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TECHNICAL MECHANICS

STATICS AND DYNAMICS

BY

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THIRD EDITION, REWRITTEN

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PREFACE

THE following paragraph is an adaptation from the preface of the first edition of this work, published ten years ago; it applies to the present edition.

This book might be described fairly as a theoretical mechanics for students of engineering. It is not comparable to books commonly called Theoretical Mechanics, generally intended for students of mathematics or physics; nor to books commonly titled Applied Mechanics which generally include a treatment of strength of materials, hydraulics, etc., for students of engineering. The title Technical Mechanics seems fairly appropriate for this book; and inasmuch as it is not otherwise used in this country, it was so adopted. On the theoretical side, practically each subject discussed herein has a direct bearing on some engineering problem. The applications were selected and presented for the purpose of illustrating a principle of mechanics and for training students in the use of such principles, — not to furnish information, except incidentally, about the structure, machine, or what not to which the application was made.

Ten years use of the book as a text in the author's classes has suggested many changes; and in recent years the need of a new collection of problems has become urgent. Accordingly, a revision was undertaken, and the effort has resulted in a practically rewritten book. Indeed the only portion of the former edition used again with little or no change is the present Appendix A. Though containing fewer pages than the old book, the new one — because of its (nearly one-third) larger printed page — contains more material than the old.

Inasmuch as Mechanics deals mainly with subjects permanent in character, the revision consists principally of changes in arrangement and presentation. Both were determined upon to a large degree by a desire to furnish an adequate course of instruction for students in engineering in one semester, "five times per week." To this end, it was necessary to sacrifice logical order of arrangement more or less. As in former editions, Statics is presented first because relatively simpler than Dynamics. Kinematics, as such, is not given a place. The chapter on Attraction and Stress has not been retained. Discussion of Friction and Efficiency has been amplified, and Dynamics has been extended to provide a quantitative explanation of simple gyroscopic action. Many solved numerical examples have been added to elucidate principles. The collection of problems to be solved by students has been completely changed.

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All of Statics except Arts. 23, 25, 26, and 27 may be mastered with no knowledge of mathematics beyond trigonometry. Calculus methods are used in Dynamics, but a good knowledge of the elements only of that branch of mathematics is presupposed. Graphical methods are used freely, as much as the algebraic in Statics.

The author is pleased to acknowledge with thanks the helpful suggestions and criticisms of the teaching staff in Mechanics at the University of Illinois; of his colleague, Professor M. O. Withey; and of Professor C. H. Burnside of Columbia University. He thanks also *American Machinist*, *Engineering Record*, and *Engineering News* for permission to copy and for gifts of cuts; and individuals and other journals named in the text for similar favors.

MADISON, WISCONSIN.
December, 1913.

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UNIV. OF CALIFORNIA

TECHNICAL MECHANICS

1. Introduction

Mechanics had its origin in the experience of ancient peoples with devices for lifting and moving heavy things. The devices included the so-called simple machines or mechanical powers; namely, the lever, the pulley, the wheel and axle, the inclined plane, the wedge and the screw. That experience probably afforded fairly definite and full knowledge of the practical advantages of these various devices, but the simple and precise mechanical principles involved in them were long unrecognized. The first recognition of such a principle marked the real beginning of the science of Mechanics.

History records that the principle of the lever is the mechanical principle first discovered, and that *Archimedes* (287–212 B.C.), famous Greek mathematician, was the discoverer. He perceived the application of this principle to the wheel and axle (continuous lever), to the pulley (movable lever), and to certain combinations or systems of pulleys and cords, one of which still bears his name. The discovery of the principle of buoyant effort on a body floating on or immersed in a fluid is due to him. Apparently no additions to these achievements of Archimedes were made during the sixteen centuries following his time.

The principle of the lever as understood by Archimedes covered only the special case of two heavy weights suspended from a horizontal bar supported at a point (fulcrum) between them. For such case he stated that the weights are inversely as the distances from the fulcrum to the points of suspension. The principle was extended to include the case of forces applied obliquely, by *Leonardo da Vinci* (1452–1519), famous Italian artist and engineer. He perceived that the efficacy of such a force depends on the distance from the fulcrum, not to the point of application of the force, but to its line of action.

The principle next discovered was that of the inclined plane, first definitely stated by *Simon Stevin* (1548–1620), Dutch mathematician and engineer. His statement of the principle was somewhat as follows: The force (acting along the plane) required to support a (frictionless) body resting upon it is to the weight of the body as the height of the plane is to its length (measured along the slope). This principle afforded the explanation of the wedge (double inclined plane) and the screw (continuous inclined plane). Stevin deduced the parallelogram law for two forces at right angles from the principle of the inclined plane; and from his study of pulleys he noted that what is gained in power is lost in speed. Thus he caught the first glimpse of two important principles, — that of the parallelogram of forces, and that of virtual velocity or work.

The first discoveries of laws of motion were made by *Galileo* (1564–1642), Italian astronomer and physicist. For 2000 years it had been believed that heavy bodies fall more rapidly than light ones. This Galileo disproved by actual trial at the leaning tower of Pisa. Next he was led to inquire about the manner in which a body falls, or how the speed changes. He made several guesses at this law, and finally verified one of them by indirect experiment and deduction. Up to Galileo's time, it was believed that rest was the natural condition for a body; and that motion was unnatural, requiring some outside cause (force) to maintain it, and ceasing only when the force ceases. Galileo perceived that motion is just as natural as rest; that motions cease not because they are unnatural, but because of some influence (force) from the outside operating to reduce the motion and eventually to destroy it. In short, he discovered the so-called first law of motion, usually credited to Newton. He invented the telescope.

Huygens (1629–1695), Dutch physicist, made some important contributions to this science. He developed the theory of the pendulum, determined the acceleration due to gravity from pendulum observations, and deduced certain theorems regarding centrifugal force. He invented the clock pendulum and escapement.

Newton (1642–1727), English mathematician and physicist, is generally regarded as the founder of Mechanics. At an early age he began an attempt to explain the motions of the planets, whose orbits and speeds were then well known, in terms of experience with more familiar motions. He succeeded in thus explaining many features of the planetary motions, and established that there are certain principles common to the motion of all bodies, celestial and terrestial. These principles are generally known as Newton's laws of motions (see index). His study of planetary motion led to other great achievements, among which may be mentioned the discovery of the law of universal gravitation, and the invention of the calculus (also invented independently by Leibnitz, German mathematician).

Since Newton, "no essentially new principle [of Mechanics] has been stated. All that has been accomplished since his day has been a deductive, formal, and mathematical development on the basis of Newton's laws."* Such development constitutes the body of knowledge which we call Mechanics, or sometimes *Rational* and *Theoretical Mechanics*, to distinguish it from Applied Mechanics. It may be defined as the science of motion, but it includes the science of rest as a relatively minor part.

* For a full and critical account of that development, see Mach's "Science of Mechanics," from which the quotation was taken, or Cox's "Mechanics" for a good but less critical account.

Adaptations of rational mechanics have played an important part in the development of the science of engineering, particularly in the departments of structures and machines. Such adaptations, together with our knowledge of friction, strength of materials, and certain properties of fluids, constitute *Applied Mechanics.* Among the pioneer workers in this field should be mentioned the following: *Coulomb* (1736–1806), *Navier* (1785–1836), *Poncelet* (1788–1867), *Morin* (1795–1880), *Saint-Vênant* (1797–1886), *Weisbach* (1806–71), *Rankine* (1820–72), *Grashof* (1826–93) and *Bauschinger* (1834–93).*

Under *Technical Mechanics,* the present author includes those principles of rational mechanics which are especially applicable in various fields of engineering, and some of our knowledge of friction. The book is divided into two parts called *Statics* and *Dynamics*. The first deals with certain of the circumstances of bodies at rest, and the second with those of bodies in motion. The certain circumstances dealt with will become apparent to the student as he progresses in the subject.

* See Keck's "Mechanik" for an account of their work and fuller list.

STATICS

CHAPTER I

COMPOSITION AND RESOLUTION OF FORCES

2. Force; Definitions

Bodies act upon each other in various ways, producing different kinds of results. Any action of one body upon another which, when exerted alone, would result in motion of the body acted upon, or in change of motion if the body is already moving, is called *force;* the word is a general term for push and pull. Our earliest notions about forces are based on our experience with forces exerted by or upon ourselves. Through this experience we have learned that a force has magnitude, place of application, and direction, sometimes called the *characteristics of a force.*

To express the magnitude of a force, we must of course compare it to some other force regarded as a unit. Many units of force are in use; the most convenient are the so-called *gravitation units.* They are the earth-pulls on our standards for measuring quantity of material (as iron, coal, grain, sugar, etc.), commonly called standards of weight.* The earth-pull on any of these standards is called by the name of the standard; thus the earth-pull on the pound standard (also any equal force) is called a *pound;* the earth-pull on the kilogram standard (also any equal force) is called a *kilogram,* etc. Since the earth-pull on any given thing varies in amount as the thing is transported from place to place, gravitation units of force are not constant with regard to place. But this variation need not be regarded in most engineering calculations because any error due to such disregard is generally smaller than errors due to other approximations in the calculations. The extreme variation in any gravitation unit is that between its magnitudes at the highest elevation on the equator and at the poles; this difference is but 0.6 per cent. For points within the United States the extreme variation equals about 0.3 per cent. For any two

* In common parlance the word *weight* is used in at least two senses. Thus, suppose that a dealer sells coal to a consumer by weight, and engages a teamster to deliver it by weight; to the consumer, the weight of each wagon load represents a certain amount of useful material, but to the teamster it represents a certain burden on his team due to the action of gravity on the coal. That is, weight suggests *material* to the one man and *earth-pull* to the other.

points on the surface of the earth, the variation equals that in the values of g in the formula

$$g = 32.0894\,(1 + 0.0052375 \sin^2 l)\,(1 - 0.0000000957\, e)$$

computed for the two places; l denotes latitude, and e elevation above sea level, in feet.

The place of application of most forces with which we shall deal is a portion of the surface of the body to which the force is applied. A notable exception is earth-pull, or gravity, which is applied not to the surface of a body but throughout the same. All such are called *distributed forces.* The places of application of some forces are very small compared to the surfaces of the bodies to which they are applied, and for many purposes these places may be regarded as points of application; any such force is called a *concentrated force.* The *line of action* of a concentrated force is a line indefinite in length, parallel to the direction of the force, and containing its point of application. A concentrated force may act along its line of action in one of two ways,—to the right or left, up or down, etc. We say that the *sense* of a force is toward the right, toward the left, up, or down as the case may be. That is, sense refers to "arrow-headedness" (see next paragraph).

Since a force is a vector quantity,* it can be represented in part by a vector (a straight line of definite length and direction), the length of the

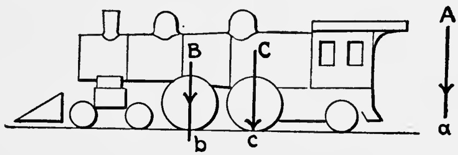


FIG. 1

vector representing the magnitude of the force according to some scale, and the direction of the vector giving the direction of the force. Thus, if the pressures of the driving wheels of the locomotive on the rails (Fig. 1) is 12 tons, then the vector Aa (0.4 inch long) represents the magnitude and direction of the pressures, the scale being one inch "equals" 30 tons. If the force to be represented is a concentrated one, as in the illustration, then the line of action also can be represented by the same vector which represents the force magnitude by drawing it through the point of application of the force. Thus the vector Bb represents magnitude, line of action, and direction of the pressure of the first driving wheel. We might extend this scheme further so as to indicate also point of application of the force by the head, say, of the vector as Cc; but we will not plan to do that because the point of application is not of importance in this subject,—Statics.

* *A vector quantity* is one having magnitude and direction, as, for example, a definite displacement of a moving point. A quantity having magnitude only, as the volume of a thing for example, is a *scalar quantity.*

This unimportance of the point of application is definitely expressed in the *principle of transmissibility* of force, which for the present purpose may be stated as follows: The effect of any force applied to a rigid body at rest is the same, no matter where in its own line of action the force is applied. The principle may be roughly verified by experiment, when the body on which the force acts is at rest, with the apparatus represented in Fig. 2; it consists of a rigid body suspended from two spring balances. The springs are elongated on account of the weight of the body, and if a force, as F, be applied at A, the springs will suffer additional elongations which in a way are a measure of the effect of the applied force. If the point of application of F be changed to B or C, the spring readings will not change; hence the effect of F will not have changed.

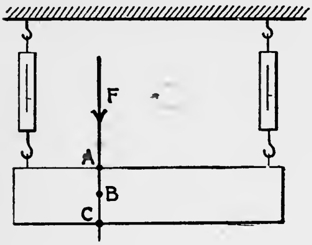


FIG. 2

Generally, when many forces are to be represented graphically and discussed, it would be well to represent each force by a line and a vector, the first to represent the line of action of the force and the second to represent the magnitude and the direction of the force. Of course the line must be drawn through the point of application of the force, but the vector may be drawn where convenient. For example, consider the forces acting on the upper end of the boom (Fig. 3) of a derrick. There are three forces; namely, a downward force at pin 1, one toward the left at pin 2, and one downward at pin 3. The lines marked *ab*, *cd*, and *ef* are the lines of action of the forces respectively; the vectors AB, CD, and EF (drawn where convenient but of proper length and direction) represent the magnitudes and directions of the forces. The scheme of notation here used — two lower-case letters on opposite sides of the line of action of a force, and the same capital letters at the ends of the vector representing its value — is in common use. Any force so marked is referred to in written statement by the two capitals used; thus the first force mentioned above would be called the force AB. The part of the drawing in which the lines of action of the forces and the body (here, a derrick-boom) are represented is called a *space diagram;* the part in which the vectors are drawn is called a *vector diagram.* The scales of these diagrams are of course different; the lengths of lines in the first represent distances, and those in the second, force magnitudes.

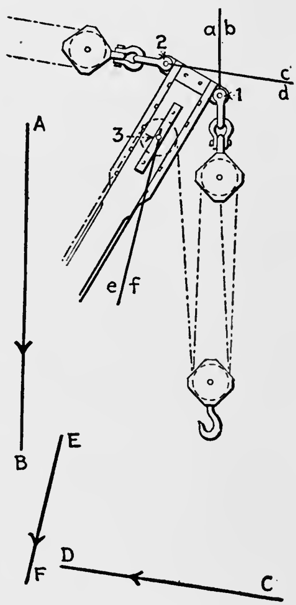


FIG. 3

Any number of forces collectively considered is called a system or a set of forces. The forces of a set are called *coplanar* if their lines of action are in the same plane, and *noncoplanar* if not in the same plane; they are called *concurrent* if their lines of action intersect in a point, and *nonconcurrent* when they do not so intersect; they are called *parallel* if their lines of action are *parallel*, and *nonparallel* if the lines of action are not parallel. Force-sets are also described in accordance with the foregoing definitions; thus, a concurrent set, a noncoplanar parallel set, etc., according as the forces of the set are concurrent, noncoplanar and parallel, etc. Force-sets can be classified in various ways, as below for example,—

Coplanar	concurrent	colinear	1
		nonparallel	2
	nonconcurrent	parallel	3
		nonparallel	4
Noncoplanar	concurrent		5
	nonconcurrent	parallel	6
		nonparallel	7

Two sets of forces acting on a rigid body are said to *balance*, when their combined effect on the rest or on the motion of that body is *nil*, so that if the body is at rest, for example, then it would remain at rest even if all the forces ceased to act. Two sets of forces acting on a rigid body are said to be *equivalent* if either set would balance the other set reversed (sense of each force changed); or, what amounts to the same thing, if each set acting singly would balance some other third set. The *resultant* of a set of forces is the single force which is equivalent to the set; or, if no single force is equivalent to the set, then the resultant is the simplest equivalent set. The resultant of a set of forces acting on a rigid body consists always of a single force or of two forces (proved later). Having given a set of forces, the process of finding a simpler equivalent set is called *composition* of the given set. The *component* of a given force is any one of a set which is equivalent to that force. Having given a force, the process of finding a set equivalent to that force is called *resolution* of the force.

The *anti-resultant* of a set of forces is the reversed resultant of the set. The *equilibrant* of a set of forces is the single force, or pair of forces if necessary, which could balance the set. Obviously the anti-resultant and the equilibrant of a set are identical.

3. Parallelogram and Triangle of Forces

The *parallelogram* and the *triangle of forces* are names of certain methods for determining (*a*) the resultant of two given concurrent forces, and (*b*) two concurrent components of a given force.

§ 1. Composition of Two Concurrent Forces.—*Parallelogram Law.*— If two forces acting upon a rigid body be represented by lines *OA* and *OB*, then their resultant is represented by the diagonal *OC* of the parallelogram

$OABC$. For example, take the two forces applied to the cap of the boom of Fig. 3 at points 1 and 2, their value being 2 and 1.2 tons respectively, let us suppose. Extending the lines of action to their intersection O (Fig. 4), then making OA = 2 tons and OB = 1.2 tons according to some convenient scale, and completing the parallelogram, we get OC, and according to the law, this line represents the resultant completely; that is, the magnitude of the resultant is OC = 2.2 tons, the line of action of the resultant is colinear with OC, and the sense of the resultant is from O to C.

The law can be verified by means of the apparatus shown in Fig. 5. It consists of a drawing board mounted in a vertical position, two pulleys, a spring balance, two weights, some cord, and a small ring. When the weights W_1 and W_2 are suspended somewhat as shown, then the ring is

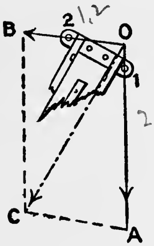


FIG. 4

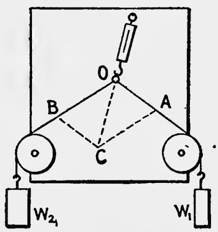


FIG. 5

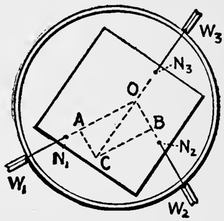


FIG. 6

subjected to three forces: pull $P_1 = W_1$, pull $P_2 = W_2$, and an upward pull P_3, the magnitude of which is indicated by the spring balance. Since P_3 is the equilibrant of P_1 and P_2, the resultant of P_1 and P_2 is equal and opposite to and colinear with P_3. It remains now to ascertain whether a construction for the resultant of P_1 and P_2 according to the parallelogram law will represent a force equal and opposite to and colinear with P_3. So we lay off OA and OB on the board, just under the strings, equal to P_1 and P_2, and complete the parallelogram $OABC$; then measure OC and compare its direction with P_3. We find that OC equals P_3 (by scale), and is colinear with P_3.

To test the law for forces having different points of application, the apparatus shown in Fig. 6 might be used; it consists of a tub of water, a floating drawing board, three smoothly running pulleys, three weights (W_1, W_2, and W_3), and three cords. Nails are driven into the drawing board at any points N_1, N_2, and N_3; the weights are then suspended by cords passing over the pulleys, and tied to the nails as shown; then if each weight is less than the sum of the other two, the board, if not too large, will move about and assume a position of rest without touching the tub. In such position, the forces acting on the board are its weight (or gravity), pressure of the water, and the three pulls (P_1, P_2, and P_3)

practically equal to W_1, W_2, and W_3 respectively. Obviously the first two forces balance each other; therefore the three pulls also balance, and so the resultant of P_1 and P_2 is equal and opposite to and colinear with P_3. We next determine the resultant R of P_1 and P_2 by the parallelogram law: extend the lines of action of the pulls P_1 and P_2 to their intersection O; from there lay off OA and OB equal (by some convenient scale) to P_1 and P_2; complete the parallelogram $OABC$. Then OC represents R; on comparison it will be found, as before, that OC is equal and opposite to and colinear with P_3, and hence OC does represent the magnitude and line of action of R. Since P_3, and hence R, passes through O (the intersection of P_1 and P_2), this experiment emphasizes the fact that the line of action of the resultant of two concurrent forces passes through their point of concurrence. The point of application of R might of course be taken anywhere in OC or its extension; for, so taken, R obviously would balance P_3.*

The Triangle Law. — If two concurrent forces acting on a rigid body be represented in magnitude and direction by AB and BC, then their resultant is represented in magnitude and direction by the side AC of the triangle

* By using accurate apparatus the foregoing tests for verifying the parallelogram law can be made very accurately. Such verifications are as satisfying to many students as "mathematical proof." What about such proof? Some writers assert that the law is fundamental, and not susceptible of deduction from anything more simple and obvious than the law itself. But many deductions or proofs have been proposed. All necessarily depend upon one or more axioms or statements whose truth is justified by experience. We give a proof based upon a principle of moments (Art. 5) which most students readily grant as axiomatic or justified by their experience. The principle is that the moment of the resultant of two concurrent forces about any point in their plane equals the algebraic sum of the moments of the two forces about the same point. Let P and Q denote the two concurrent forces and R their resultant. Suppose that P and Q act in OA and OB respectively (Fig. 7) — the body upon which they act is not represented — and let the lengths OA and OB represent the magnitudes of the forces P and Q to some scale, that is $OA \div OB = P \div Q$. $OABC$ is a parallelogram, and CD, CE, BF, and BG respectively are perpendicular to OA, OB, OA, and OC. Now the moments of P and Q about O equal zero; it follows from the principle of moments that the moment of R about O equals zero also, and hence the line of action of R passes through O. Now the area of the parallelogram is $OA \times CD$; also $OB \times CE$. Hence, $OA \div OB = CE \div CD$; and $P \div Q = CE \div CD$, or $P \times CD = Q \times CE$; that is, the moments of P and Q about C are equal. But these two moments are opposite in sign, and so their algebraic sum equals zero. It follows from the principle of moments that the moment of R about C equals zero, and hence the line of action of R passes through C. The moments of P, Q, and R about B are respectively, $P \times BF$, 0, and $R \times BG$; then, according to the principle of moments, $R \times BG = P \times BF$, or $R \div P = BF \div BG$. The area of the parallelogram is $OC \times BG$; also $OA \times BF$. Hence, $OC \div OA = BF \div BG$; and from the last proportions $R \div P = OC \div OA$; that is, OC represents R according to the same scale that OA represents P.

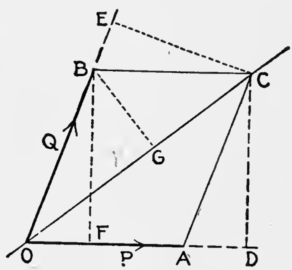


FIG. 7

ABC. For example, let two forces of 2 and 1.2 tons be applied at 1 and 2 (Fig. 8) as shown. If *AB* and *BC* be drawn anywhere in the directions of these forces, and *AB* and *BC* be made equal to the forces respectively, then *AC* gives the magnitude and the direction of the resultant; the line of action of the resultant is *ac*,—parallel to *AC* and *concurrent with the given forces*.

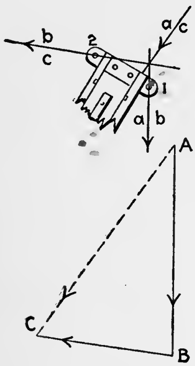


FIG. 8

The resultant of two concurrent forces can be determined without a scale drawing of a triangle or parallelogram. We sketch the triangle of forces roughly, and then solve the triangle for the length and direction of the side representing the resultant. For example, let the forces P and Q (Fig. 9)* equal 100 and 150 pounds respectively, and the angle ϕ between them be 60 degrees; required, their resultant R. Roughly, ABC is the triangle for the forces, AC representing the magnitude and direction of R, and the angle $ABC = 180° - 60° = 120°$. Then from the trigonometry of the triangle, $R^2 = 100^2 + 150^2 - 2 \times 100 \times 150 \cos 120° = 47{,}500$, or $R = 218.3$; also $\sin CAB/\sin 120° = 150/R$, or CAB (the angle α between R and P) $= 36° 35'$. Employing the foregoing method, the following general formulas may be worked out for determining the magnitude and direction of the resultant,—

$$R^2 = P^2 + Q^2 + 2\,PQ \cos \phi;$$

$$\sin \alpha = \sin \phi \cdot Q/R, \text{ and } \sin \beta = \sin \phi \cdot P/R,$$

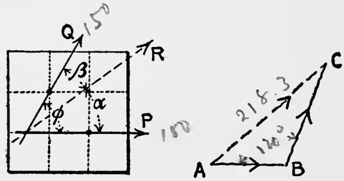


FIG. 9

where ϕ, α, and β are the angles marked in Fig. 9. When the two forces P and Q are at right angles to each other (ϕ = 90 degrees), then

$$R^2 = P^2 + Q^2, \quad \text{and} \quad \tan \alpha = Q/P.$$

§ 2. RESOLUTION OF A FORCE INTO CONCURRENT COMPONENTS can be accomplished by applying the triangle or parallelogram law inversely. Thus, let it be required to resolve the force F (Fig. 10) into two components. We draw AB anywhere equal (by some scale) and parallel to F; join any point C with A and B, and draw lines through any point in *ab* parallel to AC and BC; then AC and CB represent the magnitudes and directions, *ac* and *cb* the lines of action of two forces equivalent to F, that is, components of F. For the resultant of these two component forces is F, as shown by the tri-

* For convenience and clearness of figure, a subdivided square (or rectangle) will hereinafter represent a machine, or structure (derrick-boom, bridge, etc.), on which the forces under discussion act. If he prefers, the student might regard the subdivided square as representing a drawing board or some other definite object suggested by the square. It is important that he should have in mind the fact that forces act only on material things (bodies), and that the lines of action of the forces represented in any given figure are definitely related to the body on which the forces act.

angle law applied directly. Since C was taken at random, it is plain that a given force can be resolved into many different pairs of components.

If conditions be imposed on the components, the resolution is more or less definite. Thus, let it be required to resolve F (Fig. 11), equal to 350 pounds, into two components, one of which must act along the left-hand edge of the board and the other through the lower right-hand corner. Since the three forces must be concurrent, the second component must act through point 1; so we make AB equal and parallel to F and draw from A and B lines parallel to the two components; then AC and CB represent the values (200 and 320 pounds respectively) and the directions of the components.

An important case of resolution is that in which the components are at right angles to each other. Each is called a *rectangular component* or *resolved part* of the force. Rectangular components can generally be computed more easily than by geometrical construction. Let F (Fig. 12) be the given force to be resolved into horizontal and vertical components, the

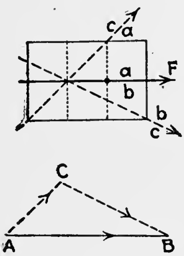


FIG. 10

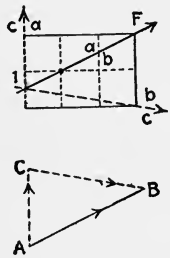


FIG. 11

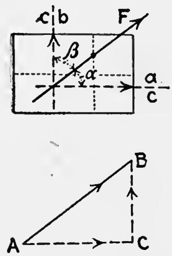


FIG. 12

angles between F and the components being α and β respectively. From ABC, a sketch of the triangle of resolution, not necessarily a scale drawing, it is plain that the desired components equal $F \cos \alpha$ and $F \cos \beta$ respectively. And always

$$\left.\begin{array}{l}\text{the rectangular com-}\\ \text{ponent of a force}\\ \text{along any line}\end{array}\right\} = \left\{\begin{array}{l}\text{the magnitude}\\ \text{of the force}\end{array}\right\} \times \left\{\begin{array}{l}\text{the cosine of the}\\ \text{acute angle between}\\ \text{the force and that line.}\end{array}\right.$$

The components of a force along two rectangular coördinate axes x and y are called the x and y components of the force respectively; they will be denoted by F_x and F_y.

4. Composition of Concurrent Forces

In the preceding article we showed how to determine the resultant of two concurrent forces; in this article we show how to determine the resultant of any number of such forces.

§ 1. COPLANAR FORCES.—*Graphical method.* By means of the parallelogram or triangle of forces (Art. 3) find the resultant R' of any two of the forces of the given set; then find the resultant R'' of any other given force and R'; then the resultant of another given force and R''; and so on until the resultant of all is found. Thus, suppose that the resultant of F_1, F_2, F_3, and F_4 (Fig. 13) is required: Taking the given forces in the order in which they are numbered, say, we first draw AB parallel to F_1 and equal to F_1 by some convenient scale, then BC in the direction of and equal to F_2; then AC gives the magnitude and direction of R', the line of action of R' passing through O parallel to AC. Next we draw CD in the direction of

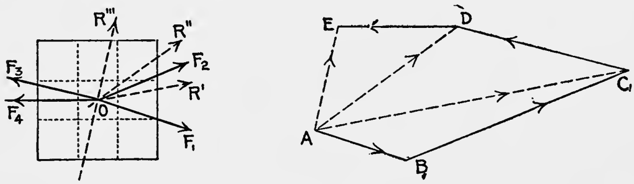


FIG. 13

F_3 and equal to F_3; then AD gives the magnitude and direction of R'', the line of action of R'' passing through O parallel to AD. Next we draw DE in the direction of and equal to F_4; then AE gives the magnitude and direction of R''', the line action of R''' passing through O parallel to AE. Of course the lines AC, AD, R', and R'' are not really essential to the solution; they were drawn here and referred to only for explanatory purposes.

The *force polygon* for a set of forces is the figure formed by drawing in succession and continuously lines which represent the magnitudes and directions of those forces. A force polygon is not necessarily a closed

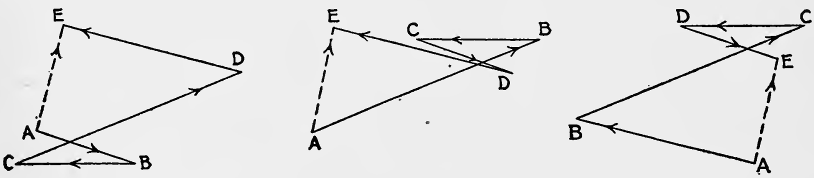


FIG. 14

figure; thus $ABCDE$, not including EA, is a force polygon for F_1, F_2, F_3, and F_4. Many force polygons can be drawn for a given set of forces, as many as there are orders of taking the forces; if there are n forces in the set, then $1 \cdot 2 \cdot 3 \cdot \quad \cdot \quad \cdot \quad \cdot n$ different force polygons can be drawn. In Fig. 14 additional polygons $ABCDE$ are shown for F_1, F_2, F_3, and F_4 of Fig. 13; the lines AE represent the magnitude and direction of R. The bare construction for determining the resultant of a set of concurrent forces can now be stated thus: Draw a polygon for the forces; join the beginning and the end of the polygon, and draw a line through the point of concurrence of the given forces parallel to the joining line; the joining line, with arrowhead pointing from the beginning to end of the force polygon, represents the magnitude and direction of the resultant, and the other line its line of action.

Algebraic Method. — Choose a pair of rectangular axes of resolution, which let us call x and y axes, with origin at the point of concurrence of the forces to be compounded; then resolve each force into its x and y components at the origin, and imagine it replaced by them; the resulting system consists of forces in the x and in the y axes; next find the resultant of the forces acting in the x axis, and the resultant of those acting in the y axis; finally, get the resultant of these two rectangular resultants; this is the resultant sought. For example, let it be required to determine the resultant of the six forces acting upon the 4 foot board shown in Fig. 15. The computations in outline are scheduled below. The values of the angles which the several forces make with the horizontal were computed from dimensions in the figure; the sum of the x components is $+3.40$, and that of the y components is -7.22 pounds. The signs of the sums indicate respectively that the x component of the resultant R acts toward the right and the y component downward; hence the resultant acts to the right and downward. The angle which R makes with the horizontal is $\tan^{-1}$ $(7.22 \div 3.40 = 2.123) = 64° \ 47'$. The value of the resultant is $R = \sqrt{3.40^2 + 7.22^2} = 7.98$ pounds.

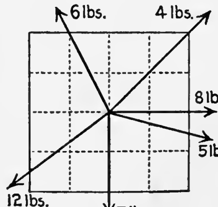


FIG. 15

F	α	$\cos \alpha$	$\sin \alpha$	F_x	F_y
8	0	1.	0.	+8.	0
4	45°	0.707	0.707	+2.83	+2.83
6	63° 26′	0.447	0.894	−2.68	+5.36
12	36° 52′	0.800	0.606	−9.60	−7.20
7	90°	0.	1.	0.00	−7.00
5	14° 2′	0.970	0.242	+4.85	−1.21
				+3.40	−7.22

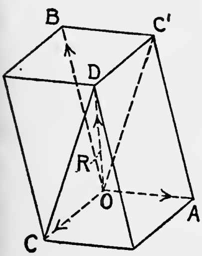


FIG. 16

§ 2. NONCOPLANAR FORCES. — Before showing how to find the resultant of any number of such forces, we explain (1) how to find the resultant of three rectangular concurrent forces (lines of action at right angles to each other), and (2) how to resolve a force into three noncoplanar rectangular components.

(1) If three noncoplanar concurrent forces acting on a rigid body be represented by OA, OB, and OC, then their resultant is represented by the diagonal OD of the parallelopiped $OABC$ (Fig. 16). (This is called the *parallelopiped law*, and $OABC$ is called a parallelopiped of forces.) For, according to the parallelogram law, OC' represents the resultant of two of the forces OA

and OB, and OD represents the resultant of OC' and the third force OC, and (hence, also) the resultant of the three given forces. This law leads to a simple algebraic method for finding the resultant when the three forces are rectangular (at right angles to each other). Thus, let F_1, F_2, and F_3 (Fig. 17) be the three forces, R their resultant, and θ_1, θ_2, and θ_3 the angles between R and the forces respectively; then

$$R^2 = F_1^2 + F_2^2 + F_3^2,$$

$$\cos\theta_1 = F_1/R, \quad \cos\theta_2 = F_2/R, \quad \cos\theta_3 = F_3/R.$$

For the resultant of F_1 and F_2 (represented by OC', Fig. 17) equals $(F_1^2 + F_2^2)^{\frac{1}{2}}$, and hence $R^2 = (F_1^2 + F_2^2) + F_3^2$; also the triangles ODA, ODB, and ODC are right-angled at A, B, and C respectively, and hence $\cos\theta_1 = OA/OD = F_1/R$, $\cos\theta_2 = OB/OD = F_2/R$, etc.

(2) A force can be resolved into three noncoplanar concurrent forces by applying the parallelopiped law inversely. Thus, let OD (Fig. 18) represent the given force F; first, construct any parallelopiped of which OD is a

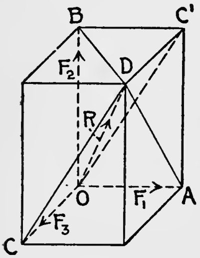


FIG. 17

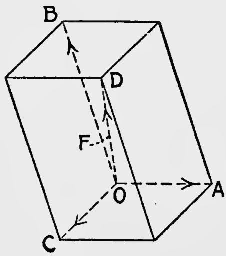


FIG. 18

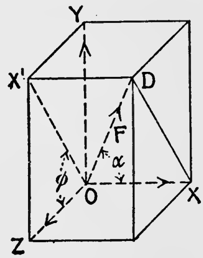


FIG. 19

diagonal; then the three edges intersecting at O represent forces equivalent to the given force because the resultant of these three forces is, according to the parallelopiped law, represented by OD. Inasmuch as many parallelopipeds can be constructed on OD as diagonal, many sets of three forces equivalent to the given force can be found.

The practical case is resolution into components along three definite rectangular axes; then there is only one set of components. The components may be found quite simply by an algebraic method: thus, let F (Fig. 19) be the force to be resolved, α, β, and γ the angles between F and the axes, and F_x, F_y, and F_z the x, y, and z components respectively; then, since OX, OY, and OZ are projections of OD on the rectangular axes,

$$F_x = F\cos\alpha, \quad F_y = F\cos\beta, \quad F_z = F\cos\gamma.$$

Sometimes the direction of the force F to be resolved is given by means of two angles, one being the angle between F and one of the desired components, and the other being the angle which the projection of F on the plane of the other two components makes with one of those two, as for instance α and ϕ (Fig. 19). Then F may be resolved best in this way: first, resolve it into two components $F \cos \alpha$ (along the x axis) and $F \sin \alpha$ (in the plane of the y and z axes), and then resolve $F \sin \alpha$ into components along the y and z axes, that is, $F \sin \alpha \sin \phi$ and $F \sin \alpha \cos \phi$.

Any number of noncoplanar concurrent forces can be compounded graphically by means of their force polygon, but this method is not practicable generally, because the polygon is not a plane one; however, it could be drawn in "plan and elevation" so as to furnish the resultant sought. The algebraic method is preferable; it is carried out as follows: First, select three rectangular axes of resolution (here called x, y, and z), with origin at the point of concurrence of the forces to be compounded; next resolve each force into its x, y, and z components, and imagine it replaced by them, thus arriving at a set consisting of forces acting in the axes; then find the resultants of the forces in the x, in the y, and in the z axis; finally, compound these three resultants, thus finding the resultant sought.

For example, let it be required to determine the resultant of the four forces acting on a 4 foot cube (Fig. 20). The forces are concurrent at O; the 10 and the 15 pound forces act through quarter points of certain edges as shown. The x, y, and z components of the 18 and 40 pound forces are obviously as scheduled adjoining. Since the 15 pound force is perpendicular to the x axis, its x component equals zero; and since the angle which that force makes with the z axis $= \tan^{-1} \frac{3}{4} = 36° 52'$, its y and z components are $15 \sin 36° 52' = 9$, and $15 \cos 36° 52' = 12$ pounds respectively as scheduled. The components of the 10 pound force were determined as follows: Since $Ya = 5$ and $YO = 4$ feet, the angle which the 10 pound force makes with the y axis is $\tan^{-1} \frac{5}{4} = 51° 20'$; the y component of the force equals $10 \cos 51° 20' = 6.25$ as scheduled, and the other rectangular component (in the zx plane) equals $10 \sin 51° 20' = 7.81$ pounds.

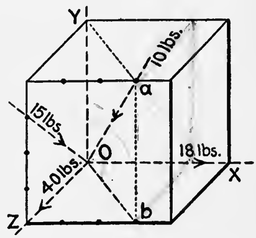


FIG. 20

F	F_x	F_y	F_z
18	18.00	0.00	0.00
40	0.00	0.00	40.00
15	0.00	−9.00	−12.00
10	−4.69	−6.25	−6.25
	+13.31	−15.25	+21.75

The angle which this component, acting in Ob, makes with the z axis equals $\tan^{-1} \frac{3}{4} = 36° 52'$; hence the x and z components of the 10 pound force equal respectively $7.81 \sin 36° 52'$, or 4.69, and $7.81 \cos 36° 52'$, or 6.25 pounds. The value of the resultant is

$$R = \sqrt{13.31^2 + 15.25^2 + 21.75^2} = 29.7 \text{ pounds.}$$

The signs of the sums of the x, y, and z components show that the resultant R acts toward the right, downwards and forward. Its angles with the x, y, and z axes are respectively: $\cos^{-1}(13.31 \div 29.7) = 63°$; $\cos^{-1}(15.25 \div 29.7) = 59°$; $\cos^{-1}(21.75 \div 29.7) = 43°$.

5. Moment of a Force; Couples*

§ 1. The MOMENT or TORQUE of a force with respect to a point is the product of the magnitude of the force and the perpendicular distance between its line of action and the point. The perpendicular distance is called the *arm* of the force with respect to that point, and the point is called an *origin* or *center of moments*. Experience suggests the notion that the moment of a force with respect to a point is a measure of the tendency of the force to rotate the body about a line through the point and perpendicular to the plane of the force and the point. Such a notion can be verified quite accurately by means of a simple apparatus represented in Fig. 21. It consists of a board mounted on a horizontal shaft, a heavy body, and the pail which can be suspended from the board; the shaft rests in ball bearings so that practically no resistance to turning is exerted at the shaft; the board, without the body and the pail, is well balanced so that gravity would not cause it to turn from any position. Now, let the pail containing shot be hung from B, C, D, etc., in succession, the amount of shot being taken so that the heavy body will be supported, OA not being horizontal necessarily. Then in each case the turning effect of the pull at B, C, or D equals the turning effect of the pull at A; hence the turning effects of the pulls at B, C, D, etc., are equal. And if the moments of these pulls (several weights of pail and shot) about O be computed, then those moments will be found equal too, and therefore moments are measures of turning effects.

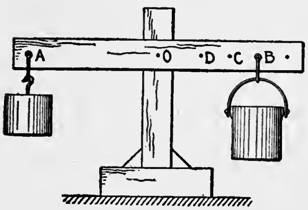


FIG. 21

It follows from the definition of moment that the *unit moment* is that of a unit force whose arm is a unit length. There are no one-word names for any of these units of moment; the units are called foot-pound, inch-ton, etc., according as the unit length and force are the foot and the pound, the inch and the ton, etc.

In a discussion involving the moments of several forces, it is generally convenient to give *signs* to the moments to indicate the directions (clockwise or anticlockwise) in which the several forces turn or tend to turn the body to which they are applied about the origin in question. In this book, clockwise rotation is regarded as negative and anti as positive, and rotations are supposed to be viewed from the reader's side of the printed page;

* See Art. 8 also.

thus the moment of the 200 pound force (Fig. 22) about O is positive and about A negative.

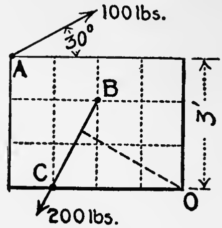


FIG. 22

Principles of Moments. — If two sets of coplanar forces are equivalent (Art. 2), then the moment-sum* for one set with respect to any point equals that for the other with respect to the same point. This will be granted as self-evident by most students; others may be convinced by this: Let S_1 and S_2 denote the two equivalent sets, and S_3 a third set (coplanar with S_1 and S_2) which could balance S_1, and hence also S_2. Since S_1 and S_3 would balance, they together would not turn the body on which they act about any line; hence the moment-sums for S_1 and S_3 with respect to any point O in the plane are equal in value but opposite in sign. Likewise, the moment-sums for S_2 and S_3 with respect to O are equal in value but opposite in sign. Therefore the moment-sums for S_1 and S_2 with respect to O, being equal to the same thing, are equal. It follows from the foregoing principle that the moment-sum for any set of coplanar forces with respect to any point in their plane equals the moment of their resultant with respect to that point. Also, the moment of a force with respect to a point equals the moment-sum of its components with respect to the same point, the components to be coplanar with the given force and the point.

When the moment of a force about a certain point must be computed, and the arm of the force with respect to that point cannot be easily determined, then the desired moment can be computed, more easily perhaps, from components of the force, by aid of the preceding principle. Thus, let it be required to compute the moment of the 100 pound force (Fig. 22) about O. The horizontal and vertical components of the force are respectively, 86.6 and 50 pounds; imagining them applied at A makes their arms 3 and 4 feet respectively, hence the desired moment equals $-(86.6 \times 3) - (50 \times 4) = -459.8$ foot-pounds. Sometimes the components can be applied (in imagination) so that one passes through the origin of moments; then its moment equals zero and the desired moment equals the moment of the other component. Thus the horizontal and vertical components of the 200 pound force equal 89.4 and 178.8 pounds respectively; imagining them applied at C, their arms are 0 and 3 feet, and so the desired moment equals $178.8 \times 3 = 536.4$ foot-pounds.†

* "Moment-sum" means the algebraic sum of the moments of the forces of a system; torque of a set of forces means the same thing.

† Some writers regard the parallelogram law (for forces) as fundamental, and deduce the principle of moments from the law. We give such a deduction of the principle for two forces, — namely, the moment of the resultant of two concurrent forces about any point in their plane equals the algebraic sum of the moments of the forces about the same point (Varignon's theorem). The theorem can be extended easily to any number of coplanar forces, thus proving the principle.

Let P and Q (Fig. 24) be two concurrent forces acting — on a body not shown — in

§ 2. COUPLES (see also Art. 8).—Two equal and parallel forces which are opposite in sense may advantageously be considered together; so considered, they are called a *couple*. By *arm* of the couple is meant the perpendicular distance between the lines of action of the forces; by *plane of the couple* is meant the plane of those lines of action. The *moment* or *torque* of the couple about any point in the plane of the couple is the algebraic sum of the moments of the forces about that same point. This sum, or moment, including the sign, is the same for all origins in the plane. For, let F and F (Fig. 23) be the forces of the couple, AB being the arm; then the moments of the couple with respect to the origins a, b, and c are respectively: $-F(Aa) + F(Ba) = F(AB)$; $+F(Ab) - F(Bb) = F(AB)$; and $+F(Ac) + F(Bc) = F(AB)$. Since a, b, and c represent all possible origins, the proposition is proved. Since the origin of moments is immaterial, no reference is made to it in speaking of the moment of a couple. Moreover, the value of the moment is computed more simply than as stated in the definition, by multiplying the common magnitude of the forces and the arm of the couple. As just shown, the moment of the couple FF is positive; that is, the couple would turn the body, if free, counterclockwise about any axis perpendicular to the plane of the couple; obviously, the moment of the couple GG is negative. *Sense* of a couple refers to the way in which the couple would turn a body; thus we speak of a clockwise sense or a counterclockwise sense.

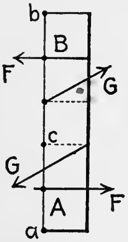


FIG. 23

Two coplaner couples whose moments (including sign) *are equal are equivalent.* The following proof is in two parts; namely, (1) when the four forces of the couples are nonparallel, and (2) when they are parallel. (1) Let P_1P_2 (Fig. 25) be one given couple, and Q_1Q_2 the other, their arms being p and q respectively; then $Pp = Qq$. We will show that the P couple would balance the reversed Q couple; it will follow that the given couples are equivalent. The area of the parallelogram $ABCD = ADp = ACq$; therefore, since $Pp = Qq$, $AD/AC = P/Q$. That is, the sides of the parallelogram represent the forces P and Q; and so the diagonal

the lines OA and OB respectively, and let R be their resultant. Also, let $OABC$ be a parallelogram for the forces, D any origin of moments, and α, β, and γ the angles between OD and the forces respectively as marked. Now $P \sin \alpha$, $Q \sin \beta$, and $R \sin \gamma$ are the values of the components of P, Q, and R at right angles to OD, and it is clear from the figure that $P \sin \alpha + Q \sin \beta = R \sin \gamma$. Multiplying through by OD, we get $P \cdot OD \sin \alpha + Q \cdot OD \sin \beta = R \cdot OD \sin \gamma$. But these terms are the moments of P, Q, and R respectively; hence the moment of R equals the sum of the moments of P and Q as stated. When D is between P and Q, then a slight modification in the proof is necessary.

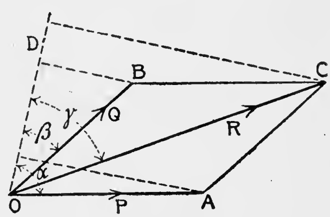


FIG. 24

AB represents the resultant of Q_1 reversed and P_1, and the diagonal BA represents the resultant of Q_2 reversed and P_2. Since the resultants are equal, opposite, and colinear they balance, and so the P couple and the reversed Q couple balance. Hence, etc. (2) When P_1, P_2, Q_1 and Q_2 are parallel, and the moments of the two couples are equal, then each couple is equivalent to some third couple, the forces of which intersect P_1, P_2, Q_1, and Q_2, according to (1). Therefore they are equivalent to each other.

§ 3. A FORCE AND A COUPLE. — *The resultant of a coplaner force and couple is a single force; the resultant is equal and parallel to the force, and its moment about any point on the given force equals the moment of the couple.* Proof follows:

Let F (Fig. 26) be the given force, and P_1P_2 the given couple. (If the forces of the given couple are parallel to F, then imagine the couple shifted

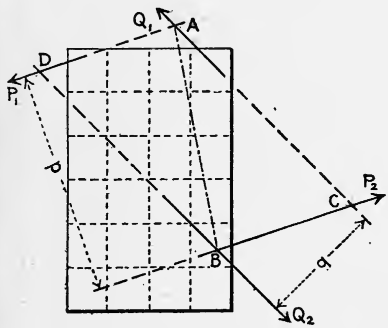


FIG. 25

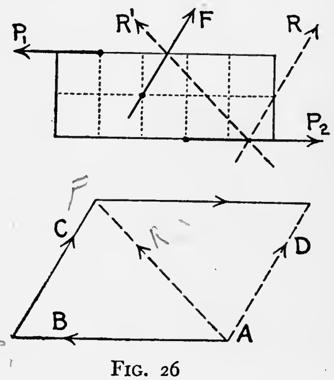


FIG. 26

until they are not so parallel.) Now suppose that AB and BC represent the magnitudes and directions of P_1 and F respectively; then AC represents the magnitude and direction of the resultant of those two forces. (The line of action of the resultant is R', parallel to AC and through the intersection of P_1 and F.) Let CD equal AB; then AD represents the magnitude and direction of the resultant of R' and P_2, and hence of the three forces P_1, F, and P_2. But AD is equal and parallel to BC; hence this final resultant is equal and parallel to F. (The line of action of this final resultant is R, parallel to BC and through the intersection of R' and P_2.) Since R is equivalent to F, P_1, and P_2, its moment about any point of F equals the sum of the moments of F, P_1, and P_2 about that point; but F has no moment about such point, and hence the moment of R equals the sum of the moments of P_1 and P_2 (the moment of the couple).

It follows from the foregoing that *a force R can be resolved into a force equal and parallel to R, and a couple whose moment equals that of R about any point on the component force.* Thus the moment of the couple component depends on the line of action chosen for the force component. Independent proof of this proposition follows:

Let R (Fig. 27) be the force to be resolved, and O a point through which the line of action of the force component is to pass. First we resolve R into two concurrent components, one of which passes through O; take any point on R (as a) for the point of concurrence and any direction (as ab) for the line of action of the second component. These components we call C_1 and C_2 respectively. To determine C_1 and C_2, we draw AB to represent R, and AC and BC parallel to C_1 and C_2 respectively; then $AC = C_1$, and $CB = C_2$. Next we resolve C_1 at O into two components parallel to C_2 and R, which components we call C_3 and C_4 respectively. To determine C_3 and C_4, we draw from A a line parallel to C_3 and from C a line parallel to C_4, and so locate D; then $AD = C_3$, and $DC = C_4$. Obviously now C_2, C_3 and C_4 are equivalent to R, that is, they are components of R; and as required C_4 passes through O, and C_2 and C_3 (equal, parallel, and opposite) constitute a couple. Moreover, according to the principle of moments, the moment of R about any point on C_4 equals that of C_2, C_3, and C_4 about that point; but the moment of C_4 equals zero, hence, etc.*

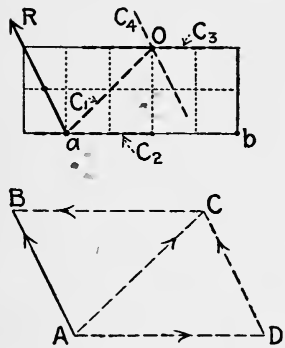


FIG. 27

concurrent forces use

6. Graphical Composition of Coplanar Nonconcurrent Forces

§ 1. FIRST METHOD. — When the forces to be compounded are not parallel nor nearly so, then we compound any two of the forces, next their resultant and the third force, that resultant and the fourth force, and so on until the resultant of all the forces has been found. For example, consider the forces acting on the retaining wall shown in section in Fig. 28;

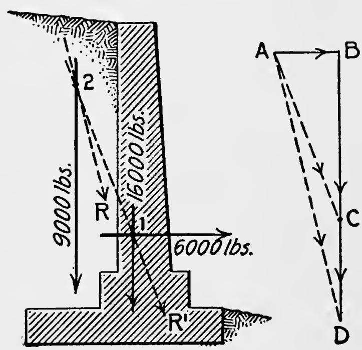


FIG. 28

* (1) *Composition of a Force and a Couple* and (2) *Resolution of a Force into a Force and a Couple* can be performed also as follows (student should supply figure): (1) Replace the couple by an equivalent couple whose forces equal the given force, and place the couple so that one of its forces is colinear with and opposite to the given force. These two forces balance; the other force of the new couple remains, and it is the resultant sought. (Study of the steps in the process shows that the resultant force is equal and parallel to the original force, and that the moment of the resultant about a point on the line of action of the original force equals the moment of the couple.) (2) Apply two forces at the given point equal and parallel to the given force and opposite to each other. These two forces along with the given force can be grouped into a force and a couple, and they (the force and couple) are the components sought. (Study of the steps of the process shows that the component force is equal and parallel to the given force, and the moment of the couple equals that of the given force about the given point.)

mag. of force) comput[illegible] twice

they consist of its own weight (16,000 pounds per foot of length), the earth pressure on the back (6000 pounds), that on the top of the base (9000 pounds), and that on the bottom of the base. The resultant of the first three forces will now be determined. We draw AB and BC to represent the 6000 and the 16,000 pound forces, and then join A and C; AC represents the magnitude and direction of the resultant of the two forces, and the line marked R' (parallel to AC and through point 1) is the line of action of that resultant. We next draw CD to represent the 9000 pound force, and join A and D; AD represents the magnitude and direction of the resultant of R' and 9000 (and hence also of the three given forces), and the line marked R (through point 2 and parallel to AD) is the line of action of that resultant.

