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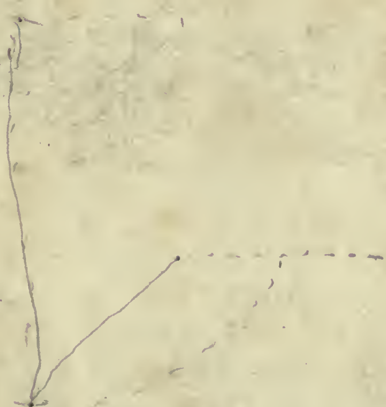

ROBINSON'S MATHEMATICAL SERIES.

ELEMENTS

OF

GEOMETRY,

AND

PLANE AND SPHERICAL TRIGONOMETRY;

WITH

NUMEROUS PRACTICAL PROBLEMS.

BY

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AUTHOR OF A FULL COURSE OF MATHEMATICS.

NEW YORK:
IVISON, PHINNEY & CO., 48 AND 50 WALKER ST.
CHICAGO: S. C. GRIGGS & CO., 39 AND 41 LAKE ST.
BOSTON: BROWN & TAGGARD. PHILADELPHIA: SOWER, BARNES & CO., AND J. B. LIPPINCOTT & CO. CINCINNATI: MOORE, WILSTACH, KEYS & CO. SAVANNAH: J. M. COOPER & CO. ST. LOUIS: KEITH & WOODS. NEW ORLEANS: E. R. STEVENS & CO. DETROIT: RAYMOND & LAPHAM. BALTIMORE: CUSHING & BAILEY.
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PREFACE.

In the preparation of this work, the Author's previous treatise, "Elements of Geometry, Plane and Spherical Trigonometry, and Conic Sections," has formed the ground-work of construction. But in adapting the work to the present advanced state of Mathematical education in our best Institutions, it was found necessary to so alter the plan, and the arrangement of subjects, as to make this essentially a new work. The demonstrations of propositions have undergone radical changes, many new propositions have been introduced, and the number of Practical Problems greatly increased, so that the work is now believed to be as full and complete as could be desired in an elementary treatise.

In view of the fact that the Seventh Book is so much larger than the others, it may be asked why it is not divided into two? We answer, that classifications and divisions are based upon differences, and that the differences seized upon for this purpose must be determined by the nature of the properties and relations we wish to investigate. There is such a close resemblance between the geometrical properties of the polyedrons and the round bodies, and the demonstrations relating to the former require such slight modifications to become applicable to the latter, that there seems no sufficient reason for separating into two Books that part of Geometry which treats of them.

The subject of Spherical Geometry, which has been much extended in the present edition, is placed as before, as an introduction to Spherical Trigonometry. The propriety of this arrangement may be questioned by some; but it is believed that much of the difficulty which the student meets in mastering the propositions of Spherical Trigonometry, arises from the fact that he is not sufficiently familiar with the geometry of the surface of the sphere; and that, by having the propositions of Spherical Geometry fresh in his mind when he begins the study of Spherical Trigonometry, he will be as little embarrassed with it as with Plane Trigonometry.

Both author and teacher must yield to the demands of the age, and by a judicious combination of the abstract and the concrete, the theoretical and the practical, make the student feel that what he learns with perhaps painful effort at first, may be made available in important applications.

In teaching Geometry and Trigonometry, questions should be asked, extra problems given, and original demonstrations required when the proper occasions arise; but care should be taken that the pupil's powers are not over-tasked. By helping him through his difficulties in such a way that he shall be scarcely conscious of having received assistance, he will be encouraged to make new and greater efforts, and will finally acquire a fondness for a study that may have been highly repugnant to him in the beginning.

A demonstration that is easily followed and comprehended by one, may be obscure and difficult to another; hence the advantage that will sometimes be gained by giving two or more demonstrations of the same proposition. When the student perceives that the same results may frequently be reached by processes entirely different, he will be stimulated to independent exertion, and in no respect can the teacher better exhibit his tact than in directing and encouraging such efforts.

Instances will be found throughout the work in which the more important propositions are twice and three times demonstrated; and as the methods of demonstration are in each case quite different, it is believed that extra space has not been thus occupied unprofitably.

Practical rules with applications will be found throughout the work, and in addition to these, there are in both the Geometry and the Trigonometry, full collections of carefully selected Practical Problems. These are given to exercise the powers and test the proficiency of the pupil, and when he has mastered the most or all of them, it is not likely that he will rest satisfied with present acquisition, but conscious of augmented strength and certain of reward, he will enter new fields of investigation.

The Author has been aided, in the preparation of the present work, by J. F. Quinby, A. M., of the University of Rochester, N. Y., late Professor of Mathematics in the United States Military Academy at West Point, and J. H. French, LL. D., of Syracuse, New York. The thorough Scholarship, and long and successful experience of these gentlemen in the class-room, rendered them eminently qualified for the task; and to them the public are indebted for much that is valuable, both in the matter and arrangement of this treatise.

October, 1860.

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GEOMETRY.

DEFINITIONS.

1. Geometry is the science which treats of position, and of the forms, measurements, mutual relations, and properties of limited portions of *space.*

SPACE extends without limit in all directions, and contains all bodies.

2. A **Point** is mere position, and has no magnitude.

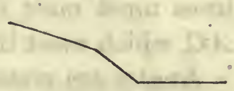
3. Extension is a term employed to denote that property of bodies by virtue of which they occupy definite portions of space. The dimensions of extension are *length, breadth,* and *thickness.*

4. A **Line** is that which has extension in length only. The extremities of a line are points.

5. A **Right** or **Straight Line** is one all of whose parts lie in the same direction.

6. A **Curved Line** is one whose consecutive parts, however small, do not lie in the same direction.

7. A **Broken** or **Crooked Line** is composed of several straight lines, joined one to another successively, and extending in different directions.



When the word *line* is used, a straight line is to be understood, unless otherwise expressed.

8. A **Surface** or **Superficies** is that which has extension in length and breadth only.

9. A **Plane Surface,** or a **Plane,** is a surface such that if any two of its points be joined by a straight line, every point of this line will lie in the surface.

10. A **Curved Surface** is one which is neither a plane, nor composed of plane surfaces.

11. A **Plane Angle**, or simply an **Angle**, is the difference in the direction of two lines proceeding from the same point.

The other angles treated of in geometry will be named and defined in their proper connections.

12. A **Volume**, **Solid**, or **Body**, is that which has extension in length, breadth, and thickness.

These terms are used in a sense purely abstract, to denote mere space — whether occupied by matter or not, being a question with which geometry is not concerned.

Lines, Surfaces, Angles, and Volumes constitute the different kinds of quantity called *geometrical magnitudes*.

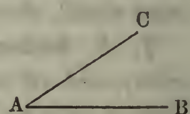
13. **Parallel Lines** are lines which have the same direction.

Hence parallel lines can never meet, however far they may be produced; for two lines taking the same direction cannot approach or recede from each other.

Two parallel lines cannot be drawn from the same point; for if parallel, they must coincide and form one line.

PLANE ANGLES.

To make an angle apparent, the two lines must meet in a point, as AB and AC, which meet in the point A.



Angles are measured by degrees.

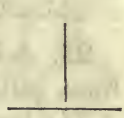
14. A **Degree** is one of the three hundred and sixty equal parts of the space about a point in a plane.

If, in the above figure, we suppose AC to coincide with AB, there will be but one line, and no angle; but if AB retain its position, and AC begin to revolve about the point A, an angle will be formed, and its magnitude will be expressed by that number of the

360 equal spaces about the point *A*, which is contained between *AB* and *AC*.

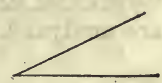
Angles are distinguished in respect to magnitude by the terms Right, Acute, and Obtuse Angles.

15. A **Right Angle** is that formed by one line meeting another, so as to make equal angles with that other.



The lines forming a right angle are *perpendicular to each other*.

16. An **Acute Angle** is less than a right angle.



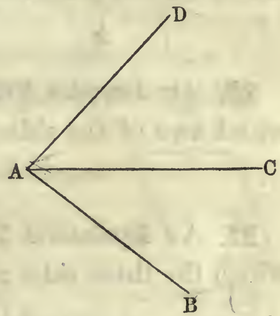
17. An **Obtuse Angle** is greater than a right angle.

Obtuse and acute angles are also called *oblique angles;* and lines which are neither parallel nor perpendicular to each other are called *oblique lines.*

18. The **Vertex** or **Apex** of an angle is the point in which the including lines meet.

19. An angle is commonly designated by a letter at its vertex; but when two or more angles have their vertices at the same point, they cannot be thus distinguished.

For example, when the three lines *AB*, *AC*, and *AD* meet in the common point *A*, we designate either of the angles formed, by three letters, placing that at the vertex between those at the opposite extremities of the including lines. Thus, we say, the angle *BAC*, etc.



20. Complements. — Two angles are said to be complements of each other, when their sum is equal to one right angle.

21. Supplements. — Two angles are said to be supplements of each other, when their sum is equal to two right angles.

PLANE FIGURES.

22. A **Plane Figure,** in geometry, is a portion of a plane bounded by straight or curved lines, or by both combined.

23. A **Polygon** is a plane figure bounded by straight lines, called the sides of the polygon.

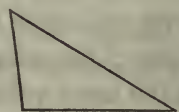
The least number of sides that can bound a polygon is three, and by the figure thus bounded all other polygons are analyzed.

FIGURES OF THREE SIDES.

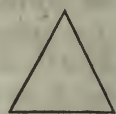
24. A **Triangle** is a polygon having three sides and three angles.

Tri is a Latin prefix signifying three; hence a Triangle is literally a figure containing three angles. Triangles are denominated from the relations both of their sides and angles.

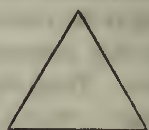
25. A **Scalene Triangle** is one in which no two sides are equal.



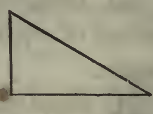
26. An **Isosceles Triangle** is one in which two of the sides are equal.



27. An **Equilateral Triangle** is one in which the three sides are equal.



28. A **Right-Angled Triangle** is one which has one of the angles a right angle.



29. An **Obtuse-Angled Triangle** is one having an obtuse angle.

30. An **Acute-Angled Triangle** is one in which each angle is acute.

31. An **Equiangular Triangle** is one having its three angles equal.

Equiangular triangles are also equilateral, and vice versa.

FIGURES OF FOUR SIDES.

32. A **Quadrilateral** is a polygon having four sides and four angles.

33. A **Parallelogram** is a quadrilateral which has its opposite sides parallel.

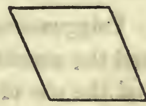
Parallelograms are denominated from the relations both of their sides and angles.

34. A **Rectangle** is a parallelogram having its angles right angles.

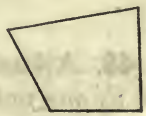
35. A **Square** is an equilateral rectangle.

36. A **Rhomboid** is an oblique-angled parallelogram.

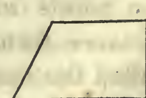
37. A **Rhombus** is an equilateral rhomboid.



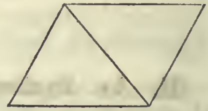
38. A **Trapezium** is a quadrilateral having no two sides parallel.



39. A **Trapezoid** is a quadrilateral in which two opposite sides are parallel, and the other two oblique.



40. Polygons bounded by a greater number of sides than four are denominated only by the number of sides. A polygon of five sides is called a *Pentagon*, of six a *Hexagon*, of seven a *Heptagon*, of eight an *Octagon*, of nine a *Nonagon*, etc.



41. **Diagonals** of a polygon are lines joining the vertices of angles not adjacent.

42. The **Perimeter** of a polygon is its boundary considered as a whole.

43. The **Base** of a polygon is the side upon which the polygon is supposed to stand.

44. The **Altitude** of a polygon is the perpendicular distance between the base and a side or angle opposite the base.

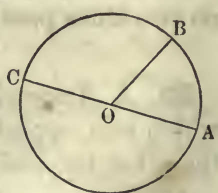
45. **Equal Magnitudes** are those which are not only equal in all their parts, but which also, when applied the one to the other, will coincide throughout their whole extent.

46. **Equivalent Magnitudes** are those which, though they do not admit of coincidence when applied the one to the other, still have common measures, and are therefore numerically equal.

47. **Similar Figures** have equal angles, and the same number of sides.

Polygons may be similar without being equal; that is, the angles and the number of sides may be equal, and the *length* of the sides and the *size* of the figures unequal.

THE CIRCLE.



48. A **Circle** is a plane figure bounded by one uniformly curved line, all of the points in which are at the same distance from a certain point within, called the *Center*.

49. The **Circumference** of a circle is the curved line that bounds it.

50. The **Diameter** of a circle is a line passing through its center, and terminating at both ends in the circumference.

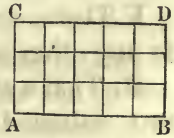
51. The **Radius** of a circle is a line extending from its center to any point in the circumference. It is one half of the diameter. All the diameters of a circle are equal, as are also all the radii.

52. An **Arc** of a circle is any portion of the circumference.

53. An angle having its vertex at the center of a circle is measured by the arc intercepted by its sides. Thus, the arc AB measures the angle AOB; and in general, to compare different angles, we have but to compare the arcs, included by their sides, of the equal circles having their centers at the vertices of the angles.

UNITS OF MEASURE.

54. The **Numerical Expression of a Magnitude** is a number expressing how many times it contains a magnitude of the same kind, and of known value, assumed as a unit. For lines, the measuring unit is any straight line of fixed value, as an inch, a foot, a rod, etc.; and for surfaces, the measuring unit is a square whose side may be any linear unit, as an inch, a foot, a mile, etc. The linear unit being arbitrary, the surface unit is equally so; and its selection is determined by considerations of convenience and propriety.



For example, the parallelogram $ABDC$ is measured by the number of *linear units* in CD, multiplied by the number of *linear units* in AC or BD; the product is the *square units* in $ABDC$. For, conceive CD to be composed of any number of equal parts—say five—and each part some unit of linear measure, and AC composed of three such units; from each point of division on CD draw lines parallel to AC, and from each point of division on AC draw lines parallel to CD or AB; then it is as obvious as an axiom that the parallelogram will contain $5 \times 3 = 15$ square units. Hence, to find the areas of right-angled parallelograms, *multiply the base by the altitude.*

EXPLANATION OF TERMS.

55. An **Axiom** is a self-evident truth, not only too simple to require, *but too simple to admit of, demonstration.*

56. A **Proposition** is something which is either proposed to be done, or to be demonstrated, and is either a problem or a theorem.

57. A **Problem** is something proposed to be done.

58. A **Theorem** is something proposed to be demonstrated.

59. A **Hypothesis** is a supposition made with a view to draw from it some consequence which establishes the truth or falsehood of a proposition, or solves a problem.

60. A **Lemma** is something which is premised, or demonstrated, in order to render what follows more easy.

61. A **Corollary** is a consequent truth derived immediately from some preceding truth or demonstration.

62. A **Scholium** is a remark or observation made upon something going before it.

63. A **Postulate** is a problem, the solution of which is self-evident.

POSTULATES.

Let it be granted—

I. That a straight line can be drawn from any one point to any other point;

II. That a straight line can be produced to any distance, or terminated at any point;

III. That the circumference of a circle can be described about any center, at any distance from that center.

AXIOMS.

1. *Things which are equal to the same thing are equal to each other.*

2. *When equals are added to equals the wholes are equal.*

3. *When equals are taken from equals the remainders are equal.*

4. *When equals are added to unequals the wholes are unequal.*

5. *When equals are taken from unequals the remainders are unequal.*

6. *Things which are double of the same thing, or equal things, are equal to each other.*

7. *Things which are halves of the same thing, or of equal things, are equal to each other.*

8. *The whole is greater than any of its parts.*

9. *Every whole is equal to all its parts taken together.*

10. *Things which coincide, or fill the same space, are identical, or mutually equal in all their parts.*

11. *All right angles are equal to one another.*

12. *A straight line is the shortest distance between two points.*

13. *Two straight lines cannot inclose a space.*

ABBREVIATIONS.

The common algebraic signs are used in this work, and demonstrations are sometimes made through the medium of equations; and it is so necessary that the student in geometry should understand some of the more simple operations of algebra, that we assume that he is acquainted with the use of the signs. As the terms circle, angle, triangle, hypothesis, axiom, theorem, corollary, and definition, are constantly occurring in a course of geometry, we shall abbreviate them as shown in the following list:

Addition is expressed by	$+$
Subtraction " "	$-$
Multiplication " "	$\times$
Equality and Equivalency are expressed by .	$=$
Greater than, is expressed by	$>$
Less than, " " . . .	$<$
Thus: *B* is greater than *A*, is written .	$B>A$
B is less than *A*, " " .	$B<A$
A circle is expressed by	$\odot$
An angle " "	∟
A right angle is expressed by	R.∟
Degrees, minutes, and seconds, are expressed by	$^{\circ}\ '\ ''$
A triangle is expressed by	$\triangle$
The term Hypothesis is expressed by . .	(Hy.)
" Axiom " " . .	(Ax.)
" Theorem " ". . .	(Th.)
" Corollary " " . .	(Cor.)
" Definition " " . .	(def.)
" Perpendicular is expressed by .	$\perp$
The difference of two quantities, when it is not known which is the greater, is expressed by the symbol	$\sim$

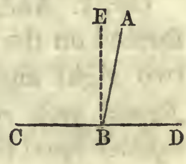
Thus ; the difference between *A* and *B* is written $A \sim B$.

BOOK I.

OF STRAIGHT LINES, ANGLES, AND POLYGONS.

THEOREM I.

When one straight line meets another, not at its extremity, the two angles thus formed are two right angles, or they are together equal to two right angles.



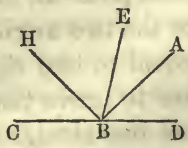
Let AB meet CD, and if AB is perpendicular to CD, it does not incline to either extremity of CD. In that case, the angle ABD is equal to the angle ABC, and is a right angle, by Definition 15.

But if these angles are unequal, we are to show that their sum is equal to two right angles. Conceive the dotted line BE to be drawn from the point B, so as not to incline to either side of CD; then, by Def. 15, the angles CBE and EBD are right angles; but the angles CBA and ABD make the same sum, or fill the same angular space, as the two angles CBE and EBD, and are, consequently, equal to two right angles. Hence the theorem; *when one straight line meets another, not at its extremity, the sum of the two angles is equal to two right angles.*

Cor. Hence, the two angles ABC and ABD are supplementary to each other, (Def. 21).

THEOREM II.

From any point in a straight line, not at its extremity, the sum of all the angles that can be formed on the same side of the line is equal to two right angles.

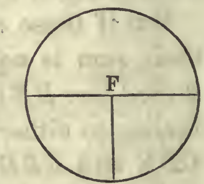


Let CD be any line, and B any point in it.

We are to show that the sum of all the angles which can be formed at B, on one side of CD, will be equal to two right angles.

By Th. 1, any two supplementary angles, as *ABD*, *ABC*, are together equal to two right angles. And since the angular space about the point *B* is neither increased nor diminished by the number of lines drawn from that point, the sum of all the angles *DBA*, *ABE*, *EBH*, *HBC*, fills the same spaces as any two angles *HBD*, *HBC*. Hence the theorem; *from any point in a line, the sum of all the angles that can be formed on the same side of the line is equal to two right angles.*

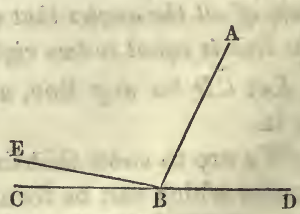
Cor. 1. And, as the sum of all the angles that can be formed on the other side of the line, *CD*, is also equal to two right angles; therefore, *all the angles that can be formed quite round a point,* B, *by any number of lines, are together equal to four right angles.*



Cor. 2. Hence, also, the whole circumference of a circle, being the sum of the measures of all the angles that can be made about the center *F*, (Def. 53), is the measure of four right angles; consequently, a semicircle, or 180°, is the measure of two right angles; and a quadrant, or 90°, is the measure of one right angle.

THEOREM III.

If one straight line meets two other straight lines at a common point, forming two angles, which together are equal to two right angles, the two straight lines are one and the same line.



Let the line *AB* meet the lines *BD* and *BE* at the common point *B*, making the sum of the two angles *ABD*, *ABE*, equal to two right angles; we are to prove that *DB* and *BE* are one straight line.

If DB and BE are not in the same line, produce DB to C, thus forming one line, DBC.

Now by Th. 1, $ABD + ABC$ must be equal to two right angles. But by hypothesis, $ABD + ABE$ is equal to two right angles.

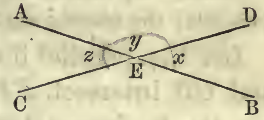
Therefore, $ABD + ABC$ is equal to $ABD + ABE$, (Ax. 1). From each of these equals take away the common angle ABD, and the angle ABC will be equal to ABE, (Ax. 5). That is, the line BE must coincide with BC, and they will be in fact one and the same line, and they cannot be separated as is represented in the figure.

Hence the theorem; *if one line meets two other lines at a common point, forming two angles which together are equal to two right angles, the two lines are one and the same line.*

THEOREM IV.

If two straight lines intersect each other, the opposite or vertical angles must be equal.

If AB and CD intersect each other at E, we are to demonstrate that the angle AEC is equal to the vertical angle DEB; and the angle AED, to the vertical angle CEB.



As AB is one line met by DE, another line, the two angles AED and DEB, on the same side of AB, are equal to two right angles, (Th. 1). Also, because CD is a right line, and AE meets it, the two angles AEC and AED are together equal to two right angles.

Therefore, $AED + DEB = AEC + AED$. (Ax. 1.)

If from these equals we take away the common angle AED, the remaining angle DEB must be equal to the remaining angle AEC, (Ax. 3). In like manner, we can prove that AED is equal to CEB. Hence the theorem; *if the two lines intersect each other, the vertical angles must be equal.*

Second Demonstration.

By Def. 11, the angle *DEB* is the difference in the direction of the lines *ED* and *EB*; and the angle *AEC* is the difference in the direction of the lines *EC* and *EA*.

But *ED* is opposite in direction to *EC*; and *EB* is opposite in direction to *EA*.

Hence, the difference in the direction of *ED* and *EB* is the same as that of *EC* and *EA*, as is obvious by inspection.

Therefore, the angle *DEB* is equal to its opposite *AEC*.

In like manner, we may prove *AED* = *CEB*.

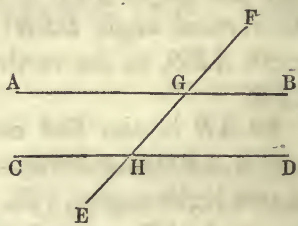
Hence the theorem; *if two lines intersect each other, the vertical angles must be equal.*

THEOREM V.

If a straight line intersects two parallel lines, the sum of the two interior angles on the same side of the intersecting line is equal to two right angles.

[Note.—By *interior* angles, we mean angles which lie between the parallels; the *exterior* angles are those not between the parallels.]

Let the parallel lines *AB* and *CD* intersect *EF*; then we are to demonstrate that the angles *BGH* + *GHD* = 2 R. ∟



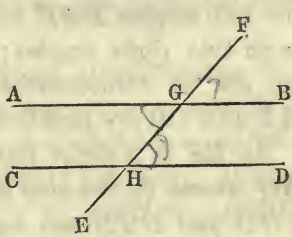
Because *GB* and *HD* are parallel, they are equally inclined to the line *EF*, or have the same difference of direction from that line. Therefore, ∟ *FGB* = ∟ *GHD*. To each of these equals add the ∟ *BGH*, and we have *FGB* + *BGH* = *GHD* + *BGH*.

But by Th. 1, the first member of this equation is equal to two right angles; and the second member is the sum of the two angles between the parallels. Hence the theorem; *if a line intersects two parallel lines, the sum of the two interior angles on the same side of the intersecting line must be equal to two right angles.*

Scholium.—As AB and CD are parallel lines, and EF is a line intersecting them, AB and EF must make equal angles to those made by CD and EF. That is, the angles about the point G must be equal to the corresponding angles about the point H.

THEOREM VI.

If a line intersects two parallel lines, the alternate interior angles are equal.



Let AB and CD be parallels, intersected by EF at H and G. Then we are to prove that the angle AGH is equal to the alternate angle GHD, and $CHG = HGB$.

By Th. 5, $\angle BGH + \angle GHD =$ two right angles. Also, by Th. 1, $\angle AGH + \angle BGH =$ two right angles. From these equals take away the common angle BGH, and $\angle GHD$ will be left, equal to $\angle AGH$, (Ax. 3). In like manner, we can prove that the angle CHG is equal to the angle HGB. Hence the theorem; *if a line intersects two parallel lines, the alternate interior angles are equal.*

Cor. 1. Since $\angle AGH = \angle FGB$,
and $\angle AGH = \angle GHD$;
Therefore, $\angle FGB = \angle GHD$ (Ax. 1).
Also, $\angle AGF + \angle AGH = 2$ R. $\angle$, (Th. 1),
and $\angle CHG + \angle AGH = 2$ R. $\angle$, (Th. 5);
Therefore,
$\angle AGF + \angle AGH = \angle CHG + \angle AGH$, (Ax. 1);
and $\angle AGF = \angle CHG$, (Ax. 3).

That is, *the exterior angle is equal to the interior opposite angle on the same side of the intersecting line.*

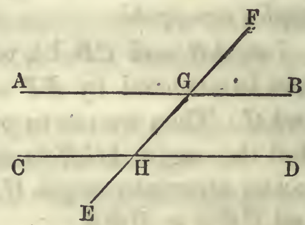
Cor. 2. Since $\angle AGH = \angle FGB$,
and $\angle AGH = \angle CHE$;
Therefore, $\angle FGB = \angle CHE$.
In the same manner it may be shown that

$$\angle AGF = \angle EHD.$$

Hence, *the alternate exterior angles are equal.*

THEOREM VII.

If a line intersects two other lines, making the sum of the two interior angles on the same side of the intersecting line equal to two right angles, the two straight lines are parallel.



Let the line EF intersect the lines AB and CD, making the two angles $BGH + GHD$ = to two right angles; then we are to demonstrate that AB and CD are parallel.

As EF is a right line and BG meets it, the two angles FGB and BGH are together equal to two right angles, (Th. 1). But by hypothesis, the angles, BGH and GHD, are together equal to two right angles. From these two equals take away the common angle BGH, and the remaining angles FGB and GHD must be equal, (Ax. 3). Now, because GB and HD make equal angles with the same line EF, they must extend in the same direction; and lines having the same direction are parallel, (Def. 13). Hence the theorem; *if a line intersects two other lines, making the sum of the two interior angles on the same side of the intersecting line equal to two right angles, the two lines must be parallel.*

Cor. 1. If a line intersects two other lines, making the alternate interior angles equal, the two lines intersected must be parallel.

Suppose the $\angle AGH = \angle GHD$. Adding $\angle HGB$ to each, we have

$$\angle AGH + \angle HGB = \angle GHD + \angle HGB.$$

but the first member of this equation, that is, $\angle AGH + \angle HGB$, is equal to two right angles; hence the second member is also equal to the same; and by the theorem, the lines AB and CD are parallel.

Cor. 2. If a line intersects two other lines, making the opposite exterior and interior angles equal, the two lines intersected must be parallel.

Suppose the $\angle FGB = \angle GHD$. Adding the $\angle HGB$ to each, we have

$$\angle FGB + \angle HGB = \angle GHD + HGB.$$

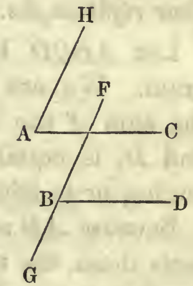
But the first member of this equation is equal to two right angles; hence the second member is also equal to two right angles; and by the theorem, the lines AB and CD are parallel.

Cor. 3. If a line intersects two other lines, making the alternate exterior angles equal, the lines must be parallel.

Suppose $\angle BGF = \angle CHE$, and $\angle AGF = \angle DHE$.
By Th. 4, $\angle BGF = \angle AGH$, and $\angle CHE = \angle DHG$.
And since $\angle BGF = \angle CHE$, $\angle AGH = \angle DHG$.
That is, the alternate interior angles are equal; and hence (by Cor. 1) the two lines are parallel.

THEOREM VIII.

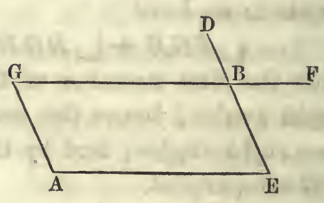
If two angles have their sides parallel, the two angles will be either equal or supplementary.



Let AC be parallel to BD, and AH parallel to BF or to BG. Then we are to prove that the angle DBF is equal to the angle CAH, and that the angle DBG is supplementary to the angle A. The angle CAH is formed by the difference in the direction of AC and AH; and the angle DBF is formed by the difference in the direction of BD and BF. But AC and AH have the same directions as BD and BF, because they are respectively parallel. Therefore, by Def. 11, $\angle CAH = \angle DBF$. But the line BG has the same direction as BF, and the angle DBG is supplementary to DBF. Hence the theorem; *angles whose sides are parallel, form either equal or supplementary angles.*

THEOREM IX.

The opposite angles of any parallelogram are equal.



Let $AEBG$ be a parallelogram. Then we are to prove that the angle GBE is equal to its opposite angle A.

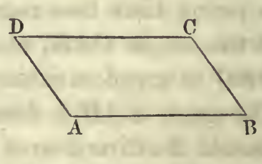
Produce EB to D, and GB to F; then, since BD is parallel to AG, and BF to AE, the angle DBF is equal to the angle A, (Th. 8).

But the angles GBE and DBF, being vertical, are equal, (Th. 4). Therefore, the opposite angles GBE and A, of the parallelogram $AEBG$, are equal.

In like manner, we can prove the angle E equal to the angle G. Hence the theorem; *the opposite angles of any parallelogram are equal.*

THEOREM X.

The sum of the angles of any parallelogram is equal to four right angles.



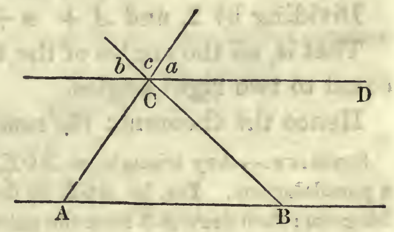
Let $ABCD$ be a parallelogram. We are to prove that the sum of the angles A, B, C and D, is equal to four right angles, or to 360°.

Because AD and BC are parallel lines, and AB intersects them, the two interior angles A and B are together equal to two right angles, (Th. 5). And because CD intersects the same parallels, the two interior angles C and D are also together equal to two right angles. By addition, we have the sum of the four interior angles of the parallelogram $ABCD$, equal to four right angles. Hence the theorem; *the sum of the angles of any parallelogram is equal to four right angles.*

THEOREM XI.

The sum of the three angles of any triangle is equal to two right angles.

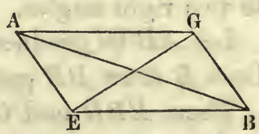
Let ABC be a triangle, and through its vertex C draw a line parallel to the base AB, and produce the sides AC and BC. Then the angles A and a, being exterior and interior opposite angles on the same side of two parallels, are equal, (Th. 6, Cor. 1).



For like reasons, $\angle B = \angle b$. And the angles C and c, being vertical angles, are also equal, (Th. 4). Therefore, the angles A, B, C are equal to the angles a, b, c respectively. But the angles around the point C, on the upper side of the parallel CD, are equal to two right angles, (by Th. 1). Hence the theorem; *the sum of the three angles, etc.*

Second Demonstration.

Let $AEBG$ be a parallelogram. Draw the diagonal GE; then the parallelogram is divided into two triangles, and the opposite angles G and E are mutually divided by the diagonal GE.



Because GB and AE are parallel, the *alternate interior* angles BGE and GEA are equal, (Th. 6). Designate each of these by b.

In like manner, because EB and AG are parallel, the alternate interior angles, BEG and EGA, are equal. Designate each of these by a.

Now we are to prove that the three angles B, b, and a, and also that the three angles A, a, and b, are equal to two right angles.

Because A and B are opposite angles of a parallelogram, they are equal, (Th. 9), and $\angle A + \angle B = 2\angle A$.

And all the interior angles of the parallelogram are equal to four right angles, (Th. 10).

Therefore, $2A + 2a + 2b = 4$ right angles.

Dividing by 2, and $A + a + b = 2$ "

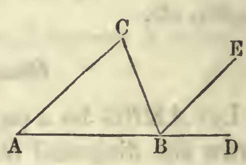
That is, all the angles of the triangle AGE are together equal to two right angles.

Hence the theorem; *the sum of the three angles, etc.*

SCHOLIUM.—Any triangle, as AGE, may be conceived to be part of a parallelogram. For, let AGE be drawn independently of the parallelogram; then draw EB from the point E parallel to AG, and through the point G draw GB parallel to AE, and a parallelogram will be formed embracing the triangle; and thus the sum of the three angles of any triangle is proved equal to two right angles.

This truth is so fundamental, important, and practical, as to require special attention; we therefore give a

Third Demonstration.



Let ABC be a triangle. Then we are to show that the angles A, C, and ABC, are together equal to two right angles.

Let AB be produced to D, and from B draw BE parallel to AC.

Then, EBD and CAB being exterior and interior opposite angles on the same side of the line AD, are equal, (Th. 6, Cor. 1). Also, CBE and ACB, being alternate angles, are equal, (Th. 6).

By addition, observing that $\angle CBE$, added to $\angle EBD$, must make $\angle CBD$, we have

$$\angle CBD = \angle A + \angle C. \qquad (1.)$$

To each of these equals add the angle CBA, and we shall have

$$\angle CBA + \angle CBD = \angle A + \angle C + \angle CBA.$$

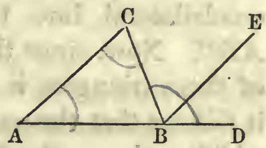
But (by Th. 1), the sum of the first two is equal to two right angles; therefore, the three angles, A, C, and CBA, are together equal to two right angles.

Hence the theorem; *the sum of the three angles, etc.*

THEOREM XII.

If any side of a triangle is produced, the exterior angle is equal to the sum of the two interior opposite angles.

Let ABC be a triangle. Produce AB to D; and we are to prove that the angle CBD is equal to the sum of the two angles A and C.



We establish this theorem by a course of reasoning in all respects the same as that by which we obtained Eq. (1.), third demonstration, (Th. 11).

Cor. 1. Since the exterior angle of any triangle is equal to the sum of the two interior opposite angles, therefore it is greater than either one of them.

Cor. 2. If two angles in one triangle be equal to two angles in another triangle, the third angles will also be equal, each to each, (Ax. 3); that is, the two triangles will be mutually equiangular.

Cor. 3. If one angle in a triangle be equal to one angle in another, the sum of the remaining angles in the one will also be equal to the sum of the remaining angles in the other, (Ax. 3).

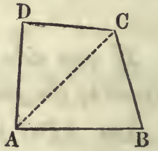
Cor. 4. If one angle of a triangle be a right angle, the sum of the other two will be equal to a right angle, and each of them singly will be acute, or less than a right angle.

Cor. 5. The two smaller angles of every triangle are acute, or each is less than a right angle.

Cor. 6. All the angles of a triangle may be acute, but no triangle can have more than one right or one obtuse angle.

THEOREM XIII.

In any quadrilateral, the sum of the four interior angles is equal to four right angles.



Let $ABCD$ be a quadrilateral; then we are to prove that the sum of the four interior angles, that is $A + B + C + D$, is equal to four right angles.

Draw the diagonal AC, dividing the quadrilateral into two triangles, ABC, ADC. Now, since the sum of the three angles of each of these triangles is equal to two right angles, (Th. 11), it follows that the sum of all the angles of both triangles which make up the four angles of the quadrilateral, must be equal to four right angles, (Ax. 2).

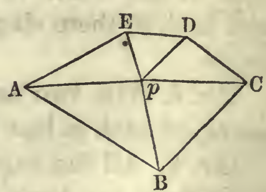
Hence the theorem; *in any quadrilateral, etc.*

Cor. 1. Hence, if three of the angles of a quadrilateral are right angles, the fourth will also be a right angle.

Cor. 2. If the sum of two of the four angles be equal to two right angles, the sum of the remaining two will also be equal to two right angles. And, if the sum of either two of the angles be less than two right angles, the sum of the other two angles will be greater than two right angles.

THEOREM XIV.

In any polygon, the sum of all the interior angles is equal to twice as many right angles, less four, as the figure has sides.



Let $ABCDE$ be any polygon; we are to prove that the sum of all its interior angles, $A + B + C + D + E$, is equal to twice as many right angles, less four, as the figure has sides.

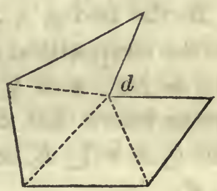
From any point, p, within the figure, draw lines pA, pB, pC, etc., to all the angles, thus dividing the polygon into as many triangles as it has sides. Now, the sum of the three angles of each of these triangles is equal to two right angles, (Th. 11); and the sum of the angles of all the triangles must be equal to twice as many right angles as the figure has sides. But the sum of these angles contains the sum of four right angles about the point *p;* taking these away, and the remainder is the sum of the interior angles of the figure. Therefore, the sum must be equal to twice as many right angles, less four, as the figure has sides.

Hence the theorem; *in any polygon, etc.*

From this Theorem is derived the rule for finding the sum of the interior angles of any right-lined figure:

Subtract 2 *from the number of sides, and multiply the remainder by* 2*; the product will be the number of right angles.*

Thus, if the number of sides be represented by S, the number of right angles will be represented by (2S — 4).

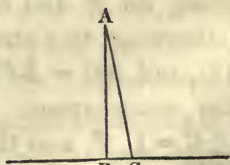


The Theorem is not varied in case of a re-entrant angle, as represented at *d*, in the figure *ABCDEF.*

Draw lines from the angle *d* to the several opposite angles, making as many triangles as the figure has sides, *less two*, and the sum of the three angles of each triangle equals two right angles.

THEOREM XV.

From any point without a straight line, but one perpendicular can be drawn to that line.



From the point *A* let us suppose it possible that two perpendiculars, *AB* and *AC*, can be drawn. Now, because *AB* is a supposed perpendicular, the angle *ABC* is a right angle; and because *AC* is a supposed perpendicular, the angle ACB is also a right angle; and if two angles of the triangle ABC are together equal to two right angles, the third angle, BAC, must be infinitely small, or zero; but this is impossible, for it requires the sum of the three angles of a triangle to make two right angles, (Th. 11). Therefore, the lines AB and AC must be identical, or but one perpendicular.

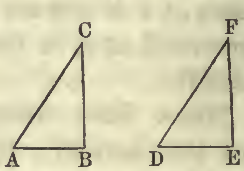
Hence the theorem; *from any point without a straight line, etc.*

Cor. At a given point in a straight line but one perpendicular can be erected to that line; for, if there could be two perpendiculars, we should have unequal right angles, which is impossible.

THEOREM XVI.

Two triangles which have two sides and the included angle in the one, equal to two sides and the included angle in the other, each to each, are equal in all respects.

In the two $\triangle$'s, ABC and DEF, on the supposition that $AB = DE$, $AC = DF$, and $\angle A = \angle D$, we are to prove that BC must $= EF$, the $\angle B = \angle E$, and the $\angle C = \angle F$.



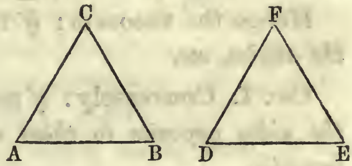
Conceive the $\triangle ABC$ cut out of the paper, taken up, and placed on the $\triangle DEF$ in such a manner that the point A shall fall on the point D, and the line AB on the line DE; then the point B will fall on the point E, because the lines are equal. Now, as the $\angle A = \angle D$, the line AC must take the same direction as DF, and fall on DF; and as $AC = DF$, the point C will fall on F. B being on E and C on F, BC must be exactly on EF, (otherwise, two straight lines would enclose a space, Ax. 13), and $BC = EF$, and the two magnitudes exactly fill the same space. Therefore, $BC = EF$, $\angle B = \angle E$, $\angle C = \angle F$, and the two $\triangle$'s are equal, (Ax. 9).

Hence the theorem; *two triangles which have two sides, etc.*

THEOREM XVII.

When two triangles have a side and two adjacent angles in the one, equal to a side and two adjacent angles in the other, each to each, the two triangles are equal in all respects.

In two $\triangle$'s, as ABC and DEF, on the supposition that $BC = EF$, $\angle B = \angle E$, and $\angle C = \angle F$, we are to prove that $AB = DE$, $AC = DF$, and $\angle A = \angle D$.



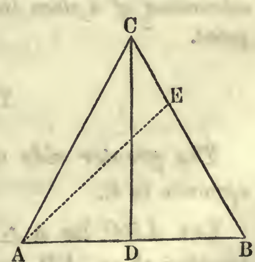
Conceive the $\triangle$ ABC taken up and placed on the $\triangle$ DEF, so that the side BC shall exactly coincide with its equal side EF; now, because the angle B is equal to the angle E, the line BA will take the direction of ED, and will fall exactly upon it; and because the angle C is equal to the angle F, the line CA will take the direction of FD, and fall exactly upon it; and the two lines BA and CA, exactly coinciding with the two lines ED and FD, the point A will fall on D, and the two magnitudes will exactly fill the same space; therefore, by Ax. 10, they are equal, and $AB = DE$, $AC = DF$, and the $\angle A = \angle D$.

Hence the theorem; *when two triangles have a side and two adjacent angles in the one, equal to, etc.*

THEOREM XVIII.

If two sides of a triangle are equal, the angles opposite to these sides are also equal.

Let ABC be a triangle; and on the supposition that $AC = BC$, we are to prove that the $\angle A$ = the $\angle B$.



Conceive the angle C divided into two equal angles by the line CD; then we have two $\triangle$'s, ADC and BDC, which have the two sides, AC and CD of the one, equal to the two sides, CB and CD of the other; and the included angle ACD, of the one, equal to the included angle BCD of the other: therefore, (Th. 16), $AD = BD$, and the angle A, opposite to CD of the one triangle, is equal to the angle B, opposite to CD of the other triangle; that is, $\angle A = \angle B$.

Hence the theorem; *if two sides of a triangle are equal, the angles, etc.*

Cor. 1. Conversely: *if two angles of a triangle are equal, the sides opposite to them are equal, and the triangle is isosceles.*

For, if AC is not equal to BC, suppose BC to be the greater, and make $BE = AE$; then will $\triangle AEB$ be isosceles, and $\angle EAB = \angle EBA$; hence $\angle EAB = \angle CAB$, or a part is equal to the whole, which is absurd; therefore, CB cannot be greater than AC, that is, neither of the sides AC, BC, can be greater than the other, and consequently they are equal.

Cor. 2. As the two triangles, ACD and BCD, are in all respects equal, the line which bisects the angle included between the equal sides of an isosceles $\triangle$ also bisects the base, and is perpendicular to the base.

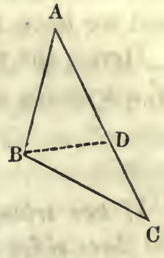
Scholium 1.—If in the perpendicular DC, any other point than C be taken, and lines be drawn to the extremities A and B, such lines will be equal, as is evident from Th. 16; hence, we may announce this truth: *Any point in a perpendicular drawn from the middle of a line, is at equal distances from the two extremities of the line.*

Scholium 2.—Since two points determine the position of a line, it follows, *that the line which connects two points equally distant from the extremities of a given line, is perpendicular to this line at its middle point.*

THEOREM XIX.

The greater side of every triangle has the greater angle opposite to it.

Let ABC be a $\triangle$; and on the supposition that AC is greater than AB, we are to prove that the angle ABC is greater than the $\angle C$. From AC, the greater of the two sides, take AD, equal to the less side AB, and draw BD, thus making two triangles of the original triangle. As $AB = AD$, the $\angle ADB =$ the $\angle ABD$, (Th. 18).



But the $\angle ADB$ is the exterior angle of the $\triangle BDC$, and is therefore greater than C, (Th. 12); that is, the $\angle ABD$ is greater than the angle C. Much more, then, is the angle ABC greater than the angle C.

Hence the theorem; *the greater side of every triangle, etc.*

Cor. Conversely: *the greater angle of any triangle has the greater side opposite to it.*

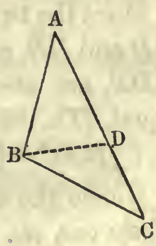
In the triangle ABC, let the angle B be greater than the angle A; then is the side AC greater than the side BC.

For, if $BC = AC$, the angle A must be equal to the angle B, (Th. 18), which is contrary to the hypothesis; and if $BC > AC$, the angle A must be greater than the angle B, by what is above proved, which is also contrary to the hypothesis; hence BC can be neither equal to, nor greater, than AC; it is therefore less than AC.

THEOREM XX.

The difference between any two sides of a triangle is less than the third side.

Let ABC be a $\triangle$, in which AC is greater than AB; then we are to prove that $AC - AB$ is less than BC.



On AC, the greater of the two sides, lay off AD equal to AB.

Now, as a straight line is the shortest distance between two points, we have

$$AB + BC > AC. \qquad (1)$$

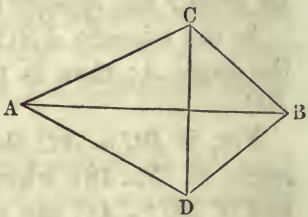
From these unequals subtract the equals $AB = AD$, and we have $BC > AC - AB$. (Ax. 5).

Hence the theorem; *the difference between any two sides of a triangle, etc.*

THEOREM XXI.

If two triangles have the three sides of the one equal to the three sides of the other, each to each, the two triangles are equal, and the equal angles are opposite the equal sides.

In two triangles, as ABC and ABD, on the supposition that the side AB of the one = the side AB of the other, $AC = AD$, and $BC = BD$, we are to demonstrate that $\llcorner ACB = \llcorner ADB$, $\llcorner BAC = \llcorner BAD$, and $\llcorner ABC = \llcorner ABD$.



Conceive the two triangles to be joined together by their longest equal sides, and draw the line CD.

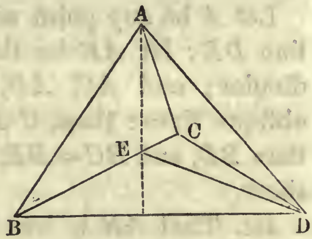
Then, in the triangle ACD, because AC is equal to AD, the angle ACD is equal to the angle ADC, (Th. 18). In like manner, in the triangle BCD, because BC is equal to BD, the angle BCD is equal to the angle BDC. Now, the angle ACD being equal to the angle ADC, and the angle BCD to the angle BDC, $\llcorner ACD + \llcorner BCD = \llcorner ADC + \llcorner BDC$, (Ax. 2); that is, the whole angle ACB is equal to the whole angle ADB.

Since the two sides AC and CB are equal to the two sides AD and DB, each to each, and their included angles ACB, ADB, are also equal, the two triangles ABC, ABD, are equal, (Th. 16), and have their other angles equal; that is, $\llcorner BAC = \llcorner BAD$, and $\llcorner ABC = \llcorner ABD$.

Hence the theorem; *if two triangles have the three sides of the one, etc.*

THEOREM XXII.

If two triangles have two sides of the one equal to two sides of the other, each to each, and the included angles unequal, the third sides will be unequal, and the greater third side will belong to the triangle which has the greater included angle.



In the two △'s, ABC and ACD, let AB and AC of the one △ be equal to AD and AC of the other △, and the angle BAC greater than the angle DAC; we are to prove that the side BC is greater than the side CD.

Conceive the two △'s joined together by their shorter equal sides, and draw the line BD. Now, as $AB = AD$, ABD is an isosceles △. From the vertex A, draw a line bisecting the angle BAD. This line must be perpendicular to the base BD, (Th. 18, Cor. 1). Since the ∠ BAC is greater than the ∠ DAC, this line must meet BC, and will not meet CD. From the point E, where the perpendicular meets BC, draw ED.

Now $BE = DE$, (Th. 18, Scholium 1).

Add EC to each; then $BC = DE + EC$.

But $DE + EC$ is greater than DC.

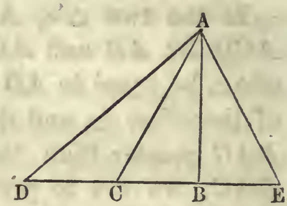
Therefore $BC > DC$.

Hence the theorem; *if two triangles have two sides of one equal to two sides of the other, etc.*

Cor. Any point out of the perpendicular drawn from the middle point of a line, is unequally distant from the extremities of the line.

THEOREM XXIII.

A perpendicular is the shortest line that can be drawn from any point to a straight line; and if other lines be drawn from the same point to the same straight line, the longer line will be at a greater distance from the perpendicular; and lines at equal distances from the perpendicular, on opposite sides, are equal.



Let A be any point without the line DE; let AB be the perpendicular; and AC, AD, and AE oblique lines: then, if BC is less than BD, and $BC=BE$, we are to show,

1st. That AB is less than AC.
2d. That AC is less than AD. 3d. That $AC=AE$.

1st. In the triangle ABC, as AB is perpendicular to BC, the angle ABC is a right angle; $\angle C + \angle BAC =$ another right angle, (Th. 11); and the angle BCA is less than a right angle; and, as the greater side is always opposite the greater angle, AB is less than AC; and AC may be any line not identical with AB; therefore a perpendicular is the shortest line that can be drawn from A to the line DE.

2d. As the two angles, ACB and ACD, are together equal to two right angles, (Th. 1), and ACB is less than a right angle, ACD must be greater than a right angle; consequently, the $\angle D$ is less than a right angle; and, in the $\triangle ACD$, AD is greater than AC, or AC is less than AD, (Th. 19).

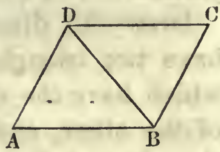
3d. In the $\triangle$'s ABC and ABE, AB is common, $CB=BE$, and the angles at B are right angles; therefore, $AC=AE$, (Th. 16).

Hence the theorem; *a perpendicular is the shortest line, etc.*

Cor. Conversely: if two equal oblique lines be drawn from the same point to a given straight line, they will meet the line at equal distances from the foot of the perpendicular drawn from that point to the given line.

THEOREM XXIV.

The opposite sides, and also the opposite angles of any parallellogram, are equal.



Let $ABCD$ be a parallelogram. Then we are to show that $AB = DC$, $AD = BC$, $\angle A = \angle C$, and $\angle ADC = \angle ABC$.

Draw a diagonal, as BD; now, because AB and DC are parallel, the alternate angles ABD and BDC are equal, (Th. 6). For the same reason, as AD and BC are parallel, the angles ADB and DBC are equal. Now, in the two triangles ABD and BCD, the side BD is common,

$$\text{the } \angle ADB = \angle DBC \qquad (1)$$
$$\text{and } \angle BDC = \angle ABD \qquad (2)$$

Therefore, the angle $A =$ the angle C, (Th. 11), and the two △'s are equal in all respects, (Th. 18); that is, the sides opposite the equal angles are equal; or, $AB = DC$, and $AD = BC$. By adding equations (1) and (2), we have the angle $ADC =$ the angle ABC, (Ax. 2).

Hence the theorem; *the opposite sides, and the opposite angles, etc.*

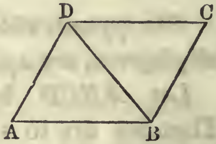
Cor. 1. As the sum of all the angles of the quadrilateral is equal to four right angles, and the angle A is always equal to the opposite angle C; therefore, if A is a right angle, C is also a right angle, and the figure is a rectangle.

Cor. 2. As the angle ABC, added to the angle A, gives the same sum as the angles of the △ ADB; therefore, the two adjacent angles of a parallelogram are together equal to two right angles. This corresponds to Th. 13, Cor. 2.

THEOREM XXV.

If the opposite sides of a quadrilateral are equal, they are also parallel, and the figure is a parallelogram.

Let $ABDC$ be any quadrilateral; on the supposition that $AD = BC$, and $AB = DC$, we are to prove that AD is parallel to BC, and AB parallel to DC.



Draw the diagonal BD; we now have two triangles, ABD and BCD, which have the side BD common, AD of the one $= BC$ of the other, and AB of the one $= CD$ of the other; therefore the two △'s are equal, (Th. 21), and the angles opposite the equal sides are equal; that is, the angle $ADB =$ the angle CBD; but these are alternate angles; and, therefore, AD is parallel to BC, (Th. 7); and because the angle $ABD =$ the angle BDC, AB is parallel to CD, and the figure is a parallelogram.

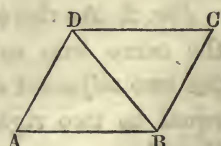
Hence the theorem; *if the opposite sides of a quadrilateral, etc.*

Cor. This theorem, and also Th. 24, proves that the two △'s which make up the parallelogram are equal; and the same would be true if we drew the diagonal from A to C; *therefore, the diagonal of any parallelogram bisects the parallelogram.*

THEOREM XXVI.

The lines which join the corresponding extremities of two equal and parallel strait lines, are themselves equal and parallel; and the figure thus formed is a parallelogram.

On the supposition that AB is equal and parallel to DC, we are to prove that AD is equal and parallel to BC; and that the figure is a parallelogram.



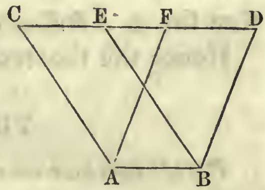
Draw the diagonal BD; now, since

AB and *DC* are parallel, and *BD* joins them, the alternate angles *ABD* and *BDC* are equal; and since the side *AB* = the side *DC*, and the side *BD* is common to the two △'s *ABD* and *CDB*, therefore the two triangles are equal, (Th. 16); that is, *AD* = *BC*, the angle *A* = *C*, and the ∟ *ADB* = the ∟ *DBC*; also *AD* is parallel to *BC*; and the figure is a parallelogram.

Hence the theorem; *the lines which join the corresponding extremities, etc.*

THEOREM XXVII.

Parallelograms on the same base, and between the same parallels, are equivalent, or equal in respect to area or surface.



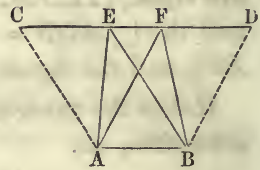
Let *ABEC* and *ABDF* be two parallelograms on the same base *AB*, and between the same parallels *AB* and *CD*; we are to prove that these two parallelograms are equal.

Now, *CE* and *FD* are equal, because they are each equal to *AB*, (Th. 24); and, if from the whole line *CD* we take, in succession, *CE* and *FD*, there will remain *ED* = *CF*, (Ax. 3); but *BE* = *AC*, and *AF* = *BD*, (Th. 24); hence we have two △'s, *CAF* and *EBD*, which have the three sides of the one equal to the three sides of the other, each to each; therefore, the two △'s are equal, (Th. 21). If, from the whole figure *ABDC*, we take away the △ *CAF*, the parallelogram *ABDF* will remain; and if from the whole figure we take away the other △ *EBD*, the parallelogram *ABEC* will remain. Therefore, (Ax. 3), the parallelogram *ABDF* = the parallelogram *ABEC*.

Hence the theorem; *Parallelograms on the same base, etc.*

THEOREM XXVIII.

Triangles on the same base and between the same parallels are equivalent.



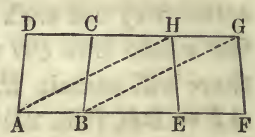
Let the two △'s ABE and ABF have the same base AB, and be between the same parallels AB and CD; then we are to prove that they are equal in surface.

From B draw the line BD, parallel to AF; and from A draw the line AC, parallel to BE; and produce EF, if necessary, to C and D; now the parallelogram $ABDF =$ the parallelogram $ABEC$, (Th. 27). But the △ ABE is one half the parallelogram $ABEC$, and the △ ABF is one half the parallelogram $ABDF$; and halves of equals are equal, (Ax. 7); therefore the △ $ABE =$ the △ ABF.

Hence the theorem; *triangles on the same base, etc.*

THEOREM XXIX.

Parallelograms on equal bases, and between the same parallels, are equal in area.



Let $ABCD$ and $EFGH$, be two parallelograms on equal bases, AB and EF, and between the same parallels, AF and DG; then we are to prove that they are equal in area.

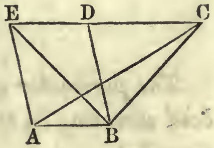
$AB = EF = HG$; but lines which join equal and parallel lines, are themselves equal and parallel, (Th. 26); therefore, if AH and BG be drawn, the figure $ABGH$ is a parallelogram = to the parallelogram $ABCD$, (Th. 27); and if we turn the whole figure over, the two parallelograms, $GHEF$ and $GHAB$, will stand on the same base, GH, and between the same parallels; therefore, $GHEF = GHAB$, and consequently $ABCD = EFGH$, (Ax. 1).

Hence the theorem; *Parallelograms on equal bases, etc.*

Cor. Triangles on equal bases, and between the same parallels, are equal in area. For, draw *BD* and *EG*; the △ *ABD* is one half of the parallelogram *AC*, and the △ *EFG* is one half of the equivalent parallelogram *FH*; therefore, the △ *ABD* = the △ *EFG*, (Ax. 7).

THEOREM XXX.

If a triangle and a parallelogram are upon the same or equal bases, and between the same parallels, the triangle is equivalent to one half the parallelogram.



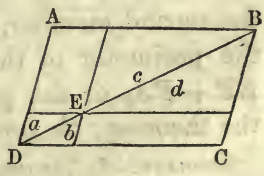
Let *ABC* be a △, and *ABDE* a parallelogram, on the same base *AB*, and between the same parallels; then we are to prove that the △ *ABC* is equivalent to one half of the parallelogram *ABDE*.

Draw the diagonal *EB* to the parallelogram; now, because the two △'s *ABC* and *ABE* are on the same base, and between the same parallels, they are equivalent, (Th. 28); but the △ *ABE* is one half the parallelogram *ABDE*, (Th. 25, Cor.); therefore the △ *ABC* is equivalent to one half of the same parallelogram, (Ax. 7).

Hence the theorem; *if a triangle and a parallelogram, etc.*

THEOREM XXXI.

The complementary parallelograms described about any point in the diagonal of any parallelogram, are equivalent to each other.



Let *AC* be a parallelogram, and *BD* its diagonal; take any point, as *E*, in the diagonal, and through this point draw lines parallel to the sides of the parallelogram, thus forming four parallelograms.

We are now to prove that the complementary parallelograms, *AE* and *EC*, are equivalent.

By (Th. 25, Cor.) we learn that the $\triangle ABD = \triangle DBC$. Also by the same Cor., $\triangle a = \triangle b$, and $\triangle c = \triangle d$; therefore by addition

$$\triangle a + \triangle c = \triangle b + \triangle d.$$

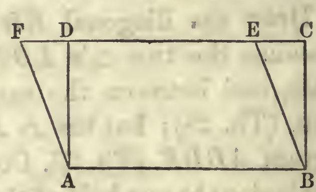
Now, from the whole $\triangle ABD$ take $\triangle a + \triangle c$, and from the whole $\triangle DBC$ take the equal sum, $\triangle b + \triangle d$, and the remaining parallelograms AE and EC are equivalent, (Ax. 3).

Hence the theorem; *the complementary parallelograms, etc.*

THEOREM XXXII.

The perimeter of a rectangle is less than that of any rhomboid standing on the same base, and included between the same parallels.

Let $ABCD$ be a rectangle, and $ABEF$ a rhomboid having the same base, and their opposite sides in the same line parallel to the base.



We are now to prove that the perimeter $ABCDA$ is less than $ABEFA$.

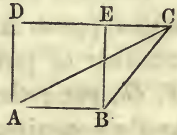
Because AD is a perpendicular from A to the line DE, and AF an oblique line, AD is less than AF, (Th. 23). For the same reason BC is less than BE; hence $AD + BC < AF + BE$. Adding the sum, $AB + DC$, to the first member of this inequality, and its equal $AB + FE$ to the second member, we have $AB + BC + CD + DA$, or the perimeter of the rectangle, less than $AB + BE + EF + FA$, or the perimeter of the rhomboid. Hence the theorem; *the perimeter of a rectangle*, etc.

SCHOLIUM.—In Theorem 30 it is shown that the triangles ABC, ABE, and DBE, are equal in area, and that each is equal to one half the parallelogram $ABDE$. This parallelogram also has the same area as the rectangle having an equal base and altitude.

Thus far, areas have been considered only relatively and in the abstract. We will now explain how we may pass to the absolute measures, or, more properly, to the numerical expressions for areas.

THEOREM XXXIII.

The area of any plane triangle is measured by the product of its base by one half its altitude; or one half its base by its altitude, or one half the product of its base by its altitude.

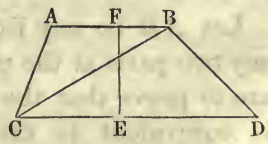


Let ABC represent any triangle, AB its base, and AD, at right angles to AB, its altitude; now we are to show that the area of ABC is equal to the product of AB by one half of AD; or one half of AB by AD; or one half of the product of AB by AD.

On AB construct the rectangle $ABED$; and the area of this rectangle is measured by AB into AD (Def. 54); but the area of the $\triangle$ ABC is equivalent to one half this rectangle, (Th. 30). Therefore, the area of the $\triangle$ is measured by $\frac{1}{2}\ AB \times AD$, or one half the product of its base by its altitude. Hence the theorem; *the area of any plane triangle, etc.*

THEOREM XXXIV.

The area of a trapezoid is measured by one half the sum of its parallel sides multiplied by the perpendicular distance between them.

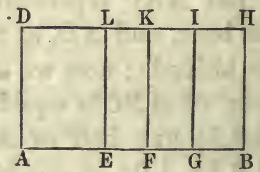


Let $ABDC$ represent any trapezoid; draw the diagonal BC, dividing it into two triangles, ABC and BCD: CD is the base of one triangle, and AB may be considered as the base of the other; and EF is the common altitude of the two triangles.

Now, by Th. 33, the area of the triangle $BCD = \frac{1}{2}\ CD \times EF$; and the area of the $\triangle$ $ABC = \frac{1}{2}AB \times EF$; but by addition, the area of the two △'s, or of the trapezoid, is equal to $\frac{1}{2}(AB+CD)\times EF$. Hence the theorem; *the area of a trapezoid, etc.*

THEOREM XXXV.

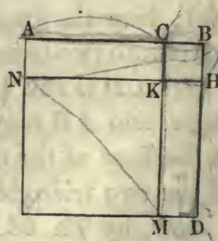
If one of two lines is divided into any number of parts, the rectangle contained by the two lines is equal to the sum of the several rectangles contained by the undivided line and the several parts of the divided line.



Let AB and AD be two lines, and suppose AB divided into any number of parts at the points E, F, G, etc.; then the whole rectangle contained by the two lines is AH, which is measured by AB into AD. But the rectangle AL is measured by AE into AD; the rectangle EK is measured by EF into EL, which is equal to EF into AD; and so of all the other partial rectangles; and the truth of the proposition is as obvious as that a whole is equal to the sum of all its parts. Hence the theorem; *if one of two lines is divided, etc.*

THEOREM XXXVI.

If a straight line is divided into any two parts, the square described on the whole line is equivalent to the sum of the squares described on the two parts plus twice the rectangle contained by the parts.



Let AB be any line divided into any two parts at the point C; now we are to prove that the square on AB is equivalent to the sum of the squares on AC and CB plus twice the rectangle contained by AC and CB.

On AB describe the square AD. Through the point C draw CM, parallel to BD; take $BH = BC$, and through H draw HKN, parallel to AB. We now have CH, the square on CB, by direct construction.

As $AB = BD$, and $CB = BH$, by subtraction, $AB - CB = BD - BH$; or $AC = HD$. But $NK = AC$, being opposite sides of a parallelogram; and for the same reason, $KM = HD$. Therefore, (Ax. 1), $NK = KM$, and the figure NM is a square on NK, equal to a square on AC. But the whole square on AB is composed of the two squares CH, NM, and the two complements or rectangles AK and KD; and each of these latter is AC in length, and BC in width; and each has for its measure AC into CB; therefore the whole square on AB is equivalent to $AC^2 + BC^2 + 2AC \times CB$.

Hence the theorem; *if a straight line is divided into any two parts, etc.*

This theorem may be proved algebraically, thus:

Let w represent any whole right line divided into any two parts a and b; then we shall have the equation

$$w = a + b$$

By squaring, $w^2 = a^2 + b^2 + 2ab$.

Cor. If $a = b$, then $w^2 = 4a^2$; that is, the square described on any line is four times the square described on one half of it.

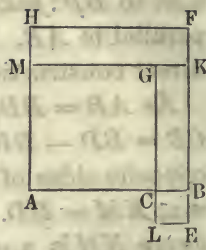
THEOREM XXXVII.

The square described on the difference of two lines is equivalent to the sum of the squares described on the two lines diminished by twice the rectangle contained by the lines.

Let AB represent the greater of two lines, CB the less line, and AC their difference.

We are now to prove that the square described on AC is equivalent to the sum of the squares on AB and BC diminished by twice the rectangle contained by AB and BC.

Conceive the square AF to be described on AB, and



the square BL on CB; on AC describe the square $ACGM$, and produce MG to K.

As $GC = AC$, and $CL = CB$, by addition, $(GC + CL)$, or GL, is equal to $AC + CB$, or AB. Therefore, the rectangle GE is AB in length, and CB in width, and is measured by $AB \times BC$.

Also $AH = AB$, and $AM = AC$; by subtraction, $MH = CB$; and as $MK = AB$, the rectangle HK is AB in length, and CB in width, and is measured by $AB \times BC$; and the two rectangles GE and HK are together equivalent to $2AB \times BC$.

Now, the squares on AB and BC make the whole figure $AHFELC$; and from this whole figure, or these two squares, take away the two rectangles HK and GE, and the square on AC only will remain; that is,

$$AC^2 = AB^2 + BC^2 - 2AB \times BC.$$

Hence the theorem; *the square described on the difference of two lines, etc.*

This theorem may be proved algebraically, thus:

Let a represent the greater of two lines, b the less, and d their difference; then we must have this equation:

$$d = a - b$$

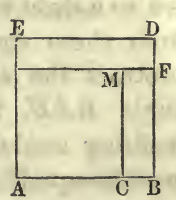
By squaring, $d^2 = a^2 + b^2 - 2ab$.

Cor. If $d = b$, then $d = \frac{a}{2}$, and $d^2 = \frac{a^2}{4}$; that is, the square described on one half of any line is equivalent to one fourth of the square described on the whole line.

THEOREM XXXVIII.

The difference of the squares described on any two lines is equivalent to the rectangle contained by the sum and difference of the lines.

Let AB be the greater of two lines, and AC the less, and on these lines describe the squares AD, AM; then, the difference of the squares on AB and AC is the two rectangles EF and FC. We are now to show that the measure of these rectangles may be expressed by $(AB + AC) \times (AB - AC)$.



The length of the rectangle EF is ED, or its equal AB; and the length of the rectangle FC is MC, or its equal AC; therefore, the length of the two together (if we conceive them put between the same parallel lines) will be $AB + AC$; and the common width is CB, which is equal to $AB - AC$; therefore, $\overline{AB}^2 - \overline{AC}^2 = (AB + AC) \times (AB - AC)$.

Hence the theorem; *the difference of the squares described on any two lines, etc.*

This theorem may be proved algebraically: thus,

Let a represent one line, and b another;

Then $a + b$ is their sum, and $a - b$ their difference;

and $$(a + b) \times (a - b) = a^2 - b^2.$$

THEOREM XXXIX.

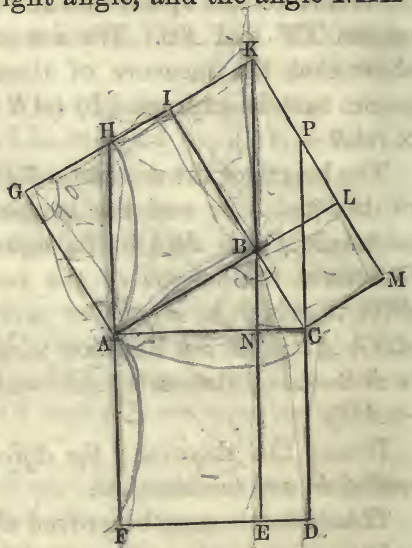
The square described on the hypotenuse of any right-angled triangle is equivalent to the sum of the squares described on the other two sides.

Let ABC represent any right-angled triangle, the right angle at B; we are to prove that the square on AC is equivalent to the sum of two squares; one on AB, the other on BC.

On the three sides of the triangle describe the three squares, AD, AI, and BM. Through the point B, draw BNE perpendicular to AC, and produce it to meet the line GI in K; also produce AF to meet GI in H, and ML to meet the point in K.

REMARK. — That the lines, GI and ML, produced, meet at the point K, may be readily shown. As the proof of this fact is not necessary for the demonstration, it is left as an exercise for the learner.

The angle BAG is a right angle, and the angle NAH is also a right angle; if from these equals we subtract the common angle BAH, the remaining angle, BAC, must be equal to the remaining angle GAH. The angle G is a right angle, equal to the angle ABC; and $AB = AG$; therefore, the two △'s ABC and AGH are equal, and $AH = AC$. But $AC = AF$; therefore, $AH = AF$. Now, the two parallelograms, AE and $AHKB$ are equivalent, because they are upon equal bases, and between the same parallels, FH and EK, (Th. 27).



But the square AI, and the parallelogram $AHKB$, are equivalent, because they are on the same base, AB, and between the same parallels, AB and GK; therefore, the square AI, and the parallelogram AE, being each equivalent to the same parallelogram $AHKB$, are equivalent to each other, (Ax. 1). In the same manner we may prove that the square BD is equivalent to the rectangle ND; therefore, by addition, the two squares, AI and BM, are equivalent to the two parallelograms, AE and ND, or to the square AD.

Hence the theorem; *the square described on the hypotenuse of a right-angled triangle, etc.*

Cor. If two right-angled triangles have the hypotenuse, and a side of the one equal to the hypotenuse and a side of the other, each to each, the two triangles are equal.

Let ABC and AGH be the two $\triangle$'s, in which we suppose $AC = AH$, and $BC = GH$; then will $AG = AB$.

For, we have $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$,

or, by transposing, $\overline{AC}^2 - \overline{BC}^2 = \overline{AB}^2$,

and $\overline{AH}^2 = \overline{AG}^2 + \overline{GH}^2$,

or, by transposing, $\overline{AH}^2 - \overline{GH}^2 = \overline{AG}^2$.

But by the hypothesis $\overline{AC}^2 - \overline{BC}^2 = \overline{AH}^2 - \overline{GH}^2$;

hence, $\overline{AB}^2 = \overline{AG}^2$, or, $AB = AG$.

SCHOLIUM.—The two sides, AB and BC, may vary, while AC remains constant. AB may be equal to BC; then the point N will be in the middle of AC. When AB is very near the length of AC, and BC very small, then the point N falls very near to C. Now as AE and AD are right-angled parallelograms, their areas are measured by the product of their bases by their altitudes; and it is evident that, as they have the same altitude, these areas will vary directly as their bases AN and NC; hence the squares on AB and BC, which are equivalent to those rectangles, vary as the lines AN and NC.

The following outline of the demonstration of this proposition is presented as a useful disciplinary exercise for the student.

We employ the same figure, in which no change is made except to draw through C the line CP, parallel to BK.

The first step is to prove the equality of the triangles AGH and ABC, whence $AH = AC$. But $AC = AF$; therefore $AH = AF$.

The parallelograms $AFEN$ and $AHKB$ are equivalent. Also, the parallelogram $AHKB =$ the square $ABIG$, (Th. 27), and the parallelogram $KBCP = NEDC =$ square $BCML$. Now, by adding the equals

$$AFEN = ABIG$$
$$NEDC = BCML$$

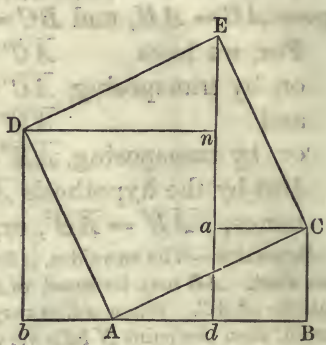
we obtain $AFDC = ABIG + BCML$.

That is, the square on AC is equivalent to the sum of the squares on AB and BC.

The great practical importance of this theorem, in the extent and variety of its applications, and the frequency of its use in establishing subsequent propositions, render it necessary that the student should master it completely. To secure this end, we present a

Second Demonstration.

Let ABC be a triangle right-angled at B. On the hypotenuse AC, describe the square $ACED$. From D and E let fall the perpendiculars Db and Ed, on AB and AB produced. Draw Dn and Ca, making right angles with Ed.



We give an outline only of the demonstration, requiring the pupil to make it complete.

First Part.—Prove the four triangles ABC, AbD, DnE, and EaC, equal to each other.

The proof is as follows: The △'s ABC and DnE are equal, because the angles of the one are equal to the angles of the other, each to each, and the hypotenuse AC of the one, is equal to the hypotenuse DE of the other. In like manner, it may be shown that the △'s AbD and EaC are equal.

Now, the sum of the three angles about A, is equal to the sum of the three angles of the △ ABC; and if, from the first sum, we take $\llcorner DAC + \llcorner CAB$, and from the second we take $\llcorner B + \llcorner CAB = \llcorner DAC + \llcorner CAB$, the remaining angles are equal; that is, $\llcorner DbA$ is equal to $\llcorner ACB$; hence the △'s ABC and DbA have their angles equal, each to each; and since $AC = DA$, the △'s are themselves equal, and the four triangles ABC, AbD, DnE, and EaC, are equal to each other.

Second.—Prove that the square $bDnd$ is equal to a square on AB. The square $BdaC$ is obviously on BC.

Third.—The area of the whole figure is equal to the square on AC, and the area of two of the four equal right-angled triangles.

Also, the area of the whole figure is equal to two other squares, *bDnd* and *daCB*, and two of the four equal triangles, *DnE* and *EaC*.

Omitting or subtracting the areas of two of the four right-angled triangles, in each of the two expressions for the area of the whole figure, there will remain the square on *AC*, equal to the sum of the two squares, *Dndb* and *daCB*.

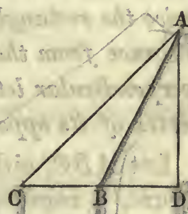
That is, $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$.

Hence the theorem; *the square described on the hypotenuse of a right-angled triangle, etc.*

SCHOLIUM.—Hence, to find the hypotenuse of a right-angled triangle, *extract the square root of the sum of the squares of the two sides about the right angle.*

THEOREM XL.

In any obtuse-angled triangle, the square on the side opposite the obtuse angle is greater than the sum of the squares on the other two sides, by twice the rectangle contained by either side about the obtuse angle, and the part of this side produced to meet the perpendicular drawn to it from the vertex of the opposite angle.



Let *ABC* be any triangle in which the angle at *B* is obtuse. Produce either side about the obtuse angle, as *CB*, and from *A* draw *AD* perpendicular to *CB*, meeting it produced at *D*.

It is obvious that $CD = CB + BD$.

By squaring, $\overline{CD}^2 = \overline{CB}^2 + 2CB \times BD + \overline{BD}^2$, (Th. 36).

Adding $\overline{AD}^2$ to each member of this equation, we have

$$\overline{AD}^2 + \overline{CD}^2 = \overline{CB}^2 + \overline{BD}^2 + \overline{AD}^2 + 2CB \times BD.$$

But, (Th. 39), the first member of the last equation is equal to $\overline{AC}^2$, and

$$\overline{BD}^2 + \overline{AD}^2 = \overline{AB}^2.$$

Therefore, this equation becomes

$$\overline{AC}^2 = \overline{CB}^2 + \overline{AB}^2 + 2CB \times BD.$$

That is, the square on AC is equivalent to the sum of the squares on CB and AB, increased by twice the rectangle contained by CB and BD.

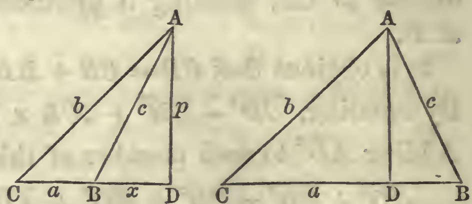
Hence the theorem; *in any obtuse-angled triangle, the square on the side opposite the obtuse angle, etc.*

Scholium.—Conceive AB to turn about the point A, its intersection with CD gradually approaching D. The last equation above will be true, however near this intersection is to D, and when it falls upon D the triangle becomes right-angled.

In this case the line BD reduces to zero, and the equation becomes $\overline{AC}^2 = \overline{CB}^2 + \overline{AB}^2$, in which CB and AB are now the base and perpendicular of a right-angled triangle. This agrees with Theorem 39, as it should, since we used the property of the right-angled triangle established in Theorem 39 to demonstrate this proposition; and in the equation which expresses a property of the obtuse-angled triangle, we have introduced a supposition which changes it into one which is right-angled.

THEOREM XLI.

In any triangle, the square on a side opposite an acute angle is less than the sum of the squares on the other two sides, by twice the rectangle contained by either of these sides, and the distance from the vertex of the acute angle to the foot of the perpendicular let fall on this side, or side produced, from the vertex of its opposite angle.



Let ABC, either figure, represent any triangle; C an acute angle, CB the base, and AD the perpendicular, which falls either without or on the base. Now we are to prove that

$$AB^2 = \overline{CB}^2 + \overline{AC}^2 - 2CB \times CD.$$

From the first figure we get $BD = CD - CB$ (1)
and from the second $BD = CB - CD$ (2)

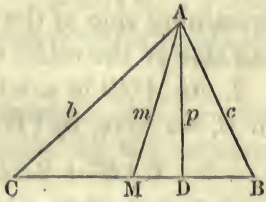
Either one of these equations will give, (Th. 37),

$$\overline{BD}^2 = \overline{CD}^2 + \overline{CB}^2 - 2CD \times CB.$$

Adding $\overline{AD}^2$ to each member and reducing, we obtain, (Th. 39), $\overline{AB}^2 = \overline{AC}^2 + \overline{CB}^2 - 2CB \times CD$, which proves the proposition. Hence the theorem.

THEOREM XLII.

If in any triangle a line be drawn from any angle to the middle of the opposite side, twice the square of this line, together with twice the square of one half the side bisected, will be equivalent to the sum of the squares of the other two sides.



Let ABC be a triangle, and M the middle point of its base.

Then we are to prove that $2\overline{AM}^2 + 2\overline{CM}^2 = \overline{AC}^2 + \overline{AB}^2$.

Draw AD perpendicular to the base, and make $AD = p$, $AC = b$, $AB = c$, $CB = 2a$, $AM = m$, and $MD = x$; then $CM = a$, $CD = a + x$, $DB = a - x$.

Now by, (Th. 39), we have the two following equations:

$$p^2 + (a - x)^2 = c^2 \qquad (1)$$
$$p^2 + (a + x)^2 = b^2 \qquad (2)$$

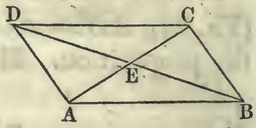
By addition, $2p^2 + 2x^2 + 2a^2 = b^2 + c^2$. But $p^2 + x^2 = m^2$.

Therefore, $2m^2 + 2a^2 = b^2 + c^2$.

This equation is the algebraic enunciation of the theorem.

THEOREM XLIII.

The two diagonals of any parallelogram bisect each other; and the sum of their squares is equivalent to the sum of the squares of the four sides of the parallelogram.



Let $ABCD$ be any parallelogram, and AC and BD its diagonals.

We are now to prove,

1st. That $AE = EC$, and $DE = EB$.

2d. That $\overline{AC}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{AD}^2$.

1. The two triangles ABE and CDE are equal, because $AB = CD$, the angle $ABE =$ the alternate angle CDE, and the vertical angles at E are equal; therefore, AE, the side opposite the angle ABE, is equal to CE, the side opposite the equal angle CDE; also EB, the remaining side of the one $\triangle$, is equal to ED, the remaining side of the other triangle.

2. As ACD is a triangle whose base, AC, is bisected in E, we have, by (Th. 42),

$$2\overline{AE}^2 + 2\overline{ED}^2 = \overline{AD}^2 + \overline{DC}^2 \quad (1)$$

And as ACB is a triangle whose base, AC, is bisected in E, we have

$$2\overline{AE}^2 + 2\overline{EB}^2 = \overline{AB}^2 + \overline{BC}^2 \quad (2)$$

By adding equations (1) and (2), and observing that

$\overline{EB}^2 = \overline{ED}^2$, we have

$$4\overline{AE}^2 + 4\overline{ED}^2 = \overline{AD}^2 + \overline{DC}^2 + \overline{AB}^2 + \overline{BC}^2$$

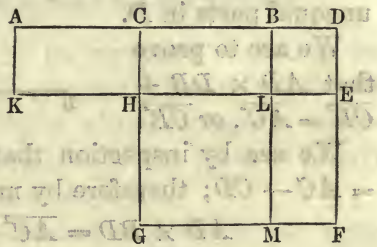
But, four times the square of the half of a line is equivalent to the square of the whole line, (Th. 36, Corollary); therefore $4\overline{AE}^2 = \overline{AC}^2$, and $4\overline{ED}^2 = \overline{DB}^2$; and by substituting these values, we have

$$\overline{AC}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{DC}^2 + \overline{AD}^2,$$

which equation conforms to the enunciation of the theorem.

THEOREM XLIV.

If a line be bisected and produced, the rectangle contained by the whole line and the part produced, together with the square of one half the bisected line, will be equivalent to the square on a line made up of the line produced and one half the bisected line.



Let AB be any line, bisected in C and produced to D. On CD describe the square CF, and on BD describe the square BE.

The sides of the square BE being produced, the square GL will be formed. Also, complete the construction of the rectangle $ADEK$.

Then we are to prove that the rectangle, AE, and the square, GL, are together equivalent to the square, $CDFG$.

The two complementary rectangles, CL and LF, are equal, (Th. 31). But $CL = AH$, the line AB being bisected at C; therefore AL is equal to the sum of the two complementary rectangles of the square CF. To AL add the square BE, and the whole rectangle, AE, will be equal to the two rectangles CE and EM. To each of these equals add HM, or the square on HL or its equal CB, and we have rectangle $AE +$ square $HM = \overline{CD}^2$; but rectangle $AE = AD \times BD$, and square $HM = \overline{CB}^2$. Hence the theorem, etc.

SCHOLIUM.—If we represent AB by $2a$, and BD by x, then $AD = 2a + x$, and $AD \times BD = 2ax + x^2$. But $\overline{CB}^2 = a^2$; adding this equation to the preceding, member to member, we get $AD \times BD + \overline{CB}^2 = a^2 + 2ax + x^2 = \overline{a + x}^2$. But $CD = a + x$; hence this equation is equivalent to the equation $AD \times DB + \overline{CB}^2 = \overline{CD}^2$, which is the algebraic proof of the theorem.

THEOREM XLV.

If a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the two unequal parts together with the square of the line between the points of division, will be equivalent to the square on one half the line.

Let AB be a line bisected in C, and divided into two unequal parts in D.

We are to prove that $AD \times DB + \overline{CD}^2 = \overline{AC}^2$, or $\overline{CB}^2$.

We see by inspection that $AD = AC + CD$, and $BD = AC - CD$; therefore by multiplication we have

$$AD \times BD = \overline{AC}^2 - \overline{CD}^2, \text{ (Th. 38)}.$$

By adding $\overline{CD}^2$ to each of these equals, we obtain

$$AD \times BD + \overline{CD}^2 = \overline{AC}^2$$

Hence the theorem.

BOOK II.

PROPORTION.

DEFINITIONS AND EXPLANATIONS.

The word Proportion, in its common meaning, denotes that *general relation* or symmetry existing between the different parts of an object which renders it agreeable to our taste, and conformable to our ideas of beauty or utility; but in a mathematical sense,

1. Proportion is the numerical relation which one quantity bears to another of the same kind.

As the magnitudes compared must be of the same kind, proportion in geometry can be only that of *a line to a line, a surface to a surface, an angle to an angle*, or *a volume to a volume.*

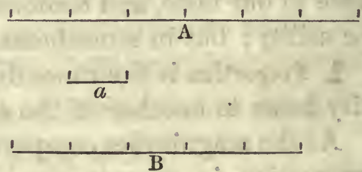
2. Ratio is a term by which the number which measures the proportion between two magnitudes is designated, and is the quotient obtained by dividing the one by the other. Thus, the ratio of A to B is $\frac{B}{A}$, or $A : B$, in which A is called the *antecedent*, and B the *consequent.* If, therefore, the magnitude A be assumed as the unit or standard, this quotient is the numerical value of B expressed in terms of this unit.

It is to be remarked that this principle lies at the foundation of the method of representing quantities by numbers. For example, when we say that a body weighs twenty-five pounds, it is implied that the weight of this body has been compared, directly or indirectly, with that of the standard, one pound. And so of geometrical magnitudes; when a line, a surface, or a volume is said to be fifteen linear, superficial, or cubical feet, it is understood that it has been referred to its particular unit, and found to contain it fifteen times; that is, fifteen is the ratio of the unit to the magnitude.

When two magnitudes are referred to the same unit, the ratio of the numbers expressing them will be the ratio of the magnitudes themselves.

Thus, if A and B have a common unit, a, which is contained in A, m times, and in B, n times, then $A = ma$ and $B = na$, and $\frac{B}{A} = \frac{na}{ma} = \frac{n}{m}$.

To illustrate, let the line A contain the line a six times, and let the line B contain the same line a five times: then $A=6a$ and $B=5a$, which give $\frac{B}{A} = \frac{5a}{6a} = \frac{5}{6}$.



3. A **Proportion** is a formal statement of the equality of two ratios.

Thus, if we have the four magnitudes A, B, C and D, such that $\frac{B}{A} = \frac{D}{C}$, this relation is expressed by the proportion $A : B :: C : D$, or $A : B = C : D$, the first of which is read, A is to B as C is to D; and the second, the ratio of A to B is equal to that of C to D.

4. The **Terms** of a proportion are the magnitudes, or more properly the representatives of the magnitudes compared.

5. The **Extremes** of a proportion are its first and fourth terms.

6. The **Means** of a proportion are its second and third terms.

7. A **Couplet** consists of the two terms of a ratio. The first and second terms of a proportion are called the *first couplet*, and the third and fourth terms are called the *second couplet.*

8. The **Antecedents** of a proportion are its first and third terms.

9. The **Consequents** of a proportion are its second and fourth terms.

In expressing the equality of ratios in the form of a proportion, we may make the denominators the antecedents, and the numerators the consequents, or the reverse, without affecting the relation between the magnitudes. It is, however, a matter of some little importance to the beginner to adopt a uniform rule for writing the terms of the ratios in the proportion; and we shall always, unless otherwise stated, make the denominators of the ratios the antecedents, and the numerators the consequents.*

10. **Equimultiples** of magnitudes are the products arising from multiplying the magnitudes by the same number. Thus, the products, Am and Bm, are equimultiples of A and B.

11. A **Mean Proportional** between two magnitudes is a magnitude which will form with the two a proportion, when it is made a consequent to the first ratio, and an antecedent to the second. Thus, if we have three magnitudes A, B, and C, such that $A : B :: B : C$, B is a mean proportional between A and C.

12. Two magnitudes are *reciprocally*, or *inversely* proportional when, in undergoing changes in value, one is multiplied and the other is divided by the same number. Thus, if A and B be two magnitudes, so related that when A becomes mA, B becomes $\frac{B}{m}$, A and B are said to be inversely proportional.

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* For discussion of the two methods of expressing Ratio, see University Algebra.

13. A *Proportion is taken inversely* when the antecedents are made the consequents and the consequents the antecedents. Thus

14. A *Proportion is taken alternately, or by alternation,* when the antecedents are made one couplet and the consequents the other.

15. Mutually Equiangular Polygons have the same number of angles, those of the one equal to those of the others, each to each, and the angles like placed.

16. Similar Polygons are such as are mutually equiangular, and have the sides about the equal angles, taken in the same order, proportional.

17. Homologous Angles in similar polygons are those which are equal and like placed; and

18. The **Homologous Sides** are those which are like disposed about the homologous angles.

THEOREM I.

If the first and second of four magnitudes are equal, and also the third and fourth, the four magnitudes may form a proportion.

Let A, B, C, and D represent four magnitudes, such that $A = B$ and $C = D$; we are to prove that $A : B :: C : D$.

Now, by hypothesis, A is equal to B, and their ratio is therefore 1; and since, by hypothesis, C is equal to D, their ratio is also 1.

Hence, the ratio of A to B is equal to that of C to D; and, (by Def. 3),

$$A : B :: C : D.$$

Therefore, four magnitudes which are equal, two and two, constitute a proportion.

THEOREM II.

If four magnitudes constitute a proportion, the product of the extremes is equal to the product of the means.

Let the four magnitudes A, B, C, and D form the proportion $A : B :: C : D$; we are to prove that $A \times D = B \times C$.

The ratio of A to B is expressed by $\frac{B}{A} = r$.

The ratio of C to D is expressed by $\frac{D}{C} = r$.

Hence, (Ax. 1), $\frac{B}{A} = \frac{D}{C}$.

Multiplying these equals each by $A \times C$, and we have

$$B \times C = A \times D.$$

Hence the theorem; *if four magnitudes are in proportion, etc.*

Cor. 1. Conversely: *If we have the product of two magnitudes equal to the product of two other magnitudes, they will constitute a proportion of which either of the two may be made the extremes and the other two the means.*

Let the magnitudes $B \times C = A \times D$. Dividing both members of the equation by $A \times C$, and we have $\frac{B}{A} = \frac{D}{C}$.

Hence the proportion $A : B :: C : D$.

Cor. 2. If we divide both members of the equation

$$A \times D = B \times C \quad \text{by } A,$$

we have $$D = \frac{B \times C}{A}.$$

That is, to find the fourth term of a proportion, *multiply the second and third terms together and divide the product by the first term.* This is the Rule of Three of Arithmetic.

This equation shows that any one of the four terms can be found by a like process, *provided* the other three are given.

THEOREM III.

If three magnitudes are continued proportionals, the product of the extremes is equal to the square of the mean.

Let A, B, and C represent the three magnitudes:

Then $A : B :: B : C$, (by Def. 11).

But, (by Th. 2), the product of the extremes is equal to the product of the means; that is, $A \times C = B^2$.

Hence the theorem; *if three magnitudes, etc.*

THEOREM IV.

Equimultiples of any two magnitudes have the same ratio as the magnitudes themselves; and the magnitudes and their equimultiples may therefore form a proportion.

Let A and B represent two magnitudes, and mA and mB their equimultiples.

Then we are to prove that $A : B :: mA : mB$.

The ratio of A to B is $\frac{B}{A}$, and of mA to mB is $\frac{mB}{mA} = \frac{B}{A}$, the same ratio.

Hence the theorem; *equimultiples of any two magnitudes, etc.*

THEOREM V.

If four magnitudes are proportional, they will be proportional when taken inversely.

If $A : B :: mA : mB$, then $B : A :: mB : mA$;

For in either case, the product of the extremes and means are manifestly equal; or the ratio of the couplets is the same.

Hence the theorem; *if four quantities are proportional, etc.*

THEOREM VI.

Magnitudes which are proportional to the same proportionals, are proportional to each other.

If $A : B = P : Q$ and $a : b = P : Q$ } Then we are to prove that $A : B = a : b$.

From the 1st proportion, $\frac{B}{A} = \frac{Q}{P}$;

From the 2d " $\frac{b}{a} = \frac{Q}{P}$;

Therefore, by (Ax. 1), $\frac{B}{A} = \frac{b}{a}$, or $A : B = a : b$.

Hence the theorem; *magnitudes which are proportional to the same proportionals, etc.*

Cor. 1. This principle may be extended through any number of proportionals.

Cor. 2. *If the ratio of an antecedent and consequent of one proportion is equal to the ratio of an antecedent and consequent of another proportion, the remaining terms of the two proportions are proportional.*

For, if $A : B :: C : D$

and $M : N :: P : Q$

in which $\frac{B}{A} = \frac{N}{M}$, then $\frac{D}{C} = \frac{Q}{P}$;

hence $C : D :: P : Q$.

THEOREM VII.

If any number of magnitudes are proportional, any one of the antecedents will be to its consequent as the sum of all the antecedents is to the sum of all the consequents.

Let A, B, C, D, E, etc., represent the several magnitudes which give the proportions

$$A : B :: C : D$$
$$A : B :: E : F$$
$$A : B :: G : H, \text{ etc., etc.}$$

To which we may annex the identical proportion,

$$A : B :: A : B.$$

Now, (by Th. 2), these proportions give the following equations,

$$A \times D = B \times C$$
$$A \times F = B \times E$$
$$A \times H = B \times G$$
$$A \times B = B \times A, \text{ etc. etc.}$$

From which, by addition, there results the equation,

$$A(B + D + F + H, \text{ etc.}) = B(A + C + E + G, \text{ etc.})$$

But the sums $B + D + F$, etc., and $A + C + E$, etc., may be separately regarded as single magnitudes; therefore, (Th. 2),

$$A : B :: A + C + E + G, \text{ etc. } : B + D + F + H, \text{ etc.}$$

Hence the theorem; *if any number of magnitudes are proportional, etc.*

THEOREM VIII.

If four magnitudes constitute a proportion, the first will be to the sum of the first and second as the third is to the sum of the third and fourth.

By hypothesis, $A : B :: C : D$; then we are to prove that $A : A + B :: C : C + D$.

By the given proportion, $\frac{B}{A} = \frac{D}{C}$.

Adding unity to both members, and reducing them to the form of a fraction, we have $\frac{B + A}{A} = \frac{D + C}{C}$. Changing this equation into its equivalent proportional form, we have

$$A : A + B :: C : C + D.$$

Hence the theorem; *if four magnitudes constitute a proportion, etc.*

Cor. If we subtract each member of the equation $\frac{B}{A} =$

$\frac{D}{C}$ from unity, and reduce as before, we shall have

$$A : A - B :: C : C - D.$$

Hence also; *if four magnitudes constitute a proportion, the first is to the difference between the first and second, as the third is to the difference between the third and fourth.*

THEOREM IX.

If four magnitudes are proportional, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.

Let A, B, C, and D be the four magnitudes which give the proportion

$$A : B :: C : D;$$

we are then to prove that they will also give the proportion

$$A + B : A - B :: C + D : C - D.$$

By Th. 8 we have $A : A + B = C : C + D$.
Also by Scholium, same Th., $A : A - B = C : C - D$.

Now, if we change the order of the means in these proportions, which may be done, since the products of extremes and means remain the same, we shall have

$$A : C = A + B : C + D.$$
$$A : C = A - B : C - D.$$

Hence, (Th. 6), we have

$$A + B : C + D = A - B : C - D.$$

Or, $A + B : A - B = C + D : C - D$.

Hence the theorem; *if four magnitudes are proportional, etc.*

THEOREM X.

If four magnitudes are proportional, like powers or like roots of the same magnitudes are also proportional.

If the four magnitudes, A, B, C, and D, give the proportion

$$A : B :: C : D,$$

we are to prove that

$$A^n : B^n :: C^n : D^n.$$

The hypothesis gives the equation $\frac{B}{A} = \frac{D}{C}$. Raising both members of this equation to the nth power, we have $\frac{B^n}{A^n} = \frac{D^n}{C^n}$, which, expressed in its equivalent proportional form, gives

$$A^n : B^n :: C^n : D^n.$$

If n is a *whole number*, the terms of the given proportion are each raised to a power; but if n is a fraction having unity for its numerator, and a *whole number* for its denominator, like roots of each are taken.

As the terms of the proportion may be first raised to like powers, and then like roots of the resulting proportion be taken, n may be any number whatever.

Hence the theorem; *if four magnitudes, etc.*

THEOREM XI.

If four magnitudes are proportional, and also four others, the products which arise from multiplying the first four by the second four, term by term, are also proportional.

Admitting that $A : B :: C : D$,

and $X : Y :: M : N$,

We are to show that $AX : BY :: CM : DN$.

From the first proportion, $\frac{B}{A} = \frac{D}{C}$;

From the second, $\frac{Y}{X} = \frac{N}{M}$.

Multiply these equations, member by member, and

$$\frac{BY}{AX} = \frac{DN}{CM};$$

Or, $AX : BY :: CM : DN$.

The same would be true in any number of proportions.

Hence the theorem; *if four magnitudes are, etc.*

THEOREM XII.

If four magnitudes are proportional, and also four others, the quotients which arise from dividing the first four by the second four, term by term, are proportional.

By hypothesis, $A : B :: C : D$,
and $X : Y :: M : N$.
Multiply extremes and means, $AD = CB$, (1)
and $XN = MY$. (2)
Divide (1) by (2), and $\frac{A}{X} \times \frac{D}{N} = \frac{C}{M} \times \frac{B}{Y}$.

Convert these four factors, which make two equal products, into a proportion, and we have

$$\frac{A}{X} : \frac{B}{Y} :: \frac{C}{M} : \frac{D}{N}.$$

By comparing this with the given proportions, we find it is composed of the quotients of the several terms of the first proportion, divided by the corresponding terms of the second.

Hence the theorem; *if four magnitudes are proportional, etc.*

THEOREM XIII.

If four magnitudes are proportional, we may multiply the first couplet, the second couplet, the antecedents or the consequents, or divide them by the same quantity, and the results will be proportional in every case.

Let the four magnitudes A, B, C, and D give the proportion $A : B :: C : D$. By multiplying the extremes and means we have

$$A.D = B.C \quad (1)$$

Multiply both members of this equation by any number, as a, and we have

$$aA.D = aB.C$$

By converting this equation into a proportion in four different ways, as follows:

$$aA : aB :: C : D$$
$$A : B :: aC : aD$$
$$aA : B :: aC : D$$
$$A : aB :: C : aD$$

resuming the original equation, (1), and dividing both members by a, we have

$$\frac{A.D}{a} = \frac{B.C}{a}$$

This equation may also be converted into a proportion in four different ways, with the following results:

$$\frac{A}{a} : \frac{B}{a} :: C : D$$
$$A : B :: \frac{C}{a} : \frac{D}{a}$$
$$\frac{A}{a} : B :: \frac{C}{a} : D$$
$$A : \frac{B}{a} :: C : \frac{D}{a}.$$

Hence the theorem; *if four magnitudes are in proportion, etc.*

THEOREM XIV.

If three magnitudes are in proportion, the first is to the third as the square of the first is to the square of the second.

Let A, B, and C, be three proportionals.

Then we are to prove that $A : C = A^2 : B^2$

By (Th. 3) $AC = B^2$

Multiply this equation by the numeral value of A, and we have $A^2C = AB^2$

This equation gives the following proportion:

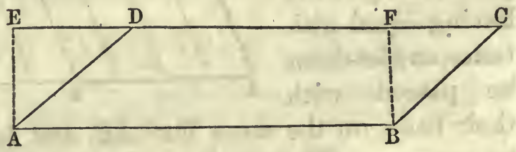
$$A : C = A^2 : B^2.$$

Hence the theorem.

REMARK.—It is now proposed to make an application of the preceding abstract principles of proportion, in geometrical investigations

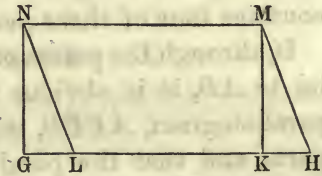
THEOREM XV.

If two parallelograms are equal in area, the base and perpendicular of either may be made the extremes of a proportion, of which the base and perpendicular of the other are the means.



Let $ABCD$, and $NLHM$, be two parallelograms having equal areas, by hypothesis; then we are to prove that

$$AB : LN :: MK : BF,$$



in which MK and BF are the altitudes or perpendiculars of the parallelograms.

This proportion is true, if the product of the extremes is equal to the product of the means; that is, if the equation

$AB.BF = LN.MK$ is true.

But $AB.BF$ is the measure of the rectangle $ABFE$, (B.I., Th. 32, Scholium), and this rectangle is equal in area to the parallelegram $ABCD$, (B. I., Th. 27).

In the same manner, we may prove that $LN.MK$ is the measure of the parallelogram $NLHM$. But these two parallelograms have equal areas by hypothesis.

Therefore, $AB.BF = LN.MK$ is a true equation, and (Th. 2, Cor. 1), gives the proportion

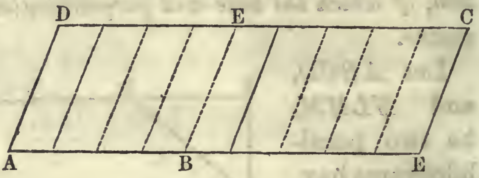
$$AB : LN :: MK : BF.$$

Hence the theorem; *if two parallelograms are equal in area, etc.*

THEOREM XVI.

Parallelograms having equal altitudes are to each other as their bases.

Since parallelograms having equal bases and equal altitudes are equal in area, however much their angles may differ, we can suppose the two parallelograms under consideration to be mutually equiangular, without in the least impairing the generality of this theorem. Therefore, let *ABCD* and *AEFD* be two parallelograms having equal altitudes, and let them be placed with their bases on the same line *AE*, and let the side, *AD*, be common. First suppose their bases commensurable, and that *AE* being divided into nine equal parts, *AB* contains four of those parts.



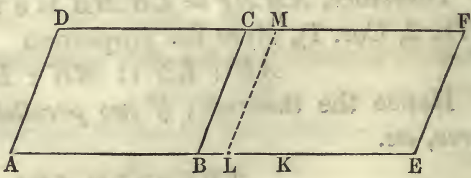
If, through the points of division, lines be drawn parallel to *AD*, it is obvious that the whole figure, or the parallelogram, *AEFD*, will be divided into nine equal parts, and that the parallelogram, *ABCD*, will be composed of four of those parts.

Therefore, *ABCD* : *AEFD* :: *AB* : *AE* :: 4 : 9.

Whatever be the whole numbers having to each other the ratio of the lines *AB* and *AE*, the reasoning would remain the same, and the proportion is established when the bases are commensurable. But if the bases are not to each other in the ratio of any two whole numbers, it remains still to be shown that

$$AEFD : ABCD :: AE : AB \quad (1)$$

If this proportion is not true, there must be a line greater or less than *AB*, to which *AE* will have the same ratio that *AEFD* has to *ABCD*.



Suppose the fourth proportional greater than *AB*, as *AK*, then,

$$AEFD : ABCD :: AE : AK \quad (2).$$

If we now divide the line AE into equal parts, each less than the line BK, one point of division, at least, will fall between B and K. Let L be such point, and draw LM parallel to BC.

This construction makes AE and AL commensurable; and by what has been already demonstrated, we have

$$AEFD : ALMD :: AE : AL. \quad (3)$$

Inverting the means in proportions (2) and (3), they become

$$AEFD : AE :: ABCD : AK;$$

and $$AEFD : AE :: ALMD : AL.$$

Hence, (Th. 6),

$$ABCD : AK :: ALMD : AL.$$

By inverting the means in this last proportion, we have

$$ABCD : ALMD :: AK : AL.$$

But AK is, by hypothesis, greater than AL; hence, if this proportion is true, $ABCD$ must be greater than $ALMD$; but on the contrary it is less. We therefore conclude that the supposition, that the fourth proportional, AK, is greater than AB, from which alone this absurd proportion results, is itself absurd.

In a similar manner it can be proved absurd to suppose the fourth proportional less than AB.

Therefore the fourth term of the proportion (1) can be neither less nor greater than AB; it is then AB itself, and parallelograms having equal altitudes are to each other as their bases, whether these bases are commensurable or not.

Hence the theorem; *Parallelograms having equal bases, etc.*

Cor. 1. Since a triangle is one half of a parallelogram having the same base as the triangle and an equal altitude, and as the halves of magnitudes have the same ratio as their wholes; therefore, *Triangles having the same or equal altitudes are to each other as their bases.*

Cor. 2. Any triangle has the same area as a right-angled triangle having the same base and an equal altitude; and as either side about the right angle of a right-angled triangle may be taken as the base, it follows that

Two triangles having the same or equal bases are to each other as their altitudes.

Cor. 3. Since either side of a parallelogram may be taken as its base, it follows from this theorem that

Parallelograms having equal bases are to each other as their altitudes.

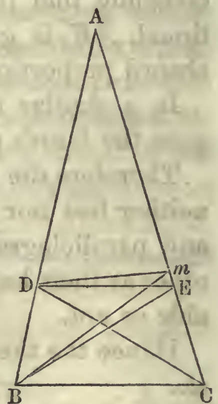
THEOREM XVII.

If lines are drawn cutting the sides, or the sides produced, of a triangle proportionally, such secant lines are parallel to the base of the triangle; and conversely, *lines drawn parallel to the base of a triangle cut the sides, or the sides produced, proportionally.*

Let ABC be any triangle, and draw the line DE dividing the sides AB and AC into parts which give the proportion

$$AD : DB :: AE : EC.$$

We are to prove that DE is parallel to BC.



If DE is not a parallel through the point D to the line BC, suppose Dm to be that parallel; and draw the lines DC and Bm.

Now, the two triangles ADm and mDC, have the same altitude, since they have a common vertex, D, and their bases in the same line, AC; hence, they are to each other as their bases, Am and mC, (Th. 16, Cor. 1).

That is, $\triangle ADm : \triangle mDC :: Am : mC$,
Also, $\triangle AmD : \triangle DmB :: AD : DB$.

But, since Dm is supposed parallel to BC, the triangles DBm and DCm have equal areas, because they are on the same base and between the same parallels, (Th. 28, B. I).

Therefore the terms of the first couplets in the two preceding proportions are equal each to each, and consequently the terms of the second couplets are also proportional,(Th. 6).

That is, $AD : DB :: Am : mC$
But $AD : DB :: AE : EC$ by hypothesis.

Hence we again have two proportions having the first couplets, the same in both, and we therefore have

$$AE : EC :: Am : mC$$

By alternation this becomes

$$AE : Am :: EC : mC$$

That is, AE is to Am, a greater magnitude is to a less, as EC is to mC, a less to a greater, which is absurd. Had we supposed the point m to fall between E and C, our conclusion would have been equally absurd; hence the suppositions which have led to these absurd results are themselves absurd, and the line drawn through the point D parallel to BC must intersect AC in the point E. Therefore the parallel and the line DE are one and the same line.

Conversely: If DE be drawn parallel to the base of the triangle, then will

$$AD : DB :: AE : EC$$

For as before,

$$\triangle ADE : \triangle EDC :: AE : EC$$

and $\triangle DEB : \triangle AED :: DB : AD$

Multiplying the corresponding terms of these proportions, and omitting the common factor, $\triangle\ ADE$, in the first couplet, we have

$$\triangle\ DEB : \triangle\ EDC :: AE \times DB : EC \times AD.$$

But the $\triangle$'s DEB and EDC have equal areas, (Th. 28, B. I); hence $AE \times DB = EC \times AD$, which in the form of a proportion is

$$AE : EC :: AD : DB$$

or,

$$AD : DB :: AE : EC$$

and therefore the line parallel to the base of the triangle, divides the sides proportionally.

It is evident that the reasoning would remain the same, had we conceived ADE to be the triangle and the sides to be produced to the points B and C.

Hence the theorem; *if lines are drawn cutting the sides, etc.*

Cor. 1. Because DE is parallel to BC, and intersects the sides AB and AC, the angles ADE and ABC are equal. For the same reason the angles AED and ACB are equal, and the $\triangle$'s ADE and ABC are equiangular.

Let us now take up the triangle ADE, and place it on ABC; the angle ADE falling on $\angle B$, the side AD on the side AB, and the side DE on the side BC.

Now, since the angle A is common, and the angles AED and ACB are equal, the side AE of the $\triangle\ ADE$, in its new position, will be parallel to the side AC of the $\triangle\ ABC$.

