Solutions
of the
Cambridge Problems

ERRATA.

Page 11, last line *for* $-\frac{1}{a}$, *read* $-\frac{1}{a}\cos a x$.

—— 14, to the end of last line but one add $+\frac{\pi}{6}$.

—— 15, to the end of lines 2, 4, 5, 6 add $+\frac{\pi}{6}$.

In line 7 insert $+\frac{\pi}{6}$ within the bracket.

—— 16, line 5 from bottom, *for* x_{x+1}, *read* u_{x+1}.

—— 24, *for* (velocity), *read* (velocity)^2.

—— 32, last line but two, *for* gives, *read* give.

—— 33, line 4 from bottom, *for* rectangular, *read* polar.

—— 46, line 10 from bottom, *for* $c^{\cos x}$, *read* $k\,c^{\cos x}$.

—— 55, line 6 from bottom, *for* $-\frac{a}{m t}\,.\sin m t$, *read* $-\frac{a}{m}\sin m t + a t$.

—— 86, line 8, *for* $\sqrt{1} + x^2$, *read* $\sqrt{(1 + x^2)}$.

—— 89, line 7, *for* expressions, *read* expression.

line 18 *for* $-2 z_1$, *read* $C z_1$.

—— 91, *for* functions of M, *read* functions of m.

—— 95, line 12, *for* $-\frac{x'}{y'}$, *read* $\therefore \frac{x'}{y'}$.

PREFACE.

It was intended, in the original plan of this Work, to give the Solutions of the Morning and Evening Problems proposed by the Moderators in the form they are usually given at their Examinations. It appeared, however, that the benefits to be derived from solutions of the Problems of past Examinations would be materially increased by putting them (whenever the case admitted of it,) in such forms as might permit a more extensive application of their uses. Accordingly, many of them have been generalised, and explanations added, as each particular case seemed to require: on the other hand, many little 'steps' and simple details have been omitted which are capable of being supplied by any reader acquainted with the first elements of the science; it being considered, that their only effect would be to unnecessarily augment the size, and consequently to increase the price of the Work. For similar reasons, the use of Figures or Diagrams has been avoided, and whenever a reference has been made to them, it has been accompanied by a description sufficiently clear to enable any one, accustomed to the Books of Euclid, to delineate them without difficulty.

With an object similar to that which suggested the present Work, the Author has in progress, *A Collection of Mechanical Problems, with their Solutions,* on a plan similar to that of the valuable collection, given by Mr. Peacock, of Examples of the Differential and Integral Calculus.

The Author takes this opportunity of acknowledging the kindness of the Moderators in permitting the publication of their Problems.

SOLUTIONS

OF THE

CAMBRIDGE PROBLEMS.

JANUARY 1830.

MONDAY MORNING.

1. If n be a whole number, prove that

$$\frac{n^3 + 5n}{6} \text{ is also a whole number.}$$

Since every number is either divisible by 6, or leaves one of the remainders 1, 2, 3, 4, 5;

$\therefore$ every number is of one of the forms

$$6m,\ 6m + 1,\ 6m + 2,\ 6m + 3,\ 6m + 4,\ 6m + 5;$$

$\therefore$ every cube number is of one of the forms

$(6m)^3$ which is of the form $6m$

$(6m + 1)^3$ .... $6m + 1$

$(6m + 2)^3$ .... $6m + 2$

$(6m + 3)^3$ .... $6m + 3$

$(6m + 4)^3$ .... $6m + 4$

$(6m + 5)^3$ .... $6m + 5$

which appears, by expanding each of the cube forms;

$\therefore$ as every number, when divided by 6, leaves the same remainder as its cube divided by 6,

$$\text{if we make } n^3 = 6p + r$$
$$\text{we may also make } n = 6q + r;$$

$\therefore$ Multiplying 2nd equation by 5, and adding it to the first;

$$\begin{aligned} n^3 + 5n &= 6p + r + 5.6q + 5r \\ &= 6p + 5.6q + 6r \\ &= 6s. \end{aligned}$$

Q. E. D.

2. The ratio between the area of an equilateral and equiangular decagon described about a circle, and that of another within the same circle, is equal to

$$\cfrac{8}{7 + \cfrac{1}{4 + \cfrac{1}{4 + \dots}}}$$

Dividing each of the decagons into ten isosceles triangles, we have

Area of circumscribed decagon $= 10 . \tan 18^\circ$

Area of inscribed decagon $= 10 . \sin 18^\circ . \cos 18^\circ$

$$\therefore \text{ ratio of 1st to 2nd } = \frac{\tan 18^\circ}{\sin 18^\circ . \cos 18^\circ}$$
$$= \frac{1}{\cos^2 18^\circ}$$
$$= \frac{8}{5 + \sqrt{5}}$$

Now, the approximate integral value of $\sqrt{5}$ being 2, we obtain successively,

$$\sqrt{5} = 2 + \frac{\sqrt{5}-2}{1} = 2 + \frac{1}{\sqrt{5}+2} = 2 + \frac{1}{\frac{\sqrt{5}+2}{1}}$$

$$\frac{\sqrt{5}+2}{1} = 4 + \frac{\sqrt{5}-2}{1} = 4 + \frac{1}{\sqrt{5}+2} = 4 + \frac{1}{4} + \ldots.$$

$$\therefore \frac{8}{5+\sqrt{5}} = \frac{8}{5+2+\frac{1}{4+\frac{1}{4+\ldots}}}$$

$$= \frac{8}{7+\frac{1}{4+\frac{1}{4+\ldots\ldots}}}$$

Q. E. D.

3. If a and b be the sides of a plane triangle, A and B their opposite angles, then will

$$\text{hyp. log } b - \text{hyp. log } a = \cos 2A - \cos 2B$$

$$+ \frac{1}{2}(\cos 4A - \cos 4B) + \frac{1}{3}(\cos 6A - \cos 6B) + \ldots$$

Since in any plane triangle, $\frac{b}{a} = \frac{\sin B}{\sin A}$;

$$\therefore \frac{b^2}{a^2} = \frac{\sin^2 B}{\sin^2 A}$$

$$= \frac{(e^{B\sqrt{-1}} - e^{-B\sqrt{-1}})^2}{(e^{A\sqrt{-1}} - e^{-A\sqrt{-1}})^2}$$

$$= \frac{1 - e^{2B\sqrt{-1}} - e^{-2B\sqrt{-1}} + 1}{1 - e^{2A\sqrt{-1}} - e^{-2A\sqrt{-1}} + 1}$$

$$= \frac{1 - e^{2B\sqrt{-1}} - e^{-2B\sqrt{-1}} + e^{2B\sqrt{-1}} \cdot e^{-2B\sqrt{-1}}}{1 - e^{2A\sqrt{-1}} - e^{-2A\sqrt{-1}} + e^{2A\sqrt{-1}} \cdot e^{-2A\sqrt{-1}}}$$

$$= \frac{(1 - e^{2B\sqrt{-1}})(1 - e^{-2B\sqrt{-1}})}{(1 - e^{2A\sqrt{-1}})(1 - e^{-2A\sqrt{-1}})}$$

$$\therefore\ 2\,(\text{hyp. log}\ b - \text{hyp. log}\ a)$$

$$= -\text{hyp. log}\,(1 - e^{2A\sqrt{-1}}) - \text{hyp. log}\,(1 - e^{-2A\sqrt{-1}})$$

$$+\ \text{hyp. log}\,(1 - e^{2B\sqrt{-1}}) + \text{hyp. log}\,(1 - e^{-2B\sqrt{-1}})$$

$$= (e^{2A\sqrt{-1}} + e^{-2A\sqrt{-1}}) + \frac{1}{2}(e^{4A\sqrt{-1}} + e^{-4A\sqrt{-1}})$$

$$+ \frac{1}{3}(e^{6A\sqrt{-1}} + e^{-6A\sqrt{-1}}) + \dots$$

$$- (e^{2B\sqrt{-1}} + e^{-2B\sqrt{-1}}) - \frac{1}{2}(e^{4B\sqrt{-1}} + e^{-4B\sqrt{-1}})$$

$$- \frac{1}{3}(e^{6B\sqrt{-1}} + e^{-6B\sqrt{-1}}) - \dots$$

$\therefore$ (dividing by 2)

$$\text{hyp. log}\ b - \text{hyp. log}\ a = (\cos 2A - \cos 2B)$$

$$+ \frac{1}{2}(\cos 4A - \cos 4B) + \frac{1}{3}3(\cos 6A - \cos 6B) + \dots.$$

Q. E. D.

4. Of all spherical triangles which have the same base and equal perpendiculars from the vertex to the base, shew that the isosceles has the greatest vertical angle; and, from the result prove, that the same is true in plane triangles.

From the vertical $\angle C$ of triangle ABC let fall perpendicular CD upon the base AB.

$$\text{Let}\ \angle ACD = \alpha \quad \therefore \quad \angle DCB = C - \alpha$$

$$AD = \theta \quad \therefore \qquad DB = c - \theta$$

$$CD = \delta$$

By Napier's Rules (making δ the middle part) we have in triangles ACD, DCB

$$\sin\delta \,.\, \tan\alpha = \tan\theta$$

$$\sin\delta \,.\, \tan(C - \alpha) = \tan(c - \theta)$$

Reducing 2nd equation in order to substitute in it the value of tan α, found from the 1st, there results

$$\sin\delta \,.\, \tan C - \sin\delta \,.\, \tan\alpha = \tan(c-\theta) \,.\, (1 + \tan C \,.\, \tan\alpha)$$

$$\therefore \tan C = \frac{\sin\delta \,.\, \tan\alpha + \tan(c-\theta)}{\sin\delta - \tan\alpha \,.\, \tan(c-\theta)}$$

$$= \frac{\tan\theta + \tan(c-\theta)}{\sin^2\delta - \tan\theta \,.\, \tan(c-\theta)} \cdot \sin\delta.$$

$$= \frac{\sin\theta \,.\, \cos(c-\theta) + \cos\theta \,.\, \sin(c-\theta)}{\sin^2\delta \cos\theta \,.\, \cos(c-\theta) - \sin\theta \,.\, \sin(c-\theta)} \,.\, \sin.\, \delta.$$

(multiplying numerator and denominator by $\cos\theta \,.\, \cos(c-\theta)$)

$$= \frac{\sin c \,.\, \sin\delta}{\cos c - \cos^2\delta \,.\, \cos\theta \,.\, \cos(c-\theta)} \text{ a maximum;}$$

$$\therefore \frac{d}{d\theta} \cdot \left\{ (\cos\theta \,.\, \cos(c-\theta) \right\} = 0,$$

$$\text{or } -\sin\theta \,.\, \cos(c-\theta) + \cos\theta \,.\, \sin(c-\theta) = 0$$

$$\text{or } \quad \sin(c - 2\theta) = 0,$$

$$\therefore c - 2\theta = 0,$$

$$\therefore \theta = \frac{c}{2}; \; \therefore \perp CD \text{ bisects the base.}$$

Now, supposing the radius of the sphere to become indefinitely great, while the distance between the points A, B remains invariable; the triangle will tend continually to become plane. But, for any value of the radius, however great, we shall always have $c - 2\theta = 0$, in case of C being a maximum; and as this is true without any limit, it will also be true when the triangle becomes plane.

Q. E. D.

5. Having given

$$\log 8801 = 3.9445320$$
$$\log 8802 = 3.9445814$$
$$\log 8804 = 3.9446800$$
$$\log 8805 = 3.9447294$$

find log 8803.

$v_0, v_1, v_2; v_3$, being 4 equidistant values of any function, we have

$$\Delta^4 v_0 = v_4 - 4v_3 + 6v_2 - 4v_1 + v_0 = 0;$$

$$\therefore\ v_2 = \frac{4(v_1 + v_3) - (v_0 + v_4)}{6};$$

But, in this case, $v_0 = \log 8801$ $\quad v_2 = \log 8803$

$v_1 = \log 8802$ $\quad v_3 = \log 8804$

$\therefore\ v_2 = 3.9446307$

6. Two straight lines which are always tangents to a given parabola, are so inclined to the axis of x, that the sum of the co-tangents of the angles which they make with that axis is constant; prove that the locus of their intersections is a straight line parallel to the axis.

Let a = latus rectum of the given parabola.
(α, β), (α', β') the co-ordinates of two corresponding points of contact.

Then, the equations of two corresponding tangents being

$$Y - \beta = m\ (X - \alpha)$$
$$Y - \beta' = m'\ (X - \alpha')$$

which, by substituting for m and m' their values, become

$$2\,Y\beta = a X + a\,\alpha$$
$$2\,Y\beta' = a X + a\,\alpha'$$

we have, by subtraction,

$$2\ Y(\beta - \beta') = a\,(\alpha - \alpha'). \qquad (A)$$

But $\frac{1}{m} + \frac{1}{m'} = \frac{2\beta}{a} + \frac{2\beta'}{a} = c$ by hypothesis.

or $\qquad 2(\beta + \beta') = a\,c \qquad (B)$

$\therefore\ 4\ Y\ (\beta^2 - \beta'^2) = a^2 c\ (\alpha - \alpha') \qquad (C)$

by multiplying (A) by (B)

But since $\beta^2 = a\alpha$

$$\beta'^2 = a\alpha'$$

and $\therefore\ \beta^2 - \beta'^2 = a(\alpha - \alpha')$

$\therefore$ (C) becomes

$$4\, a\, Y(\alpha - \alpha') = a^2 c(\alpha - \alpha')$$

or $4\, Y = a c$

the equation to a straight line parallel to the axis of x.

7. Find the surface in which the tangent plane always cuts the axis of z at distances from the origin proportional to $\frac{1}{z^n}$, and when $n = 1$ give to the arbitrary function that particular form which will produce the equation to the ellipsoid.

In the equation of the tangent plane,

$$z - z' = p(x - x') + q(y - y')$$

we have at the same time

$$x' = 0,\ y' = 0,\ z' = \frac{c^{n+1}}{z^n},$$

by the conditions of the problem; so that in this case, it will be

$$p\,x + q\,y = \frac{z^{n+1} - c^{n+1}}{z^n}$$

a partial differential equation of the form

$$Pp + Qq = R.$$

Now, by the theory of such equations,

if we call the integrals of the equations,

$$P\,dy - Q\,dx = 0$$
$$P\,dz - R\,dx = 0$$

respectively (α) & (β); we shall have

$$\beta = \phi(\alpha).$$

But $P\,dy - Q\,dx = x\,dy - y\,dx = 0$;

$$\therefore \alpha = \frac{y}{x}.$$

Also $P\,dz - R\,dx = x\,dz - \frac{z^{n+1} - c^{n+1}}{z^n}\,dx = 0$,

$$\text{or } (n+1)\,\frac{z^n\,dz}{z^{n+1} - c^{n+1}} - (n+1)\,dx = 0;$$

$$\therefore \beta = \text{hyp. log.} \left\{ \frac{z^{n+1} - c^{n+1}}{x^{n+1}} \right\}$$

$$\therefore \text{hyp. log.} \left\{ \frac{z^{n+1} - c^{n+1}}{x^{n+1}} \right\} = \phi\left(\frac{y}{x}\right),$$

or, when $n = 1$,

$$z^2 - c^2 = x^2 \,.\, \varepsilon^{\phi\left(\frac{y}{x}\right)},$$

$$= x^2 \,.\, \psi\left(\frac{y}{x}\right) \text{ suppose.}$$

Making $\therefore \psi\left(\frac{y}{x}\right) = -\frac{c^2}{a^2} - \frac{c^2}{b^2}\cdot\frac{y^2}{x^2}$

there results $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$;

the equation to the ellipsoid, which may easily be shewn to possess the required property.*

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* It may be here remarked, that giving to any one of the variables (z for instance) a particular value (γ), and assigning the form of the arbitrary function of $\frac{y}{x}$, is equivalent to subjecting the surface to pass through a given plane curve (in this case an ellipse), parallel to the plane of x, y, and at a distance from it equal to γ. But supposing z to remain indeterminate, and,

8. $\left.\begin{array}{l} x = a z \\ y = b z \end{array}\right\}$ (1) and $\left.\begin{array}{l} x^2 + y^2 = 2 c x \\ x^2 + y^2 = m^2 z^2 \end{array}\right\}$ (2)

are the equations to a straight line and curve of double curvature; find the equation to the surface generated by a straight line, moving always parallel to the plane of $x y$, and passing through the straight line and the curve.

The curve of double curvature is formed by the intersection of the surfaces of a cone and cylinder, the vertex of the cone being the origin, and its axis coinciding with that of z; while the axis of the cylinder is parallel to that of z, and at a distance from it $= c$ the radius of its circular base.

Now, since we have only to consider those points in the surfaces that are common to both of them, we may consider the cotemporary values of x and y to be common to both of the equations, and may therefore make $2 c x = m^2 z^2$.

Let now $\left.\begin{array}{l} y = \beta x + \gamma \\ z = \delta \end{array}\right\}$

be the equations to the generating line in one of its positions.

But, as this line is to pass through a point where, from equations, (1) $z = \delta$, $y = b \delta$, $x = a \delta$, these equations must become, by this condition,

$$\left.\begin{array}{l} y - b \delta = \beta (x - a \delta) \\ z = \delta \end{array}\right\} \quad (3)$$

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assigning, as before, the form of the arbitrary function, amounts to making the solution represent a surface of which all the sections parallel to the plane of $x y$ shall be ellipses. The introduction of the constants in the arbitrary function determines the species, magnitude, and position of one of the sections. This, together with the other condition of the problem, completely determines the surface.

Also, since the same line is to pass through a point where, from equations (2), $z = \delta$, $y = \frac{m\delta}{2c}\sqrt{4c^2 - m^2\delta^2}$, $x = \frac{m^2\delta^2}{2c}$,

$$\frac{m\delta}{2c} \cdot \sqrt{4c^2 - m^2\delta^2} - b\delta = \beta\left(\frac{m^2\delta^2}{2c} - a\delta\right).$$

Substituting the value of β found from this last equation in 1st of equations (3), and putting z for δ, we have finally

$$\frac{x - az}{y - bz} = \frac{2ac - m^2 z}{2bc - m\sqrt{4c^2 - m^2 z^2}},$$

an equation to a surface of the 4th order, consisting of four sheets, or two *ungular* figures, one above and the other below the plane of $x\,y$. These figures will not be equal nor similar, unless the line represented by equations (1) be supposed to coincide with the axis of z; that is, unless $a = 0$, and $b = 0$; in which case the surface has for its equation

$$\frac{mz}{\sqrt{4c^2 - m^2 z^2}} = \frac{x}{y},$$

which is of the well-known form

$$z = \phi\left(\frac{x}{y}\right).$$ *(See Peacock's Examples, p.* 440.*)*

9. Integrate

(1) $$\frac{d^2 y}{dx^2} + a^2 y = \varepsilon^x \cos ax.$$

(2) $$z - px - qy = m(x + y + z).$$

(1). The solution of the equation

$$\frac{d^2 y}{dx^2} + a^2 y = 0$$

being $y = c \,.\cos ax + c' \,.\sin ax$, will also be that of the more general equation,

$$\frac{d^2 y}{dx^2} + a^2 y = X,$$

if c and c', instead of being constants, be functions of x.

The forms of these functions may be determined in the following manner:

Differentiating $y = c \,.\, \cos\, a\,x + c' \,.\, \sin\, a\,x$ on the above supposition, gives

$$\frac{d\,y}{d\,x} = -\,c\,a \,.\, \sin a\,x + c'\,a \,.\cos\, a\,x + \cos\, a\,x \,.\, \frac{d\,c}{d\,x}$$

$$+ \sin\, a\,x \,.\, \frac{d\,c'}{d\,x} = -\,c\,a \,.\, \sin\, a\,x + c'\,a \,.\, \cos\, a\,x,$$

making $\cos\, a\,x \,.\, \frac{d\,c}{d\,x} + \sin\, a\,x \,.\, \frac{d\,c'}{d\,x} = 0.$ (A)

Differentiating again,

$$\frac{d^2\,y}{d\,x^2} = -\,a^2\,(c \,.\cos\, a\,x + c'\,\sin\, a\,x)$$

$$-\,a \,.\, \sin\, a\,x \,.\, \frac{d\,c}{d\,x} + a\,\cos\, a\,x \,.\, \frac{d\,c'}{d\,x}.$$

or $\frac{d^2\,y}{d\,x^2} + a^2\,y = -\,a\,\sin\, a\,x \,.\, \frac{d\,c}{d\,x} + a\,\cos\, a\,x \,.\, \frac{d\,c'}{d\,x}$

$$= X;$$

$$\therefore\; a\,\cos\, a\,x \,.\, d\,c' - a\,\sin\, a\,x \,.\, d\,c = X\,d\,x.$$

This last equation, combined with (A), gives

$$d\,c = -\,\frac{1}{a} \,.\, X\,\sin\, a\,x \,.\, d\,x$$

$$d\,c' = \;\frac{1}{a} \,.\, X \cos a\,x \,.\, d\,x;$$

$$\therefore\; c = C - \frac{1}{a}\int X\,\sin\, a\,x \,.\, d\,x,$$

$$c' = C' + \frac{1}{a}\int X\,\cos\, a\,x \,.\, d\,x.$$

Substituting these expressions for c, c' in that for y, gives

$$y = C\,\cos\, a\,x + C'\,\sin\, a\,x + \frac{1}{a} \,.\, \sin\, a\,x \int X\,\cos\, a\,x \,.\, d\,x$$

$$-\,\frac{1}{a} \,.\, \int X\,\sin\, a\,x \,.\, d\,x,$$

but $X = \varepsilon^x \cos a x$;

$$\therefore \sin a x \int X \cos a x \, d x - \cos a x \int X \sin a x \, . d x$$

$$= \frac{1}{2} \varepsilon^x \sin a x + \frac{\varepsilon^x}{2(1 + 4a^2)} \left\{ 2 a \cos a x - \sin a x \right\};$$

$\therefore$ making $C \cos a x + C' \sin a x = A \cos (a x + B)$,

$$y = A \cos (a x + B) + \frac{\varepsilon^x}{1 + 4a^2} \left\{ 2 a \sin a x + \cos a x \right\}.$$

(2) $\quad z - p x - q y = m (x + y + z).$

This equation is of the same kind as that in Problem 7.

$\therefore$ in the equations $P d y - Q d x = 0$,

$P d z - R d x = 0$,

we have $P = x$, $Q = y$, $R = -(m - 1) z - m(x + y)$;

$\therefore$ from the first of these, we obtain $\alpha = \frac{y}{x}$,

which makes $R = -(m - 1) z - m(1 + \alpha) x$: so that the 2d of the above equations becomes

$$x d z + (m - 1) z d x + m(1 + \alpha) x d x = 0$$

or, making $z = u x$;

$$m d x (1 + \alpha + u) + x d u = 0;$$

$$\therefore m \frac{d x}{x} + \frac{d u}{1 + \alpha + u} = 0;$$

$$\therefore \beta = \log x^m (1 + \alpha + u)$$

$$= \log x^{m-1} (x + y + z);$$

$$\therefore x^{m-1} (x + y + z) = \varepsilon^{\phi\left(\frac{y}{x}\right)}$$

10. Solve the following equations of differences:

(1) $\Delta x + \Delta^2 x + \Delta^3 x = x^3$,

(2) $u_x . u_{x+1} + u_x . u_{x+2} + u_{x+1} . u_{x+2} = m^2$.

(1). This is equivalent to finding a series of numbers, such, that the sum of the first terms of the 1st, 2nd, and 3rd orders of differences may equal the cube of the 1st term of the series.

If then $x, x_1, x_2, x_3 \ldots\ldots x_n$. be the several values of x which compose the series, they may be determined from the general expression

$$x_n = x + \frac{n}{1}\Delta x + \frac{n(n-1)}{1.2}\Delta^2 x + \frac{n(n-1)(n-2)}{1.2.3}\Delta^3 x + \ldots$$

by giving successively to n as many values 1, 2, 3, &c., (and therefore obtaining as many simple equations) as may be necessary. In these equations, however, the values of Δx, $\Delta^2 x$, $\Delta^3 x$, &c. will remain indeterminate; but it is easy to see, by a process of reasoning similar to that employed in integrating an equation of differences of the 3rd order, that three of these values must be arbitrary; and in general, as many as are denoted by the order of the equation to be integrated. We may, therefore, obtain an indefinite number of series which answer the required conditions.

Thus, if we had $\Delta x = 4$, $\Delta^2 x = 3$, $\Delta^3 x = 1$,

$$\text{then } \Delta x + \Delta^2 x + \Delta^3 x = x^3 = 8.$$

$$\therefore x = 2.$$

and the general expression gives

$$x_1 = x + \Delta x = 6,$$

$$x_2 = x + 2\Delta x + \Delta^2 x = 13,$$

$$x_3 = x + 3\Delta x + 3\Delta^2 x + \Delta^3 x = 24;$$

$\therefore$ the required series is 2, 6, 13, 24, &c.

The same result might have been obtained by integrating the several terms of the proposed equation, on the supposition that the successive increments of the variable x were not constant.

This would have led to a numerical equation in x, of which the roots would be terms of the required series, indeterminate as before, since three terms of the resulting equation would have arbitrary co-efficients.