It may be noted that the magnitude and the direction of the resultant is found just as for concurrent forces (Art. 4). For nonconcurrent forces it is necessary to draw the lines of action of the intermediate resultants (R', R'', etc.), in order to find the line of action of the final resultant, lines which are unnecessary when compounding concurrent forces.

When the forces are parallel or nearly so, the foregoing method fails because there is no accessible intersection of the lines of action of two given forces through which to draw the line of action of the first resultant. This difficulty can be met as follows: Introduce into the given system two equal, opposite, and colinear forces, which will not change the resultant, taking their common line of action somewhat across those of the given forces; then use the first method, compounding first any pair of forces whose intersection is accessible, etc.

§ 2. SECOND METHOD, applicable to any coplanar forces. — We first resolve each force into two concurrent components, resolving in such a way

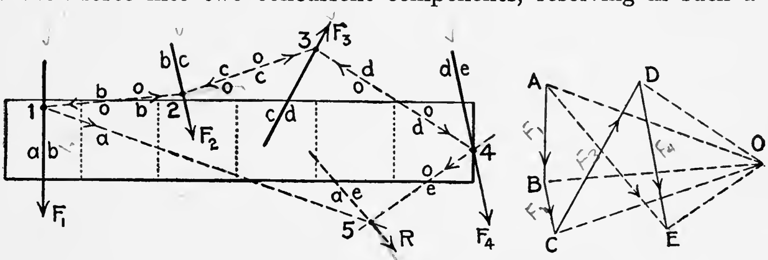


FIG. 29

that these components, excepting one of the first force and one of the last force, balance or destroy each other; these two remaining components are, in general, concurrent, and so we readily find their resultant, which is also the resultant of the given forces. For example, let F_1, F_2, F_3, and F_4 (Fig. 29) be the forces to be compounded. First we draw a force polygon for the given forces, taking them in any convenient order, as $ABCDE$; then we take any convenient point O as the common vertex of the triangles of resolution. AO and OB represent two components of F_1 in magnitude and direction, BO and OC two components of F_2, etc.; thus this resolution gives several pairs of equal and opposite components, OB and BO, OC and CO, OD and DO. The components of F_1 are taken to act through point 1; those of F_2 through 2, those of F_3 through 3, etc., the first point, 1, being taken at pleasure on F_1, point 2 where ob intersects F_2, point 3 where oc intersects F_3, etc. Thus the components OB and BO are colinear and they balance; likewise OC and CO, and OD and DO. Only the first and last components AO and OE remain; their resultant is represented by AE in magnitude and direction, and its line of action is ae (parallel to AE through the intersection of ao and oe).

The common vertex of the triangles of resolution O (Fig. 29) is the *pole* of the force polygon; the lines from the pole to the vertexes of the force polygon, OA, OB, OC, etc., are *rays*; the line of action of the several forces, oa, ob, oc, etc., are *strings* which, considered collectively, is the *string* or *funicular polygon* (also called *equilibrium polygon*, especially when the given forces are balanced or in equilibrium). The rays are sometimes referred to by number, OA being the first, OB the second, etc.; likewise the strings.

In using this second method, the beginner had best reason out the various steps of the construction somewhat as in the foregoing. After some practice he might use the following aids: (1) The two strings intersecting on the line of action of any force are parallel to the rays drawn to the ends of that side of the force polygon corresponding to that force, thus the strings intersecting on bc are ob and oc. (2) The string which joins points in the lines of action of any two forces is parallel to the ray which is drawn to the common point of the two sides of the force polygon corresponding to those forces, or, the string joining points on bc and cd is parallel to OC. (3) The bare construction in the second method is simply this: Draw a force and a string polygon for the forces, then draw a line from the beginning to the end of the force polygon and a parallel line through the intersection of the first and last strings; the first line represents the magnitude and direction of the resultant (sense being from the beginning to the end of the force polygon), and the second line is the line of action of the resultant.

This second method is not so simple in principle as the first, but in the second there is more opportunity for varying the construction to keep the drawing within convenient limits; thus the pole may be shifted, and the starting point of the string polygon may be taken anywhere on any of the given forces. Though many string polygons may be drawn for a given set of forces, all determine the same line of action of the resultant; that is, the intersections of the first and last strings of all string polygons lie on one straight line, the line of action of the resultant.

§ 3. WHEN THE FORCE POLYGON CLOSES.—It may seem, at first thought, that the resultant vanishes, or is zero; in general, this conclusion would be wrong, the system actually reducing to a couple. Thus, let F_1, F_2, F_3, and F_4 (Fig. 30) be a force-set whose force polygon $ABCDE$ closes; using the first method for compounding, we find that the resultant R'' of the first three forces is given by AD in magnitude and in direction, and *ad* is its line of action; hence R'' is equal, opposite, and parallel to F_4, and so the given force-set reduces to a couple (R'', F_4). The arm of this couple is the perpendicular distance between F_4 and R'', and so the moment of the couple is the product of F_4 (or R'') and the arm (according to the scale of the space diagram); the sense of the couple, clockwise, is apparent from the relative positions and directions of the forces of the couple as seen in the space diagram. In Fig. 31 the composition has been made by the second method; the system reduces to the two components AO (acting in *ao*) and OE (acting in *oe*). These components are equal, opposite, and parallel, and so the given force-set reduces to a couple. The arm of the couple is the perpendicular distance between the first and last strings, *ao* and *oe*; the moment of the couple is the product of OA or EO (according

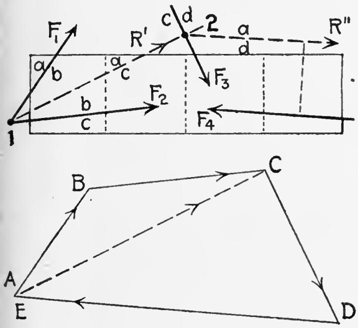


FIG. 30

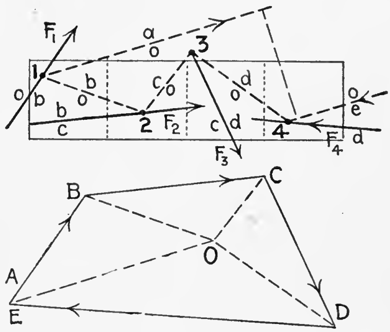


FIG. 31

to the scale of the force diagram) and the arm (according to the scale of the space diagram); the sense is apparent from the space diagram.

The length of the arm and the magnitude of the forces of the couple depend on the order in which the forces are taken in the force polygon, in the first method; and upon the position chosen for the pole O, in the second method. But the moment of the couple is independent of all these variations. This fact may be verified by actually compounding a certain force-set (whose force polygon closes) in several ways, making all these different variations and thus arriving at different couples. The couples are all equivalent to the same force-set and so equivalent to each other, and hence their moments are equal (Art. 5).

7. Algebraic Composition of Coplanar Nonconcurrent Forces

§ 1. PARALLEL FORCES.—If the forces be given sign, those in either direction being called positive and those in the other negative, then the algebraic sum of the forces gives the magnitude and sense of the resultant, the sign of the sum indicating the sense of the resultant. According to the principle of moments (Art. 5), the moment of the resultant about any point equals the algebraic sum of the moments of the forces about that point, and this requirement fixes the position or line of action of the resultant. Fo example, let us find the resultant of the four forces acting on a 10 foo board, as shown in Fig. 32. Calling upward forces positive, their alge braic sum is $+20-40-50+30=-40$; hence the resultant equals 4 pounds and acts downward. The algebraic sum of the moments of th forces about the left end of th board, say, is $0-120-350+270=$ -200 foot-pounds, and hence th moment of the resultant also is -20 foot-pounds; this fixes the arm of th resultant, $200 \div 40$, or 5 feet. Sinc the resultant acts downwards and its moment about the origin is negative its line of action must be to the right of the origin, 5 feet.

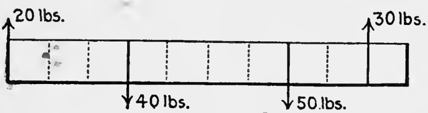


FIG. 32

To find the resultant of two parallel forces (a common problem), w may of course use the general method just explained, but the followin special results are worth noting. We distinguish two cases: (1) the tw forces are alike in sense; (2) they are opposite. In (1) the resultant equal the sum of the forces and agrees with them in sense; in (2) the resultan equals the difference between the two forces and agrees with the larger in sense. In order that the moment of the resultant R may equal the sum of the moments of the forces, P and Q (Fig. 33), then, in case (1), R must lie between the forces, and in case (2) outside of them and adjacent to the larger force (assumed to be P in the figure) Furthermore, if the distances from R to P and Q be called p and q respec tively, and that between P and Q be a, then in either case, $Rp = Qa$ an $Rq = Pa$, or

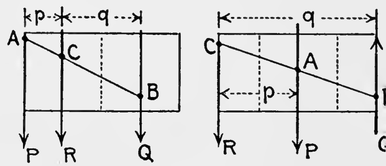


FIG. 33

$$p = Qa/R \quad \text{and} \quad q = Pa/R$$

either of which definitely fixes the position of R. Also for either case, P $= Qq$ or $P/Q = q/p$; hence

$$P/Q = BC/AC,$$

that is, the line of action the resultant of two parallel forces divides any secan intersecting their lines of action into two segments which are inversely pro portional to the two forces.

If the algebraic sum of a set of parallel forces equals zero, then it may appear to the student that their resultant vanishes or is zero; this doe not follow, but the resultant actually is in general a couple. For the re sultant of all but one of the given forces is a single force equal, opposite and parallel to the omitted one; but these two are not in general colinear and so they constitute a couple, the resultant of the system. The coupl arrived at depends on which one of the given forces is omitted, but th moment of the couple does not, for that couple is the resultant of the set and the moment equals the algebraic sum of the moments of given forces, a definite quantity. For example, let us find the resultant of the five forces acting on a 10 foot board, as shown in Fig. 34. Their algebraic sum is zero, and so their resultant is, presumably, a couple. Compounding all but the 40 pound force, we find that their resultant equals 40 pounds, acts downward, 7.5 feet to the right of the left end of the board, and so the resultant is a couple whose moment is $+ (40 \times 2.5) = +100$ foot-pounds.

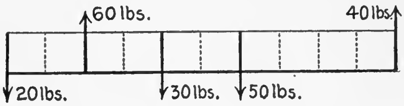


FIG. 34

Instead of actually determining the forces of the resultant couple as explained, it is usually sufficient to determine the moment of the resultant couple; this moment equals the algebraic sum of the moments of the given forces about any point. Thus, in the preceding example, after ascertaining that the resultant is a couple, we compute the moment-sum for the given forces, with moment origin at the middle of the board, say, or $(20 \times 5) - (60 \times 3) + (30 \times 1) - (50 \times 1) + (40 \times 5) = +100$ foot-pounds; and then conclude that any couple whose moment equals $+100$ foot-pounds may be regarded as the resultant of the system.

§ 2. NONPARALLEL FORCES.—As shown in Art. 6, the resultant is in general a single force, given in magnitude and direction by the line joining the beginning and end of the force polygon for the forces. It follows, therefore, that the component of that resultant force along any line equals the algebraic sum of the components of the given forces along that line. From this principle we can get the components of the resultant along any two rectangular axes; and from these components the magnitude and direction of the resultant itself can be readily determined by obvious means. According to the principle of moments (Art. 5), the moment of the resultant about any point must equal the sum of the moments of the given forces about that point; and this requirement fixes the position or line of action of the resultant. For example, let us find the resultant of the six forces acting on a board, 4 by 4 feet, as shown in Fig. 35. The angles which the forces make with the horizontal and the arms of the forces with respect to the center of the board are recorded in columns 2 and 3 of the schedule on page 26; they could be computed trigonometrically or could be scaled from a larger drawing. The x and y components of the several forces are recorded in columns 4 and 5 respectively, and the moments of the forces with respect to the center of the board in column 6. The algebraic sums of the x and the y components are $+3.40$ and -7.22 pounds respectively; hence $R = \sqrt{3.40^2 + 7.22^2} = 7.98$ pounds. The signs of the sums indicate that R acts toward the right and downward; the angle which R makes with the

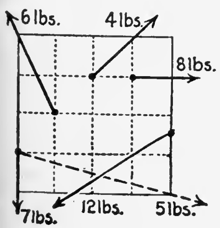


FIG. 35

horizontal is $\tan^{-1}(7.22 \div 3.40)$, or $64° 47'$. The sum of the moments is -14.14 foot-pounds; and, since the moment of R also equals -14.14, R lies on the right-hand side of the origin of moments (the moment being negative), and its arm is $14.14 \div 7.98 = 1.77$ feet. Thus, R has been completely determined.

1	2	3	4	5	6	7	8	9	10
F	α	a	F_x	F_y	M	a_1	a_2	M_1	M_2
8	0	1.00	+8.00	0.00	− 8.00	1	..	− 8.00	0
4	45°	0.71	+2.83	+2.83	− 2.84	1	0	− 2.83	0
6	63° 25′	0.89	−2.68	+5.36	− 5.37	0	1	0	− 5.36
7	90°	2.00	0.00	−7.00	+14.00	..	2	0	+14.00
12	36° 52′	1.60	−9.60	−7.20	−19.20	2	0	−19.20	0
5	14° 2′	1.45	+4.85	−1.21	+ 7.27	1	2	+ 4.85	+ 2.42
			+3.40	−7.22	−14.14			−25.18	+11.06

The moment-sum (-14.14) may be determined also — more simply in this example — by adding the moments of the x and y components of the several forces (Art. 5). This method requires that we take definite lines of action for all the components. Of course any force and its x and y components must be concurrent (Art. 3). We take the heavy dots (Fig. 35), as the points of concurrence; then the arms of the x components of the forces respectively with respect to the center of the board are as recorded in column 7, and the arms of the y components are as recorded in column 8. The moments of the x components and of the y components are recorded in columns 9 and 10 respectively. The sums of the moments of the x components and of the y components are -25.18 and $+11.06$ foot-pounds respectively, and hence the moment-sum for the given forces is $-25.18 + 11.06 = -14.12$.

If ΣF_x and ΣF_y for a force-set both equal zero, then any force polygon for the forces would close; hence the resultant of the forces is a couple (Art. 6). Any couple whose moment equals the algebraic sum of the moments of the given forces about any point may be regarded as the resultant. For example, let us find the resultant of six forces acting on a drawing board 4 by 4 feet, as shown in Fig. 36. Inspection shows that the algebraic sums of the x and the y components equal zero, and so the resultant is a couple. Taking moments about the lower left-hand corner, we get $\Sigma M = -60 \times 1.414 - 20 \times 1.414 - 18 \times 3 - 12 \times 1 = -179$ foot-pounds, and this is also the value of the moment of the resultant couple.

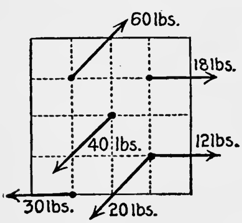


FIG. 36

If the forces consist of a number of couples, then $\Sigma F_x = \Sigma F_y = 0$, and the resultant is a couple. But the sum of the moments of these forces about any point equals the sum of the moments of the couples; hence any couple whose moment equals the sum of the moments of the given couples may be regarded as the resultant.

8. Moment of a Force; Couples *

§ 1. MOMENT ABOUT A LINE. — Art. 5 relates to moments of forces and to couples with special reference to coplanar forces and couples. In some discussions on noncoplanar forces it is convenient to make use of the moment or torgue of a force with respect to a line; this is defined as the product of the component of the force perpendicular to the line — the other component being parallel to it — and the distance from the line to the perpendicular component, or to the force (the distances being equal). For example, let F (Fig. 37), acting on a body not shown, be the force, and LL' the line, or *axis of moments* as it is called. MN is any plane perpendicular to the axis, represented to make the figure plain. $OACB$ is a parallelogram with OC (representing F) as diagonal, and sides perpendicular and parallel to LL'; then OA and OB represent the perpendicular and parallel components (F_1 and F_2) referred to; and the moment of F about LL' is the product of F_1 and PL.

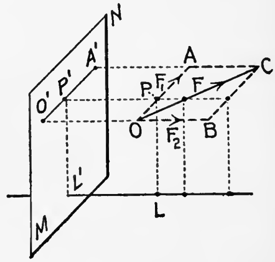


FIG. 37

The moment of a force with respect to a line is a measure of the tendency of the force to turn the body to which the force is applied about that line. Thus, when the force is parallel to the line the moment is zero, and obviously the force has no tendency to turn the body about the line. Again, when the force is perpendicular to the line the moment of the force about the line equals the product of the force and the perpendicular distance from the line to the force, and it is shown in Art. 5 that this product measures the tendency of the force to turn the body about the line. Finally, when the force F is not parallel nor perpendicular to the axis of moments (Fig. 37), then F_1 and F_2 together are equivalent to F, and their combined turning effect equals that of F. But F_2 has no turning effect; therefore that of F_1 and that of F are equal. But it was explained that $F_1 \times LP$ (the moment of F_1) measures the turning effect of F_1, and so it also measures the turning effect of F.

In a discussion involving moments of several forces about a line, it is generally convenient to give signs to the moments to indicate the directions (clockwise or counter) in which the several forces would turn the body about the line if it were free to rotate about that line. Whether a given rotation is clockwise or counter depends on the point of view; in a particular discussion a point of view should be assumed on the line or axis of moments and outside of the body, so that all rotations would be seen looking in the same direction. When the axis of moments is also an axis of

* See Art. 5 also.

coördinates, then it is customary to view rotations about that axis from the positive end of the coördinate axis, looking in the negative direction.

Principle of Moments. — If two sets of forces are equivalent (Art. 2), then the moment-sum for one set with respect to any line equals the moment-sum for the other set with respect to the same line. This will be granted as self-evident by most students; others may consider this: Let S_1 and S_2 denote the two equivalent sets of forces, and S_3 a third set which would balance S_1 and hence also S_2. Since S_1 and S_3 would balance, they would not turn the body on which they act about any line; hence the moment-sums for S_1 and S_3 with respect to any line are equal in value but opposite in sign. Likewise, the moment-sums for S_2 and S_3 with respect to that same line are equal in value and opposite in sign. The moment-sums for S_1 and S_2 being equal to the same thing, are therefore equal.

It follows from the preceding that the moment-sum for any set of forces with respect to a given line equals the moment of the resultant of those forces with respect to the same line. Also, the moment of a force about any line equals the moment-sum of its components with respect to the same line. This last principle suggests a second method for computing the moment of a force with respect to a line, more simple than the first method in some cases: Resolve the force into three rectangular components, one of which is parallel to the axis of moments; compute the moment of each of the other two components about the axis, and add the moments algebraically; this sum equals the moment of the given force. For an example, we compute the moment of a 100 pound force which acts upon a 4 foot cube as shown in Fig. 38, with respect to those edges marked X, Y, and Z. The x, y, and z components of the force are 37.2, 74.2, and 55.7 pounds respectively (see Art. 4); these components must be concurrent with the given force. Taking A as the point of concurrence, the moments are computed as follows: $-74.2 \times 4 + 55.7 \times 4 = -74$; $-37.2 \times 4 - 55.7 \times 2 = -260$; and $37.2 \times 4 + 74.2 \times 2 = 297$ foot-pounds. With point of concurrence taken at B or at any other point in AB, the same result would be obtained for the moment.

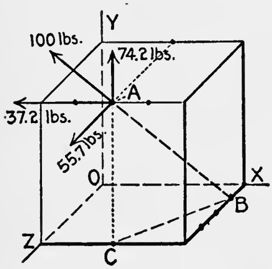


FIG. 38

§ 2. COUPLES (see also Art. 5). — *Two couples whose planes are parallel and whose moments, or torques, and senses are the same are equivalent.* Proof of this proposition for coplanar couples is given in Art. 5; proof for noncoplanar couples follows. Let P_1 and P_2 (Fig. 39) be the forces of one couple, Q_1 and Q_2 (not shown) the forces of the other, and p and q the arms of the couples respectively; then by supposition $Pp = Qq$. According to Art. 5, the Q couple can be replaced by a couple in its own plane provided that the moment and sense of the new couple equals that of the Q couple. Let S_1 and S_2 be the forces of that replacing couple, S_1 and S_2 being chosen parallel and equal to P_1 and P_2; then the arm ab of the S couple equals p, and $abcd$ is a parallelogram. We now show that the P couple would balance the reversed S couple; it will follow that the P and S couples are equivalent, and hence also the P and Q couples. The resultant R' of P_1 and $-S_2$ (S_2 reversed) equals the resultant R'' of P_2 and $-S_1$ (S_1 reversed), and R' and R'' are parallel and opposite in sense. Moreover, R' lies midway between P_1 and S_2, and R'' lies midway between P_2 and S_1; therefore each resultant acts through the center of the parallelogram $abcd$, and hence they are colinear. The resultants therefore balance, and hence the four forces P_1, P_2, $-S_1$, $-S_2$ do also. Therefore, etc.

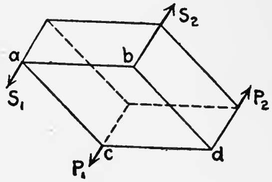


FIG. 39

The resultant of any number of couples is a couple. Proofs of this proposition for the case of coplanar couples are given in Arts. 6 and 7. For the case of noncoplanar parallel couples: The given couples can be replaced by equivalent ones respectively, all in some one plane; the resultant of these is a couple, and hence the resultant of the given ones is also a couple. For the case of nonparallel couples: Imagine each of the two couples to be replaced by an equivalent couple, and let the four forces of the replacing couples be equal; furthermore, imagine the two new couples so placed (in their respective planes) that a force of one couple will balance a force of the other. See Fig. 40 (perspective), which shows the two replacing couples, there marked F_1F_2 and F_3F_4; α is the angle between the planes of the couples. Since F_2 and F_4 balance, F_1 and F_3, constituting a couple, are equivalent to F_1, F_2, F_3 and F_4 and hence to the two original couples.

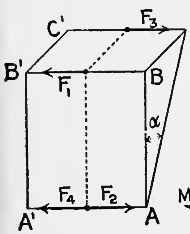


FIG. 40

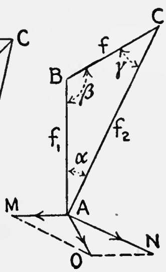


FIG. 41

The resultant of any coplanar or parallel couples can be determined very simply; the resultant is any couple parallel to the given couples, its moment being equal to the algebraic sum of the moments of the given couples. The resultant of nonparallel couples can be determined best from their vectors* by means of this proposition, — *The vector of the resultant of any number of couples equals the sum of the vectors of those couples.* Proof: Consider first two couples, say the two whose resultant was found in the preceding paragraph. Let ABC (Fig. 41) be an end view of Fig. 40, looking along the line AA'; that is, ABC of

* The vector of a given couple is perpendicular to the plane of the couple (exact position of vector immaterial); its length is equal to the moment of the couple according to some scale understood; and its sense agrees with the sense (rotation) of the couple according to some rule of agreement, as for example the following: Imagine the vector to be a right-handed screw turning with the couple; then the arrowhead on the vector must point in the direction in which the screw advances.

Fig. 41 is ABC of Fig. 40 in true proportions. Then AM (perpendicular to AB), AN (perpendicular to AC), and AO (perpendicular to BC) are respectively the vectors of the two given couples and their resultant, provided that the lengths of the vectors are proportional to the moments of the couples Ff_1, Ff_2 and Ff; let the lengths be in that proportion. Vector AO is the sum of the vectors AM and AN, provided that $OMAN$ is a parallelogram; we now show that it is a parallelogram. Angle $MAO = \beta$; since in the triangle MAO and ABC two sides are proportional each to each and the included angles are equal, the triangles are similar; it follows that OM is perpendicular to AC, or parallel to AN. From similar reasoning, it follows that ON is perpendicular to AB, or parallel to AM. Hence $OMAN$ is a parallelogram. Obviously, if the proposition holds for two couples, it holds for any number.

Composition of three couples whose planes are mutually at right angles is an important special case. We take the three planes as coördinate planes, and call the couples whose planes are perpendicular to the x, y, and z axes C_x, C_y, and C_z respectively, their vectors v_x, v_y and v_z, and the resultant couple C and its vector v. Then $v = (v_x^2 + v_y^2 + v_z^2)^{\frac{1}{2}}$; hence

$$C = (C_x^2 + C_y^2 + C_z^2)^{\frac{1}{2}}.$$

Also, if ϕ_1, ϕ_2, and ϕ_3 denote the direction angles of v, then $\cos\phi_1 = v_x/v$, $\cos\phi_2 = v_y/v$, and $\cos\phi_3 = v_z/v$; hence

$$\cos\phi_1 = C_x/C, \quad \cos\phi_2 = C_y/C, \quad \cos\phi_3 = C_z/C.$$

It follows from the preceding that a couple may be equivalent to two or more couples, which are therefore components of that couple; also, to resolve a couple we have only to resolve its vector, the component vectors being the vectors of the component couples. The resolution of a couple into three components whose planes are mutually at right angles is an important special case. Let C be the couple to be resolved and v its vector, and denote the direction angles of the vector by α, β, and γ, the coördinate planes having been taken to coincide with the planes of the desired component couples. Let C_x, C_y, and C_z denote the component couples, which are perpendicular to the x, y and z axes respectively, and v_x, v_y and v_z the corresponding vectors. Then $v_x = v\cos\alpha$, $v_y = v\cos\beta$, and $v_z = v\cos\gamma$; hence,

$$C_x = C\cos\alpha, \quad C_y = C\cos\beta, \quad C_z = C\cos\gamma.$$

9. Noncoplanar Nonconcurrent Forces

§ 1. PARALLEL FORCES.—It is shown in Art. 7 that the resultant of any two parallel forces is parallel to those forces, and that its magnitude and sense are given by the algebraic sum of the forces, the sense being given by the sign of the sum. It follows that the resultant of any number of parallel forces, not coplanar necessarily, is parallel to the forces, and that its magnitude and sense are given by the algebraic sum of the forces (all forces of the same sense having one sign, and those of the opposite sense having the opposite sign). The line of action of the resultant may be fixed by means of the arms of the resultant with respect to two rectangular axes, each perpendicular to the forces. Such arms can be computed readily from the principle that the moment of the resultant about any axis equals the algebraic sum of the moments of the forces about the same axis.

For an example, we find the resultant of four forces which referred to a set of rectangular axes are described as follows: They are parallel to the z-axis; their magnitudes are recorded in the first column of the schedule

F	x	y	M_x	M_y
+40	−4	+2	+80	+160
−30	−3	−2	+60	−90
−20	+6	+5	−100	+120
+60	+7	−8	−480	−420
+50			−440	−230

adjoining, the signs indicating the senses of the forces; their lines of action pierce the xy plane in points whose coördinates are recorded in the second and third columns. The algebraic sum of the forces is +50, and so the magnitude of the resultant is 50 pounds, and it acts in the positive z direction. The moments of the forces about the x and y axes are recorded in the last two columns respectively. The algebraic sums of these moments are −440 and −230 foot-pounds, as indicated. These sums are also the values of the moments of the resultant with respect to those axes; hence the resultant (acting in the positive direction) is above the x axis and to the right of the y axis at distances equal to $440 \div 50$ or 8.8, and $230 \div 50 = 4.6$ feet respectively.

If the algebraic sum of the forces equals zero, then their resultant is, in general, a couple. For the resultant of all the forces but one is a single force equal and opposite to that one; and, in general, that resultant force and the omitted force would not be colinear, and so they constitute a couple as stated.

§ 2. NONPARALLEL FORCES. — A set of noncoplanar, nonconcurrent, nonparallel forces may be compounded, in general, into a force acting through any arbitrary chosen point and a couple. Proof follows: As explained in Art. 5, each force of the given system may be resolved into and be replaced by a force acting through the chosen point and a couple. Supposing such a replacement made for each given force, then the new system consists of a set of concurrent forces at the chosen point and a set of couples; but the resultant of the concurrent forces is a single force acting through the chosen point (Art. 4), and the resultant of the couples is a single couple (Art. 8). This force and couple respectively will be denoted by R and C.

We now show in detail how to determine R and C. Let F_1, F_2, F_3, etc. (Fig. 42, only F_1 shown), be the forces of the given system acting on a body not shown; O the point through which R is to pass; and OX, OY and OZ any convenient axes of reference. Let P_1 and Q_1, acting at O (Fig. 42), be equal and parallel to F_1; similarily, let P_2 and Q_2 (not shown) act at O, and be equal and parallel to F_2; etc. Then the force P_1 and the couple F_1Q_1 (Fig. 43) are equivalent to F_1 (Fig. 42); the force P_2 and the couple F_2Q_2 are equivalent to F_2; etc. Now the axial components of P_1,

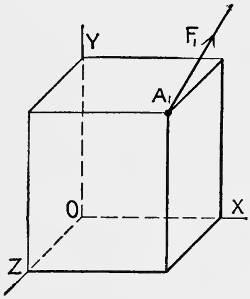


FIG. 42

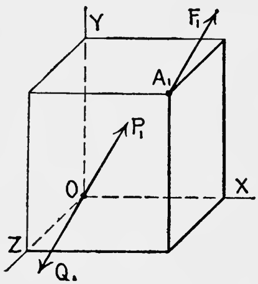


FIG. 43

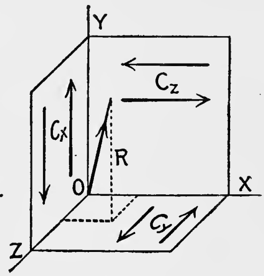


FIG. 44

P_2, P_3, etc. (the concurrent forces), are respectively equal to the axial components of F_1, F_2, F_3, etc. (the given forces); hence if ΣF_x, ΣF_y and ΣF_z denote the algebraic sums of the x, y, and z components of the given forces, then $R_x = \Sigma F_x$, $R_y = \Sigma F_y$, and $R_z = \Sigma F_z$; also

$$R^2 = (\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2.$$

And if θ_1, θ_2, and θ_3 denote the direction angles of R, then

$$\cos\theta_1 = \Sigma F_x/R, \qquad \cos\theta_2 = \Sigma F_y/R, \qquad \cos\theta_3 = \Sigma F_z/R.$$

These formulas determine R. To determine C: Imagine it resolved into three components whose planes are respectively perpendicular to the x, y, and z axes (Art. 8), and denote the components and their moments by C_x, C_y, and C_z (Fig. 44). Since the system R, C_x, C_y, and C_z is equivalent to the given system, their moments about any line are equal; hence $C_x = \Sigma M_x$, $C_y = \Sigma M_y$, and $C_z = \Sigma M_z$, where ΣM_x, ΣM_y, and ΣM_z denote the moment-sums for the given system with respect to the x, y, and z axes respectively. Also, according to Art. 8,

$$C^2 = (\Sigma M_x)^2 + (\Sigma M_y)^2 + (\Sigma M_z)^2;$$

and if ϕ_1, ϕ_2 and ϕ_3 denote the direction angles of the vector representing C, then

$$\cos\phi_1 = \Sigma M_x/C, \qquad \cos\phi_2 = \Sigma M_y/C, \qquad \cos\phi_3 = \Sigma M_z/C.$$

In general, R and C may be compounded into two noncoplanar forces. For, as explained in Art. 8, C may be shifted about without change of effect if only the direction of its plane be unchanged; assume such shift until one of the forces of C intersects R; then that force and R may be compounded into a single force R'; there remain R' and the second force of C, and obviously R' and that force are not coplanar. These two cannot be compounded; they are the simplest set equivalent to the given system, and therefore constitute the resultant of the given system. If the plane of C happens to be parallel to R, then C and R can be compounded into a single force, and the resultant of the given system is a single force. For shifting C about until C and R become coplanar, then they may be compounded readily into a single force (Art. 5).

In general, the system of forces has a torque about every line through O. There is one line which is of prime importance, the line about which the torque is greatest. The torque of the forces about that line is called *the* or resultant torque of the system (for the chosen point O). Since R has no moment about a line through O, the torque of the system about any such line equals the torque of C about that line. But the torque of C is greatest about a line perpendicular to the plane of C; this is the important line mentioned. The direction of this line is given by equations (4), and the resultant torque of the system by equations (3). The system of forces has no torque about a line through O parallel to the plane of C, (perpendicular to the line or axis of resultant torque) since R and C have no torque about such line.

CHAPTER II

FORCES IN EQUILIBRIUM

10. Principles of Equilibrium

§ 1. General Conditions of Equilibrium.—It is convenient in some discussions to distinguish forces as "external" or "internal," meaning by *external force* one which is exerted on the body under discussion by some other body, and by *internal force* one which is exerted on a part of the body under discussion by another part. (The word body is used here in a broad sense to denote any definite portion of matter, as a locomotive, a bridge, the steam in a boiler, the water in a pond, etc.) For illustration, consider the crude crane in Fig. 45. It consists of three main members (*AB*, *CD* and *DE*), a pulley, a winding drum and a hoisting chain; it is supported at *A* (ceiling) and at *B* (floor). The external forces acting on the crane consist of the weight of all the parts (exerted by the earth), the pull down on the hook (exerted by the load), the supporting force at *A* (exerted by the ceiling), and the supporting force at *B* (exerted by the floor). The members exert forces upon each other where they come together, but these are internal forces with reference to the whole crane. With reference to the crane post *AB*, the external forces are its weight, the supporting force at *A*, that at *B*, the pressures on it at *E*, *C*, and the drum. All these are exerted on the post by something else, and so are properly called external forces. Any two adjacent portions of the post, as the upper and lower halves, exert forces on each other, and these forces are internal with reference to the post.

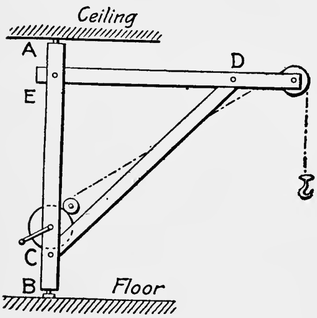


Fig. 45

All the external forces acting on a body at rest constitute a balanced system, and such system is said to be in *equilibrium*. Obviously, the resultant of such a system is *nil*, and this fact is sometimes called the general condition of equilibrium for any kind of a force system. The general condition implies subordinate conditions; thus, for any system whatever,

(*A*) *the algebraic sum of the (rectangular) components of all the forces along any line equals zero*, and

(*B*) *the algebraic sum of the moments of all the forces about any line equals zero.*

By means of (A) and (B) we can write many equations for any system in equilibrium. Thus, for a coplanar concurrent system, (A) gives $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma F_u = 0$, etc., where x, y, u, etc., are axes of resolution; and (B) gives $\Sigma M_a = 0$, $\Sigma M_b = 0$, $\Sigma M_c = 0$, etc., where a, b, c, etc., are origins of moments in the plane of the forces. Not all of such equilibrium equations are independent, however; that is, certain ones follow from the others. Thus, if $\Sigma F_x = 0$ for any coplanar concurrent system, then ΣF_y does not necessarily equal zero, but if also $\Sigma F_y = 0$, then the resultant equals zero, and it follows that $\Sigma F_u = 0$. That is, $\Sigma F_x = 0$ and $\Sigma F_y = 0$ are two independent equations, but any third similar equation (as $\Sigma F_u = 0$) is not independent of them. The independent equations or conditions of equilibrium for any particular kind of force system are such as are necessary and sufficient to insure a vanishing resultant. We will now deduce these independent conditions of equilibrium for the various classes or kinds of force systems.

(i) *Colinear Forces.*—There is one condition of equilibrium. It can be stated in several forms; namely,

$$(1)\ \Sigma F = 0 \quad \text{or} \quad (2)\ \Sigma M_a = 0.$$

Form (1) states that the algebraic sum of the forces equals zero; (2) that the algebraic sum of the moments of all the forces about any point (not on their common line of action) equals zero. On the graphical basis, the condition of equilibrium is that the force polygon for the forces (degenerated into a straight line in this case) is a closed one. For if $\Sigma F = 0$, or $\Sigma M = 0$, or the force polygon closes, then there is no resultant.

(ii) *Coplanar Concurrent Forces.*—There are two independent algebraic conditions of equilibrium. They can be expressed in three forms; namely,

$$(1)\ \Sigma F_x = \Sigma F_y = 0, \quad (2)\ \Sigma F = \Sigma M_a = 0, \quad \text{or} \quad (3)\ \Sigma M_a = \Sigma M_b = 0.$$

Form (1) states that the algebraic sums of the components of the forces along two lines x and y (in the plane of the forces) equal zero; (2) that the algebraic sum of the components of the forces along any line (as x), and the algebraic sum of the moments of all the forces about any point, each equal zero (the point a to be in the plane of the forces, and the line joining a and O, their point of concurrence, to be inclined to the x axis); and (3) that the algebraic sums of the moments of all the forces about two points (not colinear with the point of concurrence of the forces) equal zero. For in any case the resultant is zero, as will be seen from this: (1) According to Art. 4, the resultant of the system, if there is one, is a single force R, given by $R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$; and hence if $\Sigma F_x = 0$ and $\Sigma F_y = 0$, R must equal zero. (2) If $\Sigma F_x = 0$, then the resultant, if there is one, must be perpendicular to the x axis; and if $\Sigma M_a = 0$, then the moment of R about a equals zero, which requires that $R = 0$. (3) The resultant, if there is one, must pass through the point of concurrence O of the given forces; if $\Sigma M_a = 0$

then R must pass through a also; if $\Sigma M_b = 0$, then R must equal zero, b not being on Oa.

The graphical condition of equilibrium is that the force polygon for the forces closes. For, if it does close, then there is no resultant.

(iii) *Coplanar Nonconcurrent Parallel Forces.*—There are two independent algebraic conditions of equilibrium. They can be expressed in two forms; namely,

$$(1)\ \Sigma F = \Sigma M = 0 \quad \text{or} \quad (2)\ \Sigma M_a = \Sigma M_b = 0$$

Form (1) states that the algebraic sum of the forces and the algebraic sum of the moments of the forces about any point (in the plane of the forces) equal zero; (2) that the algebraic sums of the moments of the forces about two points equal zero, the line joining the origins not to be parallel to the forces. For either set of conditions is necessary and sufficient to make the resultant zero, as may be shown thus: In Art. 7 it is shown that the resultant, if there is one, is a single force or a couple. And (1), if $\Sigma F = 0$, then the resultant is not a force, and if $\Sigma M = 0$, then it is not a couple; and hence there is no resultant. (2) If $\Sigma M_a = 0$, the resultant is not a couple but a force, which passes through a; if also $\Sigma M_b = 0$, then the moment of the resultant force about b must be zero, and that requires that the force equals zero.

There are two graphical conditions of equilibrium, namely, a force and a string polygon for the forces must close. For if a force polygon closes, then the resultant, if there is one, is a couple; if a string polygon closes, then the resultant is not a couple.

(iv) *Coplanar Nonconcurrent Nonparallel Forces.*—There are three independent algebraic conditions of equilibrium. They can be stated in three forms; namely,

$$(1)\ \Sigma F_x = \Sigma F_y = \Sigma M_a = 0;$$

$$(2)\ \Sigma F_x = \Sigma M_a = \Sigma M_b = 0;$$

and

$$(3)\ \Sigma M_a = \Sigma M_b = \Sigma M_c = 0.$$

Form (1) states that the algebraic sums of the components of all the forces along two lines and the algebraic sum of the moments of the forces about any point equal zero, the lines and points to be in the plane of the forces; (2) that the algebraic sums of the components of the forces along any line x and the algebraic sums of the moments of the forces about two points, a and b, equal zero, the line x and that joining a and b not to be at right angles; and (3) that the algebraic sums of the moments of the forces about three points, a, b, and c, equal zero, the points not to be colinear. For any set of these conditions is necessary and just sufficient to make the resultant vanish as may be shown, thus: The resultant, if there is one, is a single force or a single couple (Art. 7). And (1) if $\Sigma F_x = \Sigma F_x = 0$, then the resultant is not force, and if $\Sigma M = 0$, it is not a couple; and hence there is no resultant. (2) If $\Sigma F_x = 0$, the resultant is a force R perpendicular to the x axis or a couple; if $\Sigma M_a = 0$, it is not a couple, but a force passing through a (and perpendicular to the x axis); if also $\Sigma M_b = 0$, then the moment of that force about b must equal zero, and hence the force must equal zero. (3) If $\Sigma M_a = 0$, the resultant, if there is one, is not a couple but a force passing through a; if $\Sigma M_b = 0$, that resultant passes through b; if also $\Sigma M_c = 0$, then the resultant force must equal zero.

There are two graphical conditions, just like those for parallel coplanar nonconcurrent forces; namely, a force and a string polygon must close. For if a force polygon closes, then the resultant, if there is one, is not a force but a couple; if a string polygon closes, then the resultant is not a couple, and so there is no resultant (see Art. 6).

(v) *Noncoplanar Concurrent Forces.* — There are three independent algebraic conditions of equilibrium. The convenient form is

$$\Sigma F_x = \Sigma F_y = \Sigma F_z = 0;$$

that is, the algebraic sums of the components of all the forces along three rectangular axes, x, y, and z, equal zero. For as shown in Art. 4, the resultant, if there is one, equals $\sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}$, and so if the conditions stated are fulfilled then the resultant equals zero.

(vi) *Noncoplanar Parallel Forces.* There are three independent algebraic conditions of equilibrium. There are two convenient forms; namely,

$$(1)\ \Sigma F = \Sigma M_1 = \Sigma M_2 = 0, \quad \text{and}\ (2)\ \Sigma M_1 = \Sigma M_2 = \Sigma M_3 = 0.$$

Form (1) states that the algebraic sum of the forces and the algebraic sums of the moment of the forces about two lines perpendicular to the forces but not parallel to each other equal zero; (2) that the algebraic sums of the moments about three coplanar nonconcurrent nonparallel lines perpendicular to the forces equal zero. For (1) if $\Sigma F = 0$, the resultant is not a force; if $\Sigma M_1 = 0$, the resultant is a couple whose plane is parallel to the first line or axis of moments (and to the forces); and if $\Sigma M_2 = 0$, then the plane of the couple must also be parallel to the second axis; but all these conditions of parallelism cannot be fulfilled unless the two forces of the couple are colinear, in which case the two forces balance, so that there is really no resultant. (2) If $\Sigma M_1 = \Sigma M_2 = 0$, then the resultant must be a force passing through the intersection of lines 1 and 2; if $\Sigma M_3 = 0$, then that force must equal zero; that is, the three conditions make the resultant vanish.

(vii) *Noncoplanar Nonconcurrent Nonparallel Forces.* — There are six independent algebraic conditions of equilibrium, namely,

$$\Sigma F_x = \Sigma F_y = \Sigma F_z = \Sigma M_x = \Sigma M_y = \Sigma M_z = 0;$$

that is, the algebraic sums of the components of all the forces along three lines and the algebraic sums of the moments of the forces about three noncoplanar axes equal zero. (It is generally most convenient to take the three lines and the three axes at right angles to each other.) For the resultant of the system, if there is one, is always reducible to a single force and a single couple (Art. 9); if $\Sigma F_x = \Sigma F_y = \Sigma F_z = 0$, the single force equals zero, and $\Sigma M_x = \Sigma M_y = \Sigma M_z = 0$, if then the couple vanishes, and so there is no resultant.

If every force in the given system (in equilibrium) be represented by a vector, and all these vectors be projected on three rectangular coördinate planes, then the three sets of projections represent three force systems, and each is in equilibrium (proved below). In some cases it may be more convenient to deal with these projected systems. In general, each furnishes three conditions or equations of equilibrium, making nine in all; but there are duplicates among the nine, and only six are independent. To prove the foregoing, let F (Fig. 46) be one of the forces of the system in equilibrium and P its point of application (on a body not shown). A, B, and C are projections of the vector F on the xy, yz, and zx planes respectively. Obviously, the x and y components of A equal F_x and F_y respectively; the y and z components of B equal F_y and F_z respectively, and the z and x components of C equal F_z and F_x respectively, as indicated. Since the given system is in equilibrium,

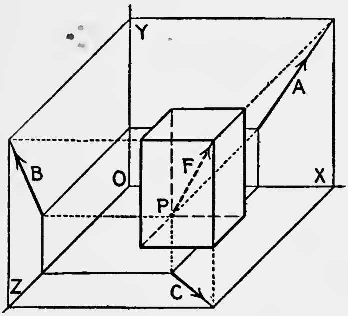


FIG. 46

$$(1)\quad \Sigma F_x = 0, \qquad (4)\quad \Sigma M_x = \Sigma(F_z y - F_y z) = 0,$$
$$(2)\quad \Sigma F_y = 0, \qquad (5)\quad \Sigma M_y = \Sigma(F_x z - F_z x) = 0, \quad \text{and}$$
$$(3)\quad \Sigma F_z = 0, \qquad (6)\quad \Sigma M_z = \Sigma(F_y x - F_x y) = 0.$$

Now ΣF_x is also the sum of the x components of the A-system; ΣF_y is also the sum of the y components of the A-system; and $\Sigma(F_y x - F_x y)$ is also the sum of the moments of the A forces about O. Hence (1), (2), and (6) are conditions which assert the equilibrium of the A-system. For similar reasons (2), (3), and (4) assert the equilibrium of the B-system and (1), (3), (5) assert the equilibrium of the C-system.

§ **2.** SPECIAL CONDITIONS OF EQUILIBRIUM, depending on number of forces in the system.—(1) A single force cannot be in equilibrium. (2) If two forces are in equilibrium, then obviously they must be colinear, equal, and opposite. (3) If three forces are in equilibrium, then they must be coplanar, and concurrent or parallel. Proof: Let the three forces be called F_1, F_2, and F_3; since F_1 and F_2 balance F_3, F_1 and F_2 have a single force resultant R colinear with F_3; since F_1 and F_2 have a resultant colinear with F_3, they lie in a plane with F_3. If F_1 and F_2 are concurrent, then R is concurrent with them and hence F_3 also; if F_1 and F_2 are parallel, then R and hence F_3 is parallel to them. When the three forces are concurrent, then each is proportional to the sine of either angle between the other two (Lami's theorem); that is,

$$\frac{F_1}{\sin\alpha' = \sin\alpha''} = \frac{F_2}{\sin\beta' = \sin\beta''} = \frac{F_3}{\sin\gamma' = \sin\gamma''},$$

where F_1, F_2, and F_3 are the forces, α' and α'' the angles between F_2 and F_3, β' and β'' those between F_1 and F_3, and γ' and γ'' those between F_1 and F_2 (see Fig. 47). For it follows from the triangle of forces, $ABCA$ (in which AB, BC, and CD represent F_1, F_2, and F_3 respectively), that $AB/\sin BCA = BC/\sin CAB = CA/\sin ABC$. But $BCA = \alpha'$, $CAB = \beta'$, and $ABC = \gamma'$, also α' and α'', β' and β'', γ' and γ'', are supplementary, hence $\sin\alpha' = \sin\alpha''$, etc., etc. When the three forces are parallel, then the two outer ones act in the same direction and the middle one in the opposite direction, and the moments of any two of the forces about a point on the third are equal in magnitude and opposite in sense, or sign. (4) When four coplanar forces are in equilibrium, then the resultant R of any two of the forces balances the other two. Hence, (*a*) if the first two are concurrent and the second two also, then the R passes through the two points of concurrence; (*b*) if either two are concurrent and the other two parallel, then the resultant R of the first pair acts through the point of concurrence and is parallel to the second pair; (*c*) if all four forces are parallel, then R is parallel to the forces. Principles (*a*) and (*b*) are useful in graphical analysis of four-force systems.

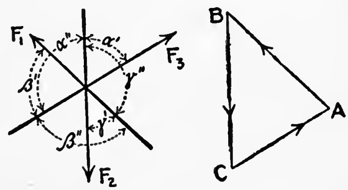


FIG. 47

§ 3. SUMMARY.—The *algebraic conditions* of equilibrium explained in detail in the foregoing are brought together here for convenience of reference.

Coplanar Forces.

Colinear, $\Sigma F = 0$; or $\Sigma M = 0$.

Concurrent, $\Sigma F_x = \Sigma F_y = 0$; or $\Sigma F_x = \Sigma M_a = 0$; or $\Sigma M_a = \Sigma M_b = 0$.

Parallel, $\Sigma F = \Sigma M = 0$; or $\Sigma M_a = \Sigma M_b = 0$.

Nonconcurrent nonparallel, $\Sigma F_x = \Sigma F_y = \Sigma M = 0$; or

$$\Sigma F_x = \Sigma M_a = \Sigma M_b = 0; \text{ or } \Sigma M_a = \Sigma M_b = \Sigma M_c = 0.$$

Noncoplanar Forces.

Concurrent, $\Sigma F_x = \Sigma F_y = \Sigma F_z = 0$.

Parallel, $\Sigma F = \Sigma M_1 = \Sigma M_2 = 0$; or $\Sigma M_1 = \Sigma M_2 = \Sigma M_3 = 0$.

Nonconcurrent nonparallel, $\Sigma F_x = \Sigma F_y = \Sigma F_z = \Sigma M_x = \Sigma M_y = \Sigma M_z = 0$.

The *graphical conditions* of equilibrium for coplanar systems: for concurrent forces, the force polygon closes; for nonconcurrent forces, the force and the string polygon close. There are graphical conditions of equilibrium for noncoplanar forces, but their usefulness is very limited, and they are therefore not given here.