But we have the proportion

$$AD : AE :: AB : AC$$

Placing the angle ADE on the angle ABC, and reasoning as before, we shall have the proportion

$$AD : DE :: AB : BC$$

And in like manner it may be shown that

$$AE : ED :: AC : CB$$

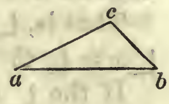
That is, *the sides about the equal angles of equiangular triangles, taken in the same order, are proportional, and the triangles are similar*, (Def. 16).

Cor. 2. *Two triangles having an angle in one equal to an angle in the other, and the sides about these equal angles proportional, are equiangular and similar.*

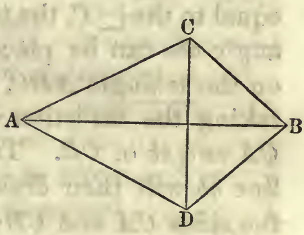
For, if the smaller triangle be placed on the larger, the equal angles of the triangles coinciding, then will the sides opposite these angles be parallel, and the triangles will therefore be equiangular and similar.

THEOREM XVIII.

If any triangle have its sides respectively proportional to the like or homologous sides of another triangle, each to each, then the two triangles will be equiangular and similar.



Let the triangle abc have its sides proportional to the triangle ABC; that is, ac to AC as cb to CB, and ac to AC as ab to AB; then we are to prove that the $\triangle$'s, abc and ABC, are equiangular and similar.



On the other side of the base, AB, and from A, conceive the angle BAD to be drawn = to the $\angle a$; and from the point B, conceive the angle ABD to be drawn = to the $\angle b$. Then the third $\angle D$ must be = to the third $\angle c$, (B. I, Th. 12, Cor. 2); and the $\triangle ABD$ will be equiangular to the $\triangle abc$ by construction.

Therefore, $ac : ab = AD : AB$

By hypothesis, $ac : ab = AC : AB$

Hence, $AD : AB = AC : AB$, (Th. 6).

In this last proportion the consequents are equal; therefore, the antecedents are equal: that is,

$$AD = AC$$

In the same manner we may prove that

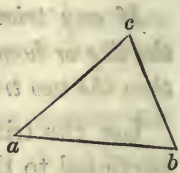
$$BD = CB$$

But AB is common to the two triangles; therefore, the three sides of the $\triangle\ ABD$ are respectively equal to the three sides of the $\triangle\ ABC$, and the two $\triangle$'s are equal, (B. I, Th. 21).

But the $\triangle$'s ABD, and abc, are equiangular by construction; therefore, the $\triangle$'s, ABC, and abc, are also equiangular and similar.

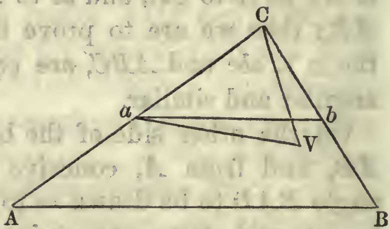
Hence the theorem; *if any triangle have its sides, etc.*

Second Demonstration.



Let abc and ABC be two triangles whose sides are respectively proportional; then will the triangles be equiangular and similar.

That is, $\angle a = \angle A$, $\angle b = \angle B$, and $\angle c = \angle C$.



If the $\angle c$ be in fact equal to the $\angle C$, the triangle abc can be placed on the triangle ABC, ca taking the direction of CA and cb of CB. The line ab will then divide the sides CA and CB proportionally, and will therefore be parallel to AB, and the triangles will be equiangular and similar, (Th. 17).

But if the $\angle c$ be not equal to the $\angle C$, then place ac on AC as before, the point c falling on C. Under the present supposition cb will not fall on CB, but will take another direction, CV, on one side or the other of CB. Make CV equal to cb and draw aV.

Now, the $\triangle\ abc$ is represented in magnitude and position by the $\triangle\ aVC$; and if, through the point a, the line ab be drawn parallel to AB, we shall have

$$Ca : CA :: ab : AB;$$

but by (Hy.) $\quad Ca : CA :: aV : AB.$

Hence, (Th. 6),

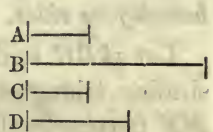
$$ab : AB :: aV : AB;$$

which requires that $ab = aV$, but (Th. 22, B. I) ab can not be equal to aV; hence the last proportion is absurd, and the supposition that the ∟ c is not equal to the ∟ C, which leads to this result, is also absurd. Therefore, the ∟ c is equal to the ∟ C, and the triangles are equiangular and similar.

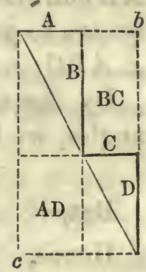
Hence the theorem; *if any triangle have its sides, etc.*

THEOREM XIX.

If four straight lines are in proportion, the rectangle contained by the lines which constitute the extremes, is equivalent to that contained by those which constitute the means of the proportion.



Let A, B, C, D, represent the four lines; then we are to show, geometrically, that $A \times D = B \times C$.



Place A and B at right angles to each other, and draw the hypotenuse. Also place C and D at right angles to each other, and draw the hypotenuse. Then bring the two triangles together, so that C shall be at right angles to B, as represented in the figure.

Now, these two △'s have each a R. ∟, and the sides about the equal angles are proportional; that is, $A : B :: C : D$; therefore, (Th. 18), the two △'s are equiangular, and the acute angles which meet at the extremities of B and C, are together equal to one right angle, and the lines B and C are so placed as to make another right angle; therefore, also, the extremities of A, B, C, and D, are in one right line, (Th. 3, B. I), and that line is the diagonal of the parallelogram *bc*. By Th. 31, B. I, the complementary parallelograms about this diagonal are equal; but, one of these parallelograms is B in length, and C in width, and the other is D in length and A in width; therefore,

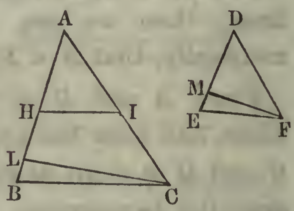
$$B \times C = A \times D.$$

Hence the theorem; *if four straight lines are in proportion, etc.*

Cor. When $B = C$, then $A \times D = B^2$, and B is the mean proportional between A and D. That is, if three straight lines are in proportion, the rectangle contained by the first and third lines is equivalent to the square described on the second line.

THEOREM XX.

Similar triangles are to one another as the squares of their homologous sides.



Let ABC and DEF be two similar triangles, and LC and MF perpendiculars to the sides AB and DE respectively. Then we are to prove that

$\triangle ABC : \triangle DEF = AB^2 : DE^2.$

By the similarity of the triangles, we have,

$$AB : DE = LC : MF$$

But, $AB : DE = AB : DE$

Hence, $\overline{AB}^2 : \overline{DE}^2 = AB \times LC : DE \times MF.$

But, (by Th. 30, B. I), $AB \times LC$ is double the area of the $\triangle ABC$, and $DE \times MF$ is double the area of the $\triangle DEF$.

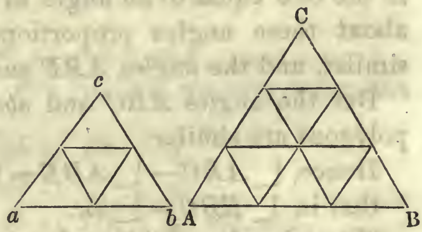
Therefore, $\triangle ABC : \triangle DEF :: AB \times LC : DE \times MF$

And, (Th. 6), $\triangle ABC : \triangle DEF = \overline{AB}^2 : \overline{DE}^2.$

Hence the theorem; *similar triangles are to one another, etc.*

The following illustration will enable the learner fully to comprehend this important theorem, and it will also serve to impress it upon his memory.

Let *abc* and *ABC* represent two equiangular triangles. Suppose the length of the side *ac* to be two units, and the length of the corresponding side *AC* to be three units.



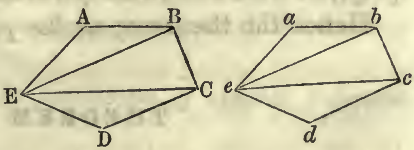
Now, drawing lines through the points of division of the sides *ac* and *AC*, parallel to the other sides of the triangles, we see that the smaller triangle is composed of four equal triangles, while the larger contains nine such triangles. That is,

the sides of the triangles are as 2 : 3,
and their areas are as $4 : 9 = 2^2 : 3^2$.

THEOREM XXI.

Similar polygons may be divided into the same number of triangles; and to each triangle in one of the polygons there will be a corresponding triangle in the other polygon, these triangles being similar and similarly situated.

Let *ABCDE* and *abcde* be two similar polygons. Now it is obvious that we can divide each polygon into as many triangles as the figure has sides, *less two;* and as the polygons have the same number of sides, the diagonals drawn from the vertices of the homologous angles will divide them into the same number of triangles.



Since the polygons are similar, the angles EAB and eab, are equal, and

$$EA : AB :: ea : ab.$$

Hence the two triangles, EAB and eab, having an angle in the one equal to an angle in the other, and the sides about these angles proportional, are equiangular and similar, and the angles ABE and abe are equal.

But the angles ABC and abc are equal, because the polygons are similar.

Hence, $\llcorner ABC - \llcorner ABE = \llcorner abc - \llcorner abe$;

that is, $\llcorner EBC = \llcorner ebc$.

The triangles, EAB and eab, being similar, their homologous sides give the proportion,

$$AB : BE :: ab : be; \quad (1)$$

and since the polygons are similar, the sides about the equal angles B and b are proportional, and we have

$$AB : BC :: ab : bc;$$

or, $$BC : AB :: bc : ab. \quad (2)$$

Multiplying proportions (1) and (2), term by term, and omitting in the result the factor AB common to the terms of the first couplet, and the factor ab common to the terms of the second, we have

$$BC : BE :: bc : be.$$

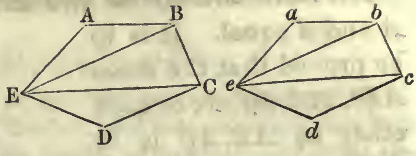
Hence the $\triangle$'s EBC and ebc are equiangular and similar; and thus we may compare all of the triangles of one polygon with those like placed in the other.

Hence the theorem; *similar polygons may be divided, etc.*

THEOREM XXII.

The perimeters of similar polygons are to one another as their homologous sides; and their areas are to one another as the squares of their homologous sides.

Let $ABCDE$ and $abcde$ be two similar polygons; then we are to prove that AB is to the sum of all the sides



of the polygon $ABCD$, as ab is to the sum of all the sides of the polygon $abcd$.

We have the identical proportion

$$AB : ab :: AB : ab;$$

and since the polygons are similar, we may write the following:

$$AB : ab :: BC : bc$$
$$AB : ab :: CD : cd$$
$$AB : ab :: DE : de, \text{ etc. etc.}$$

Hence, (Th. 7),

$AB : ab : AB+BC+CD+DE$, etc.; $ab+bc+cd+de$, etc.

Therefore, the perimeters of similar polygons are to one another as their homologous sides. This is the first part of the theorem.

Since the polygons are similar, the triangles EAB, eab, are similar, and if the triangle EAB is a part expressed by the fraction $\frac{1}{n}$, of the polygon to which it belongs, the triangle eab is a like part of the other polygon.

Therefore, $EAB : eab :: ABCDEA : abcdea$.

But, (Th. 20), $EAB : eab :: \overline{AB}^2 : \overline{ab}^2$.

Therefore, (Th. 6),

$$ABCDEA : abcdea :: \overline{AB}^2 : \overline{ab}^2.$$

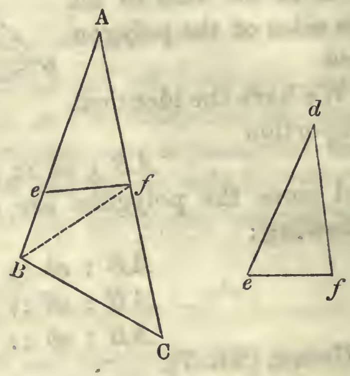
Therefore, the similar polygons are to one another as the squares on their homologous sides. This is the second part of the theorem.

Hence the theorem; *the perimeters of similar polygons are to one another, etc.*

THEOREM XXIII.

Two triangles which have an angle in the one equal to an angle in the other, are to each other as the rectangle of the sides about the equal angles.

Let ABC and def be two triangles having the angles A and d equal. It is to be proved that the areas ABC and def are to each other as $AB.AC$ is to $de.df$.



Conceive the triangle def placed on the triangle ABC, so that d shall fall on A, and de on AB; then df will fall on AC, because the ∟'s A and d are equal. On AB, lay off Ae, equal to de; and on AC, lay off Af, equal to df, and draw ef. The triangle Aef will then be equal to the triangle def. Join B and f.

Now, as triangles having the same altitude are to each other as their bases, (Th. 16, Cor. 1), we have

$$Aef : ABf :: Ae : AB$$

also,
$$ABf : ABC :: Af : AC$$

Multiplying these proportions together, term by term, omitting from the result ABf, a factor common to the terms of the first couplet, we have

$$Aef : ABC :: Ae \,.\, Af : AB \,.\, AC$$

But Aef is equal to def, Ae to de, and Af to df; therefore,

$$def : ABC :: de \,.\, df : AB \,.\, AC$$

Hence the theorem; *two triangles which have an angle, etc.*

SCHOLIUM.—If we suppose that

$$AB : AC :: de : df,$$

the two triangles will be similar; and if we multiply the terms of the first couplet of this proportion by AC, and the terms of the second couplet by df, we shall have

$$AB \,.\, AC : \overline{AC}^2 :: de \,.\, df : \overline{df}^2$$

or,
$$AB \,.\, AC : de \,.\, df :: \overline{AC}^2 : \overline{df}^2$$

Comparing this with the last proportion in this theorem, and we have, (Th. 6);

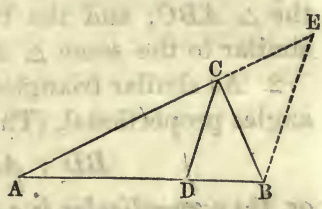
$$def : ABC :: \overline{df}^2 : \overline{AC}^2$$

REMARK. — This scholium is therefore another demonstration of Theorem 20, and hence that theorem need not necessarily have been made a distinct proposition. We require no stronger proof of the certainty of geometrical truth, than the fact that, however different the processes by which we arrive at these truths, we are never led into inconsistencies; but whenever our conclusions can be compared, they are found to harmonize with each completely, provided our premises are true and our reasoning logical.

It is hoped that the student will lose no opportunity to exercise his powers, and test his skill and knowledge, in seeking original demonstrations of theorems, and in deducing consequences and conclusions from those already established.

THEOREM XXIV.

If the vertical angle of a triangle be bisected, the bisecting line will cut the base into segments proportional to the adjacent sides of the triangle.



Let ABC be any triangle, and the vertical angle, C, be bisected by the straight line CD. Then we are to prove that

$$AD : DB = AC : CB.$$

Produce AC to E, making $CE = CB$, and draw EB. The exterior angle ACB, of the $\triangle$ CEB, is equal to the two angles E, and CBE; but the angle $E = CBE$, because $CB = CE$, and the triangle is isosceles; therefore the angle ACD, the half of the angle ACB, is equal to the angle E, and DC and BE are parallel, (Cor., Th. 7, B. I).

Now, as ABE is a triangle, and CD is parallel to BE, we have $AD : DB = AC : CE$ or CB, (Th. 17).

Hence the theorem; *if the vertical angle of a triangle be bisected, etc.*

THEOREM XXV.

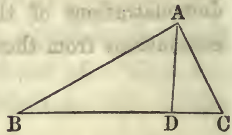
If from the right angle of a right-angled triangle, a perpendicular is drawn to the hypotenuse;

1. *The perpendicular divides the triangle into two similar triangles, each of which is similar to the whole triangle.*

2. *The perpendicular is a mean proportional between the segments of the hypotenuse.*

3. *The segments of the hypotenuse are in proportion to the squares on the adjacent sides of the triangle.*

4. *The sum of the squares on the two sides is equivalent to the square on the hypotenuse.*



Let BAC be a triangle, right angled at A; and draw AD perpendicular to BC.

1. The two $\triangle$'s, ABC and ABD, have the common angle, B, and the right angle BAC = the right angle BDA; therefore, the third $\angle C = \angle BAD$, and the two $\triangle$'s are equiangular, and similar. In the same manner we prove the $\triangle ADC$ similar to the $\triangle ABC$; and the two triangles, ABD, ADC, being similar to the same $\triangle ABC$, are similar to each other.

2. As similar triangles have the sides about the equal angles proportional, (Th. 17), we have

$$BD : AD :: AD : CD;$$

or, *the perpendicular is a mean proportional between the segments of the hypotenuse.*

3. Again, $BC : BA :: BA : BD$
hence, $\overline{BA}^2 = BC.BD$ (1)
also, $BC : CA :: CA : CD$
hence, $\overline{CA}^2 = BC.CD$ (2)

Dividing Eq. (1) by Eq. (2), member by member, we obtain

$$\frac{\overline{BA}^2}{\overline{CA}^2} = \frac{BD}{CD}$$

which, in the form of a proportion, is

$$\overline{CA}^2 : \overline{BA}^2 :: CD : BD;$$

that is, *the segments of the hypotenuse are proportional to the squares on the adjacent sides.*

4. By the addition of (1) and (2), we have

$$\overline{BA}^2 + \overline{CA}^2 = BC(BD + CD) = \overline{BC}^2;$$

that is, *the sum of the squares on the sides about the right angle is equivalent to the square on the hypotenuse.* This is another demonstration of Theorem 39, B. I.

Hence the theorem; *if from the right angle of a right-angled triangle, etc.*

BOOK III.

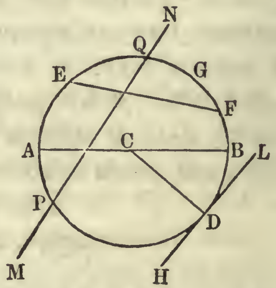
OF THE CIRCLE, AND THE INVESTIGATION OF THEOREMS DEPENDENT ON ITS PROPERTIES.

DEFINITIONS.

1. * A **Curved Line** is one whose consecutive parts, however small, do not lie in the same direction.

2. A **Circle** is a plane figure bounded by one uniformly curved line, all of the points of which are at the same distance from a certain point within, called the *center*.

3. The **Circumference** of a circle is the curved line that bounds it.



4. The **Diameter** of a circle is a line passing through the center, and terminating at both extremities in the circumference. Thus, in the figure, C is the center of the circle, the curved line $AGBD$ is the circumference, and AB is a diameter.

5. The **Radius** of a circle is a line extending from the center to any point in the circumference. Thus, CD is a radius of the circle.

6. An **Arc** of a circle is any portion of the circumference.

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* The first six of the above definitions have been before given among the general definitions of Geometry, but it was deemed advisable to reinsert them here.

7. A **Chord** of a circle is the line connecting the extremities of an arc.

8. A **Segment** of a circle is the portion of the circle on either side of a chord.

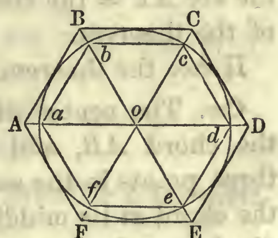
Thus, in the last figure, *EGF* is an arc, and *EF* is a chord of the circle, and the spaces bounded by the chord *EF*, and the two arcs *EGF* and *EDF*, into which it divides the circumference, are segments.

9. A **Tangent** to a circle is a line which, meeting the circumference at any point, will not cut it on being produced. The point in which the tangent meets the circumference is called the *point of tangency.*

10. A **Secant** to a circle is a line which meets the circumference in two points, and lies a part within and a part without the circumference.

11. A **Sector** of a circle is a portion of the circle included between any two radii and their intercepted arc.

Thus, in the last figure, the line *HL*, which meets the circumference at the point *D*, but does not cut it, is a tangent, *D* being the point of tangency; and the line *MN*, which meets the circumference at the points *P* and *Q*, and lies a portion within and a portion without the circle, is a secant. The area bounded by the arc *BD*, and the two radii *CB*, *CD*, is a sector of the circle.



12. A **Circumscribed Polygon** is one all of whose sides are tangent to the circumference of the circle; and conversely, the circle is then said to be *inscribed* in the polygon.

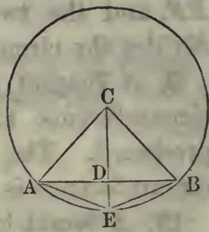
13. An **Inscribed Polygon** is one the vertices of whose angles are all formed in the circumference of the circle; and conversely, the circle is then said to be *circumscribed* about the polygon.

14. A **Regular Polygon** is one which is both equiangular and equilateral.

The last three definitions are illustrated by the last figure.

THEOREM I.

Any radius perpendicular to a chord, bisects the chord, and also the arc of the chord.



Let AB be a chord, C the center of the circle, and CE a radius perpendicular to AB; then we are to prove that $AD = BD$, and $AE = EB$.

Since C is the center of the circle, $AC = BC$, CD is common to the two △'s ACD and BCD, and the angles at D are right angles; therefore the two △'s ADC and BDC are equal, and $AD = DB$, which proves the first part of the theorem.

Now, as $AD = DB$, and DE is common to the two spaces, ADE and BDE, and the angles at D are right angles, if we conceive the sector CBE turned over and placed on CAE, CE retaining its position, the point B will fall on the point A, because $AD = BD$ and $AC = BC$; then the arc BE will fall on the arc AE; otherwise there would be points in one or the other arc unequally distant from the center, which is impossible; therefore, the arc $AE =$ the arc EB, which proves the second part of the theorem.

Hence the theorem.

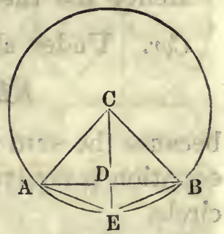
Cor. The center of the circle, the middle point of the chord AB, and of the subtended arc AEB, are three points in the same straight line perpendicular to the chord at its middle point. Now as but one perpendicular can be drawn to a line from a given point in that line, it follows:

1st. That the radius drawn to the middle point of any arc bisects, and is perpendicular to, the chord of the arc.

2d. That the perpendicular to the chord at its middle point passes through the center of the circle and the middle of the subtended arc.

THEOREM II.

Equal angles at the center of a circle are subtended by equal chords.

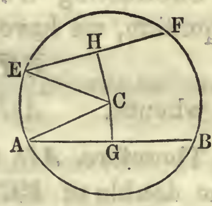


Let the angle $ACE =$ the angle BEC; then the two isosceles triangles, ACE, and ECB, are equal in all respects, and $AE = EB$.

Hence the theorem.

THEOREM III.

In the same circle, or in equal circles, equal chords are equally distant from the center.



Let AB and EF be equal chords, and C the center of the circle. From C, draw CG and CH, perpendicular to the respective chords. These perpendiculars will bisect the chords, (Th. 1), and we shall have $AG = EH$. We are now to prove that $CG = CH$.

Since the △'s ECH and ACG are right-angled, we have, (Th. 39, B. I),

$$\overline{EH}^2 + \overline{HC}^2 = \overline{EC}^2$$

and,
$$\overline{AG}^2 + \overline{GC}^2 = \overline{AC}^2.$$

By subtracting these equations, member from member, we find that

$$\overline{EH}^2 - \overline{AG}^2 + \overline{HC}^2 - \overline{GC}^2 = \overline{EC}^2 - \overline{AC}^2 \quad (1)$$

But the chords are equal by hypothesis, hence their halves, EH and AG, are equal; also $EC = AC$, being radii of the circle. Wherefore,

$$\overline{EH}^2 - \overline{AG}^2 = 0$$

and, $$\overline{EC}^2 - \overline{AC}^2 = 0.$$

These values in Equation (1) reduce it to

$$\overline{HC}^2 - \overline{GC}^2 = 0$$

or, $$\overline{HC}^2 = \overline{GC}^2$$

and, $$HC = GC.$$

Hence the theorem.

Cor. Under all circumstances we have

$$\overline{EH}^2 + \overline{HC}^2 = \overline{AG}^2 + \overline{GC}^2,$$

because the sum of the squares in either member of the equation is equivalent to the square of the radius of the circle.

Now, if we suppose HC greater than GC, then will $\overline{HC}^2$ be greater than $\overline{GC}^2$. Let the difference of these squares be represented by d.

Subtracting $\overline{GC}^2$ from both members of the above equation, we have

$$\overline{EH}^2 + d = \overline{AG}^2$$

whence, $\overline{AG}^2 > \overline{EH}^2$, and $AG > EH$.

Therefore, AB, the double of AG, is greater than EF, the double of EH; that is, *of two chords in the same or equal circles, the one nearer the center is the greater.*

The equation, $\overline{EH}^2 + \overline{HC}^2 = \overline{AG}^2 + \overline{GC}^2$, being true, whatever be the position of the chords, we may suppose GC to have any value between 0 and AC, the radius of the circle.

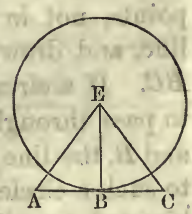
When GC becomes zero, the equation reduces to

$$\overline{EH}^2 + \overline{HC}^2 = \overline{AG}^2 = R^2;$$

that is, under this supposition, AG coincides with AC, and AB becomes the diameter of the circle, *the greatest chord that can be drawn in it.*

THEOREM IV.

A line tangent to the circumference of a circle is at right angles with the radius drawn to the point of contact.



Let AC be a line tangent to the circle at the point B, and draw the radius, EB, and the lines, AE and CE.

Now, we are to prove that EB is perpendicular to AC. Because B is the only point in the line AC which meets the circle, (Def. 9, B. II), any other line, as AE or CE, must be greater than EB; therefore, EB is the shortest line that can be drawn from the point E to the line AC; and EB is the perpendicular to AC, (Th. 23, B. I).

Hence the theorem.

THEOREM V.

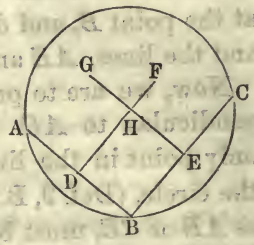
In the same circle, or in equal circles, equal chords subtend or stand on equal portions of the circumference.

Conceive two equal circles, and two equal chords drawn within them. Then, conceive one circle taken up and placed upon the other, center upon center, in such a position that the two equal chords will fall on, and exactly coincide with, each other; the circles must also coincide, because they are equal; and the two arcs of the two circles on either side of the equal chords must also coincide, or the circles could not coincide; and magnitudes which coincide, or exactly fill the same space, are in all respects equal, (Ax. 10).

Hence the theorem.

THEOREM VI.

Through three given points, not in the same straight line, one circumference can be made to pass, and but one.



Let *A*, *B*, and *C* be three given points, not in the same straight line, and draw the lines *AB* and *BC*. If a circumference is made to pass through the two points *A* and *B*, the line *AB* will be a chord to such a circle; and if a chord is bisected by a line at right angles, the bisecting line will pass through the center of the circle, (Cor., Th. 1); therefore, if we bisect the line *AB*, and draw *DF*, perpendicular to *N*, at the point of bisection, any circumference that can pass through the points, *A* and *B*, must have its center somewhere in the line *DF*. And if we draw *EG* at right angles to *BC* at its middle point, any circumference that can pass through the points *B* and *C* must have its center somewhere in the line *EG*. Now, if the two lines, *DF* and *EG*, meet in a common point, that point will be a center, about which a circumference can be drawn to pass through the three points, *A*, *B*, and *C*, and *DF* and *EG* will meet in every case, unless they are parallel; but they are not parallel, for if they were, it would follow (Th. 5, B. I) that, since *DF* is intersected at right angles by the line *AB*, it must also be intersected at right angles by the line *BC*, having a direction different from that of *AB*; which is impossible, (Th. 7, B. I).

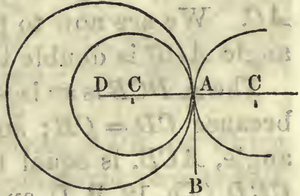
Therefore the two lines will meet; and, with the point *H*, at which they meet, as a center, and $HB = HA = HC$ as a radius, one circumference, and but one, can be made to pass through the three given points.

Hence the theorem.

THEOREM VII.

If two circles touch each other, either internally or externally, the two centers and the point of contact will be in one right line.

Let two circles touch each other internally, as represented at A, and conceive AB to be a tangent at the common point A. Now, if a line, perpendicular to AB, be drawn from the point A, it must pass through the center of each circle, (Th. 4);



and as but one perpendicular can be drawn to a line at a given point in it, A, C, and D, the point of contact and the two centers must be in one and the same line.

Next, let two circles touch each other externally, and from the point of contact conceive the common tangent, AB, to be drawn.

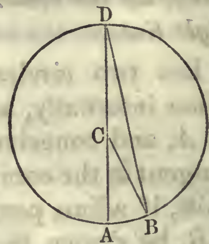
Then a line, AC, perpendicular to AB, will pass through the center of one circle, (Th. 4), and a perpendicular, AD, from the same point, A, will pass through the center of the other circle; hence, BAC and BAD are together equal to two right angles; therefore CAD is one continued straight line, (Th. 3, B. I).

Cor. When two circles touch each other internally, the distance between their centers is equal to the difference of their radii; and when they touch each other externally, the distance between their centers is equal to the sum of their radii.

THEOREM VIII.

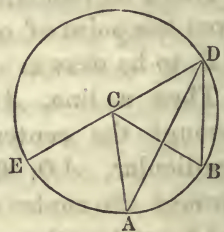
An angle at the circumference of any circle is measured by one half the arc on which it stands.

In this work it is taken as an axiom that any angle whose vertex is at the center of a circle, is measured by the arc on which it stands; and we now proceed to prove that when the arcs are equal, the angle at the circumference is equal to one half the angle at the center.



Let ACB be an angle at the center, and D an angle at the circumference, and at first suppose D in a line with AC. We are now to prove that the angle ACB is double the angle D.

The $\triangle DCB$ is an isosceles triangle, because $CD = CB$; and its exterior angle, ACB, is equal to the two interior angles, D, and CBD, (Th. 12, B. I), and since these two angles are equal to each other, the angle ACB is double the angle at D. But ACB is measured by the arc AB; therefore the angle D is measured by one half the arc AB.



Next, suppose D not in a line with AC, but at any point in the circumference, except on AB; produce DC to E.

Now, by the first part of this theorem,

the angle $ECB = 2EDB$,
also, $ECA = 2EDA$,
by subtraction, $ACB = 2ADB$.

But ACB is measured by the arc AB; therefore ADB or the angle D, is measured by one half of the same arc. Hence the theorem.

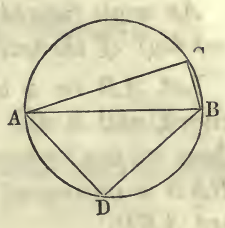
THEOREM IX.

An angle in a semicircle is a right angle; an angle in a segment greater than a semicircle is less than a right angle; and an angle in a segment less than a semicircle is greater than a right angle.

If the angle ACB is in a semicircle, the opposite segment, ADB, on which it stands, is also a semicircle; and the angle ACB is measured by one half the arc ADB.

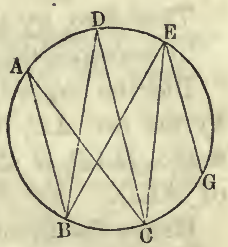
(Th. 8); that is, one half of 180°, or 90°, which is the measure of a right angle.

If the angle *ACB* is in a segment greater than a semicircle, then the opposite segment is less than a semicircle, and the measure of the angle is less than one half of 180°, or less than a right angle. If the angle *ACB* is in a segment less than a semicircle, then the opposite segment, *ADB*, on which the angle stands, is greater than a semicircle, and its half is greater than 90°; and, consequently, the angle is greater than a right angle.



Hence the theorem.

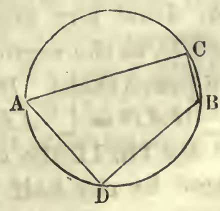
Cor. Angles at the circumference, and standing on the same arc of a circle, are equal to one another; for all angles, as *BAC*, *BDC*, *BEC*, are equal, because each is measured by one half of the arc *BC*. Also, if the angle *BEC* is equal to *CEG*, then the arcs *BC* and *CG* are equal, because their halves are the measures of equal angles.



THEOREM X.

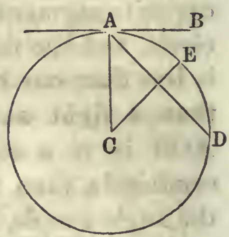
The sum of two opposite angles of any quadrilateral inscribed in a circle, is equal to two right angles.

Let *ACBD* represent any quadrilateral inscribed in a circle. The angle *ACB* has for its measure, one half of the arc *ADB*, and the angle *ADB* has for its measure, one half of the arc *ACB*; therefore, by addition, the sum of the two opposite angles at *C* and *D*, are together measured by one half of the whole circumference, or by 180 degrees, = two right angles. Hence the theorem.



THEOREM XI.

An angle formed by a tangent and a chord is measured by one half of the intercepted arc.



Let AB be a tangent, and AD a chord, and A the point of contact; then we are to prove that the angle BAD is measured by one half of the arc AED.

From A draw the radius AC; and from the center, C, draw CE perpendicular to AD.

The $\angle BAD + \angle DAC = 90°$, (Th. 4).

Also, $\angle C + \angle DAC = 90°$, (Cor. 4, Th. 12, B. I).

Therefore, by subtraction, $BAD - C = 0$;

by transposition, the angle $BAD = C$.

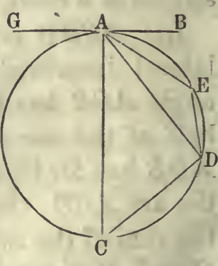
But the angle C, at the center of the circle, is measured by the arc AE, the half of AED; therefore, the equal angle, BAD, is also measured by the arc AE, the half of AED.

Hence the theorem.

See Th. 13, for another proof.

THEOREM XII.

An angle formed by a tangent and a chord, is equal to an angle in the opposite segment of the circle.



Let AB be a tangent, and AD a chord, and from the point of contact, A, draw any angles, as ACD, and AED, in the segments. Then we are to prove that $\angle BAD = \angle ACD$, and $\angle GAD = \angle AED$.

By Th. 11, the angle BAD is measured by one half the arc AED; and as the angle ACD is measured by one half of the same arc, (Th. 8), we have $\angle BAD = \angle ACD$.

Again, as $AEDC$ is a quadrilateral, inscribed in a circle, the sum of the opposite angles,

$$ACD + AED = 2 \text{ right angles.} \quad \text{(Th. 10).}$$

Also, the sum of the angles

$$BAD + DAG = 2 \text{ right angles.} \quad \text{(Th. 1, B. I).}$$

By subtraction (and observing that BAD has just been proved equal to ACD), we have,

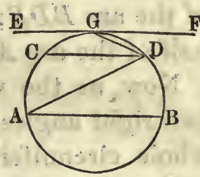
$$AED - DAG = 0.$$

Or, by transposition, $AED = DAG$.

Hence the theorem.

THEOREM XIII.

Arcs of the circumference of a circle intercepted by parallel chords, or by a tangent and a parallel chord, are equal.



Let AB and CD be parallel chords, and draw the diagonal, AD; now, because AB and CD are parallel, the angle DAB = the angle ADC (Th. 6, B. I); but the angle DAB has for its measure, one half of the arc BD; and the angle ADC has for its measure, one half of the arc AC, (Th. 8); and because the angles are equal, the arcs are equal; that is, the arc BD = the arc AC.

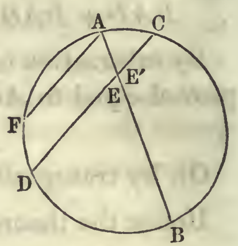
Next, let EF be a tangent, parallel to a chord, CD, and from the point of contact, G, draw GD.

Since EF and CD are parallel, the angle CDG = the angle DGF. But the angle CDG has for its measure, one-half of the arc CG, (Th. 8); and the angle DGF has for its measure, one half of the arc GD, (Th. 11); therefore, these equal measures of equals must be equal; that is, the arc CG = the arc GD.

Hence the theorem.

THEOREM XIV.

When two chords intersect each other within a circle, the angle thus formed is measured by one half the sum of the two intercepted arcs.



Let AB and CD intersect each other within the circle, forming the two angles, E and E', with their equal vertical angles.

Then, we are to prove that the angle E is measured by one half the sum of the arcs AC and BD; and the angle E' is measured by one half the sum of the arcs AD and CB.

First, draw AF parallel to CD, and FD will be equal to AC, (Th. 13); then, by reason of the parallels, $\angle BAF = \angle E$. But the angle BAF is measured by one half of the arc BDF; that is, one half of the arc BD plus one half of the arc AC.

Now, as the sum of the angles E and E' is equal to two right angles, that sum is measured by one half the whole circumference.

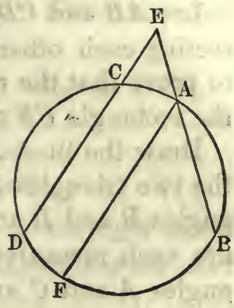
But the angle E, alone, as we have just proved, is measured by one half the sum of the arcs BD and AC; therefore, the other angle, E', is measured by one half the sum of the other parts of the circumference,

$$AD + CB.$$

Hence the theorem.

THEOREM XV.

When two secants intersect, or meet each other without a circle, the angle thus formed is measured by one half the difference of the intercepted arcs.

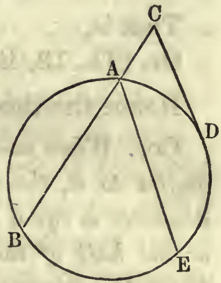


Let DE and BE be two secants meeting at E; and draw AF parallel to CD. Then, by reason of the parallels, the angle E, made by the intersection of the two secants, is equal to the angle BAF. But the angle BAF is measured by one half the arc BF; that is, by one half the difference between the arcs BD and AC.

Hence the theorem.

THEOREM XVI.

The angle formed by a secant and a tangent is measured by one half the difference of the intercepted arc.



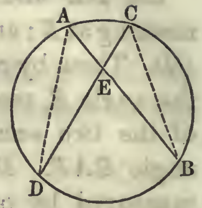
Let BC be a secant, and CD a tangent, meeting at C. We are to prove that the angle formed at C, is measured by one half the difference of the arcs BD and DA.

From A, draw AE parallel to CD; then the arc AD = the arc DE; $BD \dot{-} DE = BE$; and the $\angle BAE = \angle C$. But the angle BAE is measured by one half the arc BE, (Th. 8,) that is, by one half the difference between the arcs BD and AD; therefore, the equal angle, C, is measured by one half the arc BE.

Hence the theorem.

THEOREM XVII.

When two chords intersect each other in a circle, the rectangle contained by the segments of the one, will be equivalent to the rectangle contained by the segments of the other.



Let AB and CD be two chords intersecting each other in E. Then we are to prove that the rectangle $AE \times EB =$ the rectangle $CE \times ED$.

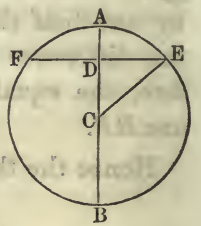
Draw the lines AD and CB, forming the two triangles AED and CEB. The angles B and D are equal, because they are each measured by one half the arc, AC. Also the angles A and C are equal, because each is measured by one half the arc, DB; and $\angle AED = \angle CEB$, because they are vertical angles; hence, the triangles, AED and CEB, are equiangular and similar. But equiangular triangles have their sides about the equal angles proportional, (Cor. 1, Th. 17, B. II); therefore, AE and ED, about the angle E, are proportional to CE and EB, about the same or equal angle.

That is, $AE : ED :: CE : EB$;

Or, (Th. 19, B. II), $AE \times EB = CE \times ED$.

Hence the theorem.

Cor. *When one chord is a diameter, and the other at right angles to it, the rectangle contained by the segments of the diameter is equal to the square of one half the other chord; or one half of the bisected chord is a mean proportional between the segments of the diameter.*



For, $AD \times DB = FD \times DE$. But, if AB passes through the center, C, at right angles to FE, then $FD = DE$ (Th. 1); and in the place of FD, write its equal, DE, in the last equation, and we have

$$AD \times DB = \overline{DE}^2,$$

or, (Th. 3, B. II), $AD : DE :: DE : DB$.

Put, $DE = x$, $CD = y$, and $CE = R$, the radius of the circle.

Then $AD = R - y$, and $DB = R + y$. With this notation,

$$AD \times DB = DE^2$$

becomes, $$(R - y)(R + y) = x^2$$

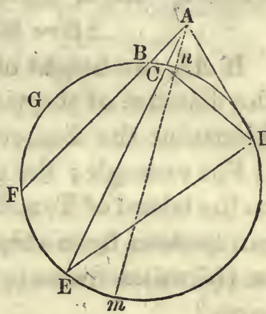
or, $$R^2 - y^2 = x^2$$

or, $$R^2 = x^2 + y^2$$

That is, *the square of the hypotenuse of the right-angled triangle*, DCE, *is equal to the sum of the squares of the other two sides.*

THEOREM XVIII.

If from a point without a circle, a tangent line be drawn to the circumference, and also any secant line terminating in the concave arc, the square of the tangent will be equivalent to the rectangle contained by the whole secant and its external segment.



Let A be a point without the circle DEG, and let AD be a tangent and AE any secant line.

Then we are to prove that

$$AC \times AE = \overline{AD}^2.$$

In the two triangles, ADE and ADC, the angles ADC and AED are equal, since each is measured by one half of the same arc, DC; the angle A is common to the two triangles; their third angles are therefore equal, and the triangles are equiangular and similar.

Their homologous sides give the proportion

$$AE : AD :: AD : AC$$

whence, $$AE \times AC = \overline{AD}^2$$

Hence the theorem.

Cor. If AE and AF are two secant lines drawn from the same point without the circumference, we shall have

$$AC \times AE = \overline{AD}^2$$

and, $$AB \times AF = \overline{AD}^2$$

hence, $$AC \times AE = AB \times AF,$$

which, in the form of a proportion, gives

$$AC : AF :: AB : AE.$$

That is, *the secants are reciprocally proportional to their external segments.*

SCHOLIUM. — By means of this theorem we can determine the diameter of a circle, when we know the length of a tangent drawn from a point without, and the external segment of the secant, which, drawn from the same point, passes through the center of the circle.

Let Am be a secant passing through the center, and suppose the tangent AD to be 20, and the external segment, An, of the secant to be 2. Then, if D denote the diameter, we shall have

$$Am = 2 + D,$$

whence, $$Am \times An = 2\,(2 + D) = 4 + 2D = (20)^2 = 400,$$

$$2D = 396, \text{ and } D = 198.$$

If An, the height of a mountain on the earth, and AD, the distance of the visible sea horizon, be given, we may determine the diameter of the earth.

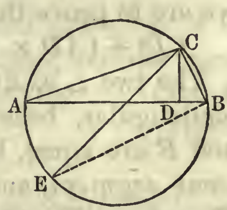
For example; the perpendicular height of a mountain on the island of Teneriffe is about 3 miles, and its summit can be seen from ships when they are known to be 154 or 155 miles distant; what then is the diameter of the earth?

Designate, as before, the diameter by D. Then $Am = 3 + D$, and $Am \times An = 9 + 3D$. $AD = 154, 5$; hence, $9 + 3D = (154, 5)^2 = 23870.\ 25$, from which we find $D = 7953.73$, which differs but little from the true diameter of the earth.

One source of error, in this mode of computing the diameter of the earth, is atmospheric refraction, the explanation of which does not belong here.

THEOREM XIX.

If a circle be described about a triangle, the rectangle contained by two sides of the triangle is equivalent to the rectangle contained by the perpendicular let fall on the third side, and the diameter of the circumscribing circle.



Let ABC be a triangle, AC and CB, the sides, CD the perpendicular let fall on the base AB, and CE the diameter of the circumscribing circle. Then we are to prove that

$$AC \times CB = CE \times CD.$$

The two △'s, ACD and CEB, are equiangular, because $\angle A = \angle E$, both being measured by the half of the arc CB; also, ADC is a right angle, and is equal to CBE, an angle in a semicircle, and therefore a right angle; hence, the third angle, $ACD = \angle BCE$, (Th. 12, Cor. 2, B. I). Therefore, (Cor., Th. 17, B. II),

$$AC : CD :: CE : CB$$

and, $$AC \times BC = CE \times CD.$$

Hence the theorem; *if a circle, etc.*

Cor. The continued product of three sides of a triangle is equal to twice the area of the triangle into the diameter of its circumscribing circle.

Multiplying both members of the last equation by AB, and we have,

$$AC \times BC \times AB = CE \times (AB \times CD).$$

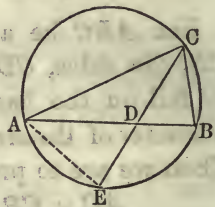
But CE is the diameter of the circle, and $(AB \times CD)$ = twice the area of the triangle;

Therefore, $AC \times CB \times AB =$ diameter multiplied by twice the area of the triangle.

THEOREM XX.

The square of a line bisecting any angle of a triangle, together with the rectangle of the segments into which it cuts the opposite side, is equivalent to the rectangle of the two sides including the bisected angle.

Let ABC be a triangle, and CD a line bisecting the angle C. Then we are to prove that



$$CD^2 + (AD \times DB) = AC \times CB.$$

The two △'s, ACE and CDB, are equiangular, because the angles E and B are equal, both being in the same segment, and the ∟ $ACE = BCD$, by hypothesis. Therefore, (Th. 17, Cor. 1, B. II),

$$AC : CE :: CD : CB.$$

But it is obvious that $CE = CD + DE$, and by substituting this value of CE, in the proportion, we have,

$$AC : CD + DE :: CD : CB.$$

By multiplying extremes and means,

$$\overline{CD}^2 + (DE \times CD) = AC \times CB.$$

But by (Th. 17),

$$DE \times CD = AD \times DB,$$

and substituting, we have,

$$\overline{CD}^2 + (AD \times DB) = AC \times CB.$$

Hence the theorem.

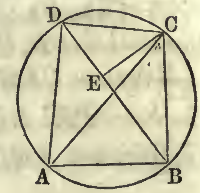
THEOREM XXI.

The rectangle contained by the two diagonals of any quadrilateral inscribed in a circle, is equivalent to the sum of the two rectangles contained by the opposite sides of the quadrilateral.

Let $ABCD$ be a quadrilateral inscribed in a circle; then we are to prove that

$$AC \times BD = (AB \times DC) + (AD \times BC).$$

From C, draw CE, making the angle DCE equal to the angle ACB; and as the angle BAC is equal to the angle CDE, both being in the same segment, therefore, the two triangles, DEC and ABC, are equiangular, and we have (Th. 17, Cor. 1, B. II),



$$AB : AC :: DE : DC \quad (1)$$

The two △'s, ADC and BEC, are equiangular; for the $\angle DAC = \angle EBC$, both being in the same segment; and the $\angle DCA = \angle ECB$, for $DCE = BCA$; to each of these add the angle ECA, and $DCA = ECB$; therefore, (Th. 17, Cor. 1, B. II),

$$AD : AC :: BE : BC \quad (2).$$

By multiplying the extremes and means in proportions (1) and (2), and adding the resulting equations, we have,

$$(AB \times DC) + (AD \times BC) = (DE + BE) \times AC.$$

But, $DE + BE = BD$; therefore,

$$(AB \times DC) + (AD \times BC) = AC \times BD.$$

Cor. When two adjacent sides of the quadrilateral are equal, as AB and BC, then the resulting equation is,

$$(AB \times DC) + (AB \times AD) = AC \times BD;$$

or, $AB \times (DC + AD) = AC \times BD$;

or, $AB : AC :: BD : DC + AD$.

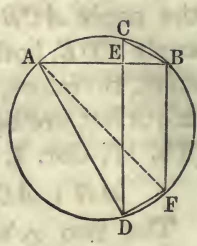
That is, *one of the two equal sides of the quadrilateral is to the adjoining diagonal, as the transverse diagonal is to the sum of the two unequal sides.*

THEOREM XXII.

If two chords intersect each other at right angles in a circle, the sum of the squares of the four segments thus formed is equivalent to the square of the diameter of the circle.

Let AB and CD be two chords, intersecting each other at right angles. Draw BF parallel to ED, and draw DF and AF. Now, we are to prove that

$$\overline{AE}^2 + \overline{EB}^2 + \overline{EC}^2 + \overline{ED}^2 = \overline{AF}^2.$$



As BF is parallel to ED, ABF is a right angle, and therefore AF is a diameter, (Th. 9). Also, because BF is parallel to CD, $CB = DF$, (Th. 13).

Because CEB is a right angle,

$$\overline{CE}^2 + \overline{EB}^2 = \overline{CB}^2 = \overline{DF}^2.$$

Because AED is a right angle,

$$\overline{AE}^2 + \overline{ED}^2 = \overline{AD}^2.$$

Adding these two equations, we have,

$$\overline{CE}^2 + \overline{EB}^2 + \overline{AE}^2 + \overline{ED}^2 = \overline{DF}^2 + \overline{AD}^2.$$

But, as AF is a diameter, and ADF a right angle, (Th. 9),

$$\overline{DF}^2 + \overline{AD}^2 = \overline{AF}^2;$$

therefore, $\overline{CE}^2 + \overline{EB}^2 + \overline{AE}^2 + \overline{ED}^2 = \overline{AF}^2.$

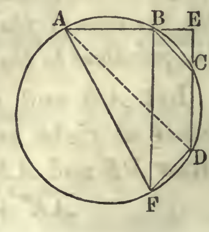
Hence the theorem.

SCHOLIUM.—If two chords intersect each other at right angles, in a circle, and their opposite extremities be joined, the two chords thus formed may make two sides of a right-angled triangle, of which the diameter of the circle is the hypotenuse.

For, AD is one of these chords, and CB is the other; and we have shown that $CB = DF$; and AD and DF are two sides of a right-angled triangle, of which AF is the hypotenuse; therefore, AD and CB may be considered the two sides of a right-angled triangle, and AF its hypotenuse.

THEOREM XXIII.

If two secants intersect each other at right angles, the sum of their squares, increased by the sum of the squares of the two segments without the circle, will be equivalent to the square of the diameter of the circle.



Let AE and ED be two secants intersecting at right angles at the point E. From B, draw BF parallel to CD, and draw AF and AD. Now we are to prove that

$$\overline{EA}^2 + \overline{ED}^2 + \overline{EB}^2 + \overline{EC}^2 = \overline{AF}^2.$$

Because BF is parallel to CD, ABF is a right angle, and consequently AF is a diameter, and $BC=DF$; and because AF is a diameter, ADF is a right angle. As AED is a right angle,

$$\overline{AE}^2+\overline{ED}^2=\overline{AD}^2$$

Also, $$\overline{EB}^2+\overline{EC}^2=\overline{BC}^2=\overline{DF}^2$$

By addition, $\overline{AE}^2+\overline{ED}^2+\overline{EB}^2+\overline{EC}^2=\overline{AD}^2+\overline{DF}^2=\overline{AF}^2$

Hence the theorem.

THEOREM XXIV.

If perpendiculars be drawn to each of the sides of a plane triangle, they will, when sufficiently produced, meet in a common point.

The three angular points of a triangle are not in the same straight line; consequently one circumference, and but one, may be made to pass through them.

Conceive a triangle to be thus circumscribed. The sides of the triangle then become chords of the circumscribing circle, and they are bisected by the perpendicular radii, (Th. 6).

Conversely: The perpendiculars bisecting the three sides of a triangle will meet in a common point, and that point will be the center of the circumscribing circle.

Hence the theorem.

THEOREM XXV.

The sums of the opposite sides of a quadrilateral circumscribing a circle are equal.

Let $ABCD$ be a quadrilateral circumscribed about a circle, whose center is O. Then we are to prove that

$$AB+DC=AD+BC.$$

From the center of the circle draw OE and OF to the points of contact of the sides AB and BC. Then,

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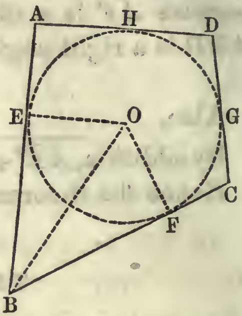
the two right-angled triangles, OEB and OFB, are equal, because they have the hypotenuse OB common, and the side $OF = OE$; therefore, $BE = BF$, (Cor., Th. 23, B. I).



In like manner we can prove that

$AE = AH$, $CF = CG$, and $DG = DH$.

Now, taking the equation $BE = BF$, and adding to its first member CG, and to its second the equal line CF. we have,

$$BE + CG = BF + CF \quad (1)$$

The equation $AE = AH$, by adding to its first member DG, and to the second the equal line, DH, gives

$$AE + DG = AH + DH \quad (2)$$

By the addition of (1) and (2), we find that

$$BE + AE + CG + DG = BF + CF + AH + DH.$$

That is, $\quad AB + CD + BC + AD.$

Hence the theorem.

BOOK IV.

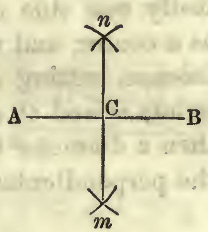
PROBLEMS.

In this section, we have, in most instances, merely shown the construction of the problem, and referred to the theorem or theorems that the student may use, to prove that the object is attained by the construction.

In obscure and difficult problems, however, we have gone through the demonstration as though it were a theorem.

PROBLEM I.

To bisect a given finite straight line.

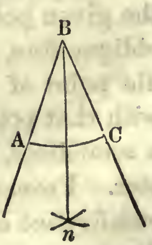


Let AB be the given line, and from its extremities, A and B, with any radius greater than one half of AB, (Postulate 3), describe arcs, cutting each other in n and m. Draw the line nm; and C, where it cuts AB, will be the middle of the given line.

Proof, (B. I, Th. 18, Sch. 2).

PROBLEM II.

To bisect a given angle.

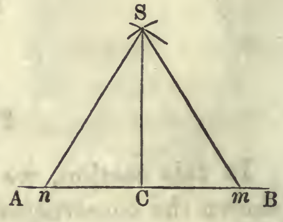


Let ABC be the given angle. With any radius, and B as a center, describe the arc AC. From A and C, as centers, with a radius greater than one half of AC, describe arcs, intersecting in n; join B and n; the joining line will bisect the given angle.

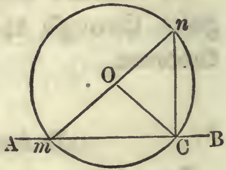
Proof, (Th. 21, B. I).

PROBLEM III.

From a given point in a given line, to draw a perpendicular to that line.



Let AB be the given line, and C the given point. Take n and m, equal distances on opposite sides of C; and with the points m and n, as centers, and any radius greater than nC or mC, describe arcs cutting each other in S. Draw SC, and it will be the perpendicular required. Proof, (B. I, Th. 18, Sch. 2).

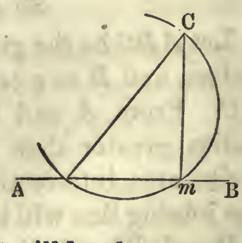


The following is another method, which is preferable, when the given point, C, is at or near the end of the line.

Take any point, O, which is manifestly one side of the perpendicular, as a center, and with OC as a radius, describe a circumference, cutting AB in m and C. Draw mn through the points m and O, and meeting the arc again in n; mn is then a diameter to the circle. Draw Cn, and it will be the perpendicular required. Proof, (Th. 9, B. III).

PROBLEM IV.

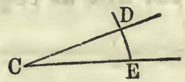
From a given point without a line, to draw a perpendicular to that line.



Let AB be the given line, and C the given point. From C draw any oblique line, as Cn. Find the middle point of Cn by Problem 1, and with that point, as a center, describe a semicircle, having Cn as a diameter. From m, where this semi-circumference cuts AB, draw Cm, and it will be the perpendicular required. Proof, (Th. 9, B. III).

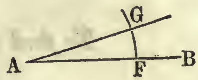
PROBLEM V.

At a given point in a line, to construct an angle equal to a given angle.



Let A be the point given in the line AB, and DCE the given angle.

With C as a center, and any radius, CE, draw the arc ED.



With A as a center, and the radius $AF = CE$, describe an indefinite arc; and with F as a center, and FG as a radius, equal to ED, describe an arc, cutting the other arc in G, and draw AG; GAF will be the angle required. Proof, (Th. 5, B. III).

PROBLEM VI.

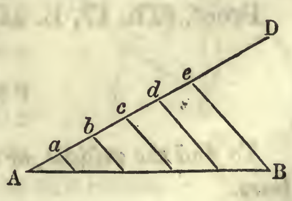
From a given point, to draw a line parallel to a given line.

Let A be the given point, and BC the given line. Draw AC, making an angle, ACB; and from the given point, A, in the line AC, draw the angle $CAD = ACB$, by Problem 5.

Since AD and BC make the same angle with AC, they are, therefore, parallel, (B. I, Th. 7, Cor. 1).

PROBLEM VII.

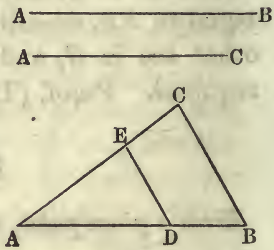
To divide a given line into any number of equal parts.



Let AB represent the given line, and let it be required to divide it into any number of equal parts, say five. From one end of the line A, draw AD, indefinite in both length and position. Take any convenient distance in the dividers, as Aa, and set it off on the line AD, thus making the parts Aa, ab, bc, etc., equal. Through the last point, e, draw EB, and through the points a, b, c, and d, draw parallels to eB, by Problem 6; these parallels will divide the line as required. Proof, (Th. 17, Book II).

PROBLEM VIII.

To find a third proportional to two given lines.

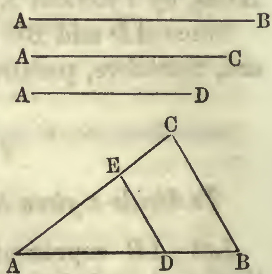


Let AB and AC be any two lines. Place them at any angle, and draw CB. On the greater line, AB, take $AD = AC$, and through D, draw DE parallel to BC; AE is the third proportional required.

Proof, (Th. 17, B. II).

PROBLEM IX.

To find a fourth proportional to three given lines.



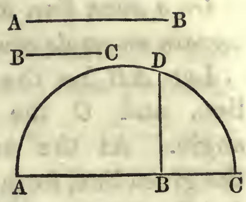
Let AB, AC, AD, represent the three given lines. Place the first two at any angle, as BAC, and draw BC. On AB place AD, and from the point D, draw DE parallel to BC, by Problem 6; AE will be the fourth proportional required.

Proof, (Th. 17, B. II).

PROBLEM X.

To find the middle, or mean proportional, between two given lines.

Place AB and BC in one right line, and on AC, as a diameter, describe a semicircle, (Postulate 3), and from the point B, draw BD at right angles to AC, (Problem 3); BD is the mean proportional required.

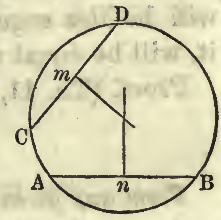


Proof, (B. III, Th. 17, Cor.).

PROBLEM XI.

To find the center of a given circle.

Draw any two chords in the given circle, as AB and CD, and from the middle points, m and n, draw perpendiculars to AB and CD; the point at which these two perpendiculars intersect will be the center of the circle.

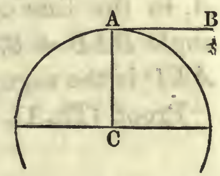


Proof, (B. III, Th. 1, Cor.).

PROBLEM XII.

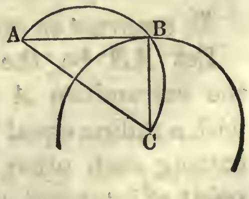
To draw a tangent to a given circle, from a given point, either in or without the circumference of the circle.

When the given point is in the circumference, as A, draw the radius AC, and from the point A, draw AB perpendicular to AC; AB is the tangent required.



Proof, (Th. 4, B. III).

When the given point is without the circle, as A, draw AC to the center of the circle; on AC, as a diameter, describe a semicircle; and from B, where the semi-circumference cuts the given circumference, draw AB, and it will be tangent to the circle.

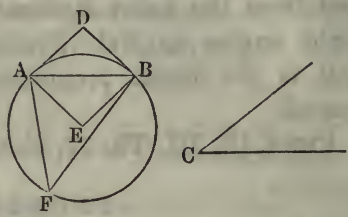


Proof, (Th. 9, B. III), and, (Th. 4, B. III).

PROBLEM XIII.

On a given line, to describe a segment of a circle, that shall contain an angle equal to a given angle.

Let *AB* be the given line, and *C* the given angle. At the ends of the given line, form angles *DAB*, *DBA*, each equal to the given angle, *C*. Then draw *AE* and *BE* perpendiculars to *AD* and *BD*; and with *E* as a center, and *EA*, or *EB*, as a radius, describe a circle; then *AFB* will be the segment required, as any angle *F*, made in it, will be equal to the given angle, *C*.

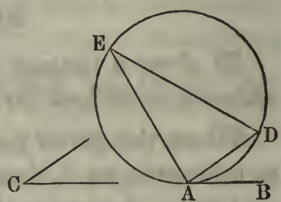


Proof, (Th. 11, B. III), and (Th. 8, B. III).

PROBLEM XIV.

From any given circle to cut a segment, that shall contain a given angle.

Let *C* be the given angle. Take any point, as *A*, in the circumference, and from that point draw the tangent *AB*; and from the point *A*, in the line *AB*, construct the angle *BAD* = *C*, (Problem 5), and *AED* is the segment required.

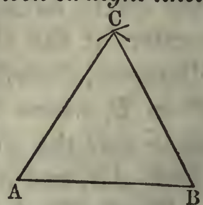


Proof, (Th. 11, B. III), and (Th. 8, B. III).

PROBLEM XV.

To construct an equilateral triangle on a given straight line.

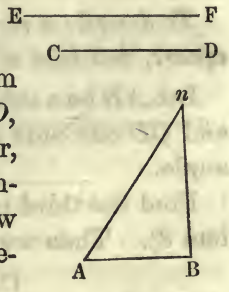
Let *AB* be the given line; from the extremities *A* and *B*, as centers, with a radius equal to *AB*, describe arcs cutting each other at *C*. From *C*, the point of intersection, draw *CA* and *CB*; *ABC* will be the triangle required.



The construction is a sufficient demonstration. Or, (Ax. 1).

PROBLEM XVI.

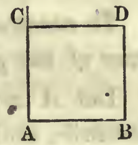
To construct a triangle, having its three sides equal to three given lines, any two of which shall be greater than the third.



Let AB, CD, and EF, represent the three lines. Take any one of them, as AB, to be one side of the triangle. From A, as a center, with a radius equal to CD, describe an arc; and from B, as a center, with a radius equal to EF, describe another arc, cutting the former in n. Draw An and Bn, and AnB will be the △ required. Proof, (Ax. 1).

PROBLEM XVII.

To describe a square on a given line.

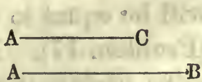


Let AB be the given line; and from the extremities, A and B, draw AC and BD perpendicular to AB. (Problem 3.)

From A, as a center, with AB as radius, strike an arc across the perpendicular at C; and from C draw CD parallel to AB; $ACDB$ is the square required. Proof, (Th. 26, B. I).

PROBLEM XVIII.

To construct a rectangle, or a parallelogram, whose adjacent sides are equal to two given lines.

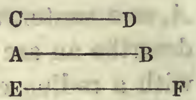


Let AB and AC be the two given lines. From the extremities of one line, draw perpendiculars to that line, as in the last problem; and from these perpendiculars, cut off portions equal to the other line; and, by a parallel, complete the figure.

When the figure is to be a parallelogram, with oblique angles, describe the angles by Problem 5. Proof, (Th. 26, B. I).

PROBLEM XIX.

To describe a rectangle that shall be equivalent to a given square, and have a side equal to a given line.



Let AB be a side of the given square, and CD one side of the required rectangle.

Find the third proportional, EF, to CD and AB, (Problem 8). Then we shall have

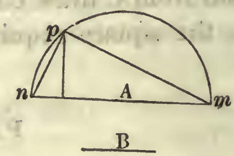
$$CD : AB :: AB : EF.$$

Construct a rectangle with the two given lines, CD and EF, (Problem 18), and it will be equal to the given square, (Th. 3, B. II).

PROBLEM XX.

To construct a square that shall be equivalent to the difference of two given squares.

Let A represent a side of the greater of two given squares, and B a side of the less square.

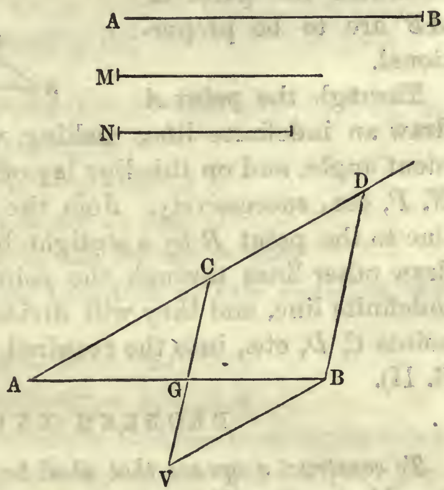


On A, as a diameter, describe a semicircle, and from one extremity, p, as a center, with a radius equal to B, describe an arc, n, and, from the point where it cuts the circumference, draw mn and np; np is the side of a square, which, when constructed, will be equal to the difference of the two given squares, (Problem 17). Proof, (Th. 9, B. III, and Th. 36, B. I.)

To construct a square equivalent to the sum of two given squares, we have only to draw through any point two lines at right angles, and lay off on one a distance equal to the side of one of the squares, and on the other a distance equal to the side of the other. The straight line connecting the extremities of these lines will be the side of the required square, (Th. 36, B. I).

PROBLEM XXI.

To divide a given line into two parts, which shall be in the ratio of two other given lines.



Let AB be the line to be divided, and M and N the lines having the ratio of the required parts of AB. From the extremity A draw AD, making any angle with AB, and take $AC = M$, and $CD = N$. Join the points D and B by a straight line, and through C draw CG parallel to BD. Then will the point G divide the line AB into parts having the required ratio. (Proof, Th. 17, B. II).

Or, having drawn AD, lay off $AC = M$, and through B draw BV parallel to AD, making it equal to N, and join C and V by a line cutting AB in the point G.

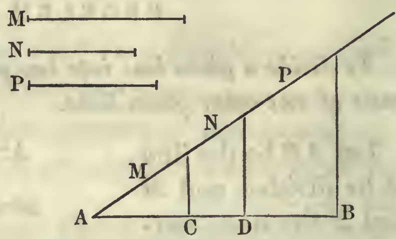
Then the two triangles ACG and GBV are equiangular and similar, and their homologous sides give the proportion,

$$AG : GB : AC :: BV :: M : N$$

The line AB is therefore divided, at the point G, into parts which are in the ratio of the lines M and N.

PROBLEM XXII.

To divide a given line into any number of parts, having to each other the ratios of other given lines.

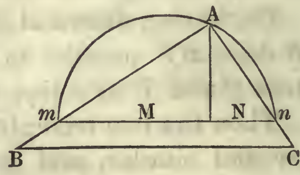


Let AB be the given line to be divided, and M, N, P, etc., the lines to which the parts of AB are to be proportional.

Through the point A draw an indefinite line, making, with AB, any convenient angle, and on this line lay off from A the lines M, N, P, etc., successively. Join the extremity of the last line to the point B by a straight line, parallel to which draw other lines through the points of division of the indefinite line, and they will divide the line AB at the points C, D, etc., into the required parts. (Proof, Th. 17, B. II).

PROBLEM XXIII.

To construct a square that shall be to a given square, as a line, M, *to a line,* N.



Place M and N in a line, and on the sum describe a semicircle. From the point where the two lines meet, draw a perpendicular to meet the circumference in A. Draw Am and An, and produce them indefinitely. On Am or Am produced, take $AB =$ to the side of the given square; and from B, draw BC parallel to mn; AC is a side of the required square.

For, $\overline{Am}^2 : \overline{An}^2 :: \overline{AB}^2 : \overline{AC}^2$, (Th. 17, B. II).

Also, $\overline{Am}^2 : \overline{An}^2 :: M : N$, (Th. 25, B. II. Sch.).

Therefore, $\overline{AB}^2 : \overline{AC}^2 :: M : N$, (Th. 6, B. II).

PROBLEM XXIV.

To cut a line into extreme and mean ratio; that is, so that the whole line shall be to the greater part, as that greater part is to the less.

REMARK.—The geometrical solution of this problem is not immediately apparent, but it is at once suggested by the form of the equation, which a simple algebraic analysis of its conditions leads to.

Represent the line to be divided by $2a$, the greater part by x, and consequently the other, or less part, by $2a - x$.

Now, the given line and its two parts are required, to satisfy the following proportion:

$$2a : x :: x : 2a - x$$

whence, $x^2 = 4a^2 - 2ax$

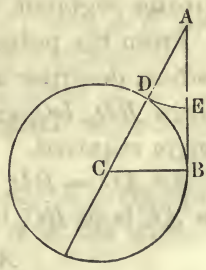
By transposition, $x^2 + 2ax = 4a^2 = (2a)^2$

If we add a^2 to both members of this equation, we shall have,

$$x^2 + 2ax + a^2 = (2a)^2 + a^2$$

or, $(x + a)^2 = (2a)^2 + a^2$

This last equation indicates that the lines represented by $(x + a)$, $2a$, and a, are the three sides of a right-angled triangle, of which $(x + a)$ is the hypotenuse, the given line, $2a$, one of the sides, and its half, a, the other.



Therefore, let AB represent the given line, and from the extremity, B, draw BC at right angles to AB, and make it equal to one half of AB.

With C, as a center, and radius CB, describe a circle. Draw AC and produce it to F. With A as a center and AD as a radius, describe the arc DE; this arc will divide the line AB, as required.

We are now to prove that

$$AB : AE :: AE : EB$$

By Scholium to Th. 18, B. III, we have,

$$AF \times AD = \overline{AB}^2$$

or, $$AF : AB :: AB : AD$$

Then, (by Cor., Th. 8, Book II), we may have,

$$(AF - AB) : AB :: (AB - AD) : AD$$

Since $CB = \frac{1}{2}AB = \frac{1}{2}DF$; therefore, $AB = DF$.

Hence, $AF - AB = AF - DF = AD = AE$.

Therefore, $AE : AB :: EB : AE$

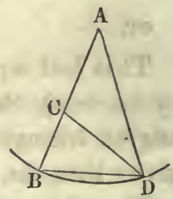
By taking the extremes for the means, we have,

$$AB : AE :: AE : EB.$$

PROBLEM XXV.

To describe an isosceles triangle, having its two equal angles each double the third angle, and the equal sides of any given length.

Let AB be one of the equal sides of the required triangle; and from the point A, with the radius AB, describe an arc, BD.



Divide the line AB into extreme and mean ratio by the last problem, and suppose C the point of division, and AC the greater segment.

From the point B, with AC, the greater segment, as a radius, describe another arc, cutting the arc BD in D. Draw BD, DC, and DA. The triangle ABD is the triangle required.

As $AC = BD$, by construction; and as AB is to AC as AC is to BC, by the division of AB; therefore

$$AB : BD :: BD : BC$$

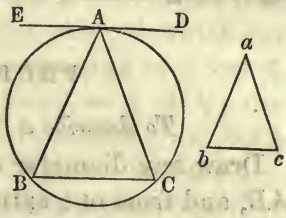
Now, as the terms of this proportion are the sides of the two triangles about the common angle, B, it follows, (Cor. 2, Th. 17, B. II), that the two triangles, ABD and

BDC, are equiangular; but the triangle *ABD* is isosceles; therefore, *BDC* is isosceles also, and $BD = DC$; but $BD = AC$: hence, $DC = AC$, (Ax. 1), and the triangle *ACD* is isosceles, and the $\angle CDA = \angle A$. But the exterior angle, $BCD = CDA + A$, (Th. 12, B. I). Therefore, $\angle BCD$, or its equal $\angle B = \angle CDA + \angle A$; or the angle $B = 2\angle A$. Hence, the triangle *ABD* has each of its angles, at the base, double of the third angle.

SCHOLIUM.—As the two angles, at the base of the triangle *ABD*, are equal, and each is double the angle *A*, it follows that the sum of the three angles is *five times* the angle *A*. But, as the three angles of every triangle are always equal to two right angles, or 180°, the angle *A* must be one fifth of two right angles, or 36°; therefore, *BD* is a chord of 36°, when *AB* is a radius to the circle; and ten such chords would extend exactly round the circle, or would form a decagon.

PROBLEM XXVI.

Within a given circle to inscribe a triangle, equiangular to a given triangle.



Let *ABC* be the circle, and *abc* the given triangle. From any point, as *A*, draw *ED* tangent to the given circle at *A*, (Problem 12).

From the point *A*, in the line *AD*, lay off the angle $DAC =$ the angle *b*, (Problem 5), and the angle $EAB =$ the angle *c*, and draw *BC*.

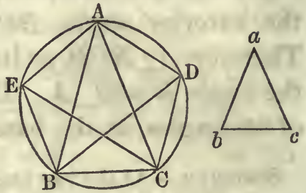
The triangle *ABC* is inscribed in the circle; it is equiangular to the triangle *abc*, and hence it is the triangle required.

Proof, (Th. 12, B. III).

PROBLEM XXVII.

To describe a regular pentagon in a given circle.

1st. Describe an isosceles triangle, *abc*, having each of the equal angles, *b* and *c*, double the third angle, *a*, by Problem 25.



2d. Inscribe the triangle, *ABC*, in the given circle, equiangular to the triangle *abc*, by Problem 26; then each of the angles, *B* and *C*, is double the angle *A*.

3d. Bisect the angles *B* and *C*, by the lines *BD* and *CE*, (Problem 2), and draw *AE*, *EB*, *CD*, *DA*; and the figure *AEBCD* is the pentagon required.

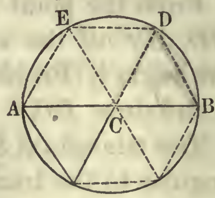
By construction, the angles *BAC*, *ABD*, *DBC*, *BCE*, *ECA*, are all equal; therefore, (B. III, Th. 9, Scho.), the arcs, *BC*, *AD*, *DC*, *AE*, and *EB*, are all equal; and if the arcs are equal, the chords *AE*, *EB*, etc., are equal.

SCHOLIUM.—The arc subtended by one of the sides of a regular pentagon, being one fifth of the whole circumference, is equal to $\frac{360°}{5}=72°$.

PROBLEM XXVIII.

To describe a regular hexagon in a circle.

Draw any diameter of the circle, as *AB*, and from one extremity, *B*, draw *BD* equal to *BC*, the radius of the circle. The arc, *BD*, will be one sixth part of the whole circumference, and the chord *BD* will be a side of the regular polygon of six sides.



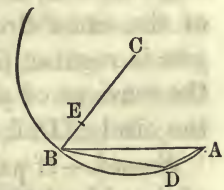
In the △ *CBD*, as $CB = CD$, and $BD = CB$ by construction, the △ is equilateral, and of course equiangular.

Since the sum of the three angles of every △ is equal to two right angles, or to 180 degrees, when the three angles are equal to one another, each one of them must be 60 degrees; but 60 degrees is a sixth part of 360 degrees, the whole number of degrees in a circle; therefore, the arc whose chord is equal to the radius, is a sixth part of the circumference; and, if a polygon of six equal sides be inscribed in a circle, each side will be equal to the radius.

SCHOLIUM. — Hence, as *BD* is the chord of 60°, and equal to *BC* or *CD*, we say generally, *that the chord of* 60° *is equal to radius.*

PROBLEM XXIX.

To find the side of a regular polygon of fifteen sides, which may be inscribed in any given circle.

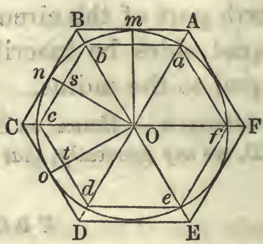


Let *CB* be the radius of the given circle; divide it into extreme and mean ratio, (Problem 24), and make *BD* equal to *CE*, the greater part; then *BD* will be a side of a regular polygon of ten sides, (Scholium to Problem 25). Draw *BA* = to *CB*, and it will be a side of a polygon of six sides. Draw *DA*, and that line must be the side of a polygon which corresponds to the arc of the circle expressed by $\frac{1}{6}$ less $\frac{1}{10}$, of the whole circumference; or $\frac{1}{6} - \frac{1}{10} = \frac{4}{60} = \frac{1}{15}$; that is, one-fifteenth of the whole circumference; or, *DA* is a side of a regular polygon of 15 sides. But the 15th part of 360° is 24°; hence the side of a regular inscribed polygon of fifteen sides is the chord of an arc of 24°.

PROBLEM XXX.

In a given circle to inscribe a regular polygon of any number of sides, and then to circumscribe the circle by a similar polygon.

Let the circumference of the circle, whose center is *O*, be divided into any number of equal arcs, as *AmB*, B*nC*, *CoD*, etc.; then will the polygon *abcde*, etc., bounded by the chords of these arcs, be regular and inscribed; and the polygon *ABCDE*, etc., bounded by the tangents to these arcs at their middle points *m*, *n*, *o*, etc, be a similar circumscribed polygon.



First. — The polygon *abcde*, etc., is equilateral, because its sides are the chords of equal arcs of the same circle, (Th. 5, B. III); and it is equiangular, because its angles are inscribed in equal segments of the same circle, (Th. 8, B. III). Therefore the polygon is regular, (Def. 14, B. III), and it is inscribed, since the vertices of all its angles are in the circumference of the circle, (Def. 13, B. III).

Second.—If we draw the radius to the point of tangency of the side *AB* of the circumscribed polygon, this radius is perpendicular to *AB*, (Th. 4, B. III), and also to the chord *ab*, (B. III, Th. 1, Cor.); hence *AB* is parallel to *ab*, and for the same reason *BC* is parallel to *bc*; therefore the angle *ABC* is equal to the angle *abc*, (Th. 8, B. I). In like manner we may prove the other angles of the circumscribed polygon, each equal to the corresponding angle of the inscribed polygon. These polygons are therefore mutually equiangular.

Again, if we draw the radii *Om* and *On*, and the line *OB*, the two △'s thus formed are right-angled, the one at *m* and the other at *n*, the side *OB* is common and *Om* is equal to *On*; hence the difference of the squares described on *OB* and *Om* is equivalent to the difference of the squares described on *OB* and *On*. But the first difference is equivalent to the square described on *Bm*, and the second difference is equivalent to the square described on Bn; hence Bm is equal to Bn, and the two right-angled triangles are equal, (Th. 20, B. I), the angle BOm opposite the side Bm being equal to the angle BOn, opposite the equal side Bn. The line OB therefore passes through the middle point of the arc mbn; but because m and n are the middle points of the equal arcs amb and bnc, the vertex of the angle abc is also at the middle point of the arc mbn. Hence the line OB, drawn from the center of the circle to the vertex of the angle ABC, also passes through the vertex of the angle abc. By precisely the same process of reasoning, we may prove that OC passes through the point c, OD through the point d, etc.; hence the lines joining the center with the vertices of the angles of the circumscribed polygon, pass through the vertices of the corresponding angles of the inscribed polygon; and conversely, the radii drawn to the vertices of the angles of the inscribed polygon, when produced, pass through the vertices of the corresponding angles of the circumscribed polygon.

Now, since ab is parallel to AB, the similar △'s abO and ABO, give the proportion

$$Ob : OB :: ab : AB,$$

and the △'s, bcO and BCO, give the proportion

$$Ob : OB :: bc : BC.$$

As these two proportions have an antecedent and consequent, the same in both, we have, (Th. 6, B. II),

$$ab : AB :: bc : BC.$$

In like manner we may prove that

$$bc : BC :: cd : CD, \text{ etc., etc.}$$

The two polygons are therefore not only equiangular, but the sides about the equal angles, taken in the same order, are proportional; they are therefore similar, (Def. 16, B. II).

Cor. 1. To inscribe any regular polygon in a circle, we have only to divide the circumference into as many equal parts as the polygon is to have sides, and to draw the chords of the arcs; hence, in a given circle, it is possible to inscribe regular polygons of any number of sides whatever. Having constructed any such polygon in a given circle, it is evident, that by changing the radius of the circle without changing the number of sides of the polygon, it may be made to represent any regular polygon of the same name, and it will still be inscribed in a circle. As this reasoning is applicable to regular polygons of whatever number of sides, it follows, that *any regular polygon may be circumscribed by the circumference of a circle.*

Cor. 2. Since ab, bc, cd, etc., are equal chords of the same circle, they are at the same distance from the center, (Th. 3, B. III); hence, if with O as a center, and Ot, the distance of one of these chords from that point, as a radius, a circumference be described, it will touch all of these chords at their middle points. It follows, therefore, that *a circle may be inscribed within any regular polygon.*

Scholium.—The center, O, of the circle, may be taken as the *center* of both the inscribed and circumscribed polygons; and the angle AOB, included between lines drawn from the center to the extremities of one of the sides AB, is called *the angle at the center.* The perpendicular drawn from the center to one of the sides is called the *Apothem* of the polygon.

Cor. 3. The angle at the center of any regular polygon is equal to four right angles divided by the number of sides of the polygon. Thus, if n be the number of sides of the polygon, the angle at the center will be expressed by $\frac{360°}{n}$.

Cor. 4. If the arcs subtended by the sides of any regular inscribed polygon be bisected, and the chords of these semi-arcs be drawn, we shall have a regular inscribed polygon of double the number of sides. Thus, from the square we may pass successively to regular inscribed polygons of 8, 16, 32, etc., sides. To get the corresponding circumscribed polygons, we have merely to draw tangents at the middle points of the arcs subtended by the sides of the inscribed polygons.

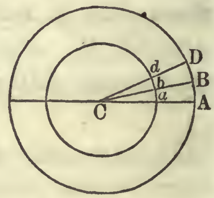
Cor. 5. It is plain that each inscribed polygon is but a part of one having twice the number of sides, while each circumscribed polygon is but a part of one having one half the number of sides.

BOOK V.

ON THE PROPORTIONALITIES AND MEASUREMENT OF POLYGONS AND CIRCLES.

PROPOSITION I.—THEOREM.

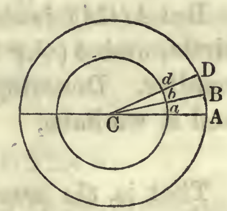
The area of any circle is equal to the product of its radius by one half of its circumference.



Let CA be the radius of a circle, and AB a very small portion of its circumference; then ACB will be a sector. We may conceive the whole circle made up of a great number of such sectors; and when each sector is very small, the arcs AB, BD, etc., each one taken separately, may be considered a right line; and the sectors CAB, CBD, etc., will be triangles. The triangle, ACB, is measured by the product of the base, AC, multiplied into one half the altitude, AB, (Th. 33, Book I); and the triangle BCD is measured by the product of BC, or its equal, AC, into one half BD; then the area, or measure of the two triangles, or sectors, is the product of AC, multiplied by one half of AB plus one half of BD, and so on for all the sectors that compose the circle; therefore, the area of the circle is measured *by the product of the radius into one half the circumference.*

PROPOSITION II.—THEOREM.

Circumferences of circles are to one another as their radii, and their areas are to one another as the squares of their radii.



Let CA be the radius of a circle, and Ca the radius of another circle. Conceive the two circles to be so placed upon each other so as to have a common center.

Let AB be such a certain definite portion of the circumference of the larger circle, that m times AB will represent that circumference.

But whatever part AB is of the greater circumference, the same part ab is of the smaller; for the two circles have the same number of degrees, and are of course susceptible of division into the same number of sectors. But by proportional triangles we have,

$$CA : Ca :: AB : ab$$

Multiply the last couplet by m, (Th. 4, B. II), and we have

$$CA : Ca :: mAB : mab.$$

That is, *the radius of one circle is to the radius of another, as the circumference of the one is to the circumference of the other.*

To prove the second part of the theorem, let C represent the area of the larger circle, and c that of the smaller; now, whatever part the sector CAB is of the circle C, the sector Cab is the corresponding part of the circle c.

That is, $C : c :: CAB : Cab$,

but, $CAB : Cab :: (CA)^2 : (Ca)^2$, (Th. 20, B. II).

Therefore, $C : c :: (CA)^2 : (Ca)^2$, (Th. 6, B. II).

That is, *the area of one circle is to the area of another, as* *the square of the radius of the one is to the square of the radius of the other.*

Hence the theorem.

Cor. If $C : c :: (CA)^2 : (Ca)^2$,
then, $C : c :: 4\,(Ca)^2 : 4\,(Ca)^2$.

But $4\,(CA)^2$ is the square of the diameter of the larger circle, and $4\,(Ca)^2$ is the square of the diameter of the smaller. Denoting these diameters respectively by D and d, we have,

$$C : c :: D^2 : d^2.$$

That is, *the areas of any two circles are to each other, as the squares of their diameters.*

Scholium. — As the circumference of every circle, great or small, is assumed to be the measure of 360 degrees, if we conceive the circumference to be divided into 360 equal parts, and one such part represented by AB on one circle, or ab on the other, AB and ab will be very near straight lines, and the length of such a line as AB will be greater or less, according to the radius of the circle; but its *absolute* length *cannot* be determined until we know the *absolute relation* between the diameter of a circle and its circumference.

PROPOSITION III.—THEOREM.

When the radius of a circle is unity, its area and semi-circumference are numerically equal.

Let R represent the radius of any circle, and the Greek letter, π, the half circumference of a circle whose radius is unity. Since circumferences are to each other as their radii, when the radius is R, the semi-circumference will be expressed by πR.

Let m denote the area of the circle of which R is the radius; then, by Theorem 1, we shall have, for the area of this circle, $\pi R^2 = m$, which, when $R = 1$, reduces to $\pi = m$.

This equation is to be interpreted as meaning that the semi-circumference contains its unit, the radius, as many times as the area of the circle contains its unit, the square of the radius.

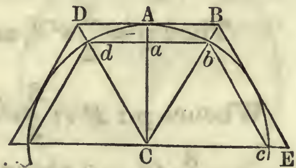
REMARK. — The celebrated problem of squaring the circle has for its object to find a line, the square on which will be equivalent to the area of a circle of a given diameter; or, in other words, it proposes to find the ratio between the area of a circle and the square of its radius.

An approximate solution only of this problem has been as yet discovered, but the approximation is so close that the exact solution is no longer a question of any practical importance.

PROPOSITION IV.—PROBLEM.

Given, the radius of a circle unity, to find the areas of regular inscribed and circumscribed hexagons.

Conceive a circle described with the radius CA, and in this circle inscribe a regular polygon of six sides (Prob. 28, B. IV), and each side will be equal to the radius CA; hence, the whole *perimeter* of this polygon must be six times the radius of the circle, or three times the diameter. The chord bd is bisected by CA. Produce Cb and Cd, and through the point A, draw BD parallel to bd; BD will then be a side of a regular polygon of six sides, circumscribed about the circle, and we can compute the length of this line, BD, as follows: The two triangles, Cbd and CBD, are equiangular, by construction; therefore,



$$Ca : bd :: CA : BD.$$

Now, let us assume $CA = CD =$ the radius of the circle, equal unity; then $bd = 1$, and the preceding proportion becomes

$$Ca : 1 :: 1 : BD \qquad (1)$$

In the right-angled triangle Cad, we have,

$$Ca^2 + ad^2 = Cd^2, \qquad \text{(Th. 39, B. I).}$$

That is, $\quad Ca^2 + \frac{1}{4} = 1$, because $Cd = 1$, and $ad = \frac{1}{2}$.

Whence, $Ca = \frac{1}{2}\sqrt{3}$. This value of Ca, substituted in proportion (1), gives

$$\frac{1}{2}\sqrt{3} : 1 :: 1 : BD; \text{ hence, } BD = \frac{2}{\sqrt{3}}.$$

But the area of the triangle Cbd is equal to $bd (= 1,)$ multiplied by $\frac{1}{2}Ca = \frac{1}{4}\sqrt{3}$; and the area of the triangle CBD is equal to BD multiplied by $\frac{1}{2}CA$.

Whence, area, $Cbd = \frac{1}{4}\sqrt{3}$,

and, area, $CBD = \frac{1}{\sqrt{3}}$.

But the area of the inscribed polygon is six times that of the triangle Cbd, and the area of the circumscribed polygon is six times that of the triangle CBD.

Let the area of the inscribed polygon be represented by p, and that of the circumscribed polygon by P.

$$\text{Then } p = \frac{3}{2}\sqrt{3}, \text{ and } P = \frac{6}{\sqrt{3}} = \frac{2 \times 3}{\sqrt{3}} = 2\sqrt{3}.$$

$$\text{Whence } p : P :: \frac{3}{2}\sqrt{3} : 2\sqrt{3} :: \frac{3}{2} : 2 :: 3 : 4 :: 9 : 12$$

$$p = \frac{3}{2}\sqrt{3} = 2.59807621. \quad P = 2\sqrt{3} = 3.46410161.$$

Now, it is obvious that the *area* of the circle must be included between the areas of these two polygons, and not far from, but somewhat greater than, their half sum, which is 3.03 +; and this may be regarded as the first approximate value of the area of the circle to the radius unity.

PROPOSITION V.—PROBLEM.

Given, the areas of two regular polygons of the same number of sides, the one inscribed in and the other circumscribed about, the same circle, to find the areas of regular inscribed and circumscribed polygons of double the number of sides.

Let p represent the area of the given inscribed polygon, and P that of the circumscribed polygon of the same number of sides. Also denote by p' the area of the inscribed polygon of double the number of sides; and by P' that of the corresponding circumscribed polygon. Now, if the arc KAL be some exact part, as one-fourth, one fifth, etc., of the circumference of the circle, of which C is the center and CA the radius, then will KL be the side of a regular inscribed polygon, and the triangle KCL will be the same part of the whole polygon that the arc KAL is of the whole circumference, and the triangle CDB will be a like part of the circumscribed polygon. Draw CA to the point of tangency, and bisect the angles ACB and ACD, by the lines CG and CE, and draw KA.

It is plain that the triangle ACK is an exact part of the inscribed polygon of double the number of sides, and that the $\triangle\ ECG$ is a like part of the circumscribed polygon of double the number of sides. Represent the area of the $\triangle\ LCK$ by a, and the area of the $\triangle\ BCD$ by b, that of the $\triangle\ ACK$ by x, and that of the $\triangle\ ECG$ by y, and suppose the $\triangle$'s, KCL and DBC, to be each the nth part of their respective polygons.

Then, $\quad na = p, \quad nb = P, \quad 2nx = p',$

and, $\quad 2ny = P'$;

But, by (Th. 33, B. I), we have

$$CM \;.\; MK = a \qquad (1)$$

$$CA \;.\; AD = b \qquad (2)$$

$$CA \;.\; MK = 2x \qquad (3)$$

Multiplying equations (1) and (2), member by member, we have

$$(CM \;.\; AD) \times (CA \;.\; MK) = ab \qquad (4)$$

From the similar △'s CMK and CAD, we have

$$CM : MK :: CA : AD$$

whence $$CM \,.\, AD = CA \,.\, MK$$

But from equation (3) we see that each member of this last equation is equal to $2x$; hence equation (4) becomes

$$2x \,.\, 2x = ab$$

If we multiply both members of this by $n^2 = n \,.\, n$, we shall have

$$4n^2x^2 = na.nb = p.P$$

or, taking the square root of both members,

$$2nx = \sqrt{p.P}$$

That is, *the area of the inscribed polygon of double the number of sides is a mean proportional between the areas of the given inscribed and circumscribed polygons* p *and* P.

Again, since CE bisects the angle ACD, we have, by, (Th. 24, B. II),

$$AE : ED :: CA : CD$$
$$:: CM : CK$$
$$:: CM : CA$$

hence, $AE : AE + ED :: CM : CM + CA.$

Multiplying the first couplet of this proportion by CA, and the second by MK, observing that $AE + ED = AD$, we shall have

$$AE.CA : AD.CA :: CM.MK : (CM + CA)\,MK.$$

But $AE.CA$ measures the area of the △ CEG, which we have called y, $AD.CA =$ △ $CBD = b$, $CM.MK =$ △ $CKL = a$, and $(CM + CA)MK =$ △ CMK, and $CAK = a + 2x$, as is seen from equations (1) and (3). Therefore the above proportion becomes

$$y : b :: a : a + 2x.$$

Multiplying the first couplet by $2n$, and the second by n, we shall have

$$2ny : 2nb :: na : na + 2nx$$

That is, $$P' : 2P :: p : p + p'$$

whence, $$P' = \frac{2Pp}{p + p'}$$

and as the value of p' has been previously found equal to $\sqrt{Pp}$, the value of P' is known from this last equation, and the problem is completely solved.

PROPOSITION VI.—PROBLEM.

To determine the approximate numerical value of the area of a circle, when the radius is unity.

We have now found, (Prob. 4), the areas of regular inscribed and circumscribed hexagons, when the radius of the circle is taken as the unit; and Prob. 5 gives us formulæ for computing from these the areas of regular inscribed and circumscribed polygons of twelve sides, and from these we may again pass to polygons of twenty-four sides, and so on, without limit. Now, it is evident that, as the number of sides of the inscribed polygon is increased, the polygon itself will increase, gradually approaching the circle, which it can never surpass. And it is equally evident that, as the number of sides of the circumscribed polygon is increased, the polygon itself will decrease, gradually approaching the circle, less than which it can never become.

The circle being included between any two corresponding inscribed and circumscribed polygons, it will differ from either less than they differ from each other; and the area of either polygon may then be taken as the area of the circle, from which it will differ by an amount less than the difference between the polygons.

It is also plain that, as the areas of the polygons approach equality, their perimeters will approach coincidence with each other, and with the circumference of the circle.

Assuming the areas already found for the inscribed and circumscribed hexagons, and applying the formulæ of Prob. 5 to them and to the successive results obtained, we may construct the following table:

NUMBER OF SIDES.	INSCRIBED POLYGONS.	CIRCUMSCRIBED POLYGONS.
6	$\frac{3}{2}\sqrt{3} = 2.59807621$	$2\sqrt{3} = 3.46410161$
12	$3 = 3.0000000$	$\frac{12}{2+\sqrt{3}} = 3.2153904$
24	$\frac{6}{\sqrt{2+\sqrt{3}}} = 3.1058286$	3.1596602
48	3.1326287	3.1460863
96	3.1393554	3.1427106
192	3.1410328	3.1418712
384	3.1414519	3.1416616
768	3.1415568	3.1416092
1536	3.1415829	3.1415963
3072	3.1415895	3.1415929
6144	3.1415912	3.1415927

Thus we have found, that when the radius of a circle is 1, the semi-circumference must be more than 3.1415912, and less than 3.1415927; and this is as accurate as can be determined with the small number of decimals here used. To be more accurate we must have more decimal places, and go through a very tedious mechanical operation; but this is not necessary, for the result is well known, and is 3.1415926535897, *plus* other decimal places to the 100th, without termination. This result was discovered through the aid of an infinite series in the Differential and Integral Calculus.

The number, 3.1416, is the one generally used in practice, as it is much more convenient than a greater number of decimals, and it is sufficiently accurate for all ordinary purposes.

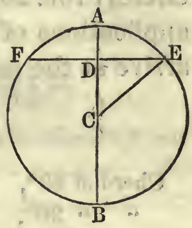
In analytical expressions it has become a general custom with mathematicians to represent this number by the Greek letter π, and, therefore, when any diameter of a circle is represented by D, the circumference of the same circle must be πD. If the radius of a circle is represented by R, the circumference must be represented by $2\pi R$.

SCHOLIUM. — The side of a regular inscribed hexagon subtends an arc of 60°, and the side of a regular polygon of twelve sides subtends an arc of 30°; and so on, the length of the arc subtended by the sides of the polygons, varying inversely with the number of sides.

Angles are measured by the arcs of circles included between their sides; they may also be measured by the chords of these arcs, or rather by the half chords called *sines* in Trigonometry. For this purpose, it becomes necessary to know the length of the chord of every possible arc of a circle.

PROPOSITION VII.—PROBLEM.

Given, the chord of any arc, to find the chord of one half that arc, the radius of the circle being unity.



Let FE be the given chord, and draw the radii CA and CE, the first perpendicular to FE, and the second to its extremity, E.

Denote FE by $2c$, and the chord of the half arc AE by x.

Then, in the right-angled triangle, DCE, we have $\overline{DC}^2 = \overline{CE}^2 - \overline{DE}^2$. Whence, since $CE = 1$, $DC = \sqrt{1 - c^2}$.

If from $CA = 1$ we subtract DC, we shall have AD. That is, $AD = 1 - \sqrt{1 - c^2}$; but $\overline{AD}^2 + \overline{DE}^2 = \overline{AE}^2$, and $\overline{AD}^2 = 2 - 2\sqrt{1 - c^2 - c^2}$. Adding to the first member of this last equation $\overline{DE}^2$, and to the second its value c^2, we have

$$\overline{AD}^2 + \overline{DB}^2 = 2\sqrt{1 - c^2}.$$

Whence, $AE = \sqrt{2 - 2\sqrt{1 - c^2}}$, the value sought.

By applying this formula successively to any known chord, we can find the chord of one half the arc, that of half of the half, and so on, to the chords of the most minute arcs.

Application.

The greatest chord in a circle is its diameter, which is 2 when the radius is 1; therefore, we may commence by making $2c = 2$, and $c = 1$.

Then, $AE = \sqrt{2 - \sqrt{1 - c^2}} = \sqrt{2 - 2\sqrt{1 - 1}} = \sqrt{2} =$ 1.41421356, which is the chord of 90°.

Now make $2c = 1.41421356$, and $c = .70710678 = \frac{1}{2}\sqrt{2}$. We shall then have,

chord of $45° = \sqrt{2 - 2\sqrt{5}} = \sqrt{2 - 1.41421356} = \sqrt{.58578644} = .7653+$.

Again, placing $2c = .7653+$, and applying the formula, we would obtain the chord of 22° 30′, and from this the chord of 11° 15′, and so on, as far as we please.

We may take, for another starting point, the chord of 60°, which is known to be equal to the radius of the circle,(Prob. 26, B. IV). If, as above, we make successive applications of the formula, putting first $2c = 1$, we shall arrive at the results in the following

TABLE.

Chord of	60°,	$= \frac{1}{6}$	of a	circumference,	1.0000000000
"	" 30°,	$= \frac{1}{12}$	"	"	.5176380902
"	" 15°,	$= \frac{1}{24}$	"	"	.2610523842
"	" 7° 30′,	$= \frac{1}{48}$	"	"	.1308062583
"	" 3° 45′,	$= \frac{1}{96}$	"	"	.0654381655
"	" 1° 52′ 30″,	$= \frac{1}{192}$	"	"	.0327234632
"	" 56′ 15″,	$= \frac{1}{384}$	"	"	.0163622792
"	" 28′ 7″ 30‴,	$= \frac{1}{768}$	"	"	.0081812080
"	" 14′ 3″ 45‴,	$= \frac{1}{1536}$	"	"	.0040906112
"	" 7′ 1″ 52½‴,	$= \frac{1}{3072}$	"	"	.0020453068
	etc.	etc.			

It is obvious that an arc so small as seven minutes of a degree can differ but very little from its chord; therefore, if we take .002045307 to be the true value of the $\frac{1}{3072}$ of the circumference, the whole circumference must be the product of .002045307 by 3072, which is 6.283183104 = circumference whose radius is unity. The half of this, 3.141592552, is the semi-circumference, the more exact value of which, as stated, (Prop. 6), is 3.141592653.

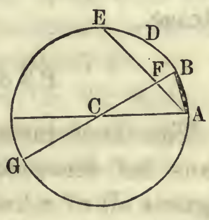
The value of the half circumference being now determined, if that of any arc whatever be required, we have merely to divide 3.141592, etc., by 10800, the number of minutes in a semi-circumference, and multiply the quotient by the number of minutes in the arc whose length is required.

But this investigation has been carried far enough for our present purposes. It will be resumed under the subject of Trigonometry.

We insert the following beautiful theorem for the trisection of an arc, although not necessary for practical application. Those not acquainted with cubic equations may omit it.

PROPOSITION VIII.—THEOREM.

Given, the chord of any arc, to determine the chord of one third of such arc.



Let AE be the given chord, and conceive its arc divided into three equal parts, as represented by AB, BD, and DE.

Through the center draw BCG, and draw AB. The two △'s, CAB and ABF, are equiangular; for, the angle FAB, being at the circumference, is measured by one half the arc BE, which is equal to AB, and the angle BCA, being at the center, is measured by the arc AB; therefore, the angle FAB = the angle BCA; but the angle CBA or FBA, is common to both triangles; therefore, the third angle, CAB, of the one triangle, is equal to the third angle, AFB, of the other, (Th. 12, B. I, Cor. 2), and the two triangles are equiangular and similar.

But the $\triangle$ ACB is isosceles; therefore, the $\triangle$ AFB is also isosceles, and $AB=AF$, and we have the following proportions:

$$CA : AB :: AB : BF.$$

Now, let $AE=c$, $AB=x$, $AC=1$. Then $AF=x$, and $EF=c-x$, and the proportion becomes,

$$1 : x :: x : BF. \quad \text{Hence, } BF=x^2.$$

Also, $$FG=2-x^2.$$

As AE and BG are two chords intersecting each other at the point F, we have,

$$GF \times FB = AF \times FE, \text{ (Th. 17, B. III).}$$

That is, $(2-x^2)\,x^2 = x\,(c-x)$;

or, $$x^3-3x=-c.$$

If we suppose the arc AE to be 60 degrees, then $c=1$, and the equation becomes $x^3-3x=-1$; a cubic equation, easily resolved by Horner's method, (Robinson's Algebra, University Ed., Art. 193), giving $x=.347296+$, the chord of 20°. This again may be taken for the value of c, and a second solution will give the chord of 6° 40′, and so on, trisecting successively as many times as we please.

PRACTICAL PROBLEMS.

The theorems and problems with which we have been thus far occupied, relate to plane figures; that is, to figures all of whose parts are situated in the same plane. It yet remains for us to investigate the intersections and relative positions of planes; the relations and positions of lines with reference to planes in which they are not contained; and the measurements, relations, and properties of solids, or volumes. But before we proceed to this, it is deemed advisable to give some practical problems for the purpose of exercising the powers of the student, and of fixing in his mind those general geometrical principles with which we must now suppose him to be acquainted.

1. The base of an isosceles triangle is 6, and the opposite angle is 60°; required the length of each of the other two equal sides, and the number of degrees in each of the other angles.

2. One angle of a right-angled triangle is 30°; what is the other angle? Also, the least side is 12, what is the hypotenuse?

Ans. { The hypotenuse is 24, the double of the least side. Why?

3. The perpendicular distance between two parallel lines is 10; what angles must a line of 20 make with these parallels to extend exactly from the one to the other? *Ans.* The angles must be 60° and 120°.

4. The perpendicular distance between two parallels is 20 feet, and a line is drawn across them at an angle of 45°; what is its length between the parallels?

Ans. $20\sqrt{2}$.

5. Two parallels are 8 feet asunder, and from a point in one of the parallels two lines are drawn to meet the other; the length of one of these lines is 10 feet, and that of the other 15 feet; what is the distance between the points at which they meet the other parallel?

Ans. 6.69 ft., or 18.69 ft. (See Th. 39, B. I).

6. Two parallels are 12 feet asunder, and from a point on one of them two lines, the one 20 feet and the other 18 feet in length, are drawn to the other parallel; what is the distance between the two lines on the other parallel, and what is the area of the triangle so formed?

Ans. { The distance on the other parallel is 29.416 feet, or 2.584 feet; and the area of the triangle is 176.496, or 15.504 square feet.

7. The diameter of a circle is 12, and a chord of the

circle is 4; what is the length of the perpendicular drawn from the center to this chord? (See Th. 3, B. III).

Ans. $4\sqrt{2}$.

8. Two parallel chords in a circle were measured and found to be 8 feet each, and their distance asunder was 6 feet; what was the radius of the circle?

Ans. 5 feet.

9. Two chords on opposite sides of the center of a circle are parallel, and one of them has a length of 16 and the other of 12 feet, the distance between them being 14 feet. What is the diameter of the circle?

Ans. 20 feet.

10. An isosceles triangle has its two equal sides, 15 each, and its base 10. What must be the altitude of a right-angled triangle on the same base, and having an equal area?

11. From the extremities of the base of any triangle, draw lines bisecting the other sides; these two lines intersecting within the triangle, will form another triangle on the same base. How will the area of this new triangle compare with that of the whole triangle?

Ans. Their areas will be as 3 to 1.

12. Two parallel chords on the same side of the center of a circle, whose diameter is 32, are measured and found to be, the one 20, and the other 8. How far are they asunder? *Ans.* $\sqrt{240} - \sqrt{156} = 3 +$.

If we suppose the two chords to be on opposite sides of the center, their distance apart will then be $\sqrt{240} + \sqrt{156} = 15.49 + 12.49 = 27.98$.

13. The longer of the two parallel sides of a trapezoid is 12, the shorter 8, and their distance asunder 5. What is the area of the trapezoid? and if we produce the two inclined sides until they meet, what will be the area of the triangle so formed?

Ans. Area of trapezoid, 50; area of triangle, 40; area of triangle and trapezoid, 90.

14. The base of a triangle is 697, one of the sides is 534, and the other 813. If a line be drawn bisecting the angle opposite the base, into what two parts will the bisecting line divide the base? (See Th. 25, B. II).

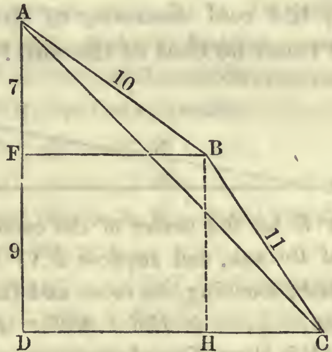
Ans. { The greater part will be 420.634; The less " " 276.316.

15. Draw three horizontal parallels, making the distance between the two upper parallels 7, and that between the middle and lower parallels 9; then place between the upper parallels a line equal to 10, and from the point in which it meets the middle parallel draw to the lower a line equal to 11, and join the point in which this last line meets the lower parallel, with the point in the upper parallel, from which the line 10 was drawn. Required the length of this line, and the area of the triangle formed by it and the two lines 10 and 11.

The adjoining figure will illustrate. Let A be the point on the upper parallel from which the line 10 is drawn. Then, $AF = 7$, $AB = 10$, $FB = \sqrt{100 - 49} = \sqrt{51}$.

$BH = FD = 9$, $BC = 11$, $HC = \sqrt{121 - 81} = \sqrt{40}$.

Whence, $DC = \sqrt{51} + \sqrt{40}$.



$$\overline{AC}^2 = (\sqrt{51} + \sqrt{40})^2 + (16)^2;\ AC = 20.89,\ Ans.$$

The area of the triangle, ABC, can be determined by first finding the area of the trapezoid, $ABHD$, then the area of the triangle, BHC, and from their sum subtracting the area of the triangle, ADC.

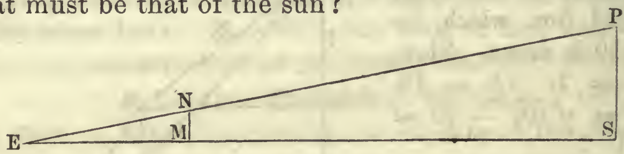
16. Construct a triangle on a base of 400, one of the angles at the base being 80°, and the other 70°; and

determine the third angle, and the area of the triangle thus constructed.

Ans. The third angle is 30°, and as nearly as our *scale of equal parts* can determine for us, the side opposite the angle 80° is 787, and that opposite 70° is 740.

The exact solution of problems like the last, except in a few particular cases, requires a knowledge of certain lines depending on the angles of the triangle. The properties and values of these lines are investigated in trigonometry; and as we are not yet supposed to be acquainted with them, we must be content with the approximate solutions obtained by the constructions and measurements made with the plane scale.

17. If we call the mean radius of the earth 1, the mean distance of the moon will be 60; and as the mean distance of the sun is 400 times the distance of the moon, its distance will be 400 times 60. The sun and moon appear to have the same diameter; supposing, then, the real diameter of the moon to be 2160 miles, what must be that of the sun?