(2). To integrate

$$u_x \, . \, u_{x+1} + u_x \, . \, u_{x+2} + u_{x+1} \, . \, u_{x+2} = m^2.$$

Assume $u_x = m \tan v_x$;

$$\therefore \tan v_x \, . \tan v_{x+1} + \tan v_x \, . \tan v_{x+2} + \tan v_{x+1} \, . \tan_{x+2} = 1,$$

$$\therefore \tan v_x = \frac{1 - \tan v_{x+1} \, . \tan v_{x+2}}{\tan v_{x+1} + \tan v_{x+2}},$$

$$= \cot (v_{x+1} + v_{x+2});$$

$$\therefore v_x = \frac{\pi}{2} - (v_{x+1} + v_{x+2}),$$

$$\text{or } v_{x+2} + v_{x+1} + v_x = \frac{\pi}{2}.$$

To integrate this equation, assume $v_x = a^x + K$;

$$\therefore \text{ it becomes } a^{x+2} + a^{x+1} + a^x = \frac{\pi}{2} - 3K,$$

$$\text{or } a^2 + a + 1 = 0;$$

$$\text{making } \frac{\pi}{2} - 3K = 0, \text{ or } K = \frac{\pi}{6}.$$

And since the values of a are

$$-\frac{1}{2} + \frac{\sqrt{-1}.\sqrt{3}}{2}, \; -\frac{1}{2} - \frac{\sqrt{-1}\,.\sqrt{3}}{2};$$

we have

$$v_x = C\left(-\frac{1}{2} + \frac{\sqrt{-1}\cdot\sqrt{3}}{2}\right)^x + C'\left(-\frac{1}{2} - \frac{\sqrt{-1}\,.\sqrt{3}}{2}\right)^x$$

But $-\frac{1}{2} = \cos\frac{2\pi}{3}$ and $\frac{\sqrt{3}}{2} = \sin\frac{2\pi}{3}$;

$$\therefore\ v_x = C\left(\cos\frac{2}{3}\pi + \sqrt{-1}\,.\sin\frac{2}{3}\pi\right)^x$$

$$+ C'\left(\cos\frac{2}{3}\pi - \sqrt{-1}\,.\sin\frac{2}{3}\pi\right)^x$$

$$= C\left(\cos\frac{2\pi x}{3} + \sqrt{-1}\,.\sin\frac{2\pi x}{3}\right)$$

$$+ C'\left(\cos\frac{2\pi x}{3} - \sqrt{-1}\,.\sin\frac{2\pi x}{3}\right)$$

$$= (C+C')\cos\frac{2\pi x}{3} + \sqrt{-1}\,(C-C')\sin\frac{2\pi x}{3}$$

$$= \quad C_1\cos\frac{2\pi x}{3} + C_2\,.\sin\frac{2\pi x}{3};$$

$$\therefore\ u_x = m\tan.\left\{C_1\cos\frac{2\pi x}{3} + C_2\sin\frac{2\pi x}{3}\right\}.$$

11. Sum the following series:

$$\frac{1}{1.3} + \frac{1}{4.6} + \frac{1}{7.9} + \frac{1}{10.12} + \ldots\ldots \text{ ad infinitum.}$$

$$\sec\theta.\cos\theta + 4(\sec\theta)^2\cos 2\theta + 13(\sec\theta)^3\cos 3\theta,$$

$$+ 40(\sec\theta)^4\cos 4\theta + \ldots \text{ \&c. to } n \text{ terms.}$$

Let $s = 1 + x^3 + x^6 + x^9 + \ldots.$ ad inf. $= \dfrac{1}{1 - x^3}$;

$$\therefore \int s\,dx = \frac{x}{1} + \frac{x^4}{4} + \frac{x^7}{7} + \frac{x^{10}}{10} + \ldots\ldots$$

$$\therefore \int x\,dx \int s\,dx = \frac{x^3}{1.3} + \frac{x^6}{4.6} + \frac{x^9}{7.9} + \frac{x^{12}}{10.12} + \ldots\ldots$$

Then, if S be the value of $\int x\,dx \int s\,dx$. between the limits $x = 0$ and $x = 1$, it will be the sum required.

$$\text{But} \int x\,dx \int s\,dx = \frac{x^2}{2}\int s\,dx - \frac{1}{2}\int s\,x^2\,dx$$

$$= \frac{x^2}{2}\int \frac{dx}{1-x^3} + \frac{1}{2.3}\log(1-x^3)$$

$$= \frac{x^2}{2}\int \frac{dx}{1-x^3} + \frac{1}{2.3}\log(1-x) + \frac{1}{3}.\log\sqrt{1+x+x^2}.$$

$$\text{Also} \int \frac{dx}{1-x^3} = -\frac{1}{3}\log(1-x) + \frac{1}{3}\log\sqrt{1+x+x^2}$$

$$+ \frac{1}{3\sqrt{3}}.\tan^{-1}\left\{\frac{x\sqrt{3}}{2+x}\right\};$$

$$\therefore \int x\,dx \int s\,dx = \frac{1}{6}\log(1-x)^{1-x^2}$$

$$+ \frac{1}{3}\left(1+\frac{x^2}{6}\right)\log\sqrt{1+x+x^2} + \frac{x^2}{6\sqrt{3}}.\tan^{-1}\left\{\frac{x\sqrt{3}}{2+x}\right\};$$

$$\therefore\ S = \frac{7}{18}.\log\sqrt{3} + \frac{1}{\sqrt{3}}.\frac{\pi}{36}.$$

(2). The co-efficient of $(\sec\theta)^n.\cos n\theta$, being the general term of the recurring series

$$1 + 4 + 13 + 40 + \ldots.$$

in which the scale of relation is $4 - 3$,

we have, for determining it, the equation of differences

$$u_{x+2} = 4\,x_{x+1} - 3\,u_x;$$

$\therefore$ making $u_x = a^x$, leads to the equation

$$a^2 - 4a + 3 = 0:$$

$\therefore$ the roots of this being 1 and 3, there results

$$u_x = C + 3^x.C'.$$

$$\left.\begin{array}{l}\text{But } u_0 = 0 = C + C' \\ u_1 = 1 = C + 3\,C'\end{array}\right\} \therefore\ C = -\frac{1}{2}$$

$$C' = +\frac{1}{2}$$

$$\therefore\ u_x = \frac{3^x - 1}{2}.$$

The general term (u_n) of the proposed series thus becomes

$$u_n = \frac{1}{2}(3^n - 1)\ (\sec\theta)^n.\cos n\,\theta.$$

Let now $\sec\theta = z$,

$$2\cos\theta = v + w,\ \left(\text{where } w = \frac{1}{v}\right);$$

$$\therefore\ 2\cos n\,\theta = v^n + w^n;$$

$$\therefore\ u_n = \frac{1}{4}(3^n - 1)\left\{(v\,z)^n + (w\,z)^n\right\},$$

$$= \frac{1}{4}(3^n - 1)(a^n + b^n),\quad (\text{making } v\,z = a, \quad w\,z = b).$$

$$= \frac{1}{4}\left\{(3\,a)^n - a^n\right\} + \frac{1}{4}\left\{(3\,b)^n - b^n\right\};$$

$$\therefore\ \Delta S_n = u_{n+1} = \frac{1}{4}\left\{3\,a\,.(3\,a)^n - a\,.\,a^n\right\} + \frac{1}{4}\left\{3\,b\,.(3\,b)^n - b\,.\,b^n\right\}.$$

$$\text{Now } \Sigma\left\{3\,a.(3\,a)^n - a.a^n\right\} = \frac{(3\,a)^{n+1}}{3\,a - 1} - \frac{a^{n+1}}{a - 1} + C$$

$$= \frac{3^{n+1}\,a^{n+1} - 3\,a}{3\,a - 1} - \frac{a^{n+1} - a}{a - 1};$$

$$\therefore\ S_n = \frac{1}{4}\left\{\frac{3^{n+1}\,a^{n+1} - 3\,a}{3\,a - 1} + \frac{3^{n+1}\,b^{n+1} - 3\,b}{3\,b - 1}\right\}$$

$$- \frac{1}{4}\left\{\frac{a^{n+1} - a}{a - 1} + \frac{b^{n+1} - b}{b - 1}\right\}$$

$$= \frac{1}{4}\left\{\frac{3^{n+2}\,a\,b\,(a^n + b^n) - 3^{n+1}(a^{n+1} + b^{n+1}) + 3(a+b) - 2\,.\,3^2\,a\,b}{3^2\,a\,b - 3\,(a + b) + 1}\right\}$$

$$- \frac{1}{4}\left\{\frac{a\,b\,(a^n + b^n) - (a^{n+1} + b^{n+1}) + (a + b) - 2\,a\,b}{a\,b - (a + b) + 1}\right\}.$$

But, since $a\,b = z^2 = (\sec\theta)^2$,

$$(a^n + b^n) = 2\,z^n \cos n\,\theta = 2\,(\sec\theta)^n\,.\cos n\,\theta,$$

$$\&\text{c.} = \&\text{c.};$$

by substituting these values, we have, finally,

$$S_n = \frac{1}{2}\left\{\frac{3^{n+2}\,(\sec\theta)^{n+2}\cos n\,\theta - 3^{n+1}\,(\sec\theta)^{n+1}\cos(n+1)\theta + 3\sec\theta\cos\theta - 3^2\,(\sec\theta)^2}{3^2\,(\sec\theta)^2 - 2\,.\,3\sec\theta\,.\cos\theta + 1}\right\}$$

$$- \frac{1}{2}\left\{\frac{(\sec\theta)^{n+2}\cos n\,\theta - (\sec\theta)^{n+1}\cos(n+1)\,\theta + \sec\theta\,.\cos\theta - (\sec\theta)^2}{(\sec\theta)^2 - 2\sec\theta\,.\cos\theta + 1}\right\}$$

12. Find $\sin x$ from the equation

$$\sin x\,.\cos x + a\sin^2 x = b,$$

and shew its use in the solution of the following problem. To determine how much the azimuth of a known star on the horizon is affected by refraction.

$$\text{Let } \sin x = u;$$

$$\therefore\; u\sqrt{1 - u^2} + a\,u^2 = b;$$

or, transposing, squaring, and arranging,

$$u^4 - \frac{2\,a\,b + 1}{a^2 + 1}\,.\,u^2 + \frac{b^2}{a^2 + 1} = 0.$$

Whence, by proceeding as in a quadratic equation,

$$\sin x = \frac{1}{\sqrt{\{2\,(a^2+1)\}}}\,.\sqrt{\left\{2\,a\,b + 1 \pm \sqrt{\{1 - 4\,(b^2 - a\,b)\}}\right\}}.$$

Let Z, P, S denote respectively the zenith, pole, and star's place; the star being below the horizon by a vertical distance $= R$ the horizontal refraction.

Draw the meridian $EZPO$ cutting the horizon in points E, O.

Let the vertical circle ZS cut the horizon in A, and take, in EA, a point B, such, that $PB =$ star's true polar distance. Join, by great circles, BS, BZ, PS, PA.

Now A being the point at which the star appears to rise, and B the point at which it really rises; the $\angle BPA$ expresses the difference between the true azimuth and that affected by refraction.

Let $PO = \lambda$, $\angle PZS = z$, $\angle OPS = h$,
$PB = 90° - \Delta$, $\angle PZB = z + \delta z$, $\angle OPB = h + \delta h$;
δz and δh being very small angles.

The triangle ZPS gives

$$\cos PZS = \frac{\cos PS - \cos ZS \cdot \cos ZP}{\sin ZS \cdot \sin ZP},$$

$$= \frac{\sin \Delta}{\cos R \cdot \cos \lambda} + \tan R \cdot \tan \lambda.$$

Also in the triangle ZPB,

$$\cos PZB = \frac{\sin \Delta}{\cos \lambda};$$

$$\therefore \cos PZS - \cos PZB = \tan R \cdot \tan \lambda + \frac{\sin \Delta}{\cos R \cdot \cos \lambda} - \frac{\sin \Delta}{\cos \lambda}$$

$$= \tan R \cdot \tan \lambda + \frac{\sin \Delta - \sin \Delta \cdot \cos R}{\cos R \cdot \cos \lambda}$$

$$= \frac{\sin R \cdot \sin \lambda + 2 \sin \Delta \cdot \sin^2 \frac{1}{2} R}{\cos R \cdot \cos \lambda}.$$

But $\cos PZS - \cos PZB$

$$= \sin \frac{1}{2} (PZB - PZS) \cdot \sin \frac{1}{2} (PZB + PZS)$$

$$= 2 \sin \frac{1}{2} \delta z \cdot \sin \left(z + \frac{1}{2} \delta z\right)$$

$$= 2 \sin z \cdot \sin \frac{1}{2} \delta z \cdot \cos \frac{1}{2} \delta z + 2 \cos z \sin^2 \frac{1}{2} \delta z;$$

$$\therefore \sin\frac{1}{2}\delta z \,.\, \cos\frac{1}{2}\delta z + \cot z \,.\, \sin^2\frac{1}{2}\delta z$$

$$= \frac{\sin R \,.\, \sin\lambda + 2\sin\Delta \,.\, \sin^2\frac{1}{2}R}{2\cos R \,.\, \cos\lambda \,.\, \sin z},$$

which, by making $\frac{1}{2}\delta z = x$,

$$\cot z = a,$$

$$\frac{\sin R \sin\lambda + 2\sin\Delta \,.\, \sin^2\frac{1}{2}R}{2\cos R \,.\, \cos\lambda \,.\, \sin z} = b,$$

becomes, $\sin x \,.\, \cos x + a\sin^2 x = b$.

13. A right-angled triangle vibrates in its own plane about an axis passing through its vertex, find the length of the isochronous simple pendulum; and if one of the sides be slightly diminished and the other as much increased, determine the variation of the pendulum.

The triangle ABC being right-angled at C, draw CE bisecting, and CD perpendicular, to base AB.

Take, in AB, any point P, and another point P' very near to it; and, with centre C, and radius CP, describe the circular arc PQ cutting CP' in Q.

Then, retaining the usual notation for the side and angles of ABC, let

$$CD = k, \ \angle DCP = \theta,$$

$$CP = r, \ \angle PCP' = \delta\theta.$$

Now the momentum of inertia of the sector $PCQ = \frac{r^4 \,.\, \delta\theta}{4}$;

$\therefore$ if u = moment of inertia of triangle ABC, we have

$$\frac{du}{d\theta} = \frac{r^4}{4} = \frac{k^4}{4\cos^4\theta};$$

$$\therefore\ u = \frac{k^4}{4}\int\frac{d\,\theta}{\cos^4\theta}$$

$$= \frac{k^4}{4}\cdot\left\{\int\frac{\sin^2\theta}{\cos^4\theta}\cdot d\,\theta + \int\frac{\cos^2\theta}{\cos^4\theta}\cdot d\,\theta\right\},$$

$$= \frac{k^4}{4}\left\{\int\tan^2\theta\,.\,d\,(\tan\theta) + \int d\,(\tan\theta)\right\},$$

$$= \frac{k^4}{4}\left\{\frac{\tan^3\theta}{3} + \tan\theta\right\} \quad \begin{cases}\theta = -\angle\, DCA = -B,\\ \theta = +\angle\, DCB = +A,\end{cases}$$

$$= \frac{k^4}{4}\left\{\frac{1}{3}(\tan^3 A + \tan^3 B) + \tan A + \tan B\right\}.$$

But. $\tan A = \frac{a}{b}$, $\tan B = \frac{b}{a}$: also, since twice area

$$ABC = a\,b = k\,c,\ k^4 = \frac{a^4\,b^4}{c^4} = \frac{a^4\,b^4}{(a^2 + b^2)^2};$$

$$\therefore\ u = \frac{a^4\,b^4}{4\,(a^2 + b^2)^2}\left\{\frac{1}{3}\left(\frac{a^3}{b^3} + \frac{b^3}{a^3}\right) + \frac{a}{b} + \frac{b}{a}\right\}$$

$$= \frac{1}{12}\,a\,b\,(a^2 + b^2),$$

$$= \frac{1}{12}\cdot a\,b\,c^2.$$

Now, if L = length of isochronous pendulum, taking the point G in CE, such that

$$CG = \frac{2}{3}\,CE = \frac{2}{3}\cdot\frac{c}{2} = \frac{c}{3};$$

$$L = \frac{u}{\text{area } ABC \times CG} = \frac{c}{2}.$$

Now, let δL be the variation of L corresponding to the small variations δa, δb of the sides a, b;

$$\therefore \delta L = \frac{dL}{da} \cdot \delta a + \frac{dL}{db} \cdot \delta b$$

$$= \left(\frac{dL}{da} - \frac{dL}{db}\right) \delta a;$$

since, by hypothesis, $\delta b = - \delta a$.

$$\text{But } \frac{dL}{da} = \frac{1}{2} \frac{dc}{da} = \frac{a}{\sqrt{(a^2 + b^2)}} = \frac{a}{c},$$

$$\text{and } \frac{dL}{db} = \frac{1}{2} \cdot \frac{dc}{db} = \frac{b}{\sqrt{(a^2 + b^2)}} = \frac{b}{c};$$

$$\therefore \delta L = \frac{a - b}{c} . \delta a .$$

14. If a hemisphere and paraboloid of equal bases and similar materials have their bases cemented together, the whole solid will rest on a horizontal plane on any point of the spherical surface, if the altitude of the paraboloid $= a\sqrt{\frac{3}{2}}$, (a) being the radius of the hemisphere.

Since the body rests on any point of the spherical surface, its centre of gravity must always lie in the vertical line, passing through the point of contact of the hemisphere with the horizontal plane.

Therefore the centre of gravity of the body is the centre of the spherical surface;

therefore mass of paraboloid × distance of its centre of gravity from centre of sphere = mass of hemisphere × distance of its centre of gravity from centre of sphere.

$$\text{that is, } \frac{1}{2}\pi a^2 h \times \frac{1}{3}h = \frac{2}{3}\pi a^3 \times \frac{3}{8}a;$$

(h being the altitude of the paraboloid)

$$\text{or, } \frac{2}{3}h^2 = a^2;$$

$$\therefore h = a\sqrt{\frac{3}{2}}.$$

Q. E. D.

15. Prove that the eye cannot be achromatic for objects at all distances.*

16. A body is acted on by two forces, the one repulsive and varying as the distance from a given point, the other constant and acting in parallel lines. Determine the motion of the body.

The centre of repulsive force being the origin of co-ordinates, let the constant force act always in the direction of y.

$$\text{Let absolute central force} = m,$$
$$\text{constant force} = m k;$$

$\therefore$ resolving these forces in directions of x and y,

$$\frac{d^2 x}{d t^2} = m x,$$

$$\frac{d^2 y}{d t^2} = m(y + k).$$

Multiplying 1st of these equations by $2\,dx$, and 2nd by $2\,dy$ and integrating, gives

$$\frac{d x^2}{d t^2} = m(x^2 - a^2),$$

$$\frac{d y^2}{d t^2} = m\left\{(y + k)^2 - (k + b)^2\right\},$$

* See the Appendix.

dt being constant, and (a) (b) being co-ordinates of the point from which the body begins to move by the action of these forces.

$\therefore$ combining the last two equations,

$$\frac{d\,(y+k)}{\sqrt{\{(y+k)^2-(k+b)^2\}}}=\frac{d\,x}{\sqrt{(x^2-a^2)}}\cdot\ \{\text{since } d\,y=d(y+k)\},$$

$\therefore$ by the integration of this last, we have, for the path of the body,

$$y+k+\sqrt{\{(y+k)^2-(k+b)^2\}}=c\,\{x+\sqrt{(x^2-a^2)}\}:$$

c being an arbitrary constant, which is determined by considering that a and b are corresponding values of x and y; a condition which gives $c=\frac{k+b}{a}$.

17. A body falls towards a centre of force which varies as $\frac{1}{D^3}$, in a medium of which the density varies as (velocity)². Prove that at any distance (r) from the centre,

$$(\text{velocity})=\frac{m}{h}\left\{1-\varepsilon^{-h\left(\frac{1}{r^2}-\frac{1}{a^2}\right)}\right\},$$

where $m=$ force at distance 1, $h=$ density at distance 1, and $a=$ distance from centre at the beginning of motion.

Let P represent the variable central force, k the density, v the velocity.

Then, the resistance being as $k\,v^2$, may, for similar and equal bodies, be considered *equal* to it: so that the equation

$v\,d\,v=f\,d\,s$ becomes, in this case,

$$v\,d\,v=-P\,d\,r+k\,v^2\,d\,r,$$

$$\text{or } v\,d\,v-k\,v^2\,d\,r=-P\,d\,r.$$

But, observing that

$$d.v^2\epsilon^{-\int 2kdr} = 2v\,dv\,\epsilon^{-\int 2kdr} - 2kv^2 dr\,\epsilon^{-\int 2kdr},$$

we may make the last equation integrable by multiplying it by $2\epsilon^{-\int 2kdr}$;

∴ multiplying and integrating, we have

$$v^2\epsilon^{-\int 2kdr} = -2\int P\,dr\,\epsilon^{-\int 2kdr} + C:$$

or, multiplying again by $\epsilon^{-\int 2kdr}$,

$$v^2 = C\epsilon^{2\int kdr} - 2\epsilon^{2\int kdr}\int P\,dr\,\epsilon^{-2\int kdr}.$$

But $P = \frac{m}{r^3}$, and $k = \frac{h}{r^3}$,

$$\therefore \int k\,dr = -\frac{1}{2}\cdot\frac{h}{r^2};$$

$$\therefore \epsilon^{-2\int kdr} = \epsilon^{\frac{h}{r^2}}$$

Also,
$$-2\int P\,dr\,\epsilon^{-2\int kdr} = -2m\int\frac{dr}{r^3}.\epsilon^{\frac{h}{r^2}}$$

$$= \frac{m}{h}\int d\left(\frac{h}{r^2}\right).\epsilon^{\frac{h}{r^2}}$$

$$= \frac{m}{h}.\epsilon^{\frac{h}{r^2}}.$$

$$\therefore v^2 = C\epsilon^{-\frac{h}{r^2}} + \frac{m}{h}.$$

But when $v = 0$, $r = a$;

$$\therefore 0 = C\epsilon^{-\frac{h}{a^2}} + \frac{m}{h},$$

$$\therefore C = -\frac{m}{h}\epsilon^{\frac{h}{a^2}},$$

$$\therefore v^2 = \frac{m}{h} - \frac{m}{h} \varepsilon^{\frac{h}{a^2}} \cdot \varepsilon^{-\frac{h}{r^2}},$$

$$= \frac{m}{h} \left\{ 1 - \varepsilon^{-h\left(\frac{1}{r^2} - \frac{1}{a^2}\right)} \right\}.$$

Q. E. D.

18. A uniform rod vibrates in a medium, the resistance of which varies as the velocity; find the time of one of its small oscillations.

Since the velocity of any point of the rod is as its distance from the point of suspension, the whole effect of the resistance is that of a set of parallel forces, varying as that distance, and acting perpendicularly to the rod. These resistances, therefore, are equivalent to a single force equal to their sum, and applied at a certain point which may be called the *centre of resistance*.

Let h = resistance to a unit of length of the rod at a distance = 1 from centre of suspension, and for a velocity = 1:

a = length of rod,

δ = distance of centre of resistance from that of suspension;

$\therefore$ resistance at a distance $x = h\,x$;

$$\therefore \text{ whole resistance} = \int h\,x\,dx,$$

$$= \frac{h\,a^2}{2} \quad \left\{ \begin{matrix} x = o \\ x = a \end{matrix} \right\}.$$

Also, by theory of parallel forces, $\delta = \dfrac{\int h\,x^2\,dx}{\int h\,x\,dx}$,

$$= \frac{2\,a}{3} \quad \left\{ \begin{matrix} x = 0 \\ x = a \end{matrix} \right\}.$$

This result shews, that the centre of resistance coincides with that of oscillation; which reduces the problem to finding the time of a small vibration of a point in a circular arc

$\left(\text{radius} = \frac{2}{3}a\right)$, the resistance being equal to $\frac{h\,a^2}{2} \cdot$ velocity.

Now, (Whewell's Dynamics, p. 206), $T = \frac{\pi}{\sqrt{(f - k^2)}}$, where T is the time of an oscillation in any arc of a cycloid and $2k =$ resistance for velocity 1, $\therefore$ T will be time required,* if for $2k$ we substitute

$\frac{h a^2}{2}$, or make $k = \frac{h a^2}{4}$, and $f = \frac{3g}{2a}$ (see page 101).

$$\therefore T = \frac{\pi}{\sqrt{\left\{\frac{3g}{2a} - \frac{h^2 a^4}{16}\right\}}}.$$

SATURDAY EVENING.

1. If $x = m \cdot \tan(z - nx)$ where x is small compared with z,

$$\text{prove that } x = \frac{m}{2} \frac{\sin 2z}{mn + \cos^2 z} \text{ very nearly.}$$

If u be a function of x, we have, by Maclaurin's Theorem,

$$u = (u)_0 + x\left(\frac{du}{dx}\right)_0,$$

very nearly when x is small: $(u)_0$ denoting the particular value of u when $x = 0$.

