel to AA'.

We will assume, without proof,

As. 9. **If C is a closed convex cylindrical surface, the plane P (Th. 46) will intersect the cylindrical surface in the two elements AA' and BB' and in no other points (Fig. 70).**

77. COR. **Any section of a closed convex cylinder made by a plane containing an element and one other point of the cylindrical surface is a parallelogram.**

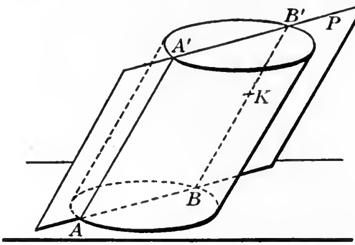


FIG. 71

Analysis:

I. To prove $ABB'A'$ a ▱, prove each side parallel to its opposite.

II. To prove $AB \parallel A'B'$ . . . . .

III. To prove $AA' \parallel BB'$, prove BB' an element of the cylinder (Th. 46 and As. 9).

Ex. 1. If a plane contains two elements of a convex cylinder, the section of the cylinder formed by this plane is a parallelogram.

Ex. 2. If a plane contains two elements of a cylinder, it is parallel to the elements of the cylinder.

Ex. 3. If two planes each containing two elements of a cylinder intersect, the line of intersection is parallel to the elements of the cylinder.

Ex. 4. Are the two previous exercises true if the planes contain only one element of the cylinder?

RIGHT CIRCULAR CYLINDERS

78. Since the form of a cylindrical space depends upon the form of its right section, cylinders may be named from the form of their right sections, thus: a cylinder is called a **circular cylinder** if its right section is a circle; it is called an **elliptical cylinder** if its right section is an ellipse.

NOTE. Transverse sections of circular cylinders that are not right sections are ellipses. This fact can be illustrated by means of a broomstick cut so that the section is not a right section. An elliptical cylinder may be cut by a plane so that the section is a circle. In higher mathematics other kinds of cylinders are studied, such as hyperbolic and parabolic cylinders.

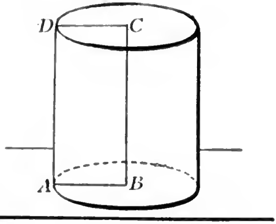


FIG. 72

If the base of a cylinder is a right section, the cylinder is said to be a **right cylinder** (Fig. 72). A cylinder that is not a right cylinder is called an **oblique cylinder** (Fig. 73).

A right cylinder whose base is a circle is called a **right circular cylinder.**

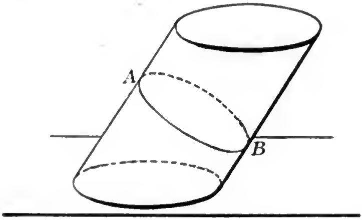


FIG. 73

A right circular cylinder may be generated by the revolution of a rectangle about one side as an axis. For this reason a right circular cylinder is sometimes called a **cylinder of revolution** (Fig. 72).

The axis about which the rectangle revolves is called the **axis** of the cylinder. It is the line joining the centers of the bases. The sides of the rectangle perpendicular to the axis generate the bases of the cylinder. The side parallel to the axis generates the cylindrical surface.

Ex. 1. What is a cylindrical surface of revolution? How may it be considered a locus?

Ex. 2. Every section of a right convex cylinder made by a plane containing an element and one other point of the cylindrical surface is a rectangle.

Ex. 3. Every element of a right cylinder is equal to the altitude.

Ex. 4. Every section of a right circular cylinder made by a plane parallel to the base is a circle.

PYRAMIDS

DEFINITIONS

79. A closed pyramidal surface is said to inclose a **pyramidal space.** A pyramidal space is composed of two parts called nappes which correspond to the two nappes of the pyramidal surface.

A solid bounded by one nappe of a closed pyramidal surface and a transverse section is called a **pyramid** (see Fig. 74). It is a portion of a pyramidal space.

The vertex of the pyramidal surface is called the **vertex** of the pyramid. The transverse section is called the **base** of the pyramid. The pyramidal surface and its elements are called respectively the **lateral surface** and the **lateral edges** of the pyramid.

The perpendicular from the vertex to the base is called the **altitude** of the pyramid.

GENERAL PROPERTIES OF PYRAMIDS

80. Theorem 47. **If a pyramid is cut by a plane parallel to the base,**

I. **The edges and altitude are divided proportionally.**

II. **The section is a polygon similar to the base.**

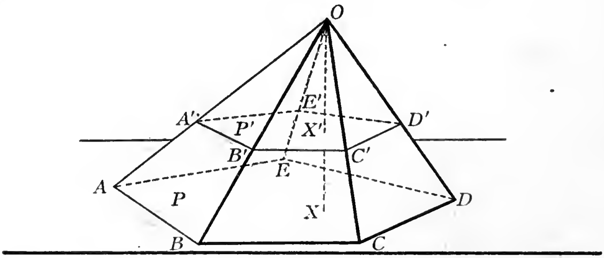


Fig. 74

Suggestion for I. Pass a plane through O parallel to P and P'. Use Th. 30.

Analysis for II:

I. To prove $P \sim P'$, prove

$a.$ $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, etc.

$b.$ $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'} =$ etc.

II. To prove $\angle A = \angle A'$, prove $AE \parallel A'E'$, $AB \parallel A'B'$.

III. To prove $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$, prove $\dfrac{AB}{A'B'} = \dfrac{OB}{OB'} = \dfrac{BC}{B'C'}$.

Cor. I. **If a pyramid is cut by a plane parallel to the base, the ratio of the area of the section to the area of the base equals the ratio of the squares of their distances from the vertex.**

Analysis: To prove $\dfrac{\text{area } P}{\text{area } P'} = \dfrac{\overline{OX}^2}{\overline{OX'}^2}$, prove $\dfrac{\text{area } P}{\text{area } P'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$,

and $\dfrac{\overline{AB}^2}{\overline{A'B'}^2} = \dfrac{\overline{OA}^2}{\overline{OA'}^2} = \dfrac{\overline{OX}^2}{\overline{OX'}^2}$ (*Plane Geometry*, Th. 127) (Fig. 74).

Cor. II. **If two pyramids have equal altitudes and equal bases, sections parallel to the bases at equal distances from the vertices are equal.**

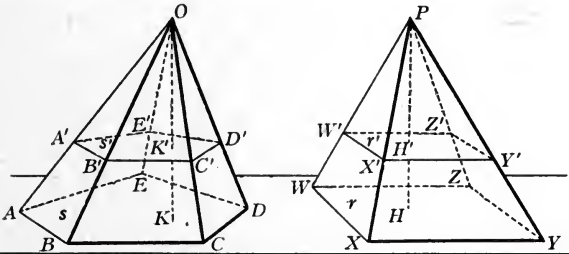


Fig. 75

Analysis: To prove area s' = area r', prove $\dfrac{\text{area } s'}{\text{area } s} = \dfrac{\overline{OK'}^2}{\overline{OK}^2}$

and $\dfrac{\text{area } r'}{\text{area } r} = \dfrac{\overline{PH'}^2}{\overline{PH}^2}$ (see *Plane Geometry*, Th. 92).

Ex. 1. At what distance from the vertex of a pyramid must a plane be passed parallel to the base so that the area of the section may be one-half the area of the base?

Ex. 2. If a plane is passed parallel to the base of a pyramid and through the mid-point of the altitude, what is the ratio of the area of the section to the area of the base?

Ex. 3. It is stated in physics that the intensity of illumination received on a screen is inversely proportional to the square of the distance of the screen from the source of light. Show that this is an application of Cor. I, Th. 47.

REGULAR PYRAMIDS

81. A pyramid is said to be **regular** if its base is a regular polygon and the altitude passes through the center of the base (Fig. 76).

THEOREM 48. **The lateral edges of a regular pyramid are equal.**

THEOREM 49. **The lateral faces of a regular pyramid are congruent isosceles triangles.**

THEOREM 50. **The altitudes of all of the lateral faces of a regular pyramid are equal.**

The altitude of any one of the lateral faces of a regular pyramid is called the **slant height of the pyramid.**

Ex. 1. Any section of a regular square pyramid made by a plane containing the altitude is an isosceles triangle.

Ex. 2. The altitudes of the face triangles of a regular pyramid meet the base at the points of tangency of the circle inscribed in the base (Fig. 76).

Suggestion. Join O, the foot of the altitude of the pyramid, with Y, the point of tangency of AB. Join YX and prove YX the altitude of $\triangle ABX$.

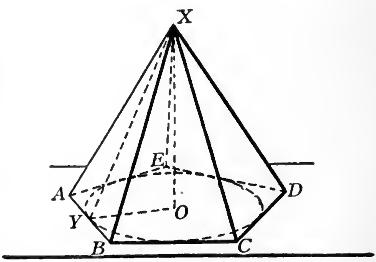


FIG. 76

FRUSTUMS OF PYRAMIDS

82. The portion of a pyramid included between the base and a section oblique to the base cutting all of the lateral edges is called a **truncated pyramid** (Fig. 77).

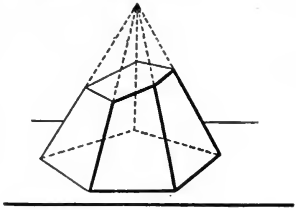


FIG. 77

The portion of a pyramid included between the base and a section parallel to the base and cutting all of the lateral edges is called a **frustum** of a pyramid (Fig. 78).

The **altitude** of a frustum of a pyramid is the perpendicular distance between the bases (YY', Fig. 78).

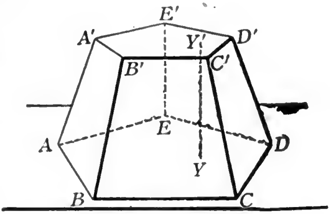


FIG. 78

THEOREM 51. **The lateral faces of a frustum of a pyramid are trapezoids.**

THEOREM 52. **The lateral faces of a frustum of a regular pyramid are congruent isosceles trapezoids.**

THEOREM 53. **The altitudes of the faces of a frustum of a regular pyramid are equal.**

The altitude of any face of a frustum of a regular pyramid is called the **slant height** of the frustum.

Ex. 1. The sum of the segments connecting the mid-points of the lateral edges of a frustum of a pyramid is one-half the sum of the perimeters of the bases.

Ex. 2. Prove that the medians referred to in the previous exercise are co-planar.

Suggestion. Let X be the mid-point of AA'. Through X pass a plane parallel to $ABCDE$. Prove that the medians of the several lateral faces lie in this plane (Fig. 79).

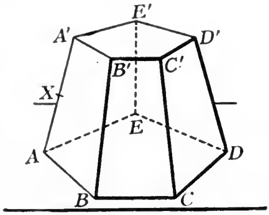


FIG. 79

CONES

DEFINITIONS

83. A closed conical surface is said to inclose a **conical space.** A conical space is composed of two parts called nappes, which correspond to the two nappes of the conical surface.

A solid bounded by one nappe of a closed conical surface and a transverse section is called a **cone** (Fig. 80). It is a portion of a conical space.

The vertex of the conical surface is called the **vertex** of the cone. The transverse section is called the **base** of the cone. The conical surface and its elements are called respectively the **lateral surface** and the **elements** of the cone.

The perpendicular from the vertex to the base is called the **altitude** of the cone.

GENERAL PROPERTIES OF CONES

84. THEOREM 54. **If the base of a cone is a circle, every section parallel to the base is also a circle.**

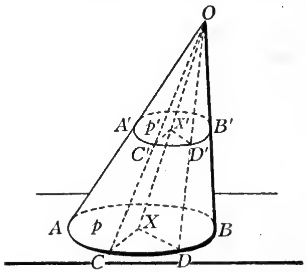


FIG. 80

Analysis and construction:

I. To prove p' a circle, prove that all points on p' are equally distant from a point within p'.

II. $\therefore$ choose C' and D' any two points on p', and prove C' and D' equally distant from a point (X') within p'.

III. To find X', join O and X, the center of $\odot p$. Let OX cut p' at X'.

IV. $\therefore$ join $C'X'$ and $D'X'$ and prove $C'X'=D'X'$.

V. To prove $C'X'=D'X'$, draw the elements OC' and OD'; let the planes $C'OX'$ and $D'OX'$ cut p in CX and DX respectively, and prove

$$\frac{C'X'}{CX}=\frac{OX'}{OX}, \text{ and } \frac{D'X'}{DX}=\frac{OX'}{OX}.$$

Cor. I. **If the base of a cone is a circle, the area of a section parallel to the base is to the area of the base as the square of the distance of the section from the vertex is to the square of the altitude.**

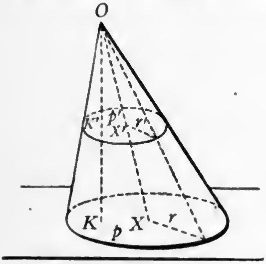


Fig. 81

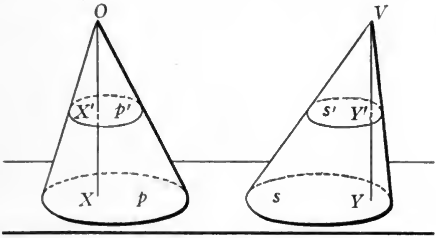


Fig. 82

Analysis: To prove $\frac{\text{area } p'}{\text{area } p}=\frac{\overline{OK'}^2}{\overline{OK}^2}$, prove

$$\frac{\text{area } p'}{\text{area } p}=\frac{r'^2}{r^2}=\frac{\overline{OX'}^2}{\overline{OX}^2}=\frac{\overline{OK'}^2}{\overline{OK}^2} \text{ (Fig. 81).}$$

Cor. II. **If two cones have equal circular bases and equal altitudes, areas of sections at equal distances from the vertices are equal.**

Analysis: To prove $p'=s'$, prove $\frac{\text{area } p'}{\text{area } p}=\frac{\overline{OX'}^2}{\overline{OX}^2}$, and

$$\frac{\text{area } s'}{\text{area } s}=\frac{\overline{OY'}^2}{\overline{VY}^8}. \text{ (Fig. 82).}$$

Exercise. Investigate Th. 54 and its two corollaries if the plane fo ming section p cuts the other nappe of the conical surface.

85. THEOREM 55. **If a plane contains one element of a conical surface and one other point of that surface, then it contains the element through that point.**

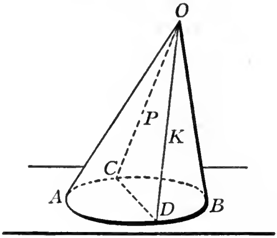


FIG. 83

Hypothesis: OAB is a conical surface cut by the plane P which contains the element OC and the point K of the conical surface.

Conclusion: The plane P contains the element OD through K.

Analysis: To prove that P contains the element OD through K, show that

a. Points K and O are both in P.

b. Points K and O determine an element.

If the conical surface of a cone is convex, the cone is said to be a **convex cone.** Unless otherwise stated, all cones referred to will be considered convex cones.

We will assume, without proof,

As. 10. **If *C* is a closed convex conical surface, the plane *P* (Th. 55) will intersect the conical surface in the two elements *CO* and *DO* and in no other point** (Fig. 83).

COR. **If a plane contains one element of a conical surface and one other point of that surface, the section of the cone made by the plane is a triangle.**

Exercise. Any section of a cone made by a plane that passes through the vertex and intersects the base is a triangle.

RIGHT CIRCULAR CONES

86. If a cone has a circular section such that the line from the vertex to the center of the section is perpendicular to the plane of that section, the cone is called a **circular cone.**

If the base of a cone is a circle, and if the line from the vertex to the center of the base is perpendicular to the base, the cone is called a **right circular cone.**

A circular cone that is not a right circular cone is called an **oblique circular cone.**

THEOREM 56. **All elements of a right circular cone are equal.**

An element of a right circular cone is called its **slant height.**

A right circular cone may be generated by the revolution of a right triangle about one leg as an axis. For this reason a right circular cone is sometimes called a **cone of revolution** (Fig. 84).

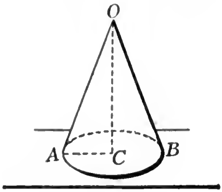


FIG. 84

The axis about which the triangle revolves is called the **axis** of the cone. The other leg generates the base of the cone. The hypotenuse generates the conical surface.

The axis of a right circular cone is, therefore, the segment from the vertex to the center of the base; it is perpendicular to the base and is the altitude of the cone.

Ex. 1. What is a conical surface of revolution? Why may it be considered a locus?

Ex. 2. A section of a right circular cone made by a plane containing the vertex and intersecting the base is an isosceles triangle.

Ex. 3. What kind of a triangle is the section referred to in the previous exercise if the cone is an oblique circular cone?

Ex. 4. Can a section of an oblique circular cone made by a plane which contains the vertex and intersects the base be an isosceles triangle? How? Give proof.

NOTE. The various forms of the sections of a right circular cone are interesting. The complete study of them is beyond this book.

For each case shown in the figure the construction of the cutting plane and the form of the section are given below:

No. 1. Contains the vertex and intersects the base. A **triangle.**

No. 2. Parallel to the base. A **circle.**

No. 3. Cuts all of the elements of the same nappe, but not parallel to the base. An **ellipse.**

No. 4. Parallel to an element. A **parabola.**

No. 5. Perpendicular to the base. The plane cuts both nappes. The section has two parts. A **hyperbola.**

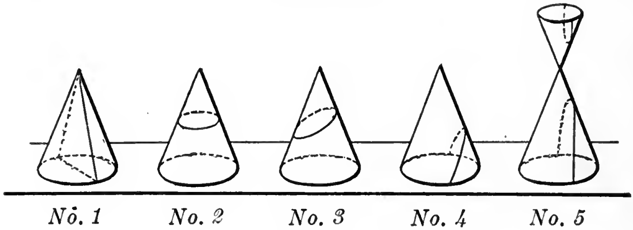


FIG. 85

FRUSTUMS OF CONES

87. The portion of a cone included between the base and a transverse section oblique to the base is called a **truncated cone** (Fig. 86).

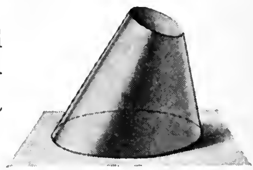


FIG. 86

The portion of a cone included between the base and a transverse section parallel to the base is called the **frustum** of a cone (Fig. 87).

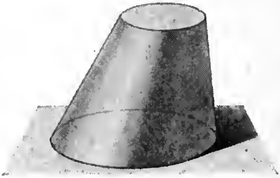


FIG. 87

THEOREM 57. **The elements of a frustum of a right circular cone are equal.**

An element of a frustum of a right circular cone is called the **slant height** of the frustum.

Ex. 1. Any section of a frustum of a cone made by a plane containing an element and one other point in the conical surface is a trapezoid.

Ex. 2. Show that if the frustum referred to in the previous exercise had been the frustum of a right circular cone the section would have been an isosceles trapezoid.

Ex. 3. The perimeter of a section of a frustum of a cone made by a plane parallel to the bases and midway between them is equal to one-half the sum of the perimeters of the two bases (Fig. 88).

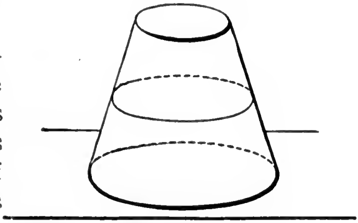


FIG. 88

A frustum of a right circular cone may be generated by the revolution of an isosceles trapezoid about the segment joining the mid-points of the bases as an axis. For this reason a frustum of a right circular cone may be called a **frustum of revolution.**

The axis about which the trapezoid revolves is called the **axis** of the frustum. The bases of the trapezoid generate the bases of the frustum.

REGULAR POLYHEDRONS

88. A polyhedron is said to be **regular** if its faces are congruent regular polygons and if its polyhedral angles are congruent.

Polyhedrons may be named from the number of faces: A polyhedron of 4 faces is called a **tetrahedron;** of 6 faces, a **hexahedron;** of 8 faces, an **octahedron;** of 12 faces, a **dodecahedron;** of 20 faces, an **icosahedron.**

89. THEOREM 58. **Not more than five regular polyhedrons are possible.**

Outline of proof:

A. *In general.*

I. The faces of the regular polyhedrons must be equilateral triangles, squares, regular pentagons, or some other regular polygons.

II. At least three faces must meet at each vertex to form a polyhedral angle.

III. The sum of the face angles of each polyhedral angle must be less than 360°.

B. *When the faces are equilateral triangles.*

How many degrees in each angle of an equilateral triangle?

What would be the sum of the face angles of a polyhedral angle if 3 equilateral triangles met at the vertex? If 4 such triangles met at a vertex? If 5? If 6 or more? Tabulate the results.

How many regular polyhedrons could be formed whose faces are equilateral triangles? How many triangles meet at each vertex?

C. *When the faces are squares.*

What would be the sum of the face angles of a polyhedral angle if 3 squares met at the vertex? If 4 squares met at the vertex? If more than 4 squares were used?

How many regular polyhedrons could be formed whose faces are squares? How many squares would meet at each vertex?

D. *When the faces are regular pentagons.*

In the same way, how many regular polyhedrons could be formed whose faces would be regular pentagons? How many pentagons would meet at each vertex?

E. *When the faces are other regular polygons.*

Show that in this case no regular polyhedron could be formed.

Exercise. Tabulate the results of your study of Th. 58, showing in your table

(1) the kinds of polygons used as faces
(2) the number of faces that meet at each vertex
(3) the sum of the face angles at each vertex
(4) the total number of faces
(5) the name of the polyhedron

90. There *are* five regular polyhedrons. This fact may be verified by models made by the pupil.

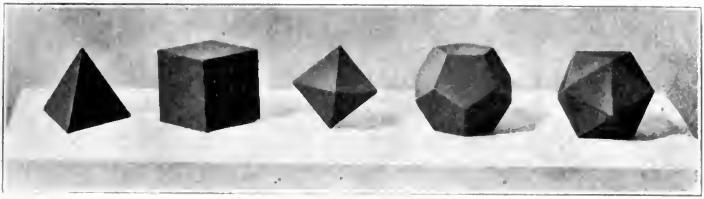


FIG. 89

Fig. 89 shows these polyhedrons, and Fig. 90 shows the patterns for making them.

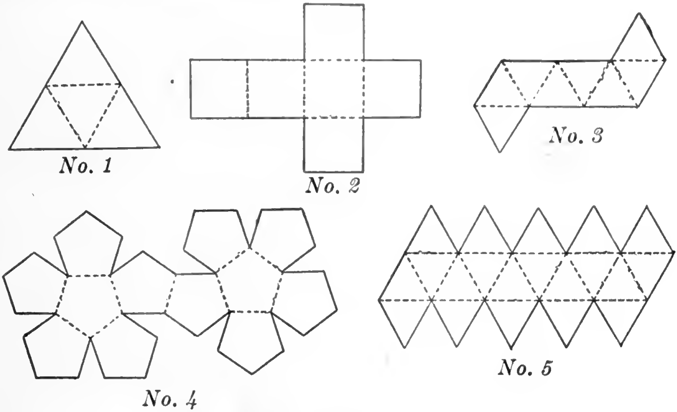
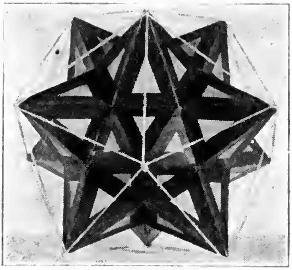


FIG. 90

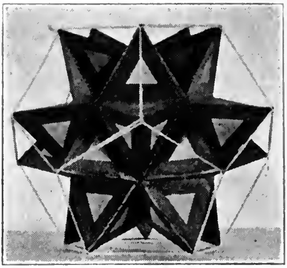
NOTE. When making the models, cut out the patterns, cutting only halfway through along the dotted lines. Fold along the dotted lines and fasten the edges together with gummed paper.

91. Note. **Star polyhedrons.** Fig. 91 shows two star polyhedrons.

To make No. 1, construct a regular dodecahedron and paste a pyramid upon each face. To construct the pyramids, draw a regular pentagon congruent with one of the faces of the dodecahedron; extend the sides of the pentagon, forming a pentagram-star; turn up the triangles to form the pyramid whose base is the regular pentagon.



No. 1



No. 2

Fig. 91

To make No. 2, paste on each face of the regular icosahedron a regular tetrahedron one face of which is congruent with the faces of the icosahedron.

The strings joining the vertices of these pyramids form what has been called the **case** of the star polyhedron. The case of No. 1 is the regular icosahedron. The case of No. 2 is the regular dodecahedron. The regular polyhedron used as the basis has been called the **core** of the star polyhedron.

SUPPLEMENTARY EXERCISES

92. 1. Name a solid which has for faces (1) four equilateral triangles, (2) two triangles and three parallelograms, (3) two triangles and three rectangles, (4) one square and four triangles, (5) six parallelograms, (6) two parallelograms and four rectangles, (7) six rectangles, (8) eight equilateral triangles.

2. Construct a parallelepiped that has its edges on three given skew lines.

3. Given three segments equal to the three edges of a rectangular parallelepiped that meet in one vertex. Construct a segment equal to the diagonal.

4. What is the form of a section of a prism made by a plane which is (1) parallel to a lateral edge and intersects the base; (2) parallel to one lateral face and intersects the base? Give proof.

5. What is the answer to Ex. 4 if the prism is a right prism?

6. What is the section of a parallelepiped made by a plane located as given below: (1) if the parallelepiped is oblique; (2) if it is right; (3) if it is rectangular:

a. Passes through two diagonally opposite edges?

b. Cuts four parallel edges?

c. Is perpendicular to a face and cuts four parallel edges?

d. Is perpendicular to an edge and cuts four parallel edges?

7. How can you cut a cube so that the section will be (1) a parallelogram, (2) a rectangle, (3) a square, (4) an equilateral triangle, (5) an isosceles triangle, (6) a regular hexagon?

Analysis for 6 (Fig. 92):

I. To prove $UVWXYZ$ a regular hexagon, prove

(1) U, V, W, X, Y, Z are coplanar.

(2) $UVWXYZ$ is equilateral.

(3) $UVWXYZ$ can be inscribed in a circle.

II. To prove I, prove VY, UX, and WZ each $\perp$ FD at O (the mid-point of FD).

III. $\therefore$ show (1) that VY lies in plane $EBCH$; (2) that O and F are equally distant from the extremities of VY.

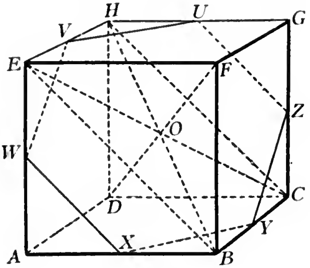


Fig. 92

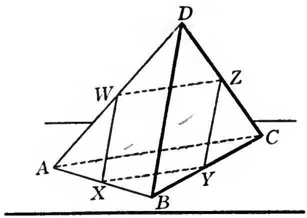


Fig. 93

8. The section of a tetrahedron made by a plane parallel to two non-intersecting edges is a parallelogram (Fig. 93).

9. Find the sum of the plane angles of the dihedral angles whose edges are the lateral edges of a triangular prism.

Suggestion. Construct a right section of the prism.

10. What is the sum of the plane angles referred to in Ex. 9 if the prism is quadrangular instead of triangular? Can the exercise be solved for any other kind of prisms?

11. If the bases of a cylinder are circles, a line parallel to an element of a cylinder containing the center of one base contains also the center of the other base.

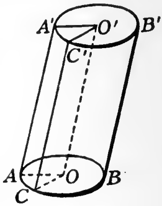


FIG. 94

Analysis and construction (Fig. 94):

I. To prove O' the center of $\odot\ A'B'$, prove that any two points in $\odot\ A'B'$ are equally distant from O'.

II. That is, if A' and C' are any two points in $\odot\ A'B'$, prove $O'A'=O'C'$.

III. $\therefore$ draw the elements through A' and C' meeting base AB in A and C respectively, and prove $A'O'=AO=OC=O'C'$.

If the bases of a cylinder are circles, the line joining the centers of the bases is called the **axis** of the cylinder.

12. If the bases of a cylinder are circles, the axis of the cylinder is parallel to the elements.

Suggestion. Use Ex. 11 and prove that the axis coincides with a line that is parallel to the elements.

13. In every cylinder whose bases are circles there is one set of parallel planes that cut the cylinder in sections that are rectangles.

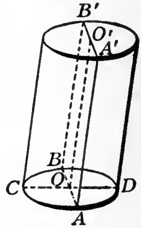


FIG. 95

Suggestion (Fig. 95). Let OO' be the axis of the cylinder. Construct CD, the projection of OO' upon the lower base. Draw $AB \perp CD$ at O. Pass a plane through AB and the element through A.

To prove $A'A \perp AB$, prove $OO' \perp AB$, and $OO' \parallel AA'$.

14. How many vertices and how many edges has a tetrahedron? A regular octahedron? A prism whose base has 5 sides? 6 sides? n sides?

15. Verify the following formula for the six regular polyhedrons:

$$F+V=E+2,$$

?

where F, V, and E are the number of faces, vertices, and edges respectively.

NOTE. The formula above is known as Euler's Theorem. It is true for all convex polyhedrons. A proof is too difficult for this book.

16. Construct a regular tetrahedron (Fig. 96).

Suggestion. Construct the equilateral triangle ABC. Let O be the center of the circle circumscribed about $\triangle ABC$. Construct $OX \perp$ plane of ABC at O. Find X so that $XB = AB$.

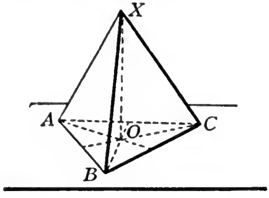


FIG. 96

17. Construct a regular octahedron (Fig. 97).

Suggestion. Construct the square $ABCD$. Let O be the intersection of the diagonals AC and BD. Construct $EF \perp ABCD$ at O. Take $OE = OF = OA$. Join EA, EB, EC, etc.

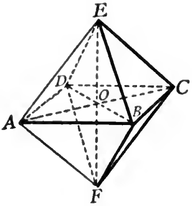


FIG. 97

18. The altitude of a regular tetrahedron meets the base at the intersection of the medians.

19. If an edge of a regular tetrahedron is E, the altitude is $\frac{E}{3}\sqrt{6}$.

20. The segments joining in order the mid-points of two pairs of opposite sides of a regular tetrahedron form a square (Fig. 98).

21. The mid-points of the edges of a regular tetrahedron are the vertices of a regular octahedron.

22. Inscribe a regular octahedron in a cube.

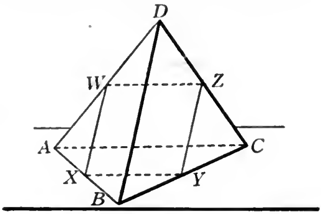


FIG. 98

CHAPTER III

THE SPHERE

INTRODUCTORY

DEFINITIONS

93. A **spherical surface** has been defined as the locus of points in space at a given distance from a given point.

The given point is called the **center.**

The given distance is called the **radius.**

A **sphere** is a solid bounded by a spherical surface.

The center and the radius of the spherical surface are called respectively the **center** and the **radius** of the sphere.