11. Coplanar Concurrent Forces in Equilibrium

§ 1. The general principles of equilibrium for such forces are explained in Art. 10 under (ii). We now show how to apply the principles in two particular problems.

Typical Problem (i). — A system of coplanar concurrent forces is in equilibrium, and all the forces except two are wholly known; the lines of action of these are known, and their magnitudes and senses are to be determined. The graphical method is generally the simplest for solving this problem; but if there are only three forces in the system, or if the angle between the two unknown forces is 90 degrees, then the algebraic method is simple.

To solve graphically, we draw a force polygon for all the forces, and make it close since they are in equilibrium; in doing so the desired unknowns will be determined. For example, consider the forces acting on the pin O of the bridge truss partially represented in Fig. 48. (A pin

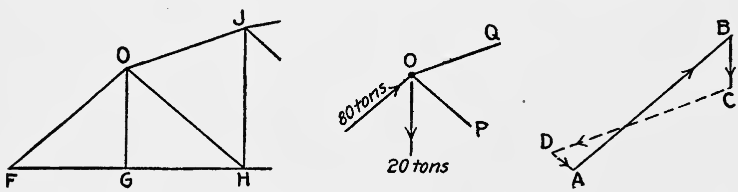


FIG. 48

passes through holes in the members, OF, OG, OH, and OJ, thus fastening them together at O.) There are four forces acting on this pin, one exerted by each member named, and they constitute a system in equilibrium. (Strictly, there is a fifth force in the system, the weight of the pin, but that is small compared to the others and is negligible.) These four forces are coplanar and concurrent. We assume that they act in the directions of the members respectively (generally not far from the fact) as shown; furthermore, we will suppose that the magnitudes and directions of two of the forces have been determined somehow. Now to determine the other two, P and Q, completely: We draw AB to represent the 80 ton force according to some convenient scale; and BC to represent 20 tons; then from C, a line parallel to Q, and from A, a line parallel to P, and mark their intersection D. Then CD and DA represent the magnitudes Q and P respectively; and, since the arrowheads in the closed vector polygon must be confluent, Q acts in the direction CD and P in the direction DA. There are other possible force polygons, each giving the same result as the one explained.

To solve this problem algebraically we may employ any one of the three sets of equations or conditions of equilibrium (Art. 10); namely,

$$\Sigma F_x = \Sigma F_y = 0, \ \Sigma F_x = \Sigma M_a = 0, \text{ or } \Sigma M_a = \Sigma M_b = 0.$$

Taking the first set and assuming* senses for P and Q (Fig. 49), we get

$$\Sigma F_x = Q \cos 20° + P \cos 40° + 80 \cos 40° = 0, \text{ and}$$

$$\Sigma F_y = -20 + Q \sin 20° - P \sin 40° + 80 \sin 40° = 0;$$

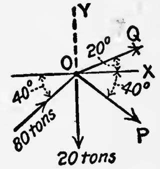


FIG. 49

solving these equations simultaneously for P and Q, we get $P = 10.04$ and $Q = -73.3$ tons.

When the system is a three-force system, then the special condition, $F_1/\sin\alpha = F_2/\sin\beta = F_3/\sin\gamma$ (Art. 10), is, in general, the simplest to apply. (F_1, F_2 and F_3 denote the forces, and α either angle between F_2 and F_3, β either angle between F_3 and F_1, and γ either angle between F_1 and F_2.)

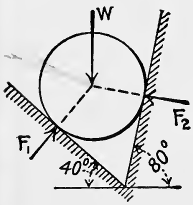


FIG. 50

To illustrate, we discuss the forces acting upon a cylinder which lies in a trough formed by two smooth † inclined planes (Fig. 50). There are three forces acting on the cylinder; namely, its own weight (100 pounds), and the two supporting forces F_1 and F_2. Since the planes are smooth F_1 and F_2 act normally, and hence through the center of the cylinder as shown. It follows from the geometry of the figure that the acute angle between F_1 and $W = 40°$, that between F_2 and $W = 80°$, and that between F_1 and $F_2 = 60°$; hence $F_1/\sin 80° = F_2/\sin 40° = 100/\sin 60°$, or $F_1 = 113.7$ and $F_2 = 74.2$ pounds.

* Whenever a force whose sense is unknown is to be entered in a resolution or moment equation, a sense should be assumed for that force and adhered to in the solution of the equation. The correct sense is indicated by the sign of the computed value of that force; a positive sign indicates that the sense assumed is correct and a negative sign that the sense assumed is wrong. Senses found to be wrong are corrected in the figures of the book, by a short line across the assumed arrowhead (Fig. 49).

† When two bodies are in contact, and they exert forces upon each other (equal and opposite), the forces are, in general, inclined to the surface of contact, assumed plane for the moment. The components of either of the forces mentioned along and perpendicular to the surface of contact are called *friction* and *normal pressure* respectively. Fig. 51 furnishes the simplest illustration; it represents a heavy body A supported by a rough surface B, and subjected to a push P. The surface B exerts a force R on A (inclined as shown), and the horizontal and vertical components of R are the friction and the normal pressure exerted by B on A. Obviously, this friction is the resistance which B offers to the tendency of A to slide over B. So long as there is only tendency to sliding, this friction equals the push P. Experience has shown that the friction is a maximum just as sliding impends, and also that the smoother the surfaces of contact, the smaller is the force required to cause sliding, and hence the smaller this maximum resistance to sliding. We are thus led to the conception of a perfectly smooth surface as one

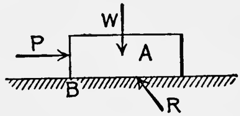


FIG. 51

Typical Problem (ii). A system of coplanar concurrent forces is in equilibrium and all except one are wholly known; the magnitude and direction of this one are required. To solve this problem we might determine the resultant of the wholly known forces; this resultant reversed is the desired force. But the problem may also be solved by means of principles of equilibrium, that is, by applying the appropriate conditions of equilibrium to the entire system of forces. To illustrate, we determine the value and direction of the tension in the cord* (Fig. 52) which supports a ring from which a body W is suspended, the ring being subjected to a force P as shown. The forces acting on the ring are W, P, and the pull of the long cord (equal to the tension), and these three forces are in equilibrium. To solve graphically, we draw AB to represent W, and BC to represent P; then CA represents the desired pull or tension. To solve algebraically, we call the desired force F and its inclination to the vertical θ. Then, using the conditions $\Sigma F_x = 0$ and $\Sigma F_y = 0$, we get $20 \cos 30° - F \sin \theta = 0$ and $-100 + F \cos \theta + 20 \sin 30° = 0$; these solved simultaneously give $F = 91.6$ pounds, and $\theta = 10° 54'$.

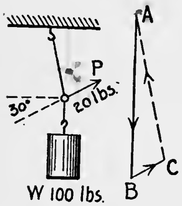


FIG. 52

FIG. 53

As another example, we determine the force which the inclined plane (Fig. 54) exerts on the body A when it is subjected to a pull $P = 20$ pounds, the plane being so rough that motion does not ensue. The weight of A (100 pounds), P, and the reaction R of the plane are in equilibrium; hence, using θ to denote the inclination of R to the plane, and resolving along the plane and normal to it, we get

$$20 - 100 \sin 30 + R \cos \theta = 0, \text{ and } R \sin \theta - 100 \cos 30° = 0.$$

Solving these simultaneously, we get $R = 91.7$ pounds, and $\theta = 70° 53'$.

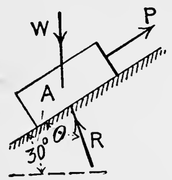


FIG. 54

which can offer no frictional resistance, only normal reaction. Such a surface is of course ideal, but there are surfaces which are nearly perfectly smooth. For brevity we will call these smooth, and those whose resistance to sliding is to be taken into account will be called rough.

If the surface of contact between two bodies is curved, then we speak of the friction and normal pressure at any elementary portion of the contact, meaning the tangential and normal components of the pressure at that element. If the contact between two bodies is small, practically a point, and they exert forces R upon each other there, then normal pressure means the component of R at right angles to the plane which is tangent to the surfaces at the contact, and friction means the component along that plane. If one or both the bodies is smooth, then any pressure exerted between the two at any point of the contact is directed along the normal there. (For fuller discussion of friction see Chapter IV.)

* "*Tension in a cord*" refers to the forces which two parts of a taut cord exert upon each other. Suppose that AB (Fig. 53) is a cord subjected to equal pulls at its ends, and imagine a

§ 2. Many machines and other devices consist of parts (members) more or less intimately connected, and, in general, these parts exert forces upon each other when the machine is in service. To determine these forces seems a complicated problem to most beginners. And yet in many instances the whole problem can be resolved into several simpler ones, often like typical problem (i), which may be solved in turn and thus furnish values of the desired forces. In this connection it will be convenient to designate a member of any device as a *one-force piece* if only one force acts upon it; as a *two-force piece* if only two forces act upon it; etc. Obviously, a one-force piece cannot be at rest. If a two-force piece is at rest, then the two forces acting upon it must be equal, opposite, and colinear; each force acts in the line joining their points of application, and the reactions which the piece exerts (upon the members which act upon it) also act along the same line.* If a three-force piece is at rest, then the three forces are coplanar, and concurrent or parallel (Art. 10, § 2). If a four-force piece is at rest, then the resultant of any pair of the four balances the other pair. We now illustrate how to resolve the apparently difficult problem into several simpler ones.

Example. The crab-tongs represented in Fig. 55 consist of six pieces fastened together by pins B,B', C,C', and D; angle ABC = 100 degrees,

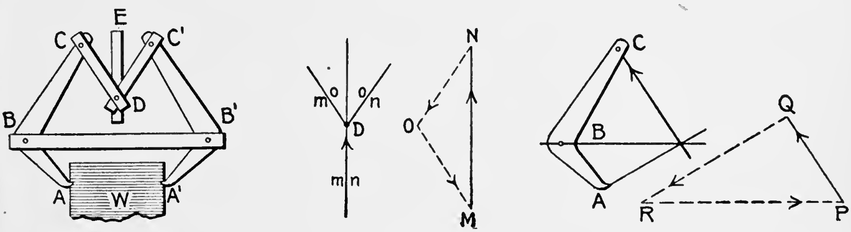


Fig. 55

AB = 1 foot, BC = 1 foot 9 inches, CD = 1 foot, and BB' = 3 feet. Required the forces which act on each piece when the tongs suspend a stone W whose weight = 1000 pounds, and width AA' = 1 foot 6 inches. Apparently, the trigonometric relations between the parts are not simple; so we will solve graphically, and first we draw (or lay out) the tongs to scale. Obviously, the supporting force at E equals 1000 pounds (weight of

plane of separation at any place C between the ends of the cord. Since the part AC is in equilibrium, there is a force acting upon it at its right end equal and opposite to P'; this force is exerted by the part BC. Similarly, there is a force acting upon BC at its left end equal and opposite to P''; this force is exerted by the part AC. These two equal and opposite forces at C hold the parts AC and BC together. By magnitude of the tension is meant the magnitude of either of the forces.

* *Action and reaction* are equal, opposite, and colinear if they are concentrated. This is a brief statement of Newton's Third Law of Motion, and it means that when one body exerts a force upon another body then the latter also exerts one on the former, and the two forces are equal in magnitude and opposite in direction. By action is meant either of these two forces and by reaction the other one.

tongs neglected). The pin D is acted upon by DE, DC and DC', and, since each of these is a two-force piece, the forces upon the pin act along DE, DC, and DC', as shown at center. The first force equals 1000 pounds and acts upwards; determination of the other two presents typical problem (i). So we draw MN to represent the 1000 pound force, and from M and N lines parallel to the other two, thus fixing O; then NO and OM represent the magnitudes of the two forces (620 pounds). It follows that DC and DC' are subjected to end pushes or compressions of 620 pounds. CBA is a three-force piece, the forces being applied at C, B and A. The first acts parallel to CD as shown and equals 620 pounds; the second is exerted by the two-force piece BB', and hence acts along BB'; and the third must be concurrent with the first two and so acts along the straight line through A. Determination of the two unknown forces presents typical problem (i). So we draw PQ to represent the 620 pound force, and lines from P and Q parallel to the other two, thus fixing R; then QR represents the force at A (950 pounds), and RP that at B (1315 pounds). It follows that the piece BB' is subjected to end pulls of 1315 pounds.

12. Coplanar Parallel Forces in Equilibrium

§ 1. Principles of equilibrium for a system of forces of this kind are developed in Art. 10 under (iii); we now show how to apply them to a common problem. (For typical problems i and ii see Art. 11.)

Typical Problem (iii). A system of coplanar parallel forces is in equilibrium, and all the forces except two are wholly known; the lines of action of these two are known and their magnitudes and senses are required.

The algebraic method is the better one, by far, for solving the problem. There are two sets of conditions of equilibrium available; namely, (1) $\Sigma F = \Sigma M = 0$, that is, the algebraic sum of the forces and the algebraic sum of the moments of the forces each equal zero; and (2) $\Sigma M_a = \Sigma M_b = 0$, that is, the moment-sums for two different origins equal zero, the line joining the origins not to be parallel to the forces. Either set will furnish a solution of the problem. The second set is recommended, and the origins of moments a and b should be taken on the lines of action of the two unknown forces. For example, consider the beam represented in Fig. 56 under the action of three loads (its own weight neglected), and supported at A and B; required, the reactions of the two supports. The five forces just mentioned constitute a system in equilibrium; therefore, taking moment origins on R_1 and R_2 respectively, and assuming that R_1 and R_2 act upwards, we get

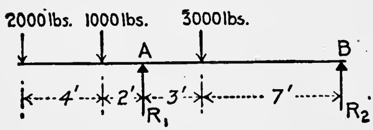


FIG. 56

$$\Sigma M_1 = 2000 \times 6 + 1000 \times 2 - 3000 \times 3 + R_2 \times 10 = 0,$$

and $$\Sigma M_2 = 2000 \times 16 + 1000 \times 12 + 3000 \times 7 - R_1 \times 10 = 0.$$

The first gives $R_2 = -500$ pounds, and the second $R_1 = 6500$; the negative sign means that R_2 acts downward on the beam and not upward, as assumed. As a check on the solution we try whether $\Sigma F = 0$; thus,

$$-2000 - 1000 - 3000 + 6500 - 500 = 0.$$

The graphical solution of the foregoing problem is based on the conditions that the force and the string polygon for the forces close; the process of constructing and closing the polygons determines the unknown forces. To illustrate we take the beam shown in Figs. 56 and 57 and determine the reactions. First, the force polygon should be drawn as far as possible, the knowns represented first, thus AB, BC, and CD (Fig. 58) representing the 2000, the 1000, and 3000 pound forces respectively; then the lines of action should be lettered to correspond, ab, bc, and cd (Fig. 57). If R_2, say, is taken next, it would be lettered DE, and R_1 would be EA, since the force

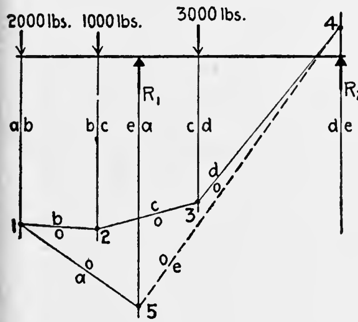


FIG. 57

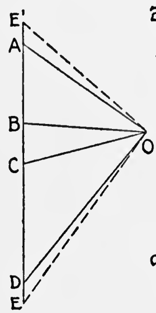


FIG. 58

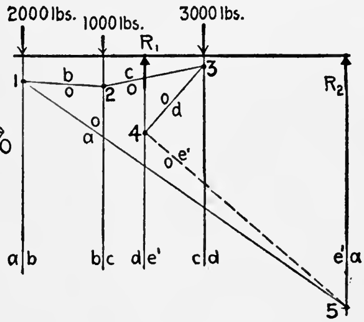


FIG. 59

polygon for all must close. It remains now to locate E; this can be done by means of the string polygon. (At this point it may be well for the reader to recall the significance of the strings of a string polygon; see Art. 6.) The polygon may be started at any point on any of the lines of action of the forces of the system; if it be started at 1 (on ab), then strings oa and ob must be drawn through that point; oc must be drawn from 2 (where ob cuts bc), od from 3 (where oc cuts cd), and oe from 4 (where od cuts de) and from 5 (where oa cuts ea); hence the closing string oe passes through 4 and 5. Finally, the ray OE, parallel to oe, is drawn, thus determining E; DE represents R_2, and EA R_1. Fig. 59 shows another solution; R_1 is taken as the fourth force DE', and R_2 as the fifth $E'A$.

§ 2. We take this opportunity to mention a class of problems on forces in equilibrium, not parallel necessarily, which cannot be solved by the principles of statics alone, and are therefore called *statically indeterminate problems*. A beam resting on more than two supports furnishes a simple illustration; thus, let it be required to determine the reactions of the supports (A, B, and C) on the beam represented in Fig. 60, due to the two loads. If not already warned of the difficulty in this problem, some students would probably write moment equations for the forces in equilibrium (P_1, P_2, R_1, R_2, and R_3), with moment origins at A, B, and C, and then attempt to solve the equations simultaneously for the three unknowns. Such attempt would fail, even though each equation would be correct, because the three would not be independent — there being only two conditions of equilibrium for a system of the kind under consideration (Art. 10 under iv) — and so the three equations would not determine the three unknowns. Doubters are advised to try to determine R_1, R_2, and R_3 in this way in the simple case where the spans and the loads are equal, and the loads are applied at the centers of the spans.

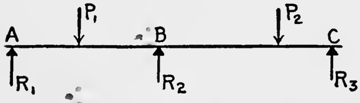


FIG. 60

How may one determine whether a given problem (a force system in equilibrium with some unknowns required) is statically determinate or indeterminate? A complete answer to the question is beyond the scope of this book; we may remark, however, that statically indeterminate problems commonly arise in connection with structures which have redundant or superfluous parts or supports, by which is meant that some of the parts or supports are not strictly necessary for the equilibrium of the structure. For example, in Fig. 60 one support is superfluous, since the beam on two supports would, if strong enough, support the load. No statically indeterminate problems are given in this book without notice; but the student may meet a force system in equilibrium containing many unknowns, and he is now reminded that it is futile to write out more equilibrium equations than there are algebraic conditions of equilibrium for the system under consideration (Art. 10), with the expectation that the equations if solved will determine the unknowns. And so it is well to know the number of conditions of equilibrium for each class of force systems.

13. Coplanar Nonconcurrent Nonparallel Forces

Principles of equilibrium for a force system of this kind are developed in Art. 10 under (iv). Their use will be explained now by applying them to two particular common problems.

§ 1. *Typical Problem* (iv). — A system of coplanar nonconcurrent nonparallel forces is in equilibrium, and all except two are wholly known; only the line of action of one of these two and a point in that of the other are known, and it is required that these two be determined completely.

The algebraic solution of this problem can be effected by means of any one of these sets of equilibrium equations:

$$\Sigma F_x = \Sigma F_y = \Sigma M = 0; \ \Sigma F_x = \Sigma M_a = \Sigma M_b = 0; \ \Sigma M_a = \Sigma M_b = \Sigma M_c = 0.$$

For an example, consider the roof truss represented in Fig. 61. It sustains two loads, 35,000 (weight of roof and truss) and 50,000 pounds (wind pressure). The left end of the truss merely rests on a wall, but the right end is fastened to a wall; therefore the reaction of the left-hand wall must be vertical, but that of the other may be inclined. Let it be required to determine these reactions. We call the left reaction A, the right one B, and the inclination of B to the horizontal θ. Then the first set of equilibrium equations gives $\Sigma M_b = +35{,}000 \times 45 + 50{,}000 \times (60 \cos 30°) - A \times 90 = 0$, or $A = 46{,}400$ pounds. $\Sigma F_x = -B \cos \theta + 50{,}000 \sin 30° = 0$, and $\Sigma F_y = +B \sin \theta - 50{,}000 \cos 30° - 35{,}000 + 46{,}400 = 0$; these solved simultaneously give $B = 40{,}500$ pounds and $\theta = 51° \ 54'$.

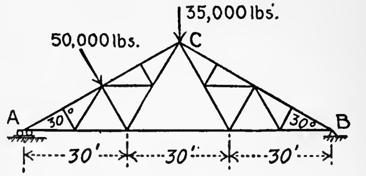


Fig. 61

For algebraic solutions, it is generally advisable to imagine the second unknown force, whose point of application is known, to be replaced by two (unknown) components. Then the problem is in the form of typical problem (v) (see next page). Thus, in the preceding example the unknowns would be A and, instead of B and θ, B_x and B_y. After finding B_x and B_y, one could easily get B and θ.

The graphical solution of this problem is effected by drawing the force and the string polygons, making both close since the force system is in equilibrium. To illustrate we use the preceding example. We first draw the polygon ABC (Fig. 62) for the known forces, and continue it with a line through C parallel to the left-hand reaction. The end of that line, as yet unknown, is to be marked D; that point once determined, then DA will represent the right-hand reaction. To find D we must construct a string polygon; so we next mark the lines of action of the several forces to agree with the notation in the force polygon, choose a pole O, and draw the rays OA, OB, and OC. To make use of the known point 1 of the fourth force (right-hand reaction), the string polygon must be begun at that point. The string oa is the one to draw through that point (to ab), and then ob and oc as shown. The string od must pass through points 1 and 4, and so is determined. Next we draw the ray OD (parallel to od), and thus determine D (the intersection of CD and OD).

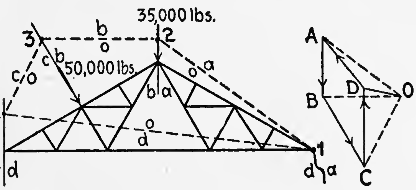


Fig. 62

The following special graphical method is simpler in principle than the preceding method: Let R = the resultant of the wholly known forces, P = the force whose line of action is known, and Q = the force whose point of application is known. Find R, and then imagine the wholly known forces replaced by R; R, P, and Q would be in equilibrium. Now a balanced three-force system is concurrent or parallel (Art. 10, § 2); hence if R intersects P, then Q acts through that point of intersection, and if R is parallel to P, then Q is also. If the three forces are concurrent, then determine P and Q from the force triangle for the three forces as explained in Art. 11; if they are parallel, determine P and Q as explained in Art. 12. To illustrate, we use the data of the foregoing example. First we draw AB and BC (Fig. 63), to represent the two loads; then AC represents the magnitude and direction of their resultant R. The line of action of R is ac, parallel to AC and passing through the intersection of ab and bc. (When the wholly known forces are nonconcurrent it is necessary to construct a string polygon to find a point in the line of action of R, see Art. 6.) We next extend the lines of action of R and P, and join their intersection with the point of application of Q; this line is the line of action of Q. Finally we complete the force triangle $ACDA$ for R, P, and Q; then $CD = P$ and $DA = Q$.

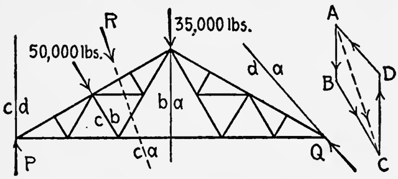


FIG. 63

§ 2. *Typical Problem* (v). — A system of coplanar nonconcurrent nonparallel forces is in equilibrium, and all the forces except three are wholly known; only the lines of action of these three are known, and their magnitudes and senses are required.*

The algebraic solution of this problem can be effected by means of any one of these three sets of equilibrium equations:

$$\Sigma F_x = \Sigma F_y = \Sigma M = 0;\ \Sigma F_x = \Sigma M_a = \Sigma M_b = 0;\ \text{or}\ \Sigma M_a = \Sigma M_b = \Sigma M_c = 0.$$

For example, consider the crane represented in Fig. 64. It consists of a post AB, a boom CD, and a brace EF; the post rests in a depression in the floor below, and against the side of a hole in the floor above. The external forces acting on the crane consist of the load W (8 tons), the weights of the parts named (0.8, 0.9, and 1.1 tons respectively), and the reactions of the floors. The upper floor exerts a single horizontal force on the post; the lower floor exerts two forces on the post, one horizontal and one vertical. Let it be required to determine the magnitudes of these reactions. The entire external system of forces just described is in equilibrium. Calling the reactions A, B_x, and B_y respectively, then the first set of equilibrium equations become: $\Sigma M_A = -8 \times 20 - 0.9 \times 11 - 1.1 \times 7 + B_x \times 18 = 0$, or $B_x = 9.86$; $\Sigma F_x = 9.86 - A = 0$, or $A = 9.86$; $\Sigma F_y = B_y - 8.0 - 0.8 - 0.9 - 1.1 = 0$, or $B_y = 10.8$ tons.

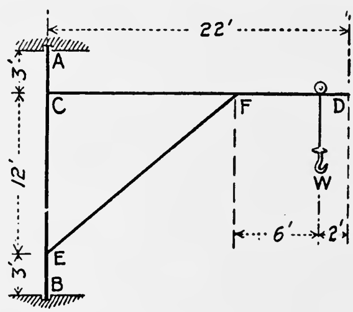


FIG. 64

* If the three unknown forces are concurrent or parallel, the problem is indeterminate.

The general graphical solution is carried out as follows: Let P, Q, and S stand for the three forces whose lines of action only are known. Imagine any two of these, say P and Q, replaced by their resultant R'; one point in that resultant is known, the intersection of P and Q. Then S, R', and the known forces would be in equilibrium, and the given problem has been transformed to typical problem iv. So we first determine S and R', as explained in § 1, and then resolve R' into two components parallel to P and Q; these components are P and Q. To illustrate, we take the preceding example, and we call the two lower reactions P and Q, and the upper one S (Fig. 65). The resultant R' of P and Q passes through the lower end of the post. We draw the polygon $ABCDE$ for the knowns, and continue it with a line parallel to S. The as yet unknown end of that line is to be marked F; that point once determined, then FA will represent R', since the polygon for all the forces must close. To find F we must construct a string polygon; so we mark the lines of action of the several forces to agree with the notation in the force polygon, choose a pole O, and draw rays OA, OB, OC, OD, and OE. The string polygon must be begun at the lower end of the post, the point of application of FA or R'. The strings to pass through that point are *of* and *oa* (Art. 6), and so we draw *oa* to *ab*; then *ob*, *oc*, *od*, and *oe* as shown. Now point 1 is in *of*, and point 6 is also; therefore *of* is determined. The ray OF is drawn next (parallel to *of*), thus determining F; then EF and FA represent S and R', as already stated. Finally we draw through F a vertical and through A a horizontal; then FG and GA represent the vertical and horizontal reactions (P and Q) of the lower floor.

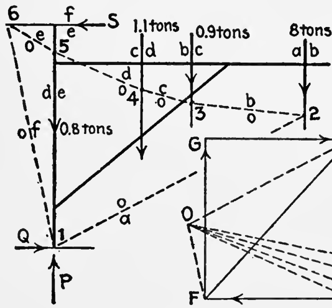


FIG. 65

The following special graphical method is simpler in principle than the preceding: First we determine the resultant R of the wholly known forces; R and the three partly unknown forces (P, Q, and S) would be in equilibrium. The special condition of equilibrium for four such forces is that the resultant R' of any pair as P and Q balances the other pair; hence R' and the other pair (R and S) are in equilibrium, and so must be concurrent or parallel. Next we solve the system R', R, and S (if concurrent by Art. 11, and if parallel by Art. 12). Finally we complete the force polygon for R, S, P, and Q. For an illus-

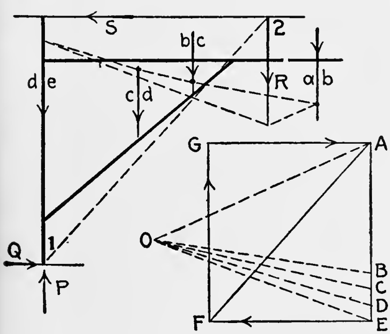


FIG. 66

tration we take the preceding example. Let the two lower reactions be called P and Q, and the upper one S (Fig. 66). The resultant R of the loads is 10.8 tons acting as shown (construction for R is indicated). The resultant R' acts through point 1; and, since R and S are concurrent at point 2, R' acts through point 2 also. We now draw the force triangle $AEFA$ for R, S, and R', AE representing R; then EF represents S. Finally we draw lines from A and F parallel to Q and P, thus fixing G; and then FG represents P, and GA represents Q.

14. Noncoplanar Forces in Equilibrium

§ 1. The principles of equilibrium for noncoplanar forces are set forth in Art. 10 under (v), (vi), and (vii). The three following illustrations deal with concurrent, parallel, and nonconcurrent nonparallel forces respectively.

(i) A heavy body W (Fig. 67) weighing 1000 pounds is suspended from a ring over the center of a street 60 feet wide; the ring is supported by three ropes OA, OB, and OC; A and B are points on the face of a building as shown, and C is a point on the face of a building (not shown) on the opposite side of the street, OC being perpendicular to the face of the buildings. Values of the tensions in the ropes are required. There are four forces acting on the ring,— the pull of 1000 pounds, and the pulls of the three ropes which we call L, M, and N respectively; this system is concurrent. To determine the unknown forces in it, we use the conditions that the algebraic sums of the components along three rectangular axes equal zero; as axes we choose a vertical line and two horizontal lines, one parallel and one transverse to the street.

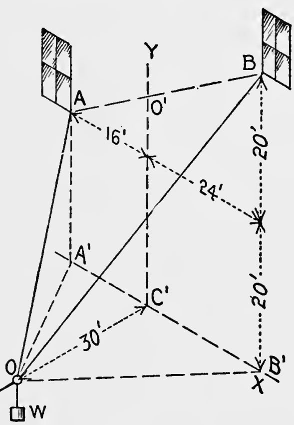


FIG. 67

To get the components of L, M, and N, we need values of certain angles: $A'OC' = \tan^{-1} A'C'/OC' = 28° \ 4'$; $AOA' = \tan^{-1} AA'/OA' = 30° \ 28'$; $B'OC' = \tan^{-1} B'C'/OC' = 38° \ 40'$; $BOB' = \tan^{-1} BB'/OB' = 46° \ 11'$. The x, y, and z components, respectively, of L are $L \cos 30° \ 28' \sin 28° \ 4' = 0.405 \ L$, $L \sin 30° \ 28' = 0.507 \ L$, and $L \cos 30° \ 28' \cos 28° \ 4' = 0.760 \ L$; of M they are $M \cos 46° \ 11' \sin 38° \ 40' = 0.4325 \ M$, $M \sin 46° \ 11' = 0.721 \ M$, and $M \cos 46° \ 11' \cos 38° \ 40' = 0.5405 \ M$; of N they are 0, 0, and N; of the 1000-pound pull they are 0, 1000, and 0. The algebraic sums of the x, y, and z components are

$$-0.405 \ L + 0.4325 \ M + 0 + 0 = 0,$$
$$+0.507 \ L + 0.721 \ M + 0 - 1000 = 0,$$
$$-0.760 \ L - 0.5405 \ M + N + 0 = 0.$$

Solving these equations simultaneously, we find that $L = 846$, $M = 792$, and $N = 1072$ pounds.

(ii) A body weighing 1000 pounds is suspended from the ceiling of a room by means of three vertical ropes; the points of attachment at the ceiling lie at the vertices of an equilateral triangle ABC (Fig. 68) whose sides are 10 feet long; W is the projection of the center of gravity of the body upon the ceiling. The tension in each rope is required. We call the tensions in the ropes fastened at A, B, and C, respectively, L, M, and N. The four forces acting on the body constitute a parallel system; the conditions of equilibrium for such are that the sums of the moments of the forces about any three coplanar nonparallel axes perpendicular to the forces equal zero. The lines AB, BC, and CA are good lines to choose as axes of moments. With respect to these lines the moment equations are respectively, $N \times 8.66 - 1000 \times 2.10 = 0$, $L \times 8.66 - 1000 \times 4.15 = 0$, and $M \times 8.66 - 1000 \times 2.41 = 0$, 8.66 being the altitude of the triangle. Solution of these equations shows that $L = 479$, $M = 278$, and $N = 243$ pounds.

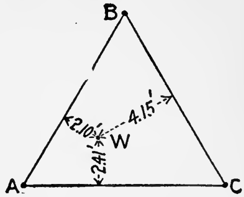


FIG. 68

(iii) Fig. 69 shows a velocipede crane. The crane can be run along on a single rail below, tipping being prevented by two overhead rails which guide a horizontal wheel mounted on the top of the crane post. The crane weighs 1.25 tons, and it is balanced so that its center of gravity is in the axis of the post. We will now show how to determine the supporting forces (exerted by the rails) when the crane supports a load of 1.5 tons and the jib is swung out at right angles to the rails toward the left (Fig. 70).

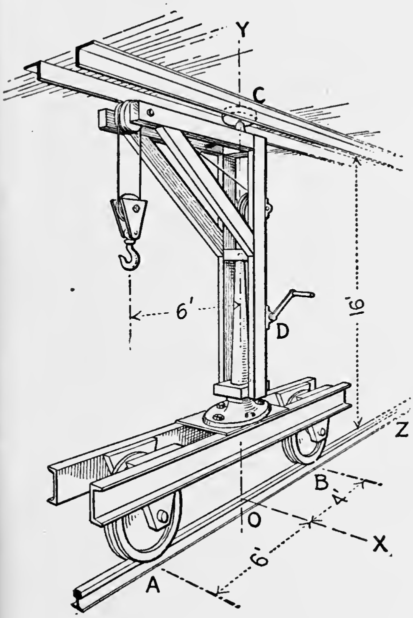


FIG. 69

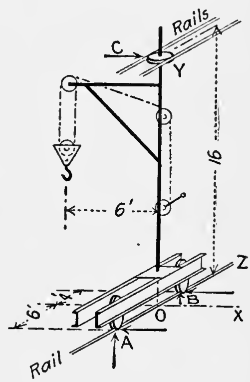


FIG. 70

There are three supporting forces or reactions, one on each wheel. Since the lower rail is level, the crane does not tend to roll, and there is no reaction of the rails in their direction. The reaction of the upper rail is directed horizontally and evidently as shown; the reaction on each lower wheel has two components as shown. We call these component reactions A_x, A_y, B_x, and B_y, and the upper reaction C. The external system of forces acting on the entire crane consist of the reactions named, the weight of the crane, and the load. For noncoplanar nonconcurrent nonparallel systems there are, in general, six conditions of equilibrium, but this system has only five because there are no "z forces" (see the figure). The five conditions of equilibrium are

$$\Sigma F_x = A_x + B_x - C = 0; \qquad (1)$$
$$\Sigma F_y = +A_y + B_y - 1.25 - 1.5 = 0; \qquad (2)$$
$$\Sigma M_x = B_y \times 10 - 1.25 \times 6 - 1.5 \times 6 = 0; \qquad (3)$$
$$\Sigma M_y = -C \times 6 + B_x \times 10 = 0; \qquad (4)$$
$$\Sigma M_z = -C \times 16 + 1.5 \times 6 = 0. \qquad (5)$$

From (5) it follows that $C = 5.625$ tons; from (4), that $B_x = 3.375$ tons; from (1), that $A_x = 2.25$ tons; from (3), that $B_y = 1.65$ tons; and from (2), that $A_y = 1.10$ tons.

We now give another solution, making use of the principle that if the forces of a system in equilibrium be represented by vectors, then the projection of the vectors on any plane represents a force system also in equilibrium (see Art. 10 under (vii)). Fig. 71 shows such projections on the x-y, y-z, and z-x planes of Fig. 70. From the y-z projection (side elevation), $\Sigma M_A = B_y \times 10 - 2.75 \times 6 = 0$, or $B_y = 1.65$ tons; and $\Sigma M_B = -A_y \times 10 + 2.75 \times 4 = 0$, or $A_y = 1.10$ tons. From the x-y projection (end elevation), $\Sigma M_A = C \times 16 - 1.5 \times 6 = 0$, or $C = 5.625$ tons. From the z-x projection (plan), $\Sigma M_A = -B_x \times 10 + 5.625 \times 6 = 0$, or $B_x = 2.375$ tons; and $\Sigma M_B = -A_x \times 10 + 5.625 \times 4 = 0$, or $A_x = 2.25$ tons.

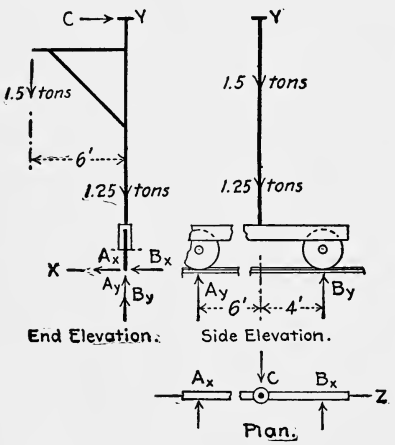


FIG. 71

§ 2. A noncoplanar system can generally be solved by means of an equivalent coplanar system. This indirect method is regarded as simpler than the direct one when the forces of the noncoplanar system are nonparallel. The two following examples will illustrate.

For one example we use the data of example (i). Instead of ropes OA and OB (Fig. 67), imagine a rope OO' in the plane of those ropes, and also in the same vertical plane with COC'. Such a rope fastened to O and to the building at O' would help to support the ring in its place, and would leave the tension in OC unchanged. Thus the ring would be acted upon by three forces,—1000, N, and the pull P of the new rope (Fig. 72). A force tri-

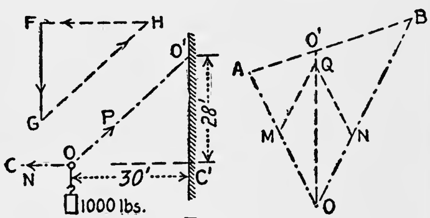


FIG. 72

angle, $FGHF$, for these forces shows that the pull $N = 1070$ and $P = 1460$ pounds. We next lay out the ropes OA, OB, and OO' in their true relations, and then we resolve the pull 1460 in the imaginary rope into components along the real ropes. Thus we lay off OQ equal to 1460, and then on the diagonal OQ complete the parallelogram $OMQN$; and find OM and ON, representing the tensions in the real ropes, 860 and 790 pounds.

For another illustration we take a tripod (Fig. 73), shown in plan and elevation. The requirement is to determine the forces acting at the top of each leg of the tripod due to a load of 1000 pounds. On account of this load, each leg is under the action of two forces, one applied at each end of that leg, and so those two forces act along the axis of the leg. We imagine a single leg in the plane of any two, and in the same vertical plane with the third, to replace the two; thus OD to replace OA and OB. Then there would be three forces applied to the pin at O, namely, the load 1000 pounds, and the supporting forces exerted by OC and OD. So we draw a force triangle for these three forces $FGHF$; it shows that the push of OC is $GH = 565$, and that of OD is $HF = 650$ pounds. Next we lay out the other pair of legs and the imaginary one in their true relation $O''A''$, $O''B''$, and $O''D''$, and make $O''P = HF = 650$ pounds; then resolve $O''P$ into two components along the pair $O''A''$ and $O''B''$ by means of a parallelogram $O''MPN$. Thus we find that $O''M$ and $O''N$ represent the pushes of AO and BO, or 340 pounds.

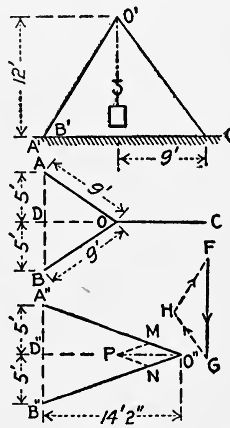


FIG. 73

CHAPTER III

SIMPLE STRUCTURES

15. Simple Frameworks (Truss Type)

§ 1. The frames herein considered consist of straight members, and the axes of all the members lie in one plane; such are called *plane frames*, and the plane of the axes is called the plane of the frame. In order to make the axes of all members lie in one plane, and the truss symmetrical with respect to that plane, some of the members must be made in parts or with forked ends. For example see Fig. 74, which shows plan and elevation of a joint of a frame at which four members are pinned together, one vertical (double), one diagonal D (single), and two horizontals H_1 and H_2 (each double).

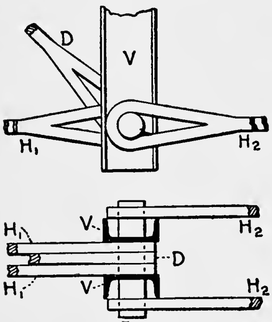


Fig. 74

Wooden members are generally bolted together with more or less mortising; steel members are riveted together or joined by pins through holes in the members, the axes of pins and holes being perpendicular to the plane of the frame. All frames here considered are assumed to be of the pin-connected type; and, furthermore, it is assumed that each member connects only two joints, that is, extends from one joint to another but not also to a third one.

In such pin-connected frames, the lines of action of the pin pressures (forces exerted by pins on the members) are in or parallel to the plane of the frame. Thus, the resultant pressure of the pin on the diagonal member D (Fig. 74) is clearly in the plane; the pin exerts on the vertical member two forces which, on account of the symmetrical arrangement, are equal, parallel, and equally distant from the plane, and therefore the resultant of these two forces lies in the plane; and obviously the resultant of the forces exerted by a pin on each horizontal member lies in the plane. Thus all resultant pin pressures will be regarded as lying in the same plane, and we will have only coplanar forces to deal with in the present connection. We assume that the pins are practically frictionless; in that case each pin pressure acts practically normally to the surface of the pin, and so the line of action of each pressure cuts the axis of the corresponding pin.

In this and the following articles we assume that the loads are applied to the frame at its joints only, and in such manner that the line of action of each load cuts the axis of the pin at the joint. Then each member, if its own weight is neglected, is subjected to forces (pin pressures and loads) at its two pin holes only, somewhat as shown in Fig. 75 or Fig. 76, where P' and P'' denote pin pressures and L' and L'' loads. Let R' denote the resultant of P' and L', and R'' the resultant of P'' and L''. Since R' and R'' balance, each acts along

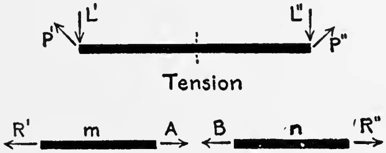


FIG. 75

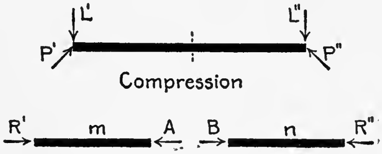


FIG. 76

the axis of the member, and hence each member is under simple tension or compression. Any two parts of the member, as m and n, exert equal and opposite forces upon each other; A (Figs. 75 and 76) denotes the force exerted on m by n, and B that exerted on n by m. Since A balances R', and B balances R'', A and B also act along the axis of the member. And obviously, if R' and R'' are pushes (the member in compression), then A and B are pushes; and if R' and R'' are pulls (member is in tension), then A and B are pulls. And conversely, if A and B are pushes, then the member is in compression; and if pulls, then in tension. By *stress* in a member is meant either of the two forces which two portions, as m and n, exert upon each other.* We are now ready to explain a method for determining the stresses in the members of a simple truss due to given loads; we begin with an

Example. — Fig. 77 represents a truss supported at each end; the angles equal 60 degrees; it sustains two loads of 2000 pounds each and one of 1000 pounds. First, it is necessary to ascertain the values of the reactions A and B. Since all the external forces acting on the truss (loads and reactions) are in equilibrium, $\Sigma M_A = B \times 40 - 2000 \times 30 - 1000 \times 10 - 2000 \times 20 = 0$, or $B = 2750$ pounds; and $\Sigma M_B = -A \times 40 + 1000 \times 30 + 2000 \times 10 + 2000 \times 20 = 0$, or $A = 2250$. $\Sigma F = 2250 + 2750 - 2000 - 2000 - 1000 = 0$, which result checks the computed values of A and B.

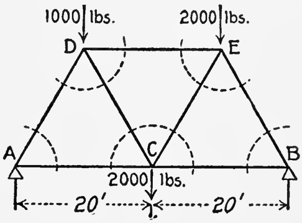


FIG. 77

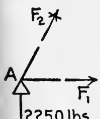


FIG. 78

We now direct our attention to the joint A, the small part of truss near A (or "pass a section" about A and consider that part of the truss within the section), and then note all the forces acting on that part (see Fig. 78). There are three such forces, — the reaction 2250

* The term stress is defined variously. Some writers use it to designate the forces which any two different bodies or any two parts of the same body exert upon each other; that is they use it as a general term for an "action and reaction" (Art. 11). Most engineers, however, use the term in a more restricted sense to designate the force which one part of a body exerts upon an adjacent part at the surface of division.

pounds, and the two forces exerted upon the part under consideration by the remainder of the truss; they are marked F_1 and F_2, and both are assumed to be pulls.* This part of the truss, as well as every other part, is at rest, and so the three forces are in equilibrium. Determination of the unknown forces F_1 and F_2 presents typical problem (i) (Art. 11). We choose the algebraic method for solving: $\Sigma F_y = F_2 \sin 60° + 2250 = 0$, or $F_2 = -2600$; the negative sign indicates that F_2 is really a push, that is, the stress is compressive. $\Sigma F_x = F_1 - 2600 \cos 60° = 0$, or $F_1 = +1300$; the positive sign indicates that the stress is tensile. Passing a section around B, and considering the forces acting on the part of the truss within the section (or "considering forces at joint B"), we get Fig. 79. The forces are the reaction 2750 pounds and the two forces exerted on the part under consideration by the remainder of the truss; they are marked F_3 and F_4 and are assumed to be pulls. Solution of this three-force system shows that $F_3 = +1588$ (tension), and $F_4 = -3177$ (compression).

Next we might discuss joint C, D, or E and determine two more stresses. Fig. 80 represents joint C and the forces acting upon it so far as known. Stress

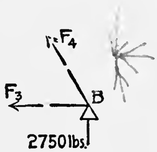


FIG. 79

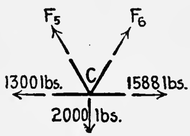


FIG. 80

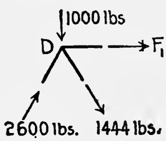


FIG. 81

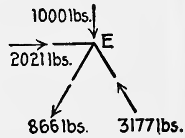


FIG. 82

in CA was determined to be a tension of 1300 pounds; therefore the part of CA not shown in the figure exerts a pull of 1300 on the part shown as indicated. Similarly, the part of CB not shown in the figure exerts a pull of 1588 on the part shown as indicated; F_5 and F_6 are assumed to be pulls. Solution of this five-force system shows that $F_5 = +1444$ (tension), and $F_6 = +866$ (tension). Taking joint D next, we get Fig. 81, four forces acting on the joint (the load, and the three forces exerted on the joint by the remainder of the truss). DA was found to be under a compression of 2600 pounds, hence the part of DA not shown in the figure acts on the part shown as indicated; CD was found to be under a tension of 1444 pounds, hence the part of DC not shown in the figure acts on the part shown as indicated; F_7 is assumed to be a pull. $\Sigma F_x = 0$ shows that $F_7 = -2021$ (compression); and writing out ΣF_y we find that it equals zero, which is a fair check on the computation. Fig. 82 represents joint E and all the forces acting upon it, as already determined. If $\Sigma F_x = 0$ and $\Sigma F_y = 0$ for those forces, then the check on the preceding computations is satisfactory.

* In simple trusses the kind of stress (tension or compression) in any member is apparent. When the kind is not apparent, we might follow the suggestion in the footnote, page 41. But for uniformity we will always assume the force to be a pull. Then, according to the footnote, the force is actually a pull or a push (and the stress is tensile or compressive), according as its computed value is positive or negative.

Directions.—The foregoing method for "analyzing a truss" (determining the stresses in its members) can be formulated into brief directions as follows: (1) Determine the reactions (supporting forces) on the truss if possible. (2) Consider a joint at which there are only two unknown forces, and then determine those two. (3) Repeat (2) again and again until all stresses have been determined. (These directions do not provide for a certain contingency which may arise; see § 2 for a case and directions for meeting it.)

We now give illustration of truss analysis by this method but omitting the computations; they should be supplied by the student. The truss shown in Fig. 83 will be used; it is supported at each end, and supports three loads of 5000 pounds as shown. Obviously each reaction equals one-half the total load. On joint A there are three forces (the reaction, and the stresses in AD and AE); solving that force system we find that the first stress = 15,000 pounds compression, and the second = 13,000 tension. On joint D there are four forces (the load 5000 pounds, the stress in AD = 15,000 pounds, and the stresses in DE and DC unknown); solving that system, we find that the stress in DE = 4335 pounds compression, and that in DC = 12,500 compression. On joint E there are four forces (the stress in AE = 13,000 pounds, the stress in DE = 4335 pounds, and the stresses in EC and EG unknown); solving the system, we find that the stress in EC = 4335 pounds tension, and that in EG = 8667 tension.

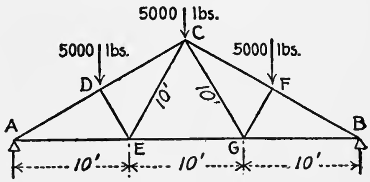


FIG. 83

§ 2. We now explain the contingency or difficulty mentioned in the foregoing directions and how to meet it; the truss shown in Fig. 84 furnishes an illustration. Following the directions, we determine the reactions R_1 and R_2, 2800 pounds and 2400. Then we take joint A, and find stresses in AB and AH to be 3960 (compression) and 2800 (tension) respectively; next we take joint G, and find stresses in GF and GI to be 3400 (compression) and 2400 (tension) respectively. No joint remains at which there are only two unknown stresses, and the difficulty is already met. Now if in some way we could ascertain the stress in almost any other member, then we could continue to apply the rule. For example, if we knew the stress in HB, HJ, or HI, then consideration of joint H would determine the two unknown stresses there; consideration of joint B would give stresses in BJ and BC; consideration of joint C would give stresses in CJ and CD, etc. Now there is a way to

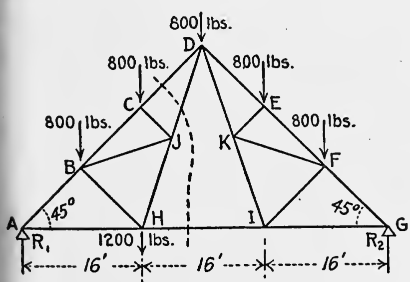


FIG. 84

ascertain the stresses in CD, JD, and HI, — by passing a section through those members, and solving the force system acting upon either portion of the truss. Fig. 85 represents the left-hand portion and all the forces acting upon it; namely, the three loads, the left reaction, and the forces which the right-hand part exerts (S_1, S_2, and S_3, assumed to be pulls). Solution of this force system presents typical problem (v) (Art. 13). To determine S_1, for example, we take moments about the intersection of S_2 and S_3 (or joint D), and find $S_1 = 1600$ pounds tension. Then having determined S_1 we proceed as in the foregoing examples.

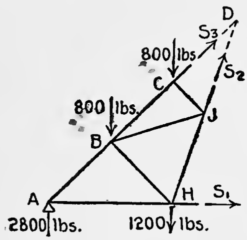


FIG. 85

In order to determine the stress in any particular member of a truss the following *direction* may be tried: Imagine the truss separated into two distinct parts ("pass a section" through the truss); pass it in such a way that the member under consideration is one of the members cut by the section, and so that the system of forces acting on one of the two parts is solvable for the desired stress; then solve the system for the desired stress. (The system of forces acting on one part of the truss consists of the loads and reactions on that part, and the forces, or stresses, which the other part exerts upon it. In plane trusses this system is always coplanar; it can be solved if it is concurrent with not more than two unknowns, or if it is non-concurrent with not more than three unknowns, provided that the three unknowns are not parallel nor concurrent.)