Let $u = m \cdot \tan(z - nx)$, $\therefore (u_0) = m \tan z$;

$$\therefore \frac{du}{dx} = -\frac{mn}{\cos^2(z - nx)}, \therefore \left(\frac{du}{dx}\right)_0 = -\frac{mn}{\cos^2 z}.$$

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* This will be obvious, by considering that the only property of the cycloid made use of in the article alluded to, is, that the accelerating force in a cycloid is as the length of the arc from the lowest point, which is also true in a circle when the arcs described are small.

$$\therefore u = x = m \tan z - \frac{m\,n}{\cos^2 z} \cdot x,$$

$$\therefore \left(1 + \frac{m\,n}{\cos^2 z}\right) x = m \tan z:$$

$$\therefore x = m \frac{\tan z \cdot \cos^2 z}{m\,n + \cos^2 z}$$

$$= \frac{m}{2} \cdot \frac{\sin 2z}{m\,n + \cos^2 z}.$$

Q. E. D.

2. If $-P_{m-p} \cdot x^{m-p}$, $-P_{m-q} \cdot x^{m-q}$, $-P_{m-r} \cdot x^{m-r}$, $-$ &c. be the negative terms of an equation of m dimensions, then will the greatest root of this equation be less than the sum of the two greatest of the quantities $P^{\frac{1}{p}}_{m-p}$, $P^{\frac{1}{q}}_{m-q}$, &c.

Let $P^{\frac{1}{p}}_{m-p} = P$, $P^{\frac{1}{q}}_{m-q} = Q$, &c. $=$ &c.;

then, transposing all the negative terms to the right hand side of the equation, it may be put in the form

$$x^m + \ldots\ldots = P^p x^{m-p} + Q^q x^{m-q} + R^r x^{m-r} + \ldots\ldots$$

Suppose P to be the greatest, and Q the next greatest, of the quantities P, Q, R, &c.

If $P > x$, *a fortiori*, $P + Q > x$.

If $P < x$, $\frac{P}{x}$ is a proper fraction.

$$\therefore \frac{P^p}{x^p} < \frac{P}{x};$$

$$\therefore P^p x^{m-p} < P x^{m-1},$$

$$< P x^{m-1} + Q^q x^{m-q} + Q^r x^{m-r} + \ldots\ldots,$$

$$(a\ fort.) < P x^{m-1} + Q^2 x^{m-2} + Q^3 x^{m-3} + \ldots\ldots Q^{m-1} x + Q^m,$$

$$< P x^{m-1} + Q^2 . \frac{x^{m-1} - Q x^{m-1}}{x - Q},$$

$$< \frac{P x^m - PQ x^{m-1} + Q^2 x^{m-1} - Q^{m-1}}{x - Q},$$

$$< \frac{P x^m}{x - Q}. \text{ For } PQ > Q^2.$$

$$\therefore x^m + \ldots < \frac{P x^m}{x - Q},$$

$$\therefore (a\, fort.) x^m < \frac{P x^m}{x - Q};$$

$$\therefore x < P + Q.$$

Q. E. D.

3. a and b are respectively the first term and common difference of an arithmetic series,

S_n the sum of n terms,
S_{n+1} .... $(n + 1)$ terms,
&c. .... &c.

prove that $S_n + S_{n+1} + S_{n+2} + \ldots$ to n terms.

$$= (3n - 1) . n . \frac{a}{2} + (7n - 2) . (n - 1) . n . \frac{b}{6} .$$

Let S'_x represent the sum of x terms of the series

$$S_n + S_{n+1} + \ldots, \text{ where } S_n = \frac{n}{2} \left\{ 2a + (n - 1) b \right\}.$$

$$\text{Hence } \Delta S'_x = S'_{n+x} = (n + x) a + (n + x)(n + x - 1) \frac{b}{2};$$

$$\therefore S'_x = \Sigma S'_{n+x} = (n + x)(n + x - 1) \frac{a}{2}$$

$$+ (n + x)(n + x - 1)(n + x - 2) \frac{b}{6} + C.$$

But $S'_x = 0$, when $x = 0$;

$$\therefore S'_0 = n (n - 1) \frac{a}{2} + n (n - 1)(n - 2) \frac{b}{6} + C = 0,$$

$$\therefore C = -n(n-1)\frac{a}{2} - n(n-1)(n-2)\frac{b}{6}.$$

Substituting this value of C, and making $x = n$, we have

$$S^1{}_n = \{2n(2n-1) - n(n-1)\}\frac{a}{2}$$
$$+ \{2n(2n-1)(2n-2) - n(n-1)(n-2)\}\frac{b}{6},$$
$$= (3n-1)n \cdot \frac{a}{2} + (7n-2)(n-1)n \cdot \frac{b}{6}.$$

Q. E. D.

4. A vertical prismatic column, the horizontal section of which is an equilateral and equiangular pentagon, is cut by a given plane; find the sides and angles of the section.

Conceive the prism to be inscribed in a cylinder, so that the horizontal sections of the two solids may be regular pentagons inscribed in circles, and the oblique sections irregular pentatagons inscribed in ellipses.

Let AB be the diameter of the circular section APB; C being the centre, and P any one of the angular points of the polygon.

Draw AD, meeting the vertical side BE of the cylinder in any point D. Then, if the oblique section be perpendicular to the plane ABE, AD may be the axis major of the circumscribing ellipse, and the angle DAB the angle which the oblique section makes with the horizontal section. Draw PQ vertical, and meeting the circumference of the ellipse AQD in Q: then, as all the points Q are in the same plane, the height of any one of them above the section APB will be as its distance from a tangent plane, to the cylinder, passing through A; that is, (letting fall the perpendicular PN on AB) as AN, which is as the versed sine of angle ACP.

Let $\angle\ DAB = \alpha,\ AC = a,$
$\angle\ ACP = \beta,\ PQ = z.$

Then, since $\frac{BD}{AB} = \frac{PQ}{AN}$, $\tan \alpha = \frac{z}{2\, a \,.\, \text{ver sin}\, \beta}$,

or $z = 2\, a \,.\, \tan \alpha \,.\, \text{ver sin}\, \beta.$

Therefore, if $P_1, P_2, \ldots\ldots P_n$ be the angular points of a regular polygon of n sides, inscribed in the circle APB, and $Q_1, Q_2, \ldots\ldots\ldots Q_n$ the corresponding points of the irregular polygon inscribed in the ellipse:

making $\angle\, PCP_1 = \frac{2\pi}{n}$, $\angle\, PCP_2 = \frac{4\pi}{n}, \ldots\ldots \angle\, PCP_m = \frac{2m\pi}{n}$,

and $\quad P_1 Q_1 = z_1, \ldots\ldots P_m Q_m = z_m,$

we shall have, in general,

$$z_m = 2\, a \,.\, \tan \alpha \,.\, \text{ver sin} \left(\beta + \frac{2m\pi}{n}\right).$$

So that, $h = a \sin \frac{\pi}{n}$ being the side of the regular polygon, the corresponding side of the irregular polygon will be

$$Q_m Q_{m+1} = \sqrt{\left\{h^2 + (z_{m+1} - z_m)^2\right\}}.$$

In the above particular case, $n = 5$; and giving m the several values 1, 2, 3, 4, 5, we obtain the several expressions for the respective lengths of the sides of the oblique section.

To find the angles, we have only to calculate, from the general form, the line $Q_m\ Q_{m+2}$, and then, from the three sides of the triangle $Q_m\ Q_{m+1}\ Q_{m+2}$ to find one angle.

Having obtained, in this manner, $n - 1$ equations, and adding the condition that the sum of the angles of the section $= (n - 2)\, \pi$, the angles are completely determined.

This solution, which is perfectly general, admits of very considerable simplification in particular cases.

5. Eliminate by differentiation $f\left(\frac{y}{x}\right)$ and $\phi\ (x\,y)$ from the equation

$$z = x f\left(\frac{y}{x}\right) + \phi\ (y\,x).$$

Let $\frac{d\,z}{d\,x} = p,\ \frac{d\,z}{d\,y} = q;$

$$\frac{d\,p}{d\,x} = r,\ \frac{d\,p}{d\,y} = \frac{d\,q}{d\,x} = s,\ \frac{d\,q}{d\,y} = t;$$

∴ taking the partial differential co-efficients of

$$z = x\,f\left(\frac{y}{x}\right) + \phi\ (y\,x), \qquad (1)$$

$$q = f'\left(\frac{y}{x}\right) + x\,\phi'\,(y\,x), \qquad (2)$$

$$p = f\left(\frac{y}{x}\right) - \frac{y}{x} f'\left(\frac{y}{x}\right) + y\,\phi'(y\,x)\,; \qquad (3)$$

∴ multiplying (2) by y, and (3) by x, and adding,

$$p\,x + q\,y = x\,f\left(\frac{y}{x}\right) + 2\,x\,y\,\phi'\,(y\,x)\,;$$

∴ subtracting (1) from this,

$$p\,x + q\,y - z = 2\,x\,y\,\phi'\,(y\,x) - \phi(y\,x). \qquad (4)$$

Differentiating (4) with respect to x and y, gives

$$r\,x + s\,y = y\,\phi'(y\,x) + 2\,x\,y^2\,\phi''\,(y\,x), \qquad (5)$$

$$s\,x + t\,y = x\,\phi'\,(y\,x) + 2\,x^2\,y\,\phi''\,(y\,x). \qquad (6)$$

∴ multiplying (5) by x, and (6) by y, and subtracting the last from the last but one of the results, gives

$$r\,x^2 - t\,y^2 = 0,$$

$$\text{or } x^2\,\frac{d^2\,z}{d\,x^2} - y^2\,\frac{d^2\,z}{d\,y^2} = 0.$$

6. Find the locus of the intersections of the tangents of an hyperbola with the perpendiculars upon them from the centre: determine its maximum ordinate, its area, and the angles at which it intersects the axis.

The general equation to the tangent being $y - y' = \frac{d\,y'}{d\,x'}$ $(x - x')$, that of a perpendicular upon it from the origin is $y = -\frac{d\,x'}{d\,y'}\,x$; where x, y denote the co-ordinates of any point in the curve touched, and x', y' those of any point in the tangent;

$\therefore$ the equation of the hyperbola being

$$a^2 y'^2 = b^2 (x'^2 - a^2), \qquad (1).$$

those of its tangent and perpendicular from the centre are found, from the above forms, to be respectively,

$$a^2 y\, y' - b^2 x\, x' + a^2 b^2 = 0, \qquad (2).$$

$$b^2 x' y + a^2 y' x = 0. \qquad (3).$$

Eliminating, therefore, x' and y' from equations (1) (2) and (3) there results the equation

$$(y^2 + x^2)^2 = a^2 x^2 - b^2 y^2,$$

which, as it comprises all the values of x and y which can be common to equations (2) and (3), or the co-ordinates of all the points where an intersection can take place between the lines they represent, is the locus required.

This, when referred to rectangular co-ordinates, by making

$$x = r \cos\theta,\ y = r \sin\theta, \text{ becomes}$$

$$r^2 = (a^2 + b^2) \cos^2\theta - b^2, \qquad (4).$$

$$\text{or } r^2 = \frac{a^2 - b^2}{2} + \frac{a^2 + b^2}{2} \cos 2\,\theta. \qquad (5).$$

Now, y or $r \sin\theta$ will be a maximum when y^2 or $r^2 \sin^2\theta$ is a maximum;

$$\therefore r^2 \sin^2\theta = \frac{a^2 - b^2}{4} - \frac{a^2 + b^2}{4} \cos^2 2\theta + \frac{b^2}{2} \cos 2\theta = \text{a maximum.}$$

$$\therefore \frac{d}{d\theta} . (r^2 \sin^2\theta) = 0,$$

$$\text{or } (a^2 + b^2) \cos 2\theta - b^2 = 0;$$

$$\therefore \cos 2\theta = \frac{b^2}{a^2 + b^2}.$$

This value of $\cos 2\theta$, substituted in the above expression for $r^2 \sin^2\theta$, gives the value of the greatest ordinate

$$= \frac{a^2}{2\sqrt{(a^2 + b^2)}}.$$

To find the angles at which the curve cuts the axis, we have

$$\frac{r\,d\theta}{dr} = -\frac{2\,r^2}{(a^2 + b^2) \sin 2\theta},$$

which becomes infinite when $\theta = 0$ or 180°,

for both of which values of θ, $r = a$;

$\therefore$ the curve cuts the extremities of the axis major of the hyperbola at right angles.

Also, from the equation (4), when

$$r = 0, \ \cos\theta = \pm \frac{b}{\sqrt{(a^2 + b^2)}},$$

at which angles, the radius vector becomes a tangent to the curve at its intersections with the axis at the pole, and limits all the angles which the radii can make with the axis.

Lastly, as the curve is symmetrical on each side of the axes of x and y,

$$\text{its whole area} = 4 \int \frac{r^2\,d\theta}{2} = 2 \int r^2\,d\theta,$$

$$= (a^2 - b^2) . \tan^{-1}\frac{a}{b} + a\,b;$$

the limits of θ being from 0 to $\sin^{-1}\dfrac{a}{\sqrt{(a^2+b^2)}}$ or $\tan^{-1}\dfrac{a}{b}$.

7. Having given the two first terms of the expansion of $(a^2 + b^2 + 2\,a\,b . \cos\theta)^{-\frac{1}{2}}$ in a series of the form $A_0 + A_1 . \cos\theta + A_2 . \cos 2\theta + \&c.$ shew how, from them the two first terms of the expansion of

$$(a^2 + b^2 + 2\,a\,b\cos\theta)^{-\frac{3}{2}}$$

may be determined.*

8. If S_1 represent the sum† of the ordinates in the quadrant of a circle whose radius is 1,

S_2 represent the sum of their squares,
S_3 .... cubes,
&c. .... &c.

prove that $S_{n-1} . S_n = \dfrac{3}{n+1} . S_1 . S_2$.

Since the magnitude of each sine depends upon its distance (x) from the centre of the circle, we may make this distance the independent variable.

$$\therefore \frac{d\,S_m}{d\,x} = \sin^m\theta,$$

$\therefore\; S_m = \int d\,x \sin^m\theta = -\int d\,\theta . \sin^{m+1}\theta$, since $x = \cos\theta$; the sign $\int$ being supposed, throughout this problem, to denote an integral taken between the limit $\theta = 0$, and $\theta = \dfrac{\pi}{2}$.

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* See Woodhouse's Physical Astronomy, p. 294.

† The *limit* of the sum.

But, in general, $\int \sin^m \theta = -\frac{\sin^{m-1}\theta \, . \cos\theta}{m}$

$$+ \frac{m-1}{m} . \int d\theta \, . \sin^{m-2}\theta;$$

$$\therefore S_m = \frac{\sin^m \theta \, . \cos\theta}{m+1} - \frac{m}{m+1} . \int d\,\theta \, . \sin^{m-1}\theta.$$

Therefore, giving, successively, to m the values

n, $n-1$, 1, 2, there result

$$S_{n-1} \, . \, S_n = \frac{n\,(n-1)\,\ldots\ldots 3\,.\,2\,.\,1}{(n+1)\,n\,\ldots\ldots 3\,.\,2\,.\,1} \, . \, \frac{\pi}{2},$$

$$S_1 \, . \, S_2 = \frac{\pi}{2\,.\,3}:$$

which expressions are the same whether n be even or odd, since, if n be even, $n - 1$ is odd, and *vice versa* ;

$$\therefore \frac{S_{n-1} \, . \, S_n}{S_1 \, . \, S_2} = \frac{3}{n+1}, \text{ or } S_{n-1} \, . \, S_n = \frac{3}{n+1} \, . \, S_1 \, . \, S_2.$$

Q. E. D.

9. A straight line, revolving in its own plane about a given point, intersects a curve line in two points; find the curve when the rectangle of the lines intercepted between the given point and the points of intersection is constant.

Making the given point the origin, let (x, y), (x', y') be the respective co-ordinates of the two points of intersection, r, r' their respective distances from the origin.

Assume $y = \phi\,(r)$, and $\therefore y' = \phi\,(r')$; also, to generalize the problem, let r' be any given function of r, or let $r' = \alpha\,(r)$. Then, from similar triangles,

$$\frac{y'}{r'} = \frac{y}{r}, \text{ or } \frac{\phi\,\{\alpha\,(r)\}}{\alpha\,(r)} = \frac{\phi\,(r)}{r};$$

from which equation the form of $\phi\,(r)$ is to be determined.

Now, it is evident that this last equation will be satisfied if we can find such a function $f(r) = \frac{\phi(r)}{r}$, that, when $a(r)$ is substituted for r, it may not be changed; that is, if we can find solutions of the equation $f\{a(r)\} = f(r)$.

But the solution of this equation is $f(r)$* $= \chi\{r, a(r)\}$, where χ denotes an arbitrary but symmetrical function of r and $a(r)$, when a is such that $a\{a(r)\} = r$.

This condition is fulfilled in the above problem, since

$$r\,r' = a^2, \text{ or } a(r) = \frac{a^2}{r};$$

and therefore the equation comprising all the curves which satisfy the required condition is

$$\begin{aligned} y &= \phi(r), \\ &= r f(r), \left(\text{since } f(r) = \frac{\phi(r)}{r}\right) \\ &= r \cdot \chi\left(r, \frac{a^2}{r}\right). \end{aligned}$$

One of the most obvious cases of this is when

$$\chi\left(r, \frac{a^2}{r}\right) = \frac{1}{2b}\left(r + \frac{a^2}{r}\right),$$

$$\text{and } \therefore 2by = r^2 + a^2,$$

$$= x^2 + y^2 + a^2,$$

an equation to the circle.

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* Babbage on Functional Equations, p. 12. See also Philosophical Transactions for 1815.

10. Prove that if the tangent plane to any curve surface make with the three co-ordinate planes the least possible volume, the distance of the intersections of the plane and axes from the origin are respectively $3x$, $3y$, and $3z$, x, y and z being the co-ordinates of the point of contact.

Let a, b, c be the respective distances of the intersections of the plane and axes of x, y, z: this plane will then have for its equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \quad (A).$$

Now, since the volume cut off by the plane, and which is expressed by $\frac{a\,b\,c}{6}$, is to be a maximum; $\therefore$ $a\,b\,c$ (which let $= u$) will be a maximum. Hence, determining c from (A), and

$$\therefore \text{ making } u = \frac{a^2\,b^2\,z}{a\,b - b\,x - a\,y};$$

the theory of maxima and minima gives the equations

$$\frac{d\,u}{d\,a} = 0, \quad \frac{d\,u}{d\,b} = 0,$$

$$\text{and } \therefore \; 2\,(a\,b - b\,x - a\,y) - a\,(b - y) = 0,$$

$$2\,(a\,b - b\,x - a\,y) - b\,(a - x) = 0.$$

From these last two equations, we find $a = 3\,x$, and $b = 3\,y$; which values, substituted in (A) give $c = 3\,z$.

Q. E. D.

11. A plane is so moved as always to cut off from a given paraboloid of revolution equal volumes; determine the equation to the surface to which it is always a tangent.

Let the indefinite straight line AX be the axis of the generating parabola APQ; the vertex being at A. Draw the chord

QVQ' parallel to the tangent at any point P, draw PV parallel to AX, and from P let fall the perpendicular NP on AX.

Suppose now, the plane that cuts off equal volumes from the paraboloid, to be perpendicular to the plane APQ, and to pass through QQ': then (Hustler's Conic Sections, p. 65.) the section will be an ellipse, whose major axis is QQ', and whose semi-axis minor is a mean proportional between the latus rectum of the parabola and PV.

Let the latus rectum $= a$, $AN = z$, constant volume $= c$.
parameter at $P = p$, $\angle PVA = \theta$,

Now, since the volume of any segment of a paraboloid, cut off by a plane, $= \frac{1}{2}$ its circumscribing cylinder,

$$= \text{area of elliptic section} \times \text{altitude of cylinder};$$

$$\therefore c = \frac{\pi}{2} \sqrt{(a \,.\, PV \,.\, QV)} \,.\, PV \sin\theta,$$

$$= \frac{\pi}{2} \sqrt{(ap)} \,.\, \sin\theta \,.\, PV.$$

$$\text{But } p = a + 4z, \text{ and } \sin\theta = \sqrt{\frac{a}{a+4z}};$$

$$\therefore c = \frac{\pi a}{2} \,.\, PV^2, \text{ or } PV = \sqrt{\frac{2c}{\pi a}}.$$

Hence, it appears that if c be constant, PV is constant; and $\therefore$ that a plane cutting off a given volume from the paraboloid is parallel to a tangent plane at any point P, and at a given distance PV from it, in the direction of the axis AX.

If therefore the equation to the surface of the given paraboloid be $x^2 + y^2 = az$, and consequently, that of its tangent plane $2(xx' + yy') = az + az'$; the tangent plane to the surface required will have for its equation

$$2(xx' + yy') = a\left\{z - \sqrt{\frac{2c}{\pi a}}\right\} + az'.$$

∴ comparing this last equation with the two preceding, we have, for the surface required, the equation

$$x^2 + y^2 = a \left\{ z - \sqrt{\frac{2c}{\pi a}} \right\};$$

which is that of a paraboloid equal, and similarly situated to the given paraboloid; the distance between the vertices being $= \sqrt{\frac{2c}{\pi a}}$.

12. Integrate the following differentials and differential equations :

$$\frac{dx}{x^4 + 1}, \quad \frac{dx}{x^4 \sqrt{1 - x^2}}, \quad \frac{dx}{\sqrt{a - x} - \sqrt{x}}$$

$$(x^2 + y^2)\, dx + x^2 y\, dy = 0, \quad (1 + x) \cdot \frac{d^2 y}{dx^2} + a \cdot \frac{dy}{dx} = 0,$$

$$1 + p^2 + q^2 = m^2,$$

and also the following equations of differences :

$$f(x^2) - f(x) = m, \text{ and } u_x \cdot u_{\pi + x} = k^2.$$

To integrate $\frac{dx}{x^4 + 1}$,

since $x^4 + 1 = (x^2 + x\sqrt{2} + 1)(x^2 - x\sqrt{2} + 1)$,

we may assume $\frac{1}{x^4 + 1} = \frac{Ax + B}{x^2 + x\sqrt{2} + 1} + \frac{A'x + B'}{x^2 - x\sqrt{2} + 1}$.

Clearing the equation of fractions, arranging according to the powers of x, and transposing all the terms to one side, give

$$(A + A')\, x^3 + (A'\sqrt{2} - A\sqrt{2} + B + B')\, x^2$$
$$+ (A + A' + B'\sqrt{2} - B\sqrt{2})\, x + B + B' - 1 = 0.$$

∴ making each of the co-efficients of $x = 0$, and determining A, A', B, B' from the four equations thus produced; there result

$$A = \frac{1}{2\sqrt{2}}, \quad A' = -\frac{1}{2\sqrt{2}}, \quad B = B' = \frac{1}{2},$$

and ∴ $\frac{1}{x^4 + 1} = \frac{1}{2\sqrt{2}} \cdot \frac{x + \sqrt{2}}{x^2 + x\sqrt{2} + 1} - \frac{1}{2\sqrt{2}} \cdot \frac{x - \sqrt{2}}{x^2 - x\sqrt{2} + 1}$.