A segment through the center of the sphere and terminating in the spherical surface is called a **diameter** of the sphere.

ASSUMPTIONS CONCERNING SPHERES

94. We will assume the following:

As. 11. Spheres with equal radii or with equal diameters are congruent.

As. 12. All radii of the same sphere or of congruent spheres are equal.

As. 13. All diameters of the same or of congruent spheres are equal.

As. 14. A diameter of a sphere is twice the radius of that sphere.

As. 15. A sphere is located definitely if its center and its radius are known.

As. 16. A segment joining a point within a sphere and the center is shorter than the radius.

As. 17. If a segment that has one end at the center of the sphere is shorter than the radius, it lies wholly within the sphere.

As. 18. A segment joining a point without a sphere and the center is longer than the radius.

As. 19. If a segment that has one end at the center of the sphere is longer than the radius, it extends without the sphere and cuts the sphere but once.

As. 20. If a straight line passes through a point within a sphere, it intersects the sphere in two and only two points.

As. 21. If a straight line passes through a point without the sphere, it may (1) have no point in common with the sphere, (2) have one point in common with the sphere, or (3) it may intersect the sphere in two and only two points.

As. 22. A plane through a point within a sphere intersects the sphere in a closed curve.

As. 23. A plane through a point without a sphere may (1) have no point in common with the sphere, (2) have one point in common with the sphere, or (3) may intersect the sphere in a closed curve.

95. The following preliminary theorem is evident:

THEOREM 59. **If a plane is passed through the center of the sphere, the intersection of the plane and the spherical surface is a circle.**

TANGENTS TO SPHERES

FUNDAMENTAL THEOREM

96. A line or a plane that has one point in common with the sphere, but does not intersect it, is said to be **tangent** to the sphere.

The point which the tangent line or plane has in common with the sphere is called the **point of contact** or the **point of tangency.**

97. THEOREM 60. **A plane that is perpendicular to a radius of a sphere at its outer extremity is tangent to the sphere.**

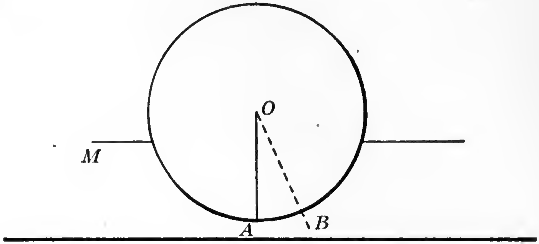


FIG. 99

Hypothesis: O is any sphere with center O. OA is any radius of sphere O. Plane M is perpendicular to OA at A.

Conclusion: Plane M is tangent to sphere O.

Analysis and construction:

I. To prove plane M tangent to sphere O at A, prove that all points in M, except A, lie outside the sphere.

II. $\therefore$ take B any point in M except A, join B and O, and prove that B lies outside the sphere.

III. $\therefore$ prove $OB > OA$. (See Th. 14 and As. 19.)

THEOREM 61. **A plane that is tangent to a sphere is perpendicular to the radius of the sphere at the point of contact.**

Analysis and construction: Use an indirect proof. If OA is not $\perp M$, suppose some other line as $OB \perp M$ and show that the supposition that $OB \perp M$ contradicts the hypothesis.

Outline of proof:

I. If $OB \perp M$, $OB < OA$.

II. If $OB < OA$, point B is within the sphere.

III. If B is within the sphere, M is not tangent to the sphere (As. 22).

IV. But M is given tangent to the sphere.

V. $\therefore$ OA is $\perp M$.

Ex. 1. Are Ths. 60 and 61 true for straight lines instead of planes? Give proof.

Ex. 2. If a plane is tangent to a sphere, every line in the plane drawn through the point of contact is tangent to the sphere.

Ex. 3. All lines tangent to a sphere at a given point on the sphere lie in the plane tangent to the sphere at that point.

Ex. 4. If two lines are tangent to a sphere at the same point, the plane of these lines is tangent to the sphere at that point.

Ex. 5. Show how to construct a line that shall be tangent to a given sphere and shall contain a given point. Discuss three cases.

Suggestion. Reduce to a plane geometry construction by passing any plane through the given point and the center of the sphere.

Ex. 6. Answer Ex. 5 for a plane tangent to the sphere instead of for a line tangent to the sphere.

Ex. 7. All segments tangent to a sphere from a given point without the sphere are equal.

CIRCUMSCRIBED AND INSCRIBED SPHERES

98. If all the faces of a polyhedron are tangent to a sphere, the polyhedron is said to be **circumscribed about the sphere** and the sphere is said to be **inscribed in the polyhedron.**

If all the vertices of a polyhedron lie in a given spherical surface, the polyhedron is said to be **inscribed in the sphere** and the sphere is said to be **circumscribed about the polyhedron.**

DETERMINATION OF SPHERES

99. The following exercise is preliminary and may be quoted as a theorem in proving Th. 62.

Exercise. Lines that are perpendicular to two intersecting planes cannot be parallel.

Analysis: Use indirect proof.

I. Show that if $a \parallel b$, M would be $\parallel N$.
II. To show that M would be $\parallel N$, show M and N would both be $\perp a$.

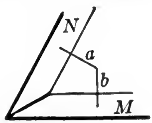


FIG. 100

THEOREM 62. **One and only one spherical surface can be passed through four given points that are not co-planar.**

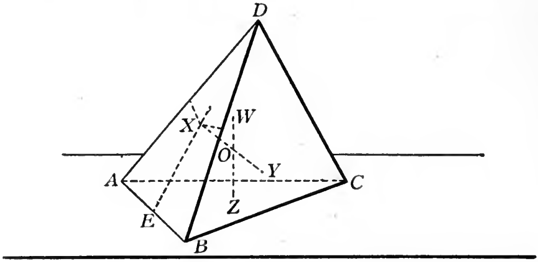


FIG. 101

Hypothesis: $ABCD$ is any tetrahedron.

Conclusion:

a. A sphere can be passed through points A, B, C, and D.

b. Only one sphere can be passed through A, B, C, and D.

Analysis in general for a:

I. It is necessary to prove that there is a point equally distant from A, B, C, and D.

II. $\therefore$ find the locus of points equally distant from A, B, and C and the locus of points equally distant from A, B, and D. Prove that these loci intersect.

Construction:

I. To find the locus of points equally distant from A, B, and D, find the perpendicular to the plane of A, B, and D at the center of the circle circumscribing $\triangle ABD$. Let this locus be XY.

II. In the same way find the locus of points equally distant from A, B, and C. Let this locus be ZW.

NOTE. To prove XY and ZW intersect, prove that they lie in the same plane and are not parallel. $\therefore$ prove XY and ZW are in a plane $\perp AB$ at its mid-point.

Analysis for b: Show that there is only one point equally distant from points A, B, C, and D.

CIRCLES OF SPHERES

FUNDAMENTAL THEOREM

100. THEOREM 63. **Every section of a spherical surface made by a plane is a circle.**

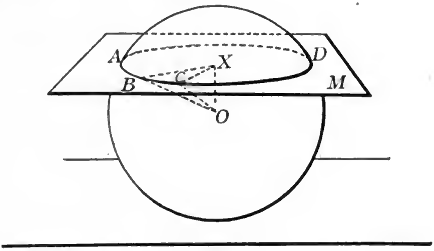


FIG. 102

Hypothesis: O is the center of the given sphere cut by plane M in section $ABCD$.

Conclusion: The section $ABCD$ is a circle.

Analysis and construction:

I. To prove $ABCD$ a circle, prove that every point in $ABCD$ is equally distant from some point within.

II. $\therefore$ draw a perpendicular from O to plane M meeting plane M at point X. Join X with B and C, any two points in $ABCD$, and prove $BX = CX$.

III. $\therefore$ join BO and CO and prove $\triangle BXO = \triangle CXO$.

NOTE. Since we have seen in Th. 59 that a plane passing through the center of the sphere is a circle, it is only necessary in Th. 63 to prove that the section of the spherical surface is a circle if the plane does not pass through the center.

DEFINITIONS

101. A section of a sphere made by a plane through the center of the sphere is called a **great circle.**

A section of a sphere made by a plane that does not pass through the center of the sphere is called a **small circle.**

The diameter of a sphere perpendicular to the plane of a circle of a sphere is called the **axis of that circle.**

The ends of axis of a circle of a sphere are called the **poles of that circle.**

THEOREMS CONCERNING CIRCLES ON SPHERES

102. THEOREM 64. **The axis of a circle of a sphere passes through the center of that circle.**

Suggestion. Th. 64 is a corollary of Th. 63. In Th. 63, OX was constructed from $O \perp$ the plane of the circle and meeting the plane at X. The point X was proved to be the center of the circle. Why is OX the axis of the circle? Use Fig. 103, No. 1.

THEOREM 65. **All great circles on a given sphere are congruent.**

THEOREM 66. **Any two great circles on a sphere bisect each other.**

Suggestion. Show that the intersection of the planes of the two great circles is a diameter of each circle.

THEOREM 67. **Every great circle on a sphere bisects the sphere.**

THEOREM 68. **Three points on a sphere no two of which are at the ends of a diameter determine a small circle of a sphere.**

THEOREM 69. **Two points on a sphere not the ends of a diameter determine a great circle of a sphere.**

The length of the minor arc of a great circle on a sphere joining two given points on the sphere is called the **spherical distance** between the two given points. This is the shortest distance between the two points on the spherical surface.

Ex. 1. How many great circles of a sphere can pass through the opposite ends of a diameter? Why?

Ex. 2. How many small circles of a sphere can pass through two points on a sphere if the two points are (1) the opposite ends of a diameter; (2) not the opposite ends of a diameter?

POLAR DISTANCE

103. THEOREM 70. **Every point on a circle on a sphere is equally distant from each of its poles.**

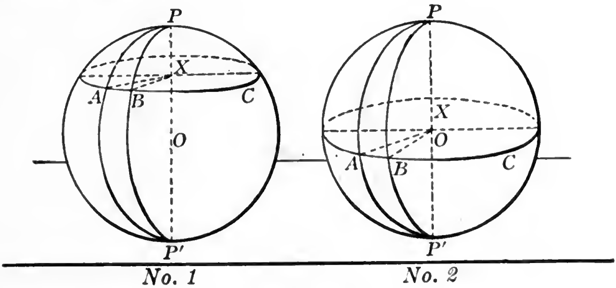


FIG. 103

Hypothesis: O is any sphere. C is a circle on the sphere. P and P' are the poles of circle C. A and B are any two points on circle C.

Conclusion: A and B are equally distant from P and also from P'.

Analysis and construction:

A. I. To prove A and B equally distant from P, draw minor arcs of great circles joining AP and BP and prove $\widehat{AP} = \widehat{BP}$.

II. $\therefore$ draw $\overline{PA}$ and $\overline{PB}$ and prove $\overline{AP} = \overline{BP}$.

B. I. To prove A and B equally distant from P', draw minor arcs of great circles joining AP' and BP' and prove $\overline{AP'} = \overline{BP'}$.

The spherical distance from the nearer pole of a circle of a sphere to any point on the circle is called the **polar distance** of the circle.

COR. **The polar distance of a great circle of a sphere is a quadrant, or an arc of 90°.**

NOTE. The proof of Th. 70 shows that the theorem is true if distance means either (1) rectilinear distance or (2) spherical distance.

NOTE. In order that a true conception of much of the work in this chapter may be obtained a slated globe should be used in class, and the pupil should prepare his lessons with a ball on which lines may be drawn. To draw a circle on a sphere, it is evident that its pole and the polar distance or the linear distance must be known.

104. THEOREM 71. **If a point is a quadrant's distance from each of two points in a sphere, it is the pole of the great circle passing through these two points.**

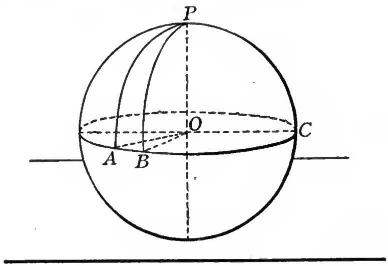


FIG. 104

Hypothesis: O is any sphere. $\overset{\frown}{PA}$ and $\overset{\frown}{PB}$ are quadrants.

Conclusion: P is the pole of the great circle ABC through points A and B.

Analysis and construction:

I. To prove P the pole of great circle ABC, join PO and prove $PO \perp$ plane ABC.

II. To prove $PO \perp$ plane ABC, join OA and BO and prove $\angle POA$ and $\angle POB$ right angles.

MISCELLANEOUS EXERCISES

105. 1. If a plane is tangent to a sphere, the plane of every great circle through the point of contact is perpendicular to the tangent plane.

2. A line perpendicular to the plane of a circle of a sphere at the center of the circle passes through the center of the sphere.

3. A segment joining the center of a circle on a sphere with the center of the sphere is perpendicular to the plane of the circle.

4. If the planes of two great circles on a sphere are perpendicular to each other, each circle passes through the pole of the other.

5. Points A and B are at the opposite ends of a diameter of a sphere, and a given point P is at a quadrant's distance from both A and B. Of which great circle or circles through A and B is P the pole?

6. Into how many parts do three great circles of a sphere divide the surface of the sphere? Discuss two cases.

7. The sections of a sphere made by parallel planes have the same poles.

8. State and prove the converse to Ex. 7.

9. If two planes cut a sphere at equal distances from the center of the sphere, the circles thus formed are congruent.

Analysis and construction:

I. To prove the circles congruent, prove their diameters equal.

II. Let OA and OB be the perpendiculars from the center of the sphere to the planes of the two circles. The plane of OA and OB will cut the planes of the two circles in diameters. Use plane geometry.

10. State and prove the converse to Ex. 9.

11. On the same sphere or on congruent spheres congruent circles have equal polar distances.

12. State and prove the converse to Ex. 11.

13. The plane of a circle of a sphere is perpendicular to the planes of all great circles passing through its poles.

14. A plane that bisects at right angles a chord of a sphere contains the center of the sphere.

15. Show that a sphere can be inscribed in a given cube. Show that one can be circumscribed about a given cube.

16. Given a segment equal to the edge of a cube. Construct on a plane a segment equal to the radius of the inscribed sphere, and one equal to the radius of the circumscribed sphere.

17. It can be proved that the six planes bisecting the dihedral angles of any tetrahedron meet in a point. Show from this how to inscribe a sphere in any given tetrahedron.

SPHERICAL ANGLES

106. If two great circle arcs on a sphere intersect, they are said to form a **spherical angle.** The point of intersection is called the **vertex** of the angle. The two great circle arcs are called the **sides** of the angle.

The **measure** of a spherical angle is defined as the measure of the plane angle made by the tangents to the two arcs at their intersection.

107. THEOREM 72. **A spherical angle has the same measure as (1) the dihedral angle formed by the planes of its sides, and (2) the arc of a great circle drawn with its vertex as a pole and included between its sides.**

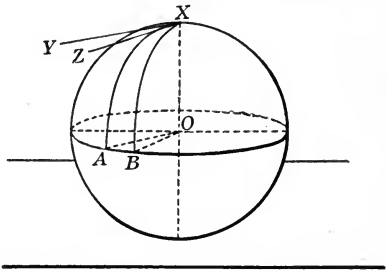


FIG. 105

Hypothesis: AXB is a spherical angle formed by the two great circle arcs AX and BX on sphere O. The planes of AX and BX intersect, forming the dihedral angle whose edge is XO. $\widehat{AB}$ is the arc of a great circle drawn with X as pole and included between the sides of $\angle AXB$. XY and XZ are tangents to $\widehat{AX}$ and $\widehat{BX}$ at X.

Conclusion:

a. $\angle AXB$ has the same measure as dihedral angle XO.

b. $\angle AXB$ has the same measure as $\widehat{AB}$.

Analysis for a: To prove that $\angle AXB$ has the same measure as dihedral angle XO, prove that $\angle YXZ$ is the plane angle of the dihedral angle XO.

Analysis and construction for b:

I. To prove that $\angle AXB$ has the same measure as $\widehat{AB}$, join AO and BO and prove $\angle YXZ = \angle AOB$, and $\angle AOB$ has the same measure as $\widehat{AB}$.

II. To prove $\angle YXZ = \angle AOB$, prove $XY \parallel OA$, and $XZ \parallel OB$.

III. $\therefore$ prove OA and $OB \perp XO$ at O.

COR. I. **Two great circle arcs are perpendicular to each other if their planes are perpendicular to each other.**

COR. II. **If two great circle arcs are perpendicular to each other, then each passes through the pole of the other.**

Ex. 1. If an arc of one great circle passes through the pole of the second, the two arcs are perpendicular to each other.

Ex. 2. If one great circle arc passes through the pole of a second, then the second passes through the pole of the first.

Ex. 3. If two points on a sphere are a quadrant's distance apart, then each is the pole of one great circle passing through the other.

SPHERICAL TRIANGLES

LINES ON SPHERES

108. We have stated that the shortest distance between two points on the surface of a sphere is the minor arc of a great circle passing through these points. This fact is important for two reasons.

I. It is important theoretically. Arcs of great circles on a sphere take the place in our work of straight lines on a plane. As we study figures made up of straight lines on a plane in plane geometry, so we are to study figures made up of arcs of great circles on a sphere in solid geometry. The geometry of these figures can be studied by the use of the same methods as those used in studying triangles and polygons on a plane, but this method is out of place here.

II. It is important practically. Use is made of it in sea sailing and airplane work. To obtain the answers to the following questions, use a small geographical globe. Suppose the earth to be a perfect sphere.

Ex. 1. What kinds of circles are the parallels of latitude on the earth? What kinds of circles are the meridians? What kind of a circle is the equator.

Ex. 2. Of what circle is the North Pole of the earth the pole?

Ex. 3. What is the shortest route from the Pacific end of the Panama Canal to Japan? Stretch a string between the two points on the globe.

Ex. 4. What is the shortest route from New York to England? From Cape Town, South Africa, to Melbourne, Australia?

Ex. 5. What is meant by the statement that there are no lines on a sphere that correspond to parallel straight lines on a plane?

Ex. 6. Why are the parallels of latitude on the earth called parallels?

DEFINITIONS

109. A closed figure formed of two or more arcs of great circles on a sphere, no one of which is greater than a semi-great circle, is called a **spherical polygon.**

NOTE. Spherical polygons can be constructed in which one or more sides are greater than a semi-great circle. Such polygons are called **general spherical polygons** and will not be considered in this book.

The intersection of the arcs are called the **vertices** of the polygon. In Fig. 106, A, B, C, and D are the vertices of the spherical polygon $ABCD$.

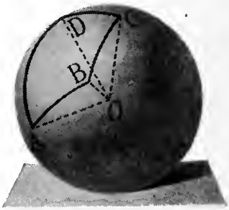


FIG. 106

The arcs are called the **sides** of the spherical polygon. In Fig. 106, $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, and $\overset{\frown}{DA}$ are the sides of the polygon $ABCD$.

The spherical angles formed by the arcs are called the **angles** of the spherical polygon. What are the angles of the spherical polygon $ABCD$ in Fig. 106?

A spherical polygon of two sides is called a **lune.** In this case the sides of the spherical polygon are semi-great circles. In Fig. 107, the figure formed by $\widehat{ACB}$ and $\widehat{ADB}$ is a lune. There is, obviously, no figure on a plane that corresponds to a lune on the sphere.

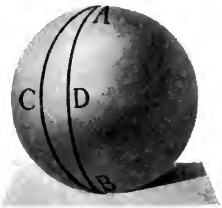


FIG. 107

A spherical polygon of three sides is called a **spherical triangle.** In Fig. 108, ABC is a spherical triangle.

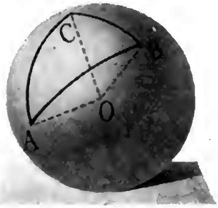


FIG. 108

Spherical triangles are **isosceles** or **equilateral** as in plane geometry.

A **convex spherical polygon** is one in which if any side is extended the whole polygon lies on one side of the extended arc. Unless otherwise stated, convex spherical polygons are intended.

Exercise. Using a slated globe or a ball of some kind on which marks can be made, draw a spherical triangle with one side greater than a semi-great circle. Is it concave or convex? Draw one in which two sides are greater than semi-great circles. Can you draw one in which two sides are semi-great circles? Why?

SPHERICAL POLYGONS AND CENTRAL POLYHEDRAL ANGLES

110. A polyhedral angle with its vertex at the center of the sphere cuts the surface of the sphere in a convex spherical polygon. Conversely, if the vertices of a convex spherical polygon are joined to the center of the sphere, a polyhedral angle is formed with its vertex at the center of the sphere. This polyhedral angle is called the **corresponding central polyhedral angle** of the spherical polygon.

NOTE. The theorems that we have studied about polyhedral angles are for convex polyhedral angles only. This is one reason why we are confining our study of spherical polygons to those whose sides are less than 180° and to convex figures.

To each side of the spherical polygon corresponds one face angle of the polyhedral angle, namely: that face angle that is subtended by the side of the polygon. To each angle of the spherical polygon corresponds one dihedral angle of the polyhedral angle, namely: that dihedral angle which is formed by the planes of the sides of the spherical angle. The next two theorems follow at once.

THEOREM 73. **Any side of a spherical polygon has the same measure as the corresponding face angle of the corresponding central polyhedral angle.** (See Fig. 109.)

Suggestion. As the sides of a spherical polygon are arcs, they may be measured in degrees.

THEOREM 74. **Any angle of a spherical polygon has the same measure as the corresponding dihedral angle of the corresponding central polyhedral angle.** (See Fig. 109.)

Suggestion. This is a restatement of what theorem?

SOME PROPERTIES OF SPHERICAL TRIANGLES

111. THEOREM 75. **The sum of two sides of a spherical triangle is greater than the third side.**

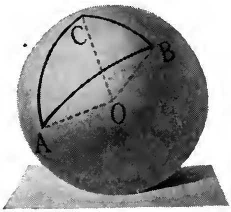


FIG. 109

Suggestion. The proof follows at once from Ths. 73 and 36

Some theorems concerning spherical polygons bear a certain peculiar relation to the corresponding theorems concerning polyhedral angles. This relation is called **reciprocal** and follows at once from Ths. 73 and 74. When the measures of arcs and angles are concerned, one theorem may be obtained from the other by the substitution of the measure of the sides of the polygon and the measure of the corresponding face angles of the polyhedral angle for each other, or the measure of the spherical angles and the measure of the corresponding dihedral angles for each other.

THEOREM 76. **The sum of the sides of a spherical polygon is less than a great circle.**

Ex. 1. Describe the corresponding central trihedral angle of a spherical triangle that has (1) two sides equal; (2) three sides equal; (3) two angles equal; (4) three angles equal. Give reasons.

Ex. 2. Describe the spherical triangle that would correspond to a central trihedral angle that has three right dihedral angles. Can you draw this triangle on the surface of a given sphere such as a slated globe?

Ex. 3. Can you answer Ex. 2 if the initial trihedral angle has only two right dihedral angles? Two obtuse dihedral angles?

POLAR TRIANGLES

112. If A, B, and C are the vertices of a spherical triangle, and a, b, and c are, respectively, the sides opposite these vertices, and if A' is that pole of side a that is on the same side of a as the vertex A, B' of b, and C' of c, then $A'B'C'$ is called the **polar triangle** of $\triangle ABC$. In Fig. 110, A' and A'' are the poles of the side a in $\triangle ABC$. A and A' are on the same side of a. Then A' is one vertex of the polar triangle of $\triangle ABC$. In the same way the other vertices are C' and B'.

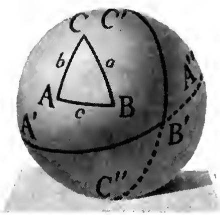


FIG. 110

113. THEOREM 77. **If one spherical triangle is the polar of a second, then reciprocally the second is the polar of the first.**

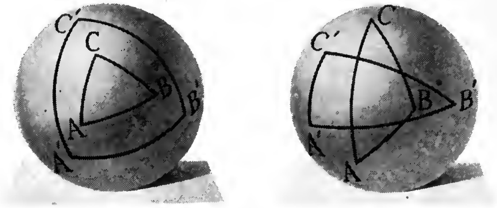


FIG. 111

Hypothesis: ABC is a spherical triangle. $A'B'C'$ is the polar triangle of $\triangle ABC$.

Conclusion: ABC is the polar triangle of $\triangle A'B'C'$.

Analysis:

I. To prove ABC the polar of $\triangle A'B'C'$, prove A the pole of $B'C'$, B the pole of $A'C'$, and C the pole of $B'A'$.

II. To prove A the pole of $B'C'$, prove A a quadrant's distance from B' and C'.

Outline of proof:

B' is the pole of AC.
$\therefore$ $B'A$ is a quadrant.
C' is the pole of AB.
$\therefore$ $C'A$ is a quadrant.
$\therefore$ A is the pole of $B'C'$.

Similarly

B is the pole of $A'C'$.
C is the pole of $B'A'$.
$\therefore$ ABC is the polar of $A'B'C'$.

Ex. 1. Show how to construct on a ball or spherical blackboard two polar spherical triangles.

Ex. 2. Construct on a ball or spherical blackboard the polar of a spherical triangle that has each side less than a quadrant.

Ex. 3. Construct as in Ex. 2 the polar of a triangle that has
a. One side greater and two sides less than a quadrant.
b. Two sides greater and one side less than a quadrant.
c. Each side greater than a quadrant.
d. One side a quadrant.
e. Two sides quadrants.
f. Three sides quadrants.

Ex. 4. Describe the corresponding central trihedral angle of each triangle mentioned in Ex. 3 and of each polar.

114. THEOREM 78. **In two polar spherical triangles the sum of the measures of any angle of one and that side of the other of which it is a pole is 180°.**

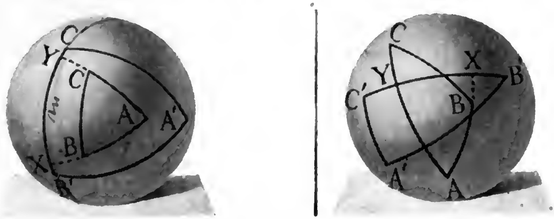


FIG. 112

Hypothesis: ABC and $A'B'C'$ are two polar spherical triangles. Let A represent the measure of $\angle A$, and a' the measure of $\widehat{B'C'}$.

Conclusion: $A+a'=180°$.

Analysis and construction:

I. To prove $A+a'=180°$, compare A with an arc which is the supplement of a'.

II. $\therefore$ continue $\widehat{AB}$ and $\widehat{AC}$ to meet $C'B'$ in X and Y respectively. Let m represent the measure of XY.

Prove (1) $A=m$.

(2) $m+a'=180°$.

III. To prove $m+a'=180°$, prove

(1) $C'X+B'Y=C'B'+XY=m+a'$.

(2) $C'X+B'Y=180°$.

IV. $\therefore$ prove $C'X=90°$, and $B'Y=90°$.

COR. I. **On the same sphere or on congruent spheres, if two spherical triangles are mutually equilateral, their polars are mutually equiangular** (Fig. 113).

COR. II. **On the same sphere or on congruent spheres, if two spherical triangles are mutually equiangular, their polars are mutually equilateral** (Fig. 113).

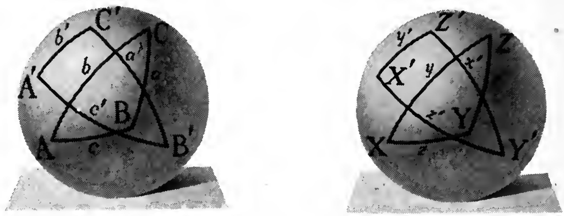


FIG. 113

Exercise. The sides of a spherical triangle are 65°, 115°, and 120°. Find the number of degrees in each angle (1) of its polar, (2) of each of the triangles formed by the intersection of the great circles of which the sides of the polar are arcs.

THE SUM OF THE ANGLES OF A SPHERICAL TRIANGLE

115. THEOREM 79. **The sum of the angles of a spherical triangle is more than two and less than six right angles.**

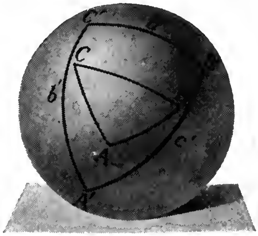


FIG. 114

Hypothesis: $\triangle ABC$ is any spherical triangle. A, B, and C represent the measures of $\angle s\ A$, B, and C respectively.

Conclusion: $A+B+C>2$ rt. $\angle s$ and <6 rt. $\angle s$.

Analysis and construction: To find the value of $A+B+C$, construct the polar of $\triangle ABC$. Let a', b', and c' represent the measures of the sides of the polar. Compare $A+B+C$ with $a'+b'+c'$.

Outline of proof:

I. $A+a'=180°$.
$B+b'=180°$.
$C+c'=180°$.
$\therefore\ A+B+C+a'+b'+c'=540°$ or 6 rt. $\measuredangle$,
but $a'+b'+c'>0$.
$\therefore\ A+B+C<6$ rt. $\measuredangle$.

II. Again, $a'+b'+c'<4$ rt. $\measuredangle$.
$\therefore\ A+B+C>2$ rt. $\measuredangle$.

In giving the reasons note the application of the inequality assumptions.

Ex. 1. On a ball or a spherical blackboard draw a spherical triangle with two right angles; with three right angles; with two obtuse angles.

A spherical triangle with two right angles is called a **birectangular spherical triangle.** One with three right angles is called a **trirectangular spherical triangle.**

The number of degrees by which the sum of the angles of a spherical triangle exceeds 180° is called the **spherical excess** of the triangle.

Ex. 2. What can you say concerning the sum of the measures of the dihedral angles of a trihedral angle? Why? What do you know concerning the sum of the face angles of a trihedral angle?

Ex. 3. In a birectangular spherical triangle the sides opposite the right angles are quadrants.

Ex. 4. The sides of a trirectangular spherical triangle are quadrants.

Ex. 5. The polar triangle of a birectangular spherical triangle is birectangular.

Ex. 6. What is the polar of a trirectangular spherical triangle? Why?

CONGRUENT AND SYMMETRIC SPHERICAL TRIANGLES

116. Two spherical triangles on the same sphere or on congruent spheres are said to be **congruent** if the sides and angles of one are equal respectively to the sides and angles of the other and arranged in the same order.

As. 24. **Two spherical triangles or two trihedral angles congruent to a third are congruent to each other.**

The next two theorems follow at once from the definition above. See the remark on p. 91 concerning the proof of Th. 75.

What are congruent trihedral angles? (§52.)

Theorem 80. **On the same sphere or on congruent spheres, if two spherical triangles are congruent, the corresponding central trihedral angles are congruent** (Fig. 115).

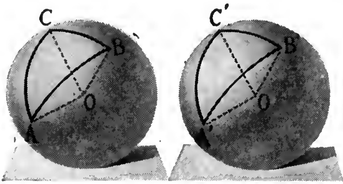


Fig. 115

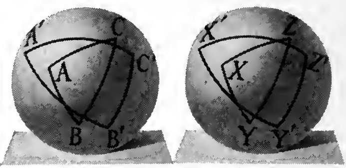


Fig. 116

Theorem 81. **In the same sphere or in congruent spheres, if two central trihedral angles are congruent, the corresponding spherical triangles are congruent** (Fig. 115).