Let E be the center of the earth, M that of the moon, and S that of the sun, and suppose ENP to be a line from the center of the earth, touching the moon and the sun.

Then, $EM : MN :: ES : SP$;

but MN is the radius of the moon, and SP that of the sun. Multiplying the consequents by 2, the above proportion becomes

$$EM : 2MN :: ES : 2SP;$$

or in numbers, $60 : 2160 :: 400 \times 60 : 2SP$;

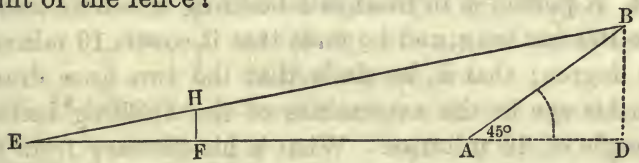
whence, $2SP$ = sun's diameter = 864000 miles, *Ans.*

18. In Problem 15, suppose BC to be drawn on the other side of BH, what, then, will be the value of AC, and what the area of the triangle ACB?

Ans. $AC = 16{,}021$;
Area of triangle, $8\sqrt{51}$, very nearly.

19. A man standing 40 feet from a building which was 24 feet wide, observed that when he closed one eye, the width of the building just eclipsed or hid from view 90 rods of fence which was parallel to the width of the building; what was the distance from the eye of the observer to the fence? *Ans.* 2475 feet.

20. Taking the same data as in the last problem, except that we will now suppose the direction of the fence to be inclined at an angle of 45° to the side of the building which we see; what, in this case, must be the distance between the eye of the observer and the remoter point of the fence?



Let HF be the width of the house, E the position of the eye, and AB that of the fence. Draw BD perpendicular to EA produced; then, since the triangle ABD is right-angled and isosceles, we have $AD = DB$, and $2\overline{AD}^2 = \overline{AB}^2 = (90)^2$; $BD = 63.64$ rods, and the similar triangles EFH and EDB give the proportion

$$HF : EF :: BD : ED = 1750.1 \text{ feet};$$

and from this we find

$$\overline{EB}^2 = \overline{ED}^2 + \overline{BD}^2 = (63.64 \times \tfrac{33}{2})^2 + (1750.1)^2$$

Whence $EB = 2040.94 +$ *Ans.*

21. In a right-angled triangle, ABC, we have $AB = 493$, $AC = 1425$, and $BC = 1338$; it is required to divide this triangle into parts by a line parallel to AB, whose areas are to each other as 1 is to 3. How will the sides AC and BC be divided by this line? (See Th. 20, B. II).

Ans. Into equal parts.

22. In a right-angled triangle, ABC, right-angled at B, the base AB is 320, and the angle A is 60°; required the remaining angle and the other sides.

Ans. $\begin{cases} \text{The angle } C = 30°; \\ AC = 640;\ BC = 554.24. \end{cases}$

23. A hunter, wishing to determine his distance from a village in sight, took a point and from it laid off two lines in the direction of two steeples, which he supposed equally distant from him, and which he knew to be 100 rods asunder. At the distance of 50 feet on each line from the common point, he measured the distance between the lines, and found it to be 5 feet 8 inches. How far was he from the steeples?

5 ft. 8 in. : 100 rods :: 50 ft. : distance.

or, $68 : 100 \times \frac{33}{2} \times 12 :: 50 :$ distance. *Ans.* { 14,559 feet, or nearly 3 miles.

24. A person is in front of a building which he knows to be 160 feet long, and he finds that it covers 10 minutes of a degree; that is, he finds that the two lines drawn from his eye to the extremities of the building include an angle of 10 minutes. What is his distance from the building?

Ans. { 50,672 feet, or nearly 10 miles.

REMARK.—The questions of distance, with which we are at present occupied, depend for their solution on the properties of similar triangles. In the preceding example we apparently have but one triangle, but we have in fact two; the second being formed by the distances *unity* on the lines drawn from the eye of the observer, and the line which connects the extremities of these units of distance. This last line may be regarded as the chord of the arc 10 minutes to the radius unity. We have seen that the length of the arc 180° to the radius 1, is 3.1415926; hence the chord of 1° or 60′ is 0.017455, and of 10′ it must be 0.0029088. Therefore, by similar triangles, we have

$$0.0029088 : 160 :: 1 : Ans. = \frac{160000}{2.9088}.$$

25. In the triangle, ABC, we have given the angles $A = 32°$, and $B = 84°$. The side AB is produced, and the exterior angle CBD thus formed, is bisected by the line BE, and the angle A is also bisected by the line AE, BE and AE meeting in the point E. What is the angle C, and what is the relation between the angles C and E?

Ans. $C = 64°$; $E = \frac{1}{2}C$.

26. Suppose a line to be drawn in any direction between two parallels. Bisect the two interior angles thus formed on either side of the connecting line, and prove that the bisecting lines meet each other at right angles, and that they are the sides of a right-angled triangle of which the line connecting the parallels is the hypotenuse.

27. If the two diagonals of a trapezoid be drawn, show that two similar triangles will be formed, the parallel sides of the trapezoid being homologous sides of the triangles. What will be the relative areas of these triangles?

Ans. The triangles will be to each other as the squares on the parallel sides of the trapezoid.

28. If from the extremities of the base of any triangle, lines be drawn to any point within the triangle, forming with the base another triangle; how will the vertical angle in this last triangle compare with that in the original triangle?

Ans. It will be as much greater than the angle in the original triangle as the sum of angles at the base of the new triangle is less than the sum of those at the base of the first.

29. The two parallel sides of a trapezoid are 12 and 20, respectively, and their perpendicular distance is 8. If a line whose length is 14.5 be drawn between the inclined sides and parallel to the parallel sides, what is the area of the trapezoid, and what the area of each part, respectively, into which the trapezoid is divided?

Ans.
Area of the whole, 128 square units;
" smaller part, $33\frac{1}{8}$ "
" larger " $94\frac{7}{8}$ "
Dividing line at the distance of $2\frac{1}{2}$ from shorter parallel side.

30. If we assume the diameter of the earth to be

7956 miles, and the eye of an observer be 40 feet above the level of the sea, how far distant will an object be, that is just visible on the earth's surface. (Employ Th. 18, B. III, after reducing miles to feet.)

Ans. 40992 feet = 7 miles 4032 feet.

31. The diameter of a circle is 4; what is the area of the inscribed equilateral triangle? *Ans.* $3\sqrt{3}$.

32. Three brothers, whose residences are at the vertices of a triangular area, the sides of which are severally 10, 11, and 12 chains, wish to dig a well which shall be at the same distance from the residence of each. Determine the point for the well, and its distance from their residences.

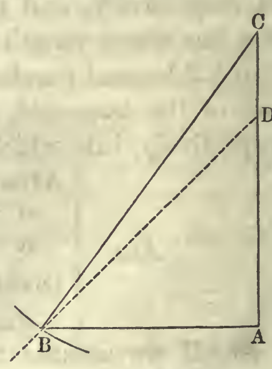
REMARK. — Construct a triangle, the sides of which are, respectively, 10, 11, and 12. The sides of this triangle will be the chords of a circle whose radius is the required distance. To find the center of this circle, bisect either two of the sides of the triangle by perpendiculars, and their intersection will be the center of the circle, and the location of the well.

Ans. The well is distant 6.25 chains, nearly, from each residence.

33. The base of an isosceles triangle is 12, and the equal sides are 20 each. What is the length of the perpendicular from the vertex to the base; and what the area of the triangle?

Ans. Perpendicular, 19.07; area, $(19.07) \times 6$.

34. The hypotenuse of a right-angled triangle is 45 inches, and the difference between the two sides is 8.45 inches. Construct the triangle.



Suppose the triangle drawn and represented by ABC, DC being the difference between the two sides.

Now, by inspection, we discover the steps to be taken for the construction of the triangle As $AD = AB$, the angle ADB, must be equal to the angle DBA, and each equal to 45°.

Therefore, draw any line, AC, and from an assumed point in it as D, draw BD, making the angle $ADB = 45°$. Take from a scale of equal parts, 8.45 inches, and lay them off from D to C, and with C as a center, and $CB = 45$ inches as a radius, describe an arc cutting BD in B. Draw CB, and from B, draw BA at right angles to AC; then is ABC the triangle sought.

Ans. $AB = 27.3$; $AC = 35.76$, when carefully constructed.

35. Taking the same triangle as in the last problem, if we draw a line bisecting the right angle, where will it meet the hypotenuse?

Ans. 19.5 from B; and 25.5 from C.

36. The diameters of the hind and fore wheels of a carriage, are 5 and 4 feet, respectively; and their centers are 6 feet asunder. At what distance from the fore wheels will the line, passing through their centers, meet the ground, which is supposed level? *Ans.* 24 feet.

37. If the hypotenuse of a right-angled triangle is 35, and the side of its inscribed square 12, what are its sides?

Ans. 28 and 21.

38. What are the sides of a right-angled triangle having the least hypotenuse, in which if a square be inscribed, its side will be 12?

Ans. { The sides are equal to 24 each, and the least hypotenuse is double the diagonal of the square.

39. The radius of a circle is 25; what is the area of a sector of 50°?

REMARK. — First find the length of an arc of 50° in a circle whose radius is unity. Then 25 times that will be the length of an arc of the same number of degrees in a circle of which the radius is 25.

$$\text{Length of arc } 1° \text{ radius unity} = \frac{3.14159269}{180}.$$

$$\text{“} \quad \text{“ } 50° \quad \text{“} \quad \text{“} \quad = \frac{1.04719763}{6} \times 5.$$

$$\text{Area of sector} = \frac{1.04719763}{6} \times 125 \times \frac{25}{2} = 54.541, \textit{ Ans.}$$

BOOK VI.

ON THE INTERSECTIONS OF PLANES, AND THE RELATIVE POSITIONS OF PLANES AND OF PLANES AND LINES.

DEFINITIONS.

A **Plane** has been already defined to be a surface, such that the straight line which joins any two of its points will lie entirely in that surface. (Def. 9, page 9.)

1. The **Intersection** or **Common Section** of two planes is the line in which they meet.

2. A **Perpendicular to a Plane** is a line which makes right angles with every line drawn in the plane through the point in which the perpendicular meets it; and, conversely, the plane is perpendicular to the line. The point in which the perpendicular meets the plane is called the *foot* of the perpendicular.

3. A **Diedral Angle** is the separation or divergence of two planes proceeding from a common line, and is measured by the angle included between two lines drawn one in each plane, perpendicular to their common section at the same point.

The common section of the two planes is called the *edge* of the angle, and the planes are its *faces*.

4. **Two Planes are perpendicular to each other,** when their diedral angle is a right angle.

5. A **Straight Line is parallel to a plane,** when it will not meet the plane, however far produced.

6. Two Planes are parallel, when they will not intersect, however far produced in all directions.

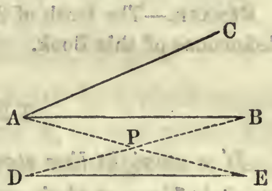
7. A **Solid** or **Polyedral Angle** is the separation or divergence of three or more plane angles, proceeding from a common point, the two sides of each of the plane angles being the edges of diedral angles formed by these plane angles.

The common point from which the plane angles proceed is called the *vertex* of the solid angle, and the intersection of its bounding planes are called its *edges*.

8. A **Triedral Angle** is a solid angle formed by three plane angles.

THEOREM I.

Two straight lines which intersect each other, two parallel straight lines, and three points not in the same straight line, will severally determine the position of a plane.

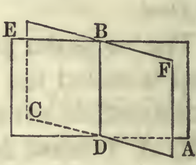


Let AB and AC be two lines intersecting each other at the point A; then will these lines determine a plane. For, conceive a plane to be passed through AB, and turned about AB as an axis until it contains the point C in the line AC. The plane, in this position, contains the lines AB and AC, and will contain them in no other. Again, let AB and DE be two parallel straight lines, and take at pleasure two points, A and B, in the one, and two points, D and E, in the other, and draw AE and BD. These last lines, from what precedes, determine the position of a plane which contains the points A, B, D, and E. And again, if A, B, and C be three points not in the same straight line, and we draw the lines AB and AC, it follows, from the first part of this proposition, that these points fix the plane.

Cor. A straight line and a point out of it determine the position of a plane.

THEOREM II.

If two planes meet each other, their common points will be found in, and form one straight line.

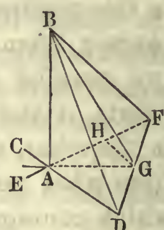


Let B and D be any two of the points common to the two planes, and join these points by the straight line BD; then will BD contain all the points common to the two planes, and be their intersection. For, suppose the planes have a common point out of the line BD; then, (Cor. Th. 1), since a straight line and a point out of it determine a plane, there would be two planes determined by this one line and single point out of it, which is absurd. Hence the common section of two planes is a straight line.

REMARK.—The truth of this proposition is implicitly assumed in the definitions of this Book.

THEOREM III.

If a straight line stand at right angles to each of two other straight lines at their point of intersection, it will be at right angles to the plane of those lines.



Let AB stand at right angles to EF and CD, at their point of intersection A. Then AB will be at right angles to any other line drawn through A in the plane, passing through EF, CD, and, of course, at right angles to the plane itself. (Def. 2.)

Through A, draw any line, AG, in the plane EF, CD, and from any point G, draw GH parallel to AD. Take $HF = AH$, and join F and G and produce FG to D. Because HG is parallel to AD, we have

$$FH : HA :: FG : GD.$$

But, in this proportion, the first couplet is a ratio of equality; therefore the last couplet is also a ratio of equality,

That is, $FG = GD$, or the line FD is bisected in G.

Draw BD, BG, and BF.

Now, in the triangle AFD, as the base FD is bisected in G, we have,

$$\overline{AF}^2 + \overline{AD}^2 = 2\overline{AG}^2 + 2\overline{GF}^2 \quad (1) \quad \text{(Th. 42, B. I).}$$

Also, as DF is the base of the $\triangle$ BDF, we have by the same theorem,

$$\overline{BF}^2 + \overline{BD}^2 = 2\overline{BG}^2 + 2\overline{GF}^2 \qquad (2)$$

By subtracting (1) from (2), and observing that $\overline{BF}^2 - \overline{AF}^2 = \overline{AB}^2$, because BAF is a right angle; and $\overline{BD}^2 - \overline{AD}^2 = \overline{AB}^2$, because BAD is a right angle, we shall have,

$$\overline{AB}^2 + \overline{AB}^2 = 2\overline{BG}^2 - 2\overline{AG}^2.$$

Dividing by 2, and transposing $\overline{AG}^2$, and we have,

$$\overline{AB}^2 + \overline{AG}^2 = \overline{BG}^2.$$

This last equation shows that BAG is a right angle. But AG is any line drawn through A, in the plane EF, CD; therefore AB is at right angles to any line in the plane, and, of course, at right angles to the plane itself.

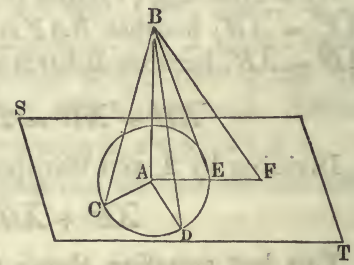
Cor. 1. The perpendicular BA is shorter than any of the oblique lines BF, BG, or BD, drawn from the point B to the plane; hence it is the shortest distance from a point to a plane.

Cor. 2. But one perpendicular can be erected to a plane from a given point in the plane; for, if there could be two, the plane of these perpendiculars would intersect the given plane in some line, as AG, and both the perpendiculars would be at right angles to this intersection at the same point, which is impossible.

Cor. 3. But one perpendicular can be let fall from a given point out of a plane on the plane; for, if there can be two, let BG and BA be such perpendiculars, then would the triangle BAG be right angled at both A and G, which is impossible.

THEOREM IV.

If from any point of a perpendicular to a plane, oblique lines be drawn to different points in the plane, those oblique lines which meet the plane at equal distances from the foot of the perpendicular are equal; and those which meet the plane at unequal distances from the foot of the perpendicular are unequal, the greater distances corresponding to the longer oblique lines.



Take any point B in the perpendicular BA to the plane ST, and draw the oblique lines BC, BD, and BE, the points C, D, and E, being equally distant from A, the foot of the perpendicular. Produce AE to F, and draw BF; then will $BC = BD = BE$, and $BF > BE$.

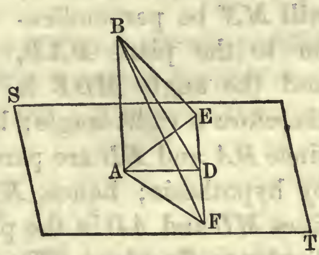
For, the triangles BAC, BAD, and BAE are all right-angled at A, the side BA is common, and $AC = AD = AE$ by construction, hence, (Th. 23, B. I), $BC = BD = BE$. Moreover, since $AF > AE$, the oblique line $BF > BE$.

Cor. If any number of equal oblique lines be drawn from the point B to the plane, they will all meet the plane in the circumference of a circle having the foot of the perpendicular for its center. It follows from this, that, if three points be taken in a plane equally distant from a point out of it, the center of the circumference passing through these three points will be the foot of the perpendicular drawn from the point to the plane.

THEOREM V.

The line which joins any point of a perpendicular to a plane, with the point in which a line in the plane is intersected, at right angles, by a line through the foot of the perpendicular, will be at right angles to the line in the plane.

Let AB be perpendicular to the plane ST, and AD a line through its foot at right angles to EF, a line in the plane. Connect D with any point, as B, of the perpendicular; and BD will be perpendicular to EF.



Make $DF = DE$, and join B to the points E, D, and F. Since $DE = DF$, and the angles at D are right angles, the oblique lines, AE and AF, are equal; and, since $AE = AF$, we have, (Th. 4), $BE = BF$; therefore the line BD has its two points, B and D, equally distant from the extremities E and F of the line EF, and hence BD is perpendicular to EF at its middle point D.

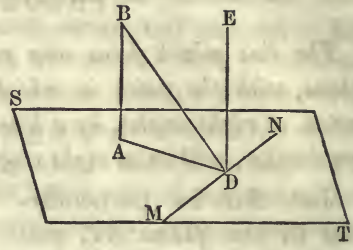
Cor. Since FD is perpendicular to the two lines AD and BD at their intersection, it is perpendicular to their plane ADB, (Th. 3).

Scholium. — The inclination of a line to a plane is measured by the angle included between the given line and the line which joins the point in which it meets the plane and the foot of the perpendicular drawn from any point of the line to the plane; thus, the angle BFA is the inclination of the line BF to the plane ST.

THEOREM VI.

If either of two parallels is perpendicular to a plane, the other is also perpendicular to the plane.

Let BA and ED be two parallels, of which one, BA, is perpendicular to the plane ST; then will the other also be perpendicular to the same plane.



The two parallels determine a plane which intersects the given plane in AD; through D draw MN perpendicular to AD; then, (Cor., Th. 5,) will MN be perpendicular to the plane BAD, and the angle MDE is therefore a right angle; but EDA is also a right angle, since BA and ED are parallel, and BAD is a right angle by hypothesis; hence, ED is perpendicular to the two lines MD and AD in the plane ST; it is therefore perpendicular to the plane, (Th. 3).

Cor. 1. The converse of this proposition is also true, that is, *if two straight lines are both perpendicular to the same plane, the lines are parallel.*

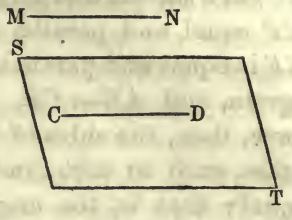
For, suppose BA and ED to be two perpendiculars; if not parallel, draw through D a parallel to BA, and this last line will be perpendicular to the plane; but ED is a perpendicular by hypothesis, and we should have two perpendiculars erected to the plane at the same point, which is impossible, (Cor. 2, Th. 3).

Cor. 2. If two lines lying in the same plane are each parallel to a third line not in the same plane, the two lines are parallel. For, pass a plane perpendicular to the third line, and it will be perpendicular to each of the others; hence they are parallel.

THEOREM VII.

A straight line is parallel to a plane, when it is parallel to a line in the plane.

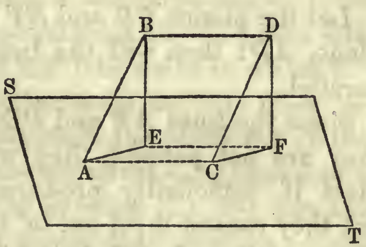
Suppose the line MN to be parallel to the line CD, in the plane ST; then will MN be parallel to the plane ST



For, *CD* being in the plane *ST*, and at the same time parallel to *MN*, it must be the intersection of the plane of these parallels with the plane *ST*; hence, if *MN* meet the plane *ST*, it must do so in the line *CD*, or *CD* produced; but *MN* and *CD* are parallel, and cannot meet; therefore *MN*, however far produced, can have no point in the plane *ST*, and hence, (Def. 5), it is parallel to this plane.

THEOREM VIII.

If two lines are parallel, they will be equally inclined to any given plane.



Let *AB* and *CD* be two parallels, and *ST* any plane met by them in the points *A* and *C*; then will the lines *AB* and *CD* be equally inclined to the plane *ST*.

For, take any distance, *AB*, on one of these parallels, and make *CD* = *AB*, and draw *AC* and *BD*. From the points *B* and *D* let fall the perpendiculars, *BE* and *DF*, on the plane; join their feet by the line *EF*, and draw *AE* and *CF*.

Now, since *AB* is equal and parallel to *CD*, *ABDC* is a parallelogram, and *BD* is equal and parallel to *AC*, and *BD* is parallel to the plane *ST*, (Th. 7); and, since *BE* and *DF* are both perpendicular to this plane, they are parallel; but *BD* and *EF* are in the plane of these parallels; and as *EF* is in the plane *ST*, and *BD* is parallel to this plane, these two lines must be parallel and equal, and *BDFE* is also a parallelogram. Now, we have shown that BD is equal and parallel to AC, and EF equal and parallel to BD; hence, (Cor. 2, Th. 6), EF is equal and parallel to AC, and $ACFE$ is a parallelogram, and $AE = CF$. The triangles ABE and CDF have, then, the sides of the one equal to the sides of the other, each to each, and their angles are consequently equal; that is, the angle BAE is equal to the angle DCF; but these angles measure the inclination of the lines AB and CD to the plane ST, (Scholium, Th. 5).

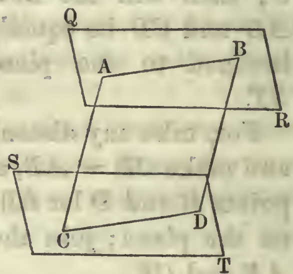
SCHOLIUM. — The converse of this proposition is not generally true; that is, straight lines equally inclined to the same plane are not necessarily parallel.

THEOREM IX.

The intersections of two parallel planes by a third plane, are parallel.

Let the planes QR and ST be intersected by the third plane, AD: then will the intersections, AB and CD, be parallel.

Since the lines AB and CD are in the same plane, if they are not parallel, they will meet if sufficiently produced; but they cannot meet out of the planes QR and ST, in which they are respectively found; therefore, any point common to the lines, must be at the same time common to the planes; and since the planes are parallel, they have no common point, and the lines, therefore, do not intersect; hence they are parallel.

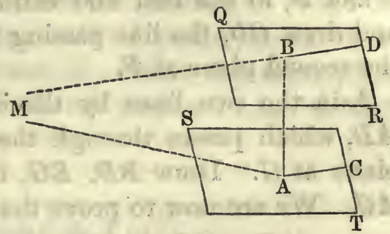


THEOREM X.

If two planes are perpendicular to the same straight line, they are parallel to each other.

Let QR and ST be two planes, perpendicular to the line AB; then will these planes be parallel.

For, if not parallel, suppose M to be a point in their line of intersection, and from this point draw lines to the extremities of the perpendicular AB, thus forming a triangle, MAB. Now, since the line AB is perpendicular to both



planes, it is perpendicular to each of the lines MA and MB, drawn through its feet in the planes, (Def. 2); hence, the triangle has two right angles, which is impossible; the planes cannot therefore meet in any point as M, and are consequently parallel.

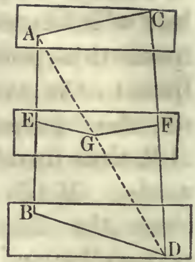
Cor. Conversely: *The straight line which is perpendicular to one of the parallel planes, is also perpendicular to the other.* For, if AB be perpendicular to the plane QR, draw in the other plane, through the point in which the perpendicular meets it, any line, as AC. The plane of the lines AB and AC will intersect the plane QR in the line BD; and since the planes are parallel by hypothesis, the lines AC and BD must be parallel, (Th. 9); but the angle DBA is a right angle; hence, BAC must be a right angle, and the line BA is perpendicular to any line whatever drawn in the plane through the point A; BA is therefore perpendicular to the plane ST.

THEOREM XI.

If two straight lines be drawn in any direction through parallel planes, the planes will cut the lines proportionally.

Conceive three planes to be parallel, as represented in the figure, and take any points, A and B, in the first and third planes, and draw AB, the line passing through the second plane at E.

Also, take any other two points, as C and D, in the first and third planes, and draw CD, the line passing through the second plane at F.



Join the two lines by the diagonal AD, which passes through the second plane at G. Draw BD, EG, GF, and AC. We are now to prove that,

$$AE : EB :: CF : FD.$$

For the sake of perspicuity, put $AG = X$, and $GD = Y$.

As the planes are parallel, BD is parallel EG; then, in the two triangles ABD and AEG, we have, (Th. 17, B. II);

$$AE : EB :: X : Y.$$

Also, as the planes are parallel, GF is parallel to AC, and we have,

$$CF : FD :: X : Y.$$

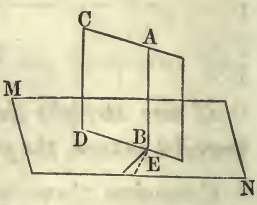
By comparing the proportions, and applying Th. 6, B. II, we have

$$AE : EB :: CF : FD.$$

THEOREM XII.

If a straight line is perpendicular to a plane, all planes passing through that line will be perpendicular to the plane.

Let MN be a plane, and AB a perpendicular to it. Let BC be any other plane, passing through AB; this plane will be perpendicular to MN.



Let BD be the common intersection of the two planes, and from the point B, draw BE at right angles to DB.

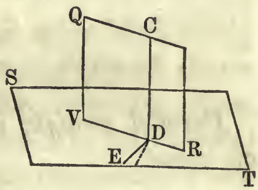
Then, as AB is perpendicular to the plane MN, it is perpendicular to every line in that plane, passing through

B; (Def. 2,); therefore, *ABE* is a right angle. But the angle *ABE*, (Def. 3), measures the inclination of the two planes; therefore, the plane *CB* is perpendicular to the plane *MN*; and thus we can show that any other plane, passing through *AB*, will be perpendicular to *MN*. Hence the theorem.

THEOREM XIII.

If two planes are perpendicular to each other, and a line be drawn in one of them perpendicular to their common intersection, it will be perpendicular to the other plane.

Let the two planes, *QR* and *ST*, be perpendicular to each other, and draw in *QR* the line *CD* at right angles to their common intersection, *RV*; then will this line be perpendicular to the plane *ST*.

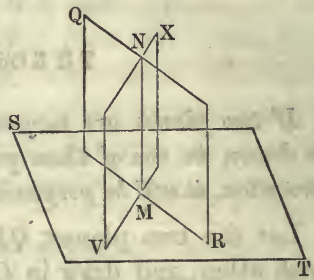


In the plane *ST* draw *ED*, perpendicular to *VR* at the point *D*. Then, since the planes *QR* and *ST* are perpendicular to each other, the angle *CDE* is a right angle, and *CD* is perpendicular to the two lines, *ED* and *VR*, passing through its foot in the plane *ST*. *CD* is therefore perpendicular to the plane *ST*, (Th. 3).

Cor. Conversely: if we erect a perpendicular to the plane *ST*, at any point, *D*, of its intersection with the plane *QR*, this perpendicular will lie in the plane *QR*. For, if it be not in this plane, we can draw in the plane the line *CD*, at right angles to *VR*; and, from what has been shown above, *CD* is perpendicular to the plane *ST*, and we should thus have two perpendiculars erected to the plane, *ST*, at the same point, which is impossible, (Cor. 2, Th. 3).

THEOREM XIV.

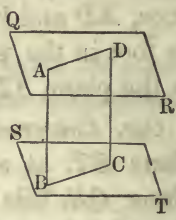
The common intersection of two planes, both of which are perpendicular to a third plane, will also be perpendicular to the third plane.



Let *MN* be the common intersection of the two planes, *QR* and *VX*, both of which are perpendicular to the plane *ST*; then will *MN* be perpendicular to the plane *ST*. For, if we erect a perpendicular to the plane *ST*, at the point *M*, it will lie in both planes at the same time, (Cor. Th. 13); and this perpendicular must therefore be their intersection. Hence the theorem.

THEOREM XV.

Parallel straight lines included between parallel planes, are equal.



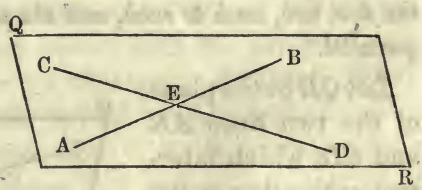
Let *AB* and *DC* be two parallel lines, included by the two parallel planes, *QR* and *ST*; then will $AB = DC$.

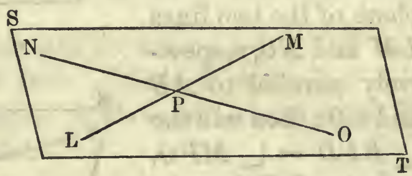
For, the plane *AC*, of the parallel lines, intersects the planes, *QR* and *ST*, in the parallel lines, *AD* and *BC*, (Th. 9); hence *ABDC* is a parallelogram, and its opposite sides, *AB* and *DC*, are equal.

Cor. It follows from this proposition, that parallel planes are everywhere equally distant; for, two perpendiculars drawn at pleasure between the two planes are parallel lines, (Cor. 1, Th. 6), and hence are equal; but these perpendiculars measure the distance between the planes.

THEOREM XVI.

Two planes are parallel when two lines not parallel, lying in the one, are respectively parallel to two lines lying in the other.





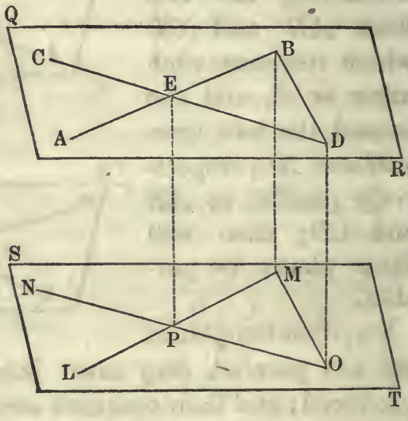
Let *QR* and *ST* be two planes, the first containing the two lines *AB* and *CD* which intersect each other at *E*, and the second the two lines *LM* and *NO*, respectively parallel to *AB* and *CD*; then will these planes be parallel.

For, if the two planes are not parallel, they must intersect when sufficiently produced; and their common section lying in both planes at the same time, would be a line of the plane *QR*. Now, the lines *AB* and *CD* intersect each other by hypothesis; hence one or both of them must meet the common section of the two planes. Suppose *AB* to meet this common section; then, since *AB* and *LM* are parallel, they determine a plane, and *AB* cannot meet the plane *ST* in a point out of the line *LM*; but *AB* and *LM* being parallel, have no common point. Hence, neither *AB* nor *CD* can meet the common section of the two planes; that is, they have no common section, and are therefore parallel.

Cor. Since two lines which intersect each other, determine a plane, it follows from this proposition, that *the plane of two intersecting lines is parallel to the plane of two other intersecting lines respectively parallel to the first lines.*

THEOREM XVII.

When two intersecting lines are respectively parallel to two other intersecting lines lying in a different plane, the angles formed by the last two lines will be equal to those formed by the first two, each to each, and the planes of the angles will be parallel.



Let *QR* be the plane of the two lines *AB* and *CD*, which intersect each other at the point *E*, and *ST* the plane of the two lines *LM* and *NO*, respectively parallel to *AB* and *CD*; then will the $\angle BED = \angle MPO$, and $\angle BEC = \angle MPN$, etc., and the planes *QR* and *ST* will be parallel.

That the plane of one set of angles is parallel to that of the other, follows from the Corollary to Theorem 16; we have then only to show that the angles are equal, each to each.

Take any points, *B* and *D*, on the lines *AB* and *CD*, and draw *BD*. Lay off *PM*, equal to and in the same direction with *EB*, and *PO*, equal to and in the same direction with *ED*, and draw *MO*. Now, since the planes *QR* and *ST* are parallel, and *ED* is equal and parallel to *PO*, *EDOP* is a parallelogram, and *DO* is equal and parallel to *EP*. For the same reason, *BM* is equal and parallel to *EP*; therefore, *BDOM* is a parallelogram, and *MO* is equal and parallel to *BD*. Hence the $\triangle$'s, *EBD* and *PMO*, have the sides of the one equal to the sides of the other, each to each; they are therefore equal, and the $\angle MPO$ = the $\angle BED$. In the same manner it can be proved that $\angle BEC = \angle MPN$, etc.

Cor. 1. The plane of the parallels AB and LM is intersected by the plane of the parallels CD and NO, in the line EP. Now, EB and ED are the intersections of these two planes with the plane QR, and PM and PO are the intersections of the same planes with the parallel plane ST. It has just been proved that the $\angle BED = \angle MPO$. Hence, *if the diedral angle formed by two planes, be cut by two parallel planes, the intersections of the faces of the diedral angle with one of these planes will include an angle equal to that included by the intersections of the faces with the other plane.*

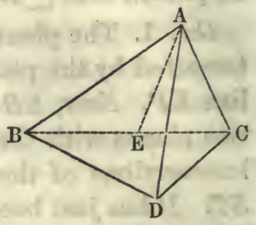
Cor. 2. *The opposite triangles formed by joining the corresponding extremities of three equal and parallel straight lines lying in different planes, will be equal and the planes of the triangles will be parallel.*

Let EP, BM, and DO, be three equal and parallel straight lines lying in different planes. By joining their corresponding extremities, we have the triangles EBD and PMO. Now, since EP and BM are equal and parallel, $EBMP$ is a parallelogram, and EB is equal and parallel to PM; in the same manner, we show that ED is equal and parallel to PO, and BD to MO; hence the triangles are equal, having the three sides of the one, respectively, equal to the three sides of the other. That their planes are parallel, follows from Cor., Theorem 16.

THEOREM XVIII.

Any one of the three plane angles bounding a triedral angle, is less than the sum of the other two.

Let A be the vertex of a solid angle, bounded by the three plane angles, BAC, BAD, and DAC; then will any one of these three angles be less than the sum of the other two. To establish this proposition, we have only to compare the greatest of the three angles with the sum of the other two.



Suppose, then, BAC to be the greatest angle, and draw in its plane the line AE, making the angle CAE equal to the angle CAD. On AE, take any point, E, and through it draw the line CEB. Take AD, equal to AE, and draw BD and DC.

Now, the two triangles, CAD and CAE, having two sides and the included angle of the one equal to the two sides and included angle of the other, each to each, are equal, and $CE = CD$; but in the triangle, BDC, $BC < BD + DC$. Taking EC from the first member of this inequality, and its equal, DC, from the second, we have, $BE < BD$. In the triangles, BAE and BAD, BA is common, and $AE = AD$ by construction; but the third side, BD, in the one, is greater than the third side, BE, in the other; hence, the angle BAD is greater than the angle BAE, (Th. 22, B. I); that is, $\angle BAE < \angle BAD$; adding the $\angle EAC$ to the first member of this inequality, and its equal, the $\angle DAC$, to the other, we have

$$\angle BAE + \angle EAC < \angle BAD + \angle DAC.$$

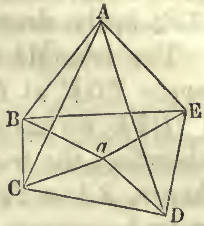
And, as the $\angle BAC$ is made up of the angles BAE and EAC, we have, as enunciated,

$$\angle BAC < \angle BAD + \angle DAC.$$

THEOREM XIX.

The sum of the plane angles forming any solid angle, is always less than four right angles.

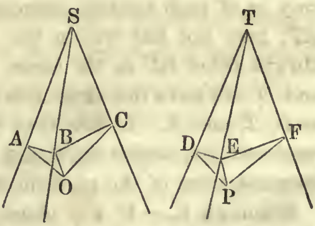
Let the planes which form the solid angle at A, be cut by another plane, which we may call the plane of the base, $BCDE$. Take any point, a, in this plane, and draw aB, aC, aD, aE, etc., thus making as many triangles on the plane of the base as there are triangular planes forming the solid angle *A*. Now, since the sum of the angles of every $\triangle$ is two right angles, the sum of all the angles of the $\triangle$'s which have their vertex in *A*, is equal to the sum of all angles of the $\triangle$'s which have their vertex in *a*. But, the angles $BCA + ACD$, are, together, greater than the angles $BCa + aCD$, or *BCD*, by the last proposition. That is, the sum of all the angles at the bases of the $\triangle$'s which have their vertex in *A*, is greater than the sum of all the angles at the bases of the $\triangle$'s which have their vertex in *a*. Therefore, the sum of all the angles at *a* is greater than the sum of all the angles at *A*; but the sum of all the angles at *a* is equal to four right angles; therefore, the sum of all the angles at *A* is less than four right angles.



THEOREM XX.

If two solid angles are formed by three plane angles respectively equal to each other, the planes which contain the equal angles will be equally inclined to each other.

Let the $\angle ASC =$ the $\angle DTF$, the $\angle ASB =$ the $\angle DTE$, and the $\angle BSC =$ the $\angle ETF$; then will the inclination of the planes, *ASC*, *ASB*, be equal to that of the planes, *DTF*, *DTE*.



Having taken *SB* at pleasure, draw *BO* perpendicular to the plane *ASC*; from the point *O*, at which that perpendicular meets the plane, draw *OA* and *OC*, perpendicular to *SA* and *SC*; draw *AB* and *BC*; next take $TE = SB$, and draw *EP* perpendicular to the plane *DTF*; from the point *P*, draw *PD* and *PF*, perpendicular to *TD* and *TF*; lastly, draw *DE* and *EF*.

The triangle *SAB*, is right-angled at *A*, and the triangle *TDE*, at *D*, (Th. 5); and since the $\angle ASB =$ the $\angle DTE$, we have $\angle SBA = \angle TED$; likewise, $SB = TE$; therefore, the triangle *SAB* is equal to the triangle *TDE*; hence, $SA = TD$, and $AB = DE$. In like manner it may be shown that $SC = TF$, and $BC = EF$. That granted, the quadrilateral *SAOC* is equal to the quadrilateral *TDPF*; for, place the angle *ASC* upon its equal, *DTF*, and because $SA = TD$, and $SC = TF$, the point *A* will fall on *D*, and the point *C* on *F*; and, at the same time, *AO*, which is perpendicular to *SA*, will fall on *PD*, which is perpendicular to *TD*, and, in like manner, *OC* on *PF*; wherefore, the point *O* will fall on the point *P*, and *AO* will be equal to *DP*. But the triangles, *AOB*, *DPE*, are right angled at *O* and *P*; the hypotenuse $AB = DE$, and the side $AO = DP$; hence, those triangles are equal, (Cor, Th. 39, B. I), and $\angle AOB = \angle PDE$. The angle *OAB* is the inclination of the two planes, *ASB*, *ASC*; the angle *PDE* is that of the two planes, *DTE*, *DTF*; consequently, those two inclinations are equal to each other.

Hence the theorem.

Scholium 1.—The angles which form the solid angles at *S* and *T*, may be of such relative magnitudes, that the perpendiculars, *BO* and *EP*, may not fall within the bases, *ASC* and *DTF*; but they will always either fall on the bases, or on the planes of the bases produced, and *O* will have the same relative situation to *A*, *S*, and *C*, as *P* has to *D*, *T*, and *F*. In case that *O* and *P* fall on the planes of the bases produced, the angles *BCO* and *EFP*, would be obtuse angles; but the demonstration of the problem would not be varied in the least.

Scholium 2.—If the plane angles bounding one of the triedral angles be equal to those of the other, each to each, and also be similarly arranged about the triedral angles, these solid angles will be absolutely equal. For it was shown, in the course of the above demonstration, that the quadrilaterals, *SAOC* and *TDPF*, were equal; and on being applied, the point *O* falls on the point *P*; and since the triangles *AOB* and *DPE* are equal, the perpendiculars *OB* and *PE* are also equal. Now, because the plane angles are like arranged about the triedral angles, these perpendiculars lie in the same direction; hence the point B will fall on the point E, and the solid angles will exactly coincide.

Scholium 3.—When the planes of the equal angles are not like disposed about the triedral angles, it would not be possible to make these triedral angles coincide; and still it would be true that the planes of the equal angles are equally inclined to each other. Hence, these triedral angles have the plane and diedral angles of the one, equal to the plane and diedral angles of the other, each to each, without having of themselves that absolute equality which admits of superposition. Magnitudes which are thus equal in all their component parts, but will not coincide, when applied the one to the other, are said to be *symmetrically equal.* Thus, two triedral angles, bounded by plane angles equal each to each, but not like placed, are *symmetrical triedral angles.*

BOOK VII.

SOLID GEOMETRY.

DEFINITIONS.

1. A **Polyedron** is a solid, or volume, bounded on all sides by planes. The bounding planes are called the *faces* of the polyedron, and their intersections are its *edges*.

2. A **Prism** is a polyedron, having two of its faces, called *bases*, equal polygons, whose planes and homologous sides are parallel. The other, or *lateral faces*, are parallelograms, and constitute the *convex surface* of the prism.

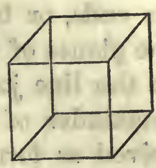
The bases of a prism are distinguished by the terms, *upper* and *lower;* and the *altitude* of the prism is the perpendicular distance between its bases.

Prisms are denominated *triangular*, *quadrangular*, *pentangular*, *etc.*, according as their bases are *triangles*, *quadrilaterals*, *pentagons*, *etc.*

3. A **Right Prism** is one in which the planes of the lateral faces are perpendicular to the planes of the bases.

4. A **Parallelopipedon** is a prism whose bases are parallelograms.

5. A **Rectangular Parallelopipedon** is a right parallelopipedon, with rectangular bases.

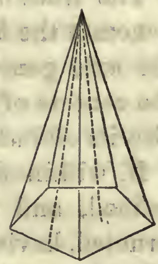


6. A **Cube** or **Hexaedron** is a rectangular parallelopipedon, whose faces are all equal squares.

7. A **Diagonal of a Polyedron** is a straight line joining the vertices of two solid angles not adjacent.

8. **Similar Polyedrons** are those which are bounded by the same number of similar polygons like placed, and whose solid angles are equal each to each.

Similar parts, whether faces, edges, diagonals, or angles, similarly placed in similar polyedrons, are termed *homologous.*



9. A **Pyramid** is a polyedron, having for one of its faces, called the *base,* any polygon whatever, and for its other faces triangles having a common vertex, the sides opposite which, in the several triangles, being the sides of the base of the pyramid.

10. The **Vertex** of a pyramid is the common vertex of the triangular faces.

11. The **Altitude** of a pyramid is the perpendicular distance from its vertex to the plane of its base.

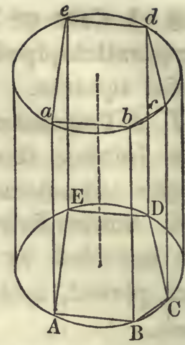
12. A **Right Pyramid** is one whose base is a regular polygon, and whose vertex is in the perpendicular to the base at its center. This perpendicular is called the *axis* of the pyramid.

13. The **Slant Height** of a right pyramid is the perpendicular distance from the vertex to one of the sides of the base.

14. The **Frustum of a Pyramid** is a portion of the pyramid included between its base and a section made by a plane parallel to the base.

Pyramids, like prisms, are named from the forms of their bases.

15. A **Cylinder** is a body, having for its ends, or bases, two equal circles, the planes of which are perpendicular to the line joining their centers; the remainder of its surface may be conceived as formed by the motion of a line, which constantly touches the circumferences of the bases, while it remains parallel to the line which joins their centers.



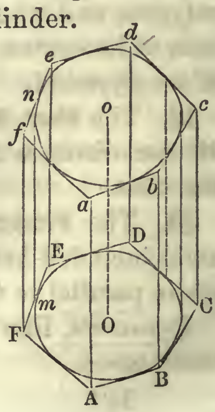
We may otherwise define the cylinder as a body generated by the revolution of a rectangle about one of its sides as an immovable axis.

The sides of the rectangle perpendicular to the axis generate the *bases* of the cylinder; and the side opposite the axis generates its *convex surface*. The line joining the centers of the bases of the cylinder is its *axis*, and is also its *altitude*.

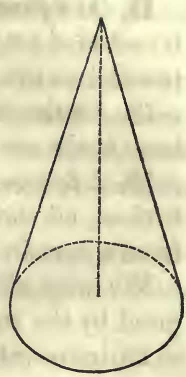
If, within the base of a cylinder, any polygon be inscribed, and on it, as a base, a right prism be constructed, having for its altitude that of the cylinder, such prism is said to be *inscribed in the cylinder*, and the cylinder is said to *circumscribe the prism*.

Thus, in the last figure, *ABCDEc* is an inscribed prism, and it is plain that all its lateral edges are contained in the convex surface of the cylinder.

If, about the base of a cylinder, any polygon be circumscribed, and on it, as a base, a right prism be constructed, having for its altitude that of the cylinder, such prism is said to be *circumscribed about the cylinder*, and the cylinder is said to be *inscribed in the prism*.



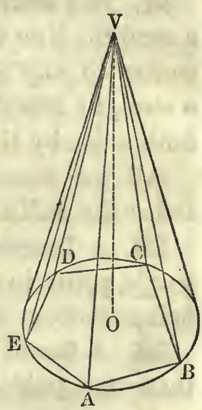
Thus, *ABCDEFc* is a circumscribed prism; and it is plain that the line, *mn*, which joins the points of tangency of the sides, *EF* and *ef*, with the circumferences of the bases of the cylinder, is common to the convex surfaces of the cylinder and prism.



16. A **Cone** is a body bounded by a circle and the surface generated by the motion of a straight line, which constantly passes through a point in the perpendicular to the plane of the circle at its center, and the different points in its circumference.

The cone may be otherwise defined as a body generated by the revolution of a right-angled triangle about one of its sides as an immovable axis. The other side of the triangle will generate the *base* of the cone, while the hypotenuse generates the *convex surface.*

The side about which the generating triangle revolves is the *axis* of the cone, and is at the same time its *altitude.*



If, within the base of the cone, any polygon be inscribed, and on it, as a base, a pyramid be constructed, having for its vertex that of the cone, such pyramid is said to be *inscribed in the cone,* and the cone is said to *circumscribe the pyramid.*

Thus, in the accompanying figure, *V—ABCDE*, is an inscribed pyramid, and it is plain that all its lateral edges are contained in the convex surface of the cone.

If, about the base of a cone, any polygon be circumscribed, and on it, as a base, a pyramid be constructed, having for its vertex that of the cone, such pyramid is said to be *circumscribed about the cone,* and the cone is said to be *inscribed in the pyramid.*

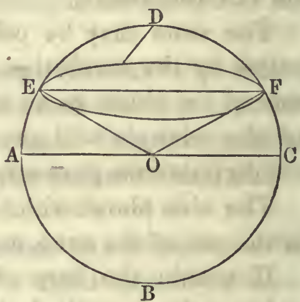
17. The **Frustum of a Cone** is the portion of the cone that is included between its base and a section made by a plane parallel to the base.

18. **Similar Cylinders,** and also **Similar Cones,** are such as have their axes proportional to the radii of their bases.

19. A **Sphere** is a body bounded by one uniformly-curved surface, all the points of which are at the same distance from a certain point within, called the *center*.

We may otherwise define the sphere as a body generated by the revolution of a semicircle about its diameter as an immovable axis.

20. A **Spherical Sector** is that portion of a sphere which is included between the surfaces of two cones having their vertices at the center of the sphere. Or, it is that portion of the sphere which is generated by a sector of the generating semicircle.



21. The **Radius of a Sphere** is a straight line drawn from the center to any point in the surface; and the *diameter* is a straight line drawn through the center, and limited on both sides by the surface.

All the diameters of a sphere are equal, each being twice the radius.

22. A **Tangent Plane** to a sphere is one which has a single point in the surface of the sphere, all the others being without it.

23. A **Secant Plane** to a sphere is one which has more than one point in the surface of the sphere, and lies partly within and partly without it.

Assuming, what will presently be proved, that the intersection of a sphere by a plane is a circle,

24. A **Small Circle** of a sphere is one whose plane does not pass through its center; and

25. A **Great Circle** of a sphere is one whose plane passes through the center of the sphere.

26. A **Zone** of a sphere is the portion of its surface included between the circumferences of any two of its parallel circles, called the *bases* of the zone. When the plane of one of these circles becomes tangent to the sphere, the zone has a single base.

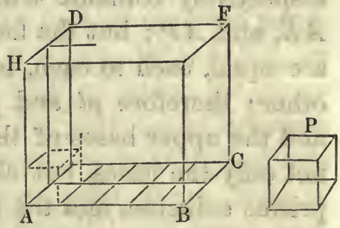
27. A **Spherical Segment** is a portion of the volume of a sphere included between any two of its parallel circles, called the *bases* of the segment.

The altitude of a zone, or of a segment, of a sphere, is the perpendicular distance between the planes of its bases.

28. The area of a surface is measured by the product of its *length* and *breadth*, and these dimensions are always conceived to be exactly at right angles to each other.

29. In a similar manner, solids are measured by the product of their *length*, *breadth*, and *height*, when all their dimensions are at right angles to each other.

The product of the length and breadth of a solid, is the measure of the *surface* of its base.



Let P, in the annexed figure, represent the measuring unit, and AF the rectangular solid to be measured.

A side of P is one unit in length, one in breadth, and one in height; one inch, one foot, one yard, or any other unit that may be taken.

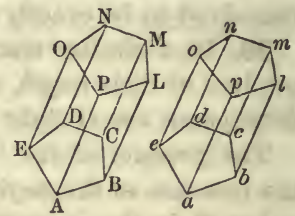
Then, $\quad 1 \times 1 \times 1 = 1$, the *unit cube*.

Now, if the base of the solid, AC, is, as here represented, 5 units in length and 2 in breadth, it is obvious that $(5 \times 2 = 10)$, 10 units, each equal to P, can be placed on the base of AC, and no more; and as each of these units will occupy a unit of altitude, therefore, 2 *units* of altitude will contain 20 solid units, 3 units of altitude, 30 solid units, and so on; or, in general terms, *the number of square units in the base multiplied by the linear units in perpendicular altitude, will give the solid units in any rectangular solid.*

THEOREM I.

If the three plane faces bounding a solid angle of one prism be equal to the three plane faces bounding a solid angle of another, each to each, and similarly disposed, the prisms will be equal.

Suppose *A* and *a* to be the vertices of two solid angles, bounded by equal and similarly placed faces; then will the prisms, *ABCDE—N* and *abcde—n*, be equal.



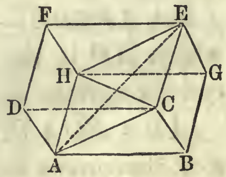
For, if we place the base, *abcde*, upon its equal, the base *ABCDE*, they will coincide; and since the solid angles, whose vertices are *A* and *a*, are equal, the lines *ab*, *ae*, and *ap*, respectively coincide with *AB*, *AE*, and *AP*; but the faces, *al* and *ao*, of the one prism, are equal, each to each, to the faces, *AL* and *AO*, of the other; therefore *pl* and *po* coincide with *PL* and *PO*, and the upper bases of the prisms also coincide: hence, not only the bases, but all the lateral faces of the two prisms coincide, and the prisms are equal.

Cor. If the two prisms are right, and have equal bases and altitudes, they are equal. For, in this case, the rectangular faces, *al* and *ao*, of the one, are respectively equal to the rectangular faces, *AL* and *AO*, of the other; and hence the three faces bounding a triedral angle in the one, are equal and like placed, to the faces bounding a triedral angle in the other.

THEOREM II.

The opposite faces of any parallelopipedon are equal, and their planes are parallel.

Let *ABCD—E* be any parallelopipedon; then will its opposite faces be equal, and their planes will be parallel.



The bases *ABCD* and *FEGH* are equal, and their planes are parallel, by definitions 2 and 4 of this Book; it remains for us, therefore, only to show that any two of the opposite lateral faces are equal and parallel.

Since all the faces of the parallelopipedon are parallelograms, *AB* is equal and parallel to *DC*, and *AH* is also equal and parallel to *DF*; hence the angles *HAB* and *FDC* are equal, and their planes are parallel, (Th. 17, B. VI), and the two parallelograms, *HABG* and *FDCE*, having two adjacent sides and the included angle of the one equal to the two adjacent sides and included angle of the other, are equal.

Cor. 1. Hence, of the six faces of the parallelopipedon, any two lying opposite may be taken as the bases.

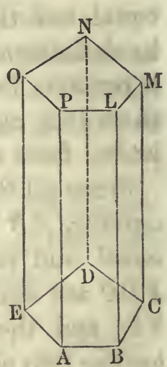
Cor. 2. The four diagonals of a parallelopipedon mutually bisect each other. For, if we draw *AC* and *HE*, we shall form the parallelogram *ACEH*, of which the diagonals are *AE* and *HC*, and these diagonals are at the same time diagonals of the parallelopipedon; but the diagonals of a parallelogram mutually bisect each other. Now, if the diagonal *FB* be drawn, it and *HC* will bisect each other, since they are diagonals of the parallelogram *FHBC*. In like manner we can show that if *DG* be drawn, it will be bisected by *AE*. Hence, the four diagonals have a common point within the parallelopipedon.

Scholium.—It is seen at once that the six faces of a parallelopipedon intersect each other in twelve edges, four of which are equal to *HA*, four to *AB*, and four to *AD*. Now, we may conceive the parallelopipedon to be bounded by the planes determined by the three lines

AH, *AB*, and *AD*, and the three planes passed through the extremities, *H*, *B*, and *D*, of these lines, parallel to the first three planes.

THEOREM III.

The convex surface of a right prism is measured by the perimeter of its base multiplied by its altitude.



Let $ABCDE-N$ be a right prism, of which AP is the altitude; then will its convex surface be measured by

$$(AB+BC+CD+DE+EA)\times AP.$$

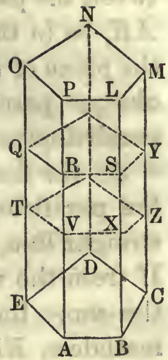
For, its convex surface is made up of the rectangles AL, BM, CN, etc., and each rectangle is measured by the product of its base by its altitude; but the altitude of each rectangle is equal to AP, the altitude of the prism; hence the convex surface of the prism is measured by the product of the sum of the bases of the rectangles, or the perimeter of the base of the prism, by the common altitude, AP.

Cor. Right prisms will have equivalent convex surfaces, when the products of the perimeters of their bases by their altitudes are respectively equal; and, generally, their convex surfaces will be to each other as the products of the perimeters of their bases by their altitudes. Hence, when their altitudes are equal, their surfaces will be as the perimeters of their bases; and when the perimeters of their bases are equal, their convex surfaces will be as their altitudes.

THEOREM IV.

The two sections of a prism made by parallel planes between its bases are equal polygons.

Let the prism $ABCDE-N$ be cut between its bases by two parallel planes, making the sections QRS, etc.,



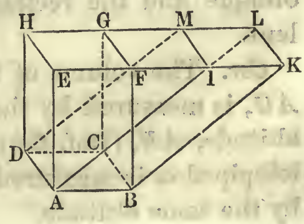
and TVX, etc.; then will these sections be equal polygons.

For, since the secant planes are parallel, their intersections, QR and TV, by the plane of the face $EAPO$ are parallel, (Th. 10, B. VI); and being included between the parallel lines, AP and EO, they are also equal. In the same manner we may prove that RS is equal and parallel to VX, and so on for the intersections of the secant planes by the other faces of the prism. Hence, these polygonal sections have the sides of the one equal to the sides of the other, each to each. The angles QRS and TVX are equal, because their sides are parallel and lie in the same direction; and in like manner we prove $\angle RSY = \angle VXZ$, and so on for the other corresponding angles of the polygons. Therefore, these polygons are both mutually equilateral and mutually equiangular, and consequently are equal.

Cor. A section of a prism made by a plane parallel to the base of the prism, is a polygon equal to the base.

THEOREM V.

Two parallelopipedons, the one rectangular and the other oblique, will be equal in volume when, having the same base and altitude, two opposite lateral faces of the one are in the planes of the corresponding lateral faces of the other.



Designating the parallelopipedons by their opposite diagonal letters, let AG be the rectangular, and AL the oblique, parallelopipedon, having the same base, AC, and of the same altitude, namely, the perpendicular distance between the parallel planes, *AC* and *EL*. Also let the face, *AK*, be in the plane of the face, *AF*, and the face, *DL*, in the plane of the face, *DG*. We are now to prove that the oblique parallelopipedon is equivalent to the rectangular parallelopipedon.

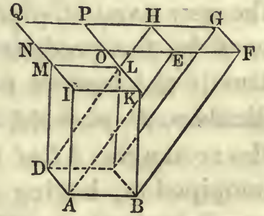
As the faces, *AF* and *AK*, are in the same plane, and the parallelopipedons have the same altitude, *EFK* is a straight line, and *EF* = *IK*, because each is equal to *AB*. If from the whole line, *EK*, we take *EF*, and then from the same line we take *IK* = *EF*, we shall have the remainders, *EI* and *FK*, equal; and since *AE* and *BF* are parallel, ∟*AEI* = ∟*BFK*; hence the △'s, *AEI* and *BFK*, are equal. Since *HE* and *MI* are both parallel to *DA*, they are parallel to each other, and *EIMH* is a parallelogram; for like reasons, *FKLG* is a parallelogram, and these parallelograms are equal, because two adjacent sides and the included angle of the one are equal to two adjacent sides and the included angle of the other. The parallelograms, *DE* and *CF*, being the opposite faces of the parallelopipedon, *AG*, are equal. Hence, the three plane faces bounding the triedral angle, *E*, of the triangular prism, *EAI—H*, are equal, each to each, and like placed, to the three plane faces bounding the triedral, *F*, of the triangular prism, *FBK—G*, and these prisms are therefore equal, (Th. 1). Now, if from the whole solid, *EABK—H*, we take the prism, *EAI—H*, there will remain the parallelopipedon, *AL*; and, if from the same solid, we take the prism, *FBK—G*, there will remain the rectangular parallelopipedon, *AG*. Therefore, the oblique and the rectangular parallelopidon are equivalent.

Cor. The volume of the rectangular parallelopipedon, *AG*, is measured by the base, *ABCD*, multiplied by the altitude, *AE*, (Def. 29); consequently, the oblique parallelopipedon is measured by the product of the same base by the same altitude.

SCHOLIUM.—If neither of the parallelopipedons is rectangular, but they still have the same base and the same altitude, and two opposite lateral faces of the one are in the planes of the corresponding lateral faces of the other, by precisely the same reasoning we could prove the parallelopipedons equivalent. Hence, in general, *any two parallelopipedons will be equal in volume when, having the same base and altitude, two opposite lateral faces of the one are in the planes of the corresponding lateral faces of the other.*

THEOREM VI.

Two parallelopipedons having equal bases and equal altitudes, are equivalent.



Let AG and AL be two parallelopipedons, having a common lower base, and their upper bases in the same plane, HF. Then will these parallelopipedons be equivalent.

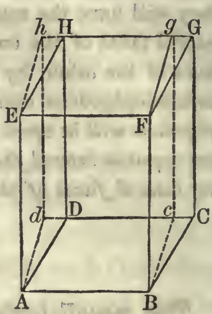
Since their upper bases are in the same plane, the lines IM, KL, EF, and HG, will intersect, when produced, and form the quadrilateral, $NOPQ$, and this quadrilateral will be a parallelogram, (Cor. 2, Th. 6, B. VI), equal to the common lower base of the two parallelopipedons. Now, if a third parallelopipedon be constructed, having BD for its lower base, and OQ for its upper base, it will be equivalent to the parallelopipedon AG, and also to the parallelopipedon AL, (Th. 5, Scholium); hence, the two given parallelopipedons, being each equivalent to the third parallelopipedon, are equivalent to each other.

Hence, *two parallelopipedons having equal bases, etc.*

THEOREM VII.

The volume of any parallelopipedon is measured by the product of its base and altitude, or the product of its three dimensions.

Let $ABCD-G$ be any parallelopipedon; then will its volume be expressed by the product of the area of its base and altitude.



If the parallelopipedon is oblique, we may construct on its base a right parallelopipedon, by erecting perpendiculars at the points A, B, C, and D, and making them each equal to the altitude of the given parallelopipedon; and the right parallelopipedon, thus constructed, will be equivalent to the given parallelopipedon, (Th. 6). Now, if the base, $ABCD$, is a rectangle, the new parallelopipedon will be rectangular, and measured by the product of its base and altitude, (Def. 16). But if the base is not rectangular, let fall the perpendiculars, Bc and Ad, on CD and CD produced, and take the rectangle $ABcd$ for the base of a rectangular parallelopipedon, having for its altitude that of the given parallelopipedon. We may now regard the rectangular face, $ABFE$, as the common base of the two parallelopipedons, Ag and AG; and, as they have a common base, and equal altitude, they are equivalent. Thus we have reduced the oblique parallelopipedon, first to an equivalent right parallelopipedon on the same base, and then the right to an equivalent rectangular parallelopipedon on an equivalent base, all having the same altitude. But the rectangular parallelopipedon, Ag, is measured by product of its base, $ABcd$, and its altitude; hence, the given and equivalent oblique parallelopipedon is measured by the product of its equivalent base and equal altitude.

Hence, *the volume of any parallelopipedon, etc.*

Cor. Since a parallelopipedon is measured by the product of its base by its altitude, it follows that *parallelopipedons of equivalent bases, and equal altitudes, are equivalent, or equal in volume.*

THEOREM VIII.

Parallelopipedons on the same, or equivalent bases, are to each other as their altitudes; and parallelopipedons having equal altitudes, are to each other as their bases.

Let P and p represent two parallelopipedons, whose bases are denoted by B and b, and altitudes by A and a, respectively.

Now, $P = B \times A$, and $p = b \times a$, (Th. 7).

But magnitudes are proportional to their numerical measures; that is,

$$P : p :: B \times A : b \times a.$$

If the bases of the parallelopipedons are equivalent, we have $B = b$; and if the altitudes are equal, we have $A = a$. Introducing these suppositions, in succession, in the above proportion, we get

$$P : p :: A : a,$$

and $$P : p :: B : b.$$

Hence the theorem; *Parallelopipedons on the same, etc.*

THEOREM IX.

Similar parallelopipedons are to each other as the cubes of their like dimensions.

Let P and p represent any two similar parallelopipedons, the altitude of the first being denoted by h, and the length and breadth of its base by l and n, respectively; and let h', l', and n', in order, denote the corresponding dimensions of the second.

Then we are to prove that

$$P : p :: n^3 : n'^3 :: l^3 : l'^3 :: h^3 : h'^3.$$

We have

$$P = lnh, \text{ and } p = l'n'h' \text{ (Th. 7)};$$

and by dividing the first of these equations by the second, member by member, we get

$$\frac{P}{p} = \frac{lnh}{l'n'h'};$$

which, reduced to a proportion, gives

$$P : p :: lnh : l'n'h'.$$

But, by reason of the similarity of the parallelopipedons, we have the proportions

$$l : l' :: n : n'$$
$$h : h' :: n : n';$$

we have also the identical proportion,

$$n : n' :: n : n'.$$

By the multiplication of these proportions, term by term, we get, (Th. 11, B. II),

$$lnh : l'n'h' :: n^3 : n'^3.$$

That is, $P : p :: n^3 : n'^3$.

By treating in the same manner the three proportions,

$$l : l' :: h : h'$$
$$n : n' :: h : h'$$
$$h : h' :: h : h',$$

we should obtain the proportion

$$P : p :: h^3 : h'^3;$$

and, by a like process, the three proportions,

$$h : h' :: l : l'$$
$$n : n' :: l : l'$$
$$l : l' :: l : l',$$

will give us the proportion

$$P : p :: l^3 : l'^3.$$

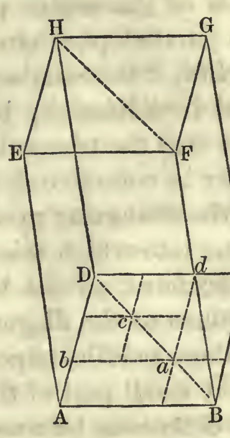
Hence the theorem; *similar parallelopipedons are to each other, etc.*

THEOREM X.

The two triangular prisms into which any parallelopipedon is divided, by a plane passing through its opposite diagonal edges, are equivalent.

Let $ABCD-F$ be a parallelopipedon, and through the diagonal edges, BF and DH, pass the plane BH, dividing the parallelopipedon into the two triangular prisms,

$ABD-E$ and $BCD-G$; then we are to prove that these prisms are equivalent. Let us divide the diagonal, BD, in which the secant plane intersects the base of the parallelopipedon, into three equal parts, a and c being the points of division. In the base, $ABCD$, construct the complementary parallelograms, aC and aA, and in the parallelogram, $badD$, construct the complementary parallelograms, cd and cb, and conceive these, together with the parallelograms, Ba, ac, cD, to be the bases of smaller parallelopipedons, having their lateral faces parallel to the lateral faces of, and their altitude equal to the altitude of, the given parallelopipedon, AG.



Now it is evident that the triangular prism, $BCD-G$, is composed of the parallelopipedons on the bases, aC and cd, and the triangular prisms, on the side of the secant plane with this prism, into which this plane divides the parallelopipedons on the bases, Ba, ac, and cD. The triangular prism, $ABD-E$, is also composed of the parallelopipedons on the bases, Aa and bc, together with the triangular prisms on the side of the secant plane with this prism, into which this plane divides the parallelopipedons on the bases, Ba, ac, and cD.

But the parallelograms, aC and aA, being complementary, are equivalent, (Th. 31, B. I); and for the same reason the parallelograms, cd and cb, are equivalent; and since parallelopipedons on equivalent bases and of equal altitudes, are equivalent, (Cor., Th. 7), we have the sum of parallelopipedons on bases aC and cd, equivalent to the sum of parallelopipedons on the bases, aA and cb. Hence, the triangular prisms, $ABD-E$ and $BCD-G$, differ in volume only by the difference which may exist between the sums of the triangular prisms on the two sides of the secant plane into which this plane divides the parallelopipedons on the bases, *Ba*, *ac*, and *cd*.

Now, if the number of equal parts into which the diagonal is divided, be indefinitely multiplied, it still holds true that the triangular prisms, *ABD—E* and *BCD—G*, differ in volume only by the difference between the sums of the triangular prisms on the two sides of the secant plane into which this plane divides the parallelopipedons constructed on the bases whose diagonals are the equal portions of the diagonal, *BD*. But in this case the sum of these parallelopipedons themselves becomes an indefinitely small part of the whole parallelopipedon, *AG*, and the difference between the parts of an indefinitely small quantity must itself be indefinitely small, or less than any assignable quantity. Therefore, the triangular prisms, *ABD—E* and *BCD—G*, differ in volume by less than any assignable volume, and are consequently equivalent.

Hence the theorem; *the two triangular prisms into which, etc.*

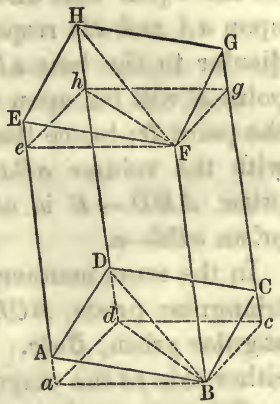
Cor. 1. Any triangular prism, as *ABD—E*, is one half the parallelopipedon having the same triedral angle, *A*, and the same edges, *AB*, *AD*, and *AE*.

Cor. 2. Since the volume of a parallelopipedon is measured by the product of its base and altitude, and the triangular prisms into which it is divided by the diagonal plane, have bases equivalent to one half the base of the parallelopipedon, and the same altitude, it follows that, *the volume of a triangular prism is measured by the product of its base and altitude.*

The above demonstration is less direct, but is thought to be more simple, than that generally found in authors, and which is here given as a

Second Demonstration.

Let $ABCD-F$ be a parallelopipedon, divided by the diagonal plane, BH, passing through the edges, BF and DH; then we are to prove that the triangular prisms, $ABD-E$ and $BCD-G$, thus formed, are equivalent.



Through the points B and F, pass planes perpendicular to the edge, BF, and produce the lateral faces of the parallelopipedon to intersect the plane through B; then the sections $Bcda$ and $Fghe$ are equal parallelograms. For, since the cutting planes are both perpendicular to BF, they are parallel, (Th. 10, B. VI); and because the opposite faces of a parallelopipedon are in parallel planes, (Th. 2), and the intersections of two parallel planes by a third plane are parallel, (Th. 9, B. VI), the sections, $Bcda$ and $Fghe$, are equal parallelograms, and may be taken as the bases of the right parallelopipedon, $Bcda-h$. But the diagonal plane divides the right parallelopipedon into the two equal triangular prisms, $aBd-e$ and $Bcd-g$, (Th. 1). We will now compare the right prism with the oblique triangular prism on the same side of the diagonal plane.

The volume $ABD-e$ is common to the two prisms, $ABD-E$ and $aBd-e$; and the volume $eFh-E$, which, added to this common part, forms the oblique triangular prism, is equal to the volume $aBd-A$, which, added to the common part, forms the right triangular prism. For, since $ABFE$ and $aBFe$ are parallelograms, $AE = ae$, and taking away the common part Ae, we have $aA = eE$; and since $BFHD$ and $BFhd$ are parallelograms, we have $DH = dh$; and from these equals taking away the common part Dh, we have $dD = hH$. Now, if the volume $eFh-H$

be applied to the volume *aBd—D*, the base *eFh* falling on the equal base *aBd*, the edges *eE* and *hH* will fall upon *aA* and *dD* respectively, because they are perpendicular to the base *aBd*, (Cor. 2, Th. 3, B. VI), and the point *E* will fall upon the point *A*, and the point *H* upon the point *D*; hence the volume *eFh—H* exactly coincides with the volume *aBd—D*, and the oblique triangular prism *ABD—E* is equivalent to the right triangular prism *aBd—e*.

In the same manner, it may be proved that the oblique triangular prism, *BCDG*, is equivalent to the right triangular prism, *Bcdg*. The oblique triangular prism on either side of the diagonal plane is, therefore, equivalent to the corresponding right triangular prism; and, as the two right triangular prisms are equal, the oblique triangular prisms are equivalent.

Hence the theorem; *the two triangular prisms, etc.*

THEOREM XI.

The volume of any prism whatever is measured by the product of the area of its base and altitude.

For, by passing planes through the homologous diagonals of the upper and lower bases of the prism, it will be divided into a number of triangular prisms, each of which is measured by the product of the area of its base and altitude. Now, as these triangular prisms all have, for their common altitude, the altitude of the given prism, when we add the measures of the triangular prism, to get that of the whole prism, we shall have, for this measure, the common altitude multiplied by the sum of the areas of the bases of the triangular prisms: that is, the product of the area of the polygonal base and the altitude of the prism.

Hence the theorem; *the volume of any prism, etc.*

Cor. If *A* denote the area of the base, and *H* the altitude of a prism, its volume will be expressed by $A \times H$. Calling this volume V, we have

$$V = A \times H.$$

Denoting by A', H', and V', in order, the area of the base, altitude, and volume of another prism, we have

$$V' = A' \times H'.$$

Dividing the first of these equations by the second, member by member, we have

$$\frac{V}{V'} = \frac{A \times H}{A' \times H'},$$

which gives the proportion,

$$V : V' :: A \times H : A' \times H'.$$

If the bases are equivalent, this proportion becomes

$$V : V' :: H : H';$$

and if the altitudes are equal, it reduces to

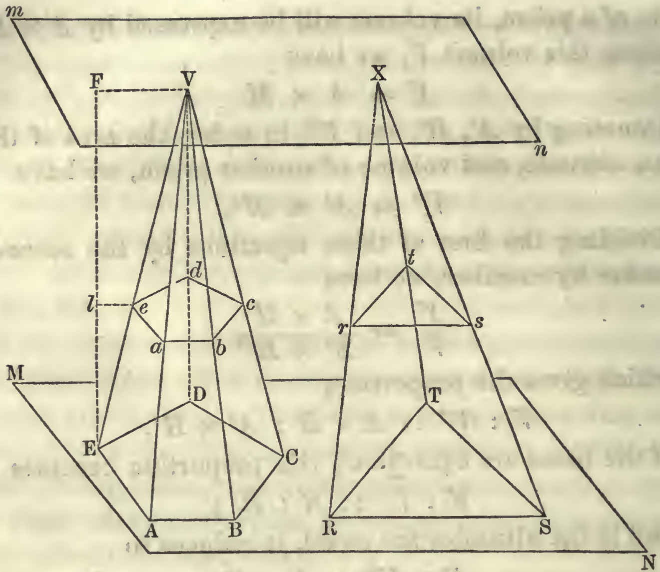
$$V : V' :: A : A'.$$

Hence, *prisms of equivalent bases are to each other as their altitudes; and prisms of equal altitudes are to each other as their bases.*

THEOREM XII.

A plane passed through a pyramid parallel to its base, divides its edges and altitude proportionally, and makes a section, which is a polygon similar to the base.

Let $ABCDE—V$ be any pyramid, whose base is in the plane, MN, and vertex in the parallel plane, mn; and let a plane be passed through the pyramid, parallel to its base, cutting its edges at the points, a, b, c, d, e, and the altitude, EF, at the point l. By joining the points, a, b, c, etc., we have the polygon formed by the intersection of the plane and the sides of the pyramid. Now, we are to prove that the edges, VA, VB, etc., and the altitude, FE, are divided proportionally at the points, a, b, etc., and l; and that the polygon, a, b, c, d, e, is similar to the base of the pyramid.



Since the cutting plane is parallel to the base of the pyramid, *ab* is parallel to *AB*, (Th. 9, B. VI); for the same reason, *bc* is parallel to *BC*, *cd* to *CD*, etc. Now, in the triangle *VAB*, because *ab* is parallel to the base *AB*, we have, (Th. 17, B. II), the proportion,

$$VA : Va :: VB : Vb.$$

In like manner, it may be shown that

$$VB : Vb :: VC : Vc,$$

and so on for the other lateral edges of the pyramid. *F* being the point in which the perpendicular from *E* pierces the plane *mn*, and *l* the point in which the parallel secant plane cuts the perpendicular, if we join the points *F* and *V*, and also the points *l* and *e* by straight lines, we have in the triangle *EFV*, the line *le* parallel to the base *FV*; hence the proportion

$$VE : Ve :: FE : Fl.$$

Therefore, the plane passed through the pyramid parallel to its base, divides the altitude into parts which have to each other the same ratio as the parts into which it divides the edges.

Again, since ab is parallel to AB, and bc to BC, the angle abc is equal to the angle ABC, (Th. 8, B. I); in the same manner we may show that each angle in the polygon, $abcde$, is equal to the corresponding angle in the polygon, $ABCDE$; therefore these polygons are mutually equiangular. But, because the triangles VBA and Vba are similar, their homologous sides give the proportion

$$Vb : VB :: ab : AB;$$

and because the triangles Vbc and VBC are similar, we also have the proportion

$$Vb : VB :: bc : BC.$$

Since the first couplet in these two proportions is the same, the second couplets are proportional, and give

$$ab : AB :: bc : BC.$$

By a like process, we can prove that

$$bc : BC :: cd : CD,$$

and that $\quad cd : CD :: de : DE,$

and so on, for the other homologous sides of the two polygons.

Hence, the two polygons are not only mutually equiangular, but the sides about the equal angles taken in the same order are proportional, and the polygons are therefore similar, (Def. 16, B. II).

Hence the theorem; *a plane passed through a pyramid, etc.*

Cor. 1. Since the areas of similar polygons are to each other as the squares of their homologous sides, (Th. 22, B. II), we have

$$\text{area } abcde : \text{area } ABCDE : \overline{ab}^2 : \overline{AB}^2.$$

But, $\quad ab : AB :: Va : VA :: Fl : FE;$

hence, $\quad \overline{ab}^2 : \overline{AB}^2 :: \overline{Fl}^2 : \overline{FE}^2:$

therefore, $\quad \text{area } abcde : \text{area } ABCDE : \overline{Fl}^2 : \overline{FE}^2.$

That is, *the area of the section made by a plane passing through a pyramid parallel to its base, is to the area of the base, as the perpendicular distance from the vertex of the pyramid to the section, is to the altitude of the pyramid.*

Cor. 2. Let $V—ABCDE$ and $X—RST$ be two pyramids, having their bases in the plane MN, and their vertices in the parallel plane mn; and suppose a plane to be passed through the two pyramids parallel to the common plane of their bases, making in the one the section $abcde$, and in the other the section rst.

Now, area $ABCDE$: area $abcde$ :: $\overline{AB}^2 : \overline{ab}^2$, (Th. 22, B. II),

and " RST : " rst :: $\overline{RS}^2 : \overline{rs}^2$.

But, $AB : ab :: VB : Vb$,

and $RS : rs :: XR : Xr$.

Because the plane which makes the sections is parallel to the planes MN and mn, we have, (Th. 11, B. VI),

$$VB : Vb :: XR : Xr;$$

therefore, (Cor. 2, Th. 6, B. II), $AB : ab :: RS : rs$.

By squaring, $\overline{AB}^2 : \overline{ab}^2 : \overline{RS}^2 : \overline{rs}^2$;

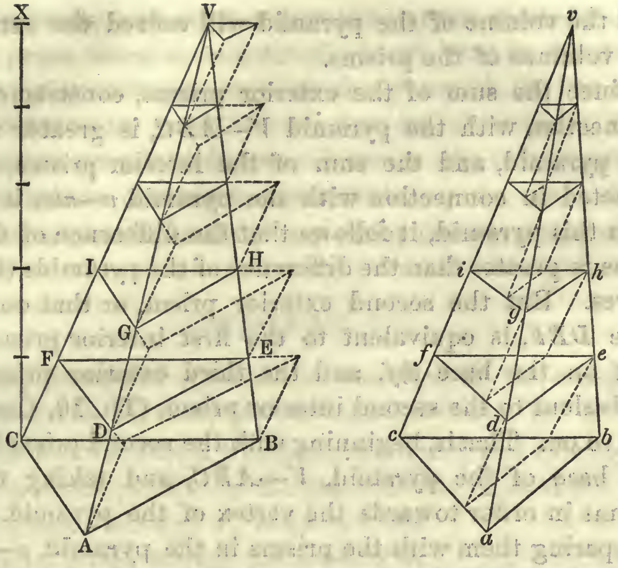
hence, area $ABCDE$: area $abcde$ :: area RST : area rst.

That is, *if two pyramids having equal altitudes, and their bases in the same plane, be cut by a plane parallel to the common plane of their bases, the areas of the sections will be proportional to the areas of the bases; and if the bases are equivalent, the sections will also be equivalent.*

THEOREM XIII.

If two triangular pyramids have equivalent bases and equal altitudes, they are equal in volume.

Let $V—ABC$ and $v—abc$ be two triangular pyramids, having the equivalent bases, ABC and abc, and let the altitude of each be equal to CX; then will these two pyramids be equivalent.



Place the bases of the pyramids on the same plane, with their vertices in the same direction, and divide the altitude into any number of equal parts. Through the points of division pass planes parallel to the plane of the bases; the corresponding sections made in the pyramids by these planes are equivalent, (Th. 12, Cor. 2); that is, the triangle *DEF* is equivalent to the triangle *def*, the triangle *GHI* to the triangle *ghi*, etc.

Now, let triangular prisms be constructed on the triangles *ABC*, *DEF*, etc., of the pyramid *V—ABC*, these prisms having their lateral edges parallel to the edge, *VC*, of the pyramid, and the equal parts of the altitude, *CX*, for their altitudes. Portions of these prisms will be exterior to the pyramid *V—ABC*, and the sum of their volumes will exceed the volume of the pyramid.

On the bases *def*, *ghi*, etc., in the other pyramid, construct interior prisms, as represented in the figure, their lateral edges being parallel to *vc*, and their altitudes also the equal parts of the altitude, *CX*. Portions of the pyramid, *v—abc*, will be exterior to these prisms, and the volume of the pyramid will exceed the sum of the volumes of the prisms.

Since the sum of the exterior prisms, constructed in connection with the pyramid *V—ABC*, is greater than the pyramid, and the sum of the interior prisms, constructed in connection with the pyramid *v—abc*, is less than this pyramid, it follows that the difference of these sums is greater than the difference of the pyramids themselves. But the second exterior prism, or that on the base *DEF*, is equivalent to the first interior prism, or that on the base *def*, and the third exterior prism is equivalent to the second interior prism, (Th. 10, Cor. 2), and so on. That is, beginning with the second prism from the base of the pyramid, *V—ABC*, and taking these prisms in order towards the vertex of the pyramid, and comparing them with the prisms in the pyramid, *v—abc*, beginning with the lowest, and taking them in order toward the vertex of this pyramid, we find that to each exterior prism of the pyramid, *V—ABC*, exclusive of the first or lowest, there is a corresponding equivalent interior prism in the pyramid, *v—abc*.

Hence the prism, *ABCDEF*, is the difference between the sum of the prisms constructed in connection with the pyramid, *V—ABC*, and the sum of the interior prisms constructed in the pyramid, *v—abc*. But the first sum being a volume greater than the pyramid, *V—ABC*, and the second sum a volume less than the pyramid, *v—abc*, it follows that the volumes of the pyramids differ by less than the prism, *ABCDEF*.

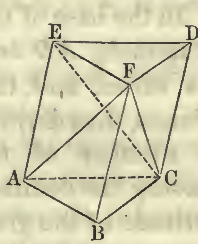
Now, however great the number of equal parts into which the altitude, *CX*, be divided, and the corresponding number of prisms constructed in connection with each pyramid, it would still be true that the difference between the volumes of the pyramids would be less than the volume of the lowest prism of the pyramid *V—ABC*; but when we make the number of equal parts into which the altitude is divided indefinitely great, the volume of this prism becomes indefinitely small: that is, the difference between the volumes of the pyramids is less than an indefinitely small volume; or, in other words, there is no assignable difference between the two pyramids, and they are, therefore, equivalent.

Hence the theorem; *if two triangular pyramids, etc.*

THEOREM XIV.

Any triangular pyramid is one third of the triangular prism having the same base and equal altitude.

Let $F—ABC$ be a triangular pyramid, and through F pass a plane parallel to the plane of the base, ABC. In this plane, through F, construct the triangle, FDE, having its sides, FD, DE, and EF, parallel and equal to BC, CA, and AB, respectively. The triangle, FDE, may be taken as the upper base of a triangular prism of which the lower base is ABC.



Now, this triangular prism is composed of the given triangular pyramid, $F—ABC$, and of the quadrangular pyramid, $F—ACDE$. This last pyramid may be divided by a plane through the three points, C, E, and F, into the two triangular pyramids, $F—DEC$ and $F—ACE$. But the pyramid, $F—DEC$, may be regarded as having the triangle, EFD, equal to the triangle, ABC, for its base, and the point, C, for its vertex. The two pyramids, $F—ABC$ and $C—DEF$, have equal bases and equal altitudes; they are therefore equivalent, (Th. 13). Again, the two pyramids, $F—DEC$ and $F—ACE$, have a common vertex, and equivalent bases in the same plane, and they are also equivalent. Therefore, the triangular prism, $ABCDEF$, is composed of three equivalent triangular pyramids, one of which is the given triangular pyramid, *F—ABC*.

Hence the theorem; *any triangular pyramid is one third of the triangular prism, etc.*

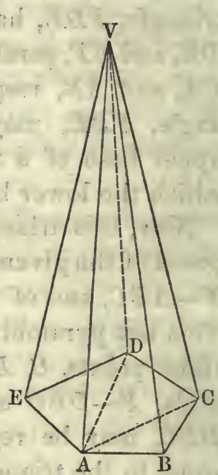
Cor. The volume of the triangular prism being measured by the product of its base and altitude, *the volume of a triangular pyramid is measured by one third of the product of its base and altitude.*

THEOREM XV.

The volume of any pyramid whatever is measured by one third of the product of its base and altitude.

Let *V—ABCDE* be any pyramid; then will its volume be measured by one third of the product of its base and altitude.

In the base of the pyramid, draw the diagonals, *AD* and *AC*, and through its vertex and these diagonals, pass planes, thus dividing the pyramid into a number of triangular pyramids having the common vertex *V*, and the altitude of the given pyramid for their common altitude.



Now, each of these triangular pyramids is measured by one third of the product of its base and altitude, (Cor., Th. 14), and their sum, which constitutes the polygonal pyramid, is therefore measured by one third of the product of the sum of the triangular bases and the common altitude; but the sum of the triangular bases constitutes the polygonal base, *ABCDE*.

Hence the theorem; *the volume of any pyramid whatever, etc.*

Cor. 1. Denote, by *B*, *H*, and *V*, respectively, the base, altitude, and volume of one pyramid, and by *B′*, *H′*, and

V', the base, altitude, and volume of another; then we shall have

$$V = \tfrac{1}{3}B \times H,$$

and

$$V' = \tfrac{1}{3}B' \times H'.$$

Dividing the first of these equations by the second, member by member, we have

$$\frac{V}{V'} = \frac{B \times H}{B' \times H'},$$

which, in the form of a proportion, gives

$$V : V' :: B \times H : B' \times H'.$$

From this proportion we deduce the following consequences:

1st. *Pyramids are to each other as the products of their bases and altitudes.*

2d. *Pyramids having equivalent bases are to each other as their altitudes.*

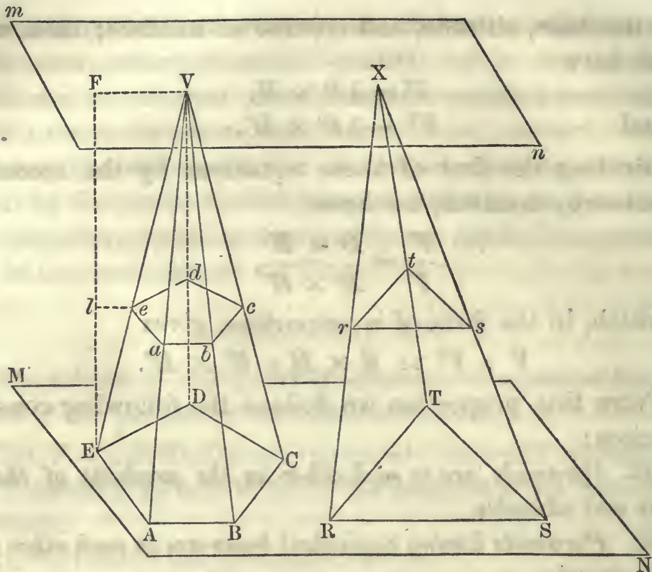
3d. *Pyramids having equal altitudes are to each other as their bases.*

Cor. 2. Since a prism is measured by the product of its base and altitude, and a pyramid by one third of the product of its base and altitude, we conclude that *any pyramid is one third of a prism having an equivalent base and equal altitude.*

THEOREM XVI.

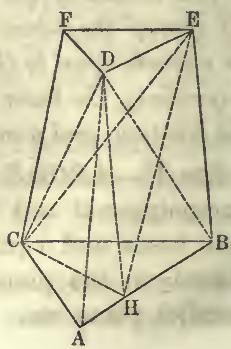
The volume of the frustum of a pyramid is equivalent to the sum of the volumes of three pyramids, each of which has an altitude equal to that of the frustum, and whose bases are, respectively, the lower base of the frustum, the upper base of the frustum, and a mean proportional between these bases.

Let V—$ABCDE$ and X—RST be two pyramids, the one polygonal and the other triangular, having equivalent bases and equal altitudes; and let their bases be placed on the plane MN, their vertices falling on the parallel plane mn. Pass through the pyramids a plane



parallel to the common plane of their bases, cutting out the sections *abcde* and *rst*; these sections are equivalent, (Th. 12, Cor. 2), and the pyramids, *V—abcde* and *X—rst*, are equivalent, (Th. 13). Now, since the pyramids, *V—ABCDE* and *X—RST*, are equivalent, if from the first we take the pyramid, *V—abcde*, and from the second, the pyramid, *X—rst*, the remainders, or the frusta, *ABCDE—a* and *RST—r*, will be equivalent.

If, then, we prove the theorem in the case of the frustum of a triangular pyramid, it will be proved for the frustum of any pyramid whatever.



Let *ABC—D* be the frustum of a triangular pyramid. Through the points *D*, *B*, and *C*, pass a plane, and through the points *D*, *C*, and *E*, pass another, thus dividing the frustum into three triangular pyramids, viz., *D—ABC*, *C—DEF*, and *D—BEC*.

Now, the first of these has, for its

base, the lower base of the frustum, and for its altitude the altitude of the frustum, since its vertex is in the upper base; the second has, for its base, the upper base of the frustum, and for its altitude the altitude of the frustum, since its vertex is in the lower base. Hence, these are two of the three pyramids required by the enunciation of the theorem; and we have now only to prove that the third is equivalent to one having, for its base, a mean proportional between the bases of the frustum, and an altitude equal to that of the frustum.

In the face $ABED$, draw HD parallel to BE, and draw HE and HC. The two pyramids, $D—BEC$ and $H—BEC$, are equivalent, since they have a common base and equal altitudes, their vertices being in the line DH, which is parallel to the plane of their common base, (Th. 7, B. VI). We may, therefore, substitute the pyramid, $H—BEC$, for the pyramid, $D—BEC$. But the triangle, BCH, may be taken as the base, and E as the vertex of this new pyramid; hence, it has the required altitude, and we must now prove that it has the required base.

The triangles, ABC and HBC, have a common vertex, and their bases in the same line; hence, (Th. 16, B. II),

$$\triangle ABC : \triangle HBC :: AB : HB :: AB : DE. \quad (1)$$

In the triangles, DEF and HBC, $\angle E = \angle B$, and $DE = HB$; hence, if DEF be applied to HBC, $\angle E$ falling on $\angle B$, and the side DE on HB, the point D will fall on H, and the triangles, in this position, will have a common vertex, H, and their bases in the same line; hence,

$$\triangle HBC : \triangle DEF :: BC : EF. \quad (2)$$

But, because the triangles, ABC and DEF, are similar, we have

$$AB : DE :: BC : EF. \quad (3)$$

From proportions (1), (2), and (3), we have, (Th. 6, B. II),

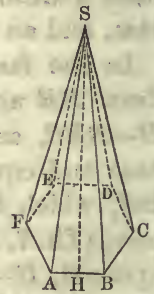
$$\triangle ABC : \triangle HBC :: \triangle HBC : \triangle DEF;$$

that is, the base, HBC, is a mean proportional between the lower and upper bases of the frustum.

Hence the theorem; *the volume of the frustum of a pyramid, etc.*

THEOREM XVII.

The convex surface of any right pyramid is measured by the perimeter of its base, multiplied by one half its slant height.



Let S—$ABCDEF$ be a right pyramid, of which SH is the slant height; then will its convex surface have, for its measure,

$$\tfrac{1}{2}SH(AB+BC+CD+DE+EF+FA).$$

Since the base is a regular polygon, and the perpendicular, drawn to its plane from S, passes through its center, the edges, SA, SB, SC, etc., are equal, (Cor. Th. 4, B. VI), and the triangles SAB, SBC, etc., are equal, and isosceles, each having an altitude equal to SH.

Now, $AB \times \tfrac{1}{2}SH$ measures the area of the triangle, SAB; and $BC \times \tfrac{1}{2}SH$ measures the area of the triangle, SBC; and so on, for the other triangular faces of the pyramid. By the addition of these different measures, we get

$$\tfrac{1}{2}SH(AB+BC+CD+DE+EF+FA),$$

as the measure of the total convex surface of the pyramid.

Hence the theorem; *the convex surface of any right pyramid, etc.*

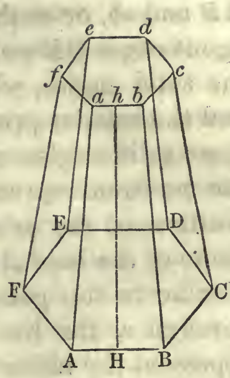
THEOREM XVIII.

The convex surface of the frustum of any right pyramid is measured by the sum of the perimeters of the two bases, multiplied by one half the slant height of the frustum.

Let $ABCDEF$—d be the frustum of a right pyramid; then will its convex surface be measured by

$$\tfrac{1}{2}Hh(AB+BC+CD+DE+EF+FA+ab+bc+cd+de+ef+fa).$$

For, the upper base, $abcdef$, of the frustum is a section of a pyramid by a plane parallel to the lower base, (Def. 14), and is, therefore, similar to the lower base, (Th. 12). But the lower base is a regular polygon, (Def. 12); hence, the upper base is also a regular polygon, of the same name; and as ab and AB are intersections of a face of the pyramid by two parallel planes, they are parallel. For the same reason, bc is parallel to BC, cd to CD, etc., and the lateral faces of the frustum are all equal trapezoids, each having an altitude equal to Hh, the slant height of the frustum.



The trapezoid $ABba$ has, for its measure, $\frac{1}{2}Hh(AB+ab)$, (Th. 34, Book I); the trapezoid $BCcb$ has, for its measure, $\frac{1}{2}Hh(BC+bc)$, and so on, for the other lateral faces of the frustum.

Adding all these measures, we find, for their sum, which is the whole convex surface of the frustum,

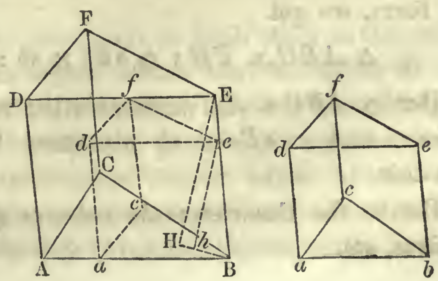
$$\tfrac{1}{2}Hh(AB+BC+CD+DE+EF+FA+ab+bc+cd+de+ef+fa).$$

Hence the theorem; *the convex surface of the frustum, etc.*

THEOREM XIX.

The volumes of similar triangular prisms are to each other as the cubes constructed on their homologous edges.

Let ABC—F and abc—f be two similar triangular prisms; then will their volumes be to each other as the cubes, whose edges are the homologous edges



AB and *ab*, or as the cubes, whose edges are the homologous edges *BE* and *be*, etc. Since the prisms are similar, the solid angles, whose vertices are *B* and *b*, are equal; and the smaller prism, when so applied to the larger that these solid angles coincide, will take, within the larger, the position represented by the dotted lines. In this position of the prisms, draw *EH* perpendicular to the plane of the base *ABC*, and join the foot of the perpendicular to the point *B*, and in the triangle *BEH* draw, through *e*, the line *eh*, parallel to *EH*; then will *EH* represent the altitude of the larger prism, and *eh* that of the smaller.

Now, as the bases *ABC* and *aBc*, are homologous faces, they are similar, and we have, (Th. 20, Book II),

$$\triangle ABC : \triangle aBc :: \overline{AB}^2 : \overline{aB}^2 \qquad (1)$$

But the △'s *BEH* and *Beh* are equiangular, and therefore similar, and their homologous sides give the proportion

$$BE : Be :: EH : eh \qquad (2)$$

and from the homologous sides of the similar faces, *ABED* and *aBed*, we also have

$$BE : Be :: AB : aB \qquad (3)$$

Proportions (2) and (3), having an antecedent and consequent the same in both, we have, (Th. 6, B. II),

$$EH : eh :: AB : aB \qquad (4)$$

By the multiplication of proportions (1) and (4), term by term, we get

$$\triangle ABC \times EH : \triangle aBc \times eh :: \overline{AB}^3 : \overline{aB}^3$$

But $\triangle ABC \times EH$ measures the volume of the larger prism, and $\triangle aBc \times eh$ measures the volume of the smaller.

Hence the theorem; *the volumes of similar triangular prisms, etc.*

Cor. 1. *The volumes of two similar prisms having any bases whatever, are to each other as the cubes constructed on their homologous edges.*

For, if planes be passed through any one of the lateral edges, and the several diagonal edges, of one of these prisms, this prism will be divided into a number of smaller triangular prisms. Taking the homologous edge of the other prism, and passing planes through it and the several diagonal edges, this prism will also be divided into the same number of smaller triangular prisms, similar to those of the first, each to each, and similarly placed.

Now, the similar smaller prisms, being triangular, are to each other as the cubes of their homologous edges; and being like parts of the larger prisms, it follows that the larger prisms are to each other as the cubes of the homologous edges of any two similar smaller prisms. But the homologous edges of the similar smaller prisms are to each other as the homologous edges of the given prisms; hence we conclude that the given prisms are to each other as the cubes of their homologous edges.

Cor. 2. *The volumes of two similar pyramids having any bases whatever, are to each other as the cubes constructed on their homologous edges.*

For, since the pyramids are similar, their bases are similar polygons; and upon them, as bases, two similar prisms may be constructed, having for their altitudes, the altitudes of their respective pyramids, and their lateral edges parallel to any two homologous lateral edges of the pyramids.

Now, these similar prisms are to each other as the cubes of their homologous edges, which may be taken as the homologous sides of their bases, or as their lateral edges, which were taken equal and parallel to any two arbitrarily assumed homologous lateral edges of the two pyramids; hence the pyramids are to each other as the cubes constructed on any two homologous edges.

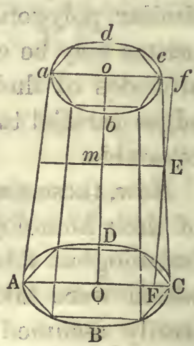
Cor. 3. *The volumes of any two similar polyedrons are to each other as the cubes constructed on their homologous edges.*

For, by passing planes through the vertices of the homologous solid angles of such polyedrons, they may both be divided into the same number of triangular pyramids, those of the one similar to those of the other, each to each, and similarly placed.

Now, any two of these similar triangular pyramids are to each other as the cubes of their homologous edges; and being like parts of their respective polyedrons, it follows that the polyedrons are to each other as the cubes of the homologous edges of any two of the similar triangular pyramids into which they may be divided. But the homologous edges of the similar triangular pyramids are to each other as the homologous edges of the polyedrons; hence the polyedrons are to each other as the cubes of their homologous edges.

THEOREM XX.

The convex surface of the frustum of a cone is measured by the product of the slant height and one half the sum of the circumferences of the bases of the frustum.



Let $ABCD—abcd$ be the frustum of a cone; then will its convex surface be measured by $Aa \times \frac{(\text{circ. } OC + \text{circ. } oc)}{2}$, in which the expression, circ. OC, denotes the circumference of the circle of which OC is the radius. Inscribe in the lower base of the frustum, a regular polygon having any number of sides, and in the upper base a similar polygon, having its sides parallel to those of the polygon in the lower base. These polygons may be taken as the bases of the frustum of a right pyramid inscribed in the frustum of the cone.

Now, however great the number of sides of the inscribed polygons, the convex surface of the frustum of the pyramid is measured by its slant height multiplied by one half the sum of the perimeters of its two bases, (Th. 18); but when we reach the limit, by making the number of sides of the polygon indefinitely great, the slant height, perimeters of the bases, and convex surface of the frustum of the pyramid become, severally, the slant height, circumferences of the bases, and convex surface of the frustum of the cone.

Hence the theorem; *the convex surface of the frustum, etc.*

Cor. 1. If we make $oc = OC$, and, consequently, circ. $oc =$ circ. OC, the frustum of the cone becomes a cylinder, and the half sum of the circumferences of the bases becomes the circumference of either base of the cylinder, and the slant height of the frustum, the altitude of the cylinder. Hence, *the convex surface of a cylinder is measured by the circumference of the base multiplied by the altitude of the cylinder.*

Cor. 2. If we make $oc = 0$, the frustum of the cone becomes a cone. Hence, *the convex surface of a cone is measured by the circumference of the base multiplied by one half the slant height of the cone.*

Cor. 3. If through E, the middle point of Cc, the line Ff be drawn parallel to Oo, and Em perpendicular to Oo, the line oc being produced, to meet Ff at f, we have, because the $\triangle$'s EFC and Efc are equal,

$$Em = \frac{OC + oc}{2}.$$

If we multiply both members of this equation by 2π, we have

$$2\pi . Em = \frac{2\pi . OC + 2\pi . oc}{2};$$

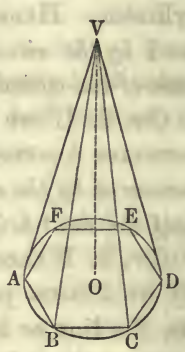
that is, circ. Em is equal to one half the sum of the circumferences of the two bases of the frustum. Hence, *the convex surface of the frustum of a cone is measured by the circumference of the section made by a plane half way between the two bases, and parallel to them, multiplied by the slant height of the frustum.*

Cor. 4. If the trapezoid, $OCco$, be revolved about Oo as an axis, the inclined side, Cc, will generate the convex surface of the frustum of a cone, of which the slant height is Cc, and the circumferences of the bases are circ. OC and circ. oc. Hence, *if a trapezoid, one of whose sides is perpendicular to the two parallel sides, be revolved about the perpendicular side as an axis, it will generate the frustum of a cone, the inclined side opposite the axis generating the convex surface, and the parallel sides the bases of the frustum.*

THEOREM XXI.

The volume of a cone is measured by the area of its base multiplied by one third of its altitude.

Let $V—ABC$, etc., be a cone; then will its volume be measured by area ABC, etc., multiplied by $\frac{1}{3}VO$.



Inscribe, in the base of the cone, any regular polygon, as $ABCDEF$, which may be taken as the base of a right pyramid, of which V is the vertex. The volume of this inscribed pyramid will have, for its measure, (Th. 15),

$$\text{polygon } ABCDEF \times \tfrac{1}{3}VO.$$

Now, however great the number of sides of the polygon inscribed in the base of the cone, it will still hold true that the pyramid of which it is the base, and whose vertex is V, will be measured by the area of the polygon, multiplied by one third of VO; but when we reach the limit, by making the number of sides indefinitely great, the polygon becomes the circle in which it is inscribed, and the pyramid becomes the cone.

Hence the theorem; *the volume of a cone, etc.*

Cor. 1. If R denote the radius of the base of a cone, and H its altitude, or axis, its volume will be expressed by

$$\tfrac{1}{3}H \times \pi R^2;$$

hence, if V and V' designate the volumes of two cones, of which R and R' are the radii of the bases, and H and H' the altitudes, we have

$$V : V' :: \tfrac{1}{3}H \times \pi R^2 : \tfrac{1}{3}H' \times \pi R'^2 :: H \times \pi R^2 : H' \times \pi R'^2.$$

From this proportion we conclude,

First. *That cones having equal altitudes are to each other as their bases.*

Second. *That cones having equal bases are to each other as their altitudes.*

Cor. 2. Retaining the notation above, we have

$$\frac{V'}{V} = \frac{H'}{H} \times \frac{R'^2}{R^2}; \quad (1)$$

and, if the two cones are similar,

$$H : H' :: R : R';$$

or, $$\frac{H'}{H} = \frac{R'}{R}; \text{ hence, } \frac{H^{2\prime}}{H^{2\prime}} = \frac{R'^2}{R^2}.$$

By substituting for the factors, in the second member of eq. (1), their values successively, and resolving into a proportion, we get

$$V : V' :: R^3 : R^{3\prime};$$

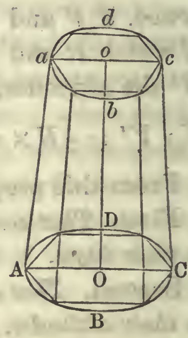
and $$V : V' :: H^{3\prime} : H'^3.$$

Hence, *similar cones are to each other as the cubes of the radii of their bases, and also as the cubes of their altitudes.*

Cor. 3. *A cone is equivalent to a pyramid having an equivalent base and an equal altitude.*

THEOREM XXII.

The volume of the frustum of a cone is equivalent to the sum of the volumes of three cones, having for their common altitude the altitude of the frustum, and for their several bases, the bases of the frustum and a mean proportional between them.



Let *ABCD—abcd* be the frustum of a cone; then will its volume be equivalent to the sum of the volumes, having *Oo* for their common altitude, and for their bases, the circles of which, *OC*, *oc*, and a mean proportional between *OC* and *oc*, are the respective radii.

Inscribe in the lower base of the frustum any regular polygon, and in the upper base a similar polygon, having its sides parallel to those of the first. These polygons may be taken as the bases of the frustum of a right pyramid inscribed in the frustum of the cone.

The volume of the frustum of the pyramid is equivalent to the sum of the volumes of three pyramids, having for their common altitude the altitude of the frustum, and for their several bases the bases of the frustum, and a mean proportional between them, (Th. 16).

Now, however great the number of sides of the polygons inscribed in the bases of the frustum of the cone, this measure for the volume of the frustum of the pyramid, of which they are the bases, still holds true; but when we reach the limit, by making the number of the sides of the polygon indefinitely great, the polygons become the circles, the frustum of the pyramid becomes the frustum of the cone, and the three partial pyramids, whose sum is equivalent to the frustum of the pyramid, become three partial cones, whose sum is equivalent to the frustum of the cone.

Hence the theorem; *the volume of the frustum of a cone, etc.*

Cor. 1. Let R denote the radius of the lower base, R' that of the upper base, and H the altitude of the frustum of a cone; then will its volume be measured, (Th. 21), by

$$\tfrac{1}{3}H \times \pi R^2 + \tfrac{1}{3}H \times \pi R'^2 + \tfrac{1}{3}H \times \pi R \times R',$$

since $\pi R \times R'$ expresses the area of a circle which is a mean proportional between the two circles, whose radii are R and R'.

Now, if the bases of the frustum become equal, or $R = R'$, the frustum becomes a cylinder, and each of the last two terms in the above expression for the volume of the frustum of a cone will be equal to the first; hence, the volume of a cylinder, of which H is the altitude, and R the radius of the base, is measured by $H \times \pi R^2$.

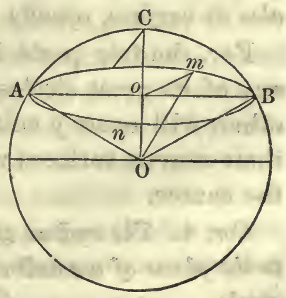
Therefore, *the volume of a cylinder is measured by the area of its base multiplied by its altitude.*

Cor. 2. By a process in all respects similar to that pursued in the case of cones, it may be shown that *similar cylinders are to each other as the cubes of the radii of their bases*, and also *as the cubes of their altitudes.*

Cor. 3. *A cylinder is equivalent to a prism having an equivalent base and an equal altitude.*

THEOREM XXIII.

If a plane be passed through a sphere, the section will be a circle.



Let O be the center of a sphere through which a plane is passed, making the section $AmBn$; then will this section be a circle.

From O let fall the perpendicular Oo upon the secant plane, and draw the radii OA, OB, and Om, to the different points in the intersection of the plane with the surface of the sphere. Now, the oblique lines OA, OB, Om, are all equal, being radii of the sphere; they therefore meet the plane at equal distances from the foot of the perpendicular Oo, (Cor., Th. 4, B. VI); hence oA, oB, om, etc., are equal: that is, all the points in the intersection of the plane with the surface of the sphere are equally distant from the point O. This intersection is therefore the circumference of a circle of which o is the center.