To make this integrable, assume, in the first fraction $x = u - \frac{1}{\sqrt{2}}$, and in the second, $x = v + \frac{1}{\sqrt{2}}$;

$$\therefore \frac{dx}{x^4+1} = \frac{1}{2\sqrt{2}} \cdot \frac{\left(u + \frac{1}{\sqrt{2}}\right)}{u^2 + \frac{1}{2}} du - \frac{1}{2\sqrt{2}} \cdot \frac{\left(v - \frac{1}{\sqrt{2}}\right)}{v^2 + \frac{1}{2}} dv,$$

$$= \frac{1}{2\sqrt{2}} \left\{ \frac{u\,du}{u^2 + \frac{1}{2}} - \frac{v\,dv}{v^2 + \frac{1}{2}} \right\} + \frac{1}{2\sqrt{2}} \left\{ \frac{\sqrt{2}\,.\,du}{1 + 2u^2} + \frac{\sqrt{2}\,.\,dv}{1 + 2v^2} \right\}.$$

$$\therefore \int \frac{dx}{x^4+1} = \frac{1}{4\sqrt{2}} \,.\log.\, \frac{1 + x\sqrt{2} + x^2}{1 - x\sqrt{2} + x^2}$$

$$+ \frac{1}{2\sqrt{2}} \left\{ \tan^{-1}(x\sqrt{2} + 1) + \tan^{-1}(x\sqrt{2} - 1) \right\}$$

$$= \frac{1}{4\sqrt{2}} \,.\log.\, \frac{1 + x\sqrt{2} + x^2}{1 - x\sqrt{2} + x^2} + \frac{1}{2\sqrt{2}} \,.\tan^{-1} \frac{x\sqrt{2}}{1 - x^2}.$$

To integrate $\frac{dx}{x^4\sqrt{(1-x^2)}}$,

Assume

$$P_3 = \frac{\sqrt{(1-x^2)}}{x^3}; \;\therefore\; \frac{dP_3}{dx} = -\frac{3\sqrt{(1-x^2)}}{x^4} - \frac{1}{x^2\sqrt{(1-x^2)}}$$

$$= -\frac{3}{x^4\sqrt{(1-x^2)}} + \frac{2}{x^2\sqrt{(1-x^2)}};$$

$$\therefore P_3 = -3\int \frac{dx}{x^4\sqrt{(1-x^2)}} + 2\int \frac{dx}{x^2\sqrt{(1-x^2)}} = \frac{\sqrt{(1-x^2)}}{x^3},$$

$$\therefore \int \frac{dx}{x^4\sqrt{(1-x^2)}} = -\frac{\sqrt{(1-x^2)}}{3x^3} + \frac{2}{3}\int \frac{dx}{\sqrt{(1-x^2)}}.$$

Again, let $P_1 = \frac{\sqrt{(1-x^2)}}{x}$; $\therefore \frac{d P_1}{d x} = - \frac{\sqrt{(1-x^2)}}{x^2}$

$$- \frac{1}{\sqrt{(1-x^2)}} = \frac{1}{x^2 \sqrt{(1-x^2)}},$$

$$\therefore P_1 = \int \frac{d x}{x^2 \sqrt{(1-x^2)}} = \frac{\sqrt{(1-x^2)}}{x},$$

$$\therefore \int \frac{d x}{x^4 \sqrt{(1-x^2)}} = - \sqrt{(1-x^2)}. \left(\frac{1}{3 x^3} + \frac{2}{3 x}\right).$$

To integrate

$$\frac{d x}{\sqrt{(a-x)} - \sqrt{x}}, \text{ assume } \sqrt{(a-x)} - \sqrt{x} = u,$$

$$\therefore x = \frac{a}{2} + \frac{1}{2} \sqrt{(2 a u^2 - u^4)},$$

$$\text{and } d x = \frac{(a - u^2) d u}{\sqrt{(2 a - u^2)}};$$

$$\therefore \int \frac{d x}{\sqrt{(a-x)} - \sqrt{x}} = \int \frac{(a-u^2) d u}{u \sqrt{(2 a - u^2)}}$$

$$= a \int \frac{d u}{u \sqrt{(2 a - u^2)}} - \int \frac{u d u}{\sqrt{(2 a - u^2)}}$$

$$= \sqrt{(2 a - u^2)} + \frac{1}{2} \sqrt{\frac{a}{2}} . \log \left\{ \frac{\sqrt{(2 a - u^2)} - \sqrt{(2 a)}}{\sqrt{(2 a - u^2)} + \sqrt{(2 a)}} \right\}$$

$$= \sqrt{\left\{ a + 2 \sqrt{(a x - x^2)} \right\}}$$

$$+ \frac{1}{2} \sqrt{\frac{a}{2}} . \log \left\{ \frac{\sqrt{\left\{a + 2 \sqrt{(a x - x^2)}\right\}} - \sqrt{(2 a)}}{\sqrt{\left\{a + 2 \sqrt{(a x - x^2)}\right\}} + \sqrt{(2 a)}} \right\}.$$

To integrate $(x^2 + y^2) d x + x^2 y d x = 0$,

assume $y^2 = 2 z$, which gives it the form

$$d z + \frac{2}{x^2} . z d z = - d x,$$

which being that of a linear equation, we have

$$z \text{ or } \frac{y^2}{2} = \varepsilon^{\frac{2}{x}} \left\{ -\int \varepsilon^{-\frac{2}{x}} dx + C \right\}.$$

But $\varepsilon^{-\frac{2}{x}} = 1 - \frac{1}{1} \cdot \frac{2}{x} + \frac{1}{1.2} \cdot \frac{2^2}{x^2} - \ldots$

$$+ (-1)^{m+1} \cdot \frac{2^{m+1}}{1.2.3 \ldots m(m+1)x^{m+1}} + \ldots;$$

m being the number of terms after the 2nd,

$$\therefore \int \varepsilon^{-\frac{2}{x}} = x - 2 \log x - \frac{1}{1^2.2} \cdot \frac{2^2}{x} + \ldots$$

$$+ (-1)^m \cdot \frac{2^{m+1}}{1.2 \ldots m^2(m+1)x^m} + \ldots;$$

$$\therefore \frac{y^2}{2} = c\varepsilon^{\frac{2}{x}} - \varepsilon^{\frac{2}{x}} \left\{ x - 2 \log x - \frac{1}{1^2.2} \cdot \frac{2^2}{x} + \ldots \right.$$

$$\left. + (-1)^m \cdot \frac{2^{m+1}}{1.2 \ldots m^2(m+1)x^m} \right\} + \ldots$$

To integrate $(1 + x)\frac{d^2 y}{dx^2} + a \cdot \frac{dy}{dx} = 0$,

$$\text{let } \frac{dy}{dx} = p, \therefore \frac{d^2 y}{dx^2} = \frac{dp}{dx};$$

$$\therefore (1 + x)\frac{dp}{dx} + ap = 0, \text{ or } \frac{dp}{p} + \frac{a\,dx}{1 + x} = 0.$$

$$\therefore \log p + a \log (1 + x) = \log c,$$

$$\text{or} \qquad p(1 + x)^a = c;$$

$$\therefore \frac{dy}{dx} = \frac{c\,dx}{(1 + x)^a} = c \cdot \frac{d(1 + x)}{(1 + x)^a},$$

$$\therefore y = C - \frac{c}{(a-1)(1 + x)^{a-1}}.$$

To integrate $1 + p^2 + q^2 = m^2$,

consider q as a function of p, x, y, z, and substitute for $\frac{dq}{dx}$ and $\frac{dq}{dz}$ in the equation of condition,

$$\frac{dp}{dy} - \frac{dq}{dx} + q\frac{dp}{dz} - p\frac{dq}{dz} = 0;$$

this will lead to a partial differential equation of four variables p, x, y, z, the integral of which expresses the value of p, and therefore of q in terms of x, y, z, and an arbitrary constant a. Substituting these values of p and q in the equation

$$dz = p\,dx + q\,dy,$$

and integrating, we shall find

$$f(x, y, z, a) = \beta = \phi(a);$$

from which a may be eliminated by differentiation, when a particular value of ϕ is assigned. (*Peacock's Examples*, p. 445.)

Now, from the proposed equation, $q = \sqrt{(m^2 - p^2 - 1)}$, and

$$\therefore \frac{dq}{dp} = -\frac{p}{\sqrt{(m^2 - p^2 - 1)}}; \text{ whence } \frac{dq}{dx} = \frac{dq}{dp}\cdot\frac{dp}{dx}$$

$$= -\frac{p}{\sqrt{(m^2 - p^2 - 1)}}\cdot\frac{dp}{dx}.$$

$$\text{Also } \frac{dq}{dz} = \frac{dq}{dp}\cdot\frac{dp}{dz} = \frac{dq}{dp}\cdot\frac{dp}{dx}\cdot\frac{1}{p} = -\frac{1}{\sqrt{(m^2 - p^2 - 1)}}\cdot\frac{dp}{dx}.$$

The above equation of condition thus becomes

$$\frac{dp}{dy} + \frac{2p}{\sqrt{(m^2 - p^2 - 1)}}\cdot\frac{dp}{dx} + \sqrt{(m^2 - p^2 - 1)}.\frac{dp}{dz} = 0;$$

$$\therefore dp = 0, \text{ or } p = a, \text{ and } \therefore q = \sqrt{(m^2 - a^2 - 1)}.$$

Substituting these values of p and q in the equation $dz - p\,dx - q\,dy = 0$, and integrating, we get

$$z - ax - \sqrt{(m^2 - a^2 - 1)}y = \phi(a);$$

and, differentiating this with respect to a,

$$-x+\frac{ay}{\sqrt{(m^2-a^2-1)}}=\phi'(a).$$

From this last equation, together with the preceding, we may eliminate a, after arbitrarily assigning the value of ϕ.

We now proceed to the solution of the equation

$$f(x^2)-f(x)=m.$$

The method, given by Laplace, for the reduction of equations of this kind to equations of Differences, being rarely met with in Elementary Treatises, a brief explanation of it may not be unacceptable.

This method applies to every equation of the form

$$f\{\alpha(x)\}+X.f(x)+X'=0, \qquad (1)$$

where $\alpha(x)$, X, and X' denote given functions of x, and $f(x)$ the function to be determined.

Let $x=u_z$, where u_z is such a function of z, that

$$\alpha(x)=u_{z+1},\ \alpha\{\alpha(x)\}=u_{z+2}\ \&c.=\&c.;$$

from which condition $u_{z+1}=\alpha(u_z)$, an equation from which u_z may be determined, since the form of α is known. By the substitution of u_z for x in equation (1), it becomes

$$f(u_{z+1})+Z.f(u_z)+Z'=0;$$

or, making $f(u_z)=v_z$, and $f(u_{z+1})=v_{z+1}$,

$$v_{z+1}+Z.v_z+Z'=0, \qquad (2),$$

Z and Z' being the same functions of z that X and X' are of x.

Lastly, determining v_z from equation (2),

we have v_z or $f(u_z)=F(z,C)$,

which determines the form of f.

In the above example, we have $a(x) = x^2$, or $u_{z+1} = u_z^2$, an equation which is satisfied by making $u_z = C^{2^z}$;

$$\therefore\ x = C^{2^z},\ \therefore\ z = \log\,(\log x^c)^{\frac{1}{\log 2}}.$$

The proposed equation thus becomes

$$f(u_{z+1}) - f(u_z) = m,$$

$$\text{or (making}\quad f(u_z) = v_z),$$

$$v_{z+1} - v_z = m,\ \therefore\ v_z = mz + C'$$

$$\text{or } f(x) = m\,.\log\,(\log x^c)^{\frac{1}{\log 2}} + C'.$$

For a more extensive application of this method, see a very ingenious paper by Mr. Herschel in Vol. I. of the Cambridge Philosophical Transactions.

To integrate the equation $u_x\,.\,u_{x+\pi} = k^2$, we may observe that

$$\cos x + \cos(\pi + x) = 0,$$

$$\therefore\ c^{\cos x}\,.\,c^{\cos(\pi + x)} = 1,$$

$$\therefore\ k^2\,.\,c^{\cos x}\,.\,c^{\cos(\pi+x)} = k^2;$$

$$\therefore\ u_x = c^{\cos x}.$$

As this value of u_x satisfies the proposed equation, and moreover contains an arbitrary constant c, it is the complete solution.

13. There are two urns A and B, the former containing three white and the latter three black balls: a ball is taken from each at the same time and put into the other, and this operation is repeated three times; what is the probability that A will contain three black and B three white balls?

In addition to the two principles given in Arts. 433, 443 of Wood's Algebra, we shall make use of the following one of Laplace.*

The probability that two events will take place, when one is dependent on the other, is the product of the probability of one of these events, by the probability that, this event having taken place, the other will take place.

After the first exchange, the urn A will contain one black and two white balls; and the urn B one white and two black balls. Then, the separate probabilities that, in the second exchange, a white ball will be taken from A, and a black one from B, being each $\frac{2}{3}$; the probability that both these events will happen will be $\frac{2}{3}\cdot\frac{2}{3}$ or $\frac{4}{9}$. Now, supposing these events to have taken place, or that the urn A now contains one white and two black balls, and the urn B, one black and two white balls; the separate probabilities, that, in the third exchange, a white ball will be taken from A, and a black one from B, are each $\frac{1}{3}$; and therefore the probability that the exchange will be made in this manner is $\frac{1}{3}\cdot\frac{1}{3}$ or $\frac{1}{9}$. Lastly, since the probabilities that the second and third exchanges will be effected in the manner above specified being respectively $\frac{4}{9}$ and $\frac{1}{9}$; we shall have (by applying the principle of Laplace) the probability that the exchanges will be made according to the conditions of the problem, $=\frac{4}{9}\cdot\frac{1}{9}$ or $\frac{4}{81}$.

14. A uniform rod rests with one of its extremities in a semi-circle whose axis is vertical, find the nature of the line supporting its other extremity so that it may rest in every position.

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* "La probabilite d'un evenement compose de deux evenements simples, est le produit de la probabilite d'un de ces evenemens, par la probabilite que cet evenement etant arrive, l'autre evenement aura lieu." *Theorie Analytique des Probabilites, p.* 181.

The conditions of equilibrium will be the same, if, for the uniform rod, we substitute two material points P, Q, connected by an inflexible straight line, and then separately consider the forces acting on P and Q. Now the forces acting on P are, 1st. gravity in a vertical direction, 2nd, the re-action of the curve on which it rests in a normal direction, and 3rd, a force or *thrust* in the direction QP in consequence of its connection with Q, which is evidently equal to the force of P on Q in the direction PQ, since the rod remains at rest. By thus taking into consideration all the forces that act on P and Q, the equations of equilibrium of the rod will be the same as for two free points, subject, however, to the condition of remaining at an invariable distance from each other: so that if X, Y denote respectively the whole forces acting on P in the directions of x and y; and X', Y' similar forces acting on Q, we shall have the four equations $X=0$, $Y=0$, $X'=0$, $Y'=0$, which together with an equation expressing the invariability of the distance of P from Q, and that of the curve on which one of the points P, Q rests, will furnish us with the complete solution of the problem.

Taking the lowest point of the semi-circle for the origin of co-ordinates, and a vertical line for the axis of x, let x, y be the co-ordinates of any point in the semi-circle, or of P; and x', y' those of any point in the required curve, or of Q. Let R, R' be the respective re-actions of the semi-circle, and the other curve on P and Q, T the *thrust* or force in direction of the rod's length, θ the variable angle which the rod makes with the horizon in any position, and L its length.

By the above principles we have for the equilibrium of P, the equations

$$X = R\frac{dy}{ds} - T\sin\theta - g = 0, \qquad (1).$$

$$Y = -R\frac{dx}{ds} + T\cos\theta = 0, \qquad (2).$$

Similarly, for Q,

$$X' = R'\frac{dy'}{ds'} + T\sin\theta - g = 0, \qquad (3).$$

$$Y' = -R'\frac{dx'}{ds'} + T\cos\theta = 0, \qquad (4).$$

The invariability of the distance of P from Q is expressed by the equation

$$(x' - x)^2 + (y' - y)^2 = L^2. \qquad (5).$$

Multiplying (1) by dx, (2) by dy, and adding, we get

$$T\cos\theta \,.\, dy - T\sin\theta \,.\, dx - g\,dx = 0.$$

Similarly, from (3) and (4),

$$T\cos\theta \,.\, dy' + T\sin\theta \,.\, dx' - g\,dx' = 0.$$

Eliminating T from the two last equations,

$$\left(\frac{dy'}{dx'} - \frac{dy}{dx}\right) . \cos\theta + 2\sin\theta = 0,$$

$$\text{or } \frac{dy'}{dx'} - \frac{dy}{dx} + 2\tan\theta = 0.^{*}$$

But $L\sin\theta = x' - x$, and $L\cos\theta = y' - y$;

$$\therefore \tan\theta = \frac{x' - x}{y' - y},$$

$$\therefore \frac{dy'}{dx'} - \frac{dy}{dx} + 2 \,.\, \frac{x' - x}{y' - y}.$$

Now, by this last equation, combined with (5), and the equation of the curve in which P rests, we may eliminate x and y; and the resulting equation will be that of the curve required.

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* This result agrees with Cor. 1. of the Problem given in p. 85. of Whewell's Mechanics.

We have, therefore, in the present case, to eliminate x and y from the equations,

$$\frac{d y'}{d x'} - \frac{a - x}{y} + 2 \cdot \frac{x' - x}{y' - y}, \qquad (\alpha)$$

$$(x' - x)^2 + (y' - y)^2 = L^2, \qquad (\beta)$$

$$y^2 = 2 a x - x^2 : \qquad (\gamma)$$

(a being the radius of the semi-circle).

But as this elimination is of a difficult nature, and our only object is to determine the nature of the curve, we may, without obtaining a final equation, construct the curve by means of the equations (α), (β), (γ), conjointly. Now the equation (α) is readily integrable for particular values of x, y and $\frac{d y}{d x}$, and the integral thus found would be the equation to a curve, which, though not the curve required, might yet be made subservient to its construction. In fact, it is evident that only an indefinitely small portion of such a curve would satisfy the conditions of the problem, and that its intersection with another immediately consecutive, and described according to the same law, would be a point in the curve required; we may, therefore, infer that the required curve is the locus of the continual intersections of all the curves derived from the equation (α) by giving all possible values to x, y, and $\frac{d y}{d x}$.

To apply this reasoning, let

$$x = h, \; y = k, \; \frac{d y}{d x} = m;$$

then, eliminating $y' - y$ by equation (β), the equation (α) becomes

$$\frac{d y'}{d x'} - m + \frac{2 (x' - h)}{\sqrt{\{L^2 - (x' - h)^2\}}} = 0,$$

∴ by integration,

$$y' = m x' + c + 2\sqrt{\{L^2 - (x' - h)^2\}}, \qquad (\delta).$$

c being an arbitrary constant which may be suppressed, as we only wish to find a curve satisfying the equation (a). This curve being an ellipse whose species, magnitude, and position, are expressed in known functions of h, the curve required is therefore the locus of the continual intersections of a set of ellipses described according to a law expressed by the equation

$$y - m x = 2\sqrt{\{L^2 - (x - h)\}},$$

in which h is supposed to have every value in succession from 0 to a.

In applying the principle of virtual velocities to this problem, we should have commenced by assuming the equations

$$X\, dx + Y dy = 0,$$
$$X' dx' + Y dy' = 0.$$

15. Having given the variation of the obliquity of the ecliptic; find the corresponding variations in right ascension and declination.

In the right-angled spherical triangle ABC, right-angled at A, let $BC = l$ be an arc of the ecliptic to which corresponds the arc $BD = a$ of the equator, the $\angle\, ABC = \omega$ being the obliquity:

$$\text{then } \cos\omega = \tan a \,.\, \cot l, \text{ or } \frac{\cos\omega}{\tan a} = \cot l.$$

If $\delta\omega$, δa denote corresponding small variations of ω and a, we may obtain the relation between them, from the above equation, in the same manner as for differentials;

$$\therefore \sin\omega \,.\, \tan a \,.\, \delta\omega + \frac{\cos\omega}{\cos^2 a} \,.\, \delta a = 0,$$

$$\text{or } \sin\omega \,.\, \sin a \,.\, \cos a \,.\, \delta\omega + \cos\omega \,.\, \delta a = 0,$$

$$\text{or } \sin 2a \,.\, \delta\omega + 2 \cot \omega \,.\, \delta a = 0,$$

$$\therefore \; \delta a = \frac{\sin 2a}{2 \cot \omega} \,.\, \delta \omega.$$

Again, since $\dfrac{\sin D}{\sin \omega} = \sin l$, ($D$ being the declination.).

$$\sin \omega \,.\, \cos D \,.\, \delta D - \sin D \,.\, \cos \omega \,.\, \delta \omega;$$

$$\therefore \; \delta D = \tan D \,.\, \cot \omega \,.\, \delta \omega.$$

16. Determine the latitude of the place of observation from observing the times of the rising of two known stars.

The times of rising are supposed to be observed by the clock, and therefore the hour angle corresponding to their interval only is known; that is, if h, h' be the respective hour angles (reckoned from the meridian below the pole) at the rising of each star, $h' - h$ is known, while h and h' are unknown.

Let δ, δ' be the respective declinations of the two stars and λ the latitude of the place; then, by Napier's rules,

$$\cos h = \tan \delta \,.\, \tan \lambda,$$

$$\cos h' = \tan \delta' \,.\, \tan \lambda;$$

$$\therefore \cos h + \cos h' = (\tan \delta + \tan \delta') \,.\, \tan \lambda,$$

$$\text{and } \cos h - \cos h' = (\tan \delta - \tan \delta') \,.\, \tan \lambda;$$

$$\text{or } 2 \cos \frac{1}{2}(h' + h) \,.\, \cos \frac{1}{2}(h' - h) = \frac{\sin(\delta + \delta')}{\cos \delta \,.\, \cos \delta'} \,.\, \tan \lambda, \quad (A)$$

$$\text{and } 2 \sin \frac{1}{2}(h' + h) \,.\, \sin \frac{1}{2}(h' - h) = \frac{\sin(\delta - \delta')}{\cos \delta \,.\, \cos \delta'} \,.\, \tan \lambda; \quad (B)$$

$\therefore$ dividing (B) by (A),

$$\tan \frac{1}{2}(h' + h) \,.\, \tan \frac{1}{2}(h' - h) = \frac{\sin(\delta - \delta')}{\sin(\delta + \delta')},$$

$\therefore$ $h' + h$ is known, and $\therefore$ from either of equations (A), (B), λ is known.

17. The axis of a given cone filled with fluid is inclined at a given angle to the horizon; find how much of the fluid will flow out and determine the pressure exercised by the remainder upon the conical surface.

Let *ABDC* be a vertical section of the cone passing through the axis *AD*; from the vertex *A* draw an indefinite horizontal line *AR* in the same plane with this section, and *CH* parallel to *AR*, meeting the upper slant side *AB* in *H*: draw *HK* perpendicular to *AD*, and meeting the lower slant side *AC* in *K*.

Let the axis $AD = h$, semi-angle at vertex $BAD = \alpha$, radius of base $BD = a$, inclination of axis to the horizon $= \beta$.

From the above construction, $HC =$ the major axis of the ellipse made by the surface of the remaining fluid, the minor axis being a mean proportional between *BC* and *HK*. (*Hustler's Conic Sections, p.* 40.)

From the triangles *HBC, HCK,* and *KAH,* we have

$$HC = \frac{2a \cdot \cos\alpha}{\sin(\beta+\alpha)}, \quad HK = 2a \cdot \frac{\sin(\beta-\alpha)}{\sin(\beta+\alpha)},$$

$$AH = \frac{a}{\sin\alpha} \cdot \frac{\sin(\beta-\alpha)}{\sin(\beta+\alpha)};$$

∴ area of the surface of the fluid

$$= \pi \cdot \frac{1}{2} HC \cdot \sqrt{(BC \cdot HK)}$$

$$= \pi a^2 \cdot \cos\alpha \cdot \frac{\{\sin(\beta-\alpha)\}^{\frac{1}{2}}}{\{\sin(\beta+\alpha)\}^{\frac{3}{2}}}.$$

Also, the altitude of the fluid above AR

$$= AH . \sin(\beta + \alpha),$$

$$= \frac{a}{\sin\alpha} . \sin(\beta - \alpha);$$

$\therefore$ volume of remaining fluid

$$= \frac{\pi a^3}{3} . \cot\alpha . \left\{\frac{\sin(\beta - \alpha)}{\sin(\beta + \alpha)}\right\}^{\frac{3}{2}},$$

$$= \frac{\pi a^2 h}{3} . \left\{\frac{\sin(\beta - \alpha)}{\sin(\beta + \alpha)}\right\}^{\frac{3}{2}};$$

$\therefore$ volume of fluid discharged

$$= \frac{\pi a^2 h}{3} . \left\{\frac{\sin^{\frac{3}{2}}(\beta + \alpha) - \sin^{\frac{3}{2}}(\beta - \alpha)}{\sin^{\frac{3}{2}}(\beta + \alpha)}\right\}.*$$

18. A body attracting with a force varying directly as the distance moves uniformly in a straight line; determine the motion of another body situated in the same plane and subject to its influence.

Referring the position of the bodies to two rectangular axes of co-ordinates, let the attracting body move along the axis of x, its co-ordinates being $x' = at$, $y' = 0$, at the end of any time t. Also, let x, y be the co-ordinates of the attracted body at the same time; then r being their mutual distance, we have the force of the attracting body $= m^2 r$; the absolute force being expressed by m^2 for the sake of avoiding irrational forms, as will

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* To find the pressure, together with a general explanation of the steps of the process used for finding the area of any surface by double integrals, see Appendix II.

presently be seen. Resolving, therefore, the force $m^2 r$ into directions of x and y, the general equations of motion give

$$\frac{d^2 x}{d t^2} + m^2 (x - a t) = 0, \qquad (1).$$

$$\frac{d^2 y}{d t^2} + m^2 y = 0, \qquad (2).$$

The integration of these by the process given in page 10, gives

$$x = C_1 . \cos m t + C_2 . \sin m t + a t, \qquad (3).$$

$$y = C . \cos m t + C' . \sin m t, \qquad (4).$$

The arbitrary constants must be determined by some hypothesis respecting the positions of the two bodies, and the velocity of the attracted body, at some given time: let us suppose then that the co-ordinates of the attracted body, at the beginning of the time t, are $x = \alpha$, $y = \beta$, and that it has no other motion than that arising from the attraction of the other body. The first hypothesis reduces the equations (3) and (4) to

$$x = \alpha , \cos m t + C_2 . \sin m t + a t,$$

$$y = \beta . \cos m t + C' . \sin m t ;$$

then since the second hypothesis gives $\frac{d x}{d t} = 0$, and $\frac{d y}{d t} = 0$, when $t = 0$, the equations (3) and (4) finally become

$$x = \alpha . \cos m t - \frac{a}{m t} . \sin m t \qquad (5).$$

$$y = \beta . \cos m t. \qquad (6).$$

Now, to find the equation of the curve described by the attracted body, we have to eliminate t between equations (5) and (6): the last gives $\cos m t = \frac{y}{\beta}$, and $\therefore \sin m t = \sqrt{\left(1 - \frac{y^2}{\beta^2}\right)}$;

$$\therefore x = \frac{a}{\beta} . y - \frac{a}{m \beta} . \sqrt{(\beta^2 - y^2)} + \frac{a}{m} . \cos^{-1} \frac{y}{\beta} .$$

Since, then, the curve is defined by a transcendental equation, it cuts the axis of x in an infinite number of points: to find any number of these points, make $y = 0$, which gives

$$x = \frac{a}{m}(\cos^{-1} 0 - 1),$$

in which the several values of

$$\cos^{-1} 0 \text{ are } \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \&c.$$

To find the times at which the attracted body cuts the path of the attracting body, make $y = 0$ in the equation $y = \beta \cos m t$, and the values of t corresponding to this are

$$\frac{\pi}{2m}, \frac{3\pi}{2m}, \frac{5\pi}{2m}, \&c.$$

To find the angles at which the curve cuts the axis, we have

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{m\beta . \sin m t}{m a . \sin m t + a . \cos m t - a},$$

which, for the above values of t, becomes $\mp \frac{m\beta}{a \mp m a}$, the upper sign being taken when the body passes from the negative to the positive side of the axis of x, and the lower sign in the contrary case.