Theorem 82. **On the same sphere or on congruent spheres, if two spherical triangles are congruent, their polar triangles are congruent** (Fig. 116). (See Th. 78.)

117. Two spherical triangles on the same sphere or on congruent spheres are said to be **symmetric** if the sides and angles of one are equal respectively to the sides and angles of the other and arranged in the opposite order.

What are symmetric trihedral angles? (§52.)

In general, two symmetrical spherical triangles or two symmetric trihedral angles cannot be made to coincide.

Ex. 1 Show how two plane figures that are symmetric with respect to a point or to a line can be made to coincide.

Ex. 2. Draw on a spherical blackboard figures to illustrate two congruent and two symmetric spherical triangles.

Ex. 3. Construct from cardboard two symmetric trihedral angles, making the faces circular sectors with equal radii.

As. 25. **Two spherical triangles or two trihedral angles symmetric to a third are congruent to each other.**

The next two theorems follow at once from the above definition and the remarks following Th. 75.

THEOREM 83. **On the same sphere or on congruent spheres, if two spherical triangles are symmetric, the corresponding central trihedral angles are symmetric** (Fig. 117).

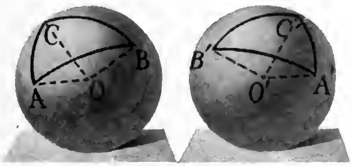


FIG. 117

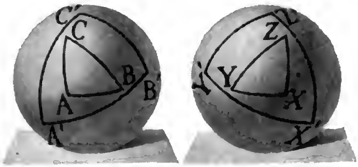


FIG. 118

THEOREM 84. **In the same sphere or in congruent spheres, if two central trihedral angles are symmetric, the corresponding spherical triangles are symmetric** (Fig. 117).

THEOREM 85. **On the same sphere or on congruent spheres, if two spherical triangles are symmetric, their polar triangles are symmetric** (Fig. 118).

118. If the edges of one trihedral angle are prolongations of the edges of another trihedral angle, the trihedral angles are said to be **vertical trihedral angles,** and the corresponding spherical triangles cut out on the sphere are said to be **vertical spherical triangles.**

THEOREM 86. **Two vertical trihedral angles are symmetric.**

THEOREM 87. **Two vertical spherical triangles are symmetric** (Fig. 119).

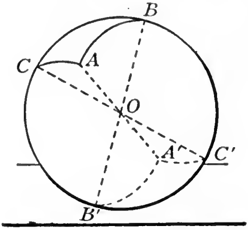


FIG. 119

119. THEOREM 88. **If two spherical triangles on the same or on congruent spheres have two sides and the included angle of one equal to two sides and the included angle of the other,**

a. **The triangles are congruent if the parts are arranged in the same order.**

b. **The triangles are symmetric if the parts are arranged in the opposite order.**

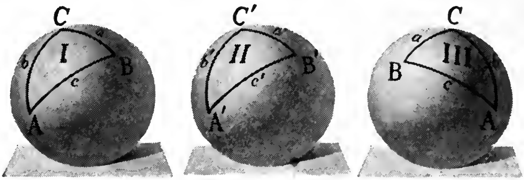


FIG. 120

Hypothesis: ABC and $A'B'C'$ are two spherical triangles in which $\angle A = \angle A'$, $b = b'$, and $c = c'$.

Conclusion:

a. If $\angle A$, b, and c are arranged in the same order as $\angle A'$, b', and c', $\triangle ABC \cong \triangle A'B'C'$ ($\triangle$I $\cong \triangle$II).

b. If $\angle A$, b, and c are arranged in the opposite order to $\angle A'$, b', and c', $\triangle ABC$ and $\triangle A'B'C'$ are symmetric ($\triangle$II symmetric to $\triangle$III).

Analysis for a: To prove $\triangle$I $\cong \triangle$II, show that they will coincide if superposed as in plane geometry.

Analysis and construction for b:

I. To prove $\triangle$III symmetric to $\triangle$II, prove $\triangle$III congruent to a triangle that is symmetric to $\triangle$II.

II. $\therefore$ construct $\triangle$IV symmetric to $\triangle$II and prove $\triangle$IV congruent to $\triangle$III.

NOTE. $\triangle$IV is not shown in the figure.

COR. **If two trihedral angles have two face angles and the included dihedral angle of one equal to two face angles and the included dihedral angle of the other, the trihedral angles are congruent if the parts are arranged in the same order, and symmetric if they are arranged in the opposite order.**

120. THEOREM 89. **If two spherical triangles on the same or on congruent spheres have two angles and the included side of one equal to two angles and the included side of the other,**

a. **The triangles are congruent if the parts are arranged in the same order.**

b. **The triangles are symmetric if the parts are arranged in the opposite order.**

Analysis for a:

I. To prove $\triangle\mathrm{I} \cong \triangle\mathrm{II}$, prove that their polars are congruent.

II. To prove the polars congruent, prove that they have two sides and the included angle of one equal to $\triangle$, etc.

Analysis for b: To prove $\triangle$III symmetric to $\triangle$II, prove that their polars are symmetric.

COR. **If two trihedral angles have two dihedral angles and the included face angle of one equal to two dihedral angles and the included face angle of the other, the trihedral angles are congruent if the parts are arranged in the same order, and symmetric if they are arranged in the opposite order.**

121. THEOREM 90. **If two spherical triangles on the same or on congruent spheres have three sides of one equal to three sides of the other, the triangles are either congruent or symmetric.**

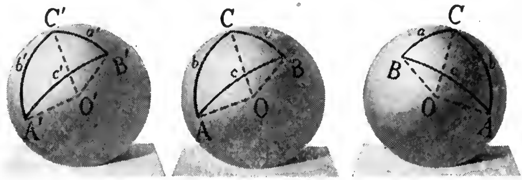


FIG. 121

Analysis: To prove $\triangle ABC$ congruent or symmetric to $\triangle A'B'C'$, prove that the corresponding central trihedral angles are congruent or symmetric. (See Th. 38, Cor.)

122. THEOREM 91. **If two spherical triangles on the same or on congruent spheres have three angles of one equal to the three angles of the other, the triangles are either congruent or symmetric.**

Analysis: To prove $\triangle ABC$ congruent or symmetric to $\triangle A'B'C'$, prove their polars congruent or symmetric.

COR. **If two trihedral angles have the three dihedral angles of one equal to the three dihedral angles of the other, the trihedral angles are either congruent or symmetric.**

ISOSCELES SPHERICAL TRIANGLES

123. THEOREM 92. **The angles opposite the equal sides of an isosceles spherical triangle are equal.**

Suggestion. Join the vertex C with X, the mid-point of the base AB, by an arc of a great circle. Prove $\angle A$ and $\angle B$ corresponding angles of symmetric spherical triangles (Fig. 122).

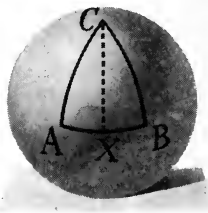


FIG. 122

THEOREM 93. **If two angles of a spherical triangle are equal, the sides opposite the equal angles are equal** (Fig. 122).

Analysis: To prove $AC=CB$, prove that $\measuredangle$ A' and B' of the polar are equal.

Ex. 1. If one of two polar spherical triangles is isosceles, the other is.

Ex. 2. If a spherical triangle is equilateral, its polar is also equilateral.

THEOREM 94. **If two symmetric spherical triangles are isosceles, they are congruent.**

FIG. 123

Hypothesis: ABC and $A'B'C'$ are two symmetric isosceles spherical triangles with $AB=A'B'$, $AC=A'C'$, $CB=C'B'$, $\angle C=\angle C'$, $\angle A=\angle A'$, $\angle B=\angle B'$.

Conclusion: $\triangle ABC \cong \triangle A'B'C'$.

Analysis: To prove $\triangle ABC \cong \triangle A'B'C'$, prove $AC=C'B'$, $\angle C=\angle C'$, $CB=C'A'$.

EQUIVALENT SPHERICAL TRIANGLES

124. We have seen that, in general, two symmetric spherical triangles cannot be made to coincide. Why? We will prove, however, that two symmetric spherical triangles are equivalent; that is, they cover the same extent of spherical surface. In general, we have

As. 26. **Two spherical polygons are equivalent if they are congruent or are made up of parts congruent in pairs.**

THEOREM 95. **Two symmetric spherical triangles are equivalent.**

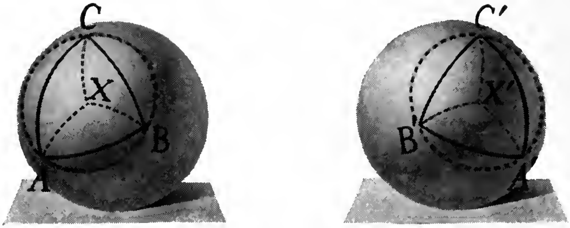


FIG. 124

Hypothesis: $\triangle ABC$ and $\triangle A'B'C'$ are two symmetric spherical triangles.

Conclusion: $\triangle ABC = \triangle A'B'C'$.

Analysis and construction:

I. To prove $\triangle ABC = \triangle A'B'C'$, prove that they can be divided into parts congruent in pairs.

II. $\therefore$ divide $\triangle ABC$ and $\triangle A'B'C'$ into isosceles spherical triangles symmetric in pairs.

III. $\therefore$ find X and X', the poles of the small circles through A, B, and C and through A', B', and C' respectively. Draw $\overset{\frown}{AX}$, $\overset{\frown}{BX}$, $\overset{\frown}{CX}$, $\overset{\frown}{A'X'}$, $\overset{\frown}{B'X'}$, $\overset{\frown}{C'X'}$, and prove $\triangle XAB$ and $\triangle X'A'B'$, $\triangle XBC$ and $\triangle X'B'C'$, also $\triangle XAC$ and $X'A'C'$, isosceles and symmetric.

IV. $\therefore$ prove $\overset{\frown}{AX} = \overset{\frown}{BX} = \overset{\frown}{CX} = \overset{\frown}{A'X'} = \overset{\frown}{B'X'} = \overset{\frown}{C'X'}$.

V. $\therefore$ prove $\odot ABC \cong \odot A'B'C'$.

VI. $\therefore$ prove plane $\triangle ABC \cong$ plane $\triangle A'B'C'$.

SUPPLEMENTARY EXERCISES

MISCELLANEOUS EXERCISES

125. 1. If two sides of a spherical triangle are quadrants, the included angle has the same measure as the third side.

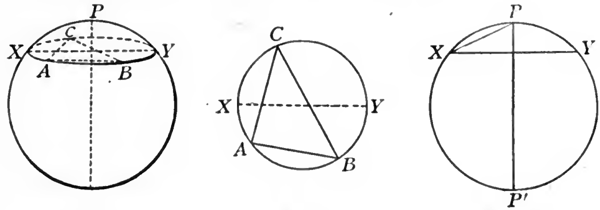
2. The spherical excess of a birectangular spherical triangle is the measure of the third angle.

3. Find the locus of poles of small circles that pass through two given points in a sphere.

4. Find the locus of poles of great circles that pass through the ends of a diameter on a sphere.

5. The central trihedral angles that correspond to two polar spherical triangles have the edges of one perpendicular to the faces of the other.

6. Find the diameter of a given material sphere.



No. 1 *No. 2* *No. 3*

FIG. 125

Analysis:

I. We can find the diameter of the sphere if we can construct on paper or blackboard a circle equal to a great circle of the sphere and find its diameter.

II. To construct a circle equal to a great circle of the sphere, locate three points on this circle. These three points may be the ends of a diameter of any small circle and its pole (Fig. 125, No. 3).

III. ∴ construct a circle equal to any small circle on the sphere.

Construction:

I. ∴ with any point, P, as a pole draw any small circle on the sphere. Draw, on paper or blackboard, $\triangle ABC$ (Fig. 125, No. 2) whose sides are equal to the chords AB, BC, CA (No. 1). (The segments AB, BC, and CA may be transferred with the dividers.) Pass a circle about points A, B, and C. This circle is equal to small circle XY of the sphere in Fig. 125, No. 1.

II. Find diameter XY of the small circle. Draw (Fig. 125, No. 3) a segment XY. With X and Y as centers and XP (Fig. 125, No. 1) as radius locate point P. Pass a circle about points X, Y, and P. This circle is equal to a great circle of the sphere, and its diameter is equal to the diameter of the sphere.

7. The intersection of two spherical surfaces is a circle (Fig. 126).

Analysis and construction (Fig. 126):

I. To prove that the intersection of spheres A and B is a circle, prove that the intersection may be generated by the revolution of a straight line that is bisected at right angles by the line of centers of the two spheres.

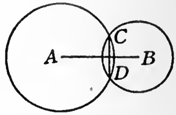


FIG. 126

II. ∴ pass any plane through the line of centers of the two spheres. This plane will cut the spheres in great circles that intersect in two points C and D. Prove AB a $\perp$ bisector of CD. (See *Plane Geometry*, Ths. 72 and 73.)

8. If two spheres intersect, the line of centers meets the spheres in points which are poles of the common circle of the two spheres.

9. The locus of points of contact of lines tangent to a given sphere from a given point without the sphere is a circle.

Suggestion. Show that this locus may be regarded as the intersection of two spheres.

10. The locus of lines tangent to a sphere from a given point without the sphere is a conical surface.

NOTE. If the elements of a conical surface are tangent to a sphere, the **conical surface** is said to be **tangent to the sphere** and is **circumscribed** about the sphere.

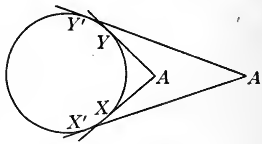


FIG. 127

Suppose (Fig. 127) that a conical surface is tangent to a given sphere and that the vertex of the conical surface recedes from the sphere. What change will take place in the circle in which the elements of the cone are tangent to the sphere? What is the limiting position of this circle? What is the limiting form of the cone? From these considerations we may infer that

(1) The locus of the points of contact of parallel tangents to a sphere is a great circle.

(2) The locus of parallel tangents to a sphere is a cylindrical surface.

If the elements of a cylindrical surface are tangent to a sphere, the **cylindrical surface** is said to be **tangent to the sphere** and is **circumscribed about the sphere.**

11. Find the locus of centers of spheres that pass through (1) two given points; (2) three given points.

12. Find the locus of centers of spheres that shall be tangent to (1) a given line at a given point; (2) a given plane at a given point.

EXERCISES ANALOGOUS TO CERTAIN PLANE GEOMETRY THEOREMS

126. NOTE. Many plane geometry theorems are true also on the surface of the sphere if we replace the straight line by the great circle arc. No plane geometry theorems that depend in any way upon parallels can be applied to the sphere. The following exercises are illustrations of some of the theorems that are true for both plane and spherical geometry. It is well for the teacher to assign these exercises in the given order, as the later ones depend in some cases on the earlier ones.

1. Give the plane geometry definitions for each of the following terms, and if possible give the definitions for the corresponding terms in spherical geometry: adjacent angles, right angles, vertical angles, supplementary angles, complementary angles.

2. Only one great circle can be drawn through a given point perpendicular to another great circle, unless the given point is the pole of the second great circle. In this latter case how many great circle arcs can be drawn?

3. Construct an arc of a great circle that is a perpendicular bisector of a given great circle arc.

4. Construct the great circle arc that bisects a given spherical angle.

5. Construct a circle circumscribed about a given spherical triangle.

6. The great circle arc that bisects the vertex angle of an isosceles spherical triangle is perpendicular to the base and bisects the base.

7. The great circle arc that passes through the vertex and the mid-point of the base of an isosceles spherical triangle is perpendicular to the base and bisects the vertex angle.

8. Are plane geometry Theorems 54 and 55 true for the spherical surface? If so, give proof.

CHAPTER IV

Areas and Volumes

AREAS OF POLYHEDRONS

GENERAL STATEMENT

127. The area of any polyhedron is the sum of the areas of its faces.

Exercise. Find the total area of a regular tetrahedron if each side is 4 in., 6 in.; if each side is s.

THE LATERAL AREA OF PRISMS

128. Theorem 96. **The lateral area of any prism is the product of the perimeter of a right section and a lateral edge.**

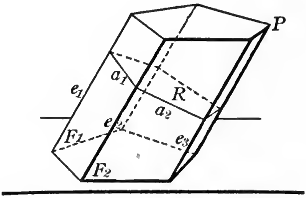


Fig. 128

Hypothesis: P is any prism. R is its right section. F_1, F_2, etc., are its faces. e_1, e_2, etc., are its edges. $a_1+a_2+\text{etc.}$ is the perimeter of R.

Conclusion: The lateral area of P is $e(a_1+a_2+\text{etc.})$.

Analysis:

I. To find the lateral area of P, find the area of each of the lateral faces and add.

II. To find the area of F_1, use e_1 as the base and prove that a_1 is the altitude.

Proof:

STATEMENTS

I. a_1, a_2, etc., are the altitudes of F_1, F_2, etc., respectively.

II. $e_1 = e_2 = e_3 =$ etc.

III. $\therefore$ area $F_1 = e_1 a_1$.
area $F_2 = e_2 a_2$.
area $F_3 = e_3 a_3$.
. . . . etc.

IV. Adding, the sum of the areas of the lateral faces is $e(a_1 + a_2 + a_3 + \text{etc.})$.

Let the pupil give all reasons. In IV use: The sum of numbers having a common factor is the common factor multiplied by the sum of the coefficients. How does this theorem apply?

COR. **The lateral area of a right prism is the product of the perimeter of the base and a lateral edge.**

THE LATERAL AREA OF A REGULAR PYRAMID

129. THEOREM 97. **The lateral area of a regular pyramid is one-half the product of the perimeter of its base and the slant height.**

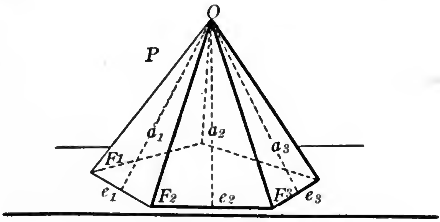


FIG. 129

Analysis: To find the lateral area of P, find the sum of the areas of the lateral faces. In the figure F_1, F_2, etc., are the faces of the pyramid. e_1, e_2, etc., are the edges of the base. a_1, a_2, etc., are the altitudes of the lateral faces.

Let the pupil give the proof. Use the proof of Th. 96 as a model. The slant height is the common factor in adding.

THE LATERAL AREA OF A FRUSTUM OF A REGULAR PYRAMID

130. THEOREM 98. **The lateral area of a frustum of a regular pyramid is one-half the product of its slant height and the sum of the perimeters of its bases.**

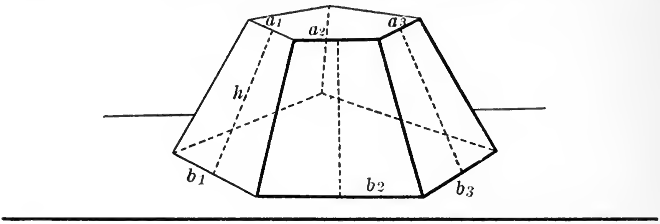


FIG. 130

Analysis: To find the lateral area of the frustum, find the area of each of the lateral faces and add.

EXERCISES INVOLVING AREAS OF POLYHEDRONS

131. 1. Find the total area of a regular triangular prism if one side of its base is 3 in. and the lateral edge is 5 in.

2. Find the total area of a regular hexagonal prism if one side of the base is 2 in. and a lateral edge is 4 in.

3. What will be the cost at 10¢ a square yard of painting the lateral surface of a tower in the form of a regular octagonal prism if each side of the base is 5 ft. and the height is 25 ft.?

4. The total area of a rectangular parallelepiped is 82 sq. in. Two dimensions are 7 in. and 2 in. Find the third dimension.

5. Find the lateral area of a regular octagonal pyramid if the slant height is 15 in. and one edge of the base is 7 in. Find the cost of gilding the same at 5¢ a square inch.

6. What is the cost at 15¢ a square foot of painting a steeple in the form of a regular square pyramid if one side of the base is 9 ft. and the height is 40 ft.?

7. Find the total area of a regular square pyramid if one edge of the base is 10 in. and one lateral edge 13 in; if one edge of the the base is a and one lateral edge e.

8. Find the total area of a regular tetrahedron whose edge is 10 in.

9. Find the total area of a regular octahedron whose edge is 7 in.

10. Find the total area of a frustum of a regular hexagonal pyramid if the sides of the bases are 2 ft. and 1 ft. 10 in. respectively and a lateral edge is 4 ft.

VOLUMES OF POLYHEDRONS

MEASURING SPACE

132. To measure the space inclosed by the surface of a polyhedron is to find how many times it contains another solid used as a unit of measure.

The **volume** of a polyhedron is the measure number of the space inclosed by the surface of the polyhedron.

While any solid might be used as a unit of volume, it is the common practice to use a cube whose edge is a unit of length. Any segment may be used as a unit of length with the corresponding units of area and of volume. It is, however, most convenient practically to use one of the recognized standard units of length; thus we have one inch, one square inch, one cubic inch; one foot, one square foot, one cubic foot; one centimeter, one square centimeter, one cubic centimeter, etc.

FUNDAMENTAL ASSUMPTION

133. As. 27. **The number of units of volume in a rectangular parallelepiped is the product of the number of linear units in three edges that meet at a common vertex.**

If V represents the volume of the rectangular parallelepiped, and a, b, and c the length of three edges that meet at a common vertex, As. 27 may be stated as a formula:

$$V = abc$$

The assumption will be discussed under two heads:

134. A. When the edges of the rectangular parallelepiped are all commensurable with a given unit of length.

In this case the unit of length can be applied an integral number of times to each edge. Suppose the unit of length is contained in one side of the base a times, and in the adjacent side of

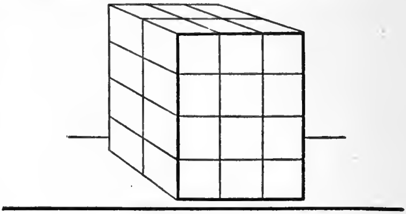


FIG. 131

the base b times; then the base of the rectangular parallelepiped can be divided into ab unit squares. The unit of length may be contained in the height of the parallelepiped c times. By planes passing through the points of division, the parallelepiped may be divided into c layers with ab unit cubes in a layer. The volume is, therefore, abc units of volume (Fig. 131).

Suppose one or more edges of the parallelepiped are not exactly divisible by the unit chosen, but are divisible by some aliquot part of the unit. In this case this part of the chosen unit may be taken as a new unit of length, and a cube whose edge is this new linear unit may be considered as the unit of volume. The assumption is then evident as above.

Let the pupil give special cases as illustrations.

135. B. When one or more edges of the rectangular parallelepiped are incommensurable with the chosen unit.

In this case it is not possible to measure one side or perhaps all sides of the parallelepiped in integral or fractional terms of the chosen unit. Since the ratio of two incommensurable segments is an irrational number, these sides may be expressed in irrational terms of the chosen unit and the volume found. This volume may be irrational or rational according to the measure number of the sides.

The following illustrations come under this case:

1. Length incommensurable expressed as an irrational number:

Length $3\sqrt{2}$, width 3, height 5, volume $45\sqrt{2}$

2. Length and width incommensurable:

a. Length $\sqrt{2}$, width $2\sqrt{3}$, height 5, volume $10\sqrt{6}$

b. Length $2\sqrt{2}$, width $3\sqrt{2}$, height 5, volume 60

Let the pupil give illustrations of cases in which all three dimensions are expressed by irrational numbers.

Since an irrational number cannot be expressed exactly as an integer or a fraction, rectangular parallelepipeds such as those mentioned in the illustrations above cannot be divided into unit cubes by planes passing through points of division as in Case A. These dimensions can, however, be expressed as approximate decimals, and these approximations may be made as close as we choose to make them to the true dimensions. From the approximate dimensions an approximate volume may be computed, and this approximate volume may be as close as we choose to make it to the true volume.

Illustration. If the length is $\sqrt{2}$, the width $\sqrt{3}$, the height 5, we may have the following approximations:

Length 1.4, width 1.7, height 5, volume 12
Length 1.41, width 1.73, height 5, volume 12.2
Length 1.414, width 1.732, height 5, volume 12.24

This process may be continued indefinitely, as the decimals expressing $\sqrt{2}$ and $\sqrt{3}$ do not terminate.

Ex. 1. Find the approximate volume of a rectangular parallelepiped correct to two decimal places, if its dimensions are $\sqrt{3}$, $2\sqrt{2}$, and $\sqrt{5}$. Find the total area.

Ex. 2. What is the answer to Ex. 1 if the dimensions of the parallelepiped are $\sqrt{3}$, $2\sqrt{2}$, and $\sqrt{6}$? Find the total area.

Ex. 3. Find the total area and the volume of a rectangular parallelepiped whose dimensions are $2\frac{3}{4}$ in., $5\frac{1}{2}$ in., and $3\frac{3}{8}$ in.

THE VOLUME OF ANY PARALLELEPIPED

136. Note. If the teacher desires, the treatment of volumes by Cavalieri's theorem (§§173–175) may be substituted for §§142 and 144. In this case §§136–141 may be omitted.

Two solids that fill the same extent of space are said to be **equivalent.** Congruent solids are the simplest examples of equivalent solids.

137. Theorem 99. **Two truncated right prisms are congruent if their right bases are congruent and three lateral edges of one are equal respectively to three corresponding lateral edges of the other, and similarly placed.**

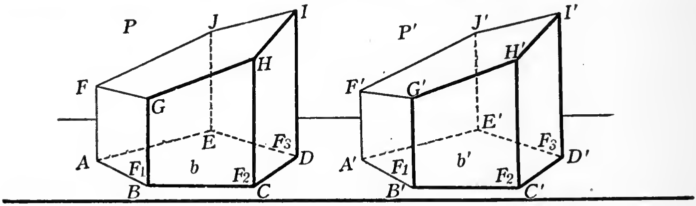


Fig. 132

Hypothesis: P and P' are two truncated right prisms with the right bases b and b' congruent, and the edges $AF = A'F'$, $BG = B'G'$, and $CH = C'H'$.

Conclusion: $P \cong P'$.

Outline of proof:

I. Base b can be made to coincide with base b'.

II. The lateral edges of P will fall along the corresponding lateral edges of P'.

III. The points F, G, and H will fall on the points F', G', and H' respectively.

IV. The plane of the upper base of P will fall on the plane of the upper base of P'.

V. Points I, J, etc., will fall on points I', J' respectively.

VI. P and P' will coincide and be congruent.

Is it necessary for the given lateral edges to be consecutive?

COR. **Two right prisms are congruent if they have congruent bases and equal altitudes.**

138. THEOREM 100. **If an oblique prism and a right prism are cut from the same prismatic space and have equal lateral edges, they are equivalent.**

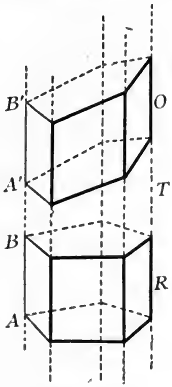


FIG. 133

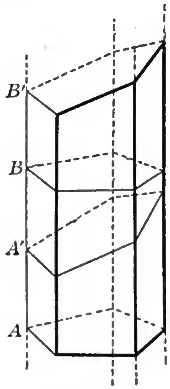


FIG. 134

Hypothesis: R is a right prism and O an oblique prism cut from the same prismatic space. The edge AB equals the edge $A'B'$ (Fig. 133).

Conclusion: $R=O$.

Analysis: To prove $R=O$, prove the truncated right prisms $R+T$ and $T+O$ congruent and subtract from each the truncated prism T.

Discussion: A complete proof of Th. 100 requires a discussion of two cases.

Case A. **When the prisms are not telescoped.**

Case B. **When the prisms are telescoped.**

Let the pupil give the proof for Case B, using Fig. 134. Notice that two solids are **equivalent** if they are sums, differences, or equal parts of equivalent solids.

COR. **Two prisms having equal edges and congruent right sections are equivalent.**

139. THEOREM 101. **The volume of any parallelepiped is the product of the area of its base and its altitude.**

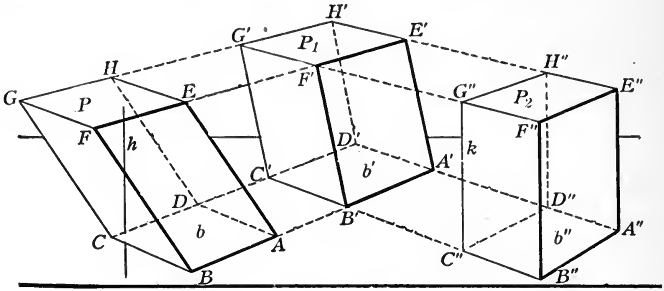


FIG. 135

Hypothesis: P is any parallelepiped, b is its base, and h its altitude.

Conclusion: The volume of $P = bh$.

Analysis in general: To prove that the volume of $P = bh$, compare P with a rectangular parallelepiped that has a base equivalent to b and an altitude equal to h.

Construction:

I. Extend the edges of P that are parallel to BA and take $B'A' = BA$. Pass planes $A'H'$ and $B'G' \perp A'B'$ at A' and B' respectively. This gives parallelepiped P_1.

II. Extend the edges of P_1 that are parallel to $B'C'$, and take $B''C'' = B'C'$. Pass planes $B''E''$ and $C''H''$ $\perp B''C''$ at B'' and C'' respectively. This gives parallelepiped P_2.

Outline of proof:

It is necessary to prove:

I. $P = P_2$.

II. P_2 is a rectangular parallelepiped.

III. $b'' = b$ and $k = h$.

$\therefore$ since vol. $P_2 = b''k$, vol. $P = b''k$; but $b'' = b$ and $k = h$.

$\therefore$ vol. $P = bh$.

Analysis for I: To prove $P = P_2$, prove $P = P_1 = P_2$. (Use Th. 100.)

Analysis for II:

a. To prove P_2 rectangular, prove that the edges $B''A''$, $B''C''$, and $B''F''$ are each $\perp$ the other two.

b. To prove $B''F'' \perp A''B''$ and $B''C''$, prove $B''F''$ $\perp$ plane of b''.

c. $\therefore$ prove planes $B''C''G''F''$ and $B''A''E''F''$ each $\perp$ $A''B''C''D''$. (See Ths. 25 and 28. Use the construction.)

Analysis for III: Prove $b = b' = b''$. (See *Plane Geometry*, Th. 116.)

THE VOLUME OF TRIANGULAR PRISMS

140. THEOREM 102. **A plane passed through two diagonally opposite edges of a parallelepiped divides the parallelepiped into two equivalent triangular prisms.**

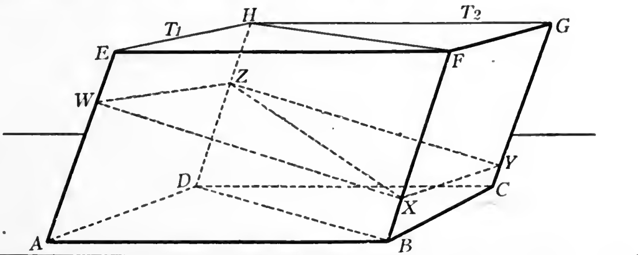


FIG. 136

Analysis:

I. To prove $T_1 = T_2$, prove that they have equal edges and congruent right sections.

II. $\therefore$ construct the right section $WXYZ$ and prove that

(1) The edges of T_1 are equal to the edges of T_2.