Hence the theorem; *if a plane be passed through a sphere, etc.*

Cor. 1. Since AB, the diameter of the section, is a chord of the sphere, it is less than the diameter of the sphere; except when the plane of the section passes through the center of the sphere, and then its diameter becomes the diameter of the sphere. Hence,

1. *All great circles of a sphere are equal.*

2. *Of two small circles of a sphere, that is the greater whose plane is the less distant from the center of the sphere.*

3. *All the small circles of a sphere whose planes are at the same distance from the center, are equal.*

Cor. 2. Since the planes of all great circles of a sphere pass through its center, the intersection of two great circles will be both a diameter of the sphere and a common diameter of the two circles. Hence, *two great circles of a sphere bisect each other.*

Cor. 3. *A great circle divides the volume of a sphere, and also its surface, equally.*

For, the two parts into which a sphere is divided by any of its great circles, on being applied the one to the other, will exactly coincide; otherwise all the points in their convex surfaces would not be equally distant from the center.

Cor. 4. *The radius of the sphere which is perpendicular to the plane of a small circle, passes through the center of the circle.*

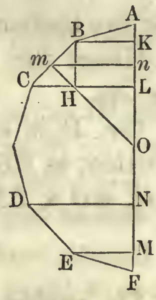
Cor. 5. *A plane passing through the extremity of a radius of a sphere, and perpendicular to it, is tangent to the sphere.*

For, if the plane intersect the sphere, the section is a circle, and all the lines drawn from the center of the sphere to points in the circumference are radii of the sphere, and are therefore equal to the radius which is perpendicular to the plane, which is impossible, (Cor. 1, Th. 3, B. VI). Hence the plane does not intersect the sphere, and has no point in its surface except the extremity of the perpendicular radius. The plane is therefore tangent to the sphere by Def 22.

THEOREM XXIV.

If the line drawn through the center and vertices of two opposite angles of a regular polygon of an even number of sides, be taken as an axis of revolution, the perimeter of either semi-polygon thus formed will generate a surface whose measure is the axis multiplied by the circumference of the inscribed circle.

Let $ABCDEF$ be a semi-polygon cut off from a regular polygon of an even number of sides by drawing the line AF through the center O, and the vertices A and F, of two opposite angles of the polygon; then will the surface generated by the perimeter of this semi-polygon revolving about AF as an axis, be measured by $AF \times$ circumference of the inscribed circle.



From m, the middle point, and the extremities B and C of the side BC, draw mn, BK, and CL, perpendicular to AF; join also m and O, and draw BH perpendicular to CL. The surface of the frustum of the cone generated by the trapezoid $BKLC$, has for its measure circ. $mn \times BC$, (Cor. 3, Th. 20). Since mO is perpendicular to BC, and mn to BH, the two $\triangle$'s, BCH and mnO, are similar, and their homologous sides give the proportion

$$mn : mO :: BH \ (= KL) : BC$$

and as circumferences are to each other as their radii, we have

$$\text{circ. } mn : \text{circ. } mO :: KL : BC$$

Hence, $\text{circ. } mn \times BC = \text{circ. } mO \times KL.$

But mO is the radius of the circle inscribed in tne polygon. Hence, the surface generated by BC during the revolution of the semi-polygon, is measured by the circumference of the inscribed circle multiplied by KL, the part of the axis included between the two perpendiculars let fall upon it from the extremities B and C. The surface generated by any other side of the semi-polygon will be measured, in like manner, by the circumference of the inscribed circle multiplied by the corresponding part of the axis.

By adding the measures of the surfaces generated by the several sides of the semi-polygon, we get

$$\text{Circ. } mO \times (AK + KL + LN + NM + MF)$$

for the measure of the whole surface.

Hence the theorem; *if the line drawn through the center, etc.*

Cor. It is evident that the surface generated by any portion, as CD and DE, of the perimeter, is measured by circ. $mO \times LM$.

THEOREM XXV.

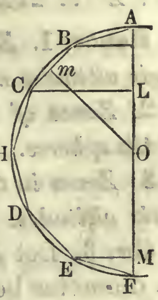
The surface of a sphere is measured by the circumference of one of its great circles multiplied by its diameter.

Let a sphere be generated by the revolution of the semi-circle, AHF, about its diameter, AF; then will the surface of the sphere be measured by

$$\text{Circ. } AO \times AF.$$

Inscribe in the semi-circle any regular semi-polygon, and let it be revolved, with the semi-circle, about the axis

AF; the surface generated by its perimeter will be measured by



Circ. $mO \times AF$, (Th. 24),

and this measure will hold true, however great the number of sides of the inscribed semi-polygon. But as the number of these sides is increased, the radius mO, of the inscribed semi-circle, increases and approaches equality with the radius, AO; and when we reach the limit, by making the number of sides indefinitely great, the radii and semi-circles become equal, and the surface generated by the perimeter of the inscribed semi-polygon becomes the surface of the sphere. Therefore, the surface of the sphere has, for its measure,

$$\text{Circ. } AO \times AF.$$

Hence the theorem; *the surface of a sphere is measured, etc.*

Cor. 1. *A zone of a sphere is measured by the circumference of a great circle of the sphere multiplied by the altitude of the zone.*

For, the surface generated by any portion, as CD and DE, of the perimeter of the inscribed semi-polygon has, for its measure, circ. $mO \times LM$, (Cor. Th. 24); and as the number of the sides of the semi-polygon increases, LM remains the same, the radius mO alone changing, and becoming, when we reach the limit, equal to AO; hence, the surface of the zone is expressed by

$$\text{Circ. } AO \times LM,$$

whether the zone have two bases, or but one.

Cor. 2. Let H and H' denote the altitudes of two zones of spheres, whose radii are R and R'; then these zones will be expressed by $2\pi R \times H$ and $2\pi R' \times H'$; and if the surfaces of the zones be denoted by Z and Z', we have

$Z : Z' :: 2\pi R \times H : 2\pi R' \times H' :: R \times H : R' \times H'$.

Hence, 1. *Zones in different spheres are to each other as their altitudes multiplied by the radii of the spheres.*

2. *Zones of equal altitudes are to each other as the radii of the spheres.*

3. *Zones in the same, or equal spheres, are to each other as their altitudes.*

Cor. 3. Let R denote the radius of a sphere; then will its diameter be expressed by $2R$, and the circumference of a great circle by $2\pi R$; hence its surface will be expressed by

$$2\pi R \times 2R = 4\pi R^2.$$

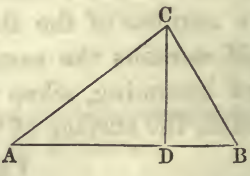
That is, *the surface of a sphere is equivalent to the area of four of its great circles.*

Cor. 4. *The surfaces of spheres are to each other as the squares of their radii.*

THEOREM XXVI.

If a triangle be revolved about either of its sides as an axis, the volume generated will be measured by one third of the product of the axis and the area of a circle, having for its radius the perpendicular let fall from the vertex of the opposite angle on the axis, or on the axis produced.

First. Let the triangle ABC, in which the perpendicular from C falls on the opposite side, AB, be revolved about AB as an axis; then will *Vol. $\triangle$ ABC have, for its measure, $\frac{1}{3}AB \times \pi\overline{CD}^2$.



The two $\triangle$'s into which $\triangle$ ABC is divided by the perpendicular DC, are right-angled, and during the revolution they will generate two cones, having for their

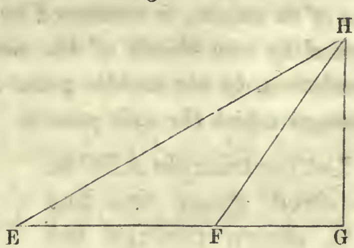
* Vol. $\triangle$ ABC, cone $\triangle$ ADC, are abbreviations for volume generated by $\triangle$ ABC, cone generated by $\triangle$ ADC; and surfaces of revolution generated by lines will hereafter be denoted by like abbreviations.

common base the circle, of which DC is the radius, and for their axes the parts DA and DB, into which AB is divided.

Now, *Cone △ ADC is measured by $\frac{1}{3}AD \times \pi\overline{DC}^2$, (Th. 21), and cone △ BDC, by $\frac{1}{3}BD \times \pi\overline{DC}^2$; but these two cones compose Vol. △ ABC; and by adding their measures, we have, for that of Vol. △ ABC,

$$\tfrac{1}{3}AD \times \pi\overline{DC}^2 + \tfrac{1}{3}BD \times \pi\overline{DC}^2 = \tfrac{1}{3}AB \times \pi\overline{DC}^2.$$

Second. Let the triangle EFG, in which the perpendicular from G falls on the opposite side EF produced, be revolved about EF as an axis; then will Vol. △ EFG



have, for its measure, $\frac{1}{3}EF \times \pi\overline{GH}^2$, GH being the perpendicular on EF produced. For, in this case it is apparent, that Vol. △ EFG is the difference between the cone △ EHG and the cone △ FHG. The first cone has, for its measure, $\frac{1}{3}EH \times \pi\overline{GH}^2$, and the second, for its measure, $\frac{1}{3}FH \times \pi\overline{GH}^2$; hence, by subtraction, we have

$$\text{Vol. } \triangle\, EFG = \tfrac{1}{3}EH \times \pi\overline{GH}^2 - \tfrac{1}{3}FH \times \pi\overline{GH}^2 = \tfrac{1}{3}EF \times \pi\overline{GH}^2.$$

Hence the theorem; *if a triangle be revolved about either of its sides, etc.*

SCHOLIUM.—If we take either of the above expressions for the measure of the volume generated by the revolution of a triangle about one of its sides, for example the last, and factor it otherwise, we have

$$\tfrac{1}{3}EF \times \pi\overline{GH}^2 = EF \times \tfrac{1}{2}GH \times \tfrac{1}{3}\pi \times 2GH = EF \times \tfrac{1}{2}GH \times \frac{2\pi \times GH}{3}.$$

Now, $EF \times \frac{1}{2}GH$ expresses the area of the triangle EFG; and $\frac{2\pi \times GH}{3}$, one third of the circumference described by the point G during the revolution.

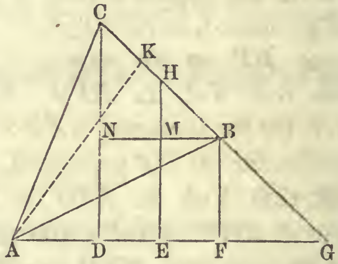
The expression, $\frac{1}{3}AB \times \pi\overline{DC}^2$, may be factored and interpreted in the

* See note on the preceding page.

same manner. Hence, we conclude that *the volume generated by the revolution of a triangle about either of its sides, is measured by the area of the triangle multiplied by one third of the circumference described in the revolution by the vertex of the angle opposite the axis.*

THEOREM XXVII.

The volume generated by the revolution of a triangle about any line lying in its plane, and passing through the vertex of one of its angles, is measured by the area of the triangle multiplied by two thirds of the circumference described, in the revolution, by the middle point of the side opposite the vertex through which the axis passes.



Let the triangle ABC be revolved about the line AG, drawn through the vertex A, and lying in the plane of the triangle, and let HE be the perpendicular let fall from H, the middle point of BC, upon the axis AG; then will Vol. $\triangle\ ABC$ have, for its measure, $\triangle\ ABC \times \frac{2}{3}$ circ. HE.

From the extremities of BC, let fall the perpendiculars BF and CD, on the axis; and from A draw AK perpendicular to BC, or BC produced, and produce CB, until it meets the axis in G.

Now, it is evident that Vol. $\triangle\ ABC$ is the difference between Vol. $\triangle\ AGC$ and Vol. $\triangle\ AGB$. But Vol. $\triangle\ AGC$ is expressed by $\triangle\ AGC \times \frac{1}{3}$ circ. CD; and Vol. $\triangle\ AGB$, by $\triangle\ AGB \times \frac{1}{3}$ circ. BF, (Scholium, Th. 26). Hence,

$$\text{Vol. } \triangle\ ABC = \triangle\ AGC \times \tfrac{1}{3} \text{ circ. } CD - \triangle\ AGB \times \tfrac{1}{3} \text{ circ. } BF.$$

Substituting for areas of $\triangle$'s, and for circumferences, their measures, we have

$$\text{Vol.}\ \triangle\ ABC = GC \times \tfrac{1}{2}AK \times \frac{2\pi.CD}{3} - GB \times \tfrac{1}{2}AK \times \frac{2\pi.BF}{3}$$

$$= GC \times \tfrac{1}{2}AK \times \frac{2\pi.CD}{3} - (GC - BC) \times \tfrac{1}{2}AK \times \frac{2\pi.BF}{3}$$

$$= GC \times \tfrac{1}{2}AK \times \frac{2\pi.CD}{3} - GC \times \tfrac{1}{2}AK \times \frac{2\pi.BF}{3} + BC \times \tfrac{1}{2}AK \times \frac{2\pi.BF}{3}$$

$$= GC \times \tfrac{1}{2}AK \times \frac{2\pi}{3}(CD - BF) + BC \times \tfrac{1}{2}AK \times \frac{2\pi.BF}{3}.$$

But BN being drawn parallel to AG, we have

$$CN = CD - BF;$$

hence, substituting this value for $CD - BF$, in the first term of the second member of the last equation, we have

$$\text{Vol.}\ \triangle\ ABC = GC \times \tfrac{1}{2}AK \times \frac{2\pi.CN}{3} + BC \times \tfrac{1}{2}AK \times \frac{2\pi.BF}{3}$$

$$= GC \times CN \times \tfrac{1}{2}AK \times \frac{2\pi}{3} + BC \times \tfrac{1}{2}AK \times \frac{2\pi.BF}{3},$$

by changing the order of factors in the first term of the second member. The homologous sides of the similar triangles, GCD and BCN, give the proportion

$$GC : CD :: BC : CN$$

whence, $\qquad GC \times CN = CD \times BC$

Substituting this value for $GC \times CN$, in the last equation above, and arranging the factors as before, it becomes

$$\text{Vol.}\ \triangle\ ABC = BC \times \tfrac{1}{2}AK \times \frac{2\pi.CD}{3} + BC \times \tfrac{1}{2}AK \times \frac{2\pi.BF}{3}$$

$$= BC \times \tfrac{1}{2}AK \times \frac{2\pi\ (CD + BF)}{3}.$$

But $CD + BF = 2HE$; hence

$$\text{Vol.}\ \triangle\ ABC = BC \times \tfrac{1}{2}AK \times \frac{4\pi.HE}{3} = BC \times \tfrac{1}{2}AK \times \tfrac{2}{3}.2\pi.HE;$$

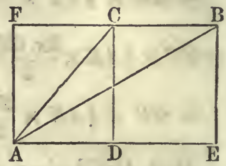
and since

$BC \times \frac{1}{2}AK = \triangle\ ABC$, and $\frac{2}{3} \times 2\pi.HE = \frac{2}{3}$ circ. HE,

this measure conforms to the enunciation.

It only remains for us to consider the case in which the axis is parallel to the base BC of the triangle. The preceding demonstration will not now apply, because it supposes BC, or BC produced, to intersect the axis.

Let the axis AE, be parallel to the base BC, of the $\triangle$ ABC. From B and C let fall on the axis the perpendiculars BE and CD.



Now it is plain that

$$\text{Vol. } \triangle\ ABC = \text{cylinder rectangle } BCDE + \text{cone } \triangle\ ADC - \text{cone } \triangle\ AEB.$$

Substituting in second member, for cylinder and cones, their measures, we have

$$\text{Vol. } \triangle ABC = DE \times \pi\overline{CD}^2 + \tfrac{1}{3}AD \times \pi\overline{CD}^2 - \tfrac{1}{3}AE \times \pi\overline{BE}^2$$
$$= \tfrac{2}{3}DE \times \pi\overline{CD}^2 + \tfrac{1}{3}DE \times \pi\overline{CD}^2 + \tfrac{1}{3}AD \times \pi\overline{CD}^2 - \tfrac{1}{3}AE \times \pi\overline{BE}^2.$$

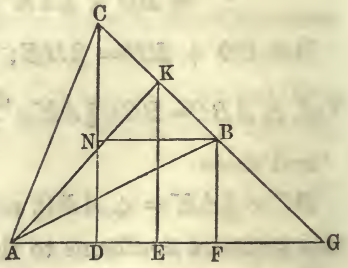
But $BE = CD$, and $\frac{1}{3}DE + \frac{1}{3}AD = \frac{1}{3}AD$. Reducing by these relations, we have

$$\text{Vol. } \triangle\ ABC = \tfrac{2}{3}DE \times \pi\overline{CD}^2 = \tfrac{1}{3}DE \times \tfrac{1}{2}CD \times 4\pi.CD$$
$$= DE \times \tfrac{1}{2}CD \times \tfrac{2}{3}.2\pi.CD = BC \times \tfrac{1}{2}CD \times \tfrac{2}{3}.2\pi.CD.$$

And, since $BC \times \frac{1}{2}CD$ expresses the area of the triangle ABC, and $\frac{2}{3}.2\pi.CD$, two thirds of the circumference described by any point of the base, this expression also conforms to the enunciation.

Hence the theorem; *the volume generated by the revolution, etc.*

Cor. If the generating triangle becomes isosceles, the perpendicular from A meets the base at its middle point. In this case, if we resume the expression



$$BC \times \tfrac{1}{2}AK \times \frac{4\pi.HE}{3},$$

it becomes

$$BC \times \tfrac{1}{2}AK \times KE \times \tfrac{4}{3}\pi.$$

But, since AK is perpendicular to BC, and KE to BN, the $\triangle$'s AKE and CBN are similar, and their homologous sides give the proportion

$$BC : BN :: AK : KE$$

whence, $\quad BC \times KE = BN \times AK$

Changing the order of factors in the last expression on the preceding page, and replacing $BC \times KE$ by its value, it becomes

$$\tfrac{1}{2}AK \times AK \times BN \times \tfrac{4}{3}\pi = \overline{AK}^2 \times BN \times \tfrac{2}{3}\pi$$

Hence,

$$\text{Vol. } \triangle ABC = \tfrac{2}{3}\pi \times \overline{AK}^2 \times BN. = \tfrac{2}{3}\pi \times \overline{AK}^2 \times DF$$

That is, *the volume generated by the revolution of an isosceles about any line drawn through its vertex and lying in the plane of the triangle, is measured by* $\tfrac{2}{3}\pi$ *times the square of the perpendicular of the triangle multiplied by the part of the axis included between the two perpendiculars let fall upon it from the extremities of the base of the triangle.*

SCHOLIUM.—If we resume the equation

$$\text{Vol. } \triangle ABC = BC \times \tfrac{1}{2}AK \times \frac{4\pi.HE}{3}$$

and change the order of the factors in the second member, it may be put under the form

$$\text{Vol. } \triangle ABC = BC \times 2\pi.HE \times \tfrac{1}{3}AK.$$

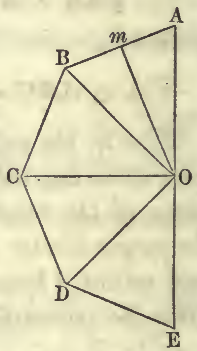
But during the revolution of the triangle, the side BC generates the surface of the frustum of a cone, which surface has for its measure

$BC \times 2\pi.HE$ (Th. 20, Cor. 3).

Hence, the above equation may be thus interpreted: *The volume generated by the revolution of a triangle about any line lying in its plane and passing through the vertex of one of its angles, is measured by the surface generated, during the revolution, by the side opposite the vertex through which the axis passes multiplied by one third of the perpendicular drawn from the vertex to that side.*

THEOREM XXVIII.

If the line drawn through the center and vertices of two opposite angles of a regular polygon, of an even number of sides, be taken as an axis of revolution, either semi-polygon thus formed will, during this revolution, generate a volume which has, for its measure, the surface generated by the perimeter of the semi-polygon multiplied by one third of its apothem.



Let $ABCDE$ be a regular semi-polygon, cut off from a regular polygon of an even number of sides, by drawing a line through the center, O, and the vertices, A and E, of two opposite angles of the polygon; then will the volume generated by the revolution of this semi-polygon about AE, as an axis, be measured by (Sur. AB + sur. BC + sur. CD + sur. DE) $\times \frac{1}{3}Om$, Om being the apothem of the polygon.

For, if from the center of O, the lines OB, OC, OD, be drawn to the vertices of the several angles of the semi-polygon, it will be divided into equal isosceles triangles, the perpendicular of each being the apothem of the polygon.

Now, the volume generated by $\triangle AOB$ has, for its measure,

Sur. $AB \times \frac{1}{3}Om$,
that by $\triangle BOC$, Sur. $BC \times \frac{1}{3}Om$,
" $\triangle COD$, Sur. $CD \times \frac{1}{3}Om$,
" $\triangle DOE$, Sur. $DE \times \frac{1}{3}Om$, (Scholium, Th. 27).

By the addition of the measures of these partial volumes, we find, for that of the whole volume,

Vol. semi-polygon $ABCDE =$ sur. perimeter $ABCDE \times \frac{1}{3}Om$,

and were the number of the sides of the semi-polygon increased or diminished, the reasoning would be in no wise changed.

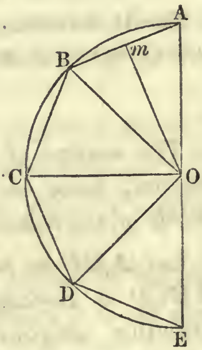
Hence the theorem; *if the line drawn through the center, etc.*

Scholium.—The volume generated by any portion of the semi-polygon, as that composed of the two isosceles $\triangle$'s BOC, COD, is measured by

$$\text{Sur. perimeter } BCD \times \tfrac{1}{3} Om.$$

THEOREM XXIX.

The volume of a sphere is measured by its surface multiplied by one third of its radius.



Let a sphere be generated by the revolution of the semicircle ACE, about its diameter, AE, as an axis; then will the volume of the sphere be measured by

$$\text{sur. semi-circ. } OA \times \tfrac{1}{3} OA.$$

For, inscribe in the semi-circle any regular semi-polygon, as $ABCDE$, and let it, together with the semi-circle, revolve about the axis AE. The semi-polygon will generate a volume which has, for its measure,

$$\text{Sur. perimeter } ABCDE \times \tfrac{1}{3} Om, \text{ (Th. 28)},$$

in which Om is the apothem of the polygon.

Now, however great the number of sides of the inscribed regular semi-polygon, this measure for the volume generated by it, will hold true; but when we reach the limit, by making the number of sides indefinitely great, the perimeter and apothem become, respectively, the semi-circumference and its radius, and the volume generated by the semi-polygon becomes that generated by the semi-circle, that is, the sphere. Therefore,

$$\text{Vol. sphere} = \text{sur. semi-circ. } OA \times \tfrac{1}{3} OA.$$

SCHOLIUM 1.—If we take any portion of the inscribed semi-polygon, as BOC, the volume generated by it is measured by sur. $BC \times \frac{1}{3} Om$, (Scholium, Th. 27); and when we pass to the limit, this volume becomes a sector, and sur. BC a zone of the sphere, which zone is the base of the sector. Hence, *the volume of a spherical sector is measured by the zone which forms its base multiplied by one third of the radius of the sphere.*

SCHOLIUM 2.—Let R denote the radius of a sphere; then will its diameter be represented by $2R$. Now, since the surface of a sphere is equivalent to the area of four of its great circles, and the area of a great circle is expressed by πR^2, we have

$$\text{Vol. sphere} = 4\pi R^2 \times \tfrac{1}{3}R = \tfrac{4}{3}\pi R^3.$$

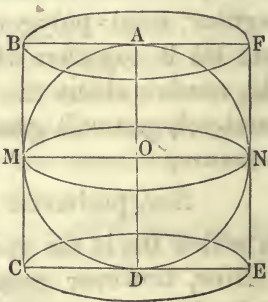
And since $R^3 = \frac{1}{8}(2R)^3$, we also have

$$\text{Vol. sphere} = \tfrac{4}{3}\pi R^3 = \tfrac{1}{6}\pi(2R)^3.$$

That is, *the volume of a sphere is measured four thirds of π times the cube of the radius, or by one sixth of π times the cube of the diameter.*

THEOREM XXX.

The surface of a sphere is equivalent to two thirds of the surface, bases included, and the volume of a sphere to two thirds of the volume, of the circumscribing cylinder.



Let AMD be a semi-circle, and $ABCD$ a rectangle formed by drawing tangents through the middle point and extremities of the semi-circumference, and let the semi-circle and rectangle be revolved together about AD as an axis. The rectangle will thus generate a cylinder circumscribed about the sphere generated by the semi-circle.

First. The diameter of the base, and the altitude of the cylinder, are each equal to the diameter of the sphere; hence the convex surface of the cylinder, being measured by the circumference of its base multiplied by its altitude, (Cor. 1, Th. 20), has the same measure as the surface of the sphere, (Th. 25). But the surface of the sphere is equivalent to four great circles, (Cor. 3, Th. 25). Hence, the convex surface of the cylinder is equivalent to four great circles; and adding to these the bases of the cylinder, also great circles, we have the whole surface of the cylinder equivalent to six great circles. Therefore, the surface of the sphere is four sixths = two thirds of the surface of the cylinder, including its bases.

Second. The volume of the cylinder, being measured by the area of the base multiplied by the altitude, (Cor. 1, Th. 22), is, in this case, measured by the area of a great circle multiplied by its diameter = four great circles multiplied by one half the radius of the sphere.

But the volume of the sphere is measured by four great circles multiplied by one third of the radius, (Scholium 2, Th. 29). Therefore,

$$\text{Vol. sphere} : \text{Vol. cylinder} :: \tfrac{1}{3} : \tfrac{1}{2} :: 2 : 3;$$

whence, $\text{Vol. sphere} = \tfrac{2}{3}\,\text{Vol. cylinder}.$

Hence the theorem; *the surface of a sphere is equivalent, etc.*

Cor. The volume of a sphere is to the volume of the circumscribed cylinder, as the surface of the sphere is to the surface of the cylinder.

Scholium.—Any polyedron circumscribing a sphere, may be regarded as composed of as many cones as the polyedron has faces, the center of the sphere being the common vertex of these cones, and the several faces of the polyedron their bases. The altitude of each cone will be a radius of the sphere; hence the volume of any one cone will be measured by the area of the face of the polyedron which forms its base, multiplied by one third of the radius of the sphere. Therefore, the aggregate of these cones, or the whole polyedron, will be measured by the surface of the sphere multiplied by one third of the radius of the sphere.

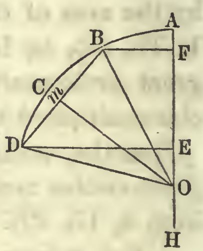
But the volume of the sphere is also measured by the surface of the sphere multiplied by one third of its radius. Hence,

$$\text{Sur. polyedron} : \text{Sur. sphere} :: \text{Vol. polyedron} : \text{Vol. sphere}.$$

That is, *the surface of any circumscribed polyedron is to the surface of the sphere, as the volume of the polyedron is to the volume of the sphere.*

THEOREM XXXI.

The volume generated by the revolution of the segment of a circle about a diameter of the circle exterior to the segment, is measured by one sixth of π times the square of the chord of the segment, multiplied by the part of the axis included between the perpendiculars let fall upon it from the extremities of the chord.



Let BCD be a segment of the circle, whose center is O, and AH a part of a diameter exterior to the segment. Draw the chord BD, and from its extremities let fall the perpendiculars, BF, DE on AH; also draw Om perpendicular to BD. The spherical sector generated by the revolution of the circular sector $BCDO$ about AH, is measured by zone $BD \times \frac{1}{3}BO$, (Scholium 1, Th. 29), $= 2\pi.BO \times EF \times \frac{1}{3}BO = \frac{2}{3}\pi\overline{BO}^2 \times EF$; and the volume generated by the isosceles triangle BOD is measured by

$$\tfrac{2}{3}\pi\overline{Om}^2 \times EF, \text{ (Cor. 1, Th. 27).}$$

The difference between these two volumes is that generated by the circular segment BCD, which has, therefore, for its measure,

$$\tfrac{2}{3}\pi EF(\overline{BO}^2 - \overline{Om}^2) = \tfrac{2}{3}\pi EF \times \overline{Bm}^2, \text{ (Th. 39, B. I).}$$

But since $Bm = \frac{1}{2}BD$, $\overline{Bm}^2 = \frac{1}{4}\overline{BD}^2$; hence, by substituting, we have

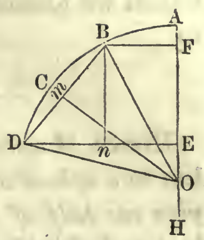
$$\text{Vol. segment } BCD = \tfrac{2}{3}\pi EF \times \tfrac{1}{4}\overline{BD}^2 = \tfrac{1}{6}\pi\overline{BD}^2 \times EF.$$

Hence the theorem.

THEOREM XXXII.

The volume of a segment of a sphere has, for its measure, the half sum of the bases of the segment multiplied by its altitude, plus the volume of a sphere which has this altitude for its diameter.

Let BCD be the arc of a circle, and BF and DE perpendiculars let fall from its extremities upon a diameter, of which AH is a part; then, if the area $BCDEF$ be revolved about AH as an axis, a spherical segment will be generated, for the volume of which it is proposed to find a measure.



The circular segment will generate a volume measured by $\frac{1}{6}\pi\overline{BD}^2 \times EF$, (Th. 31); and the frustum of the cone generated by the trapezoid $BDEF$ will have, for its measure,

$$\tfrac{1}{3}\pi\overline{BF}^2 \times EF + \tfrac{1}{3}\pi\overline{DE}^2 \times EF + \tfrac{1}{3}\pi BF \times DE \times EF, \text{(Th. 22)},$$

$$= \tfrac{1}{3}\pi EF(\overline{BF}^2 + \overline{DE}^2 + BF \times DE).$$

But the sum of these two volumes is the volume of the spherical segment, which has, therefore, for its measure,

$$\tfrac{1}{6}\pi EF\,(\overline{BD}^2 + 2\overline{BF}^2 + 2\overline{DE}^2 + 2BF \times DE)$$

From B let fall the perpendicular Bn on DE; then will

$$Dn = DE - nE = DE - BF;$$

hence, $\quad \overline{Dn}^2 = \overline{DE}^2 - 2DE \times BF + \overline{BF}^2$;

and since $\quad \overline{BD}^2 = \overline{Bn}^2 + \overline{Dn}^2 = \overline{EF}^2 + \overline{Dn}^2$,

we have $\quad \overline{BD}^2 = \overline{EF}^2 + \overline{DE}^2 + \overline{BF}^2 - 2DE \times BF.$

By substituting this value for $\overline{BD}^2$, in the above measure for the volume of the segment, we find

$$\tfrac{1}{6}\pi EF(\overline{EF}^2 + \overline{DE}^2 + \overline{BF}^2 - 2DE \times BF + 2\overline{BF}^2 + 2\overline{DE}^2 + 2BF \times DE)$$

$$= \tfrac{1}{6}\pi EF\,(\overline{EF}^2 + 3\overline{DE}^2 + 3\overline{BF}^2) = \tfrac{1}{6}\pi\overline{EF}^3 + EF\left(\frac{\pi\overline{DE}^2 + \pi\overline{BF}^2}{2}\right).$$

Which last expression conforms to the enunciation.

Hence the theorem; *the volume of a segment of a sphere, etc.*

Cor. When the segment has but one base, BF becomes zero, and EF becomes EA; and the final expression which we found for the volume of the segment reduces to

$$\frac{1}{6}\pi\overline{EA}^3 + EA \times \frac{\pi\overline{DE}^2}{2}.$$

Hence, *A spherical segment having but one base, is equivalent to a sphere whose diameter is the altitude of the segment, plus one half of a cylinder having for base and altitude the base and altitude of the segment.*

SCHOLIUM.—When the spherical segment has a single base, we may put the expression, $\frac{1}{6}\pi\overline{EA}^3 + EA \times \frac{\pi\overline{DE}^2}{2}$, under a form to indicate a convenient practical rule for computing the volume of the segment.

Thus, since the triangle DEO is right-angled, and $OE = OA - EA$, we have

$$\overline{DE}^2 = \overline{DO}^2 - \overline{OE}^2 = \overline{OA}^2 - \overline{OA}^2 + 2OA \times EA - \overline{EA}^2$$
$$= 2OA \times EA - \overline{EA}^2.$$

By substituting this value for $\overline{DE}^2$ in the expression for the volume of the segment, we find

$$\frac{1}{6}\pi\overline{EA}^3 + EA \times \frac{\pi}{2} \times (2OA \times EA - \overline{EA}^2)$$
$$= \frac{1}{6}\pi\overline{EA}^3 + \overline{EA}^2 \times \frac{\pi}{2}(2OA - EA)$$
$$= \frac{1}{6}\pi\overline{EA}^3 + \frac{1}{6}\pi.3\overline{EA}^2(2OA - EA)$$
$$= \frac{1}{6}\pi\overline{EA}^2(EA + 6.OA - 3EA)$$
$$= \frac{1}{6}\pi\overline{EA}^2(6.OA - 2EA)$$
$$= \frac{1}{3}\pi\overline{EA}^2(3OA - EA)$$

Hence, *the volume of a spherical segment, having a single base, is measured by one third of π times the square of the altitude of the segment, multiplied by the difference between three times the radius of the sphere and this altitude.*

RECAPITULATION

Of some of the principles demonstrated in this and the preceding Books.

Let R denote the radius, and D the diameter of any circle or sphere, and H the altitude of a cone, or of a segment of a sphere; then,

Circumference of a circle $= 2\pi R$.
Surface of a sphere $= 4\pi R^2$.
Zone forming the base of a segment of a sphere, $= 2\pi R \times H$.
Volume or solidity of a sphere $= \frac{4}{3}\pi R^3$, or $\frac{1}{6}\pi D^3$.
Volume of a spherical sector $= \frac{2}{3}\pi R^2 \times H$.
Volume of a cone, of which R is the radius of the base $= \frac{1}{3}\pi R^2 \times H$.
Volume of a spherical segment, of which R' is the radius of one base, and R'' the radius of the other, and whose altitude is H, $= \frac{1}{6}\pi H^3 + H\frac{(\pi R'^2 + \pi R''^2)}{2}$
If the segment has but one base, $R'' =$ zero, and the volume of the segment, $= \frac{1}{6}\pi H^3 + H.\frac{\pi R'^2}{2}$; or, $= \frac{1}{3}\pi H^2(3R - H)$.

PRACTICAL PROBLEMS.

1. The diameter of a sphere is 12 inches; how many cubic inches does it contain? *Ans.* 904.78 cu. in.

2. What is the solidity of the segment of a single base that is cut from a sphere 12 inches in diameter, the altitude of the segment being 3 inches? *Ans.* 141.371 cu. in.

3. The surface of a square is 68 square feet; what is its diameter? *Ans.* $D = 4.625$ feet.

4. If from a sphere, whose surface is 68 square feet, a segment be cut, having a depth of two feet and a single base, what is the convex surface of the segment?

Ans. 29.229+ sq. ft.

5. What is the solidity of the sphere mentioned in the two preceding examples, and what is the solidity of the segment, having a depth of two feet, and but one base?

Ans. { Solidity of sphere, 52.71 cu. in.
" " segment, 20.85 "

6. In a sphere whose diameter is 20 feet, what is the solidity of a segment, the bases of which are on the same side of the center, the first at the distance of 3 feet from it, and the second of 5 feet; and what is the solidity of a second segment of the same sphere, whose bases are also on the same side of the center, and at distances from it, the first of 5 and the second of 7 feet?

Ans. { Solidity of first segment, 525.7 cu. ft.
" " second " 400.03 "

7. If the diameter of the single base of a spherical segment be 16 inches, and the altitude of the segment 4 inches, what is its solidity? *

Ans. 435.6352 cubic inches.

8. The diameter of one base of a spherical segment is 18 inches, and that of the other base 14 inches, these bases being on opposite sides of the center of the sphere, and the distance between them 9 inches; what is the volume of the segment, and the radius of the sphere?

Ans. { Vol. seg., 2600.3 cubic inches.
Rad. of sphere, 9.4027 inches.

9. The radius of a sphere is 20, the distance from the center to the greater base of a segment is 10, and the distance from the same point to the lesser base is 16; what is the volume of the segment, the bases being on the same side of the center? *Ans.* 4297.7088.

10. If the diameter of one base of a spherical segment be 20 miles, and the diameter of the other base 12 miles, and the altitude of the segment 2 miles, what is its solidity, and what is the diameter of the sphere?

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* First find the radius of the sphere.

BOOK VIII.

PRACTICAL GEOMETRY.

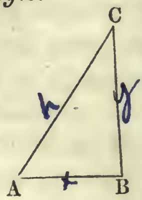
APPLICATION OF ALGEBRA TO GEOMETRY, AND ALSO PROPOSITIONS FOR ORIGINAL INVESTIGATION.

No definite rules can be given for the algebraic solution of geometrical problems. The student must, in a a great measure, depend on his own natural tact, and his power of making a skillful application of the geometrical and analytical knowledge he has thus far obtained.

The known quantities of the problem should be represented by the first letters of the alphabet, and the unknown by the final letters; and the relations between these quantities must be expressed by as many independent equations as there are unknown quantities. To obtain the equations of the problem, we draw a figure, the parts of which represent the known and unknown magnitudes, and very frequently it will be found necessary to draw auxiliary lines, by means of which we can deduce, from the conditions enunciated, others that can be more conveniently expressed by equations. In many cases the principal difficulty consists in finding, from the relations directly given in the statement, those which are ultimately expressed by the equations of the problem. Having found these equations, they are treated by the known rules of algebra, and the values of the required magnitudes determined in terms of those given.

PROBLEM I.

Given, the hypotenuse, and the sum of the other two sides of a right-angled triangle, to determine the triangle.



Let ABC be the $\triangle$. Put $CB = y$, $AB = x$, $AC = h$, and $CB + AB = s$. Then, by a given condition, we have

$$x + y = s;$$

and, $$x^2 + y^2 = h^2, \text{ (Th. 39, B. I).}$$

Reducing these two equations, and we have

$$x = \tfrac{1}{2}s \pm \tfrac{1}{2}\sqrt{2h^2 - s^2}; \qquad y = \tfrac{1}{2}s \pm \tfrac{1}{2}\sqrt{2h^2 - s^2}.$$

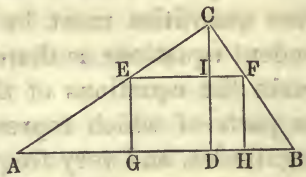
If $h = 5$ and $s = 7$, $x = 4$ or 3, and $y = 3$ or 4.

REMARK.—In place of putting x to represent one side, and y the other, we might put $(x + y)$ to represent the greater side, and $(x - y)$ the less side; then,

$$x^2 + y^2 = \frac{h^2}{2}, \text{ and } 2x = s, \text{ etc.}$$

PROBLEM II.

Given, the base and perpendicular of a triangle, to find the side of its inscribed square.



Let ABC be the $\triangle$. Put $AB = b$, the base, $CD = p$, the perpendicular.

Draw EF parallel to AB, and suppose it equal to EG, a side of the required square; and put $EF = x$.

Then, by similar $\triangle$'s, we have

$$CI : EF :: CD : AB.$$

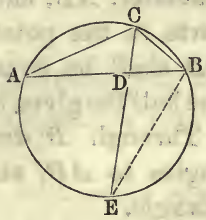
That is, $$p - x : x :: p : b.$$

Hence, $$bp - bx = px; \text{ or, } x = \frac{bp}{b + p}.$$

That is, *the side of the inscribed square is equal to the product of the base and altitude, divided by their sum.*

PROBLEM III.

In a triangle, having given the sides about the vertical angle, and the line bisecting that angle and terminating in the base, to find the base.



Let ABC be the $\triangle$, and let a circle be circumscribed about it. Divide the arc AEB into two equal parts at the point E, and draw EC. This line bisects the vertical angle, (Cor., Th. 9, B. III). Draw BE.

Put $AD = x$, $DB = y$, $AC = a$, $CB = b$, $CD = c$, and $DE = w$. The two $\triangle$'s, ADC and EBC, are equiangular; from which we have

$$w + c : b :: a : c; \text{ or, } cw + c^2 = ab; \quad (1)$$

But, as EC and AB are two chords that intersect each other in a circle, we have

$$cw = xy, \quad \text{(Th. 17, B. III).}$$

Therefore, $\quad xy + c^2 = ab. \quad (2)$

But, as CD bisects the vertical angle, we have

$$a : b :: x : y, \quad \text{(Th. 24, B. II).}$$

Or, $\quad x = \frac{ay}{b}. \quad (3)$

Hence, $\quad \frac{a}{b}y^2 + c^2 = ab; \text{ or, } y = \sqrt{b^2 - \frac{c^2b^1}{a}}$

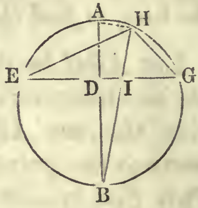
And, $\quad x = \frac{a}{b}\sqrt{b^2 - \frac{c^2b}{a}}.$

Now, as x and y are determined, the base is determined.

Remark.—Observe that equation (2) is Theorem 20, Book III.

PROBLEM IV.

To determine a triangle, from the base, the line bisecting the vertical angle, and the diameter of the circumscribing circle.



Describe the circle on the given diameter, AB, and divide it into two parts, in the point D, so that $AD \times DB$ shall be equal to the square of one half the given base, (Th. 17, B. III).

Through D draw EDG, at right angles to AB, and EG will be the given base of the triangle.

Put $\quad AD = n,\ DB = m,\ AB = d,\ DG = b.$

Then, $\quad n + m = d$, and $nm = b^2$;

and these two equations will determine n and m; therefore, we shall consider n and m as known.

Now, suppose EHG to be the required △; and draw HIB and HA. The two △'s, ABH, DBI, are equiangular; and, therefore, we have

$$AB : HB :: IB : DB.$$

But HI is a given line, that we will represent by c; and if we put $IB = w$, we shall have $HB = c + w$; then the above proportion becomes,

$$d : c + w :: w : m.$$

Now, w can be determined by a quadratic equation; and, therefore, IB is a known line.

In the right-angled △ DBI, the hypotenuse IB, and the base DB, are known; therefore, DI is known, (Th. 39, B. I); and if DI is known, EI and IG are known.

Lastly, let $EH = x$, $HG = y$, and put $EI = p$, and $IG = q$.

Then, by Theorem 20, Book III, $pq + c^2 = xy \quad (1)$

But, $\quad x : y :: p : q$ (Th. 24, B. II).

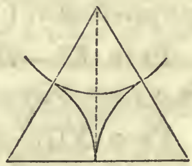
Or, $$x = \frac{py}{q} \qquad (2)$$

Now, from equations (1) and (2) we can determine x and y, the sides of the △; and thus the determination has been attained, carefully and easily, step by step.

PROBLEM V.

Three equal circles touch each other ex ernally, and thus inclose one acre of ground; what is the diameter in rods of each of these circles?

Draw three equal circles to touch each other externally, and join the three centers, thus forming a triangle. The lines joining the centers will pass through the points of contact, (Th. 7, B. III).



Let R represent the radius of these equal circles; then it is obvious that each side of this △ is equal to $2R$. The triangle is therefore equilateral, and it incloses the given area, and three equal sectors.

As the angle of each sector is one third of two right angles, the three sectors are, together, equal to a semi-circle; but the area of a semi-circle, whose radius is R, is expressed by $\frac{\pi R^2}{2}$; and the area of the whole triangle must be $\frac{\pi R^2}{2} + 160$; but the area of the △ is also equal to R multiplied by the perpendicular altitude, which is $R\sqrt{3}$.

Therefore, $$R^2\sqrt{3} = \frac{\pi R^2}{2} + 160.$$

Or, $$R^2(2\sqrt{3} - \pi) = 320.$$

$$R^2 = \frac{320}{2\sqrt{3} - 3.1415926} = \frac{3.20}{0.3225} = 992.248.$$

Hence, $R = 31.48 +$ rods, for the required result.

Problem VI.—*In a right-angled triangle, having given the base and the sum of the perpendicular and hypotenuse, to find these two sides.*

Prob. VII.—*Given, the base and altitude of a triangle, to divide it into three equal parts, by lines parallel to the base.*

Prob. VIII.—*In any equilateral △, given the length of the three perpendiculars drawn from any point within, to the three sides, to determine the sides.*

Prob. IX.—*In a right-angled triangle, having given the base,* (3), *and the difference between the hypotenuse and perpendicular,* (1), *to find both these two sides.*

Prob. X.—*In a right-angled triangle, having given the hypotenuse,* (5), *and the difference between the base and perpendicular,* (1), *to determine both these two sides.*

Prob. XI.—*Having given the area of a rectangle inscribed in a given triangle, to determine the sides of the rectangle.*

Prob. XII.—*In a triangle, having given the ratio of the two sides, together with both the segments of the base, made by a perpendicular from the vertical angle, to determine the sides of the triangle.*

Prob. XIII.—*In a triangle, having given the base, the sum of the other two sides, and the length of a line drawn from the vertical angle to the middle of the base, to find the sides of the triangle.*

Prob. XIV.—*To determine a right-angled triangle, having given the lengths of two lines drawn from the acute angles to the middle of the opposite sides.*

Prob. XV.—*To determine a right-angled triangle, having given the perimeter, and the radius of the inscribed circle.*

Prob. XVI.—*To determine a triangle, having given the base, the perpendicular, and the ratio of the two sides.*

Prob. XVII.—*To determine a right-angled triangle, having given the hypotenuse, and the side of the inscribed square.*

PROB. XVIII. — *To determine the radii of three equal circles inscribed in a given circle, and tangent to each other, and also to the circumference of the given circle.*

PROB. XIX.—*In a right-angled triangle, having given the perimeter, or sum of all the sides, and the perpendicular let fall from the right angle on the hypotenuse, to determine the triangle; that is, its sides.*

PROB. XX.—*To determine a right-angled triangle, having given the hypotenuse, and the difference of two lines drawn from the two acute angles to the center of the inscribed circle.*

PROB. XXI. — *To determine a triangle, having given the base, the perpendicular, and the difference of the two other sides.*

PROB. XXII. — *To determine a triangle, having given the base, the perpendicular, and the rectangle, or product of the two sides.*

PROB. XXIII.—*To determine a triangle, having given the lengths of three lines drawn from the three angles to the middle of the opposite sides.*

PROB. XXIV. — *In a triangle, having given all the three sides, to find the radius of the inscribed circle.*

PROB. XXV.—*To determine a right-angled triangle, having given the side of the inscribed square, and the radius of the inscribed circle.*

PROB. XXVI. — *To determine a triangle, and the radius of the inscribed circle, having given the lengths of three lines drawn from the three angles to the center of that circle.*

PROB. XXVII. — *To determine a right-angled triangle, having given the hypotenuse, and the radius of the inscribed circle.*

PROB. XXVIII.—*The lengths of two parallel chords on the same side of the center being given, and their distance apart, to determine the radius of the circle.*

PROB. XXIX. — *The lengths of two chords in the same* *circle being given, and also the difference of their distances from the center, to find the radius of the circle.*

PROB. XXX.—*The radius of a circle being given, and also the rectangle of the segments of a chord, to determine the distance of the point at which the chord is divided, from the center.*

PROB. XXXI.—*If each of the two equal sides of an isosceles triangle be represented by* a, *and the base by* 2b, *what will be the value of the radius of the inscribed circle?*

$$\text{Ans. } R = \frac{b\sqrt{a^2 - b^2}}{a + b}.$$

PROB. XXXII. — *From a point without a circle whose diameter is* d, *a line equal to* d *is drawn, terminating in the concave arc, and this line is bisected at the first point in which it meets the circumference. What is the distance of the point without from the center of the circle?*

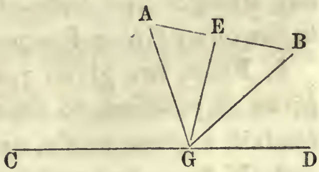
It is not deemed necessary to multiply problems in the application of algebra to geometry. The preceding will be a sufficient exercise to give the student a clear conception of the nature of such problems, and will serve as a guide for the solution of others that may be proposed to him, or that may be invented by his own ingenuity.

MISCELLANEOUS PROPOSITIONS.

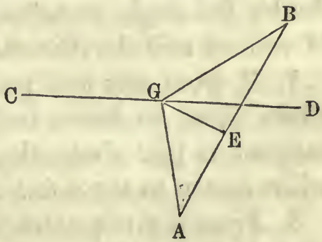
We shall conclude this book, and the subject of Geometry, by offering the following propositions, — some theorems, others problems, and some a combination of both, —not only for the purpose of impressing, by application, the geometrical principles which have now been established, but for the not less important purpose of cultivating the power of independent investigation.

After one or two propositions in which the beginner will be assisted in the analysis and construction, we shall leave him to his own resources, with the caution that a patient consideration of all the conditions in each case, and not mere trial operation, is the only process by which he can hope to reach the desired result.

1. From two given points, to draw two equal straight lines, which shall meet in the same point in a given straight line.

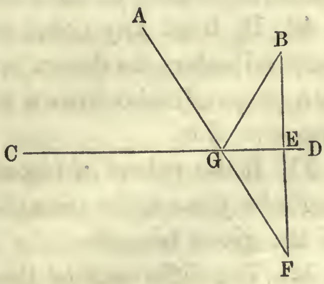


Let A and B be the given points, and CD the given straight line. Produce the perpendicular to the straight line AB at its middle point, until it meets CD in G. It is then easily proved that G is the point in CD in which the equal lines from A and B must meet. That is, that $AG = BG$.



If the points A and B were on opposite sides of CD, the directions for the construction would be the same, and we should have this figure; but the reasoning by which we prove $AG = BG$ would be unchanged.

2. From two given points on the same side of a given straight line, to draw two straight lines which shall meet in the given line, and make equal angles with it.



Let CD be the given line, and A and B the given points.

From B draw BE perpendicular to CD, and produce the perpendicular to F, making EF equal to BE; then draw AF, and from the point G, in which it intersects CD, draw GB. Now, $\angle BGE = \angle EGF = \angle AGC$. Hence, the angles BGD and AGC are equal, and the lines AG and BG meet in a common point in the line CD, and made equal angles with that line.

3. If, from a point without a circle, two straight lines be drawn to the concave part of the circumference, making equal angles with the line joining the same point and the center, the parts of these lines which are intercepted within the circle, are equal.

4. If a circle be described on the radius of another circle, any straight line drawn from the point where they meet, to the outer circumference, is bisected by the interior one.

5. From two given points on the same side of a line given in position, to draw two straight lines which shall contain a given angle, and be terminated in that line.

6. If, from any point without a circle, lines be drawn touching the circle, the angle contained by the tangents is double the angle contained by the line joining the points of contact and the diameter drawn through one of them.

7. If, from any two points in the circumference of a circle, there be drawn two straight lines to a point in a tangent to that circle, they will make the greatest angle when drawn to the point of contact.

8. From a given point within a given circle, to draw a straight line which shall make, with the circumference, an angle, less than any angle made by any other line drawn from that point.

9. If two circles cut each other, the greatest line that can be drawn through either point of intersection, is that which is parallel to the line joining their centers.

10. If, from any point within an equilateral triangle, perpendiculars be drawn to the sides, their sum is equal to a perpendicular drawn from any of the angles to the opposite side.

11. If the points of bisection of the sides of a given triangle be joined, the triangle so formed will be one fourth of the given triangle.

12. The difference of the angles at the base of any triangle, is double the angle contained by a line drawn from the vertex perpendicular to the base, and another bisecting the angle at the vertex.

13. If, from the three angles of a triangle, lines be drawn to the points of bisection of the opposite sides, these lines intersect each other in the same point.

14. The three straight lines which bisect the three angles of a triangle, meet in the same point.

15. The two triangles, formed by drawing straight lines from any point within a parallelogram to the extremities of two opposite sides, are, together, one half the parallelogram.

16. The figure formed by joining the points of bisection of the sides of a trapezium, is a parallelogram.

17. If squares be described on three sides of a right-angled triangle, and the extremities of the adjacent sides be joined, the triangles so formed are equal to the given triangle, and to each other.

18. If squares be described on the hypotenuse and sides of a right-angled triangle, and the extremities of the sides of the former, and the adjacent sides of the others, be joined, the sum of the squares of the lines joining them will be equal to five times the square of the hypotenuse.

19. The vertical angle of an oblique-angled triangle inscribed in a circle, is greater or less than a right angle, by the angle contained between the base and the diameter drawn from the extremity of the base.

20. If the base of any triangle be bisected by the diameter of its circumscribing circle, and, from the extremity of that diameter, a perpendicular be let fall upon the longer side, it will divide that side into segments, one of which will be equal to one half the sum, and the other to one half the difference, of the sides.

21. A straight line drawn from the vertex of an equilateral triangle inscribed in a circle, to any point in the opposite circumference, is equal to the sum of the two lines which are drawn from the extremities of the base to the same point.

22. The straight line bisecting any angle of a triangle

inscribed in a given circle, cuts the circumference in a point which is equi-distant from the extremities of the side opposite to the bisected angle, and from the center of a circle inscribed in the triangle.

23. If, from the center of a circle, a line be drawn to any point in the chord of an arc, the square of that line, together with the rectangle contained by the segments of the chord, will be equal to the square described on the radius.

24. If two points be taken in the diameter of a circle, equidistant from the center, the sum of the squares of the two lines drawn from these points to any point in the circumference, will be always the same.

25. If, on the diameter of a semicircle, two equal circles be described, and in the space included by the three circumferences, a circle be inscribed, its diameter will be $\frac{2}{3}$ the diameter of either of the equal circles.

26. If a perpendicular be drawn from the vertical angle of any triangle to the base, the difference of the squares of the sides is equal to the difference of the squares of the segments of the base.

27. The square described on the side of an equilateral triangle, is equal to three times the square of the radius of the circumscribing circle.

28. The sum of the sides of an isosceles triangle is less than the sum of any other triangle on the same base and between the same parallels.

29. In any triangle, given one angle, a side adjacent to the given angle, and the difference of the other two sides, to construct the triangle.

30. In any triangle, given the base, the sum of the other two sides, and the angle opposite the base, to construct the triangle.

31. In any triangle, given the base, the angle opposite to the base, and the difference of the other two sides, to construct the triangle.

PLANE TRIGONOMETRY

AND

SPHERICAL GEOMETRY AND TRIGONOMETRY.

TRIGONOMETRY.

PART I.

PLANE TRIGONOMETRY.

SECTION I.

ELEMENTARY PRINCIPLES.

TRIGONOMETRY, in its literal and restricted sense, has for its object the measurement of triangles. When it treats of plane triangles it is called *Plane Trigonometry.* In a more enlarged sense, trigonometry is the science which investigates the relations of all possible arcs of the circumference of a circle to certain straight lines, termed *trigonometrical lines* or *circular functions*, connected with and dependent on such arcs, and the relations of these trigonometrical lines to each other.

The measure of an angle is the arc of a circle intercepted between the two lines which form the angle—the center of the arc always being at the point where the two lines meet.

The arc is measured by *degrees*, *minutes*, and *seconds;* there being 360 degrees to the whole circle, 60 minutes in one degree, and 60 seconds in one minute. Degrees, minutes, and seconds, are designated by °, ′, ″; thus, 27° 14′ 21″, is read 27 degrees 14 minutes 21 seconds.

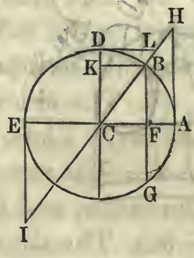
The circumferences of all circles contain the same number of degrees, but the greater the radius the greater is the absolute length of a degree. The circumference of a carriage wheel, the circumference of the earth, or the still greater and indefinite circumference of the heavens, has the same number of degrees; yet the same number of degrees in each and every circumference is the measure of precisely the same angle.

DEFINITIONS.

1. The **Complement** of an arc is 90° minus the arc.

2. The **Supplement** of an arc is 180° minus the arc.

3. The **Sine** of an angle, or of an arc, is a line drawn from one end of an arc, perpendicular to a diameter drawn through the other end. Thus, *BF* is the sine of the arc *AB*, *and also* of the arc *BDE*. *BK* is the sine of the arc *BD*.



4. The **Cosine** of an arc is the perpendicular distance from the center of the circle to the sine of the arc; or, it is the same in magnitude as the sine of the complement of the arc. Thus, *CF* is the cosine of the arc *AB*; but *CF* = *KB*, is the sine of *BD*.

5. The **Tangent** of an arc is a line touching the circle in one extremity of the arc, and continued from thence, to meet a line drawn through the center and the other extremity. Thus, *AH* is the tangent to the arc *AB*, and *DL* is the tangent of the arc *DB*.

6. The **Cotangent** of an arc is the tangent of the complement of the arc. Thus, *DL*, which is the tangent of the arc *DB*, is the cotangent of the arc *AB*.

REMARK.—The *co* is but a contraction of the word complement.

7. The **Secant** of an arc is a line drawn from the center of the circle to the extremity of the tangent. Thus, *CH* is the secant of the arc *AB*, or of its supplement *BDE*.

8. The **Cosecant** of an arc is the secant of the complement. Thus, *CL*, the secant of *BD*, is the cosecant of *AB*.

9. The **Versed Sine** of an arc is the distance from the extremity of the arc to the foot of the sine. Thus, *AF* is the versed sine of the arc *AB*, and *DK* is the versed sine of the arc *DB*.

For the sake of brevity, these technical terms are contracted thus: for sine *AB*, we write *sin. AB*; for cosine *AB*, we write *cos. AB*; for tangent *AB*, we write *tan. AB*, etc.

From the preceding definitions we deduce the following obvious consequences:

1st. That when the arc *AB* becomes insensibly small, or zero, its sine, tangent, and versed sine are also nothing, and its secant and cosine are each equal to radius.

2d. The sine and versed sine of a quadrant are each equal to the radius; its cosine is zero, and its secant and tangent are infinite.

3d. The chord of an arc is twice the sine of one half the arc. Thus, the chord, *BG*, is double the sine, *BF*.

4th. The versed sine is equal to the difference between the radius and the cosine.

5th. The sine and cosine of any arc form the two sides of a right-angled triangle, which has a radius for its hypotenuse. Thus, *CF* and *FB* are the two sides of the right-angled triangle, *CFB*.

Also, the radius and tangent always form the two sides of a right-angled triangle, which has the secant of the arc for its hypotenuse. This we observe from the right-angled triangle, *CAH*.

To express these relations analytically, we write

$$\sin.^2 + \cos.^2 = R^2 \qquad (1)$$

$$R^2 + \tan.^2 = \sec.^2 \qquad (2)$$

From the two equiangular triangles *CFB*, *CAH*, we have

$$CF : FB = CA : AH.$$

That is,

$$\cos. : \sin. = R : \tan.;\ \text{whence, } \tan. = \frac{R.\sin.}{\cos.} \qquad (3)$$

Also, $CF : CB = CA : CH.$

That is,

$$\cos. : R = R : \sec.;\ \text{whence, } \cos.\ \sec. = R^2. \qquad (4)$$

The two equiangular triangles, CAH and CDL, give

$$CA : AH = DL : DC.$$

That is,

$$R : \tan. = \cot. : R;\ \text{whence, } \tan.\ \cot. = R^2. \qquad (5)$$

Also, $CF : FB = DL : DC.$

That is,

$$\cos. : \sin. = \cot. : R;\ \text{whence, } \cos.\ R = \sin.\ \cot. \qquad (6)$$

From equations (4) and (5), we have

$$\cos.\ \sec. = \tan.\ \cot. \qquad (7)$$

Or, $\cos. : \tan. = \cot. : \sec.$

$$\text{ver. sin.} = 1 - \cos. \qquad (8)$$

The *ratios* between the various trigonometrical lines are always the same for arcs of the same number of degrees, whatever be the length of the radius; and we may, therefore, assume radius of any length to suit our convenience. The preceding equations will be more concise, and more readily applied, by making the radius equal unity. This supposition being made, we have, for equations 1 to 6, inclusive,

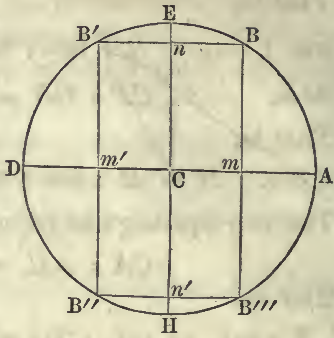
$$\sin.^2 + \cos.^2 = 1. \qquad (1)$$

$$1 + \tan.^2 = \sec.^2 \qquad (2)$$

$$\tan. = \frac{\sin.}{\cos.} \quad (3) \qquad \cos. = \frac{1}{\sec.} \quad (4)$$

$$\tan. = \frac{1}{\cot.} \quad (5) \qquad \cos. = \sin.\ \cot. \quad (6)$$

Let the circumference, $AEDH$, be divided into four equal parts by the diameters, AD and EH, the one horizontal and the other vertical. These equal parts are called *quadrants*, and they may be distinguished as the *first*, *second*, *third*, and *fourth* quadrants.



The center of the circle is taken as the origin of distances, or the zero point, and the different directions in which distances are estimated from this point are indicated by the signs + and —. If those from C to the right be marked +, those from C to the left must be marked —; and if distances from C upwards be considered plus, those from C downwards must be considered minus.

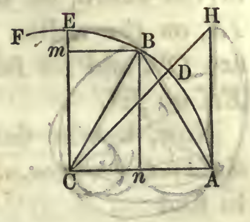
If one extremity of a varying arc be constantly at A, and the other extremity fall successively in each of the several quadrants, we may readily determine, by the above rule, the algebraic signs of the sines and cosines of all arcs from 0° to 360°. Now, since all other trigonometrical lines can be expressed in terms of the sine and cosine, it follows that the algebraic signs of all the circular functions result from those of the sine and cosine.

We shall thus find for arcs terminating in the

	sin.	cos.	tan.	cot.	sec.	cosec.	vers.
1st quadrant,	+	+	+	+	+	+	+
2d "	+	—	—	—	—	+	+
3d "	—	—	+	+	—	—	+
4th "	—	+	—	—	+	—	+

PROPOSITION I.

The chord of 60° *and the tangent of* 45° *are each equal to radius; the sine of* 30°, *the versed sine of* 60°, *and the cosine of* 60° *are each equal to one half the radius.*



With C as a center, and CA as a radius, describe the arc ABF, and from A lay off the arcs $AD = 45°$, $AB = 60°$, and $AE = 90°$; then is $EB = 30°$.

1st. The side of a regular inscribed hexagon is the radius of the circle, (Prob. 28, B. IV), and as the arc subtended by each side of the hexagon contains 60°, we have the chord of 60° equal to the radius.

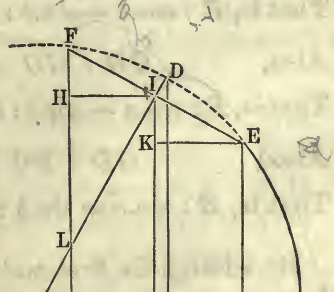
2d. The triangle CAH is right-angled at A, and the angle C is equal to 45°, being measured by the arc AD; hence the angle at H is also equal to 45°, and the triangle is isosceles. Therefore $AH = CA$ = radius of the circle.

3d. The triangle ABC is isosceles, and Bn is a perpendicular from the vertex upon the base; hence $An = nC = Bm$. But Bm is the sine of the arc BE, Cn is the cosine of the arc AB, and An is the versed sine of the same arc, and each is equal to one half the radius.

Hence the proposition; *the chord of* 60°, *etc.*

PROPOSITION II.

Given, the sine and the cosine of two arcs, to find the sine and the cosine of the sum and of the difference of the same arcs expressed by the sines and cosines of the separate arcs.



Let G be the center of the circle, CD the greater arc, and DF the less, and denote these arcs by a and b respectively.

Draw the radius GD; make the arc DE equal to the arc DF, and draw the chord EF. From F and E, the extremities, and I, the middle point of the chord, let fall the perpendiculars FM, EP, and IN, on the radius GC. Also draw DO, the sine of the arc CD, and let fall the perpendiculars IH on FM, and EK on IN.

Now, by the definition of sines and cosines, $DO = \sin.a$; $GO = \cos.a$; $FI = \sin.b$; $GI = \cos.b$. We are to find

$$FM = \sin.(a+b); \quad GM = \cos.(a+b);$$
$$EP = \sin.(a-b); \quad GP = \cos.(a-b).$$

Because IN is parallel to DO, the two △'s, GDO, GIN, are equiangular and similar. Also, the △ FHI is similar to the △ GIN; for the angles, FIG and HIN, are right angles; from these two equals, taking away the common angle HIL, we have the angle FIH = the angle GIN. The angles at H and N are right angles; therefore, the △'s FHI, GIN, and GDO, are equiangular and similar; and the side HI is homologous to IN and DO.

Again, as $FI = IE$, and IK is parallel to FM,

$$FH = IK, \text{ and } HI = KE.$$

By similar triangles we have

$$GD : DO = GI : IN.$$

That is, $R : \sin.a = \cos.b : IN$; or, $IN = \dfrac{\sin.a \cos.b}{R}$. (1)

Also, $GD : GO = FI : FH$.

That is, $R : \cos.a = \sin.b : HF$; or, $FH = \dfrac{\cos.a \sin.b}{R}$. (2)

Also, $GD : GO = GI : GN$.

That is, $R : \cos.a = \cos.b : GN$; or, $GN = \dfrac{\cos.a \cos.b}{R}$. (3)

Also, $GD : DO = FI : IH$.

That is, $R : \sin.a = \sin.b : IH$; or, $IH = \dfrac{\sin.a \sin.b}{R}$. (4)

By adding the first and second of these equations, we have

$$IN + FH = FM = \sin.(a+b).$$

That is, $\quad \sin.(a+\ ) = \dfrac{\sin.a\ \cos.b + \cos.a\ \sin.b}{R}.$

By subtracting the second from the first, since

$IN - FH = IN - IK = EP$, we have

$$\sin.(a-b) = \frac{\sin.a\ \cos.b - \cos.a\ \sin.b}{R}.$$

By subtracting the fourth from the third, we have

$GN - IH = GM = \cos.(a+b)$ for the first member.

Hence, $\quad \cos.(a+b) = \dfrac{\cos.a\ \cos.b - \sin.a\ \sin.b}{R}. \quad (5)$

By adding the third and fourth, we have

$$GN + IH = GN + NP = GP = \cos.(a-b).$$

Hence, $\cos.(a-b) = \dfrac{\cos.a\ \cos.b + \sin.a\ \sin.b}{R}. \quad (6)$

Collecting these four expressions, and considering the radius unity, we have

$$(A)\begin{cases} \sin.(a+b) = \sin.a\ \cos.b + \cos.a\ \sin.b & (7) \\ \sin.(a-b) = \sin.a\ \cos.b - \cos.a\ \sin.b & (8) \\ \cos.(a+b) = \cos.a\ \cos.b - \sin.a\ \sin.b & (9) \\ \cos.(a-b) = \cos.a\ \cos.b + \sin.a\ \sin.b & (10) \end{cases}$$

Formulæ (A) accomplish the objects of the proposition, and from these equations many useful and important deductions can be made. The following are the most essential:

By adding (7) to (8), we have (11); subtracting (8) from (7) gives (12). Also, (9) added to (10) gives (13); (9) taken from (10) gives (14).

$$(B)\begin{cases} \sin.(a+b) + \sin.(a-b) = 2\sin.a\ \cos.b & (11) \\ \sin.(a+b) - \sin.(a-b) = 2\cos.a\ \sin.b & (12) \\ \cos.(a+b) + \cos.(a-b) = 2\cos.a\ \cos.b & (13) \\ \cos.(a-b) - \cos.(a+b) = 2\sin.a\ \sin.b & (14) \end{cases}$$

If we put $a+b = A$, and $a-b = B$, then (11) becomes (15), (12) becomes (16), (13) becomes (17), and (14) becomes (18).

$$(C)\begin{cases}\sin.A+\sin.B=2\sin.\left(\frac{A+B}{2}\right)\cos.\left(\frac{A-B}{2}\right) & (15)\\ \sin.A-\sin.B=2\cos.\left(\frac{A+B}{2}\right)\sin.\left(\frac{A-B}{2}\right) & (16)\\ \cos.A+\cos.B=2\cos.\left(\frac{A+B}{2}\right)\cos.\left(\frac{A-B}{2}\right) & (17)\\ \cos.B-\cos.A=2\sin.\left(\frac{A+B}{2}\right)\sin.\left(\frac{A-B}{2}\right) & (18)\end{cases}$$

If we divide (15) by (16), (observing that $\frac{\sin.}{\cos.}=\tan.$, and $\frac{\cos.}{\sin.}=\cot.=\frac{1}{\tan.}$ as we learn by equations (6) and (5), we shall have

$$\frac{\sin.A+\sin.B}{\sin.A-\sin.B}=\frac{\sin.\left(\frac{A+B}{2}\right)}{\cos.\left(\frac{A+B}{2}\right)}\times\frac{\cos.\left(\frac{A-B}{2}\right)}{\sin.\left(\frac{A-B}{2}\right)}=\frac{\tan.\left(\frac{A+B}{2}\right)}{\tan.\left(\frac{A-B}{2}\right)} \quad (19)$$

Whence,

$$\sin.A+\sin.B : \sin.A-\sin.B=\tan.\left(\frac{A+B}{2}\right) : \tan.\left(\frac{A-B}{2}\right)$$

That is: *The sum of the sines of any two arcs is to the difference of the same sines, as the tangent of one half the sum of the same arcs is to the tangent of one half their difference.*

By operating in the same way with the different equations in formulæ (C), we find,

$$(D)\begin{cases}\frac{\sin.A+\sin.B}{\cos.A+\cos.B}=\tan.\left(\frac{A+B}{2}\right) & (20)\\ \frac{\sin.A+\sin.B}{\cos.B-\cos.A}=\cot.\left(\frac{A-B}{2}\right) & (21)\\ \frac{\sin.A-\sin.B}{\cos.A+\cos.B}=\tan.\left(\frac{A-B}{2}\right) & (22)\\ \frac{\sin.A-\sin.B}{\cos.B-\cos.A}=\cot.\left(\frac{A+B}{2}\right) & (23)\\ \frac{\cos.A+\cos.B}{\cos.B-\cos.A}=\frac{\cot.\left(\frac{A+B}{2}\right)}{\tan.\left(\frac{A-B}{2}\right)} & (24)\end{cases}$$

These equations are all true, whatever be the value of the arcs designated by A and B; we may, therefore, assign any possible value to either of them, and if in equations (20), (21), and (24), we make $B = 0$, we shall have,

$$\frac{\sin.A}{1+\cos.A} = \tan.\frac{A}{2} = \frac{1}{\cot.\frac{1}{2}A} \qquad (25)$$

$$\frac{\sin.A}{1-\cos.A} = \cot.\frac{A}{2} = \frac{1}{\tan.\frac{1}{2}A} \qquad (26)$$

$$\frac{1+\cos.A}{1-\cos.A} = \frac{\cot.\frac{1}{2}A}{\tan.\frac{1}{2}A} = \frac{1}{\tan^2.\frac{1}{2}A} \qquad (27)$$

If we now turn back to formulæ (A), and divide equation (7) by (9), and (8) by (10), observing at the same time that $\frac{\sin.}{\cos.} = \tan.$, we shall have,

$$\tan.(a+b) = \frac{\sin.a \cos.b + \cos.a \sin.b}{\cos.a \cos.b - \sin.a \sin.b}$$

$$\tan.(a-b) = \frac{\sin.a \cos.b - \cos.a \sin.b}{\cos.a \cos.b + \sin.a \sin.b}$$

By dividing the numerators and denominators of the second members of these equations by ($\cos.a \cos.b$), we find,

$$\tan.(a+b) = \frac{\frac{\sin.a \cos.b}{\cos.a \cos.b} + \frac{\cos.a \sin.b}{\cos.a \cos.b}}{\frac{\cos.a \cos.b}{\cos.a \cos.b} - \frac{\sin.a \sin.b}{\cos.a \cos.b}} = \frac{\tan.a + \tan.b}{1 - \tan.a \tan.b} \qquad (28)$$

$$\tan.(a-b) = \frac{\frac{\sin.a \cos.b}{\cos.a \cos.b} - \frac{\cos.a \sin.b}{\cos.a \cos.b}}{\frac{\cos.a \cos.b}{\cos.a \cos.b} + \frac{\sin.a \sin.b}{\cos.a \cos.b}} = \frac{\tan.a - \tan.b}{1 + \tan.a \tan.b} \qquad (29)$$

If in equation (11), formulæ (B), we make $a = b$, we shall have,

$$\sin.2a = 2\sin.a \cos.a \qquad (30)$$

Making the same hypothesis in equation (13), gives,

$$\cos.2a + 1 = 2\cos^2.a \qquad (31)$$

The same hypothesis reduces equation (14) to

$$1 - \cos.2a = 2\sin^2.a \qquad (32)$$

The same hypothesis reduces equation (28) to

$$\tan.2a = \frac{2\tan.a}{1 - \tan^2.a} \qquad (33)$$

If we substitute a for $2a$ in (31) and (32), we shall have

$$1 + \cos.a = 2\cos.^2\tfrac{1}{2}a. \qquad (34)$$

$$\text{and } 1 - \cos.a = 2\sin.^2\tfrac{1}{2}a. \qquad (35)$$

PROPOSITION III.

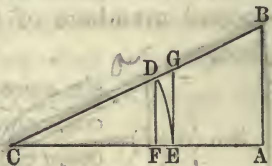
In any right-angled plane triangle, we may have the following proportions:

1st. *The hypotenuse is to either side, as the radius is to the sine of the angle opposite to that side.*

2d. *One side is to the other side, as the radius is to the tangent of the angle adjacent to the first side.*

3d. *One side is to the hypotenuse, as the radius is to the secant of the angle adjacent to that side.*

Let CAB represent any right-angled triangle, right-angled at A.



(Here, and in all cases hereafter, we shall represent the angles of a triangle by the large letters A, B, C, and the sides opposite to them, by the small letters a, b, c.)

From either acute angle, as C, take any distance, as CD, *greater* or *less* than CB, and describe the arc DE. This arc measures the angle C. From D, draw DF parallel to BA; and from E, draw EG, also parallel to BA or DF.

By the definitions of sines, tangents, secants, etc, DF is the sine of the angle C; EG is the tangent, CG the secant, and CF the cosine.

Now, by proportional triangles we have,

$$CB : BA = CD : DF \quad \text{or, } a : c = R : \sin.C$$
$$CA : AB = CE : EG \quad \text{or, } b : c = R : \tan.C$$
$$CA : CB = CE : CG \quad \text{or, } b : a = R : \sec.C$$

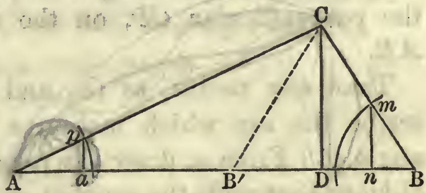
Hence the proposition.

SCHOLIUM.—If the hypotenuse of a triangle is made radius, one side is the sine of the angle opposite to it, and the other side is the cosine of the same angle. This is obvious from the triangle *CDF*.

PROPOSITION IV.

In any triangle, the sines of the angles are to one another as the sides opposite to them.

Let *ABC* be any triangle. From the points *A* and *B*, as centers, with any radius, describe the arcs measuring these angles, and draw *pa*, *CD*, and *mn*, perpendicular to *AB*.



Then, $pa = \sin.A$, and $mn = \sin.B$.

By the similar △'s, *Apa* and *ACD*, we have,

$$R : \sin.A = b : CD; \text{ or, } R(CD) = b \sin.A \quad (1)$$

By the similar △'s, *Bmn* and *BCD*, we have,

$$R : \sin.B = a : CD; \text{ or, } R(CD) = a \sin.B \quad (2)$$

By equating the second members of equations (1) and (2)

$$b \sin.A = a \sin.B.$$

Hence, $\sin.A : \sin.B = a : b$

Or, $a : b = \sin.A : \sin.B.$

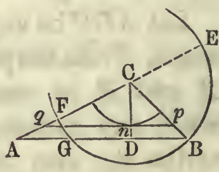
SCHOLIUM 1.—When either angle is 90°, its sine is radius.

SCHOLIUM 2.—When *CB* is less than *AC*, and the angle *B*, acute, the triangle is represented by *ACB*. When the angle *B* becomes *B′*, it is obtuse, and the triangle is *ACB′*; but the proportion is equally true with either triangle; for the angle $CB'D = CBA$, and the sine of $CB'D$ is the same as the sine of $AB'C$. In practice we can determine which of these triangles is proposed, by the side AB being greater or less than AC; or, by the angle at the vertex C being large, as ACB, or small, as ACB'.

In the solitary case in which AC, CB, and the angle A, are given, and CB less than AC, we can determine both of the △'s ACB and ACB'; and then we surely have the right one.

PROPOSITION V.

If from any angle of a triangle, a perpendicular be let fall on the opposite side, or base, the tangents of the segments of the angle are to each other as the segments of the base.



Let ABC be the triangle. Let fall the perpendicular CD, on the side AB.

Take any radius, as Cn, and describe the arc which measures the angle C. From n, draw qnp parallel to AB. Then it is obvious that np is the tangent of the angle DCB, and nq is the tangent of the angle ACD.

Now, by reason of the parallels AB and qp, we have,

$$qn : np = AD : DB$$

That is, $\tan.ACD : \tan.DCB = AD : DB.$

PROPOSITION VI.

If a perpendicular be let fall from any angle of a triangle to its opposite side or base, this base is to the sum of the other two sides, as the difference of the sides is to the difference of the segments of the base.

(See figure to Proposition 5.)

Let AB be the base, and from C, as a center, with the shorter side as radius, describe the circle, cutting AB in G, and AC in F; produce AC to E.

It is obvious that AE is the sum of the sides AC and CB, and AF is their difference.

Also, AD is one segment of the base made by the perpendicular, and $BD = DG$ is the other; therefore, the difference of the segments is AG.

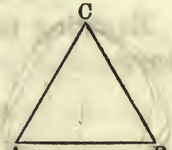
As A is a point without a circle, by Cor. Th. 18, B. III, we have

$$AE \times AF = AB \times AG$$

Hence, $$AB : AE = AF : AG.$$

PROPOSITION VII.

The sum of any two sides of a triangle is to their difference, as the tangent of one half the sum of the angles opposite to these sides, is to the tangent of one half their difference.



Let ABC be any plane triangle. Then, by Proposition 4, we have,

$$BC : AC = \sin.A : \sin.B.$$

Hence,

$$BC + AC : BC - AC = \sin.A + \sin.B : \sin.A - \sin.B \text{ (Th. 9, B. II).}$$

But,

$$\tan.\left(\frac{A+B}{2}\right) : \tan.\left(\frac{A-B}{2}\right) = \sin.A + \sin.B : \sin.A - \sin.B, \text{ (eq. (19), Trig.)}$$

Comparing the two latter proportions, (Th. 6, B. II), we have,

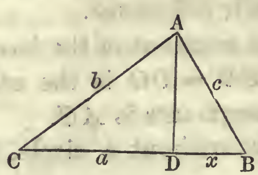
$$BC + AC : BC - AC = \tan.\left(\frac{A+B}{2}\right) : \tan.\left(\frac{A-B}{2}\right)$$

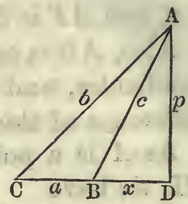
Hence the proposition.

PROPOSITION VIII.

Given, the three sides of any plane triangle, to find some relation which they must bear to the sines and cosines of the respective angles.

Let ABC be the triangle, and let the perpendicular fall either upon, or without the base, as shown in the figures.





By recurring to Th. 40, B. I, we shall find

$$CD = \frac{a^2 + b^2 - c^2}{2a}. \qquad (1)$$

Now, by Proposition 3, we have

$$R : \cos. C = b : CD.$$

Therefore, $$CD = \frac{b \cos. C}{R}. \qquad (2)$$

Equating these two values of CD, and reducing, we have

$$\cos. C = \frac{R(a^2 + b^2 - c^2)}{2ab}. \qquad (m)$$

In this expression we observe, that the part c, whose square is found in the numerator with the minus sign, is the side opposite to the angle; and that the denominator is twice the rectangle of the sides adjacent to the angle. From these observations we at once draw the following expressions for the cosine A, and cosine B:

$$\cos. A = \frac{R(b^2 + c^2 - a^2)}{2bc}. \qquad (n)$$

$$\cos. B = \frac{R(a^2 + c^2 - b^2)}{2ac}. \qquad (p)$$

As these expressions are not convenient for logarithmic computation, we modify them as follows:

If we put $2a = A$, in equation (31), we have

$$\cos. A + 1 = 2\cos.^2 \tfrac{1}{2}A.$$

In the preceding expression, (n), if we consider radius unity, and add 1 to both members, we shall have

$$\cos. A + 1 = 1 + \frac{b^2 + c^2 - a^2}{2bc}.$$

Therefore, $$2\cos.^2 \tfrac{1}{2}A = \frac{2bc + b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(b + c)^2 - a^2}{2bc}.$$

Considering $b + c$ as one quantity, and observing that $(b + c)^2 - a^2$ is the difference of *two squares*, we have

$(b+c)^2-a^2=(b+c+a)(b+c-a)$; but $(b+c-a)=b+c+a-2a$.

Hence, $$2\cos.^2 \tfrac{1}{2}A = \frac{(b + c + a)(b + c + a - 2a)}{2bc}.$$

Or, $$\cos.^2 \tfrac{1}{2}A = \frac{\left(\frac{b + c + a}{2}\right)\left(\frac{b + c + a}{2} - a\right)}{bc}.$$

By putting $\frac{a + b + c}{2} = s$, and extracting square root, the final result for radius unity is

$$\cos. \tfrac{1}{2}A = \sqrt{\frac{s(s - a)}{bc}}.$$

For any other radius we must write

$$\cos. \tfrac{1}{2}A = \sqrt{\frac{R^2 s(s - a)}{bc}}.$$

By inference, $$\cos. \tfrac{1}{2}B = \sqrt{\frac{R^2 s(s - b)}{ac}}.$$

Also, $$\cos. \tfrac{1}{2}C = \sqrt{\frac{R^2 s(s - c)}{ab}}.$$

In every triangle, the sum of the three angles is equal to 180°; and if one of the angles is small, the other two must be comparatively large; if two of them are small, the third one must be large. The greater angle is always opposite the greater side; hence, by merely inspecting the given sides, any person can decide at once which is the greater angle; and of the three preceding equations, *that one* should be taken which applies to the greater angle, whether that be the particular angle required or not; because the equations bring out the *cosines* to the angles; and the cosines to very small arcs vary so slowly, that it may be impossible to decide, with sufficient numerical accuracy, to what particular arc the cosine belongs. For instance, the cosine 9.999999, carried to the table, applies to several arcs; and, of course, we should not know which one to take; but this difficulty does not exist when the angle is large; therefore, compute the largest angle first, and then compute the other angles by Proposition 4.

But we can deduce an expression for the sine of any of the angles, as well as the cosine. It is done as follows:

EQUATIONS FOR THE SINES OF THE ANGLES.

Resuming equation (m), and considering radius unity, we have

$$\cos. C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Subtracting each member of this equation from unity, gives

$$1 - \cos. C = 1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right). \qquad (1)$$

Make $2a = C$, in equation (32); then $a = \frac{1}{2}C$, and $1 - \cos. C = 2\sin.^2 \frac{1}{2}C.$ (2)

Equating the second members of (1) and (2),

$$2\sin.^2 \tfrac{1}{2}C = \frac{2ab - a^2 - b^2 + c^2}{2ab}$$

$$= \frac{c^2 - (a - b)^2}{2ab}$$

$$= \frac{(c + b - a)(c + a - b)}{2ab}.$$

Or, $$\sin.^2 \tfrac{1}{2}C = \frac{\left(\frac{c+b-a}{2}\right)\left(\frac{c+a-b}{2}\right)}{ab}.$$

But, $\frac{c+b-a}{2} = \frac{c+b+a}{2} - a$, and $\frac{c+a-b}{2} = \frac{c+a+b}{2} - b$.

Put $\frac{a+b+c}{2} = s$, as before; then,

$$\sin. \tfrac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{ab}}.$$

By taking equation (p), and proceeding in the same manner, we have

$$\sin. \tfrac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{ac}}.$$

From (n), $\sin. \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{cb}}.$

The preceding results are for radius unity; for any other radius, we must multiply by the number of units in such radius. For the radius of the tables we write R; and if we put it under the radical sign, we must write R^2; hence, for the sines corresponding with our logarithmic table, we must write the equations thus,

$$\sin. \tfrac{1}{2}A = \sqrt{\frac{R^2(s-b)(s-c)}{bc}}.$$

$$\sin. \tfrac{1}{2}B = \sqrt{\frac{R^2(s-a)(s-c)}{ac}}.$$

$$\sin. \tfrac{1}{2}C = \sqrt{\frac{R^2(s-a)(s-b)}{ab}}.$$

A large angle should not be determined by these equations, for the same reason that a small angle should not be determined from an equation expressing the cosine.

In practice, the equations for cosine are more generally used, because more easily applied.

The formulæ which we have thus analytically developed, express nearly all the important relations between the sines, cosines, and tangents of arcs or angles; and we have also demonstrated all the theorems required for the determination of the unknown parts of any plane triangle, three of the parts of which are given, one at least being a side.

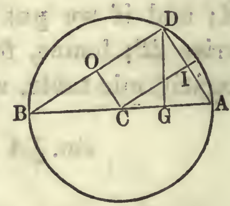
Such relations might be indefinitely multiplied, but those already established are sufficient for most practical purposes, and when others are required, no difficulty will be found in deducing them from these.

The following geometrical demonstrations of many of the preceding relations, are offered, in the belief that they will prove useful disciplinary exercises to the student.

1st. Let the arc $AD=A$; then $DG=\sin.A$; $CG=\cos.A$;

$DI=\sin.\frac{1}{2}A$; $AD=2\sin.\frac{1}{2}A$; $CI=\cos.\frac{1}{2}A$;

$CI=DO$; and $DB=2DO=2\cos.\frac{1}{2}A$.



The angle, DBA, is measured by one half the arc AD; that is, by $\frac{1}{2}A$.

Also, $ADG=DBA=\frac{1}{2}A$.

Now, in the triangle, BDG, we have

$\sin.DBG : DG=\sin.90° : BD$.

That is, $\sin.\frac{1}{2}A : \sin.A=1 : 2\cos.\frac{1}{2}A$.

Or, $\sin.A=2\sin.\frac{1}{2}A \cos.\frac{1}{2}A$;

which corresponds to equation (30).

In the same triangle,

$\sin.90° : BD=\sin.BDG : BG$; and $\sin.BDG=\cos.DBG$.

That is, $1 : 2\cos.\frac{1}{2}A=\cos.\frac{1}{2}A : 1+\cos.A$.

Or, $2\cos.^2\frac{1}{2}A=1+\cos.A$, same as equation (34).

In the triangle, DGA, we have,

$\sin.90° : AD=\sin.GDA : GA$.

That is, $1 : 2\sin.\frac{1}{2}A=\sin.\frac{1}{2}A : 1-\cos.A$.

Or, $2\sin.^2\frac{1}{2}A=1-\cos.A$, same as equation (35).

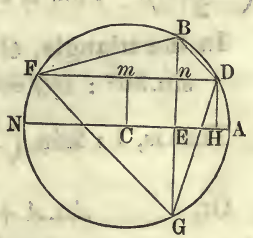
By similar triangles, we have,

$$BA : AD = AD : AG.$$

That is, $2 : 2\sin.\frac{1}{2}A = 2\sin.\frac{1}{2}A : \text{versed}\ \sin.A.$

Or, $\text{versed}\ \sin.A = 2\sin.^2\frac{1}{2}A.$

2d. From C as the center, with CA as the radius, describe a circle. Take any arc, AB, and call it A; and AD a less arc, and call it B; then BD is the difference of the two arcs, and must be designated by $(A-B)$; arc AG = arc AB; therefore,



arc $DG = A + B$; $EG = \sin.A$;

$En = \sin.B$; $Gn = \sin.A + \sin.B$;

$Bn = \sin.A - \sin.B.$

$Fm = mD = CH = \cos.B$; $mn = \cos.A$;

therefore, $Fm + mn = \cos.A + \cos.B = Fn$;

$mD - mn = \cos.B - \cos.A = nD$;

and $DG = 2\sin.\left(\frac{A+B}{2}\right).$

Because, $NF = AD$; $AB + NF = A + B$;

therefore, $180° - (A + B) = \text{arc}\ FB$;

or, $90° - \left(\frac{A+B}{2}\right) = \frac{1}{2}\text{arc}\ FB.$

But the chord, FB, is twice the sine of $\frac{1}{2}$ arc FB;

that is, $FB = 2\sin.\left(90° - \frac{A+B}{2}\right) = 2\cos.\left(\frac{A+B}{2}\right).$

The $\angle nGD = \angle BFD$, because both are measured by one half of the arc BD; that is, by $\left(\frac{A-B}{2}\right)$, and the two triangles, GnD and FnB, are similar.

The angle, GFn, is measured by $\left(\frac{A+B}{2}\right).$

In the triangle, FBG, Fn is drawn from an angle perpendicular to the opposite side; therefore, by Proposition 5, we have,

$$Gn : nB = \tan.GFn : \tan.BFn.$$

That is, $\sin.A+\sin.B : \sin.A-\sin.B = \tan.\left(\frac{A+B}{2}\right) : \tan.\left(\frac{A-B}{2}\right)$. This is equation (19).

In the triangle, GnD, we have,

$$\sin.90^\circ : DG = \sin.nDG : Gn; \ \sin.nDG = \cos.nGD.$$

That is, $1 : 2\sin.\left(\frac{A+B}{2}\right) = \cos.\left(\frac{A-B}{2}\right) : \sin.A+\sin.B.$

Or, $$\sin.A + \sin.B = 2\sin.\left(\frac{A+B}{2}\right)\cos.\left(\frac{A-B}{2}\right),$$

the same as equation (15).

3d. In the triangle, FnB, we have,

$$\sin.90 : FB = \sin.BFn : Bn.$$

That is, $1 : 2\cos.\left(\frac{A+B}{2}\right) = \sin.\left(\frac{A-B}{2}\right) : \sin.A-\sin.B.$

Or, $$\sin.A - \sin.B = 2\cos.\left(\frac{A+B}{2}\right)\sin.\left(\frac{A-B}{2}\right),$$

the same as equation (16).

4th. In the triangle, FBn, we have,

$$\sin.90 : FB = \cos.BFn : Fn.$$

That is, $1 : 2\cos.\left(\frac{A+B}{2}\right) = \cos.\left(\frac{A-B}{2}\right) : \cos.A+\cos.B.$

Or, $\cos.A + \cos.B = 2\cos.\left(\frac{A+B}{2}\right)\cos.\left(\frac{A-B}{2}\right)$, the same as equation (17).

5th. In the triangle, GnD, we have,

$$\sin.90^\circ : GD = \sin.nGD : nD.$$

That is, $1 : 2\sin.\left(\frac{A+B}{2}\right) = \sin.\left(\frac{A-B}{2}\right) : \cos.B-\cos.A$, the same as equation (18).

6th. In the triangle, FGn, we have,

$$\sin.GFn : Gn = \cos.GFn : Fn.$$

That is, $\sin.\frac{A+B}{2} : \sin.A+\sin.B = \cos.\frac{A+B}{2} : \cos.A+\cos.B$.

Or, $(\sin.A+\sin.B)\cos.\left(\frac{A+B}{2}\right) = (\cos.A+\cos.B)\sin.\left(\frac{A+B}{2}\right)$.

Or, $\frac{\sin.A+\sin.B}{\cos.A+\cos.B} = \frac{\sin.\frac{A+B}{2}}{\cos.\frac{A+B}{2}} = \tan.\left(\frac{A+B}{2}\right)$, the same as equation (20).

7th. In the triangle, FnB, we have,

$$Fn : nB :: 1 : \tan.BFn.$$

That is, $\cos.B+\cos.A : \sin.A-\sin.B :: 1 : \tan.\frac{1}{2}(A-B)$.

Or, $\frac{\sin.A-\sin.B}{\cos.A+\cos.B} = \tan.\left(\frac{A-B}{2}\right)$, the same as equation (22).

8th. In the triangle, GnD, we have,

$$Gn : nD :: 1 : \tan.nGD.$$

That is,

$$\sin.A+\sin.B : \cos.B-\cos.A :: 1 : \tan.\left(\frac{A-B}{2}\right),$$

or, $$\frac{\cos.B-\cos.A}{\sin.A+\sin.B} = \tan.\left(\frac{A-B}{2}\right).$$

NATURAL SINES, COSINES, ETC.

When the radius of the circle is taken as the unit of measure, the numerical values of the trigonometrical lines belonging to the different arcs of the quadrant, become *natural* sines, cosines, etc. They are then, in fact, but numbers expressing the number of times that these lines contain the radius of the circle in which they are taken. The tables usually contain only the sines and cosines, because these are generally sufficient for practical purposes, and the others, when required, are readily expressed in terms of them.

We proceed to explain a method for computing a table of natural sines and cosines.

It was shown, in Book V, that the linear value of the arc 180°, in a circle whose radius is unity, is

$$3.141592653.$$

This divided by 180×60, the number of minutes in 180°, will give the length of one minute of arc, which is

$$.00029088820867.$$

But there can be no sensible difference between the length of the arc 1′ and its sine; and, within narrow limits, that sine will increase directly with the arc.

Hence,

$$\begin{aligned}
\sin.\ 1' &= .0002908882. \\
\sin.\ 2' &= .0005817764. \\
\sin.\ 3' &= .0008726646. \\
\sin.\ 4' &= .0011635528. \\
\sin.\ 5' &= .0014544410. \\
\sin.\ 6' &= .0017453292. \\
\sin.\ 7' &= .0020362175. \\
\sin.\ 8' &= .0023271057. \\
\sin.\ 9' &= .0026179938. \\
\sin.\ 10' &= .0029088811.
\end{aligned}$$

Beyond this, the error which would arise from taking the arc for its sine, upon which the above proceeds, would affect the final decimal figures; and we must, therefore, continue the computation of the series by other processes. To find the values of the cosines of arcs, from 1′ to 10′, we have

$$\cos. = \sqrt{1 - \sin.^2} = 1 - \tfrac{1}{2}\sin.^2, \text{ nearly.}$$

That is, when the sines are very small fractions, as is the case for all arcs below 10′, we can find the cosine *by subtracting one half of the square of the sine from unity.*

Whence, $\cos. 1' = .9999999577.$
$\cos. 2' = .9999998308.$
$\cos. 3' = .9999993204.$
$\cos. 4' = .99999932304.$
$\cos. 5' = .99999894290.$
$\cos. 6' = .99999847753.$
$\cos. 7' = .99999792735.$
$\cos. 8' = .9999973035.$
$\cos. 9' = .9999965730.$
$\cos. 10' = .9999957703.$

The natural sines of arcs, differing by 1′, from 10′ up to 1°, may be computed from those of arcs less than 10′, by means of equation (11), group B, which is

$$\sin.(a+b) = 2\sin. a \cos. b - \sin.(a+b);$$

And when $a = b$, this equation becomes

$$\sin. 2a = 2\sin. a \cos. b. \quad \text{Eq. (30)}.$$

To find the sine of 11′, we make $a = 6'$, and $b = 5'$;

then $\sin. 11' = 2\sin. 6' \cos. 5' - \sin. 1' = .00319976913.$
$a = b = 6'$, $\sin. 12' = 2\sin. 6' \cos. 6'.$
$a = 7', b = 6'$, $\sin. 13' = 2\sin. 7' \cos. 6' - \sin. 1'.$
$a = b = 7$, $\sin. 14' = 2\sin. 7' \cos. 7'.$
$a = 8, b = 7$, $\sin. 15' = 2\sin. 8' \cos. 7' - \sin. 1'$

And so on to the

$$\sin. 30' = 2\sin. 15' \cos. 15'.$$
$$\sin. 1^\circ = \sin. 60' = 2\sin. 30' \cos. 30'.$$
$$\sin. 2^\circ = 2\sin. 1^\circ \cos. 1^\circ.$$
$$\sin. 3^\circ = 2\sin. 2^\circ \cos. 1^\circ - \sin. 1^\circ, \text{ etc., etc., etc.}$$

This process may be continued until we have found the sines and cosines of all arcs differing by 1′, from 0 to 90°, the values of the cosines being deduced successively from those of the sines by means of the formula,

$$\cos. = \sqrt{1 - \sin.^2}.$$

In this calculation, we began by assuming that, for small arcs, the sines and the arcs were sensibly equal.

It must be remembered that this is but an approximation; and although the error in the early stages of the process is not sufficient to affect any of the decimal figures which enter the tables, it will finally become so, since it is constantly increased in the operations by which the sines and cosines of the larger arcs are deduced from those of the smaller. When the error has been thus increased until it reaches the order of the last decimal unit of the table, which assigns our limit of error, we must have the means of detecting and correcting it.

This consists in calculating the sines and cosines of certain arcs by independent processes, and comparing them with those found by the above method.

We have seen, for example, (Prop. 7, B. V), that the chord of

30° = .517638090; whence, sin. 15° = .258819045.
15° = .2610523842; " " 7° 15′ = .130526192.
7° 15′ = .1308062583; " " 3° 7′ 30″ = .0654031291.

And so on to
sin. 14′ 3″ 45‴ = .004090604.
etc. etc. etc.

The following elegant method of deducing, from the sine of an arc, the sine and cosine of one half the arc, is given, assuming that the student is familiar with the simple algebraic principles upon which it depends.

Let us take the natural sine of 18°, which is .3090170, and make $x =$ sine, and y the cosine of $9° = \frac{18°}{2}$.

Then, $x^2 + y^2 = 1;$ (1)

and $2xy = .3090170$ (2); Eq. (30).

Adding, we have

$$x^2 + 2xy + y^2 = 1.3090170;$$

Taking the square root, we have

$$x + y = 1.144123. \quad (3)$$

Subtracting (2) from (1),

$$x^2 - 2xy + y^2 = .690983;$$

taking the square root,

$$x - y = -.831254^* \qquad (4)$$

Adding (3) and (4), $2x = .312869$,

hence, $x = \sin.9° \doteq .1564345$

Subtracting (4) from (3), $2y = 1.975377$

hence, $y = \cos.9° = .9876885$

Now, by making $x =$ the sine of 4° 30′, and $y =$ cosine of 4° 30′, and as before

$$x^2 + y^2 = 1$$

and $2xy = .1564345$,

we obtain the sine and cosine of 4° 30′; and another operation will give the sine and cosine 2° 15′, etc., etc.

We may in this manner compute the sines and cosines of all arcs resulting from the division of 18° by 2, and we may make their values accurate to any assigned decimal figure.

This has been carried far enough to show how a table of natural sines, etc., could be computed; but in consequence of the tedious numerical operations which the process requires, other methods are resorted to in the actual construction of the table.

The Calculus furnishes formulæ giving the values of the sines and cosines of arcs developed into rapidly converging series, and from these the sines and cosines of all arcs from 0° to 90°, can be determined with great

---

* When an arc is less than 45°, the cosine exceeds the sine; and when the arc is between 45° and 90°, the sine exceeds the cosine. Hence, when the arc is 9°, y, its cosine, exceeds x, its sine; and we therefore placed the minus sign before the second member of Eq. (4).

accuracy and with comparatively little labor. In the last two columns on each page of Table II, will be found the values thus computed of the sines and cosines of every degree and minute of a quadrant.

TRIGONOMETRICAL LINES FOR ARCS EXCEEDING 90°.

From the annexed figure, the construction of which needs no explanation, are deduced by simple inspection the results given in the following

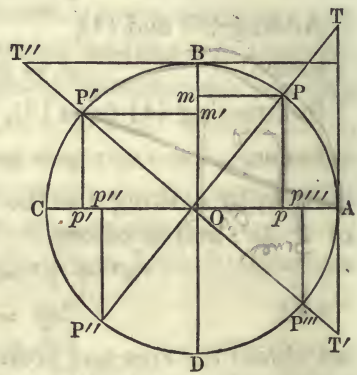


TABLE.

$90° + a°$	$270° - a°$
sin. = cos. a, cos. = − sin. a	sin. = − cos. a, cos. = − sin. a
tan. = − cot. a, cot. = − tan. a	tan. = cot. a, cot. = tan. a
sec. = − cosec. a, cosec. = sec. a	sec. = − cosec. a, cosec. = − sec. a
$180° - a°$	$270° + a°$
sin. = sin. a, cos. = − cos. a	sin. = − cos. a, cos. = sin. a
tan. = − tan. a, cot. = − cot. a	tan. = − cot. a, cot. = − tan. a
sec. = − sec. a, cosec. = cosec. a	sec. = cosec. a, cosec. = − sec. a
$180° + a°$	$360° - a°$
sin. = − sin. a, cos. = − cos. a	sin. = − sin. a, cos. = cos. a
tan. = tan. a, cot. = cot. a	tan. = − tan. a, cot. = − cot. a
sec. = − sec. a, cosec. = − cosec. a	sec. = sec. a, cosec. = − cosec. a

By means of this table, the values of the trigonometrical lines of any arc between 90° and 360°, can be expressed by those of arcs less than 90°.

If, for example, the arc is 118°, we have

$$\sin. 118° = \sin.(90° + 28°) = \cos.28°;$$
$$\tan.118° = \tan.(90° + 28°) = -\cot.28°;$$
etc., etc., etc.

For the arc 230°, we have

$$\sin. 230° = \sin.(270° - 40°) = -\cos. 40°;$$
$$\sec.230° = \sec.(270° - 40°) = -\text{cosec}.40°;$$
etc., etc., etc.

In many investigations, it becomes necessary to consider the functions of arcs greater than 360°; but since the addition of 360° any number of times to the arc a, will give an arc terminating in the extremity of a, it is obvious that the arc resulting from such addition will have the same functions as the arc a. And hence it follows that the functions of arcs, however great, may be expressed in terms of the functions of arcs less than 90°.

SECTION II.

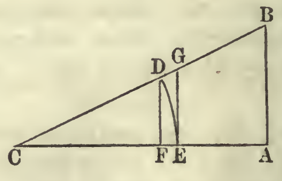
PLANE TRIGONOMETRY, PRACTICALLY APPLIED.

In the preceding section, the theory of Trigonometry has been quite fully developed, and the student should now be prepared for its various applications, were he acquainted with logarithms. But logarithms are no part of Trigonometry, and serve only to facilitate the numerical operations. Trigonometrical computations can be made without logarithms, and were so made long before the theory of logarithms was understood.

For this reason, we proceed at once to the solution of the following triangles.

1. The hypotenuse of a right-angled triangle is 21, and the base is 17; required the perpendicular and the acute angles.

Let CAB be the triangle, in which $CB = 21$, and $CA = 17$. With C as a center, and $CD = 1$ as a radius, describe the arc DE, of which the sine is DF, the tangent is EG, and the cosine is CF.



By similar triangles we have

$$CB : CA :: CD : CF;$$

that is, $\quad 21 : 17 :: 1 : \cos. C.$

Hence, $\quad \cos. C = \frac{17}{21} = .80952+.$

We must now turn to Table II, and find in the last two columns the cosine nearest to .80952, and the corresponding degrees and minutes will be the value of the angle C.

On page 56, of Tables, near the bottom of the page, and in the column with cosine at the top, we find .80953, which corresponds to 35° 56′ for the angle C. The angle B is, therefore, 54° 4′.

This Table is so arranged, that the sum of the degrees at the top and bottom of the page, added to the sum of the minutes which are found on the same horizontal line in the two side columns of the page, make 90°.

Thus, in finding the angle C, the number .80953 was found in the column with cosine at its foot. We therefore took the degrees from the bottom of the page, and the minutes were taken from the right hand column, counting upwards.

For the side AB, we have the proportion

$$CF : FD :: CA : AB;$$

or, $\cos. C : \sin. C :: 17 : AB$;

that is, $.80953 : .58708 :: 17 : AB$.

From which we find $AB = .58708 \times 17 \div .80953$;

whence, $AB = 12.328$.

If we had formed a table of natural tangents, as well as of natural sines, AB could have been found by the following proportion:

$$CE : EG :: CA : AB$$

or, $1 : \tan. C :: 17 : AB$;

whence, $AB = 17 \tan. C$.

The perpendicular AB may also be found by the proportion

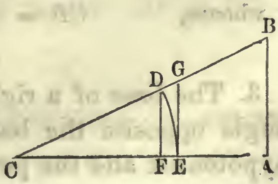
$$CD : DF :: CB : AB;$$

or, $1 : \sin. C :: 21 : AB$;

whence, $AB = 21 \sin. C = 21 \times .58708 = 12.32868$.

2. The two sides of a right-angled triangle are 150 and 125; required the hypotenuse and the acute angles.

Let CAB be the triangle, which is the same as in the preceding problem.



Then, from the similar triangles, CFD and CAB, we get

$$CF : FD :: CA : AB;$$

that is, $\cos. C : \sin. C :: 150 : 125 :: 6 : 5$,

which gives $6 \sin. C = 5 \cos. C$;

hence, $36 \sin.^2 C = 25 \cos.^2 C$.

Adding member to member, $36 \cos.^2 C = 36 \cos.^2 C$.

we have $36 (\sin.^2 C + \cos.^2 C) = 61 \cos.^2 C$.

But $\sin.^2 C + \cos.^2 C = 1$, (Eq. (1) Trigonometry);

whence, $61 \cos.^2 C = 36$;

$$\cos.^2 C = \frac{36}{61} = .5901639344;$$

and $\cos. C = .76816$, nearly.

To find the angle of which this is the cosine, we turn to page 60 of tables, and looking in the column having cosine at the head, we see that .76816 falls between .76868, which has 48′ opposite to it in the left hand column, and .76810, which has 49′ opposite to it in the same column. Now, the cosines of arcs less than 90° decrease when the arcs increase, and the converse; and while the increase of the arc is confined within the limits of 1′, the increase of the arc will be sensibly proportional to the decrease of the cosine.

$$\text{Hence,}\quad \begin{array}{r} 0.76828 \\ 0.76810 \\ \hline 18 \end{array} : \begin{array}{r} .76828 \\ .76816 \\ \hline 12 \end{array} :: 60'' : x''$$

which gives $x'' = 40''$.

The angle C is, therefore, equal to 39° 48′ 40″, and the angle $B = 90° - 39° 48' 40'' = 50° 11' 20''$.

To find CB, we have

$$CF : CD :: CA : CB;$$

or, $\cos. C : 1 :: 150 : CB$;

that is, $.78816 : 1 :: 150 : CB$;

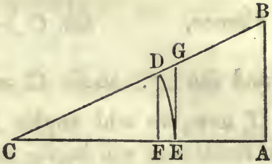
whence, $$CB = \frac{150}{.76816} = 195.27+.$$

3. The base of a right-angled triangle is 150, and the angle opposite the base is 50° 11′ 20″; required the hypotenuse and the perpendicular.

Let CAB be the triangle.

Then, (Prop. 4, Sec. I),

$\sin. 50° 11' 20'' : \sin. 90° :: 150 : CB.$



Whence,

$$CB = \frac{150}{.76816} = 195.27,$$

the same as in the preceding example.

To find AB, we have

$$CD : DF :: CB : AB;$$

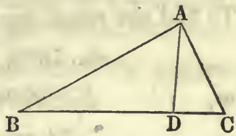
that is, $1 : \sin. C \text{ or } \cos. B :: 195.27 : AB;$

from which we find

$$AB = 195.27 \sin. 39° 48' 40'';$$

or, $AB = 125.01077.$

4. Two sides, the one 30 and the other 35, and the included angle 20°, of a triangle, are given, to find the other two angles and the third side.



Let BAC be the triangle, in which $BC = 35$, $BA = 30$, and the angle $B = 20°$. From A, the extremity of the shorter side, let fall on BC the perpendicular AD, thus dividing the triangle into the two right-angled triangles BAD and CAD.