By making $\frac{dy}{dx} = 0$, it appears that the distance of the attracted body from the axis never exceeds its primitive distance.

19. Determine the orbit described and the time of describing any angle when a body is projected round a centre of force varying as $\frac{1}{D^7}$ at an angle

whose tangent $= \frac{3^{\frac{1}{2}}}{2^{\frac{5}{6}}}$, and with a velocity which is to the velocity in a circle at the same distance :: $\sqrt{2} : \sqrt{3}$.

Let $v =$ velocity
$p =$ perpendicular on tangent } for the distance r;

$V =$ velocity
$P =$ perpendicular on tangent } for the distance R,

the body being supposed to be projected with a velocity V at a distance R, and at an angle δ.

Now, if F denote the variable central force, we have

$$v^2 = C - 2\int F\,d\,r,$$

$$= C - 2\int \frac{m}{r^7}\,.\,d\,r,$$

$$= C + \frac{m}{3\,r^6}.$$

But when $r = R$, $v = V$;

$$\therefore\ V^2 = C + \frac{m}{3\,R^6}, \text{ or } C = V^2 - \frac{m}{3\,R^6}.$$

Again, since (velocity)^2 in circle (rad $= R$)$= FR = \frac{m}{R^6}$,

$$\text{and } V^2 = \frac{2}{3}(\text{velocity})^2 \text{ in circle} = \frac{2}{3}\frac{m}{R^6};$$

$$\therefore\ C = \frac{m}{3\,R^6}.$$

$$\therefore\ v^2 = \frac{m}{3}\left\{\frac{1}{R^6} + \frac{1}{r^6}\right\}.$$

Also, $V^2 : v^2 :: p^2 : P^2$;

$$\text{or } p^2 = \frac{P^2\,V^2}{v^2} = \frac{2\,P^2\,r^6}{R^6 + r^6},$$

which is an equation to the orbit in terms of the perpendicular and distance. To have this in terms of r and an angle θ, we must put for p^2 its expression $\frac{r^4\,d\theta^2}{dr^2 + r^2\,d\theta^2}$, which then gives

$$d\theta = \frac{\sqrt{(2)}\,.\,P\,r\,dr}{\sqrt{\{r^6 - 2P^2\,.\,r^4 + R^6\}}} \qquad (A),$$

Also, since $dt = \frac{r^2\,d\theta}{h}$, where $h = PV$,

$$dt = \frac{\sqrt{(2)}\,.\,P\,r^3\,dr}{\sqrt{\{r^6 - 2P^2\,.\,r^4 + R^6\}}}: \qquad (B),$$

$$\text{where } P = R\sin\delta = \frac{R\sqrt{3}}{\sqrt{(3 + 2^{\frac{5}{3}})}}.$$

By making $r^2 = z$, the expressions (A) and (B) are reduced to

$$d\theta = \frac{P}{\sqrt{(2)}}\,.\,\frac{dz}{\sqrt{(z^3 - 2P^2\,.\,z^2 + R^6)}}, \text{ and}$$

$$dt = \frac{P}{h\sqrt{(2)}}\,.\,\frac{z\,dz}{\sqrt{(z^3 - 2P^2\,.\,z^2 + R^6)}};$$

or

$$dt = \frac{R^3}{2}\,.\,\sqrt{\left(\frac{3}{m}\right)}\,.\,\frac{z\,dz}{\sqrt{(z^3 - 2P^2\,.\,z^2 + R^6)}}.$$

The integration of these expressions, which are called *elliptic transcendants*, may be effected by the process given by Legendre, (*Exercices de Calcul. Integral*, Vol. I. Part I.) by means of elliptic arcs: it may, however, be obtained in the following manner:

Since $z^3 - 2P^2\,.\,z^2 + R^6$ is of the third degree, with respect to z, it may always be resolved into two possible factors, one of the first, and the other of the second degree: we may, therefore, make

$$z^3 - 2P^2 . z^2 + R^6 = \mu (a - \beta z + z^2)(1 - \nu z).$$

$$\text{and } \therefore \frac{dz}{\sqrt{(z^3 - 2P^2 . z^2 + R^6)}} = \frac{1}{\sqrt{\mu}} . \frac{dz}{\sqrt{a - \beta z + z^2)}} . \frac{1}{\sqrt{(1 - \nu z)}}.$$

Next, expanding $\frac{1}{\sqrt{(1 - v z)}}$ by the Binomial Theorem, and multiplying each term by the remaining part of the differential expression, we shall have to integrate a set of terms of the form

$$\frac{A z^n dz}{\sqrt{(a - \beta z + z^2)}}, \text{ in finding } \theta,$$

and a set of terms of the form

$$\frac{B z^{n+1} dz}{\sqrt{(a - \beta z + z^2)}} \text{ in finding } t;$$

the relation between t and θ is thus readily determined.

With the same law of force, and with a velocity which is to that in a circle as 1 to $\sqrt{3}$, the orbit would be the Lemniscata.

20. A body descends down the arc of a vertical catenary having its vertex at the lowest point; find the curve of ascent when the oscillations are isochronous, the two curves being so united at the lowest point as to have a common tangent.

To take the problem in its most general form, let us first suppose the curve of descent to be any given curve whatever.

Let s' be the arc of the given curve corresponding to the vertical abscissa x', measured from the lowest point, and s, x similar quantities for the curve of ascent.

Let the velocity at the lowest point be due to the altitude h; then, since the vertical ascent of the body in the required curve, is equal to its vertical descent in the given curve, the time of the whole oscillation will be expressed by

$$\frac{1}{\sqrt{(2g)}} \cdot \left\{ \int \frac{ds'}{\sqrt{(h-x')}} + \int \frac{ds}{\sqrt{(h-x)}} \right\}, \qquad (A),$$

the limits of the integrals being respectively

$$\left\{ \begin{matrix} x' = 0 \\ x' = h \end{matrix} \right\}, \text{ and } \left\{ \begin{matrix} x = 0 \\ x = h \end{matrix} \right\}.$$

Now, since, by the conditions of the problem, the time of the whole oscillation is to be the same, whatever be the value of h, the integrals in (A) must be independent of h;

$$\text{let then } ds' = \frac{dx' \sqrt{p}}{\sqrt{x'}} + P\,dx', \qquad (B),$$

$$\text{and } ds = \frac{dx \sqrt{q}}{\sqrt{x}} - Q\,dx;$$

p and q being constants, and P and Q certain functions of x' and x, respectively.

Substituting these expressions for ds' and ds in (A), the part multiplied by $\frac{1}{\sqrt{(2g)}}$ becomes

$$\int \frac{dx' \sqrt{p}}{\sqrt{(hx' - x'^2)}} + \int \frac{dx \sqrt{q}}{\sqrt{(hx - x^2)}} + \int \frac{P\,dx'}{\sqrt{(h-x')}} - \int \frac{Q\,dx}{\sqrt{(h-x)}},$$

which, when $x = h$, is to be independent of h.

Now, the two first terms of this already satisfy this condition, since, by effecting the integration, they give

$$\text{ver sin}^{-1} \frac{2x'}{h} \cdot \sqrt{p} + \text{ver sin}^{-1} \frac{2x}{h} \cdot \sqrt{q},$$

which, when $x = h$, and $x' = h$, is equal to

$$\pi \sqrt{p} + \pi \sqrt{q}:$$

if then, we so determine P and Q, that the two last terms may destroy each other, the conditions of the problem will be fulfilled.

But as both these integrals are taken between the same limits, we may make $x' = x$; so that, if P be the same function of x' that Q is of x, these two last terms will destroy each other, and the expression for the time of an oscillation will thus become independent of h.

We thus have, by adding together the equations (B),

$$d s + d s' = (\sqrt{p} + \sqrt{q}) \cdot \frac{d x}{\sqrt{x}},$$

$$\text{or } s + s' = 2 (\sqrt{p} + \sqrt{q}) \sqrt{x},$$

an equation expressing the relation between the two curves; from which it appears, that the sum of their corresponding arcs, measured from the lowest point, is proportional to the square root of their common abscissa.

Also, if a be the diameter of the generating circle of a cycloid, the arc, for the abscissa x, is $2\sqrt{(a x)}$; we may, therefore, say, that the sum of the two arcs is equal to the arc of a cycloid, for the same abscissa, the diameter of whose generating circle is $(\sqrt{p} + \sqrt{q})^2$.

To apply this to the case proposed, let the equation to the catenary be $s' = \sqrt{(2 c x' + x'^2)}$; therefore, that of the curve required is

$$s = 2\sqrt{(a x)} - \sqrt{(2 c x + x^2)}.$$

21. A corpuscle is attracted by two straight lines at right angles to each other, the particles of which attract with forces varying as $\frac{1}{D^2}$: having given the position of the corpuscle and the length of one of the lines, find the length of the other when the direction in which the corpuscle begins to move is towards their common intersection.

Let the two lines $AB = a$, $AC = b$, be measured along the axes of x and y respectively; the origin being at their intersection A.

Let α, β be the co-ordinates of the corpuscle P; then the attraction of any small portion dx of the line AB, at a distance r from P, will $= \frac{dx}{r^2}$: the resolved part of this in the direction $BA = \frac{dx}{r^2} \cdot \frac{\beta - x}{r}$;

$\therefore$ the whole attraction of a in direction BA

$$= \int \frac{(\beta - x)\, dx}{r^3} = \int \frac{(\beta - x)\, dx}{\left\{\alpha^2 + (\beta - x)^2\right\}^{\frac{3}{2}}}$$

$$= \frac{1}{\sqrt{\left\{\alpha^2 + (\beta - a)^2\right\}}} - \frac{1}{c}, \begin{Bmatrix} x = 0 \\ x = a \end{Bmatrix}$$

making $\alpha^2 + \beta^2 = c^2$.

Again the resolved part of the attraction of dx in direction $CA = \frac{\alpha\, dx}{r^3}$;

$\therefore$ the whole attraction of a in direction $CA = \int \frac{\alpha\, dx}{\left\{\alpha^2 + (\beta - x)^2\right\}}$

$$= \frac{\beta - a}{\alpha \sqrt{\left\{\alpha^2 + (\beta - a)^2\right\}}} - \frac{\beta}{\alpha c} \begin{Bmatrix} x = 0 \\ x = a \end{Bmatrix}.$$

Now, it is evident that by writing α for β, β for α, and b for a, in the above expressions for the attraction of a, we shall have similar expressions for the attraction of b.

Thus, the attraction of b in direction CA

$$= \frac{1}{\sqrt{\left\{\beta^2 + (\alpha - b)^2\right\}}} - \frac{1}{c};$$

that in direction $BA = \dfrac{\alpha - b}{\beta \sqrt{\left\{\beta^2 + (\alpha - b)^2\right\}}} - \dfrac{\alpha}{\beta c}$.

If then, X, Y denote the whole forces acting on P in directions of x and y, in order that their resultant may pass through A, we must have

$$\frac{Y}{X} = \frac{\beta}{\alpha}, \text{ or } \alpha Y - \beta X = 0:$$

that is, since X

$$= \frac{1}{c} - \frac{1}{\sqrt{\{a^2 + (\beta - \alpha)^2\}}} + \frac{\alpha}{\beta c} - \frac{\alpha - b}{\beta \sqrt{\{\beta^2 + (\alpha - b)^2\}}},$$

$$\text{and } Y = \frac{1}{c} - \frac{1}{\sqrt{\{\beta^2 + (\alpha - b)^2\}}} + \frac{\beta}{\alpha c} - \frac{\beta - a}{\alpha \sqrt{\{a^2 + (\beta^2 - a)\}}};$$

$$\frac{a}{\sqrt{\{a^2 + (\beta - a)^2\}}} = \frac{b}{\sqrt{\{\beta^2 + (\alpha - b)^2\}}},$$

$$\text{or } a^2 c^2 - 2\alpha a^2 b = b^2 c^2 - 2\beta a b^2;$$

from which equation, either of the quantities a or b may be found when the other is given.

22. A body descends in a straight line in a medium whereof the density varies as the square root of the distance from a given point, and is urged by a constant force tending to that point: find the velocity and time corresponding to a given space, supposing the resistance to vary as the density and velocity jointly.

Let t, v denote respectively the time and velocity for the distance x from the given point; let g represent the constant force, and k the density at distance 1:

Then, from the general equation $\frac{dv}{dt} = f$ we shall have

$$\frac{dv}{dt} = +g - k v x^{\frac{1}{2}}, \qquad (1),$$

$$\therefore v = C + gt - \int k x^{\frac{1}{2}} v\, dt.$$

But $v\,d\,t = d\,s = -\,d\,x$,

$$\therefore\ v = C - g\,t - \frac{2}{3} k\,x^{\frac{3}{2}}$$

Supposing that the body begins to move from a distance a, we have $v = 0$, $t = 0$, $x = a$ together, and

$$\therefore\ C = \frac{2}{3} k\,a^{\frac{3}{2}},$$

$$\therefore\ v = \frac{2}{3} k\,(a^{\frac{3}{2}} - x^{\frac{3}{2}}) - g\,t, \qquad (2).$$

The equation (2) alone is not sufficient to establish a relation between t, v, x, but, by means of another, obtained from (1), we may obtain any two of these quantities when the third is given

Since $\dfrac{d\,v}{d\,t} = -\dfrac{d}{d\,t}\left(\dfrac{d\,x}{d\,t}\right) = -\dfrac{d^2\,x}{d\,t^2}$, and $v = -\dfrac{d\,x}{d\,t}$,

equation (1) becomes

$$\frac{d^2\,x}{d\,t^2} = k\,x^{\frac{1}{2}} \cdot \frac{d\,x}{d\,t} - g, \qquad (3),$$

which may be integrated by approximation in the following manner:

Since x is a function of t, Maclaurin's Theorem, gives us

$$x = A + A_1\,t + A_2\,\frac{t^2}{1\,.\,2} + A_3\,\frac{t^3}{1\,.\,2\,.\,3} + \ldots,$$

where A, A_1, A_2, &c. denote the values of

$$x,\ \frac{d\,x}{d\,t},\ \frac{d^2\,x}{d\,t^2},\ \text{\&c. when } t = 0.$$

But when $t = 0$, $x = a$, and $A_1 = 0$;

Also, from (3) $A_2 = -\,g$.

Differentiating (3) and then dividing it by t,

$$\frac{d^3 x}{d t^3} = \frac{k}{2 x^{\frac{1}{2}}}; \ \frac{d x^2}{d t^2} + k x^{\frac{1}{2}} \frac{d^2 x}{d t^2}; \qquad (4),$$

$$\therefore A_3 = k a^{\frac{1}{2}} A_2 = - k a^{\frac{1}{2}} g.$$

Similarly, by differentiating (4) and dividing by t, we have

$$A_n = k a^{\frac{1}{2}} A_3 = - k^2 a g;$$

and we may thus obtain as many terms as we please:

$$\therefore x = a - \frac{g t^2}{2} - \frac{k a^{\frac{1}{2}} g t^3}{1 . 2 . 3} - \frac{k^2 a g t^4}{1 . 2 . 3 . 4} - \dots\dots$$

The two first terms express the space moved through *in vacuo,* which is that which should result from making $k = 0$. This equation, together with (2), completely establishes the relation between x, t, and v.

23. A body describes a circle of given radius uniformly, acted upon by two forces each varying as the distance and without the plane of the circle; find the velocity of the body and the position of the plane of its orbit.

Let the centre of the circle be the origin of three rectangular co-ordinate planes; the plane of the circle being in that of $x y$, and the two centres of force in certain positions without it. If the positions of the two centres of force be known, those of the co-ordinate planes will be unknown, and *vice versa;* but the relations between these positions being once exhibited by the conditions of the problem, either of them may be determined from the others.

Let then (x_1, y_1, z_1), (x_2, y_2, z_2) be the respective co-ordinates of the two centres of force,

μ, μ' their absolute intensities, Let also the equations of the circle be $\left.\begin{array}{l} y^2 = a^2 - x^2 \\ z = 0 \end{array}\right\}$,

and let r_1, r_2 be the distances of any point in the circle, or of the body, from the two centres of force.

Now the force from

$$\left.\begin{array}{r} 1^{\text{st}} \text{ centre on the body} = \mu\, r_1, \\ \text{from } 2^{\text{nd}} = \mu' r_2, \end{array}\right\}$$

∴ the resolved part of 1st in direction of

$$x = \mu\, r_1 \,.\, \frac{x_1 - x}{r_1} = \mu\,(x_1 - x),$$

and so for the others; so that if X, Y, Z be respectively the whole forces, soliciting the body in the directions of x, y, z, we shall have

$$\begin{aligned} X &= \mu\,(x_1 - x) + \mu'(x_2 - x), \\ Y &= \mu\,(y_1 - y) + \mu'(y_2 - y), \\ Z &= \mu\, z_1 \;+ \mu'\, z_2. \end{aligned}$$

But since the body moves in the plane of $x\,y$, the resultant of all the forces must always lie in that plane; we must, therefore, have

$$Z = 0, \text{ or } \frac{\mu}{\mu'} = -\frac{z_2}{z_1},$$

which shews, that the two centres of force must lie on different sides of the plane of the circle, and that their perpendicular distances from that plane must be inversely as their absolute intensities.

Also, since the body moves uniformly in a circle, the resultant must always pass through the centre, (Newton, Prop. 2. Lib. 1), and be of constant magnitude.

So that if the constant magnitude of the force be represented by f, we have

$$X = -\frac{f}{a}\,.\,x, \text{ and } Y = -\frac{f}{a}\,.\,y;$$

$$\text{or,}\quad \mu a (x_1 - x) + \mu' a (x_2 - x) = - f x,$$
$$\text{and}\quad \mu a (y_1 - y) + \mu' a (y_2 - y) = - f y;$$

which equations must always give the same value of f, whatever be the value of x. Thus, if $x = 0$, and $\therefore y = a$, these equations become

$$\left.\begin{aligned} &\mu x_1 + \mu' x_2 = 0, \\ &\mu (y_1 - a) + \mu' (y_2 - a) = - f. \end{aligned}\right\} \qquad (1).$$

Again, if $y = 0$, and $\therefore x = a$;

$$\left.\begin{aligned} &\mu (x_1 - a) + \mu' (x_2 - a) = - f, \\ &\mu y_1 + \mu' y_2 = 0, \end{aligned}\right\} \qquad (2).$$

The 1st of equations (1) and the 2nd of (2) give the same property of the positions of the planes of $x z$, and $y z$, as that obtained above for the plane of $x y$.

For the velocity (v) we have in general $v^2 = f a$ (Newton, Prop. 4. Lib. 1.); but, by combining the systems of equations (1), (2), we get $f = (\mu + \mu') a$;

$$\therefore\ v^2 = (\mu + \mu') a^2.$$

This result is remarkable, as it shews, that the velocity and periodic time in the circle are the same as if the two centres of force were together transferred to the centre of the circle.

24. If a body revolve in an ellipse round the focus, prove that a progressive motion of the apse will be the effect of any continual addition of force in the direction of the radius vector during the progress of the body from the higher to the lower apse, and point out the effect on the eccentricity.

It appears from Newton (Prop. 44. Lib. 1.) that if the apsides of an ellipse be progressive, while the body moves from the higher to the lower apse, it must be from the action of a force greater than that at a similar point of the quiescent ellipse;

and since, the force in the quiescent ellipse increases during the motion from the higher to the lower apse, the extra force in the moveable ellipse must increase also. Hence, the converse of this must be true, or an addition of force must *(cæteris paribus)* produce a progression of the apsides; for, by assuming the contrary, Prop. 44, would be absurd. This is without reference to the particular law by which the extra force varies, that of the inverse cube of the distance being only necessary when the species of the moveable and quiescent ellipses are required to be the same.

To shew the effect on the eccentricity, let A, a be the greatest and least apsidal distances in the quiescent ellipse; F, f the forces at those distances: then $\frac{A^2}{a^2} = \frac{f}{F}$; but since, in the case of the problem, an extra force is supposed to begin from nothing at the higher apse, and to attain to a maximum at the lower apse, f is increased while F remains the same; therefore $\frac{f}{F}$ or $\frac{A^2}{a^2}$ is increased, and hence the eccentricity, which increases with $\frac{A^2}{a^2}$, is increased.

25. Two balls connected together by an inflexible and inextensible line are constrained to move the one on a horizontal plane, the other on an inclined plane which is at liberty to move freely on the horizontal plane; find the motion of the balls and of the plane, supposing the motion of the rod to be in a vertical plane.

Let P, Q represent the masses of the two balls, W that of the inclined plane, and α the angle of its inclination with the horizon.

Taking a fixed point in the horizontal plane for the origin, let x, 0 be the co-ordinates of P, and x', y' those of Q, P being on the horizontal, and Q on the inclined plane*: let $\theta =$ the variable angle made by the line PQ with the horizon, and Tg the *thrust* or force of P on Q in the direction PQ, which is equal to that of Q on P in the direction QP.

Now the moving forces of W are 1st, the resolved part of Tg in a direction perpendicular to the plane and equal to $Tg . \sin(\alpha - \theta)$, and 2nd, the pressure arising from the weight of Q which $= Qg . \cos\alpha$. The whole accelerating force of W in the direction of x is therefore equal to

$$\frac{Tg}{W} \cdot \sin(\alpha - \theta) . \sin\alpha + \frac{Qg}{P + Q + W} \cdot \cos\alpha . \sin\alpha.$$

The vertical motion of Q is the same whether the inclined plane be moveable or not, and is therefore due to the resolved parts of the forces Tg, and $Qg \sin\alpha$ in that direction: the horizontal motion is partly due to the forces Tg, $Q\sin\alpha$, and partly to the motion of the plane in a contrary direction.

The equations of motion thus give

$$\frac{d^2 x}{dt^2} = -\frac{Tg}{P} \cdot \cos\theta, \qquad (1),$$

$$\frac{d^2 x'}{dt^2} = \frac{Tg}{Q} \cdot \cos\theta - g . \sin\alpha . \cos\alpha + \frac{Tg}{W} \cdot \sin(\alpha - \theta) . \sin\alpha$$
$$+ \frac{Qg}{P + Q + W} \cdot \cos\alpha . \sin\alpha. \qquad (2),$$

$$\frac{d^2 y'}{dt^2} = \frac{Tg}{Q} \cdot \sin\theta - g\sin^2\alpha, \qquad (3).$$

Also, if x_1 be the distance of the foot of the inclined plane from the origin,

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* The motion of the plane is supposed to be in the direction of $+ x$.

$$\frac{d^2 x_1}{d t^2} = \frac{Tg}{W} \cdot \sin(a - \theta) . \sin a + \frac{Q g}{P + Q + W} . \cos a . \sin a' \quad (4).$$

Multiplying (1) by P, (2) by Q, adding and reducing by (4),

$$P \frac{d^2 x}{d t^2} + Q \frac{d^2 x'}{d t^2} - Q \frac{d^2 x_1}{d t^2} = - Q g . \sin a . \cos a, \quad (5).$$

But, if X, Y be the co-ordinates of the common centre of gravity of P and Q,

$$P x + Q x' = (P + Q) X,$$

$$\therefore \frac{P\, d^2 x + Q\, d^2 x'}{d t^2} = (P + Q) \cdot \frac{d^2 X}{d t^2};$$

$$\therefore (P + Q) \cdot \frac{d^2 X}{d t^2} - Q \cdot \frac{d^2 x_1}{d t^2} = - Q g . \sin a . \cos a, \quad (6)$$

from which equation, if we know the motion of the plane for a given time, that of the rest of the system is determined. Or, integrating twice in succession, and supposing the system to be set in motion by gravity, (which supposition gives

$\frac{d X}{d t} = 0$, and $\frac{d x_1}{d t} = 0$, when $t = 0$,) we have

$$(P + Q) X - Q x_1 = C - \frac{Q g t^2}{4} . \sin 2 a,$$

where C is determined by the given position of the parts of the system, at a given time.

But if l = length of the line joining P, Q,

$$y = a \sin \theta,$$

$$x' = x + l \cos \theta,$$

$$x_1 = x + l \cos \theta - l \sin \theta . \cot a,$$

$$\therefore \frac{d^2 x'}{d t^2} = \frac{d^2 x}{d t^2} + l . \frac{d^2 . \cos \theta}{d t^2},$$

$$\frac{d^2 y'}{d t^2} = l . \frac{d^2 . \sin \theta}{d t^2},$$

and thus T may be expressed from (3) in terms of θ: substituting this expression in (1) and (2), putting

$$\frac{d^2 x}{d t^2} + l \cdot \frac{d^2 \cos \theta}{d t^2} \text{ for } \frac{d^2 x'}{d t^2} \text{ in (2),}$$

and then, eliminating $\frac{d^2 x}{d t^2}$ from (1) and (2), we arrive at a differential equation between θ and t. This process, together with the integration of the final equation, would occupy more space than the limits of this work admit of.

FRIDAY EVENING.