(2) The right section WXZ of T_1 is congruent to the right section XYZ of T_2.

III. $\therefore$ prove $WXYZ$ a parallelogram.

Are T_1 and T_2 congruent? Illustrate with a model.

141. THEOREM 103. **The volume of a triangular prism is the product of the area of its base and its altitude.**

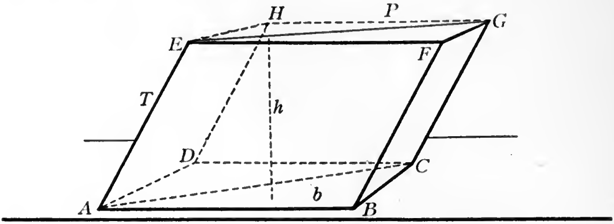


FIG. 137

Hypothesis: T is a triangular prism with base b (ABC) and altitude h.

Conclusion: The volume of $T = bh$.

Analysis in general: To find the volume of T, compare it with a parallelepiped.

Construction: Complete the parallelogram that has AB and BC for sides. Complete the parallelogram that has EF and FG for sides. Join HD.

Outline of proof:

I. $ABCD\text{–}EFGH$ is a parallelepiped (P).
II. Volume of $T = \frac{1}{2}$ volume of P.
III. Volume of $P = ABCD \cdot h$.
IV. $\therefore$ volume of $T = \frac{1}{2} ABCD \cdot h$.
V. $\triangle ABC = \frac{1}{2} ABCD = b$.
VI. $\therefore$ volume of $T = bh$.

Analysis for I:

a. To prove $ABCD\text{–}EFGH$ a parallelepiped, prove $HD \parallel AE$.
b. $\therefore$ prove $AEHD$ a $\square$.

Exercise. In Fig. 137, construct the parallelepiped by passing planes through GC and AE parallel to planes $ABFE$ and $BCGF$ respectively and extending the planes of the upper and lower bases. Prove that the solid formed is a parallelepiped.

THE VOLUME OF ANY PRISM

142. THEOREM 104. **The volume of any prism is the product of the area of its base and its altitude.**

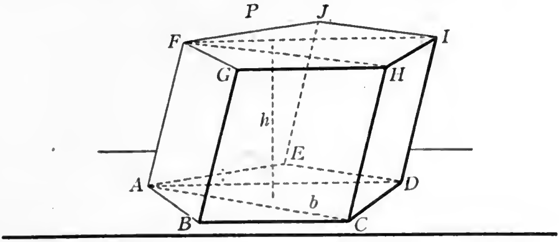


FIG. 138

Analysis and construction: To find the volume of P, divide it into triangular prisms and add their volumes.

In the proof use the theorem: The sum of numbers having a common factor is the common factor multiplied by the sum of the coefficients.

THEOREM 105. **If two prisms have equivalent bases and equal altitudes, they are equivalent.**

EXERCISES INVOLVING VOLUMES OF PRISMS

143. 1. Find the volume of a regular triangular prism if one side of the base is 15 in. and the height is 10 ft.

2. The corner of a cellar is boarded off to form a triangular coal bin. How many tons will it hold if the base is an isosceles right triangle each leg of which is 8 ft. and it is 4 ft. deep? Allow 35 cu. ft. to one ton.

3. The base of a triangular prism is an isosceles right triangle with the hypotenuse 8 in. If the height of the prism is 15 in., find its volume.

4. Find the volume of a regular hexagonal prism if its height is 10 in. and one side of the base is 3 in.

5. The base of a parallelepiped is a rhombus with one side 12 in. and one angle 60°. Find its volume if its height is 24 in.

6. A stick of timber is 8 in. × 15 in. and 20 ft. long. Find the weight if 1 cu. ft. weighs 50 lbs.

7. The space left in the basement for a coal bin is 10 ft. × 12 ft. How deep must the bin be to hold 10 tons?

8. The area of a cube is 96 sq. in. Find its volume and its diagonal.

THE VOLUME OF A TRIANGULAR PYRAMID

144. In Fig. 139, the altitude of pyramid P is divided into any number of equal parts. Through the points of division planes are passed parallel to the base of the pyramid. These planes cut the pyramid in the sections $A_1B_1C_1$, $A_2B_2C_2$, etc. On these sections as upper bases prisms are constructed by planes passing through the lines B_1C_1, B_2C_2, etc., parallel to the edge AP. The **series of prisms** thus formed is said to be **inscribed in the pyramid.** Show that the edges of these prisms are parallel to the edge PA of the pyramid.

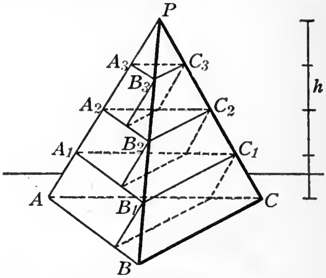


FIG. 139

It is evident that if we increase the number of divisions in the altitude we shall increase also the number of inscribed prisms, and that the number of inscribed prisms may be increased indefinitely (Fig. 140).

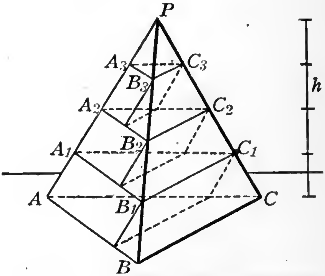


FIG. 140

It is evident also that if a series of prisms is inscribed in a pyramid, and if their number is increased indefinitely, a series of prisms will soon be obtained that can with difficulty be distinguished from the pyramid.

The following theorem can be proved in higher mathematics and will be assumed here:

As. 28. **There is a definite limit to the sum of the volumes of a series of prisms inscribed in a given pyramid if their number is increased indefinitely.**

This limit is by definition the volume of the pyramid.

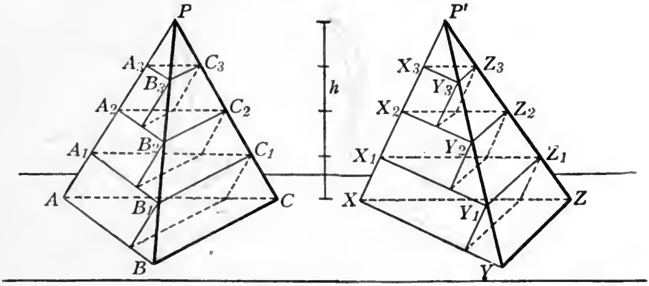


FIG. 141

Fig. 141 shows two pyramids, P and P', standing on the same plane. They have equal altitudes and equivalent bases. Suppose that the altitude h is divided into equal parts, and that through the points of division planes are passed cutting both pyramids, and that on these sections as upper bases a series of prisms is inscribed in each pyramid.

Prove that *the sum of the volumes of the prisms in P is always equal to the sum of the volumes of the prisms in P'.*

It can be proved in higher mathematics that, since the sum of the volumes of the prisms of the series in P is always equal to the sum of the volumes of the prisms in P', the limits of these two sums are equal.

We will, therefore, assume

THEOREM 106. **If two pyramids have equivalent bases and equal altitudes, they are equivalent.**

145. THEOREM 107. **Any triangular prism may be divided into three equivalent triangular pyramids.**

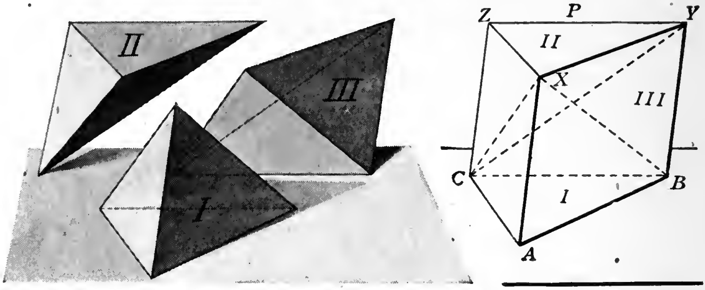


FIG. 142

Hypothesis: P is a triangular prism cut by the planes XBC and XYC, forming three triangular pyramids.

Conclusion: $X\text{–}ABC = C\text{–}XYZ = X\text{–}CBY$.

Analysis:

I. To prove pyramid I = pyramid II, prove base ABC = base XYZ, and their altitudes equal.

II. To prove pyramid II = pyramid III, regard X as the vertex of each, and CYZ and CBY as bases.

146. THEOREM 108. **The volume of a triangular pyramid is one-third the product of the area of its base and its altitude.**

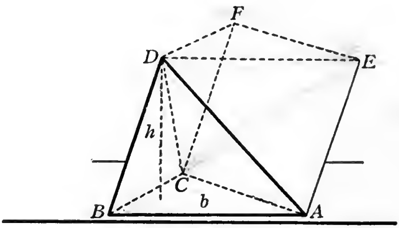


FIG. 143

Hypothesis: $D\text{–}ABC$ is a triangular pyramid with base b (ABC) and altitude h.

Conclusion: The volume of $D\text{–}ABC = \frac{1}{3}bh$.

Analysis in general: To find the volume of $D\text{–}ABC$, compare it with a triangular prism having the same base and altitude.

Construction: From A and C draw lines parallel to BD. Through D pass a plane parallel to ABC.

Outline of proof:

I. $ABC\text{–}DEF$ is a prism.
II. Volume of $D\text{–}ABC = \frac{1}{3}$ volume of $ABC\text{–}DEF$.
III. Volume of $ABC\text{–}DEF = bh$.
IV. $\therefore$ volume of $D\text{–}ABC = \frac{1}{3}bh$.

147. THEOREM 109. **The volume of any pyramid is one-third the product of the area of its base and its altitude.**

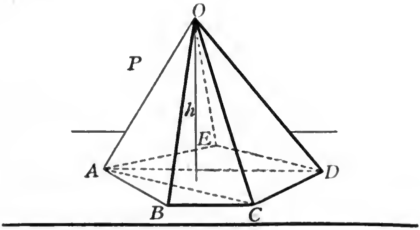


FIG. 144

Suggestion. Divide P into triangular pyramids, and add their volumes.

GENERAL FORMULA FOR THE VOLUME OF PRISMS, PYRAMIDS, AND FRUSTUMS

148. A **prismatoid** is a polyhedron that has for bases two polygons in parallel planes, and for lateral faces triangles, trapezoids, or parallelograms that have one side in common with one base and the opposite vertex or side in common with the other base. The **altitude** of a prismatoid is the perpendicular distance between the bases. The **mid-section** of a prismatoid is a section made by a plane parallel to the bases and bisecting the altitude.

THEOREM 110. **The volume of a prismatoid is one-sixth the altitude times the sum of the areas of the bases and four times the area of the mid-section.**

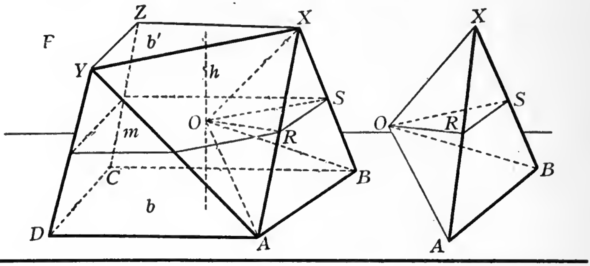


FIG. 145

Hypothesis: P is a *prismatoid* with b the lower base, b' the upper base, m the mid-section, h the altitude, and V the volume.

Conclusion: $V = \frac{1}{6}h(b + b' + 4m)$.

Analysis: To find V, divide P into triangular pyramids and add their volumes.

Construction: Join O any point in m with the vertices of b and b'. Draw one diagonal in each of these lateral faces which are trapezoids and parallelograms. O is the common vertex of a series of triangular pyramids and also of those pyramids whose bases are the upper and the lower bases of the prismatoid.

Outline of proof:

I. Vol. of $O\text{–}XYZ = \frac{1}{3} \cdot \frac{1}{2}h \cdot b' = \frac{1}{6}hb'$.

II. Vol. of $O\text{–}ABC$ etc. $= \frac{1}{3} \cdot \frac{1}{2}h \cdot b = \frac{1}{6}hb$.

III. Vol. of $O\text{–}XAB = XAB \cdot \frac{1}{3}$ altitude from O to XAB.
R and S are mid-points of AX and BX respectively.

$\therefore \quad \triangle ABX = 4 \ \triangle RSX.$

$\therefore$ vol. $O\text{–}XAB = 4$ vol. of $O\text{–}RSX$.

$O\text{–}RSX = X\text{–}ORS.$

$\therefore$ vol. $O\text{–}XAB = 4 \cdot \frac{1}{3} \cdot \frac{1}{2}h \cdot ORS = \frac{1}{6}h \cdot 4\ ORS.$

IV. The sum of the volumes of these pyramids that have O as a vertex and the lateral faces as bases is $\frac{1}{6}h \cdot 4$ m.

V. $\therefore$ the sum of the volumes of all pyramids is

$$\tfrac{1}{6}h(b+b'+4m)$$

149. Prisms, pyramids, and frustums of pyramids are special cases of prismatoids. How? Their volumes may be obtained from the volume of the prismatoid as shown below. In each case it is necessary to find the volume in terms of b and b'.

COR. I. **The volume of any prism is bh.**

Suggestion. $b=b'=m$. Then

$$V=\tfrac{1}{6}h\ (b+b+4b)=\tfrac{1}{6}h \cdot 6b=bh$$

COR. II. **The volume of any pyramid is $\frac{1}{3}bh$.**

Suggestion. $b'=0$, $m=\frac{1}{4}b$. Then

$$V=\tfrac{1}{6}h(b+b)=\tfrac{1}{3}bh$$

COR. III. **The volume of a frustum of a pyramid is**

$$V=\tfrac{1}{3}h(b+b'+\sqrt{bb'})$$

Outline of proof (Fig. 146):

$$\frac{b}{m}=\frac{a^2}{x^2} \text{ and } \frac{b'}{m}=\frac{a'^2}{x^2}.$$

$$\therefore \frac{\sqrt{b}}{\sqrt{m}}=\frac{a}{x} \text{ and } \frac{\sqrt{b'}}{\sqrt{m}}=\frac{a'}{x}.$$

$$\therefore \frac{\sqrt{b}+\sqrt{b'}}{\sqrt{m}}=\frac{a+a'}{x}=2.$$

$$\therefore \frac{b+2\sqrt{bb'}+b'}{m}=4.$$

$$\therefore 4m=b+2\sqrt{bb'}+b'.$$

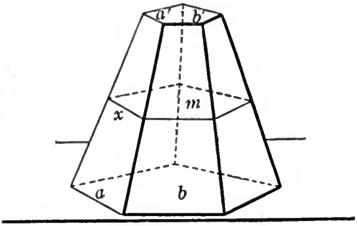


FIG. 146

Since $V=\frac{1}{6}h(b+b'+4m)$, where h is altitude.

$$V=\tfrac{1}{6}h(b+b'+b+b'+2\sqrt{bb'})$$

$$V=\tfrac{1}{3}h(b+b'+\sqrt{bb'})$$

EXERCISES INVOLVING VOLUMES OF POLYHEDRONS

150. 1. A grain bin is $2\frac{1}{2}$ ft. wide, 6 ft. long, and 4 ft. deep. How many bushels of grain will it hold? 2150.42 cu. in. = 1 bu.

2. What would be the area of the base of a grain bin 6 ft. deep to hold 250 bu.?

3. The volume of any prism is the product of the area of a right section and a lateral edge.

4. The lateral edge of a triangular prism is 15 in. The right section is an equilateral triangle 5 in. on a side. Find the volume.

5. If the dimensions of a rectangular parallelepiped are a, b, and c, write a formula for (1) the sum of the edges; (2) the diagonal; (3) the total area; (4) the volume.

6. When the rainfall is $\frac{1}{2}$ in., how many barrels of water will fall per acre? 1 cu. ft. of water $= 7\frac{1}{2}$ gal.; $31\frac{1}{2}$ gal. = 1 bbl.

7. The plane determined by one edge of a tetrahedron and the mid-point of the opposite edge divides the pyramid into two equivalent parts.

8. Lines drawn from the center of a cube to the vertices divide it into six equivalent pyramids.

9. Find the volumes of the following pyramids if

a. Base is a regular hexagon 5 cm. on a side and the altitude is 9 cm.

b. Base is a rectangle 10 in. $\times$ 14 in. and the altitude is 18 in.

10. Find the volume of a regular octahedron 8 in. on a side.

11. Find the total area and the volume of a regular hexagonal pyramid if one edge of the base is 8 in. and one lateral edge is 10 in.

12. Find a formula for the volume of a pyramid whose base is an equilateral triangle with one side e and the altitude h.

13. What is the formula called for in the previous exercise if the base is (1) a square; (2) a regular hexagon?

14. The edges of the bases of a frustum are 15 in. and 9 in. respectively, and its height is 4 ft. Find its volume if it is a frustum of (1) a regular square pyramid; (2) a regular triangular pyramid.

15. Find the volume of the pyramids from which the frustums mentioned in the previous exercise are cut.

THE MEASUREMENT OF ROUND BODIES IN GENERAL

151. NOTE. Rigorous proofs for the theorems concerning the measurement of round bodies are too difficult for this book. The following treatment will make the theorems concerning the measurement of cylinders and cones appear reasonable to the pupil. If the teacher desires, the proofs of theorems concerning volumes, based on Cavalieri's theorem (§§173, 177, 178), may be substituted for §§156 and 163.

The surface of a polygon that can be drawn on a plane may be measured by finding how many times it would contain another polygon used as a unit of measure. It is evident that the measure of a curved surface like the lateral surface of a cylinder or of a cone cannot be found in this way. We will assume, however, that the area of curved surfaces can be expressed in terms of plane units, but it will be necessary for us to define what is meant by the area of these surfaces. These definitions will be given later.

Similarly, we will assume that the measure of the space inclosed by cylinders and cones can be expressed in terms of unit cubes, and will define later what is meant by the volume of these solids.

THE MEASUREMENT OF THE CYLINDER

INSCRIBED PRISMS

152. A polygon is said to be inscribed in any curve if its vertices lie on the curve.

If the bases of a prism are inscribed in the bases of a cylinder, and the lateral edges of the prism are elements of the cylinder, the **prism** is said to be **inscribed** in the cylinder and the **cylinder** is said to be **circumscribed** about the prism (Fig. 147).

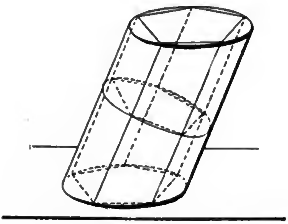


FIG. 147

We will also assume

As. 29. **Any plane that cuts all of the elements of a cylinder will cut any inscribed prism in a section which is inscribed in the section of the cylinder.**

Fig. 148 shows a triangular prism inscribed in a circular cylinder. *AD* represents the plane that cuts the cylinder and the prism in right sections. *ACE* is the right section of the prism. If the arcs between the vertices of $\triangle ACE$ are bisected, and the points of division are joined, a hexagon is inscribed in the right section of the cylinder. If elements of the cylinder are drawn through the points of division *A*, *B*, *C*, *D*, *E*, and *F*, they will form the edges of an inscribed hexagonal prism whose right section *ABCDEF* is inscribed in the right section of the cylinder. It is evident that if this process is continued the number of lateral faces of the inscribed prism can be increased indefinitely. It is to be noted that this definition applies to any closed cylinder and is independent of the number of lateral faces of the original prism.

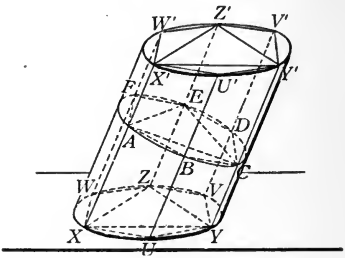


FIG. 148

THE LATERAL AREA OF CIRCULAR CYLINDERS

153. It is evident that if a prism is inscribed in a circular cylinder, and if the number of lateral faces is increased indefinitely, a prism will soon be obtained that can with difficulty be distinguished from the cylinder.

The following theorem from higher mathematics will be assumed:

THEOREM 111. **There is a definite limit to the lateral areas of a series of prisms inscribed in a circular cylinder when the number of lateral faces is increased indefinitely.**

NOTE. Theorem 111 is true no matter what is the nature of the initial prism of the series.

The **lateral area of the cylinder** is defined as the limit of the lateral areas of a series of inscribed prisms as the number of lateral faces is increased indefinitely.

Therefore, since the lateral area of a prism is the product of the perimeter of a right section and an element, no matter how many lateral faces the prism may have, we will assume

THEOREM 112. **The lateral area of a circular cylinder is the product of the perimeter of a right section and an element.**

NOTE. As proved in higher mathematics, Th. 112 is true for any cylinder. It can be used here only for circular cylinders, because in high-school mathematics we do not learn how to find the perimeters of curves other than circles.

COR. I. **If r is the radius of the right section, e an element, and L the lateral area of a circular cylinder,**

$$L = 2\pi re$$

In Cor. II, r represents the radius of the base, h represents the height, L represents the lateral area, A represents the total area.

COR. II. **In right circular cylinders**

$$L = 2\pi rh$$

$$A = 2\pi rh + 2\pi r^2 = 2\pi r\,(h + r)$$

154. NOTE 1. Since the value of π can be found only approximately, it follows that the area of circular cylinders can be found only approximately. In brief, the lateral area of an inscribed prism of a great many lateral faces is taken as an approximation to the lateral area of the cylinder.

155. NOTE 2. The lateral surface of a right circular cylinder may be **developed** or rolled out into a **rectangle**. Cut a rectangular strip of paper whose width is equal to an element of the given cylinder. Wrap the strip about the cylinder and cut it with a sharp knife so that it exactly fits about the cylinder without overlapping. Find its area. Find the development of the lateral surface of an oblique circular cylinder.

THE VOLUME OF CYLINDERS WITH CIRCULAR BASES

156. In studying the volumes of cylinders we will imagine that a series of prisms is inscribed in a cylinder with a circular base, and that the number of lateral faces of the prisms is increased indefinitely.

The following theorem from higher mathematics will be assumed:

THEOREM 113. **There is a definite limit to the volumes of a series of prisms inscribed in a cylinder with a circular base when the number of lateral faces of the prisms is increased indefinitely.**

NOTE. Theorem 113 is true no matter what is the nature of the initial prism of the series.

The **volume of the cylinder** is defined as the limit of the volumes of a series of inscribed prisms as the number of lateral faces is increased indefinitely.

Therefore, since the volume of a prism is the product of the area of the base and the altitude, no matter how many lateral faces the prism may have, we will assume

THEOREM 114. **The volume of a cylinder with a circular base is the product of the area of its base and its altitude.**

NOTE. As proved in higher mathematics, Th. 114 is true for any cylinder. It can be used here, however, only for cylinders with circular bases, because in high-school mathematics we do not learn how to find the areas of surfaces inclosed by curves other than circles.

COR. **If r is the radius of the base of a cylinder, h the altitude, and V the volume,**

$$V = \pi r^2 h$$

157. NOTE 1. The volume of cylinders can be found only approximately; in fact, the volume of an inscribed prism of a great many lateral faces is taken as an approximation to the volume of the cylinder.

158. NOTE 2. By similar processes the lateral area and the volume of cylinders can be obtained from circumscribed prisms.

THE MEASUREMENT OF THE CONE

INSCRIBED PYRAMIDS

159. If the base of a pyramid is inscribed in the base of a cone, and the lateral edges of the pyramid coincide with elements of the cone, the **pyramid** is said to be **inscribed** in the cone and the **cone** is said to be **circumscribed** about the pyramid (Fig. 149).

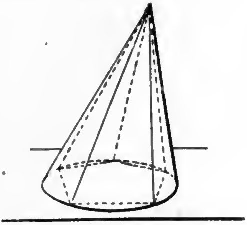


FIG. 149

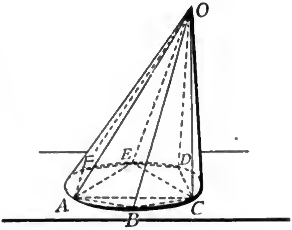


FIG. 150

Fig. 150 shows a triangular pyramid *O–ACE* inscribed in a cone. If the arcs between the vertices of $\triangle ACE$ are bisected, and the points of division are joined, a hexagon is inscribed in the base of the cone. If elements of the cone are drawn through the vertices of this hexagon, they will form the edges of an inscribed hexagonal pyramid. It is evident that if this process is continued the number of lateral faces of the pyramid may be increased indefinitely. This definition applies to any convex cone and is independent of the number of lateral faces of the original pyramid.

THE LATERAL AREA OF RIGHT CIRCULAR CONES

160. In our work we can find the lateral areas of right circular cones only. Th. 97 gives us a formula for the lateral area of regular pyramids only, and not for all pyramids. In studying the lateral areas of cones, therefore, we will imagine that a series of regular pyramids is inscribed in a right circular cone, and that the number of sides of the pyramids is increased indefinitely. It is evident that a pyramid will soon be found that can with difficulty be distinguished from the cone.

The following theorem from higher mathematics will be assumed:

THEOREM 115. **There is a definite limit to the lateral areas of a series of regular pyramids inscribed in a right circular cone when the number of lateral faces is increased indefinitely.**

The **lateral area of a right circular cone** is defined as the limit of the lateral areas of a series of regular inscribed pyramids as the number of lateral faces is increased indefinitely. Therefore, since the lateral area of a regular pyramid is one-half the product of the perimeter of the base and the slant height, no matter how many lateral faces the pyramid may have, we will assume

THEOREM 116. **The lateral area of a right circular cone is one-half the product of the perimeter of its base and its slant height.**

COR. **If r is the radius of the base, l the slant height, L the lateral area, and A the total area of a right circular cone,**

$$L = \pi r l$$

$$A = \pi r l + \pi r^2 = \pi r(r + l)$$

161. NOTE 1. The lateral areas of cones can be found only approximately; in fact, the lateral area of a regular inscribed pyramid of a great many lateral faces is taken as an approximation to the lateral area of a right circular cone.

162. NOTE 2. The lateral surface of a right circular cone may be **developed** into a **sector of a circle.** Cut out a circle whose radius is equal to the slant height of the cone. Wrap the circle about the cone and cut it so that the sector exactly fits about the cone without overlapping. Open out the paper.

If r is the radius of the circle and a the angle of the sector, the area of the sector is by plane geometry $\frac{\pi r^2 a}{360}$ or $\frac{2\pi r a}{360} \cdot \frac{1}{2} r$; that is, one-half the product of the radius and the length of its arc.

THE VOLUME OF CONES WITH CIRCULAR BASES

163. In studying the volumes of cones we will imagine that a series of pyramids is inscribed in a cone with a circular base, and that the number of lateral faces of the pyramid is increased indefinitely.

The following theorem from higher mathematics will be assumed:

THEOREM 117. **There is a definite limit to the volumes of a series of pyramids inscribed in a cone with a circular base when the number of lateral faces of the pyramid is increased indefinitely.**

NOTE. Theorem 117 is true no matter what is the nature of the initial pyramid of the series.

The **volume of the cone** is defined as the limit of the volumes of a series of inscribed pyramids as the number of lateral faces is increased indefinitely.

Therefore, since the volume of a pyramid is one-third the product of the area of the base and the altitude, no matter how many lateral faces the pyramid may have, we will assume

THEOREM 118. **The volume of a cone with a circular base is one-third the product of the area of its base and its altitude.**

NOTE. As proved in higher mathematics, Th. 118 is true for any cone. It can be used here, however, only for cones with circular bases.

COR. **If r is the radius of the base of a cone, h its altitude, and V its volume,**

$$V = \tfrac{1}{3}\pi r^2 h$$

164. NOTE 1. The volume of cones can be found only approximately; in fact, the volume of an inscribed pyramid of a great many lateral faces is taken as an approximation to the volume of the cone.

165. NOTE 2. By similar processes the lateral area and the volume of cones can be obtained from circumscribed pyramids.

THE MEASUREMENT OF FRUSTUMS OF CONES

166. The theorems for the measurement of the lateral surface and the volume of certain frustums of cones may be obtained from the corresponding theorems concerning the lateral surface and the volume of frustums of pyramids. The method is similar to that used in obtaining the theorems for the lateral surface and the volume of certain cones from the corresponding theorems concerning pyramids.

If the bases of a frustum of a pyramid are inscribed in the bases of a frustum of a cone, and the lateral edges of the frustum of the pyramid are elements of the frustum of the cone, the frustum of the pyramid is said to be inscribed in the frustum of the cone.

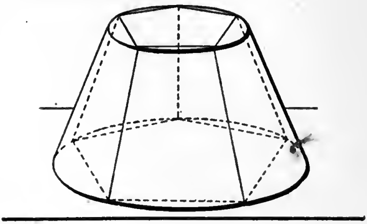


FIG. 151

By a process similar to that used on pages 125 and 126, the number of lateral faces of the inscribed frustum may be increased indefinitely. It is evident that a frustum of a pyramid will soon be obtained that can with difficulty be distinguished from the frustum of the cone.

167. The following theorems will be assumed without further discussion:

THEOREM 119. **If r_1 and r_2 are the radii of the upper and lower bases of a frustum of a right circular cone, and l is its slant height, the lateral area of the frustum is**

$$\pi l(r_1+r_2)$$

THEOREM 120. **If r^1 and r^2 are the radii of the upper and lower bases of a frustum of a cone with a circular base, and h is its altitude, the volume of the frustum is**

$$\tfrac{1}{3}h(\pi r_1^2+\pi r_2^2+\sqrt{\pi r_1^2\cdot\pi r_2^2})=\tfrac{1}{3}h\pi(r_1^2+r_2^2+r_1r_2)$$

EXERCISES INVOLVING THE AREAS AND VOLUMES OF CYLINDERS, CONES, AND FRUSTUMS

168. 1. Find the total area and the volume of a right circular cylinder 30 ft. high and 8 ft. in diameter.

2. Find the cost of digging a cistern 25 ft. deep and 6 ft. in diameter at 5¢ a cubic foot.

3. How many gallons of paint are required for the lateral surface of a cylindrical tower 100 ft. high and 30 ft. in diameter? Allow 1 gal. of paint for 500 sq. ft. of surface.

4. Find the total area of a right circular cone whose slant height is 15 in. and whose radius is 6 in.

5. Find the total area and volume of a right circular cone if its altitude is 12 in. and its diameter is 18 in.

6. Find the total area and volume of a right circular cone if its slant height is 13 in. and its radius 5 in.

7. What must be the depth of a cylindrical measure 12 in. in diameter if it holds 1 bushel? 2150.42 in. = 1 bu.

8. A cylindrical tower has a conical top. Find the total cost of painting the tower at 25¢ a square foot. The total height of the tower is 50 ft., the height of the cylindrical part is 42 ft., and the diameter is 18 ft.