Then, from the triangle BAD, we have

1st, $\sin. D : \sin. B :: BA : AD;$

or, $1 : \sin. 20° :: 30 : AD = 30 \sin. 20°.$

2d, $1 : \cos. B :: BA : BD;$

or, $1 : \cos. 20° :: 30 : BD = 30 \cos. B.$

In the table of natural sines, we find $\sin. 20° = .34202$, and the $\cos. 20° = .93969$; hence, $AD = 30 \times .34202 = 10.26060$, and $BD = 30 \times .93969 = 28.19070$, and therefore $DC = BC - BD = 6.8093$.

From the triangle CAD, we have

1st, $AC = \sqrt{\overline{AD}^2 + \overline{DC}^2} = \sqrt{(10.26)^2 + (6.9+)^2} = 12.367.$

2d, $AC : AD :: \sin. 90° : \sin. C;$

or, $12.367 : 10.26+ :: 1 : \sin. C$;

whence, $\sin. C = \frac{10.26}{12.367} = .82968.$

and the angle $C = 56° 3'$.

If, now, we add angles B and C, and take the sum from 180°, the remainder will be the angle BAC.

Hence, $\angle BAC = 180° - (56° 3' + 20°) = 103° 57'$.

5. Two sides, the one 18 and the other 24, and the angle opposite the side 24 equal to 76°, are given, to find the remaining side and the other two angles.

Let x denote the angle opposite the side 18. Then,

$$24 : 18 :: \sin. 76° : \sin. x, \text{ (Prop. 4, Trig.).}$$

or, $4 : 3 :: \sin. 76° : \sin. x.$

$$\sin. x = \tfrac{3}{4} \sin. 76° = \tfrac{3}{4} \times .97030 = .72772;$$

whence the angle opposite the side 18 is 46° 41′ 45″.

Adding this to the given angle, and taking the sum from 180°, we get 57° 17′ 15″ for the third angle.

To find the remaining side, denoted by y, we have

$$\sin. 76° : \sin. 57° 18' 15'' :: 24 : y;$$

or, $97030 : .84154 :: 24 : y.$

$$y = \frac{24 \times .84154}{.97030} = 20.815 = 3\text{d side.}$$

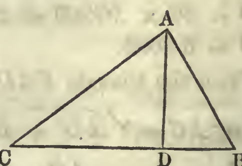
6. The three sides of a triangle are 18, 24, and 20.815; required the angles.

This problem may be solved by Prop. 6, or by Prop. 8, Trigonometry.

First. By Prop. 6.

In the triangle ABC, make $CB = 24$, $AC = 20.815$, and $AB = 18$.

Then,



$$24 : 38.815 :: 2.815 : CD - BD.$$

$$CD - BD = \frac{109.264225}{24} = 4.5527.$$

But $CD + BD = CB = 24.$

By addition, we get $2\,CD = 28.5527;$

dividing by 2, and $CD = 14.2763+.$

And hence, $BD = CB - CD = 24 - 14.2763 = 9.7237.$

In the triangle ADB, we have

$$BA : BD :: 1 : \cos. B$$

or, $18 : 9.7237 :: 1 : \cos. B = .54020$

Table II, Page 53, $\left\{ \begin{array}{l} \cos. 57°\ 18' = .54024 \\ \cos. 57°\ 19' = \underline{.54000} \end{array} \right\}$

diff. $=$ $24 : 60'' :: 4 : 10''$

hence, $\angle B = 57°\ 18'\ 10''.$

It will be observed that Examples 5 and 6 refer to the same triangle, and that in Example 5 the angle B was 57° 18′ 15″. This slight discrepancy in the results should be expected, on account of the small number of decimal places used in the computations.

Second. By Prop. 8.

Sum of the sides,	$= 62.815,$
half sum denoted by S,	$= 31.4075$
a	$= 24$
$S - a$	$= 7.4075$

Formula, $\cos. \tfrac{1}{2} A = \sqrt{\dfrac{S(S-a)}{bc}}$, radius being unity.

$$S(S-a) = 31.4075 \times 7.4075 = 232.65105625$$

$$bc = 20.815 \times 18 = 374.67$$

$$\frac{S(S-a)}{bc} = .62095 \text{ very nearly.}$$

$$\sqrt{.62095} = .78800.$$

Hence, $\cos. \tfrac{1}{2}A = .78800$, and $\tfrac{1}{2}A$ (Table II, page 59) $= 38°$ very nearly; the angle A is therefore equal to 76°, which agrees with Example 5.

7. Given, the three sides, 1425, 1338, and 493, of a triangle; required, the angle opposite the greater side, using the formula for the sine of one half an angle.

Make $a = 1425$, $b = 1338$, and $c = 493$; then the $\angle A$ is opposite the side a, and the formula is

$$\sin.^2 \tfrac{1}{2}A = \frac{(s - b)(s - c)}{bc}$$

in which s denotes the half sum of the three sides.

Then we have $s = 1628$, $s - b = 290$, $s - c = 1135$, $(s - b)(s - c) = 329150$, $bc = 659634$, $\frac{(s-b)(s-c)}{bc} = .498988$.

Hence, $\sin. \tfrac{1}{2}A = \sqrt{.498988} = .70632$.

In the table we find $\sin. 44° \, 56' \, 12'' = .70632$.

Therefore, $\tfrac{1}{2}A = 44° \, 56' \, 12''$, and $A = 89° \, 52' \, 24''$;—but little less than a right angle.

In these seven examples we have shown that it is possible to solve any plane triangle, in which three parts, one at least being a side, are given, without the aid of logarithms. But, when great accuracy is required, and the number of decimal places employed is large, the necessary multiplications and divisions, the raising to powers, and the extraction of roots, *become very tedious.* All of these operations may be performed without impairing the correctness of results, and with a great saving of labor, by means of logarithms; but, before using them, the student should be made acquainted with their nature and properties.

LOGARITHMS.

Logarithms are the exponents of the powers to which a fixed number, called the *base,* must be raised, to produce other numbers.

The exponent of a number is also a number expressing how many times the first number is taken as a factor.

Thus, let a denote any number; then a^3 indicates that a has been used three times as a factor, a^4 that it has been used four times as a factor, and a^n that it has been thus used n times.

Now, instead of calling these numbers 3, 4, — — n, exponents, we call them the logarithms of the powers a^3, a^4, — — a^n.

To multiply a^2 by a^5, we have simply to write a, giving it an exponent equal to $2 + 5$; thus, $a^2 \times a^5 = a^7$.

Hence, *the sum of the logarithms of any number of factors is equal to the logarithm of the product.*

To divide a^{12} by a^9, we have only to write a, giving it an exponent equal to $12 - 9$; thus, $a^{12} \div a^9 = a^3$; and, generally, the quotient arising from the division of a^m by a^n, is equal to a^{m-n}.

Hence, *the logarithm of a quotient is the logarithm of the dividend diminished by the logarithm of the divisor.*

If it is required to raise a number denoted by a^3, to the fifth power, we write a, giving it an exponent equal to 3×5; thus, $(a^3)^5 = a^{15}$, and, generally, $(a^n)^m = a^{nm}$.

Hence, *the logarithm of the power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

To extract the 5th root of the number a^3, we write a, giving it an exponent equal to $\frac{3}{5}$; thus, $\sqrt[5]{a^3} = a^{\frac{3}{5}}$, and, generally, to extract any root of a number, we divide the exponent of the number by the index of the root, and the quotient will be the exponent of the required root.

Hence, *the logarithm of a root of a number is equal to the quotient obtained by dividing the logarithm of the number by the index of the root.*

Now, understanding that by means of a table of logarithms we may find the numbers answering to given logarithms, with as much facility as we can find the logarithms of given numbers, we see from what precedes that multiplications, divisions, raising to powers, and the extraction of roots, may be performed by logarithms; and the utility of logarithms, in trigonometrical computations, mainly consists in the simplicity and abridgment of these operations as executed by them.

The common logarithms are those of which 10 is the base; that is, they are the exponents of 10.

Thus, $10^1 = 10$	Hence the logarithm			10	= 1.
$10^2 = 100$	"	"	"	100	= 2.
$10^3 = 1000$	"	"	"	1000	= 3.
$10^4 = 10000$	"	"	"	10000	= 4.
etc. etc.		etc.		etc.	etc.

Since $\frac{10}{10} = 1 = 10^{1-1} = 10^0$, and generally $\frac{a^m}{a^m} = a^0 = 1$, it follows that in this, as in all other systems, the logarithm of $1 = 0$.

From what precedes, it is evident that the logarithm of any number between 10 and 100 must be found between 1 and 2; that is, its logarithm is 1 plus a number less than 1; and any number between 100 and 1000, will have for its logarithm 2 plus some number less than 1, and so on. The fractional part of the logarithms of numbers are expressed decimally.

The entire number belonging to a logarithm is called its *index*. The index is never put in the tables, (except from 1 to 100), and need not be put there, because we always know what it is. It is always one less than the number of digits in the whole number. Thus, the number 3754 has 3 for the index to its logarithm, because the number consists of 4 digits; that is, *the logarithm is 3 and some decimal.*

The number 347.921 has 2 for the index of its logarithm, because the number is between 347 and 348, and 2 is the index for the logarithms of all numbers over 100, and less than 1000.

All numbers consisting of the same figures, whether integral, fractional, or mixed, have logarithms consisting of the same *decimal* part. The logarithms would differ only in their indices.

Thus, the number	7956. has	3.900695	for its log.
the number	795.6 has	2.900695	"
the number	79.56 has	1.900695	"
the number	7.956 has	0.900695	"
the number	.7956 has	—1.900695	"
the number	.07956 has	—2.900695	"

From this we perceive that we must take the logarithm out of the table for a mixed number or a decimal, the same as if the figures expressed an entire number; and then, to *prefix* the index, we must consider the *value* of the number.

The decimal part of a logarithm is always positive; but the index becomes negative when the number is a decimal; and the smaller the decimal, the greater the negative index. Hence,

To prefix the index to a decimal, count the decimal point as 1, and every cipher as 1, up to the first significant figure, and this is the negative index.

For example, find the logarithm of the decimal .0000831.

Num. .0000831; log. —5.919601.

The point is counted one, and each of the ciphers is counted one; therefore the index is *minus five.*

The smaller the decimal, the greater the negative index; and when the number becomes 0, the logarithm is *negatively infinite.*

Hence, the logarithmic sine of 0° is *negatively infinite,* however great the radius.

A number being given, to find its corresponding logarithm.

The logarithm of any number consisting of four figures, or less, is taken out of the table directly, and without the least difficulty.

Thus, to find the logarithm of the number 3725, we find 372 at the side of the table, and in the column marked 5 at the top, and opposite 372, we find .571126, for the decimal part of the logarithm.

Hence, the logarithm of 3725 is 3.571126.
the logarithm of 37250 is 4.571126.
the logarithm of 37.25 is 1.571126, etc.

Find the logarithm of the number 834785.

This number is so large that we cannot find it in the table, but we can find the numbers 8347 and 8348. The logarithms of these numbers are the same as the logarithms of the numbers 834700 and 834800, except the indices.

	834700	log.	5.921530
	834800	log.	5.921582
Difference,	100		52

Now, our proposed number, 834785, is between the two assumed numbers; and, of course, its logarithm lies between the logarithms of the two assumed numbers; and, without further comment, we may proportion it thus,

$$100 : 85 = 52 : 44.2$$

Or, $$1. : .85 = 52 : 44.2$$

Hence, for finding from the table the logarithm of a number consisting of more than four places of figures, we have the following

RULE.

Take from the table the log. of the number expressed by the the four superior figures; this, with the proper index, is the approximate logarithm. Multiply the number expressed by the remaining figures of the number, regarded as a decimal, by the tabular difference, and the product will be the correction to be added to the approximate log. to obtain the true log.

EXAMPLES.

1. What is the log. of 357.32514?

The log. of 357.3 is		2.553033
No. not included,	.2514	
Tabular diff.,	122	
	Prod., 30.6708; correction,	31
	log. sought,	2.553064
The log. of 35732.514 is		4.553064
" .035732514 "		—2.553064.

2. What is the log. of 7912532?

Approximate log.,	6.898286
$.532 \times 55 =$ correction,	29
True log. =	6.898315.

A logarithm being given, to find its corresponding number.

For example, what number corresponds to the log. 6.898315?

The index 6 shows that the entire part of the number must contain seven places of figures. With the decimal part, .898315, of the log., we turn to the table, and find the next less decimal part to be .898286, which corresponds to the superior places, 7912.

The difference between the given log. and the one next less is 29. This we divide by the tabular difference, 55, because we are working the converse of the preceding problem. Thus,

$$29 \div 55 = 52727+.$$

Place the quotient to the right of the four figures before found, and we shall have 7912527.27 for the number sought.

This example was taken from the preceding case, and the number found should have been 7912532; and so it would have been, had we used the true difference, 29.26, in place of 29.

When the numbers are large, as in this example, the result is liable to a small error, to avoid which the logarithms should contain a great number of decimal places; but the logarithms in our table contain a sufficient number of decimal places for most practical purposes.

Hence, for finding the number corresponding to any given logarithm, we have the following

RULE.

Look in the table for the decimal part of the given logarithm, and if not found, take the decimal next less, and take out the four corresponding figures.

Take the difference between the given log. and the next less in the table; divide that difference by the tabular difference, and write the quotient on the right of the four superior figures, and the result is the number sought.

Point off the whole number required by the given index.

EXAMPLES.

1. Given, the logarithm 3.743210, to find its corresponding number true to three places of decimals.

Ans. 5536.182.

2. Given, the logarithm 2.633356, to find its corresponding number true to two places of decimals.

Ans. 429.89.

3. Given, the logarithm —3.291746, to find its corresponding number. *Ans.* .0019577.

4. What number corresponds to the log. 3.233568?

Ans. 1712.25.

5. What is the number of which 1.532708 is the log.?

Ans. 34.0963.

6. Find the number whose log. is 1.067889.

Ans. 11.692.

EXPLANATION OF TABLE II.

Table I is merely a table of numbers and their corresponding logarithms, and requires no explanation other than that which has been given in connection with the subject of logarithms.

Table II, with the exception of the last two columns, which contain natural sines and cosines, is a table in which are arranged the logarithms of the numerical values of the several trigonometrical lines corresponding to the different angles in a quadrant. The values of these lines are computed to the radius 10,000,000,000, and their logarithms are nothing more than the logarithms, each increased by 10, of the natural sines, cosines, and tangents, of the same angles; because the values of these lines, for arcs of the same number of degrees taken in different circles, are directly proportional to the radii of the circles.

The natural sines are made to the radius of unity; and, of course, any particular sine is a decimal fraction, expressed by natural numbers. The logarithm of any natural sine, with its index increased by 10, will give the logarithmic sine. Thus, the natural sine of 3° is .052336.

The logarithm of this decimal is	— 2.718800
To which add	10.
The logarithmic sine of 3° is, therefore,	8.718800

In this manner we may find the logarithmic sine of any other arc, when we have the natural sine of the same arc.

If the natural sines and logarithmic sines were on the same radius, the logarithm of the natural sine would be the logarithmic sine, at once, without any increase of the index.

The radius for the logarithmic sines is arbitrarily taken so large that the index of its logarithm is 10. It might have been more or less; but, by common consent, it is settled at this value; so that the sines of the smallest arcs ever used shall not have a negative index.

In our preceding equations, sin. a, cos. a, etc., refer to *natural sines;* and by such equations we determine their values in natural numbers; and these numbers are put in Table II, under the heads of *nat. sine* and *nat. cosine*, as before observed.

When we have the sines and cosines of an arc, the tangent and cotangent are found by Eq. (3); that is,

$$\text{tan.} = \frac{R\ \text{sin.}}{\text{cos.}} \quad (6) \quad \text{cot.} = \frac{R\ \text{cos.}}{\text{sin.}};$$

and the secant is found by equation (4); that is,

$$\text{sec.} = \frac{R^2}{\text{cos.}}.$$

For example, the logarithmic sine of 6° is 9.019235, and its cosine 9.997614. From these it is required to find the logarithmic tangent, cotangent, and secant.

R sin.		19.019235
Cos.	subtract	9.997614
Tan. is		9.021621
R cos.		19.997614
Sin.	subtract	9.019235
Cotan. is		10.978379
R^2 is		20.000000
Cos.	subtract	9.997674
Secant is		10.002326

The secants and cosecants of arcs are not given in our table, because they are very little used in practice; and if any particular secant is required, it can be determined by subtracting the cosine from 20; and the cosecant can be found by subtracting the sine from 20.

The sine of every degree and minute of the quadrant is given, directly, in the table, commencing at 0°, and extending to 45°, at the head of the table; and from 45° to 90°, at the bottom of the table, increasing backward.

The same column that is marked sine, at the top, is marked cosine at the bottom; and the reason for this is apparent to any one who has examined the definitions of sines.

The difference of two consecutive logarithms is given, corresponding to *ten* seconds. Removing the decimal point one figure, will give the difference for *one* second; and if we multiply this difference by any proposed number of seconds, we shall have a difference corresponding to that number of seconds, above the logarithm corresponding to the preceding degree and minute.

For example, find the sine of 19° 17′ 22″.

The sine of 19° 17′, taken directly from the table, is	9.518829
The difference for 10″ is 60.2; for 1″, is 6.02; and	
$6.02 \times 22 =$	133
Hence, 19° 17′ 22″ sine is	9.518962

From this it will be perceived that there is no difficulty in obtaining the sine or tangent, cosine or cotangent, of any angle greater than 30′.

Conversely: Given, the logarithmic sine 9.982412, to find its corresponding arc. The sine next less in the table is 9.982404, which gives the arc 73° 48′. The difference between this and the given sine is 8, and the difference for 1″ is .61; therefore, the number of seconds corresponding to 8, must be discovered by dividing 8 by the decimal .61, which gives 13. Hence, the arc sought is 73° 48′ 13″.

These operations are too obvious to require a rule. When the arc is very small,—and such arcs as are sometimes required in Astronomy,—it is necessary to be very accurate; for this reason we omitted the difference for seconds for all arcs under 30′. Assuming that the sines and tangents of arcs under 30′ vary in the same proportion as the arcs themselves, we can find the sine or tangent of any very small arc, with great exactness, as follows:

The sine of 1′, as expressed in the table, is	6.463726
Divide this by 60; that is, subtract logarithm	1.778151
The logarithmic sine of 1″, therefore, is	4.685575
Now, for the sine of 17″, add the logarithm of 17	1.230449
Logarithmic sine of 17″, is	5.916024

In the same manner we may find the sine of any other small arc.

For example, find the sine of 14′ 21½″; that is, 861″5.

The logarithmic sine of 1″ is	4.685576
Add logarithm of 861.5,	2.935254
Logarithmic sine of 14′ 21½″,	7.620830

Two lines drawn, the one from the surface and the other from the center of the earth, to the center of the sun, make with each other an angle of 8.61″. What is the logarithmic sine of this angle?

The log. of the sine 1″ is	4.685575
Log. of 861,	0.935003
Log. sine of sun's horizontal parallax	= 5.620578

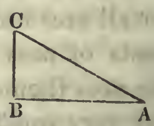
GENERAL APPLICATIONS WITH THE USE OF LOGARITHMS.

I. RIGHT-ANGLED TRIGONOMETRY.

One figure will be sufficient to represent the triangle in all of the following examples; the right angle being at *B*.

PRACTICAL PROBLEMS.

1. In a right-angled triangle, *ABC*, given the base *AB*, 1214, and the angle *A*, 51° 40′ 30″, to find the other parts.



To find BC.

Radius,	10.000000
: tan. A, 51° 40′ 30″,	10.102119
:: AB, 1214,	3.084219
: BC, 1535.8,	3.186338

REMARK.—When the first term of a logarithmic proportion is radius, the required logarithm is found by adding the second and third logarithms, rejecting 10 in the index, which is dividing by the first term.

In all cases we add the second and third logarithms together; which, in logarithms, is multiplying these terms together; and from that sum we subtract the first logarithm, whatever it may be, which is dividing by the first term.

To find AC.

Sin. C, or cos. A, 51° 40′ 30″,	9.792477
: AB, 1214,	3.084219
:: Radius,	10.000000
: AC, 1957.7,	3.291742

To find this resulting logarithm, we subtracted the first logarithm from the second, conceiving its index to be 13.

Let ABC represent any plane triangle, right-angled at B.

2. Given, AC 73.26, and the angle A, 49° 12′ 20″; required the other parts.

Ans. The angle C, 40° 47′ 40″; BC, 55.46; and AB, 47.87.

3. Given, AB 469.34, and the angle A, 51° 26′ 17″, to find the other parts.

Ans. The angle C, 38° 33′ 43″; BC, 588.7; and AC, 752.9.

4. Given, AB 493, and the angle C, 20° 14′; required, the remaining parts.

Ans. The angle A, 69° 46′; BC, 1338; and AC, 1425.5.

5. Let $AB = 331$, and the angle $A = 49° 14′$; what are the other parts?

Ans. AC, 506.9; BC, 383.9; and the angle C, 40° 46′.

6. If $AC=45$, and the angle $C=37° 22′$, what are the remaining parts?

Ans. AB, 27.31; BC, 35.76; and the angle A, 52° 38′.

7. Given, $AC = 4264.3$, and the angle $A = 56° 29' 13''$, to find the remaining parts.

Ans. AB, 2354.4; BC, 3555.4; and the angle C, 33° 30′ 47″.

8. If $AB = 44.2$, and the angle $A = 31° 12' 49''$, what are the other parts?

Ans. AC, 49.35; BC, 25.57; and the angle C, 58° 47′ 11″.

9. If $AB = 8372.1$, and $BC = 694.73$, what are the other parts?

Ans. { AC, 8400.9; the angle C, 85° 15′; and the angle A, 4° 45′.

10. If AB be 63.4, and AC be 85.72, what are the other parts?

Ans. { BC, 57.7; the angle C, 47° 42′; and the angle A, 42° 18′.

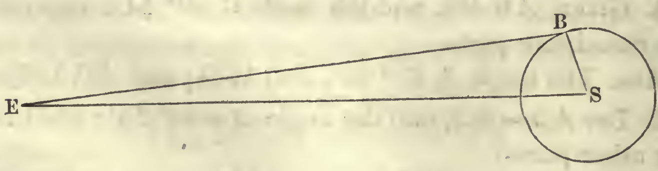
11. Given, $AC = 7269$, and $AB = 3162$, to find the other parts.

Ans. { BC, 7546; the angle C, 25° 47′ 7″; and the angle A, 64° 12′ 53″.

12. Given, $AC = 4824$, and $BC = 2412$, to find the other parts.

Ans. { The angle $A = 30° 00'$, the angle $C = 60° 00'$, and $AB = 4178$.

13. The distance between the earth and sun is 94,770,000 miles, and at that distance the semi-diameter of the sun subtends an angle of 16′ 6″. What is the diameter of the sun in miles? *Ans.* 887,700 miles.



In this example, let E be the center of the earth, S that of the sun, and EB a tangent to the sun's surface. Then the △ EBS is right-angled at B, and BS is the semi-diameter of the sun. The value of $2BS$ is required.

14. The semi-diameter of the earth is 3956 miles, and the distance of the sun 94.770000 miles. What angle will the semi-diameter of the earth subtend, as seen from the sun? *Ans.* 8.60″.

This angle is called, in astronomy, the sun's horizontal parallax. The preceding figure applies to this example, by supposing E to be the center of the sun, S that of the earth, and BS equal to 3956 miles.

15. The mean distance of the moon from the earth is 60.3 times 3960 miles, and at this distance the semi-diameter of the moon subtends an angle of 15′ 32″. What is the diameter of the moon in miles?

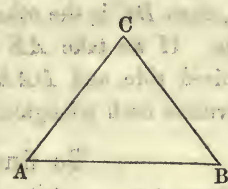
Ans. 2159 miles.

II. OBLIQUE-ANGLED TRIGONOMETRY.

PROBLEM I.

In a plane triangle, given a side and the two adjacent angles, to find the other parts.

In the triangle ABC, let $AB = 376$, the angle $A = 48° 3'$, and the angle $B = 40° 14'$, to find the other parts.



As the sum of the three angles of every triangle is always 180°, the third angle, C, must be $180° - 88° 17' = 91° 43'$.

To find AC.

Sin. 91° 43′,	9.999805
: AB, 376,	2.575188
:: sin. B 40° 14′,	9.810167
	12.385355
: AC, 243,	2.385550

Observe, that the sine of 91° 43′ is the same as the cosine of 1° 43′.

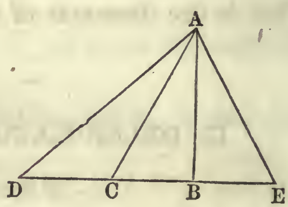
To find BC.

Sin. 91° 43′,	9.999805
: AB, 376,	2.575188
:: sin. A, 48° 3′,	9.871414
	12.446602
: sin. BC, 279.8,	2.446797

PROBLEM II.

In a plane triangle, given two sides and an angle opposite one of them, to determine the other parts.

Let AD = 1751 feet, one of the given sides; the angle $D = 31° 17' 19''$; and the side opposite, 1257.5. From these data, we are required to find the other side and the other two angles.



In this case we do not know whether AC or AE represents 1257.5, because $AC = AE$. If we take AC for the other given side, then DC is the other required side, and DAC is the vertical angle. If we take AE for the other given side, then DE is the required side, and DAE is the vertical angle. In such cases we determine both triangles.

To find the angle $E = C$.

(Prop. 4.)	$AC = AE = 1257.5$,	log.	3.099508
	: D, 31° 17′ 19″,	sin.	9.715460
	:: AD, 1751,	log.	3.243286
			12.958746
	$E = C$, 46° 18′,	sin.	9.859238

From 180° take 46° 18′, and the remainder is the angle DCA = 133° 42′.

The angle $DAC = ACE - D$, (Th. 11, B. I);
that is, $DAC = 46° 18' - 31° 17' 19'' = 15° 0' 41''$.

The angles D and E, taken from 180°, give

$$DAE = 102° 24' 41''.$$

To find *DC*.

Sin. D, 31° 17′ 19″,	log.	9.715460
: AC, 1257.5,	log.	3.099508
:: sin. DAC 15° 0′ 41″,	log.	0.413317
		12.512825
: DC, 626.86,		2.797165

To find *DE*.

Sin. D, 31° 17′ 17″,	9.715460
: AE, 1257.5,	3.099508
:: sin. DAE, 102° 24′ 41″,	9.989730
	13.089238
: DE, 2364.7,	3.373778

Remark.—To make the triangle possible, AC must not be less than AB, the sine of the angle D, when DA is made radius.

PROBLEM III.

In any plane triangle, given two sides and the included angle, to find the other parts.

Let $AD = 1751$, (see last figure), $DE = 2364.5$, and the included angle $D = 31° 17' 19''$. We are required to find AE, the angle DAE, and the angle E.

Observe that the angle E must be less than the angle DAE, because it is opposite a less side.

From	180°
Take D,	31° 17′ 19″,
Sum of the other two angles, =	148° 42′ 41″, (Th. 11, B. I),
½ sum =	74° 21′ 20″.

By Proposition 7,

$$DE + DA : DE - DA = \tan. 74° 21' 20'' : \tan. \tfrac{1}{2}(DAE - E).$$

That is,

$$4115.5 : 613.5 = \tan. 74° 21' 20'' : \tan. \tfrac{1}{2}(DAE - E).$$

Tan. 74° 21′ 20″,	10.552778
613.5,	2.787815
	13.340593
4115.5 log. (subtracted),	3.614423
tan.½(*DAEE* —) tan.28° 1′ 36″,	9.726170

But the half sum plus the half difference of any two quantities is equal to the greater of the two; and the half sum minus the half difference is equal the less.

Therefore, to	74° 21′ 20″,
Add	28° 1′ 36″,
DAE =	102° 22′ 56″,
E =	46° 19′ 45″,

To find *AE*.

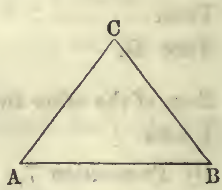
Sin. *E*, 46° 19′ 45″,	9.859323
: *DA*, 1751,	3.243286
:: sin. *D*, 31° 17′ 19″,	9.715460
	12.958746
: *AE*, 1257.2,	3.099423

PROBLEM IV.

Given, the three sides of a plane triangle, to find the angles.

Let $AC = 1751$, $CB = 1257.5$, $AB = 2364.5$, to find the angles A, B, and C.

If we take the formula for cosines, we will compute the greatest angle, which is C. To correspond with the formula,



$$\cos.\tfrac{1}{2}C = \sqrt{\frac{R^2 s(s-c)}{ab}},$$

we must take $a = 1257.5$, $b = 1751$, and $c = 2364.5$.

The half sum of these is,

$$s = 2686.5;\ \text{and}\ s - c = 322.$$

	R^2		20.000000
	$s = 2686.5$		3.429187
	$s - c = 322$		2.507856
	Numerator, log.		25.937043
	a 1257.5	3.099508	
	b 1751.	3.243286	
Denominator, log.		6.342794	6.342794
			2)19.594249
$\frac{1}{2}C =$ 51° 11′ 10″		cos.	9.797124
$C =$ 102 22 20			

The remaining angles may now be found by Problem 4.

PRACTICAL PROBLEMS.

Let ABC represent any oblique-angled triangle.

1. Given, AB 697, the angle A 81° 30′ 10″, and the angle B 40° 30′ 44″, to find the other parts.

Ans. AC, 534; BC, 813; and $\angle C$, 57° 59′ 6″.

2. If $AC = 720.8$, $\angle A = 70°$ 5′ 22″, $\angle B = 59°$ 35′ 36″, required the other parts.

Ans. AB, 643.2; BC, 785.8; and $\angle C$, 50° 19′ 2″.

3. Given, BC 980.1, the angle A 7° 6′ 26″, and the angle B 106° 2′ 23″, to find the other parts.

Ans. AB, 7284; AC, 7613.3; and $\angle C$, 66° 51′ 11″.

4. Given, AB 896.2, BC 328.4, and the angle C 113° 45′ 20″, to find the other parts.

Ans. { AC, 712; $\angle A$, 19° 35′ 48″; and $\angle B$, 46° 38′ 52″.

5. Given, $AC = 4627$, $BC = 5169$, and the angle A = 70° 25′ 12″, to find the other parts.

Ans. { AB, 4328; $\angle B$, 57° 29′ 56″; and $\angle C$, 52° 4′ 52″.

6. Given, AB 793.8, BC 481.6, and AC 500.0, to find the angles.

Ans. $\begin{cases} \angle A, 35° 15' 32''; \angle B, 36° 49' 18''; \text{ and } \angle C, \\ 107° 55' 10''. \end{cases}$

7. Given, AB 100.3, BC 100.3, and AC 100.3, to find the angles.

Ans. { The angle A, 60°; the angle B, 60°; and the angle C, 60°.

8. Given, AB 92.6, BC 46.3, and AC 71.2, to find the angles.

Ans. $\begin{cases} \angle A, 29° 17' 22''; \angle B, 48° 47' 31''; \text{ and } \angle C, \\ 101° 55' 8''. \end{cases}$

9. Given, AB 4693, BC 5124, and AC 5621, to find the angles.

Ans. $\begin{cases} \angle A, 57° 30' 28''; \angle B, 67° 42' 36''; \text{ and } \angle C, \\ 54° 46' 56''. \end{cases}$

10. Given, AB 728.1, BC 614.7, and AC 583.8, to find the angles.

Ans. $\begin{cases} \angle A = 54° 32' 52'', \angle B = 50° 40' 58'', \text{ and } \angle C \\ = 74° 46' 10''. \end{cases}$

11. Given, AB 96.74, BC 83.29, and AC 111.42, to find the angles.

Ans. $\begin{cases} \angle A = 46° 30' 45'', \angle B = 76° 3' 45'', \text{ and } \angle C \\ = 57° 25' 30''. \end{cases}$

12. Given, AB 363.4, BC 148.4, and the angle B 102° 18′ 27″, to find the other parts.

Ans. $\begin{cases} \angle A = 20° 9' 17'', \text{ the side } AC = 420.8, \text{ and } \angle C \\ = 57° 32' 16''. \end{cases}$

13. Given, AB 632, BC 494, and the angle A 20° 16′, to find the other parts, the angle C being acute.

Ans. $\begin{cases} \angle C = 26° 18' 19'', \angle B = 133° 25' 41'', \text{ and} \\ AC = 1035.86. \end{cases}$

14. Given, AB 53.9, AC 46.21, and the angle B 58° 16′, to find the other parts.

Ans. $\angle A = 38° 58'$, $\angle C = 82° 46'$, and $BC = 34.16$.

15. Given, AB 2163, BC 1672, and the angle C 112° 18′ 22″, to find the other parts.

Ans. AC, 877.2; ∟B, 22° 2′ 16″; and ∟A, 45° 39′ 22″.

16. Given, AB 496, BC 496, and the angle B 38° 16′, to find the other parts.

Ans. AC, 325.1; ∟A, 70° 52′; and ∟C, 70° 52′.

17. Given, AB 428, the angle C 49° 16′, and $(AC+BC)$ 918, to find the other parts, the angle B being obtuse.

Ans. { The angle $A = 38° 44′ 48″$, the angle $B = 91° 59′ 12″$, $AC = 564.49$, and $BC = 353.5$.

18. Given, AC 126, the angle B 29° 46′, and $(AB-BC)$ 43, to find the other parts.

Ans. { The angle $A = 55° 51′ 32″$, the angle $C = 94° 22′ 28″$, $AB = 253.05$, and $BC = 210.054$.

19. Given, AB 1269, AC 1837, and the angle A 53° 16′ 20″, to find the other parts.

Ans. { ∟$B = 83° 23′ 47″$, ∟$C = 43° 19′ 53″$, and $BC = 1482.16$.

SECTION III.

APPLICATION OF TRIGONOMETRY TO MEASURING HEIGHTS AND DISTANCES.

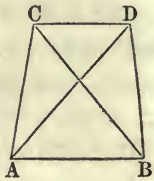
In this useful application of Trigonometry, a base line is always supposed to be measured, or given in length; and by means of a quadrant, sextant, circle, theodolite, or some other instrument for measuring angles, such angles are measured as, connected with the base line and the objects whose heights or distances it is proposed to determine, enable us to compute, from the principles of Trigonometry, what those heights or distances are.

Sometimes, particularly in marine surveying, horizontal angles are determined by the compass; but the varying effect of surrounding bodies on the needle, even in situations little removed from each other, and the general construction of the instrument itself, render it unfit to be employed in the determination of angles where anything like precision is required.

The following problems present sufficient variety, to guide the student in determining what will be the most eligible mode of proceeding, in any case that is likely to occur in practice.

PROBLEM I.

Being desirous of finding the distance between two distant objects, *C* and *D*, I measured a base, *AB*, of 384 yards, on the same horizontal plane with the objects *C* and D. At A, I found the angles $DAB = 48° \ 12'$, and $CAB = 89° \ 18'$; at B, the angles ABC 46° 14′, and ABD 87° 4′. It is required, from these data, to compute the distance between C and D.



From the angle CAB, take the angle DAB; the remainder, 41° 6′, is the angle CAD. To the angle DBA, add the angle DAB, and 44° 44′, the supplement of the sum, is the angle ADB. In the same way the angle ACB, which is the supplement of the sum of CAB and CBA, is found to be 44° 28′.

Hence, in the triangles ABC and ABD, we have

Sin. ACB, 44° 28′,	9.845405
: AB, 384 yards,	2.584331
:: sin. ABC, 46° 14′,	9.858635
	12.442966
: AC, 395.9 yards,	2.597561
Sin. ADB, 44° 44′,	9.847454
: AB, 384 yards,	2.584331
:: sin. ABD, 87° 4′,	9.999431
	12.583762
: AD, 544.9 yards,	2.736308

Then, in the triangle CAD, we have given the sides CA and AD, and the included angle CAD, to find CD; to compute which we proceed thus:

The supplement of the angle CAD, is the sum of the angles ACD and ADC;

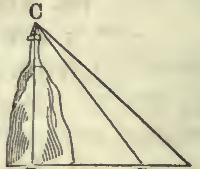
Hence, $\frac{ACD + ADC}{2} = 69° \ 27'$; and, by proportion we have,

$AD + AC$	(= 940.8)	2.937497
: $AD - AC$	(= 149)	2.173186
:: tan. $\frac{ACD + ADC}{2}$	(= 69° 27′)	10.426108
		12.599294

tan. $\frac{ACD-ADC}{2}$ (= 22 54)	9.551797
the angle *ACD*, sum, 92 21	
the angle *ADC*, diff., 46 33	
Sin. *ADC*, 46° 33′,	9.860922
: *AC*, 395.9 yards,	2.597585
:: sin. *CAD*, 41° 6′,	9.817813
	12.415398
: *CD*, 358.5 yards,	2.554476

PROBLEM II.

To determine the altitude of a lighthouse, I observed the elevation of its top above the level sand on the sea-shore, to be 15° 32′ 18″; and measuring directly from it, 638 yards along the sand, I then found its elevation to be 9° 56′ 26″. Required the height of the lighthouse.



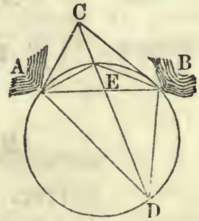
Let *CD* represent the height of the lighthouse above the level of the sand, and let *B* be the first station, and *A* the second; then the angle *CBD* is 15° 32′ 18, and the angle *CAB* is 9° 56′ 26″; therefore, the angle *ACB*, which is the difference of the angles *CBD* and *CAB*, is 5° 35′ 52″.

Hence,	Sin. *ACB*, 5° 35′ 52″,	8.989201
	: *AB*, 638,	2.804821
	:: sin. angle *A*, 9° 56′ 26″,	9.237107
		12.041928
	: *BC*, 1129.06 yards,	3.052727
	Radius,	10.000000
	· *BC*, 1129.06,	3.052727
	:: sin. *CBD*, 15° 32′ 18″,	9.427945
		12.480672
	: *DC*, 302.46 yards,	2.480672

PROBLEM III.

Coming from sea, at the point D I observed two headlands, A and B, and inland, at C, a steeple, which appeared between the headlands. I found, from a map, that the headlands were 5.35 from each other; that the distance from A to the steeple was 2.8 miles, and from B to the steeple 3.47 miles; and I found, with a sextant, that the angle ADC was 12° 15, and the angle BDC, 15° 30′. Required my distance from each of the headlands, and from the steeple.

CONSTRUCTION.



The angle between the two headlands is the sum of 15° 30′ and 12° 15′, or 27° 45′. Take double this sum, 55° 30′. Conceive AB to be the chord of a circle, and the arc on one side of it to be 55° 30′; and, of course, the other will be 304° 30′. The point D will be somewhere in the circumference of this circle. Consider that point as determined, and draw CD.

In the triangle ABC, we have all the sides, and, of course, we can find all the angles; and if the angle ACB is less than 180°— 27° 45′ = 152° 15′, then the circle cuts the line CD in a point E, and C is without the circle.

Draw AE, BE, AD, and BD. $AEBD$ is a quadrilateral in a circle, and $\angle AEB + \angle ADB = 180°$.

The $\angle ADE =$ the $\angle ABE$, because both are measured by one half the arc AE. Also, $\angle EDB = \angle EAB$, for a similar reason.

Now, in the triangle AEB, its side AB, and all its angles, are known; and from thence AE can be computed. Then, having the two sides, AC and AE, of the triangle AEC, and the included angle CAE, we can find the angle AEC, and, of course, its supplement, AED. Then, in the triangle AED, we have the side AE, and the two angles AED and ADE, from which we can find AD.

The computation, at length, is as follows:

To find AE.

Angle EAB =	15° 30′	Sin. AEB, 152° 15′,	9.668027
Angle EBA =	12° 15′	: AB, 5.35,	.728354
	27° 45′	:: sin. ABE 12° 15′	9.326700
	180°		10.055054
Angle AEB =	152° 15′	: AE, 2.438,	.387027

To find the angle BAC.

BC,	3.47		
AB,	5.35	log.	.728354
AC,	2.80	log.	.447158
	2) 11.62		1.175512
	5.81	log.	.764176
BC,	2.34	log.	.369216
			20
			21.133392
			2) 19.957880
	17° 41′ 58″	cos.	9.978940
	2		
Angle BAC =	35° 23′ 56″		
Angle EAB =	15° 30′		
Angle EAC =	19° 53′ 56″		
	180°		
	2) 160° 6′ 4″		
	80° 3′ 2″		$\frac{AEC + ACE}{2}$

To find the angles AEC and ACE.

$AC + AE$	5.238	.719165
: $AC - AE$	.362	$-$1.558709
:: tan. $\frac{AEC + ACE}{2}$	80° 3′ 2″	10.755928
		10.314637
: tan. $\frac{AEC - ACE}{2}$	21° 30′ 12″	9.595472

angle AEC, 101° 33′ 14″, sum.

angle ACE or ACD, 58° 32′ 50″, diff.
angle CDA, 12° 15′

70° 47′ 50″, supplement 109° 12′ 10″, angle CAD
35° 23′ 56″, angle CAB

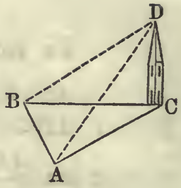
73° 48′ 14″, angle BAD

To find AD.

Sin. ADC, 12° 15′,	9.326700
: AC, 2.8,	.447158
:: sin. ACD 58° 32′ 50″,	9.930985
	10.378143
: AD 11.26 miles.	1.051443

PROBLEM IV.

The elevation of a spire at one station was 23° 50′ 17″, and the horizontal angle at this station, between the spire and another station, was 93° 4′ 20″. The horizontal angle at the latter station, between the spire and the first station, was 54° 28′ 36″, and the distance between the two stations was 416 feet. Required the height of the spire.



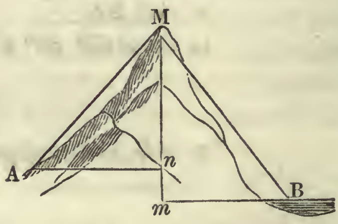
Let *CD* be the spire, *A* the first station, and *B* the second; then the vertical angle *CAD* is 23° 50′ 17″; and as the horizontal angles, *CAB* and *CBA*, are 93° 4′ 20″ and 54° 28′ 36″, respectively, the angle *ACB*, the supplement of their sum, is 32° 27′ 4″.

To find *AC*.

Sin. *BCA*, 32° 27′ 3″,	9.729634
: side *AB*, 416,	2.619093
:: sin. *ABC*, 54° 28′ 36″,	9.910560
	12.529653
: side *AC*, 631,	2.800019

To find *DC*.

Radius,	10.000000
: side *AC*, 631,	2.800019
:: tan. *DAC*, 23° 50′ 17″,	9.645270
: *DC*, 278.8,	2.445289



By the application of Problem 4, we can compute the distance between two horizontal planes, if the same object is visible from both.

For example, let *M* be a prominent tree or rock near the top of a mountain, and by observations taken at *A*, we can determine the perpendicular *Mn*. By like observations taken at *B*, we can determine the perpendicular *Mm*. The difference between these two perpendiculars is *nm*, or the difference in the elevation between the two points *A* and *B*. If the distances between *A* and *n*, or *B* and *m*, are considerable, or more than two or three miles, corrections must be made for the convexity of the earth; but for less distances such corrections are not necessary.

PRACTICAL PROBLEMS.

1. Required the height of a wall whose angle of elevation, at the distance of 463 feet, is observed to be 16° 21′. *Ans.* 135.8 feet.

2. The angle of elevation of a hill is, near its bottom, 31° 18′, and 214 yards further off, 26° 18′. Required the perpendicular height of the hill, and the distance of the perpendicular from the first station.

Ans. The height of the hill is 565.2 yards, and the distance of the perpendicular from the first station is 929.6 yards.

3. The wall of a tower which is 149.5 feet in height, makes, with a line drawn from the top of it to a distant object on the horizontal plane, an angle of 57° 21′. What is the distance of the object from the bottom of the tower? *Ans.* 233.3 feet.

4. From the top of a tower, which is 138 feet in height, I took the angle of depression of two objects standing in a direct line from the bottom of the tower, and upon the same horizontal plane with it. The depression of the nearer object was found to be 48° 10′, and that of the further, 18° 52′. What was the distance of each from the bottom of the tower?

Ans. Distance of the nearer, 123.5 feet; and of the further, 403.8 feet.

5. Being on the side of a river, and wishing to know the distance of a house on the opposite side, I measured 312 yards in a right line by the side of the river, and then found that the two angles, one at each end of this line, subtended by the other end and the house, were 31° 15′ and 86° 27′. What was the distance between each end of the line and the house? *Ans.* 351.7, and 182.8 yards.

6. Having measured a base of 260 yards in a straight line, on one bank of a river, I found that the two angles, one at each end of the line, subtended by the

other end and a tree on the opposite bank, were 40° and 80°. What was the width of the river?

Ans. 190.1 yards.

7. From an eminence of 268 feet in perpendicular height, the angle of depression of the top of a steeple which stood on the same horizontal plane, was found to be 40° 3′, and of the bottom, 56° 18′. What was the height of the steeple? *Ans.* 117.8 feet.

8. Wanting to know the distance between two objects which were separated by a morass, I measured the distance from each to a point from whence both could be seen; the distances were 1840 and 1428 yards, and the angle which, at that point, the objects subtended, was 36° 18′ 24″. Required their distance. *Ans.* 1090.85 yards.

9. From the top of a mountain, three miles in height, the visible horizon appeared depressed 2° 13′ 27″. Required the diameter of the earth, and the distance of the boundary of the visible horizon.

Ans. { Diameter of the earth, 7958 miles; distance of the horizon, 154.54 miles.

10. From a ship a headland was seen, bearing north 39° 23′ east. After sailing 20 miles north, 47° 49′ west, the same headland was observed to bear north, 87° 11′ east. Required the distance of the headland from the ship at each station.

Ans. { At first station, 19.09 miles; at the second, 26.96 miles.

11. The top of a tower, 100 feet above the level of the sea, was seen as on the surface of the sea, from the masthead of a ship, 90 feet above the water. The diameter of the earth being 7960 miles, what was the distance between the observer and the object?

Ans. 23.9 plus $\frac{1}{13}$ for refraction = 25.7 miles.

12. From the top of a tower, by the seaside, 143 feet high, it was observed that the angle of depression of a

ship's bottom, then at anchor, measured 35°; what, then, was the ship's distance from the foot of the tower?

Ans. 204.22 feet.

13. Wanting to know the breadth of a river, I measured a base of 500 yards in a straight line on one bank; and at each end of this line I found the angles subtended by the other end and a tree on the opposite bank of the river, to be 53° and 79° 12′. What, then, was the perpendicular breadth of the river? *Ans.* 529.48 yards.

14. What is the perpendicular height of a hill, its angle of elevation, taken at the bottom of it, being 46°, and 200 yards further off, on a level with the bottom, 31°? *Ans.* 286.28 yards.

15. Wanting to know the height of an inaccessible tower, at the least accessible distance from it, on the same horizontal plane, I found its angle of elevation to be 58°; then going 300 feet directly from it, I found the angle there to be only 32°; required the height of the tower, and my distance from it at the first station.

Ans. { Height, 307.53 feet.
Distance, 192.15 "

16. Two ships of war, intending to cannonade a fort, are, by the shallowness of the water, kept so far from it, that they suspect their guns cannot reach it with effect. In order, therefore, to measure the distance, they separate from each other a quarter of a mile, or 440 yards, and then each ship observes and measures the angle which the other ship and fort subtends; these angles are 83° 45′, and 85° 15′. What, then, is the distance between each ship and the fort? *Ans.* { 2292.26 yards.
2298.05 "

17. A point of land was observed by a ship, at sea, to bear east-by-south;* and after sailing north-east 12 miles,

* That is, one point south of east. A point of the compass is 11° 15′.

it was found to bear south-east-by-east. It is required to determine the place of that headland, and the ship's distance from it at the last observation.

Ans. Distance, 26.0728 miles.

18. Wishing to know my distance from an inaccessible object, O, on the opposite side of a river, and having a chain or chord for measuring distances, but no instrument for taking angles; from each of two stations, A and B, which were taken at 500 yards asunder, I measured in a direct line from the object, O, 100 yards, viz., AC and BD, each equal to 100 yards; and I found that the diagonal AD measured 550 yards, and the diagonal BC 560. What, then, was the distance of the object 0 from each station A and B?

Ans. $\begin{cases} AO, 536.25 \text{ yards.} \\ BO, 500.09 \quad \text{"} \end{cases}$

19. A navigator found, by observation, that the summit of a certain mountain, which he supposed to be 45 minutes of a degree distant, had an altitude above the sea horizon of 31′ 20″. Now, on the supposition that the earth's radius is 3956 miles, and the observer's *dip* was 4′ 15″, what was the height of the mountain?

Ans. 3960 feet.

REMARK.—This should be diminished by about one eleventh part of itself, for the influence of horizontal refraction.

20. From two ships, A and B, which are anchored in a bay, two objects, C and D, on the shore, can be seen. These objects are known to be 500 yards apart. At the ship A, the angle subtended by the objects was measured, and found to be 41° 25′; and that by the object D and the other ship was found to be 52° 12′. At the other ship, the angle subtended by the objects on shore was found to be 48° 10′; and that by the object C, and the ship A, to be 47° 40′. Required the distance between

the ships, and the distance from each ship to the objects on shore.

Ans.

Distance between ships,	395.6	yards.
From ship A to object D,	743.5	"
From ship A to object C,	467.7	"
From ship B to object D,	590.5	"

To solve this problem, suppose the distance between the ships to be 100 yards, and determine the several distances, including the distance between the objects, C and D, under this supposition; then multiply the values thus found for the required distances by the quotient obtained by dividing the given value of CD by the computed value.

PART II.

SPHERICAL GEOMETRY

AND

TRIGONOMETRY.

SECTION I.

SPHERICAL GEOMETRY.

DEFINITIONS.

1. **Spherical Geometry** has for its object the investigation of the properties, and of the relations to each other, of the portions of the surface of a sphere which are bounded by the arcs of its great circles.

2. A **Spherical Polygon** is a portion of the surface of a sphere bounded by three or more arcs of great circles, called the *sides* of the polygon.

3. The **Angles** of a spherical polygon are the angles formed by the bounding arcs, and are the same as the angles formed by the planes of these arcs.

4. A **Spherical Triangle** is a spherical polygon having but three sides, each of which is less than a semi-circumference.

5. A **Lune** is a portion of the surface of a sphere included between two great semi-circumferences having a common diameter.

6. A **Spherical Wedge, or Ungula,** is a portion of the surface of a sphere included between two great semi-circles having a common diameter.

7. A **Spherical Pyramid** is a portion of a sphere bounded by the faces of a solid angle having its vertex at the center, and the spherical polygon which these faces intercept on the surface. This spherical polygon is called the *base* of the pyramid.

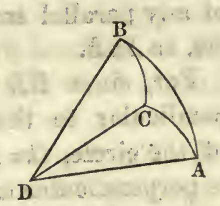
8. The **Axis** of a great circle of a sphere is that diameter of the sphere which is perpendicular to the plane of the circle. This diameter is also the axis of all small circles parallel to the great circle.

9. A **Pole** of a circle of a sphere is a point on the surface of the sphere equally distant from every point in the circumference of the circle.

10. **Supplemental,** or **Polar Triangles,** are two triangles on a sphere, so related that the vertices of the angles of either triangle are the poles of the sides of the other.

PROPOSITION I.

Any two sides of a spherical triangle are together greater than the third side.



Let AB, AC, and BC, be the three sides of the triangle, and D the center of the sphere.

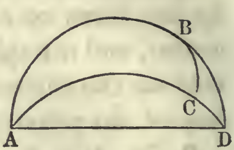
The arcs AB, AC, and BC, are measured by the angles of the planes that form the solid angle at D. But any two of these angles are together greater than the third angle, (Th. 18, B. VI). Therefore, any two sides of the triangle are, together, greater than the third side.

Hence the proposition.

PROPOSITION II.

The sum of the three sides of any spherical triangle is less than the circumference of a great circle.

Let ABC be a spherical triangle; the two sides, AB and AC, produced, will meet at the point which is diametrically opposite to A, and the arcs, ABD and ACD are together equal to a great circle. But, by the last proposition, BC is less than the two arcs, BD and DC. Therefore, $AB + BC + AC$, is less than $ABD + ACD$; that is, less than a great circle.

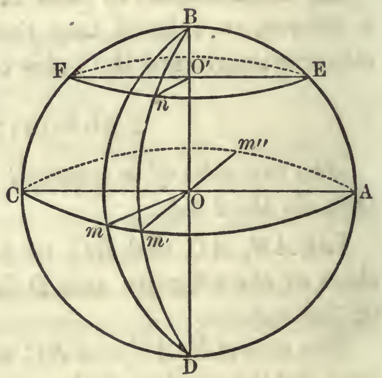


Hence the proposition.

PROPOSITION III.

The poles of a great circle of a sphere are the extremities of its axis, and these points are also the poles of all small circles parallel to the great circle.

Let O be the center of the sphere, and BD the axis of the great circle, $Cm\ Am''$; then will B and D, the extremities of the axis, be the poles of the circle, and also the poles of any parallel small circle, as FnE.



For, since BD is perpendicular to the plane of the circle, $Cm\ Am''$, it is perpendicular to the lines OA, Om', Om'', etc., passing through its foot in the plane, (Th. 3, B. VI); hence, all the arcs, Bm, Bm', etc., are quadrants, as are also the arcs Dm, Dm', etc. The points B and D are, therefore, each equally distant from all the points in the circumference, $Cm\ Am''$; hence, (Def. 9), they are its poles.

Again, since the radius, OB, is perpendicular to the plane of the circle, $Cm\ Am''$, it is also perpendicular to the plane of the parallel small circle, FnE, and passes through its center, O'. Now, the chords of the arcs, BF, Bn, BE, etc., being oblique lines, meeting the plane of the small circle at equal distances from the foot of the perpendicular, BO', are all equal, (Th. 4, B. VI); hence, the arcs themselves are equal, and B is one pole of the circle, FnE. In like manner we prove the arcs, DF, Dn, DE, etc., equal, and therefore D is the other pole of the same circle.

Hence the proposition, etc.

Cor. 1. *A point on the surface of a sphere at the distance of a quadrant from two points in the arc of a great circle, not at the extremities of a diameter, is a pole of that arc.*

For, if the arcs, Bm, Bm', are each quadrants, the angles, BOm and BOm', are each right angles; and hence, BO is perpendicular to the plane of the lines, Om and Om', which is the plane of the arc, $m\ m'$; B is therefore the pole of this arc.

Cor. 2. *The angle included between the arc of a great circle and the arc of another great circle, connecting any of its points with the pole, is a right angle.*

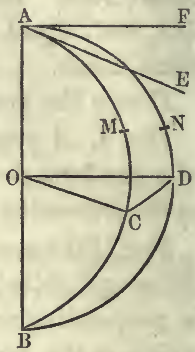
For, since the radius, BO, is perpendicular to the plane of the circle, $Cm\ Am''$, every plane passed through this radius is perpendicular to the plane of the circle; hence, the plane of the arc Bm is perpendicular to that of the arc Cm; and the angle of the arcs is that of their planes.

PROPOSITION IV.

The angle formed by two arcs of great circles which intersect each other, is equal to the angle included between the tangents to these arcs at their point of intersection, and is measured by that arc of a great circle whose pole is the vertex of the angle, which is limited by the sides of the angle or the sides produced.

Let AM and AN be two arcs intersecting at the point A, and let AE and AF be the tangents to these arcs at this point. Take AC and AD, each quadrants, and draw the arc CD, of which A is the pole, and OC and OD are the radii.

Now, since the planes of the arcs intersect in the radius OA, and AE is a tangent to one arc, and AF a tangent to the other, at the common point A, these tangents form with each other an angle which is the measure of the angle of the planes of the arcs; but the angle of the planes of the arcs is taken as the angle included by the arcs, (Def. 4).



Again, because the arcs, AC and AD, are each quadrants, the angles, AOC, AOD, are right angles; hence the radii, OC and OD, lie, the one in one face, and the other in the other face, of the diedral angle formed by the planes of the arcs, and are perpendicular to the common intersection of these faces at the same point. The angle, COD, is therefore the angle of the planes, and consequently the angle of the arcs; but the angle COD is measured by the arc CD.

Hence the proposition.

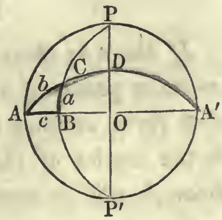
Cor. 1. Since the angles included between the arcs of great circles on a sphere, are measured by other arcs of great circles of the same sphere, we may compare such angles with each other, and construct angles equal to other angles, by processes which do not differ in principle from those by which plane angles are compared and constructed.

Cor. 2. Two arcs of great circles will form, by their intersection, four angles, the opposite or vertical ones of which will be equal, as in the case of the angles formed by the intersection of straight lines, (Th. 4, B. I).

PROPOSITION V.

The surface of a hemisphere may be divided into three right-angled and four quadrantal triangles, and one of these right-angled triangles will be so related to the other two, that two of its sides and one of its angles will be complemental to the *sides of one of them, and two of its sides supplemental to two of the sides of the other.*

Let ABC be a right-angled spherical triangle, right angled at B.



Produce the sides, AB and AC, and they will meet at A', the opposite point on the sphere. Produce BC, both ways, 90° from the point B, to P and P', which are, therefore, poles to the arc AB, (Prop. 3). Through A, P, and the center of the sphere, pass a plane, cutting the sphere into two equal parts, forming a great circle on the sphere, which great circle will be represented by the circle $PAP'A'$ in the figure. At right angles to this plane, pass another plane, cutting the sphere into two equal parts; this great circle is represented in the figure by the straight line, POP'. A and A' are the poles to the great circle, POP'; and P and P' are the poles to the great circle, ABA'.

Now, CPD is a spherical triangle, right-angled at D, and its sides CP and CD are complemental respectively to the sides BC and AC of the $\triangle\ ABC$, and its side PD is complemental to the arc DO, which measures the $\angle BAC$ of the same triangle. Again, the $\triangle\ A'BC$ is right-angled at B, and its sides $A'C$, $A'B$, are supplemental respectively to the sides AC, AB, of the $\triangle ABC$. Therefore, the three right-angled $\triangle$'s, ABC, CPD, and $A'BC$, have the required relations. In the $\triangle\ ACP$, the side AP is a quadrant, and for this reason the $\triangle$ is called a quadrantal triangle. So also, are the $\triangle$'s $A'CP$, ACP', and $P'CA'$, quadrantal triangles. Hence the proposition.

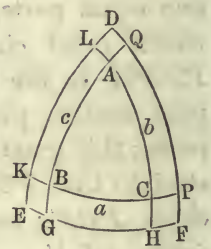
SCHOLIUM.—In every triangle there are *six* elements, three sides and three angles, called the parts of the triangle.

Now, if all the parts of the triangle ABC are known, the parts of each of the $\triangle$'s, PCD and $A'BC$, are as completely known. And when the parts of the $\triangle\ PCD$ are known, the parts of the $\triangle$'s ACP

and $A'CP$ are also known; for, the side PD measures each of the $\lfloor$'s PAC and $PA'C$, and the angle CPD, added to the right angle $A'PD$, gives the $\lfloor A'PC$, and the $\lfloor CPA$ is supplemental to this. Hence, the solution of the $\triangle ABC$ is a solution of the two right-angled and four quadrantal $\triangle$'s, which together with it make up the surface of the hemisphere.

PROPOSITION VI.

If there be three arcs of great circles whose poles are the angular points of a spherical triangle, such arcs, if produced, will form another triangle, whose sides will be supplemental to the angles of the first triangle, and the sides of the first triangle will be supplemental to the angles of the second.



Let the arcs of the three great circles be GH, PQ, KL, whose poles are respectively A, B, and C. Produce the three arcs until they meet in D, E, and F. We are now to prove that E is the pole of the arc AC; D the pole of the arc BC; F the pole to the arc AB. Also, that the side EF, is supplemental to the angle A; ED to the angle C; and DF to the angle B; and also, that the side AC is supplemental to the angle E, etc.

A pole is 90° from any point on the circumference of its great circle; and, therefore, as A is the pole of the arc GH, the point A is 90° from the point E. As C is the pole of the arc LK, C is 90° from any point in that arc; therefore, C is 90° from the point E; and E being 90° from both A and C, it is the pole of the arc AC. In the same manner, we may prove that D is the pole of BC, and F the pole of AB.

Because A is the pole of the arc GH, the arc GH measures the angle A, (Prop. 4); for a similar reason, PQ measures the angle B, and LK measures the angle C.

Because E is the pole of the arc AC, $\quad EH = 90°$

Or, $\quad EG + GH = 90°$

For a like reason, $\quad FH + GH = 90°$

Adding these two equations, and observing that $GH = A$, and afterward transposing one A, we have,

$$EG + GH + FH = 180° - A.$$

$$\left.\begin{array}{lr} \text{Or,} & EF = 180° - A \\ \text{In like manner,} & FD = 180° - B \\ \text{And,} & DE = 180° - C \end{array}\right\} \quad (a)$$

But the arc $(180° - A)$, is a supplemental arc to A, by the definition of arcs; therefore, the three sides of the triangle DEF, are supplements of the angles A, B, C, of the triangle ABC.

Again, as E is the pole of the arc AC, the whole angle E is measured by the whole arc LH.

$$\begin{array}{lr} \text{But,} & AC + CH = 90° \\ \text{Also,} & AC + AL = 90° \end{array}$$

By addition, $AC + AC + CH + AL = 180°$

By transposition, $AC + CH + AL = 180° - AC$

$$\left.\begin{array}{lr} \text{That is,} & LH, \text{ or } E = 180° - AC \\ \text{In the same manner,} & F = 180° - AB \\ \text{And,} & D = 180° - BC \end{array}\right\} \quad (b)$$

That is, the sides of the first triangle are supplemental to the angles of the second triangle.

PROPOSITION VII.

The sum of the three angles of any spherical triangle, is greater than two right angles, and less than six right angles.

Add equations (a), of the last proposition. The first member of the equation so formed will be the sum of the three sides of a spherical triangle, which sum we may designate by S. The second member will be 6 right angles (there being 2 right angles in each 180°) less the three angles A, B, and C.

That is, $S = 6 \text{ right angles} - (A + B + C)$

By Prop. 2, the sum S is less than 4 right angles;

therefore, to it add s, a sufficient quantity to make 4 right angles. Then,

$$4 \text{ right angles} = 6 \text{ right angles} - (A + B + C) + s$$

Drop or cancel 4 right angles from both members, and transpose $(A + B + C)$.

Then, $$A + B + C = 2 \text{ right angles} + s.$$

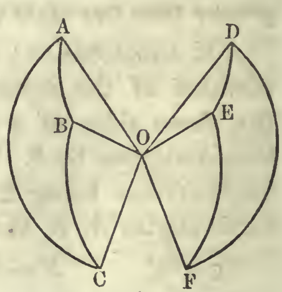
That is, the three angles of a spherical triangle make a greater sum than two right angles by the indefinite quantity s, which quantity is called the *spherical excess*, and is greater or less according to the size of the triangle.

Again, the sum of the angles is less than 6 right angles. There are but *three* angles in any triangle, and each one of them must be less than 180°, or 2 right angles. For, an angle is the inclination of two lines or two planes; and when two planes incline by 180°, the planes are parallel, or are in one and the same plane; therefore, as neither angle can be equal to 2 right angles, the three can never be equal to 6 right angles.

PROPOSITION VIII.

On the same sphere, or on equal spheres, triangles which are mutually equilateral are also mutually equiangular; and, conversely, triangles which are mutually equiangular are also mutually equilateral, equal sides lying opposite equal angles.

First.—Let ABC and DEF, in which $AB = DE$, $AC = DF$, and $BC = EF$, be two triangles on the sphere whose center is O; then will the $\angle A$, opposite the side BC, in the first triangle, be equal the $\angle D$, opposite the equal side EF, in the second; also $\angle B = \angle E$, and $\angle C = \angle F$.



For, drawing the radii to the vertices of the angles of these triangles, we may conceive O to be the common vertex of two triedral angles, one of which is bounded by the plane angles AOB, BOC, and AOC, and the other by the plane angles DOE, EOF, and DOF. But the plane angles bounding the one of these triedral angles, are equal to the plane angles bounding the other, each to each, since they are measured by the equal sides of the two triangles. The planes of the equal arcs in the two triangles are therefore equally inclined to each other, (Th. 20, B. VI); but the angles included between the planes of the arcs are equal to the angles formed by the arcs, (Def. 3).

Hence the $\angle A$, opposite the side BC, in the $\triangle ABC$, is equal to the $\angle D$, opposite the equal side EF, in the other triangle; and for a similar reason, the $\angle B = \angle E$, and the $\angle C = \angle F$.

Second.—If, in the triangles ABC and DEF, being on the same sphere whose center is O, the $\angle A = \angle D$, the $\angle B = \angle E$, and the $\angle C = \angle F$; then will the side AB, opposite the $\angle C$, in the first, be equal to the side DE, opposite the equal $\angle F$, in the second; and also the side AC equal to the side DF, and the side BC equal to the side EF.

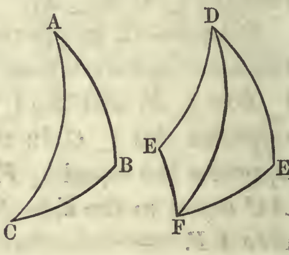
For, conceive two triangles, denoted by $A'B'C'$ and $D'E'F'$, supplemental to ABC and DEF, to be formed; then will these supplemental triangles be mutually equilateral, for their sides are measured by 180° less the opposite and equal angles of the triangles ABC and DEF, (Prop. 6); and being mutually equilateral, they are, as proved above, mutually equiangular. But the triangles ABC and DEF are supplemental to the triangles $A'B'C'$ and $D'E'F'$; and their sides are therefore measured severally by 180° less the opposite and equal angles of the triangles $A'B'C'$ and $D'E'F'$, (Prop. 6).

Hence the triangles ABC and DEF, which are mutually equiangular, are also mutually equilateral.

SCHOLIUM.—With the three arcs of great circles, AB, AC, and BC, either of the two triangles, ABC, DEF, may be formed; but it is evident that these two triangles cannot be made to coincide, though they are both mutually equilateral and mutually equiangular. Spherical triangles on the same sphere, or on equal spheres, in which the sides and angles of the one are equal to the sides and angles of the other, each to each, but are not themselves capable of superposition, are called *symmetrical triangles.*

PROPOSITION IX.

On the same sphere, or on equal spheres, triangles having two sides of the one equal to two sides of the other, each to each, and the included angles equal, have their remaining sides and angles equal.



Let ABC and DEF be two triangles, in which $AB = DE$, $AC = DF$, and the angle $A =$ the angle D; then will the side BC be equal to the side FE, the $\angle B =$ the $\angle E$, and $\angle C = \angle F$.

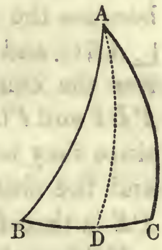
For, if DE lies on the same side of DF that AB does of AC, the two triangles, ABC and DEF, may be applied the one to the other, and they may be proved to coincide, as in the case of plane triangles. But, if DE does not lie on the same side of DF that AB does of AC, we may construct the triangle which is symmetrical with DEF; and this symmetrical triangle, when applied to the triangle ABC, will exactly coincide with it. But the triangle DEF, and the triangle symmetrical with it, are not only mutually equilateral, but also are mutually equiangular, the equal angles lying opposite the equal sides, (Prop. 8); and as the one or the other will coincide with the triangle ABC, it follows that the triangles, ABC and DEF, are either absolutely or symmetrically equal.

Cor. On the same sphere, or on equal spheres, triangles having two angles of the one equal to two angles of the other, each to each, and the included sides equal, have their remaining sides and angles equal.

For, if $\angle A = \angle D$, $\angle B = \angle E$, and side $AB =$ side DE, the triangle DEF, or the triangle symmetrical with it, will exactly coincide with $\triangle ABC$, when applied to it as in the case of plane triangles; hence, the sides and angles of the one will be equal to the sides and angles of the other, each to each.

PROPOSITION X.

In an isosceles spherical triangle, the angles opposite the equal sides are equal.



Let ABC be an isosceles spherical triangle, in which AB and AC are the equal sides; then will $\angle B = \angle C$.

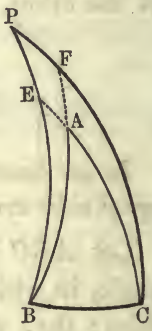
For, connect the vertex A with D, the middle point of the base, by the arc of a great circle, thus forming the two mutually equilateral triangles, ADB and ADC. They are mutually equilateral, because AD is common, $BD = DC$ by construction, and $AB = AC$ by supposition; hence they are mutually equiangular, the equal angles being opposite the equal sides, (Prop. 8). The angles B and C, being opposite the common side AD, are therefore equal.

Cor. The arc of a great circle which joins the vertex of an isosceles spherical triangle with the middle point of the base, is perpendicular to the base, and bisects the vertical angle of the triangle; and, conversely, the arc of a great circle which bisects the vertical angle of an isosceles spherical triangle, is perpendicular to, and bisects the base.

PROPOSITION XI.

If two angles of a spherical triangle are equal, the opposite sides are also equal, and the triangle is isosceles.

In the spherical triangle, ABC, let the $\angle B = \angle C$; then will the sides, AB and AC, opposite these equal angles, be equal.



For, let P be the pole of the base, BC, and draw the arcs of great circles, PB, PC; these arcs will be quadrants, and at right angles to BC, (Cor. 1, Prop. 3). Also, produce CA and BA to meet PB and PC, in the points E and F. Now, the angles, PBF and PCE, are equal, because the first is equal to 90° less the $\angle ABC$, and the second is equal to 90° less the equal $\angle ACB$; hence, the $\triangle$'s, PBF and PCE, are equal in all their parts, since they have the $\angle P$ common, the $\angle PBF = \angle PCE$, and the side PB equal to the side PC, (Cor., Prop. 9). PE is therefore equal to PF, and $\angle PEC = \angle PFB$.

Taking the equals PF and PE, from the equals PC and PB, we have the remainders, FC and EB, equal; and, from 180°, taking the $\angle$'s PFB and PEC, we have the remaining $\angle$'s, AFC and AEB, equal. Hence, the $\triangle$'s, AFC and AEB, have two angles of the one equal to two angles of the other, each to each, and the included sides equal; the remaining sides and angles are therefore equal, (Cor., Prop. 9). Therefore, AC is equal to BA, and the $\triangle$ ABC is isosceles.

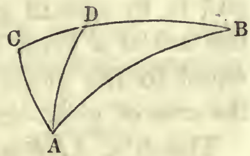
Cor. An equiangular spherical triangle is also equilateral, and the converse.

REMARK.—In this demonstration, the pole of the base, BC, is supposed to fall without the triangle, ABC. The same figure may be used for the case in which the pole falls within the triangle; the modification the demonstration then requires is so slight and obvious, that it would be superfluous to suggest it.

PROPOSITION XII.

The greater of two sides of a spherical triangle is opposite the greater angle; and, conversely, the greater of two angles of a spherical triangle is opposite the greater side.

Let ABC be a spherical triangle, in which the angle A is greater than the angle B; then is the side BC greater than the side AC.



Through A draw the arc of a great circle, AD, making, with AB, the angle BAD equal to the angle ABD. The triangle, DAB, is isosceles, and $DA = DB$, (Prop. 11).

In the $\triangle\ ACD$, $AC < CD + AD$, (Prop. 1); or, substituting for AD its equal DB, we have,

$$AC < CD + DB.$$

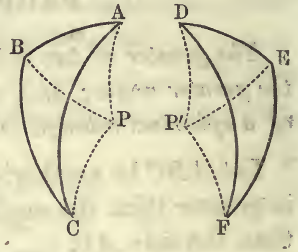
Inverting the members of the inequality, and writing CB for $CD + DB$, it becomes $CB > CA$.

Conversely; if the side CB be greater than the side CA, then is the $\angle A >$ the $\angle B$. For, if the $\angle A$ is not greater than the $\angle B$, it is either equal to it, or less than it. The $\angle A$ is not equal to the $\angle B$; for if it were, the triangle would be isosceles, and CB would be equal to CA, which is contrary to the hypothesis. The $\angle A$ is not less than the $\angle B$; for if it were, the side CB would be less than the side CA, by the first part of the proposition, which is also contrary to the hypothesis; hence, the $\angle A$ must be greater than the $\angle B$.

PROPOSITION XIII.

Two symmetrical spherical triangles are equal in area.

Let ABC and DEF be two △'s on the same sphere, having the sides and angles of the one equal to the sides and angles of the other, each to each, the triangles themselves not admitting of superposition. It is to be proved that these △'s have equal areas.



Let P be the pole of a small circle passing through the three points, ABC, and connect P with each of the points, A, B, and C, by arcs of great circles. Next, through E draw the arc of a great circle, EP', making the angle DEP' equal to the angle ABP. Take $EP' = BP$, and draw the arcs of great circles, $P'D$, $P'F$.

The △'s, ABP and DEP', are equal in all their parts, because $AB = DE$, $BP = EP'$, and the $\angle ABP = \angle DEP'$, (Prop. 9). Taking from the $\angle ABC$ the $\angle ABP$, and from the $\angle DEF$ the $\angle DEP'$, we have the remaining angles, PBC and $P'EF$, equal; and therefore the △'s, BCP and EFP', are also equal in all their parts.

Now, since the △'s, ABP and DEP', are isosceles, they will coincide when applied, as will also the △'s, BCP and EFP', for the same reason. The polygonal areas, $ABCP$ and $DEFP'$, are therefore equivalent. If from the first we take the isosceles triangle, PAC, and from the second the equal isosceles triangle, $P'DF$, the remainders, or the triangles ABC and DEF, will be equivalent.

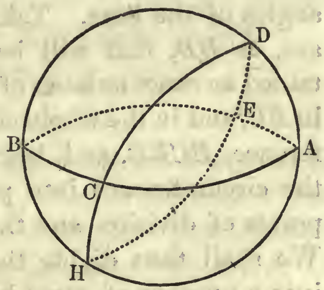
REMARK.—It is assumed in this demonstration that the pole P falls without the triangle. Were it to fall within, instead of without, no other change in the above process would be required than to add the isosceles triangles, PAC, $P'DF$, to the polygonal areas, to get the areas of the triangles, ABC, DEF.

Cor. Two spherical triangles on the same sphere, or on equal spheres, will be equivalent — 1st, when they are mutually equilateral; — 2d, when they are mutually equiangular; — 3d, when two sides of the one are equal to two sides of the other, each to each, and the included angles are equal; — 4th, when two angles of the one are equal to two angles of the other, each to each, and the included sides are equal.

PROPOSITION XIV.

If two arcs of great circles intersect each other on the surface of a hemisphere, the sum of either two of the opposite triangles thus formed will be equivalent to a lune whose angle is the corresponding angle formed by the arcs.

Let the great circle, $AEBC$, be the base of a hemisphere, on the surface of which the semi-great circumferences, BDA and CDE, intersect each other at D; then will the sum of the opposite triangles, BDC and DAE, be equivalent to the lune whose angle is BDC; and the sum of the opposite triangles, CDA and BDE, will be equivalent to the lune whose angle is CDA.



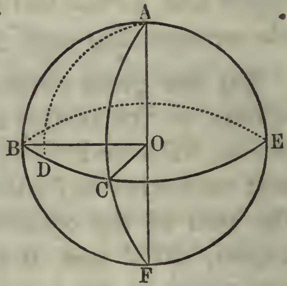
Produce the arcs, BDA and CDE, until they intersect on the opposite hemisphere at H; then, since CDE and DEH are both semi-circumferences of a great circle, they are equal. Taking from each the common part DE, we have $CD = HE$. In the same way we prove $BD = HA$, and $AE = BC$. The two triangles, BDC and HAE, are therefore mutually equilateral, and hence they are equivalent, (Prop. 13). But the two triangles, HAE and ADE, together, make up the lune

DEHAD; hence the sum of the △'s, *BDC* and *ADE*, is equivalent to the same lune.

By the same course of reasoning, we prove that the sum of the opposite △'s, *DAC* and *DBE*, is equivalent to the lune *DCHAD*, whose angle is *ADC*.

PROPOSITION XV.

The surface of a lune is to the whole surface of the sphere, as the angle of the lune is to four right angles; or, as the arc which measures that angle is to the circumference of a great circle.



Let *ABFCA* be a lune on the surface of a sphere, and *BCE* an arc of a great circle, whose poles are *A* and *F*, the vertices of the angles of the lune. The arc, *BC*, will then measure the angles of the lune. Take any arc, as *BD*, that will be contained an exact number of times in *BC*, and in the whole circumference, *BCEB*, and, beginning at *B*, divide the arc and the circumference into parts equal to *BD*, and join the points of division and the poles, by arcs of great circles. We shall thus divide the whole surface of the sphere into a number of equal lunes. Now, if the arc *BC* contains the arc *BD* m times, and the whole circumference contains this arc n times, the surface of the lune will contain m of these partial lunes, and the surface of the sphere will contain n of the same; and we shall have,

Surf. lune : surf. sphere :: $m : n$.

But, $m : n$ :: *BC* : circumference great circle;

hence, surf. lune : surf. sphere :: *BC* : cir. great circle;

or, surf. lune : surf. sphere :: $\angle BOC$: 4 right angles.

This demonstration assumes that BD is a common measure of the arc, BC, and the whole circumference. It may happen that no finite common measure can be found; but our reasoning would remain the same, even though this common measure were to become indefinitely small.

Hence the proposition.

Cor. 1. Any two lunes on the same sphere, or on equal spheres, are to each as their respective angles.

SCHOLIUM.— Spherical triangles, formed by joining the pole of an arc of a great circle with the extremities of this arc by the arcs of great circles, are isosceles, and contain two right angles. For this reason they are called *bi-rectangular*. If the base is also a quadrant, the vertex of either angle becomes the pole of the opposite side, and each angle is measured by its opposite side. The three angles are then right angles, and the triangle is for this reason called *tri-rectangular*. It is evident that the surface of a sphere contains eight of its tri-rectangular triangles.

Cor. 2. Taking the right angle as the unit of angles, and denoting the angle of a lune by A, and the surface of a tri-rectangular triangle by T, we have,

$$\text{surf. of lune} : 8T :: A : 4;$$

whence, $\quad \text{surf. of lune} = 2A \times T.$

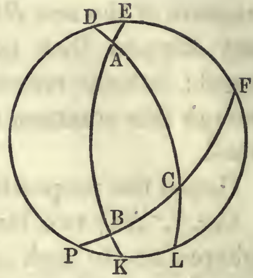
Cor. 3. A spherical ungula bears the same relation to the entire sphere, that the lune, which is the base of the ungula, bears to the surface of the sphere; and hence, any two spherical ungulas in the same sphere, or in equal spheres, are to each other as the angles of their respective lunes.

PROPOSITION XVI.

The area of a spherical triangle is measured by the excess of the sum of its angles over two right angles, multiplied by the tri-rectangular triangle.

Let ABC be a spherical triangle, and $DEFLK$ the circumference of the base of the hemisphere on which this triangle is situated.

Produce the sides of the triangle until they meet this circumference in the points, D, E, F, L, K, and P, thus forming the sets of opposite triangles, DAE, AKL; BEF, BPK; CFL, CDP.



Now, the triangles of each of these sets are together equal to a lune, whose angle is the corresponding angle of the triangle, (Prop. 14); hence we have,

$$\triangle DAE + \triangle AKL = 2A \times T, \text{(Prop. 15, Cor. 2).}$$
$$\triangle BEF + \triangle BPK = 2B \times T.$$
$$\triangle CFL + \triangle CDP = 2C \times T.$$

If the first members of these equations be added, it is evident that their sum will exceed the surface of the hemisphere by twice the triangle ABC; hence, adding these equations member to member, and substituting for the first member of the result its value, $4T + 2\triangle ABC$, we have

$$4T + 2\triangle ABC = 2A.T + 2B.T + 2C.T$$

or, $\quad 2T + \triangle ABC = A.T + B.T + C.T$

whence, $\quad \triangle ABC = A.T + B.T + C.T - 2T.$

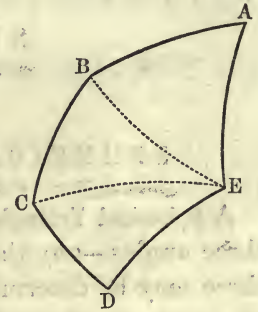
That is, $\quad \triangle ABC = (A + B + C - 2)\ T.$

But $A + B + C - 2$ is the excess of the sum of the angles of the triangle over two right angles, and T denotes the area of a tri-rectangular triangle.

Hence the proposition; *the area, etc.*

PROPOSITION XVII.

The area of any spherical polygon is measured by the excess of the sum of all its angles over two right angles, taken as many times, less two, as the polygon has sides, multiplied by the tri-rectangular triangle.



Let $ABCDE$ be a spherical polygon; then will its area be measured by the excess of the sum of the angles, A, B, C, D, and E, over two right angles taken a number of times which is two less than the number of sides, multiplied by T, the tri-rectangular triangle. Through the vertex of any of the angles, as E, and the vertices of the opposite angles, pass arcs of great circles, thus dividing the polygon into as many triangles, less two, as the polygon has sides. The sum of the angles of the several triangles will be equal to the sum of the angles of the polygon.

Now, the area of each triangle is measured by the excess of the sum of its angles over two right angles, multiplied by the tri-rectangular triangle. Hence the sum of the areas of all the triangles, or the area of the polygon, is measured by the excess of the sum of all the angles of the triangles over two right angles, taken as many times as there are triangles, multiplied by the tri-rectangular triangle. But there are as many triangles as the polygon has sides, less two.

Hence the proposition; *the area of any spherical polygon, etc.*

Cor. If S denote the sum of the angles of any spherical polygon, n the number of sides, and T the tri-rectangular triangle, the right angle being the unit of angles; the area of the polygon will be expressed by

$$[S - 2(n - 2)] \times T = (S - 2n + 4)\, T.$$

SECTION II.

SPHERICAL TRIGONOMETRY.

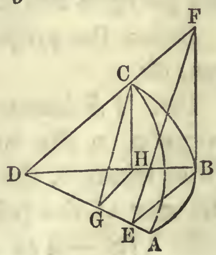
A **Spherical Triangle** contains six parts—three sides and three angles—any three of which being given, the other three may be determined.

Spherical Trigonometry has for its object to explain the different methods of computing three of the six parts of a spherical triangle, when the other three are given. It may be divided into *Right-angled* Spherical Trigonometry, and *Oblique-angled* Spherical Trigonometry; the first treating of the solution of right-angled, and the second of oblique-angled spherical triangles.

RIGHT-ANGLED SPHERICAL TRIGONOMETRY.

PROPOSITION I.

With the sines of the sides, and the tangent of ONE SIDE *of any right-angled spherical triangle, two plane triangles can be formed that will be similar, and similarly situated.*



Let ABC be a spherical triangle, right-angled at B; and let D be the center of the sphere. Because the angle CBA is a right angle, the plane CBD is perpendicular to the plane DBA. From C let fall CH, perpendicular to the plane DBA; and as the plane CBD is perpendicular to the plane DBA, CH will lie in the plane CBD, and be perpendicular to the line DB, and perpendicular to all lines that can be drawn in the plane DBA, from the point H (Def. 2, B. VI).

Draw HG perpendicular to DA, and draw GC; GC will lie wholly in the plane CDA, and CHG is a right-angled triangle, right-angled at H.

We will now demonstrate that the angle DGC is a right angle.

The right-angled $\triangle CHG$, gives $CH^2+HG^2=CG^2$ (1)

The right-angled $\triangle DGH$, gives $DG^2+HG^2=DH^2$ (2)

By subtraction, $CH^2-DG^2=CG^2-DH^2$ (3)

By transposition, $CH^2+DH^2=CG^2+DG^2$ (4)

But the first member of equation (4), is equal to CD^2, because CDH is a right-angled triangle;

Therefore, $CD^2=CG^2+DG^2$

Hence, CD is the hypotenuse of the right-angled triangle DGC, (Th. 39, B. I).

From the point B, draw BE at right angles to DA, and BF at right angles to DB, in the plane CDB extended; the point F will be in the line DC. Draw EF, and as F is in the plane CDA, and E is in the same plane, the line EF is in the plane CDA. Now we are to prove that the triangle CHG is similar to the triangle BEF, and similarly situated.

As HG and BE are both at right angles to DA, they are parallel; and as HC and BF are both at right angles to DB, they are parallel; and by reason of the parallels, the angles GHC and EBF are equal; but GHC is a right angle; therefore, EBF is also a right angle.

Now, as GH and BE are parallel, and CH and BF are also parallel, we have,

$$DH : DB = HG : BE$$

And, $$DH : DB = HC : BF$$

Therefore, $HG : BE = HC : BF$ (Th. 6, B. II),

Or, $HG : HC = BE : BF$.

Here, then, are two triangles, having an angle in the one equal to an angle in the other, and the sides about the equal angles proportional; the two triangles are therefore equiangular, (Cor. 2, Th. 17, B. II); and they are similarly situated, for their sides make equal angles at H and B with the same line, DB.

Hence the proposition.

SCHOLIUM.—By the definition of sines, cosines, and tangents, we perceive that CH is the sine of the arc BC, DH is its cosine, and BF its tangent; CG is the sine of the arc AC, and DG its cosine. Also, BE is the sine of the arc AB, and DE is the cosine of the same arc. With this figure we are prepared to demonstrate the following propositions.

PROPOSITION II.

In any right-angled spherical triangle, the sine of one side is to the tangent of the other side, as radius is to the tangent of the angle adjacent to the first-mentioned side.

Or, the sine of one side is to the tangent of the other side, as the cotangent of the angle adjacent to the first-mentioned side is to the radius.

For the sake of brevity, we will represent the angles of the triangle by A, B, C, and the sides or arcs opposite to these angles, by a, b, c; that is, a opposite A, etc.

In the right-angled plane triangle EBF, we have,

$$EB : BF = R : \tan.BEF$$

That is, $\sin.c : \tan.a = R : \tan.A$,

which agrees with the first part of the enunciation. By reference to equation (5), Section I, Plane Trigonometry, we shall find that,

$$\tan.A \cot.A = R^2;$$

therefore, $$\tan.A = \frac{R^2}{\cot.A}.$$

Substituting this value for tangent A, in the preceding proportion, and dividing the last couplet by R, we shall have,

$$\sin.c : \tan.a = 1 : \frac{R}{\cot.A}.$$

Or, $$\sin.c : \tan.a = \cot.A : R.$$

Or, $$R \sin.c = \tan.a \cot.A, \qquad (1)$$

which answers to the second part of the enunciation.

Cor. By changing the construction, drawing the tangent to AB, in place of the tangent to BC, and proceeding in a similar manner, we have,

$$R \sin.a = \tan.c \cot.C. \qquad (2)$$

PROPOSITION III.

In any right-angled spherical triangle, the sine of the right angle is to the sine of the hypotenuse, as the sine of either of the other angles is to the sine of the side opposite to that angle.

The sine of 90°, or radius, is designated by R.

In the plane triangle, CHG, we have,

$$\sin.CHG : CG = \sin.CGH : CH$$

That is, $$R : \sin.b = \sin.A : \sin.a$$

Or, $$R \sin.a = \sin.b \sin.A \qquad (3)$$

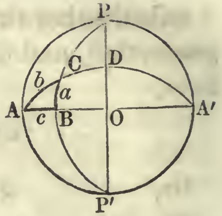
Cor. By a change in the construction of the figure, drawing a tangent to AB, etc., we shall have,

$$R : \sin.b = \sin.C : \sin.c$$

Or, $$R \sin.c = \sin.b \sin.C. \qquad (4)$$

Scholium. — Collecting the four equations taken from this and the preceding proposition, we have,

$$\left.\begin{array}{ll} (1) & R \sin.c = \tan.a \cot.A \\ (2) & R \sin.a = \tan.c \cot.C \\ (3) & R \sin.a = \sin.b \sin.A \\ (4) & R \sin.c = \sin.b \sin.C \end{array}\right\}$$



These equations refer to the right-angled triangle, ABC; but the principles are true for any right-angled spherical triangle. Let us apply them to the right-angled triangle, PDC, the complemental triangle to ABC.

Making this application, equation

(1) becomes $R\sin.CD = \tan.PD \cot.C$ $\quad (n)$

(2) becomes $R\sin.PD = \tan.CD \cot.P$ $\quad (m)$

(3) becomes $R\sin.PD = \sin.PC \sin.C$ $\quad (o)$

(4) becomes $R\sin.CD = \sin.PC \sin.P$ $\quad (p)$

By observing that $\sin.CD = \cos.AC = \cos.b$.

And that $\tan.PD = \cot.DO = \cot.A$, etc.; and by running equations (n), (m), (o), and (p), back into the triangle, ABC, we shall have,

$$\left.\begin{array}{ll}(5) & R\cos.b = \cot.A \;\cot.C \\ (6) & R\cos.A = \cot.b \;\tan.c \\ (7) & R\cos.A = \cos.a \;\sin.C \\ (8) & R\cos.b = \cos.a \;\cos.c\end{array}\right\}$$

By observing equation (6), we find that the second member refers to sides adjacent to the angle A. The same relation holds in respect to the angle C, and gives,

$$(9)\quad R\cos.C = \cot.b \;\tan.a.$$

Making the same observations on (7), we infer,

$$(10)\quad R\cos.C = \cos.c \;\sin.A.$$

Observation 1. Several of these equations can be deduced geometrically without the least difficulty. For example, take the figure to Proposition 1. The parallels in the plane, DBA, give,

$$DB : DH = DE : DG.$$

That is, $\qquad R : \cos.a = \cos.c : \cos.b.$

A result identical with equation (8), and in words it is expressed thus: *Radius is to cosine of one side, as the cosine of the other side is to the cosine of the hypotenuse.*

Observation 2. The equations numbered from (1) to (10) cover every possible case that can occur in right-angled spherical trigonometry; but the combinations are too various to be remembered, and readily applied to practical use.

We can remedy this inconvenience, by taking the *complement* of the hypotenuse, and the *complements* of the two oblique angles, in place of the arcs themselves.

Thus, b is the hypotenuse, and let b' be its complement.

Then, $b+b'=90°$; or, $b=90°-b'$; and, $\sin.b=\cos.b'$, $\cos.b=\sin.b'$; $\tan.b=\cot.b'$. In the same manner, if A' is the complement to A,

Then, $\sin.A=\cos.A'$; $\cos.A=\sin.A'$; and, $\tan.A=\cot.A'$; and similarly, $\sin.C=\cos.C'$; $\cos.C=\sin.C'$; and $\tan.C=\cot.C'$.

Substituting these values for b, A, and C, in the foregoing *ten* equations (a and c remaining the same), we have,

NAPIER'S CIRCULAR PARTS.

(11) $R\sin.c=\tan.a\tan.A'$

(12) $R\sin.a=\tan.c\tan.C'$

(13) $R\sin.a=\cos.b'\cos.A'$

(14) $R\sin.c=\cos.b'\cos.C'$

(15) $R\sin.b'=\tan.A'\tan.C'$

(16) $R\sin.A'=\tan.b'\tan.c$

(17) $R\sin.A'=\cos.a\cos.C'$

(18) $R\sin.b'=\cos.a\cos.c$

(19) $R\sin.C'=\tan.b'\tan.a$

(20) $R\sin.C'=\cos.c\cos.A'$

Omitting the consideration of the right angle, there are five parts. Each part taken as a middle part, is connected to its adjacent parts by one equation, and to its extreme parts by another equation; therefore, ten equations are required for the combinations of all the parts.

These equations are very remarkable, because the first members are all composed of radius into *some sine*, and the second members are all composed of the product of *two tangents*, or *two cosines*.

To condense these equations into words, for the purpose of assisting the memory, we will refer any one of them directly to the right-angled triangle, ABC, in the last figure.

When the right angle is left out of the question, a right-angled triangle consists of *five* parts — *three* sides, and *two* angles. Let any one of these parts be called a *middle part;* then two other parts will lie adjacent to this part, and two *opposite to it,* that is, separated from it by two other parts.

For instance, take equation (11), and call c the *middle* part; then A' and a will be adjacent parts, and C'' and b' opposite parts. Again, take a as a *middle part;* then c and C'' will be adjacent parts, and A' and b' will be opposite parts; and thus we may go round the triangle.

Take any equation from (11) to (20), and consider the middle part in the first member of the equation, and we shall find that they correspond to the two following *invariable and comprehensive rules:*

1. *The radius into the sine of the middle part is equal to the product of the tangents of the adjacent parts.*

2. *The radius into the sine of the middle part is equal to the product of the cosines of the opposite parts.*

These rules are known as Napier's Rules, because they were first given by that distinguished mathematician, who was also the inventor of logarithms.

In the application of these equations, the *accent* may be omitted if tan. be changed to cotan., sin. to cosin., etc. Thus, if equation (13) were to be employed, it would be written, in the first instance, $R \sin.a = \cos.b' \cos.A'$, to insure conformity to the rule; then, we would change it into $R \sin.a = \sin.b \sin.A$.

Remark.—We caution the pupil to be very particular to take the *complements* of the hypotenuse, and the complements of the oblique angles.

SECTION III.

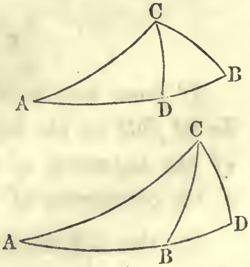
OBLIQUE-ANGLED SPHERICAL TRIGONOMETRY.

The preceding investigations have had reference to right-angled spherical trigonometry only, but the application of these principles cover oblique-angled trigonometry also; for, every oblique-angled spherical triangle may be considered as made up of the sum or difference of two right-angled spherical triangles. With this explanatory remark, we give

PROPOSITION I.

In all spherical triangles, the sines of the sides are to each other, as the sines of the angles opposite to them.

This was proved in relation to right-angled triangles in Prop. 3, Sec. II, and we now apply the principle to oblique-angled triangles.



Let ABC be the triangle, and let CD be perpendicular to AB, or to AB produced.

Then, by Prop. 3, Sec. II, we have,

$$R : \sin. AC = \sin. A : \sin. CD.$$

Also,

$$\sin. CB : R = \sin. CD : \sin. B.$$

By multiplying these two proportions together, term by term, and omitting the common factor R, in the first couplet, and the common factor, sin.CD, in the second, we have

$$\sin.CB : \sin.AC = \sin.A : \sin.B.$$

PROPOSITION II.

In any spherical triangle, if an arc of a great circle be let fall from any angle perpendicular to the opposite side as a base, or to the base produced, the cosines of the other two sides will be to each other as the cosines of the segments of the base.

By the application of equation 8, (Sec. II), to the last figure, we have,

$$R\cos.AC = \cos.AD\cos.DC$$

Similarly, $R\cos.BC = \cos.DC\cos.BD$

Dividing one of these equations by the other, omitting common factors in numerators and denominators, we have,

$$\frac{\cos.AC}{\cos.BC} = \frac{\cos.AD}{\cos.BD}$$

Or, $\cos.AC : \cos.BC = \cos.AD : \cos.BD.$

PROPOSITION III.

If from any angle of a spherical triangle, a perpendicular be let fall on the base, or on the base produced, the tangents of the segments of the base will be reciprocally proportional to the cotangents of the segments of the angle.

By the application of Equation 2, (Sec. II), to the last figure, we have,

$$R\sin.CD = \tan.AD\cot.ACD.$$

Similarly, $R \sin.CD = \tan.BD \cot.BCD$

Therefore, by equality,

$$\tan.AD \cot.ACD = \tan.BD \cot.BCD$$

Or, $\tan.AD : \tan.BD = \cot.BCD : \cot.ACD.$

PROPOSITION IV.

The same construction remaining, the cosines of the angles at the extremities of the segments of the base are to each other as the sines of the segments of the opposite angle.

Equation 7, (Sec. II), applied to the triangle ACD, gives

$$R \cos.A = \cos.CD \sin.ACD \quad (s)$$

Also, $R \cos.B = \cos.CD \sin.BCD \quad (t)$

Dividing equation (s) by (t), gives

$$\frac{\cos.A}{\cos.B} = \frac{\sin.ACD}{\sin.BCD}$$

Or, $\cos.B : \cos.A = \sin.BCD : \sin.ACD.$

PROPOSITION V.

The same construction remaining, the sines of the segments of the base are to each other as the cotangents of the adjacent angles.

Equation 1, (Sec. II), applied to the triangle ACD, gives

$$R \sin.AD = \tan.CD \cot.A \quad (s)$$

Similarly, $R \sin.BD = \tan.CD \cot.B \quad (t)$

Dividing (s) by (t), gives

$$\frac{\sin.AD}{\sin.BD} = \frac{\cot.A}{\cot.B}$$

Or, $\sin.BD : \sin.AD = \cot.B : \cot.A.$

PROPOSITION VI.

The same construction remaining, the cotangents of the two sides are to each other as the cosines of the segments of the angle.

Equation 9, (Sec. II), applied to the triangle ACD, gives

$$R \cos.ACD = \cot.AC \tan.CD \quad (s)$$

Similarly, $R \cos.BCD = \cot.BC \tan.CD \quad (t)$

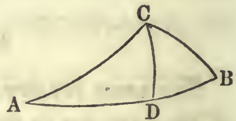
Dividing (s) by (t), gives

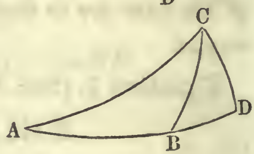
$$\frac{\cos.ACD}{\cos.BCD} = \frac{\cot.AC}{\cot.BC}$$

Or, $\cot.AC : \cot.BC = \cos.ACD : \cos.BCD.$

PROPOSITION VII.

The cosine of any side of a spherical triangle, is equal to the product of the cosines of the other two sides, plus the product of the sines of those sides multiplied by the cosine of the included angle.





Let ABC be a spherical triangle, and CD a perpendicular from the angle C on to the side AB, or on to the side AB produced. Then, by Prop. 2,

$\cos.AC : \cos.CB = \cos.AD : \cos.BD$ (1)

When CD falls within the triangle,

$$BD = (AB - AD);$$

and when CD falls without the triangle,

$$BD = (AD - AB).$$

Hence, $\cos.BD = \cos.(AD - AB)$

Now, $\cos.(AB - AD) = \cos.(AD - AB)$,

because each of them is equal to

$\cos.AB \cos.AD + \sin.AB \sin.AD$, (Eq. 10, Prop. 2, Sec. I, Plane Trig.).

This value of cos. BD, put in proportion (1), gives

$$\cos.AC : \cos.CB = \cos.AD : \cos.AB \cos.AD + \sin.AB \sin.AD \quad (2)$$

Dividing the last couplet of proportion (2) by cos.AD, observing that

$$\frac{\sin.AD}{\cos.AD} = \tan.AD,$$

and we have

$$\cos.AC : \cos.CB = 1 : \cos.AB + \sin.AB \tan.AD \quad (3)$$

By applying equation 6, (Sec. II), to the triangle ACD, taking the radius as unity, we have

$$\cos.A = \cot.AC \tan.AD \quad (k)$$

But, $\tan.AC \cot.AC = 1$, (Eq. 5, Sec. I, Plane Trig.) (l)

Multiply equation (k) by tan. AC, observing equation (l), and we have

$$\tan.AC \cos.A = \tan.AD$$

Substituting this value of tan. AD, in proportion (3), we have

$$\cos.AC : \cos.CB = 1 : \cos.AB + \sin.AB \tan.AC \cos.A \quad (4)$$

Multiplying extremes and means, gives

$$\cos.CB = \cos.AC \cos.AB + \sin.AB (\cos.AC \tan.AC) \cos.A.$$

But, $\tan.AC = \frac{\sin.AC}{\cos.AC}$, or, $\cos.AC \tan.AC = \sin.AC$.

Therefore, $\cos.CB = \cos.AC \cos.AB + \sin.AB \sin.AC \cos.A$.

If the sides opposite the angles, A, B, and C, be respectively represented by a, b, and c, this equation becomes,

$$\cos.a = \cos.b \cos.c + \sin.b \sin.c \cos.A.$$

This formula conforms to the enunciation in respect to the side a. Now, by simply writing b for a, and B for A, in the last equation, we get the formula for cos.b, which is,

$$\cos.b = \cos.a \cos.c + \sin.a \sin.c \cos.B.$$

By writing c for a, and C for A, we get the formula for cos.c, which is,

$$\cos.c = \cos.a \ \cos.b + \sin.a \ \sin.b \ \cos.C.$$

Hence, we have the three symmetrical formulæ:

$$\left.\begin{aligned} \cos.a &= \cos.b \ \cos.c + \sin.b \ \sin.c \ \cos.A \\ \cos.b &= \cos.a \ \cos.c + \sin.a \ \sin.c \ \cos.B \\ \cos.c &= \cos.a \ \cos.b + \sin.a \ \sin.b \ \cos.C \end{aligned}\right\} \quad (S)$$

From these, by simple transposition and division, we deduce the following formulæ for the cosines of the angles of any spherical triangle, viz:

$$\left.\begin{aligned} \cos.A &= \frac{\cos.a - \cos.b \ \cos.c}{\sin.b \ \sin.c} \\ \cos.B &= \frac{\cos.b - \cos.a \ \cos.c}{\sin.a \ \sin.c} \\ \cos.C &= \frac{\cos.c - \cos.a \ \cos.b}{\sin.a \ \sin.b} \end{aligned}\right\} \quad (S')$$

By means of these equations we can find the cosine of any of the three angles of a spherical triangle in terms of the functions of the sides; but in their present form they are not suited for the employment of logarithms, and we should be compelled to use a table of natural sines and cosines, and to perform tedious numerical operations, to obtain the value of the angle.

They are, however, by the following process, transformed into others well adapted to the use of logarithms.

In Eq. 34, Sec. I, Plane Trig., we have

$$1 + \cos.A = 2\cos.^2\tfrac{1}{2}A.$$

Therefore, $2\cos.^2\frac{1}{2}A = 1 + \dfrac{\cos.a - \cos.b \ \cos.c}{\sin.b \ \sin.c}$.

$$= \frac{(\sin.b \ \sin.c - \cos.b \ \cos.c) + \cos.a}{\sin.b \ \sin.c} \quad (m).$$

But, $\cos.(b + c) = \cos.b \ \cos.c = \sin.c \ \sin.b$, (Equation 9, Section I, Plane Trig.). By comparing this equation with the second member of equation (m), we perceive that equation (m) is readily reduced to

$$2\cos.^2\tfrac{1}{2}A = \frac{\cos.a - \cos.(b+c)}{\sin.b\ \sin.c}.$$

Considering $(b+c)$ as one arc, and then making application of equation (18), Plane Trigonometry, we have,

$$2\cos.^2\tfrac{1}{2}A = \frac{2\sin.\left(\frac{a+b+c}{2}\right)\sin.\left(\frac{b+c-a}{2}\right)}{\sin.b\ \sin.c}.$$

But, $\frac{b+c-a}{2} = \frac{b+c+a}{2} - a$; and if we put S to represent $\frac{b+c+a}{2}$, we shall have,

$$\cos.^2\frac{A}{2} = \frac{\sin.S\sin.(S-a)}{\sin.b\ \sin.c}.$$

Or, $$\cos.\frac{A}{2} = \sqrt{\frac{\sin.S\ \sin.(S-a)}{\sin.b\ \sin.c}}.$$

The second member of this equation gives the value of the cosine when the radius is unity. To a greater radius, the cosine would be greater; and in just the same proportion as the radius increases, all the trigonometrical lines increase; therefore, to adapt the above equation to our tables where the radius is R, we must write R in the second member, as a factor; and if we put it under the radical sign, we must write R^2.

For the other angles we shall have precisely similar equations:

That is,
$$\left.\begin{aligned}\cos.\frac{A}{2} &= \sqrt{\frac{R^2\sin.S\ \sin.(S-a)}{\sin.b\ \sin.c}}\\ \cos.\frac{B}{2} &= \sqrt{\frac{R^2\sin.S\ \sin.(S-b)}{\sin.a\ \sin.c}}\\ \cos.\frac{C}{2} &= \sqrt{\frac{R^2\sin.S\ \sin.(S-c)}{\sin.a\ \sin.b}}\end{aligned}\right\}\quad (T)$$

To deduce from formulæ (S), formulæ for the sines of the half of each of the angles of a spherical triangle, we proceed as follows:

From Eq. 35, Sec. I, Plane Trig., we have

$$2\sin.^2 \tfrac{1}{2}A = 1 - \cos.A.$$

Substituting the value of $\cos.A$, taken from formulæ (S), and we have,

$$2\sin.^2 \tfrac{1}{2}A = 1 - \frac{\cos.a - \cos.b \cos.c}{\sin.b \sin.c}.$$

$$= \frac{(\sin.b \sin.c + \cos.b \cos.c) - \cos.a}{\sin.b \sin.c}. \quad (o)$$

But, $\cos.(b \backsim c) = \sin.b \sin.c + \cos.b \cos.c$, (Eq. 10, Sec. I, Plane Trig.).

This equation reduces equation (o) to

$$2\sin.^2 \tfrac{1}{2}A = \frac{\cos.(b \backsim c) - \cos.a}{\sin.b \sin.c}.$$

Considering ($b \backsim c$) as a single arc, and applying equation 18, Sec. I, Plane Trig., we have

$$2\sin.^2 \tfrac{1}{2}A = \frac{2\sin.\left(\frac{a+b-c}{2}\right)\sin.\left(\frac{a+c-b}{2}\right)}{\sin.b \sin.c}. \quad (o')$$

But, $\frac{a+b-c}{2} = \frac{a+b+c}{2} - c = S - c$, if we put $S = \frac{a+b+c}{2}$.

Also, $\frac{a+c-b}{2} = \frac{a+b+c}{2} - b = S - b$.

Dividing equation (o') by 2, and making these substitutions, we have

$$\sin.^2 \tfrac{1}{2}A = \frac{\sin.(S-c)\sin.(S-b)}{\sin.b \sin.c},$$

when radius is unity.

When radius is R, we have

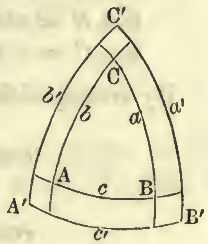
$$\left.\begin{aligned} \sin.\tfrac{1}{2}A &= \sqrt{\frac{R^2 \sin.(S-c) \sin.(S-b)}{\sin.b \sin.c}} \\ \text{Similarly,}\quad \sin.\tfrac{1}{2}B &= \sqrt{\frac{R^2 \sin.(S-a) \sin.(S-c)}{\sin.a \sin.c}} \\ \text{And,}\quad \sin.\tfrac{1}{2}C &= \sqrt{\frac{R^2 \sin.(S-a) \sin.(S-b}{\sin.a \sin.b}} \end{aligned}\right\} (U)$$

To apply to our tables, R^2 must be put under the radical sign. We shall show the application of these formulæ, and those in group (T), hereafter.

PROPOSITION VIII.

The cosine of any of the angles of a spherical triangle, is equal to the product of the sines of the other two angles multiplied by the cosine of the included side, minus the product of the cosines of these other two angles.

Let ABC be a spherical triangle, and $A'B'C'$ its supplemental or polar triangle, the angles of the first being denoted by A, B, and C, and the sides opposite these angles by a, b, c, respectively; A', B', C', a', b', c', denoting the angles and corresponding sides of the second.