1. Find the value of $\frac{1}{(\sin \theta)^2} - \frac{1}{\theta^2}$, when $\theta = 0$.

Since $\sin \theta$ changes its sign with θ, and has a limiting ratio of equality with it, we may assume

$$\sin \theta = \theta + A \theta^3 + B \theta^5 + \ldots$$

$$\therefore (\sin \theta)^2 = \theta^2 + 2 A \theta^4 + B' \theta^6 + \ldots$$

$$\therefore \frac{1}{(\sin \theta)^2} - \frac{1}{\theta^2} \text{ or } \frac{\theta^2 - (\sin \theta)^2}{\theta^2 (\sin \theta)^2} = - \frac{2 A \theta^4 + B' \theta^6 + \ldots}{\theta^4 + 2 A \theta^6 + \ldots}$$

$$= - \frac{2 A + B \theta^4 + \ldots}{1 + 2 \theta^4 + \ldots}$$

$$= - 2 A, \text{ when } \theta = 0.$$

But, by differentiating, twice in succession, the expression for $\sin \theta$, we obtain

$$- \sin \theta = 2 . 3 A \theta + 4 . 5 B \theta^3 + \ldots$$

$$\text{or, } \sin \theta = - 2 . 3 A \theta - 4 . 5 B \theta^3 - \ldots$$

comparing this with

$$\sin \theta = \theta + A \theta^3 + \ldots$$

gives $-2 \cdot 3 A = 1$, or $A = -\frac{1}{2 \cdot 3}$,

$$\therefore \frac{1}{(\sin \theta)^2} - \frac{1}{\theta^2} = \frac{1}{3}, \text{ when } \theta = 0.$$

2. A person spends in the first year m times the interest of his property, in the second $2m$ times that of the remainder, in the third $3m$ times that at the end of the second, and so on; and at the end of $2p$ years he has nothing remaining; shew, that in the p^{th} year he spends as much as he has left at the end of that year.

If P_x be his property at the end of the x^{th} year, and r the interest of £1. for one year, the interest of his property in the $(x+1)^{\text{th}}$ year will be $r P_x$;

$\therefore$ his expenditure in the $(x+1)^{\text{th}}$ year $= (x+1) m r P_x$,

$\therefore$ his property at the end of the $(x+1)^{\text{th}}$ year will be expressed by

$$P_x + r P_x - (x+1) m r P_x,$$

$$\text{or, } \{1 + r - (x+1) m r\} P_x : \qquad (1).$$

By writing $2p - 1$ for x in (1), we have, by the question,

$$\{1 + r - 2pmr\} P_{2p-1} = 0;$$

$$\therefore 1 + r = 2pmr. \qquad (2).$$

Similarly, his expenditure in the p^{th} year is $p m r P_{p-1}$, and his property at the end of that year,

$$\{1 + r - pmr\} P_{p-1},$$

which by (2) is reduced to $p m r P_{p-1}$.

3. Given

$$\left.\begin{aligned} \tan\theta + \tan\phi + \tan\psi = 1 + \frac{4}{\sqrt{3}} \\ \tan\theta\,.\,\tan\phi + \tan\theta\,.\,\tan\psi + \tan\phi\,.\,\tan\psi = 1 + \frac{4}{\sqrt{3}} \\ \tan\theta\,.\,\tan\phi\,.\,\tan\psi = 1, \end{aligned}\right\}$$

find θ, ϕ, and ψ; and sum the series

$$(\sec\theta)^2 + \left(\frac{1}{2}\sec\frac{\theta}{2}\right)^2 + \left(\frac{1}{2^2}\sec\frac{\theta}{2^2}\right)^2 + \left(\frac{1}{2^3}\sec\frac{\theta}{2^3}\right)^2 \ldots \textit{ad inf.}$$

Since the three given expressions are the co-efficients of the terms of an equation whose roots are $\tan\theta$, $\tan\phi$, $\tan\psi$, these roots may be obtained by the solution of the recurring equation

$$x^3 - \left(1 + \frac{4}{\sqrt{3}}\right)x^2 + \left(1 + \frac{4}{\sqrt{3}}\right)x - 1 = 0.$$

One root being 1, the equation divided by $x - 1 = 0$ gives

$$x^2 - \frac{4}{\sqrt{3}}\cdot x + 1 = 0,$$

of which the roots being $\sqrt{3}$ and $\frac{1}{\sqrt{3}}$,

we obtain

$$\tan\theta = 1, \qquad \therefore\ \theta = 45^\circ \text{ or } 45^\circ + m\pi,$$

$$\tan\phi = \sqrt{3}, \quad \therefore\ \phi = 60^\circ \text{ or } 60^\circ + m\pi,$$

$$\tan\psi = \frac{1}{\sqrt{3}}, \quad \therefore\ \psi = 30^\circ \text{ or } 30^\circ + m\pi.$$

Since the quantities $\tan\theta$, $\tan\phi$, $\tan\psi$ enter the given expressions symmetrically, their values just found may be interchanged, the above arrangement of them being arbitrary.

To sum the series

$$(\sec\theta)^2 + \left(\frac{1}{2}\sec\frac{\theta}{2}\right)^2 + \left(\frac{1}{2^2}\sec\frac{\theta}{2^2}\right)^2 + \ldots \textit{ad infinitum.}$$

The x^{th} term is $\left(\frac{1}{2^{x-1}}\sec\cdot\frac{\theta}{2^{x-1}}\right)^2$,

$$\therefore \Delta S_x = \frac{1}{\left(2^x \cdot \cos\frac{\theta}{2^x}\right)}.$$

But since (Herschel's Examples, p. 3. Ex. 16),

$$\Delta\frac{1}{\left(2^z \cdot \sin\frac{\theta}{2^z}\right)^2} = -\frac{1}{\left(2^{z+1} \cdot \cos\frac{\theta}{2^{z+1}}\right)^2},$$

we have, by making $z = x - 1$,

$$\Delta S_x = -\Delta\cdot\frac{1}{\left(2^{x-1} \cdot \sin\frac{\theta}{2^{x-1}}\right)^2},$$

$$\therefore S_x = C - \frac{4}{2^{2x} \cdot \sin^2\frac{\theta}{2^{x-1}}}.$$

But $S_0 = C - \frac{4}{\sin^2 2\theta} = 0$, $\therefore C = \frac{4}{\sin^2 2\theta}$;

$$\therefore S_x = \frac{4}{\sin^2 2\theta} - \frac{4}{2^{2x} \cdot \sin^2\frac{\theta}{2^{x-1}}}.$$

By making x infinite, this expression for S_x takes the form $\frac{0}{0}$: its value may be determined by considering that

$$\sin^2\frac{\theta}{2^{x-1}} = \frac{1}{2}\left(1 - \cos\frac{\theta}{2^{x-2}}\right)$$

$$= \frac{1}{2}\left\{\frac{\theta^2}{2 \cdot 2^{2x-4}} - \frac{\theta^4}{2 \cdot 4 \cdot 2^{4x-8}} + \ldots\right\}$$

$$\therefore\ 2^{2x}.\sin^2\frac{\theta}{2^{x-1}}=\frac{1}{2}\left\{8\,\theta^2-\frac{\theta^4}{2.4.2^{2x-8}}+\ldots\right\}$$
$$=4\,\theta^2 \text{ when } x \text{ is infinite;}$$
$$\therefore\ S=\frac{4}{\sin^2 2\theta}-\frac{1}{\theta^2}.$$

4. In any polygon with n sides $A_1 A_2$, $A_2 A_3$, ... respectively represented by a_1, a_2 ....

$$a_1 \sin A_1 - a_2 \sin(A_1+A_2) + a_3 \sin(A_1+A_2+A_3) - \ldots\ldots$$
$$\pm\, a_{n-1}\sin(A_1+A_2+A_3+\ldots+A_{n-1}) = 0.$$

From A_2 let fall a perpendicular $A_2 K$ upon a_n, and from the angles A_3, A_4, ... A_{n-1},
let fall the perpendiculars

$$A_3 P_3,\ A_4 P_4,\ \ldots\ A_{n-1} P_{n-1} \text{ on } A_2 K,$$

produced if necessary:

$$\text{then } A_2 K = A_2 P_3 + P_3 P_4 + \ldots + P_{n-1} K. \quad (A).$$

Let the angles made by a_2, a_3 ..... with a_n be respectively denoted by

$$(a_2, a_n),\ (a_3, a_n),\ \ldots\ldots$$

and we have

$$\left.\begin{aligned} A_2 K &= a_1 . \sin A_1, \\ A_2 P_3 &= a_2 . \sin(a_2, a_n) \\ P_3 P_4 &= a_3 . \sin(a_3, a_n) \\ &\vdots \\ P_{n-1} K &= a_{n-1} . \sin A_n . \end{aligned}\right\} \quad (B).$$

But, producing each of the sides a_2, a_3, a_4, to meet a_n, we have, from Euclid B. I. Prop. 32, and Cors.

$$(a_2, a_n) = \pi - (A_1 + A_2),$$
$$(a_3, a_n) = (a_2, a_n) + \pi - A_3,$$
$$= 2\pi - (A_1 + A_2 + A_3);$$
$$(a_4, a_n) = (a_3, a_n) + \pi - A_4,$$
$$= 3\pi - (A_1 + A_2 + A_3 + A_4);$$

and finally,

$$(a_{n-1}, a_n) \text{ or } A_n = (n-2)\pi - (A_1 + A_2 + A_3 + \ldots A_{n-1}).$$

The equations B, therefore, become

$$A_2 K = a_1 . \sin A_1,$$
$$A_2 P_3 = - a_2 . \sin (A_1 + A_2),$$
$$P_3 P_4 = + a_3 \sin (A_1 + A_2 + A_3),$$
$$\vdots$$
$$P_{n-1} K = \pm a_{n-1} . \sin (A_1 + A_2 + \ldots + A_{n-1}).$$

Substituting these values in equation (A), and transposing all the terms to one side, we have

$$a_1 \sin A_1 - a_2 \sin (A_1 + A_2) + \ldots\ldots$$
$$\pm a_{n-1} . \sin (A_1 + A_2 + \ldots A_{n-1}).$$

Q. E. D.

5. A ray of light is refracted through a prism, the angle of which is 60° and index of refraction $\sqrt{2}$, so as to undergo the least possible deviation; determine that deviation. Shew also that no ray can be directly transmitted through a prism of the same refracting power when the angle exceeds 90°.

Let $\phi = \angle$ of incidence, $\psi = \angle$ of emergence
$\phi' = \angle$ of 1st refraction, $\delta = \angle$ of deviation
$\psi' = \angle$ of 2nd refraction, $\iota = \angle$ of the prism.

From the well known properties of the prism (Coddington's Optics) we have the following relations;

$$\frac{\sin\phi}{\sin\phi'} = \sqrt{2}, \quad \frac{\sin\psi}{\sin\psi'} = \sqrt{2}, \quad \phi' + \psi' = \iota = 60°,$$
$$\delta = \phi + \psi - \iota = \phi + \psi - 60°;$$
$$\therefore \sin\psi' = \sin(60° - \phi') = \frac{\sqrt{3}}{2}\cos\phi' - \frac{1}{2} . \sin\phi';$$

$$\therefore \sin\psi = \sqrt{\frac{3}{2}} \,.\, \cos\phi' - \frac{1}{\sqrt{2}} \,.\, \sin\varphi',$$

$$= \sqrt{\frac{3}{2}} \,.\, \sqrt{(1 - \sin^2\phi')} - \frac{1}{\sqrt{2}} \,.\, \sin\phi',$$

$$= \sqrt{\frac{3}{2}} \,.\, \sqrt{\left(1 - \frac{\sin^2\phi}{2}\right)} - \frac{1}{2} \,.\, \sin\phi.$$

But in the case of minimum deviation, $\sin\psi = \sin\phi$;

$$\therefore \sin\phi = \sqrt{\frac{3}{2}} \,.\, \sqrt{\left(1 - \frac{\sin^2\phi}{2}\right)} - \frac{1}{2} \sin\phi,$$

$$\therefore \sin\phi = \pm \frac{1}{\sqrt{2}}.$$

To determine which of these two values corresponds to the minimum deviation, we must substitute them in the expression for $\frac{d^2\delta}{d\varphi^2}$, obtained from the equation $\delta = \phi + \psi - \iota$. This substitution is found to give a positive result for $\sin\phi = \frac{1}{\sqrt{2}}$, and a negative one for $\sin\phi = -\frac{1}{\sqrt{2}}$; we have, therefore, when δ is a minimum,

$$\phi = 45°, \text{ and } \therefore \delta = 45° + 45° - 60° = 30°.$$

Lastly, since $\sin\psi = \sqrt{2} \,.\, \sin(1 - \phi')$

$$= \sqrt{2} \,.\, \sin\iota \,.\, \sqrt{\left(1 - \frac{\sin^2\phi}{2}\right)} - \cos\iota \sin\phi,$$

and the last term of this expression becomes positive when ι exceeds 90°, and therefore $\sin\psi$ becomes greater than unity, no ray can, in such a case, be directly transmitted through the prism.

6. If $\left.\begin{array}{l} aX + bY + cZ = 0 \\ a_1X + b_1Y + c_1Z = 0 \end{array}\right\}$ where $X = ax + a_1x_1 + a_2$,

$$Y = bx + b_1x_1 + b_2,$$
$$Z = cx + c_1x_1 + c_2,$$

then $X^2 + Y^2 + Z^2$

$$= \frac{\{a_2 (b c_1 - b_1 c) + b_2 (a_1 c - a c_1) + c_2 (a b_1 - a_1 b)\}^2}{(b c_1 - b_1 c)^2 + (a_1 c - a c_1)^2 + (a b_1 - a_1 b)^2}.$$

Eliminating x from the 1st and 2nd, and afterwards from the 2nd and 3rd of the equations

$$X = a x + a_1 x_1 + a_2$$
$$Y = b x + b_1 x_1 + b_2$$
$$Z = c x + c_1 x_1 + c_2,$$

we have

$$a Y - b X = (a b_1 - a_1 b) x_1 + (a b_2 - a_2 b)$$
$$b Z - c Y = (b c_1 - b_1 c) x_1 + (b c_2 - b_2 c)$$

Again, eliminating x_1 from these two last equations, by multiplying the 1st of them by $b c_1 - b_1 c$, the 2nd by $b c_2 - b_2 c$, and subtracting; we have, after reducing, dividing by b, and arranging,

$$(b c_1 - b_1 c) X + (a_1 c - a c_1) Y + (a b_1 - a_1 b) Z$$
$$= a_2 (b c_1 - b_1 c) + b_2 (a_1 c - a c_1) + c_2 (a b_1 - a_1 b),$$

$$\text{or } X \left\{ (b c_1 - b_1 c) + (a_1 c - a c_1) \frac{Y}{X} + (a b_1 - a_1 b) \frac{Z}{X} \right\}$$

$$= a_2 (b c_1 - b_1 c) + b_2 (a_1 c - a c_1) + c_2 (a b_1 - a_1 b). \qquad (A).$$

But from the equations

$$a X + b Y + c Z = 0$$
$$a_1 X + b_1 Y + c_1 Z = 0,$$

$$\text{we have } \frac{Z}{X} = \frac{a b_1 - a_1 b}{b c_1 - b_1 c}, \quad \frac{Y}{X} = \frac{a_1 c - a c_1}{b c_1 - b_1 c},$$

by which the equation (A) becomes

$$X \left\{ (b c_1 - b_1 c) + \frac{(a_1 c - a c_1)^2}{b c_1 - b_1 c} + \frac{(a b_1 - a_1 b)^2}{b c_1 - b_1 c} \right\}$$
$$= a_2 (b c_1 - b_1 c) + b_2 (a_1 c - a c_1) + c_2 (a b_1 - a_1 b):$$

$$\therefore X = (b c_1 - b_1 c) \cdot \frac{a_2 (b c_1 - b_1 c) + b_2 (a_1 c - a c_1) + c_2 (a b_1 - a_1 b)}{(b c_1 - b_1 c)^2 + (a_1 c - a c_1)^2 + (a b_1 - a_1 b)^2},$$

$$= (b\,c_1 - b_1\,c)\,.\,\frac{N}{M} \text{ suppose.}$$

Similarly, $Y = (a\,c_1 - a_1\,c)\,.\,\frac{N}{M}$,

and $Z = (a\,b_1 - a_1\,b)\,.\,\frac{N}{M}$;

$\therefore X^2 + Y^2 + Z^2$

$$= \{(b\,c_1 - b_1\,c)^2 + (a_1\,c - a\,c_1)^2 + (a\,b_1 - a_1\,b)^2\}\,\frac{N^2}{M^2},$$

$$= \frac{N^2}{M}\,.$$

Q. E. D.

7. Trace the curve, the equation to which is $y = \varepsilon^{\sin x}$, and express in a series the area which recurs.

Giving to x the several values

$$0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi,\ \frac{5\pi}{2},\ 3\pi,\ \frac{7\pi}{2},\ \&c.$$

the corresponding values of y are

$$1,\ \varepsilon,\ 1,\ \frac{1}{\varepsilon},\ 1,\ \varepsilon,\ 1,\ \frac{1}{\varepsilon},\ \&c.$$

and those of $\frac{dy}{dx} = \cos x\,.\,\varepsilon^{\sin x}$

$$1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ 1,\ \&c.$$

These expressions follow in the same order, if the above values of x be made negative.

As every value of y recurs and also that of $\frac{dy}{dx}$ corresponding to it, it is evident that the curve will consist of a set of recurring figures; also from the above values of y and $\frac{dy}{dx}$, it appears that it consists of a set of equal, similar, and continuous undulations, the greatest and least distances of which, from the axis of x, are respectively ε and $\frac{1}{\varepsilon}$, and which are symmetrical on each side of the ordinates, corresponding to these greatest and least distances, and parallel to the axis of x at these distances.

We may suppose the recurring area to be contained between the greatest and least ordinates of any undulation, that is, to be measured from an ordinate $y = \varepsilon$ to an ordinate $y = \frac{1}{\varepsilon}$, or in the reverse order.

$$\text{Hence, the area} = \int \varepsilon^{\sin x}\, d\, x \begin{Bmatrix} x = \frac{\pi}{2} \\ x = \frac{3\,\pi}{2} \end{Bmatrix}.$$

But the $(m + 1)^{\text{th}}$ term of the expansion of

$$\varepsilon^{\sin x} \text{ is } \frac{\sin^m x}{1 \,.\, 2 \,.\, 3 \ldots . m}:$$

$\therefore$ the $(m + 1)^{\text{th}}$ term of the series expressing the area will be

$$\frac{1}{1 \,.\, 2 \,.\, 3 \ldots . m} \cdot \int \sin^m x \,.\, d\, x \begin{Bmatrix} x = \frac{\pi}{2} \\ x = \frac{3\,\pi}{2} \end{Bmatrix}.$$

But the integral $\int \sin^m x \,.\, d\, x$ between these limits is

$$\frac{1 \,.\, 3 \,.\, 5 \ldots .. (m - 3)\,(m - 1)}{2 \,.\, 4 \,.\, 6 \ldots .. (m - 2)\, m} \,.\, \pi$$

$$\text{or } - 2 \,.\, \frac{1 \,.\, 3 \,.\, 5 \ldots .. (m - 3)\,(m - 1)}{2 \,.\, 4 \,.\, 6 \ldots ... (m - 2)\, m},$$

according as m is an odd or even number;

$\therefore$ any even term of the required series being

$$\frac{\pi}{(2 \,.\, 4 \,.\, 6 \ldots ... (m - 2) \,.\, m)^2},$$

the following term will be

$$- \frac{2}{(2 \,.\, 4 \,.\, 6 \ldots . (m - 2) \,.\, m)^2}$$

8. A perfectly smooth rod in a vertical plane revolves uniformly round a vertical axis, and a ring placed on it is attracted to a horizontal plane by a force varying as the distance in addition to the uniform force of gravity; required the form of the rod that the ring may remain on whatever point it is placed.

Let X, Y denote respectively the forces applied to the ring in the directions of the vertical axis of x, and the horizontal axis of y; the origin being in the horizontal plane: then the principles of virtual velocities gives the equation

$$X\,dx + Y\,dy = 0.$$

But, if ω be the angular velocity of the rod, and $\therefore\ \omega^2 y$ the centrifugal force of the ring, we have

$$X = -g - mx,$$
$$Y = \omega^2 y;$$
$$\therefore\ -(g + mx)\,dx + \omega^2 y\,dy = 0,$$
$$\therefore\ C - (g + mx)^2 + m\omega^2 y^2 = 0.$$

But, if we suppose the rod to be inserted into the vertical axis in the horizontal plane, we must have $y = 0$, when $x = 0$, and $\therefore\ C - g^2 = 0$; and the equation to the form of the rod thus becomes

$$\frac{\omega^2}{m} \cdot y^2 = \frac{2g}{m} \cdot x + x^2,$$

which is that of an hyperbola, of which the major and minor semi-axes are respectively

$$\frac{g}{m} \text{ and } \frac{g}{\omega}\sqrt{\frac{1}{m}}.$$

9. An ellipse may be constructed, so that if any abscissa be taken to represent the aberration in longitude of a given star, the corresponding ordinate will

represent the aberration in latitude, co-ordinates being measured from the centre along the axes; prove this and determine the axes.

Maddy's Astronomy (pp. 130, 131), if l = star's longitude, λ = latitude, and $\odot$ = sun's longitude,

$$\text{aberr}^n. \text{ in long.} = -\frac{20''.25.\cos(\odot - l)}{\cos\lambda},$$

$$\text{aberr}^n. \text{ in lat.} = -20''.25.\sin\lambda.\sin(\odot - l):$$

$\therefore$ if $-20''.25 = a$, we have

$$x = \frac{a}{\cos\lambda}\cdot\cos(\odot - l),$$

$$y = a.\sin\lambda.\sin(\odot - l);$$

$$\therefore\ x^2 = \frac{a^2}{\cos^2\lambda}\cdot\cos^2(\odot - l),$$

$$y^2 = a^2.\sin^2\lambda.\sin^2(\odot - l),$$

$$= a^2.\sin^2\lambda - a^2\sin^2\lambda.\cos^2(\odot - l),$$

$$= a^2.\sin^2\lambda - \sin^2\lambda.\cos^2\lambda.x^2;$$

$$\therefore\ \frac{y^2}{a^2.\sin^2\lambda} + \frac{x^2}{a.\sec^2\lambda} = 1,$$

the equation to an ellipse, of which the semi-axes are $a.\sec\lambda$, and $a.\sin\lambda$.

10. The equation to the path of a projectile is

$$y = ax + \frac{g}{k^2}.\log(1 - bx), \text{ gravity } (= g)$$

acting parallel to the axis of y; shew that the resistance = k.velocity.

Since the resistance R acts in the direction of the curve, the resolved parts in the direction of x and y are

$$-R\frac{dx}{ds},\ -R\frac{dy}{ds}:$$

the equations of motion, therefore, give

$$\left.\begin{aligned} \frac{d^2 x}{d t^2} &= - R \frac{d x}{d s}, \\ \frac{d^2 y}{d t^2} &= - g - R \frac{d y}{d s}; \end{aligned}\right\} \qquad (1),$$

$$\therefore \frac{2 d x d^2 x + 2 d y d^2 y}{d t^2} = - 2 g d y - 2 R d s,$$

$$\text{or } d . \left(\frac{d s^2}{d t^2}\right) = - 2 g d y - 2 R d s, \qquad (2),$$

Also, by (1),

$$\frac{d x d^2 y - d y d^2 x}{d t^2} = - g d x,$$

$$\text{or } \frac{d x^2}{d t^2} d . \left(\frac{d y}{d x}\right) = - g d x. \qquad (3),$$

In deducing the equations (1), $d t$ was supposed constant; but as the equations (2) and (3) are true independently of any such condition, we may, in applying them, suppose $d x$ constant; effecting the differentiation in (3) according to this hypothesis, we have

$\frac{d^2 y}{d t^2} = - g$; and substituting the value of $d t$,

found from this, in (2),

$$g . d \left(\frac{d s^2}{d^2 y}\right) = 2 g d y + 2 R d s;$$

or, since $d . d s^2 = 2 d y d^2 y$,

$$2 g d y - \frac{g d s^2 d^3 y}{(d^2 y)^2} = 2 g d y + 2 R d s:$$

$$\therefore R = - \frac{g d s d^3 y}{2 (d^2 y)^2}.$$

To find the velocity, we have $\frac{d^2 y}{d t^2} = - g$, and

$$\therefore \frac{d s^2}{d t^2} = (\text{vel}^{y}.)^2 = - \frac{g\, d s^2}{d^2 y}.$$

But, in the problem,

$$y = a x + \frac{g}{k^2} \cdot \log (1 - b x),$$

$$d y = a\, d x - \frac{g b}{k^2} \cdot \frac{d x}{1 - b x},$$

$$d^2 y = - \frac{g b^2}{k^2} \cdot \frac{d x^2}{(1 - b x)^2},$$

$$d^3 y = - \frac{2 g b^3}{k^2} \cdot \frac{d x^3}{(1 - b x)^3};$$

$$\therefore R = \frac{k^2}{b} d s (1 - b x).$$

Also, $(\text{velocity})^2 = \frac{k^2}{b^2} \cdot d s^2 \cdot (1 - b x)^2$,

or velocity $= \frac{k}{b} \cdot d s (1 - b x)$;

$\therefore R = k \cdot \text{velocity}$.