Foregoing direction may be applied not only to bridge over the difficulty sometimes met in connection with directions in § 1, but also when it is desired to determine the stress in a particular member quite directly without first computing stresses in several other members. For example, let it be required to determine the stress in BC (Fig. 86), the truss being supported at its ends, span $AE = 32$ feet, rise $CG = 8$ feet, and five loads as shown. Obviously

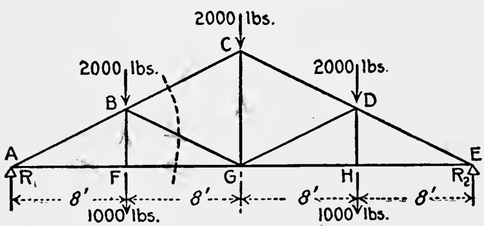


FIG. 86

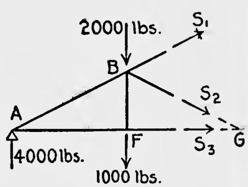


FIG. 87

each reaction equals 4000 pounds. A section cutting BC, BG, and GF gives a left-hand part of the truss with its external forces as shown in Fig. 87. The force system can be solved for the desired stress; taking moments about the intersection of S_2 and S_3 (joint G), we get $- S_1 \times 8 \times \cos 26° 34' - 4000 \times 16 + 3000 \times 8 = 0$, or $S_1 = -5600$, the negative sign indicating that S_1 is compressive and not tensile, as assumed in the moment equation.

§ 3. Warning is here given that not all trusses can be analyzed by the principles of statics alone, as in the preceding; that is to say, there are trusses that are *statically indeterminate.* Only the so-called complete or perfect trusses are always statically determinate; beside these there are incomplete trusses, and trusses with redundant members.

A pin-connected triangle (Fig. 88) is the simplest complete truss; it is indeformable and has no superfluous or redundant members. Adding two more members makes a complete truss of two triangles; and each addition of two members as shown extends the truss and leaves it complete. If m = number of members, and j = number of joints, then for a complete truss, $m = 2j - 3$. A pin-connected quadrilateral (Fig. 89) is the simplest incomplete truss; it is deformable and requires the addition of one or more members to make it complete. For an incomplete truss, $m < 2j - 3$. A pin-connected quadrilateral with two diagonal members (Fig. 90) is the simplest truss with a superfluous or redundant member; it is indeformable and would be so with any member removed. For a truss with a redundant member $m > 2j - 3$. Figs. 91, 92, and 93 are other examples of the three classes of trusses described.

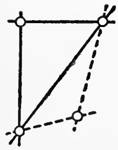


FIG. 88

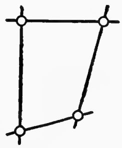


FIG. 89

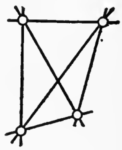


FIG. 90

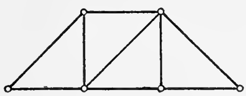


FIG. 91

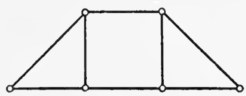


FIG. 92

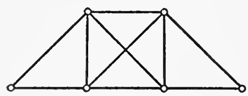


FIG. 93

In the foregoing it is assumed that the trusses are pin-connected, and that each member can sustain tension or compression as called upon by the loading. For a classification not so restricted as this one, readers are referred to standard works on Structures.*

16. Graphical Analysis of Trusses; Stress Diagrams

§ 1. Graphical methods are especially well adapted for analyzing trusses. As in the algebraic methods of the preceding article, we imagine the truss separated into two parts, and direct our attention to the external forces acting upon either part. Graphical instead of algebraical conditions of equilibrium are then applied to these forces to determine the unknowns. The *notation* for graphical work described in Art. 2 can be advantageously systemized as follows: Each triangular space in the truss diagram is marked by a lowercase letter, also the space between consecutive lines of action of the loads and reactions (Fig. 94); then the two letters on opposite sides of any line serve to

* Johnson, Bryan, and Turneaure's *Modern Framed Structures.*

designate that line, and the same capital letters are used to designate the magnitude of the corresponding force. This scheme of notation is a great help in graphical analyses of trusses.

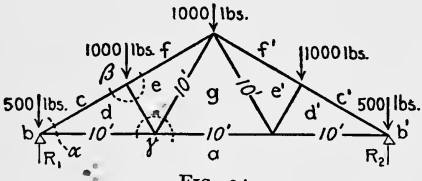


FIG. 94

As an illustration we determine the stress in each member of the truss of Fig. 94. Evidently each reaction equals one-half the load, or 2000 pounds. We "pass section" α, and consider the forces acting on the left-hand part of the truss (Fig. 95); they are the load 500 pounds, the reaction 2000 pounds, and the stresses *cd* and *da*. Since those forces are in equilibrium, their polygon closes; in constructing it, the unknowns will be determined. Beginning with the knowns, *AB* is drawn to represent 2000 pounds, *BC* to represent 500 pounds; and then a line from *A* (or *C*) parallel to the line of action of one unknown, and a line from *C* (or *A*) parallel to the other, are drawn. The last two lines determine *D* (or *D'*), and the closed polygon is *ABCDA* (or *ABCD'A*); hence the forces in the members *cd* and *ad* are represented by *CD* and *DA* (3000 and 2600 pounds) respectively. It is seen from the force polygon that *CD* is a push, and *DA* is a pull; hence the members *cd* and *ad* are in compression and tension respectfully.

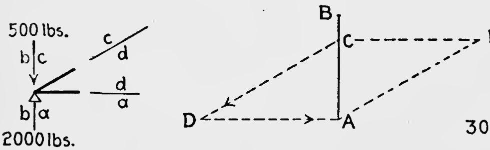


FIG. 95

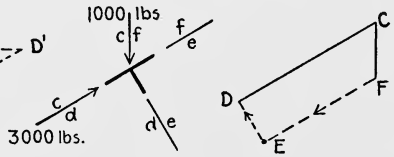


FIG. 96

We may next pass section β, and consider the forces acting on the smaller (and simpler) part of the truss (Fig. 96); they are the load 1000 pounds, the stress 3000 pounds (compressive), and the stresses *fe* and *de*. Their force polygon may be drawn thus: *DC* to represent 3000 pounds (compression), *CF* to represent 1000 pounds, a line from *F* parallel to one of the unknowns, and one from *D* parallel to the other. The last two lines determine *E*, and the force polygon is *DCFED*; hence the forces in the members *fe* and *ed* are represented by *FE* and *ED* (2500 and 866 pounds) respectively. Both members are in compression.

We next pass section γ, and consider the forces acting on the smaller part of the truss (Fig. 97); they consist of the stress 2600 pounds (tension), the stress 866 pounds (compression), and the stresses *eg* and *ga*. Their force polygon may be drawn thus: *AD* to represent 2600 pounds (tension), *DE* to represent 866 pounds (compression), a line from *E* parallel to one of the unknowns, and a line from *A* parallel to the other. The last two lines determine *G*, and the force polygon is *ADEGA*;

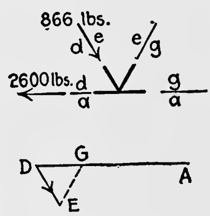


FIG. 97

hence the forces in the members *eg* and *ag* are represented by *EG* and *GA* (866 and 1732 pounds) respectively. Each member is in tension. On account of the symmetry of the truss and loading, the forces in the remaining members are now known.

In drawing the force polygon for all the external forces on the part of a truss included within a section about a joint, it will be advantageous to represent the forces in the order in which they occur about the joint. A force polygon so drawn will be called a polygon for the joint; and for brevity, if the order taken is clockwise the polygon will be called a clockwise polygon, and if counterclockwise it will be called a counterclockwise polygon. *ABCDA* (Fig. 95) is a clockwise polygon for joint *b* of Fig. 94; *ABCD'A* is a force polygon for the "forces at joint 1," but it is not a polygon for the joint, because the forces are not represented in the polygon in the order in which the forces occur about the joint. The student should draw the counterclockwise polygon for the joint, and compare with *ABCDA*.

If the polygons for all the joints of a truss are drawn separately as in the preceding illustration, then the stress in each member will have been represented twice. It is possible to combine the polygons so that it will not be necessary to represent the stress in any member more than once, thus reducing the number of lines to be drawn. Such a combination of force polygons is called a *stress diagram*. Fig. 98 is a stress diagram for the truss of Fig. 94 loaded as there shown. Comparing the part of the stress diagram consisting of solid lines with Figs. 95, 96, and 97, it is seen to be a combination of the latter three figures. It will also be observed that the polygons are all clockwise polygons; counterclockwise polygons also could be combined into a stress diagram.

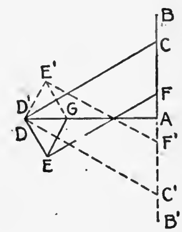


FIG. 98

Directions for constructing a stress diagram for a truss under given loads:

(1) Letter the truss diagram as already explained.

(2) Determine the reactions. (In some exceptional cases this stage may or must be omitted; also stage (3). See § 2 for two illustrations.)

(3) Construct a force polygon for all the external forces applied to the truss (loads and reactions), representing them in the order in which their points of application occur about the truss, clockwise or counterclockwise. (The part of that polygon representing the loads is called a load line.)

(4) On the sides of that polygon construct the polygons for all the joints. They must be clockwise or counterclockwise ones, according as the polygon for the loads and reactions was drawn clockwise or counterclockwise. The first polygon drawn must be for a joint at which but two members are fastened; the joints at the supports are usually such. Next the polygon is drawn for a point at which not more than two stresses are unknown; that is, of all the members fastened at that joint the forces in not more than two are unknown. Then the next joint at which not more than two stresses are unknown is con-

sidered, etc., etc. (These directions do not provide for a certain difficulty which may arise; see § 2 for a case and directions for handling it.)

To illustrate the foregoing directions we analyze the truss represented in Fig. 99; it sustains four loads (600, 1000, 1200, and 1800 pounds), and is

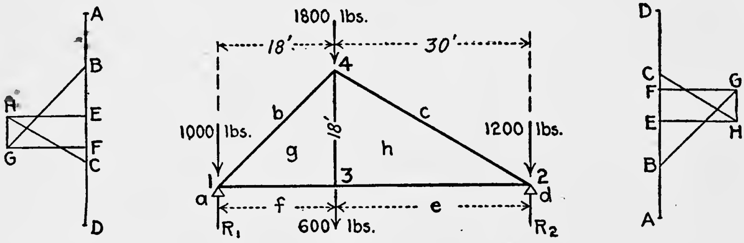


FIG. 99

supported at its ends. Supposing the reactions to have been determined, we draw the force polygon for the loads and reactions *ABCDEFA*, at the left; it is a clockwise polygon. We may begin by drawing the clockwise polygon for joint 1 or 2; for the former it is *FABGF*.* Member *bg* is therefore in compression and *gf* in tension. Next we may draw the clockwise polygon for joint 2, 3, or 4; for the joint 2 it is *CDEHC*. Member *ch* is in compression and *eh* in tension. For joint 3, the polygon is *HEFGH*, and member *gh* is in tension. If the work has been correctly and accurately done, the line *GH* is parallel to *gh*.

§ 2. There are *exceptional cases* not covered by the foregoing directions. In case the reactions cannot be determined in advance, the stress diagram can still be drawn if the truss is statically determinate. Fig. 100 represents such a case, the truss being pinned to its supports. The diagram can be constructed by drawing in succession the proper polygons (all clockwise or counterclockwise) for joints 1, 2, 3, and 4. Then, if desired, the reactions can be determined by drawing the polygons for joints 5 and 6.

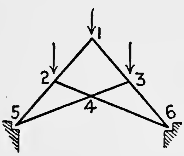


FIG. 100

Fig. 101 represents a case where the reactions can be determined at stage (2) of the analysis, but determination of the reactions is not essential for the construction of the stress diagram. The truss is supported by a shelf *A* and a tie *B*. The stress diagram can be constructed by drawing in succession proper polygons for joints 1, 2, 3, 4, and 5. The reaction at *B* is determined by the polygon for joint 5; that at *A* by the polygon for joint 6.

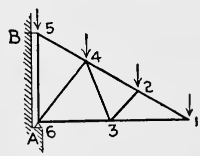


FIG. 101

* The student is urged to make sketches of the bodies (parts of truss) upon which the forces, whose polygons are being drawn, act. A force acting upon the "cut" end of a member and toward the joint is a push, and the stress in the member is compressive; if the force acts away from the joint, it is a pull, and the stress is tensile.

Fig. 102 shows a truss the analysis of which is not fully provided for in the directions. Thus, suppose that the reactions have been determined; the polygon for joint 1 may be drawn first, next that for joint 2, and then that for joint 3. Similarly the polygons for joints 1′, 2′, and 3′ can be drawn; but then no joint remains at which there are but two unknown stresses, and so no more polygons can be drawn, as yet. If in any way the number of unknown stresses at a remaining joint could be reduced to two, then the polygon for that joint could be drawn, and the stress diagram could be completed. Thus, if the stress in *ij*, *jm*, or *mf* could be determined, then the polygon for joint 4 could be drawn, and then those for 5, 6, 7, and 8.

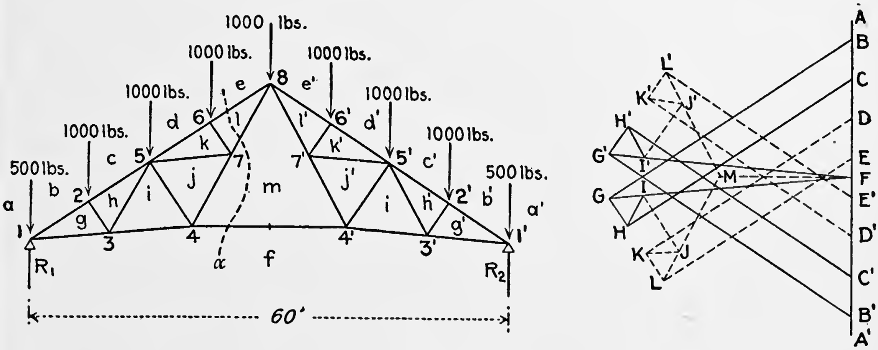


Fig. 102

The difficulty here pointed out is just like that mentioned under the directions in § 1 of the preceding article. It may be met by means of the direction in § 2 of that article, which explains how to determine the stress in a particular member quite directly and independently of any stress diagram or polygons for joints. Thus to determine the stress in *mf* we pass a section as α, and solve the external system of forces (including stresses in the members cut) which acts upon either part of the truss for the desired stress. Then we proceed with the stress diagram as already pointed out. There are other ways of meeting the difficulty presented in this form of truss, but that here explained is quite general and can be applied readily to other forms.

We will now explain this matter in detail, using the same truss. Evidently each reaction equals one-half the total load. *ABCDEE′D′C′B′A′FA* is a clockwise polygon for the loads and reactions. The polygon for joint 1 is *FABGF*; that for joint 2 is *GBCHG*; that for joint 3 is *FGHIF*. The polygons for joints 1′, 2′, and 3′ are *B′A′FG′B′*, *C′B′G′H′C′*, and *H′G′FI′H′* respectively. The forces acting on the part of the truss to the left of section α are the loads at joints 1, 2, 5, and 6, the left reaction, and the forces exerted on the left part of the truss by the right (stresses *el*, *lm*, and *mf*). This system may be solved graphically or algebraically; the algebraic method is much the simpler, arms of forces being scaled from the truss drawing. Thus to ascertain the stress *mf*, we take moments about the intersection of *el* and *lm*, and get 1000 ×

$7.5 + 1000 \times 15 + 1000 \times 22.5 + 500 \times 30 - 4000 \times 30 - (mf) \times 17.5 = 0$, or $mf = 3425$ (tension). Next we represent the stress mf in its proper place in the stress diagram at MF, and then draw the polygon for joint 4; it is $MFIJM$. Completion presents no difficulties.

17. Simple Frameworks (Crane Type)

The frames here considered, like the trusses of the preceding articles, are plane and symmetrical with respect to the plane of the frame. For example, the crane represented in Fig. 103 consists of a post MN, a boom PQ, and a brace KQ; the boom consists of two pieces between which the post and the brace lie, and the brace is forked at its lower end by means of side pieces and straddles the post. Like the trusses, these frames are assumed to be pin connected, the pins being practically frictionless. Thus each pin pressure lies in the plane of the frame, and the line of action cuts the axis of the pin.

Unlike the trusses, these frames may include a member which is pinned to others at more than two points; the loads also on these frames are applied anywhere, not at the joints necessarily. The result of these conditions is that the stress in any member of the frame is generally not a simple tension or compression, the member being bent as well as stretched or shortened. We will not attempt to determine the stresses in the members of these frames but limit the discussions to a determination of the forces which act upon each member, the pin pressures, reactions of supports, etc.

In general the pressure of a pin on a member does not act along the axis of that member. Take, for example, the brace (diagonal) (Fig. 103); it is

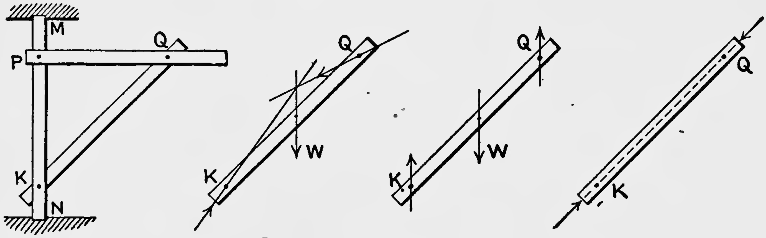


FIG. 103

acted upon by three forces, — its own weight W and the pin pressures K and Q. These three forces must be concurrent or parallel (Art. 10, § 2). If they are concurrent, then neither K nor Q is axial or else both are; but obviously both K and Q cannot be axial and then balance W, and so neither acts axially. If they are parallel, then neither K nor Q acts axially.

In some consideration of frameworks, the weights of some or all members are negligible in comparison with other forces (loads) which act upon the frame, and so we may have to do with a member acted upon by only two forces, — pin pressures. On such a member, the pin pressures do act along the axis of that member, since the pressures balance each other and so must be colinear (Fig. 103).

"Analysis of a crane" means the determination of every force (magnitude and direction) acting on each part or member due to weight of the crane or loads on it or both. The general method of procedure may be briefly summarized as follows: (1) Make a sketch of the entire crane, and represent as far as possible all the external forces acting upon it; apply the appropriate conditions of equilibrium to the force system, and then determine as many of the unknowns as possible. (2) Make a sketch of a member or of a combination as they are on the crane, and represent as far as possible all the external forces acting on it; then apply the appropriate conditions of equilibrium to the force system, and then determine as many of the unknowns as possible. (3) If other forces remain to be determined, then continue as directed in (2), bearing in mind the law of "action and reaction" (Art. 11). We will now give two examples of analysis employing both algebraic and graphic methods.

Example (i). — We analyze the crane represented in Fig. 103; the crane is supported at M and N by sockets in the ceiling and floor. $MN = 18$, $PQ = 14$, $MP = NK = 3$ feet; it bears a load of 8 tons on the boom at 16 feet from the axis of the post; weights of members neglected. Fig. 104

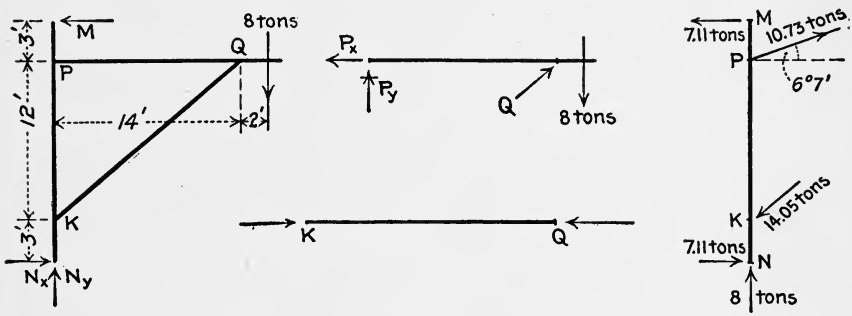


FIG. 104

(at the left) represents the entire crane with all external forces, the senses of the reactions being quite obvious. The solution of this system falls under Art. 13. $\Sigma M_N = 0$ gives $M = 7.11$ tons; since $\Sigma F_x = 0$, $N_x = 7.11$ tons; and since $\Sigma F_y = 0$, $N_y = 8$ tons. We sketch the brace KQ next. Since it is a two-force member, the pin pressures K and Q are axial, equal, and obviously have senses as shown. The common value of K and Q cannot be determined from a consideration of their equilibrium. Next we sketch the boom. Q on the boom and Q on the brace constitute an action and reaction, and so are colinear, opposite, and equal; the pressure at P is unknown in direction, and in an algebraic solution can be dealt with most easily through its components P_x and P_y, senses guessed at. Solution of this system falls under Art. 13. Since $\Sigma M_P = 0$, $Q = 14.05$ tons; since $\Sigma F_x = 0$, $P_x = 10.67$ tons; since $\Sigma F_y = 0$, $P_y = -1.14$ tons, the negative sign indicating that P_y acts downward. Finally, $P = \sqrt{(10.67^2 + 1.14^2)} = 10.73$ tons, and the inclination of P with the horizontal is $\tan^{-1}(1.14 \div 10.67) = 6^\circ\ 7'$; and now all the forces on each member are determined, those on the post being represented in the figure.

Generally, several sketches may be made and considered in several different orders, each furnishing a complete analysis. For example, we might have taken the entire crane, the boom, and the post; or the brace, the boom, and the entire crane. The student is advised to try these orders and make the analysis.

The graphic method of solving the various force systems may be carried out as follows: The system acting on the entire crane consists of four forces, and so the resultant of any pair of the four forces, as N_x and N_y, balances the other pair; therefore that resultant is concurrent with the second pair and acts in the line 1–2 (Fig. 105). So we draw the force triangle $ABCA$ for those three

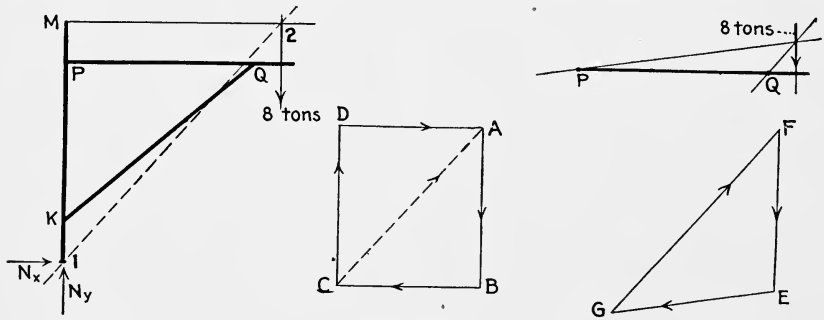


FIG. 105

forces (making AB represent 8 tons), and find that BC represents M and CA the resultant of the first pair. Next we resolve CA into components parallel to N_x and N_y, and find that CD and DA represent N_y and N_x respectively. The forces on the boom being three in number (the load, Q, and P), they must be parallel or concurrent, and because two (the load and Q) are concurrent, all must be; thus the line of action of P is determined. So we may draw the force triangle $EFGE$ for the three forces, making EF represent 8 tons; thus we find that $EG = P$ and $GF = Q$.

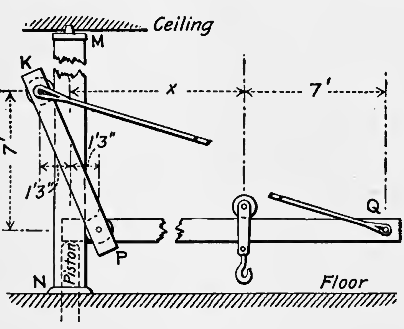


FIG. 106

Example (ii). — For another illustration, we analyze the hydraulic crane represented in Figure 106. It consists of a hollow post MN (up into which the piston can be projected) a boom PQ, and a pin-connected frame KPQ. A single roller is mounted on the pin K, and two on the pin P, so that as the piston moves the frame moves with it, all rollers rolling on the post. Thus there are twelve parts: a post, a boom, two struts KP (one on each side of the post), two ties KQ (one on each side), a pin at P, one at Q, one at K, two rollers at P and one at K. We take the load as 10 tons and $x = 15$ feet, and neglect the weights of the parts.

Fig. 107 represents the entire crane, not including the piston, with all the external forces acting upon it. $\Sigma F_x = 0$ shows that $M = N_x$, and $\Sigma M_N = 0$ shows that $M = (10 \times 15) \div h$ where h = height of post. L and N_y cannot be found from this force system; so we try the frame with rollers (Fig. 108). The external forces acting on it are the load, the piston pressure L, the post pressure R_1 against the single roller, and the resultant post pressure R_2 against the lower rollers. The solution of this system falls under Art. 13; it shows that $L = 10$ and $R_1 = R_2 = 21.4$ tons. Fig. 109 represents the boom alone and the external forces acting upon it,— the load, the piston pressure, the pin pressure Q (acting along the ties because each is a two-force member), and the pin pressure P whose direction is unknown. The solution of this system falls under Art. 13. Dealing with the unknown components of P (senses guessed at), we find from $\Sigma M_P = 0$ that $Q = 25.2$ tons; from $\Sigma F_x = 0$ that $P_x = 24$ tons; and from $\Sigma F_y = 0$ that $P_y = 7.2$ tons. The pin at K (Fig. 110) is subjected to three forces, namely, the pull of the two ties (25.2 tons), the pressure of the roller (21.4), and the force F exerted by the struts (along the axis of the struts, since each is a two-force member). From $\Sigma F_y = 0$ we find that $F = 7.8$ tons. Fig. 110 also represents the post with all the external forces acting upon it. Since $\Sigma F_y = 0$, $N_y = 0$; $\Sigma M_N = 0$ gives $M = 7.14$; and $\Sigma F_x = 0$ gives $N_x = 7.14$. We have now determined all the forces on each part; each tie is subjected to end pulls of 12.6 tons; each strut to end pushes of 3.9 tons; the pin at P to three forces as shown in the figure.

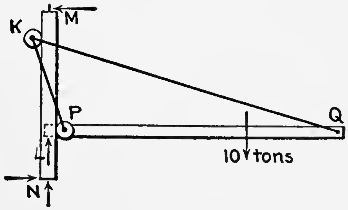


FIG. 107

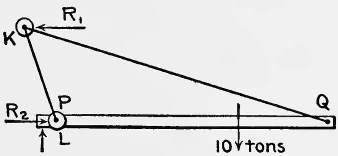


FIG. 108

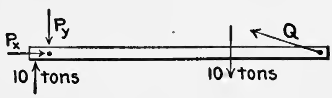


FIG. 109

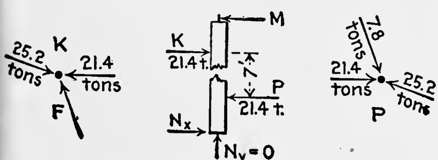


FIG. 110

The graphical solutions of the various force systems might be carried out as follows: Four forces act on the portion of the crane shown in Fig. 111,—the load 10 tons, the pressures L, R_1, and R_2. The resultant R of L and R_2 acts through their intersection and through that of R_1 and the load, hence in the line 1–2. The load, R_1, and R are in equilibrium; so we draw a closed force polygon for them as $ABCA$ (Fig. 112); AB = 10 tons, BC = 21.4, and CA

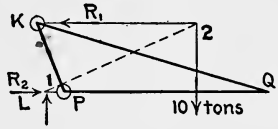


FIG. 111

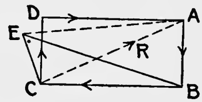


FIG. 112

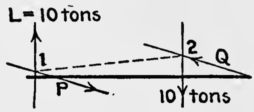


FIG. 113

represents R. Finally we resolve R into its two components; CD and DA represent L and R_2 respectively. There are four forces acting on the boom, namely, the load = 10 tons, L = 10 tons, the pin pressure P, and that at Q (Fig. 113). Obviously the pressure Q acts along the tie rod. The first pair of forces named constitute a couple; and since a couple can be balanced only by another couple, the second pair is a couple and P is parallel to Q, and the resultant of each pair therefore acts in the line 1–2. We now draw a line through B (Fig. 112) parallel to Q, and one through A parallel to 1–2; then BE represents Q and AE represents the resultant of L and P. Finally, there are three forces acting on the pin at P, namely, R_2 (or CB), $-P$ (or BE), and the pressure of the braces KP (Fig. 111). These three forces being on equilibrium, the last one is represented by EC.

Example (iii).—We now make an analysis of a crane taking into account the weights of the members. For this purpose we take the crane described in example (i) and assume that the weights of members are as follows: MN = 0.8 ton, PQ = 0.9 ton, and KQ = 1.1 tons. The load is taken, as in example (i), to be 8 tons at 16 feet out from the axis of the post, and the boom 22 feet long.

Fig. 114 shows the entire crane and all the external forces acting upon it so far as known. Determination of the unknown reactions M, N_x, and N_y presents typical problem (v) (Art. 13). From $\Sigma M_M = 0$ we get $N_x = 8.09$; from $\Sigma F_x = 0$, $M = 8.09$; and from $\Sigma F_y = 0$, $N_y = 10.8$. Fig. 115 represents the post and all the external forces acting upon it so far as known. The pressures on the post are exerted by members which are not two force members, and therefore those pressures do not act in the directions of the boom and brace. The directions of those pressures being unknown, we represent each by its (unknown) horizontal and

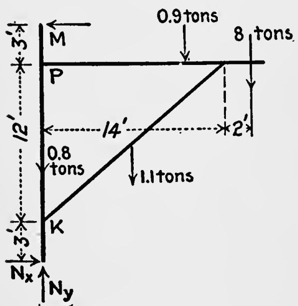


FIG. 114

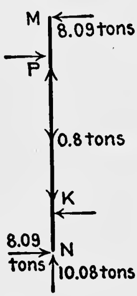


FIG. 115

vertical component. The force system acting on the post contains four unknowns, namely, P_x, P_y, K_x, and K_y. Not all of these unknowns can be determined from a study of this system alone; but two of them, P_x and K_x, can be so determined. $\Sigma M_K = 0$ gives $P_x = 12.13$, and $\Sigma F_x = 0$ gives $K_x = 12.13$ tons.

Fig. 116 shows the boom and the forces acting upon it so far as known. The direction of the pressure at Q is unknown as yet; therefore that pressure is represented by means of its (unknown) components. Determination of the

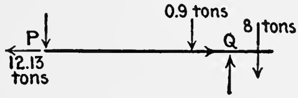


FIG. 116

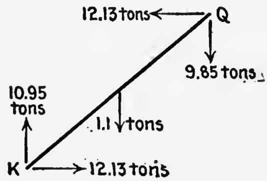


FIG. 117

unknowns in the force system presents typical problem (v). $\Sigma F_x = 0$ gives $Q_x = 12.13$ tons; $\Sigma M_Q = 0$ gives $P_y = 0.95$; and $\Sigma F_y = 0$ gives $Q_y = 9.85$. Having found the value of P_y, we find from $\Sigma F_y = 0$ for Fig. 115 that $K_y = 10.95$ tons. To check the analysis, we might supply values of the forces acting on the brace (Fig. 117), and then test whether the force system is balanced, that is, whether $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M = 0$.

18. Cranes. — *Continued*

In this article we show how to analyze three cranes, paying some attention to the forces due to the hoisting rig. Generally, a pulley is an important part of such rig. We assume here that the tensions T_1 and T_2 (Fig. 118) in the rope or chain on opposite sides of the pulley on which it bears are equal. This assumption implies perfect flexibility of rope or chain and a frictionless pin supporting the pulley. The pressure P against the pin equals the resultant of those tensions, or $2\,T \cos \frac{1}{2}\,\alpha$, and it bisects the angle between their lines of action. If the lines of action are parallel ($\alpha = 0$), $P = 2\,T$; if they are at right angles ($\alpha = 90°$), $P = 1.414\,T$.

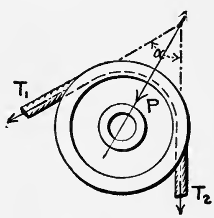


FIG. 118

Example (i). — Fig. 119 represents a crane supported in a footstep bearing at the floor and a collar bearing on the wall bracket H. The hoisting rig consists of a simple hand winch mounted on the wall at W, a chain, and pulleys as shown. Pulley at G is 12 inches in diameter; the load is one-half ton. The reactions at the supports depend on the hoisting rig, as will be seen from the following: On the entire crane, including the top pulley (Fig. 120), there are acting four forces, namely, the upper reaction H, the lower reactions P_x

and P_y, and the pressure of the chain against the pulley equivalent to two components, one-half ton each, as shown. Taking moments about the lower end, we find H to be 0.087 ton; from $\Sigma F_x = 0$ and $\Sigma F_y = 0$, we find that

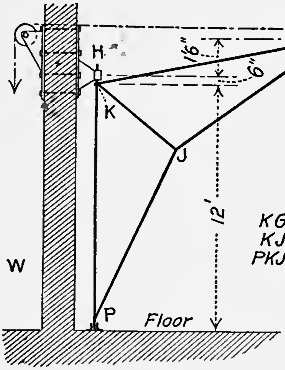


FIG. 119

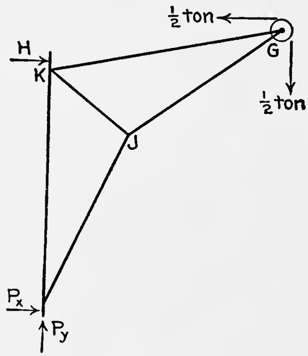


FIG. 120

$P_x = 0.413$ and $P_y = 0.5$ ton. All members except the vertical HP are simple tension or compression members. Force polygons for joints G and J show that the stresses are as follows: GK = 0.35 ton (tension); GJ = 1 ton (compression); JK = 0.57 ton (compression); JP = 1 ton (compression). Member HP is subjected to the reactions of the supports as already computed, and the following forces: a pull of 0.35 ton along KG; a push of 0.57 ton along KJ; and a push of 1 ton along PJ.

Example (ii). — Fig. 121 represents a common type of derrick. It is supported by a footstep at the bottom of post and at the top by two stiff legs

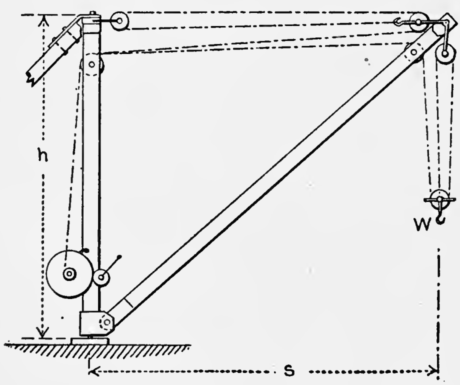


FIG. 121

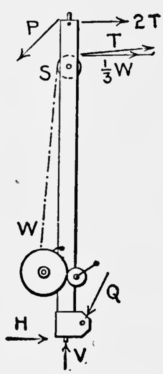


FIG. 122

which extend backward to the ground or other base; the spread (angle between their horizontal projections) being 90 degrees so that the derrick can swing about its vertical axis through 270 degrees. Sometimes the derrick is supported at the top by a collar bearing held in place by cables extending off to quite remote points on the ground.

Obviously the pull on a stiff leg is greatest when the boom is in the same plane with that leg; the pull on a cable is greatest when that cable and the boom are in the same plane and on opposite sides of the post. Let P denote this pull, and α the inclination of the cable to the horizontal or the inclination of the line joining the pivot on the post with the lower end of a stiff leg. Then taking moments of all external forces on the derrick about the footstep bearing, we get $Ph \cos \alpha = Ws$, or $P = Ws/h \cos \alpha$ (only the weight of the load being taken into account). Calling the horizontal and the vertical reactions at the footstep H and V respectively, we find that $H = Ws/h$ and $V = W + P \sin \alpha = W(1 + \tan \alpha \cdot s/h)$.

There are seven forces acting on the part shown in Fig. 122, which consists of the crane post, the winch W, the two sheaves S, and a part of the hoisting and topping ropes as shown. The forces are: H, V, and P (already explained); Q, the pressure of the boom on the post acting in a direction as yet unknown; $\frac{1}{3} W$, approximate value of the tension in the hoisting rope; T, which denotes the tension in the topping rope; and $2\ T$, exerted by the top pulley shackle. Of these seven forces, all except Q and T are already known. To find these we may proceed as follows: Take moments of all the forces about the pin at Q, and thus find T; then take horizontal and vertical components, and thus find the horizontal and vertical components of Q, and finally Q itself. The force system can be solved graphically as follows: First find the line of action of the resultant R of the two forces T and $2\ T$; then this R and the other five forces constitute a system in equilibrium, which solve for R and Q by methods explained in Art. 13; finally resolve R into its components T and $2\ T$.

Example (iii). — Fig. 123 represents a sheer leg crane. It consists of two front legs AC and BC and a back leg CD, all connected by a horizontal pin at C; the front legs are pin-supported on the ground at A and B, and the back leg is restrained at the ground by a holding-down rail and a long horizontal screw which works in a nut on the lower end D. The purpose of the screw is to move D, thus turning the front legs about AB and moving the load in and out. We will now show how

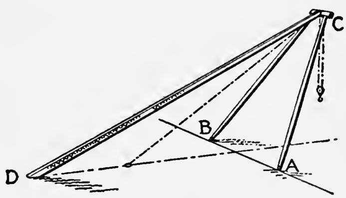


FIG. 123

to determine the pressures on the ends of the legs due to their own weights, taking the following data: lengths of front legs 160 feet, distance between their lower ends 50 feet, distance between their upper ends 10 feet, length of back stay 210 feet, weight of each front leg 44 tons, of the back leg 53 tons; we take the crane in its position of greatest overhang, 64 feet.

The external forces acting on the crane are the following (see Fig. 124): the three weights, the holding-down force D_y, the push of the screw D_x, the inward pushes A_z and B_z of the supports at A and B, and the pressures of the pins at A and B; each of these pressures is represented by two components in the figure, A_x, A_y, and B_x, B_y, respectively. There are six conditions of equilibrium for this system, namely, the sums of the components of the forces along the x, y, and z axes, and the sums of the moments about those axes equal zero. Thus,—

$$\Sigma F_x = A_x + B_x - D_x = 0 \qquad (1)$$
$$\Sigma F_y = A_y + B_y - D_y - 53 - 44 - 44 = 0 \qquad (2)$$
$$\Sigma F_z = - A_z + B_z = 0 \qquad (3)$$
$$\Sigma M_x = -A_y \times 25 + B_y \times 25 + 44 \times 15 - 44 \times 15 = 0 \qquad (4)$$
$$\Sigma M_y = A_x \times 25 - B_x \times 25 = 0 \qquad (5)$$
$$\Sigma M_z = D_y \times 87.6 + 53 \times 11.8 - 44 \times 32 \times 2 = 0 \qquad (6)$$

Equation (6) shows that $D_y = 25$ tons; (4) shows that $A_y = B_y$; from these results and (2) it follows that A_y and B_y equal 83 tons. No other unknowns can be determined from the equations; but (3) shows that $A_z = B_z$, (5) that $A_x = B_x$, and (1) that $A_x + B_x = D_x$.

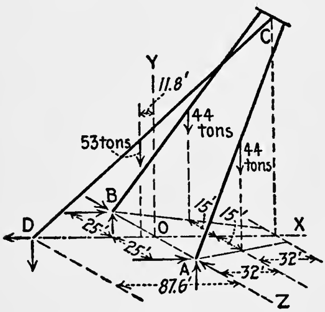


FIG. 124

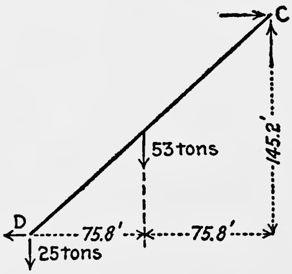


FIG. 125

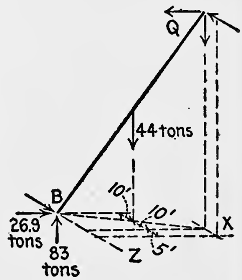


FIG. 126

To get values of these unknowns we consider the forces acting on the back leg; there are four forces, namely, the weight of the leg (53 tons), the holding-down force D_y (25 tons), the screw pressure D_x, and the pressure of the upper pin at C, represented for convenience by two components which we call C_x and C_y (Fig. 125). This system is in equilibrium and so $\Sigma M_c = 25 \times 151.6 - D_x \times 145.2 + 53 \times 75.8 = 0$, or $D_x = 53.8$ tons; $\Sigma F_x = C_x - 53.8 = 0$, or $C_x = 53.8$; and $\Sigma F_y = C_y - 25 - 53 = 0$, or $C_y = 78$. Returning now to equations (1) and (5), we find that A_x and $B_x = 26.9$. To get A_z and B_z it is necessary to discuss the forces on one of the front legs. There are three forces,— the weight 44 tons, and the pressures at the ends; each of the pressures is represented (Fig. 126) by three components, 26.9, 83, and B_z below, and Q_x, Q_y and Q_z above. The system being in equilibrium, we take moments about the vertical line through Q; thus $B_z \times 64 - 26.9 \times 20 = 0$, or $B_z = 8.41$ tons. Inspection shows that $Q_x = 26.9$, $Q_y = 39$, and $Q_z = 8.41$ tons.

The forces acting on the upper pin (at C) are represented in Fig. 127, by means of their components.

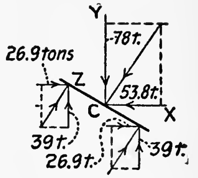


FIG. 127

We now give another solution of the foregoing example, making use of the principle that if the forces of a system in equilibrium be represented by vectors, then the projection of those vectors on any plane represents a force system also in equilibrium (Art. 10 under (vii)). Projecting the force system represented in Fig. 124 on the three coördinate planes, we get the three systems represented in Fig. 128, — side elevation, end elevation, and plan. From the side elevation, $\Sigma M_A = 0$ gives $D_y = 25$ tons; $\Sigma M_D = 0$ shows that $A_y = B_y$; and ΣF_y shows that $A_y + B_y = 166$, or A_y and $B_y = 83$ tons. No further numerical result can be obtained from these projected systems. Considering the back leg alone as before, we would find that $D_x = 53.8$ tons; then from the plan $A_x = B_x$ obviously, and $A_x + B_x = 53.8$, or A_x and $B_x = 26.9$ tons. A_z and B_z would be gotten as before.*

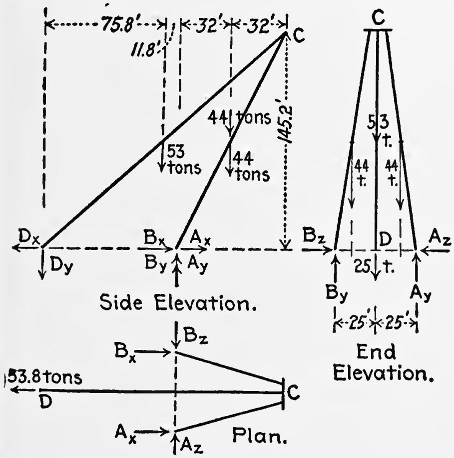


FIG. 128

* For full information on cranes, see Böttcher's book on that subject, English translation by Tolhausen.

CHAPTER IV

FRICTION

19. Definitions and General Principles

§ 1. Definitions, Etc. — When one body slides or tends to slide over another, then the sliding of the first or its tendency to slide is resisted by the second. Thus, if A (Fig. 129) is a body which slides or tends to slide toward the right over B, then B is exerting some such force as R on A, and the component of R along the surface of contact is the resistance which B offers to the sliding or tendency. Of course A exerts on B a force equal and opposite to R; either of these equal forces is called the *total reaction* between the two bodies. The component of either total reaction along the (plane) surface of contact is called *friction*, and the component of either along the normal is called *normal pressure;* they will be denoted by F and N respectively. If the surface of contact of the two bodies is not plane, the force exerted at each elementary part of the surface is the total reaction at that element, and its components in and normal to the element are the friction and the normal pressure at the element. Friction is called *kinetic* or *static* according as sliding does or does not take place. Only static friction is considered here.

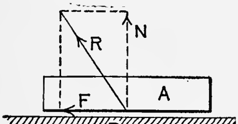


Fig. 129

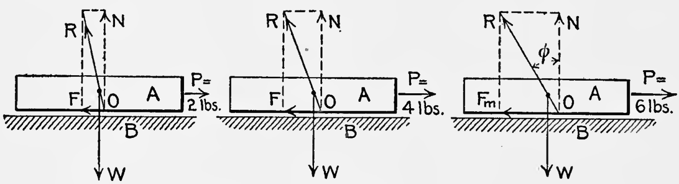


Fig. 130

The amount of static friction between two bodies depends upon the degree of the tendency to slip. Thus suppose that A (Fig. 130) is a block weighing 10 pounds, upon a horizontal surface B; that the block is subjected to a horizontal pull P, and that the pull must exceed 6 pounds to start the block. Obviously when $P = 2$ pounds say, then $F = 2$; when $P = 4$ pounds, then $F = 4$; etc., until motion begins. So long as P does not exceed 6 pounds, F equals P; that is, F is passive and changes just as P changes. The inclination of the reaction R also depends on the degree of the tendency to slip. When $P = 2$ pounds, then the angle $NOR = \tan^{-1} \frac{2}{10} = 11° 19'$; when $P = 4$ pounds,

$NOR = \tan^{-1} \frac{4}{10} = 21° 48'$; etc., until motion begins. The greatest values of the friction F and the angle NOR obtain when motion impends.

The friction corresponding to impending motion is called *limiting friction.* We will denote it by F_m, since it is a maximum value (see Fig. 130). The *coefficient of static friction* for two surfaces is the ratio of the limiting friction corresponding to any normal pressure between the surfaces and that normal pressure. We will denote it by μ; then

$$\mu = F_m/N, \text{ or } F_m = \mu N; \text{ also, } F \not> \mu N.$$

The *angle of friction* for two surfaces is the angle between the directions of the normal pressure and the total reaction when motion is impending. We will denote it by ϕ (see Fig. 130); then

$$\tan \phi = F_m/N; \text{ hence } \tan \phi = \mu.$$

If a block were placed upon an inclined plane, the inclination at which slipping would impend is called the *angle of repose* for the two rubbing surfaces; it will be denoted by ρ. The angles of friction and repose for two surfaces are equal; proof follows: Suppose that A (Fig. 131) is on the point of sliding down the incline; two forces act on A, its own weight W and the reaction R of the plane. Since A is at rest, R and W are colinear, that is, R is vertical; and since motion impends, the angle between R and the normal is the angle of friction ϕ. It follows, from the geometry of the figure, that ϕ and ρ are equal.

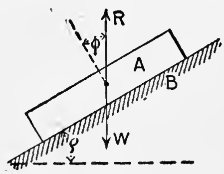


FIG. 131

The coefficient of static friction for two bodies A and B may be found in several ways: (i) Place A on B as in Fig. 130, and determine the pull P which will just start A; then $\mu = P$ divided by the weight of A. Or (ii) tilt B, and determine the inclination at which gravity will start A down; then μ equals the tangent of that angle of inclination. In either method several determinations must be made to obtain a fair average. Many experiments have been made in these ways, and it has been ascertained that coefficients of static friction depend on the nature of the materials, character of rubbing surfaces and kind of lubricant, if any be used. Early experimenters reported (Coulomb 1871, Rennie 1828, Morin 1834, and others) that the coefficient is independent of the intensity of normal pressure; and although this announcement was clearly subject to the limitation of the range of the experiments performed, yet it was generalized and long accepted as a universal law of friction. But the universality of the law has been questioned; Morin himself pointed out that length of time of contact of the two bodies influences the coefficient; and obviously the coefficient changes when the intensities of pressure get so low that a considerable part of the friction is due to adhesion, or so high as to affect the character of the surfaces in contact. Messiter and Hanson report* prac-

* *Eng. News*, 1895, Vol. 33, page 322.

tical constancy of coefficient for yellow pine and spruce. They give the following for planed or sandpapered (1) yellow pine and (2) spruce.

(1) $\mu = 0.25$ to 0.32; average $\mu = 0.29$ for 100 to 1000 lbs. per sq. in.
(2) $\mu = 0.18$ to 0.53; average $\mu = 0.42$ for 100 to 1600 lbs. per sq. in.

The variation depends on relation of grain of wood to direction of slide.

Coefficients of Static Friction

(Compiled by Rankine from experiments by Morin and others.)

Dry masonry and brickwork	0.6 to 0.7
Masonry and brickwork with damp mortar	0.74
Timber on stone	about 0.4
Iron on stone	0.3 to 0.7
Timber on timber	0.2 to 0.5
Timber on metals	0.2 to 0.6
Metals on metals	0.15 to 0.25
Masonry on dry clay	0.51
Masonry on moist clay	0.33
Earth on earth	0.25 to 1.0
Earth on earth, dry sand, clay, and mixed earth	0.38 to 0.75
Earth on earth, damp clay	1.0
Earth on earth, wet clay	0.31
Earth on earth, shingle and gravel	0.81 to 1.11

§ 2. TRACTIVE FORCE.—Let W = the weight of a body A upon a horizontal surface B (Fig. 132), μ = the coefficient of friction for the surfaces in contact, ϕ = their angle of friction, and P a force applied to the body as shown, θ being the inclination of P to the horizontal. Then the force P required to start the body to move is given by

$$P = \frac{\mu W}{\cos\theta + \mu \sin\theta} = \frac{W \sin\phi}{\cos(\theta - \phi)}.$$

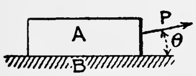


FIG. 132

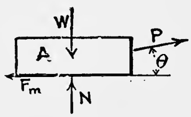


FIG. 133

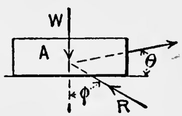


FIG. 134

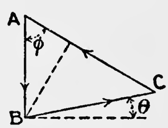


FIG. 135

The forces acting on A are P, W, and the reaction of the plane whose two components are N and (when motion impends) F_m (Fig. 133). Now $P\cos\theta = F_m$, $N = W - P\sin\theta$, and $F_m = \mu N$; these three equations solved simultaneously furnish the first stated value of P. The second value can be obtained from the first, or by solving the three-force system acting on A as represented in Fig. 134. According to Lami's theorem (Art. 10), $P/W = \sin\phi/\sin(90 + \phi - \theta)$; hence $P = W\sin\phi/\cos(\theta - \phi)$.

If the pull P is horizontal ($\theta = 0$), then $P = \mu W$. If the pull is inclined, but not too much, then the pull P required to start the body may be less than μW. In fact the least value of P obtains when $\theta = \phi$, — "the best angle of traction equals the angle of friction," — and the minimum value of the pull is $W \sin \phi$. Proofs follow: (i) Evidently $W \sin \phi \div \cos (\theta - \phi)$, the general value of P, changes as θ changes, and, for a given W and ϕ, P is least when $\cos (\theta - \phi)$ is greatest; but this greatest value is 1, and obtains when $\theta - \phi = 0$, or when $\theta = \phi$ as stated, etc. (ii) Or, let AB (Fig. 135) represent W, BC be parallel to P, and AC be parallel to R; then CA represents R. If θ be changed, then BC (and P) will change; and evidently P will be least when BC is perpendicular to CA, that is, when $\theta = \phi$. And then BC (or P) $= W \sin \phi$.