9. Find the total area and the volume of a frustum of a right circular cone if the radii of its bases are 3 in. and 10 in. respectively and its slant height is 5 in.

10. Find the total area and the volume of the solid formed by revolving a square about one side if one side is (1) 15 in.; (2) S.

11. Find the total area and the volume of the solid formed by revolving a rectangle whose sides are 5 in. and 7 in. (1) about the side 5 in.; (2) about the side 7 in.

12. What are the answers called for in Ex. 11 if the sides of the rectangle are a and b?

13. What are the answers called for in Exs. 11 and 12 if a right triangle whose legs are 7 in. and 5 in. is used instead of a rectangle?

14. Find the total area and the volume of the solid formed by revolving an isosceles triangle about its base, if the base is 8 in. and one leg is 10 in.

15. Answer Ex. 14 if the triangle is revolved about its altitude.

16. Find the total area and the volume of the solid formed by the revolution of an isosceles right triangle about the hypotenuse (1) if one leg is 6 in.; (2) if one leg is a.

17. ABC is an isosceles right triangle each of whose legs is 3 in. l is a line perpendicular to hypotenuse AB at B. Find the total area and the volume of the solid formed by revolving $\triangle ABC$ about l.

GENERAL FORMULA

LATERAL AREA OF SOLIDS OF REVOLUTION

169. THEOREM 121. **The area of the surface generated by a segment revolving about an axis in the same plane with it, but not crossing it, is the product of the projection of the line segment on the axis and the circumference of a circle whose radius is the perpendicular erected at the mid-point of the segment and terminated by the axis.**

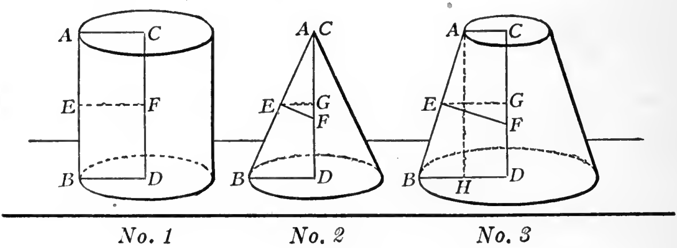


FIG. 152

Hypothesis: AB is a line segment revolving about an axis which is in the same plane with AB, but does not cross AB. CD is the projection of AB on the axis. E is the mid-point of AB. $EF \perp AB$ and is terminated by CD. Let L represent the area generated by AB.

Conclusion: $L = CD \cdot 2\pi EF$.

Case A. When AB is parallel to the axis, L is the lateral area of a right circular cylinder (Fig. 152, No. 1).

The proof is left to the pupil.

Case B. When AB meets the axis at A, L is the lateral area of a right circular cone (Fig. 152, No. 2).

For right circular cones $L = AB\pi BD$, or $AB 2\pi EG$.

To prove $L = CD \cdot 2\pi EF$, prove $CD \cdot 2\pi EF = AB \cdot 2\pi EG$.

That is, $AB \cdot EG$ must be replaced by $CD \cdot EF$.

Use similar triangles ABD and EFG.

Let the pupil complete the proof.

Case C. When AB is not parallel to the axis and does not meet the axis, L is the lateral area of a frustum of a right circular cone (Fig. 152, No. 3).

The lateral area of a frustum of a right circular cone is $\pi l(r_1+r_2)$.

Since $EG = \frac{1}{2}(r_1+r_2)$, $L = AB \cdot 2\pi EG$.

To prove $L = CD \cdot 2\pi EF$, prove $CD \cdot 2\pi EF = AB \cdot 2\pi EG$. That is, replace $AB \cdot EG$ by $CD \cdot EF$.

Let the pupil complete the proof. Draw AH from $A \perp BD$.

Ex. 1. Fig. 153 shows half of a regular hexagon inscribed in a semicircle. Find the area of the surface generated by the broken line $ABCD$ revolving about the diameter AD. The radius of the circle is 3 in.

Ex. 2. What would be the area of the surface formed in Ex. 1 if $ABCD$ were circumscribed about the semicircle?

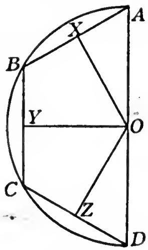


FIG. 153

170. NOTE. The lateral area of a frustum of a right circular cone may be found from its development. Show that this development is a sector of a circular ring (Fig. 154).

For the area of the sector of the circular ring we have (a is the angle of the sector):

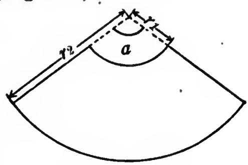


FIG. 154

$$\text{Area } = \frac{a}{360}(\pi r_2^2 - \pi r_1^2)$$

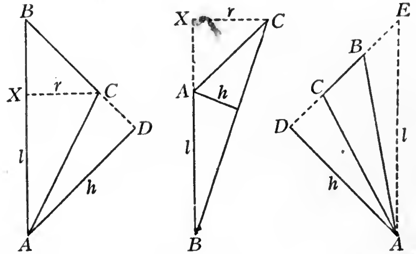
$$= \frac{\pi a}{360}(r_2^2 - r_1^2) = \frac{\pi a}{360}(r_2 - r_1)(r_2 + r_1)$$

$$= \tfrac{1}{2}(r_2 - r_1)\frac{2\pi a}{360}(r_2 + r_1) = \tfrac{1}{2}(r_2 - r_1)\left[\frac{2\pi a r_2}{360} + \frac{2\pi a r_1}{360}\right]$$

GENERAL FORMULA FOR THE VOLUMES OF SOME SOLIDS OF REVOLUTION

171. THEOREM 122. **The volume of a solid generated by a triangle that revolves about an axis in its plane, passing through one of its vertices, but not crossing it, is one-third the product of the area of the surface generated by the side opposite the fixed vertex and the corresponding altitude of the triangle.**

Case A. When one side of the triangle coincides with the axis.



No. 1 No. 2 No. 3

FIG. 155

Hypothesis: $\triangle ABC$ revolves about side AB as an axis. The axis is supposed to pass through A. BC is the side opposite A. h is the altitude from A to BC. r is the $\perp$ from C to AB. V is the volume generated by $\triangle ABC$.

Conclusion: $V = \frac{1}{3}h \cdot$ area generated by BC.

Analysis and construction (Fig. 155, Nos. 1 and 2):

I. *a.* The volume generated by $\triangle ABC$ is

(1) $\frac{1}{3}\pi r^2 BX + \frac{1}{3}\pi r^2 AX$, or $\frac{1}{3}\pi r^2 AB$.

(2) $\frac{1}{3}\pi r^2 BX - \frac{1}{3}\pi r^2 AX$, or $\frac{1}{3}\pi r^2 AB$.

b. The area generated by BC is πrBC.

$\therefore$ we are to prove $V = \frac{1}{3}h \cdot \pi rBC$.

II. $\therefore$ prove $\frac{1}{3}h \cdot \pi rBC = \frac{1}{3}\pi r^2 AB$.

III. $\therefore$ prove $h \cdot BC = r \cdot AB$.

IV. $\therefore$ prove $\dfrac{AB}{BC} = \dfrac{h}{r}$.

V. $\therefore$ prove $\triangle ABD \sim \triangle BCX$.

Ex. 1. Give the analysis and proof for the case in which B is taken as the fixed point instead of point A.

Ex. 2. Apply the theorem to the special case (Fig. 155) in which BC is perpendicular to the axis.

Case B. When no side of the triangle coincides with the axis.

Hypothesis: $\triangle ABC$ revolves about line l as an axis. Line l is in plane ABC, passes through vertex A, but does not cross the triangle. BC is the side of $\triangle ABC$ opposite vertex A. h is the altitude from A to BC. V represents the volume generated by $\triangle ABC$ (Fig. 155, No. 3).

Conclusion: $V = \frac{1}{3}h \cdot$ area generated by BC.

Analysis and construction: To find the volume generated by $\triangle ABC$, extend CB to meet the axis at E and subtract the volume generated by $\triangle EBA$ from the volume generated by $\triangle ECA$.

Outline of proof:

I. Volume generated by $\triangle ECA = \frac{1}{3}h \cdot$ area CE.

II. Volume generated by $\triangle EBA = \frac{1}{3}h \cdot$ area BE.

$\therefore$ volume generated by $\triangle BCA = \frac{1}{3}h \cdot$ area CB.

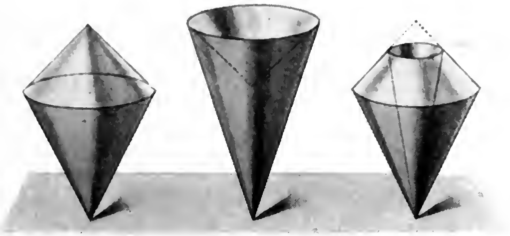


FIG. 156

Ex. 3. Apply the theorem to the special cases (Fig. 155, No. 3) in which (1) BC is parallel to l; (2) BC extended cuts l below A.

Ex. 4. Describe the figure formed in each case of Th. 122 and in Exs. 1, 2, and 3.

VOLUMES BY CAVALIERI'S THEOREM

172. NOTE. The treatment of volumes that follows may be substituted for the treatment in §§142, 144, 156, and 163 if the teacher desires.

CAVALIERI'S THEOREM

173. The following theorem is proved in higher mathematics. It will be assumed here.

THEOREM 123. **If two solids lie between parallel planes, and if the two sections made by any plane parallel to their bases are equivalent, then the solids are equivalent.**

VOLUME OF ANY PRISM

174. THEOREM 104. **The volume of any prism is the product of the area of its base and its altitude.**

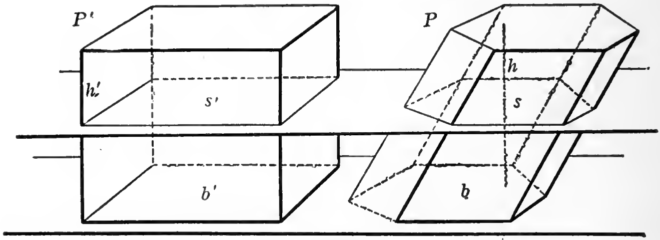


FIG. 157

Hypothesis: P is any prism with base b, altitude h, and V its volume.

Conclusion: V of $P = bh$.

Analysis in general: To find the V of P, compare P with a rectangular parallelepiped that has an equivalent base and the same altitude.

Construction: Construct P' a rectangular parallelepiped with base $b' = b$, and an edge $h' = h$. Let P' and P stand on the same plane. Pass a plane parallel to b and b' cutting both P' and P. Let s' and s be the sections of P' and P respectively.

Analysis:

I. To prove V of $P = bh$, prove $P' = P$.

II. To prove $P' = P$, prove the sections of P and P' made by planes parallel to the bases are equivalent.

III. $\therefore$ prove $s = s'$.

Outline of proof:

I. $s' = b' = b = s$. $\therefore s' = s$.

II. In the same way, the sections of P and P' made by every plane parallel to the bases are equivalent.

III. $\therefore P' = P$.

IV. V of $P' = b'h'$.

V. $\therefore V$ of $P = b'h' = bh$.

NOTE. If this proof of Th. 104 is used, §§136–141 may be omitted.

EQUIVALENT TRIANGULAR PYRAMIDS

175. THEOREM 106. **If two triangular pyramids have equivalent bases and equal altitudes, they are equivalent.**

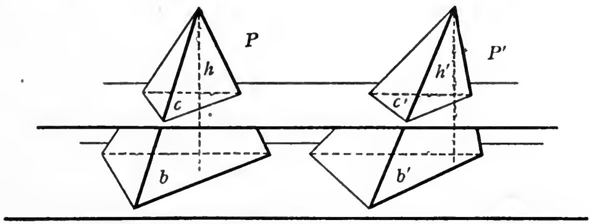


FIG. 158

Hypothesis: P and P' are two pyramids with equivalent bases b and b' and equal altitudes h and h'.

Conclusion: $P = P'$.

Analysis and construction: Suppose P and P' are standing on the same plane. Pass any plane through P and P' parallel to b and b', cutting P in section c, and P' in section c', and prove $c = c'$.

NOTE. At this point the theorems concerning the volumes of pyramids follow as given in §§145–147.

THE VOLUME OF CYLINDERS WITH CIRCULAR BASES

176. We know that it is not possible to construct with straight edge and compass, only, a triangle, square, or rectangle that has the same area as a given circle. We do know, however, that there are instruments by which these figures can be constructed. This must be assumed in the next two proofs.

177. THEOREM 114. **The volume of a cylinder with a circular base is the product of the area of its base and its altitude.**

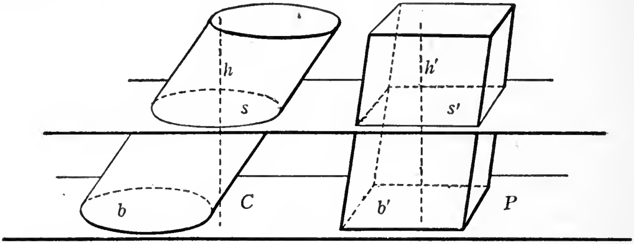


FIG. 159

Hypothesis: C is a cylinder with a circular base b; h is its altitude; V is its volume.

Conclusion: V of $C = bh$.

Analysis in general: To find V of C, compare it with a prism that has an equivalent base and the same altitude.

Let the pupil give the construction, complete the analysis, and give proof. Use §174 as a model.

THE VOLUME OF CONES WITH CIRCULAR BASES

178. THEOREM 118. **The volume of a cone with a circular base is one-third the product of the area of its base and its altitude.**

Analysis: To find the volume of the cone, compare it with a pyramid that has an equivalent base and the same altitude.

MISCELLANEOUS EXERCISES

179. 1. A sheet of paper 6 in. $\times$ 8 in. is bent into a right circular cylinder. Find the total area and the volume. (Two answers.)

2. Water is flowing through a pipe 2 in. in diameter at the rate of 150 ft. per minute. How many cubic feet is that per hour?

3. A cylindrical tomato can is $4\frac{9}{16}$ in. high and 4 in. in diameter. Find its capacity in quarts. 231 cu. in. = 1 gal.

4. A regular four-sided prism is inscribed in a right circular cylinder. Find the volume and the lateral area of the prism if the radius of the cylinder is 10 in. and its altitude is 25 in.

5. The section of a right circular cone made by a plane through the altitude is a triangle whose base angles are 45°. If the height of the cone is 6 in., find the total area and the volume.

6. The volume of an irregular body that will not absorb water may be found by placing it in water and finding the volume of the water displaced. What is the volume of a stone if, when it is dropped into a cylindrical tank 2 ft. in diameter, it causes the water in the tank to rise 2 in.?

7. A regular hexagonal pyramid is inscribed in a right circular cone whose radius is 10 in. and whose height is 24 in. Find the total area and the volume of the pyramid.

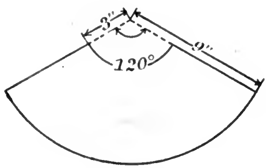


FIG. 160

8. Find the lateral area of the frustum of a cone formed by rolling the sector of the circle shown in Fig. 160. Use the data given.

9. The altitude of a cone equals the diameter of its base. Using $2r$ for the diameter, find formulas for the volume and for the total area.

10. Find a formula for the volume of a hollow column in the form of a right circular cylinder. Use h = height, t = thickness, and d = outside diameter. The column is open at both ends.

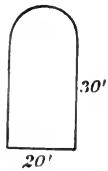


FIG. 161

11. The cross section of a straight tunnel ½ mi. long is of the form shown in Fig. 161. Find the quantity of material taken out, using the dimensions given in the diagram. Height of rectangular part, 30 ft.

SPHERICAL MEASUREMENTS

AREAS OF SPHERES

180. We have seen in plane geometry that if a regular polygon is inscribed in a circle, and if the number of sides is increased indefinitely, a polygon is soon formed which can with difficulty be distinguished from the circle. (See *Plane Geometry*, §295.) The perimeter of such a polygon may be taken as an approximation to the circumference of the circle. An approximation to the surface of the sphere may be obtained in a somewhat similar way.

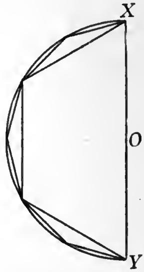


FIG. 162

Suppose a semicircle is divided into any number of equal arcs and the points of division are joined. A chain of equal chords is obtained which is half of an inscribed regular polygon, with two of its vertices at the ends of the diameter XY (Fig. 162). If we bisect the original arcs and join the points of division, half of a regular polygon of twice as many sides will be inscribed in the semicircle. This process can be continued indefinitely. If now we revolve the figure about the diameter XY as an axis, we know that the semicircle will generate a spherical surface. The chain of equal chords will generate the lateral surface of a series of cones and frustums of cones inscribed in the sphere. We have seen that if the semicircle is divided into a very great number of equal arcs, the chain of equal chords obtained can with difficulty be distinguished from the semicircle. It is evident, therefore, that if this semicircle is revolved about its diameter, the lateral surface of the sets of cones and frustums obtained can with difficulty be distinguished from the spherical surface.

The measure of the surface of one of these sets of inscribed solids may be taken as an approximation to the measure of the surface of the sphere.

181. PROBLEM. **To find the area of the surface generated by the revolution of a chain of equal chords inscribed in a semicircle as it revolves about the diameter of that semicircle as an axis.**

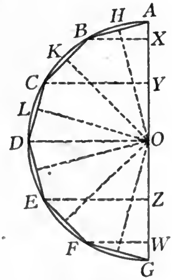


FIG. 163

Given $AB+BC+CD+$ etc., a chain of equal chords inscribed in the semicircle ADG, revolving about the diameter AG as an axis.

To find the area of the surface generated by $AB+BC+CD+$ etc.

Analysis in general:

I. To find the area of the surface generated by $AB+BC+CD+$etc., find the area of the surface generated by AB, by BC, by etc., and add the results.

II. To find the area of surfaces generated by AB, BC, CD, $+$ etc., use Th. 121.

Construction: $\therefore$ draw the $\perp$s from B, C, D, etc., to the diameter AG, meeting the diameter in X, Y, O, etc., respectively. Draw the $\perp$ bisectors of chords AB, BC, etc.

Outline of solution:

Area $AB=2\pi HO \cdot AX$.

Area $BC=2\pi KO \cdot XY$.

. . . . etc.

$HO=KO=$etc.

$\therefore$ area $(AB+BC+$etc.$)=2\pi HO(AX+XY+$etc.$)$

or area $(AB+BC+$etc.$)=2\pi HO \cdot AG$.

182. The following theorem from higher mathematics will be assumed:

THEOREM 124. **The areas of the surfaces generated by a series of chains of equal chords inscribed in the same semicircle, and revolving about the diameter of that semicircle as an axis, have a definite limit if the number of chords is increased indefinitely.**

183. Since we cannot measure any spherical surface in terms of plane units, we shall have to define what is meant by the area of a spherical surface.

The **area of a spherical surface** is defined as the limit of the areas of the surfaces generated by a series of chains of equal chords inscribed in the same semicircle, and revolving about the diameter of that semicircle as an axis as the number of chords is increased indefinitely.

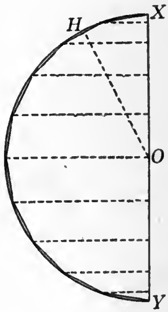


FIG. 164

184. Therefore, since the area of the surface generated by a chain of equal chords inscribed in a semicircle, and revolving about the diameter of that semicircle as an axis, is always $2\pi HO \cdot XY$ (Fig. 164), no matter how many chords there may be in the chain, we will assume

THEOREM 125. **The area of a spherical surface is $2\pi r \cdot d$ when r represents the radius of the sphere, and d its diameter.**

Since $2\pi r$ represents the circumference of a great circle of the sphere, Th. 125 may be stated:

The area of a spherical surface is the product of its diameter and the circumference of a great circle.

If we substitute $2r$ for d in the first statement in Th. 125, we have:

The area of the spherical surface $=4\pi r^2$.

Since πr^2 represents the area of a great circle, Th. 125 may be stated:

The area of a sperical surface is equal to the area of four great circles.

Ex. 1. Find the area of a sphere whose radius is 3 in.; $4\frac{1}{2}$ in.

Ex. 2. If the area of a sphere is 265 sq. in., find its radius.

Ex. 3. Find a formula for obtaining the radius of a sphere from its area.

Ex. 4. A tank consists of a cylindrical portion with hemispherical ends. The diameter of the ends equals the diameter of the cylinder, which is 2 ft. The total length is 7 ft. Find the total area.

AREAS OF ZONES

185. The portion of a spherical surface included between two parallel planes is called a **zone** (Fig. 165).

The circles in which the parallel planes cut the sphere are called the **bases** of the zone. The perpendicular distance between the parallel planes is called the **altitude** of the zone.

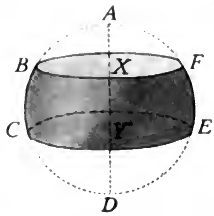


FIG. 165

If one of the planes is tangent to the sphere, the zone is called a **zone of one base.**

Just as a spherical surface is considered as generated by the revolution of a semicircle about a diameter as an axis, so a zone is considered as generated by the revolution of an arc of that semicircle about the same diameter as an axis.

In Fig. 165 the revolution of the semicircle $ABCD$ about the diameter AD generates the spherical surface. The revolution of $\overset{\frown}{AB}$ about AD generates a zone of one base; the revolution of $\overset{\frown}{BC}$ about AD generates a zone of two bases $BCEF$. XY is the altitude of zone $BCEF$. The circles BF and CE are the bases of the zone $BCEF$.

Exercise. What are the bases of the North Temperate Zone of the earth? Of the Arctic Zone?

186. COR. **The area of a zone is $2\pi r$ times the altitude of the zone.**

The corollary may be verified by the same method as was used in verifying Th. 125. The arc AD (Fig. 166) is divided into any number of equal parts, and the points of division joined. Find the area generated by $\overline{AB}+\overline{BC}+$ etc. Imagine the number of divisions to be increased indefinitely.

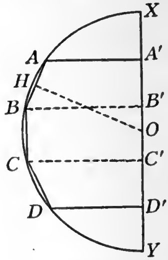


FIG. 166

Since $2\pi r$ is the circumference of a great circle, the corollary may read:

The area of a zone is the product of the altitude of the zone and the circumference of a great circle.

If h is the altitude of the zone and L its area, we have $\quad L = 2\pi rh$

Ex. 1. The radius of a sphere is 10 in. and the altitude of a zone is 6. Find the area of the sphere and of the zone.

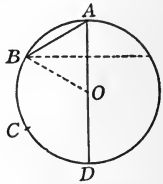


FIG. 167

Ex. 2. The diameter of a circle (Fig. 167) is 6 in. $\overset{\frown}{AB}$ subtends a central angle of 60°. Find the area of the zone of one base formed if $\overset{\frown}{AB}$ revolves about AD.

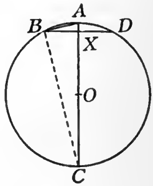


FIG. 168

Ex. 3. In Fig. 168, the diameter of circle O is 51″. The chord BD is 24″. Find the area of the zone generated if $\overset{\frown}{AB}$ revolves about AC.

Suggestion. Draw BC and use Th. 108, *Plane Geometry.*

AREAS OF LUNES

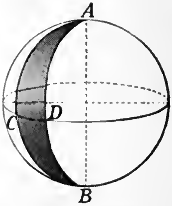


FIG. 169

187. We have defined a lune as a spherical polygon of two sides. In other words, it is a portion of a sphere between two semi-great circles (Fig. 169).

The following corollary of this definition is evident:

COR. **The two angles of a lune are equal.**

188. Since the sum of the possible adjacent spherical angles that have a common vertex on the sphere is 360°, we will assume that the surface of the sphere can be divided into 360 equal parts by great circles that have a common diameter. Any two of these semi-great circles form a lune. We will assume that the ratio of the surface of this lune to the surface of the sphere is equal to the ratio of the number of degrees in the angle of the lune to 360°. If, however, a lune is drawn at random on the sphere, its angle may be incommensurable with 360°. It can be proved that the same relation holds in this case. We have, therefore,

As. 30. **The area of a lune is to the area of a sphere as the angle of the lune is to 360°.**

If L represents the area of a lune, a the measure of its angle, and A the area of the sphere, we have $\frac{L}{A}=\frac{a}{360°}$.

Solving for L,

$$L=A\frac{a}{360}=\frac{a}{360}\cdot 4\pi r^2=\frac{a\pi r^2}{90}.$$

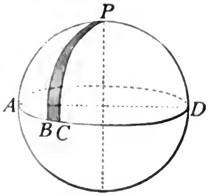


FIG. 170

189. A birectangular spherical triangle whose vertex angle is one degree is called a **spherical degree** (Fig. 170).

It is evident that a spherical degree is half a lune whose angle is one degree. A spherical degree is, therefore, $\frac{1}{720}$ of the surface of the sphere. Its area is constant for any given sphere.

The following theorem is, therefore, evident:

THEOREM 126. **The area of a lune in spherical degrees is equal to twice the measure of its angle in angle degrees.**

If L represents the area of the lune, and a the measure of its angle, $\quad$ ***$L=2a$ spherical degrees***

But a spherical degree is $\frac{1}{720}\cdot 4\pi r^2$.

$$\therefore\ L=\frac{2a}{720}\cdot 4\pi r^2=\frac{a\pi r^2}{90} \text{ as in §188}$$

AREAS OF SPHERICAL TRIANGLES

190. THEOREM 127. **The area of a spherical triangle expressed in spherical degrees is equal to the spherical excess of the triangle.**

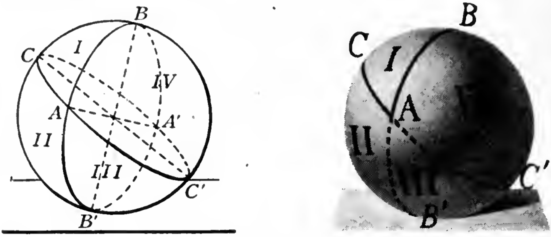


FIG. 171

Hypothesis: ABC is a spherical triangle. A, B, and C are the measures of $\angle A$, $\angle B$, and $\angle C$ respectively.

Conclusion: The area of $\triangle ABC$ is $(A+B+C-180)$ spherical degrees.

Analysis and construction:

I. To find the area of $\triangle ABC$, compare $\triangle ABC$ with the lunes whose angles are $\angle A$, $\angle B$, and $\angle C$.

II. $\therefore$ complete the great circles $ABA'B'$, $ACA'C'$, and $BCB'C'$.

Outline of proof:

I. *a.* Area of lune $CAC'B = 2C$ sph. deg.
b. Area of lune $BCB'A = 2B$ sph. deg.
c. Area of lune $ACA'B = 2A$ sph. deg.

II. *a.* Or area $\triangle\mathrm{I}$ + area $\triangle\mathrm{IV} = 2C$ sph. deg.
b. Area $\triangle\mathrm{I}$ + area $\triangle\mathrm{II} = 2B$ sph. deg.
c. Area $\triangle\mathrm{I}$ + area $\triangle CBA' = 2A$ sph. deg.

III. *a.* $\triangle CBA'$ and $\triangle\mathrm{III}$ are symmetric.
b. $\therefore \triangle CBA' = \triangle\mathrm{III}$.
c. $\therefore$ area $\triangle\mathrm{I}$ + area $\triangle\mathrm{III} = 2A$ sph. deg.

IV. Adding,
Areas $(3\triangle\mathrm{I}+\triangle\mathrm{II}+\triangle\mathrm{III}+\triangle\mathrm{IV}) = 2(A+B+C)$ sph. deg.

V. $\triangle I + \triangle II + \triangle III + \triangle IV$ make one hemisphere.
$\therefore$ area $(\triangle I + \triangle II + \triangle III + \triangle IV) = 360$ sph. deg.

VI. $\therefore$ 2 area $\triangle I + 360$ sph. deg. $= 2(A+B+C)$ sph. deg.
$\therefore$ area $\triangle I + 180$ sph. deg. $= (A+B+C)$ sph. deg.

VII. $\therefore$ area $\triangle I = (A+B+C-180)$ sph. deg.

COR. **If r represents the area of the sphere, E the spherical excess of the triangle, and A its area,**

$$A = \frac{E}{720} \cdot 4\pi r^2 = \frac{E}{180} \cdot \pi r^2$$

EXERCISES INVOLVING AREAS OF LUNES AND SPHERICAL TRIANGLES

191. 1. Find the area of a lune on a sphere if the radius of the sphere is 6 in. and the angle of the lune is 25°; 50°; 60°; 120°.

2. Find the area of a spherical triangle in spherical degrees if the angles of the triangle are:

a. 60°, 150°, 120°.
b. 120°, 75°, 150°.
c. 140°, 72°, 60°.
d. 72°, 65°, 90°.

3. Find the area of a spherical triangle on a sphere whose radius is 18 in. if the angles of the triangle are:

a. 60°, 72°, 120°.
b. 50°, 80°, 150°.
c. 45°, 90°, 105°.
d. 68°, 74°, 96°.

4. The area of a lune is 154 sq. in. Find its angle if the radius of the sphere is 8 in.

5. The area of a lune whose angle is 75° is 65 sq. in. Find the area and the radius of the sphere.

6. What is the angle of a lune whose area is $\frac{1}{5}$; $\frac{1}{3}$; $\frac{2}{7}$; $\frac{5}{9}$ the area of the sphere?

7. The area of a trirectangular spherical triangle is $\frac{1}{8}$ the surface of the sphere.

8. How many degrees in the angle of a lune if its area is equal to the area of a great circle of the sphere?

9. What portion of a sphere is covered by a spherical triangle whose angles are 68°, 72°, and 124°?

10. What is the area in spherical degrees of a spherical triangle if the sides of its polar are 30°, 75°, 54°?

THE VOLUME OF THE SPHERE

192. If a chain of a great many equal chords be inscribed in a semicircle, and this chain of chords be revolved about the diameter of this semicircle as an axis, the surface generated by the chain of chords will inclose a solid whose volume may be taken as an approximation to the volume of the sphere.

PROBLEM. **To find the volume of the solid inclosed by the surface generated when a chain of equal chords inscribed in a semicircle revolves about the diameter of that semicircle as an axis.**

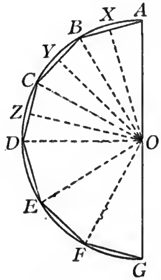


FIG. 172

Analysis and construction:

I. To find the volume inclosed by the surface generated by $AB+BC+CD+$etc., join the vertices A, B, C, D, etc., with O, and find the volume generated by $\triangle AOB$, $\triangle BOC$, $\triangle COD$, etc. Add the results.

II. $\therefore$ draw the $\perp$s from O to AB, BC, etc. (Th. 122).

Outline of solution:

I. *a*. Vol. $AOB =$ area $AB \cdot \frac{1}{3}OX$.

b. Vol. $BOC =$ area $BC \cdot \frac{1}{3}OY$.

c. Vol. $COD =$ area $CD \cdot \frac{1}{3}OZ$.