Q. E. D.

11. Find the volume of a solid the equation to which is $z = c^{-\frac{y}{c}(a^2 + x^2)}$

between $\left\{ \begin{matrix} x = 0 \\ x = \infty \end{matrix} \right\}$ and $\left\{ \begin{matrix} y = 0 \\ y = \infty \end{matrix} \right\}$

and integrate $\frac{d x}{(1 - x^{\frac{2}{3}})^{\frac{3}{2}}}$, $d \theta \cdot \frac{a + b \tan \theta}{A + B \tan \theta}$,

$$\frac{d y}{d x} \cdot \frac{d^2 y}{d x^2} + (2 x + a) \frac{d^2 y}{d x^2} + 2 \frac{d y}{d x} + 4 x + 2 a = 0.$$

Let $u =$ the volume required;

then $u = \iint z\, dx\, dy,$

$$= \int dx \int dy\, \varepsilon^{-\frac{y}{c}(a^2+x^2)},$$

$$= C - \int^{\circ} \frac{c\, dx}{a^2 + x^2} \cdot \varepsilon^{-\frac{y}{c}(a^2+x^2)}$$

Now the values of $\varepsilon^{-\frac{y}{c}(a^2+x^2)}$ when $y = 0$, and $y = \infty$, are respectively 1 and 0;

$$\therefore\ u = c \int^{\circ} \frac{dx}{a^2 + x^2}$$

$$= \frac{c}{a} \cdot \tan^{-1} \frac{x}{a} \begin{Bmatrix} x = 0 \\ x = \infty \end{Bmatrix},$$

$$= \frac{c}{a} \cdot \frac{\pi}{2}.$$

To integrate $du = \dfrac{dx}{(1 - x^{\frac{2}{3}})^{\frac{3}{2}}}$,

let $x = z^3$; $\therefore\ du = \dfrac{3 z^2\, dz}{(1 - z^2)^{\frac{3}{2}}}$;

$$\therefore\ u = 3 \int \frac{z^2\, dz}{(1 - z^2)^{\frac{3}{2}}}$$

$$= 3 \int z^2 \,.\, d \,.\, \frac{z}{\sqrt{(1 - z^2)}}$$

$$= \frac{3 z^3}{\sqrt{(1 - z^2)}} + 3 z \sqrt{1 - z^2} - 3 \sin^{-1} z,$$

$$= \frac{3 z}{\sqrt{(1 - z^2)}} - 3 \sin^{-1} z$$

$$= \frac{3 x^{\frac{1}{3}}}{\sqrt{(1 - x^{\frac{2}{3}})}} - 3 \sin^{-1} x^{\frac{1}{3}}.$$

To integrate $d\,u = d\,\theta \cdot \dfrac{a + b \tan\theta}{A + B \tan\theta}$,

$$\text{let } \tan\theta = x; \;\therefore\; d\,\theta = \frac{d\,x}{1 + x^2},$$

$$\therefore\; d\,u = \frac{(a + b\,x)\,d\,x}{(1 + x^2)\,(A + B\,x)},$$

which being a rational fraction, we may assume

$$\frac{a + b\,x}{(1 + x^2)(A + B\,x)} = \frac{K x + L}{1 + x^2} + \frac{M}{A + B\,x}; \qquad (1).$$

K, L, M being constants, to be afterwards determined.

By integration,

$$u = K \log \sqrt{1 + x^2} + \frac{M}{B} \log (A + B\,x) + L \tan^{-1} x,$$

$$= K \log \sec\theta + \frac{M}{B} \log (A + B \tan\theta) + L\,\theta.$$

Adding together the fractions on the right side of (1), and equating the numerator so found with that on the left, we get

$$a + b\,x = (BK + M)x^2 + (AK + BL)\,x + AL + M.$$

and equating the homologous terms,

$$BK + \quad M = 0,$$

$$AK + \; BL = b,$$

$$AL + \quad M = a;$$

from which equations, $K = \dfrac{A\,b - B\,a}{A^2 + B^2}$,

$$M = -B \cdot \frac{A\,b - B\,a}{A^2 + B^2}$$

$$L = \frac{a}{A} + \frac{B}{A} \cdot \frac{A\,b - B\,a}{A^2 + B^2}.$$

To integrate the equation

$$\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + (2x+a)\cdot\frac{d^2y}{dx^2} + \frac{2\,dy}{dx} + 2(2x+a) = 0;$$

we may observe, that it is readily resolvable into two factors, and may, therefore, be put in the form

$$\left\{\frac{dy}{dx} + 2x + a\right\}\left\{\frac{d^2y}{dx^2} + 2\right\} = 0,$$

$$\text{or } \left\{\frac{dy}{dx} + 2x + a\right\}\cdot\frac{d}{dx}\cdot\left\{\frac{dy}{dx} + 2x + a\right\} = 0;$$

$\therefore$ multiplying by $2\,dx$, and integrating

$$\left\{\frac{dy}{dx} + 2x + a\right\}^2 = C^2,$$

$$\text{or} \quad \frac{dy}{dx} + 2x + a = C;$$

$$\therefore y + x^2 + ax = Cx + C',$$

$$\text{or } y + x^2 + C_1 x + C_2 = 0.$$

12. A circular sector revolves through any angle round one of its extreme radii; find the centre of gravity of the solid generated, its density varying as the n^{th} power of the distance from the centre of the circle.

In any solid, referred in position and magnitude to three rectangular axes of co-ordinates, if $r =$ the distance of any point from the origin; $\theta =$ angle made by this distance with the axis of z; $\omega =$ angle made by the projection of this distance, on the plane of xy, with the axis of x; $M =$ whole mass; $\rho =$ density at distance r; then (Poisson. Mecanique, Vol. I. p. 169.)

$$M = \iiint \rho\, r^2 \sin\theta \,.\, dr \,.\, d\theta \,.\, d\omega, \qquad (1),$$

Also, if x_1, y_1, z_1 be the co-ordinates of the centre of gravity of M;

$$\left.\begin{aligned} M x_1 &= \iiint \rho r^3 . \sin^2 \theta . \cos \omega . d r d \theta d \omega, \\ M y_1 &= \iiint \rho r^3 . \sin^2 \theta . \sin \omega . d r d \theta d \omega, \\ M z_1 &= \iiint \rho r^3 . \sin \theta . \cos \theta . d r d \theta d \omega. \end{aligned}\right\} \quad (2).$$

Taking the centre of the circular sector as origin, let the extreme radius about which it revolves, be the axis of z. Let θ_1 = angle of the sector; ω_1 = angle through which it revolves; a = its radius; and let the first position of the sector be in the plane of $x z$: then as r, θ, and ω are independent of each other, and $\rho = m r^n$ a function of r only, we have the following limits of the integrations in (1) and (2):

$$\left.\begin{aligned} \theta &= 0 \\ \theta &= \theta_1 \end{aligned}\right\}, \quad \left.\begin{aligned} \omega &= 0 \\ \omega &= \omega_1 \end{aligned}\right\}, \quad \left.\begin{aligned} r &= 0 \\ r &= a \end{aligned}\right\}.$$

We thus have $M = A \omega_1 (1 - \cos \theta_1) = 2 A \omega_1 \sin^2 \frac{\theta_1}{2}$;

A denoting the value of $\int \rho r^2 d r$ or $\frac{m}{n+3} . a^{n+3}$.

Also, from (2),

$$M x_1 = A' . \sin \omega_1 \left(\frac{\theta_1}{2} - \frac{\sin 2 \theta_1}{4} \right),$$

$$M y_1 = A' . \sin^2 \frac{\omega_1}{2} \left(\theta_1 - \frac{\sin 2 \theta_1}{2} \right),$$

$$M z_1 = A' . \frac{\omega_1}{2} . \sin^2 \theta_1 ;$$

A' denoting the value of $\int \rho r^3 d r$, or $\frac{m}{n+4} a^{n+4}$.

The position of the centre of gravity is thus fully determined.

13. In the above case, supposing the angle of the sector and the angle through which it has revolved to remain the same, prove that as the radius varies the motion of the centre of gravity will be in a plane passing through the centre of the circle; find the line of motion, and the equation to the plane.

Since the expressions for each of the co-ordinates x_1, y_1, z_1 of the centre of gravity consists of two factors, one of which is a, and the other a function of ω_2 and θ_1, which are constants, we may briefly express them, by the three equations

$$x_1 = c\,a,\ y_1 = c'\,a,\ z_1 = c''\,a.$$

Eliminating a from the 1st and 3rd, and afterwards from the 2nd and 3rd of these, we have the two equations

$$c\,z_1 = c''\,x_1,\ c'\,z_1 = c''\,y_1,$$

which are those of the line of motion, which is, therefore, a straight line passing through the centre of the sector.

Also, the combination of the two last equations gives

$$x_1 + y_1 - 2\,z_1 = 0,$$

which is the equation to a plane passing through the centre.

14. Water issues from the horizontal surface of a fountain, at an angle α, with a velocity due to h, through a circular annulus of which the radius is r: V is the volume contained by the surface of the fountain, the ascending, and the descending stream; and

V' by the surface, the ascending stream and a plane touching it at the highest point: prove that $\frac{V}{V'}$ is constant when $\sin 2\alpha \propto \frac{r}{h}$.

Let O be the centre of the annulus, and, in the horizontal line OBD, let BD be the horizontal range of the stream, or the base of a parabola BAD, whose vertex is at A, and whose vertical axis is AC.

We thus have $OB = r$, and by the well-known properties of projectiles, $AC = h \sin 2\alpha$, $BD = 2\,BC = 2h \sin \alpha$; and since V may be generated by the revolution of the parabola BAD about a vertical axis passing through 0; the theorem of Guldinus gives

$$V = \frac{2}{3}\,.\,AC\,.\,BD\,.\,2\pi\,.\,OC,$$

$$= \frac{8}{3}\pi h^2 \sin^2 2\alpha\,(r + h \sin 2\alpha).$$

But, by the problem, $\sin 2\alpha \propto \frac{r}{h}$, or $h \sin 2\alpha = mr$, m being any constant:

$$\therefore\ V = \frac{8}{3} m^2 (m+1) r^3.$$

To find V', draw a horizontal line AH, meeting a vertical line drawn through O in H; then V' may be generated by the revolution of the figure $ABOH$ about the vertical axis OH.

Let G be the centre of gravity of the semi-parabola BAC, and G' that of the figure $ABOH$; draw GM, $G'M'$ perpendicular to HO, and meeting it in M, M', then

$$V' = 2\pi\,.\,G'M'\,(AC\,.\,OC - \frac{2}{3} AC\,.\,BC).$$

But, by the property of the centre of gravity,

$$G'M'\left\{AC \,.\, OC - \frac{2}{3}AC \,.\, BC\right\} + \frac{2}{3}AC \,.\, BC \,.\, GM$$

$$= \frac{1}{2}AC \,.\, OC^2,$$

or, since $GM = r + \frac{1}{4}h \sin 2\alpha$,

$$G'M'\left\{h \sin 2\alpha\,(r + h \sin 2\alpha) - \frac{2}{3}h^2 \sin^2 2\alpha\right\}$$

$$+ \frac{2}{3}h^2 \sin^2 2\alpha \left(r + \frac{3}{4}h \sin 2\alpha\right)$$

$$= \frac{1}{2}h \sin 2\alpha\,(r + h \sin 2\alpha)^2;$$

or, making $h \sin 2\alpha = m r$,

$$G'M\left\{m(m+1) - \frac{2}{3}m^2\right\} + \frac{2}{3}m^2(m+1)\,r = \frac{1}{2}m(m+1)^2 r;$$

$$\therefore\ G'M = M r,$$

M being a function of M only.

$$\therefore\ V' = 2\pi\left\{m(m+1) - \frac{2}{3}m^2\right\} M r^3,$$

$$= M' r^3 \text{ suppose.}$$

Since, therefore, it has been shown, that $V = N r$, N being a function of m only, we have

$$\frac{V}{V'} = \frac{N}{M'}, \quad \therefore \frac{V}{V'}, \text{ is constant.}$$

Q. E. D.

15. A body acted on by gravity oscillates in a curve, and a chain of given length, suspended from the horizontal ordinate where the motion commences, is divided by the ordinate at each point into two parts

proportional to the two parts of the tension at that point arising from the centrifugal force and from gravity. What is the curve?

Taking the lowest point of the curve for the origin, let h = abscissa to the ordinate from which the chain is suspended, k = length of that part of the chain that is below the lowest point of the curve, so that $h + k$ = its whole length.

Now (Whewell's Dynamics, p. 93) the two parts of the tension arising from the centrifugal force, and from gravity are respectively equal to $\frac{g\,dy}{ds}$, and $\frac{v^2}{\rho}$, v being the velocity, and ρ the radius of curvature at any point:

∴ by the question,

$$h - x : k + x :: \frac{v^2}{\rho} : \frac{g\,dy}{ds},$$

$$:: 2g(h - x) : g\rho\frac{dy}{ds},$$

$$\text{or} \qquad 1 : k + x :: 2 : \rho\frac{dy}{ds}.$$

As the above expressions are true, independently of any relation between the several differentials, we may suppose dx constant; we, therefore, have

$$\rho = -\frac{ds^3}{dx\,d^2y},$$

$$\therefore \rho\frac{dy}{ds} = -\frac{ds^2}{d^2y}\cdot\frac{dy}{dx};$$

∴ from the above proportion, making

$$\frac{dy}{dx} = p, \quad \frac{-2\,dp}{p(1 + p^2)} = \frac{dx}{x + k}$$

and, by integration,

$$\frac{c(1 + p^2)}{p^2} = x + k, \ \therefore\ p^2 = \frac{c}{x + k - c}.$$

But, supposing the oscillations to be symmetrical on each side of the axis of x, the curve has a horizontal tangent at its lowest point, and we must, therefore, have p infinite when $x = 0$: this condition gives

$$c = k, \text{ and } \therefore\ p = \sqrt{\frac{k}{x}}, \text{ or } y^2 = 4\,k\,x.$$

The curve is, therefore, a parabola whose latus rectum = four times the length of the part of the chain below the vertex.

16. A paraboloid revolving round its axis strikes a body P in a direction perpendicular to the radius, and P, being attracted to the intersection of the radius and axis by a force varying as $\frac{1}{D^2}$, after impact describes a parabola of the same dimensions as the generating one. Determine the velocity of rotation and the point of impact.

Let M be the mass of the paraboloid; k its radius of gyration, ω, ω' its respective angular velocities before and after impact, and a the distance of the point of impact from the axis.

The effect of M on P at a distance Q from the axis is the same as that of a mass $\frac{M\,k^2}{a^2}$ at that distance; if then V, v be the velocities of the point of impact before and after the impact takes place, we must have

$$v\left(P + \frac{M\,k^2}{a^2}\right) = \frac{M\,k^2}{a^2}\,.\,V,$$

or, since $v = a\,\omega'$, and $V = a\,\omega$,

$$\omega'\left(P + \frac{M\,k^2}{a^2}\right) = \frac{M\,k^2}{a^2}\,\omega$$

$$\therefore\ \omega' = \frac{M\,k^2\,\omega}{M\,k^2 + P\,a^2}.$$

Again, since P is struck in a direction perpendicular to the radius, and its path is a parabola, it must begin to move from the vertex of the parabola; therefore the distance a is equal to $\frac{1}{4}$ latus rectum or $\frac{b^2}{4h}$ if h be the altitude and b the base of the paraboloid. Also, the velocity at the vertex of a parabola, with a force tending to the focus, being that acquired by falling through $\frac{1}{4}$ latus rectum with the force at the vertex, we have

$$v^2 = \frac{m}{a^2} \cdot \frac{b^2}{4h} = \frac{4mh}{b^2}; \quad \therefore v = 2\frac{\sqrt{(mh)}}{b},$$

m denoting the absolute force.

$$\text{But } v = a\omega' = \frac{Mk^2 a\omega}{Mk^2 + Pa^2}, \text{ and } k^2 = \frac{b^2}{3};$$

$$\therefore \omega = \sqrt{\left(\frac{m}{h}\right)} \cdot \frac{16Mh^2 + 3Pb^2}{2Mb^3}.$$

17. A given opaque sphere and a given luminous paraboloid of revolution have their axes in the same line; the distance between them being known, deduce the equation to the surface of the shadow, and find the form of the shadow thrown on a given plane.

Let us first suppose the axis of the paraboloid to be of indefinite length, and it is evident that a common tangent to the generating parabola and circle will be the generating line to the surface of the shadow, and that no part of the paraboloid at a greater distance from the vertex than its point of contact with the straight line, can illuminate the sphere or affect the form or dimensions of the shadow. To determine the position of a common tangent to the parabola and circle, let a = radius of the circle, δ = distance of its centre from the vertex of the parabola, $4c$ = latus rectum of the parabola: then taking the centre of the circle for the origin, the equations of the two curves are

$$y'^2 = a^2 - x'^2,\ y^2 = 4\,c\,(x - \delta);$$

those of their tangents are

$$Y y' + X x' = a^2,$$
$$Y y = 2\,c\,X + 2\,c\,x - 4\,c\,\delta.$$

Now since these two tangents coincide, they must cut the axis in the same point and have the same inclination to it; ∴ making $Y = 0$ in each of their equations, and equating the resulting values of x, we have

$$\frac{a^2}{x'} = 2\,\delta - x \qquad (1);$$

and, since $\dfrac{d\,y'}{d\,x'} = \dfrac{d\,y}{d\,x} - \dfrac{x'}{y'} = \dfrac{2\,c}{y} \qquad (2).$

From (2) we get $x'^2 = \dfrac{4\,a^2\,c^2}{4\,c^2 + y^2}$, which being substituted in (1) gives, after reduction,

$$x^2 - \left(\frac{a^2}{c} + 4\,\delta\right) x = a^2 - \frac{a^2\,\delta}{c} + 4\,\delta^2;$$

from which equation,

$$x = \frac{1}{2}\left(\frac{a^2}{c} + 4\,\delta\right) \pm a\,\sqrt{\left\{\frac{a^2}{4\,c^2} + \frac{\delta}{c} + 1\right\}}.$$

The two roots of this equation indicate two positions of the tangent; one in which the two points of contact are on the same side, and the other in which they are on different sides of the axis: the former determining the dark shadow and the other the *penumbra* or partial shadow of the sphere: but, as the tangent in the latter case is more inclined to the axis than in the former, and, therefore, the abscissa in the former case is greater than in the latter; the greatest root of the equation determines the required point.

Subtracting δ from this value of x gives the length of the axis of a paraboloid of given latus rectum, every part of which illuminates the sphere. If, therefore, the given axis of the paraboloid be greater than

$$\frac{a^2}{2c} + \delta + a\sqrt{\left\{\frac{a^2}{4c^2} + \frac{\delta}{c} + 1\right\}} \text{ which call } k,$$

the whole paraboloid will not affect the shadow, but only a portion of it included between the vertex and a circular section at a distance $= k$ from it. If the given axis be less than k, the shadow is the same as when, for the paraboloid, is substituted a circle equal and similarly situated to its base.

The value of $\frac{dy}{dx}$ corresponding to the value of x, found above, gives the tangent of inclination of the slant side of the conical shadow to the axis. The equation to the surface is most simply exhibited by making the origin its intersection with the axis, and measuring z along that axis: the equation is, therefore, $x^2 + y^2 = kz$.

To find the form of the shadow on a given plane, see Hamilton's Analytical Geometry, p. 266.

18. In the series of quantities $A_1, A_2, A_3, \ldots\ldots$

$$\text{if } A_1 = r \tan\left(\sin\overline{\frac{2\pi}{3} + a}\right), \; A_2 = r\tan\left(\sin\overline{\frac{4\pi}{3} + a}\right),$$

and the remaining ones be derived according to the following law:

$$A_1 . A_2 . A_3 = r^2(A_1 + A_2 + A_3),\; A_2 . A_3 . A_4 = r^2(A_2 + A_3 + A_4), \ldots$$

$$\text{prove that } A_n = r\tan\left(\sin\overline{\frac{2n\pi}{3} + a}\right).$$

By the given law,

$$A_n . A_{n+1} . A_{n+2} = r^2(A_n + A_{n+1} + A_{n+2}).$$

13. In the above case, supposing the angle of the sector and the angle through which it has revolved to remain the same, prove that as the radius varies the motion of the centre of gravity will be in a plane passing through the centre of the circle; find the line of motion, and the equation to the plane.

Since the expressions for each of the co-ordinates x_1, y_1, z_1 of the centre of gravity consists of two factors, one of which is a, and the other a function of ω_2 and θ_1, which are constants, we may briefly express them, by the three equations

$$x_1 = c\,a,\ y_1 = c'\,a,\ z_1 = c''\,a.$$

Eliminating a from the 1st and 3rd, and afterwards from the 2nd and 3rd of these, we have the two equations

$$c\,z_1 = c''x_1,\ c'\,z_1 = c''y_1,$$

which are those of the line of motion, which is, therefore, a straight line passing through the centre of the sector.

Also, the combination of the two last equations gives

$$x_1 + y_1 - 2\,z_1 = 0,$$

which is the equation to a plane passing through the centre.

14. Water issues from the horizontal surface of a fountain, at an angle α, with a velocity due to h, through a circular annulus of which the radius is r: V is the volume contained by the surface of the fountain, the ascending, and the descending stream; and

V' by the surface, the ascending stream and a plane touching it at the highest point: prove that $\frac{V}{V'}$ is constant when $\sin 2a \propto \frac{r}{h}$.

Let O be the centre of the annulus, and, in the horizontal line OBD, let BD be the horizontal range of the stream, or the base of a parabola BAD, whose vertex is at A, and whose vertical axis is AC.

We thus have $OB = r$, and by the well-known properties of projectiles, $AC = h \sin 2a$, $BD = 2\,BC = 2\,h \sin a$; and since V may be generated by the revolution of the parabola BAD about a vertical axis passing through 0; the theorem of Guldinus gives

$$V = \frac{2}{3} . AC . BD . 2\pi . OC,$$

$$= \frac{8}{3} \pi h^2 \sin^2 2a (r + h \sin 2a).$$

But, by the problem, $\sin 2a \propto \frac{r}{h}$, or $h \sin 2a = m r$, m being any constant:

$$\therefore\ V = \frac{8}{3} m^2 (m+1) r^3.$$

To find V', draw a horizontal line AH, meeting a vertical line drawn through O in H; then V' may be generated by the revolution of the figure $ABOH$ about the vertical axis OH.

Let G be the centre of gravity of the semi-parabola BAC, and G' that of the figure $ABOH$; draw GM, $G'M'$ perpendicular to HO, and meeting it in M, M', then

$$V' = 2\pi . G'M' (AC . OC - \frac{2}{3} AC . BC).$$

But, by the property of the centre of gravity,

$$G'M' \left\{AC \cdot OC - \frac{2}{3} AC \cdot BC\right\} + \frac{2}{3} AC \cdot BC \cdot GM$$

$$= \frac{1}{2} AC \cdot OC^2,$$

$$\text{or, since } GM = r + \frac{1}{4} h \sin 2\alpha,$$

$$G'M' \left\{h \sin 2\alpha \, (r + h \sin 2\alpha) - \frac{2}{3} h^2 \sin^2 2\alpha\right\}$$

$$+ \frac{2}{3} h^2 \sin^2 2\alpha \left(r + \frac{3}{4} h \sin 2\alpha\right)$$

$$= \frac{1}{2} h \sin 2\alpha \, (r + h \sin 2\alpha)^2;$$

or, making $h \sin 2\alpha = m r$,

$$G'M \left\{m(m+1) - \frac{2}{3} m^2\right\} + \frac{2}{3} m^2 (m+1) r = \frac{1}{2} m (m+1)^2 r;$$

$$\therefore \; G'M = M r,$$

M being a function of M only.

$$\therefore \; V' = 2\pi \left\{m(m+1) - \frac{2}{3} m^2\right\} M r^3,$$

$$= M' r^3 \text{ suppose.}$$

Since, therefore, it has been shown, that $V = N r$, N being a function of m only, we have

$$\frac{V}{V'} = \frac{N}{M'}, \quad \therefore \frac{V}{V'}, \text{ is constant.}$$

Q. E. D.

15. A body acted on by gravity oscillates in a curve, and a chain of given length, suspended from the horizontal ordinate where the motion commences, is divided by the ordinate at each point into two parts

proportional to the two parts of the tension at that point arising from the centrifugal force and from gravity. What is the curve?

Taking the lowest point of the curve for the origin, let $h =$ abscissa to the ordinate from which the chain is suspended, $k =$ length of that part of the chain that is below the lowest point of the curve, so that $h + k =$ its whole length.

Now (Whewell's Dynamics, p. 93) the two parts of the tension arising from the centrifugal force, and from gravity are respectively equal to $\frac{g\,dy}{ds}$, and $\frac{v^2}{\rho}$, v being the velocity, and ρ the radius of curvature at any point:

∴ by the question,

$$h - x : k + x :: \frac{v^2}{\rho} : \frac{g\,dy}{ds},$$

$$:: 2g(h - x) : g\rho\frac{dy}{ds},$$

$$\text{or} \qquad 1 : k + x :: 2 : \rho\frac{dy}{ds}.$$

As the above expressions are true, independently of any relation between the several differentials, we may suppose dx constant; we, therefore, have

$$\rho = -\frac{ds^3}{dx\,d^2y},$$

$$\therefore \rho\frac{dy}{ds} = -\frac{ds^2}{d^2y}\cdot\frac{dy}{dx};$$

∴ from the above proportion, making

$$\frac{dy}{dx} = p, \quad \frac{-2\,dp}{p(1 + p^2)} = \frac{dx}{x + k}$$

and, by integration,

$$\frac{c(1 + p^2)}{p^2} = x + k, \quad \therefore p^2 = \frac{c}{x + k - c}.$$

But, supposing the oscillations to be symmetrical on each side of the axis of x, the curve has a horizontal tangent at its lowest point, and we must, therefore, have p infinite when $x=0$: this condition gives

$$c=k, \text{ and } \therefore p=\sqrt{\frac{k}{x}}, \text{ or } y^2=4\,k\,x.$$

The curve is, therefore, a parabola whose latus rectum = four times the length of the part of the chain below the vertex.