By Prop. 5, Sec. I, we have the following relations between the sides and angles of these two triangles.

$$A' = 180° - a,\ B' = 180° - b,\ C' = 180° - c;$$
$$a' = 180° - A,\ b' = 180° - B,\ c' = 180° - C.$$

The first of formulæ (S), Prop. 7, when applied to the polar triangle, gives

$$\cos.a' = \cos.b' \cos.c' + \sin.b' \sin.c' \cos.A' \quad (1)$$

which, by substituting the values of a', b', c', and A', becomes

$$\cos.(180° - A) = \cos.(180° - B)\cos.(180° - C) + \sin.(180° - B)\sin.(180° - C)\cos.(180° - a), \quad (2)$$

But,

$$\cos.(180° - A) = -\cos.A, \text{ etc., } \sin.(180° - B) = \sin.B, \text{ etc.};$$

and placing these values for their equals in eq. (2), and changing the sines of both members of the resulting equation, we get

$$\cos.A = \sin.B\,\sin.C\cos.a - \cos.B\cos.C,$$

which agrees with the enunciation.

By treating the other two of formulæ (S), Prop. 7, in the same manner, we would obtain similar values for the cosines of the other two angles of the triangle ABC; or we may get them more easily by a simple permutation of the letters A, B, C, a, etc.

Hence, we have the three equations

$$\left.\begin{aligned} \cos.A &= \sin.B\,\sin.C\cos.a - \cos.B\cos.C \\ \cos.B &= \sin.A\,\sin.C\cos.b - \cos.A\cos.C \\ \cos.C &= \sin.A\,\sin.B\cos.c - \cos.A\cos.B \end{aligned}\right\} \quad (V)$$

By transposition and division, these equations become

$$\cos.a = \frac{\cos.A + \cos.B\cos.C}{\sin.B\,\sin.C} \quad (3)$$

$$\cos.b = \frac{\cos.B + \cos.A\cos.C}{\sin.A\,\sin.C}$$

$$\cos.c = \frac{\cos.C + \cos.A\cos.B}{\sin.A\,\sin.B}$$

From these we can find formulæ to express the sine or the cosine of one half of the side of a spherical triangle, in terms of the functions of its angles; thus:

Add 1 to each member of eq. (3), and we have

$$1 + \cos.a = \frac{\cos.A + \cos.B\cos.C + \sin.B\,\sin.C}{\sin.B\,\sin.C}$$

$$= \frac{\cos.A + \cos.(B - C)}{\sin.B \ \sin.C}$$

But, $1 + \cos.a = 2\cos.^2 \tfrac{1}{2}a$; hence,

$$2\cos.^2 \tfrac{1}{2}a = \frac{\cos.A + \cos.(B - C)}{\sin.B \sin.C}$$

and since $\cos.A + \cos.(B - C) = 2\cos.\tfrac{1}{2}(A + B - C)\cos.\tfrac{1}{2}(A + C - B)$ (Eq. 17, Sec. I, Plane Trig.), we have

$$2\cos.^2 \tfrac{1}{2}a = \frac{2\cos.\tfrac{1}{2}(A + B - C)\cos.\tfrac{1}{2}(A + C - B)}{\sin.B \sin.C}$$

Make $A + B + C = 2S$; then $A + B - C = 2S - 2C$, $A + C - B = 2S - 2B$, $\tfrac{1}{2}(A + B - C) = S - C$, and $\tfrac{1}{2}(A + C - B) = S - B$; whence

$$2\cos.^2 \tfrac{1}{2}a = \frac{2\cos.(S - C)\cos.(S - B)}{\sin.B \sin.C}$$

$$\left.\begin{aligned} &\text{or,} && \cos.\tfrac{1}{2}a = \sqrt{\frac{\cos.(S - C)\cos.(S - B)}{\sin.B \sin.C}} \\ &\text{Similarly,} && \cos.\tfrac{1}{2}b = \sqrt{\frac{\cos.(S - A)\cos.(S - C)}{\sin.A \sin.C}} \\ &\text{and,} && \cos.\tfrac{1}{2}c = \sqrt{\frac{\cos.(S - A)\cos.(S - A)}{\sin.A \sin.B}} \end{aligned}\right\} (V')$$

To find the $\sin.\tfrac{1}{2}a$ in terms of the functions of the angles, we must subtract each member of eq. (3) from 1, by which we get

$$1 - \cos.a = 1 - \frac{\cos.A + \cos.B \cos.C}{\sin.B \sin.C}.$$

But, $1 - \cos.a = 2\sin.^2 \tfrac{1}{2}a$; hence we have,

$$2\sin.^2 \tfrac{1}{2}a = \frac{(\sin.B \sin.C - \cos.B \cos.C) - \cos.A}{\sin.B \sin.C}.$$

Operating upon this in a manner analogous to that by which $\cos.\tfrac{1}{2}a$ was found, we get,

$$\left.\begin{aligned}\sin.\tfrac{1}{2}a &= \left\{\frac{-\cos.S\cos.(S-A)}{\sin.B\sin.C}\right\}^{\frac{1}{2}}\\ \sin.\tfrac{1}{2}b &= \left\{\frac{-\cos.S\cos.(S-B)}{\sin.A\sin.C}\right\}^{\frac{1}{2}}\\ \sin.\tfrac{1}{2}c &= \left\{\frac{-\cos.S\cos.(S-C)}{\sin.A\sin.B}\right\}^{\frac{1}{2}}\end{aligned}\right\}\quad (W)$$

If the first equation in (W) be divided by the first in (V'), we shall have,

$$\tan.\tfrac{1}{2}a = \left\{\frac{-\cos.S\cos.(S-A)}{\cos.(S-B)\cos.(S-C)}\right\}^{\frac{1}{2}}$$

And corresponding expressions may be obtained for $\tan.\frac{1}{2}b$ and $\tan.\frac{1}{2}c$.

NAPIER'S ANALOGIES.

If the value of $\cos.c$, expressed in the third equation of group (S), Prop. 7, be substituted for $\cos.c$, in the second member of the first equation of the same group, we have,

$$\cos.a = \cos.a\cos.^2b + \sin.a\sin.b\cos.b\cos.C + \sin.b\sin.c\cos.A;$$

which, by writing for $\cos.^2b$ its equal, $1-\sin.^2b$, becomes,

$$\cos.a = \cos.a - \cos.a\sin.^2b + \sin.a\sin.b\cos.b\cos.C + \sin.b\sin.c\cos.A.$$

Or, $0 = -\cos.a\sin.^2b + \sin.a\sin.b\cos.b\cos.C + \sin.b\sin.c\cos.A.$

Dividing through by $\sin.b$, and transposing, we find,

$$\cos.A\sin.c = \cos.a\sin.b - \sin.a\cos.b\cos.C;$$

hence, $\cos.A = \dfrac{\cos.a\sin.b-\sin.a\cos.b\cos.C}{\sin.c}$. (1)

By substituting the value of $\cos.c$, in the second of the equations of group (S), Prop. 7; or, more simply, by writing B for A, and b for a, in the above value, for $\cos.A$, we obtain,

$$\cos.B = \frac{\cos.b\sin.a-\sin.b\cos.a\cos.C}{\sin.c}. \quad (2)$$

Adding equations (1) and (2), member to member, we have,

$$\cos.A + \cos.B = \frac{\sin.(a+b) - \sin.(a+b)\cos.C}{\sin.c};$$

by remembering that $\sin.a \cos.b + \cos.a \sin.b = \sin.(a+b)$. (See Eq. (7), Sec. I, Plane Trig.).

Whence, $\cos.A + \cos.B = (1 - \cos.C)\dfrac{\sin.(a+b)}{\sin.c}$. (3)

In any spherical triangle we have, (Prop. I),

$$\sin.A : \sin.B :: \sin.a : \sin.b;$$

And therefore, $\sin.A + \sin.B : \sin.B :: \sin.a + \sin.b : \sin.b$.

Hence, $\sin.A + \sin.B = \dfrac{(\sin.a + \sin.b)\sin.B}{\sin.b}$.

But, $\dfrac{\sin.B}{\sin.b} = \dfrac{\sin.C}{\sin.c}$, which value of $\dfrac{\sin.B}{\sin.b}$, in the above equation, gives

$$\sin.A + \sin.B = \frac{(\sin.a + \sin.b)\sin.C}{\sin.c}. \qquad (4)$$

Dividing equation (4) by equation (3), member by member, we obtain,

$$\frac{\sin.A + \sin.B}{\cos.A + \cos.B} = \frac{\sin.C}{1 - \cos.C} \times \frac{\sin.a + \sin.b}{\sin.(a+b)}. \qquad (5)$$

Comparing this equation with Equations (20) and (26), Sec. I, Plane Trigonometry, we see that it can be reduced to

$$\tan.\tfrac{1}{2}(A+B) = \cot.\tfrac{1}{2}C \times \frac{\sin.a + \sin.b}{\sin.(a+b)} \qquad (6)$$

Again, from the proportion,

$$\sin.A : \sin.B :: \sin.a : \sin.b,$$

we likewise have,

$$\sin.A - \sin.B : \sin.B :: \sin.a - \sin.b : \sin.b;$$

hence, $\sin.A - \sin.B = (\sin.a - \sin.b)\dfrac{\sin.B}{\sin.b} = (\sin.a - \sin.b)\dfrac{\sin.C}{\sin.c}$.

Dividing this equation by equation (3), member by member, we obtain,

$$\frac{\sin.A - \sin.B}{\cos.A + \cos.B} = \frac{\sin.C}{1 - \cos.C} \times \frac{\sin.a - \sin.b}{\sin.(a+b)}.$$

Comparing this with Equations (22) and (26), Sec. I, Plane Trigonometry, we see that it will reduce to

$$\tan.\tfrac{1}{2}(A - B) = \cot.\tfrac{1}{2}C \times \frac{\sin.a - \sin.b}{\sin.(a+b)}. \quad (7)$$

Now, $\sin.a + \sin.b = 2\sin.\left(\dfrac{a+b}{2}\right)\cos.\left(\dfrac{a-b}{2}\right)$; Eq. (15), Sec. I, Plane Trig.).

and, $\sin.(a+b) = 2\sin.\left(\dfrac{a+b}{2}\right)\cos.\left(\dfrac{a+b}{2}\right)$; Eq. (30), Sec. I, Plane Trig.).

Dividing the first of these by the second, we have

$$\frac{\sin.a + \sin.b}{\sin.(a+b)} = \frac{\cos.\left(\dfrac{a-b}{2}\right)}{\cos.\left(\dfrac{a+b}{2}\right)}$$

Writing the second member of this equation for its first member in Eq (6), that equation becomes

$$\tan.\tfrac{1}{2}(A+B) = \cot.\tfrac{1}{2}C\frac{\cos.\tfrac{1}{2}(a-b)}{\cos.\tfrac{1}{2}(a+b)}. \quad (8)$$

By a similar operation, Eq. (7) may be reduced to

$$\tan.\tfrac{1}{2}(A-B) = \cot.\tfrac{1}{2}C\frac{\sin.\tfrac{1}{2}(a-b)}{\sin.\tfrac{1}{2}(a+b)}. \quad (9)$$

Equations (8) and (9) may be resolved into the proportions

$\cos.\tfrac{1}{2}(a+b) : \cos.\tfrac{1}{2}(a-b) :: \cot.\tfrac{1}{2}C : \tan.\tfrac{1}{2}(A+B)$;

$\sin.\tfrac{1}{2}(a+b) : \sin.\tfrac{1}{2}(a-b) :: \cot.\tfrac{1}{2}C : \tan.\tfrac{1}{2}(A-B)$.

These proportions are known as Napier's 1st and 2d

Analogies, and may be advantageously used in the solution of spherical triangles, when *two sides and the included angle are given.*

When expressed in language, these proportions furnish the following rules:

1. *The cosine of the half sum of any two sides of a spherical triangle is to the cosine of the half difference of the same sides, as the cotangent of half the included angle is to the tangent of the half sum of the other two angles.*

2. *The sine of the half sum of any two sides of a spherical triangle is to the sine of the half difference of the same sides, as the cotangent of half the included angle is to the tangent of the half difference of the other two angles.*

The half sum, and the half difference of two angles of a spherical triangle, may be found by these rules, when two sides and the included angle are given; and by adding the half sum to the half difference, we get the greater of these two angles, and by subtracting the half difference from the half sum, we get the smaller. The third side may then be found by proportion.

We have analogous proportions applicable to the case in which two angles and the included side of a spherical triangle are given.

To deduce these, let us represent the angles of the triangle by A, B, and C, and the opposite sides by a, b, and c; A', B', C', a', b', c', denoting the corresponding angles and sides of the polar triangle.

Now, Eq. (9) is applicable to any spherical triangle, and when applied to the polar triangle, it becomes

$$\tan. \tfrac{1}{2}(A' - B') = \cot. \tfrac{1}{2}C' \frac{\sin. \frac{1}{2}(a' - b')}{\sin. \frac{1}{2}(a' + b')}. \qquad (n)$$

But by Prop. 6, Sec. I, Spherical Geometry, we have

$$A' = 180° - a,\ B' = 180° - b,\ C' = 180° - c,$$
$$a' = 180° - A,\ b' = 180° - B,\ c' = 180° - C.$$

Whence, $\frac{1}{2}(A' - B') = \frac{1}{2}(b - a)$, $\frac{1}{2}(a' + b') = 180° - \frac{A + B}{2}$,

$$\tfrac{1}{2}(a' - b') = \tfrac{1}{2}(B - A),\ \tfrac{1}{2}C' = 90° - \tfrac{1}{2}c.$$

By the substitution of these values in Eq. (n), that equation becomes

$$\tan. \tfrac{1}{2}(b - a) = \frac{\sin. \frac{1}{2}(B - A)}{\sin. \frac{1}{2}(A + B)} \tan. \tfrac{1}{2}c,$$

$$\text{or,} \quad \tan. \tfrac{1}{2}(a - b) = \frac{\sin. \frac{1}{2}(A - B)}{\sin. \frac{1}{2}(A + B)} \tan. \tfrac{1}{2}c, \qquad (p)$$

since $\tan. \frac{1}{2}(b - a) = - \tan. \frac{1}{2}(a - b)$, and $\sin. \frac{1}{2}(B - A = - \sin. \frac{1}{2}(A - B)$.

By applying Eq. (8) to the polar triangle, and treating the resulting equation in a manner similar to the above, we find

$$\tan. \tfrac{1}{2}(a + b) = \frac{\cos. \frac{1}{2}(A - B)}{\cos. \frac{1}{2}(A + B)} \tan. \tfrac{1}{2}c, \qquad (q)$$

Equations (p) and (q) may be resolved into the following proportions.

$$\sin. \tfrac{1}{2}(A + B) : \sin. \tfrac{1}{2}(A - B) :: \tan. \tfrac{1}{2}c : \tan. \tfrac{1}{2}(a - b);$$
$$\cos. \tfrac{1}{2}(A + B) : \cos. \tfrac{1}{2}(A - B) :: \tan. \tfrac{1}{2}c : \tan. \tfrac{1}{2}(a + b).$$

These proportions are called Napier's 3d and 4th Analogies, and when expressed in words become the following rules:

1. *The cosine of the half sum of any two angles of a spherical triangle is to the cosine of the half difference of the same angles, as the tangent of half the included side is to the tangent of the half sum of the other two sides.*

2. *The sine of the half sum of any two angles of a spherical triangle is to the sine of the half difference of the same angles, as the tangent of half the included side is to the tangent of the half difference of the other two sides.*

The half sum, and the half difference of two sides of a spherical triangle, may be found by these rules, when two angles and the included side are given; and by adding the half sum to the half difference, we get the greater of these sides, and by subtracting the half difference from the half sum, we get the smaller.

SECTION IV.

SPHERICAL TRIGONOMETRY APPLIED.

SOLUTION OF RIGHT-ANGLED SPHERICAL TRIANGLES.

A GOOD general conception of the sphere is essential to a practical knowledge of spherical trigonometry, and this conception is best obtained by the examination of an artificial globe. By tracing out upon its surface the various forms of right-angled and oblique-angled triangles, and viewing them from different points, we may soon acquire the power of making a natural representation of them on paper, which will be found of much assistance in the solution and interpretation of problems.

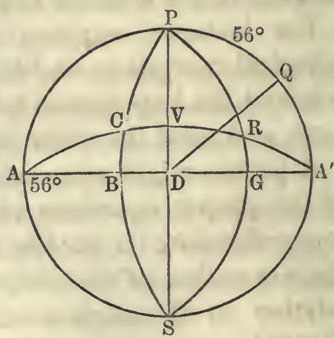
For instance, suppose one side of a right-angled spherical triangle to be 56°, and the angle between this side and the hypotenuse to be 24°. What is the hypotenuse, and what the other side and angle?

A person might solve this problem by the application of the proper equations or proportions, without really comprehending it; that is, without being able to form a distinct notion of the shape of the triangle, and of its relation to the surface of the sphere on which it is situated.

If we refer this triangle to the common geographical globe, the side 56° may be laid off on the equator, or on a meridian. In the first case, the hypotenuse will be the arc of a great circle drawn through one extremity of the side 56°, above or below the equator, and making with it an angle of 24°; the other side will be an arc of a meridian. In the second case, the side 56° falling on a meridian, the hypotenuse will be the arc of a great circle drawn through one extremity of this side, on the right or left of the meridian, and making with it an angle of 24°; the other side will be the arc of a great circle, at right angles to the meridian in which the given side lies.

Generally speaking, the apparent form of a spherical triangle, and consequently the manner of representing it on paper, will differ with the position assumed for the eye in viewing it. From whatever point we look at a sphere, its outline is a perfect circle in the axis of which the eye is situated; and when the eye is, as will be hereafter supposed, at an infinite distance, this circle will be a great circle of the sphere. All great circles of the sphere whose planes pass through the eye, will seem to be diameters of the circle which represents the outline of the sphere.

We will now suppose the eye to be in the plane of the equator, and proceed to construct our triangle on paper.



Let the great circle, $PASA'$, represent the outline of the sphere, the diameter AA' the equator, and the diameter PS the central meridian, or the meridian in whose plane the eye is situated. Let $AB = 56°$, represent the given side, and AC, making with AB the angle $BAC =$ 24°, the hypotenuse, then will BC, the arc of a meridian, be the other side at right angles to AB, and the triangle, ABC, corresponds in all respects to the given triangle.

Again, measure off 56° from P to Q, draw the radius DQ, make the arc $A'G$ equal to 24°, and draw the quadrant PRG. The triangle PQR will also represent the given triangle in every particular.

We know from the construction, that $DV, = 24°$, is greater than BC, and that AC is greater than AB, that is, greater than 56°.

In like manner, we know that $A', = 24°$, is greater than QR, and that PR is greater than PQ, because PR is more nearly equal to $PG, = 90°$, than PQ is to $PA, = 90°$.

For illustration and explanation, we also give the following example:

In a right-angled spherical triangle, there are given, the hypotenuse equal to 150° 33′ 20″, the angle at the base, 23° 27′ 29″, to find the base and the perpendicular. Let $A'BC$ in the last figure, represent the triangle in which $A'C = 150° 33' 20''$, the $\angle BA'C = 23° 27' 29''$, and the sides $A'B$ and BC are required.

This problem presents a right-angled spherical triangle, whose base and hypotenuse are each greater than 90°; and in cases of this kind, let the pupil observe, *that the base is greater than the hypotenuse*, and the oblique angle opposite the base, is greater than a right angle. In all cases, a spherical triangle and its supplemental triangle make a *lune.* It is 180° from one pole to its opposite, whatever great circle be traversed. It is 180° along the equator ABA', and also 180° along the ecliptic ACA'. The lune always gives two triangles; and when the sides of one of them are greater than 90°, we take the triangle having supplemental sides; hence in this case we operate on the triangle ABC.

AC is greater than AB, therefore $A'B$ is greater than the hypotenuse $A'C$.

The $\angle ACB$ is less than 90°; therefore, the adjacent angle $A'CB$ is greater than 90°, the two together being equal to two right angles.

These facts are technically expressed, by saying, that the sides and opposite angles are of the *same affection.**

---

* *Same affection:* that is, both greater or both less than 90°. *Different affection:* the one greater, the other less than 90°.

Now, if the two sides of a right-angled spherical triangle are of the *same affection*, the hypotenuse will be less than 90°; and if of *different affection*, the hypotenuse will be greater than 90°.

If, in every instance, we make a natural construction of the figure, and use common judgment, it will be impossible to doubt whether an arc must be taken greater or less than 90°.

We will now solve the triangle ACB. $AC = 180° - 150° 33' 20'' = 29° 26' 40''$.

To find BC, we use Eq. (3) or (13), Prop. 3, Sec. II., thus:

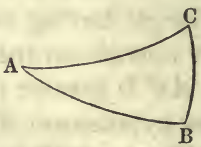
b, sin.	29° 26′ 40″	.	9.691594
A, sin.	23° 27′ 29″	.	9.599984
a, sin.	11° 17′ 7″	.	9.291578

To find AB, we use equation (1) or (11), thus:

a, tan.	11° 17′ 7″	.	9.300016
A, cot.	23° 27′ 29″	.	10.362674
c, sin.	27° 22′ 32″	.	9.662690
	180		
$A'B =$	152° 37′ 28″		

PRACTICAL PROBLEMS IN RIGHT-ANGLED SPHERICAL TRIGONOMETRY.

1. In the right-angled spherical triangle ABC, given $AB = 118° 21' 4''$, and the angle $A = 23° 40' 12''$, to find the other parts.



Ans. { AC, 116° 17′ 45″; the angle C, 100° 59′ 26″; and BC, 21° 5′ 42″.

2. In the right-angled spherical triangle ABC, given AB 53° 14′ 20″, and the angle A 91° 25′ 53″, to find the other parts.

Ans. { AC, 91° 4′ 9″; the angle C, 53° 15′ 8″; and BC, 91° 47′ 11″.

3. In the right-angled spherical triangle *ABC*, given *AB* 102° 50′ 25″, and the angle *A* 113° 14′ 37″, to find the other parts.

Ans. { *AC*, 84° 51′ 36″; the angle *C*, 101° 46″ 57″; and *BC*, 113° 46′ 27″.

4. In the right-angled spherical triangle *ABC*, given *AB* 48° 24′ 16″, and *BC* 59° 38′ 27″, to find the other parts.

Ans. { *AC*, 70° 23′ 42″; the angle *A*, 66° 20′ 40″; and the angle *C*, 52° 32′ 55″.

5. In the right-angled spherical triangle *ABC*, given *AB* 151° 23′ 9″, and *BC* 16° 35′ 14″, to find the other parts.

Ans. { *AC*, 147° 16′ 51″; the angle *C*, 117° 37′ 21″; and the angle *A*, 31° 52′ 50″.

6. In the right-angled spherical triangle *ABC*, given *AB* 73° 4′ 31″, and *AC* 86° 12′ 15,″ to find the other parts.

Ans. { *BC*, 76° 51′ 20″; the angle *A*, 77° 24′ 23″; and the angle *C*, 73° 29′ 40″.

7. In the right-angled spherical triangle *ABC*, given *AC* 118° 32′ 12″, and *AB* 47° 26′ 35″, to find the other parts.

Ans. { *BC*, 134° 56′ 20″; the angle *A*, 126° 19′ 2″; and the angle *C*, 56° 58′ 44″.

8. In the right-angled spherical triangle *ABC*, given *AB* 40° 18′ 23″, and *AC* 100′° 3′ 7″, to find the other parts.

Ans. { The angle *A*, 98° 38′ 53″; the angle *C*, 40° 4′ 6″; and *BC*, 103° 13′ 52″.

9. In the right-angled spherical triangle *ABC*, given *AC* 61° 3′ 22″, and the angle *A* 49° 28′ 12″, to find the other parts.

Ans. { *AB*, 49° 36′ 6″; the angle *C*, 60° 29′ 19″; and *BC*, 41° 41′ 32″.

10. In the right-angled spherical triangle *ABC*, given

AB 29° 12′ 50″, and the angle *C* 37° 26′ 21″, to find the other parts.

Ans. Ambiguous; the angle *A*, 65° 27′ 58″, or its supplement; *AC*, 53° 24′ 13″, or its supplement; *BC*, 46° 55′ 2″, or its supplement.

11. In the right-angled spherical triangle *ABC*, given *AB* 100° 10′ 3″, and the angle *C* 90° 14′ 20″, to find the other parts.

Ans. *AC*, 100° 9′ 55″, or its supplement; *BC*, 1° 19′ 53″, or its supplement; and the angle *A*, 1° 21′ 8″, or its supplement.

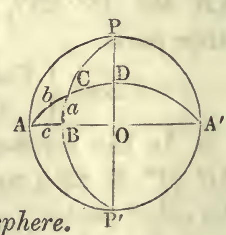
12. In the right-angled spherical triangle *ABC*, given *AB* 54° 21′ 35″, and the angle *C* 61° 2′ 15″, to find the other parts.

Ans. *BC*, 129° 28′ 28″, or its supplement; *AC*, 111° 44′ 34″, or its supplement; and the angle *A*, 123° 47′ 44″, or its supplement.

13. In the right-angled spherical triangle *ABC*, given *AB* 121° 26′ 25″, and the angle *C* 111° 14′ 37″, to find the other parts.

Ans. The angle *A*, 136° 0′ 3″, or its supplement; *AC*, 66° 15′ 38″, or its supplement; and *BC*, 140° 30′ 56″, or its supplement.

QUADRANTAL TRIANGLES.



The solution of right-angled spherical triangles includes, also, the solution of *quadrantal* triangles, as may be seen by inspecting the adjoining figure. *When we have one quadrantal triangle, we have four, which with one right-angled triangle, fill up the whole hemisphere.*

To effect the solution of either of the four quadrantal triangles, *APC*, *AP′C*, *A′PC*, or *A′P′C*, it is sufficient to solve the small right-angled spherical triangle *ABC*.

To the half lune $AP'B$, we add the triangle ABC, and we have the quadrantal triangle $AP'C$; and by subtracting the same from the equal half lune APB, we have the quadrantal triangle PAC.

When we have the side, AC, of the same triangle, we have its supplement, $A'C$, which is a side of the triangles $A'PC$, and $A'P'C$. When we have the side, CB, of the small triangle, by adding it to 90°, we have $P'C$, a side of the triangle $A'P'C$; and subtracting it from 90°, we have PC, a side of the triangles APC, and $AP'C$.

PROBLEM I.

In a quadrantal triangle, there are given the quadrantal side, 90°, *a side adjacent,* 42° 21′, *and the angle opposite this last side, equal to* 36° 31′. *Required the other parts.*

By this enumeration we cannot decide whether the triangle APC or $AP'C$, is the one required, for $AC = 42° 21'$ belongs equally to both triangles. The angle $APC = AP'C = 36° 31' = AB$. We operate wholly on the triangle ABC.

To find the angle A, call it the *middle part.*

Then, $R \cos. CAB = R \sin. PAC = \cot. AC \tan. AB$.

$\cot. AC$	=	42° 21′	.	10.040231
$\tan. AB$	=	36° 31′	.	9.869473
$\cos. CAB$	=	35° 40′ 51″		9.909704
		90°		
PAC	=	54° 19′ 9″		
$P'AC$	=	125° 40′ 51″		

To find the angle C, call it the *middle part.*

$R \cos. ACB = \sin. CAB \cos. AB$.

$\sin. CAB$	=	35° 40′ 51″		9.765869
$\cos. AB$	=	36° 31′	.	9.905085
$\cos. ACB$	=	62° 2′ 45″		9.670954
		180°		

$ACP = A'CP' = 117° 57' 15''$

To find the side BC, call it the *middle part*.

$$R \sin.BC = \tan.AB \cot.ACB.$$

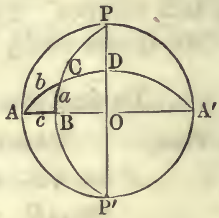
$\tan.AB$	=	36° 31′ 0″	9.869473
$\cot.ACB$	=	62° 2′ 45″	9.724835
$\sin.BC$	=	23° 8′ 11″	9.594308
		90°	
PC	=	66° 51′ 49″	
$P'C$	=	113° 8′ 11″	

We now have all the sides, and all the angles of the *four* triangles in question.

PROBLEM II.

In a quadrantal spherical triangle, having given the quadrantal side, 90°, *an adjacent side*, 115° 09′, *and the included angle*, 115° 55′, *to find the other parts.*

This enunciation clearly points out the particular triangle $A'P'C$. $A'P' = 90°$; and conceive $A'C = 115° 09'$. Then the angle $P'A'C = 115° 55' = P'D$.



From the angle $P'A'C$ take 90°, or $P'A'B$, and the remainder is the angle $OA'D = BAC = 25° 55'$.

We here again operate on the triangle ABC. $A'C$, taken from 180°, gives

$$64° 51' = AC.$$

To find BC, we call it the *middle part*.

$$\text{R} \sin.BC = \sin.AC \sin.BAC.$$

$\sin.AC$	=	64° 51′	.	9.956744
$\sin.BAC$	=	25° 55′	.	9.640544
$\sin.BC$	=	23° 18′ 19″	.	9.597288
		90°		
$P'C$	=	113° 18′ 19″		

To find AB, we call it the *middle part.*

$$R \sin.AB = \tan.BC \cot.BAC.$$

$\tan.BC$	$= 23° 18' 19''$	.	9.634251
$\cot.BAC$	$= 25° 55'$	.	9.313423
$\sin.AB$	$= 62° 26' 8''$	.	8.947674
	$180°$		
$A'B$	$= 117° 33' 52''$ = the angle $A'P'C$.		

To find the angle C, we call it the *middle part.*

$$R \cos. C = \cot.AC \tan.BC.$$

$\cot.AC$	$= 64° 51'$	.	9.671634
$\tan.BC$	$= 23° 18' 19''$	.	9.634251
$\cos. C$	$= 78°$	.	9.305885
	$180° 19' 53''$	.	
$P'CA'$	$= 101° 40' 7''$		

Thus we have found the side $P'C = 113° 18' 19''$
The angle $A'P'C = 117° 33' 52''$ } *Ans.*
" $P'CA' = 101° 40' 7''$

PRACTICAL PROBLEMS.

1. In a quadrantal triangle, given the quadrantal side, 90°, a side adjacent, 67° 3′, and the included angle, 49° 18′, to find the other parts.

Ans. { The remaining side is 53° 5′ 46″; the angle opposite the quadrantal side, 108° 32′ 27″; and the remaining angle, 60° 48′ 54″.

2. In a quadrantal triangle, given the quadrantal side, 90°, one angle adjacent, 118° 40′ 36″, and the side opposite this last-mentioned angle, 113° 2′ 28″, to find the other parts.

Ans. { The remaining side is 54° 38′ 57″; the angle opposite, 51° 2′ 35″; and the angle opposite the quadrantal side 72° 26′ 21″.

3. In a quadrantal triangle, given the quadrantal side,

90°, and the two adjacent angles, one 69° 13′ 46″, the other 72° 12′ 4″, to find the other parts.

Ans. { One of the remaining sides is 70° 8′ 39″, the other is 73° 17′ 29″, and the angle opposite the quadrantal side is 96° 13′ 23″.

4. In a quadrantal triangle, given the quadrantal side, 90°, one adjacent side, 86° 14′ 40″, and the angle opposite to that side, 37° 12′ 20″, to find the other parts.

Ans. { The remaining side is 4° 43′ 2″; the angle opposite, 2° 51′ 23″; and the angle opposite the quadrantal side, 142° 42′ 2″.

5. In a quadrantal triangle, given the quadrantal side, 90°, and the other two sides, one 118° 32′ 16″, the other 67° 48′ 40″, to find the other parts — the three angles.

Ans. { The angles are 64° 32′ 21″, 121° 3′ 40″, and 77° 11′ 6″; the greater angle opposite the greater side, of course.

6. In a quadrantal triangle, given the quadrantal side, 90°, the angle opposite, 104° 41′ 17″, and one adjacent side, 73° 21′ 6″, to find the other parts.

Ans. { Remaining side, 49° 42′ 18″; remaining angles, 47° 32′ 39″, and 67° 56′ 13″.

SOLUTION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES.

All cases of oblique-angled spherical trigonometry may be solved by right-angled Trigonometry, except two; because every oblique-angled spherical triangle is composed of the sum, or the difference, of two right-angled spherical triangles.

When a side and two of the angles, or *an angle and two of the sides are given*, to find the other parts, conform to the following directions:

Let a perpendicular be drawn from an extremity of a given side, and opposite a given angle or its supplement; this will form two right-angled spherical triangles; and one of them will have its hypotenuse and one of its adjacent angles given, from which all its other parts can be computed; and some of these parts will become as known parts to the other triangle, from which all its parts can be computed.

To facilitate these computations, we here give a summary of the practical truths demonstrated in the foregoing propositions.

1. *The sines of the sides of spherical triangles are proportional to the sines of their opposite angles.*

2. *The sines of the segments of the base, made by a perpendicular from the opposite angle, are proportional to the cotangents of their adjacent angles.*

3. *The cosines of the segments of the base are proportional to the cosines of the adjacent sides of the triangle.*

4. *The tangents of the segments of the base are reciprocally proportional to the cotangents of the segments of the vertical angle.*

5. *The cosines of the angles at the base are proportional to the sines of the corresponding segments of the vertical angle.*

6. *The cosines of the segments of the vertical angle are proportional to the cotangents of the adjoining sides of the triangle.*

The two cases in which right-angled spherical triangles are not used, are,

1st. When the three sides are given to find the angles; and,

2d. When the three angles are given to find the sides.

The first of these cases is the most important of all, and for that reason great attention has been given to it, and two series of equations, (T and U, Prop. 7, Sec. III), have been deduced to facilitate its solution.

As heretofore, let ABC represent any triangle whose angles are denoted by A, B, and C, and sides by a, b, and c; the side a being opposite $\angle A$, the side b opposite $\angle B$, etc.

EXAMPLES.

1. In the triangle ABC, $a = 70° 4' 18''$; $b = 63° 21' 27''$; and c, $59° 16' 23''$; required the angle A.

The formula for this is the first equation in group (T, Prop. 7, Sec. III), which is

$$\cos.\frac{A}{2} = \left(\frac{R^2 \sin.S \sin.(S-a)}{\sin.b \sin.c}\right)^{\frac{1}{2}}.$$

We write the second member of this equation thus:

$$\sqrt{\left(\frac{R}{\sin.b}\right)\left(\frac{R}{\sin.c}\right)(\sin.S)\sin.(S-a)},$$

showing four distinct factors under the radical.

The logarithm corresponding to $\frac{R}{\sin.b}$ is that of $\sin.b$ subtracted from 10; and of $\frac{R}{\sin.c}$ is that of $\sin.c$ subtracted from 10, which we call *sin.complement.*

$BC = a =$	70° 4′ 18″		
$AB = c =$	59° 16′ 23″	sin. com.	.065697
$AC = b =$	63° 21′ 27″	sin. com.	.048749
	2)192° 42′ 8″		
$S =$	96° 21′ 4″	sin.	9.997326
$S - a =$	26° 16′ 46″	sin.	9.646158
			2) 19.757930
$\frac{1}{2}A =$	40° 49′ 10″	cos.	9.878965
	2		
$A =$	81° 38′ 20″		

When we apply the equation to find the angle A, we write a first, at the top of the column; when we apply the equation to find the angle B, we write b at the top of the column. Thus, To find the angle B.

$$\cos.\tfrac{1}{2}B = \sqrt{\frac{R^2 \sin.S \sin.(S-b)}{\sin.a \sin.c}}$$

$$= \sqrt{\left(\frac{R}{\sin.a}\right)\left(\frac{R}{\sin.c}\right)(\sin.S)\sin.(S-b)}$$

$b =$	63° 21′ 27″		
$c =$	59° 16′ 23″	sin.com. .	.065697
$a =$	70° 4′ 18″	sin.com. .	.026875
	2)192° 42′ 8″		
$S =$	96° 21′ 4″	sin. . .	9.997326
$S - b =$	32° 59′ 37″	sin. . .	9.736034
			2) 19.825872
$\tfrac{1}{2}B =$	35° 4′ 49″	cos. . .	9.912936
	2		
$B =$	70° 9′ 38″		

By the other equation in formulæ (T, Prop. 7, Sec. III), we can find the angle C; but, for the sake of variety, we will find the angle C by the application of the third equation in formulæ (U, Prop. 7, Sec. III).

$$\sin.\tfrac{1}{2}C = \sqrt{\frac{R^2 \sin.(S-b)\sin.(S-a)}{\sin.b \sin.a}}$$

$$= \sqrt{\left(\frac{R}{\sin.b}\right)\left(\frac{R}{\sin.a}\right)\sin.(S-b)\sin.(S-a)}$$

$c =$	59° 16′ 23″		
$a =$	70° 4′ 18″	sin.com.	.026817
$b =$	63° 21′ 27″	sin.com.	.048479
	2)192° 42′ 8″		
$S =$	96° 21′ 4″		
$S - a =$	26° 16′ 46″	sin. .	9.646158
$S - b =$	32° 59′ 37″	sin. .	9.736034
			2) 19.457488
$\tfrac{1}{2}C =$	32° 23′ 17″	sin. .	9.778744
	2		
$C =$	64° 46′ 34″		

To show the harmony and practical utility of these two sets of equations, we will find the angle A, from the equation

$$\sin.\tfrac{1}{2}A = \sqrt{\left(\frac{R}{\sin.b}\right)\left(\frac{R}{\sin.c}\right)\sin.(S-b)\sin.(S-c)}.$$

$a =$	70° 4′ 18″		
$b =$	63° 21′ 27″	sin.com.	.048749
$c =$	59° 16′ 23″	sin.com.	.065697
2)	192° 42′ 8″		
$S =$	96° 21′ 4″		
$S-b =$	32° 59′ 37″	sin.	9.736034
$S-c =$	37° 4′ 41″	sin.	9.780247
			2) 19.630727
$\tfrac{1}{2}A =$	40° 49′ 10″	sin.	9.815363
	2		
$A =$	81° 38′ 20″		

2. In a spherical triangle ABC, given the angle A, 38° 19′ 18″; the angle B, 48° 0′ 10″; and the angle C, 121° 8′ 6″; to find the sides a, b, c.

By passing to the triangle polar to this, we have, (Prop. 6, Sec. I, Spherical Geometry),

$A =$	38° 19′ 18″	supplement	141° 40′ 42″
$B =$	48° 0′ 10″	supplement	131° 59′ 50″
$C =$	121° 8′ 6″	supplement	58° 51′ 54″

We now find the angles to the spherical triangle, the sides of which are these supplements.

Thus,	141° 40′ 42″		
	131° 59′ 50″	sin.com.	.128909
	58° 51′ 54″	sin.com.	.067551
2)	332° 32′ 26″		
	166° 16′ 13″	sin.	9.375375
	24° 35′ 31″	sin.	9.619253
			2) 19.191088
	66° 47′ 37½″	cos.	9.595544

$$\begin{array}{rl} & 60° \ 47' \ 37\tfrac{1}{2}'' \\ & \underline{\qquad\quad 2''} \\ \text{angle} = & 121° \ 35' \ 15'' \\ \text{supp.} = & 58° \ 24' \ 45'' = a \text{ of the original triangle.} \end{array}$$

In the same manner we find $b = 60° \ 14' \ 25''$; $c = 89° \ 1' \ 14''$.

It is perhaps better to avoid this indirect process of computing the sides of a spherical triangle when the angles are given, by the application of the equations in group V' or W, Prop. 8, Sec. III. We will illustrate their use by applying the second equation in group (W), for computing the side b. This equation is

$$\sin.\tfrac{1}{2}b = \left(\frac{-\cos.S \ \cos.(S-B)}{\sin.A \ \sin.C}\right)^{\frac{1}{2}}$$

$$\begin{array}{rl} A = & 38° \ 19' \ 18'' \\ B = & 48° \ \ 0' \ 10'' \\ C = & 121° \ \ 8' \ \ 6'' \\ 2\,) & 207° \ 27' \ 34'' \end{array}$$

$$\begin{array}{rlr} S = & 103° \ 43' \ 47'' & -\cos.S = +\sin.13° \ 43' \ 47'' = 9.375376 \\ B = & 48° \ \ 0' \ 10'' & \cos.(S-B) = 55° \ 43' \ 37'' = 9.750612 \\ (S-B) = & 55° \ 43' \ 37'' & 2\,)\,19.125988 \\ & & \text{square root} = 9.562994 \end{array}$$

$$\begin{array}{rl} \sin.A = & 38° \ 19' \ 18'' = 9.792445 \\ \sin.C = & 121° \ \ 8' \ \ 6'' = 9.932443 \\ & 2\,)\,19.724888 \\ & \text{square root} = 9.862444 = 9.862444 \end{array}$$

$$\begin{array}{rr} & \text{diff.} -1.700550 \\ \text{Add 10, for radius of the table,} & 10 \\ \text{Tabular } \sin.\tfrac{1}{2}b = \ 30° \ \ 7' \ 14'' = & 9.700550 \\ & 2 \\ b = & 60° \ 14' \ 28'', \text{ nearly.} \end{array}$$

PRACTICAL PROBLEMS.

1. In any triangle, ABC, whose sides are a, b, c, given $b = 118° \ 2' \ 14''$, $c = 120° \ 18' \ 33''$, and the included angle $A = 27° \ 22' \ 34''$, to find the other parts.

Ans. $\begin{cases} a = 23° \ 57' \ 13'', \text{ angle } B = 91° \ 26' \ 44, \text{ and } C = \\ 102° \ 5' \ 54''. \end{cases}$

2. Given, $A = 81° \ 38' \ 17''$, $B = 70° \ 9' \ 38''$, and $C = 64° \ 46' \ 32''$, to find the sides a, b, c.

Ans. $\begin{cases} a = 70° \ 4' \ 18'', b = 63° \ 21' \ 27'', \text{ and } c = 59° \ 16' \\ 23''. \end{cases}$

3. Given, the three sides, $a = 93° \ 27' \ 34''$, $b = 100° \ 4' \ 26''$, and $c = 96° \ 14' \ 50''$, to find the angles A, B, and C.

Ans. $\begin{cases} A = 94° \ 39' \ 4'', B = 100° \ 32' \ 19'', \text{ and } C = 96° \\ 58' \ 36''. \end{cases}$

4. Given, two sides, $b = 84° \ 16'$, $c = 81° \ 12'$, and the angle $C = 80° \ 28'$, to find the other parts.

Ans. The result is ambiguous, for we may consider the angle B as acute or obtuse. If the angle B is acute, then $A = 97° \ 13' \ 45''$, $B = 83° \ 11' \ 24''$, and $a = 96° \ 13' \ 33''$. If B is obtuse, then $A = 21° \ 16' \ 44''$, $B = 96° \ 48' \ 36''$, and $a = 21° \ 19' \ 29''$.

5. Given, one side, $c = 64° \ 26'$, and the angles adjacent, $A = 49°$, and $B = 52°$, to find the other parts.

Ans. $\begin{cases} b = 45° \ 56' \ 46'', a = 43° \ 29' \ 49'', \text{ and } C = 98° \\ 28' \ 5''. \end{cases}$

6. Given, the three sides, $a = 90°$, $b = 90°$, $c = 90°$, to find the angles A, B, and C.

Ans. $A = 90°$, $B = 90°$, and $C = 90°$.

7. Given, the two sides, $a = 77° \ 25' \ 11''$, $c = 128° \ 13' \ 47''$, and the angle $C = 131° \ 11' \ 12''$, to find the other parts.

Ans. $\begin{cases} b = 84° \ 29' \ 24'', A = 69° \ 14', \text{ and } B = 72° \ 28' \\ 46''. \end{cases}$

8. Given, the three sides, $a = 68° \ 34' \ 13''$, $b = 59° \ 21' \ 18''$, and $c = 112° \ 16' \ 32''$, to find the angles A, B, and C.

Ans. $\begin{cases} A = 45° \ 26' \ 12'', B = 41° \ 11' \ 6'', C = 134° \ 54' \\ 27''. \end{cases}$

9. Given, $a = 89° 21' 37''$, $b = 97° 18' 39''$, $c = 86° 53' 46''$, to find A, B, and C.

Ans. $A = 88° 57' 20''$, $B = 97° 21' 26''$, $C = 86° 47' 17''$.

10. Given, $a = 31° 26' 41''$, $c = 43° 22' 13''$, and the angle $A = 12° 16'$, to find the other parts.

Ans. Ambiguous; $b = 73° 7' 35''$, or $12° 17' 39''$; angle $B = 115° 0' 31''$, or $47° 1' 36''$; $C = 16° 14' 27''$, or $163° 45' 33''$.

11. In a triangle, ABC, we have the angle $A = 56° 18' 40''$, $B = 39° 10' 38''$; AD, one of the segments of the base, is $32° 54' 16''$. The point D falls upon the base AB, and the angle C is obtuse. Required the sides of the triangle and the angle C.

Ans. $C = 135° 47' 56''$, $c = 123° 4' 56''$, $a = 90° 8' 17''$, $b = 49° 23' 41''$.

12. Given, $A = 80° 10' 10''$, $B = 58° 48' 36''$, $C = 91° 52' 42''$, to find a, b, and c.

Ans. $a = 79° 38' 21''$, $b = 58° 39' 16''$, $c = 86° 12' 52''$.

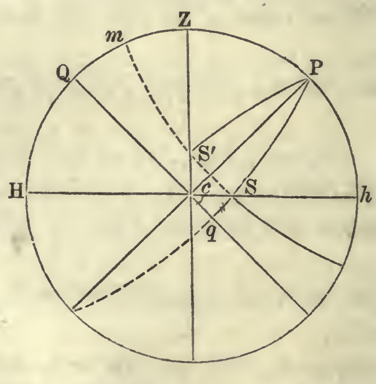
SECTION V.

APPLICATIONS OF SPHERICAL TRIGONOMETRY TO ASTRONOMY AND GEOGRAPHY.

SPHERICAL TRIGONOMETRY APPLIED TO ASTRONOMY.

Spherical Trigonometry becomes a science of incalculable importance in its connection with geography, navigation, and astronomy; for neither of these subjects can be understood without it; and to stimulate the student to a study of the science, we here attempt to give him a glimpse at some of its points of application.

Let the lines in the annexed figure represent circles in the heavens above and around us.



Let Z be the zenith, or the point just overhead, Hch the horizon, PZH the meridian in the heavens, and P the pole of the earth's equator; then Ph is the latitude of the observer, and PZ is the co.latitude. Qcq is a portion of the equator, and the dotted, curved line, $mS'S$, parallel to the equator, is the parallel of the sun's declination at some particular time; and in this figure the sun's declination is supposed to be north. By the revolution of the earth on its axis, the sun is apparently brought from the horizon, at *S*, to the meridian, at *m*; and from thence it is carried down on the same curve, on the other side of the meridian; and this apparent motion of the sun (or of any other celestial body,) makes angles at the pole *P*, which are in direct proportion to their times of description.

The apparent straight line, *Zc*, is what is denominated, in astronomy, the *prime vertical;* that is, the east and west line through the zenith, passing through the *east* and *west* points in the horizon.

When the latitude of the place is north, and the declination is also north, as is represented in this figure, the sun rises and sets on the horizon to the north of the east and west points, and the distance is measured by the arc, *cS*, on the horizon.

This arc can be found by means of the right-angled spherical triangle *cqS*, right-angled at *q*. *Sq* is the sun's declination, and the angle *Scq* is equal to the *co.latitude* of the place; for the angle *Pch* is the latitude, and the angle *Scq* is its complement.

The side *cq*, a portion of the equator, measures the angle *cPq*, the time of the sun's rising or setting before or after *six o'clock*, apparent time. Thus we perceive that this little triangle, *cSq*, is a very important one.

When the sun is exactly *east* or *west*, it can be determined by the triangle *ZPS′*; the side *PZ* is known, being the co.latitude; the angle *PZS′* is a right angle, and the side *PS′* is the sun's polar distance. Here, then, are the hypotenuse and side of a right-angled spherical triangle given, from which the other parts can be computed. The angle *ZPS′* is the time from noon, and the side *ZS′* is the sun's zenith distance at that time.

The following problems are given, to illustrate the important applications that can be made of the right-angled triangle *cqS*.

PRACTICAL PROBLEMS.

1. At what time will the sun rise and set in Lat. 48° N., when its declination is 21° N.?

In this problem, we must make $qS=21°$, $Ph=48°=$the angle Pch. Then the angle $Scq=42°$. It is required to find the arc cq, and convert it into time at the rate of four minutes to a degree. This will give the apparent time after six o'clock that the sun sets, and the apparent time before six o'clock that the sun rises, (no allowance being made for refraction).

Making cq the middle part, we have

$$R \sin.cq = \tan.21° \tan.48°$$

$$\tan.21° = 9.584177$$

$$\tan.48° = 10.045563$$

$\sin.cq = 25° 14' 5'' = 25.2346°$ 9.629740, rejecting 10.

4

$1^h\ 40^m\ 56^s$

Adding to 6^h

Sun sets P. M., $7^h\ 40^m\ 56^s$, apparent time,

From 6^h

Taking $1^h\ 40^m\ 56^s$

Sun rises A. M., $4^h\ 19^m\ 4^s$, apparent time.

From this we derive the following rule for finding the apparent time of sunrise and sunset, assuming that the declination undergoes no change in the interval between these instants, which we may do without much error.

RULE.

To the logarithmic tangent of the sun's declination, add the logarithmic tangent of the latitude of the observer; and, after rejecting ten from the result, find from the tables the arc of which this is the logarithmic sine, and convert it into time at the rate of 4 *minutes to a degree.*

This time, added to 6 *o'clock, will give the time of sunset, and, subtracted from* 6 *o'clock, will give the time of sunrise,* *when the latitude and declination are both north or both south; but when one is north, and the other south, the addition gives the time of sunrise, and the subtraction the time of sunset.*

2. At what time will the sun set when its declination is 23° 12′ N., and the latitude of the place is 42° 40′ N.?

Ans. $7^h\ 33^m\ 8^s$, apparent time.

3. What will be the time of sunset for places whose latitude is 42° 40′ N., when the sun's declination is 15° 21′ south? *Ans.* $5^h\ 1^m\ 20^s$, apparent time.

4. What will be the time of sunrise and sunset for places whose latitude is 52° 30′ N., when the sun's declination is 18° 42′ south?

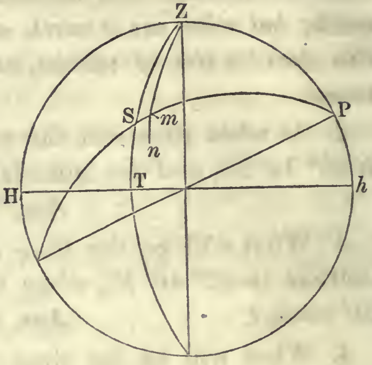
Ans. $\left\{\begin{array}{l}\text{Rises } 7^h\ 44^m\ 42^s, \\ \text{Sets } 4^h\ 15^m\ 18^s,\end{array}\right\}$ apparent time.

5. What will be the time of sunset and of sunrise at St. Petersburgh, in lat. 59° 56′, north, when the sun's declination is 23° 24′, north? What will be its amplitude at these instants? Also, at what hours will it be due east and west, and what will be its altitude at such times?

Ans.
- Sun sets at $9^h\ 13^m\ 30^s$ P.M. } apparent
- Sun rises at $2^h\ 46^m\ 30^s$ A.M. } time.
- Sun rises N. of east } 52° 25′ 30″
- Sun sets N. of west }
- Sun is east at $6^h\ 58^m$ A.M.
- Sun is west at $5^h\ 2^m$ P.M.
- Alt. when east and west is 27° 19′.

ON THE APPLICATION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES.

One of the most important problems in navigation and astronomy, is the determination of the formula for time. This problem will be understood by the triangle PZS. When the sun is on the meridian, it is then apparent noon. When not on the meridian, we can determine the interval from noon, by means of the triangle PZS; for we can know all its sides; and the angle at P, changed into time at the rate of 15° to one hour, will give the time from apparent noon, when any particular altitude, as TS, may have been observed. PS is known, by the sun's declination at about the time; and PZ is known, if the observer knows his latitude.



Having these three sides, we can always find the sought angle at the pole, by the equations already given in formulæ (T, or U, Prop. 7, Sec. III); but these formulæ require the use of the *co.latitude* and the *co.altitude*, and the practical navigator is very averse to taking the trouble of finding the complements of arcs, when he is quite certain that formulæ can be made, comprising but the arcs themselves.

The practical man, also, *very properly* demands the most concise practical results. No matter how much labor is spent in theorizing, provided we arrive at practical brevity; and for the especial accommodation of seamen, the following formula for finding time has been deduced.

From the symmetrical formulæ (s'), Prop. 7, Sec. III, we have,

$$\cos.P = \frac{\cos.ZS - \cos.PZ \cos.PS}{\sin.PZ \sin.PS}$$

Now, in place of $\cos.ZS$, we take $\sin.ST$, which is, in fact, the same thing; and in place of cos.PZ, we take sin.lat., which is also the same.

In short, let $A =$ the altitude of the sun, $L =$ the latitude of the observer, and $D =$ the sun's polar distance.

Then, $\cos.P = \frac{\sin.A - \sin.L \cos.D}{\cos.L \sin.D}$

But, $2\sin.^2 \frac{1}{2}P = 1 - \cos.P$. (See Eq. 32, Prop. 2, Sec. I, Plane Trig.)

Therefore,

$$2\sin.^2 \tfrac{1}{2}P = 1 - \frac{\sin.A - \sin.L \cos.D}{\cos.L \sin.D}$$

$$= \frac{(\cos.L \sin.D + \sin.L \cos.D) - \sin.A}{\cos.L \sin.D}$$

$$= \frac{\sin.(L + D) - \sin.A}{\cos.L \sin.D}.$$

Considering $(L + D)$ as a single arc, and (applying Equation 16, Sec. I, Plane Trig.), we have, after dividing by 2,

$$\sin.\tfrac{1}{2}P = \frac{\cos.\left(\frac{L + D + A}{2}\right) \sin.\left(\frac{L + D - A}{2}\right).}{\cos.L \sin.D}$$

But, $\frac{L + D - A}{2} = \frac{L + D + A}{2} - A,$

and if we assume $S = \frac{L + D + A}{2},$

we shall have, $\sin.^2 \frac{1}{2}P = \frac{\cos.S \sin.(S - A)}{\cos.L \sin.D}$

Or, $\sin.\frac{1}{2}P = \sqrt{\frac{\cos.S \sin.(S - A)}{\cos.L \sin.D}}.$

This is the final result, when the radius is unity; and when the radius is greater by R, then the sin.$\frac{1}{2}P$ will be greater by R; and, therefore, the value of this sine, corresponding to our tables, is,

$$\sin.\tfrac{1}{2}P = \sqrt{\left(\frac{R}{\cos.L}\right)\left(\frac{R}{\sin.D}\right) \cos.S \sin.(S - A)}.$$

PRACTICAL PROBLEMS.

1. In lat. 39° 6′ 20″ North, when the sun's declination was 12° 3′ 10″ North, the true altitude* of the sun's center was observed to be 30° 10′ 40″, *rising*. What was the apparent time?

Alt.	30° 10′ 30″		
Lat.	39° 6′ 20″	cos.com.	.110146
P.D.	77° 56′ 50″	sin.com.	.009680
2)	147° 13′ 40″		
$S =$	73° 36′ 50″	cos.	9.450416
$(S - A) =$	43° 26′ 20″	sin.	9.837299
		2)	19.407541
	30° 22′ 5″	sin.	9.703770
	2		
$P =$	60° 44′ 10″		

This angle, converted into time at the rate of 15° to one hour, or 4 minutes to 1°, gives $4^h\ 2^m\ 56^s$ from apparent noon; and as the sun was rising, it was before noon or

$$7^h\ 57^m\ 4^s \text{ A.M.}$$

If to this the equation of time were applied, we should have the mean time; and if such time were compared with that of a clock or watch, we could determine its error. A good observer, with a good instrument, can, in this manner, determine the local time within 4 or 5 seconds.

2. In lat. 40° 21′ North, the true altitude of the sun, in the forenoon, was found to be 36° 12′, when the declina-

* The instrument used, the manner of taking the altitude, its correction for refraction, semi-diameter, and other practical or circumstantial details, do not belong to a work of this kind, but to a work on Practical Astronomy or Navigation.

tion of the sun was 3° 20′ South. What was the apparent time? *Ans.* $9^h\ 43^m\ 44^s$ A. M.

3. In latitude 21° 2′ South, when the sun's declination was 18° 32′ North, the true altitude, in the afternoon, was found to be 40° 8′. What was the apparent time of day? *Ans.* $2^h\ 2^m$ P. M.

SPHERICAL TRIGONOMETRY APPLIED TO GEOGRAPHY.

If we wish to find the shortest distance between two places over the surface of the earth, when the distance is considerable, we must employ Spherical Trigonometry.

Suppose the least distance between Rome and New Orleans is required; we would first find the distance in degrees and parts of a degree, and then multiply that distance by the number of miles in one degree.

In the solution of this problem, it is supposed that we have the latitude and longitude of both places. Then the distances, in degrees, from the north pole of the earth to Rome and to New Orleans are the two sides of a spherical triangle, the difference of longitude of the two places is the angle at the pole included between these sides, and the problem is, to determine the third side of a spherical triangle, when we have two sides and the included angle given.

Let P be the north pole, R the position of Rome, and N that of New Orleans.

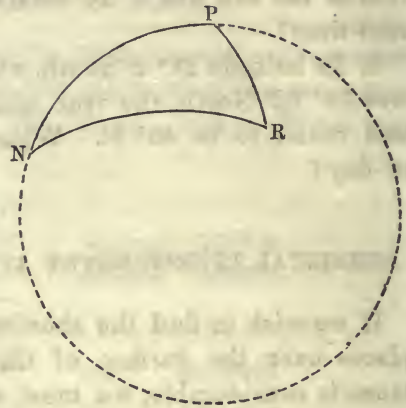
	Lat.	Long.
New Orleans,	29° 57′ 30″ N.	90° W.
Rome,	41° 53′ 54″ N.	12° 28′ 40″ E.

Whence, $PR = 48°\ 6'\ 6''$,

$$PN = 60°\ 2'\ 30''.$$

$$\text{Angle } NPR = 102°\ 28'\ 40''.$$

We now employ Napier's 1st and 2d Analogies, and find the distance, in degrees, to be 101° 31′ 30″. This reduced to miles, at the rate of 69.16 miles to the degree, will make the distance 7021.469 miles.



The angle at N is 47° 49′, and at R, 59° 35′ 40″.

The third side of a spherical triangle can be found by a single formula, as we shall see by inspecting formulæ (S') Prop. 7, Sec. III.

Let C be the included angle, and c the unknown side opposite; then,

$$\cos. C = \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b}.$$

Adding 1 to each member, and reducing, observing at the same time that $1 + \cos. C = 2\cos.^2 \tfrac{1}{2}C$, we have,

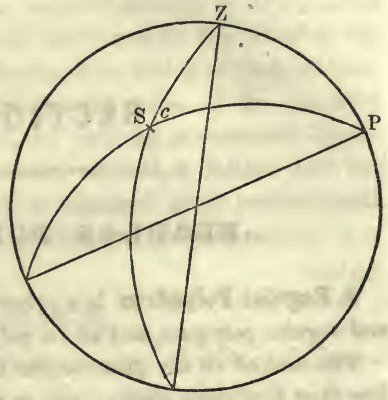
$$2\cos.^2 \tfrac{1}{2}C = \frac{\sin. a \sin. b - \cos. a \cos. b + \cos. c}{\sin. a \sin. b}.$$

Whence, $2\cos.^2 \tfrac{1}{2}C \sin. a \sin. b = \cos. c - \cos. (a + b)$;

or, $\cos. c = \cos. (a + b) + 2\cos.^2 \tfrac{1}{2}C \sin. a \sin. b$.

The second member of this equation is the algebraic sum of two decimal fractions, and expresses the value of the natural cosine of the side sought.

This case of Spherical Trigonometry, namely, that in which two sides and the included angle are given, to find the third side, is very extensively used in practical astronomy, in finding the angular distance of the moon from the sun, stars, and planets. For this purpose, the right ascension and declination of each body must be found for the same moment of absolute time. Their difference in right ascension gives the included angle, P, at the celestial pole. The declination subtracted from 90°, if it be north, and added to 90°, if it be south, will give the sides, PZ and PS.



In the following examples, we give the right ascension and declination of the bodies, and from these the student is required to compute the distance between them.

The right ascensions are given in time. Their difference must be changed to degrees for the included angle.

June 24, 1860.

MEAN TIME GREENWICH.

	MOON'S		JUPITER'S		
	R. A.	*Dec.*	*R. A.*	*Dec.*	*Distance.*
	h. m. s.	° ′ ″	h. m. s.	° ′ ″	° ′ ″
At noon,	10 51 36.5	3 33 24 N.	8 4 27.6	20 51 36.8 N.	44 8 12
" 3 h.,	10 58 1	2 47 43	8 4 34.2	20 51 17.8	45 53 47
" 6 h.,	11 4 24.6	1 59 56.2	8 4 40.8	20 50 58.7	47 39 18
" 9 h.,	11 10 47.6	1 12 6	8 4 47.2	20 50 39.6	49 24 43

October 6, 1860.

	☽ *R. A.*	*Dec.*	⊙ *R. A.*	*Dec.*	*Distance.*
	h. m. s.	° ′ ″	h. m. s.	° ′ ″	° ′ ″
At noon,	5 41 21.8	26 8 0 N.	12 49 27.4	5 18 31 S.	107 37 2
" 3 h.,	5 48 30.1	26 3 20	12 49 54.8	5 20 13.7	106 8 19
" 6 h.,	5 55 40	25 57 19	12 50 22.2	5 21 56.4	104 39 19
" 9 h.,	6 2 50.5	25 49 58	12 50 49.6	5 23 38.1	103 10 0
" 12 h.,	6 10 1.2	25 41 15.8	12 51 11.9	5 25 20.8	101 40 23

SECTION VI.

REGULAR POLYEDRONS.

A **Regular Polyedron** is a polyedron having all its faces equal and regular polygons, and all its polyedral angles equal.

The sum of all the plane angles bounding any polyedral angle is less than four right angles; and as the angle of the equilateral triangle is $\frac{2}{3}$ of a right angle, we have $\frac{2}{3}\times 3<4$, $\frac{2}{3}\times 4<4$, and $\frac{2}{3}\times 5<4$; but $\frac{2}{3}\times 6=4$, $\frac{2}{3}\times 7>4$, and so on. Hence, it follows that three, and only three, polyedral angles may be formed, having the equilateral triangle for faces; namely, a triedral angle and polyedral angles of four and of five faces.

There are, therefore, three distinct regular polyedrons bounded by the equilateral triangle.

1. The **Tetraedron,** having four faces and four solid angles.
2. The **Octaedron,** having eight faces and six solid angles.
3. The **Icosaedron,** having twenty faces and twenty solid angles.

With right plane angles we can form only a triedral angle; hence, with equal squares we may bound a solid having six faces and eight equal triedral angles. This solid is called the **Hexaedron.**

The angle of the regular pentagon being $\frac{6}{5}$ of a right angle, we have $\frac{6}{5}\times 3<4$; but $\frac{6}{5}\times 4>4$; hence, with plane angles equal to those of the regular pentagon, we can form only a triedral angle. The solid bounded by twelve regular pentagons, and having twenty solid angles, is called the **Dodecaedron.**

There are, then, but five regular polyedrons, viz.: The *tetraedron*, the *octaedron*, and the *icosaedron*, each of which has the equilateral triangle for faces; the *hexaedron*, whose faces are equal squares, and the *dodecaedron*, whose faces are equal regular pentagons.

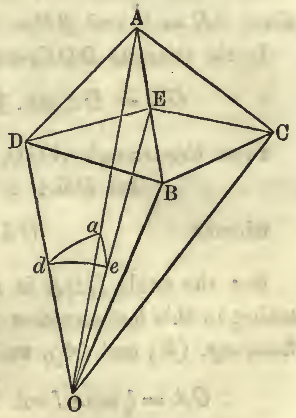
It is obvious that a sphere may be circumscribed about, or inscribed within, any of these regular solids, and conversely: and that these spheres will have a common center, which may also be taken as the *center* of the polyedron.

Any regular polyedron may be regarded as made up of a number of regular pyramids, whose bases are severally the faces of the polyedron, and whose common vertex is its center. Each of these pyramids will have, for its altitude, the radius of the inscribed sphere; and since the volume of the pyramid is measured by one third of the product of its base and altitude, it follows that the volume of any regular polyedron is measured by its surface multiplied by one third of the radius of the inscribed sphere.

PROBLEM.

Given, the name of a regular polyedron, and the side of the bounding polygon, to find the inclination of its faces; the radii of the inscribed and circumscribed spheres; the area of its surface; and its volume.

Let AB be the intersection of two adjacent faces of the polyedron, and C and D the centers of these faces, O being the center of the polyedron. Draw the radii, OC and OD, of the inscribed, and the radii OA and OB, of the circumscribed sphere; also from C and D let fall the perpendiculars CE and DE, on the edge AB, and draw OE; then will the angle DEC measure the inclination of the faces of the polyedron, and the angle DEO is one half of this inclination.



Let I denote the inclination of the faces, m the number of faces which meet to form a polyedral angle, n the number of sides in each face, and suppose the edge of the polyedron to be unity.

The surface of the sphere of which O is the center, and radius unity, will form, by its intersections with the planes, AOE, AOD, DOE, the right-angled spherical triangle dae, right-angled at e. In the right-angled triangle DEO, the angle DOE is equal to

$$90° - DEO = 90° - \tfrac{1}{2}I,$$

and is measured by the arc *de*. The angle *dae*, of the spherical triangle, is equal to $\frac{360°}{2m}$, and the angle $ade = \frac{360°}{2n}$.

Now, by Napier's Rules we have

$$\cos.dae = \sin.ade \cos.de.$$

or, $$\cos.de = \frac{\cos.dae}{\sin.ade}; \qquad (1)$$

and, $$\cos.ad = \cot.dae \cot.ade \qquad (2)$$

Substituting in eq. (1), for the angles *dae* and *ade*, their values, we find

$$\sin.\tfrac{1}{2}I = \frac{\cos.\frac{360°}{2m}}{\sin.\frac{360°}{2n}} \qquad (3)$$

Equation (3) gives the value of the sine of one half of the inclination of the planes; and by means of this equation we may readily find the radii of the inscribed and circumscribed spheres.

In the triangle *BED*, we have

$$DE = BE \cot.BDE = \tfrac{1}{2}\cot.\frac{360°}{2n},$$

since $AB = 1$, and $BE = \frac{1}{2}AB$.

In the triangle *DOE*, we have

$$OD = DE \tan.\tfrac{1}{2}I = \tfrac{1}{2}\cot.\frac{360°}{2n}\tan.\tfrac{1}{2}I \qquad (4)$$

From the triangle *AOD*, we find

$$\cos.DOA : 1 :: OD : OA$$

whence $$OA = \frac{OD}{\cos.DOA}$$

But the angle *DOA* is measured by the arc *ad*; hence, substituting in this last equation the values of cos.*DOA* and *OD*, taken from eqs. (2) and (4), we have

$$OA = \tfrac{1}{2}\tan.\tfrac{1}{2}I \cot.\frac{360°}{2n} \times \frac{1}{\cot.\frac{360°}{2m}} \times \frac{1}{\cot.\frac{360°}{2n}}$$

$$= \tfrac{1}{2}\tan.\tfrac{1}{2}I \tan.\frac{360°}{2m}, \qquad (5)$$

by writing tan. for $\frac{1}{\cot.}$, and reducing.

Equation (4) gives the value of OD, the radius of the inscribed sphere, and equation (5) gives that of OA, the radius of the circumscribed sphere. The area of one of the faces of the polyedron is equal to one half of the apothegm multiplied by the perimeter. The apothegm, as found above, is equal to $\frac{1}{2}\cot.\frac{360°}{2n}$; hence, we have $\frac{1}{2}n \times \frac{1}{2}\cot.\frac{360°}{2n}$, for the area of one of the faces; and multiplying this by the number of faces of the polyedron, we will have the expression for its entire area. The expression for the surface multiplied by one third of the radius of the inscribed sphere, gives the measure of the volume of the polyedron.

In what precedes, we have supposed the edge of the polyedron to be unity. Having found the radii of the inscribed and circumscribed spheres, the surfaces, and the volumes of such polyedrons, to determine the radii, surfaces, and volumes of regular polyedrons having any edge whatever, we have merely to remember that the homologous dimensions of similar bodies are proportional; their surfaces are as the squares of these dimensions; and their volumes as the cubes of the same.

Formula (3) gives, for the inclination of the adjacent faces of

The Tetraedron,	70° 31′ 42″
" Hexaedron,	90° 00′ 00″
" Octaedron,	109° 28′ 18″
" Dodecaedron,	116° 33′ 54″
" Icosaedron,	138° 11′ 23″

The subjoined table gives the surfaces and volumes of the regular polyedrons, when the edge is unity.

	Surfaces.	Volumes.
Tetraedron,	1.7320508	0.1178513
Hexaedron,	6.0000000	1.0000000
Octaedron,	3.4641016	0.4714045
Dodecaedron,	20.6457288	7.6631189
Icosaedron,	8.6602540	2.1816950

LOGARITHMIC TABLES;

ALSO A TABLE OF

NATURAL AND LOGARITHMIC

SINES, COSINES, AND TANGENTS,

TO EVERY MINUTE OF THE QUADRANT.

LOGARITHMS OF NUMBERS

FROM

1 TO 10000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0 000000	26	1 414973	51	1 707570	76	1 880814
2	0 301030	27	1 431364	52	1 716003	77	1 886491
3	0 477121	28	1 447158	53	1 724276	78	1 892095
4	0 602060	29	1 462398	54	1 732394	79	1 897627
5	0 698970	30	1 477121	55	1 740363	80	1 903090
6	0 778151	31	1 491362	56	1 748188	81	1 908485
7	0 845098	32	1 505150	57	1 755875	82	1 913814
8	0 903090	33	1 518514	58	1 763428	83	1 919078
9	0 954243	34	1 531479	59	1 770852	84	1 924279
10	1 000000	35	1 544068	60	1 778151	85	1 929419
11	1 041393	36	1 556303	61	1 785330	86	1 934498
12	1 079181	37	1 568202	62	1 792392	87	1 939519
13	1 113943	38	1 579784	63	1 799341	88	1 944483
14	1 146128	39	1 591065	64	1 806180	89	1 949390
15	1 176091	40	1 602060	65	1 812913	90	1 954243
16	1 204120	41	1 612784	66	1 819544	91	1 959041
17	1 230449	42	1 623249	67	1 826075	92	1 963788
18	1 255273	43	1 633468	68	1 832509	93	1 968483
19	1 278754	44	1 643453	69	1 838849	94	1 973128
20	1 301030	45	1 653213	70	1 845098	95	1 977724
21	1 322219	46	1 662578	71	1 851258	96	1 982271
22	1 342423	47	1 672098	72	1 857333	97	1 986772
23	1 361728	48	1 681241	73	1 863323	98	1 991226
24	1 380211	49	1 690196	74	1 869232	99	1 995635
25	1 397940	50	1 698970	75	1 875061	100	2 000000

NOTE. In the following table, in the last nine columns of each page, where the first or leading figures change from 9's to 0's, points or dots are now introduced instead of the 0's through the rest of the line, to catch the eye, and to indicate that from thence the corresponding natural number in the first column stands in the *next lower line*, and its annexed first two figures of the Logarithms in the second column.

N.	0	1	2	3	4	5	6	7	8	9
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891
101	4321	4750	5181	5609	6038	6466	6894	7321	7748	8174
102	8600	9026	9451	9876	.300	.724	1147	1570	1993	2415
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616
104	7033	7451	7868	8284	8700	9116	9532	9947	.361	.775
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978
107	9384	9789	.195	.600	1004	1408	1812	2216	2619	3021
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028
109	7426	7825	8223	8620	9017	9414	9811	.207	.602	.998
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830
112	9218	9606	9993	.380	.766	1153	1538	1924	2309	2694
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	.320
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815
117	8186	8557	8928	9298	9668	..38	.407	.776	1145	1514
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819
120	9181	9543	9904	.266	.626	.987	1347	1707	2067	2426
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552
123	9905	.258	.611	.963	1315	1667	2018	2370	2721	3071
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	1026
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	.253
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609
130	3943	4277	4611	4944	5278	5611	5943	6276	6608	6940
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	0245
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	..12
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564
138	9879	.194	.508	.822	1136	1450	1763	2076	2389	2702
139	143015	3327	3630	3951	4263	4574	4885	5196	5507	5818
140	6128	6438	6748	7058	7367	7676	7985	8294	8603	8911
141	9219	9527	9835	.142	.449	.756	1063	1370	1676	1982
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061
144	8362	8664	8965	9266	9567	9868	.168	.469	.769	1068
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802

N.	0	1	2	3	4	5	6	7	8	9
150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689
151	8977	9264	9552	9839	.126	.413	.699	.985	1272	1558
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	..51
					281					
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623
157	5899	6176	6453	6729	7005	7281	7556	7832	8107	8382
158	8657	8932	9206	9481	9755	..29	.303	.577	.850	1124
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848
					273					
160	4120	4391	4663	4934	5204	5475	5746	6016	6286	6556
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247
162	9515	9783	..51	.319	.586	.853	1121	1388	1654	1921
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221
					264					
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	.193
					257					
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795
173	8046	8297	8548	8799	9049	9299	9550	9800	..50	.300
174	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790
					249					
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	.176
178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031
					242					
180	5273	5514	5755	5996	6237	6477	6718	6958	7198	7439
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937
					235					
185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279
186	9513	9746	9980	.213	.446	.679	.912	1144	1377	1609
187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525
					229					
190	8754	8982	9211	9439	9667	9895	.123	.351	.578	.806
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812
					224					
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635
199	8853	9071	9289	9507	9725	9943	.161	.378	.595	.813

N.	0	1	2	3	4	5	6	7	8	9
200	301030	1247	1464	1681	1898	2114	2331	2547	2764	2980
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417
204	9630	9843	..56	.268	.481	.693	.906	1118	1330	1542
					212					
205	311754	1966	2177	2389	2600	2812	3023	3234	3445	3656
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012
					207					
210	2219	2426	2633	2839	3046	3252	3458	3665	3871	4077
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176
213	8380	8583	8787	8991	9194	9398	9601	9805	...8	.211
214	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236
					202					
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257
218	8456	8656	8855	9054	9253	9451	9650	9849	..47	.246
219	340444	0642	0841	1039	1237	1435	1632	1830	2028	2225
					198					
220	2423	2620	2817	3014	3212	3409	3606	3802	3999	4196
221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157
222	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110
223	8305	8500	8694	8889	9083	9278	9472	9666	9860	..54
224	350248	0442	0636	0829	1023	1216	1410	1603	1796	1989
					193					
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646
229	9835	..25	.215	.404	.593	.783	.972	1161	1350	1539
					190					
230	361728	1917	2105	2294	2482	2671	2859	3048	3236	3424
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030
234	9216	9401	9587	9772	9958	.143	.328	.513	.698	.883
					185					
235	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	..30
					182					
240	380211	0392	0573	0754	0934	1115	1296	1476	1656	1837
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212
244	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989
					178					
245	9166	9343	9520	9698	9875	..51	.228	.405	.582	.759
246	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766

N.	0	1	2	3	4	5	6	7	8	9
250	397940	8114	8287	8461	8634	8808	8981	9154	9328	9501
251	9674	9847	..20	.192	.365	.538	.711	.883	1056	1228
252	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370
					171					
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764
257	9933	.102	.271	.440	.609	.777	.946	1114	1283	1451
258	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132
259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806
260	4973	5140	5307	5474	5641	5808	5974	6141	6308	6474
261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135
262	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791
263	9956	.121	.286	.451	.616	.781	.945	1110	1275	1439
264	421604	1788	1933	2097	2261	2426	2590	2754	2918	3082
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718
266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973
268	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591
269	9752	9914	..75	.236	.398	.559	.720	.881	1042	1203
270	431364	1525	1685	1846	2007	2167	2328	2488	2649	2809
271	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409
272	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004
273	6163	6322	6481	6640	6800	6957	7116	7275	7433	7592
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175
					158					
275	9333	9491	9648	9806	9964	.122	.279	.437	.594	.752
276	440909	1066	1224	1381	1538	1695	1852	2009	2166	2323
277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889
278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449
279	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003
280	7158	7313	7468	7623	7778	7933	8088	8242	8397	8552
281	8706	8861	9015	9170	9324	9478	9633	9787	9941	..95
282	450249	0403	0557	0711	0865	1018	1172	1326	1479	1633
283	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242
288	9392	9543	9694	9845	9995	.146	.296	.447	.597	.748
289	460898	1048	1198	1348	1499	1649	1799	1948	2098	2248
290	2398	2548	2697	2847	2997	3146	3296	3445	3594	3744
291	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234
292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719
293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200
294	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675
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296	471292	1438	1585	1732	1878	2025	2171	2318	2464	2610
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526
299	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976

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302	480007	0151	0294	0438	0582	0725	0869	1012	1156	1299
303	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731
304	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157
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305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579
306	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997
307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410
308	8551	8692	8833	8974	9114	9255	9396	9537	9667	9818
309	9959	..99	.239	.380	.520	.661	.801	.941	1081	1222
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312	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406
313	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791
314	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173
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318	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655
319	3791	3927	4063	4199	4335	4471	4607	4743	1878	5014
320	5150	5286	5421	5557	5693	5828	5964	6099	6234	6370
321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721
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326	3218	3351	3484	3617	3750	3883	4016	4149	4282	4414
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328	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064
329	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382
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334	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915
335	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210
336	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501
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339	530200	0328	0456	0584	0712	0840	0968	1096	1223	1351
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341	2754	2882	3009	3136	3264	3391	3518	3645	3772	3899
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344	6558	6685	6811	6937	7060	7189	7315	7441	7567	7693
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352	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652
353	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881
354	9003	9126	9249	9371	9494	9616	9739	9861	9984	.196
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355	550228	0351	0473	0595	0717	0840	0962	1084	1206	1328
356	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547
357	2668	2790	2911	3033	3155	3276	3393	3519	3640	3762
358	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973
359	5094	5215	5346	5457	5578	5699	5820	5940	6061	6182
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361	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589
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365	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362
366	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548
367	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730
368	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909
369	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084
370	8202	8319	8436	8554	8671	8788	8905	9023	9140	9257
371	9374	9491	9608	9725	9882	9959	..76	.193	.309	.426
372	570543	0660	0776	0893	1010	1126	1243	1359	1476	1592
373	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755
374	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915
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375	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072
376	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226
377	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377
378	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525
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380	9784	9898	..12	.126	.241	.355	.469	.583	.697	.811
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383	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218
384	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348
385	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475
386	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599
387	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720
388	8832	8944	9056	9167	9279	9391	9503	9615	9726	9834
389	9950	..61	.173	.284	.396	.507	.619	.730	.842	.953
390	591065	1176	1287	1399	1510	1621	1732	1843	1955	2066
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392	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282
393	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386
394	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487
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395	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586
396	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681
397	8791	8900	9009	9119	9228	9337	9446	.556	9666	9774
398	9883	9992	.101	.210	.319	.428	.537	.646	755	.864
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402	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197
403	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274
404	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348
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405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419
406	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488
407	9594	9701	9808	9914	..21	.128	.234	.341	.447	.554
408	610660	0767	0873	0979	1086	1192	1298	1405	1511	1617
409	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678
410	2784	2890	2996	3102	3207	3313	3419	3525	3630	3736
411	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792
412	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845
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414	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943
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416	9293	9198	9302	9406	9511	9615	9719	9824	9928	..32
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419	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146
420	3249	3353	3456	3559	3663	3766	3869	3973	4076	4179
421	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210
422	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238
423	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263
424	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287
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425	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308
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427	630428	0530	0631	0733	0835	0936	1038	1139	1241	1342
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429	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367
430	3468	3569	3670	3771	3872	3973	4074	4175	4276	4376
431	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383
432	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388
433	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390
434	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389
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436	9486	9586	9686	9785	9885	9984	..84	.183	.283	.382
437	640481	0581	0680	0779	0879	0978	1077	1177	1276	1375
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439	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354
440	3453	3551	3650	3749	3847	3946	4044	4143	4242	4340
441	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324
442	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306
443	6404	6502	6600	6698	6796	6894	6992	7039	7187	7285
444	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262
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446	9335	9432	9530	9627	9724	9821	9919	..16	.113	.210
447	650308	0405	0502	0599	0696	0793	0890	0987	1084	1181
448	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150
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452	5138	5235	5331	5427	5526	5619	5715	5810	5906	6002
453	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960
454	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916
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455	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870
456	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821
457	9916	..11	.106	.201	.296	.391	.486	.581	.676	.771
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459	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663
460	2758	2852	2947	3041	3135	3230	3324	3418	3512	3607
461	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548
462	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487
463	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424
464	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360
465	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293
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468	670241	0339	0431	0524	0617	0710	0802	0895	0988	1080
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474	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602
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476	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427
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479	680336	0426	0517	0607	0698	0789	0879	0970	1060	1151
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484	4854	4935	5025	5114	5204	5294	5383	5473	5563	5652
485	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547
486	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440
487	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331
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493	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639
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496	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269
497	6356	5444	6531	6618	6706	6793	6880	6968	7055	7142
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502	700704	0790	0877	0963	1050	1136	1222	1309	1395	1482
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504	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205
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505	3291	3377	3463	3549	3635	3721	3807	3895	3979	4065
506	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922
507	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778
508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632
509	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485
510	7570	7655	7740	7826	7910	7996	8081	8166	8251	8336
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512	9270	9355	9440	9524	9609	9694	9779	9863	9948	..33
513	710117	0202	0287	0371	0456	0540	0625	0710	0794	0879
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515	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566
516	2650	2734	2818	2902	2986	3070	3154	3238	3326	3407
517	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246
518	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084
519	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920
520	6003	6087	6170	6254	6337	6421	6504	6588	6671	6754
521	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587
522	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419
523	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248
524	9331	9414	9497	9580	9663	9745	9828	9911	9994	..77
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529	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194
530	4276	4358	4440	4522	4604	4685	4767	4849	4931	5013
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532	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646
533	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460
534	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273
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536	9165	9246	9327	9403	9489	9570	9651	9732	9813	9893
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540	2394	2474	2555	2635	2715	2796	2876	2956	3037	3117
541	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919
542	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720
543	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519
544	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317
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545	6397	6476	6556	6636	6715	6795	6874	6954	7034	7113
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547	7987	8037	8146	8225	8305	8384	8463	8543	8622	8701
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552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2646
553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431
554	3510	3558	3667	3745	3823	3902	3980	4058	4136	4215
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555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556
558	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334
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563	750508	0586	0663	0740	0817	0894	0971	1048	1125	1202
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567	3582	3660	3736	3813	3889	3966	4042	4119	4195	4272
568	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036
569	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799
570	5875	5951	6027	6103	6180	6256	6332	6408	6484	6560
571	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320
572	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079
573	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836
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584	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082
585	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823
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588	9377	9451	9525	9599	9673	9746	9820	9894	9968	..42
589	770115	0189	0263	0336	0410	0484	0557	0631	0705	0778
590	0852	0926	0999	1073	1146	1220	1293	1367	1440	1514
591	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248
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594	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444
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596	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902
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598	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354
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605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401
606	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117
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608	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546
609	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259
610	5330	5401	5472	5543	5615	5686	5757	5828	5899	5970
611	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680
612	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390
613	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098
614	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804
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616	9581	9651	9722	9792	9863	9933	...4	..74	.144	.215
617	790285	0356	0426	0496	0567	0637	0707	0778	0848	0918
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619	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322
620	2392	2462	2532	2602	2672	2742	2812	2882	2952	3022
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622	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418
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624	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811
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625	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505
626	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198
627	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890
628	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582
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633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021
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635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389
636	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071
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638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433
639	5501	5669	5637	5705	5773	5841	5908	5976	6044	6112
640	6180	6248	6316	6384	6451	6519	6587	6655	6723	6790
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648	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178
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652	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847
653	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511
654	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175
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656	6904	6970	7036	7102	7169	7233	7301	7367	7433	7499
657	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160
658	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820
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663	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103
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665	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409
666	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061
667	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711
668	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361
669	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010
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674	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239
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679	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445
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682	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357
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685	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261
686	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894
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689	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786
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694	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922
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696	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170
697	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793
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702	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894
703	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511
704	7573	7634	7676	7758	7819	7831	7943	8004	8066	8128
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706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358
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713	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637
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715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852
716	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459
717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064
718	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668
719	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272
720	7332	7393	7453	7513	7574	7634	7694	7755	7815	7875
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727	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072
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730	3323	3382	3442	3501	3561	3620	3680	3739	3799	3858
731	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452
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734	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228
735	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819
736	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409
737	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998
738	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586
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742	870404	0462	0521	0579	0638	0696	0755	0813	0872	0930
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744	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098
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751	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160
752	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737
753	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314
754	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889
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755	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464
756	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039
757	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612
758	9669	9726	9784	9841	9898	9956	..13	..70	.127	.185
759	880242	0299	0356	0413	0471	0528	0580	0642	0699	0756
760	0814	0871	0928	0985	1042	1099	1156	1213	1271	1328
761	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898
762	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468
763	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037
764	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605
765	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172
766	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739
767	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305
768	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870
769	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434
770	6491	6547	6604	6660	6716	6773	6829	6885	6942	6998
771	7054	7111	7167	7233	7280	7336	7392	7449	7505	7561
772	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123
773	8179	8236	8292	8348	8404	8460	8516	8573	8629	8655
774	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246
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775	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806
776	9862	9918	0974	..30	..86	.141	.197	.253	.309	.365
777	890421	0477	0533	0589	0645	0700	0756	0812	0868	0924
778	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482
779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039
780	2095	2150	2206	2262	2317	2373	2429	2484	2540	2595
781	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151
782	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706
783	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261
784	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814
785	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367
786	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920
787	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471
788	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022
789	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572
790	7627	7683	7737	7792	7847	7902	7957	8012	8067	8122
791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670
792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218
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794	9821	9875	9930	9985	..39	..94	.149	.203	.258	.312
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797	1458	1513	1567	1622	1676	1736	1785	1840	1894	1948
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492
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802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742
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805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431
810	8485	8539	8592	8646	8699	8753	8807	8860	8914	8967
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812	9556	9610	9663	9716	9770	9823	9877	9930	9984	..37
813	910091	0144	0197	0251	0304	0358	0411	0464	0518	0571
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815	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637
816	1690	1743	1797	1850	1903	1956	2009	2063	2115	2169
817	2222	2275	2323	2381	2435	2488	2541	2594	2645	2700
818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231
819	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761
820	3814	3867	3920	3973	4026	4079	4132	4184	4237	4290
821	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819
822	4872	4925	4977	5030	5083	5136	5189	5241	5594	5347
823	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875
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825	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927
826	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453
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829	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026
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831	9601	9653	9706	9758	9810	9862	9914	9967	..19	..71
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833	0645	0697	0749	0801	0853	0905	0958	1010	1062	1114
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835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154
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837	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192
838	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710
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840	4279	4331	4383	4434	4486	4538	4589	4641	4693	4744
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776
843	5828	5874	5931	5982	6034	6085	6137	6188	6240	6291
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845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319
846	7370	7422	7473	7524	7576	7627	7678	7730	7783	7832
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848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857
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854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915
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855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437
858	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943
859	3993	4044	4094	4145	4195	4246	4269	4347	4397	4448
860	4498	4549	4599	4650	4700	4751	4801	4852	4902	4953
861	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457
862	5507	5558	5603	5658	5709	5759	5809	5860	5910	5960
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463
864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966
865	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468
866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969
867	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470
868	8520	8570	8620	8670	8720	8770	8820	8870	8919	8970
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871	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467
872	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964
873	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950
877	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433
880	4483	4532	4581	4631	4680	4729	4779	4828	4877	4927
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8365
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341
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891	9878	9926	9975	..24	..73	.121	.170	.219	.267	.316
892	950365	0414	0462	0511	0560	0608	0657	0705	0754	0803
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775
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896	2308	2356	2405	2453	2502	2550	2599	2647	5696	2744
897	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711
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902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601
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905	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080
906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994
910	9041	9089	9137	9185	9232	9280	9328	9375	9423	9471
911	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947
912	9995	..42	..90	.138	.185	.233	.280	.328	.376	.423
913	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899
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915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741
920	3788	3835	3882	3929	3977	4024	4071	4118	4165	4212
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436
930	8483	8530	8576	8623	8670	8716	8763	8810	8856	8903
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835
933	9882	9928	9975	..21	..68	.114	.161	.207	.254	.300
934	970347	0393	0440	0486	0533	0579	0626	0672	0719	0765
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936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082
940	3128	3174	3220	3266	3313	3359	3405	3451	3497	3543
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386
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945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304
947	6350	6396	6442	6488	6533	6579	6925	6671	6717	6763
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220
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951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958
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955	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226
960	2271	2316	2362	2407	2452	2497	2543	2588	2633	2678
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382
967	5426	5471	5516	5561	5606	5651	5699	5741	5786	5830
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727
970	6772	6817	6861	6906	6951	6996	7040	7085	7130	7175
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960
975	9005	9049	9093	9138	9183	9227	9272	9316	9361	9405
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850
977	9895	9939	9983	..28	..72	.117	.161	.206	.250	.294
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979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182
980	1226	1270	1315	1359	1403	1448	1492	1536	1580	1625
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713
988	4757	4801	4845	4886	4933	4977	5021	5065	5108	5152
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591
990	5635	5679	5723	5767	5811	5854	5898	5942	5986	6030
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779
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995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652
997	8695	8739	8792	8826	8869	8913	8956	9000	9043	9087
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957

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1	6.463726		000000		6.463726		13.536274	00029	100000	59
2	764756		000000		764756		235244	00058	100000	58
3	940847		000000		940847		059153	00087	100000	57
4	7.065786		000000		7.065786		12.934214	00116	100000	56
5	162696		000000		162696		837304	00145	100000	55
6	241877		9.999999		241878		758122	00175	100000	54
7	308824		999999		308825		691175	00204	100000	53
8	366816		999999		366817		633183	00233	100000	52
9	417968		999999		417970		582030	00262	100000	51
10	463725		999998		463727		536273	00291	100000	50
11	7.505118		9.999998		7.505120		12.494880	00320	99999	49
12	542906		999997		542909		457091	00349	99999	48
13	577668		999997		577672		422328	00378	99999	47
14	609853		999996		609857		390143	00407	99999	46
15	639816		999996		639820		360180	00436	99999	45
16	667845		999995		667849		332151	00465	99999	44
17	694173		999995		694179		305821	00495	99999	43
18	718997		999994		719003		280997	00524	99999	42
19	742477		999993		742484		257516	00553	99998	41
20	764754		999993		764761		235239	00582	99998	40
21	7.785943		9.999992		7.785951		12.214049	00611	99998	39
22	806146		999991		806155		193845	00640	99998	38
23	825451		999990		825460		174540	00669	99998	37
24	843934		999989		843944		156056	00698	99998	36
25	861663		999988		861674		138326	00727	99997	35
26	878695		999988		878708		121292	00756	99997	34
27	895085		999987		895099		104901	00785	99997	33
28	910879		999986		910894		089106	00814	99997	32
29	926119		999985		926134		073866	00844	99996	31
30	940842		999983		940858		059142	00873	99996	30
31	7.955082	2298	9.999982	0.2	7.955100	2298	12.044900	00902	99996	29
32	968870	2227	999981	0.2	968889	2227	031111	00931	99996	28
33	982233	2161	999980	0.2	982253	2161	017747	00960	99995	27
34	995198	2098	999979	0.2	995219	2098	004781	00989	99995	26
35	8.007787	2039	999977	0.2	8.007809	2039	11.992191	01018	99995	25
36	020021	1983	999976	0.2	020045	1983	979955	01047	99995	24
37	031919	1930	999975	0.2	031945	1930	968055	01076	99994	23
38	043501	1880	999973	0.2	043527	1880	956473	01105	99994	22
39	054781	1832	999972	0.2	054809	1833	945191	01134	99994	21
40	065776	1787	999971	0.2	065806	1787	934194	01164	99993	20
41	8.076500	1744	9.999969	0.2	8.076531	1744	11.923469	01193	99993	19
42	086965	1703	999968	0.2	086997	1703	913003	01222	99993	18
43	097183	1664	999966	0.2	097217	1664	902783	01251	99992	17
44	107167	1626	999964	0.3	107202	1627	892797	01280	99992	16
45	116926	1591	999963	0.3	116963	1591	883037	01309	99991	15
46	126471	1557	999961	0.3	126510	1557	873490	01338	99991	14
47	135810	1524	999959	0.3	135851	1524	864149	01367	99991	13
48	144953	1492	999958	0.3	144996	1493	855004	01396	99990	12
49	153907	1462	999956	0.3	153952	1463	846048	01425	99990	11
50	162681	1433	999954	0.3	162727	1434	837273	01454	99989	10
51	8.171280	1405	9.999952	0.3	8.171328	1406	11.828672	01483	99989	9
52	179713	1379	999950	0.3	179763	1379	820237	01513	99989	8
53	187985	1353	999948	0.3	188036	1353	811964	01542	99988	7
54	196102	1328	999946	0.3	196156	1328	803844	01571	99988	6
55	204070	1304	999944	0.3	204126	1304	795874	01600	99987	5
56	211895	1281	999942	0.4	211953	1281	788047	01629	99987	4
57	219581	1259	999940	0.4	219641	1259	780359	01658	99986	3
58	227134	1237	999938	0.4	227195	1238	772805	01687	99986	2
59	234557	1216	999936	0.4	234621	1217	765379	01716	99985	1
60	241855		999934		241921		758079	01745	99985	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

89 Degrees.

′	Sine	D 10″	Cosine.	D. 10″	Tang.	D 10″	Cotang.	N. sine.	N. cos.	
0	8.241855	1196	9.999934	0.4	8.241921	1197	11.758079	01745	99985	60
1	249033	1177	999932	0.4	249102	1177	750898	01774	99984	59
2	256094	1158	999929	0.4	256165	1158	743835	01803	99984	58
3	263042	1140	999927	0.4	263115	1140	736885	01832	99983	57
4	269881	1122	999925	0.4	269956	1122	730044	01862	99983	56
5	276614	1105	999922	0.4	276691	1105	723309	01891	99982	55
6	283243	1088	999920	0.4	283323	1089	716677	01920	99982	54
7	289773	1072	999918	0.4	289856	1073	710144	01949	99981	53
8	296207	1056	999915	0.4	296292	1057	703708	01978	99980	52
9	302546	1041	999913	0.4	302634	1042	697366	02007	99980	51
10	308794	1027	999910	0.4	308884	1027	691116	02036	99979	50
11	8.314954	1012	9.999907	0.4	8.315046	1013	11.684954	02065	99979	49
12	321027	998	999905	0.4	321122	999	678878	02094	99978	48
13	327016	985	999902	0.4	327114	985	672886	02123	99977	47
14	332924	971	999899	0.5	333025	972	666975	02152	99977	46
15	338753	959	999897	0.5	338856	959	661144	02181	99976	45
16	344504	946	999894	0.5	344610	946	655390	02211	99976	44
17	350181	934	999891	0.5	350289	934	649711	02240	99975	43
18	355783	922	999888	0.5	355895	922	644105	02269	99974	42
19	361315	910	999885	0.5	361430	911	638570	02298	99974	41
20	366777	899	999882	0.5	366895	899	633105	02327	99973	40
21	8.372171	888	9.999879	0.5	8.372292	888	11.627708	02356	99972	39
22	377499	877	999876	0.5	377622	879	622378	02385	99972	38
23	382762	867	999873	0.5	382889	867	617111	02414	99971	37
24	387962	856	999870	0.5	388092	857	611908	02443	99970	36
25	393101	846	999867	0.5	393234	847	606766	02472	99969	35
26	398179	837	999864	0.5	398315	837	601685	02501	99969	34
27	403199	827	999861	0.5	403338	828	596662	02530	99968	33
28	408161	818	999858	0.5	408304	818	591696	02560	99967	32
29	413068	809	999854	0.5	413213	809	586787	02589	99966	31
30	417919	800	999851	0.6	418068	800	581932	02618	99966	30
31	8.422717	791	9.999848	0.6	8.422869	791	11.577131	02647	99965	29
32	427462	782	999844	0.6	427618	783	572382	02676	99964	28
33	432156	774	999841	0.6	432315	774	567685	02705	99963	27
34	436800	766	999838	0.6	436962	766	563038	02734	99963	26
35	441394	758	999834	0.6	441560	758	558440	02763	99962	25
36	445941	750	999831	0.6	446110	750	553890	02792	99961	24
37	450440	742	999827	0.6	450613	743	549387	02821	99960	23
38	454893	735	999823	0.6	455070	735	544930	02850	99959	22
39	459301	727	999820	0.6	459481	728	540519	02879	99959	21
40	463665	720	999816	0.6	463849	720	536151	02908	99958	20
41	8.467985	712	9.999812	0.6	8.468172	713	11.531828	02938	99957	19
42	472263	706	999809	0.6	472454	707	527546	02967	99956	18
43	476498	699	999805	0.6	476693	700	523307	02996	99955	17
44	480693	692	999801	0.6	480892	693	519108	03025	99954	16
45	484848	686	999797	0.7	485050	686	514950	03054	99953	15
46	488963	679	999793	0.7	489170	680	510830	03083	99952	14
47	493040	673	999790	0.7	493250	674	506750	03112	99952	13
48	497078	667	999786	0.7	497293	668	502707	03141	99951	12
49	501080	661	999782	0.7	501298	661	498702	03170	99950	11
50	505045	655	999778	0.7	505267	655	494733	03199	99949	10
51	8.508974	649	9.999774	0.7	8.509200	650	11.490800	03228	99948	9
52	512867	643	999769	0.7	513098	644	486902	03257	99947	8
53	516726	637	999765	0.7	516961	638	483039	03286	99946	7
54	520551	632	999761	0.7	520790	633	479210	03316	99945	6
55	524343	626	999757	0.7	524586	627	475414	03345	99944	5
56	528102	621	999753	0.7	528349	622	471651	03374	99943	4
57	531828	616	999748	0.7	532080	616	467920	03403	99942	3
58	535523	611	999744	0.7	535779	611	464221	03432	99941	2
59	539186	605	999740	0.7	539447	606	460553	03461	99940	1
60	542819		999735		543084		456916	03490	99939	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

88 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	8.542819	600	9.999735	0.7	8.543084	602	11.456916	03490	99939	60
1	546422	595	999731	0.7	546691	596	453309	03519	99938	59
2	549995	591	999726	0.7	550268	591	449732	03548	99937	58
3	553539	586	999722	0.8	553817	587	446183	03577	99936	57
4	557054	581	999717	0.8	557336	582	442664	03606	99935	56
5	560540	576	999713	0.8	560828	577	439172	03635	99934	55
6	563999	572	999708	0.8	564291	573	435709	03664	99933	54
7	567431	567	999704	0.8	567727	568	432273	03693	99932	53
8	570836	563	999699	0.8	571137	564	428863	03723	99931	52
9	574214	559	999694	0.8	574520	559	425480	03752	99930	51
10	577566	554	999689	0.8	577877	555	422123	03781	99929	50
11	8.580892	550	9.999685	0.8	8.581208	551	11.418792	03810	99927	49
12	584193	546	999680	0.8	584514	547	415486	03839	99926	48
13	587469	542	999675	0.8	587795	543	412205	03868	99925	47
14	590721	538	999670	0.8	591051	539	408949	03897	99924	46
15	593948	534	999665	0.8	594283	535	405717	03926	99923	45
16	597152	530	999660	0.8	597492	531	402508	03955	99922	44
17	600332	526	999655	0.8	600677	527	399323	03984	99921	43
18	603489	522	999650	0.8	603839	523	396161	04013	99919	42
19	606623	519	999645	0.8	606978	519	393022	04042	99918	41
20	609734	515	999640	0.9	610094	516	389906	04071	99917	40
21	8.612823	511	9.999635	0.9	8.613189	512	11.386811	04100	99916	39
22	615891	508	999629	0.9	616262	508	383738	03129	99915	38
23	618937	504	999324	0.9	619313	505	380687	04159	99913	37
24	621962	501	999619	0.9	622343	501	377657	04188	99912	36
25	624965	497	999614	0.9	625352	498	374648	04217	99911	35
26	627948	494	999608	0.9	628340	495	371660	04246	99910	34
27	630911	490	999603	0.9	631308	491	368692	04275	99909	33
28	633854	487	999597	0.9	634256	488	365744	04304	99907	32
29	636776	484	999592	0.9	637184	485	362816	04333	99906	31
30	639680	481	999586	0.9	640093	482	359907	04362	99905	30
31	8.642563	477	9.999581	0.9	8.642982	478	11.357018	04391	99904	29
32	645428	474	999575	0.9	645853	475	354147	04420	99902	28
33	648274	471	999570	0.9	648704	472	351296	04449	99901	27
34	651102	468	999564	0.9	651537	469	348463	04478	99900	26
35	653911	465	999558	1.0	654352	466	345648	04507	99898	25
36	656702	462	999553	1.0	657149	463	342851	04536	99897	24
37	659475	459	999547	1.0	659928	460	340072	04565	99896	23
38	662230	456	999541	1.0	662689	457	337311	04594	99894	22
39	664968	453	999535	1.0	665433	454	334567	04623	99893	21
40	667689	451	999529	1.0	668160	453	331840	04653	99892	20
41	8.670393	448	9.999524	1.0	8.670870	449	11.329130	04682	99890	19
42	673080	445	999518	1.0	673563	446	326437	04711	99889	18
43	675751	442	999512	1.0	676239	443	323761	04740	99888	17
44	678405	440	999506	1.0	678900	442	321100	04769	99886	16
45	681043	437	999500	1.0	681544	438	318456	04798	99885	15
46	683665	434	999493	1.0	684172	435	315828	04827	99883	14
47	686272	432	999487	1.0	6 6784	433	313216	04856	99882	13
48	688863	429	999481	1.0	689381	430	310619	04885	99881	12
49	691438	427	999475	1.0	691963	428	308037	04914	99879	11
50	693998	424	999469	1.0	694529	425	305471	04943	99878	10
51	8.696543	422	9.999463	1.1	8.697081	423	11.302919	04972	99876	9
52	699073	419	999456	1.1	699617	420	300383	05001	99875	8
53	701589	417	999450	1.1	702139	418	297861	05030	99873	7
54	704090	414	999443	1.1	704246	415	295354	05059	99872	6
55	706577	412	999437	1.1	707140	413	292860	05088	99870	5
56	709049	410	999431	1.1	709618	411	290382	05117	99869	4
57	711507	407	999424	1.1	702083	408	287917	05146	99867	3
58	713952	405	999418	1.1	714534	406	285465	05175	99866	2
59	716383	403	999411	1.1	716972	404	283028	05205	99864	1
60	718800		999404		719396		280604	05234	99863	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	8.718800	401	9.999404	1.1	8.719396	402	11.280604	05234	99863	60
1	721204	398	999398	1.1	721806	399	278194	05263	99861	59
2	723595	396	999391	1.1	724204	397	275796	05292	99860	58
3	725972	394	999384	1.1	726588	395	273412	05321	99858	57
4	728337	392	999378	1.1	728959	393	271041	05350	99857	56
5	730688	390	999371	1.1	731317	391	268683	05379	99855	55
6	733027	388	999364	1.2	733663	389	266337	05408	99854	54
7	735354	386	999357	1.2	735996	387	264004	05437	99852	53
8	737667	384	999350	1.2	738317	385	261683	05466	99851	52
9	739969	382	999343	1.2	740626	383	259374	05495	99849	51
10	742259	380	999336	1.2	742922	381	257078	05524	99847	50
11	8.744536	378	9.999329	1.2	8.745207	379	11.254793	05553	99846	49
12	746802	376	999322	1.2	747479	377	252521	05582	99844	48
13	749055	374	999315	1.2	749740	375	250260	05611	99842	47
14	751297	372	999308	1.2	751989	373	248011	05640	99841	46
15	753528	370	999301	1.2	754227	371	245773	05669	99839	45
16	755747	368	999294	1.2	756453	369	243547	05698	99838	44
17	757955	366	999286	1.2	758668	367	241332	05727	99836	43
18	760151	364	999279	1.2	760872	365	239128	05756	99834	42
19	762337	362	999272	1.2	763065	364	236935	05785	99833	41
20	764511	361	999265	1.2	765246	362	234754	05814	99831	40
21	8.766675	359	9.999257	1.2	8.767417	360	11.232583	05844	99829	39
22	768828	357	999250	1.3	769578	358	230422	05873	99827	38
23	770970	355	999242	1.3	771727	356	228273	05902	99826	37
24	773101	353	999235	1.3	773866	355	226134	05931	99824	36
25	775223	352	999227	1.3	775995	353	224005	05960	99822	35
26	777333	350	999220	1.3	778114	351	221886	05989	99821	34
27	779434	348	999212	1.3	780222	350	219778	06018	99819	33
28	781524	347	999205	1.3	782320	348	217680	06047	99817	32
29	783605	345	999197	1.3	784408	346	215592	06076	99815	31
30	785675	343	999189	1.3	786486	345	213514	06105	99813	30
31	8.787736	342	9.999181	1.3	8.788554	343	11.211446	06134	99812	29
32	789787	340	999174	1.3	790613	341	209387	06163	99810	28
33	791828	339	999166	1.3	792662	340	207338	06192	99808	27
34	793859	337	999158	1.3	794701	338	205299	06221	99806	26
35	795881	335	999150	1.3	796731	337	203269	06250	99804	25
36	797894	334	999142	1.3	798752	335	201248	06279	99803	24
37	799897	332	999134	1.3	800763	334	199237	06308	99801	23
38	801892	331	999126	1.3	802765	332	197235	06337	99799	22
39	803876	329	999118	1.3	804858	331	195242	06366	99797	21
40	805852	328	999110	1.3	806742	329	193258	06395	99795	20
41	8.807819	326	9.999102	1.3	8.808717	328	11.191283	06424	99793	19
42	809777	325	999094	1.4	810683	326	189317	06453	99792	18
43	811726	323	999086	1.4	812641	325	187359	06482	99790	17
44	813667	322	999077	1.4	814589	323	185411	06511	99788	16
45	815599	320	999069	1.4	816529	322	183471	06540	99786	15
46	817522	319	999061	1.4	818461	320	181539	06569	99784	14
47	819436	318	999053	1.4	820384	319	179616	06598	99782	13
48	821343	316	999044	1.4	822298	318	177702	06627	99780	12
49	823240	315	999036	1.4	824205	316	175795	06656	99778	11
50	825130	313	999027	1.4	826103	315	173897	06685	99776	10
51	8.827011	312	9.999019	1.4	8.827992	314	11.172008	06714	99774	9
52	828884	311	999010	1.4	829874	312	170126	06743	99772	8
53	830749	309	999002	1.4	831748	311	168252	06773	99770	7
54	832607	308	998993	1.4	833613	310	166387	06802	99768	6
55	834456	307	998984	1.4	835471	308	164529	06831	99766	5
56	836297	306	998976	1.4	837321	307	162679	06860	99764	4
57	838130	304	998967	1.5	839163	306	160837	06889	99762	3
58	839956	303	998958	1.5	840998	304	159002	06918	99760	2
59	841774	302	998950	1.5	842825	303	157175	06947	99758	1
60	843585		998941		844644		155356	06976	99756	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