§ 3. TEST FOR REST OR MOTION. — A body is supported so that it can slip and is subjected to given forces; it is required to ascertain whether those forces do cause slipping, and the value of the friction is desired. We assume that the body is at rest, and determine the friction F and the normal pressure N from conditions or equations of equilibrium; then we compare F with μN. If F is less than μN, there is no motion and the computed value of F is correct; if F is greater than μN, then there is motion and the friction is kinetic, its value being less than μN. For example, consider a block of material weighing 100 pounds supported on a horizontal surface, the coefficient of friction being $\frac{1}{2}$, and imagine a down push of 200 pounds applied to the block at an angle of 30 degrees with the vertical. $N = 100 + 200 \cos 30 = 273.2$, and for rest, $F = 200 \sin 30 = 100$; $\mu N = \frac{1}{2} \times 273.2 = 136.6$, and this is the greatest frictional resistance which the support can offer so long as $N = 273.2$. Only 100 pounds are required to prevent motion, and so the body is at rest under the action of friction of that required value.

Or, to test for rest or motion, we may make use of the so-called *cone of friction* for the two bodies in contact, which may be described thus: Let P (Fig. 136 or 137) denote the resultant of all the forces applied to or acting on the body A (whose state is to be investigated) but not including the total reaction of the supporting body B; O the point where P cuts the surface of contact between A and B, and DOC equal the angle of friction; then the cone generated by revolving OC about OD is the cone of friction. If the line of action of the resultant P does not fall outside the cone (Fig. 136), then there is no slipping; if it does fall outside (Fig. 137), then there is slipping. Proof follows: As already pointed out, the direction of the total reaction R on a body, which tends to slide over another, depends on the degree of the tendency; the greater the tendency, the greater the inclination of R from the normal; but the inclination has a limit, that limit being equal to the angle of friction, and it

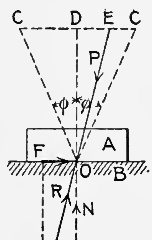


FIG. 136

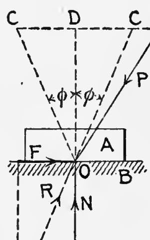


FIG. 137

obtains when slipping impends. Therefore when P acts within the cone or along an element of it, then R can incline and completely oppose P (Fig. 136), no matter how large P may be. When P falls outside the cone, R can incline only to an element, and the friction cannot successfully oppose the component of P which tends to move the body (Fig. 137). In the preceding example P is the resultant of the weight of the block 100 pounds, and the applied push 200 pounds. That resultant makes an angle of $10° 33'$ with W or the normal. The angle of friction is $\tan^{-1} \frac{1}{2}$, or $26° 34'$; hence P falls inside the cone and, according to the principle of the cone, motion does not ensue.

As another application of the cone principle consider Fig. 138, which represents (in plan and elevation) a type of simple hanger. It consists of a fixed vertical rod and a horizontal piece which is forked; there is a hole in each part of the fork so that the piece can be slipped over the rod as shown in the elevation. The hanger, if properly made, will not slip down along the rod on account of its own weight or that of a load unless it be hung quite close to the fork. The mechanics of the device may be explained as follows: Obviously the rod reacts on the hanger at O_1 and O_2. When slipping impends at these points, the reactions act along O_1C_1 and O_2C_2 inclined to the normals an amount equal to the angle of friction ϕ as shown.

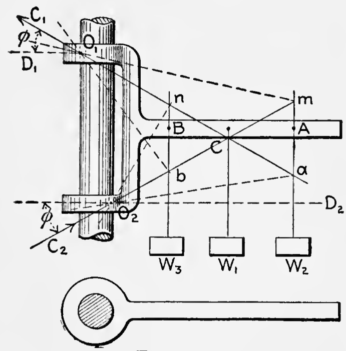


FIG. 138

The hanger being at rest (by supposition), the third force acting upon it (the load, weight of hanger neglected) must be concurrent with these two reactions; hence to just put the hanger on the point of slipping, the load must be hung from a point in the vertical through C. If the load is hung out beyond C, as at A, the hanger will not slip. For suppose slipping to impend at O_1, then R at O_1 would act along O_1C_1, and R and W would concur at a. To preserve equilibrium, R at O_2 must also act through a, which is possible, since O_2a is within the cone. Or suppose slipping to impend at O_2, then R at O_2 would act along O_2C_2, and R and W would concur at m. To preserve equilibrium, R at O_1 must also act through m which is possible. In similar manner, it can be shown that a load hung between the rod and C, as at B, would cause slipping.

20. Friction in Some Mechanical Devices

§ 1. INCLINED PLANE.—Let α = the inclination of the plane to the horizontal (Fig. 139), ρ = angle of repose for the plane and a particular body upon it, ϕ = their angle of friction, μ = coefficient of friction, W = weight of the body, and θ = angle between the push or pull P and the incline. (i) The pull P required to start the body up the plane is given by

$$P_1 = W \sin(\alpha + \phi)/\cos(\theta - \phi)$$

as can be shown by means of Lami's theorem (Art. 10) applied to the three forces acting on the body (P, W, and the reaction R of the plane). Thus $P_1/W = \sin(\alpha + \phi)/\sin(90 - \phi + \theta)$; hence, etc. P_1 is a minimum (for given W, α, and ϕ) when $\theta = \phi$; then its value is $W \sin(\alpha + \phi)$. For, it is obvious that P_1 is least when $\sin(90 - \phi + \theta)$ is greatest, that is, when $\phi = \theta$. (ii) When the inclination of the plane is greater than the angle of repose ($\alpha > \rho = \phi$), then the body would slip down unless prevented by a suitable force. The pull P required to prevent the slipping down is given by

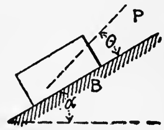


FIG. 139

$$P_2 = W \sin(\alpha - \phi)/\cos(\theta + \phi).$$

P_2 is a minimum when $\theta = -\phi$; then its value is $W \times \sin(\alpha - \phi)$. (iii) When the inclination of the plane is less than the angle of repose ($\alpha < \rho = \phi$), then the body would not slip down on account of its own weight. The push P required to start the body down is given by

$$P_3 = W \sin(\phi - \alpha)/\cos(\phi + \theta).$$

P_3 is a minimum when $\theta = -\phi$; then its value is $W \sin(\phi - \alpha)$.

When the force P acts along the plane ($\theta = 0$), then the values of P_1, P_2, and P_3 are respectively,

$$W\frac{\sin(\alpha + \phi)}{\cos\phi}, \quad W\frac{\sin(\alpha - \phi)}{\cos\phi}, \quad W\frac{\sin(\phi - \alpha)}{\cos\phi}.$$

§ 2. WEDGE. — In order that the force P (Fig. 140) may start the wedge inward to overcome the load W, the friction at the three rubbing surfaces must be overcome also. If the three rubbing contacts are equally rough and ϕ = their common angle of friction, then the force P required to start the wedge inward is given by

$$P_1 = W \tan(2\phi + \alpha).$$

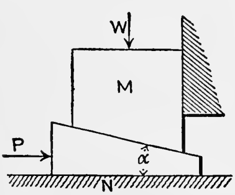


FIG. 140

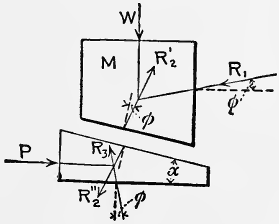


FIG. 141

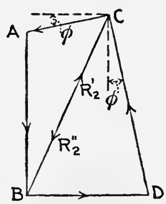


FIG. 142

Fig. 141 represents the three forces W, R_1, and R_2' acting on the block M; also the three forces R_2'' ($= R_2'$), R_3, and P acting on the wedge. The angles which R_1, R_2, and R_3 make with their normal components equal ϕ, since motion impends, by supposition. In Fig. 142, $ABCA$ is a triangle for the forces acting on M, AB representing W; and $CBDC$ is a triangle for the forces acting on the wedge. The given formula for P_1 may be derived from these triangles by solving for BD, which represents P_1. From the first triangle $(R_2' = R_2'')/W = \cos\phi/\sin(90 - \phi - \alpha - \phi)$, or $R_2' = R_2'' = W\cos\phi/\cos(2\phi + \alpha)$; from the second triangle $P_1/(R_2' = R_2'') = \sin(2\phi + \alpha)/\cos\phi$. Therefore $P_1 = (R_2' = R_2'')\sin(2\phi + \alpha)/\cos\phi = W\tan(2\phi + \alpha)$.

If the wedge angle α is less than 2ϕ, the wedge will not slip out under any load W even when there is no push P; that is, the wedge is self-locking. The force required to pull the wedge out, that is to lower the load W, must equal

$$W\tan(2\phi - \alpha), \qquad \text{when} \quad \alpha > \phi \text{ (guide at right of } M\text{)},$$

or

$$W\sin(2\phi - \alpha) \div \cos\alpha, \quad \text{when} \quad \alpha < \phi \text{ (guide at left of } M\text{)}.$$

In order that the force Q (Fig. 143) may overcome the resistances W, the frictional resistances at the four contacts must be overcome also. If the con-

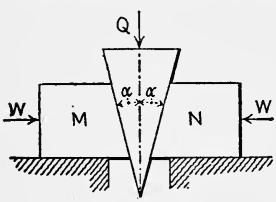


FIG. 143

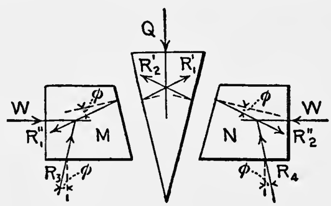


FIG. 144

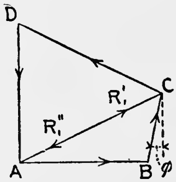


FIG. 145

tacts are equally rough and ϕ = their common angle of friction, then the force necessary to start the wedge down is given by

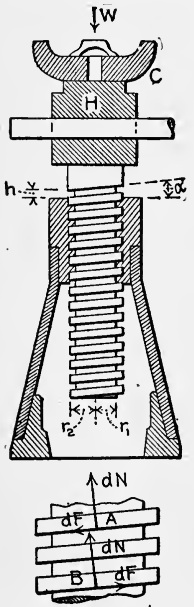


FIG. 146

$$Q_1 = \frac{2W}{\cot(\phi + \alpha) - \tan\phi}.$$

Fig. 144 represents the forces Q, R_1', and R_2' acting on the wedge, and the forces acting on M and N. Each of the reactions R makes with its normal component an angle equal to ϕ (motion impending). In Fig. 145, $ABCA$ is a triangle for the forces acting on M, AB representing W. $ACDA$ is a triangle for the forces acting on the wedge. The given formula for Q_1 can be derived from these triangles by solving them for DA, which represents Q_1.

If the wedge angle 2α is less than 2ϕ, then the wedge would not slip out under any pressures W even when there is no push Q; that is, the wedge is self-locking. The force required to pull the wedge out (M and N guided above) is given by

$$Q_2 = \frac{2W}{\cot(\phi - \alpha) + \tan\phi}.$$

§ 3. SCREW.—Fig. 146 represents a simple jackscrew much used for raising and lowering heavy loads through short distances. In the simpler forms, the screw is turned by means of a lever stuck through a hole in the head H of the screw. There is frictional resistance between the screw and the nut, also between the cap C and the head of the screw, unless the load can turn with the screw. Let P = the (horizontal) force applied to the lever; a = the arm of P with respect to the axis of the screw; W = load on the cap; r = mean radius of the screw, $\frac{1}{2}(r_1 + r_2)$; α = pitch angle = $\tan^{-1}(h \div 2\pi r)$; and ϕ = angle of friction = $\tan^{-1}\mu$, where μ = coefficient of friction. Disregarding the friction between the cap and head of the screw, the moment required to raise the load (or move the screw against W) is given by

$$P_1 a = Wr \tan(\phi + \alpha).$$

If the pitch angle is less than the angle of friction, the load would not turn the screw; that is, the screw is self-locking. The moment required to lower the load (or move the screw with W) is given by

$$P_2 a = Wr \tan(\phi - \alpha).$$

Jackscrews are always made self-locking, the pitch angle α being between 4 and 6 degrees generally. With α = 4 degrees and ϕ = 6 degrees (μ = 0.1),

$$P_1 a = 0.18\, Wr \quad \text{and} \quad P_2 a = 0.035\, Wr.$$

Derivation of formulas for P_1 and P_2: At each point of contact between the screw and nut, the latter exerts a pressure whose normal and tangential component we call dN and dF respectively. (1) When the screw tends to rise, dF acts downward on the screw as shown at A; the vertical components of the forces dN and dF are, everywhere, $dN \cos\alpha$ and $dF \sin\alpha$, and their horizontal components are $dN \sin\alpha$ and $dF \cos\alpha$. Taking the sum of the vertical components of all the forces acting on the screw and the sum of the moments of the forces about the axis of the screw, we get

$$\Sigma F_y = -W - \Sigma(dF \cdot \sin\alpha) + \Sigma(dN \cdot \cos\alpha) = 0, \text{ and}$$
$$\Sigma M = Pa - \Sigma(dF \cdot \cos\alpha \cdot r) - \Sigma(dN \cdot \sin\alpha \cdot r) = 0.$$

When slipping impends, $dF = \mu dN$ and P becomes P_1; substituting these values in the two equations gives

$$W = (\cos\alpha - \mu \sin\alpha)\Sigma dN, \quad \text{and} \quad P_1 a = r(\sin\alpha + \mu\cos\alpha)\Sigma dN.$$

Division of these and substitution of $\tan\phi$ for μ furnishes the value of P_1. (2) When the load is lowered dF is changed in direction as at B; also the direction of the pull on the lever is changed. Therefore, changing P_1 to $-P_2$ and ϕ to $-\phi$ in the formula for P_1, we get the formula for P_2.

To allow for the friction between the cap and the head of the screw, let μ = the coefficient of friction, and R = the effective arm of the friction there with respect to the axis of the screw. (If the surface of contact between the cap and the head were flat and a full circle, R would equal two-thirds the radius of the circle. But the contact is generally a hollow circle, as in Fig. 146, and then R is practically equal to the mean radius.) The friction moment at the cap is μWR;

(1) for raising the load, $Pa = Wr \tan(\phi + \alpha) + \mu WR$,
(2) for lowering the load, $Pa = Wr \tan(\phi - \alpha) + \mu WR$.

§ 4. JOURNAL IN WORN BEARING. — Fig. 147 represents, in section, a journal in a worn bearing, wear much exaggerated; the contact between the two is along a line practically. When the journal is about to turn clockwise and slip, then the bearing exerts a reaction R', making an angle ϕ (the angle of friction for the surfaces in contact) with the normal ON; when the journal is about to turn counterclockwise and slip, then the bearing exerts a reaction R'' inclined at an angle ϕ with ON, but on the other side. If the radius of the journal is r, then the perpendicular from the center to R' and R'' equals $r \sin \phi$, and the circle of radius $r \sin \phi$ with center at the center of the cross section of the journal is tangent to R' and R''. This circle is called the *friction circle* for journal and bearing. For smooth contacts $\sin \phi$ nearly equals $\tan \phi$ or μ, and hence the radius of the circle practically equals μr.

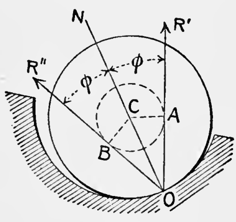


FIG. 147

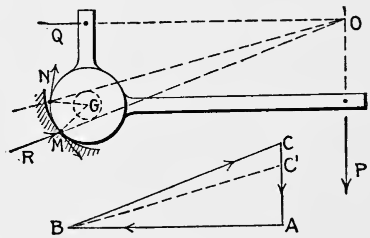


FIG. 148

We use the friction circle as an aid to fix upon the line of action of the reaction between journal and bearing when motion impends; the line is tangent to the circle. For example, consider the bell crank shown in Fig. 148, the journal being $1\frac{1}{4}$ inches in diameter and the coefficient of friction 0.3; the requirement is to determine the least force P, acting as shown, which will overcome Q (that is, start the bell crank to turn clockwise), and the pressure on the bearing then. The radius of the friction circle is $\frac{5}{8} \sin \tan^{-1} 0.3 = 0.18$ inch. Since there are but three forces acting on the bell crank (P, Q, and R), they are concurrent, that is, R acts through O; but R is also tangent to the circle as shown, and so its line of action is known. To determine the values of P and R, we draw AB to represent Q by some scale, and lines through A and B parallel to P and R to their intersection C; then BC and CA represent the magnitudes and directions of R and P respectively.

(Which one of the two tangent lines to take can be determined by trial. Thus, trying ON, the contact between journal and bearing would be at N, and the tangential or frictional component of the pressure on the journal would be as shown, not consistent with the assumed tendency to slipping. Obviously the other tangent is the correct one, and on investigating for the friction component of R when acting at M we find that such component is consistent with the assumed tendency to slip.)

The force P which would just permit Q to start the bell crank to turn counterclockwise could be determined in a similar way. Then R would act along the tangent ON, and P would be represented by $C'A$. When P has any value between $C'A$ and CA, then slipping does not impend, and the line of action of R cuts the friction circle.

When a link L (Fig. 149) of a machine or structure is pinned to other parts or members, and there is slipping or tendency to slipping at the pins, then the pressure exerted by each pin on the link does not necessarily act through the center of the pinhole there. If slipping impends, then the line of action of the pressure is tangent to the friction circle; and if the link is a two-force member (only the two pin pressures acting on it), then the two pressures are colinear and must act along a line which is tangent to both friction circles. Which one of the four tangents to take in a given case depends upon the direction of the tendency to slipping at each pin, and whether the link is under tension or compression. To ascertain the correct tangent, try any one as the line of action of the two pin pressures R, and then investigate the R's for their frictional components to ascertain whether the directions of those components are consistent with the directions of slip; only one tangent will satisfy all

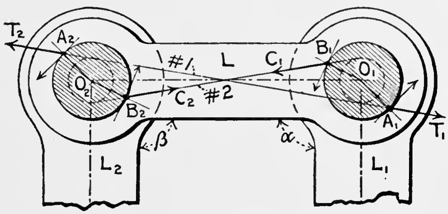


FIG. 149

the conditions for a given case. For example, suppose that the tendency is for α to increase and β to decrease; if the pressures put the link under tension, then the two pressures act along tangent number 1 at points A_1 and A_2, and if the pins put the link under compression then the two pressures act along tangent number 2 at points B_1 and B_2.

The deviations of the various tangents (lines of action of the pin pressures) from the axis of the link depend on the diameter of the friction circle and the length of the link. Generally the diameter is so small compared to the length of the link that the deviation is small, and one may safely take the axis of the link as the line of action of the pin pressures so long as the link is at rest and for all states of tendency to slip.

§ 5. BELT OR COIL FRICTION. — Fig. 150 represents a cylinder about a part of which a belt or rope is wrapped. If the cylinder is not very smooth, then the pulls P_1 and P_2 may be quite unequal without causing slipping over the cylinder, as may be easily verified by trial. When slipping impends, then the ratio of these pulls depends on the coefficient of friction and on the angle of wrap. If P_2 = the larger pull, μ = the coefficient of friction, α = the angle of lap expressed in radians, and e = base of the Napierian system of logarithms (2.718), then as proved below,

$$P_2 \div P_1 = e^{\mu\alpha}.$$

For a given value of P_1, P_2 increases very rapidly with α as shown by Fig. 151, which is the polar graph of the foregoing equation, P_2 and α being the variables, $e = 2.718$, μ taken as $\frac{1}{4}$, and $P_1 = OA$. The following table gives values of the ratio P_2/P_1 for three values of the coefficient of friction and for twelve values of the angle of lap.

MAXIMUM RATIOS P_2/P_1 (Slipping Impending)

$\frac{\alpha}{2\pi}$	μ			$\frac{\alpha}{\pi}$	μ		
	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
0.1	1.17	1.23	1.37	0.7	3.00	4.33	9.00
0.2	1.37	1.51	1.87	0.8	3.51	5.34	12.34
0.3	1.60	1.87	2.57	0.9	4.11	6.58	16.90
0.4	1.87	2.31	3.51	1.0	4.81	8.12	23.14
0.5	2.19	2.85	4.81	2.0	23.	66.	535.
0.6	2.57	3.51	6.59	3.0	111.	535.	12,390.

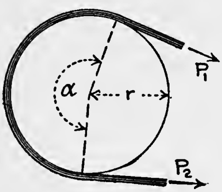


FIG. 150

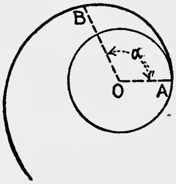


FIG. 151

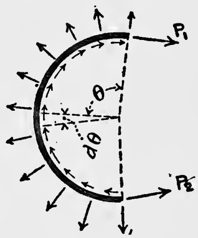


FIG. 152

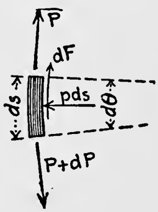


FIG. 153

Derivation of Formula.—The forces acting upon the part of the belt in contact with the cylinder consist of the tensions P_1 and P_2, the normal pressure, and the friction (Fig. 152). Let p denote the normal pressure per unit length of arc; then the normal pressure on any part whose length is ds (enlarged in Fig. 153) is pds. The friction on that part may be called dF, and the tensions P and $P + dP$. Since the part is at rest, $pds = 2\,P \sin \frac{1}{2}\,d\theta = Pd\theta$, or $p = P/r$; that is, the normal pressure per unit length at any point of the contact equals the belt tension there divided by the radius of the cylinder. When slipping impends, $dF = \mu pds$, and since $dF = dP$,

$$dP = \mu \frac{P}{r}\,ds, \text{ or } \frac{dP}{P} = \mu \frac{ds}{r} = \mu d\theta.$$

Integration gives $\left[\log_e P\right]_{P_1}^{P_2} = \mu\left[\theta\right]_0^{\alpha}$; hence,

$$\log_e P_2 - \log_e P_1 = \mu\alpha, \text{ or } P_2 = P_1 e^{\mu\alpha}.$$

For an example consider the band-brake shown in Fig. 154. It consists of a rope or other band wrapped part way around a brake wheel W, the two ends of the band being fastened to the brake lever L; the lever is pivoted at Q. Obviously any force as P tightens the band, and if the wheel tends to turn (on account of some turning force, not shown), then P induces friction between wheel and band. We will now show how great a frictional moment (origin in the axis of the wheel) the force P can induce. Let M = the moment, P_2 = the larger tension in the brake band (on the side as marked when the wheel tends to rotate as indicated), P_1 = the smaller tension, r = radius of the wheel, a_1 = arm of P_1 with respect to Q, a_2 = arm of P_2, and a = arm of P. Consideration of the forces acting on the brake-strap shows that $M = (P_2 - P_1)r$; consideration of forces acting on the lever shows that $Pa = P_1a_1 + P_2a_2$. For a given P, M is greatest when slipping impends, and then $P_2 \div P_1 = e^{\mu\alpha}$. These three equations solved simultaneously show that

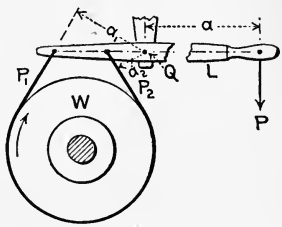


FIG. 154

$$M = Pa\,(e^{\mu\alpha} - 1)r \div (a_2e^{\mu\alpha} + a_1).$$

For example, let $P = 75$ pounds, $a = 10$ feet, $\mu = \frac{1}{4}$, $\alpha = 320°$ ($= 5.5$ radians), $r = 3$ feet, $a_1 = 2$ feet, and $a_2 = 9$ inches. Then $\alpha \div 2\pi$ = about 9, and $e^{\mu\alpha} = 4.115$ (see table on preceding page); and

$$M = 75 \times 6\,(4.11 - 1)\,3 \div (\tfrac{3}{4} \times 4.11 + 2) = 765 \text{ foot-pounds.}$$

CHAPTER V

CENTER OF GRAVITY

21. Center of Gravity of Bodies

§ 1. It is shown in Art. 7 that the resultant of two parallel forces F_1 and F_2 acting at two points A and B of any body cuts the line AB in a point P so that $AP/PB = F_2/F_1$ (Fig. 155). This proportion fixes the position of P, and since the proportion is independent of the angle between AB and the forces, P is also, so independent. Therefore if AB were a rod and F_1 and F_2 the weights of two bodies suspended from A and B, then the resultant R of F_1 and F_2 would always pass through the same point even if the tilt of the rod were changed slowly so as to leave the suspending strings parallel. Furthermore, if three parallel forces be applied at definite points A, B, and C of a body (Fig. 155), and if R denotes the resultant of F_1 and F_2 as before and R' the resultant of R and F_3 (and so also the resultant of F_1, F_2, and F_3), then $CP'/PP' = R/F_3$.

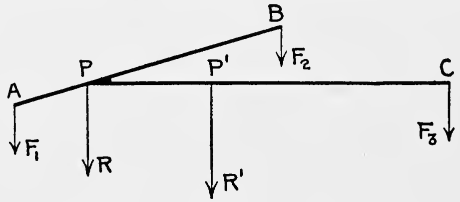


FIG. 155

This proportion fixes P' (in CP), and it is independent of the angle between the forces and the plane of ABC. Therefore if AB and CP be two rods rigidly fastened at P, and F_1, F_2, and F_3 the weights of bodies suspended from A B, and C, then the resultant of the three forces would always pass through P if the rods were slowly turned about leaving the strings parallel. And so if any number of parallel forces have definite points of application on a rigid body, the resultant of the forces always passes through some one definite point of the body, or of its extension, when the body is turned about so as not to disturb the parallelism of the forces. This unique point is called the *center* or *centroid* of the parallel forces.

The forces of gravity on all the constituent particles of a body constitute a parallel force system having definite points of application; therefore all those forces have a centroid. That is, the resultant of the forces of gravity on all the particles of a body (its weight) always passes through some one definite point of the body, or of its extension, no matter how the body is turned about;

this point is called the *center of gravity* of the body. The positions of the centers of gravity of many regular bodies are given in Art. 24, and methods for determining those centers of gravity are explained in Art. 23.

We now show how to locate the center of gravity of a body (or of a collection of bodies) which consists of simple parts whose weights and centers of gravity are known. Let A, B, C, etc. (Fig. 156), be the centers of gravity of certain parts of a body (not shown); W_1, W_2, W_3, etc., the weights of those parts; x_1, y_1, z_1, the coördinates of A; x_2, y_2, z_2, the coördinates of B, etc. Also let W denote the weight of the whole body, Q its center of gravity, and $\bar{x}, \bar{y}, \bar{z}$, the coördinates of Q. Since W is the resultant of W_1, W_2, W_3, etc., the moment of W about any line equals the algebraic sum of the moments of W_1, W_2, W_3, etc., about the same line (Art. 8). Thus, taking moments about the y-axis, we get

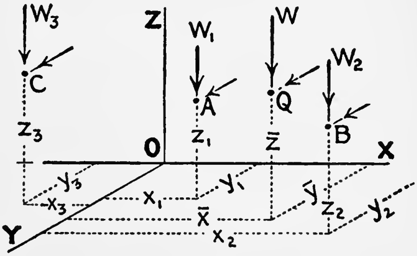


FIG. 156

$$W\bar{x} = W_1x_1 + W_2x_2 + W_3x_3 + \ . \ . \ .,$$

from which equation $\bar{x}$ can be determined. Similarly, by taking moments about the x-axis we can get $\bar{y}$. To get $\bar{z}$, we imagine the body turned until the y-axis is vertical, — the coördinate axes are assumed fixed to the body, — and then take moments about the x-axis; or, what comes to the same thing, we imagine the forces of gravity (W_1, W_2, W_3, etc.) all turned about their respective points of application until they become parallel to the y-axis, and then take moments with respect to the x-axis.

A name for the product of the weight of the body and the ordinate of its center of gravity with respect to a plane will prove convenient; we will call such product the *moment of the body* with respect to the plane.* Then the equations mentioned can be rendered in the form of a proposition as follows: *The moment of a body with respect to any plane equals the algebraic sum of the moments of its parts with respect to that same plane.*

(i) As an example we determine the coördinates of the center of gravity of a slender wire 43 inches long bent as represented by the heavy line in Fig. 157. If the weight of the wire per unit length is w, say, then the weights of the several straight portions beginning at the left are as listed in the schedule under W. The coördinates of the respective centers of gravity are listed under x, y, and z; and the moments of the parts with respect to the yz, zx, and xy planes in the last three columns respectively. The coördinates of the center

* This moment does not of course have anything to do with turning effect like the ordinary moment of a force (with respect to a line or point). To distinguish these moments, the first is sometimes called a statical moment, not very appropriately, however. See also Art. 22 for other statical moments.

of gravity of the whole wire are: $\bar{x} = 177.5\,w \div 43\,w = 4.13$ in.; $\bar{y} = 148\,w \div 43\,w = 3.44$ in.; $\bar{z} = 192\,w \div 43\,w = 4.47$ in.

W	x	y	z	Wx	Wy	Wz
$5\,w$	-2.5	0	8	$-12.5\,w$	$0\,w$	$40\,w$
6	0.0	3	8	00.0	18	48
8	0.0	6	4	00.0	48	32
10	5.0	6	0	50.0	60	00
10	10.0	3	4	100.0	30	40
4	10.0	-2	8	40.0	-8	32
$43\,w$				$177.5\,w$	$148\,w$	$192\,w$

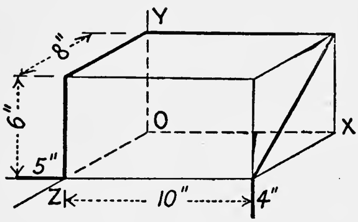


FIG. 157

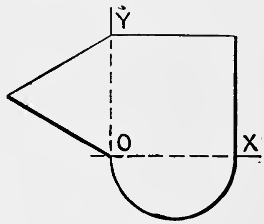


FIG. 158

(ii) As another example, we determine the center of gravity of a flat sheet of tin consisting of three parts (Fig. 158), namely, a square, a semicircle, and an equilateral triangle. If s = the side of the square, and w = the weight of the tin per unit area, then the weights of the parts are as scheduled under W. Obviously, the center of gravity of the square is at the intersection of the diagonals; in Art. 24 it is explained that the center of gravity of a semicircle is $4\,r/3\,\pi$ distant from the center of the circle where r = radius, and that the center of gravity of a triangle is on the medians of the triangle and $\frac{1}{3}\,a$ distant from any base where a = altitude measured to that base. Hence the coördinates of the centers of gravity of the parts are as scheduled under x and y. The moments of the parts with respect to the yz and zx planes are scheduled in the last two columns respectively; hence, for the whole sheet of tin $\bar{x} = 0.571\,s^3w \div 1.826\,s^2w = 0.313\,s$, and $\bar{y} = 0.633\,s^3w \div 1.826\,s^2w = 0.347\,s$.

Part.	W	x	y	Wx	Wy
Square.	$1.000\,s^2w$	$0.50\,s$	$0.50\,s$	$.500\,s^3w$	$.500\,s^3w$
Semicircle.	.393	.50	$-.212$	.196	$-.083$
Triangle.	.433	$-.289$	.500	$-.125$	.216
	$1.826\,s^2w$			$.571\,s^3w$	$.633\,s^3w$

§ 2. TO DETERMINE THE CENTER OF GRAVITY OF THE REMAINDER OF A SIMPLE BODY from which one or more simple parts have been taken, we may use the principles of moments in modified form thus: The moment of the remainder of a body with respect to any plane equals the moment of the whole minus the moments of the parts taken away.

(iii) As an example, we determine the center of gravity of a cylinder of cast iron (specific weight 450 pounds per cubic foot) with a conical recess in one end and a cylindrical hole in the other, shown in section in Fig. 159. The weights of the complete solid cylinder, of the cone, and of the small cylinder, all as of cast iron, are given under W. The coördinates of the center of gravity of the solid cylinder and of the parts are given under x and y (see Art. 24 for information on cone), and the moments with respect to the yz and zx planes are given in the last two columns. The weight of the actual piece of cast iron is $327.5 - (41 + 26.2) = 260.3$ pounds; the moments of the piece equal $1637.5 - (205.0 + 78.6) = 1353.9$ and $2620 - (61.5 + 314.4) = 2244.1$ inch pounds respectively. For the piece of cast iron, therefore, $\bar{x} = 1353.9 \div 260.3 = 5.2$, and $\bar{y} = 2244.1 \div 260.3 = 8.6$ inches.

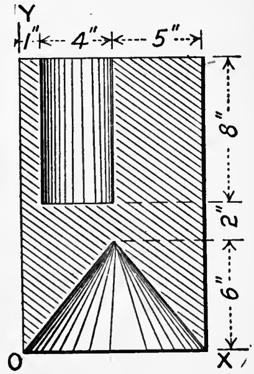


FIG. 159

Part.	W	x	y	Wx	Wy
Cylinder	327.5	5	8	1637.5	2620.0
Cone	41.0	5	1.5	205.0	61.5
Hole	26.2	3	12	78.6	314.4

§ 3. EXPERIMENTAL METHODS may be resorted to for finding the center of gravity of a body so irregular that the foregoing methods cannot be applied.

(1) Method of Suspension: The body is suspended from one point of it, and the direction of the suspending cord is then marked in some way on the body; the operation is repeated for another point of suspension. Since the center of gravity is in the two lines or directions so fixed in the body, it is at their intersection.

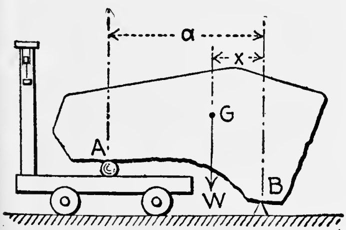


FIG. 160

(2) Method of Balancing: The body is balanced on a straight-edge, and the vertical plane containing the edge is marked on the body; the operation is repeated for two more balancing positions of the body. Since the center of gravity is in the three planes so fixed in the body, it is at their common point. This method is readily applied to a body in the form of a thin plane plate; for such only two balancings are necessary.

(3) Method of Weighing: The weight W of the body is determined, and then it is supported on a knife-edge B (Fig. 160) and on a point support which rests upon a platform scale; the reaction W' of the point support is weighed, and the horizontal distance a of the point from the knife-edge is measured;

then the horizontal distance from the center of gravity to the knife-edge is $W'a/W$. In this manner the horizontal distances of the center of gravity from several knife-edge supports can be got and the center of gravity located.

The distance of the center of gravity of a body from the plane through three points of the body can be determined if the body can be supported at the points and if certain weighings can be performed as described. Let A, B, and C (behind B and not shown) be three such points of the body (Fig. 161); a = distance of A from the line joining B and C; W = weight of the body; W' = weight recorded by the scale when A, B, and C are at the same level as shown

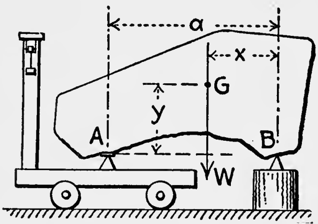


FIG. 161

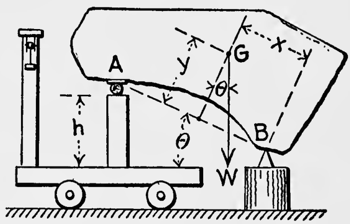


FIG. 162

in Fig. 161, and W'' = weight recorded by the scale when A is higher than B and C by any amount h (Fig. 162). Then the distance y of the center of gravity from the plane ABC is given by

$$y = \frac{\sqrt{a^2 - h^2}}{h} \frac{W' - W''}{W} a.$$

Proof: From the first position it is plain that $W'a = Wx$; from the second it follows that $W''a \cos\theta = W(x \cos\theta - y \sin\theta)$. Solving these simultaneously we get $y = (W' - W'')(a \cot\theta)/W$; but $\cot\theta = \sqrt{a^2 - h^2} \div h$, hence, etc.

22. Centroids of Lines, Surfaces, and Solids

§ 1. Lines, surfaces, and (geometric) solids have no weight, and therefore they have no center of gravity in the strict sense of the term as defined in the preceding article. However, we do speak of the center of gravity of those geometric conceptions; and we mean by the term, the center of gravity of the line, surface, or volume materialized, that is, conceived as a homogeneous slender wire, thin plate, or body, respectively. The center of gravity of a line, surface, or solid is sometimes spoken of as the center of gravity of the length (of the line), area (of the surface), and volume (of the solid). The term *centroid* has been proposed as a substitute for center of gravity when applied to lines, surfaces, and solids as being more appropriate; the new term is given preference in this book.

If a given line, surface, or solid is imagined as materialized, then we can apply the principle of moments (Art. 21) to it. Thus, if W = the weight of the whole materialized line, surface, or solid, W_1, W_2, W_3, etc., = the weights of all the parts into which we imagine it divided, $\bar{x}$ = the coördinate of the center of gravity of the whole with reference to some convenient reference plane, and x_1, x_2, x_3, etc. = the coördinates of the centers of gravity of all the parts respectively, then

$$W\bar{x} = W_1x_1 + W_2x_2 + W_3x_3 + \cdots.$$

But the weights W, W_1, W_2, W_3, etc., are proportional to the respective lengths (L, L_1, L_2, L_3, etc.) or areas (A, A_1, A_2, A_3, etc.) or volumes (V, V_1, V_2, V_3, etc.), as the case may be; and therefore it follows from the preceding equations that

$$\begin{aligned} &\text{for lines,} && L\bar{x} = L_1x_1 + L_2x_2 + L_3x_3 + \cdots, \\ &\text{for surfaces,} && A\bar{x} = A_1x_1 + A_2x_2 + A_3x_3 + \cdots, \\ &\text{and for solids,} && V\bar{x} = V_1x_1 + V_2x_2 + V_3x_3 + \cdots. \end{aligned}$$

The foregoing formulas can be rendered conveniently in a single statement of words or proposition by means of a new term which we now define. The *moment of a line, surface, or solid* with respect to a plane is the product of the length of the line, area of the surface, or volume of the solid and the coördinate of the centroid of the line, surface, or solid with respect to that plane. (The moment of a plane line or surface with respect to a plane perpendicular to the plane of the line or surface is also called its moment with respect to the line of intersection of the two planes.) The proposition or principle of moments, then, is this: The moment of a line, surface, or solid with respect to any plane equals the algebraic sum of the moments of the parts of that line, surface, or solid into which we imagine the whole divided, with respect to that same plane.* The principle of moments can be used to determine the centroids of all geometrical bodies which can be divided up into parts whose magnitudes and centroids are known. Three examples follow:

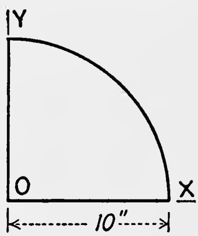


FIG. 163

(i) Let it be required to locate the centroid of the line represented (heavily) in Fig. 163, the curved portion being a circular arc; given that each coördinate of the centroid of the arc is 6.366 inches (Art. 24). Let $\bar{x}$ denote the x coördinate of the centroid of the line; then taking moments about OY (the length of the line = 35.7 inches), $35.7\,\bar{x} = 10 \times 0 + 10 \times 5 + 15.71 \times 6.366$, or $\bar{x} = 4.20$ inches. Obviously, the y coördinate also equals 4.20 inches.

(ii) Let it be required to locate the centroid of the shaded area in Fig. 164, which represents the cross section of a "channel" (a form of steel beam much used in construction). We consider the section as divided into a rectangle, 0.40 by 15 inches, and two trapezoids. The distance of the centroid of either trapezoid from its longer base is given by $3\,(0.90 + 0.80) \div 3\,(0.90 + 0.40) = 1.31$ inches (Art. 24). The second column of the adjoining schedule gives the areas of the parts; the third, the centroidal coördinates with respect to the base

* Of course these moments have nothing to do with turning effects like the moment of a force with respect to a line or a point. To distinguish these moments, the former are sometimes called *statical moments*, not very appropriately, however.

of the section; and the last, the moments with respect to that base. The distance of the centroid of the entire section from the base is $7.70 \div 9.8 = 0.79$ inch.

Part.	A	y	Ay
Rectangle..............	6.0	0.20	1.20
Two trapezoids.........	3.8	1.71	6.50
	9.8		7.70

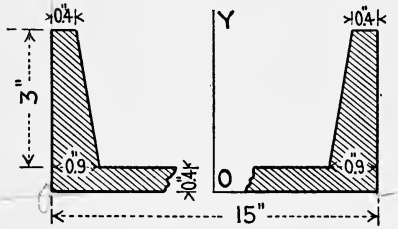


FIG. 164

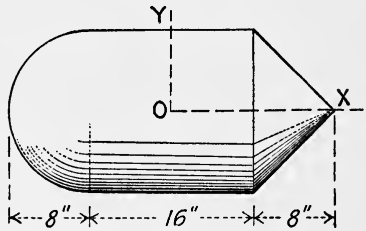


FIG. 165

(iii) Let it be required to locate the centroid of a solid consisting of a cone, a cylinder and a hemisphere as represented in Fig. 165; given that the centroid of the cone is 2 inches from its base, and that of the hemisphere is 3 inches from its base (Art. 24). The volumes of the parts, the x coördinates of their centroids, and the moments with respect to the yz plane are as recorded in the adjoining schedule. The total volume is 4825.5 in^3 and the algebraic sum of the moments of the parts is -6433 in^4; therefore $\bar{x} = -6433 \div 4825.5 = -1.33$ inches, the negative sign indicating that the centroid of the whole solid is to the left of O.

Part	V	x	M
Cone..................	536.2	10	5,362
Cylinder..............	3217.0	0	0
Hemisphere............	1072.3	-11	$-11,795$
	4825.5		$-6,433$

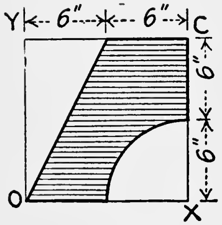


FIG. 166

§ **2.** TO DETERMINE THE CENTROID OF THE REMAINDER OF A SIMPLE FIGURE from which one or more simple parts have been taken, we may use the principle of moments in the following modified form. The moment of the remainder of a figure with respect to any plane equals the moment of the whole minus the moments of the parts taken away. For example, let it be required to determine the centroid of the shaded area in Fig. 166, the part of the square remaining after the triangle and the quadrant have been taken away; given that the centroid of the triangle is 2 inches from OY and 4 inches from YC, and that the centroid of the quadrant is 2.54 inches from OX and CX (see Art. 24). The areas, centroidal coördinates, and moments appear in the adjoining schedule. The area of the shaded portion is $144 - (36 + 28.27) = 79.73$ square inches, and the moments of the shaded part with respect to the y and x axes are $864 - (72 + 266.9) = 525.1$ and $864 - (288 + 71.8) = 494.2$ cubic inches respectively. Therefore $\bar{x} = 525.1 \div 79.73 = 6.59$, and $\bar{y} = 494.2 \div 79.73 = 6.20$ inches.

Part	A	x	y	Ax	Ay
Square	144	6	6	864	864
Triangle	36	2	8	−72	−288
Quadrant	28.27	9.44	2.54	−266.9	−71.8
				525.1	494.2

23. Centroids Determined by Integration

§ 1. If it is desired to locate the centroid of a line, a surface, or a solid which cannot be divided into a finite number of simple parts whose lengths, areas, or volumes and centroids are known, and if the line, surface, or solid is "mathematically regular," then we imagine the line, surface, or solid divided into an infinitely great number of parts, and apply the principle of moments. To find the sum of the moments of all these elementary parts involves an integration. Thus, let L = the length of a line, $\bar{x}$ = the x coördinate of its centroid, dL_1, dL_2, dL_3, etc. = the lengths of elementary portions of the line, and x_1, x_2, x_3, etc. = the x coördinates of the centroids of those portions respectively,

then $$L\bar{x} = dL_1 \cdot x_1 + dL_2 \cdot x_2 + dL_3 \cdot x_3 + \cdots = \int dL \cdot x,$$

in which dL stands for any of the elementary lengths and x for the x coördinate of the centroid of that dL. Similarly, for areas and volumes; and thus we have these formulas:

$$(1)\ L\bar{x} = \int dL \cdot x; \quad (2)\ A\bar{x} = \int dA \cdot x; \quad (3)\ V\bar{x} = \int dV \cdot x,$$

and corresponding ones for $\bar{y}$ and $\bar{z}$ (the y and z coördinates of the centroid).

These formulas can be used to determine $\bar{x}$, $\bar{y}$, and $\bar{z}$ if the form of the line, surface, or solid is such that the integrations can be performed. In any particular case, limits of integration must be assigned so that all elementary portions are included in the integration (summation).* Six examples illustrating their use follow:

* *The centroid is a mean point.* The ordinate from any plane to the centroid of a line, surface, or solid equals the mean of the ordinates of all the equal elementary portions of the line, surface, or solid, it being understood that the mean takes into account signs of the ordinates. For, let x_1, x_2, x_3, etc., be the ordinates of the elementary portions and n the number

(i) Circular arc; radius $= r$ and central angle $= 2\,\alpha$ (Fig. 167). The radius which bisects the central angle is a line of symmetry, therefore the centroid is on that line; if that line is taken as x axis, then $\bar{y} = 0$. The length of the arc $= 2\,r\alpha$ (α expressed in radians), $dL = r\,d\phi$, and $x = r\cos\phi$; therefore formula (1) becomes

$$2\,r\alpha\bar{x} = \int_{-\alpha}^{+\alpha} r\,d\phi \cdot r\cos\phi = r^2\int_{-\alpha}^{+\alpha}\cos\phi\,d\phi = 2\,r^2\sin\alpha; \text{ or } \bar{x} = (r\sin\alpha) \div \alpha.$$

(ii) The preceding problem will now be solved without using polar coördinates. Since $x^2 + y^2 = r^2$, $x\,dx + y\,dy = 0$, or $dy = -(x\,dx)/y$. Hence

$$dL = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + x^2/y^2} = dx\,r/y = dx\,r/\sqrt{r^2 - x^2},$$

and $$2\,r\alpha\bar{x} = \int x\,dL = 2\,r\int_{r\cos\alpha}^{r}\frac{x\,dx}{\sqrt{r^2 - x^2}} = 2\,r^2\sin\alpha; \text{ etc.}$$

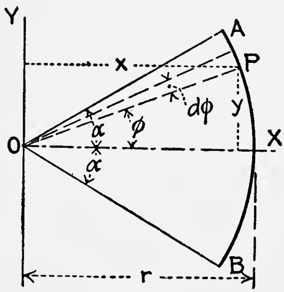


FIG. 167

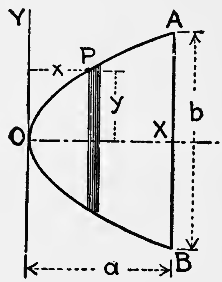


FIG. 168

(iii) The parabolic segment $AOBA$ (Fig. 168); altitude $= a$ and base $= b$. Evidently the axis of the parabola is a line of symmetry, and therefore it contains the centroid. If that line be taken as the x axis, then $\bar{y} = 0$. Let x and y be the coördinates of any point P on the parabola; then the area of the elementary portion shaded is $2\,y\,dx$. Since the area of the segment is $\frac{2}{3}\,ab$, and the equation of the parabola is $4\,ay^2 = b^2x$, formula (2) becomes

$$\frac{2}{3}ab\bar{x} = \int_0^a 2\,y\,dx \cdot x = \frac{b}{\sqrt{a}}\int_0^a x^{\frac{3}{2}}\,dx = \frac{2}{5}ba^2;$$

and $$\bar{x} = \frac{2}{5}ba^2 \div \frac{2}{3}ab = \frac{3}{5}a.$$

of them (infinite); then the mean ordinate is $(x_1 + x_2 + x_3 + \cdots) \div n$; also, let Q = the length, area, or volume of the line, surface, or solid, and dQ = the length, area, or volume of the equal elementary portions; then the mean ordinate equals

$$\frac{(x_1 + x_2 + \cdots)dQ}{ndQ} = \frac{\int dQ \cdot x}{Q} = \bar{x}.$$

(iv) Circular sector (Fig. 169); radius $= r$ and central angle $= 2\,\alpha$. The radius which bisects the central angle is evidently a line of symmetry, and so the centroid is on that line. If that line is taken as x axis, then $\bar{y} = 0$. The area of the sector equals $r^2\alpha$, α expressed in radians; $dA = \rho\,d\phi \cdot d\rho$, where $\rho = OP$ and P is any point in the sector. Therefore formula (2) becomes

$$r^2\alpha\bar{x} = \int_0^r \int_{-\alpha}^{+\alpha} \rho\,d\phi\,d\rho \cdot x = \int_0^r \int_{-\alpha}^{+\alpha} \rho\,d\phi\,d\rho \cdot \rho\cos\phi;$$

and
$$\bar{x} = \left(\frac{2}{3}r^3 \sin\alpha\right) \div \left(r^2\alpha\right) = \frac{2\,r\sin\alpha}{3\,\alpha}$$

(v) Conical or pyramidal solid; altitude $= a$ (Fig. 170). We take the origin of coördinates at the apex, and the x axis perpendicular to the base; $OMNO$ represents the projection of the cone or pyramid on the XY plane. We imagine the solid divided into plates or laminas parallel to the base; if the area of the base is called A, say, then the area of the lamina represented is Ax^2/a^2, and the volume of the lamina is $dx \cdot Ax^2/a^2$. And since the volume of the solid is $\frac{1}{3}Aa$, formula (3) becomes

$$\frac{1}{3}Aa\bar{x} = \int_0^a (dx \cdot Ax^2/a^2)\,x = A/a^2 \int_0^a x^3 dx = \frac{Aa^2}{4};$$

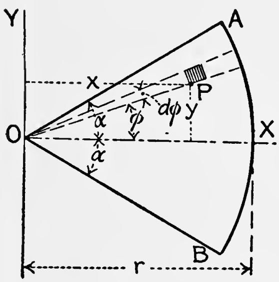


FIG. 169

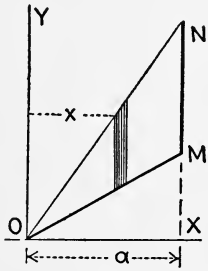


FIG. 170

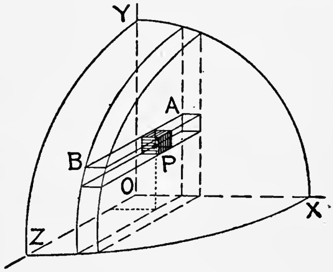


FIG. 171

hence, $\bar{x} = \frac{3}{4}\,a$, that is, the perpendicular distance from the centroid to the base equals one-fourth the altitude. Evidently, the centroid of every lamina lies on the line joining the apex and the centroid of the base; therefore the centroid of all the laminas (that is, the solid) lies on that line.