16. A paraboloid revolving round its axis strikes a body P in a direction perpendicular to the radius, and P, being attracted to the intersection of the radius and axis by a force varying as $\frac{1}{D^2}$, after impact describes a parabola of the same dimensions as the generating one. Determine the velocity of rotation and the point of impact.

Let M be the mass of the paraboloid; k its radius of gyration, ω, ω' its respective angular velocities before and after impact, and a the distance of the point of impact from the axis.

The effect of M on P at a distance Q from the axis is the same as that of a mass $\frac{M k^2}{a^2}$ at that distance; if then V, v be the velocities of the point of impact before and after the impact takes place, we must have

$$v\left(P+\frac{M k^2}{a^2}\right)=\frac{M k^2}{a^2}\,.\,V,$$

or, since $v=a\,\omega'$, and $V=a\,\omega$,

$$\omega'\left(P+\frac{M k^2}{a^2}\right)=\frac{M k^2}{a^2}\,\omega$$

$$\therefore \omega'=\frac{M k^2\,\omega}{M k^2+P a^2}.$$

Again, since P is struck in a direction perpendicular to the radius, and its path is a parabola, it must begin to move from the vertex of the parabola; therefore the distance a is equal to $\frac{1}{4}$ latus rectum or $\frac{b^2}{4h}$ if h be the altitude and b the base of the paraboloid. Also, the velocity at the vertex of a parabola, with a force tending to the focus, being that acquired by falling through $\frac{1}{4}$ latus rectum with the force at the vertex, we have

$$v^2 = \frac{m}{a^2} \cdot \frac{b^2}{4h} = \frac{4mh}{b^2}; \therefore v = 2\frac{\sqrt{(mh)}}{b},$$

m denoting the absolute force.

$$\text{But } v = a\omega' = \frac{Mk^2 a\omega}{Mk^2 + Pa^2}, \text{ and } k^2 = \frac{b^2}{3};$$

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17. A given opaque sphere and a given luminous paraboloid of revolution have their axes in the same line; the distance between them being known, deduce the equation to the surface of the shadow, and find the form of the shadow thrown on a given plane.

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$$y'^2 = a^2 - x'^2,\ y^2 = 4\,c\,(x - \delta);$$

those of their tangents are

$$Y y' + X x' = a^2,$$
$$Y y = 2\,c\,X + 2\,c\,x - 4\,c\,\delta.$$

Now since these two tangents coincide, they must cut the axis in the same point and have the same inclination to it; $\therefore$ making $Y = 0$ in each of their equations, and equating the resulting values of x, we have

$$\frac{a^2}{x'} = 2\,\delta - x \qquad (1);$$

and, since $\dfrac{d\,y'}{d\,x'} = \dfrac{d\,y}{d\,x} - \dfrac{x'}{y'} = \dfrac{2\,c}{y}$ (2).

From (2) we get $x'^2 = \dfrac{4\,a^2\,c^2}{4\,c^2 + y^2}$, which being substituted in (1) gives, after reduction,

$$x^2 - \left(\frac{a^2}{c} + 4\,\delta\right) x = a^2 - \frac{a^2\,\delta}{c} + 4\,\delta^2;$$

from which equation,

$$x = \frac{1}{2}\left(\frac{a^2}{c} + 4\,\delta\right) \pm a\sqrt{\left\{\frac{a^2}{4\,c^2} + \frac{\delta}{c} + 1\right\}}.$$

The two roots of this equation indicate two positions of the tangent; one in which the two points of contact are on the same side, and the other in which they are on different sides of the axis: the former determining the dark shadow and the other the *penumbra* or partial shadow of the sphere: but, as the tangent in the latter case is more inclined to the axis than in the former, and, therefore, the abscissa in the former case is greater than in the latter; the greatest root of the equation determines the required point.

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$$\frac{a^2}{2c} + \delta + a\sqrt{\left\{\frac{a^2}{4c^2} + \frac{\delta}{c} + 1\right\}} \text{ which call } k,$$

the whole paraboloid will not affect the shadow, but only a portion of it included between the vertex and a circular section at a distance $= k$ from it. If the given axis be less than k, the shadow is the same as when, for the paraboloid, is substituted a circle equal and similarly situated to its base.

The value of $\frac{dy}{dx}$ corresponding to the value of x, found above, gives the tangent of inclination of the slant side of the conical shadow to the axis. The equation to the surface is most simply exhibited by making the origin its intersection with the axis, and measuring z along that axis: the equation is, therefore, $x^2 + y^2 = kz$.

To find the form of the shadow on a given plane, see Hamilton's Analytical Geometry, p. 266.

18. In the series of quantities $A_1, A_2, A_3, \ldots\ldots$

$$\text{if } A_1 = r\tan\left(\sin\overline{\frac{2\pi}{3} + a}\right), \; A_2 = r\tan\left(\sin\overline{\frac{4\pi}{3} + a}\right),$$

and the remaining ones be derived according to the following law:

$$A_1 . A_2 . A_3 = r^2(A_1 + A_2 + A_3),\; A_2 . A_3 . A_4 = r^2(A_2 + A_3 + A_4), \ldots$$

$$\text{prove that } A_n = r\tan\left(\sin\overline{\frac{2n\pi}{3} + a}\right).$$

By the given law,

$$A_n . A_{n+1} . A_{n+2} = r^2(A_n + A_{n+1} + A_{n+2}).$$

Assume $A_n = r \tan u_n$;

$\therefore \tan u_n . \tan u_{n+1} . \tan u_{n+2} = \tan u_n + \tan u_{n+1} + \tan u_{n+2}$;

$$\therefore u_{n+2} + u_{n+1} + u_n = 0;$$

which equation of differences, being integrated as in page 14, gives

$$u_n = C \cos \frac{2n\pi}{3} + C' \sin \frac{2n\pi}{3},$$

$$= C_1 \sin \left(\frac{2n\pi}{3} + C_2\right):$$

$$\therefore A_n = r \tan \left\{ C_1 \sin \left(\frac{2n\pi}{3} + C_2\right)\right\},$$

which, compared with the two given forms of A_1, A_2,

gives $C_1 = 1$, $C_2 = a$:

$$\therefore A_n = r . \tan \left\{\sin \left(\frac{2n\pi}{3} + a\right)\right\}.$$

19. If $Ay^3 + Bxy^2 + Cxy + Dx^2 + Ey + Fx = 0$ be the equation to a curve; $By^2 + Cy + 2Dx + F = 0$ is the equation to a parabola which bisects all the chords parallel to the axis of x; $2By^2 + Cy + 2Dx + F = 0$ and $Cy + 2Dx + F = 0$ are equations to a parabola and straight line which are asymptotes to the curve; and the two parabolas and the straight line have a common point of contact in the bisection of that chord which passes through the origin.

Dividing every term of the equation by D, we may put it in the form

$$ay^3 + bxy^2 + cxy + x^2 + ey + fx = 0;$$

and solving this equation with respect to x, we have

$$x = -\frac{by^2 + cy + f}{2} \pm \sqrt{\left\{\frac{(by^2 + cy + f)^2}{4} - ay^3 - ey\right\}} \quad (1).$$

Now, since the value of x consists of two parts, one rational, and having one value for each value of y; the other irrational, and having two equal values, with opposite signs, for each value of y; it is clear, that the part

$$x = -\frac{by^2 + cy + f}{2} \qquad (2)$$

would express the equation to a line from which two points of the curve are equally distant in the directions of $\pm x$; and which, therefore, bisects all the chords parallel to the axis of x.

But, restoring the values of b, c, f, transposing all the terms to one side, and clearing it of fractions, (2) becomes

$$By^2 + Cy + 2Dx + F = 0, \qquad (3).$$

Q. E. D.

Since the irrational part of (1) expresses half the length of any one of the chords, we may find how far the curve recedes from the parabola (3) at an infinite distance from the axis of x, by making y infinite in that part, which reduces it to $\pm \frac{by^2}{2}$: the corresponding value of x is

$$x = -\frac{by^2 + cy + f}{2} \pm \frac{by^2}{2};$$

∴ taking the lower and upper signs in succession, we have the equations

$$x = -\frac{2by^2 + cy + f}{2},$$

$$x = -\frac{cy + f}{2};$$

or $$2By^2 + Cy + 2Dx + F = 0, \qquad (4),$$

$$Cy + 2Dx + F = 0, \qquad (5),$$

which express the relations between x and y, when y is infinite.

But since, in general, the equations (4) and (5) respectively represent a parabola and straight line, and coincide with the equation to the given curve when y is infinite; this parabola and straight line must be asymptotes to the given curve.

Q. E. D.

To find where the curves defined by equations (3) and (4), have a common point, subtract (3) from (4), and we have

$$y = 0, \text{ and } x = -\frac{F}{2D} = -\frac{f}{2}$$

for the co-ordinates of this point.

Similarly, the equations (4) and (5) give

$$y = 0, \text{ and } x = -\frac{F}{2D} = -\frac{f}{2}$$

for the common point of the lines they represent.

Putting $y = 0$ in equation (1), the values of x are 0 and $-f$; $\therefore$ $y = 0$ and $x = -\frac{f}{2}$ are the co-ordinates of the point of bisection of that chord which passes through the origin.

Q. E. D.

20. If a pendulum of length l vibrate in a small circular arc in a medium of which the resistance $= k v^2$ to velocity v, and if s be the arc described from the commencement of a vibration to the point where the velocity is greatest when the friction at the axis of suspension is taken into account, and s' the corresponding arc when the friction is neglected, prove that

$$\varepsilon^{2ks'} - \varepsilon^{2ks} = f \cdot \frac{2kl}{g},$$

f being the constant effect of friction and g gravity.

Since, in a small circular arc, the effective force of gravity, at any point, is as the length of the arc measured from the lowest point, if z be the length of the arc, the force may be expressed by $m z$:

$$\therefore\ v dv = -(f + m z - k v^2)\, d z.$$

$$\text{Let } v^2 = 2 u,\ \therefore\ v dv = d u;$$

$$\therefore\ du = -(f + m z - 2 k u)\, d z:$$

Multiply by ε^{-2kz},

$$\therefore\ d u \,.\, \varepsilon^{-2kz} - u \,.\, \varepsilon^{-2kz} \,.\, 2 k\, d z = - f\, \varepsilon^{-2kz} d z - m\, \varepsilon^{-2kz} z\, d z;$$

whence, by integration,

$$u\, \varepsilon^{-2kz} = C + \frac{f}{2k} \cdot \varepsilon^{-2kz} + \frac{m z}{2k} \varepsilon^{-2kz} + \frac{m\, \varepsilon^{-2kz}}{4 k^2};$$

$$\therefore\ u = C \varepsilon^{2kz} + \frac{m}{4 k^2} (2 k z + 1) + \frac{f}{2k}.$$

Suppose now, that, at the beginning of the motion $z = a$; or that $u = 0$, when $z = a$;

$$\therefore\ 0 = C \varepsilon^{2ka} + \frac{m}{4 k^2} (2 k a + 1) + \frac{f}{2k},$$

$$\text{or } C = - \frac{m}{4 k^2} (2 k a + 1)\, \varepsilon^{-2ka} - \frac{f}{2k} \varepsilon^{-2ka},$$

$$\text{and } C \varepsilon^{2kz} = - \frac{m}{4 k^2} (2 k a + 1)\, \varepsilon^{-2k(a-z)} - \frac{f}{2k} \varepsilon^{-2k(a-z)};$$

$$\therefore\ u = \frac{m}{4 k^2} \left\{ 2 k z + 1 - (2 k a + 1)\, \varepsilon^{-2k(a-z)} \right\}$$

$$+ \frac{f}{2k} \left\{ 1 - \varepsilon^{-2k(a-z)} \right\}.$$

To find where the velocity is greatest, we must make

$\dfrac{d v}{d z} = 0$, and $\therefore\ \dfrac{d u}{d z} = 0$; also at this point $a - z = s'$:

$$\therefore \frac{m}{2k}\left\{1 - (2ka + 1)\varepsilon^{-2ks'}\right\} - f\varepsilon^{-2ks'} = 0:$$

$$\therefore \varepsilon^{-2ks'} = 2ka + 1 + f \cdot \frac{2k}{m}.$$

But if x be the abscissa to the circular arc z (radius $= l$), the force of gravity down the arc is

$$g\frac{dx}{dz} = g\frac{\sqrt{2lx - x^2}}{l} = \frac{gz}{l}$$

when x is small; that is

$$mz = \frac{gz}{l}, \text{ or } m = \frac{g}{l}:$$

$$\therefore \varepsilon^{-2ks'} = 2ka + 1 + f \cdot \frac{2kl}{g};$$

and when $f = 0$, $s' = s$;

$$\therefore \varepsilon^{-2kz} = 2ka + 1;$$

$$\therefore \varepsilon^{-2ks'} - \varepsilon^{-2ks} = f \cdot \frac{2kl}{g}.$$

Q. E. D.

21. Two planets P_1, P_2 revolve in circular orbits at the distances r_1, r_2 from the Sun, and when they appear stationary to one another, cot P_2's elongation seen from $P_1 = \frac{1}{2}\tan\theta$; shew that $\frac{r_1}{r_2} = \frac{1}{2}\tan\frac{\theta}{2} \cdot \tan\theta$.

If L_1, L_2 be the heliocentric longitudes of P_1, P_2, and λ the longitude of P_2 seen from P_1, we have (Maddy's Astronomy, p. 191.)

$$\tan\lambda = \frac{r_1 \sin L_1 - r_2 \sin L_2}{r_1 \cos L_1 - r_2 \cos L_2}:$$

but as the longitude of either planet may be taken at pleasure, we may suppose $L_1 = 0$, in which case the elongation of P_2 seen from $P_1 = \pi - \lambda$;

∴ tangent of elongation

$$= \tan(\pi - \lambda) = -\tan\lambda = \frac{r_2 \sin L_2}{r_1 - r_2 \cdot \cos L_2}$$

$$= \frac{\sin L_2}{m - \cos L_2}, \text{ making } \frac{r_1}{r_2} = m.$$

We, therefore, have

$$\frac{1}{2}\tan\theta = \frac{m - \cos L_2}{\sin L_2}. \qquad (A)$$

But (Maddy's Astronomy, p. 192) when the planets appear stationary to one another,

$$\cos L_2 = \frac{m + m^{\frac{1}{2}}}{1 + m^{\frac{3}{2}}};$$

$$\therefore \sin L_2 = \frac{(m^2 - 1)\sqrt{(1 + m)}}{1 + m^{\frac{3}{2}}};$$

and substituting these values in (A), we get

$$\frac{1}{2}\tan\theta = m^{\frac{1}{2}}(m + 1)^{\frac{1}{2}}; \qquad (B)$$

$$\therefore 1 + \tan^2\theta = (2m + 1)^2;$$

$$\therefore \sec\theta = \frac{1}{\cos\theta} = 2m + 1:$$

∴ subtracting and adding 1 successively to both sides of this equation, and dividing the 1st result by the 2nd, we get

$$\frac{1 - \cos\theta}{1 + \cos\theta} = \frac{m}{m + 1}, \text{ or } \tan\frac{\theta}{2} = \frac{m^{\frac{1}{2}}}{(m + 1)^{\frac{1}{2}}},$$

which multiplied by equation (B) gives

$$m = \frac{r_1}{r_2} = \frac{1}{2}\tan\frac{\theta}{2} \cdot \tan\theta.$$

Q. E. D.

22. $\left.\begin{array}{l} Ax + By + Cz = 0 \\ A'x + B'y + C'z = 0 \end{array}\right\}$ are the equations to the planes in which two planets move. Apply them to find the inclination of the orbits to one another, in terms of their inclinations to the ecliptic and of the longitudes of their ascending nodes, the ecliptic being in the plane of x and y.

Let λ, λ' be the longitudes of the nodes; θ, θ' the inclinations of the orbits to the ecliptic; ϕ their inclination to one another; and let the longitude be reckoned from the axis of x:

then on the plane of $x\,y$ we have

$$y = -\frac{A}{B}x, \quad \therefore \tan\lambda = -\frac{A}{B};$$

$$y = -\frac{A'}{B'}x, \quad \therefore \tan\lambda' = -\frac{A'}{B'}.$$

Also, $$\cos\theta = \pm\frac{C}{\sqrt{(A^2 + B^2 + C^2)}},$$

$$\cos\theta' = \pm\frac{C'}{\sqrt{(A'^2 + B'^2 + C'^2)}},$$

$$\cos\phi = \frac{AA' + BB' + CC'}{\sqrt{(A^2 + B^2 + C^2)}\sqrt{(A'^2 + B'^2 + C'^2)}}.$$

(Hamilton's Analytical Geometry.)

But, from the above expressions for $\cos\theta$, $\tan\lambda$,

$$\sin^2\theta = 1 - \cos^2\theta = \frac{A^2 + B^2}{A^2 + B^2 + C^2};$$

$$\therefore \tan^2\theta = \frac{A^2 + B^2}{C^2}: \text{ also } \sec^2\lambda = \frac{A^2 + B^2}{B^2};$$

$$\therefore \frac{B}{C} = \frac{\tan\theta}{\tan\lambda}. \quad \text{Similarly, } \frac{B'}{C'} = \frac{\tan\theta'}{\tan\lambda'}.$$

Hence $\frac{BB'}{CC'} = \frac{\tan\theta \, . \, \tan\theta'}{\tan\lambda \, . \, \tan\lambda'}$, and $\frac{AA'}{CC'} = \tan\theta \, . \, \tan\theta'$;

by adding together these two last equations, and then adding 1 to the resulting equation, we get

$$\frac{AA' + BB' + CC'}{CC'} = \frac{\tan\theta . \tan\theta' + \tan\lambda . \tan\lambda' + \tan\theta . \tan\theta' \tan\lambda . \tan\lambda'}{\tan\lambda \, . \, \tan\lambda'}.$$

Finally, since

$$CC' = \cos\theta \, . \, \cos\theta' \sqrt{(A^2 + B^2 + C^2)} \sqrt{(A'^2 + B'^2 + C'^2)},$$

there results

$$\cos\phi = \frac{\tan\theta . \tan\theta' + \tan\lambda \, . \, \tan\lambda' + \tan\theta . \tan\theta' . \tan\lambda . \tan\lambda'}{\tan\lambda \, . \, \tan\lambda' \, . \, \sec\theta \, . \, \sec\theta}.$$

23. If the Earth be an oblate spheroid of small ellipticity with semi-axes a and b, the ratio of the mean density to that at the surface is

$$\frac{3}{k^2 a^2}\left(1 - \frac{k(4a - b)}{3 \tan k b}\right) \text{very nearly,}$$

assuming the density to be uniform throughout each spheroidal stratum at the same distance from the Earth's surface, and to vary as $\frac{\sin k r}{r}$ at different distances, where k is a constant quantity and r the polar semi-axis of the surface of equal density.

The mean density of the Earth is the density of a homogeneous spheroid of equal mass and volume; it is therefore expressed by $\frac{M}{V}$, M denoting the Earth's whole mass, and V its volume.

All the spheroidal strata being of equal ellipticity (ε), let R denote the surface of any one of them whose polar semi-axis is r; then

$$M = 2\int R \, \frac{\sin k\,r}{r} \, d\,r \, . \begin{Bmatrix} r = 0 \\ r = b \end{Bmatrix} .$$

The surface R being generated by the revolution of an ellipse about its minor axis, the equation to this ellipse will be

$$y^2 = (1 + \varepsilon)^2 (r^2 - x^2).$$

$$\text{Hence } \frac{d\,R}{d\,x} = 2\,\pi\,y\,\sqrt{\left(1 + \frac{d\,y^2}{d\,x^2}\right)},$$

$$= 2\pi (1 + \varepsilon) \sqrt{\left\{r^2 - x^2 + (1 + \varepsilon)^2 x^2\right\}},$$

$$= 2\pi (1 + \varepsilon) \sqrt{\left\{r^2 + 2\,\varepsilon\,x^2\right\}} \text{ nearly,}$$

$$= 2\,\pi\,r\,(1 + \varepsilon)\left(1 + \frac{\varepsilon\,x^2}{r^2}\right),$$

expanding and neglecting powers of ε above the first;

$\therefore$ integrating, and making $x = r$,

$$R = 2\,\pi\,(1 + \varepsilon)\left(1 + \frac{\varepsilon}{3}\right) r^2$$

$$= 2\,\pi\left(1 + \frac{4\,\varepsilon}{3}\right) r^2 \text{ nearly;}$$

$$\therefore\ M = 4\left(1 + \frac{4}{3}\varepsilon\right)\int \sin k\,r \, . \, r\,d\,r,$$

$$= \frac{4\,\pi}{k^2}\left(1 + \frac{4\,\varepsilon}{3}\right)(\sin k\,b - k\,b \, . \cos k\,b).$$

$$\text{Also, } V = \frac{4\,\pi\,a^2\,b}{3};$$

$$\therefore\ \frac{M}{V} = \frac{3}{k^2\,a^2\,b}\left(1 + \frac{4}{3}\varepsilon\right)\left\{\sin k\,b - k\,b\,\cos k\,b\right\};$$

and the density at the surface being $\dfrac{\sin k\,b}{b}$;

the ratio required

$$= \frac{3}{k^2 a^2}\left(1 + \frac{4\varepsilon}{3}\right)\left(1 - \frac{k b}{\tan k b}\right),$$

$$= \frac{3}{k^2 a^2}\left\{1 - \left(1 + \frac{4\varepsilon}{3}\right)\frac{k b}{\tan k b}\right\} \text{ nearly.}$$

$$\text{But } 1 + \frac{4\varepsilon}{3} = 1 + \frac{4}{3b}(a - b) = \frac{4a - b}{3b};$$

$$\therefore \text{ the ratio} = \frac{3}{k^2 a^2}\left(1 - \frac{k(4a - b)}{3 \tan k b}\right) \text{nearly.}$$

Q. E. D.

When $a = b$, the above expression agrees with that given in Professor Airy's Tracts, p. 110.

24. Explain the theory of the interferences of light, and determine the colour, origin, and intensity of a ray resulting from the interference of two similar rays, differing in origin and intensity.

On this subject see Dr. Young's Essay on the Theory of Light; Encyclopædia Metropolitana, Article Light; also an abridged translation of the same by Quetelet and Verhülst, a small supplement to the Encyclopædia Britannica, and Biot's Traite de Physique.

APPENDIX I.

(*To page* 23.)

If L', L'' be the powers of the two lenses of a double object-glass at a distance $= t$ from each other, p', p'' their dispersive powers, and D the distance of the object; then, that the combination may be achromatic we must have

$$\left\{1 - t\,(L' + D)\right\}^2 + \frac{p'}{p''} \cdot \frac{L'}{L''} = 0.$$

" Such is the condition of achromaticity. Since it depends on D, it appears, that if the lenses of an object-glass be not close together, it will cease to be achromatic for near objects, however perfectly the colour be corrected for distant ones.

The eye, therefore, cannot be achromatic for objects at all distances, its lenses being of great thickness compared to their focal lengths; and, therefore, although in contact at their adjacent surfaces, yet having considerable intervals between others."

Encyclopædia Metropolitana, Part 19, arts. 479, 480, Light.

APPENDIX II.

(*To page* 54.)

It is shown, in works on the Integral Calculus, that the area of any surface, defined by an equation referring it to three rectangular co-ordinate planes, is expressed by

$$\iint dx\,dy\,\sqrt{(1 + p^2 + q^2)},$$

the integrals being taken between proper limits. Now

$$dx\,dy\,\sqrt{(1 + p^2 + q^2)}$$

expresses the area of an indefinitely small element of the surface intercepted by planes parallel to the co-ordinate planes, and the integration with respect to y gives the area of one of the elementary zones into which the surface is supposed to be divided by planes parallel to the plane of $y\,z$: the subsequent integration with respect to x gives the sum of all these zones or the area of the surface.

In page 54 it is required to find the pressure of a fluid on the surface of an oblique cone; for which purpose we will first express the surface by an equation referring it to three rectangular co-ordinate planes, placing the origin in the vertex, and the principal sections in the vertical plane of $x\,y$. Since all the horizontal sections of the cone are similar ellipses whose major axes vary as their heights from the plane of $y\,z$, or as x; also, since the distances of their centres from the axis of x are as x; the equation to any one of them will be

$$z^2 = c^2 \left\{ m^2 x^2 - (y - n\,x)^2 \right\};$$

c being the constant ratio of the minor to the major axis, and m and n constants readily determined from the formulæ in pages 53 and 54: the equation thus found evidently defines the conical surface.

Differentiating successively with respect to x and y, we have

$$z p = c^2 \{m^2 x + n (y - n x)\},$$

$$z q = - c^2 (y - n x);$$

whence $d x d y \sqrt{1 + p^2 + q^2}$ is readily expressed in terms of x and y, and being integrated with respect to y gives the area of an elementary elliptic zone at an indeterminate distance x from the plane of $y z$: the limits of the integral are evidently the extremities of the minor axis of the ellipse; that is, it must be taken from $y = - c m x$ to $y = + c m x$. Multiplying the result by $g \rho (h - x) d x$, we obtain the pressure on the elementary zone at the depth $h - x$; ρ being the density of the fluid and g gravity. The integral of the expression last found taken between the limits $x = 0$ and $x = h$ gives the whole pressure required. We have restricted ourselves to a bare explanation of the process, as the integrations are of a more complicated nature than the value of the Problem would have warranted us in performing.

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