86 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	8.843585	300	9.998941	1.5	8.844644	302	11.155356	06976	99756	60
1	845387	299	998932	1.5	846455	301	153545	07005	99754	59
2	847183	298	998923	1.5	848260	299	151740	07034	99752	58
3	848971	297	998914	1.5	850057	298	149943	07063	99750	57
4	850751	295	998905	1.5	851846	297	148154	07092	99748	56
5	852525	294	998896	1.5	853628	296	146372	07121	99746	55
6	854291	293	998887	1.5	855403	295	144597	07150	99744	54
7	856049	292	998878	1.5	857171	293	142829	07179	99742	53
8	857801	291	998869	1.5	858932	292	141068	07208	99740	52
9	859546	290	998860	1.5	860686	291	139314	07237	99738	51
10	861283	288	998851	1.5	862433	290	137567	07266	99736	50
11	8.863014	287	9.998841	1.5	8.864173	289	11.135827	07295	99734	49
12	864738	286	998832	1.5	865906	288	134094	07324	99731	48
13	866455	285	998823	1.6	867632	287	132368	07353	99729	47
14	868165	284	998813	1.6	869351	285	130649	07382	99727	46
15	869868	283	998804	1.6	871064	284	128936	07411	99725	45
16	871565	282	998795	1.6	872770	283	127230	07440	99723	44
17	873255	281	998785	1.6	874469	282	125531	07469	99721	43
18	874938	279	998776	1.6	876162	281	123838	07498	99719	42
19	876615	279	998766	1.6	877849	280	122151	07527	99716	41
20	878285	277	998757	1.6	879529	279	120471	07556	99714	40
21	8.879949	276	9.998747	1.6	8.881202	278	11.118798	07585	99712	39
22	881607	275	998738	1.6	882869	277	117131	07614	99710	38
23	883258	274	998728	1.6	884530	276	115470	07643	99708	37
24	884903	273	998718	1.6	886185	275	113815	07672	99705	36
25	886542	272	998708	1.6	887833	274	112167	07701	99703	35
26	888174	271	998699	1.6	889476	273	110524	07730	99701	34
27	889801	270	998689	1.6	891112	272	108888	07759	99699	33
28	891421	269	998679	1.6	892742	271	107258	07788	99696	32
29	893035	268	998669	1.7	894366	270	105634	07817	99694	31
30	894643	267	998659	1.7	895984	269	104016	07846	99692	30
31	8.896246	266	9.998649	1.7	8.897596	268	11.102404	07875	99689	29
32	897842	265	998639	1.7	899203	267	100797	07904	99687	28
33	899432	264	998629	1.7	900803	266	099197	07933	99685	27
34	901017	263	998619	1.7	902398	265	097602	07962	99683	26
35	902596	262	998609	1.7	903987	264	096013	07991	99680	25
36	904169	261	998599	1.7	905570	263	094430	08020	99678	24
37	905736	260	998589	1.7	907147	262	092853	08049	99676	23
38	907297	259	998578	1.7	908719	261	091281	08078	99673	22
39	908853	258	998568	1.7	910285	260	089715	08107	99671	21
40	910404	257	998558	1.7	911846	259	088154	08136	99668	20
41	8.911949	257	9.998548	1.7	8.913401	258	11.086599	08165	99666	19
42	913488	256	998537	1.7	914951	257	085049	08194	99664	18
43	915022	255	998527	1.7	916495	256	083505	08223	99661	17
44	916550	254	998516	1.8	918034	256	081966	08252	99659	16
45	918073	253	998506	1.8	919568	255	080432	08281	99657	15
46	919591	252	998495	1.8	921096	254	078904	08310	99654	14
47	921103	251	998485	1.8	922619	253	077381	08339	99652	13
48	922610	250	998474	1.8	924136	252	075864	08368	99649	12
49	924112	249	998464	1.8	925649	251	074351	08397	99647	11
50	925609	249	998453	1.8	927156	250	072844	08426	99644	10
51	8.927100	248	9.998442	1.8	8.928658	249	11.071342	08455	99642	9
52	928587	247	998431	1.8	930155	249	069845	08484	99639	8
53	930068	246	998421	1.8	931647	248	068353	08513	99637	7
54	931544	245	998410	1.8	933134	247	066866	08542	99635	6
55	933015	244	998399	1.8	934616	246	065384	08571	99632	5
56	934481	243	998388	1.8	936093	245	063907	08600	99630	4
57	935942	243	998377	1.8	937565	244	062435	08629	99627	3
58	937398	242	998366	1.8	939032	244	060968	08658	99625	2
59	938850	241	998355	1.8	940494	243	059506	08687	99622	1
60	940296		998344		941952		058048	08716	99619	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

85 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	8.940296	240	9.998344	1.9	8.941952	242	11.058048	08716	99619	60
1	941738	239	998333	1.9	943404	241	056596	08745	99617	59
2	943174	239	998322	1.9	944852	240	055148	08774	99614	58
3	944606	238	998311	1.9	946295	240	053705	08803	99612	57
4	946034	237	998300	1.9	947734	239	052266	08831	99609	56
5	947456	236	998289	1.9	949168	238	050832	08860	99607	55
6	948874	235	998277	1.9	950597	237	049403	08889	99604	54
7	950287	235	998266	1.9	952021	237	047979	08918	99602	53
8	951693	234	998255	1.9	953441	236	046559	08947	99599	52
9	953100	233	998243	1.9	954856	235	045144	08976	99596	51
10	954499	232	998232	1.9	956267	234	043733	09005	99594	50
11	8.955894	232	9.998220	1.9	8.957674	234	11.042326	09034	99591	49
12	957284	231	998209	1.9	959075	233	040925	09063	99588	48
13	958670	230	998197	1.9	960473	232	039527	09092	99586	47
14	960052	229	998186	1.9	961866	231	038134	09121	99583	46
15	961429	229	998174	1.9	963255	231	036745	09150	99580	45
16	962801	228	998163	1.9	964639	230	035361	09179	99578	44
17	964170	227	998151	1.9	966019	229	033981	09208	99575	43
18	965534	227	998139	2.0	967394	229	032606	09237	99572	42
19	966893	226	998128	2.0	968766	228	031234	09266	99570	41
20	968249	225	998116	2.0	970133	227	029867	09295	99567	40
21	8.969600	224	9.998104	2.0	8.971496	226	11.028504	09324	99564	39
22	970947	224	998092	2.0	972855	226	027145	09353	99562	38
23	972289	223	998080	2.0	974209	225	025791	09382	99559	37
24	973628	222	998068	2.0	975560	224	024440	09411	99556	36
25	974962	222	998056	2.0	976906	224	023094	09440	99553	35
26	976293	221	998044	2.0	978248	223	021752	09469	99551	34
27	977619	220	998032	2.0	979586	222	020414	09498	99548	33
28	978941	220	998020	2.0	980921	222	019079	09527	99545	32
29	980259	219	998008	2.0	982251	221	017749	09556	99542	31
30	981573	218	997996	2.0	983577	220	016423	09585	99540	30
31	8.982883	218	9.997984	2.0	8.984899	220	11.015101	09614	99537	29
32	984189	217	997972	2.0	986217	219	013783	09642	99534	28
33	985491	216	997959	2.0	987532	218	012468	09671	99531	27
34	986789	216	997947	2.0	988842	218	011158	09700	99528	26
35	988083	215	997935	2.1	990149	217	009851	09729	99526	25
36	989374	214	997922	2.1	991451	216	008549	09758	99523	24
37	990660	214	997910	2.1	992750	216	007250	09787	99520	23
38	991943	213	997897	2.1	994045	215	005955	09816	99517	22
39	993222	212	997885	2.1	995337	215	004663	09845	99514	21
40	994497	212	997872	2.1	996624	214	003376	09874	99511	20
41	8.995768	211	9.997860	2.1	8.997908	213	11.002092	09903	99508	19
42	997036	211	997847	2.1	999188	213	000812	09932	99506	18
43	998299	210	997835	2.1	9.000465	212	10.999535	09961	99503	17
44	999560	209	997822	2.1	001738	211	998262	09990	99500	16
45	9.000816	209	997809	2.1	003007	211	996993	10019	99497	15
46	002059	208	997797	2.1	004272	210	995728	10048	99494	14
47	003318	208	997784	2.1	005534	210	994466	10077	99491	13
48	004563	207	997771	2.1	006792	209	993208	10106	99488	12
49	005805	206	997758	2.1	008047	208	991953	10135	99485	11
50	007044	206	997745	2.1	009298	208	990702	10164	99482	10
51	9.008278	205	9.997732	2.1	9.010546	207	10.989454	10192	99479	9
52	009510	205	997719	2.1	011790	207	988210	10221	99476	8
53	010737	204	997706	2.1	013031	206	686969	10250	99473	7
54	011962	203	997693	2.2	014268	206	985732	10279	99470	6
55	013182	203	997680	2.2	015502	205	984498	10308	99467	5
56	014400	202	997667	2.2	016732	204	983268	10337	99464	4
57	015613	202	997654	2.2	017959	204	983041	10366	99461	3
58	016824	201	997641	2.2	019183	203	980817	10395	99458	2
59	018031	201	997628	2.2	020403	203	979597	10424	99455	1
60	019235		997614		021620		978380	10453	99452	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.019235	200	9.997614	2.2	9.021620	202	10.978380	10453	99452	60
1	020435	199	997601	2.2	022834	202	977166	10482	99449	59
2	021632	199	997588	2.2	024044	201	975956	10511	99446	58
3	022825	198	997574	2.2	025251	201	974749	10540	99443	57
4	024016	198	997561	2.2	026455	200	973545	10569	99440	56
5	025203	197	997547	2.2	027655	199	972345	10597	99437	55
6	026386	197	997534	2.3	028852	199	971148	10626	99434	54
7	027567	196	997520	2.3	030046	198	969954	10655	99431	53
8	028744	196	997507	2.3	031237	198	968763	10684	99428	52
9	029918	195	997493	2.3	032425	197	967575	10713	99424	51
10	031089	195	997480	2.3	033609	197	966391	10742	99421	50
11	9.032257	194	9.997466	2.3	9.034791	196	10.965209	10771	99418	49
12	033421	194	997452	2.3	035969	196	964031	10800	99415	48
13	034582	193	997439	2.3	037144	195	962856	10829	99412	47
14	035741	192	997425	2.3	038316	195	961684	10858	99409	46
15	036896	192	997411	2.3	039485	194	960515	10887	99406	45
16	038048	191	997397	2.3	040651	194	959349	10916	99402	44
17	039197	191	997383	2.3	041813	193	958187	10945	99399	43
18	040342	190	997369	2.3	042973	193	957027	10973	99396	42
19	041485	190	997355	2.3	044130	192	955870	11002	99393	41
20	042625	189	997341	2.3	045284	192	954716	11031	99390	40
21	9.043762	189	9.997327	2.4	9.046434	191	10.953566	11060	99386	39
22	044895	180	997313	2.4	047582	191	952418	11089	99383	38
23	046026	188	997299	2.4	048727	190	951273	11118	99380	37
24	047154	187	997285	2.4	049869	190	950131	11147	99377	36
25	048279	187	997271	2.4	051008	189	948992	11176	99374	35
26	049400	186	997257	2.4	052144	189	947856	11205	99370	34
27	050519	186	997242	2.4	053277	188	946723	11234	99367	33
28	051635	185	997228	2.4	054407	188	945593	11263	99364	32
29	052749	185	997214	2.4	055535	187	944465	11291	99360	31
30	053859	184	997199	2.4	056659	187	943341	11320	99357	30
31	9.054966	184	9.997185	2.4	9.057781	186	10.942219	11349	99354	29
32	056071	184	997170	2.4	058900	186	941100	11378	99351	28
33	057172	183	997156	2.4	060016	185	939984	11407	99347	27
34	058271	183	997141	2.4	061130	185	938870	11436	99344	26
35	059367	182	997127	2.4	062240	185	937760	11465	99341	25
36	060460	182	997112	2.4	063348	184	936652	11494	99337	24
37	061551	181	997098	2.4	064453	184	935547	11523	99334	23
38	062639	181	997083	2.5	065556	183	934444	11552	99331	22
39	063724	180	997068	2.5	066655	183	933345	11580	99327	21
40	064806	180	997053	2.5	067752	182	932248	11609	99324	20
41	9.065885	179	9.997039	2.5	9.068846	182	10.931154	11638	99320	19
42	066962	179	997024	2.5	069038	181	930062	11667	99317	18
43	068036	179	997009	2.5	071027	181	928973	11696	99314	17
44	069107	178	996994	2.5	072113	181	927887	11725	99310	16
45	070176	178	996979	2.5	073197	180	926803	11754	99307	15
46	071242	177	996964	2.5	074278	180	925722	11783	99303	14
47	072306	177	996949	2.5	075356	179	924644	11812	99300	13
48	073366	176	996934	2.5	076432	179	923568	11840	99297	12
49	074424	176	996919	2.5	077505	178	922495	11869	99293	11
50	075480	175	996904	2.5	078576	178	921424	11898	99290	10
51	9.076533	175	9.996889	2.5	9.079644	178	10.920356	11927	99286	9
52	077583	175	996874	2.5	080710	177	919290	11956	99283	8
53	078631	174	996858	2.5	081773	177	918227	11985	99279	7
54	079676	174	996843	2.5	082833	176	917167	12014	99276	6
55	080719	173	996828	2.5	083891	176	916109	12043	99272	5
56	081759	173	996812	2.6	084947	175	915053	12071	99269	4
57	082797	172	996797	2.6	086000	175	914000	12100	99265	3
58	083832	172	996782	2.6	087050	175	912950	12129	99262	2
59	084864	172	996766	2.6	088098	174	911902	12158	99258	1
60	085894		996751		089144		910856	12187	99255	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

83 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.085894	171	9.996751	2.6	9.089144	174	10.910856	12187	99255	60
1	086922	171	996735	2.6	090187	173	909813	12216	99251	59
2	087947	170	996720	2.6	091228	173	908772	12245	99248	58
3	088970	170	996704	2.6	092266	173	907734	12274	99244	57
4	089990	170	996688	2.6	093302	172	906698	12302	99240	56
5	091008	169	996673	2.6	094336	172	905664	12331	99237	55
6	092024	169	996657	2.6	095367	171	904633	12360	99233	54
7	093037	168	996641	2.6	096395	171	903605	12389	99230	53
8	094047	168	996625	2.6	097422	171	902578	12418	99226	52
9	095056	168	996610	2.6	098446	170	901554	12447	99222	51
10	096062	167	996594	2.6	099468	170	900532	12476	99219	50
11	9.097065	167	9.996578	2.7	9.100487	169	10.899513	12504	99215	49
12	098066	166	996562	2.7	101504	169	898496	12533	99211	48
13	099065	166	996546	2.7	102519	169	897481	12562	99208	47
14	100062	166	996530	2.7	103532	168	896468	12591	99204	46
15	101056	165	996514	2.7	104542	168	895458	12620	99200	45
16	102048	165	996498	2.7	105550	168	894450	12649	99197	44
17	103037	164	996482	2.7	106556	167	893444	12678	99193	43
18	104025	164	996465	2.7	107559	167	892441	12706	99189	42
19	105010	164	996449	2.7	108560	166	891440	12735	99186	41
20	105992	163	996433	2.7	109559	166	890441	12764	99182	40
21	9.106973	163	9.996417	2.7	9.110556	166	10.889444	12793	99178	39
22	107951	163	996400	2.7	111551	165	888449	12822	99175	38
23	108927	162	996384	2.7	112543	165	887457	12851	99171	37
24	109901	162	996368	2.7	113533	165	886467	12880	99167	36
25	110873	162	996351	2.7	114521	164	885479	12908	99163	35
26	111842	161	996335	2.7	115507	164	884493	12937	99160	34
27	112809	161	996318	2.7	116491	164	883509	12966	99156	33
28	113774	160	996302	2.8	117472	163	882528	12995	99152	32
29	114737	160	996285	2.8	118452	163	881548	13024	99148	31
30	115698	160	996269	2.8	119429	162	880571	13053	99144	30
31	9.116656	159	9.996252	2.8	9.120404	162	10.879596	13081	99141	29
32	117613	159	996235	2.8	121377	162	878623	13110	99137	28
33	118567	159	996219	2.8	122348	161	877652	13139	99133	27
34	119519	158	996202	2.8	123317	161	876683	13168	99129	26
35	120469	158	996185	2.8	124284	161	875716	13197	99125	25
36	121417	158	996168	2.8	125249	160	874751	13226	99122	24
37	122362	157	996151	2.8	126211	160	873789	13254	99118	23
38	123306	157	996134	2.8	127172	160	872828	13283	99114	22
39	124248	157	996117	2.8	128130	159	871870	13312	99110	21
40	125187	156	996100	2.8	129087	159	870913	13341	99106	20
41	9.126125	156	9.996083	2.9	9.130041	159	10.869959	13370	99102	19
42	127060	156	996066	2.9	130994	158	869006	13399	99098	18
43	127993	155	996049	2.9	131944	158	868056	13427	99094	17
44	128925	155	996032	2.9	132893	158	867107	13456	99091	16
45	129854	154	996015	2.9	133839	157	866161	13485	99087	15
46	130781	154	995998	2.9	134784	157	865216	13514	99083	14
47	131706	154	995980	2.9	135726	157	864274	13543	99079	13
48	132630	153	995963	2.9	136667	156	863333	13572	99075	12
49	133551	153	995946	2.9	137605	156	862395	13600	99071	11
50	134470	153	995928	2.9	138542	156	861458	13629	99067	10
51	9.135387	152	9.995911	2.9	9.139476	155	10.860524	13658	99063	9
52	136303	152	995894	2.9	140409	155	859591	13687	99059	8
53	137216	152	995876	2.9	141340	155	858660	13716	99055	7
54	138128	152	995859	2.9	142269	154	857731	13744	99051	6
55	139037	151	995841	2.9	143196	154	856804	13773	99047	5
56	139944	151	995823	2.9	144121	154	855879	13802	99043	4
57	140850	151	995806	2.9	145044	153	854956	13831	99039	3
58	141754	150	995788	2.9	145966	153	854034	13860	99035	2
59	142655	150	995771	2.9	146885	153	853115	13889	99031	1
60	143555		995753		147803		852197	13917	99027	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.143555	150	9.995753	3.0	9.147803	153	10.852197	13917	99027	60
1	144453	149	995735	3.0	148718	152	851282	13946	99023	59
2	145349	149	995717	3.0	149632	152	850368	13975	99019	58
3	146243	149	995699	3.0	150544	152	849456	14004	99015	57
4	147136	148	995681	3.0	151454	151	848546	14033	99011	56
5	148026	148	995664	3.0	152363	151	847637	14061	99006	55
6	148915	148	995646	3.0	153269	151	846731	14090	99002	54
7	149802	147	995628	3.0	154174	150	845826	14119	98998	53
8	150686	147	995610	3.0	155077	150	844923	14148	98994	52
9	151569	147	995591	3.0	155978	150	844022	14177	98990	51
10	152451	147	995573	3.0	156877	150	843123	14205	98986	50
11	9.153330	146	9.995555	3.0	9.157775	149	10.842225	14234	98982	49
12	154208	146	995537	3.0	158671	149	841329	14263	98978	48
13	155083	146	995519	3.0	159565	149	840435	14292	98973	47
14	155957	145	995501	3.1	160457	148	839543	14320	98969	46
15	156830	145	995482	3.1	161347	148	838653	14349	98965	45
16	157700	145	995464	3.1	162236	148	837764	14378	98961	44
17	158569	144	995446	3.1	163123	148	836877	14407	98957	43
18	159435	144	995427	3.1	164008	147	835992	14436	98953	42
19	160301	144	995409	3.1	164892	147	835108	14464	98948	41
20	161164	144	995390	3.1	165774	147	834226	14493	98944	40
21	9.162025	143	9.995372	3.1	9.166654	146	10.833346	14522	98940	39
22	162885	143	995353	3.1	167532	146	832468	14551	98936	38
23	163743	143	995334	3.1	168409	146	831591	14580	98931	37
24	164600	142	995316	3.1	169284	145	830716	14608	98927	36
25	165454	142	995297	3.1	170157	145	829843	14637	98923	35
26	166307	142	995278	3.1	171029	145	828971	14666	98919	34
27	167159	142	995260	3.1	171899	145	828101	14695	98914	33
28	168008	141	995241	3.2	172767	144	827233	14723	98910	32
29	168856	141	995222	3.2	173634	144	826366	14752	98906	31
30	169702	141	995203	3.2	174499	144	825501	14781	98902	30
31	9.170547	140	9.995184	3.2	9.175362	144	10.824638	14810	98897	29
32	171389	140	995165	3.2	176224	143	823776	14838	98893	28
33	172230	140	995146	3.2	177084	143	822916	14867	98889	27
34	173070	140	995127	3.2	177942	143	822058	14896	98884	26
35	173908	139	995108	3.2	178799	142	821201	14925	98880	25
36	174744	139	995089	3.2	179655	142	820345	14954	98876	24
37	175578	139	995070	3.2	180508	142	819492	14982	98871	23
38	176411	139	995051	3.2	181360	142	818640	15011	98867	22
39	177242	138	995032	3.2	182211	141	817789	15040	98863	21
40	178072	138	995013	3.2	183059	141	816941	15069	98858	20
41	9.178900	138	9.994993	3.2	9.183907	141	10.816093	15097	98854	19
42	179726	137	994974	3.2	184752	141	815248	15126	98849	18
43	180551	137	994955	3.2	185597	140	814403	15155	98845	17
44	181374	137	994935	3.2	186439	140	813561	15184	98841	16
45	182196	137	994916	3.3	187280	140	812720	15212	98836	15
46	183016	136	994896	3.3	188120	140	811880	15241	98832	14
47	183834	136	994877	3.3	188958	139	811042	15270	98827	13
48	184651	136	994857	3.3	189794	139	810206	15299	98823	12
49	185466	136	994838	3.3	190629	139	809371	15327	98818	11
50	186280	135	994818	3.3	191462	139	808538	15356	98814	10
51	9.187092	135	9.994798	3.3	9.192294	138	10.807706	15385	98809	9
52	187903	135	994779	3.3	193124	138	806876	15414	98805	8
53	188712	135	994759	3.3	193953	138	806047	15442	98800	7
54	189519	134	994739	3.3	194780	138	805220	15471	98796	6
55	190325	134	994719	3.3	195606	137	804394	15500	98791	5
56	191130	134	994700	3.3	196430	137	803570	15529	98787	4
57	191933	134	994680	3.3	197253	137	802747	15557	98782	3
58	192734	133	994660	3.3	198074	137	801926	15586	98778	2
59	193534	133	994640	3.3	198894	136	801106	15615	98773	1
60	194332		994620		199713		800287	15643	98769	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

81 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.194332	133	9.994620	3.3	9.199713	136	10.800287	15643	98769	60
1	195129	133	994600	3.3	200529	136	799471	15672	98764	59
2	195925	132	994580	3.3	201345	136	798655	15701	98760	58
3	196719	132	994560	3.4	202159	135	797841	15730	98755	57
4	197511	132	994540	3.4	202971	135	797029	15758	98751	56
5	198302	132	994519	3.4	203782	135	796218	15787	98746	55
6	199091	131	994499	3.4	204592	135	795408	15816	98741	54
7	199879	131	994479	3.4	205400	134	794600	15845	98737	53
8	200666	131	994459	3.4	206207	134	793793	15873	98732	52
9	201451	131	994438	3.4	207013	134	792987	15902	98728	51
10	202234	130	994418	3.4	207817	134	792183	15931	98723	50
11	9.203017	130	9.994397	3.4	9.208619	133	10.791381	15959	98718	49
12	203797	130	994377	3.4	209420	133	790580	15988	98714	48
13	204577	130	994357	3.4	210220	133	789780	16017	98709	47
14	205354	129	994336	3.4	211018	133	788982	16046	98704	46
15	206131	129	994316	3.4	211815	133	788185	16074	98700	45
16	206906	129	994295	3.4	212611	132	787389	16103	98695	44
17	207679	129	994274	3.5	213405	132	786595	16132	98690	43
18	208452	128	994254	3.5	214198	132	785802	16160	98686	42
19	209222	128	994233	3.5	214989	132	735011	16189	98681	41
20	209992	128	994212	3.5	215780	131	784220	16218	98676	40
21	9.210760	128	9.994191	3.5	9.216568	131	10.783432	16246	98671	39
22	211526	127	994171	3.5	217356	131	782644	16275	98667	38
23	212291	127	994150	3.5	218142	131	781858	16304	98662	37
24	213055	127	994129	3.5	218926	130	781074	16333	98657	36
25	213818	127	994108	3.5	219710	130	780290	16361	98652	35
26	214579	127	994087	3.5	220492	130	779508	16390	98648	34
27	215338	126	994066	3.5	221272	130	778728	16419	98643	33
28	216097	126	994045	3.5	222052	130	777948	16447	98638	32
29	216854	126	994024	3.5	222830	129	777170	16476	98633	31
30	217609	126	994003	3.5	223605	129	776394	16505	98629	30
31	9.218363	125	9.993981	3.5	9.224382	129	10.775618	16533	98624	29
32	219116	125	993960	3.5	225156	129	774844	16562	98619	28
33	219868	125	993939	3.5	225929	129	774071	16591	98614	27
34	220618	125	993918	3.5	226700	128	773300	16620	98609	26
35	221367	125	993896	3.6	227471	128	772529	16648	98604	25
36	222115	124	993875	3.6	228239	128	771761	16677	98600	24
37	222861	124	993854	3.6	229007	128	770993	16706	98595	23
38	223606	124	993832	3.6	229773	127	770227	16734	98590	22
39	224349	124	993811	3.6	230539	127	769461	16763	98585	21
40	225092	123	993789	3.6	231302	127	768698	16792	98580	20
41	9.225833	123	9.993768	3.6	9.232065	127	10.767935	16820	98575	19
42	226573	123	993746	3.6	232826	127	767174	16849	98570	18
43	227311	123	993725	3.6	233586	126	766414	16878	98565	17
44	228048	123	993703	3.6	234345	126	765655	16906	98561	16
45	228784	122	993681	3.6	235103	126	764897	16935	98556	15
46	229518	122	993660	3.6	235859	126	764141	16964	98551	14
47	230252	122	993638	3.6	236614	126	763386	16992	98546	13
48	230984	122	993616	3.6	237368	125	762632	17021	98541	12
49	231714	122	993594	3.7	238120	125	761880	17050	98536	11
50	232444	121	993572	3.7	238872	125	761128	17078	98531	10
51	9.233172	121	9.993550	3.7	9.239622	125	10.760378	17107	98526	9
52	233899	121	994528	3.7	240371	125	759629	17136	98521	8
53	234625	121	993506	3.7	241118	124	758882	17164	98516	7
54	235349	120	993484	3.7	241865	124	758135	17193	98511	6
55	236073	120	993462	3.7	242610	124	757390	17222	98506	5
56	236795	120	993440	3.7	243354	124	756646	17250	98501	4
57	237515	120	993418	3.7	244097	124	755903	17279	98496	3
58	238235	120	-993396	3.7	244839	123	755161	17308	98491	2
59	238953	119	993374	3.7	245579	123	754421	17336	98486	1
60	239670		993351		246319		753681	17365	98481	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N.sine.	N. cos.	
0	9.239670	119	9.993351	3.7	9.246319	123	10.753681	17365	98481	60
1	240386	119	993329	3.7	247057	123	752943	17393	98476	59
2	241101	119	993307	3.7	247794	123	752206	17422	98471	58
3	241814	119	993285	3.7	248530	122	751470	17451	98466	57
4	242526	118	993262	3.7	249264	122	750736	17479	98461	56
5	243237	118	993240	3.7	249998	122	750002	17508	98455	55
6	243947	118	993217	3.8	250730	122	749270	17537	98450	54
7	244656	118	993195	3.8	251461	122	748539	17565	98445	53
8	245363	118	993172	3.8	252191	121	747809	17594	98440	52
9	246069	117	993149	3.8	252920	121	747080	17623	98435	51
10	246775	117	993127	3.8	253648	121	746352	17651	98430	50
11	9.247478	117	9.993104	3.8	9.254374	121	10.745626	17680	98425	49
12	248181	117	993081	3.8	255100	121	744900	17708	98420	48
13	248883	117	993059	3.8	255824	120	744176	17737	98414	47
14	249583	116	993036	3.8	256547	120	743453	17766	98409	46
15	250282	116	993013	3.8	257269	120	742731	17794	98404	45
16	250980	116	992990	3.8	257990	120	742010	17823	98399	44
17	251677	116	992967	3.8	258710	120	741290	17852	98394	43
18	252373	116	992944	3.8	259429	120	740571	17880	98389	42
19	253067	116	992921	3.8	260146	119	739854	17909	98383	41
20	253761	115	992898	3.8	260863	119	739137	17937	98378	40
21	9.254453	115	9.992875	3.8	9.261578	119	10.738422	17966	98373	39
22	255144	115	992852	3.8	262292	119	737708	17995	98368	38
23	255834	115	992829	3.9	263005	119	736995	18023	98362	37
24	256523	115	992806	3.9	263717	118	736283	18052	98357	36
25	257211	114	992783	3.9	264428	118	735572	18081	98352	35
26	257898	114	992759	3.9	265138	118	734862	18109	98347	34
27	258583	114	992736	3.9	265847	118	734153	18138	98341	33
28	259268	114	992713	3.9	266555	118	733445	18166	98336	32
29	259951	114	992690	3.9	267261	118	732739	18195	98331	31
30	260633	113	992666	3.9	267967	117	732033	18224	98325	30
31	9.261314	113	9.992643	3.9	9.268671	117	10.731329	18252	98320	29
32	261994	113	992619	3.9	269375	117	730625	18281	98315	28
33	262673	113	992596	3.9	270077	117	729923	18309	98310	27
34	263351	113	992572	3.9	270779	117	729221	18338	98304	26
35	264027	113	992549	3.9	271479	116	728521	18367	98299	25
36	264703	112	992525	3.9	272178	116	727822	18395	98294	24
37	265377	112	992501	3.9	272876	116	727124	18424	98288	23
38	266051	112	992478	4.0	273573	116	726427	18452	98283	22
39	266723	112	992454	4.0	274269	116	725731	18481	98277	21
40	267395	112	992430	4.0	274964	116	725036	18509	98272	20
41	9.268065	111	9.992406	4.0	9.275658	115	10.724342	18538	98267	19
42	268734	111	992382	4.0	276351	115	723649	18567	98261	18
43	269402	111	992359	4.0	277043	115	722957	18595	98256	17
44	270069	111	992335	4.0	277734	115	722266	18624	98250	16
45	270735	111	992311	4.0	278424	115	721576	18652	98245	15
46	271400	111	992287	4.0	279113	115	720887	18681	98240	14
47	272064	110	992263	4.0	279801	114	720199	18710	98234	13
48	272726	110	992239	4.0	280488	114	719512	18738	98229	12
49	273388	110	992214	4.0	281174	114	718826	18767	98223	11
50	274049	110	992190	4.0	281858	114	718142	18795	98218	10
51	9.274708	110	9.992166	4.0	9.282542	114	10.717458	18824	98212	9
52	275367	110	992142	4.0	283225	114	716775	18852	98207	8
53	276024	109	992117	4.1	283907	113	716093	18881	98201	7
54	276681	109	992093	4.1	284588	113	715412	18910	98196	6
55	277337	109	992069	4.1	285268	113	714732	18938	98190	5
56	277991	109	992044	4.1	285947	113	714053	18967	98185	4
57	278644	109	992020	4.1	286624	113	713376	18995	98179	3
58	279297	109	991996	4.1	287301	113	712699	19024	98174	2
59	279948	108	991971	4.1	287977	112	712023	19052	98168	1
60	280599		991947		288652		711348	19081	98163	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.280599	108	9.991947	4.1	9.288652	112	10.711348	19081	98163	60
1	281248	108	991922	4.1	289326	112	710674	19109	98157	59
2	281897	108	991897	4.1	289999	112	710001	19138	98152	58
3	282544	108	991873	4.1	290671	112	709329	19167	98146	57
4	283190	108	991848	4.1	291342	112	708658	19195	98140	56
5	283836	107	991823	4.1	292013	111	707987	19224	98135	55
6	284480	107	991799	4.1	292682	111	707318	19252	98129	54
7	285124	107	991774	4.2	293350	111	706650	19281	98124	53
8	285766	107	991749	4.2	294017	111	705983	19309	98118	52
9	286408	107	991724	4.2	294684	111	705316	19338	98112	51
10	287048	107	991699	4.2	295349	111	704651	19366	98107	50
11	9.287687	106	9.991674	4.2	9.296013	111	10.703987	19395	98101	49
12	288326	106	991649	4.2	296677	110	703323	19423	98096	48
13	288964	106	991624	4.2	297339	110	702661	19452	98090	47
14	289600	106	991599	4.2	298001	110	701999	19481	98084	46
15	290236	106	991574	4.2	298662	110	701338	19509	98079	45
16	290870	106	991549	4.2	299322	110	700678	19538	98073	44
17	291504	105	991524	4.2	299980	110	700020	19566	98067	43
18	292137	105	991498	4.2	300638	109	699362	19595	98061	42
19	292768	105	991473	4.2	301295	109	698705	19623	98056	41
20	293399	105	991448	4.2	301951	109	698049	19652	98050	40
21	9.294029	105	9.991422	4.2	9.302607	109	10.697393	19680	98044	39
22	294658	105	991397	4.2	303261	109	696739	19709	98039	38
23	295286	104	991372	4.3	303914	109	696086	19737	98033	37
24	295913	104	991346	4.3	304567	109	695433	19766	98027	36
25	296539	104	991321	4.3	305218	108	694782	19794	98021	35
26	297164	104	991295	4.3	305869	108	694131	19823	98016	34
27	297788	104	991270	4.3	306519	108	693481	19851	98010	33
28	298412	104	991244	4.3	307168	108	692832	19880	98004	32
29	299034	104	991218	4.3	307815	108	692185	19908	97998	31
30	299655	103	991193	4.3	308463	108	691537	19937	97992	30
31	9.300276	103	9.991167	4.3	9.309109	107	10.690891	19965	97987	29
32	300895	103	991141	4.3	309754	107	690246	19994	97981	28
33	301514	103	991115	4.3	310398	107	689602	20022	97975	27
34	302132	103	991090	4.3	311042	107	688958	20051	97969	26
35	302748	103	991064	4.3	311685	107	688315	20079	97963	25
36	303364	102	991038	4.3	312327	107	687673	20108	97958	24
37	303979	102	991012	4.3	312967	107	687033	20136	97952	23
38	304593	102	990986	4.3	313608	106	686392	20165	97946	22
39	305207	102	990960	4.3	314247	106	685753	20193	97940	21
40	305819	102	990934	4.4	314885	106	685115	20222	97934	20
41	9.306430	102	9.990908	4.4	9.315523	106	10.684477	20250	97928	19
42	307041	102	990882	4.4	316159	106	683841	20279	97922	18
43	307650	101	990855	4.4	316795	106	683205	20307	97916	17
44	308259	101	990829	4.4	317430	106	682570	20336	97910	16
45	308867	101	990803	4.4	318064	105	681936	20364	97905	15
46	309474	101	990777	4.4	318697	105	681303	20393	97899	14
47	310080	101	990750	4.4	319329	105	680671	20421	97893	13
48	310685	101	990724	4.4	319961	105	680039	20450	97887	12
49	311289	100	990697	4.4	320592	105	679408	20478	97881	11
50	311893	100	990671	4.4	321222	105	678778	20507	97875	10
51	9.312495	100	9.990644	4.4	9.321851	105	10.678149	20535	97869	9
52	313097	100	990618	4.4	322479	104	677521	20563	97863	8
53	313698	100	990591	4.4	323106	104	676894	20592	97857	7
54	314297	100	990565	4.4	323733	104	676267	20620	97851	6
55	314897	100	990538	4.4	324358	104	675642	20649	97845	5
56	315495	100	990511	4.5	324983	104	675017	20677	97839	4
57	316092	99	990485	4.5	325607	104	674393	20706	97833	3
58	316689	99	990458	4.5	326231	104	673769	20734	97827	2
59	317284	99	990431	4.5	326853	104	673147	20763	97821	1
60	317879		990404		327475		672525	20791	97815	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

78 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.317879	99.0	9.990404	4.5	9.327474	103	10.672526	20791	97815	60
1	318473	98.8	990378	4.5	328005	103	671905	20820	97809	59
2	319066	98.7	990351	4.5	328715	103	671285	20848	97803	58
3	319658	98.6	990324	4.5	329334	103	670666	20877	97797	57
4	320249	98.4	990297	4.5	329953	103	670047	20905	97791	56
5	320840	98.3	990270	4.5	330570	103	669430	20933	97784	55
6	321430	98.2	990243	4.5	331187	103	668813	20962	97778	54
7	322019	98.0	990215	4.5	331803	102	668197	20990	97772	53
8	322607	97.9	990188	4.5	332418	102	667582	21019	97766	52
9	323194	97.7	990161	4.5	333033	102	666967	21047	97760	51
10	323780	97.6	990134	4.5	333646	102	666354	21076	97754	50
11	9.324366	97.5	9.990107	4.6	9.334259	102	10.665741	21104	97748	49
12	324950	97.3	990079	4.6	334871	102	665129	21132	97742	48
13	325534	97.2	990052	4.6	335482	102	664518	21161	97735	47
14	326117	97.0	990025	4.6	336093	102	663907	21189	97729	46
15	326700	96.9	989997	4.6	336702	101	663298	21218	97723	45
16	327281	96.8	989970	4.6	337311	101	662689	21246	97717	44
17	327862	96.6	989942	4.6	337919	101	662081	21275	97711	43
18	328442	96.5	989915	4.6	338527	101	661473	21303	97705	42
19	329021	96.4	989887	4.6	339133	101	660867	21331	97698	41
20	329599	96.2	989860	4.6	339739	101	660261	21360	97692	40
21	9.330176	96.1	9.989832	4.6	9.340344	101	10.659656	21388	97686	39
22	330753	96.0	989804	4.6	340948	101	659052	21417	97680	38
23	331329	95.8	989777	4.6	341552	100	658448	21445	97673	37
24	331903	95.7	989749	4.7	342155	100	657845	21474	97667	36
25	332478	95.6	989721	4.7	342757	100	657243	21502	97661	35
26	333051	95.4	989693	4.7	343358	100	656642	21530	97655	34
27	333624	95.3	989665	4.7	343958	100	656042	21559	97648	33
28	334195	95.2	989637	4.7	344558	100	655442	21587	97642	32
29	334766	95.0	989609	4.7	345157	100	654843	21616	97636	31
30	335337	94.9	989582	4.7	345755	100	654245	21644	97630	30
31	9.335906	94.8	9.989553	4.7	9.346353	99.4	10.653647	21672	97623	29
32	336475	94.6	989525	4.7	346949	99.3	653051	21701	97617	28
33	337043	94.5	989497	4.7	347545	99.2	652455	21729	97611	27
34	337610	94.4	989469	4.7	348141	99.1	651859	21758	97604	26
35	338176	94.3	989441	4.7	348735	99.0	651265	21786	97598	25
36	338742	94.1	989413	4.7	349329	98.8	650671	21814	97592	24
37	339306	94.0	989384	4.7	349922	98.7	650078	21843	97585	23
38	339871	93.9	989356	4.7	350514	98.6	649486	21871	97579	22
39	340434	93.7	989328	4.7	351106	98.5	648894	21899	97573	21
40	340996	93.6	989300	4.7	351697	98.3	648303	21928	97566	20
41	9.341558	93.5	9.989271	4.7	9.352287	98.2	10.647713	21956	97560	19
42	342119	93.4	989243	4.7	352876	98.1	647124	21985	97553	18
43	342679	93.2	989214	4.7	353465	98.0	646535	22013	97547	17
44	343239	93.1	989186	4.7	354053	97.9	645947	22041	97541	16
45	343797	93.0	989157	4.7	354640	97.7	645360	22070	97534	15
46	344355	92.9	989128	4.8	355227	97.6	644773	22098	97528	14
47	344912	92.7	989100	4.8	355813	97.5	644187	22126	97521	13
48	345469	92.6	989071	4.8	356398	97.4	643602	22155	97515	12
49	346024	92.5	989042	4.8	356982	97.3	643018	22183	97508	11
50	346579	92.4	989014	4.8	357566	97.1	642434	22212	97502	10
51	9.347134	92.2	9.988985	4.8	9.358149	97.0	10.641851	22240	97496	9
52	347687	92.1	988956	4.8	358731	96.9	641269	22268	97489	8
53	348240	92.0	988927	4.8	359313	96.8	640687	22297	97483	7
54	348792	91.9	988898	4.8	359893	96.7	640107	22325	97476	6
55	349343	91.7	988869	4.8	360474	96.6	639526	22353	97470	5
56	349893	91.6	988840	4.8	361053	96.5	638947	22382	97463	4
57	350443	91.5	988811	4.9	361632	96.3	638368	22410	97457	3
58	350992	91.4	988782	4.9	362210	96.2	637790	22438	97450	2
59	351540	91.3	988753	4.9	362787	96.1	637213	22467	97444	1
60	352088		988724		363364		636636	22495	97437	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

77 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.352088	91.1	9.988724	4.9	9.363364	96.0	10.636636	22495	97437	60
1	352635	91.0	988695	4.9	363940	95.9	636060	22523	97430	59
2	353181	90.9	988666	4.9	364515	95.8	635485	22552	97424	58
3	353726	90.8	988636	4.9	365090	95.7	634910	22580	97417	57
4	354271	90.7	988607	4.9	365664	95.5	634336	22608	97411	56
5	354815	90.5	988578	4.9	366237	95.4	633763	22637	97404	55
6	355358	90.4	988548	4.9	366810	95.3	633190	22665	97398	54
7	355901	90.3	988519	4.9	367382	95.2	632618	22693	97391	53
8	356443	90.2	988489	4.9	367953	95.1	632047	22722	97384	52
9	356984	90.1	988460	4.9	368524	95.0	631476	22750	97378	51
10	357524	89.9	988430	4.9	369094	94.9	630906	22778	97371	50
11	9.358064	89.8	9.988401	4.9	9.369663	94.8	10.630337	22807	97365	49
12	358603	89.7	988371	4.9	370232	94.6	629768	22835	97358	48
13	359141	89.6	988342	4.9	370799	94.5	629201	22863	97351	47
14	359678	89.5	988312	5.0	371367	94.4	628633	22892	97345	46
15	360215	89.3	988282	5.0	371933	94.3	628067	22920	97338	45
16	360752	89.2	988252	5.0	372499	94.2	627501	22948	97331	44
17	361287	89.1	988223	5.0	373064	94.1	626936	22977	97325	43
18	361822	89.0	988193	5.0	373629	94.0	626371	23005	97318	42
19	362356	88.9	988163	5.0	374193	93.9	625807	23033	97311	41
20	362889	88.8	988133	5.0	374756	93.8	625244	23062	97304	40
21	9.363422	88.7	9.988103	5.0	9.375319	93.7	10.624681	23090	97298	39
22	363954	88.5	988073	5.0	375881	93.5	624119	23118	97291	38
23	364485	88.4	988043	5.0	376442	93.4	623558	23146	97284	37
24	365016	88.3	988013	5.0	377003	93.3	622997	23175	97278	36
25	365546	88.2	987983	5.0	377563	93.2	622437	23203	97271	35
26	366075	88.1	987953	5.0	378122	93.1	621878	23231	97264	34
27	366604	88.0	987922	5.0	378681	93.0	621319	23260	97257	33
28	367131	87.9	987892	5.0	379239	92.9	620761	23288	97251	32
29	367659	87.7	987862	5.0	379797	92.8	620203	23316	97244	31
30	368185	87.6	987832	5.1	380354	92.7	619646	23345	97237	30
31	9.368711	87.5	9.987801	5.1	9.380910	92.6	10.619090	23373	97230	29
32	369236	87.4	987771	5.1	381466	92.5	618534	23401	97223	28
33	369761	87.3	987740	5.1	382020	92.4	617980	23429	97217	27
34	370285	87.2	987710	5.1	382575	92.3	617425	23458	97210	26
35	370808	87.1	987679	5.1	383129	92.2	616871	23486	97203	25
36	371330	87.0	987649	5.1	383682	93.1	616318	23514	97196	24
37	371852	86.9	987618	5.1	384234	92.0	615766	23542	97189	23
38	372373	86.7	987588	5.1	384786	91.9	615214	23571	97182	22
39	372894	86.6	987557	5.1	385337	91.8	614663	23599	97176	21
40	373414	86.5	987526	5.1	385888	91.7	614112	23627	97169	20
41	9.373933	86.4	9.987496	5.1	9.386438	91.5	10.613562	23656	97162	19
42	374452	86.3	987465	5.1	386987	91.4	613013	23684	97155	18
43	374970	86.2	987434	5.1	387536	91.3	612464	23712	97148	17
44	375487	86.1	987403	5.2	388084	91.2	611916	23740	97141	16
45	376003	86.0	987372	5.2	388631	91.1	611369	23769	97134	15
46	376519	85.9	987341	5.2	389178	91.0	610822	23797	97127	14
47	377035	85.8	987310	5.2	389724	90.9	610276	23825	97120	13
48	377549	85.7	987279	5.2	390270	90.8	609730	23853	97113	12
49	378063	85.6	987248	5.2	390815	90.7	609185	23882	97106	11
50	378577	85.4	987217	5.2	391360	90.6	608640	23910	97100	10
51	9.379089	85.3	9.987186	5.2	9.391903	90.5	10.608097	23938	97093	9
52	379601	85.2	987155	5.2	392447	90.4	607553	23966	97086	8
53	380113	85.1	987124	5.2	392989	90.3	607011	23995	97079	7
54	380624	85.0	987092	5.2	393531	90.2	606469	24023	97072	6
55	381134	84.9	987061	5.2	394073	90.1	605927	24051	97065	5
56	381643	84.8	987030	5.2	394614	90.0	605386	24079	97058	4
57	382152	84.7	986998	5.2	395154	89.9	604846	24108	97051	3
58	382661	84.6	986967	5.2	395694	89.8	604306	24136	97044	2
59	383168	84.5	986936	5.2	396233	89.7	603767	24164	97037	1
60	383675		986904		396771		603229	24192	97030	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.383675	84.4	9.986904	5.2	9.396771	89.6	10.603229	24192	97030	60
1	384182	84.3	986873	5.3	397309	89.6	602691	24220	97023	59
2	384687	84.2	986841	5.3	397846	89.5	602154	24249	97015	58
3	385192	84.1	986809	5.3	398383	89.4	601617	24277	97008	57
4	385697	84.0	986778	5.3	398919	89.3	601081	24305	97001	56
5	386201	83.9	986746	5.3	399455	89.2	600545	24333	96994	55
6	386704	83.8	986714	5.3	399990	89.1	600010	24362	96987	54
7	387207	83.7	986683	5.3	400524	89.0	599476	24390	96980	53
8	387709	83.6	986651	5.3	401058	88.9	598942	24418	96973	52
9	388210	83.5	986619	5.3	401591	88.8	598409	24446	96966	51
10	388711	83.4	986587	5.3	402124	88.7	597876	24474	96959	50
11	9.389211	83.3	9.986555	5.3	9.402656	88.6	10.597344	24503	96952	49
12	389711	83.2	986523	5.3	403187	88.5	596813	24531	96945	48
13	390210	83.1	986491	5.3	403718	88.4	596282	24559	96937	47
14	390708	83.0	986459	5.3	404249	88.3	595751	24587	96930	46
15	391206	82.8	986427	5.3	404778	88.2	595222	24615	96923	45
16	391703	82.7	986395	5.3	405308	88.1	594692	24644	96916	44
17	392199	82.6	986363	5.4	405836	88.0	594164	24672	96909	43
18	392695	82.5	986331	5.4	406364	87.9	593636	24700	96902	42
19	393191	82.4	986299	5.4	406892	87.8	593108	24728	96894	41
20	393685	82.3	986266	5.4	407419	87.7	592581	24756	96887	40
21	9.394179	82.2	9.986234	5.4	9.407945	87.6	10.592055	24784	96880	39
22	394673	82.1	986202	5.4	408471	87.5	591529	24813	96873	38
23	395166	82.0	986169	5.4	408997	87.4	591003	24841	96866	37
24	395658	81.9	986137	5.4	409521	87.4	590479	24869	96858	36
25	396150	81.8	986104	5.4	410045	87.3	589955	24897	96851	35
26	396641	81.7	986072	5.4	410569	87.2	589431	24925	96844	34
27	397132	81.7	986039	5.4	411092	87.1	588908	24954	96837	33
28	397621	81.6	986007	5.4	411615	87.0	588385	24982	96829	32
29	398111	81.5	985974	5.4	412137	86.9	587863	25010	96822	31
30	398600	81.4	985942	5.4	412658	86.8	587342	25038	96815	30
31	9.399088	81.3	9.985909	5.5	9.413179	86.7	10.586821	25066	96807	29
32	399575	81.2	985876	5.5	413699	86.6	586301	25094	96800	28
33	400062	81.1	985843	5.5	414219	86.5	585781	25122	96793	27
34	400549	81.0	985811	5.5	414738	86.4	585262	25151	96786	26
35	401035	80.9	985778	5.5	415257	86.4	584743	25179	96778	25
36	401520	80.8	985745	5.5	415775	86.3	584225	25207	96771	24
37	402005	80.7	985712	5.5	416293	86.2	583707	25235	96764	23
38	402489	80.6	985679	5.5	416810	86.1	583190	25263	96756	22
39	402972	80.5	985646	5.5	417326	86.0	582674	25291	96749	21
40	403455	80.4	985613	5.5	417842	85.9	582158	25320	96742	20
41	9.403938	80.3	9.985580	5.5	9.418358	85.8	10.581642	25348	96734	19
42	404420	80.2	985547	5.5	418873	85.7	581127	25376	96727	18
43	404901	80.1	985514	5.5	419387	85.6	580613	25404	96719	17
44	405382	80.0	985480	5.5	419901	85.5	580099	25432	96712	16
45	405862	79.9	985447	5.5	420415	85.5	579585	25460	96705	15
46	406341	79.8	985414	5.6	420927	85.4	579073	25488	96697	14
47	406820	79.7	985380	5.6	421440	85.3	578560	25516	96690	13
48	407299	79.6	985347	5.6	421952	85.2	578048	25545	96682	12
49	407777	79.5	985314	5.6	422463	85.1	577537	25573	96675	11
50	408254	79.4	985280	5.6	422974	85.0	577026	25601	96667	10
51	9.408731	79.4	9.985247	5.6	9.423484	84.9	10.576516	25629	96660	9
52	409207	79.3	985213	5.6	423993	84.8	576007	25657	96653	8
53	409682	79.2	985180	5.6	424503	84.8	575497	25685	96645	7
54	410157	79.1	985146	5.6	425011	84.7	574989	25713	96638	6
55	410632	79.0	985113	5.6	425519	84.6	574481	25741	96630	5
56	411106	78.9	985079	5.6	426027	84.5	573973	25766	96623	4
57	411579	78.8	985045	5.6	426534	84.4	573466	25798	96615	3
58	412052	78.7	985011	5.6	427041	84.3	572959	25826	96608	2
59	412524	78.6	984978	5.6	427547	84.3	572453	25854	96600	1
60	412996		984944		428052		571948	25882	96593	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

75 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine	N. cos	
0	9.412993	78.5	9.984944	5.7	9.428052	84.2	10.571948	25882	96593	60
1	413467	78.4	984910	5.7	428557	84.1	571443	25910	96585	59
2	413938	78.3	984876	5.7	429062	84.0	570938	25938	96578	58
3	414408	78.3	984842	5.7	429566	83.9	570434	25966	96570	57
4	414878	78.2	984808	5.7	430070	83.8	569930	25994	96562	56
5	415347	78.1	984774	5.7	430573	83.8	569427	26022	96555	55
6	415815	78.0	984740	5.7	431075	83.7	568925	26050	96547	54
7	416283	77.9	984706	5.7	431577	83.6	568423	26079	96540	53
8	416751	77.8	984672	5.7	432079	83.5	567921	26107	96532	52
9	417217	77.7	984637	5.7	432580	83.4	567420	26135	96524	51
10	417684	77.6	984603	5.7	433080	83.3	566920	26163	96517	50
11	9.418150	77.5	9.984569	5.7	9.433580	83.2	10.566420	26191	96509	49
12	418615	77.4	984535	5.7	434080	83.2	565920	26219	96502	48
13	419079	77.3	984500	5.7	434579	83.1	565421	26247	96494	47
14	419544	77.3	984466	5.7	435078	83.0	564922	26275	96486	46
15	420007	77.2	984432	5.8	435576	82.9	564424	26303	96479	45
16	420470	77.1	984397	5.8	436073	82.8	563927	26331	96471	44
17	420933	77.0	984363	5.8	436570	82.8	563430	26359	96463	43
18	421395	76.9	984328	5.8	437067	82.7	562933	26387	96456	42
19	421857	76.8	984294	5.8	437563	82.6	562437	26415	96448	41
20	422318	76.7	984259	5.8	438059	82.5	561941	26443	96440	40
21	9.422778	76.7	9.984224	5.8	9.438554	82.4	10.561446	26471	96433	39
22	423238	76.6	984190	5.8	439048	82.3	560952	26500	96425	38
23	423697	76.5	984155	5.8	439543	82.3	560457	26528	96417	37
24	424156	76.4	984120	5.8	440036	82.2	559964	26556	96410	36
25	424615	76.3	984085	5.8	440529	82.1	559471	26584	96402	35
26	425073	76.2	984050	5.8	441022	82.0	558978	26612	96394	34
27	425530	76.1	984015	5.8	441514	81.9	558486	26640	96386	33
28	425987	76.0	983981	5.8	442006	81.9	557994	26668	96379	32
29	426443	76.0	983946	5.8	442497	81.8	557503	26696	96371	31
30	426899	75.9	983911	5.8	442988	81.7	557012	26724	96363	30
31	9.427354	75.8	9.983875	5.8	9.443479	81.6	10.556521	26752	96355	29
32	427809	75.7	983840	5.9	443968	81.6	556032	26780	96347	28
33	428263	75.6	983805	5.9	444458	81.5	555542	26808	96340	27
34	428717	75.5	983770	5.9	444947	81.4	555053	26836	96332	26
35	429170	75.4	983735	5.9	445435	81.3	554565	26864	96324	25
36	429623	75.3	983700	5.9	445923	81.2	554077	26892	96316	24
37	430075	75.2	983664	5.9	446411	81.2	553589	26920	96308	23
38	430527	75.2	983629	5.9	446898	81.1	553102	26948	96301	22
39	430978	75.1	983594	5.9	447384	81.0	552616	26976	96293	21
40	431429	75.0	983558	5.9	447870	80.9	552130	27004	96285	20
41	9.431879	74.9	9.983523	5.9	9.448356	80.9	10.551644	27032	96277	19
42	432329	74.9	983487	5.9	448841	80.8	551159	27060	96269	18
43	432778	74.8	983452	5.9	449326	80.7	550674	27088	96261	17
44	433226	74.7	983416	5.9	449810	80.6	550190	27116	96253	16
45	433675	74 6	983381	5.9	450294	80.6	549706	27144	96246	15
46	434122	74.5	983345	5.9	450777	80.5	549223	27172	96238	14
47	434569	74.4	983309	5.9	451260	80.4	548740	27200	96230	13
48	435016	74.4	983273	6.0	451743	80.3	548257	27228	96222	12
49	435462	74.3	983238	6.0	452225	80.2	547775	27256	96214	11
50	435908	74.2	983202	6.0	452706	80.2	547294	27284	96206	10
51	9.436353	74.1	9.983166	6.0	9.453187	80.1	10.546813	27312	96198	9
52	436798	74.0	983130	6.0	453668	80.0	546332	27340	96190	8
53	437242	74.0	983094	6.0	454148	79.9	545852	27368	96182	7
54	437686	73.9	983058	6.0	454628	79.9	545372	27396	96174	6
55	438129	73.8	983022	6.0	455107	79.8	544893	27424	96166	5
56	438572	73.7	982986	6.0	455586	79.7	544414	27452	96158	4
57	439014	73.6	982950	6.0	456064	79.6	543936	27480	96150	3
58	439456	73.6	982914	6.0	456542	79.6	543458	27508	96142	2
59	439897	73.5	982878	6.0	457019	79.5	542981	27536	96134	1
60	440338		982842		457496		542504	27564	96126	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

74 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.440338	73.4	9.982842	6.0	9.457496	79.4	10.542504	27564	96126	60
1	440778	73.3	982805	6.0	457973	79.3	542027	27592	96118	59
2	441218	73.2	982769	6.1	458449	79.3	541551	27620	96110	58
3	441658	73.1	982733	6.1	458925	79 2	541075	27648	96102	57
4	442096	73.1	982696	6.1	459400	79.1	540600	27676	96094	56
5	442535	73.0	982660	6.1	459875	79.0	540125	27704	96086	55
6	442973	72.9	982624	6.1	460349	79.0	539651	27731	96078	54
7	443410	72.8	982587	6.1	460823	78.9	539177	27759	96070	53
8	443847	72.7	982551	6.1	461297	78.8	538703	27787	96062	52
9	444284	72.7	982514	6.1	461770	78.9	538230	27815	96054	51
10	444720	72.6	982477	6.1	462242	78.7	537758	27843	96046	50
11	9.445155	72.5	9.982441	6.1	9.462714	78.6	10.537286	27871	96037	49
12	445590	72.4	982404	6.1	463186	78.5	536814	27899	96029	48
13	446025	72.3	982367	6.1	463658	78.5	536342	27927	96021	47
14	446459	72.3	982331	6.1	464129	78.4	535871	27955	96013	46
15	446893	72.2	982294	6.1	464599	78.3	535401	27983	96005	45
16	447326	72.1	982257	6.1	465069	78.3	534931	28011	95997	44
17	447759	72.0	982220	6.2	465539	78.2	534461	28039	95989	43
18	448191	72.0	982183	6.2	466008	78.1	533992	28067	95981	42
19	448623	71.9	982146	6.2	466476	78.0	533524	28095	95972	41
20	449054	71.8	982109	6.2	466945	78.0	533055	28123	95964	40
21	9.449485	71.7	9.982072	6.2	9.467413	77.9	10.532587	28150	95956	39
22	449915	71.6	982035	6.2	467880	77.8	532120	28178	95948	38
23	450345	71.6	981998	6.2	468347	77.8	531653	28206	95940	37
24	450775	71.5	981961	6.2	468814	77.7	531186	28234	95931	36
25	451204	71.4	981924	6.2	469280	77.6	530720	28262	95923	35
26	451632	71.3	981886	6.2	469746	77.5	530254	28290	95915	34
27	452060	71.3	981849	6.2	470211	77.5	529789	28318	95907	33
28	452488	71.2	981812	6.2	470676	77.4	529324	28346	95898	32
29	452915	71.1	981774	6.2	471141	77.3	528859	28374	95890	31
30	453342	71.0	981737	6.2	471605	77.3	528395	28402	95882	30
31	9.453768	71.0	9.981699	6.3	9.472068	77.2	10.527932	28429	95874	29
32	454194	70.9	981662	6.3	472532	77.1	527468	28457	95865	28
33	454619	70.8	981625	6.3	472995	77.1	527005	28485	95857	27
34	455044	70.7	981587	6.3	473457	77.0	526543	28513	95849	26
35	455469	70.7	981549	6.3	473919	76.9	526081	28541	95841	25
36	455893	70.6	981512	6.3	474381	76.9	525619	28569	95832	24
37	456316	70.5	981474	6.3	474842	76.8	525158	28597	95824	23
38	456739	70.4	981436	6.3	475303	76.7	524697	28625	95816	22
39	457162	70.4	981399	6.3	475763	76.7	524237	28652	95807	21
40	457584	70.3	981361	6.3	476223	76.6	523777	28680	95799	20
41	9.458006	70.2	9.981323	6.3	9.476683	76.5	10.523317	28708	95791	19
42	458427	70.1	981285	6.3	477142	76.5	522858	28736	95782	18
43	458848	70.1	981247	6.3	477601	76.4	522399	28764	95774	17
44	459268	70.0	981209	6.3	478059	76.3	521941	28792	95766	16
45	459688	69.9	981171	6.3	478517	76.3	521483	28820	95757	15
46	460108	69.8	981133	6.4	478975	76.2	521025	28847	95749	14
47	460527	69.8	981095	6.4	479432	76.1	520568	28875	95740	13
48	460946	69.7	981057	6.4	479889	76.1	520111	28903	95732	12
49	461364	69.6	981019	6.4	480345	76.0	519655	28931	95724	11
50	461782	69.5	980981	6.4	480801	75.9	519199	28959	95715	10
51	9.462199	69.5	9.980942	6.4	9.481257	75.9	10.518743	28987	95707	9
52	462616	69.4	980904	6.4	481712	75.8	518288	29015	95698	8
53	463032	69.3	980866	6.4	482167	75.7	517833	29042	95690	7
54	463448	69.3	980827	6.4	482621	75.7	517379	29070	95681	6
55	463864	69.2	980789	6.4	483075	75.6	516925	29098	95673	5
56	464279	69.1	980750	6.4	483529	75.5	516471	29126	95664	4
57	464694	69.0	980712	6.4	483982	75.5	516018	29154	95656	3
58	465108	69.0	980673	6.4	484435	75.4	515565	29182	95647	2
59	465522	68.9	980635	6.4	484887	75.3	515113	29209	95639	1
60	465935		980596		485339		514661	29247	95630	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

73 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.465935	68.8	9.980596	6.4	9.485339	75.3	10.514661	29237	95630	60
1	466348	68.8	980558	6.4	485791	75.2	514209	29265	95622	59
2	466761	68.7	980519	6.5	486242	75.1	513758	29293	95613	58
3	467173	68.6	980480	6.5	486693	75.1	513307	29321	95605	57
4	467585	68.5	980442	6.5	487143	75.0	512857	29348	95596	56
5	467996	68.5	980403	6.5	487593	74.9	512407	29376	95588	55
6	468407	68.4	980364	6.5	488043	74.9	511957	29404	95579	54
7	468817	68.3	980325	6.5	488492	74.8	511508	29432	95571	53
8	469227	68.3	980286	6.5	488941	74.7	511059	29460	95562	52
9	469637	68.2	980247	6.5	489390	74.7	510610	29487	95554	51
10	470046	68.1	980208	6.5	489838	74.6	510162	29515	95545	50
11	9.470455	68.0	9.980169	6.5	9.490286	74.6	10.509714	29543	95536	49
12	470863	68.0	980130	6.5	490733	74.5	509267	29571	95528	48
13	471271	67.9	980091	6.5	491180	74.4	508820	29599	95519	47
14	471679	67.8	980052	6.5	491627	74.4	508373	29626	95511	46
15	472086	67.8	980012	6.5	492073	74.3	507927	29654	95502	45
16	472492	67.7	979973	6.5	492519	74.3	507481	29682	95493	44
17	472898	67.6	979934	6.6	492965	74.2	507035	29710	95485	43
18	473304	67.6	979895	6.6	493410	74.1	506590	29737	95476	42
19	473710	67.5	979855	6.6	493854	74.0	506146	29765	95467	41
20	474115	67.4	979816	6.6	494299	74.0	505701	29793	95459	40
21	9.474519	67.4	9.979776	6.6	9.494743	74.0	10.505257	29821	95450	39
22	474923	67.3	979737	6.6	495186	73.9	504814	29849	95441	38
23	475327	67.2	979697	6.6	495630	73.8	504370	29876	95433	37
24	475730	67.2	979658	6.6	496073	73.7	503927	29904	95424	36
25	476133	67.1	979618	6.6	496515	73.7	503485	29932	95415	35
26	476536	67.0	979579	6.6	496957	73.6	503043	29960	95407	34
27	476938	66.9	979539	6.6	497399	73.6	502601	2998[illegible]	95398	33
28	477340	66.9	979499	6.6	497841	73.5	502159	30015	95389	32
29	477741	66.8	979459	6.6	468282	73.4	501718	30043	95380	31
30	478142	66.7	979420	6.6	498722	73.4	501278	30071	95372	30
31	9.478542	66.7	9.979380	6.6	9.499163	73.3	10.500837	30098	95363	29
32	478942	66.6	979340	6.6	499603	73.3	500397	30126	95354	28
33	479342	66.5	979300	6.7	500042	73.2	499958	30154	95345	27
34	479741	66.5	979260	6.7	500481	73.1	499519	30182	95337	26
35	480140	66.4	979220	6.7	500920	73.1	499080	30209	95328	25
36	480539	66.3	979180	6.7	501359	73.0	498641	30237	95319	24
37	480937	66.3	979140	6.7	501797	73.0	498203	30265	95310	23
38	481334	66.2	979100	6.7	502235	72.9	497765	30292	95301	22
39	481731	66.1	979059	6.7	502672	72.8	497328	30320	95293	21
40	482128	66.1	979019	6.7	503109	72.8	496891	30348	95284	20
41	9.482525	66.0	9.978979	6.7	9.503546	72.7	10.496454	30376	95275	19
42	482921	65.9	978939	6.7	503982	72.7	496018	30403	95266	18
43	483316	65.9	978898	6.7	504418	72.6	495582	30431	95257	17
44	483712	65.8	978858	6.7	504854	72.5	495146	30459	95248	16
45	484107	65.7	978817	6.7	505289	72.5	494711	30486	95240	15
46	484501	65.7	978777	6.7	505724	72.4	494276	30514	95231	14
47	484895	65.6	978736	6.7	506159	72.4	493841	30542	95222	13
48	485289	65.5	978696	6.8	506593	72.3	493407	30570	95213	12
49	485682	65.5	978655	6.8	507027	72.2	492973	30597	95204	11
50	486075	65.4	978615	6.8	507460	72.2	492540	30625	95195	10
51	9.486467	66.3	9.978574	6.8	9.507893	72.1	10.492107	30653	95186	9
52	486860	65.3	978533	6.8	508326	72.1	491674	30680	95177	8
53	487251	65.2	978493	6.8	508759	72.0	491241	30708	95168	7
54	487643	65.1	978452	6.8	509191	71.9	490809	30736	95159	6
55	488034	65.1	978411	6.8	509622	71.9	490378	30763	95150	5
56	488424	65.0	978370	6.8	510054	71.8	489946	30791	95142	4
57	488814	65.0	978329	6.8	510485	71.8	489515	30819	95133	3
58	489204	64.9	978288	6.8	510916	71.7	489084	30846	95124	2
59	489593	64.8	978247	6.8	511346	71.6	488654	30874	95115	1
60	489982		978206		511776		488224	30902	95106	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.489982	64.8	9.978206	6.8	9.511776	71.6	10.488224	30902	95106	60
1	490371	64.8	978165	6.8	512206	71.6	487794	30929	95097	59
2	490759	64.7	978124	6.8	512635	71.5	487365	30957	95088	58
3	491147	64.6	978083	6.9	513064	71.4	486936	30985	95079	57
4	491535	64.6	978042	6.9	513493	71.4	486507	31012	95070	56
5	491922	64.5	978001	6.9	513921	71.3	486079	31040	95061	55
6	492308	64.4	977959	6.9	514349	71.3	485651	31068	95052	54
7	492695	64.4	977918	6.9	514777	71.2	485223	31095	95043	53
8	493081	64.3	977877	6.9	515204	71.2	484796	31123	95033	52
9	493466	64.2	977835	6.9	515631	71.1	484369	31151	95024	51
10	493851	64.2	977794	6.9	516057	71.0	483943	31178	95015	50
11	9.494236	64.1	9.977752	6.9	9.516484	71.0	10.483516	31206	95006	49
12	494621	64.1	977711	6.9	516910	70.9	483090	31233	94997	48
13	495005	64.0	977669	6.9	517335	70.9	482665	31261	94988	47
14	495388	63.9	977628	6.9	517761	70.8	482239	31289	94979	46
15	495772	63.9	977586	6.9	518185	70.8	481815	31316	94970	45
16	496154	63.8	977544	7.0	518610	70.7	481390	31344	94961	44
17	496537	63.7	977503	7.0	519034	70.6	480966	31372	94952	43
18	496919	63.7	977461	7.0	519458	70.6	480542	31399	94943	42
19	497301	63.6	977419	7.0	519882	70.5	480118	31427	94933	41
20	497682	63.6	977377	7.0	520305	70.5	479695	31454	94924	40
21	9.498064	63.5	9.977335	7.0	9.520728	70.4	10.479272	31482	94915	39
22	498444	63.4	977293	7.0	521151	70.3	478849	31510	94906	38
23	498825	63.4	977251	7.0	521573	70.3	478427	31537	94897	37
24	499204	63.3	977209	7.0	521995	70.3	478005	31565	94888	36
25	499584	63.2	977167	7.0	522417	70.2	477583	31593	94878	35
26	499963	63.2	977125	7.0	522838	70.2	477162	31620	94869	34
27	500342	63.1	977083	7.0	523259	70.1	476741	31648	94860	33
28	500721	63.1	977041	7.0	523680	70.1	476320	31675	94851	32
29	501099	63.0	976999	7.0	524100	70.0	475900	31703	94842	31
30	501476	62.9	976957	7.0	524520	69.9	475480	31730	94832	30
31	9.501854	62.9	9.976914	7.0	9.524939	69.9	10.475061	31758	94823	29
32	502231	62.8	976872	7.1	525359	69.8	474641	31786	94814	28
33	502607	62.8	976830	7.1	525778	69.8	474222	31813	94805	27
34	502984	62.7	976787	7.1	526197	69.7	473803	31841	94795	26
35	503360	62.6	976745	7.1	526615	69.7	473385	31868	94786	25
36	503735	62.6	976702	7.1	527033	69.6	472967	31896	94777	24
37	504110	62.5	976660	7.1	527451	69.6	472549	31923	94768	23
38	504485	62.5	976617	7.1	527868	69.5	472132	31951	94758	22
39	504860	62.4	976574	7.1	528285	69.5	471715	31979	94749	21
40	505234	62.3	976532	7.1	528702	69.4	471298	32006	94740	20
41	9.505608	62.3	9.976489	7.1	9.529119	69.3	10.470881	32034	94730	19
42	505981	62.2	976446	7.1	529535	69.3	470465	32061	94721	18
43	506354	62.2	976404	7.1	529950	69.3	470050	32089	94712	17
44	506727	62.1	976361	7.1	530366	69.2	469634	32116	94702	16
45	507099	62.0	976318	7.1	530781	69.1	469219	32144	94693	15
46	507471	62.0	976275	7.1	531196	69.1	468804	32171	94684	14
47	507843	61.9	976232	7.2	531611	69.0	468389	32199	94674	13
48	508214	61.9	976189	7.2	532025	69.0	467975	32227	94665	12
49	508585	61.8	976146	7.2	532439	68.9	467561	32250	94656	11
50	508956	61.8	976103	7.2	532853	68.9	467147	32282	94646	10
51	9.509326	61.7	9.976060	7.2	9.533266	68.8	10.466734	32309	94637	9
52	509696	61.6	976017	7.2	533679	68.8	466321	32337	94627	8
53	510065	61.6	975974	7.2	534092	68.7	465908	32364	94618	7
54	510434	61.5	975930	7.2	534504	68.7	465496	32392	94609	6
55	510803	61.5	975887	7.2	534916	68.6	465084	32419	94599	5
56	511172	61.4	975844	7.2	535328	68.6	464672	32447	94590	4
57	511540	61.3	975800	7.2	535739	68.5	464261	32474	94580	3
58	511907	61.3	975757	7.2	536150	68.5	463850	32502	94571	2
59	512275	61.2	975714	7.2	536561	68.4	463439	32529	94561	1
60	512642		975670		536972		463028	32557	94552	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.512642	61.2	9.975670	7.3	9.536972	68.4	10.463028	32557	94552	60
1	513009	61.1	975627	7.3	537382	68.3	462618	32584	94542	59
2	513375	61.1	975583	7.3	537792	68.3	462208	32612	94533	58
3	513741	61.0	975539	7.3	538202	68.2	461798	32639	94523	57
4	514107	60.9	975496	7.3	538611	68.2	461389	32667	94514	56
5	514472	60.9	975452	7.3	539020	68.1	460980	32694	94504	55
6	514837	60.8	975408	7.3	539429	68.1	460571	32722	94495	54
7	515202	60.8	975365	7.3	539837	68.0	460163	32749	94485	53
8	515566	60.7	975321	7.3	540245	68.0	459755	32777	94476	52
9	515930	60.7	975277	7.3	540653	67.9	459347	32804	94466	51
10	516294	60.6	975233	7.3	541061	67.9	458939	32832	94457	50
11	9.516657	60.5	9.975189	7.3	9.541468	67.8	10.458532	32859	94447	49
12	517020	60.5	975145	7.3	541875	67.8	458125	32887	94438	48
13	517382	60.4	975101	7.3	542281	67.7	457719	32914	94428	47
14	517745	60.4	975057	7.3	542688	67.7	457312	32942	94418	46
15	518107	60.3	975013	7.3	543094	67.6	456906	32969	94409	45
16	518468	60.3	974969	7.4	543499	67.6	456501	32997	94399	44
17	518829	60.2	974925	7.4	543905	67.5	456095	33024	94390	43
18	519190	60.1	974880	7.4	544310	67.5	455690	33051	94380	42
19	519551	60.1	974836	7.4	544715	67.4	455285	33079	94370	41
20	519911	60.0	974792	7.4	545119	67.4	454881	33106	94361	40
21	9.520271	60.0	9.974748	7.4	9.545524	67.3	10.454476	33134	94351	39
22	520631	59.9	974703	7.4	545928	67.3	454072	33161	94342	38
23	520990	59.9	974659	7.4	546331	67.2	453669	33189	94332	37
24	521349	59.8	974614	7.4	546735	67.2	453265	33216	94322	36
25	521707	59.8	974570	7.4	547138	67.1	452862	33244	94313	35
26	522066	59.7	974525	7.4	547540	67.1	452460	33271	94303	34
27	522424	59.6	974481	7.4	547943	67.0	452057	33298	94293	33
28	522781	59.6	974436	7.4	548345	67.0	451655	33326	94284	32
29	523138	59.5	974391	7.4	548747	66.9	451253	33353	94274	31
30	523495	59.5	974347	7.5	549149	66.9	450851	33381	94264	30
31	9.523852	59.4	9.974302	7.5	9.549550	66.8	10.450450	33408	94254	29
32	524208	59.4	974257	7.5	549951	66.8	450049	33436	94245	28
33	524564	59.3	974212	7.5	550352	66.7	449648	33463	94235	27
34	524920	59.3	974167	7.5	550752	66.7	449248	33490	94225	26
35	525275	59.2	974122	7.5	551152	66.6	448848	33518	94215	25
36	525630	59.1	974077	7.5	551552	66.6	448448	33545	94206	24
37	525984	59.1	974032	7.5	551952	66.5	448048	33573	94196	23
38	526339	59.0	973987	7.5	552351	66.5	447649	33600	94186	22
39	526693	59.0	973942	7.5	552750	66.5	447250	33627	94176	21
40	527046	58.9	973897	7.5	553149	66.4	446851	33655	94167	20
41	9.527400	58.9	9.973852	7.5	9.553548	66.4	10.446452	33682	94157	19
42	527753	58.8	973807	7.5	553946	66.3	446054	33710	94147	18
43	528105	58.8	973761	7.5	554344	66.3	445656	33737	94137	17
44	528458	58.7	973716	7.6	554741	66.2	445259	33764	94127	16
45	528810	58.7	973671	7.6	555139	66.2	444861	33792	94118	15
46	529161	58.6	973625	7.6	555536	66.1	444464	33819	94108	14
47	529513	58.6	973580	7.6	555933	66.1	444067	33846	94098	13
48	529864	58.5	973535	7.6	556329	66.0	443671	33874	94088	12
49	530215	58.5	973489	7.6	556725	66.0	443275	33901	94078	11
50	530565	58.4	973444	7.6	557121	65.9	442879	33929	94068	10
51	9.530915	58.4	9.973398	7.6	9.557517	65.9	10.442483	33956	94058	9
52	531265	58.3	973352	7.6	557913	65.9	442087	33983	94049	8
53	531614	58.2	973307	7.6	558308	65.8	441692	34011	94039	7
54	531963	58.2	973261	7.6	558702	65.8	441298	34038	94029	6
55	532312	58.1	973215	7.6	559097	65.7	440903	34065	94019	5
56	532661	58.1	973169	7.6	559491	65.7	440509	34093	94009	4
57	533009	58.0	973124	7.6	559885	65.6	440115	34120	93999	3
58	533357	58.0	973078	7.6	560279	65.6	439721	34147	93989	2
59	533704	57.9	973032	7.7	560673	65.5	439327	34175	93979	1
60	534052		972986		561066		438934	34202	93969	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10	Cotang.	N. sine.	N. cos.	
0	9.534052	57.8	9.972986	7.7	9.561066	65.5	10.438934	34202	93969	60
1	534399	57.7	972940	7.7	561459	65.4	438541	34229	93959	59
2	534745	57.7	972894	7.7	561851	65.4	438149	34257	93949	58
3	535092	57.7	972848	7.7	562244	65.3	437756	34284	93939	57
4	535438	57.6	972802	7.7	562636	65.3	437364	34311	93929	56
5	535783	57.6	972755	7.7	563028	65.3	436972	34339	93919	55
6	536129	57.5	972709	7.7	563419	65.2	436581	34366	93909	54
7	536474	57.4	972663	7.7	563811	65.2	436189	34393	93899	53
8	536818	57.4	972617	7.7	564202	65.1	435798	34421	93889	52
9	537163	57.3	972570	7.7	564592	65.1	435408	34448	93879	51
10	537507	57.3	972524	7.7	564983	65.0	435017	34475	93869	50
11	9.537851	57.2	9.972478	7.7	9.565373	65.0	10.434627	34503	93859	49
12	538194	57.2	972431	7.8	565763	64.9	434237	34530	93849	48
13	538538	57.1	972385	7.8	566153	64.9	433847	34557	93839	47
14	538880	57.1	972338	7.8	566542	64.9	433458	34584	93829	46
15	539223	57.0	972291	7.8	566932	64.8	433068	34612	93819	45
16	539565	57.0	972245	7.8	567320	64.8	432680	34639	93809	44
17	539907	56.9	972198	7.8	567709	64.7	432291	34666	93799	43
18	540249	56.9	972151	7.8	568098	64.7	431902	34694	93789	42
19	540590	56.8	972105	7.8	568486	64.6	431514	34721	93779	41
20	540931	56.8	972058	7.8	568873	64.6	431127	34748	93769	40
21	9.541272	56.7	9.972011	7.8	9.569261	64.5	10.430739	34775	93759	39
22	541613	56.7	971964	7.8	569648	64.5	430352	34803	93748	38
23	541953	56.6	971917	7.8	570035	64.5	429965	34830	93738	37
24	542293	56.6	971870	7.8	570422	64.4	429578	34857	93728	36
25	542632	56.5	971823	7.8	570809	64.4	429191	34884	93718	35
26	542971	56.5	971776	7.8	571195	64.3	428805	34912	93708	34
27	543310	56.4	971729	7.9	571581	64.3	428419	34939	93698	33
28	543649	56.4	971682	7.9	571967	64.2	428033	34966	93688	32
29	543987	56.3	971635	7.9	572352	64.2	427648	34993	93677	31
30	544325	56.3	971588	7.9	572738	64.2	427262	35021	93667	30
31	9.544663	56.2	9.971540	7.9	9.573123	64.1	10.426877	35048	93657	29
32	545000	56.2	971493	7.9	573507	64.1	426493	35075	93647	28
33	545338	56.1	971446	7.9	573892	64.0	426108	35102	93637	27
34	545674	56.1	971398	7.9	574276	64.0	425724	35130	93626	26
35	546011	56.0	971351	7.9	574660	63.9	425340	35157	93616	25
36	546347	56.0	971303	7.9	575044	63.9	424956	35184	93606	24
37	546683	55.9	971256	7.9	575427	63.9	424573	35211	93596	23
38	547019	55.9	971208	7.9	575810	63.8	424190	35239	93585	22
39	547354	55.8	971161	7.9	576193	63.8	423807	35266	93575	21
40	547689	55.8	971113	7.9	576576	63.7	423424	35293	93565	20
41	9.548024	55.7	9.971066	8.0	9.576958	63.7	10.423041	35320	93555	19
42	548359	55.7	971018	8.0	577341	63.6	422659	35347	93544	18
43	548693	55.6	970970	8.0	577723	63.6	422277	35375	93534	17
44	549027	55.6	970922	8.0	578104	63.6	421896	35402	93524	16
45	549360	55.5	970874	8.0	578486	63.5	421514	35429	93514	15
46	549693	55.5	970827	8.0	578867	63.5	421133	35456	93503	14
47	550026	55.4	970779	8.0	579248	63.4	420752	35484	93493	13
48	550359	55.4	970731	8.0	579629	63.4	420371	35511	93483	12
49	550692	55.3	970683	8.0	580009	63.4	419991	35538	93472	11
50	551024	55.3	970635	8.0	580389	63.3	419611	35565	93462	10
51	9.551356	55.2	9.970586	8.0	9.580769	63.3	10.419231	35592	93452	9
52	551687	55.2	970538	8.0	581149	63.2	418851	35619	93441	8
53	552018	55.2	970490	8.0	581528	63.2	418472	35647	93431	7
54	552349	55.1	970442	8.0	581907	63.2	418093	35674	93420	6
55	552680	55.1	970394	8.0	582286	63.1	417714	35701	93410	5
56	553010	55.0	970345	8.1	582665	63.1	417335	35728	93400	4
57	553341	55.0	970297	8.1	583043	63.0	416957	35755	93389	3
58	553670	54.9	970249	8.1	583422	63.0	416578	35782	93379	2
59	554000	54.9	970200	8.1	583800	62.9	416200	35810	93368	1
60	554329		970152		584177		415823	35837	93358	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.554329	54.8	9.970152	8.1	9.584177	62.9	10.415823	35837	93358	60
1	554658	54.8	970103	8.1	584555	62.9	415445	35864	93348	59
2	554987	54.7	970055	8.1	584932	62.8	415068	35891	93337	58
3	555315	54.7	970006	8.1	585309	62.8	414691	35918	93327	57
4	555643	54.6	969957	8.1	585686	62.7	414314	35945	93316	56
5	555971	54.6	969909	8.1	586062	62.7	413938	35972	93306	55
6	556299	54.5	969860	8.1	586439	62.7	413561	36000	93295	54
7	556626	54.5	969811	8.1	586815	62.6	413185	36027	93285	53
8	556953	54.4	969762	8.1	587190	62.6	412810	36054	93274	52
9	557280	54.4	969714	8.1	587566	62.5	412434	36081	93264	51
10	557606	54.3	969665	8.1	587941	62.5	412059	36108	93253	50
11	9.557932	54.3	9.969616	8.2	9.588316	62.5	10.411684	36135	93243	49
12	558258	54.3	969567	8.2	588691	62.4	411309	36162	93232	48
13	558583	54.2	969518	8.2	589066	62.4	410934	36190	93222	47
14	558909	54.2	969469	8.2	589440	62.3	410560	36217	93211	46
15	559234	54.1	969420	8.2	589814	62.3	410186	36244	93201	45
16	559558	54.1	969370	8.2	590188	62.3	409812	36271	93190	44
17	559883	54.0	969321	8.2	590562	62.2	409438	36298	93180	43
18	560207	54.0	969272	8.2	590935	62.2	409065	36325	93169	42
19	560531	53.9	969223	8.2	591308	62.2	408692	36352	93159	41
20	560855	53.9	969173	8.2	591681	62.1	408319	36379	93148	40
21	9.561178	53.8	9.969124	8.2	9.592054	62.1	10.407946	36406	93137	39
22	561501	53.8	969075	8.2	592426	62.0	407574	36434	93127	38
23	561824	53.7	969025	8.2	592798	62.0	407202	36461	93116	37
24	562146	53.7	968976	8.2	593170	61.9	406829	36488	93106	36
25	562468	53.6	968926	8.3	593542	61.9	406458	36515	93095	35
26	562790	53.6	968877	8.3	593914	61.8	406086	36542	93084	34
27	563112	53.6	968827	8.3	594285	61.8	405715	36569	93074	33
28	563433	53.5	968777	8.3	594656	61.8	405344	36596	93063	32
29	563755	53.5	968728	8.3	595027	61.7	404973	36623	93052	31
30	564075	53.4	968678	8.3	595398	61.7	404602	36650	93042	30
31	9.564396	53.4	9.968628	8.3	9.595768	61.7	10.404232	36677	93031	29
32	564716	53.3	968578	8.3	596138	61.6	403862	36704	93020	28
33	565036	53.3	968528	8.3	596508	61.6	403492	36731	93010	27
34	565356	53.2	968479	8.3	596878	61.6	403122	36758	92999	26
35	565676	53.2	968429	8.3	597247	61.5	402753	36785	92988	25
36	565995	53.1	968379	8.3	597616	61.5	402384	36812	92978	24
37	566314	53.1	968329	8.3	597985	61.5	402015	36839	92967	23
38	566632	53.1	968278	8.3	598354	61.4	401646	36867	92956	22
39	566951	53.0	968228	8.4	598722	61.4	401278	36894	92945	21
40	567269	53.0	968178	8.4	599091	61.3	400909	36921	92935	20
41	9.567587	52.9	9.968128	8.4	9.599459	61.3	10.400541	36948	92926	19
42	567904	52.9	968078	8.4	599827	61.3	400173	36975	92913	18
43	568222	52.8	968027	8.4	600194	61.2	399806	37002	92902	17
44	568539	52.8	967977	8.4	600562	61.2	399438	37029	92892	16
45	568856	52.8	967927	8.4	600929	61.1	399071	37056	92881	15
46	569172	52.7	967876	8.4	601296	61.1	398704	37083	92870	14
47	569488	52.7	967826	8.4	601662	61.1	398338	37110	92859	13
48	569804	52.6	967775	8,4	602029	61.0	397971	37137	92849	12
49	570120	52.6	967725	8.4	602395	61.0	397605	37164	92838	11
50	570435	52.5	967674	8.4	602761	61.0	397239	37191	92827	10
51	9.570751	52.5	9.967624	8.4	9.603127	60.9	10.396873	37218	92816	9
52	571066	52.4	967573	8.4	603493	60.9	396507	37245	92805	8
53	571380	52.4	967522	8.5	603858	60.9	396142	37272	92794	7
54	571695	52.3	967471	8.5	604223	60.8	395777	37299	92784	6
55	572009	52.3	967421	8.5	604588	60.8	395412	37326	92773	5
56	572323	52.3	967370	8.5	604953	60.7	395047	37353	92762	4
57	572636	52.2	967319	8.5	605317	60.7	394683	37380	92751	3
58	572950	52.2	967268	8.5	605682	60.7	394318	37407	92740	2
59	573263	52.1	967217	8.5	606046	60.6	393954	37434	92729	1
60	573575		967166		606410		393590	37461	92718	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.573575	52.1	9.967166	8.5	9.606410	60.6	10.393590	37461	92718	60
1	573888	52.0	967115	8.5	606773	60.6	393227	37488	92707	59
2	574200	52.0	967064	8.5	607137	60.5	392863	37515	92697	58
3	574512	51.9	967013	8.5	607500	60.5	392500	37542	92686	57
4	574824	51.9	966961	8.5	607863	60.4	392137	37569	92675	56
5	575136	51.9	966910	8.5	608225	60.4	391775	37595	92664	55
6	575447	51.8	966859	8.5	608588	60.4	391412	37622	92653	54
7	575758	51.8	966808	8.5	608950	60.3	391050	37649	92642	53
8	576069	51.7	966756	8.6	609312	60.3	390688	37676	92631	52
9	576379	51.7	966705	8.6	609674	60.3	390326	37703	92620	51
10	576689	51.6	966653	8.6	610036	60.2	389964	37730	92609	50
11	9.576999	51.6	9.966602	8.6	9.610397	60.2	10.389603	37757	92598	49
12	577309	51.6	966550	8.6	610759	60.2	389241	37784	92587	48
13	577618	51.5	966499	8.6	611120	60.1	388880	37811	92576	47
14	577927	51.5	966447	8.6	611480	60.1	388520	37838	92565	46
15	578236	51.4	966395	8.6	611841	60.1	388159	37865	92554	45
16	578545	51.4	966344	8.6	612201	60.0	387799	37892	92543	44
17	578853	51.3	966292	8.6	612561	60.0	387439	37919	92532	43
18	579162	51.3	966240	8.6	612921	60.0	387079	37946	92521	42
19	579470	51.3	966188	8.6	613281	59.9	386719	37973	92510	41
20	579777	51.2	966136	8.6	613641	59.9	386359	37999	92499	40
21	9.580085	51.2	9.966085	8.7	9.614000	59.8	10.386000	38026	92488	39
22	580392	51.1	966033	8.7	614359	59.8	385641	38053	92477	38
23	580699	51.1	965981	8.7	614718	59.8	385282	38080	92466	37
24	581005	51.1	965928	8.7	615077	59.7	384923	38107	92455	36
25	581312	51.0	965876	8.7	615435	59.7	384565	38134	92444	35
26	581618	51.0	965824	8.7	615793	59.7	384207	38161	92432	34
27	581924	50.9	965772	8.7	616151	59.6	383849	38188	92421	33
28	582229	50.9	965720	8.7	616509	59.6	383491	38215	92410	32
29	582535	50.9	965668	8.7	616867	59.6	383133	38241	92399	31
30	582840	50.8	965615	8.7	617224	59.5	382776	38268	92388	30
31	9.583145	50.8	9.965563	8.7	9.617582	59.5	10.382418	38295	92377	29
32	583449	50.7	965511	8.7	617939	59.5	382061	38322	92366	28
33	583754	50.7	965458	8.7	618295	59.4	381705	38349	92355	27
34	584058	50.6	965406	8.7	618652	59.4	381348	38376	92343	26
35	584361	50.6	965353	8.8	619008	59.4	380992	38403	92332	25
36	584665	50.6	965301	8.8	619364	59.3	380636	38430	92321	24
37	584968	50.5	965248	8.8	619721	59.3	380279	38456	92310	23
38	585272	50.5	965195	8.8	620076	59.3	379924	38483	92299	22
39	585574	50.4	965143	8.8	620432	59.2	379568	38510	92287	21
40	585877	50.4	965090	8.8	620787	59.2	379213	38537	92276	20
41	9.586179	50.3	9.965037	8.8	9.621142	59.2	10.378858	38564	92265	19
42	586482	50.3	964984	8.8	621497	59.1	378503	38591	92254	18
43	586783	50.3	964931	8.8	621852	59.1	378148	38617	92243	17
44	587085	50.2	964879	8.8	622207	59.0	377793	38644	92231	16
45	587386	50.2	964826	8.8	622561	59.0	377439	38671	92220	15
46	587688	50.1	964773	8.8	622915	59.0	377085	38698	92209	14
47	587989	50.1	964719	8.8	623269	58.9	376731	38725	92198	13
48	588289	50.1	964666	8.9	623623	58.9	376377	38752	92186	12
49	588590	50.0	964613	8.9	623976	58.9	376024	38778	92175	11
50	588890	50.0	964560	8.9	624330	58.8	375670	38805	92164	10
51	9.589190	49.9	9.964507	8.9	9.624683	58.8	10.375317	38832	92152	9
52	589489	49.9	964454	8.9	625036	58.8	374964	38859	92141	8
53	589789	49.9	964400	8.9	625388	58.7	374612	38886	92130	7
54	590088	49.8	964347	8.9	625741	58.7	374259	38912	92119	6
55	590387	49.8	964294	8.9	626093	58.7	373907	38939	92107	5
56	590686	49.7	964240	8.9	626445	58.6	373555	38966	92096	4
57	590984	49.7	964187	8.9	626797	58.6	373203	38993	92085	3
58	591282	49.7	964133	8.9	627149	58.6	372851	39020	92073	2
59	591580	49.6	964080	8.9	627501	58.5	372499	39046	92062	1
60	591878		964026		627852		372148	39073	92050	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.591878	49.6	9.964026	8.9	9.627852	58.5	10.372148	39073	92050	60
1	592176	49.5	963972	8.9	628203	58.5	371797	39100	92039	59
2	592473	49.5	963919	8.9	628554	58.5	371446	39127	92028	58
3	592770	49.5	963865	9.0	628905	58.4	371095	39153	92016	57
4	593067	49.4	963811	9.0	629255	58.4	370745	39180	92005	56
5	593363	49.4	963757	9.0	629606	58.3	370394	39207	91994	55
6	593659	49.3	963704	9.0	629956	58.3	370044	39234	91982	54
7	593955	49.3	963650	9.0	630306	58.3	369694	39260	91971	53
8	594251	49.3	963596	9.0	630656	58.3	369344	39287	91959	52
9	594547	49.2	963542	9.0	631005	58.2	368995	39314	91948	51
10	594842	49.2	963488	9.0	631355	58.2	368645	39341	91936	50
11	9.595137	49.1	9.963434	9.0	9.631704	58.2	10.368296	39367	91925	49
12	595432	49.1	963379	9.0	632053	58.1	367947	39394	91914	48
13	595727	49.1	963325	9.0	632401	58.1	367599	39421	91902	47
14	596021	49.0	963271	9.0	632750	58.1	367250	39448	91891	46
15	596315	49.0	963217	9.0	633098	58.0	366902	39474	91879	45
16	596609	48.9	963163	9.0	633447	58.0	366553	39501	91868	44
17	596903	48.9	963108	9.1	633795	58.0	366205	39528	91856	43
18	597196	48.9	963054	9.1	634143	57.9	365857	39555	91845	42
19	597490	48.8	962999	9.1	634490	57.9	365510	39581	91833	41
20	597783	48.8	962945	9.1	634838	57.9	365162	39608	91822	40
21	9.598075	48.7	9.962890	9.1	9.635185	57.8	10.364815	39635	91810	39
22	598368	48.7	962836	9.1	635532	57.8	364468	39661	91799	38
23	598660	48.7	962781	9.1	635879	57.8	364121	39688	91787	37
24	598952	48.6	962727	9.1	636226	57.7	363774	39715	91775	36
25	599244	48.6	962672	9.1	636572	57.7	363428	39741	91764	35
26	599536	48.5	962617	9.1	636919	57.7	363081	39768	91752	34
27	599827	48.5	962562	9.1	637265	57.7	362735	39795	91741	33
28	600118	48.5	962508	9.1	637611	57.6	362389	39822	91729	32
29	600409	48.4	962453	9.1	637956	57.6	362044	39848	91718	31
30	600700	48.4	962398	9 2	638302	57.6	361698	39875	91706	30
31	9.600990	48.4	9.962343	9.2	9.638647	57.5	10.361353	39902	91694	29
32	601280	48.3	962288	9.2	638992	57.5	361008	39928	91683	28
33	601570	48.3	962233	9.2	639337	57.5	360663	39955	91671	27
34	601860	48.2	962178	9.2	639682	57.4	360318	39982	91660	26
35	602150	48.2	962123	9.2	640027	57.4	359973	40008	91648	25
36	602439	48.2	962067	9.2	640371	57.4	359629	40035	91636	24
37	602728	48.1	962012	9.2	640716	57.3	359284	40062	91625	23
38	603017	48.1	961957	9.2	641060	57.3	358940	40088	91613	22
39	603305	48.1	961902	9.2	641404	57.3	358596	40115	91601	21
40	603594	48.0	961846	9.2	641747	57.2	358253	40141	91590	20
41	9.603882	48.0	9.961791	9.2	9.642091	57.2	10.357909	40168	91578	19
42	604170	47.9	961735	9.2	642434	57.2	357566	40195	91566	18
43	604457	47.9	961680	9.2	642777	57.2	357223	40221	91555	17
44	604745	47.9	961624	9.3	643120	57.1	356880	40248	91543	16
45	605032	47 8	961569	9.3	643463	57.1	356537	40275	91531	15
46	605319	47.8	961513	9.3	643806	57.1	356194	40301	91519	14
47	605606	47.8	961458	9.3	644148	57.0	355852	40328	91508	13
48	605892	47.7	961402	9.3	644490	57.0	355510	40355	91496	12
49	606179	47.7	961346	9.3	644832	57.0	355168	40381	91484	11
50	606465	47.6	961290	9.3	645174	56.9	354826	40408	91472	10
51	9.606751	47.6	9.961235	9.3	9.645516	56.9	10.354484	40434	91461	9
52	607036	47.6	961179	9.3	645857	56.9	354143	40461	91449	8
53	607322	47.5	961123	9.3	646199	56.9	353801	40488	91437	7
54	607607	47.5	961067	9.3	646540	56.8	353460	40514	91425	6
55	607892	47.4	961011	9.3	646881	56.8	353119	40541	91414	5
56	608177	47.4	960955	9.3	647222	56.8	352778	40567	91402	4
57	608461	47.4	960899	9.3	647562	56.7	352438	40594	91390	3
58	608745	47.3	960843	9.4	647903	56.7	352097	40621	91378	2
59	609029	47.3	960786	9.4	648243	56.7	351757	40647	91366	1
60	609313		960730		648583		351417	40674	91355	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.609313	47.3	9.960730	9.4	9.648583	55.6	10.351417	40674	91355	60
1	609597	47.2	960674	9.4	648923	56.6	351077	40700	91343	59
2	609880	47.2	960618	9.4	649263	56.6	350737	40727	91331	58
3	610164	47.2	960561	9.4	649602	56 6	350398	40753	91319	57
4	610447	47.1	960505	9.4	649942	56.5	350058	40780	91307	56
5	610729	47.1	960448	9.4	650281	56.5	349719	40806	91295	55
6	611012	47.0	960392	9.4	650620	59.5	349380	40833	91283	54
7	611294	47.0	960335	9.4	650959	56.4	349041	40860	91272	53
8	611576	47.0	960279	9.4	651297	56.4	348703	40886	91260	52
9	611858	46.9	960222	9.4	651636	56.4	348364	40913	91248	51
10	612140	46.9	960165	9.4	651974	56.3	348026	40939	91236	50
11	9.612421	46.9	9.960109	9.5	9.652312	56.3	10.347688	40966	91224	49
12	612702	46.8	960052	9.5	652650	56.3	347350	40992	91212	48
13	612983	46.8	959995	9.5	652988	56.3	347012	41019	91200	47
14	613264	46.7	959938	9.5	653326	56.2	346674	41045	91188	46
15	613545	46.7	959882	9.5	653663	56.2	346337	41072	91176	45
16	613825	46.7	959825	9.5	654000	56.2	346000	41098	91164	44
17	614105	46.6	959768	9.5	654337	56.1	345663	41125	91152	43
18	614385	46.6	959711	9.5	654174	56.1	345326	41151	91140	42
19	614665	46.6	959654	9.5	655011	56.1	344989	41178	91128	41
20	614944	46.5	959596	9.5	655348	56.1	344652	41204	91116	40
21	9.615223	46.5	9.959539	9.5	9.655684	56.0	10.344316	41231	91104	39
22	615502	46.5	959482	9.5	656020	56.0	343980	41257	91092	38
23	615781	46.4	959425	9.5	656356	56.0	343644	41284	91080	37
24	616060	46.4	959368	9.5	656692	55.9	343308	41310	91068	36
25	616338	46.4	959310	9.6	657028	55.9	342972	41337	91056	35
26	616616	46.3	959253	9.6	657364	55.9	342636	41363	91044	34
27	616894	46.3	959195	9.6	657699	55.9	342301	41390	91032	33
28	617172	46.2	959138	9.6	658034	55.8	341966	41416	91020	32
29	617450	46.2	959081	9.6	658369	55.8	341631	41443	91008	31
30	617727	46.2	959023	9.6	658704	55.8	341296	41469	90996	30
31	9.618004	46.1	9.958965	9.6	9.659039	55.8	10.340961	41496	90984	29
32	618281	46.1	958908	9.6	659373	55.7	340627	41522	90972	28
33	618558	46.1	958850	9.6	659708	55.7	340292	41549	90960	27
34	618834	46.0	958792	9.6	660042	55.7	339958	41575	90948	26
35	619110	46.0	958734	9.6	660376	55.7	339624	41602	90936	25
36	619386	46.0	958677	9.6	660710	55.6	339290	41628	90924	24
37	619662	45.9	958619	9.6	661043	55.6	338957	41655	90911	23
38	619938	45.9	958561	9.6	661377	55.6	338623	41681	90899	22
39	620213	45.9	958503	9.7	661710	55.5	338290	41707	90887	21
40	620488	45.8	958445	9.7	662043	55.5	337957	41734	90875	20
41	9.620763	45.8	9.958387	9.7	9.662376	55.5	10.337624	41760	90863	19
42	621038	45.7	958329	9.7	662709	55.4	337291	41787	90851	18
43	621313	45.7	958271	9.7	663042	55.4	336958	41813	90839	17
44	621587	45.7	958213	9.7	663375	55.4	336625	41840	90826	16
45	621861	45.6	958154	9.7	663707	55.4	336293	41866	90814	15
46	622135	45.6	958096	9.7	664039	55.3	335961	41892	90802	14
47	622409	45.6	958038	9.7	664371	55.3	335629	41919	90790	13
48	622682	45.5	957979	9.7	664703	55.3	335297	41945	90778	12
49	622956	45.5	957921	9.7	665035	55.3	334965	41972	90766	11
50	623229	45.5	957863	9.7	665366	55.2	334634	41998	90753	10
51	9.623512	45.4	9.957804	9.7	9.665697	55.2	10.334303	42024	90741	9
52	623774	45.4	957746	9.8	666029	55.2	333971	42051	90729	8
53	624047	45.4	957687	9.8	666360	55.1	333620	42077	90717	7
54	624319	45.3	957628	9.8	666691	55.1	333309	42104	90704	6
55	624591	45.3	957570	9.8	667021	55.1	332979	42130	90692	5
56	624863	46.3	957511	9.8	667352	55.1	332648	42156	90680	4
57	625135	45.2	957452	9.8	667682	55.0	332318	42183	90668	3
58	625406	45.2	957393	9.8	668013	55.0	331987	42209	90655	2
59	625677	45.2	957335	9.8	668343	55.0	331657	42235	90643	1
60	625948		957276		668672		331328	42262	90631	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.625948	45.1	9.957276	9.8	9.668673	55.0	10.331327	42262	90631	60
1	626219	45.1	957217	9.8	669002	54.9	330998	42288	90613	59
2	626490	45.1	957158	9.8	669332	54.9	330668	42315	90606	58
3	626760	45.0	957099	9.8	669661	54.9	330339	42341	90594	57
4	627030	45.0	957040	9.8	669991	54.8	330009	42367	90582	56
5	627300	45.0	956981	9.8	670320	54.8	329680	42394	90569	55
6	627570	44.9	956921	9.9	670649	54.8	329351	42420	90557	54
7	627840	44.9	956862	9.9	670977	54.8	329023	42446	90545	53
8	628109	44.9	956803	9.9	671306	54.7	328694	42473	90532	52
9	628378	44.8	956744	9.9	671634	54.7	328366	42499	90520	51
10	628647	44.8	956684	9.9	671963	54.7	328037	42525	90507	50
11	9.628916	44.7	9.956625	9.9	9.672291	54.7	10.327709	42552	90495	49
12	629185	44.7	956566	9.9	672619	54.6	327381	42578	90483	48
13	629453	44.7	956506	9.9	672947	54.6	327053	42604	90470	47
14	629721	44.6	956447	9.9	673274	54.6	326726	42631	90458	46
15	629989	44.6	956387	9.9	673602	54.6	326398	42657	90446	45
16	630257	44.6	956327	9.9	673929	54.5	326071	42683	90433	44
17	630524	44.6	956268	9.9	674257	54.5	325743	42709	90421	43
18	630792	44.5	956208	10.0	674584	54.5	325416	42736	90408	42
19	631059	44.5	956148	10.0	674910	54.4	325090	42762	90396	41
20	631326	44.5	956089	10.0	675237	54.4	324763	42788	90383	40
21	9.631593	44.4	9.956029	10.0	9.675564	54.4	10.324436	42815	90371	39
22	631859	44.4	955969	10.0	675890	54.4	324110	42841	90358	38
23	632125	44.4	955909	10.0	676216	54.3	323784	42867	90346	37
24	632392	44.3	955849	10.0	676543	54.3	323457	42894	90334	36
25	632658	44.3	955789	10.0	676869	54.3	323131	42920	90321	35
26	632923	44.3	955729	10.0	677194	54.3	322806	42946	90309	34
27	633189	44.2	955669	10.0	677520	54.2	322480	42972	90296	33
28	633454	44.2	955609	10.0	677846	54.2	322154	42999	90284	32
29	633719	44.2	955548	10.0	678171	54.2	321829	43025	90271	31
30	633984	44.1	955488	10.0	678496	54.2	321504	43051	90259	30
31	9.634249	44.1	9.955428	10.1	9.678821	54.1	10.321179	43077	90246	29
32	634514	44.0	955368	10.1	679146	54.1	320854	43104	90233	28
33	634778	44.0	955307	10.1	679471	54.1	320529	43130	90221	27
34	635042	44.0	955247	10.1	679795	54.1	320205	43156	90208	26
35	635306	43.9	955186	10.1	680120	54.0	319880	43182	90196	25
36	635570	43.9	955126	10.1	680444	54.0	319556	43209	90183	24
37	635834	43.9	955035	10.1	680768	54.0	319232	43235	90171	23
38	636097	43.8	955005	10.1	681092	54.0	318908	43261	90158	22
39	636360	43.8	954944	10.1	681416	53.9	318584	43287	90146	21
40	636623	43.8	954883	10.1	681740	53.9	318260	43313	90133	20
41	9.636886	43.7	9.954823	10.1	9.682063	53.9	10.317937	43340	90120	19
42	637148	43.7	954762	10.1	682387	53.9	317613	43366	90108	18
43	637411	43.7	954701	10.1	682710	53.8	317290	43392	90095	17
44	637673	43.7	954640	10.1	683033	53.8	316967	43418	90082	16
45	637935	43.6	954579	10.1	683356	53.8	316644	43445	90070	15
46	638197	43.6	954518	10.2	683679	53.8	316321	43471	90057	14
47	638458	43.6	954457	10.2	684001	53.7	315999	43497	90045	13
48	638720	43.5	954396	10.2	684324	53.7	315676	43523	90032	12
49	638981	43.5	954335	10.2	684646	53.7	315354	43549	90019	11
50	639242	43.5	954274	10.2	684968	53.7	315032	43575	90007	10
51	9.639503	43.4	9.954213	10.2	9.685290	53.6	10.314710	43602	89994	9
52	639764	43.4	954152	10.2	685612	53.6	314388	43628	89981	8
53	640024	43.4	954090	10.2	685934	53.6	314066	43654	89968	7
54	640284	43.3	954029	10.2	686255	53.6	313745	43680	89956	6
55	640544	43.3	953968	10.2	686577	53.5	313423	43706	89943	5
56	640304	43.3	953906	10.2	686898	53.5	313102	43733	89930	4
57	641064	43.2	953845	10.2	687219	53.5	312781	43759	89918	3
58	641324	43.2	953783	10.2	687540	53.5	312460	43785	89905	2
59	641584	43.2	953722	10.3	687861	53.4	312139	43811	89892	1
60	641842		953660		688182		311818	43837	89879	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