(vi) Octant of a sphere; radius $= r$ (Fig. 171). Obviously $\bar{x} = \bar{y} = \bar{z}$; $\bar{x}$ is given by

$$V\bar{x} = \int_0^r \int_0^{(r^2-x^2)^{\frac{1}{2}}} \int_0^{(r^2-x^2-y^2)^{\frac{1}{2}}} (dx\,dy\,dz)x.$$

Evaluating the integral and substituting for V its value, $\frac{1}{6}\pi r^3$, we find that $\bar{x} = \frac{3}{8}\,r$.

§ 2. SURFACES AND SOLIDS OF REVOLUTION. — For surfaces, we use formula (2) and select as element the surface described by an elementary part of the generating curve MN (Fig. 172). Let the x axis be taken coincident with the axis of revolution; then the area described by a part of the generating curve of length ds is $2\pi y\,ds$. The centroid of this area is in the x axis, and its x coördinate is the x in the figure; hence if A stands for the area of the surface of revolution,

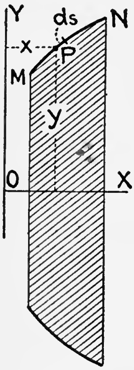


FIG. 172

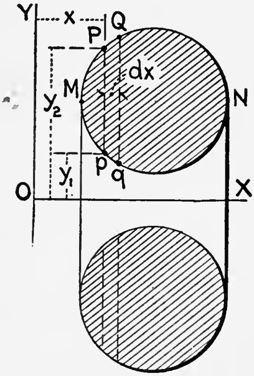


FIG. 173

$$A\bar{x} = 2\pi \int y\,ds \cdot x, \quad \text{or} \quad \bar{x} = \frac{2\pi}{A}\int xy\,ds.$$

The limits of integration must be assigned so that each product $xy\,ds$ will be included in the integration.

For solids, we use formula (3), and take as element that volume generated by an elementary part of the generating plane MPN (Fig. 173) which is included between two lines perpendicular to the axis of revolution. Thus, if the x axis is taken coincident with the axis of revolution, then $PQqp$ generates the elementary volume, or $dV = \pi(y_2^2 - y_1^2)\,dx$. Now the centroid of this elementary volume is in the x axis, and its x coördinate is the x in the figure; hence if V denotes the volume of the solid of revolution, then

$$V\bar{x} = \pi \int (y_2^2 - y_1^2)\,dx \cdot x, \quad \text{or} \quad \bar{x} = \frac{\pi}{V}\int (y_2^2 - y_1^2)x\,dx.$$

The limits of integration are to be assigned so that each product $(y_2^2 - y_1^2)x\,dx$ will be included in the integration.*

* *Theorems of Pappus and Guldinus.* — These relate primarily to the determination of the area and volume of a solid of revolution; they involve the centroid of the generating curve or plane, and are therefore mentioned in this place. (1) The first theorem states that the area of a surface of revolution generated by a plane curve revolved about a line in its plane equals the product of the length of the curve and the circumference of the circle described by the centroid of the curve. Proof: Let MN (Fig. 172) be the generating curve, L = length of the curve, $\bar{y}$ = the ordinate of the centroid of L from the axis of revolution, and A = area of the surface generated. Then

$$A = \int 2\pi y\,dL \text{ and } \bar{y}L = \int y\,dL.$$

Combining these equations we get $A = L\,2\pi\bar{y}$, which is the proposition in mathematical form. (2) The second theorem states that the volume of a solid of revolution generated by a plane figure revolved about an axis in the plane equals the product of the area of the figure and the circumference of the circle described by its centroid. Proof: Let MPN (Fig. 173) be the generating plane, a = area of the plane, $\bar{y}$ = the ordinate of the centroid of a from the axis of revolution, and V = volume of the solid generated. Then

$$V = \int \pi(y_2^2 - y_1^2)\,dx, \text{ and from eq. (2), } a\bar{y} = \int (y_2 - y_1)\,dx\,\tfrac{1}{2}(y_2 + y_1).$$

Combining these equations we get $V = a\,2\pi\bar{y}$, which is the proposition in mathematical form.

To illustrate, we determine the area of the surface generated by revolving the circular arc ABC (Fig. 175) about AC, and the volume of the solid generated by revolving the figure

For an illustration imagine the quadrant XY (Fig. 174) rotated about OX so as to generate a hemisphere. The positions of the centroids of the surface and solid generated could be computed as follows: (i) The area of the hemisphere is $2\pi r^2$, $x = r \sin\phi$, $y = r\cos\phi$, and $ds = r\,d\phi$; hence for the area

$$\bar{x} = (2\pi \div 2\pi r^2)\int xy\,ds = r\int_0^{\frac{\pi}{2}} \sin\phi\cos\phi\,d\phi = \tfrac{1}{2}r.$$

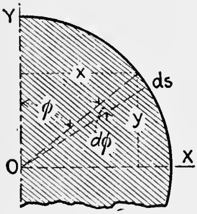


FIG. 174

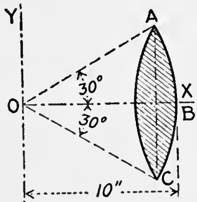


FIG. 175

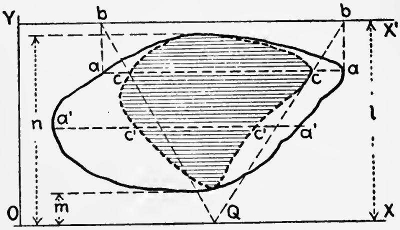


FIG. 176

(ii) The volume of the hemisphere is $\frac{2}{3}\pi r^3$, $y_2 = r\cos\phi$, $x = r\sin\phi$, and $dx = r\cos\phi\,d\phi$; hence for the volume

$$\bar{x} = (\pi \div \tfrac{2}{3}\pi r^3)\int (y_2^2 - 0)\,x\,dx = (3\,r/2)\int_0^{\frac{\pi}{2}} \cos^3\phi\sin\phi\,d\phi = \tfrac{3}{8}r.$$

§ 3. The centroid of an *irregular plane surface or figure* can be determined graphically or experimentally. The graphical method requires the use of a planimeter or other device for measuring an area. Let $aaa'a'$ (Fig. 176) be the figure whose centroid is to be located. (i) Take OX and YX' on opposite sides of the figure at any convenient distance l apart. (ii) Project any width of the figure as aa on YX'; connect projections bb with any point on OX as Q, and note the intersections cc. (iii) Repeat (ii) for other widths as $a'a'$, and then connect all points like c by a smooth curve. (iv) Measure the area A' included by this curve, and the area A of the given figure. Then $A'l$ is the (statical) moment of A with respect to OX (proof follows), and hence the distance from OX to the centroid is $\bar{y} = A'l \div A$. Proof: Let w = any width of the figure as aa, and w' the corresponding width cc of the derived curve; then the moment of A with respect to OX is

$$\int_m^n yw\,dy = \int_m^n lw'\,dy = l\int_m^n w'\,dy = lA'.$$

To determine the centroid experimentally cut a piece of stiff cardboard into the shape of the irregular figure, and find its center of gravity by balancing as explained in Art. 21; this point locates the centroid sought.

$ABCA$ about AC. The length of the arc = 10.47 inches; the distance of its centroid from AC = 0.89 inch (Art. 24); hence $A = 10.47 \times 2\pi \times 0.89 = 58.5$ square inches. The area of $ABCA$ = 9.06 square inches; the distance of its centroid from AC = 0.54 inch; hence $V = 0.06 \times 2\pi \times 0.54 = 30.7$ cubic inches.

The theorems $A = L\,2\pi y$ and $V = a\,2\pi\bar{y}$ can be used also for computing $\bar{y}$ if A and L, or V and a (as the case may be), are known.

24. Centroids of Some Lines, Surfaces, and Solids

Circular Arc (Fig. 177). — C is the centroid; its distance from the center is $(r \sin \alpha)/\alpha$, the divisor α to be expressed in radians (1 degree = 0.0175 radian); the distance is also rc/s, where s = arc. If the arc is flat then the distance of its centroid from the chord is nearly $\frac{2}{3}h$; the discrepancy is less than one-half per cent for arcs whose central angle 2α is less than 60 degrees.

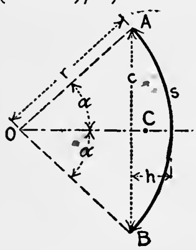


FIG. 177

When the arc is a semicircle, then the distance from the centroid to the center is $2r/\pi = 0.6366r$. When the arc is a quadrant, then the distance to the center is $2r\sqrt{2}/\pi = 0.9003r$, and the distance to the radii OA and OB is $2r/\pi = 0.6366r$.

Triangle. — The centroid is at the intersection of the medians; its perpendicular distance from any side equals one-third the altitude of the triangle measured from that side. If x_1, x_2, and x_3 are the coördinates of the vertexes with respect to any plane and $\bar{x}$ the coördinate of the centroid, then $\bar{x} = (x_1 + x_2 + x_3)$.

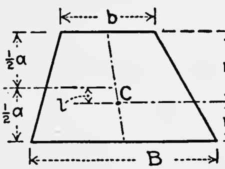


FIG. 178

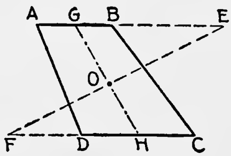


FIG. 179

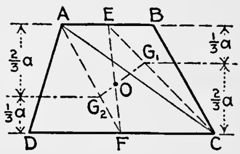


FIG. 180

Trapezoid. — The centroid is on the median (line joining the middle points of the parallel sides) (Fig. 178);

$$l = \frac{(B - b)a}{6(B + b)}, \quad m = \frac{(2B + b)a}{3(B + b)}, \quad n = \frac{(B + 2b)a}{3(B + b)}.$$

Two geometrical constructions for locating position on the median follow: (1) Extend AE (Fig. 179) so that $BE = CD$, and in the opposite direction extend CD so that $DF = AB$; the intersection of FE and the median GH is the centroid sought. (2) Divide the trapezoid (Fig. 180) into triangles by a diagonal as AC; find the centroids G_1 and G_2 of the triangles (construction indicated in the figure); the intersection G_1G_2 with the median EF is the centroid sought.

Quadrilateral. — (1) Divide the quadrilateral into triangles by a diagonal AC (Fig. 181) and find their centroids G_1 and G_2; divide it into triangles by the other diagonal and find their centroids G_3 and G_4; the intersections of the lines G_1G_2 and G_3G_4 is the centroid sought. (2) Divide the sides into thirds (Fig. 182), and draw lines through the third points as shown; these lines form a parallelogram whose diagonals intersect at the centroid of the quadrilateral.

Sector of a Circle (Fig. 183). — C is the centroid; its distance from the center is $\frac{2}{3}\cdot(r \sin \alpha)/\alpha$, the divisor α to be expressed in radians (1 degree = 0.0175 radian); the distance also equals $\frac{2}{3} rc/s$, where s = arc.

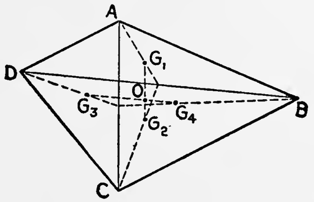


FIG. 181

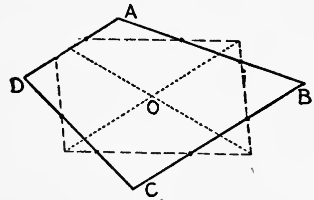


FIG. 182

When the sector is a quadrant, then the distance of the centroid from the center is $4\sqrt{2}\,r/3\pi = 0.6002\,r$; and the distance to the radii OA and OB is $4\,r/3\,\pi = 0.4242\,r$. For a semicircle the distance from diameter to centroid is $4\,r/3\,\pi = 0.4242\,r$.

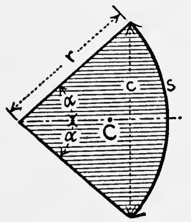


FIG. 183

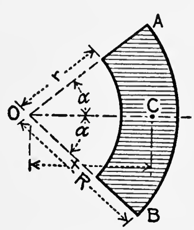


FIG. 184

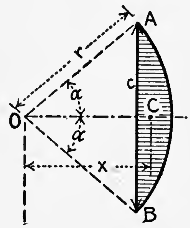


FIG. 185

Sector of a Circular Ring (Fig. 184). — The distance from the centroid to the center is

$$\frac{2}{3}\,\frac{R^3 - r^3}{R^2 - r^2}\,\frac{\sin \alpha}{\alpha},$$

the divisor α to be expressed in radians (1 degree = 0.0175 radian).

Segment of a Circle (Fig. 185). — The distance from the centroid to the center is

$$\frac{c^3}{12\,A} = \frac{2\,r^3 \sin^3 \alpha}{3\,A},$$

where A denotes the area of the segment. $A = r^2\,(2\,\alpha - \sin 2\,\alpha)$, the first α to be expressed in radians (1 degree = 0.0175 radian).

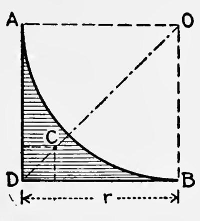


FIG. 186

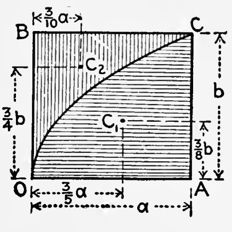


FIG. 187

The Area Shaded in Fig. 186, included between a quadrant and the tangents at its extremities. The distance of the centroid from the bounding tangents is 0.223 r, and the distance to their intersection is 0.315 r.

Parabolic Segments (Fig. 187). — C_1 and C_2 are the centroids of the shaded parts. Their distances from the sides of the inclosing rectangle $(a \times b)$ are marked in the figure.

Elliptic Segment (Fig. 188). — The centroid of the segment $XAAX$ coincides with the centroid of the segment $XaaX$ of the circumscribed circle; the centroid of the segment $YBBY$ coincides with the centroid of the $YbbY$ of the inscribed circle.

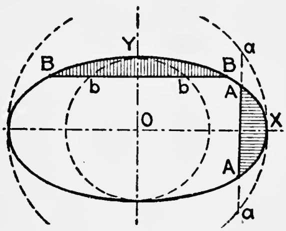


FIG. 188

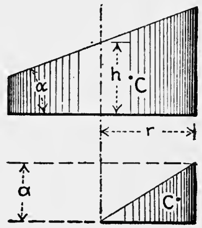


FIG. 189

Right Circular Cylinder (Fig. 189). — C is the centroid; its distance from the axis of the cylinder is $\frac{1}{4}(r^2 \tan \alpha)/h$, and its distance from the base is $\frac{1}{2}h + \frac{1}{8}(r^2 \tan^2 \alpha)/h$. When the oblique top cuts the base in a diameter of the base (lower part of Fig. 189), then the distance from the centroid to the axis is $\frac{3}{16}\pi r$, and to the base $\frac{3}{32}\pi a$.

Cone and Pyramid. — The centroid of the surface (not including base) is on a line joining the apex with the centroid of the perimeter of the base at a distance of two-thirds the length of that line from the apex. The centroid of the solid cone or pyramid is on the lines joining the apex with centroid of the base at a distance of three-fourths the length of that line from the apex.

Frustum of a Circular Cone. — Let R = radius larger base, r = radius smaller, a = altitude. The distance of the centroid of the curved surface from larger base is $\frac{1}{3}a(R + 2r)/(R + r)$; from smaller base $\frac{1}{3}a(2R + r)/(R + r)$; from a plane midway between bases $\frac{1}{6}a(R - r)/(R + r)$. The distance from the centroid of the solid frustum to the larger base is

$$\frac{1}{4}a(R^2 + 2Rr + 3r^2)/(R^2 + Rr + r^2).$$

Frustum of a Pyramid. — If the frustum has regular bases, let R and r be the lengths of sides of the larger and smaller bases, and h the altitude; then the distance of the centroid of the surface (not including bases) from the larger base is $\frac{1}{3}h(R + 2r)/(R + r)$. Whether the bases are regular or not, let A and a = the areas of the large and small bases and h the altitude; then the distance of the centroid of the solid from the larger base is

$$\frac{1}{4}h(A + 2\sqrt{Aa} + 3a)/(A + \sqrt{Aa} + a).$$

Obelisk and Wedge (Fig. 190). — The distance from the centroid to the base AB is

$$\tfrac{1}{2}\, h\,(AB + Ab + aB + 3\, ab)/(2\, AB + Ab + aB + 2\, ab).$$

If $b = 0$ the solid is a wedge, and the distance from the centroid to the base is

$$\tfrac{1}{2}\, h\,(A + a)/(2\, A + a).$$

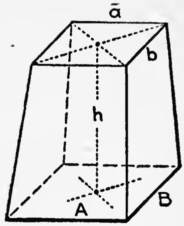


FIG. 190

Sphere. — The centroid of any zone (surface) of a sphere (Fig. 191) is midway between the bases. The distance of the centroid of a segment (solid) (Fig. 192) from the base is $\frac{1}{4}\, h(4\, r - h)/(3\, r - h)$; when $h = r$ (hemisphere) then the distance is $\frac{3}{8}\, r$. The distance of the centroid of a sector (solid) (Fig. 193) from the center of the sphere is $\frac{3}{8}\,(1 + \cos\alpha)\, r = \frac{3}{8}\,(2\, r - h)$.

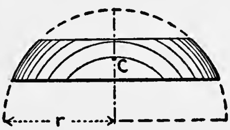


FIG. 191

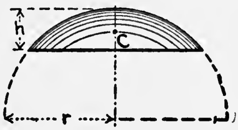


FIG. 192

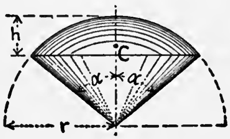


FIG. 193

Ellipsoid. — Let the three axes be taken as x, y, and z coördinate axes, and a, b, and c to denote the semi-lengths of the corresponding axes of the ellipsoid; the centroid of one octant of the solid is given by $\bar{x} = \frac{3}{8}\, a$, $\bar{y} = \frac{3}{8}\, b$, and $\bar{z} = \frac{3}{8}\, c$.

Paraboloid of Revolution, formed by revolving a parabola about its axis. Let h = height of the paraboloid, the distance from its apex to the base, then the distance from the centroid of the solid to the base is $\frac{1}{3}\, h$.

CHAPTER VI

SUSPENDED CABLES (WIRE, CHAIN, ETC.)

25. Parabolic Cable

§1. Symmetrical Case. — When a cable is suspended from two points and it sustains loads uniformly spaced along the horizontal and spaced so closely that the loading is practically continuous, then the curve assumed by the cable is a parabolic arc as will now be shown. The symmetrical case (points of suspension at same level) will be considered first. Let AOB (Fig. 194) be

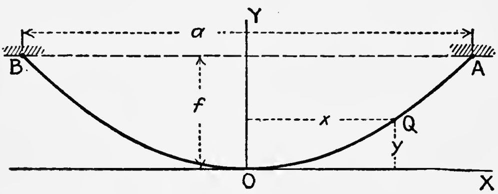


Fig. 194

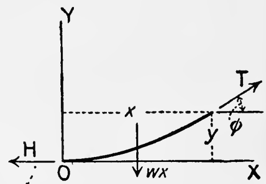


Fig. 195

the cable suspended from A and B, w = load per unit (horizontal) length, a = span AB, f = sag, H = tension in cable at lowest point, and T = tension at any other point Q (coördinates x and y). The forces acting on the portion OQ are H, T, and the distributed load wx (Fig. 195); this load acts at mid-length of x. Since the forces are in equilibrium, their moment-sum equals zero for any origin of moments; hence moments about Q give $Hy = wx(x/2)$,

$$x^2 = \frac{2H}{w}y, \qquad \text{or} \qquad y = \frac{w}{2H}x^2. \tag{1}$$

This is the standard form of the equation of a parabola; the axis of the parabola coincides with the y axis, and the vertex is at O. If we substitute for x and y their values for the point A ($x = a/2$, and $y = f$), then we get $a^2/4 = (2H/w)f$, or $H = wa^2/8f$; hence equation (1) may be written

$$x^2 = \frac{a^2}{4f}y, \qquad \text{or} \qquad y = \frac{4f}{a^2}x^2. \tag{2}$$

A formula for the tension T at any point Q may be arrived at as follows: Let ϕ = slope of the curve at Q; then it is plain from Fig. 195 that $T \sin\phi = wx$, and $T \cos\phi = H$. Squaring and adding gives

$$T^2 = w^2x^2 + H^2 = w^2x^2 + w^2a^4/64f^2. \tag{3}$$

At the points of suspension, $x = a/2$, and the value of T at that point is

$$H\left(1 + 16\frac{f^2}{a^2}\right)^{\frac{1}{2}} = \tfrac{1}{2}\,wa\left(1 + \frac{a^2}{16 f^2}\right)^{\frac{1}{2}}. \tag{4}$$

The adjoining table gives values of T/wa for various values of f/a, the sag ratio (denoted by n in the table).

$n=$	1.0	0.5	0.25	0.125	0.1	0.05	0.01
$T/wa=$	0.515	0.559	0.707	1.118	1.346	2.550	12.81

The *length of cable* for any span a and sag f or sag ratio $n = f/a$.—Let l = length of cable AB and ds = length of an elementary portion; then as in all plane curves, $ds^2 = dx^2 + dy^2 = [1 + (dy/dx)^2]\,dx^2$, or

$$ds = [1 + (dy/dx)^2]^{\frac{1}{2}}\,dx.$$

From the equation of the curve (2), $dy/dx = (8f/a^2)\,x = 8\,nx/a$; hence

$$ds = (1 + 64\,n^2x^2/a^2)\,dx.$$

Integrating between proper limits ($-\frac{1}{2}l$ and $+\frac{1}{2}l$ for s, and $-\frac{1}{2}a$ and $+\frac{1}{2}a$ for x), we get

$$l = a\left(\tfrac{1}{2}\left(1 + 16\,n^2\right)^{\frac{1}{2}} + \frac{1}{8\,n}\log_e\left[4\,n + \left(1 + 16\,n^2\right)^{\frac{1}{2}}\right]\right). \tag{5}$$

An approximate formula for l, much more convenient to use than the foregoing one, may be deduced as follows: Expanding the coefficient of dx above by the binomial theorem, we get

$$ds = \left(1 + 32\,n^2\frac{x^2}{a^2} - 512\,n^4\frac{x^4}{a^4} + \cdots\right)dx;$$

and integrating between limits as before we find that

$$l = a\left(1 + \frac{8}{3}n^2 - \frac{32}{5}n^4 + \cdots\right). \tag{6}$$

The following table gives values of l/a by the exact and approximate formulas, equations (5) and (6) respectively, for several sag ratios $n = f/a$.

$n=$	1.0	0.5	0.25	0.125	0.1	0.05	1.01
l/a exact	2.3234	1.4789	1.1478	1.0402	1.0260	1.0066	1.0003
l/a approximate			1.1417	1.0401	1.0260	1.0066	1.0003

§2. UNSYMMETRICAL CASE.—By this is meant a cable suspended from two points not at the same level; see Fig. 196, where ACB represents a cable suspended from A and B. In this case, also, the cable hangs in the arc of a parabola as will be proved presently. Let a = horizontal distance between points of supports (as in §1), b = vertical distance between the points, θ = angle which AB makes with horizontal ($= \tan^{-1} b/a$), x and y = coördinates of any point Q of the cable as shown, T = tension at the highest point, V' and H' respectively = the two components of T along AY and AB. There are three forces acting on the part AQ, — its load wx, the

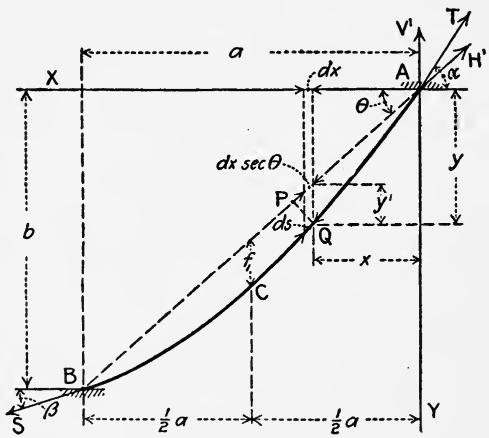


FIG. 196

tension T, and the tension at Q. The moment-sum for these three forces for any origin equals zero; with Q as origin

$$- wx \cdot x/2 + V'x - H'\,(QP) = 0, \text{ or } V'x - wx^2/2 = H'\,(y \cos\theta - x \sin\theta). \quad (1)$$

This is the equation of a parabola with the axis parallel to the y coördinate axis.

To express the equation of the curve in terms of the dimensions a, b, and the vertical sag f_1 under the middle point of the chord AB:—The forces acting on the entire cable consist of the load wa, the tension at A, and that at B. Their moment-sum with origin at B is

$$wa \cdot a/2 - V'a = 0; \quad \text{hence} \quad V' = wa/2. \quad (2)$$

The forces on the upper half AC consist of the load $wa/2$, the tension at A, and that at C. Their moment-sum with origin at C is

$$\frac{wa}{2}\frac{a}{4} - V'\frac{a}{2} + H'f_1 \cos\theta = 0; \quad \text{hence} \quad H' = \frac{wa^2}{8f_1 \cos\theta}. \quad (3)$$

Substituting these values of V' and H' in (1) gives

$$\frac{4f_1x}{a^2}(a - x) = y - x \tan\theta, \quad \text{or} \quad y = (4f_1 + b)\frac{x}{a} - 4f_1\frac{x^2}{a^2}. \quad (4)$$

The vertical distance of any point as Q below the chord AB is $y - \tan\theta$; hence if we let y' denote that distance, the foregoing equation can be put into the more convenient form

$$y' = \frac{4f_1x}{a^2}(a - x). \quad (5)$$

The value of the slope at any point of the curve is (differentiating equation 4)

$$\frac{dy}{dx} = 4\frac{f_1}{a} + \frac{b}{a} - \frac{8 f_1}{a}\frac{x}{a}.$$

Let α and β = the slope angles at A and B respectively (where $x = 0$, and $x = a$); then

$$\tan\alpha = (b + 4f_1)/a, \quad \text{and} \quad \tan\beta = (b - 4f_1)/a. \tag{6}$$

Let x_0 and y_0 = the coördinates of the vertex of the parabola (where $dy/dx = 0$); then

$$x_0 = a\,(b + 4f_1)/8f_1, \quad \text{and} \quad y_0 = (b + 4f_1)^2/16f_1. \tag{7}$$

Let H and V respectively = the horizontal and vertical components of T. Then (see Fig. 196)

$$H = H'\cos\theta = wa^2/8f_1, \text{ and}$$
$$V = V' + H'\sin\theta = \tfrac{1}{2}\,wa\,(1 + a\tan\theta/4f_1);$$

and since $T^2 = H^2 + V^2$, we find that

$$T = \frac{wa^2}{8f_1}\left[1 + \left(\tan\theta + \frac{4f_1}{a}\right)^2\right]^{\frac{1}{2}} = \tfrac{1}{2}\,wa\left[\frac{1}{16\,n_1^2} + \frac{\sin\theta}{2\,n_1} + 1\right]^{\frac{1}{2}}, \tag{8}$$

where n_1 = sag ratio $f_1 \div AB = f_1 \div a\sec\theta$. The last expression shows that for given w, a, and n_1, the tension T increases as the angle θ is made larger; also that for given w, a, and θ, T increases as n_1 is made smaller.*

Length of the Parabolic Arc AB (Fig. 196). — Let a_1 = the length of the chord AB, n_1 = sag ratio $f_1 \div a_1$, and l_1 = length of the arc AB. Also let ds = length of an elementary portion of the arc; then

$$ds = [1 + (dy/dx)^2]^{\frac{1}{2}}\,dx.$$

From the equation of the curve (4), we can get

$$\frac{dy}{dx} = \frac{4f_1}{a}\left(1 - \frac{2x}{a}\right) + \tan\theta = \left[4\,n_1\left(1 - \frac{2x}{a}\right) + \sin\theta\right]\sec\theta. \tag{9}$$

This last value of dy/dx substituted in the foregoing expression for ds gives

$$ds = \left\{1 + 8\,n_1\left(1 - 2\frac{x}{a}\right)\left[2\,n_1\left(1 - 2\frac{x}{a}\right) + \sin\theta\right]\right\}^{\frac{1}{2}}\sec\theta\,dx.$$

Now this equation is in the form $ds = (1 + X)\sec\theta\,dx$, where

$$X = 8\,n_1\,(1 - 2\,x/a)\,[2\,n_1\,(1 - 2\,x/a) + \sin\theta].$$

* Let MN (Fig. 197) represent the load on the cable AB, and let PO and QO be parallel to the tangents at A and B (Fig. 196) respectively; then $OMNO$ is a force triangle for the three forces acting on the cable AB, and OM represents T and ON represents S. It is plain from the figure that $OR\tan\alpha - OR\tan\beta = MN$, or $OR(\tan\alpha - \tan\beta) = MN$. But $OR = T\cos\alpha = S\cos\beta$, and $MN = wa$; hence

$$T\cos\alpha\,(\tan\alpha - \tan\beta) = wa = S\cos\beta\,(\tan\alpha - \tan\beta).$$

Substituting the values of $\tan\alpha$ and $\tan\beta$ given by equation (6), we find that

$$T\cos\alpha = wa^2/8f_1 = S\cos\beta.$$

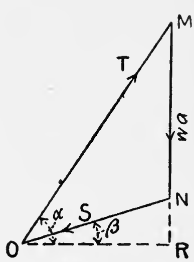


FIG. 197

Unless the sag is relatively large ds and $\sec\theta\, dx$ are nearly equal at all points along the curve (see Fig. 196); hence $(1 + X)$ is nearly equal to 1 at all points, which means that X is small compared to 1. Therefore, we may expand $(1 + X)$ by the binomial theorem, and drop all terms except the first few without serious error. Thus we have as a close approximation $ds = (1 + \frac{1}{2}X - \frac{1}{8}X^2)\sec\theta\, dx$,

and $$l_1 = \int_0^a (1 + \tfrac{1}{2}X - \tfrac{1}{8}X^2)\sec\theta\, dx.$$

Substituting for X and X^2 their values, and integrating we finally get

$$l_1 = a_1\left(1 + \tfrac{8}{3}\cos^2\theta \cdot n_1^2 - \tfrac{32}{5}n_1^4\right). \tag{10}$$

If the approximation made in the derivation of formula (10) is not permissible in a given case, then one might determine the exact length of the cable AB somewhat as follows when a, b, and f_1 are given: We first locate the vertex O of the parabola of which the cable is a part from equation (7). The vertex will be found either between A and B, on the cable (Fig. 198), or

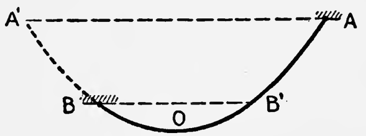


FIG. 198

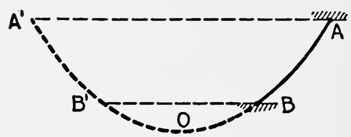


FIG. 199

beyond B (Fig. 199). Then we determine the length of the arcs AOA' and BOB' by means of equation (5), §1, and finally the length l_1 of the arc AB from

$$l_1 = \tfrac{1}{2}AOA' + \tfrac{1}{2}BOB' \text{ for Fig. 198, or } l_1 = \tfrac{1}{2}AOA' - \tfrac{1}{2}BOB' \text{ for Fig. 199.}$$

For example take $a = 800$ feet, $b = 300$ feet, and $f_1 = 200$ feet. Let x_0 and y_0 = the coördinates of the vertex. From equation (7)

$$x_0 = \frac{(300 + 4 \times 200)\,800}{8 \times 200} = 550, \text{ and } y_0 = \frac{(300 + 4 \times 200)^2}{16 \times 200} = 378.5.$$

Hence the cable hangs as shown in Fig. 200. The length $AA' = 1348.6$ feet according to (5), ($a = 1100$ and $n = 378.5 \div 1100$); the length $BB' = 530.9$

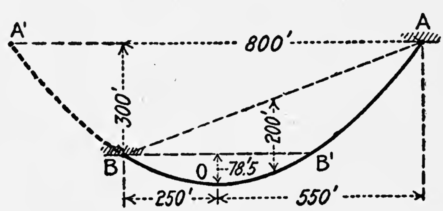


FIG. 200

feet according to (5), ($a = 500$ and $n = 78.5 \div 500$). Hence $AB = \frac{1}{2} \times 1348.6 + \frac{1}{2} \times 530.9 = 939.8$ feet.

26. Catenary Cable.

§1. SYMMETRICAL CASE.—A chain or flexible cable suspended from two points and hanging freely under its own weight or a load uniformly distributed along its length assumes a curve called (common) catenary. Let A and B (Fig. 201) be the points of suspension of such a cable, C its lowest point, Q any other point of the cable, s = the length CQ, H = tension at C, T = tension at Q, ϕ = slope of the curve at Q, w = weight of load per unit length of cable, and c = a length so that $cw = H$ or $c = H/w$. The forces acting on CQ are H, T, and ws. Since they are in equilibrium, $T \cos \phi = H$, and $T \sin \phi = ws$; hence $\tan \phi = ws/H = s/c$. But $\tan \phi = dy/dx$, therefore

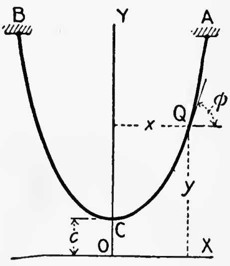


FIG. 201

$$dy/dx = s/c. \tag{1}$$

Now since $ds^2 = dx^2 + dy^2$, $(ds/dy)^2 = (dx/dy)^2 + 1$ and $(ds/dx)^2 = 1 + (dy/dx)^2$; also

$$\left(\frac{ds}{dy}\right)^2 = \frac{c^2}{s^2} + 1 = \frac{c^2 + s^2}{s^2} \quad \text{and} \quad \left(\frac{ds}{dx}\right)^2 = 1 + \frac{s^2}{c^2} = \frac{c^2 + s^2}{c^2}. \tag{2}$$

Integrating the first one of these equations we get $y = (c^2 + s^2)^{\frac{1}{2}} + A$ where A is a constant of integration. But $y = c$ where $s = 0$, therefore $A = 0$, and hence

$$y^2 = c^2 + s^2, \qquad \text{or} \qquad s^2 = y^2 - c^2. \tag{3}$$

Integrating the second differential equation we get

$$x = c \log_e\left[\frac{s}{c} \pm \sqrt{\left(\frac{s}{c}\right)^2 + 1}\right] = c \sinh^{-1}\frac{s}{c}, \tag{4}$$

the constant of integration being zero ($x = 0$ when $s = 0$). From (3)

$$s = \tfrac{1}{2} c\,(e^{x/c} - e^{-x/c}) = c \sinh\frac{x}{c}. \tag{5}$$

To obtain the cartesian equation of the catenary we combine (3) and (4) or (3) and (5) so as to eliminate s. Thus squaring (5) and comparing with (3) we get easily

$$y = \tfrac{1}{2} c\,(e^{x/c} + e^{-x/c}) = c \cosh\frac{x}{c}, \tag{6}$$

or

$$x = c \log_e\left[\frac{y}{c} \pm \sqrt{\left(\frac{y}{c}\right)^2 - 1}\right] = c \cosh^{-1}\frac{y}{c}. \tag{7}$$

The slope angle ϕ at any point in terms of the coördinates of the point (x,y,s) is given by

$$\tan \phi = s/c = \tfrac{1}{2}\,(e^{x/c} - e^{-x/c}) = \sinh\,(x/c). \tag{8}$$

See equations (1) and (5). And from equations (2) and (3), we get

$$\sin\phi = s/y \quad \text{and} \quad \cos\phi = c/y. \tag{9}$$

It follows from the equilibrium equation $T \sin\phi = ws$ and (9), that

$$T = wy, \tag{10}$$

that is, the tension at any point Q equals the weight of a length of cable reaching from Q to the directrix OX. Hence T increases from C to A. According to the definition of c

$$H = wc. \tag{11}$$

In passing, it may be noted that since $T\cos\phi = H$, the horizontal component of the tension at any point $Q = wc$, constant for a given suspended cable.

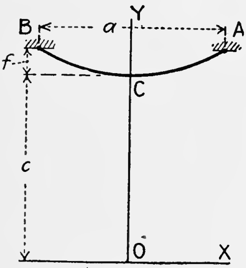


FIG. 202

As in the preceding article, let a = span AB (Fig. 202), f = sag, and l = length of cable ACB. Any two of the three dimensions a, l, and f determine the catenary, as will be shown presently. For the point A, $x = \frac{1}{2}a$, $y = f + c$, and $s = \frac{1}{2}l$. Hence substituting in equations (3), (4), and (6) respectively we get

$$(f+c)^2 = c^2 + \tfrac{1}{4}l^2, \quad \text{or} \quad c/f = \tfrac{1}{8}(l/f)^2 - \tfrac{1}{2}. \tag{3'}$$

$$\tfrac{1}{2}a = c\sinh^{-1}(\tfrac{1}{2}l/c), \quad \text{or} \quad \tfrac{1}{2}a/c = \sinh^{-1}(\tfrac{1}{2}l/c). \tag{4'}$$

and

$$f + c = c\cosh(\tfrac{1}{2}a/c), \quad \text{or} \quad 1 + (f/c) = \cosh(\tfrac{1}{2}a/c). \tag{6'}$$

When l and f are given (3′) gives c, and then a may be gotten from (4′) or (6′). When a and f are given (6′) determines c but the equation cannot be solved directly, — only by trial or by some similar method; having thus determined c, l may be gotten from (3′) or (4′). When a and l are given, (4′) determines c (solution by trial), and then f may be gotten from (3′) or (6′).

Inasmuch as these trial methods are generally long, computations on some catenary problems may be facilitated by means of diagrams. In Fig. 203 the curves marked A give the relation between f/a and l/a for values of f/a from 0 to 0.5 and (corresponding) values of l/a from 1 to about 1.50. For example, let a = 800 feet and f = 160 feet. Then $f/a = 0.20$, and the corresponding ordinate (over $f/a = 0.20$) to curve A reads 1.10; hence $l/a = 1.10$, and $l = 800 \times 1.10 = 880$ feet (length of cable).

Most practical catenary problems involve the strength of the wire or cable and the load per unit length of wire. For such problems we have, in addition to (3′), (4′) and (6′),

$$T = w(f + c), \quad \text{or} \quad T/w = f + c, \tag{11'}$$

where T = the greatest tension (at the points of support), which should of course not exceed the strength of the wire. Most of these problems can be solved by trial only, unless a diagram is available. For example, given the strength T of a wire, the load per unit length w, and the span a; required the proper length of wire l. Here

$$T/wa = f/a + c/a. \tag{11''}$$

This equation and (6′) contain only two unknown quantities f and c, and the two equations determine f and c. But they can be solved only by trial. After f and c have been ascertained, then l may be computed from (3′) directly. The curves marked B in Fig. 203 show the relation between f/a and T/wa.

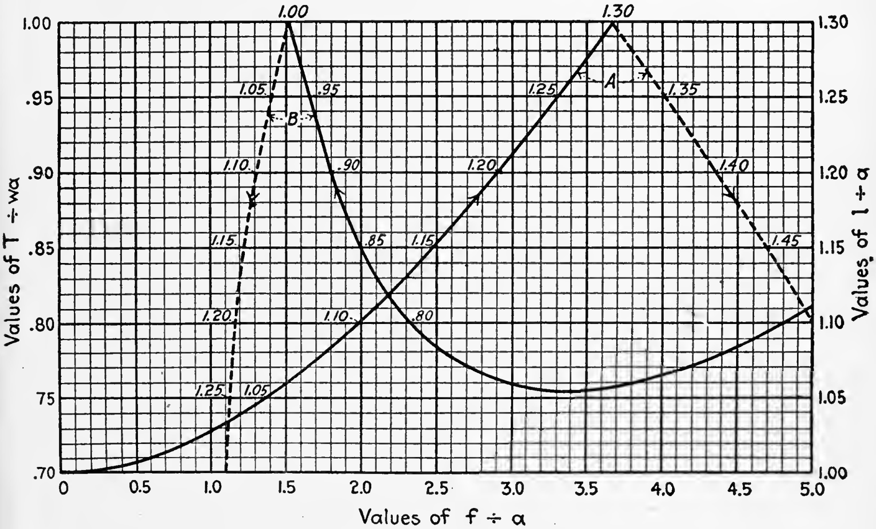


FIG. 203

Thus if $f/a = 0.20$, as in the preceding illustration, then the corresponding ordinate to curve B (over $f/a = 0.20$) reads 0.85; hence $T/wa = 0.85$ and $T = 0.85\,wa$*.

* Fig. 203 was prepared from plate II of Mr. Thomas' paper mentioned in the footnote at the end of this chapter. (For cases of relatively small sag ratios, see that plate.) The curves marked A might be prepared as follows: (i) Assume different values of l/f; (ii) compute the corresponding values of c/f by means of (3′); (iii) compute values of l/c from $l/c = (l/f) \div (c/f)$; (iv) compute values of a/c from (4′); (v) compute values of l/a from $l/a = (l/c) \div (a/c)$; (vi) compute values of f/a from $f/a = (l/a) \div (l/f)$. Finally plot values of f/a and l/a. (The adjoining schedule gives the computed values for $l/f = 5$ as an illustration.)

i	ii	iii	iv	v	vi	vii	viii
l/f	c/f	l/c	a/c	l/a	f/a	c/a	T/wa
5	2.6250	1.9048	1.6946	1.1237	0.2247	0.5900	0.8147

To plot curves B we would continue the computation, first computing values of c/a from the values of a/c already obtained, and then values of T/wa from $T/wa = f/a + c/a$. Finally values in columns vi and viii of the schedule when plotted furnish the desired graph.

§ 2. UNSYMMETRICAL CASE (points of suspension not on same level). — The cable uniformly loaded along its length hangs in an arc of a catenary. The vertex C may be on the cable (between the points of suspension A and B) as in Fig. 204, or beyond the lower point of suspension as in Fig. 205. In either

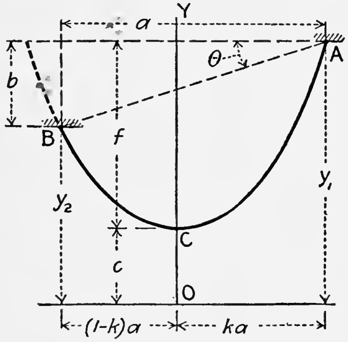


FIG. 204

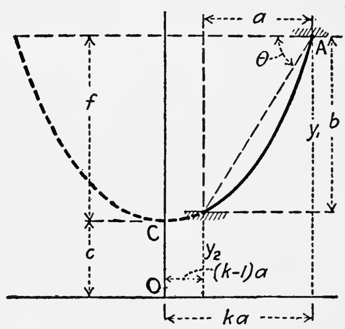


FIG. 205

figure, a = the horizontal distance between A and B, b = the vertical distance, θ = angle between AB and the horizontal, f = sag or vertical distance AC, and l = arc AB. Most problems in this case as in the symmetrical case can be solved only by a trial method; hence diagrams are practically necessary in this case also.

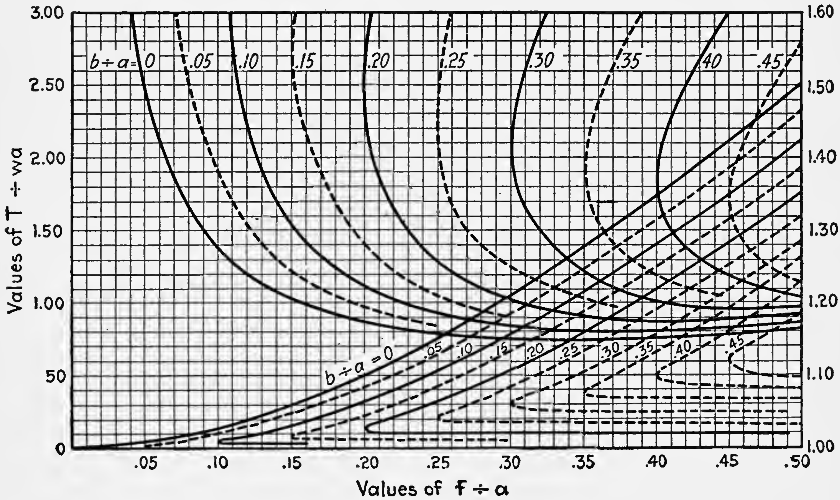


FIG. 206

In Fig. 206 there are two groups of curves relating to this unsymmetrical case; the group occupying the lower right-hand portion consists of graphs showing the relation between f/a and l/a (values at right-hand margin) for ten values of b/a (slope of AB, Figs. 204 and 205). The other group consists of graphs showing the relation between f/a and T/wa (values at left-hand margin) for the same ten slopes (T = tension at higher point of support and w = weight of cable per unit length). To illustrate, let a = 200 feet, b = 40 feet, l = 240 feet, and w = 2 pounds per foot. On the curve for $b/a = 0.20$ in the lower group, we find the point whose ordinate $l/a = 1.20$ and note that the abscissa of that point is $f/a = 0.385$. Hence $f = 200 \times 0.385 = 77$ feet. On the curve for $b/a = 0.20$ of the upper group, we find the point whose abscissa is 0.385 and note that its ordinate $T/wa = 0.90$. Hence $T = 2 \times 200 \times 0.90 = 360$ pounds.*

* Fig. 206 was made from certain of the (more extensive) figures in Mr. Robertson's paper mentioned in the footnote at the end of this chapter. The following is an explanation of a method for the construction of such a figure. Let l_1 = arc AC and l_2 = arc BC (Figs. 204 and 205); also let y_1 and y_2 = ordinates of A and B respectively, and ka and $(1 - k)a$ = the abscissas of A and B. Then for A, $x = ka$, $y = y_1$ and $s = l_1$; for B, $x = (k - 1)a$, $y = y_2$. and $s = l_2$. Hence, substituting in equations (3) and (6) of §1, we get

$$\left(\frac{l_1}{c}\right)^2 = \left(\frac{y_1}{c}\right)^2 - 1, \quad \text{and} \quad \left(\frac{l_2}{c}\right)^2 = \left(\frac{y_2}{c}\right)^2 - 1, \tag{3'}$$

$$\frac{y_1}{c} = \cosh k\frac{a}{c}, \quad \text{and} \quad \frac{y_2}{c} = \cosh (k - 1)\frac{a}{c}. \tag{6'}$$

At the higher point of support A (Fig. 203 or 204), $y = y_1$; hence according to equation (11) §1, the tension at that point $= wy_1$, and

$$\frac{T}{wa} = \frac{y_1}{a} = \left(\frac{y_1}{c}\right) \div \left(\frac{a}{c}\right).$$

This equation, (3′) and (6′), constitute the basis of the method. We first assume a value of k, say 0.6, and different values of a/c (say 0.02, 0.04, etc.); then (i) compute values of y_1/c and y_2/c from (6′) corresponding to those values of a/c (and $k = 0.6$); (ii) compute the values of l_1/c and l_2/c from (3′) to correspond; (iii) compute values of l/c from $l/c = (l_1/c) + (l_2/c)$; (iv) compute values of f/c from $f/c = (y_1/c) - 1$. Finally compute: b/a from $b/a = [(y_1/c) - (y_2/c)] \div (a/c)$ (see Figs. 203 and 204); l/a from $l/a = (l/c) \div (a/c)$; f/a from $f/a = (f/c) \div (a/c)$; and T/wa from $T/wa = (y_1/c) \div (a/c)$. (See schedule.)

i	ii	iii	iv	v	vi	vii	viii	ix	x	xi
a/c	y_1/c	y_2/c	l_1/c	l_2/c	l/c	f/c	b/a	l/a	f/a	T/wa

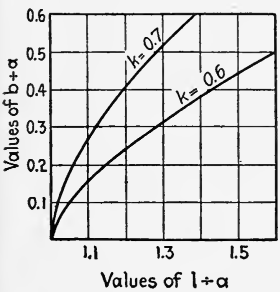


FIG. 207

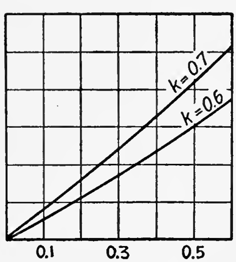


FIG. 208

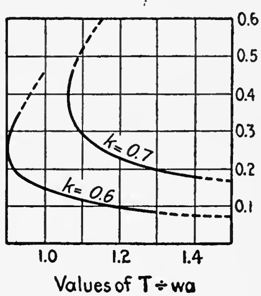


FIG. 209

We now plot values of b/a and l/a as in Fig. 207, and get a curve (marked $k = 0.6$); plot values of b/a and f/a as in Fig. 208, and get a curve (marked $k = 0.6$); plot values of b/a and T/wa as in Fig. 209, and get a curve (marked $k = 0.6$). In a similar manner we take

§3. APPROXIMATE SOLUTIONS OF CATENARY PROBLEMS. — If the cable is suspended from two points at the same level and the sag is small compared to the span so that the slope of the catenary is small at every point, then the load (weight) per unit length of span is nearly constant and equal to the weight of the cable per unit length. Hence the catenary coincides very nearly with a parabola of the given span and sag, and the formulas and results of the preceding article §1 (symmetrical case) may be applied to the case here under consideration without serious error.

That the catenary agrees closely with a parabola can be shown otherwise as follows: Expanding the exponentials in the equation of the catenary, (6) §1, we get

$$e^{x/c} = 1 + \frac{x}{c} + \frac{x^2}{2c^2} + \frac{x^3}{3c^3} + \cdots \quad \text{and} \quad e^{-x/c} = 1 - \frac{x}{c} + \frac{x^2}{2c^2} - \frac{x^3}{3c^3} + \cdots;$$

hence the equation of the catenary may be written

$$y = \frac{c}{2}\left(2 + \frac{x^2}{c^2} + \cdots\right).$$

Neglecting the higher powers of the small quantity $\frac{x}{c}$, we have as close approximations

$$y = c + x^2/2c, \quad \text{or} \quad x^2 = 2c(y - c).$$

These are equations of a parabola whose axis coincides with the y coördinate axis and vertex c distant above the origin of coördinates.

If the supports A and B are not at the same level (Fig. 196) and the sag f of the cable is small compared to the distance between the points of support, then the slope of the catenary is nearly constant and the load per unit length of horizontal distance is nearly constant ($w \sec\theta$, where w = weight of cable per unit length, and θ = angle BAX). Hence the catenary coincides very

$k = 0.7$ say, and make computations i, ii, iii, etc., as described; then plot three more curves (Figs. 207, 208 and 209). Then we repeat for still other values of k.

From the three sets of curves (Figs. 207, 208, and 209) we pick out sets of corresponding values of l/a, f/a and T/wa for the several values of b/a. Thus for $b/a = 0.2$, we find the adjoining tabulated values from the curves.

$b/a = 0.20$			
k	l/a	f/a	T/wa
0.6	1.15	0.35	0.91
0.7	1.06	0.25	1.30
etc.	etc.	etc.	etc.

Then we plot these values of f/a and l/a and get the curve marked $b/a = 0.20$ in lower group, Fig. 206; also we plot values of f/a and T/wa and get the curve marked $b/a = 0.20$ in the upper group of Fig. 206. In a similar way we tabulate and plot for other values of b/a (0.30, 0.40, etc.) and complete the diagram (Fig. 206).

nearly with a parabolic arc of the given (oblique) chord AB and sag f_1, and the formulas of the preceding article §2 (unsymmetrical case) may be applied to the cable under consideration without serious error, it being understood that w of §2 = (weight of cable per unit length) $\times \sec\theta$.