. . . . etc.

II. *a*. Adding, since $OX = OY = OZ$,

b. Vol. $(AOB + BOC + COD +$ etc.$) =$

$\frac{1}{3}OX \cdot$ area$(AB+BC+CD+$etc.$)$.

The solution obtained to the foregoing problem may be translated thus:

THEOREM 128. **The volume inclosed by the surface generated when a chain of equal chords inscribed in a semicircle revolves about the diameter of that semicircle as an axis is the product of the area of that surface and one-third the radius of the semicircle.**

193. The following theorem from higher mathematics will be assumed:

THEOREM 129. **The volumes of the solids inclosed by the surfaces generated when a series of chains of equal chords inscribed in a semicircle revolves about the diameter of that semicircle as an axis has a definite limit if the number of chords in the chain is increased indefinitely.**

194. Since we cannot measure the space inclosed by any sphere in terms of cubic units, we shall have to define what is meant by the volume of a sphere.

The **volume of a sphere** is defined as the limit of the volumes of the solids inclosed by the surfaces generated when a series of chains of equal chords inscribed in a semicircle revolves about the diameter of that semicircle as an axis, as the number of chords in the chain is increased indefinitely.

195. Therefore, since the volume inclosed by the surface generated when a chain of equal chords inscribed in a semicircle revolves about the diameter of that semicircle as an axis is always the product of the area of that surface and one-third the radius, we will assume

THEOREM 130. **The volume of a sphere is the product of the area of its surface and one-third its radius.**

If r represents the radius of the sphere, V its volume, and A the area of its surface, we have

$$V = \tfrac{1}{3}Ar \text{ or } V = \tfrac{1}{3}r \cdot 4\pi r^2 = \tfrac{4}{3}\pi r^3.$$

Ex. 1. Find the volume of a sphere whose radius is 2 in.; 3 in.; 7 in.

Ex. 2. How many iron balls 2 in. in diameter can be made from one 15 in. in diameter?

Ex. 3. Find the volume of a spherical shell $\frac{1}{2}$ in. thick if its outer diameter is 6 in.

Ex. 4. Make a formula for the volume of a spherical shell if its thickness is h and its outer diameter d.

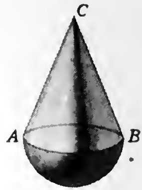


FIG. 173

Ex. 5. A solid is in the form of a right circular cone with hemisphere on its base (Fig. 173). If the diameter of the base is 6 in. and the slant height of the cone 8 in., find the total area and the volume.

VOLUMES OF SPHERICAL SECTORS

196. The solid generated by a circular sector revolving about a diameter that does not cross the sector is called a **spherical sector** (Fig. 174).

FIG. 174

COR. **The volume of a spherical sector is the product of the area of the zone which forms its base and one-third of the radius of the sphere.**

The corollary may be verified as follows:

Describe on the arc of the circular sector any number of equal chords (Fig. 175). Join the points of division on the arc with the center of the circle, thus inscribing in the sector a series of congruent triangles. Find the volume generated when the series of triangles revolves about the diameter, and proceed as in the discussion for Th. 130.

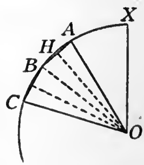


FIG. 175

If r represents the radius of the sphere, A the area of the zone, and V the volume of the spherical sector, we have

$$V = \tfrac{1}{3}Ar$$

Since $A = 2\pi rh$, $\quad V = \tfrac{2}{3}\pi r^2 h$

where h is the altitude of the zone.

VOLUMES OF SPHERES BY CAVALIERI'S THEOREM

197. THEOREM 131. **Any section of a sphere is constantly equal to that of the solid between the circumscribed cylinder of revolution and a double cone of revolution inscribed in the cylinder.**

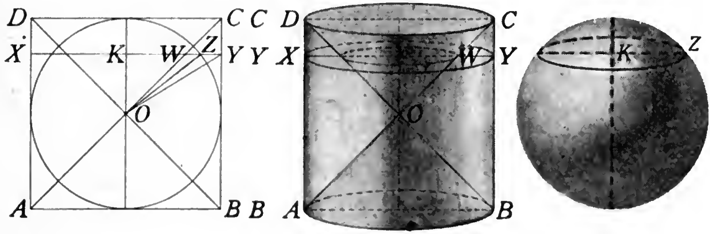


FIG. 176

Hypothesis: O is any sphere, $ABCD$ the circumscribed cylinder of revolution, and $AOB\text{–}COD$ a double cone of revolution inscribed in the cylinder. XY is a plane cutting the three solids. KW, KZ, and KY are the radii of the circles cut from the cone, the sphere, and the cylinder.

Conclusion: Area of circle KZ = area of ring WY.

Outline of proof:

$$
\begin{aligned}
\text{Area of ring } WY &= \pi\overline{KY}^2 - \pi\overline{KW}^2 \\
&= \pi(\overline{KY}^2 - \overline{KW}^2) \\
&= \pi(\overline{KY}^2 - \overline{KO}^2) \text{ (prove } KO = KW) \\
&= \pi(\overline{OZ}^2 - \overline{KO}^2) \text{ (prove } KY = OZ) \\
&= \pi\overline{KZ}^2 = \text{area of circle } KZ.
\end{aligned}
$$

THEOREM 130. **The volume of the sphere is $\frac{4}{3}\pi r^3$.**

Suggestion. It follows at once from Cavalieri's Theorem that the volume of the sphere is the difference between the volumes of the circumscribed cylinder of revolution and a double cone inscribed in the cylinder.

$\therefore$ vol. of sphere $= \pi r^2h - \frac{1}{3}\pi r^2h = \frac{2}{3}\pi r^2h$. But $h = 2r$.

$$\therefore\ V = \tfrac{4}{3}\pi r^3$$

VOLUMES OF SPHERICAL WEDGES AND SPHERICAL PYRAMIDS

198. A solid bounded by a lune and the plane of its sides is called a **spherical wedge.**

A solid bounded by a spherical polygon and the planes of its sides is called a **spherical pyramid.**

The volume of a spherical wedge or of a spherical pyramid may be obtained from the following assumption:

As. 31. **The ratio of the volume of a spherical wedge or of a spherical pyramid to the volume of the sphere equals the ratio of the area of the base of the wedge or of the pyramid to the area of the sphere.**

If A is the area of the base of the wedge or of the pyramid and V its volume,

$$\frac{A}{4\pi r^2}=\frac{V}{\frac{4}{3}\pi r^3}.$$

$$\therefore V=\tfrac{1}{3}rA.$$

Since the area of a lune is $\frac{a}{90}\pi r^2$,

Cor. I. **The volume of the spherical wedge is** $\frac{a}{270}\pi r^3$.

Since the area of a spherical triangle is $\frac{E}{180}\pi r^2$,

Cor. II. **The volume of the spherical pyramid is** $\frac{E}{540}\pi r^3$.

Ex. 1. Find the volume of a spherical wedge cut from a sphere whose radius is 6 in. if the angle of the lune is 72°.

Ex. 2. Find the volume of a spherical pyramid whose base is a spherical triangle with angles 156°, 94°, and 128°, cut from a sphere 8 in. in diameter.

A **spherical cone** is something like a spherical pyramid. Its base is a zone of one base.

Ex. 3. Make a formula for the volume of a spherical cone. As. 31 holds.

VOLUMES OF SPHERICAL SEGMENTS

199. A portion of a sphere included between two parallel planes is a **spherical segment.**

THEOREM 132. **If a and b represent the radii of the bases of a spherical segment, h its altitude, r the radius of the sphere, and V the volume of the segment, then**

$$V = \tfrac{1}{2}\pi h(a^2+b^2)+\tfrac{1}{6}\pi h^3$$

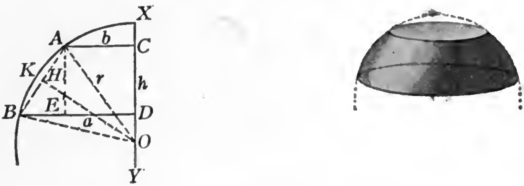


FIG. 177

Analysis:

I. Find the volume of the frustum generated by $CAHBD$ and add the volume generated by $AKBH$.

II. To find the volume generated by $AKBH$, subtract the volume generated by $\triangle ABO$ from the volume generated by the circular sector $AKBO$.

Outline of proof:

I. $$\text{Vol. of sector} = \tfrac{2}{3}\pi r^2 h.$$

II. $$\text{Vol. of } \triangle ABO = \tfrac{1}{3}OH \cdot \text{area } AB.$$

$$\text{But area } AB = 2\pi \cdot OH \cdot h.$$

$$\therefore \text{ vol. of } \triangle ABO = \tfrac{2}{3}\pi \overline{OH}^2 h.$$

III. $$\therefore \text{ vol. of } AKBH = \tfrac{2}{3}\pi h(r^2-\overline{OH}^2).$$

$$\text{From } \triangle AHO, (r^2-\overline{OH}^2) = \tfrac{1}{4}\overline{AB}^2.$$

$$\therefore \text{vol. of } AKBH = \tfrac{1}{6}\pi h \overline{AB}^2.$$

$$\text{From } \triangle ABE,\ \overline{AB}^2 = h^2+(a-b)^2.$$

$$\therefore \text{ vol. } AKBH = \tfrac{1}{6}\pi h(h^2+a^2+b^2-2ab).$$

IV. $$\text{Vol. of frustum} = \tfrac{1}{3}\pi h(a^2+b^2+ab).$$

V. $$\therefore \text{ vol. of sph. seg.} = \tfrac{1}{6}\pi h(h^2+3a^2+3b^2)$$
$$= \tfrac{1}{6}\pi h^3+\tfrac{1}{2}\pi h(a^2+b^2).$$

SUMMARY AND SUPPLEMENTARY EXERCISES

FORMULAE OBTAINED

200. NOTE. In the formulae below L = lateral area, A = total area, V = volume, b = base, r = radius, h = height, l = slant height, p = perimeter, e = edge of prisms, elements of cylinders, spherical excess of spherical triangles, a = angle.

A. PRISMS:

L (any prism) = p of rt. sec. $\cdot\, e$.
L (rt. prisms) = p of base $\cdot\, e$.
V (any prism) = bh.

B. PYRAMIDS:

L (regular) = $\frac{1}{2}$(p of base) $\cdot\, l$.
V (any pyramid) = $\frac{1}{3}bh$.

C. PRISMATOIDS:

$V = \frac{1}{6}h(b+b'+4m)$.

D. FRUSTUMS OF PYRAMIDS:

L (regular) = $\frac{1}{2}l$(sum of p of bases).
$V = \frac{1}{3}h(b+b'+\sqrt{bb'})$

E. CYLINDERS:

L (circular) = p of rt. sec. $\cdot\, e$.
L (rt. circular) = $2\pi rh$.
A (rt. circular) = $2\pi r(r+h)$.
V (circular base) = πr^2h.

F. CONES:

L (rt. circular) = πrl.
A (rt. circular) = $\pi r(r+l)$.
V (circular base) = $\frac{1}{3}\pi r^2h$.

G. FRUSTUMS OF CONES:

L (rt. circular) = $\pi l(r_1+r_2)$.
V (circular base) = $\frac{1}{3}\pi h(r_1^2+r_2^2+r_1r_2)$.

H. Area generated by a segment revolving about an axis = projection of segment on axis times 2π perpendicular from mid-point of segment to axis.

I. Volume of solid formed by revolving a triangle about an axis $= \frac{1}{3}$ area generated by revolving one side about a fixed vertex times the corresponding altitude.

J. SPHERES:

A of sphere $= 4\pi r^3$.

A of zone $= 2\pi rh$.

A of lune $= 2a$ sph. deg. $= \dfrac{a\pi r^2}{90}$.

A of sph. triangle $= e$ sph. deg. $= \dfrac{e\pi r^2}{180}$.

V of sphere $= \frac{4}{3}\pi r^3$.

V of sph. sector $= \frac{2}{3}\pi r^2 h$.

V of sph. wedge $= \frac{a}{270}\pi r^3$.

V of sph. pyramid $= \frac{e}{540}\pi r^3$.

V of sph. segment $= \frac{1}{2}\pi h(a^2+b^2) + \frac{1}{6}\pi h^3$ when a and b are the radii of the bases of the segment.

NUMERICAL PROBLEMS

201. 1. Find the number of cubic yards of concrete required for the foundation walls of a house 24 ft. $\times$ 20 ft. The walls are to be 10 in. thick and 8 ft. high. No deductions for windows.

2. Two congruent regular square pyramids stand on opposite sides of the same base. The distance between their vertices is 16 cm. The diagonal of the common base is 16 cm. Find the total area and the volume.

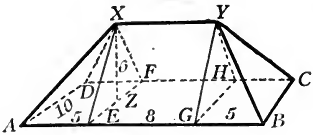


FIG. 178

3. Find the volume of the wedge shown in Fig. 178. The base $ABCD$ is a rectangle 10 in. $\times$ 18 in. XEF and YGH are right sections and are isosceles triangles. Use the data given. XZ is $\perp$ base.

4. Solve the previous exercise by the prismatoid formula.

5. Find the formula for the volume of a wedge if the dimensions of the base are a and b, the altitude is h, and the edge XY (Fig. 178) is c.

6. Find the volume of a regular octahedron if one edge is 6 in.; if one edge is E.

7. Find the volume of a regular tetrahedron if one edge is 6 in.; if one edge is E.

8. A berry box is in the form of a frustum of a regular square pryamid 5 in. square at the top and $4\frac{1}{2}$ in. square at the bottom. What should be the depth of the box if it holds a quart? Use 1 dry qt. = 67.2 cu. in.

9. A pail is in the form of a frustum of a right circular cone the radii of whose bases are 12 in. and 10 in. respectively. Find the depth of the pail if it holds $2\frac{1}{2}$ gal. 1 gal = 231 cu. in.

10. Fig. 179 shows a cylindrical tank partly filled with water. The tank is 6 ft. long and 4 ft. in diameter. If the greatest depth of the water is 3 ft., find the number of gallons of water in the tank.

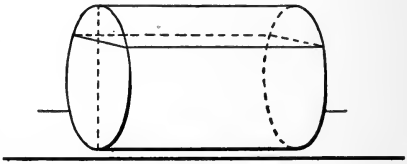


FIG. 179

NOTE. Problems like Ex. 10 cannot be solved without trigonometry except in special cases.

11. Find the total area and the volume of the solid formed by revolving the square $ABCD$ (Fig. 180) about line l, if l is $\perp$ diagonal AC at C. Use data given.

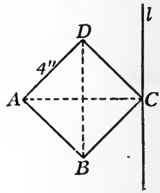


FIG. 180

12. Find the total area and the volume of the solid formed by revolving the equilateral triangle ABC (Fig. 181) about line l, if l is $\perp$ the altitude DB at point B. Use data given.

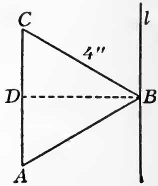


FIG. 181

13. How many shot $\frac{1}{2}$ in. in diameter can be made from a cylindrical bar 10 in. long and $\frac{3}{8}$ in. in diameter?

14. Find the weight of a spherical shell of iron if its outside diameter is 6 in. and it is $\frac{7}{8}$ of an inch thick. Use 1 cu. in. of iron weighs 0.28 lb.

15. Given a cube whose side is 2 in. Find the area and the volume of (1) the inscribed sphere; (2) the circumscribed sphere.

16. Find the radius of a sphere that is equal in volume to two spheres whose radii are (1) 2 in. and 5 in. respectively; (2) R and r respectively.

17. Find the area of a trirectangular spherical triangle on a sphere whose radius is 18 in.

18. If the earth is a perfect sphere, prove that one-half of its surface lies within 30° of the equator.

19. What is the area in spherical degrees of a spherical triangle if the sides of its polar are 72°, 96°, and 40°?

20. Find each angle of an equilateral spherical triangle if its area is one-third the area of the sphere.

21. Find the angle of a lune if its area is two-ninths the area of the sphere.

22. Find the angle of a lune that is equivalent to a spherical triangle each angle of which is 84°.

23. The area of a lune is 154 sq. in. Find its angle if the radius of the sphere is 9 in.

24. The area of a lune whose angle is 48° is 54 sq. in. Find the radius and the area of the sphere.

25. How many degrees are there in the angle of a lune if its area is equal to ⅓ the area of a great circle of the sphere.

26. The polar distance of a small circle on a sphere is 60°. The radius of the sphere is 15 in.Find the radius of the circle, the distance of the plane of the circle from the center of the sphere, and the area of the zone of one base cut off by the circle.

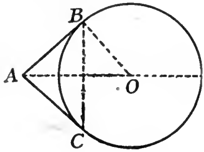


FIG. 182

27. The circumference of a great circle of a sphere is 8 ft. Find the radius of the sphere; also its area and volume.

28. Given a sphere whose center is O, and A a point of light outside of sphere O. Find the area illuminated by A. Let the radius of the sphere be 10 ft., and OA be 15 ft. (Fig. 182).

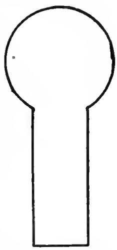


FIG. 183

29. Fig. 183 shows a post which is capped by a portion of a sphere. The height of the post without the sphere is 3 ft. Its diameter is 15 in. The radius of the sphere is 15 in. Find the total volume.

30. The water tank shown in Fig. 184 consists of a cylinder with a hemisphere below and a cone above. The diameter is 20 ft. and the height of the cylinder is 32 ft. The conical roof is 10 ft. high. Find the capacity of the tank in gallons. 231 cu. in. = 1 gal. How much paint will be required to paint the outside of the tank, allowing one gallon of paint for 500 sq. ft. of surface?

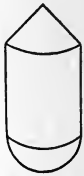


FIG. 184

31. A hemispherical cap of aluminum is 3 in. in diameter. Find the diameter of the blank from which it is pressed.

NOTE. The **blank** is a circular piece of metal cut from a flat sheet. This is pressed into the desired form by means of a **die.** Assume that the area of the blank is equal to the area of the finished article.

32. Find the diameter of the blank if the cap in the preceding exercise has a flat ring ½ in. around it.

33. A spherical shell of iron whose outer diameter is 1 ft. is filled with lead. Find the thickness of the iron if the filled shell weighs twice the unfilled shell. A cubic inch of iron weighs 4.2 oz.; a cubic inch of lead weighs 6.6 oz.

34. What is the ratio of the surface of a sphere to the entire surface of its hemisphere?

35. A plane is drawn tangent to the inner of two concentric spheres 8 in. and 12 in. in diameter. Find the circumference of the circle cut out on the outer sphere.

36. What is the altitude of a zone of a sphere which equals a trirectangular triangle in area?

37. The diameter of a right circular cylinder equals its height. Find its dimensions if its capacity is one gallon. 231 cu. in. = 1 gal.

38. The area of a sphere is equal to the lateral area of the circumscribed cylinder of revolution.

39. With the compasses open 4 in. a circle is drawn on a 12-in. globe. Find the circumference of the circle, and the area of the zone of one base cut off.

40. A cylindrical bore is made through a sphere. If the radius of the sphere is 6 in., and the diameter of the bore 6 in., find the entire area of the part removed.

EXERCISES INVOLVING PROOFS

202. 1. The volume of a triangular prism is the product of one lateral face and one-half the distance of that face from the opposite edge.

2. The volume of a regular prism is the product of the lateral area and one-half the altitude of the base.

3. Any plane passed through the center of a parallelepiped divides it into two equivalent parts.

4. A truncated triangular prism is equivalent to three pyramids whose common base is the lower base of the prism and whose vertices are the vertices of the upper base (Fig. 185).

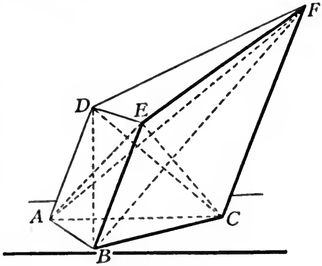


FIG. 185

Analysis:

I. $ABC\text{-}DEF = E\text{-}ABC + E\text{-}ACD + E\text{-}CDF$.

II. $E\text{-}ABC$ is one of the required pyramids.

III. Prove $E\text{-}ACD = D\text{-}ABC$ by proving both equivalent to $B\text{-}ACD$.

IV. Prove $E\text{-}CDF = F\text{-}ABC$ by proving $F\text{-}ABC = B\text{-}ACF = B\text{-}CFD$ $= E\text{-}CFD$.

5. The volume of a truncated right triangular prism is the product of the area of its base and one-third the sum of the lateral edges. (See Ex. 4.)

6. The volume of any truncated triangular prism is the product of the area of a right section and one-third the sum of its lateral edges.

Suggestion. The right section divides it into two truncated right prisms.

7. The area of a zone of one base is equal to the area of a circle whose radius is the generating arc.

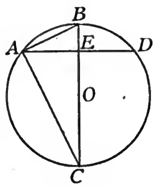


FIG. 186

Suggestion. Since the area of the zone generated by AB (Fig. 186) is $2\pi rBE$, and the area of the circle whose radius is AB is $\pi\overline{AB}^2$, prove that $2\pi rBE = \pi\overline{AB}^2$. $\therefore$ prove $\pi BC\cdot BE = \pi\overline{AB}^2$or $BCBE\cdot \quad = \overline{AB}^2$.

CHAPTER V

SIMILARITY AND SYMMETRY

SIMILARITY

TESTS FOR SIMILAR SOLIDS

203. Two **polyhedrons** are said to be **similar** if they have the same number of faces respectively similar, and similarly placed, and their corresponding polyhedral angles equal.

For convenience, corresponding parts will be lettered alike; for example, AB and $A'B'$ are corresponding edges.

204. THEOREM 133. **Two tetrahedrons are similar if three faces meeting at a common vertex in one are respectively similar to three faces meeting at a common vertex of the other, and arranged in the same order.**

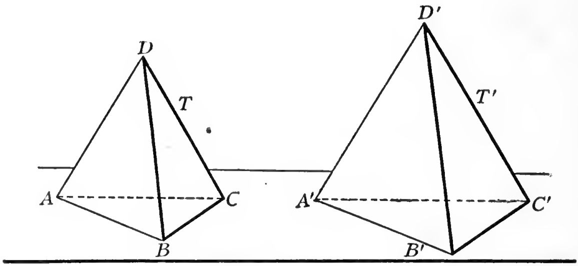


FIG. 187

Hypothesis: T and T' are two tetrahedrons with $ADB \backsim A'D'B'$, $BDC \backsim B'D'C'$, and $ADC \backsim A'D'C'$.

Conclusion: $T \backsim T'$.

I. To prove $T \backsim T'$, prove $\triangle ABC \backsim \triangle A'B'C'$, and the trihedral angles at A, B, C, and D equal respectively to the trihedral angles at A', B', C', and D'.

Analysis:

II. To prove the trihedral angle at A equal to the trihedral angle at A', prove the face angles of one equal to the face angles of the other, and arranged in the same order.

III. $\therefore$ prove $\triangle ABC \sim \triangle A'B'C'$.

IV. $\therefore$ prove $\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CA}{C'A'}$.

205. If the segments that join the vertices of a given polyhedron with a given point are divided in the same ratio from the given point, and if the points of division are joined in the same order as the vertices of the given polyhedron, the polyhedron so formed and the given polyhedron are **radially placed.** The given point may be called the **radial center.**

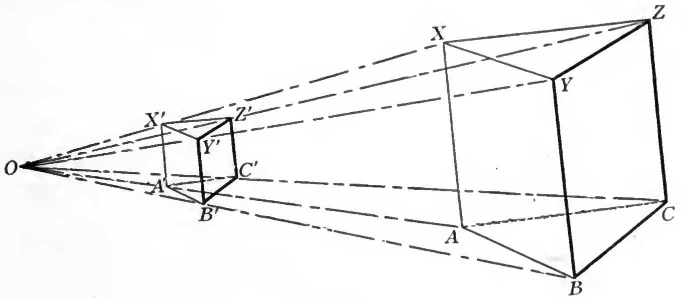


FIG. 188

The radial center of the two radially placed polyhedrons may be within or without the polyhedrons. If it is without them, the two polyhedrons may be on the same or on opposite sides of the center. Let the pupil draw figures for each case.

206. THEOREM 134. **Two dihedral angles are equal if the faces of one are parallel to the faces of the other and extend in the same direction from their edges.**

For proof see suggestion to Th. 24, p. 23.

207. THEOREM 135. **If two polyhedrons are radially placed, and are on the same side of the radial center, the polyhedrons are similar.**

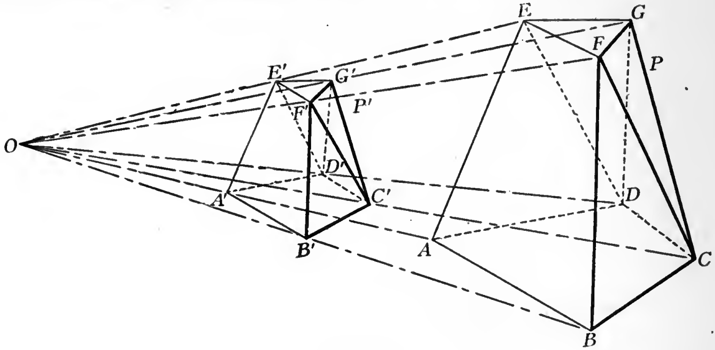


FIG. 189

Hypothesis: P and P' are two polyhedrons radially placed on the same side of the radical center O.

Conclusion: $P \backsim P'$.

Analysis:

I. To prove $P \backsim P'$, prove
  (1) The faces of P similar to the corresponding faces of P'.
  (2) The polyhedral angles of P equal to the corresponding polyhedral angles of P'.
  (3) The parts are similarly placed.

II. To prove face $ABFE \backsim$ face $A'B'F'E'$, prove . . . .

III. To prove the polyhedral angle at B equal to the polyhedral angle at B', prove that the face angles and the dihedral angles of one are equal respectively to the corresponding parts of the other.

IV. Verify that the parts are arranged in the same order.

When two polyhedrons are radially placed, the radial center is called the **center of similitude.**

The ratio of the distances of corresponding vertices from the center of similitude is called the **ratio of similitude.**

208. Theorem 136. **If two polyhedrons are radially placed, with the center of similitude within the polyhedrons, the polyhedrons are similar.**

209. Theorem 137. **If two polyhedrons are radially placed, and are on opposite sides of the radial center, the faces of one are similar to the corresponding faces of the other, the polyhedral angles of one are equal to the corresponding polyhedral angles of the other, but the parts are arranged in the reverse order.**

210. Two polyhedrons in the position called for in Th. 135 are said to be in the **direct radial position** and are said to be **directly similar.** If, however, the two polyhedrons are in the position called for in Th. 137, they are said to be in the **inverse radial position** and are said to be **inversely similar.**

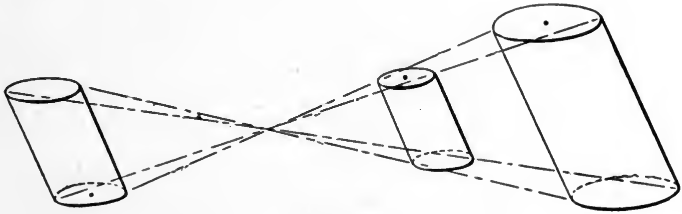


Fig. 190

Although the proofs are too difficult for this book, the statements above apply to all kinds of solids. These definitions may be used as fundamental definitions of similar solids (Figs. 190 and 191).

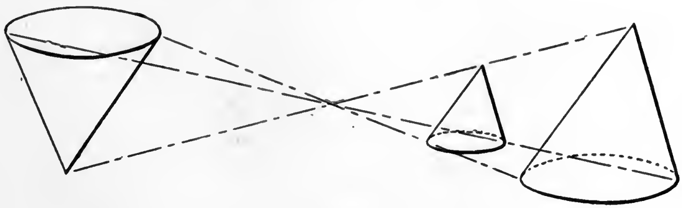


Fig. 191

PROPERTIES OF SIMILAR SOLIDS

211. As. 32. **Two solids similar to a third are similar to each other.**

Theorem 138. **Two similar polyhedrons may be placed in the direct radial position.**

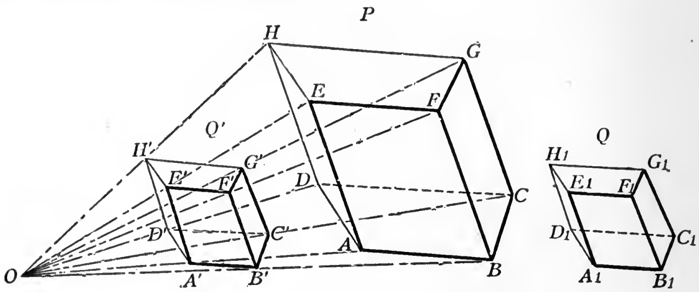


Fig. 192

Hypothesis: P and Q are two similar polyhedrons.

Conclusion: P and Q can be placed in the direct radial position.

Analysis and construction:

I. It is necessary to prove Q congruent to a polyhedron that is in the direct radial position with P.

II. $\therefore$ draw rays to any point O, and construct Q' in direct radial position with P so that $\dfrac{OA}{OA'}=\dfrac{AB}{A'B'}$.

III. To prove $Q \cong Q'$, prove the polyhedral angles of Q equal to the corresponding polyhedral angles of Q', and the faces of Q congruent to the corresponding faces of Q'.

IV. To prove the corresponding polyhedral angles equal, prove $Q \backsim Q'$.

V. To prove $A_1B_1F_1E_1 \cong A'B'F'E'$, prove $A_1B_1F_1E_1 \backsim A'B'F'E'$ and $A_1B_1 = A'B'$.

VI. $\therefore$ prove $\dfrac{AB}{A_1B_1}=\dfrac{OA}{OA_1}=\dfrac{AB}{A'B'}$.

212. THEOREM 139. **Two similar polyhedrons can be divided into the same number of tetrahedrons similar each to each and similarly placed.**

Suggestion. Pass a plane through ACF and $A_1C_1F_1$ in P and Q (Fig. 192) and show that the tetrahedrons cut from P and Q are similar. Continue in this way until P and Q are completely divided into tetrahedrons.

213. THEOREM 140. **If two polyhedrons are similar, the ratio of any two corresponding edges equals the ratio of similitude.**

Suggestion. Place the two similar polyhedrons in the direct radial position.

214. NOTE. It is also true that if two polyhedrons are similar the ratio of any two corresponding segments equals the ratio of similitude. We will prove here only one special case under this theorem. This special case is Th. 141 and is needed in the proof to Th. 143. Let the pupil name other special cases.