64 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.641842	43.1	9.953660	10.3	9.688182	53.4	10.311818	43837	89879	60
1	642101	43.1	953599	10.3	688502	53.4	311498	43863	89867	59
2	642360	43.1	953537	10.3	688823	53.4	311177	43889	89854	58
3	642618	43.0	953475	10.3	689143	53.3	310857	43916	89841	57
4	642877	43.0	953413	10.3	689463	53.3	310537	43942	89828	56
5	643135	43.0	953352	10.3	689783	53.3	310217	43968	89816	55
6	643393	43.0	953290	10.3	690103	53.3	309897	43994	89803	54
7	643650	42.9	953228	10.3	690423	53.3	309577	44020	89790	53
8	643908	42.9	953166	10.3	690742	53.2	309258	44046	89777	52
9	644165	42.9	953104	10.3	691062	53.2	308938	44072	89764	51
10	644423	42.8	953042	10.3	691381	53.2	308619	44098	89752	50
11	9.644680	42.8	9.952980	10.4	9.691700	53.1	10.308300	44124	89739	49
12	644936	42.8	952918	10.4	692019	53.1	307981	44151	89726	48
13	645193	42.7	952855	10.4	692338	53.1	307662	44177	89713	47
14	645450	42.7	952793	10.4	692656	53.1	307344	44203	89700	46
15	645706	42.7	952731	10.4	692975	53.1	307025	44229	89687	45
16	645962	42.6	952669	10.4	693293	53.0	306707	44255	89674	44
17	646218	42.6	952606	10.4	693612	53.0	306388	44281	89662	43
18	646474	42.6	952544	10.4	693930	53.0	306070	44307	89649	42
19	646729	42.5	952481	10.4	694248	53.0	305752	44333	89636	41
20	646984	42.5	952419	10.4	694566	52.9	305434	44359	89623	40
21	9.647240	42.5	9.952356	10.4	9.694883	52.9	10.305117	44385	89610	39
22	647494	42.4	952294	10.4	695201	52.9	304799	44411	89597	38
23	647749	42.4	952231	10.4	695518	52.9	304482	44437	89584	37
24	648004	42.4	952168	10.5	695836	52.9	304164	44464	89571	36
25	648258	42.4	952106	10.5	696153	52.8	303847	44490	89558	35
26	648512	42.3	952043	10.5	696470	52.8	303530	44516	89545	34
27	648766	42.3	951980	10.5	696787	52.8	303213	44542	89532	33
28	649020	42.3	951917	10.5	697103	52.8	302897	44568	89519	32
29	649274	42.2	951854	10.5	697420	52.7	302580	44594	89506	31
30	649527	42.2	951791	10.5	697736	52.7	302264	44620	89493	30
31	9.649781	42.2	9.951728	10.5	9.698053	52.7	10.301947	44646	89480	29
32	650034	42.2	951665	10.5	698369	52.7	301631	44672	89467	28
33	650287	42.1	951602	10.5	698685	52.6	301315	44698	89454	27
34	650539	42.1	951539	10.5	699001	52.6	300999	44724	89441	26
35	650792	42.1	951476	10.5	699316	52.6	300684	44750	89428	25
36	651044	42.0	951412	10.5	699632	52.6	300368	44776	89415	24
37	651297	42.0	951349	10.6	699947	52.6	300053	44802	89402	23
38	651549	42.0	951286	10.6	700263	52.5	299737	44828	89389	22
39	651800	41.9	951222	10.6	700578	52.5	299422	44854	89376	21
40	652052	41.9	951159	10.6	700893	52.5	299107	44880	89363	20
41	9.652304	41.9	9.951096	10.6	9.701208	52.4	10.298792	44906	89350	19
42	652555	41.8	951032	10.6	701523	52.4	298477	44932	89337	18
43	652806	41.8	950968	10.6	701837	52.4	298163	44958	89324	17
44	653057	41.8	950905	10.6	702152	52.4	297848	44984	89311	16
45	653308	41.8	950841	10.6	702466	52.4	297534	45010	89298	15
46	653558	41.7	950778	10.6	702780	52.3	297220	45036	89285	14
47	653808	41.7	950714	10.6	703095	52.3	296905	45062	89272	13
48	654059	41.7	950650	10.6	703409	52.3	296591	45088	89259	12
49	654309	41.6	950586	10.6	703723	52.3	296277	45114	89245	11
50	654558	41.6	950522	10.7	704036	52.2	295964	45140	89232	10
51	9.654808	41.6	9.950458	10.7	9.704350	52.2	10.295650	45166	89219	9
52	655058	41.6	950394	10.7	704663	52.2	295337	45192	89206	8
53	655307	41.5	950330	10.7	704977	52.2	295023	45218	89193	7
54	655556	41.5	950266	10.7	705290	52.2	294710	45243	89180	6
55	655805	41.5	950202	10.7	705603	52.1	294397	45269	89167	5
56	656054	41.4	950138	10.7	705916	52.1	294084	45295	89153	4
57	656302	41.4	950074	10.7	706228	52.1	293772	45321	89140	3
58	656551	41.4	950010	10.7	706541	52.1	293459	45347	89127	2
59	656799	41.3	949945	10.7	706854	52.1	293146	45373	89114	1
60	657047		949881		707166		292834	45399	89101	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.657047	41.3	9.949881	10.7	9.707166	52.0	10.292834	45399	89101	60
1	657295	41.3	949816	10.7	707478	52.0	292522	45425	89087	59
2	657542	41.2	949752	10.7	707790	52.0	292210	45451	89074	58
3	657790	41.2	949688	10.8	708102	52.0	291898	45477	89061	57
4	658037	41.2	949623	10.8	708414	51.9	291586	45503	89048	56
5	658284	41.2	949558	10.8	708726	51.9	291274	45529	89035	55
6	658531	41.1	949494	10.8	709037	51.9	290963	45554	89021	54
7	658778	41.1	949429	10.8	709349	51.9	290651	45580	89008	53
8	659025	41.1	949364	10.8	709660	51.9	290340	45606	88995	52
9	659271	41.0	949300	10.8	709971	51.8	290029	45632	88981	51
10	659517	41.0	949235	10.8	710282	51.8	289718	45658	88968	50
11	9.659763	41.0	9.949170	10.8	9.710593	51.8	10.289407	45684	88955	49
12	660009	40.9	949105	10.8	710904	51.8	289096	45710	88942	48
13	660255	40.9	949040	10.8	711215	51.8	288785	45736	88928	47
14	660501	40.9	948975	10.8	711525	51.7	288475	45762	88915	46
15	660746	40.9	948910	10.8	711836	51.7	288164	45787	88902	45
16	660991	40.8	948845	10.8	712146	51.7	287854	45813	88888	44
17	661236	40.8	948780	10.9	712456	51.7	287544	45839	88875	43
18	661481	40.8	948715	10.9	712766	51.6	287234	45865	88862	42
19	661726	40.7	948650	10.9	713076	51.6	286924	45891	88848	41
20	661970	40.7	948584	10.9	713386	51.6	286614	45917	88835	40
21	9.662214	40.7	9.948519	10.9	9.713696	51.6	10.286304	45942	88822	39
22	662459	40.7	948454	10.9	714005	51.6	285995	45968	88808	38
23	662703	40.6	948388	10.9	714314	51.5	285686	45994	88795	37
24	662946	40.6	948323	10.9	714624	51.5	285376	46020	88782	36
25	663190	40.6	948257	10.9	714933	51.5	285067	46046	88768	35
26	663433	40.5	948192	10.9	715242	51.5	284758	46072	88755	34
27	663677	40.5	948126	10.9	715551	51.4	284449	46097	88741	33
28	663920	40.5	948060	10.9	715860	51.4	284140	46123	88728	32
29	664163	40.5	947995	11.0	716168	51.4	283832	46149	88715	31
30	664406	40.4	947929	11.0	716477	51.4	283523	46175	88701	30
31	9.664648	40.4	9.947863	11.0	9.716785	51.4	10.283215	46201	88688	29
32	664891	40.4	947797	11.0	717093	51.3	282907	46226	88674	28
33	665133	40.3	947731	11.0	717401	51.3	282599	46252	88661	27
34	665375	40.3	947665	11.0	717709	51.3	282291	46278	88647	26
35	665617	40.3	947600	11.0	718017	51.3	281983	46304	88634	25
36	665859	40.2	947533	11.0	718325	51.3	281675	46330	88620	24
37	666100	40.2	947467	11.0	718633	51.2	281367	46355	88607	23
38	666342	40.2	947401	11.0	718940	51.2	281060	46381	88593	22
39	666583	40.2	947335	11.0	719248	51.2	280752	46407	88580	21
40	666824	40.1	947269	11.0	719555	51.2	280445	46433	88566	20
41	9.667065	40.1	9.947203	11.0	9.719862	51.2	10.280138	46458	88553	19
42	667305	40.1	947136	11.1	720169	51.1	279831	46484	88539	18
43	667546	40.1	947070	11.1	720476	51.1	279524	46510	88526	17
44	667786	40.0	947004	11.1	720783	51.1	279217	46536	88512	16
45	668027	40 0	946937	11.1	721089	51.1	278911	46561	88499	15
46	668267	40.0	946871	11.1	721396	51.1	278604	46587	88485	14
47	668506	39.9	946804	11.1	721702	51.0	278298	46613	88472	13
48	668746	39.9	946738	11.1	722009	51.0	277991	46639	88458	12
49	668986	39.9	946671	11.1	722315	51.0	277685	46664	88445	11
50	669225	39.9	946604	11.1	722621	51.0	277379	46690	88431	10
51	9.669464	39.8	9.946538	11.1	9.722927	51.0	10.277073	46716	88417	9
52	669703	39.8	946471	11.1	723232	50.9	276768	46742	88404	8
53	669942	39.8	946404	11.1	723538	50.9	276462	46767	88390	7
54	670181	39.7	946337	11.1	723844	50.9	276156	46793	88377	6
55	670419	39.7	946270	11.2	724149	50.9	275851	46819	88363	5
56	670658	39.7	946203	11.2	724454	50.9	275546	46844	88349	4
57	670896	39.7	946136	11.2	724759	50.8	275241	46870	88336	3
58	671134	39.6	946069	11.2	725065	50.8	274935	46896	88322	2
59	671372	39.6	946002	11.2	725369	50.8	274631	46921	88308	1
60	671609		945935		725674		274326	46947	88295	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.671609	39.6	9.945935	11.2	9.725674	50.8	10.274326	46947	88295	60
1	671847	39.5	945868	11.2	725979	50.8	274021	46973	88281	59
2	672084	39.5	945800	11.2	726284	50 7	273716	46999	88267	58
3	672321	39.5	945733	11.2	726588	50.7	273412	47024	88254	57
4	672558	39.5	945666	11.2	726892	50.7	273108	47050	88240	56
5	672795	39.4	945598	11.2	727197	50.7	272803	47076	88226	55
6	673032	39.4	945531	11.2	727501	50.7	272499	47101	88213	54
7	673268	39.4	945464	11.3	727805	50.6	272195	47127	88199	53
8	673505	39.4	945396	11.3	728109	50.6	271891	47153	88185	52
9	673741	39.3	945328	11.3	728412	50.6	271588	47178	88172	51
10	673977	39.3	945261	11.3	728716	50.6	271284	47204	88158	50
11	9.674213	39.3	9.945193	11.3	9.729020	50.6	10.270980	47229	88144	49
12	674448	39.2	945125	11.3	729323	50.5	270677	47255	88130	48
13	674684	39.2	945058	11.3	729626	50.5	270374	47281	88117	47
14	674919	39.2	944990	11.3	729929	50.5	270071	47306	88103	46
15	675155	39.2	944922	11.3	730233	50.5	269767	47332	88089	45
16	675390	39.1	944854	11.3	730535	50.5	269465	47358	88075	44
17	675624	39.1	944786	11.3	730838	50.4	269162	47383	88062	43
18	675859	39.1	944718	11.3	731141	50.4	268859	47409	88048	42
19	676094	39.1	944650	11.3	731444	50.4	268556	47434	88034	41
20	676328	39.0	944582	11.4	731746	50.4	268254	47460	88020	40
21	9.676562	39.0	9.944514	11.4	9.732048	50.4	10.267952	47486	88006	39
22	676796	39.0	944446	11.4	732351	50.3	267649	47511	87993	38
23	677030	39.0	944377	11.4	732653	50.3	267347	47537	87979	37
24	677264	38.9	944309	11.4	732955	50.3	267045	47562	87965	36
25	677498	38.9	944241	11.4	733257	50.3	266743	47588	87951	35
26	677731	38.9	944172	11.4	733558	50.3	266442	47614	87937	34
27	677964	38.8	944104	11.4	733860	50.2	266140	47639	87923	33
28	678197	38.8	944036	11.4	734162	50.2	265838	47665	87909	32
29	678430	38.8	943967	11.4	734463	50.2	265537	47690	87896	31
30	678663	38.8	943899	11.4	734764	50.2	265236	47716	87882	30
31	9.678895	38.7	9.943830	11.4	9.735066	50.2	10.264934	47741	87868	29
32	679128	38.7	943761	11.4	735367	50.2	264633	47767	87854	28
33	679360	38.7	943693	11.5	735668	50.1	264332	47793	87840	27
34	679592	38.7	943624	11.5	735969	50.1	264031	47818	87826	26
35	679824	38.6	943555	11.5	736269	50.1	263731	47844	87812	25
36	680056	38.6	943486	11.5	736570	50.1	263430	47869	87798	24
37	680288	38.6	943417	11.5	736871	50.1	263129	47895	87784	23
38	680519	38.5	943348	11.5	737171	50.0	262829	47920	87770	22
39	680750	38.5	943279	11.5	737471	50.0	262529	47946	87756	21
40	680982	38.5	943210	11.5	737771	50.0	262229	47971	87743	20
41	9.681213	38.5	9.943141	11.5	9.738071	50.0	10.261929	47997	87729	19
42	681443	38.4	943072	11.5	738371	50.0	261629	48022	87715	18
43	681674	38.4	943003	11.5	738671	49.9	261329	48048	87701	17
44	681905	38.4	942934	11.5	738971	49.9	261029	48073	87687	16
45	682135	38.4	942864	11.5	739271	49.9	260729	48099	87673	15
46	682365	38.3	942795	11.6	739570	49.9	260430	48124	87659	14
47	682595	38.3	942726	11.6	739870	49.9	260130	48150	87645	13
48	682825	38.3	942656	11.6	740169	49.9	259831	48175	87631	12
49	683055	38.3	942587	11.6	740468	49.8	259532	48201	87617	11
50	683284	38.2	942517	11.6	740767	49.8	259233	48226	87603	10
51	9.683514	38.2	9.942448	11.6	9.741066	49.8	10.258934	48252	87589	9
52	683743	38.2	942378	11.6	741365	49.8	258635	48277	87575	8
53	683972	38.2	942308	11.6	741664	49.8	258336	48303	87561	7
54	684201	38.1	942239	11.6	741962	49.7	258038	48328	87546	6
55	684430	38.1	942169	11.6	742261	49.7	257739	48354	87532	5
56	684658	38.1	942099	11.6	742559	49.7	257441	48379	87518	4
57	684887	38.0	942029	11.6	742858	49.7	257142	48405	87504	3
58	685115	38.0	941959	11.6	743156	49.7	256844	48430	87490	2
59	685343	38.0	941889	11.7	743454	49.7	256546	48456	87476	1
60	685571		941819		743752		256248	48481	87462	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

61 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.685571	38.0	9.941819	11.7	9.743752	49.6	10.256248	48481	87462	60
1	685799	37.9	941749	11.7	744050	49.6	255950	48506	87448	59
2	686027	37.9	941679	11.7	744348	49.6	255652	48532	87434	58
3	686254	37.9	941609	11.7	744645	49.6	255355	48557	87420	57
4	686482	37.9	941539	11.7	744943	49.6	255057	48583	87406	56
5	686709	37.8	941469	11.7	745240	49.6	254760	48608	87391	55
6	686936	37.8	941398	11.7	745538	49.5	254462	48634	87377	54
7	687163	37.8	941328	11.7	745835	49.5	254165	48659	87363	53
8	687389	37.8	941258	11.7	746132	49.5	253868	48684	87349	52
9	687616	37.7	941187	11.7	746429	49.5	253571	48710	87335	51
10	687843	37.7	941117	11.7	746726	49.5	253274	48735	87321	50
11	9.688069	37.7	9.941046	11.8	9.747023	49.4	10.252977	48761	87306	49
12	688295	37.7	940975	11.8	747319	49.4	252681	48786	87292	48
13	688521	37.6	940905	11.8	747616	49.4	252384	48811	87278	47
14	688747	37.6	940834	11.8	747913	49.4	252087	48837	87264	46
15	688972	37.6	940763	11.8	748209	49.4	251791	48862	87250	45
16	689198	37.6	940693	11.8	748505	49.3	251495	48888	87235	44
17	689423	37.5	940622	11.8	748801	49.3	251199	48913	87221	43
18	689648	37.5	940551	11.8	749097	49.3	250903	48938	87207	42
19	689873	37.5	940480	11.8	749393	49.3	250607	48964	87193	41
20	690098	37.5	940409	11.8	749689	49.3	250311	48989	87178	40
21	9.690323	37.4	9.940338	11.8	9.749985	49.3	10.250015	49014	87164	39
22	690548	37.4	940267	11.8	750281	49.2	249719	49040	87150	38
23	690772	37.4	940196	11.8	750576	49.2	249424	49065	87136	37
24	690996	37.4	940125	11.9	750872	49.2	249128	49090	87121	36
25	691220	37.3	940054	11.9	751167	49.2	248833	49116	87107	35
26	691444	37.3	939982	11.9	751462	49.2	248538	49141	87093	34
27	691668	37.3	939911	11.9	751757	49.2	248243	49166	87079	33
28	691892	37.3	939840	11.9	752052	49.1	247948	49192	87064	32
29	692115	37.2	939768	11.9	752347	49.1	247653	49217	87050	31
30	692339	37.2	939697	11.9	752642	49.1	247358	49242	87036	30
31	9.692562	37.2	9.939625	11.9	9.752937	49.1	10.247063	49268	87021	29
32	692785	37.1	939554	11.9	753231	49.1	246769	49293	87007	28
33	693008	37.1	939482	11.9	753526	49.1	246474	49318	86993	27
34	693231	37.1	939410	11.9	753820	49.0	246180	49344	86978	26
35	693453	37.1	939339	11.9	754115	49.0	245885	49369	86964	25
36	693676	37.0	939267	12.0	754409	49.0	245591	49394	86949	24
37	693898	37.0	939195	12.0	754703	49.0	245297	49419	86935	23
38	694120	37.0	939123	12.0	754997	49.0	245003	49445	86921	22
39	694342	37.0	939052	12.0	755291	49.0	244709	49470	86906	21
40	694564	36.9	938980	12.0	755585	48.9	244415	49495	86892	20
41	9.694786	36.9	9.938908	12.0	9.755878	48.9	10.244122	49521	86878	19
42	695007	36.9	938836	12.0	756172	48.9	243828	49546	86863	18
43	695229	36.9	938763	12.0	756465	48.9	243535	49571	86849	17
44	695450	36.8	938691	12.0	756759	48.9	243241	49596	86834	16
45	695671	36.8	938619	12.0	757052	48.9	242948	49622	86820	15
46	695892	36.8	938547	12.0	757345	48.8	242655	49647	86805	14
47	696113	36.8	938475	12.0	757638	48.8	242362	49672	86791	13
48	696334	36.7	938402	12.1	757931	48.8	242069	49697	86777	12
49	696554	36.7	938330	12.1	758224	48.8	241776	49723	86762	11
50	696775	36.7	938258	12.1	758517	48.8	241483	49748	86748	10
51	9.696995	36.7	9.938185	12.1	9.758810	48.8	10.241190	49773	86733	9
52	697215	36.6	938113	12.1	759102	48.7	240898	49798	86719	8
53	697435	36.6	938040	12.1	759395	48.7	240605	49824	86704	7
54	697654	36.6	937967	12.1	759687	48.7	240313	49849	86690	6
55	697874	36.6	937895	12.1	759979	48.7	240021	49874	86675	5
56	698094	36.5	937822	12.1	760272	48.7	239728	49899	86661	4
57	698313	36.5	937749	12.1	760564	48.7	239436	49924	86646	3
58	698532	36.5	937676	12.1	760856	48.6	239144	49950	86632	2
59	698751	36.5	937604	12.1	761148	48.6	238852	49975	86617	1
60	698970		937531		761439		238561	50000	86603	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.698970	36.4	9.937531	12.1	9.761439	48.6	10.238561	50000	86603	60
1	699189	36.4	937458	12.2	761731	48.6	238269	50025	86588	59
2	699407	36.4	937385	12.2	762023	48.6	237977	50050	86573	58
3	699626	36.4	937312	12.2	762314	48.6	237686	50076	86559	57
4	699844	36.3	937238	12.2	762606	48.5	237394	50101	86544	56
5	700062	36.3	937165	12.2	762897	48.5	237103	50126	86530	55
6	700280	36.3	937092	12.2	763188	48.5	236812	50151	86515	54
7	700498	36.3	937019	12.2	763479	48.5	236521	50176	86501	53
8	700716	36.3	936946	12.2	763770	48.5	236230	50201	86486	52
9	700933	36.2	936872	12.2	764061	48.5	235939	50227	86471	51
10	701151	36.2	936799	12.2	764352	48.4	235648	50252	86457	50
11	9.701368	36.2	9.936725	12.2	9.764643	48.4	10.235357	50277	86442	49
12	701585	36.2	936652	12.3	764933	48.4	235067	50302	86427	48
13	701802	36.1	936578	12.3	765224	48.4	234776	50327	86413	47
14	702019	36.1	936505	12.3	765514	48.4	234486	50352	86398	46
15	702236	36.1	936431	12.3	765805	48.4	234195	50377	86384	45
16	702452	36.1	936357	12.3	766095	48.4	233905	50403	86369	44
17	702669	36.0	936284	12.3	766385	48.3	233615	50428	86354	43
18	702885	36.0	936210	12.3	766675	48.3	233325	50453	86340	42
19	703101	36.0	936136	12.3	766965	48.3	233035	50478	86325	41
20	703317	36.0	936062	12.3	767255	48.3	232745	50503	86310	40
21	9.703533	35.9	9.935988	12.3	9.767545	48.3	10.232455	50528	86295	39
22	703749	35.9	935914	12.3	767834	48.3	232166	50553	86281	38
23	703964	35.9	935840	12.3	768124	48.2	231876	50578	86266	37
24	704179	35.9	935766	12.4	768413	48.2	231587	50603	86251	36
25	704395	35.9	935692	12.4	768703	48.2	231297	50628	86237	35
26	704610	35.8	935618	12.4	768992	48,2	231008	50654	86222	34
27	704825	35.8	935543	12.4	769281	48.2	230719	50679	86207	33
28	705040	35.8	935469	12.4	769570	48.2	230430	50704	86192	32
29	705254	35.8	935395	12.4	769860	48.1	230140	50729	86178	31
30	705469	35.7	935320	12.4	770148	48.1	229852	50754	86163	30
31	9.705683	35.7	9.935246	12.4	9.770437	48.1	10.229563	50779	86148	29
32	705898	35.7	935171	12.4	770726	48.1	229274	50804	86133	28
33	706112	35.7	935097	12.4	771015	48.1	228985	50829	86119	27
34	706326	35.6	935022	12.4	771303	48.1	228697	50854	86104	26
35	706539	35.6	934948	12.4	771592	48.1	228408	50879	86089	25
36	706753	35.6	934873	12.4	771880	48.0	228120	50904	86074	24
37	706967	35.6	934798	12.5	772168	48.0	227832	50929	86059	23
38	707180	35.5	934723	12.5	772457	48.0	227543	50954	86045	22
39	707393	35.5	934649	12.5	772745	48.0	227255	50979	86030	21
40	707606	35.5	934574	12.5	773033	48.0	226967	51004	86015	20
41	9.707819	35.5	9.934499	12.5	9.773321	48.0	10.226679	51029	86000	19
42	708032	35.4	934424	12.5	773608	47.9	226392	51054	85985	18
43	708245	35.4	934349	12.5	773896	47.9	226104	51079	85970	17
44	708458	35.4	934274	12.5	774184	47.9	225816	51104	85956	16
45	708670	35.4	934199	12.5	774471	47.9	225529	51129	85941	15
46	708882	35.3	934123	12.5	774759	47.9	225241	51154	85926	14
47	709094	35.3	934048	12.5	775046	47.9	224954	51179	85911	13
48	709306	35.3	933973	12.5	775333	47.9	224667	51204	85896	12
49	709518	35.3	933898	12.6	775621	47.8	224379	51229	85881	11
50	709730	35.3	933822	12.6	775908	47.8	224092	51254	85866	10
51	9.709941	35.2	9.933747	12.6	9.776195	47.8	10.223805	51279	85851	9
52	710153	35.2	933671	12.6	776482	47.8	223518	51304	85836	8
53	710364	35.2	933596	12.6	776769	47.8	223231	51329	85821	7
54	710575	35.2	933520	12.6	777055	47.8	222945	51354	85806	6
55	710786	35.1	933445	12.6	777342	47.8	222658	51379	85792	5
56	710967	35.1	933369	12.6	777628	47.7	222372	51404	85777	4
57	711208	35.1	933293	12.6	777915	47.7	222085	51429	85762	3
58	711419	35.1	933217	12.6	778201	47.7	221799	51454	85747	2
59	711629	35.0	933141	12.6	778487	47.7	221512	51479	85732	1
60	711839		933056		778774		221226	51504	85717	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

59 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N.sine.	N. cos.	
0	9.711839	35.0	9.933036	12.6	9.778774	47.7	10.221226	51504	85717	60
1	712050	35.0	932990	12.7	779060	47.7	220940	51529	85702	59
2	712260	35.0	932914	12.7	779346	47.6	220654	51554	85687	58
3	712469	34.9	932838	12.7	779632	47.6	220368	51579	85672	57
4	712679	34.9	932762	12.7	779918	47.6	220082	51604	85657	56
5	712889	34.9	932685	12.7	780203	47.6	219797	51628	85642	55
6	713098	34.9	932609	12.7	780489	47.6	219511	51653	85627	54
7	713308	34.9	932533	12.7	780775	47.6	219225	51678	85612	53
8	713517	34.8	932457	12.7	781060	47.6	218940	51703	85597	52
9	713726	34.8	932380	12.7	781346	47.5	218654	51728	85582	51
10	713935	34.8	932304	12.7	781631	47.5	218369	51753	85567	50
11	9.714144	34.8	9.932228	12.7	9.781916	47.5	10.218084	51778	85551	49
12	714352	34.7	932151	12.7	782201	47.5	217799	51803	85536	48
13	714561	34.7	932075	12.8	782486	47.5	217514	51828	85521	47
14	714769	34.7	931998	12.8	782771	47.5	217229	51852	85506	46
15	714978	34.7	931921	12.8	783056	47.5	216944	51877	85491	45
16	715186	34.7	931845	12.8	783341	47.5	216659	51902	85476	44
17	715394	34.6	931768	12.8	783626	47.4	216374	51927	85461	43
18	715602	34.6	931691	12.8	783910	47.4	216090	51952	85446	42
19	715809	34.6	931614	12.8	784195	47.4	215805	51977	85431	41
20	716017	34.6	931537	12.8	784479	47.4	215521	52002	85416	40
21	9.716224	34.5	9.931460	12.8	9.784764	47.4	10.215236	52026	85401	39
22	716432	34.5	931383	12.8	785048	47.4	214952	52051	85385	38
23	716639	34.5	931306	12.8	785332	47.3	214668	52076	85370	37
24	716846	34.5	931229	12.9	785616	47.3	214384	52101	85355	36
25	717053	34.5	931152	12.9	785900	47.3	214100	52126	85340	35
26	717259	34.4	931075	12.9	786184	47.3	213816	52151	85325	34
27	717466	34.4	930998	12.9	786468	47.3	213532	52175	85310	33
28	717673	34.4	930921	12.9	786752	47.3	213248	52200	85294	32
29	717879	34.4	930843	12.9	787036	47.3	212964	52225	85279	31
30	718085	34.3	930766	12.9	787319	47.2	212681	52250	85264	30
31	9.718291	34.3	9.930688	12.9	9.787603	47.2	10.212397	52275	85249	29
32	718497	34.3	930611	12.9	787886	47.2	212114	52299	85234	28
33	718703	34.3	930533	12.9	788170	47.2	211830	52324	85218	27
34	718909	34.3	930456	12.9	788453	47.2	211547	52349	85203	26
35	719114	34.2	930378	12.9	788736	47.2	211264	52374	85188	25
36	719320	34.2	930300	13.0	789019	47.2	210981	52399	85173	24
37	719525	34.2	930223	13.0	789302	47.1	210698	52423	85157	23
38	719730	34.2	930145	13.0	789585	47.1	210415	52448	85142	22
39	719935	34.1	930067	13.0	789868	47.1	210132	52473	85127	21
40	720140	34.1	929989	13.0	790151	47.1	209849	52498	85112	20
41	9.720345	34.1	9.929911	13.0	9.790433	47.1	10.209567	52522	85096	19
42	720549	34.1	929833	13.0	790716	47.1	209284	52547	85081	18
43	720754	34.0	929755	13.0	790999	47.1	209001	52572	85066	17
44	720958	34.0	929677	13.0	791281	47.1	208719	52597	85051	16
45	721162	34.0	929599	13.0	791563	47.0	208437	52621	85035	15
46	721366	34.0	929521	13.0	791846	47.0	208154	52646	85020	14
47	721570	34.0	929442	13.0	792128	47.0	207872	52671	85005	13
48	721774	33.9	929364	13.1	792410	47.0	207590	52696	84989	12
49	721978	33.9	929286	13.1	792692	47.0	207308	52720	84974	11
50	722181	33.9	929207	13.1	792974	47.0	207026	52745	84959	10
51	9.722385	33.9	9.929129	13.1	9.793256	47.0	10.206744	52770	84943	9
52	722588	33.9	929050	13.1	793538	46.9	206462	52794	84928	8
53	722791	33.8	928972	13.1	793819	46.9	206181	52819	84913	7
54	722994	33.8	928893	13.1	794101	46.9	205899	52844	84897	6
55	723197	33.8	928815	13.1	794383	46.9	205617	52869	84882	5
56	723400	33.8	928736	13.1	794664	46.9	205336	52893	84866	4
57	723603	33.7	928657	13.1	794945	46.9	205055	52918	84851	3
58	723805	33.7	928578	13.1	795227	46.9	204773	52943	84836	2
59	724007	33.7	928499	13.1	795508	46.8	204492	52967	84820	1
60	724210		928420		795789		204211	52992	84805	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

58 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.724210	33.7	9.928420	13.2	9.795789	46.8	10.204211	52992	84805	60
1	724412	33.7	928342	13.2	796070	46.8	203930	53017	84789	59
2	724614	33.6	928263	13.2	796351	46.8	203649	53041	84774	58
3	724816	33.6	928183	13.2	796632	46.8	203368	53066	84759	57
4	725017	33.6	928104	13.2	796913	46.8	203087	53091	84743	56
5	725219	33.6	928025	13.2	797194	46.8	202806	53115	84728	55
6	725420	33.5	927946	13.2	797475	46.8	202525	53140	84712	54
7	725622	33.5	927867	13.2	797755	46.8	202245	53164	84697	53
8	725823	33.5	927787	13.2	798036	46.7	201964	53189	84681	52
9	726024	33.5	927708	13.2	798316	46.7	201684	53214	84666	51
10	726225	33.5	927629	13.2	798596	46.7	201404	53238	84650	50
11	9.726426	33.4	9.927549	13.2	9.798877	46.7	10.201123	53263	84635	49
12	726626	33.4	927470	13.3	799157	46.7	200843	53288	84619	48
13	726827	33.4	927390	13.3	799437	46.7	200563	53312	84604	47
14	727027	33.4	927310	13.3	799717	46.7	200283	53337	84588	46
15	727228	33.4	927231	13.3	799997	46.6	200003	53361	84573	45
16	727428	33.3	927151	13.3	800277	46.6	199723	53386	84557	44
17	727628	33.3	927071	13.3	800557	46.6	199443	53411	84542	43
18	727828	33.3	926991	13.3	800836	46.6	199164	53435	84526	42
19	728027	33.3	926911	13.3	801116	46.6	198884	53460	84511	41
20	728227	33.3	926831	13.3	801396	46.6	198604	53484	84495	40
21	9.728427	33.2	9.926751	13.3	9.801675	46.6	10.198325	53509	84480	39
22	728626	33.2	926671	13.3	801955	46.6	198045	53534	84464	38
23	728825	33.2	926591	13.3	802234	46.5	197766	53558	84448	37
24	729024	33.2	926511	13.4	802513	46.5	197487	53583	84433	36
25	729223	33.1	926431	13.4	802792	46.5	197208	53607	84417	35
26	729422	33.1	926351	13.4	803072	46.5	196928	53632	84402	34
27	729621	33.1	926270	13.4	803351	46.5	196649	53656	84386	33
28	729820	33.1	926190	13.4	803630	46.5	196370	53681	84370	32
29	730018	33.0	926110	13.4	803908	46.5	196092	53705	84355	31
30	730216	33.0	926029	13.4	804187	46.5	195813	53730	84339	30
31	9.730415	33.0	9.925949	13.4	9.804466	46.4	10.195534	53754	84324	29
32	730613	33.0	925868	13.4	804745	46.4	195255	53779	84308	28
33	730811	33.0	925788	13.4	805023	46.4	194977	53804	84292	27
34	731009	32.9	925707	13.4	805302	46.4	194698	53828	84277	26
35	731206	32.9	925626	13.4	805580	46.4	194420	53853	84261	25
36	731404	32.9	925545	13.5	805859	46.4	194141	53877	84245	24
37	731602	32.9	925465	13.5	806137	46.4	193863	53902	84230	23
38	731799	32.9	925384	13.5	806415	46.3	193585	53926	84214	22
39	731996	32.8	925303	13.5	806693	46.3	193307	53951	84198	21
40	732193	32.8	925222	13.5	806971	46.3	193029	53975	84182	20
41	9.732390	32.8	9.925141	13.5	9.807249	46.3	10.192751	54000	84167	19
42	732587	32.8	925060	13.5	807527	46.3	192473	54024	84151	18
43	732784	32.8	924979	13.5	807805	46.3	192195	54049	84135	17
44	732980	32.7	924897	13.5	808083	46.3	191917	54073	84120	16
45	733177	32.7	924816	13.5	808361	46.3	191639	54097	84104	15
46	733373	32.7	924735	13.6	808638	46.2	191362	54122	84088	14
47	733569	32.7	924654	13.6	808916	46.2	191084	54146	84072	13
48	733765	32.7	924572	13.6	809193	46.2	190807	54171	84057	12
49	733961	32.6	924491	13.6	809471	46.2	190529	54195	84041	11
50	734157	32.6	924409	13.6	809748	46.2	190252	54220	84025	10
51	9.734353	32.6	9.924328	13.6	9.810025	46.2	10.189975	54244	84009	9
52	734549	32.6	924246	13.6	810302	46.2	189698	54269	83994	8
53	734744	32.5	924164	13.6	810580	46.2	189420	54293	83978	7
54	734939	32.5	924083	13.6	810857	46.2	189143	54317	83962	6
55	735135	32.5	924001	13.6	811134	46.1	188866	54342	83946	5
56	735330	32.5	923919	13.6	811410	46.1	188590	54366	83930	4
57	735525	32.5	923837	13.6	811687	46.1	188313	54391	83915	3
58	735719	32.4	923755	13.7	811964	46.1	188036	54415	83899	2
59	735914	32.4	923673	13.7	812241	46.1	187759	54440	83883	1
60	736109		923591		812517		187483	54464	83867	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.736109	32.4	9.923591	13.7	9.812517	46.1	10.187482	54464	83867	60
1	736303	32.4	923509	13.7	812794	46.1	187206	54488	83851	59
2	736498	32.4	923427	13.7	813070	46 1	186930	54513	83835	58
3	736692	32.3	923345	13.7	813347	46.0	186653	54537	83819	57
4	736886	32.3	923263	13.7	813623	46.0	186377	54561	83804	56
5	737080	32.3	923181	13.7	813899	46.0	186101	54586	83788	55
6	737274	32.3	923098	13.7	814175	46.0	185825	54610	83772	54
7	737467	32.3	923016	13.7	814452	46.0	185548	54635	83756	53
8	737661	32.2	922933	13.7	814728	46.0	185272	54659	83740	52
9	737855	32.2	922851	13.7	815004	46.0	184996	54683	83724	51
10	738048	32.2	922768	13.8	815279	46.0	184721	54708	83708	50
11	9.738241	32.2	9.922686	13.8	9.815555	45.9	10.184445	54732	83692	49
12	738434	32.2	922603	13.8	815831	45.9	184169	54756	83676	48
13	738627	32.1	922520	13.8	816107	45.9	183893	54781	83660	47
14	738820	32.1	922438	13.8	816382	45.9	183618	54805	83645	46
15	739013	32.1	922355	13.8	816658	45.9	183342	54829	83629	45
16	739206	32.1	922272	13.8	816933	45.9	183067	54854	83613	44
17	739398	32.1	922189	13.8	817209	45.9	182791	54878	83597	43
18	739590	32.0	922106	13.8	817484	45.9	182516	54902	83581	42
19	739783	32.0	922023	13.8	817759	45.9	182241	54927	83565	41
20	739975	32.0	921940	13.8	818035	45.8	181965	54951	83549	40
21	9.740167	32.0	9.921857	13.9	9.818310	45.8	10.181690	54975	83533	39
22	740359	32.0	921774	13.9	818585	45.8	181415	54999	83517	38
23	740550	31.9	921691	13.9	818860	45.8	181140	55024	83501	37
24	740742	31.9	921607	13.9	819135	45.8	180865	55048	83485	36
25	740934	31.9	921524	13.9	819410	45.8	180590	55072	83469	35
26	741125	31.9	921441	13.9	819684	45.8	180316	55097	83453	34
27	741316	31.9	921357	13.9	819959	45.8	180041	55121	83437	33
28	741508	31.8	921274	13.9	820234	45.8	179766	55145	83421	32
29	741699	31.8	921190	13.9	820508	45.7	179492	55169	83405	31
30	741889	31.8	921107	13.9	820783	45.7	179217	55194	83389	30
31	9.742080	31.8	9.921023	13.9	9.821057	45.7	10.178943	55218	83373	29
32	742271	31.8	920939	14.0	821332	45.7	178668	55242	83356	28
33	742462	31.7	920856	14.0	821606	45.7	178394	55266	83340	27
34	742652	31.7	920772	14.0	821880	45.7	178120	55291	83324	26
35	742842	31.7	920688	14.0	822154	45.7	177846	55315	83308	25
36	743033	31.7	920604	14.0	822429	45.7	177571	55339	83292	24
37	743223	31.7	920520	14.0	822703	45.7	177297	55363	83276	23
38	743413	31.6	920436	14.0	822977	45.6	177023	55388	83260	22
39	743602	31.6	920352	14.0	823250	45.6	176750	55412	83244	21
40	743792	31.6	920268	14.0	823524	45.6	176476	55436	83228	20
41	9.743982	31.6	9.920184	14.0	9.823798	45.6	10.176202	55460	83212	19
42	744171	31.6	920099	14.0	824072	45.6	175928	55484	83195	18
43	744361	31.5	920015	14.0	824345	45.6	175655	55509	83179	17
44	744550	31.5	919931	14.1	824619	45.6	175381	55533	83163	16
45	744739	31.5	919846	14.1	824893	45.6	175107	55557	83147	15
46	744928	31.5	919762	14.1	825166	45.6	174834	55581	83131	14
47	745117	31.5	919677	14.1	825439	45.5	174561	55605	83115	13
48	745306	31.4	919593	14.1	825713	45.5	174287	55630	83098	12
49	745494	31.4	919508	14.1	825986	45.5	174014	55654	83082	11
50	745683	31.4	919424	14.1	826259	45.5	173741	55678	83066	10
51	9.745871	31.4	9.919339	14.1	9.826532	45.5	10.173468	55702	83050	9
52	746059	31.4	919254	14.1	826805	45.5	173195	55726	83034	8
53	746248	31.3	919169	14.1	827078	45.5	172922	55750	83017	7
54	746436	31.3	919085	14.1	827351	45.5	172649	55775	83001	6
55	746624	31.3	919000	14.1	827624	45.5	172376	55799	82985	5
56	746812	31.3	918915	14.2	827897	45.4	172103	55823	82969	4
57	746999	31.3	918830	14.2	828170	45.4	171830	55847	82953	3
58	747187	31.2	918745	14.2	828442	45.4	171558	55871	82936	2
59	747374	31.2	918659	14.2	828715	45.4	171285	55895	82920	1
60	747562		918574		828987		171013	55919	82904	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

56 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine	N. cos.	
0	9.747562	31.2	9.918574	14.2	9.828987	45.4	10.171013	55919	82904	60
1	747749	31.2	918489	14.2	829260	45.4	170740	55943	82887	59
2	747936	31.2	918404	14.2	829532	45.4	170468	55968	82871	58
3	748123	31.1	918318	14.2	829805	45.4	170195	55992	82855	57
4	748310	31.1	918233	14.2	830077	45.4	169923	56016	82839	56
5	748497	31.1	918147	14.2	830349	45.3	169651	56040	82822	55
6	748683	31.1	918062	14.2	830621	45.3	169379	56064	82806	54
7	748870	31.1	917976	14.3	830893	45.3	169107	56088	82790	53
8	749056	31.0	917891	14.3	831165	45.3	168835	56112	82773	52
9	749243	31.0	917805	14.3	831437	45.3	168563	56136	82757	51
10	749426	31.0	917719	14.3	831709	45.3	168291	56160	82741	50
11	9.749615	31.0	9.917634	14.3	9.831981	45.3	10.168019	56184	82724	49
12	749801	31.0	917548	14.3	832253	45.3	167747	56208	82708	48
13	749987	30.9	917462	14.3	832525	45.3	167475	56232	82692	47
14	750172	30.9	917376	14.3	832796	45.3	167204	56256	82675	46
15	750358	30.9	917290	14.3	833068	45.2	166932	56280	82659	45
16	750543	30.9	917204	14.3	833339	45.2	166661	56305	82643	44
17	750729	30.9	917118	14.4	833611	45.2	166389	56329	82626	43
18	750914	30.8	917032	14.4	833882	45.2	166118	56353	82610	42
19	751099	30.8	916946	14.4	834154	45.2	165846	56377	82593	41
20	751284	30.8	916859	14.4	834425	45.2	165575	56401	82577	40
21	9.751469	30.8	9.916773	14.4	9.834696	45.2	10.165304	56425	82561	39
22	751654	30.8	916687	14.4	834967	45.2	165033	56449	82544	38
23	751839	30.8	916600	14.4	835238	45.2	164762	56473	82528	37
24	752023	30.7	916514	14.4	835509	45.2	164491	56497	82511	36
25	752208	30.7	916427	14.4	835780	45.1	164220	56521	82495	35
26	752392	30.7	916341	14.4	836051	45.1	163949	56545	82478	34
27	752576	30.7	916254	14.4	836322	45.1	163678	56569	82462	33
28	752760	30.7	916167	14.5	836593	45.1	163407	56593	82446	32
29	752944	30.6	916081	14.5	836864	45.1	163136	56617	82429	31
30	753128	30.6	915994	14.5	837134	45.1	162866	56641	82413	30
31	9.753312	30.6	9.915907	14.5	9.837405	45.1	10.162595	56665	82396	29
32	753495	30.6	915820	14.5	837675	45.1	162325	56689	82380	28
33	753679	30.6	915733	14.5	837946	45.1	162054	56713	82363	27
34	753862	30.5	915646	14.5	838216	45.1	161784	56736	82347	26
35	754046	30.5	915559	14.5	838487	45.0	161513	56760	82330	25
36	754229	30.5	915472	14.5	838757	45.0	161243	56784	82314	24
37	754412	30 5	915385	14.5	839027	45.0	160973	56808	82297	23
38	754595	30.5	915297	14.5	839297	45.0	160703	56832	82281	22
39	754778	30.4	915210	14.5	839568	45.0	160432	56856	82264	21
40	754960	30.4	915123	14.6	839838	45.0	160162	56880	82248	20
41	9.755143	30.4	9.915035	14.6	9.840108	45.0	10.159892	56904	82231	19
42	755326	30.4	914948	14.6	840378	45.0	159622	56928	82214	18
43	755508	30.4	914860	14.6	840647	45.0	159353	56952	82198	17
44	755690	30.4	914773	14.6	840917	44.9	159083	56976	82181	16
45	755872	30.3	914685	14.6	841187	44.9	158813	57000	82165	15
46	756054	30.3	914598	14.6	841457	44.9	158543	57024	82148	14
47	756236	30.3	914510	14.6	841726	44.9	158274	57047	82132	13
48	756418	30.3	914422	14.6	841996	44.9	158004	57071	82115	12
49	756600	30.3	914334	14.6	842266	44.9	157734	57095	82098	11
50	756782	30.2	914246	14.7	842535	44.9	157465	57119	82082	10
51	9.756963	30.2	9.914158	14.7	9.842805	44.9	10.157195	57143	82065	9
52	757144	30.2	914070	14.7	843074	44.9	156926	57167	82048	8
53	757326	30.2	913982	14.7	843343	44.9	156657	57191	82032	7
54	757507	30.2	913894	14.7	843612	44.9	156388	57215	82015	6
55	757688	30.1	913806	14.7	843882	44.8	156118	57238	81999	5
56	757869	30.1	913718	14.7	844151	44.8	155849	57262	81982	4
57	758050	30.1	913630	14.7	844420	44.8	155580	57286	81965	3
58	758230	30.1	913541	14.7	844689	44.8	155311	57310	81949	2
59	758411	30.1	913453	14.7	844958	44.8	155042	57334	81932	1
60	758591		913365		845227		154773	57358	81915	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.758591	30.1	9.913365	14.7	9.845227	44.8	10.154773	57358	81915	60
1	758772	30.0	913276	14.7	845496	44.8	154504	57381	81899	59
2	758952	30.0	913187	14.8	845764	44.8	154236	57405	81882	58
3	759132	30.0	913099	14.8	846033	44.8	153967	57429	81865	57
4	759312	30.0	913010	14.8	846302	44.8	153698	57453	81848	56
5	759492	30.0	912922	14.8	846570	44.7	153430	57477	81832	55
6	759672	29.9	912833	14.8	846839	44.7	153161	57501	81815	54
7	759852	29.9	912744	14.8	847107	44.7	152893	57524	81798	53
8	760031	29.9	912655	14.8	847376	44.7	152624	57548	81782	52
9	760211	29.9	912566	14.8	847644	44.7	152356	57572	81765	51
10	760390	29.9	912477	14.8	847913	44.7	152087	57596	81748	50
11	9.760569	29.8	9.912388	14.8	9.848181	44.7	10.151819	57619	81731	49
12	760748	29.8	912299	14.9	848449	44.7	151551	57643	81714	48
13	760927	29.8	912210	14.9	848717	44.7	151283	57667	81698	47
14	761106	29.8	912121	14.9	848986	44.7	151014	57691	81681	46
15	761285	29.8	912031	14.9	849254	44.7	150746	57715	81664	45
16	761464	29.8	911942	14.9	849522	44.7	150478	57738	81647	44
17	761642	29.7	911853	14.9	849790	44.6	150210	57762	81631	43
18	761821	29.7	911763	14.9	850058	44.6	149942	57786	81614	42
19	761999	29.7	911674	14.9	850325	44.6	149675	57810	81597	41
20	762177	29.7	911584	14.9	850593	44.6	149407	57833	81580	40
21	9.762356	29.7	9.911495	14.9	9.850861	44.6	10.149139	57857	81563	39
22	762534	29.6	911405	14.9	851129	44.6	148871	57881	81546	38
23	762712	29.6	911315	15.0	851396	44.6	148604	57904	81530	37
24	762889	29.6	911226	15.0	851664	44.6	148336	57928	81513	36
25	763067	29.6	911136	15.0	851931	44.6	148069	57952	81496	35
26	763245	29.6	911046	15.0	852199	44.6	147801	57976	81479	34
27	763422	29.6	910956	15.0	852466	44.6	147534	57999	81462	33
28	763600	29.5	910866	15.0	852733	44.5	147267	58023	81445	32
29	763777	29.5	910776	15.0	853001	44.5	146999	58047	81428	31
30	763954	29.5	910686	15.0	853268	44.5	146732	58070	81412	30
31	9.764131	29.5	9.910596	15.0	9.853535	44.5	10.146465	58094	81395	29
32	764308	29.5	910506	15.0	853802	44.5	146198	58118	81378	28
33	764485	29.4	910415	15.0	854069	44.5	145931	58141	81361	27
34	764662	29.4	910325	15.1	854336	44.5	145664	58165	81344	26
35	764838	29.4	910235	15.1	854603	44.5	145397	58189	81327	25
36	765015	29.4	910144	15.1	854870	44.5	145130	58212	81310	24
37	765191	29.4	910054	15.1	855137	44.5	144863	58236	81293	23
38	765367	29.4	909963	15.1	855404	44.5	144596	58260	81276	22
39	765544	29.3	909873	15.1	855671	44.4	144329	58283	81259	21
40	765720	29.3	909782	15.1	855938	44.4	144062	58307	81242	20
41	9.765896	29.3	9.909691	15.1	9.856204	44.4	10.143796	58330	81225	19
42	766072	29.3	909601	15.1	856471	44.4	143529	58354	81208	18
43	766247	29.3	909510	15.1	856737	44.4	143263	58378	81191	17
44	766423	29.3	909419	15.1	857004	44.4	142996	58401	81174	16
45	766598	29.2	909328	15.2	857270	44.4	142730	58425	81157	15
46	766774	29.2	909237	15.2	857537	44.4	142463	58449	81140	14
47	766949	29.2	909146	15.2	857803	44.4	142197	58472	81123	13
48	767124	29.2	909055	15.2	858069	44.4	141931	58496	81106	12
49	767300	29.2	908964	15.2	858336	44.4	141664	58519	81089	11
50	767475	29.1	908873	15.2	858602	44.3	141398	58543	81072	10
51	9.767649	29.1	9.908781	15.2	9.858868	44.3	10.141132	58567	81055	9
52	767824	29.1	908690	15.2	859134	44.3	140866	58590	81038	8
53	767999	29.1	908599	15.2	859400	44.3	140600	58614	81021	7
54	768173	29.1	908507	15.2	859666	44.3	140334	58637	81004	6
55	768348	29.0	908416	15.3	859932	44.3	140068	58661	80987	5
56	768522	29.0	908324	15.3	860198	44.3	139802	58684	80970	4
57	768697	29.0	908233	15.3	860464	44.3	139536	58708	80953	3
58	768871	29.0	908141	15.3	860730	44.3	139270	58731	80936	2
59	769045	29.0	908049	15.3	860995	44.3	139005	58755	80919	1
60	769219		907958		861261		138739	58779	80902	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

54 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.769219	29.0	9.907958	15.3	9.861261	44.3	10.138739	58779	80902	60
1	769393	28.9	907866	15.3	861527	44.3	138473	58802	80885	59
2	769566	28.9	907774	15.3	861792	44.2	138208	58826	80867	58
3	769740	28.9	907682	15.3	862058	44.2	137942	58849	80850	57
4	769913	28.9	907590	15.3	862323	44.2	137677	58873	80833	56
5	770087	28.9	907498	15.3	862589	44.2	137411	58896	80816	55
6	770260	28.8	907406	15.3	862854	44.2	137146	58920	80799	54
7	770433	28.8	907314	15.4	863119	44.2	136881	58943	80782	53
8	770606	28.8	907222	15.4	863385	44.2	136615	58967	80765	52
9	770779	28.8	907129	15.4	863650	44.2	136350	58990	80748	51
10	770952	28.8	907037	15.4	863915	44.2	136085	59014	80730	50
11	9.771125	28.8	9.906945	15.4	9.864180	44.2	10.135820	59037	80713	49
12	771298	28.7	906852	15.4	864445	44.2	135555	59061	80696	48
13	771470	28.7	906760	15.4	864710	44.2	135290	59084	80679	47
14	771643	28.7	906667	15.4	864975	44.1	135025	59108	80662	46
15	771815	28.7	906575	15.4	865240	44.1	134760	59131	80644	45
16	771987	28.7	906482	15.4	865505	44.1	134495	59154	80627	44
17	772159	28.7	906389	15.5	865770	44.1	134230	59178	80610	43
18	772331	28.6	906296	15.5	866035	44.1	133965	59201	80593	42
19	772503	28.6	906204	15.5	866300	44.1	133700	59225	80576	41
20	772675	28.6	906111	15.5	866564	44.1	133436	59248	80558	40
21	9.772847	28.6	9.906018	15.5	9.866829	44.1	10.133171	59272	80541	39
22	773018	28.6	905925	15.5	867094	44.1	132906	59295	80524	38
23	773190	28.6	905832	15.5	867358	44.1	132642	59318	80507	37
24	773361	28.5	905739	15.5	867623	44.1	132377	59342	80489	36
25	773533	28.5	905645	15.5	867887	44.1	132113	59365	80472	35
26	773704	28.5	905552	15.5	868152	44.0	131848	59389	80455	34
27	773875	28.5	905459	15.5	868416	44.0	131584	59412	80438	33
28	774046	28.5	905366	15.6	868680	44.0	131320	59436	80422	32
29	774217	28.5	905272	15.6	868945	44.0	131055	59459	80403	31
30	774388	28.4	905179	15.6	869209	44.0	130791	59482	80386	30
31	9.774558	28.4	9.905085	15.6	9.869473	44.0	10.130527	59506	80368	29
32	774729	28.4	904992	15.6	869737	44.0	130263	59529	80351	28
33	774899	28.4	904898	15.6	870001	44.0	129999	59552	80334	27
34	775070	28.4	904804	15.6	870265	44.0	129735	59576	80316	26
35	775240	28.4	904711	15.6	870529	44.0	129471	59599	80299	25
36	775410	28.3	904617	15.6	870793	44.0	129207	59622	80282	24
37	775580	28.3	904523	15.6	871057	44.0	128943	59646	80264	23
38	775750	28.3	904429	15.7	871321	44.0	128679	59669	80247	22
39	775920	28.3	904335	15.7	871585	44.0	128415	59693	80230	21
40	776090	28.3	904241	15.7	871849	43.9	128151	59716	80212	20
41	9.776259	28.3	9.904147	15.7	9.872112	43.9	10.127888	59739	80195	19
42	776429	28.2	904053	15.7	872376	43.9	127624	59763	80178	18
43	776598	28.2	903959	15.7	872640	43.9	127360	59786	80160	17
44	776768	28.2	903864	15.7	872903	43.9	127097	59809	80143	16
45	776937	28.2	903770	15.7	873167	43.9	126833	59832	80125	15
46	777106	28.2	903676	15.7	873430	43.9	126570	59856	80108	14
47	777275	28.1	903581	15.7	873694	43.9	126306	59879	80091	13
48	777444	28.1	903487	15.7	873957	43.9	126043	59902	80073	12
49	777613	28.1	903392	15.8	874220	43.9	125780	59926	80056	11
50	777781	28.1	903298	15.8	874484	43.9	125516	59949	80038	10
51	9.777950	28.1	9.903202	15.8	9.874747	43.9	10.125253	59972	80021	9
52	778119	28.1	903108	15.8	875010	43.9	124990	59995	80003	8
53	778287	28.0	903014	15.8	875273	43.8	124727	60019	79986	7
54	778455	28.0	902919	15.8	875536	43.8	124464	60042	79968	6
55	778624	28.0	902824	15.8	875800	43.8	124200	60065	79951	5
56	778792	28.0	902729	15.8	876063	43.8	123937	60089	79934	4
57	778960	28.0	902634	15.8	876326	43.8	123674	60112	79916	3
58	779128	28.0	902539	15.9	876589	43.8	123411	60135	79899	2
59	779295	27.9	902444	15.9	876851	43.8	123149	60158	79881	1
60	779463		902349		877114		122886	60182	79864	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.779463	27.9	9.902349	15.9	9.877114	43.8	10.122886	60182	79864	60
1	779631	27.9	902253	15.9	877377	43.8	122623	60205	79846	59
2	779798	27.9	902158	15.9	877640	43.8	122360	60228	79829	58
3	779966	27.9	902063	15.9	877903	43.8	122097	60251	79811	57
4	780133	27.9	901967	15.9	878165	43.8	121835	60274	79793	56
5	780300	27.8	901872	15.9	878428	43.8	121572	60298	79776	55
6	780467	27.8	901776	15.9	878691	43.8	121309	60321	79758	54
7	780634	27.8	901681	15.9	878953	43.7	121047	60344	79741	53
8	780801	27.8	901585	15.9	879216	43.7	120784	60367	79723	52
9	780968	27.8	901490	15.9	879478	43.7	120522	60390	79706	51
10	781134	27.8	901394	16.0	879741	43.7	120259	60414	79688	50
11	9.781301	27.7	9.901298	16.0	9.880003	43.7	10.119997	60437	79671	49
12	781468	27.7	901202	16.0	880265	43.7	119735	60460	79658	48
13	781634	27.7	901106	16.0	880528	43.7	119472	60483	79635	47
14	781800	27.7	901010	16.0	880790	43.7	119210	60506	79618	46
15	781966	27.7	900914	16.0	881052	43.7	118948	60529	79600	45
16	782132	27.7	900818	16.0	881314	43.7	118686	60553	79583	44
17	782298	27.6	900722	16.0	881576	43.7	118424	60576	79565	43
18	782464	27.6	900626	16.0	881839	43.7	118161	60599	79547	42
19	782630	27.6	900529	16.0	882101	43.7	117899	60622	79530	41
20	782796	27.6	900433	16.1	882363	43.6	117637	60645	79512	40
21	9.782961	27.6	9.900337	16.1	9.882625	43.6	10.117375	60668	79494	39
22	783127	27.6	900242	16.1	882887	43.6	117113	60691	79477	38
23	783282	27.5	900144	16.1	883148	43.6	116852	60714	79459	37
24	783458	27.5	900047	16.1	883410	43.6	116590	60738	79441	36
25	783623	27.5	899951	16.1	883672	43.6	116328	60761	79424	35
26	783788	27.5	899854	16.1	883934	43.6	116066	60784	79406	34
27	783953	27.5	899757	16.1	884196	43.6	115804	60807	79388	33
28	784118	27.5	899660	16.1	884457	43.6	115543	60830	79371	32
20	784282	27.4	899564	16.1	884719	43.6	115281	60853	79353	31
39	784447	27.4	899467	16.2	884980	43.6	115020	60876	79335	30
31	9.784612	27.4	9.899370	16.2	9.885242	43.6	10.114758	60899	79318	29
32	784776	27.4	899273	16.2	885503	43.6	114497	60922	79300	28
33	784941	27.4	899176	16.2	885765	43.6	114235	60945	79282	27
34	785105	27.4	899078	16.2	886026	43.6	113974	60968	79264	26
35	785269	27.3	898981	16.2	886288	43.6	113712	60991	79247	25
36	785433	27.3	898884	16.2	886549	43.5	113451	61015	79229	24
37	785597	27.3	898787	16.2	886810	43.5	113190	61038	79211	23
38	785761	27.3	898689	16.2	887072	43.5	112928	61061	79193	22
39	785925	27.3	898592	16.2	887333	43.5	112667	61084	79176	21
40	786089	27.3	898494	16.3	887594	43.5	112406	61107	79158	20
41	9.786252	27.2	9.898397	16.3	9.887855	43.5	10.112145	61130	79140	19
42	786416	27.2	898299	16.3	888116	43.5	111884	61153	79122	18
43	786579	27.2	898202	16.3	888377	43.5	111623	61176	79105	17
44	786742	27.2	898104	16.3	888639	43.5	111361	61199	79087	16
45	786906	27.2	898006	16.3	888900	43.5	111100	61222	79069	15
46	787069	27.2	897908	16.3	889160	43.5	110840	61245	79051	14
47	787232	27.1	897810	16.3	889421	43.5	110579	61268	79033	13
48	787395	27.1	897712	16.3	889682	43.5	110318	61291	79016	12
49	787557	27.1	897614	16.3	889943	43.5	110057	61314	78998	11
50	787720	27.1	897516	16.3	890204	43.4	109796	61337	78980	10
51	9.787883	27.1	9.897418	16.4	9.890465	43.4	10.109535	61360	78962	9
52	788045	27.1	897320	16.4	890725	43.4	109275	61383	78944	8
53	788208	27.1	897222	16.4	890986	43.4	109014	61406	78926	7
54	788370	27.0	897123	16.4	891247	43.4	108753	61429	78908	6
55	788532	27.0	897025	16.4	891507	43.4	108493	61451	78891	5
56	788694	27.0	896926	16.4	891768	43.4	108232	61474	78873	4
57	788856	27.0	896828	16.4	892028	43.4	107972	61497	78855	3
58	789018	27.0	896729	16.4	892289	43.4	107711	61520	78837	2
59	789180	27.0	896631	16.4	892549	43.4	107451	61543	78819	1
60	789342		896532		892810		107190	61566	78801	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.789342	26.9	9.896532	16.4	9.892810	43.4	10.107190	61566	78801	60
1	789504	26.9	896433	16.5	893070	43.4	106930	61589	78783	59
2	789665	26.9	896335	16.5	893331	43.4	106669	61612	78765	58
3	789827	26.9	896236	16.5	893591	43.4	106409	61635	78747	57
4	789988	26.9	896137	16.5	893851	43.4	106149	61658	78729	56
5	790149	26.9	896038	16.5	894111	43.4	105889	61681	78711	55
6	790310	26.8	895939	16.5	894371	43.4	105629	61704	78694	54
7	790471	26.8	895840	16.5	894632	43.3	105368	61726	78676	53
8	790632	26.8	895741	16.5	894892	43.3	105108	61749	78658	52
9	790793	26.8	895641	16.5	895152	43.3	104848	61772	78640	51
10	790954	26.8	895542	16.5	895412	43.3	104588	61795	78622	50
11	9.791115	26.8	9.895443	16.6	9.895672	43.3	10.104328	61818	78604	49
12	791275	26.7	895343	16.6	895932	43.3	104068	61841	78586	48
13	791436	26.7	895244	16.6	896192	43.3	103808	61864	78568	47
14	791596	26.7	895145	16.6	896452	43.3	103548	61887	78550	46
15	791757	26.7	895045	16.6	896712	43.3	103288	61909	78532	45
16	791917	26.7	894945	16.6	896971	43.3	103029	61932	78514	44
17	792077	26.7	894846	16.6	897231	43.3	102769	61955	78496	43
18	792237	26.6	894746	16.6	897491	43.3	102509	61978	78478	42
19	792397	26.6	894646	16.6	897751	43.3	102249	62001	78460	41
20	792557	26.6	894546	16.6	898010	43.3	101990	62024	78442	40
21	9.792716	26.6	9.894446	16.7	9.898270	43.3	10.101730	62046	78424	39
22	792876	26.6	894346	16.7	898530	43.3	101470	62069	78405	38
23	793035	26.6	894246	16.7	898789	43.3	101211	62092	78387	37
24	793195	26.5	894146	16.7	899049	43.2	100951	62115	78369	36
25	793354	26.5	894046	16.7	899308	43.2	100692	62138	78351	35
26	793514	26.5	893946	16.7	899568	43.2	100432	62160	78333	34
27	793673	26.5	893846	16.7	899827	43.2	100173	62183	78315	33
28	793832	26.5	893745	16.7	900086	43.2	099914	62206	78297	32
29	793991	26.5	893645	16.7	900346	43.2	099654	62229	78279	31
30	794150	26.4	893544	16.7	900605	43.2	099395	62251	78261	30
31	9.794308	26.4	9.893444	16.8	9.900864	43.2	10.099136	62274	78243	29
32	794467	26.4	893343	16.8	901124	43.2	098876	62297	78225	28
33	794626	26.4	893243	16.8	901383	43.2	098617	62320	78206	27
34	794784	26.4	893142	16.8	901642	43.2	098358	62342	78188	26
35	794942	26.4	893041	16.8	901901	43.2	098099	62365	78170	25
36	795101	26.4	892940	16.8	902160	43.2	097840	62388	78152	24
37	795259	26.3	892839	16.8	902419	43.2	097581	62411	78134	23
38	795417	26.3	892739	16.8	902679	43.2	097321	62433	78116	22
39	795575	26.3	892638	16.8	902938	43.2	097062	62456	78098	21
40	795733	26.3	892536	16.8	903197	43.1	096803	62479	78079	20
41	9.795891	26.3	9.892435	16.9	9.903455	43.1	10.096545	62502	78061	19
42	796049	26.3	892334	16.9	903714	43.1	096286	62524	78043	18
43	796206	26.3	892233	16.9	903973	43.1	096027	62547	78025	17
44	796364	26.2	892132	16.9	904232	43.1	095768	62570	78007	16
45	796521	26.2	892030	16.9	904491	43.1	095509	62592	77988	15
46	796679	26.2	891929	16.9	904750	43.1	095250	62615	77970	14
47	796836	26.2	891827	16.9	905008	43.1	094992	62638	77952	13
48	796993	26.2	891726	16.9	905267	43.1	094733	62660	77934	12
49	797150	26.1	891624	16.9	905526	43.1	094474	62683	77916	11
50	797307	26.1	891523	17.0	905784	43.1	094216	62706	77897	10
51	9.797464	26.1	9.891421	17.0	9.906043	43.1	10.093957	62728	77879	9
52	797621	26.1	891319	17.0	906302	43.1	093698	62751	77861	8
53	797777	26.1	891217	17.0	906560	43.1	093440	62774	77843	7
54	797934	26.1	891115	17.0	906819	43.1	093181	62796	77824	6
55	798091	26.1	891013	17.0	907077	43.1	092923	62819	77806	5
56	798247	26.1	890911	17.0	907336	43.1	092664	62842	77788	4
57	798403	26.0	890809	17.0	907594	43.1	092406	62864	77769	3
58	798560	26.0	890707	17.0	907852	43.1	092148	62887	77751	2
59	798716	26.0	890605	17.0	908111	43.0	091889	62909	77733	1
60	798872		890503		908369		091631	62932	77715	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.798772	26.0	9.890503	17.0	9.908369	43.0	10.091631	62932	77715	60
1	799028	26.0	890400	17.1	908628	43.0	091372	62955	77696	59
2	799184	26.0	890298	17.1	908886	43.0	091114	62977	77678	58
3	799339	25.9	890195	17.1	909144	43.0	090856	63000	77660	57
4	799495	25.9	890093	17.1	909402	43.0	090598	63022	77641	56
5	799651	25.9	889990	17.1	909660	43.0	090340	63045	77623	55
6	799806	25.9	889888	17.1	909918	43.0	090082	63068	77605	54
7	799962	25.9	889785	17.1	910177	43.0	089823	63090	77586	53
8	800117	25.9	889682	17.1	910435	43.0	089565	63113	77568	52
9	800272	25.8	889579	17.1	910693	43.0	089307	63135	77550	51
10	800427	25.8	889477	17.1	910951	43.0	089049	63158	77531	50
11	9.800582	25.8	9.889374	17.2	9.911209	43.0	10.088791	93180	77513	49
12	800737	25.8	889271	17.2	911467	43.0	088533	63203	77494	48
13	800892	25.8	889168	17.2	911724	43.0	088276	63225	77476	47
14	801047	25.8	889064	17.2	911982	43.0	088018	63248	77458	46
15	801201	25.8	888961	17.2	912240	43.0	087760	63271	77439	45
16	801356	25.7	888858	17.2	912498	43.0	087502	63293	77421	44
17	801511	25.7	888755	17.2	912756	43.0	087244	63316	77402	43
18	801665	25.7	888651	17.2	913014	42.9	086986	63338	77384	42
19	801819	25.7	888548	17.2	913271	42.9	086729	63361	77366	41
20	801973	25.7	888444	17.3	913529	42.9	086471	63383	77347	40
21	9.802128	25.7	9.888341	17.3	9.913787	42.9	10.086213	63406	77329	39
22	802282	25.6	888237	17.3	914044	42.9	085956	63428	77310	38
23	802436	25.6	888134	17.3	914302	42.9	085698	63451	77292	37
24	802589	25.6	888030	17.3	914560	42.9	085440	63473	77273	36
25	802743	25.6	887926	17.3	914817	42.9	085183	63496	77255	35
26	802897	25.6	887822	17.3	915075	42.9	084925	63518	77236	34
27	803050	25.6	887718	17.3	915332	42.9	084668	63540	77218	33
28	803204	25.6	887614	17.3	915590	42.9	084410	63563	77199	32
29	803357	25.5	887510	17.3	915847	42.9	084153	63585	77181	31
30	803511	25.5	887406	17.4	916104	42.9	083896	63608	77162	30
31	9.803664	25.5	9.887302	17.4	9.916362	42.9	10.083638	63630	77144	29
32	803817	25.5	887198	17.4	916619	42.9	083381	63653	77125	28
33	803970	25.5	887093	17.4	916877	42.9	083123	63675	77107	27
34	804123	25.5	886989	17.4	917134	42.9	082866	63698	77088	26
35	804276	25.4	886885	17.4	917391	42.9	082609	63720	77070	25
36	804428	25.4	886780	17.4	917648	42.9	082352	63742	77051	24
37	804581	25.4	886676	17.4	917905	42.9	082095	63765	77033	23
38	804734	25.4	886571	17.4	918163	42.8	081837	63787	77014	22
39	804886	25.4	886466	17.4	918420	42.8	081580	63810	76996	21
40	805039	25.4	886362	17.5	918677	42.8	081323	63832	76977	20
41	9.805191	25.4	9.886257	17.5	9.918934	42.8	10.081066	63854	76959	19
42	805343	25.3	886152	17.5	919191	42.8	080809	63877	76940	18
43	805495	25.3	886047	17.5	919448	42.8	080552	63899	76921	17
44	805647	25.3	885942	17.5	919705	42.8	080295	63922	76903	16
45	805799	25.3	885837	17.5	919962	42.8	080038	63944	76884	15
46	805951	25.3	885732	17.5	920219	42.8	079781	63966	76866	14
47	806103	25.3	885627	17.5	920476	42.8	079524	63989	76847	13
48	806254	25.3	885522	17.5	920733	42.8	079267	64011	76828	12
49	806406	25.2	885416	17.5	920990	42.8	079010	64033	76810	11
50	806557	25.2	885311	17.6	921247	42.8	078753	64056	76791	10
51	9.806709	25.2	9.885205	17.6	9.921503	42.8	10.078497	64078	76772	9
52	806860	25.2	885100	17.6	921760	42.8	078240	64100	76754	8
53	807011	25.2	884994	17.6	922017	42.8	077983	64123	76735	7
54	807163	25.2	884889	17.6	922274	42.8	077726	64145	76717	6
55	807314	25.2	884783	17.6	922530	42.8	077470	64167	76698	5
56	807465	25.1	884677	17.6	922787	42.8	077213	64190	76679	4
57	807615	25.1	884572	17.6	923044	42.8	076956	64212	76661	3
58	807766	25.1	884466	17.6	923300	42.8	076700	64234	76642	2
59	807917	25.1	884360	17.6	923557	42.7	076443	64256	76623	1
60	808067		884254		923813		076187	64279	76604	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N.sine.	N. cos.	
0	9.808067	25.1	9.884254	17.7	9.923813	42.7	10.076187	64279	76604	60
1	808218	25.1	884148	17.7	924070	42.7	075930	64301	76586	59
2	808368	25.1	884042	17.7	924327	42.7	075673	64323	76567	58
3	808519	25.0	883936	17.7	924583	42.7	075417	64346	76548	57
4	808669	25.0	883829	17.7	924840	42.7	075160	64368	76530	56
5	808819	25.0	883723	17.7	925096	42.7	074904	64390	76511	55
6	808969	25.0	883617	17.7	925352	42.7	074648	64412	76492	54
7	809119	25.0	883510	17.7	925609	42.7	074391	64435	76473	53
8	809269	25.0	883404	17.7	925865	42.7	074135	64457	76455	52
9	809419	24.9	883297	17.8	926122	42.7	073878	64479	76436	51
10	809569	24.9	883191	17.8	926378	42.7	073622	64501	76417	50
11	9.809718	24.9	9.883084	17.8	9.926634	42.7	10.073366	64524	76398	49
12	809868	24.9	882977	17.8	926890	42.7	073110	64546	76380	48
13	810017	24.9	882871	17.8	927147	42.7	072853	64568	76361	47
14	810167	24.9	882764	17.8	927403	42.7	072597	64590	76342	46
15	810316	24.8	882657	17.8	927659	42.7	072341	64612	76323	45
16	810465	24.8	882550	17.8	927915	42.7	072085	64635	76304	44
17	810614	24.8	882443	17.8	928171	42.7	071829	64657	76286	43
18	810763	24.8	882336	17.9	928427	42.7	071573	64679	76267	42
19	810912	24.8	882229	17.9	928683	42.7	071317	64701	76248	41
20	811061	24.8	882121	17.9	928940	42.7	071060	64723	76229	40
21	9.811210	24.8	9.882014	17.9	9.929196	42.7	10.070804	64746	76210	39
22	811358	24.7	881907	17.9	929452	42.7	070548	64768	76192	38
23	811507	24.7	881799	17.9	929708	42.7	070292	64790	76173	37
24	811655	24.7	881692	17.9	929964	42.6	070036	64812	76154	36
25	811804	24.7	881584	17.9	930220	42.6	069780	64834	76135	35
26	811952	24.7	881477	17.9	930475	42.6	069525	64856	76116	34
27	812100	24.7	881369	17.9	930731	42.6	069269	64878	76097	33
28	812248	24.7	881261	18.0	930987	42.6	069013	64901	76078	32
29	812396	24.6	881153	18.0	931243	42.6	068757	64923	76059	31
30	812544	24.6	881046	18.0	931499	42.6	068501	64945	76041	30
31	9.812692	24.6	9.880938	18.0	9.931755	42.6	10.068245	64967	76022	29
32	812840	24.6	880830	18.0	932010	42.6	067990	64989	76003	28
33	812988	24.6	880722	18.0	932266	42.6	067734	65011	75984	27
34	813135	24.6	880613	18.0	932522	42.6	067478	65033	75965	26
35	813283	24.6	880505	18.0	932778	42.6	067222	65055	75946	25
36	813430	24.5	880397	18.0	933033	42.6	066967	65077	75927	24
37	813578	24.5	880289	18.1	933289	42.6	066711	65100	75908	23
38	813725	24.5	880180	18.1	933545	42.6	066455	65122	75889	22
39	813872	24.5	880072	18.1	933800	42.6	066200	65144	75870	21
40	814019	24.5	879963	18.1	934056	42.6	065944	65166	75851	20
41	9.814166	24.5	9.879855	18.1	9.934311	42.6	10.065689	65188	75832	19
42	814313	24.5	879746	18.1	934567	42.6	065433	65210	75813	18
43	814460	24.4	879637	18.1	934823	42.6	065177	65232	75794	17
44	814607	24.4	879529	18.1	935078	42.6	064922	65254	75775	16
45	814753	24.4	879420	18.1	935333	42.6	064667	65276	75756	15
46	814900	24.4	879311	18.1	935589	42.6	064411	65298	75738	14
47	815046	24.4	879202	18.2	935844	42.6	064156	65320	75719	13
48	815193	24.4	879093	18.2	936100	42.6	063900	65342	75700	12
49	815339	24.4	878984	18.2	936355	42.6	063645	65364	75680	11
50	815485	24.3	878875	18.2	936610	42.6	063390	65386	75661	10
51	9.815631	24.3	9.878766	18.2	9.936866	42.5	10.063134	65408	75642	9
52	815778	24.3	878656	18.2	937121	42.5	062879	65430	75623	8
53	815924	24.3	878547	18.2	937376	42.5	062624	65452	75604	7
54	816069	24.3	878438	18.2	937632	42.5	062368	65474	75585	6
55	816215	24.3	878328	18.2	937887	42.5	062113	65496	75566	5
56	816361	24.3	878219	18.3	938142	42.5	061858	65518	75547	4
57	816507	24.2	878109	18.3	938398	42.5	061602	65540	75528	3
58	816652	24.2	877999	18.3	938653	42.5	061347	65562	75509	2
59	816798	24.2	877890	18.3	938908	42.5	061092	65584	75490	1
60	816943		877780		939163		060837	65606	75471	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.816943	24.2	9.877780	18.3	9.939163	42.5	10.060837	65606	75471	60
1	817088	24.2	877670	18.3	939418	42.5	060582	65628	75452	59
2	817233	24.2	877560	18.3	939673	42.5	060327	65650	75433	58
3	817379	24.2	877450	18.3	939928	42.5	060072	65672	75414	57
4	817524	24.1	877340	18.3	940183	42.5	059817	65694	75395	56
5	817668	24.1	877230	18.4	940438	42.5	059562	65716	75375	55
6	817813	24.1	877120	18.4	940694	42.5	059306	65738	75356	54
7	817958	24.1	877010	18.4	940949	42.5	059051	65759	75337	53
8	818103	24.1	876899	18.4	941204	42.5	058796	65781	75318	52
9	818247	24.1	876789	18.4	941458	42.5	058542	65803	75299	51
10	818392	24.1	876678	18.4	941714	42.5	058286	65825	75280	50
11	9.818536	24.0	9.876568	18.4	9.941968	42.5	10.058032	65847	75261	49
12	818681	24.0	876457	18.4	942223	42.5	057777	65869	75241	48
13	818825	24.0	876347	18.4	942478	42.5	057522	65891	75222	47
14	818969	24.0	876236	18.5	942733	42.5	057267	65913	75203	46
15	819113	24.0	876125	18.5	942988	42.5	057012	65935	75184	45
16	819257	24.0	876014	18.5	943243	42.5	056757	65956	75165	44
17	819401	24.0	875904	18.5	943498	42.5	056502	65978	75146	43
18	819545	23.9	875793	18.5	943752	42.5	056248	66000	75126	42
19	819689	23.9	875682	18.5	944007	42.5	055993	66022	75107	41
20	819832	23.9	875571	18.5	944252	42.5	055738	66044	75088	40
21	9.819976	23.9	9.875459	18.5	9.944517	42.5	10.055483	66066	75069	39
22	820120	23.9	875348	18.5	944771	42.4	055229	66088	75050	38
23	820263	23.9	875237	18.5	945026	42.4	054974	66109	75030	37
24	820406	23.9	875126	18.6	945281	42.4	054719	66131	75011	36
25	820550	23.8	875014	18.6	945535	42.4	054465	66153	74992	35
26	820693	23.8	874903	18.6	945790	42.4	054210	66175	74973	34
27	820836	23.8	874791	18.6	946045	42.4	053955	66197	74953	33
28	820979	23.8	874680	18.6	946299	42.4	053701	66218	74934	32
29	821122	23.8	874568	18.6	946554	42.4	053446	66240	74915	31
30	821265	23.8	874456	18.6	946808	42.4	053192	66262	74896	30
31	9.821407	23.8	9.874344	18.6	9.947063	42.4	10.052937	66284	74876	29
32	821550	23.8	874232	18.7	947318	42.4	052682	66306	74857	28
33	821693	23.7	874121	18.7	947572	42.4	052428	66327	74838	27
34	821835	23.7	874009	18.7	947826	42.4	052174	66349	74818	26
35	821977	23.7	873896	18.7	948081	42.4	051919	66371	74799	25
36	822120	23.7	873784	18.7	948336	42.4	051664	66393	74780	24
37	822262	23.7	873672	18.7	948590	42.4	051410	66414	74760	23
38	822404	23.7	873560	18.7	948844	42.4	051156	66436	74741	22
39	822546	23.7	873448	18.7	949099	42.4	050901	66458	74722	21
40	822688	23.6	873335	18.7	949353	42.4	050647	66480	74703	20
41	9.822830	23.6	9.873223	18.7	9.949607	42.4	10.050393	66501	74683	19
42	822972	23.6	873110	18.8	949862	42.4	050138	66523	74663	18
43	823114	23.6	872998	18.8	950116	42.4	049884	66545	74644	17
44	823255	23.6	872885	18.8	950370	42.4	049630	66566	74625	16
45	823397	23.6	872772	18.8	950625	42.4	049375	66588	74606	15
46	823539	23.6	872659	18.8	950879	42.4	049121	66610	74586	14
47	823680	23.5	872547	18.8	951133	42.4	048867	66632	74567	13
48	823821	23.5	872434	18.8	951388	42.4	048612	66653	74548	12
49	823963	23.5	872321	18.8	951642	42.4	048358	66675	74522	11
50	824104	23.5	872208	18.8	951896	42.4	048104	66697	74509	10
51	9.824245	23.5	9.872095	18.9	9.952150	42.4	10.047850	66718	74489	9
52	824386	23.5	871981	18.9	952405	42.4	047595	66740	74470	8
53	824527	23.5	871868	18.9	952659	42.4	047341	66762	74451	7
54	824668	23.4	871755	18.9	952913	42.4	047087	66783	74431	6
55	824808	23.4	871641	18.9	953167	42.3	046833	66805	74412	5
56	824949	23.4	871528	18.9	953421	42.3	046579	66827	74392	4
57	825090	23.4	871414	18.9	953675	42.3	046325	66848	74373	3
58	825230	23.4	871301	18.9	953929	42.3	046071	66870	74353	2
59	825371	23.4	871187	18.9	954183	42.3	045817	66891	74334	1
60	825511		871073		954437		045563	66913	74314	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos	N.sine.	′

′	Sine.	D. 10′	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.825511	23.4	9.871073	19.0	9.954437	42.3	10.045563	66913	74314	60
1	825651	23.3	870960	19.0	954691	42.3	045309	66935	74295	59
2	825791	23.3	870846	19.0	954945	42.3	045055	66956	74276	58
3	825931	23.3	870732	19.0	955200	42.3	044800	66978	74256	57
4	826071	23.3	870618	19.0	955454	42.3	044546	66999	74237	56
5	826211	23.3	870504	19.0	955707	42.3	044293	67021	74217	55
6	826351	23.3	870390	19.0	955961	42.3	044039	67043	74198	54
7	826491	23.3	870276	19.0	956215	42.3	043785	67064	74178	53
8	826631	23.3	870161	19.0	956469	42.3	043531	67086	74159	52
9	826770	23.2	870047	19.1	956723	42.3	043277	67107	74139	51
10	826910	23.2	869933	19.1	956977	42.3	043023	67129	74120	50
11	9.827049	23.2	9.869818	19.1	9.957231	42.3	10.042769	67151	74100	49
12	827189	23.2	869704	19.1	957485	42.3	042515	67172	74080	48
13	827328	23.2	869589	19.1	957739	42.3	042261	67194	74061	47
14	827467	23.2	869474	19.1	957993	42.3	042007	67215	74041	46
15	827606	23.2	869360	19.1	958246	42.3	041754	67237	74022	45
16	827745	23.2	869245	19.1	958500	42.3	041500	67258	74002	44
17	827884	23.1	869130	19.1	958754	42.3	041246	67280	73983	43
18	828023	23.1	869015	19.2	959008	42.3	040992	67301	73963	42
19	828162	23.1	868900	19.2	959262	42.3	040738	67323	73944	41
20	828301	23.1	868785	19.2	959516	42.3	040484	67344	73924	40
21	9.828439	23.1	9.868670	19.2	9.959769	42.3	10.040231	67366	73904	39
22	828578	23.1	868555	19.2	960023	42.3	039977	67387	73885	38
23	828716	23.1	868440	19.2	960277	42.3	039723	67409	73865	37
24	828855	23.0	868324	19.2	960531	42.3	039469	67430	73846	36
25	828993	23.0	868209	19.2	960784	42.3	039216	67452	73826	35
26	829131	23.0	868093	19.2	961038	42.3	038962	67473	73806	34
27	829269	23.0	867978	19.3	961291	42.3	038709	67495	73787	33
28	829407	23.0	867862	19.3	961545	42.3	038455	67516	73767	32
29	829545	23.0	867747	19.3	961799	42.3	038201	67538	73747	31
30	829683	23.0	867631	19.3	962052	42.3	037948	67559	73728	30
31	9.829821	22.9	9.867515	19.3	9.962306	42.3	10.037694	67580	73708	29
32	829959	22.9	867399	19.3	962560	42.3	037440	67602	73688	28
33	830097	22.9	867283	19.3	962813	42.3	037187	67623	73669	27
34	830234	22.9	867167	19.3	963067	42.3	036933	67645	73649	26
35	830372	22.9	867051	19.3	963320	42.3	036680	67666	73629	25
36	830509	22.9	866935	19.4	963574	42.3	036426	67688	73610	24
37	830646	22.9	866819	19.4	963827	42.3	036173	67709	73590	23
38	830784	22.9	866703	19.4	964081	42.3	035919	67730	73570	22
39	830921	22.8	866586	19.4	964335	42.3	035665	67752	73551	21
40	831058	22.8	866470	19.4	964588	42.2	035412	67773	73531	20
41	9.831195	22.8	9.866353	19.4	9.964842	42.2	10.035158	67795	73511	19
42	831332	22.8	866237	19.4	965095	42.2	034905	67816	73491	18
43	831469	22.8	866120	19.4	965349	42.2	034651	67837	73472	17
44	831606	22.8	866004	19.5	965602	42.2	034398	67859	73452	16
45	831742	22.8	865887	19.5	965855	42.2	034145	67880	73432	15
46	831879	22.8	865770	19.5	966109	42.2	033891	67901	73413	14
47	832015	22.7	865653	19.5	966362	42.2	033638	67923	73393	13
48	832152	22.7	865536	19.5	966616	42.2	033384	67944	73373	12
49	832288	22.7	865419	19.5	966869	42.2	033131	67965	73353	11
50	832425	22.7	865302	19.5	967123	42.2	032877	67987	73333	10
51	9.832561	22.7	9.865185	19.5	9.967376	42.2	10.032624	68008	73314	9
52	832697	22.7	865068	19.5	967629	42.2	032371	68029	73294	8
53	832833	22.7	864950	19.5	967883	42.2	032117	68051	73274	7
54	832969	22.6	864833	19.6	968136	42.2	031864	68072	73254	6
55	833105	22.6	864716	19.6	968389	42.2	031611	68093	73234	5
56	833241	22.6	864598	19.6	968643	42.2	031357	68115	73215	4
57	833377	22.6	864481	19.6	968896	42.2	031104	68136	73195	3
58	833512	22.6	864363	19.6	969149	42.2	030851	68157	73175	2
59	833648	22.6	864245	19.6	969403	42.2	030597	68179	73155	1
60	833783		864127		969656		030344	68200	73135	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.833783	22.6	9.864127	19.6	9.969656	42.2	10.030344	68200	73135	60
1	833919	22.5	864010	19.6	969909	42.2	030091	68221	73116	59
2	834054	22.5	863892	19.7	970162	42.2	029838	68242	73096	58
3	834189	22.5	863774	19.7	970416	42.2	029584	68264	73076	57
4	834325	22.5	863656	19.7	970669	42.2	029331	68285	73056	56
5	834460	22.5	863538	19.7	970922	42.2	029078	68306	73036	55
6	834595	22.5	863419	19.7	971175	42.2	028825	68327	73016	54
7	834730	22.5	863301	19.7	971429	42.2	028571	68349	72996	53
8	834865	22.5	863183	19.7	971682	42.2	028318	68370	72976	52
9	834999	22.4	863064	19.7	971935	42.2	028065	68391	72957	51
10	835134	22.4	862946	19.8	972188	42.2	027812	68412	72937	50
11	9.835269	22.4	9.862827	19.8	9.972441	42.2	10.027559	68434	72917	49
12	835403	22.4	862709	19.8	972694	42.2	027306	68455	72897	48
13	835538	22.4	862590	19.8	972948	42.2	027052	68476	72877	47
14	835672	22.4	862471	19.8	973201	42.2	026799	68497	72857	46
15	835807	22.4	862353	19.8	973454	42.2	026546	68518	72837	45
16	835941	22.4	862234	19.8	973707	42.2	026293	68539	72817	44
17	836075	22.3	862115	19.8	973960	42.2	026040	68561	72797	43
18	836209	22.3	861996	19.8	974213	42.2	025787	68582	72777	42
19	836343	22.3	861877	19.8	974466	42.2	025534	68603	72757	41
20	836477	22.3	861758	19.9	974719	42.2	025281	68624	72737	40
21	9.836611	22.3	9.861638	19.9	9.974973	42.2	10.025027	68645	72717	39
22	836745	22.3	861519	19.9	975226	42.2	024774	68666	72697	38
23	836878	22.3	861400	19.9	975479	42.2	024521	68688	72677	37
24	837012	22.2	861280	19.9	975732	42.2	024268	68709	72657	36
25	837146	22.2	861161	19.9	975985	42.2	024015	68730	72637	35
26	837279	22.2	861041	19.9	976238	42.2	023762	68751	72617	34
27	837412	22.2	860922	19.9	976491	42.2	023509	68772	72597	33
28	837546	22.2	860802	19.9	976744	42.2	023256	68793	72577	32
29	837679	22.2	860682	20.0	976997	42.2	023003	68814	72557	31
30	837812	22.2	860562	20.0	977250	42.2	022750	68835	72537	30
31	9.837945	22.2	9.860442	20.0	9.977503	42.2	10.022497	68857	72517	29
32	838078	22.1	860322	20.0	977756	42.2	022244	68878	72497	28
33	838211	22.1	860202	20.0	978009	42.2	021991	68899	72477	27
34	838344	22.1	860082	20.0	978262	42.2	021738	68920	72457	26
35	838477	22.1	859962	20.0	978515	42.2	021485	68941	72437	25
36	838610	22.1	859842	20.0	978768	42.2	021232	68962	72417	24
37	838742	22.1	859721	20.1	979021	42.2	020979	68983	72397	23
38	838875	22.1	859601	20.1	979274	42.2	020726	69004	72377	22
39	839007	22.1	859480	20.1	979527	42.2	020473	69025	72357	21
40	839140	22.0	859360	20.1	979780	42.2	020220	69046	72337	20
41	9.839272	22.0	9.859239	20.1	9.980033	42.2	10.019967	69067	72317	19
42	839404	22.0	859119	20.1	980286	42.2	019714	69088	72297	18
43	839536	22.0	858998	20.1	980538	42.2	019462	69109	72277	17
44	839668	22.0	858877	20.1	980791	42.1	019209	69130	72257	16
45	839800	22.0	858756	20.2	981044	42.1	018956	69151	72236	15
46	839932	22.0	858635	20.2	981297	42.1	018703	69172	72216	14
47	840064	21.9	858514	20.2	981550	42.1	018450	69193	72196	13
48	840196	21.9	858393	20.2	981803	42.1	018197	69214	72176	12
49	840328	21.9	858272	20.2	982056	42.1	017944	69235	72156	11
50	840459	21.9	858151	20.2	982309	42.1	017691	69256	72136	10
51	9.840591	21.9	9.858029	20.2	9.982562	42.1	10.017438	69277	72116	9
52	840722	21.9	857908	20.2	982814	42.1	017186	69298	72095	8
53	840854	21.9	857786	20.2	983067	42.1	016933	69319	72075	7
54	840985	21.9	857665	20.3	933320	42.1	016680	69340	72055	6
55	841116	21.8	857543	20.3	983573	42.1	016427	69361	72035	5
56	841247	21.8	857422	20.3	983826	42.1	016174	69382	72015	4
57	841378	21.8	857300	20.3	934079	42.1	015921	69403	71995	3
58	841509	21.8	857178	20.3	984331	42.1	015669	69424	71974	2
59	841640	21.8	857056	20.3	984584	42.1	015416	69445	71954	1
60	841771		856934		984837		015163	69466	71934	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N. sine.	′

46 Degrees.

′	Sine.	D. 10″	Cosine.	D. 10″	Tang.	D. 10″	Cotang.	N. sine.	N. cos.	
0	9.841771	21.8	9.856934	20.3	9.984837	42.1	10.015163	69466	71934	60
1	841902	21.8	856812	20.3	985090	42.1	014910	69487	71914	59
2	842033	21.8	856690	20.4	985343	42.1	014657	69508	71894	58
3	842163	21.7	856568	20.4	985596	42.1	014404	69529	71873	57
4	842294	21.7	856446	20.4	985848	42.1	014152	69549	71853	56
5	842424	21.7	856323	20.4	986101	42.1	013899	69570	71833	55
6	842555	21.7	856201	20.4	986354	42.1	013646	69591	71813	54
7	842685	21.7	856078	20.4	986607	42.1	013393	69612	71792	53
8	842815	21.7	855956	20.4	986860	42.1	013140	69633	71772	52
9	842946	21.7	855833	20.4	987112	42.1	012888	69654	71752	51
10	843076	21.7	855711	20.5	987365	42.1	012635	69675	71732	50
11	9.843206	21.6	9.855588	20.5	9.987618	42.1	10.012382	69696	71711	49
12	843336	21.6	855465	20.5	987871	42.1	012129	69717	71691	48
13	843466	21.6	855342	20.5	988123	42.1	011877	69737	71671	47
14	843595	21.6	855219	20.5	988376	42.1	011624	69758	71650	46
15	843725	21.6	855096	20.5	988629	42.1	011371	69779	71630	45
16	843855	21.6	854973	20.5	988882	42.1	011118	69800	71610	44
17	843984	21.6	854850	20.5	989134	42.1	010866	69821	71590	43
18	844114	21.5	854727	20.6	989387	42.1	010613	69842	71569	42
19	844243	21.5	854603	20.6	989640	42.1	010360	69862	71549	41
20	844372	21.5	854480	20.6	989893	42.1	010107	69883	71529	40
21	9.844502	21.5	9.854356	20.6	9.990145	42.1	10.009855	69904	71508	39
22	844631	21.5	854233	20.6	990398	42.1	009602	69925	71488	38
23	844760	21.5	854109	20.6	990651	42.1	009349	69946	71468	37
24	844889	21.5	853986	20.6	990903	42.1	009097	69966	71447	36
25	845018	21.5	853862	20.6	991156	42.1	008844	69987	71427	35
26	845147	21.5	853738	20.6	991409	42.1	008591	70008	71407	34
27	845276	21.4	853614	20.7	991662	42.1	008338	70029	71386	33
28	845405	21.4	853490	20.7	991914	42.1	008086	70049	71366	32
29	845533	21.4	853366	20.7	992167	42.1	007833	70070	71345	31
30	845662	21.4	853242	20.7	992420	42.1	007580	70091	71325	30
31	9.845790	21.4	9.853118	20.7	9.992672	42.1	10.007328	70112	71305	29
32	845919	21.4	852994	20.7	992925	42.1	007075	70132	71284	28
33	846047	21.4	852869	20.7	993178	42.1	006822	70153	71264	27
34	846175	21.4	852745	20.7	993430	42.1	006570	70174	71243	26
35	846304	21.4	852620	20.7	993683	42.1	006317	70195	71223	25
36	846432	21.3	852496	20.8	993936	42.1	006064	70215	71203	24
37	846560	21.3	852371	20.8	994189	42.1	005811	70236	71182	23
38	846688	21.3	852247	20.8	994441	42.1	005559	70257	71162	22
39	846816	21.3	852122	20.8	994694	42.1	005306	70277	71141	21
40	846944	21.3	851997	20.8	994947	42.1	005053	70298	71121	20
41	9.847071	21.3	9.851872	20.8	9.995199	42.1	10.004801	70319	71100	19
42	847199	21.3	851747	20.8	995452	42.1	004548	70339	71080	18
43	847327	21.3	851622	20.8	995705	42.1	004295	70360	71059	17
44	847454	21.2	851497	20.9	995957	42.1	004043	70381	71039	16
45	847582	21.2	851372	20.9	996210	42.1	003790	70401	71019	15
46	847709	21.2	851246	20.9	996463	42.1	003537	70422	70998	14
47	847836	21.2	851121	20.9	996715	42.1	003285	70443	70978	13
48	847964	21.2	850996	20.9	996968	42.1	003032	70463	70957	12
49	848091	21.2	850870	20.9	997221	42.1	002779	70484	70937	11
50	848218	21.2	850745	20.9	997473	42.1	002527	70505	70916	10
51	9.848345	21.2	9.850619	20.9	9.997726	42.1	10.002274	70525	70896	9
52	848472	21.1	850493	21.0	997979	42.1	002021	70546	70875	8
53	848599	21.1	850368	21.0	998231	42.1	001769	70567	70855	7
54	848726	21.1	850242	21.0	998484	42.1	001516	70587	70834	6
55	848852	21.1	850116	21.0	998737	42.1	001263	70608	70813	5
56	848979	21.1	849990	21.0	998989	42.1	001011	70628	70793	4
57	849106	21.1	849864	21.0	999242	42.1	000758	70649	70772	3
58	849232	21.1	849738	21.0	999495	42.1	000505	70670	70752	2
59	849359	21.1	849611	21.0	999748	42.1	000253	70690	70731	1
60	849485		849485		10.000000		000000	70711	70711	0
	Cosine.		Sine.		Cotang.		Tang.	N. cos.	N.sine.	′

TABLE III.

LOGARITHMS OF NUMBERS.

From 1 to 200,

INCLUDING TWELVE DECIMAL PLACES.

N.	Log.	N.	Log.	N.	Log.
1	000000 000000	41	612783 856720	81	908485 018879
2	301029 995664	42	623249 290398	82	913813 852384
3	477121 254720	43	633468 455580	83	919078 092376
4	602059 991328	44	643452 676486	84	924279 286062
5	698970 004336	45	653212 513775	85	929418 925714
6	778151 250384	46	662757 831682	86	934498 451244
7	845098 040014	47	672097 857926	87	939519 252619
8	903089 986992	48	681241 237376	88	944482 672150
9	954242 509439	49	690196 080028	89	949390 006645
10	Same as to 1.	50	Same as to 5.	90	Same as to 9.
11	041392 685158	51	707570 176098	91	959041 392321
12	079181 246048	52	716003 343635	92	963787 827346
13	113943 352307	53	724275 869601	93	968482 948554
14	146128 035678	54	732393 759823	94	973127 853600
15	176091 259056	55	740362 689494	95	977723 605889
16	204119 982656	56	748188 027006	96	982271 233040
17	230448 921378	57	755874 855672	97	986771 734266
18	255272 505103	58	763427 993563	98	991226 075692
19	278753 600953	59	770852 011642	99	995635 194598
20	Same as to 2.	60	Same as to 6	100	Same as to 10.
21	322219 2947	61	785329 835011	101	004321 373783
22	342422 680822	62	792391 699498	102	008600 171762
23	361727 836018	63	799340 549453	103	012837 224705
24	380211 241712	64	806179 973984	104	017033 339299
25	397940 008672	65	812913 356643	105	021189 299070
26	414973 347971	66	819543 935542	106	025305 865265
27	431363 764159	67	826074 802701	107	029383 777685
28	447158 031342	68	832508 912706	108	033423 755487
29	462397 997899	69	838849 090737	109	037426 497941
30	Same as to 3.	70	Same as to 7.	110	Same as to 11.
31	491361 693834	71	851258 348719	111	045322 978787
32	505149 978320	72	857332 496431	112	049218 022670
33	518513 939878	73	863322 860120	113	053078 443483
34	531478 917042	74	869231 719731	114	056904 851336
35	544068 044350	75	875061 263392	115	060697 840354
36	556302 500767	76	880813 592281	116	064457 989227
37	568201 724067	77	886490 725172	117	068185 861746
38	579783 596617	78	892094 602690	118	071882 007306
39	591064 607026	79	897627 091290	119	075546 961393
40	Same as to 4.	80	Same as to 8.	120	Same as to 12.

N.	Log.	N.	Log.	N.	Log
121	082785 370316	148	170261 715395	175	243038 048686
122	086359 830675	149	173186 268412	176	245512 667814
123	089905 111439	150	176091 259056	177	247973 266362
124	093421 685162	151	178976 947293	178	250420 002309
125	096910 013008	152	181843 587945	179	252853 030980
126	100370 545118	153	184691 430818	180	255272 505103
127	103803 720956	154	187520 720836	181	257678 574869
128	107209 969648	155	190331 698170	182	260071 387985
129	110589 710299	156	193124 588354	183	262451 089730
130	Same as to 13.	157	195899 652409	184	264817 823010
131	117271 295656	158	198657 086954	185	267171 728403
132	120573 931206	159	201397 124320	186	269512 944218
133	123851 640967	160	204119 982656	187	271841 606536
134	127104 798365	161	206825 876032	188	274157 849264
135	130333 768495	162	209515 014543	189	276461 804173
136	133538 908370	163	212187 604404	190	278753 600953
137	136720 567156	164	214843 848048	191	281033 367248
138	139879 086401	165	217483 944214	192	283301 228704
139	143014 800254	166	220108 088040	193	285557 309008
140	146128 035678	167	222716 471148	194	287801 729930
141	149219 112655	168	225309 281726	195	290034 611362
142	152288 344383	169	227886 704614	196	292256 071356
143	155336 037465	170	230448 921378	197	294466 226162
144	158362 492095	171	232996 110392	198	296665 190262
145	161368 002235	172	235528 446908	199	298853 076410
146	164352 855784	173	238046 103129		
147	167317 334748	174	240549 248283		

LOGARITHMS OF THE PRIME NUMBERS

From 200 to 1543,

INCLUDING TWELVE DECIMAL PLACES.

N.	Log.	N.	Log.	N.	Log.
201	303196 057420	277	442479 769064	379	578639 209968
203	307496 037913	281	448706 319905	383	583198 773968
207	315970 345457	283	451786 435524	389	589949 601326
209	320146 286111	293	466867 620354	397	598790 506763
211	324282 455298	307	487138 375477	401	603144 372620
223	348304 863048	311	492760 389027	409	611723 308007
227	356025 857193	313	495544 337546	419	622214 022966
229	359835 482340	317	501059 262218	421	624282 095836
233	367355 921026	331	519827 993776	431	634477 270161
239	378397 900948	337	527629 900871	433	636487 896353
241	382017 042575	347	540329 474791	439	642424 520242
251	399673 721481	349	542825 426959	443	646403 726223
257	409933 123331	353	547774 705388	449	652246 341003
263	419955 748490	359	555094 448578	457	659916 200070
269	429752 280002	367	564666 064252	461	663700 925390
271	432969 290874	373	571708 831809	463	665580 991018

N.	Log.	N.	Log.	N.	Log.
467	669316 880566	821	914343 157119	1171	068556 895072
479	680335 513414	823	915399 835212	1181	072249 807613
487	687528 961215	827	917505 509553	1187	074450 718955
491	691081 492123	829	918554 530550	1193	076640 443670
499	698100 545623	839	923761 960829	1201	0.9543 007385
503	701567 985056	853	930949 031168	1213	083860 800845
509	706717 782337	857	932980 821923	1217	085290 578210
521	716837 723300	859	933993 163831	1223	087426 458017
523	718501 688867	863	936010 795715	1229	089551 882866
541	733197 265107	877	942999 593356	1231	090258 052912
547	737987 326333	881	944975 908412	1237	092369 699609
557	745855 195174	883	945960 703578	1249	096562 438356
563	750508 394851	887	947923 619832	1259	100025 729204
569	755112 266395	907	957607 287060	1277	106190 896808
571	756636 108245	911	959518 376973	1279	106870 542460
577	761175 813156	919	963315 511386	1283	108226 656362
587	768638 101248	929	968015 713994	1289	110252 917337
593	773054 693364	937	971739 590888	1291	110926 242517
599	777426 822389	941	973589 623427	1297	112939 986066
601	778874 472002	947	976349 979003	1301	114277 296540
607	783138 691075	953	979092 900638	1303	114944 415712
613	787460 474518	967	985426 474083	1307	116275 587564
617	790285 164033	971	987219 229908	1319	120244 795568
619	791690 649020	977	989894 563719	1321	120902 817604
631	800029 359244	983	992553 517832	1327	122870 922849
641	806858 029519	991	996073 654485	1361	133858 125188
643	808210 972924	997	998695 158312	1367	135768 514554
647	810904 280669	1009	003891 166237	1373	137670 537223
653	814913 181275	1013	005609 445360	1381	140193 678544
659	818885 414594	1019	008174 184006	1399	145817 714122
661	810201 459486	1021	009025 742087	1409	148910 994096
673	828015 064224	1031	013258 665284	1423	153204 896557
677	830588 668685	1033	014100 321520	1427	154424 012366
683	834420 703682	1039	016615 547557	1429	155032 228774
691	839478 047374	1049	020775 488194	1433	156246 402184
701	845718 017967	1051	021602 716028	1439	158060 793919
709	850646 235183	1061	025715 383901	1447	160468 531109
719	856728 890383	1063	026533 264523	1451	161667 412427
727	861534 410859	1069	028977 705209	1453	162265 614286
733	865103 974742	1087	036229 544086	1459	164055 291883
739	868644 488395	1091	037824 750588	1471	167612 672629
743	870988 813761	1093	038620 161950	1481	170555 058512
751	855639 937004	1097	040206 627575	1483	171141 151014
757	879095 879500	1103	042595 512440	1487	172310 968489
761	881384 656771	1109	044931 546149	1489	172894 731332
769	885926 339801	1117	048053 173116	1493	174059 807708
773	888179 493918	1123	050379 756261	1499	175801 632866
787	895974 732359	1129	052693 941925	1511	179264 464329
797	901458 321396	1151	061075 323630	1523	182699 903324
809	907948 521612	1153	061829 307295	1531	184975 190807
811	909020 854211	1163	065579 714728	1543	188365 926053

AUXILIARY LOGARITHMS.

N.	Log.		N.	Log.	
1.009	003891166237	A	1.0009	000390689248	B
1.008	003460532110		1.0008	000347296684	
1.007	003029470554		1.0007	000303899784	
1.006	002598080685		1.0006	000260498547	
1.005	002166061756		1.0005	000217092970	
1.004	001733712775		1.0004	000173683057	
1.003	001300933020		1.0003	000130268804	
1.002	000867721529		1.0002	000086850211	
1.001	000434077479		1.0001	000043427277	

C

N.	Log.	N.	Log.
1.00009	000039083266	1.000009	000003908628
1.00008	000034740691	1.000008	000003474338
1.00007	000030398072	1.000007	000003040047
1.00006	000026055410	1.000006	000002605756
1.00005	000021712704	1.000005	000002171464
1.00004	000017371430	1.000004	000001737173
1.00003	000013028638	1.000003	000001302880
1.00002	000008685802	1.000002	000000868587
1.00001	000004342923	1.000001	000000434294

N.	Log.	
1.0000001	000000043429	(n)
1.00000001	000000004343	(o)
1.000000001	000000000434	(p)
1.0000000001	000000000043	(q)

m=0.4342944819 log. —1.637784298.

By the preceding tables — and the auxiliaries *A*, *B*, and *C*, we can find the logarithm of any number, true to at least ten decimal places.

But some may prefer to use the following direct formula, which may be found in any of the standard works on algebra:

$$\text{Log. } (z+1) = \text{log.} z + 0.8685889638 \left(\frac{1}{2z+1} \right)$$

The result will be true to twelve decimal places, if z be over 2000.

The log. of composite numbers can be determined by the combination of logarithms, already in the table, and the prime numbers from the formula.

Thus, the number 3083 is a prime number, find its logarithm.

We first find the log. of the number 3082. By factoring, we discover that this is the product of 46 into 67.

Log. 46,	1.6627578316
Log. 67,	1.8260748027
Log. 3082	3.4888326343

$$\text{Log. } 3083 = 3.4888326343 + \frac{0.8685889638}{6165}$$

NUMBERS AND THEIR LOGARITHMS,

OFTEN USED IN COMPUTATIONS.

		Log.
Circumference of a circle to dia. 1 / Surface of a sphere to diameter 1 / Area of a circle to *radius* 1	=3.14159265	0.4971499
Area of a circle to diameter 1	= .7853982	—1.8950899
Capacity of a sphere to diameter 1	= .5235988	—1.7189986
Capacity of a sphere to radius 1	=4.1887902	0.6220886
Arc of any circle equal to the radius	=57°29578	1.7581226
Arc equal to radius expressed in sec.	=206264″8	5.3144251
Length of a degree, (radius unity)	=.01745329	—2.2418773
12 hours expressed in seconds,	= 43200	4.6354837
Complement of the same,	=0.00002315	—5.3645163
360 degrees expressed in seconds,	= 1296000	6.1126050

A gallon of distilled water, when the temperature is 62° Fahrenheit, and Barometer 30 inches, is $277.\frac{274}{1000}$ cubic inches.

$\sqrt{277.274} = 16.651542$ nearly.

$\sqrt{\frac{277.274}{.775398}} = 18.78925284$ $\sqrt{231} = 15.198684.$

$\sqrt{282} = 16.792855.$

$\sqrt{\frac{282.}{.785398}} = 18.948708.$

The French Metre=3.2808992, English *feet* linear measure, =39.3707904 inches, the length of a pendulum vibrating seconds.