27. Cables with Concentrated Loads

§1. WEIGHT OF CABLE NEGLIGIBLE. — Let Fig. 210 represent a cable ACB suspended from two given points A and B, C being a given point from which a load is suspended. If the cable can be "laid out" in a drawing, the tensions in AC and BC can be determined easily by constructing the force triangle $PQRP$ for the load W and the two tensions. $PQ = W$ according to some convenient scale; PR and QR (parallel to AC and BC respectively) represent the tensions in AC and BC. Or, if one wishes to avoid graphical methods the two tensions (T_1 and T_2) may be computed by solving the triangle algebraically. Such solution would give

$$T_1 = W\cos\beta/\sin(\alpha+\beta) \qquad \text{and} \qquad T_2 = W\cos\alpha/\sin(\alpha+\beta),$$

where α and β are the angles which AC and BC make with the horizontal (Fig. 210).

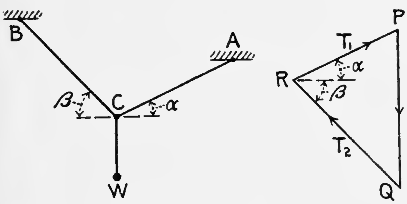


FIG. 210

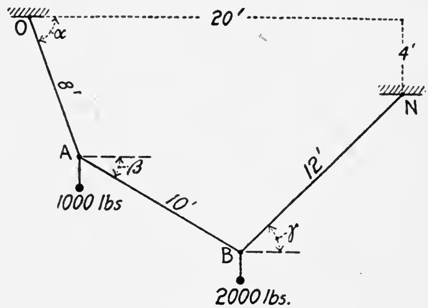


FIG. 211

When several bodies are suspended from given points on the cable, the cable takes up a definite position, but it is not easy to determine the slopes of the segments of the cable and the tensions. The difficulty lies in the algebraic computation. For example, consider the case represented in Fig. 211. The given data are shown in the figure; the lengths are drawn to scale, but the inclinations of the segments of the cable may not be correct, being unknown as yet. Let the inclinations be called α, β, and γ as shown; and T_1, T_2, and T_3 = the tensions in OA, AB, and BN respectively. At each point of suspension of a load (A or B) there are three forces acting; at A, the load 1000 pounds, T_1, and T_2, and at B, the load 2000 pounds, T_2, and T_3. Consideration of forces at A and of those at B gives respectively

$$T_1\cos\alpha = T_2\cos\beta \qquad \text{and} \qquad T_1\sin\alpha - T_2\sin\beta = 1000$$
$$T_2\cos\beta = T_3\cos\gamma \qquad \text{and} \qquad T_2\sin\beta + T_3\sin\gamma = 2000.$$

It is plain from the geometry of the figure that

$$8 \cos \alpha + 10 \cos \beta + 12 \cos \gamma = 20, \quad \text{and} \quad 8 \sin \alpha + 10 \sin \beta - 12 \sin \gamma = 4.$$

These six equations may be solved simultaneously for the six unknowns (T_1, T_2, T_3, α, β, and γ); the actual solution is not simple. For similar cases with more than two loads, the work of solving the equations increases rapidly with increasing number of loads.

Suspended loads can be chosen so as to hold points of suspension (A, B, etc.) in certain definite positions. For instance let it be required to determine W_1, W_2, etc., to hold a cable in the position shown in Fig. 212. We

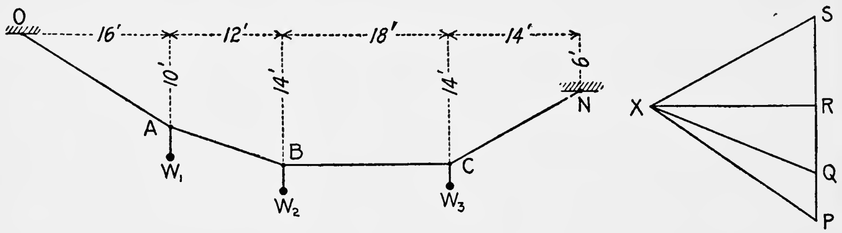


FIG. 212

may assume any value for one of the weights and then compute the values of the others. Thus taking $W_1 = 1000$ pounds say, then we compute the tension in AB from a force triangle for the three forces acting at A. $PQXP$ is such a triangle, where $PQ = 1000$ pounds (according to any convenient scale) and PX and QX are parallel to OA and AB respectively; then XQ represents the tension in AB. The next step is to find the value of W_2 which corresponds to such tension in AB; so we draw a force triangle for the three forces acting at B one of which is the determined tension in AB. This force triangle is $XQRX$, and so QR represents W_2 and RX represents the tension in BC. Finally, we draw the force triangle $XRSX$ for the three forces acting at C, one of which is the determined tension in BC, and thus find that W_3 is represented by RS. Obviously any three weights W_1, W_2, and W_3 in the proportion of PQ, QR, and RS would hold the cable in the specified position.

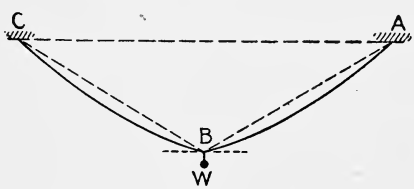


FIG. 213

§2. WEIGHT OF CABLE NOT NEGLIGIBLE. — It is assumed in the following discussion that the cable segments are quite flat so that they are practically parabolic arcs (see preceding article §3). Then the weight of any segment of the cable is practically the same as the weight of a length equal to the chord of the segment. Let ABC (Fig. 213) be a cable supported at A and C, a load being suspended from the cable at its middle point B. Given the span $AC = 2a$, the length of the cable $= 2l$, the weight of the cable per unit length $= w$, and the load $= W$; required the sag (depth of B below AC) and the tension at A. This (apparently) simple problem is determinate but practically unsolvable on account of algebraic difficulties. The equations are easily set up. Thus let $a_1 =$ the (unknown) length of chord AB, $f_1 =$ the sag of the cable below the chord as in Fig. 196, $S =$ the tension and $\beta =$ the slope of the cable at B (Fig. 196). Then according to equations (10) and (6) respectively of Art. 25, § 2

$$\frac{l}{a} = l + \frac{8}{3}\left(\frac{a}{a_1}\right)^2\left(\frac{f_1}{a_1}\right)^2 \tag{1}$$

and

$$\tan\beta = \frac{\sqrt{a_1^2 - a^2}}{a} - \frac{4f_1}{a}. \tag{2}$$

According to the footnote, on page 105.

$$S\cos\beta = (wa_1/a)\,a^2/8f_1. \tag{3}$$

From the three forces acting at B (W, S, and S), it is plain that

$$2\,S\sin\beta = W. \tag{4}$$

These four equations determine the unknowns appearing in them, a_1, f_1, S, and β. Thus by division, the last two give $\tan\beta = 4\,Wf_1/wa^2$; equating the two values of $\tan\beta$ and transforming, we get

$$\frac{a}{a_1}\frac{4\,W}{wa}\frac{f_1}{a_1} = \sqrt{1 - \left(\frac{a}{a_1}\right)^2} - 4\frac{f_1}{a_1}. \tag{5}$$

This equation and (1) contain only two unknowns, the ratios (a/a_1) and (f_1/a_1), and the equations determine the ratios. Supposing the ratios determined we may find a_1 since a is given, and then f_1. Exact simultaneous solution of equations (1) and (5) is impossible, but each equation may be graphed and then the coördinates of their intersection would be the desired values of a/a_1 and f_1/a_1.

The converse of the preceding problem is much simpler. It is this: Given the span $AC = 2\,a$, the chord $AB = a_1$, the sag f_1, and the weight of the cable per unit length w; required the load W. Equations (2), (3), and (4) give in succession β, S, and W. Equation (1) gives the length l.*

* For other information on the subjects of this chapter, particularly as related to electric transmission lines, see the following: University of Illinois Bulletin, No. 11 (1912) by A. Gruell; Transactions American Institute of Electric Engineers, Vol. 30 (1911), papers by Wm. L. Robertson, Percy H. Thomas, and Harold Pender and H. F. Thompson. These papers contain extensive tables and diagrams, and discuss effects of temperature changes.

DYNAMICS

DYNAMICS

CHAPTER VII

RECTILINEAR MOTION

28. Velocity and Acceleration

§ 1. VELOCITY. — When a point moves so that it traverses or describes equal distances in all equal intervals of time then it is said to move uniformly, and we call the motion uniform. All other motions we call nonuniform. By *velocity* of a moving point is meant the time-rate at which the point is moving, or describing distance. To express the magnitude of any velocity we must of course compare that velocity to some particular velocity as a standard or unit. Any velocity — that of light, for example — might be taken as standard; but it is more convenient to take the velocity of a point moving uniformly and describing a unit of length in a unit of time for a standard. Thus, we use the foot per second, mile per hour, etc. The word *per* in these names of velocity-units is quite commonly replaced by the solidus sign /; thus, foot per second, mile per hour, etc., are abbreviated to ft/sec, mi/hr, etc.*

In any uniform motion the velocity may be computed by dividing the distance traversed in any interval of time by the interval. Thus, if $v =$ the velocity, $\Delta s =$ the distance traversed, and $\Delta t =$ the interval of time, then

$$v = \Delta s/\Delta t. \tag{1}$$

In any nonuniform motion the rate of moving is not constant but changes continuously, as we all realize. Not all, however, have a clear notion of the value of the rate, or velocity, at a particular instant of the motion. To bring this matter up definitely, let us consider the following example: — In a certain launching, the ship moved through the distances given after s in the adjoining schedule in the times given after t.

$t =$ 0	2	4	6	8	10	12	14	16 seconds;
$s =$ 0	3.4	9.3	17.3	27.4	39.6	53.4	69.4	88.0 feet.

Any displacement divided by the time required for that displacement we regard as the *average velocity* for that time; thus, in the last 8 seconds the displacement is 60.6 feet, and 60.6 ÷ 8 = 7.57 ft/sec is the average velocity for that time. (Obviously a constant velocity of this value would produce a displacement of 60.6 feet in 8 seconds.) In the adjoining table we have listed

* For dimensions of a unit velocity, see Appendix A.

Δt (secs.)	Δs (ft.)	v_a (ft/sec)	Δv_a (ft/sec)
8 to 16=8	60.6	7.57	0.57
8 to 14=6	42.0	7.00	0.50
8 to 12=4	26.0	6.50	0.40
8 to 10=2	12.2	6.10	

the displacements (under Δs) for the intervals 8 to 16, 8 to 14, etc. (under Δt); and also the average velocities (under v_a) for these intervals, respectively, and the decrements in the average velocity (under Δv_a). Apparently, the average velocity for the intervals 8 to 9, 8 to $8\frac{1}{2}$, 8 to $8\frac{1}{4}$, etc., continues to decrease, approaching a definite limit as the interval of time approaches zero. The column of decrements (0.57, 0.50, 0.40) suggests that the next decrement is about 0.25, and hence the limit about $6.10 - 0.25 = 5.85$ feet per second. The exact limit is the value of the rate at which the ship is moving at the time 8 seconds.

Summarizing now:—Let Δs = distance traversed in any interval Δt, and v_a = average velocity for that interval; then in any kind of rectilinear motion

$$v_a = \Delta s/\Delta t. \tag{2}$$

The value of the velocity at a particular instant of the interval is the limiting value of the average velocity for the interval, as the interval is taken smaller and smaller but always including the particular instant. In the calculus notation, this limit is ds/dt; hence if v = velocity, then

$$v = ds/dt. \tag{3}$$

Here s means the (varying) distance of the moving point from any fixed point in the path.

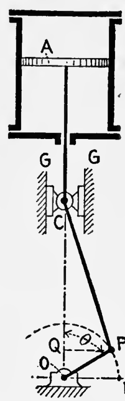


FIG. 218

Formula (3) can be used for finding the value of v in any motion in which the relation between s and t is known. Thus, suppose that a point is known to move so that its distance (in feet) from the starting point always equals four times the square of the time (in minutes) from starting, that is, $s = 4\,t^2$; then $v = ds/dt = 8\,t$. This is the general formula for v in this motion; that is, the formula holds for all instants and all positions of the moving point. Thus, at the instant $t = 2$ minutes, $v = 8 \times 2 = 16$ feet per minute; at the position $s = 100$ feet, $t = \sqrt{100 \div 4} = 5$ minutes and $v = 8 \times 5 = 40$ feet per minute.

For another example of the use of formula (3) we consider a "crank and connecting-rod mechanism" (Fig. 218). OP is the crank, mounted on its shaft at O, PC is the connecting rod, C is the crosshead; (A is a piston and AC the piston rod). When the crank is rotated the crosshead is constrained to move in a straight line by the guides G. We will now find a general formula for the velocity of the crosshead (and piston) when the crank rotates uniformly. Let $r =$ the length of crank, $l =$ length of connecting rod, $c = r/l$, $n =$ number of revolutions of the crank per unit time (assumed constant), $\omega =$ angle in radians described by crank per unit time ($\omega = 2\pi n$), $s =$ varying distance of the crosshead from its highest position, $\theta =$ the "crank angle" QOP, and $t =$ time required for the crank to describe the angle θ ($= \omega t = 2\pi n t$). Obviously, there is a definite relation between s and θ (or t), and this relation we need in order to get ds/dt or v. When the crosshead is in its highest position, its distance from O equals $l + r$; hence for any position, $s = (l + r) - CQ \mp OQ$, $\mp OQ$ according as the crank OP is above or below OX. Now $CQ = (l^2 - r^2 \sin^2\theta)^{\frac{1}{2}} = l\,(1 - c^2 \sin^2\theta)^{\frac{1}{2}}$, and $OQ = \pm r \cos\theta$; hence

$$s = (l + r) - l\,(1 - c^2 \sin^2\theta)^{\frac{1}{2}} - r\cos\theta.$$

Differentiating the expression for s with respect to t, we get ds/dt, or v; and remembering that $d\theta/dt = \omega$, we can easily reduce the result to

$$v = r\omega\left(\sin\theta + \frac{c \sin 2\theta}{2\,(1 - c^2\sin^2\theta)^{\frac{1}{2}}}\right).$$

From this general formula we can get the value of v for any particular case. Thus, let $r = 10$ inches, $l = 30$ inches (then $c = \frac{1}{3}$), and $n = 100$ revolutions per minute ($\omega = 2\pi\,100 = 628$ radians per minute). When the crank is at OP_0 say, $\theta = 90^\circ$ and the formula gives $v = 6280$ inches per minute $= 523$ feet per minute.

The expression ds/dt in equation (3) may be positive or negative; therefore v must be regarded as having the same sign that ds/dt has. Now ds/dt is positive when s increases algebraically, and ds/dt is negative when s decreases algebraically; hence the sign of the velocity of a moving point at any instant is positive or negative according as s is increasing or decreasing then, that is the sign is the same as that of the direction in which the point is moving then.

When the mathematical relation between s and t is unknown, then equation (3) cannot be used to determine the velocity at a particular instant. But if the displacements of the moving point are known for a number of known intervals beginning or terminating at the instant in question, then a fair approximation to the desired velocity can be obtained from the values of the average velocity for those intervals as explained in the launching illustration preceding. One may determine the limit of the average velocities approximately by graphical methods. Thus, in Fig. 219 we have plotted the average velocities of the launching example in a manner which is obvious and then joined the plotted points by a smooth curve; this curve was extended, as seemed best, to the vertical through point 8. The ordinate $8\,A$ represents approximately the limit sought, that is the velocity at the 8th second. Another graphical method is explained in § 2 of the following article.

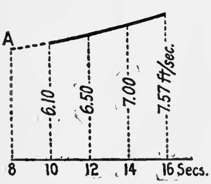


FIG. 219

§ 2. ACCELERATION. — A nonuniform motion is said to be accelerated, and the moving point is said to have acceleration. If the velocity changes uniformly, that is by equal amounts in all equal intervals of time, the motion is uniformly accelerated; if the velocity does not change uniformly, then the motion is nonuniformly accelerated.

By acceleration of a moving point is meant the rate at which its velocity is changing. To express the magnitude of any acceleration we must compare that acceleration to some particular acceleration as a standard or unit. Any rate of velocity-change — that of a freely falling body, for example — might be taken as a unit of acceleration but it is more convenient to take the acceleration of a point whose velocity changes uniformly by one unit (of velocity) in one unit of time. Thus, we have the foot-per-second per second, the mile-per-hour per second, etc. And, abbreviating the word per as before, these would be written ft/sec/sec (also written ft/sec ^{2}), mi/hr/sec, etc.*

In a *uniformly accelerated motion* (u.a.m.), the acceleration may be computed by dividing the velocity-change which takes place in any interval of time by the length of the interval. Thus, if a = acceleration, Δv = the velocity-change and Δt = the interval, then

$$a = \Delta v/\Delta t. \tag{4}$$

In a *nonuniformly accelerated motion* the rate of change of the velocity is not constant but it varies continuously from instant to instant. To arrive at a definite notion of the value of the rate or acceleration at a particular instant, let us consider an example. The adjoining schedule gives values of velocity and time taken from a "starting test" of an electric street railway car.

$t = 0$	1	2	3	4	5	6	7	8	9	10	seconds;
$v = 0$	2.8	5.3	7.7	9.9	11.9	13.7	15.2	16.4	17.3	18.0	miles per hour.

Any velocity-change divided by the time required for the change we regard as the *average acceleration* for that time; thus, during the first six seconds the velocity-change is 13.7 miles per hour, and 13.7 ÷ 6 = 2.28 miles per hour per second is the average acceleration for the first six seconds. (Obviously a uniform acceleration of this value would produce in six seconds a velocity-change of 13.7 miles per hour.)

Δt (secs.)	Δv (mi./hr.)	a_a (mi/hr/sec)	Δa_a (mi/hr/sec)
0 to 6=6	13.7	2.28	
1 to 6=5	10.9	2.18	0.10
2 to 6=4	8.4	2.10	0.08
3 to 6=3	6.0	2.00	0.10
4 to 6=2	3.8	1.90	0.10
5 to 6=1	1.8	1.80	0.10

* For dimensions of a unit acceleration, see Appendix A.

In the adjoining table we have listed the velocity-changes (under Δv) for the intervals 0 to 6, 1 to 6, 2 to 6, etc.; and also the average acceleration (under a_a) for the same intervals, and the decrements in the average acceleration under Δa_a. Obviously the average acceleration for the intervals $5\frac{1}{2}$ to 6, $5\frac{3}{4}$ to 6, etc., continues to decrease, approaching a definite limit as the interval approaches zero. The series of decrements, 0.10, 0.08, 0.10, 0.10, 0.10, suggests that the next (full) decrement is about 0.10, and hence the limit about $1.80 - 0.10 = 1.70$ miles per hour per second. The exact limit is the value of the rate of change of velocity, or the acceleration, when $t = 6$ seconds.

Summarizing now: — Let $\Delta v =$ the velocity-change in any interval of time Δt, and $a_a =$ average acceleration for that interval, then in any kind of rectilinear motion

$$a_a = \Delta v/\Delta t. \tag{5}$$

The true value of the acceleration at a particular instant of the interval is the limiting value of the average acceleration as the time interval is taken smaller and smaller but always including the particular instant; or in the calculus notation

$$a = dv/dt = d^2s/dt^2. \tag{6}$$

Formulas (6), respectively, can be used for finding the value of a in any rectilinear motion if the relation between v and t or s and t are known. Thus suppose that a point is known to move in a straight line so that the velocity (in miles per hour) always equals one-tenth of the square of the time (in seconds) from the start, that is $v = 0.1\,t^2$; then $a = dv/dt = 0.2\,t$. This is the general formula for a in this motion; for instance, at 3 seconds after starting $a = 0.6$ miles per hour per second.

For another example of the use of equation (6), we consider the motion of the crosshead of the crank and connecting-rod mechanism described in § 1. There we found that

$$v = r\omega\left(\sin\theta + \frac{c\sin 2\theta}{2(1 - c^2\sin^2\theta)^{\frac{1}{2}}}\right).$$

Differentiating this with respect to t and remembering that ω is constant, we get dv/dt or

$$a = r\omega^2\left(\cos\theta + \frac{c\cos 2\theta + c^3\sin^4\theta}{(1 - c^2\sin^2\theta)^{\frac{3}{2}}}\right).$$

From this general formula, we can get the value of a for any special case. Thus as in § 1, let $r = 10$ inches, $l = 30$ inches (then $c = \frac{1}{3}$), and $n = 100$ revolutions per minute ($\omega = 2\pi\,100 = 628$ radians per minute). When the crank is at OP_0 (Fig. 218), then $\theta = 90$ and the formula gives $a = -2220$ inches per minute per minute $= -185$ feet per minute per minute. The negative sign means that the acceleration is upward (negative), that is, the velocity is decreasing (at $\theta = 90°$).

The expression dv/dt, equation (6), may be positive or negative; therefore a must be regarded as having the same sign as has dv/dt. Now dv/dt is positive when the velocity increases algebraically, and dv/dt is negative when the velocity decreases algebraically; therefore the sign of the acceleration of a moving point at any instant is positive or negative according as the velocity is increasing or decreasing (algebraically) then.

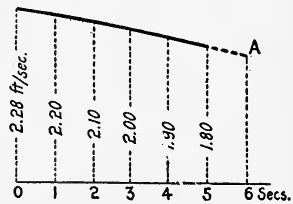


FIG. 220

When the mathematical relation between v and t (and s and t) are unknown, then equation (6) cannot be used to determine the acceleration at a particular instant. But if the values of the velocity are known at a number of known instants near the instant in question, then a fair approximation to the acceleration desired can be obtained from the values of the average acceleration for intervals beginning or terminating at the instant in question, as explained in the car-starting example preceding. Fig. 220 shows a construction for determining the limit of the average acceleration in the example referred to. The ordinate $6A$ represents the limit approximately. Another graphic method is explained in the next article under § 2.

Note on Rate of a Scalar Quantity. — The foregoing explanations of two particular rates (velocity and acceleration) will now be generalized so that hereafter we will not need to derive the expressions or formulas for other rates which will come up for discussion.

By a *scalar quantity* is meant one which has magnitude only, not direction also. An amount of money, the volume of a thing, the population of a city, etc., are scalar quantities. Let x and y denote two scalar quantities which are related to each other so that any change in one produces a change in the other. If all equal changes in x produce equal changes in y, then y is said to vary uniformly with respect to x and y is called a uniform variable. If all equal changes in x produce unequal changes in y, then y is said to vary nonuniformly and y is called a nonuniform variable.

If *y is a uniform scalar,* then the graph which represents the relation between x and y is a straight line obviously, as for example in Fig. 221 where y_1 and y_2 respectively denote values of y corresponding to x_1 and x_2 (values of x). The meaning of "rate of y with respect to x" or "x-rate of y" is quite generally understood; it is the change in y per unit change in x. The value of the rate is computed by dividing any change in y by the corresponding change in x. Thus, if Δx and Δy = corresponding changes in x and y ($x_2 - x_1$ and $y_2 - y_1$), and r = x-rate of y, then

$$r = \Delta y/\Delta x.$$

Evidently r is the same for all values of x, that is, the rate of a uniform scalar is constant.

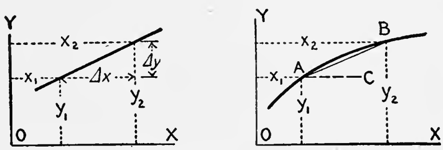


FIG. 221

If *y is a nonuniform scalar* then the graph which represents the relation between x and y is a curved line, as for example in Fig. 221 where y_1 and y_2 represent values of y which correspond to x_1 and x_2 respectively. Any change in y divided by the corresponding change in x

is commonly called the average rate of y with respect to x for the range $x_2 - x_1$. Thus, if r_a = average x-rate of y for the range Δx ($= x_2 - x_1$) in x, then

$$r_a = \Delta y/\Delta x.$$

The average rate is represented by the slope of the chord AB, for $\tan BAC = \Delta y/\Delta x$. The value of the average x-rate of y depends on the amount of the range Δx. It approaches a definite value as Δx is taken smaller and smaller, x_2 approaching x_1 for instance. This limiting value is taken as the true or instantaneous rate of y at the value $y = y_1$ (or $x = x_1$). Thus, if r = x-rate of y at any value of y, then

$$r = \lim (\Delta y/\Delta x) = dy/dx.$$

The x-rate of y at $y = y_1$ is represented by the limit of the slope of the chord AB as B approaches A, that is, by the slope of the tangent at A.

By means of the foregoing formula, we can determine the x-rate of y provided that we know the precise relation between x and y, that is, the equation $y = f(x)$. In case we do not know this equation but do know values of y corresponding to several values of x, then we can determine the x-rate of y at one of the values of x approximately. This approximate value can be obtained from the average rates for ranges of x which begin or terminate at the value of x for which the rate is desired as already explained in some of the preceding examples.

§ 3. FEATURES OF A MOTION DETERMINED BY INTEGRATION. — In the preceding article we showed how to determine the velocity from the distance-time (s-t) law, and the acceleration from the velocity-time (v-t) law. The process, in each case, is one of differentiation. By means of the reverse process, integration, one may determine the s-t from the v-t law, and the v-t from the a-t law. We explain further by means of examples.

Suppose that a point moves in a straight line according to the law $v = 60\,t + 4$. In all cases of rectilinear motion $v = ds/dt$, or $ds = v\,dt$; hence in the present instance, $ds = (60\,t + 4)\,dt$. Integration gives $s = 30\,t^2 + 4\,t + C$, where C is a constant to be determined from "initial conditions." Let us suppose that s is reckoned from the place where the moving point is when $t = 0$, or that $s = 0$ when $t = 0$; then substituting these (simultaneous) values of s and t in the equation containing C, we get $0 = 0 + 0 + C$, or $C = 0$. Hence the s-t law is $s = 30\,t^2 + 4\,t$. We might have integrated "between limits," thus

$$\int_0^s ds = \int_0^t (60\,t + 4)\,dt, \quad \text{or} \quad s = 30\,t^2 + 4\,t,$$

the lower limits being the simultaneous values of s and t from the given initial conditions.

For another example, we will suppose that a point moves in a straight line so that $a = \cos t$, initial conditions being $v = 4$ when $t = 0$.

In all cases of rectilinear motion $a = dv/dt$, or $dv = a\,dt$; hence, in this instance, $dv = \cos t\,dt$. Integration gives $v = \sin t + C$. Substituting the (initial) simultaneous values of v and t in this equation we find $4 = 0 + C$, or $C = 4$; hence $v = \sin t + 4$ is the law sought. Or, integrating between limits we get

$$\int_4^v dv = \int_0^t \cos t\,dt, \quad \text{or} \quad v - 4 = \sin t.$$

If the a-s law for a motion is given, then the v-s law can be found by integrating $v\,dv = a\,ds$, which follows from $a = dv/dt = (dv/ds)\,(ds/dt) = (dv/ds)\,v$. Thus, suppose that in a rectilinear motion $a = 2s + 3$, initial conditions being $v = 10$ when $s = 4$; then

$$\int v\,dv = \int (2s + 3)\,ds, \quad \text{or} \quad \tfrac{1}{2}v^2 = s^2 + 3s + C.$$

Initial values substituted in the last equation give $C = 22$, and hence $\frac{1}{2}v^2 = s^2 + 3s + 22$.

The formulas $a = dv/dt$ and $v = ds/dt$ can be used to get "time." These can be written $dt = (1/a)\,dv$ and $dt = (1/v)\,ds$; hence by integration

$$t = \int_{v_1}^{v_2} \frac{1}{a}\,dv, \quad \text{and} \quad t = \int_{s_1}^{s_2} \frac{1}{v}\,ds.$$

These respectively give the time required for v to change from v_1 to v_2, and for s to change from s_1 to s_2.

Since $$a\,ds = v\,dv,\ \tfrac{1}{2}(v_2^2 - v_1^2) = \int_{s_1}^{s_2} a\,ds,$$

where v_1 and v_2 are values of the velocity when $s = s_1$ and $s = s_2$ respectively.

Uniformly Accelerated Motion. — Let a = the value of the (constant) acceleration, and v_0 = the velocity at the instant from which time is reckoned, and s_0 = the distance of the moving point from the origin at that instant. (Sometimes v_0 and s_0 are called initial velocity and initial distance, respectively.) Since a is constant, integration of $a = dv/dt$ gives at once $v = at + C_1$, and from the initial conditions ($v = v_0$ when $t = 0$), $C_1 = v_0$, hence

$$v = at + v_0. \tag{1}$$

From $v = ds/dt = at + v_0$ we find by integration that $s = \frac{1}{2}at^2 + v_0t + C_2$, and the initial conditions ($s = s_0$ when $t = 0$) make $C_2 = s_0$; hence

$$s = \tfrac{1}{2}at^2 + v_0t + s_0. \tag{2}$$

Eliminating t between (1) and (2) we find that

$$2a(s - s_0) = v^2 - v_0^2. \tag{3}$$

If the initial velocity and distance = 0, then

$$v = at, \quad s = \tfrac{1}{2}at^2, \quad \text{and} \quad 2as = v^2. \tag{4}$$

Although uniformly accelerated motions are important practically, the student is advised not to make a special effort to memorize the foregoing formulas (1, 2, 3, and their special forms, 4). But, if he will memorize them, then he should also remember that they are for a special motion, constant acceleration. All students ought to be able to discuss a uniformly accelerated motion nonmathematically — by means of elementary notions somewhat as in the following example: The velocity of a certain train can be reduced by braking from 40 to 20 miles per hour in a distance of 1600 feet. In what distance would braking stop the train from 40 miles per hour, supposing the retardation to be the same at all velocities? Since the velocity changes uniformly, the average velocity during the reduction from 40 to 20 miles per hour equals one-half of 40 + 20 or 30 miles per hour; and the time required for the reduction of velocity or travel of 1600 feet (= 0.303 miles) is 0.303 ÷ 30 = 0.0101 hours, or 36.4 seconds. The time required to stop the train from 40 miles per hour would be twice 36.4 or 72.8 seconds; and, inasmuch as the average velocity during the stoppage would be one-half of (40 + 0) = 20 miles per hour or 29.3 feet per second, the distance travelled in the 72.8 seconds would be 29.3 × 72.8 = 2133 feet.

29. Motion Graphs

The features of rectilinear motion, discussed in the preceding article, can be represented nicely by certain curves described in the following:

A *distance-time* (*s-t*) *graph* for any motion is a curve drawn "upon" a pair of rectangular reference axes so that the coördinates of any point on the curve represent corresponding, or simultaneous, values of s and t, where t = the time elapsed from some instant of reckoning (usually taken at the instant of starting), and s = the distance of the moving point from some fixed point chosen as origin (usually taken at the place of starting). Fig. 222 is the *s-t* graph for the launching mentioned in § 1 of the preceding article.

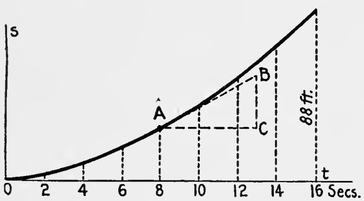


FIG. 222

Since the slope of the *s-t* graph is proportional to ds/dt and $v = ds/dt$, the slope at any point of the graph represents the velocity at the corresponding instant, according to some scale. The slope scale depends on the scales used for plotting the *s-t* graph. Thus, in Fig. 222 the scales are 1 inch of ordinate = 100 feet and 1 inch of abscissa = 10 seconds, hence, a slope of unity = 100 (feet) ÷ 10 (seconds) = 10 (feet per second). Thus, the velocity at t = 8 seconds, where the slope is $BC \div AC = 0.54$, is 5.4 feet per second. Instead of interpreting the slope in this way, that is by a slope scale, we might determine the velocity as follows: draw the tangent line at the point A in question, drop a perpendicular from any point B in the tangent to the horizontal through A, measure CA and CB according to the proper scales and compute the ratio $BC \div AC$ (as measured); this ratio equals the desired velocity. Thus, in Fig. 222, AC = 5 seconds, CB = 27 feet, and $v = 27 \div 5 = 5.4$ feet per second.*

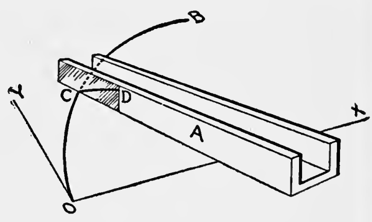


FIG. 223

* Several instruments have been devised recently for drawing a tangent to a plane curve. A very simple one is represented in Fig. 223. It consists of a metal straight-edge A with a portion of one side polished to a mirror. OB represents a curve on a piece of paper across which

The *velocity-time* (v-t) *graph* for any rectilinear motion is a curve drawn upon a pair of rectangular reference axes so that the coördinates of any point of the curve represent corresponding, or simultaneous, values of the velocity v and time t. The curve in Fig. 224 is a v-t graph for the car-starting test

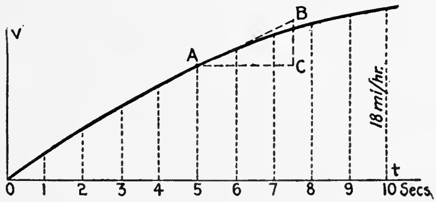


FIG. 224

mentioned in § 2 of the preceding article. The slope of the v-t graph at any point represents the acceleration at the corresponding instant. To actually determine the acceleration from the graph, the slope must be interpreted by proper scale or be computed in a manner analogous to that explained in the foregoing under distance-time graph. Thus, at the fifth second, the acceleration is represented by the slope of the tangent at A; since $AC = 2.5$ seconds and $CB = 4.8$ miles per hour, the acceleration $= 4.8 \div 2.5 = 1.92$ miles per hour per second.

The "area under the curve" (between the curve, the time axis, and any two ordinates) represents the displacement for the interval of time represented by the distance between the ordinates. Proof: Let m = velocity scale-number and n = time scale-number, that is

unit ordinate (inch) = m units of velocity (feet per second);
unit abscissa (inch) = n units of time (seconds).

Thus, let x and y be the lengths (inches) of the coördinates of any point P of a v-t curve (Fig. 225); then the corresponding values of v and t are my and nx.

the straight-edge is laid at random but so that a portion of the curve is reflected from the mirror. The image CD and the curve CO are not smoothly continuous; there is a cusp at C. But if the instrument be turned about C until the cusp disappears, the curve merging smoothly into its image, then the straight-edge A is normal to the curve OB at C. Having located the normal at C, it is easy to draw the tangent. The principle of this instrument is the basis of Wagener's derivator (see Gramberg's *Technische Messungen*) by means of which the slope of a curve at any point can be read from a graduated arc (with vernier) without drawing the tangent or normal.

Guillery's "aphegraphe" is another instrument for drawing a tangent to a plane curve. A metal strip or batten must first be fitted to the curve before the instrument proper can be applied. For full description of the aphegraphe, see *Mem. Soc. Ing. Civ. de France*, Bull. for April, 1911, where M. Guillery also explains how he applied his instrument to determine the acceleration-time curves for several mechanisms, and in particular the a-t curve for the "tup" of an impact testing machine during a blow.

Further let t_1 and t_2 = the times corresponding to x_1 and x_2 and to s_1 and s_2, the values of s (space); and A = area. Then

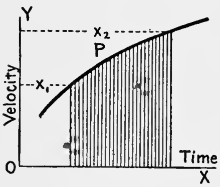


FIG. 225

$$A = \int_{x_1}^{x_2} y\,dx = \int_{t_1}^{t_2} \frac{v}{m}\frac{dt}{n} = \frac{1}{mn}\int_{t_1}^{t_2} v\,dt = \frac{s_2 - s_1}{mn},$$

(see preceding article); and $A\,(mn) = s_2 - s_1$. Hence (mn) is the scale-number for interpretating the area. Thus, in Fig. 224, one-inch ordinate = 20 miles per hour = 29.3 feet per second, and one-inch abscissa = 5 seconds; hence one square inch = 29.3 (feet per second) × 5 (seconds) = 1465 feet. The area may be interpreted more directly by multiplying the average ordinate measured by the scale of ordinates (hence equal to the average velocity for the time interval) by the length of the interval. Thus, in Fig. 224 the average ordinate represents 10.9 miles per hour = 16 feet per second, and the time interval is 10 seconds, hence the displacement is 160 feet.

The *acceleration-time* (*a-t*) *graph* for any rectilinear motion is a curve drawn upon a pair of rectangular reference axes so that the coördinates of any point of the curve represent corresponding, or simultaneous, values of the acceleration a and the time t. The "area under the curve" represents the velocity-change for the time interval represented by the distance between the ordinates. For the area under the curve is given by

$$\int_{t_1}^{t_2} a\,dt, \quad \text{and} \quad v_2 - v_1 = \int_{t_1}^{t_2} a\,dt$$

(see preceding article). To determine the numerical value of the velocity-change, the area must be interpreted by scale or be computed in a manner analogous to that explained in the foregoing under velocity-time graph.

The *velocity-distance* (*v-s*) *graph* for a rectilinear motion is a curve drawn upon a pair of rectangular axes so that the coördinates of any point of the curve represent corresponding, or simultaneous, values of the velocity v and distance s. Fig. 226 is the *v-s* graph for an air-brake test on a passenger train.* The subnormal at any point of the graph represents the acceleration at the corresponding instant. For, any subnormal as BC is given by $AC \tan BAC = v\,dv/ds$, and from the preceding article $a = dv/dt = (dv/ds)\,(ds/dt) = v\,dv/ds$; hence $BC = a$. To actually determine the value of a from a subnormal we must use the proper scale, depending on the scales used for plotting the *v-s* graph. For Fig. 228 one-inch ordinate = 50 miles per hour, and one-inch abscissa = 1000 feet = 0.19 mile; hence the subnormal scale is one inch = $50^2 \div 0.19$ = 13,150 miles per hour per hour = 3.65 miles per hour per second. The subnormal BC

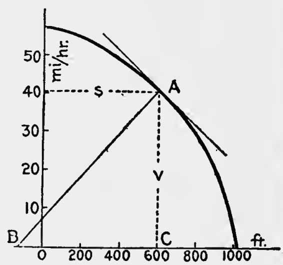


FIG. 226

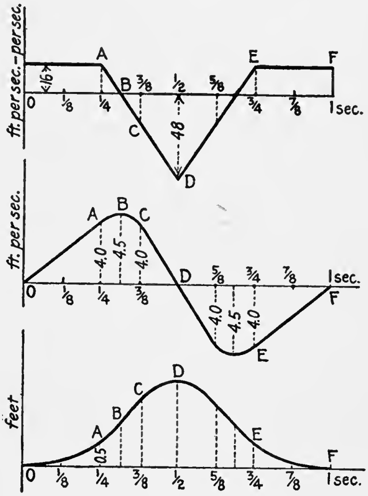
* "Air-brake Tests — Westinghouse." Page 297.

= 0.72 inch; hence the (negative) acceleration at A (when the train had made 600 feet from the place where braking began) was $0.72 \times 3.65 = 2.63$ miles per hour per second.

The *acceleration-distance* (a-s) *graph* for a rectilinear motion is a curve drawn upon a pair of rectangular axes so that the coördinates of any point on the graph represent corresponding, or simultaneous, values of a and s. Any "area under the curve" represents one-half the change in the velocity-square for the corresponding displacement. For the area is given by

$$\int_{s_1}^{s_2} a\,ds = \tfrac{1}{2}\,(v_2^2 - v_1^2), \text{ see § 3 preceding article.}$$

Example. — A mechanism is to be designed for producing a rectilinear motion whose acceleration-time graph is shown in Fig. 227. There are three distinct laws of acceleration. In the first and last quarter seconds the acceleration is constant and equals 16 feet per second per second; in the second quarter the acceleration decreases uniformly from 16 to −48; and in the third it increases uniformly from −48 to 16. Preliminary to the design it is necessary to find the distance-time law; this we proceed to do, but first we get the velocity-time and distance-time graphs approximately.



FIGS. 227, 228, 229

During the first quarter of a second the velocity changes uniformly, and the change is $16 \times \frac{1}{4} = 4$ feet per second; and if the initial velocity is zero, then OA (Fig. 228) is the velocity-time graph for the first quarter second. Since the velocity changes uniformly in the first quarter second, the average velocity equals $\frac{1}{2}(0 + 4) = 2$ feet per second, and the displacement during the quarter $= 2 \times \frac{1}{4} = \frac{1}{2}$ foot. If the initial distance is zero then 0 and A (Fig. 229) are points on the distance-time graph. In a similar way intermediate points could be computed.

In the second quarter the acceleration varies uniformly. The average acceleration for the interval from $\frac{1}{4}$ to $\frac{5}{16}$ second is 8 feet per second per second; hence the velocity-change for that interval is $8 \times \frac{1}{16} = \frac{1}{2}$ foot per second, and the velocity at $t = \frac{5}{16}$ is $4 + \frac{1}{2} = 4.5$ feet per second, and B (Fig. 228) is a point on the velocity-time graph. In a similar way, C, D, and intermediate points could be determined. This portion AD is curved, and the average velocity for any interval cannot be ascertained so simply. But estimating the average ordinate for the third eighth of a second to be 4.4, then the displacement for that interval is $4.4 \times \frac{1}{8} = 0.55$ feet, and C (Fig. 229) is another point on the distance-time graph. In a similar way we might determine other points approximately. Determination of the graphs for the third and fourth quarter seconds by this method presents no difficulties, so we pass on to a second (mathematical) determination of the graphs.

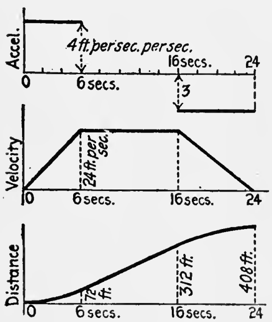
In the first quarter, $dv/dt = 16$, or $dv = 16\,dt$; hence $v = 16\,t + C$. But in accordance with initial conditions assumed, $v = 0$ when $t = 0$; hence $C = 0$, and $v = 16\,t$ is the equation of the velocity-time graph for the first quarter. From that equation we find for $t = \frac{1}{4}$, $v = 4$ as before. Since $v = ds/dt$, $ds = v\,dt = 16\,t\,dt$, and $s = 8\,t^2 + C$. In accordance with initial conditions assumed, $s = 0$ when $t = 0$; hence $C = 0$, and $s = 8\,t^2$ is the equation of the distance-time graph for the first quarter. From that equation we find for $t = \frac{1}{4}$, $s = \frac{1}{2}$ as before; at $t = \frac{1}{8}$, $s = \frac{1}{8}$ foot; etc.

In the second quarter, $a = 80 - 256\,t$, equation of AD (Fig. 227); hence $dv = (80 - 256\,t)\,dt$ or $v = 80\,t - 128\,t^2 + C$. We found that $v = 4$ when $t = \frac{1}{4}$; therefore $4 = 80 \times \frac{1}{4} - 128 \times \frac{1}{16} + C$, or $C = -8$, and $v = 80\,t - 128\,t^2 - 8$ is the equation of the velocity-time graph for the second quarter. Continuing, $ds/dt = 80\,t - 128\,t^2 - 8$, or $s = 40\,t^2 - 42\frac{2}{3}\,t^3 - 8\,t + C$; but $s = \frac{1}{2}$ when $t = \frac{1}{4}$, hence $C = \frac{2}{3}$, and $s = 40\,t^2 - 42\frac{2}{3}\,t^3 - 8\,t + \frac{2}{3}$ is the equation of the distance-time graph for the second quarter.

The equations of the graphs for the remaining quarters could be obtained in a similar way. Care must be taken in determining the constants of integration; use no value of t (and corresponding value of v or s) which does not fall within the period to which the equation under consideration pertains. The remaining equations are —

For the third quarter	*For the fourth quarter*
$a = -176 + 256\,t$	$a = 16,$
$v = -176\,t + 128\,t^2 + 56$	$v = 16\,t - 16,$
$s = -88\,t^2 + 42\frac{2}{3}\,t^3 + 56\,t - 10$	$s = 8\,t^2 - 16\,t + 8.$

Graphs for Uniformly Accelerated Motion. — Fig. 230 shows the acceleration-time graph for a rectilinear motion; in the first six seconds $a = 4$ feet per second per second, in the next ten seconds $a = 0$, and in the last 8 seconds $a = -3$ feet per second per second (the negative sign meaning retardation). Fig. 231 shows the corresponding velocity-time graph, it being assumed that there is no initial velocity. Fig. 232 shows the corresponding distance-time graph, initial distance being taken as zero. Fig. 233 shows the a-s and v-s graphs; AB-CD-EF is the former, and $OGHJ$ is the latter.



FIGS. 230, 231, 232

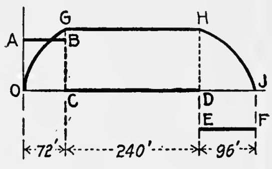


FIG. 233

30. Simple Harmonic Motion and a Similar One

§ 1. SIMPLE HARMONIC MOTION (S.H.M.). — If a point moves uniformly along the circumference of a circle then the motion of the projection of that point on any diameter is called a simple harmonic motion. Obviously the projection (Q) moves to and fro in its path, and travels the length of the diameter twice while the point (P) in the circumference goes once around. By *amplitude* of the s.h.m. is meant one-half the length of the path of Q, equal to the radius of the circle. By *frequency* of the s.h.m. is meant the number of complete (to and fro) oscillations of the moving point Q per unit time, equal to the number of excursions of P around the circumference per unit time. By *period* of the s.h.m. is meant the time required for one complete to and fro oscillation of the moving point Q, equal to the time required for one excursion of P around the circle. By *displacement* of the moving point Q is meant its distance from the center of the path; it is regarded as positive or negative according as Q is on the positive or negative side of the center.

Let us now consider a simple harmonic motion to ascertain approximately its nature. Suppose that the circle (Fig. 234) to be the path of P, and the

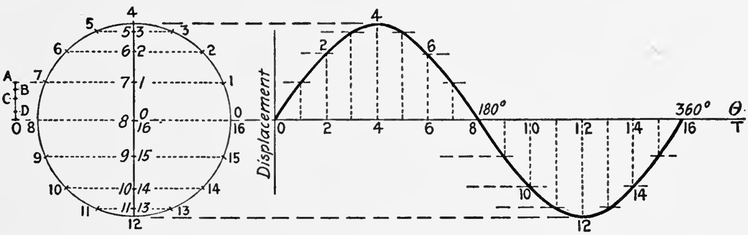


FIG. 234

vertical diameter, say, the path of Q. The y-t (space-time) and the y-θ graphs for the motion of Q can be constructed very easily. We mark any number, say sixteen, equidistant positions of P, and number them consecutively and also the positions of Q to correspond (Fig. 234). Then on an extension of the horizontal diameter we lay off any convenient length oT to represent 360° or the period, and divide oT into sixteen equal parts numbering the points of division as shown. Finally we project points 0, 1, 2, etc., of the circle upon the verticals through the corresponding points 0, 1, 2, etc., of oT. These projections are on the y-t or y-θ graph. The slope of the graph at any point represents the velocity of Q at the corresponding instant; hence the velocity of Q is greatest at the middle of its path, and equals zero at the ends of the path. The arcs 0-1, 1-2, 2-3, etc., are equal, and are therefore described by the moving point P in equal intervals of time. The lengths 0-1, 1-2, 2-3, etc., in the diameter are described by Q in the same equal intervals of time; hence the average velocities of Q for those intervals are proportional to those distances. For comparison, the distances were laid off from O; OA = 0-1, OB = 1-2, OC = 2-3, and OD = 3-4. Carefully comparing these distances, we see that the numerical difference between successive average velocities increases; hence the acceleration increases in value as Q moves from 0 to 4. In fact the acceleration is zero when Q is at the middle, and greatest when at either end of its path (proved below).

We now examine s.h.m. more carefully, using the following notation:—

r = amplitude (radius of the circle),
n = frequency,
$\omega = 2\pi n$ (abbreviation), ω being angle in radians swept out per unit time by OP,
x, y, or s = displacement,
t = time after some convenient origin as described later,
v = velocity of the s.h.m. at the time t,
a = acceleration of the s.h.m. at time t.

When time t is reckoned from the instant when Q was at middle of its path and moving in positive direction. — Suppose the circle (Fig. 235) to be the path of P, which moves in sense indicated by arrow, and let us consider the motion of the projection of P on the vertical diameter, from now on called V instead of Q. Let θ = angle XOP; then, since t is time elapsed since P was at X, $\theta = 2\pi nt = \omega t$ and $d\theta/dt = \omega$. It is plain from the figure that $y = r \sin\theta$; and since $v = dy/dt = r (d\theta/dt) \cos\theta$,

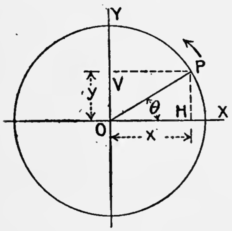


FIG. 235

$$v = r\omega\cos\theta = r\omega\cos\omega t.$$

These are (general) formulas for v in terms of θ and t respectively.

Since $\cos\theta = \sin(\theta + \frac{1}{2}\pi)$, $v = r\omega\sin(\theta + \frac{1}{2}\pi)$. This formula for v suggests an easy method for drawing a v-θ graph, showing how the velocity varies with θ, and hence with t. First we draw an auxiliary circle with radius equal to $2\pi rn$ according to any convenient scale; divide the circumference into any convenient number of equal parts, as sixteen; and number the points of division as in Fig. 236, that is 90° ahead of the numbers in Fig. 234. On an

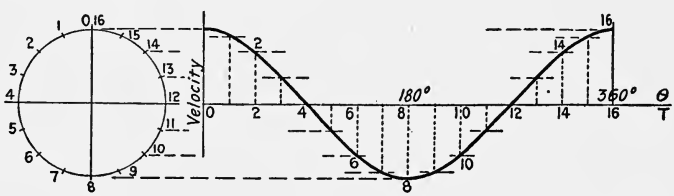


FIG. 236

extension of the horizontal diameter we lay off oT to represent 360°, and subdivide this into the same number of equal parts (sixteen), numbering as shown;

then 01, 02, etc., represent $\theta = 22\frac{1}{2}°$, $\theta = 45°$, etc. Finally we project points 0, 1, 2, etc., of the circle toward the right to meet corresponding vertical lines through points 0, 1, 2, etc., of oT. These points of meeting are on the v-θ graph, for the coördinates of any point on the curve are corresponding, or simultaneous, values of θ and $r\omega \sin(\theta + \frac{1}{2}\pi)$, or v.

Inspection of the v-θ graph verifies what was said about the acceleration. It shows clearly that the velocity of the moving point V (Fig. 235) changes more rapidly when V is near the ends of its path than when near the center; hence the acceleration of V is greater near the ends than near the center. Since the v-θ graph is also a v-t graph, the slopes of the graph represent, to proper scale, values of the (varying) acceleration. The curve is steepest when $\theta = 90°$ and $270°$ (when V is at the ends of its path), and horizontal where $\theta = 0$ and $180°$ (when V is at the center of its path); hence again the acceleration is greatest at the ends of the path, and zero at the center. When the moving point V is approaching the center of its path — from either side — then V is getting up speed, and hence the acceleration of V is directed toward the center; when V is receding from the center, then V is slowing down, and hence the acceleration is directed toward the center. Therefore the acceleration is always directed toward the center.