THEOREM 141. **If two tetrahedrons are similar, the ratio of corresponding altitudes equals the ratio of similitude.**

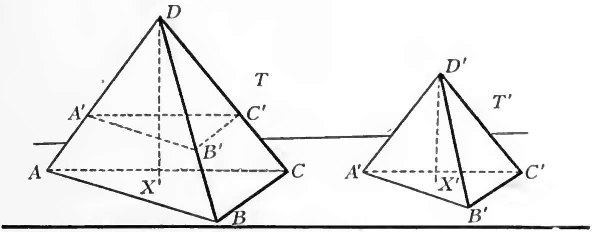


FIG. 193

Analysis and construction:

I. To prove $\frac{DX}{D'X'}=\frac{AB}{A'B'}$, prove $\frac{DX}{D'X'}=\frac{AD}{A'D'}=\frac{AB}{A'B'}$.

II. To prove $\frac{DX}{D'X'}=\frac{DA}{D'A'}$, place T' on T so that the trihedral angle D' fits upon the trihedral angle D and prove (1) $A'B'C'$ parallel to ABC; (2) $D'X'$ falls on DX.

215. THEOREM 142. **If two polyhedrons are similar, the ratio of the areas of corresponding faces equals the square of the ratio of similitude.**

Show that this reduces to a plane geometry theorem.

216. THEOREM 143. **If two tetrahedrons are similar, the ratio of the volumes equals the cube of the ratio of similitude.**

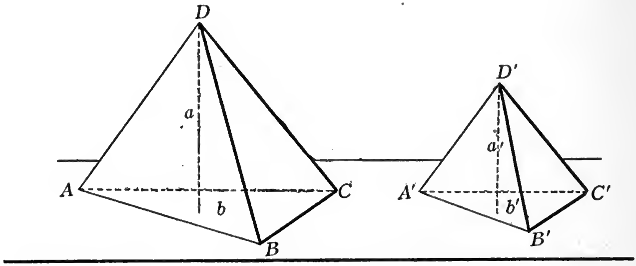


FIG. 194

Hypothesis: $D-ABC \sim D-A'B'C'$, b and b' represent the bases, a and a' the altitudes, and V and V' the volumes.

Conclusion: $\dfrac{V}{V'}=\dfrac{\overline{AB}^3}{\overline{A'B'}^3}$.

Analysis: To prove $\dfrac{V}{V'}=\dfrac{\overline{AB}^3}{\overline{A'B'}^3}$, prove (1) $\dfrac{V}{V'}=\dfrac{ab}{a'b'}$; (2) $\dfrac{a}{a'}=\dfrac{AB}{A'B'}$; (3) $\dfrac{b}{b'}=\dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.

Outline of proof:

I. $\dfrac{V}{V'}=\dfrac{\frac{1}{3}ab}{\frac{1}{3}a'b'}=\dfrac{ab}{a'b'}$.

II. $\dfrac{a}{a'}=\dfrac{AB}{A'B'}$. $\dfrac{b}{b'}=\dfrac{\overline{AB}^2}{\overline{A'B'}^2}$. $\therefore \dfrac{ab}{a'b'}=\dfrac{\overline{AB}^3}{\overline{A'B'}^3}$.

III. $\therefore \dfrac{V}{V'}=\dfrac{\overline{AB}^3}{\overline{A'B'}^3}$

THEOREM 144. **If two polyhedrons are similar, the ratio of the volumes is equal to the cube of the ratio of similitude.**

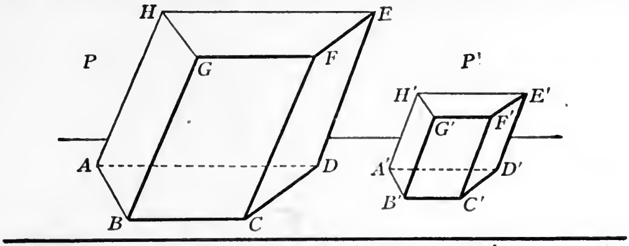


FIG. 195

Hypothesis: $P \sim P'$. V and V' represent the volumes.

Conclusion: $\dfrac{V}{V'} = \dfrac{\overline{AB}^3}{\overline{A'B'}^3}$.

Analysis and construction:

I. To prove $\dfrac{V}{V'} = \dfrac{\overline{AB}^3}{\overline{A'B'}^3}$, divide P and P' into similar tetrahedrons T_1 and T_1', T_2 and T_2', T_3 and T_3', etc., and prove $\dfrac{T_1 + T_2 + T_3, \text{ etc.}}{T_1' + T_2' + T_3', \text{ etc.}} = \dfrac{\overline{AB}^3}{\overline{A'B'}^3}$.

II. $\therefore$ prove $\dfrac{T_1}{T_1'} = \dfrac{\overline{AB}^3}{\overline{A'B'}^3}$, $\dfrac{T_2}{T_2'} = \dfrac{\overline{AB}^3}{\overline{A'B'}^3}$, etc.

Suggestion. See Th. 124, *Plane Geometry.*

Ex. 1. Are regular polyhedrons of the same number of faces similar? Why?

Ex. 2. The bases of two similar pyramids are in the ratio of 4:9. What is the ratio of their volumes? Of their altitudes?

Ex. 3. The dimensions of a box are 6, 8, and 12. Find the dimensions of a similar box that will hold twice as much.

217. Two **cylinders** or two **cones of revolution** are said to be **similar** if they are generated by the revolution of similar figures revolving about corresponding sides.

THEOREM 145. **The lateral areas or the total areas of two similar cylinders of revolution have the same ratio as the squares of their radii or the squares of their altitudes.**

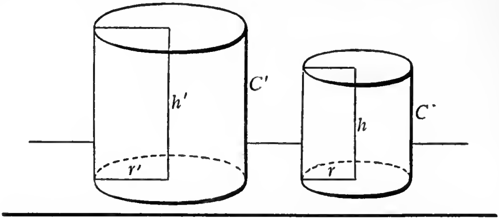


FIG. 196

Hypothesis: C and C' are two similar cylinders of revolution; r and r' represent the radii, h and h' the altitudes, L and L' the lateral areas, and A and A' the total areas.

Conclusion:

I. $\dfrac{L}{L'}=\dfrac{r^2}{r'^2}=\dfrac{h^2}{h'^2}.$

II. $\dfrac{A}{A'}=\dfrac{r^2}{r'^2}=\dfrac{h^2}{h'^2}.$

Outline of proof:

I. 1. $\dfrac{L}{L'}=\dfrac{2\pi rh}{2\pi r'h'}=\dfrac{rh}{r'h'}.$

2. Since $\dfrac{r}{r'}=\dfrac{h}{h'}$, $\therefore \dfrac{rh}{r'h'}=\dfrac{r^2}{r'^2}=\dfrac{h^2}{h'^2}.$

3. $\therefore \dfrac{L}{L'}=\dfrac{r^2}{r'^2}=\dfrac{h^2}{h'^2}.$

II. 1. $\dfrac{A}{A'}=\dfrac{2\pi r(h+r)}{2\pi r'(h'+r')}=\dfrac{r(h+r)}{r'(h'+r')}.$

2. Since $\dfrac{r}{r'}=\dfrac{h}{h'}$, $\therefore \dfrac{r}{r'}=\dfrac{h+r}{h'+r'}.$

3. $\therefore \dfrac{r(h+r)}{r'(h'+r')}=\dfrac{r^2}{r'^2}=\dfrac{h^2}{h'^2}.$

4. $\therefore \dfrac{A}{A'}=\dfrac{r^2}{r'^2}=\dfrac{h^2}{h'^2}.$

NOTE for II. 2. See *Plane Geometry*, Th. 124.

THEOREM 146. **The lateral areas or the total areas of two similar cones of revolution have the same ratio as the squares of the radii or the squares of the altitudes.**

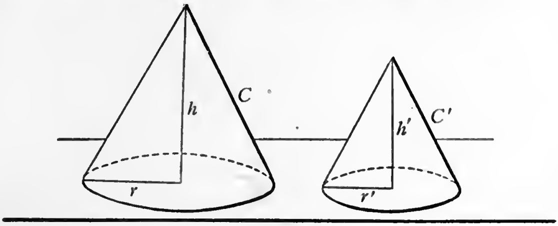


FIG. 197

THEOREM 147. **The volumes of two similar cylinders of revolution have the same ratio as the cubes of the radii or the cubes of the altitudes** (Fig. 196).

Outline of proof:

I. $\frac{V}{V'}=\frac{\pi r^2h}{\pi r'^2h'^2}=\frac{r^2h}{r'^2h'^2}.$

II. Since $\frac{r}{r'}=\frac{h}{h'}$, $\therefore\ \frac{r^2}{r'^2}=\frac{h^2}{h'^2}$ and $\frac{r^3}{r'^3}=\frac{h^3}{h'^3}.$

III. $\therefore\ \frac{r^2h}{r'^2h'}=\frac{h^3}{h'^3}=\frac{r^3}{r'^3}.$

IV. $\therefore\ \frac{V}{V'}=\frac{r^3}{r'^3}=\frac{h^3}{h'^3}.$

THEOREM 148. **The volumes of two similar cones of revolution have the same ratio as the cubes of the radii or the cubes of the altitudes** (Fig. 197).

218. Any two spheres are similar.

THEOREM 149. **The areas of two spheres have the same ratio as the squares of their radii.**

THEOREM 150. **The volumes of two spheres have the same ratio as the cubes of their radii.**

Exercise. The radii of two spheres are in the ratio of 2 : 5. The sum of their volumes is 3994 cu. in. Find the volumes.

219. The fundamental definition for similar solids is given in §210. The following facts are true for all similar solids regardless of shape:

I. The ratio of corresponding segments is equal to the ratio of similitude.

II. The ratio of the areas of corresponding surfaces equals the square of the ratio of similitude.

III. The ratio of the volumes is equal to the cube of the ratio of similitude.

Only special cases have been proved in this book.

SYMMETRY

220. We have defined two polyhedrons as symmetric if the parts of one are equal respectively to the parts of the other, but arranged in the reverse order.

THEOREM 151. **If two polyhedrons are in the inverse radial position, and the ratio of similitude is one, the polyhedrons are symmetric with respect to the center of similitude.**

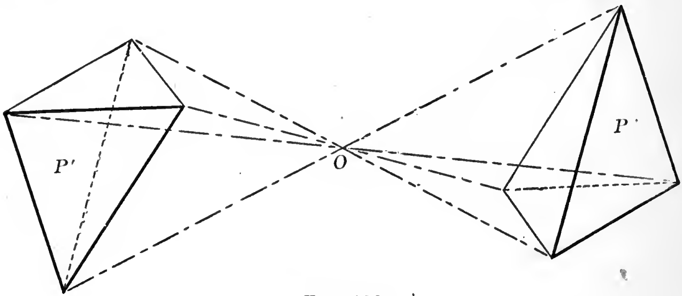


FIG. 198

Analysis: To prove that P and P' are symmetric, prove that the parts of one are equal respectively to the parts of the other, but arranged in the reverse order.

By a method similar to that used in proving Th. 138, we may prove that if two polyhedrons are symmetric they may be placed in the inverse radial position.

221. Two solids may be so situated on opposite sides of a plane that the plane bisects at right angles all segments joining corresponding points of the two solids. We can prove that in this case the two solids are symmetric. They are said to be symmetric with respect to the plane. We can prove also that if two solids are symmetric they can be placed on opposite sides of a plane as indicated above.

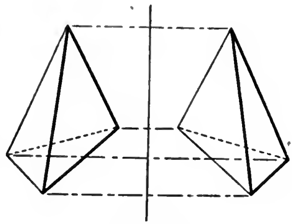


FIG. 199

It is to be noted that two figures in plane geometry that are symmetric with respect to a point or to a line can be made to coincide, but that two solids that are symmetric with respect to a point or to a plane cannot be made to coincide.

Two solids may be so situated in regard to a line that the line bisects at right angles all segments joining corresponding points of the two solids. In this case the solids are said to be symmetrically situated with regard to the line. The two solids are, however, congruent, and can be made to coincide.

MISCELLANEOUS EXERCISES

222. 1. Name as many solids as you can that have a center of symmetry. Tell what is the center of symmetry in each case.

2. Name as many solids as you can that have a plane of symmetry. Tell what is the plane of symmetry in each case.

3. Give an everyday illustration of figures that are symmetric with respect to a plane.

4. In shipping goods, which would be more economical of the material of which the packing boxes are made: (1) to use two boxes of the same dimensions or (2) to use one box similar to the first that will hold twice as much? Why?

NOTES ON ARITHMETIC AND ALGEBRA

FRACTIONS

223. The following fundamental law of fractions underlies all operations that involve fractions!

Multiplying or dividing numerator and denominator of a fraction by the same number does not alter the value of the fraction.

A. **The sum or difference of two or more fractions** that have a common denominator is the sum or difference of the numerators divided by the common denominator.

Two or more fractions that have not a common denominator must be reduced to a common denominator before adding or subtracting. To reduce fractions to a common denominator, apply the fundamental law given above.

Add and subtract the following:

1. $\frac{5}{12}+\frac{7}{18}$ 3. $\frac{a}{b}+\frac{b}{c}$ 5. $4-\frac{x}{y}+\frac{4x^2}{4y^2}$

2. $\frac{11}{24}+\frac{13}{36}-\frac{5}{6}$ 4. $\frac{a}{bc}+\frac{b}{ac}-\frac{c}{ab}$ 6. $\frac{x}{x+1}-\frac{1}{x-1}$

B. **The product of two fractions** is the product of the numerators divided by the product of the denominators. Where possible, divide numerator and denominator by common factors.

Multiply the following:

1. $\frac{3}{16}\times\frac{8}{15}$ 2. $\frac{3a}{b}\times\frac{b^2}{c}$ 3. $\frac{x^2-1}{a-b}\times\frac{(a-b)^2}{x-1}$

C. **The quotient of one fraction divided by a second** is the product of the first multiplied by the reciprocal of the second.

Divide the following:

1. $\frac{11}{20} \div \frac{33}{25}$ 2. $\frac{3a^2b}{8b^3} \div 6ab$ 3. $\frac{a^2-ab}{16ab^2} \div \frac{a-b}{2a}$

SQUARE ROOTS

224. The rule for square root is based on the algebraic formula $(a+b)^2 = a^2+2ab+b^2$. Notice that $a^2+2ab+b^2$ may be written $a^2+b(2a+b)$. The method is illustrated below:

Illustration 1. Find $\sqrt{694.563}$

$$
\begin{array}{rrl}
 & 694.56,3 & \underline{|26.3} \\
 & 4 & \\
2(20)=40 & \overline{294} & \\
40+6=46 & 276 & \\
\hline
2(260)=520 & 1856 & \\
520+3=523 & 1569 & \\
\hline
 & 287 &
\end{array}
$$

In the illustration above, notice:

(1) The number was divided into periods of two figures each, counting to the left and to the right from the decimal point.

(2) The largest square under 6 is 4. The 4 was subtracted from 6 and the next period annexed. This gave a remainder of 294. The square root of 4, or 2, was written as the first figure in the root.

(3) A zero was placed after the 2, making 20. The 20 was doubled, making 40. The 40 is used as a trial divisor for the remainder 294. The next figure of the root is either 6 or 7. The 6 is added to the 40, making 46. The 46 is multiplied by 6, giving 276. The 276 is subtracted from 294, leaving 18. The next period is annexed, giving 1856.

(4) The process above is repeated at each step of the work; thus, a zero is placed after 26 and the result doubled, giving the 520. The work is then continued as above.

In general we may say: Annex a zero to the part of the root already found and double the result. To this result add the next figure of the root. Multiply the result by the last figure of the root found.

Show that this statement may be regarded as a translation of $b(2a+b)$ in the formula $(a+b)^2=a^2+b(2a+b)$.

Find the square root of the following:

1. 1369	4. 106276	7. 6
2. 3744	5. 3	8. 15
3. 2304	6. 5	9. 7

Because of the frequent occurrence of the square roots of 2 and 3 in geometry work, the application of the following law should be noted:

225. The square root of a product is the product of the square roots of the factors.

Illustration 2. $36=4\times 9 \quad \therefore \sqrt{36}=\sqrt{4}\times\sqrt{9}$

This law is used most conveniently for inexact square roots when one factor is a perfect square.

Illustration 3. $18=9\times 2 \quad \therefore \sqrt{18}=\sqrt{9}\times\sqrt{2}=3\sqrt{2}$
Notice that $\sqrt{2}$ occurs when the side of a square and a diagonal of the square are used in the same exercise.

Illustration 4. $12=4\times 3 \therefore \quad \sqrt{12}=\sqrt{4}\times\sqrt{3}=2\sqrt{3}$
Notice that $\sqrt{3}$ occurs when the side of an equilateral triangle and its altitude occur in the same exercise.

Illustration 5. $20=4\times 5 \quad \therefore \sqrt{20}=\sqrt{4}\times\sqrt{5}=2\sqrt{5}$
The $\sqrt{5}$ occurs in connection with the regular decagon and pentagon.

Find the value of the following correct to three decimal places. Apply the law given above.

1. $\sqrt{8}$	4. $\sqrt{108}$	7. $\sqrt{128}$	10. $\sqrt{150}$	13. $\sqrt{54}$
2. $\sqrt{18}$	5. $\sqrt{32}$	8. $\sqrt{75}$	11. $\sqrt{125}$	14. $\sqrt{45}$
3. $\sqrt{27}$	6. $\sqrt{80}$	9. $\sqrt{320}$	12. $\sqrt{98}$	15. $\sqrt{180}$

16. $\sqrt{20}$	18. $\sqrt{48}$	20. $\sqrt{50}$	22. $\sqrt{288}$	24. $\sqrt{243}$
17. $\sqrt{96}$	19. $\sqrt{72}$	21. $\sqrt{300}$	23. $\sqrt{162}$	25. $\sqrt{242}$

226. The square root of a fraction.

A. If the denominator is a perfect square: Find the square root of the numerator and of the denominator separately and divide the first result by the second.

Illustration 6. $\sqrt{\frac{144}{625}}=\frac{12}{25}$

Illustration 7. $\sqrt{\frac{3}{4}}=\frac{1}{2}\sqrt{3}$

B. If the denominator is not a perfect square, two methods are suggested:

(1) The fraction may be reduced to a decimal and the square root of the result found.

(2) Numerator and denominator may be multiplied by some number that will make the denominator a perfect square and method A above used.

Illustration 8. To find $\sqrt{\frac{1}{3}}$ either

1. Find square root of $.333333+$; or
2. Write $\frac{1}{3}=\frac{3}{9}$ and use $\frac{1}{3}\sqrt{3}$.

Find the value of the following correct to three decimal places:

1. $\sqrt{\frac{7}{25}}$	5. $\sqrt{\frac{2}{5}}$	9. $\sqrt{\frac{4}{5}}$	13. $\sqrt{\frac{5}{6}}$
2. $\sqrt{\frac{11}{16}}$	6. $\sqrt{\frac{3}{7}}$	10. $\sqrt{\frac{9}{11}}$	14. $\sqrt{\frac{9}{20}}$
3. $\sqrt{\frac{2}{9}}$	7. $\sqrt{\frac{5}{8}}$	11. $\sqrt{\frac{7}{18}}$	15. $\sqrt{\frac{3}{8}}$
4. $\sqrt{\frac{1}{5}}$	8. $\sqrt{\frac{5}{12}}$	12. $\sqrt{\frac{2}{3}}$	16. $\sqrt{\frac{7}{27}}$

17. $\sqrt{\frac{1}{\pi}}$ to four decimal places.

EQUATIONS

227. The method of solving linear equations is illustrated below:

Illustration 1. Solve for x: $\frac{3x+1}{4}-2=4-\frac{x+1}{3}$

Multiply both sides by the L. C. M. of the denominators	$3(3x+1)-24=48-4(x+1)$
performing multiplications.	$9x+3-24=48-4x-4$
Combining terms	$9x-21=44-4x$
Add $+21$ and $+4x$ to each side	$13x=65$
Divide both sides by 13	$x=5$

Solve the following equations:

1. $\frac{x-2}{3}=\frac{x-4}{5}$
2. $\frac{2x-7}{7}-\frac{3+2x}{4}=2$
3. $\frac{4}{x-5}=\frac{3}{2x+7}$
4. $\frac{3-4x}{8}-\frac{5-2x}{3}=1-\frac{x}{4}$

228. If an equation contains both the first and the second powers of the unknown, two methods of solution are suggested.

A. The equation may be solved by factoring.

Illustration 2. Solve for x: $x^2-x=20$

$$x^2-x-20=0$$
$$(x-5)\ (x+4)=0$$
$$x-5=0 \qquad x+4=0$$
$$x=5 \qquad x=4$$

Notice that to solve an equation by factoring, one member of the equation must be zero.

B. The equation may be solved by completing the square.

Illustration 3. Solve for x: $3x^2-5x=7$ . . . . (1)

Divide both sides by 3 . . . . $x^2-\frac{5x}{3}=\frac{7}{3}$ . . . . (2)

Add the square of $(\frac{1}{2}\cdot\frac{5}{3})$ to each side . . . . $x^2-\frac{5x}{3}+(\frac{1}{2}\cdot\frac{5}{3})^2=\frac{7}{3}+\frac{25}{36}$ . . (3)

Take the square root of each side

of the equation . . . . . $x-\frac{5}{6}=\pm\frac{\sqrt{109}}{36}=\pm\frac{1}{6}\sqrt{109}$. (4)

$$x-\frac{5}{6}=\pm\frac{10.44}{6} \quad . \quad . \quad . \quad . \quad (5)$$

$$x=\frac{5}{6}+\frac{10.44}{6} \qquad\qquad x=\frac{5}{6}-\frac{10.44}{6}$$

$$=\frac{15.44}{6} \qquad\qquad =-\frac{5.44}{6}$$

$$=2.57+ \qquad\qquad =-.90+$$

Notice in (3), $(\frac{1}{2}\cdot\frac{5}{3})^2$ is added to the left side to make the left side a perfect square. It is added to the right side to preserve the balance of the equation. $(\frac{1}{2}\cdot\frac{5}{3})^2$, or $\frac{25}{36}$, is obtained by squaring half the coefficient of x. Notice that in step (2) the equation is divided by 3 to make the first term x^2, which is a perfect square.

Solve the following equations:

1. $x^2+3x=18$
2. $2x^2-x=15$
3. $3x^2-11x=2$
4. $2x^2+5x=17$

229. To solve a system of equations consisting of two equations containing two unknowns, eliminate one of the unknowns and solve the resulting equation for the other.

A. When both equations are of the first degree, eliminate by addition or subtraction.

Illustration 4. Solve for x and y: $\begin{cases}5x-4y=6.5\\7x+5y=38.25\end{cases}$

$5x-4y=6.5$ . . . . (1)

$7x+5y=38.25$ . . . . (2)

$35x-28y=45.5$ . . . . $(1)\times 7$

$35x+25y=191.25$ . . . . $(2)\times 5$

$-53y=-145.75$ . . . . . Subtract the third

$y=2.75$ equation from the second

Notice that x may be found by multiplying equation (1) by 5 and equation (2) by 4 and adding the results or by substituting 2.75 for y in either equation (1) or (2) and solving the result for x.

B. When one of the equations is of the first degree and one of the second, solve the first-degree equation for one of the unknowns in terms of the other unknown and substitute in the other equation.

Illustration 5. Solve for x and y: $\begin{cases} x+y=8 \quad . \quad . \quad . \quad . \quad (1) \\ x^2+y^2=34 \quad . \quad . \quad . \quad . \quad (2) \end{cases}$

Solve (1) for x, $\qquad x=8-y \quad . \quad . \quad . \quad (3)$

Substitute $8-y$ for x in (2) $\qquad (8-y)^2+y^2=34$

$$64-16y+2y^2=34$$
$$2y^2-16y+30=0$$
$$y^2-8y+15=0$$
$$(y-5)\,(y-3)=0$$
$$y=5 \text{ and } y=3$$

To find x, substitute the values of y in (3):

$y=5$	$y=3$
$x=8-y$	$x=8-5$
$=8-5$	$=8-3$
$=3$	$=5$

The solutions are $\begin{cases} x=3 \\ y=5 \end{cases}$ $\qquad \begin{cases} x=5 \\ y=3 \end{cases}$

Solve the following systems for x and y:

1. $2x+3y=16$, $\quad 5x-2y=21$
2. $5x-2y=14$, $\quad xy=20$
3. $2x^2+y^2=57$, $\quad x-y=1$

TABLES

230. TABLE OF SQUARE ROOTS

TABLE OF SQUARE ROOTS OF NUMBERS FROM 0 TO '99

	0	1	2	3	4	5	6	7	8	9
0	0.000	1.000	1.414	1.732	2.000	2.236	2.449	2.646	2.828	3.000
1	3.162	3.317	3.464	3.606	3.742	3.873	4.000	4.123	4.243	4.359
2	4.472	4.583	4.690	4.796	4.899	5.000	5.099	5.196	5.292	5.385
3	5.477	5.568	5.657	5.745	5.831	5.916	6.000	6.083	6.164	6.245
4	6.325	6.403	6.481	6.557	6.633	6.708	6.782	6.856	6.928	7.000
5	7.071	7.141	7.211	7.280	7.348	7.416	7.483	7.550	7.616	7.681
6	7.746	7.810	7.874	7.937	8.000	8.062	8.124	8.185	8.246	8.307
7	8.367	8.426	8.485	8.544	8.602	8.660	8.718	8.775	8.832	8.888
8	8.944	9.000	9.055	9.110	9.165	9.220	9.274	9.327	9.381	9.434
9	9.487	9.539	9.592	9.644	9.695	9.747	9.798	9.849	9.894	9.950

CUBE ROOT

231. The rule for cube root is based on the algebraic formula

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$

Notice that

$$a^3+3a^2b+3ab^2+b^3$$

may be written

$$a^3+b(3a^2+3ab+b^2)$$

The method is illustrated below:

Illustration: Find $\sqrt[3]{41523.629}$

$$
\begin{array}{lr|l}
 & & 41{,}523.629 \ \underline{|34.6+} \\
 & & 27 \\ \hline
3\times(30)^2=3\times900= & 2700 & 14523 \\
3\times30\times4 = & 360 & \\
4^2 = & 16 & \\ \hline
 & 3076 & 12304 \\ \hline
 & & 2219629 \\
3\times(340)^2=3\times115600= & & \\
 & 346800 & \\
3\times340\times6 = & 6120 & \\
6^2= & 36 & \\ \hline
 & 352956 & 2117736 \\ \hline
 & & 101893
\end{array}
$$

232. TABLE OF CUBE ROOTS

0	0	1	2	3	4	5	6	7	8	9
0	0	1	1.260	1.442	1.587	1.710	1.817	1.913	2.000	2.080
1	2.154	2.224	2.289	2.351	2.410	2.466	2.520	2.571	2.621	2.668
2	2.714	2.759	2.802	2.844	2.885	2.924	2.963	3.000	3.037	3.072
3	3.107	3.141	3.175	3.208	3.240	3.271	3.302	3.332	3.362	3.391
4	3.420	3.448	3.476	3.503	3.530	3.557	3.583	3.609	3.634	3.659
5	3.684	3.708	3.733	3.756	3.780	3.803	3.826	3.849	3.871	3.893
6	3.915	3.937	3.958	3.979	4.000	4.021	4.041	4.062	4.082	4.102
7	4.121	4.141	4.160	4.179	4.198	4.217	4.236	4.254	4.273	4.291
8	4.309	4.327	4.345	4.362	4.380	4.397	4.414	4.431	4.448	4.465
9	4.481	4.498	4.514	4.531	4.547	4.563	4.579	4.595	4.610	4.626

233. TABLE OF SINES, COSINES, AND TANGENTS

Deg.	Sine	Cosine	Tangent	Deg.	Sine	Cosine	Tangent
1	.017	.999	.017	46	.719	.695	1.036
2	.035	.999	.035	47	.731	.682	1.072
3	.052	.999	.052	48	.743	.669	1.111
4	.070	.998	.070	49	.755	.656	1.150
5	.087	.996	.087	50	.766	.643	1.192
6	.105	.995	.105	51	.777	.629	1.235
7	.122	.993	.123	52	.788	.616	1.280
8	.139	.990	.141	53	.799	.602	1.327
9	.156	.988	.158	54	.809	.588	1.376
10	.174	.985	.176	55	.819	.574	1.428
11	.191	.982	.194	56	.829	.559	1.483
12	.208	.978	.213	57	.839	.545	1.540
13	.225	.974	.231	58	.848	.530	1.600
14	.242	.970	.249	59	.857	.515	1.664
15	.259	.966	.268	60	.866	.500	1.732
16	.276	.961	.287	61	.875	.485	1.804
17	.292	.956	.306	62	.883	.469	1.881
18	.309	.951	.325	63	.891	.454	1.963
19	.326	.946	.344	64	.899	.438	2.050
20	.342	.940	.364	65	.906	.423	2.144
21	.358	.934	.384	66	.914	.407	2.246
22	.375	.927	.404	67	.921	.391	2.356
23	.391	.921	.424	68	.927	.375	2.475
24	.407	.914	.445	69	.934	.358	2.605
25	.423	.906	.466	70	.940	.342	2.747
26	.438	.899	.488	71	.946	.326	2.904
27	.454	.891	.510	72	.951	.309	3.078
28	.469	.883	.532	73	.956	.292	3.271
29	.485	.875	.554	74	.961	.276	3.487
30	.500	.866	.577	75	.966	.259	3.732
31	.515	.857	.601	76	.970	.242	4.011
32	.530	.848	.625	77	.974	.225	4.331
33	.545	.839	.649	78	.978	.208	4.705
34	.559	.829	.675	79	.982	.191	5.145
35	.574	.819	.700	80	.985	.174	5.671
36	.588	.809	.727	81	.988	.156	6.314
37	.602	.799	.754	82	.990	.139	7.115
38	.616	.788	.781	83	.993	.122	8.144
39	.629	.777	.810	84	.995	.105	9.514
40	.643	.766	.839	85	.996	.087	11.430
41	.656	.755	.869	86	.998	.070	14.301
42	.669	.743	.900	87	.999	.052	19.081
43	.682	.731	.933	88	.999	.035	28.636
44	.695	.719	.966	89	.999	.017	57.290
45	.707	.707	1.000				

UNITS OF MEASURE

234. Units of Length

ENGLISH

12 inches (in.) = 1 foot (ft.)
3 feet = 1 yard (yd.)
$5\frac{1}{2}$ yards = 1 rod (rd.)
320 rods or 5280 ft. = 1 mile (mi.)

METRIC

10 centimeters (cm.) = 1 decimeter (dm.)
10 decimeters = 1 meter (m.)
1000 meters = 1 kilometer (km.)
1 meter = 39.37 in.
1 kilometer = .62 of a mile
1 foot = 30.48 centimeters
1 mile = 1.6093 kilometers

235. Units of Surface

ENGLISH

144 square inches (sq. in.) = 1 square foot (sq. ft.)
9 square feet = 1 square yard (sq. yd.)
$30\frac{1}{4}$ square yards = 1 square rod (sq. rd.)
160 square rods = 1 acre (A.)
4840 square yards = 1 acre (A.)
640 acres = 1 square mile (sq. mi.)

METRIC

100 square centimeters = 1 square decimeter
100 square decimeters = 1 square meter

236. Units of Volume

ENGLISH	METRIC
1728 cu. in. = 1 cu. ft.	1000 cu. mm. = 1 cu. cm.
27 cu. ft. = 1 cu. yd.	1000 cu. cm. = 1 cu. dm.
128 cu. ft. = 1 cord (of wood)	1000 cu. dm. = 1 cu. m.

237. Units of Capacity

ENGLISH

Dry Measure

2 pints (pt.) = 1 quart (qt.)
8 quarts = 1 peck (pk.)
4 pecks = 1 bushel (bu.)
1 bushel = 2150.42 cu. in.

Liquid Measure

4 gills (gi.) = 1 pint (pt.)
2 pints = 1 quart (qt.)
4 quarts = 1 gallon (gal.)
1 gallon = 231 cu. in.
1 cu. ft. of water weighs 62.4 lbs.

METRIC

1 liter = 1 cu. dm.
= 1000 cu. cm.
= 61.02 cu. in.
= 1.0567 liquid quarts
1 quart = 9463 liters
= 946.3 cu. cm.

238. Units of Weight

ENGLISH

Avoirdupois Weight

16 ounces (oz.) = 1 pound (lb.)
100 pounds = 1 hundredweight (cwt.)
2000 pounds = 1 ton (T.)
2240 pounds = 1 long ton

Troy Weight

24 grains (gr.) = 1 pennyweight (pwt. or dwt.)
20 pwt. = 1 Troy ounce
12 Troy ounces = 1 Troy pound
7000 grains = 1 avoirdupois lb.
5760 grains = 1 Troy lb.

METRIC

1 gram = weight of 1 cu. cm. of water at 39.2° F.
1000 grams = 1 kilogram (kg.)
1 kg. = 2.2046 lbs.
1 lb. = .45359 kg.
= 453.59 g.

OUTLINE OF PLANE GEOMETRY

INTRODUCTORY AND GENERAL

When a ray starts from a point in a straight line and forms two congruent angles, the angles are called **right angles,** and the ray is said to be **perpendicular** to the line.

Two angles are called **complementary** if their sum is an angle of 90°. Each of two complementary angles is called the **complement** of the other.

Two angles are said to be **supplementary** if their sum is an angle of 180°. Each of two supplementary angles is called the **supplement** of the other.

As. 1. Two different straight lines can intersect in only one point, or two intersecting straight lines locate a point.

As. 2. A segment can have only one mid-point.

As. 3. Only one segment can be drawn between two points, or a segment is located definitely if its extremities are given.

As. 4. Only one ray can be drawn having a given origin and passing through a second given point.

As. 5. Only one ray can be drawn bisecting a given angle.

As. 6. Only one straight line can pass through two given points.

As. 7. Only one perpendicular can be drawn to a line from a given point in the line.

As. 8. Only one perpendicular can be drawn to a line from a given point not in the line.

As. 9. Circles with equal radii are congruent.

As. 10. Congruent circles have equal radii.

As. 11. All straight angles are equal.

As. 12. All right angles are equal.

As. 13. Complements of equal angles are equal.

As. 14. Supplements of equal angles are equal.

As. 15. Vertical angles are equal.

As. 16. If a ray starts from a point in a straight line, the sum of the two adjacent angles formed on one side of the line is 180°, or a straight angle.

As. 17. The sum of the adjacent angles on one side of a straight line formed by any number of rays having a common origin on the line is 180°, or a straight angle.

As. 18. The sum of the adjacent angles formed by a number of rays from the same origin is 360°, or a perigon.

As. 19. If two supplementary angles are adjacent, their exterior sides are collinear.

As. 20. If equal segments (or angles) are added to equal segments (or angles), the results are equal segments (or angles).

As. 21. If equal segments (or angles) are subtracted from equal segments (or angles), the results are equal segments (or angles).

As. 22. If equal segments (or angles) are multiplied by the same number, the results are equal segments (or angles).

As. 23. If equal segments (or angles) are divided by the same number, the results are equal segments (or angles).

As. 24. Segments (or angles) that are equal to the same segment (or angle) are equal.

As. 25. Equal segments (or angles) may be substituted for equal segments (or angles).

CONGRUENT TRIANGLES

Any two figures that can be made to coincide are called **congruent figures.**

As. 26. Any figure can be moved about in space without changing either its size or its shape.

As. 27. Figures congruent to the same figure are congruent to each other.

Theorem 1. If two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle, the triangles are congruent in all corresponding parts and are called congruent triangles.

Theorem 2. If two angles and the included side of one triangle are equal to two angles and the included side of another triangle, the triangles are congruent in all corresponding parts and are called congruent triangles.

Theorem 4. If three sides of one triangle are equal to three sides of another triangle, the triangles are congruent.

Theorem 22. Two right triangles are congruent if the hypotenuse and an acute angle of one are equal to the hypotenuse and an acute angle of the other.

Theorem 23. Two right triangles are congruent if the hypotenuse and a side of one are equal to the hypotenuse and a side of the other.

Cor. If a perpendicular is erected to a straight line, equal segments drawn from the same point in the perpendicular cut off equal distances from the foot of the perpendicular.

Theorem 5. If a perpendicular be erected to a straight line, oblique segments drawn from the same point in the perpendicular cutting the straight line at equal distances from the foot of the perpendicular are equal.

PARALLELS AND ANGLES

As. 30. Only one line can be drawn through a given point parallel to a given line.

Theorem 9. If two straight lines in the same plane are cut by a third straight line so that the alternate interior angles are equal, the two straight lines are parallel.

Theorem 10. If two straight lines in the same plane are cut by a third straight line so that one pair of corresponding angles are equal, the two straight lines are parallel.

Theorem 11. If two straight lines in the same plane are cut by a third straight line so that the interior angles on the same side of the transversal are supplements, the two straight lines are parallel.

Theorem 12. Two straight lines in the same plane perpendicular to the same straight line are parallel.

Theorem 13. Two lines parallel to a third line are parallel to each other.

Theorem 14. If two parallel lines are cut by a third straight line, the alternate interior angles are equal.

Theorem 15. If two parallel lines are cut by a third straight line, the corresponding angles are equal.

Theorem 16. If two parallel lines are cut by a third straight line, the interior angles on the same side of the transversal are supplements of each other.

Theorem 17. A line which is perpendicular to one of two parallels is perpendicular to the other.

ANGLE SUMS

Theorem 18. The sum of the interior angles of a triangle is two right angles.

Cor. I. Each angle of an equilateral triangle is 60°.

Cor. II. If two angles of one triangle are equal respectively to two angles of a second triangle, the third angles are equal.

Cor. III. The acute angles of a right triangle are complements of each other.

Theorem 19. The exterior angle of a triangle is equal to the sum of the two non-adjacent interior angles.

Theorem 20. The sum of the interior angles of a polygon of n sides is $2(n-2)$ right angles.

THEOREM 21. The sum of the exterior angles of a polygon of n sides is four right angles.

ISOSCELES TRIANGLES

A triangle that has at least two sides equal is called an **isosceles triangle.**

THEOREM 3. The angles opposite the equal sides of an isosceles triangle are equal.

COR. An equilateral triangle has all of its angles equal; that is, it is equiangular.

THEOREM 24. If two angles of a triangle are equal, the triangle is isosceles.

THEOREM 6. The bisector of the vertex angle of an isosceles triangle is the perpendicular bisector of the base.

THEOREM 7. The segment which joins the vertex of an isosceles triangle with the mid-point of the base bisects the vertex angle and is perpendicular to the base

THEOREM 25. A segment from the vertex of an isosceles triangle perpendicular to the base bisects the base and the vertex angle.

THEOREM 26. The bisector of the vertex angle of an isosceles triangle is an axis of symmetry of the triangle.

SYMMETRY

A figure is said to be **symmetric with respect to a line as an axis** if one part coincides with the remainder when it is folded on that line as an axis. Two figures are said to be **symmetric with respect to a line as an axis** if one figure coincides with the other when the plane in which it lies is folded on that line as an axis.

Such a figure or such figures are said to have **axial symmetry.**

A figure is said to be **symmetric with respect to a point as a center** if one part of the figure coincides with the remainder when it is rotated through an angle of 180° about the point as a center. Two figures are said to be **symmetric with respect to a point as a center** if one figure coincides with the other when it is rotated through an angle of 180° about the point as a center.

Such a figure or such figures are said to have **central symmetry.**

THEOREM 27. Two polygons are symmetric with respect to an axis if the vertices of one are symmetric to the corresponding vertices of the other.

THEOREM 28. Two polygons are symmetric with respect to a center if the vertices of one are symmetric to the corresponding vertices of the other.

THEOREM 29. Any figure that has two axes of symmetry at right angles to each other has the intersection of the axes as a center of symmetry.

QUADRILATERALS

A quadrilateral with each side parallel to its opposite is called a **parallelogram.**

The perpendicular distance between the bases of a parallelogram is called the **altitude** of the parallelogram.

A quadrilateral with but one pair of parallel sides is called a **trapezoid.**

If a trapezoid has its non-parallel sides equal, it is called an **isosceles trapezoid.**

A parallelogram with one right angle is called a **rectangle.**

A parallelogram with two consecutive sides equal is called a **rhombus.**

A rectangle with two consecutive sides equal is called a **square.**

The segment joining the mid-points of two opposite sides of a quadrilateral is called a **median** of the quadrilateral.

THEOREM 30. Each diagonal of a parallelogram divides it into two congruent triangles.

THEOREM 31. The opposite sides of a parallelogram are equal.

THEOREM 32. The opposite angles of a parallelogram are equal.

THEOREM 33. The diagonals of a parallelogram bisect each other.

THEOREM 34. The intersection of the diagonals of a parallelogram is the center of symmetry of the parallelogram.

THEOREM 35. Two parallelograms are congruent if two sides and the included angle of one are equal to two sides and the included angle of the other.

THEOREM 36. If a quadrilateral has one side equal and parallel to its opposite, it is a parallelogram.

THEOREM 37. If a quadrilateral has each side equal to its opposite, it is a parallelogram.

THEOREM 38. If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.

THEOREM 39. Segments of parallels intercepted between parallel lines are equal.

THEOREM 40. Segments of perpendiculars intercepted between parallel lines are equal.

THEOREM 41. The diagonals of a kite are perpendicular to each other, and the one which is an axis of symmetry bisects the other.

THEOREM 42. All the angles of a rectangle are right angles.

THEOREM 43. All the sides of a rhombus are equal.

THEOREM 44. The diagonals of a rhombus are perpendicular to each other and bisect the angles through which they pass.

PARALLELS AND TRANSVERSALS

THEOREM 45. If a series of parallels cuts off equal segments on one transversal, it cuts off equal segments on all transversals.

THEOREM 46. A segment parallel to the base of a triangle and bisecting one side is equal to half the base.

THEOREM 47. A segment parallel to the base of a triangle and bisecting one side bisects the other side also.

THEOREM 48. A segment bisecting two sides of a triangle is parallel to the third side.

The segment joining any vertex of a triangle with the mid-point of the opposite side is called a **median** of the triangle.

THEOREM 50. The median from the vertex of the right angle of a right triangle to the hypotenuse is one-half the hypotenuse.

THEOREM 51. The segment joining the mid-points of the non-parallel sides of a trapezoid is parallel to the bases.

THEOREM 52. The segment joining the mid-points of the non-parallel sides of a trapezoid is equal to one-half the sum of the bases.

INEQUALITIES

As. 28. If one angle or segment is greater than a second and the second is equal to or greater than a third, then the first is greater than the third.

As. 29. The whole is greater than any of its parts.

As. 31. If equal segments (or angles) are added to unequal segments (or angles), the resulting segments (or angles) are unequal in the same order.

As. 32. If equal segments (or angles) are subtracted from unequal segments (or angles), the resulting segments (or angles) are unequal in the same order.

As. 33. If unequal segments (or angles) are added to unequal segments (or angles), the greater to the greater and the lesser to the lesser, the resulting segments (or angles) are unequal in the same order.

As. 34. If unequal segments (or angles) are subtracted from equal segments (or angles), the resulting segments (or angles) are unequal in the opposite order.

As. 35. If unequal segments (or angles) are multiplied by the same number, the resulting segments (or angles) are unequal in the same order.

As. 36. If unequal segments (or angles) are divided by the same number, the resulting segments (or angles) are unequal in the same order.

As. 37. The sum of two sides of a triangle is greater than the third.

As. 38. The difference between two sides of a triangle is less than the third side.

THEOREM 8. An exterior angle of a triangle is greater than either of the non-adjacent interior angles.

THEOREM 53. If from a point within a triangle segments are drawn to the extremities of one side, their sum is less than the sum of the other two sides of the triangle.

THEOREM 54. If one angle of a triangle is greater than a second, the side opposite the first angle is greater than the side opposite the second angle.

THEOREM 55. If one side of a triangle is greater than a second, the angle opposite the greater side is greater than the angle opposite the lesser side.

THEOREM 56. The perpendicular is the shortest segment from a point to a straight line.

THEOREM 57. If from a point in a perpendicular to a straight line two oblique segments are drawn cutting the straight line at unequal distances from the foot of the perpendicular, the more remote is the greater.

THEOREM 58. If from a point in a perpendicular to a straight line two unequal oblique segments are drawn, the greater cuts the straight line at the greater distance from the foot of the perpendicular.

THEOREM 59. If two triangles have two sides of one equal to two sides of the other, but the included angle of one greater than the included angle of the other, the third side of the first is greater than the third side of the second.

THEOREM 60. If two triangles have two sides of one equal to two sides of the other, but the third side of one greater than the third side of the other, the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

CIRCLES

A closed curved line every point of which is equally distant from a given point in the same plane is called a **circle.** The given point is called the **center** of the circle.

As. 39. The diameter of a circle is twice its radius.

As. 40. A circle is located definitely if its center and its radius are known.

As. 41. If a line passes through a point within a circle, the line and the circle intersect in two and *only* two points.

As. 42. Every diameter bisects the circle.

As. 43. A circle is symmetric with respect to any diameter as an axis and with respect to its center as a center.

As. 44. Between the same two points on a circle there is one and *only* one minor arc of the circle, provided these points are not the ends of a diameter.

As. 45. A segment joining a point within a circle and the center is shorter than the radius.

As. 46. If a segment that has one end at the center of a circle is shorter than the radius, it lies wholly within the circle.

As. 47. A segment joining a point without a circle and the center is longer than the radius.

As. 48. If a segment that has one end at the center of a circle is longer than the radius, it extends without the circle and cuts the circle but once.

As. 49. In the same circle or in congruent circles equal central angles intercept equal minor arcs.

As. 50. In the same circle or in congruent circles equal minor arcs intercept equal central angles.

As. 54. In the same circle, or in congruent circles, if two central angles are unequal, the minor arc subtended by the greater angle is greater than the minor arc subtended by the lesser angle.

As. 55. In the same circle or in congruent circles, if two minor arcs are unequal, the angle subtended by the greater arc is greater than the angle subtended by the lesser arc.

CIRCLES AND RELATED LINES

A line that touches a circle at one point but does not cut it is called a **tangent** to the circle. This definition is the **fundamental test for tangents.**

The point at which the tangent touches the circle is called the **point of contact** or the **point of tangency** of the tangent.

THEOREM 61. In the same circle or in congruent circles

A. Equal chords intercept equal central angles.

B. Equal central angles intercept equal chords.

THEOREM 62. In the same circle or in congruent circles

A. Equal chords have equal minor arcs.

B. Equal minor arcs have equal chords.

THEOREM 63. A radius perpendicular to a chord bisects the chord and its arc.

THEOREM 64. The perpendicular bisector of a chord passes through the center of the circle.

THEOREM 65. One and only one circle can be drawn through three non-collinear points.

THEOREM 66. If in the same circle or in congruent circles perpendiculars from the center to two chords are equal, the chords are equal.

THEOREM 67. In the same circle or in congruent circles perpendiculars from the center to two equal chords are equal.

THEOREM 68. A line which is perpendicular to a radius at its outer extremity is a tangent to the circle.

THEOREM 69. A tangent to a circle is perpendicular to the radius drawn to the point of contact.

THEOREM 70. If two tangents meet at a point without a circle, the distances from the intersection to the points of tangency are equal.

THEOREM 71. A perpendicular to a tangent at the point of contact passes through the center of the circle.

MEASUREMENT OF ANGLES

As. 53. The measure of a central angle and its intercepted arc are expressed by the same number, or a central angle is measured by its intercepted arc.

An angle is said to be **inscribed in a circle** if its vertex is on the circle and its sides are chords of the circle.

The arc cut off between the sides of an inscribed angle is called its **intercepted arc.**

THEOREM 77. An inscribed angle is measured by one-half its intercepted arc.

COR. I. Inscribed angles measured by the same or by equal arcs are equal, and, conversely, arcs that measure equal inscribed angles are equal.

COR. II. An angle inscribed in a semicircle is a right angle.

COR. III. Inscribed angles are supplementary if the sum of their intercepted arcs is 360°.

THEOREM 78. An angle formed by two chords intersecting within a circle is measured by one-half the sum of the intercepted arcs.

THEOREM 79. An angle formed by two secants intersecting without a circle is measured by one-half the difference of the intercepted arcs.

THEOREM 80. Parallel chords intercept equal arcs on a circle.

THEOREM 81. An angle formed by a tangent and a chord is measured by one-half its intercepted arc.

THEOREM 82. If a chord and a tangent are parallel, they cut off equal arcs.

THEOREM 83. An angle formed by a secant and a tangent is measured by one-half the difference of the intercepted arcs.

THEOREM 84. An angle formed by two tangents is measured by one-half the difference of the intercepted arcs.

TWO CIRCLES

As. 51. The line of centers of two circles is an axis of symmetry of the two circles.

THEOREM 72. If two circles intersect in one point not on the line of centers, they intersect in two points.

COR. If two circles intersect, the points of intersection are symmetric points.

As. 52. Two circles cannot intersect at more than two points.

THEOREM 73. If any two circles intersect, the line of centers is the perpendicular bisector of the common chord.

THEOREM 74. If two congruent circles intersect, the common chord is an axis of symmetry of the figure.

COR. If two congruent circles intersect, the segment joining the centers and the common chord are perpendicular bisectors of each other.

Two circles are said to be **tangent** if they have but one common point. They may be tangent internally or tangent externally.

THEOREM 75. If two circles meet at a point on their line of centers, the circles are tangent.

COR. I. If the segment joining the centers of two circles is equal to the sum of the radii, the circles are tangent externally.

COR. II. If the segment joining the centers of two circles is equal to the difference between the radii, the circles are tangent internally.

THEOREM 76. If two circles are tangent, the point of contact is on the line of centers.

LOCI

A point which moves so as to fulfill some given requirement is called a **variable point.**

The path of a point which moves so as to fulfill some given requirement is called a **locus.**

A line or group of lines is called a locus if they contain all points which fulfill some given requirement and contain no other points.

THEOREM 85. The bisector of an angle is the locus of points equally distant from the sides of the angle.

THEOREM 86. The perpendicular bisector of a segment is the locus of a point equally distant from the ends of the segment.

COR. If two points are each equally distant from the extremities of a segment, the line passing through these points is the perpendicular bisector of the segment.

THEOREM 87. The perpendicular bisectors of the sides of a triangle are concurrent at a point which is equally distant from the vertices.

A perpendicular from any vertex of a triangle to the opposite side is called an **altitude** of the triangle.

THEOREM 88. The altitudes of a triangle are concurrent.

THEOREM 89. The bisectors of the angles of a triangle are concurrent at a point equally distant from the sides of the triangle.

THEOREM 49. The medians of a triangle are concurrent in a point that is two-thirds the distance from each vertex to the mid-point of the opposite side.

Each triangle has four sets of concurrent lines. The intersection of each set has a special name as shown below.

I. The medians.................... **centroid** or **center of gravity**
II. Perpendicular bisectors of the sides............. **circumcenter**
III. The altitudes.................................. **orthocenter**
IV. Bisectors of the angles............................ **incenter**

RATIOS

As. 56. Multiplying or dividing both terms of a ratio by the same number does not change the value of the ratio.

As. 57. Ratios equal to the same ratio are equal.

As. 58. Equal ratios may be substituted for equal ratios.

THEOREM 90. If four numbers are in proportion, the product of the means is equal to the product of the extremes.

THEOREM 91. If the product of two numbers equals the product of two other numbers, either pair of factors may be made the extremes and the other pair the means of a proportion.

THEOREM 92. If three terms of one proportion are equal respectively to three corresponding terms of another proportion, the fourth terms are equal.

THEOREM 93. If four numbers are in proportion, the first is to the third as the second is to the fourth; that is, they are in proportion by mean alternation.

THEOREM 94. If four numbers are in proportion, the fourth is to the second as the third is to the first; that is, they are in proportion by extreme alternation.

THEOREM 95. If four numbers are in proportion, the second is to the first as the fourth is to the third; that is, they are in proportion by inversion.

THEOREM 96. If four numbers are in proportion, the first plus the second is to the second as the third plus the fourth is to the fourth; that is, they are in proportion by addition. This is sometimes called proportion by composition.

THEOREM 97. If four numbers are in proportion, the first minus the second is to the second as the third minus the fourth is to the fourth; that is, they are in proportion by subtraction. This is sometimes called proportion by division.

THEOREM 124. In a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

THEOREM 98. If three parallels cut two transversals, the segments on one transversal have the same ratio as the corresponding segments on the other transversal.

THEOREM 99. If a line is parallel to the base of a triangle, the ratio of the segments on one side equals the ratio of the corresponding segments on the other side.

COR. If a line is parallel to the base of a triangle, one side is to either of its segments as the other side is to its corresponding segment.

THEOREM 100. If a line divides the sides of a triangle so that one side is to one segment as a second side is to its corresponding segment, the line is parallel to the third side of the triangle.

COR. If a line divides the sides of a triangle so that the ratio of the segments on one side is equal to the ratio of the segments on the other, the line is parallel to the third side of the triangle.

THEOREM 101. If two triangles have the angles of one respectively equal to the angles of the other, the corresponding sides have equal ratios.

THEOREM 102. Two mutually equiangular triangles are similar.

If the product of two segments equals the square of a third segment, the last segment is called a **mean proportional** between the other two. In $b^2 = cm$, b is a mean proportional between c and m.

Our **fundamental methods for proving ratios equal** are:

1. By parallels and transversals.
2. By similar triangles.

Before either of these methods can be applied it is often necessary to find a third ratio to which each of the given ratios can be proved equal.

THEOREM 103. If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.

THEOREM 104. If two secants intersect without a circle, the product of one secant and its external segment is equal to the product of the other secant and its external segment.

THEOREM 105. If a secant and a tangent meet without a circle, the tangent is a mean proportional between the whole secant and its external segment.

THEOREM 106. The bisector of an angle of a triangle divides the opposite side internally into segments that have the same ratio as the other two sides of the triangle.

THEOREM 107. The bisector of an exterior angle of a triangle divides the opposite side externally into segments that have the same ratio as the other two sides of the triangle.

THEOREM 108. If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse, the perpendicular is a mean proportional between the segments of the hypotenuse.

THEOREM 109. If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse, either leg is a mean proportional between the whole hypotenuse and the segment adjacent to that leg.

THEOREM 110. The sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse.

THEOREM 111. If one side of a square is s, its diagonal is $s\sqrt{2}$. If the diagonal of a square is d, the side is $\frac{1}{2}d\sqrt{2}$.

THEOREM 112. If one side of an equilateral triangle is s, its altitude is $\frac{1}{2}s\sqrt{3}$. If the altitude is a, one side of the equilateral triangle is $\frac{2}{3}a\sqrt{3}$.

SIMILAR FIGURES

Similar polygons are polygons that have

1. The angles of one equal to the corresponding angles of the other, and
2. Corresponding sides proportional.

The **ratio of similitude** of two similar polygons is the ratio of any two corresponding sides.

If the segments which join the vertices of a polygon with a given point are divided in the same ratio from the given point and the points of division joined in the same order as the vertices of the polygon, the polygon so formed and the given polygon are **radially** placed. The point may be without the polygon, or within the polygon. The radial point is called the **center of similitude** of the polygons.

THEOREM 102. Two mutually equiangular triangles are similar.

THEOREM 119. Two triangles are similar if an angle of one is equal to an angle of the other and the ratios of the including sides are equal.

THEOREM 120. Two triangles are similar if the corresponding sides have equal ratios.

THEOREM 121. Two polygons are similar if diagonals drawn from two corresponding vertices divide the polygons into the same number of triangles similar each to each and similarly placed.

THEOREM 122. If two polygons are similar, diagonals drawn from two corresponding vertices divide the polygon into the same number of triangles similar each to each and similarly placed.

THEOREM 123. The ratio of corresponding altitudes of two similar triangles equals the ratio of the bases.

THEOREM 125. The ratio of the perimeters of two similar triangles is equal to the ratio of similitude.

THEOREM 126. The areas of two similar triangles have the same ratio as the squares of the bases or the squares of the altitudes.

THEOREM 127. The areas of two similar polygons have the same ratio as the squares of two corresponding sides.

REGULAR POLYGONS

A polygon of 3 sides is called a **triangle.**

A polygon of 4 sides is called a **quadrilateral.**

A polygon of 5 sides is called a **pentagon.**

A polygon of 6 sides is called a **hexagon.**

A polygon of 7 sides is called a **heptagon.**

A polygon of 8 sides is called an **octagon.**

A polygon of 10 sides is called a **decagon.**

A polygon of 12 sides is called a **duodecagon.**

A polygon of 15 sides is called a **pentadecagon.**

A polygon with all of its sides and all of its angles equal is a **regular** polygon.

A polygon is said to be **inscribed** in a circle if its vertices are on the circle and its sides are chords of the circle. In this case the circle is said to be **circumscribed** about the polygon. A polygon is said to be **circumscribed** about a circle if its sides are tangent to the circle. In this case the circle is said to be **inscribed** in the polygon.

THEOREM 128. If a circle is divided into n equal arcs, the chords joining the points of division form a regular polygon.

THEOREM 129. If a circle is divided into n equal arcs, the tangents drawn to the points of division form a regular polygon.

To construct regular 4-, 8-, or 16-sided polygons, construct two perpendicular diameters.

To construct regular 3-, 6-, or 12-sided polygons, construct a central angle of 60° by means of an equilateral triangle.

To construct regular 5-, 10-, or 15-sided polygons, divide the radius of the circle into extreme and mean ratio.

THEOREM 130. A circle can be circumscribed about any regular polygon.

COR. The radius of the circumscribed circle of a regular polygon bisects the angle through whose vertex it passes.

THEOREM 131. A circle can be inscribed in any regular polygon.

The center of the circumscribed and of the inscribed circle of a regular polygon is called the **center** of the polygon.

The radius of the circumscribed circle of a regular polygon is called the **radius** of the polygon.

The radius of the inscribed circle of a regular polygon is called the **apothem** of the polygon.

By the **central angle** of a regular polygon is meant the angle between two consecutive radii.

COR. I. The central angle of a regular polygon of n sides is $^1/_n$ of $360°$.

COR. II. The radius of a regular polygon bisects the angle between two consecutive apothems, and the apothem bisects the angle between two consecutive radii.

COR. III. The radius of a regular polygon bisects the arc between the points of contact of the inscribed circle.

THEOREM 132. Each angle of a regular polygon of n sides is $\frac{2n-4}{n}$ rt. $\measuredangle$.

THEOREM 133. The area of a regular polygon is one-half the product of the perimeter and the apothem.

THEOREM 134. Two regular polygons of the same number of sides are similar.

THEOREM 135. If two regular polygons have the same number of sides, the ratio of the perimeters is equal to the ratio of the radii or of the apothems.

COR. The ratio of the perimeter to the diameter of the inscribed or of the circumscribed circle is the same for all regular polygons of the same number of sides.

THEOREM 136. If two regular polygons have the same number of sides, the ratio of the areas is equal to the ratio of the squares of the radii or of the apothems.

MEASUREMENT AND EQUIVALENCE

To measure a segment is to find the number of times that it contains another segment which is taken as a unit.

The number found is called the **measure number**, the **measure**, or the **length** of the segment.

Two segments are said to be **commensurable** if they can be measured exactly by a common unit of measure.

Two segments are said to be **incommensurable** if there is no common unit that will measure each exactly.

An **irrational number** is a number that cannot be expressed as an integer or as the quotient of two integers.

To **measure the surface inclosed by the sides of a polygon** is to find how many times it contains another surface chosen as a unit of measure.

The **area of a polygon** is the measure number of the surface of the polygon.

Two polygons that cover the same extent of surface are called **equivalent polygons.** The symbol $(=)$ is used for equivalence.

As. 59. If equivalent polygons are added to equivalent polygons, the results are equivalent polygons.

As. 60. If equivalent polygons are subtracted from equivalent polygons, the results are equivalent polygons.

As. 61. If equivalent polygons are divided into the same number of equivalent polygons, each part of one is equivalent to any part of the other.

As. 62. Polygons equivalent to the same polygon or to equivalent polygons are equivalent.

Any two polygons are equivalent if they are sums, differences, or equal parts of equivalent polygons.

As. 63. The number of units of area in a rectangle is equal to the product of the number of units of length in the base and altitude.

THEOREM 113. The area of a parallelogram is the product of the base and altitude.

THEOREM 114. The area of a triangle is one-half the product of the base and altitude.

THEOREM 115. The area of a trapezoid is equal to one-half the product of the altitude and the sum of the bases.

THEOREM 116. Two parallelograms or two triangles are equivalent if

1. They have equal bases and are between the same parallels.
2. $a=a'$ and $b=b'$.
3. $ab=a'b'$.

THEOREM 117. If a triangle and a parallelogram have equal bases and equal altitudes, the triangle is equivalent to half the parallelogram.

THEOREM 118. The square constructed on the hypotenuse of a right triangle is equivalent to the sum of the squares constructed on the other two sides.

The **length or the circumference of a circle** is defined as the limit of the perimeters of a series of polygons inscribed in or circumscribed about a circle as the number of sides is increased indefinitely.

As. 64. The limit of the perimeters of a series of regular polygons inscribed in or circumscribed about the same circle as the number of sides is increased indefinitely is the same.

This limit does not depend upon the number of sides of the initial polygon nor upon the method of increasing the number of sides. This limit is πd.

THEOREM 137. The circumference of a circle of diameter d is πd.

The **area of a circle** is defined as the limit of the areas of a series of inscribed or circumscribed regular polygons as the number of sides is increased indefinitely.

As. 65. The limit of the areas of a series of regular polygons inscribed in or circumscribed about the same circle as the number of sides is increased indefinitely is the same. This limit is one-half the product of the radius of the circle and its circumference.

THEOREM 138. The area of a circle is one-half the product of its radius and circumference.

A **sector** of a circle is a figure bounded by two radii and the subtended arc.

As. 66. The area of a sector has the same ratio to the area of a circle of which it is a part as the angle of the sector has to four right angles.

THEOREM 139. The ratio of the circumference to the diameter is the same for all circles.

THEOREM 140. The ratio of the circumferences of two circles equals the ratio of their diameters or of their radii.

THEOREM 141. The ratio of the areas of two circles equals the ratio of the squares of their radii or of their diameters.

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