A general formula for acceleration in a s.h.m. will now be derived. We take the motion of V (Fig. 235) for that purpose, and let a = the acceleration at any time t. Now $a = dv/dt$, $v = r\omega \cos\theta$, and $d\theta/dt = \omega$; hence

$$a = -r\omega^2 \sin\theta = -r\omega^2 \sin\omega t.$$

These are (general) formulas for a in terms of θ and t respectively.

Since $\sin\theta = -\sin(\theta + \pi)$, $a = r\omega^2 \sin(\theta + \pi)$. This last formula suggests an easy method for drawing the a-θ graph, showing how a varies with θ, and hence also with t. First we draw an auxiliary circle (Fig. 237) with

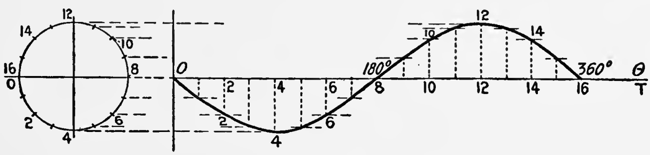


FIG. 237

radius equal to $r\omega^2$ according to any convenient scale; divide the circumference into any convenient number of equal parts, say sixteen; and number them as in the figure, that is 180° ahead of the numbers in Fig. 234. On an extension of the horizontal diameter we lay off OT to represent 360°, and subdivide OT into sixteen equal parts numbering as shown; then 01, 02, etc., represent $\theta = 22\frac{1}{2}°$, $\theta = 45°$, etc. Finally, we project points 0, 1, 2, etc., of the circle horizontally to meet the corresponding vertical lines through points 0, 1, 2, etc., of the line OT. These points of meeting are on the a-θ graph, for the coördinates of any point on the curve are corresponding, or simultaneous, values of θ and $r\omega^2 \sin(\theta + \pi)$, or a.

In Fig. 238 the foregoing described distance, velocity, and acceleration graphs are superimposed; the solid curve is the y-θ graph, the dashed curve the v-θ graph, and dot-dash curve is the a-θ graph.

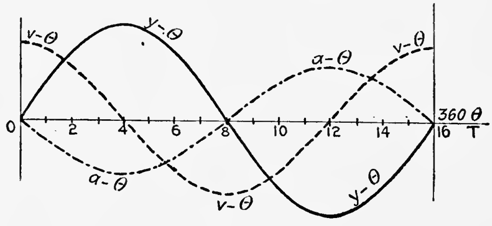


FIG. 238

Time dated from the instant when Q was at the positive end of its path. — We might continue to regard the s.h.m. as taking place in the vertical diameter of Fig. 235, reckoning time from the instant when P was at Y. It will be more convenient to consider the motion of the projection of P on the horizontal diameter; then we measure θ and t as before. In this case, it is easy to show that

$$x = r \cos\theta = r \cos\omega t;$$
$$v = -\, r\omega \sin\theta = -\, r\omega \sin\omega t;$$
$$a = -\, r\omega^2 \cos\theta = -\, r\omega^2 \cos\omega t.$$

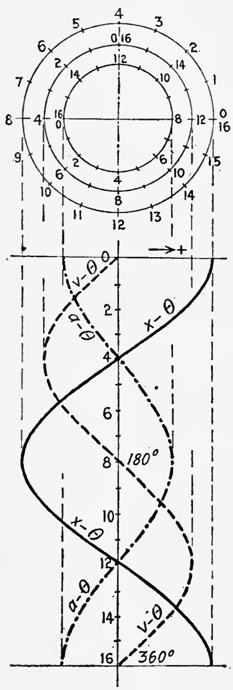


FIG. 239

Fig. 239 shows the distance, velocity, and acceleration-time graphs for a s.h.m. regarded in this way.

Time dated from instant when Q was anywhere. — Let t be reckoned from instant when P (Fig. 240) was at some point as P_0, and let $\theta = P_0OP$ and $\epsilon = XOP_0$. This latter angle is called angle of lead — but angle of lag when P_0 is below OX. Now $XOP = \theta + \epsilon = \omega t + \epsilon$. In the s.h.m. executed by V,

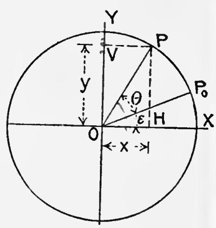


FIG. 240

$$y = r \sin(\theta + \epsilon);$$
$$v = r\omega \cos(\theta + \epsilon);$$
$$a = -\, r\omega^2 \sin(\theta + \epsilon).$$

In the simple harmonic motion executed by H,

$$x = r \cos(\theta + \epsilon);$$
$$v = -\, r\omega \sin(\theta + \epsilon);$$
$$a = -\, r\omega^2 \cos(\theta + \epsilon).$$

Formulas for Velocity and Acceleration in Terms of Displacement. — These do not depend on the way in which time is reckoned. Referring to the foregoing formulas we see that

$$v = \omega\sqrt{r^2 - s^2} = \omega r\sqrt{1 - (s/r)^2},$$
$$a = -\omega^2 s,$$

where s stands for displacement x or y.

The graph of $v = \omega\sqrt{r^2 - s^2}$ is the velocity-displacement graph for any s.h.m.; it is an ellipse. Fig. 241 shows that graph for the motion of the projection of P on the horizontal diameter of the circle. When P is where indicated say, the velocity of H is represented by the ordinate HV. The graph of $a = -\omega^2 s$ is the acceleration-displacement graph; it is a straight line. The diagonal line in the figure is the a-s graph for the motion of H. The acceleration of H is represented by the ordinate HA.

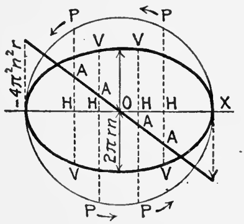


FIG. 241

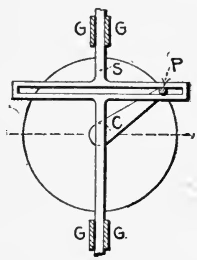


FIG. 242

Mechanism for Producing a Simple Harmonic Motion. — The mechanism represented in Fig. 242 consists of a crank C and a slotted slider S. The slider is constrained by fixed guides G so that it can be moved to and fro only (vertically in this figure). The crank-pin P projects through the slot of the slider; hence if the crank be turned, the crank-pin presses against and moves the slider. If the crank be turned uniformly then every point of the slider executes a simple harmonic motion.

§ 2. CRANK AND CONNECTING-ROD MECHANISM (Fig. 243). — When the crank is rotating uniformly, the motion of the crosshead (and piston) resem-

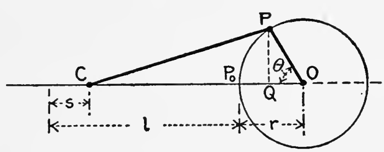


FIG. 243

bles a simple harmonic one quite closely, as will be shown presently. Exact formulas for the position, velocity, and acceleration of the crosshead for any position of the crank were derived in Art. 28. The formulas are not simple.

The following approximate formulas (1, 2, and 3) are simpler and quite accurate, as will be shown.

As in Art. 28, let r = length of crank, l = length of connecting rod, $c = r/l$, n = number of revolutions of crank per unit time (assumed constant), ω = angle in radians described by crank per unit time ($\omega = 2\pi n$), s = the varying distance of the crosshead from its position most remote from the crank, θ = the crank angle P_0OP, and t = time required for the crank to describe the angle θ ($\theta = \omega t = 2\pi nt$). It follows from the geometry of the figure, as explained in Art. 28, that

$$s = (l + r) - l\,(1 - c^2 \sin^2 \theta)^{\frac{1}{2}} - r \cos \theta.$$

Now $(1 - c^2 \sin^2 \theta)^{\frac{1}{2}} = 1 - \frac{1}{2} c^2 \sin^2 \theta - \frac{1}{8} c^4 \sin^4 \theta -$ etc. (binomial expansion). And since c is generally less than $\frac{1}{3}$, the third and succeeding terms in the series are very small and negligible; hence we have as a good approximation

$$l\,(1 - c^2 \sin^2 \theta)^{\frac{1}{2}} = l\,(1 - \tfrac{1}{2} c^2 \sin^2 \theta) = l\,(1 - \tfrac{1}{4} c^2 + \tfrac{1}{4} c^2 \cos 2\,\theta),$$

and

$$s = r\,(1 - \cos \theta) + \tfrac{1}{4}\,cr\,(1 - \cos 2\,\theta). \qquad (1)$$

Now if we differentiate this with respect to t, we get ds/dt or v (velocity of the crosshead), and remembering that $d\theta/dt = \omega$, we finally get

$$v = r\omega\,(\sin \theta + \tfrac{1}{2}\,c \sin 2\,\theta). \qquad (2)$$

Differentiating again and remembering that ω is constant we get dv/dt or

$$a = r\omega^2\,(\cos \theta + c \cos 2\,\theta). \qquad (3)$$

Because of our way of measuring s, the positive direction is from the cylinder toward the crank. Positive velocity v means that the crosshead is moving toward the crank, and positive acceleration a means that velocity toward the crank is being added to the velocity.

In order to furnish a comparison between the foregoing approximate formulas and the exact ones of Art. 28, we give in the adjoining table the values of a for the case $c = 3\frac{1}{2}$ for a few values of the crank angle θ (Fig. 243).

θ	a, exact	a, approx.
0°	$+1.286\ r\omega^2$	$+1.286\ r\omega^2$
30	$+1.015$	$+1.009$
60	$+0.357$	$+0.357$
90	-0.298	-0.286
120	-0.643	-0.643
150	-0.717	-0.723
180	-0.714	-0.714

To find the acceleration of the crosshead when the crank is at the "head-end dead-center" (crank at OP_0) we put $\theta = 0$, and find from either the exact or approximate formula that

$$a = r\omega^2\,(1 + c).$$

To find the acceleration at the "crank-end dead-center" we put $\theta = 180°$, and find from either the exact or approximate formula that

$$a = -r\omega^2(1 - c).$$

To show that the motion of the crosshead C is approximately simple harmonic we show that its motion resembles the motion of Q (Fig. 243) which is a simple harmonic one. In Fig. 244 we have marked nine corresponding positions of Q and C. Thus points 0 to 8 are the positions of Q when the crank angles are $0°$, $22\frac{1}{2}°$, $45°$, etc., and points O, I, II, III, etc., are the corresponding positions of C. In the lower part of Fig. 244 the paths of Q and C (with the points 1, 2, 3 and I, II, III, marked upon them) have been brought together for comparison. It is seen that the actual distances described by Q and C in any interval of time are nearly the same, and so the motion of C is nearly the same as that of Q.

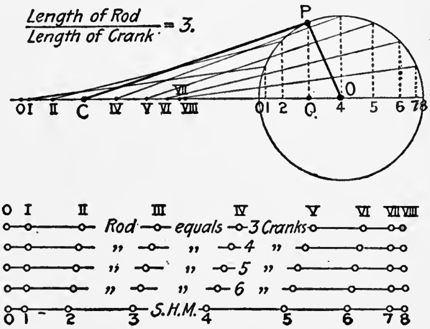


FIG. 244

The three intermediate lines in the figure are paths of C with points corresponding to 1, 2, 3, etc., for three other lengths of connecting rod. And we see that the longer the rod the more nearly is the motion of the crosshead simply harmonic.

To arrive at a more complete comparison of the motions of C and Q, we will derive the formulas for the position, velocity, and acceleration of Q corresponding to equations (1), (2), and (3). The variable distance of Q from P_0 (Fig. 243) we will call z, then

$$z = r(1 - \cos\theta). \tag{4}$$

Differentiating with respect to t, we get for velocity of Q

$$v = r\omega \sin\theta, \tag{5}$$

and differentiating again we get for acceleration of Q

$$a = r\omega^2 \cos\theta. \tag{6}$$

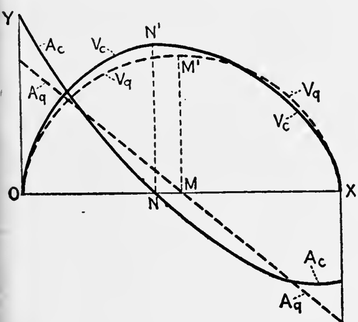


FIG. 245

Now compare (1) and (4), (2) and (5), and (3) and (6) and note that the formulas for the motion of C contain an "extra" term. Each of these terms depends on c ($= r/l$), or on the "obliquity" of the connecting rod (maximum inclination of the rod to the line of stroke OC). The smaller c (the longer the rod in comparison with the crank), the smaller are the extra terms, and so the longer the rod the more nearly is the motion of the crosshead a simple harmonic one.

Fig. 245 presents a comparison of the motion of the crosshead C and the motion of Q. The solid lines refer to the first motion and the dashed lines to the second. V_c is the velocity-distance (v-s) graph and A_c is the acceleration-distance (a-s) graph for the motion of C. V_q is the velocity-distance graph and A_q is the acceleration-distance graph for the motion of Q. The graphs for C were drawn for a connecting rod three cranks long ($c = 1 \div 3$). For longer rods the graphs for C would come much nearer the graphs for Q.

31. Motion and Force

The preceding discussion of motion deals, for the most part, with displacement, velocity, and acceleration; it does not refer at all to the forces acting upon the moving bodies. In this article we explain in what manner any rectilinear motion of a rigid body depends upon the forces acting upon it.

§ 1. FIRST VIEW AND FORM OF THE FUNDAMENTAL PRINCIPLE. — In Art. 2 it is explained that the units of force most used by engineers are the so-called *gravitation units*, equal to the earth-pulls on certain things called standards of weight. These units have slightly different values at different places; thus we have the London pound-force, the New York pound-force, etc. Some writers define the pound-force as any force equal to the earth's attraction on the standard pound weight at London or at sea level in latitude 45°, thus making the unit force invariable or an "absolute" one. Besides these units there are others; see § 2 of this article.

In Art. 2, we explained also that the word weight is used in at least two senses in common parlance (see footnote, page 4). But we will continue to use it in a single sense, to connote the earth-pull on a body, and we employ a separate word (mass, see § 2 of this article) to connote the amount of substance or stuff in a body. Our two weighing devices, beam-scale and spring-scale, differ in a certain feature which is worth noting here. A beam-scale measures the weight (earth-pull) of a body in terms of the local unit of force, say the pound force for the place where the weighing is done; a spring-scale measures the weight of a body in terms of an invariable unit, say the particular pound force for which the scale was graduated. A beam-scale will not detect the change in the weight of a body with change of place because the magnitude of the unit (pull on the poise) changes just as the weight of the body changes. A spring-scale if sufficiently accurate will detect change in weight with change of place.

First-hand knowledge of the relation between motion and the forces acting on the moving body must rest on observation or experiment. Let us consider a simple case of motion, that of a falling body. The motion takes place under the action of the weight of the body and the resistance of the surrounding air. But if the falling body is quite dense, the air resistance is negligible compared to the weight until the velocity becomes quite large. Observations have shown that such a body falls with a constant acceleration of about 32 feet per second per second at moderate velocities, and we infer that any force equal to the weight of the body would, if acting alone on that body, produce an acceleration of the value stated.

We are now led to inquire what is the effect on a body of an applied force of some other magnitude, say a force equal to double its weight or one-half its weight? If we could intensify or dilute the earth-pull upon a body by a (gravity) lens or screen, then we could make a body fall under a force differing from its own weight and ascertain the answer to our question by observing the fall. Unfortunately for our purpose, we cannot so concentrate or dilute the force of gravity but we can dilute it indirectly by means of an "Atwood machine," designed for that purpose. The essential parts of that machine are a light pulley P mounted on a smooth horizontal axle (Fig. 246), some blocks of metal which can be suspended as shown by a light flexible cord, and a timing device for getting the acceleration of A and B when the system is allowed to move. Neglecting the small influence of the pulley, axle, and cord, we regard A and B as the body moved and the difference in their weights $(W_b - W_a)$ as the driving force. Experiments with this machine show that A and B move with constant acceleration, and when runs are made with various driving forces — all metal pieces being used each time — then *the accelerations in the different runs are directly proportional to the driving forces.* In this machine the driving force can be made very small but it cannot be made larger than the weight of all the metal pieces. It would seem that the force-acceleration relation stated holds even for driving forces larger than the weight of the body moved; and we assume that when *any forces are applied successively to the same body so as to make it move in a straight line, then the accelerations are proportional to the forces respectively.* Or, if F and F' = the magnitudes of two forces applied to any body in succession, and a and a' = the accelerations respectively, then

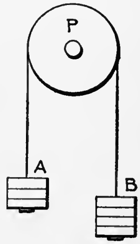


Fig. 246

$$F/F' = a/a'.$$

If W = the weight of the body, g = the acceleration due to gravity (W), F and a as above, then the foregoing principle gives also $F/W = a/g$, or as it is more commonly written, $F = (W/g)\,a$.

Generally, a moving body is under the influence of more than one force. When the *body moves in a straight line, the resultant of all the forces acting upon it is a single force acting in the direction of the acceleration* (proved in Art. 35). Therefore *the resultant has no component at right angles to the line of motion;* or, *the algebraic sum of the components of all the forces acting on the body along any line at right angles to the path equals zero.* Thus, if the path is taken as an x axis and two lines at right angles to each other and to the path as y and z axes, then

$$\Sigma F_y = 0, \quad \Sigma F_z = 0, \quad \text{and} \quad \Sigma F_x = R,$$

where ΣF_x, ΣF_y, and ΣF_z stand for the algebraic sums of the x, y, and z components of all the forces acting on the body, and R denotes their resultant. Furthermore, as proved in Art. 35,

$$R = \frac{W}{g} a. \tag{1}$$

Any unit of force may be used for R and W in equation (1), and any unit for g and a. When a gravitational unit of force is used — such are most convenient in engineering calculations — then, strictly, the numerical value of g used should correspond to the "locality" of the unit-force used. That is, when one is about to make a calculation by means of equation (1), implying the New York pound-force say, then he should use for g its value for New York. As already stated, the variation in g is negligible in most engineering calculations, and we generally use 32.2 feet per second per second or even 32 for simplicity. Non-gravitational units, the dyne for example, may be used in equation (1). But when such units are preferred, then equation (2) is to be preferred in place of equation (1).

Examples. — When a body moves in a straight line and if all the forces acting on it are known so that R can be computed, then the acceleration can be determined easily by means of equation (1). If the acceleration is known then we can determine R easily, and from R we can find out something about the forces acting on the body.

1. A (Fig. 247) represents a body being dragged along a rough horizontal surface B by a pull P acting as shown. Suppose that the body weighs 100 pounds, $P = 40$ pounds, and the friction resistance $= 10$ pounds. We will find the acceleration of A and the normal component of the force exerted between A and B. The forces acting on A are represented in Fig. 248, N denoting the normal component of the reaction of B on A, friction being the other component. Resolving at right angles to the path, we get $N + 40 \sin 20° = 100$, or $N = 86.3$ pounds. Resolving along the path, we get $R = 40 \cos 20° - 10 = 27.6$. Equation (1) gives $27.6 = (100 \div 32.2)\, a$, or $a = 8.9$ feet per second per second.

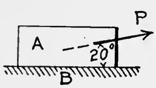


FIG. 247

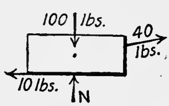


FIG. 248

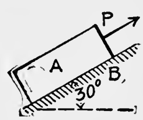


FIG. 249

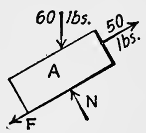


FIG. 250

2. A (Fig. 249) represents a body being dragged up the rough inclined plane B by a pull P equal to 50 pounds; A weighs 60 pounds and the coefficient of friction for A and B is $\frac{1}{4}$. We determine the acceleration. Three forces act on A, namely the weight, the pull, and the reaction of B. The last force is represented by two components (N and F) in Fig. 250. Resolving at right angles to the path, we get $N = 60 \cos 30° = 52$; hence $F = 52 \div 4 = 13$ pounds. Resolving along the path, we get $R = 50 - 13 - 60 \sin 30° = 7$ pounds; hence $7 = (60 \div 32.2)\, a$, or $a = 3.75$ feet per second per second.

3. A certain passenger elevator gets up speed at the rate of 4 feet per second per second, and can be stopped at the rate of 8 feet per second per second. We discuss the pressure on the shoes of a standing passenger weighing 160 pounds, during an ascent. The forces acting on the man are his own weight and the pressure P of the floor on his shoes (upward). During acceleration the resultant of these forces is upward, hence P is larger than 160 pounds and $R = P - 160$. Equation (1) becomes $P - 160 = (160 \div 32) \times 4 = 20$, or $P = 180$ pounds. During the next period, constant speed, $a = 0$ and $P = 160$. During retardation the acceleration is downward and hence R also. Therefore $R = 160 - P = (160 \div 32) \times 8 = 40$, or $P = 120$ pounds.

4. We determine the reaction of the car (Fig. 251) on A during the period of getting up speed at the rate of 2 feet per second per second; A weighs 1000 pounds. We suppose the floor of the car so rough that A does not slip. There are two forces acting on A (Fig. 252), its own weight and the pressure P of the floor. This latter force must be inclined as shown to furnish a component on A in the direction of the acceleration. Resolving at right angles to the path, we get $P \cos \theta = 1000$; resolving along the path, we get $R = P \sin \theta = (1000 \div 32.2) \times 2$. Solving these two simultaneously we find that $P = 1002$ pounds and $\theta = 3° 33'$. (The horizontal component of P is friction. To prevent slipping the floor must be rough enough to furnish such a resistance.)

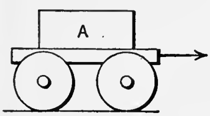


FIG. 251

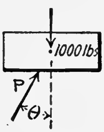


FIG. 252

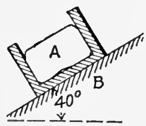


FIG. 253

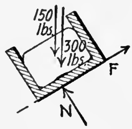


FIG. 254

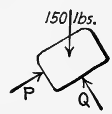


FIG. 255

5. A box (Fig. 253) containing a body A slides down a rough inclined plane B whose inclination is 40°. The box weighs 300 pounds, A weighs 150 pounds, the coefficient of kinetic friction "between" box and plane is $\frac{1}{5}$, and A is perfectly smooth. We determine the acceleration of A and box, and the pressures between them. Fig. 254 represents all external forces acting on A and box, the reaction of the plane B being represented by two components, F and N. Resolving at right angles to the path, we get $N = 150 \cos 40° + 300 \cos 40° = 345$, hence $F = 345 \div 5 = 69$. Resolving along the path, we get $R = 300 \sin 40° + 150 \sin 40° - 69 = 220 = (450 \div 32.2)\, a$, or $a = 15.75$ feet per second per second. Fig. 255 represents all the forces acting on A, where P and Q are the pressures exerted by the front and bottom of the box respectively. Resolving along the direction of the motion we get $R = 150 \sin 40° - P = (150 \div 32.2)\, 15.75$, or $P = 23.1$. Resolving along the normal we get $Q = 150 \cos 40° = 115$ pounds.

6. A body slides down a plane under the influence of gravity and the reaction of the plane only; required the acceleration. Let W = weight of the body; μ = coefficient of friction; α = inclination of the plane; N = normal

pressure; and F = friction. Then resolving forces normally to the path we get $N = W \cos \alpha$; therefore $F = \mu N = \mu W \cos \alpha$. Resolving along the path we get $R = W \sin \alpha - F = W (\sin \alpha - \mu \cos \alpha) = (W \div g) a$, or

$$a = g (\sin \alpha - \mu \cos \alpha).$$

If the plane is perfectly smooth $\mu = 0$, and $a = g \sin \alpha$.

§ 2. SECOND VIEW AND FORM OF THE FUNDAMENTAL PRINCIPLE.—Physicists avoid the (common) double meaning of the word weight by employing the word mass to connote amount of material, substance, or stuff, in a body, and weight to connote the earth-pull on the body. Such usage is followed in this book. Material is measured in different ways; for example, liquids generally by gallon, earthwork by cubic yard, cloth by square yard, brick by thousand, iron by ton, etc. But mass of a body means substance as measured by a beam-scale. Our standards of mass (commonly and legally called "standards of weight") are the pound and the kilogram. These are certain pieces of metal preserved in London and Paris respectively. The mass of a body, measured as just explained, does not change with change of locality, and this is in accordance with our conception of material, substance, or stuff.

The force-acceleration relation, $F = (W/g) a$, can be put into an alternative form which is preferable from some points of view. Thus suppose that two bodies whose weights at the same place are W_1 and W_2 are subjected to equal forces F; let g = the acceleration due to gravity at the place and a_1 and a_2 = the accelerations produced by the two forces F. Then $F = (W_1/g) a_1 = W_2/g) a_2$, or

$$a_1/a_2 = W_2/W_1.$$

That is, the accelerations of the two bodies are inversely as their weights at the same place; and since the masses of two bodies are proportional to the weights (at the same place), the accelerations of the two bodies are inversely proportional to their masses. This relation and that between the accelerations produced in a body by two different forces acting singly can be expressed in one statement as follows:—*Whenever a force acts upon a body so as to make it move in a straight line, then the acceleration produced is proportional to the force directly and to the mass of the body inversely, or* $a \propto F \div m$. This proportionality can be put into the form of an equation,

$$F = Kma,$$

where K is a proportionality factor whose value depends on the units used for expressing magnitudes of F, m, and a. This is the alternative form mentioned.

We may fix the value of K in two ways:—(1) choose units of F, m, and a at pleasure, and deduce the value of K; or (2) choose a value of K and units for any two of the quantities F, m, and a, and then deduce the proper unit for the third quantity. On plan (1) we choose, for example, the pound-force, the pound-mass, and the foot per second per second as units for F, m, and a, and then determine K by reference to any motion in which F, m, and a are known. The motion of a falling body is such a one. Thus when a body "weighing" say 10 pounds falls, then $F = 10$ pounds, $m = 10$ pounds, and $a =$ about 32.2 feet per second per second, and we have $10 = K \times 10 \times 32.2$, or $K = 1 \div 32.2$. On plan (2) we take K equal to unity for simplicity, and then (i) choose units of m and a at pleasure, and deduce the proper unit of F; or (ii) choose units of F and a at pleasure, and deduce the proper unit of m. (i) Physicists take the gram as unit mass, and the centimeter per second per second as unit of acceleration; then the corresponding unit of force ($K = 1$) is such a force as would give to the gram an acceleration of one centimeter per second per second. They call this force the *dyne.* (ii) If we take the pound as unit of force, the foot per second per second as unit of acceleration, then the corresponding unit of mass ($K = 1$) is such a mass which will sustain an acceleration of one foot per second per second under the action of a force of one pound. This unit of mass has no generally accepted name, but it is sometimes called "engineers' unit of mass," also "slug" and "gee-pound."

A set of units for which $K = 1$ is called a *systematic set of units,* also a *kinetic set.* We will always use systematic units and thus always have $F = ma$, or when several forces make a body move in a straight line,

$$R = ma. \tag{2}$$

where R denotes the resultant of those forces. For a falling body $R = W$ and $a = g$; thus when systematic units are used

$$W = mg, \quad \text{or} \quad m = W/g. \tag{3}$$

Therefore $R = (W/g)\,a$ as in § 1.

To arrive at a notion of the magnitude of the unfamiliar units dyne (force) and slug (mass), let us consider the well-known force-mass-acceleration relation in the case of a falling body. A body whose mass is one gram, falling at Paris, falls under the action of a force (earth-pull) of one Paris gram, and has an acceleration of 981 centimeters per second per second. Hence a force of 0.001019 ($= 1 \div 981$) Paris grams would give to a body whose mass is one gram an acceleration of one centimeter per second per second. Therefore that force is the dyne, that is

$$1 \text{ dyne} = 0.001019 \text{ Paris grams (force).}$$

A body whose mass is one pound, falling at London, falls under the action of a force (earth-pull) of one London pound, and has an acceleration of 32.2 feet per second per second. Hence a force of one London pound would give to a body whose mass is 32.2 pounds an acceleration of one foot per second per second. Therefore, that mass is the slug, that is

$$1 \text{ slug} = 32.2 \text{ pounds (mass).}$$

CHAPTER VIII

CURVILINEAR MOTION

32. Velocity and Acceleration

§ 1. Velocity. — In common parlance, velocity of a moving point at a certain instant means the rate at which the point is describing distance then. So understood, velocity has magnitude and sign only, and is therefore a scalar quantity. In the preceding chapter (on rectilinear motion) we used the word in this sense; in the present chapter we use the word in a broader sense — so that it is a vector quantity whose magnitude is the rate at which the moving point is describing distance at the instant in question and whose direction is the same as that of the motion then.

If s = the (varying) distance of the moving point from some fixed origin in the path, the distance being measured along the path, then the magnitude of the velocity at any instant equals the value of ds/dt for that instant. Or if v = magnitude of velocity,

$$v = ds/dt.$$

If the point is moving uniformly, then the rate at which distance is described is constant, and is given by $\Delta s/\Delta t$, where Δs is the distance described in any interval Δt. The direction of the motion at any instant (and the direction of the velocity there) is along the tangent to the path at the position of the moving point at that instant. To illustrate, imagine a 10-foot wheel mounted on a horizontal axis which points north and south, and suppose that the wheel is rotating at 180 revolutions per minute clockwise when viewed from the south. When a certain point on the rim is in its highest position then the velocity of the point has a magnitude of $2\pi\, 5 \times 180 = 5655$ feet per minute, and the direction of the velocity is horizontal from west to east.

The magnitude part of a velocity is called *speed* by some writers; we follow this usage. Thus in the preceding illustration the speed is 5655 feet per minute; while the wheel turns, the speed of the point is constant but the velocity changes in direction.

§ 2. Acceleration. — The acceleration of a moving point at any instant is the rate at which its velocity — not speed — is changing then. If V denotes the (varying) velocity of a moving point and v the (varying) speed, then the definition states that the acceleration is dV/dt and not dv/dt. Inasmuch as most readers are unfamiliar with the rate of a vector quantity — the rate chapters in most books on differential calculus deal with rates of scalar quantities only — we explain in considerable detail just what is meant by the rate of change of a velocity, but first we explain for subsequent use a motion graph called

Hodograph. — This is a curve which shows how the velocity of a moving point varies. It is constructed by laying off vectors from a point to represent successive velocities, and then the free ends of the vectors are joined by a smooth curve. The curve is the hodograph for the motion. Thus, suppose that $ABCD$ (Fig. 256) is the path of a moving point P, and that the vectors at A, B, C, and D represent the velocities of P when at A, B, C, and D respectively. If $O'A'$, $O'B'$, $O'C'$, and $O'D'$ (Fig. 257) are drawn (from any point O')

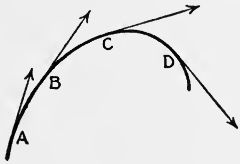


FIG. 256

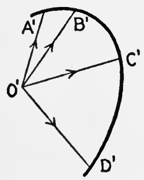


FIG. 257

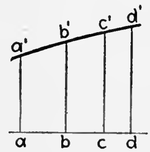


FIG. 258

to represent the velocities respectively, then the curve $A'B'C'D'$ is the hodograph for the motion of P from A to D. The increment or change in the velocity while P moves from A to D say is represented by the vector $A'D'$ (in magnitude and direction). The change in the speed = length $O'D'$ — length $O'A'$. (The hodograph should not be confused with the speed-time curve. The latter is represented in Fig. 258 where ab, bc, and cd represent the times required for P to move from A to B, B to C, and C to D respectively, and the ordinates over a, b, c, and d represent the speeds at A, B, C, and D.)

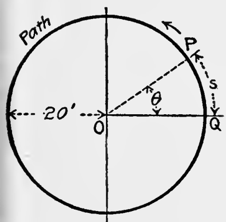


FIG. 259

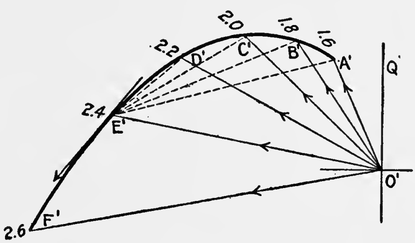


FIG. 260

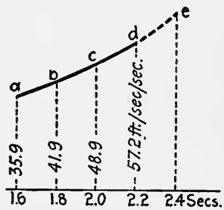


FIG. 261

We are now ready to explain the meaning of rate of change of velocity; we base our explanation on a simple case of curvilinear motion. Suppose that a point P starts at Q (Fig. 259) and describes the circle shown in such a way that the distance traversed (in feet) equals double the cube of the time after starting (in seconds), or $s = 2\,t^3$. Required the acceleration say, when $t = 2.4$ seconds, or $s = 2 \times 2.4^3 = 27.65$ feet. The curve in Fig. 260 is the hodograph of the motion for the interval from $t = 1.6$ to $t = 2.6$, containing the instant in question. It was constructed from the adjoining schedule, computed from $s = 2t^3$, $\theta = s/20$ (radians) $= 2.865\,s$ (degrees), and $v = ds/dt = 6t^2$.

t (sec.)	s (ft.)	θ (deg.)	v (ft/sec)
1.6	8.192	23.5	15.36
1.8	11.664	33.4	19.44
2.0	16.000	45.8	24.00
2.2	21.296	61.0	29.04
2.4	27.648	79.2	34.56
2.6	35.152	100.7	40.56

Vectors $O'A'$, $O'B'$, $O'C'$, etc., represent the velocities of P when $t = 1.6$, 1.8, 2.0, etc., as marked. Vectors $A'E'$, $B'E'$, $C'E'$, and $D'E'$ represent the velocity-increments for the intervals 1.6 to 2.4, 1.8 to 2.4, 2.0 to 2.4, and 2.2 to 2.4 seconds respectively. The magnitudes of these increments were scaled from the original hodograph drawing (the scale of which was one inch = 5 feet per second) and are recorded in the adjoining schedule under ΔV.

Δt (sec.)	ΔV (ft/sec)	$\Delta V/\Delta t$ (ft/sec/sec)	Δv (ft/sec)	$\Delta v/\Delta t$ (ft/sec/sec)
1.6 to 2.4=0.8	28.75	35.9	19.20	24.0
1.8 to 2.4=0.6	25.15	41.9	15.12	25.2
2.0 to 2.4=0.4	19.55	48.9	10.56	26.4
2.2 to 2.4=0.2	11.45	57.2	5.52	27.6

Now the magnitude of the average acceleration for the interval 1.6 to 2.4 seconds is $28.75 \div 0.8 = 35.9$ feet per second per second, and the direction of that average acceleration is $A'E'$. The magnitudes of the average accelerations for the intervals 1.8 to 2.4, 2.0 to 2.4 and 2.2 to 2.4 also are given under $\Delta V/\Delta t$; the directions of those average accelerations are respectively $B'E'$, $C'E'$, and $D'E'$.

Now the acceleration when $t = 2.4$ seconds is the limit of these average accelerations, as Δt is taken smaller and smaller but always terminating at $t = 2.4$. The magnitude of this limit, which is the magnitude of the acceleration sought, is the limit of the magnitudes of the average accelerations. This limit can be found approximately by plotting as in Fig. 261, where ordinates equal to the computed average accelerations were erected at the proper points on the time-base, thus determining the solid curve *abcd*. Any other ordinate represents an average acceleration for an interval terminating at $t = 2.4$ seconds, thus the ordinate at 2.1 represents the average acceleration for the interval 2.1 to 2.4. The curve *abcd* may be extended a short distance without error, and therefore the ordinate over $t = 2.4$ is approximately the limit of the values 35.9, 41.9, etc., and it represents closely the magnitude of the acceleration at $t = 2.4$ seconds. This ordinate scales 66.5 feet per second per second on the original drawing already mentioned. The direction of this acceleration is the limit of the directions of the average acceleration, and obviously this limit is the tangent to the hodograph at E'. On the original drawing the angle between this tangent and the horizontal is 24 degrees.

For emphasis by contrast we will determine the way in which the *speed* changes during the motion under consideration. Speed-increments are listed under Δv in the schedule; average rates of change of speed for the respective time-intervals are listed under $\Delta v/\Delta t$. The limiting value of these averages, as Δt is taken smaller and smaller but always terminating at $t = 2.4$, is about 28 feet per second per second, and this is the rate at which the speed changes (dv/dt) at $t = 2.4$ seconds.

We now generalize the foregoing. Let AB (Fig. 262) be the path of a moving point P, and let $O'A'$ and $O'B'$ be the velocities of P when at A and B respectively. Then vector $A'B'$ is the velocity-increment for the interval Δt while P moves from A to B; (chord $A'B'$) $\div \Delta t$ is the magnitude of the average acceleration for the interval, and the direction $A'B'$ is the direction of the average acceleration. The magnitude of the (instantaneous) acceleration of P when passing A is the limit of (chord $A'B'$) $\div \Delta t$, as B is taken closer and closer to A; and the direction of the acceleration is the limit of the direction of $A'B'$ as B approaches A, or B' approaches A'. Now lim (chord $A'B'$) $\div \Delta t =$ lim (arc $A'B'$) $\div \Delta t = ds'/dt$ where ds' is the elementary portion of the hodograph at A', and s' is the distance of P' (the point in the hodograph corresponding to P) from any fixed origin on the hodograph; and the limiting direction of the chord $A'B'$ is the tangent at A'. Finally, the acceleration of P is a vector quantity whose magnitude is ds'/dt and whose direction is that of the tangent to the hodograph at the point P' corresponding to P. This final result can be viewed differently: Since the magnitude of the velocity of $P' = ds'/dt$ and its direction is along the tangent to the hodograph, the acceleration of P is the same as the velocity of its corresponding point P', it being understood that s' (distance on the hodograph) must be interpreted by the scale of the hodograph diagram, where distances represent velocity. The student should note that the acceleration of P is not directed along the tangent to the path but always toward the concave side of the path.

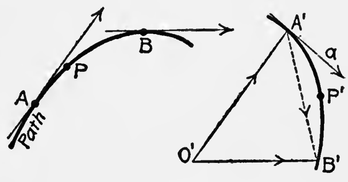


FIG. 262

As an example of the use of our final result, that the acceleration of P is given by the velocity of its corresponding point in the hodograph, we determine the acceleration of a point which describes a circle at a constant speed. Let P (Fig. 263) be the point, r = radius of the circle, and v = the speed of P. The hodograph is a circle whose radius equals v; A' corresponds to A and P' to P; and hence $A'O'P'$ equals θ. We measure the distance s (traversed by P) from A, and s' (traversed by P') from A'. Then $s'/v = s/r$, or $s' = sv/r$. Now the velocity of P' equals $ds'/dt = (ds/dt)\,(v/r) = v^2/r$, and the velocity of P' is directed along the tangent at P' (parallel to the radius OP); hence the acceleration of P is directed from P to O and its magnitude is v^2/r.

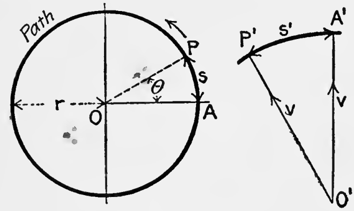


FIG. 263

The method for determining acceleration used in the preceding example is difficult to apply in most motions. Why then was the method developed at length? To make plain the meaning of acceleration in curvilinear motion and particularly to show students, in an elementary way, that acceleration in curvilinear motion does not equal dv/dt and is not directed along the tangent to the path in general. Thus in the preceding example it was found that the magnitude of the acceleration is v^2/r, whereas $dv/dt = 0$ since v is constant; also it was found that the acceleration is directed along the normal to the path. In the motion discussed at length (where $s = 2\,t^3$), it was found that the magnitude of the acceleration when $t = 2.4$ seconds is about 66.5 feet per second per second; but, since $v = ds/dt = 6\,t^2$, $dv/dt = 12\,t = 28.8$ for $t = 2.4$.*

33. Components of Velocity and Acceleration

§ 1. COMPONENTS OF VELOCITY. — Velocity, like any other vector quantity, can be resolved into components. For our purpose components parallel to axes of coördinates (as x, y, and z) are most useful; such components are called

* *Note on Rate of Change of a Vector Quantity.* — We shall have to deal with the rates of vector quantities other than velocity. Therefore we now generalize our notions on the rate of this vector quantity (velocity) just arrived at so as to prepare for the rates of these other vector quantities for future use. Let OA, OB, OC, etc. (Fig. 264), represent successive values of any vector ρ, in magnitude and direction, vector OB representing ρ at time t_1, OB at time t_2, OC at time t_3, etc. The change in ρ during the intervals t_1 to t_2, t_1 to t_3, t_1 to t_4, etc., are represented by the vectors AB, AC, AD, etc. The average rate of change in the vector ρ during any of these intervals may be found by dividing the change by the time; thus for the interval t_1 to t_2 the average rate $= AB \div (t_2 - t_1)$, and this rate is a vector whose direction is AB. For the interval t_1 to t_3, the average rate $= AC \div (t_3 - t_1)$ and the direction of the rate is AC. In general, both the magnitude and the direction of the average rate of a vector depends on the length of the interval for which the average rate is taken or computed. By true or instantaneous rate of change of the vector at the time t_1, say, is meant the limit of the average rate $AB \div (t_2 - t_1)$ as t_2 is taken closer and closer to t_1. The magnitude of this limit = limit of chord $AB \div (t_2 - t_1)$ = limit of arc $AB \div (t_2 - t_1) = dS/dt$ where dS = elementary portion of the arc; the direction of the limit is the direction of the tangent to the arc at A.

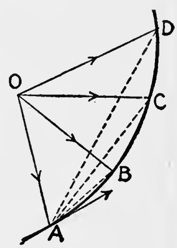


FIG. 264

Imagine a point P to move in the curve AD so that the vector OP represents the vector ρ at each instant. The velocity of $P = dS/dt$ and its direction at any instant is tangent to the curve at the point where P is at the instant; hence the time-rate of ρ is the same as the velocity of P (the moving end of ρ).

Axial Components. — Let x, y, and z = the (changing) coördinates of a moving point P, and v_x, v_y, and v_z = the components of the velocity of P parallel to the x, y, and z coördinate axes respectively; then

$$v_x = dx/dt, \qquad v_y = dy/dt, \qquad v_z = dz/dt.$$

(Proof follows.) These formulas state that each axial component of the velocity at any instant equals the rate at which the corresponding coördinate of the moving point is changing then. In the following derivation of the formulas we assume for simplicity that the path of the moving point is a plane curve — in the xy plane; proof can be extended readily to include the case of a tortuous, or twisted, path. Let P (Fig. 265) be the moving point, v = the magnitude of the velocity of P, and α = the angle which the tangent at P makes with the x axis. Then $v_x = v \cos \alpha$, and $v_y = v \sin \alpha$. But $v = ds/dt$, $\cos \alpha = dx/ds$, and $\sin \alpha = dy/ds$; hence

$$v_x = \frac{ds}{dt}\frac{dx}{ds} = \frac{dx}{dt}, \quad \text{and} \quad v_y = \frac{ds}{dt}\frac{dy}{ds} = \frac{dy}{dt}.$$

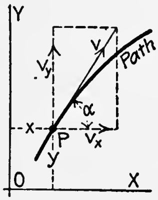


FIG. 265

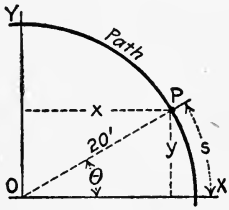


FIG. 266

For an example, we determine the x and y components of the velocity of a point P which moves in the circle of Fig. 266 according to the law $s = 2\,t^3$, s being in feet and t in seconds. (This is the motion discussed at length in the preceding article.) It is plain from the figure that $x = 20 \cos \theta = 20 \cos (s/20) = 20 \cos (0.1t^3)$; hence

$$v_x = -\,20 \sin (0.1t^3)\, 0.3\,t^2 = -\,6\,t^2 \sin (0.1t^3).$$

When $t = 2$ seconds, say, $v_x = -6 \times 4 \sin (0.8 \text{ radians}) = -\,24 \sin 45.8° = -\,17.2$ feet per second. The negative sign means that the component of the velocity is directed toward the left. In a similar way it can be shown that $v_y = 6\,t^2 \cos (0.1t^3)$.

Other Components. — The velocity of a moving point P is directed along the tangent to its path at the point where P is at the instant under consideration; hence, the tangential component of the velocity equals the velocity itself, and the velocity has no normal component (along the normal to the path). For formulas for components of velocity along and perpendicular to the radius-vector of the moving point see Hoskins' "Theoretical Mechanics," Ziwet's, or any other standard work on that subject.

§ 2. COMPONENTS OF ACCELERATION. — Acceleration is a vector quantity, and can be resolved into components therefore. The most useful components for our purposes are: — (1) Those parallel to axes of coördinates (x, y, and z), called axial components; (2) those parallel to the tangent and normal to the path of the moving point at the place where the point is at the instant in question.*

Axial Components. — Let a_x, a_y, and a_z = the axial components of the acceleration of a moving point P, and as in § 1 let v_x, v_y, and v_z = the (varying) axial components of the velocity of P, then

$$a_x = dv_x/dt, \qquad a_y = dv_y/dt, \qquad a_z = dv_z/dt.$$

(Two proofs follow.) These formulas state that each axial component of the acceleration of P at any instant equals the rate at which the corresponding axial component of the velocity of P is changing then. Since $v_x = dx/dt$, $v_y = dy/dt$, and $v_z = dz/dt$, we have also

$$a_x = d^2x/dt^2, \qquad a_y = d^2y/dt^2, \qquad a_z = d^2z/dt^2.$$

In the following proof it is assumed for simplicity, that the path of the moving point is a plane curve — in the xy plane. The proof can be extended readily to include the case of a tortuous or twisted path. Let P (Fig. 267) be the moving point. Fig. 268 shows the hodograph for the motion; P' is the

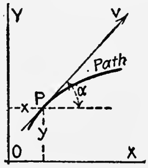


FIG. 267

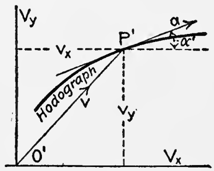


FIG. 268

point "corresponding" to P (see Art. 32 under hodograph), and the direction of the acceleration of P is tangent to the hodograph at P' as indicated. Let a = the magnitude of the acceleration, and α' = the angle between the acceleration and the x axis. Then $a_x = a \cos \alpha'$ and $a_y = a \sin \alpha'$. But $a = ds'/dt$, where ds' denotes elementary length on the hodograph (see preceding article); and since the coördinates of P' are v_x and v_y, $\cos \alpha' = dv_x/ds'$, and $\sin \alpha' = dv_y/ds'$. Hence

$$a_x = \frac{ds'}{dt}\frac{dv_x}{ds'} = \frac{dv_x}{dt}, \text{ and } a_y = \frac{ds'}{dt}\frac{dv_y}{ds'} = \frac{dv_y}{dt}.\dagger$$

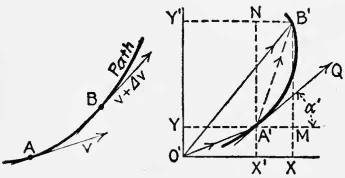


FIG. 269

* For discussion of components along and perpendicular to the radius-vector drawn from any fixed origin to the moving point see texts referred to in § 1.

† The following is an alternative proof: — Let AB (Fig. 269) be a portion of the path of the moving point P, and let $O'A'$ and $O'B'$ represent the velocities of P when at A and B. Then $A'B'$ represents the change in the velocity while P moves from A to B, and $A'M$ and $A'N$ represent the x and y components of this velocity-change. Let $A'Q$, tangent to the hodograph at A', represent the acceleration of P when at A. Then

$$a_x = a \cos \alpha' = \lim \frac{A'B'}{\Delta t} \lim (\cos B'A'M) = \lim \frac{A'B' \cos B'A'M}{\Delta t} = \lim \frac{A'M}{\Delta t}.$$

But $A'M = O'X - O'X'$ = increment in the x component of the velocity = Δv_x; hence $a_x = \lim (\Delta v_x/\Delta t) = dv_x/dt$. In a similar way one could prove that $a_y = dv_y/dt$.

For an example we determine the x and y components of the acceleration in the motion of the preceding example (see Fig. 266). In that example it was shown that the general value of the x component of the velocity (true for any instant) is $v_x = -6t^2 \sin(0.1t^3)$; hence

$$dv_x/dt \text{ (or } a_x) = -12t\sin(0.1t^3) - 1.8t^4\cos(0.1t^3).$$

And when $t = 2.4$ seconds, say, $a_x = -29.4$ feet per second per second. In a similar way the value of a_y can be found from the general expresssion for v_y.

Tangential and Normal Components.—They will be denoted by a_t and a_n respectively; other notation as before, and r = radius of curvature of the path at the point occupied by the moving point at the instant in question. Then

$$a_t = dv/dt = d^2s/dt^2, \qquad \text{and} \qquad a_n = v^2/r.$$

(Two proofs follow.) These formulas respectively state that a_t = the rate at which the speed (magnitude of the velocity) changes, and that a_n is proportional to the square of the speed directly and to the radius of curvature inversely. Where the speed is increasing, dv/dt is positive and a_t has the same direction as the velocity; where the speed is decreasing, dv/dt is negative and a_t is opposite to the velocity in direction. The normal acceleration a_n is always directed from the moving point toward the center of curvature.

Let AB (Fig. 270) be the path of a moving point P, v = velocity of P at A, and $v + \Delta v$ = velocity of P at B. Also let $O'A'$ be equal and parallel to v, and $O'B'$ be equal and parallel to $v + \Delta v$; then A' and B' are on the hodograph.

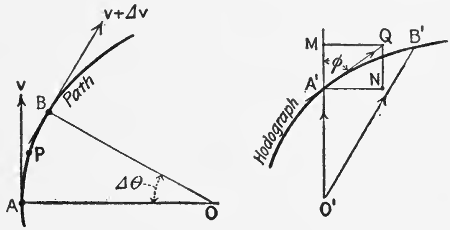


FIG. 270

The acceleration a of P when at A is parallel to the tangent $A'Q$. Let $A'Q$ = the acceleration; then $A'M$ and $A'N$ respectively represent the tangential and normal components of a. Hence

$$a_t = a\cos\phi = (ds'/dt)\cos\phi,$$

and

$$a_n = a\sin\phi = (ds'/dt)\sin\phi.$$

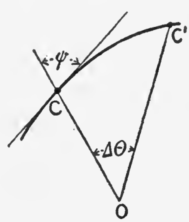


FIG. 271

For use in the foregoing equations we now recall certain formulas from calculus. In the language of that subject we call OC and OC' (Fig. 271) the radius vectors of C and C', we denote them by ρ and ρ' (or $\rho + \Delta\rho$), the arc CC' by Δl, and the angle COC' by $\Delta\theta$. It is shown in standard works on calculus that $\sin\psi = \rho\, d\theta/dl$ and $\cos\psi = d\rho/dl$. Now these formulas from calculus when applied to the curve under consideration, hodograph (Fig. 270), become $\sin \phi = v\,d\theta/ds'$ and $\cos \phi = dv/ds'$. Therefore

$$a_t = \frac{ds'}{dt}\frac{dv}{ds'} = \frac{dv}{dt}, \text{ and } a_n = \frac{ds'}{dt}\,v\frac{d\theta}{ds'} = v\frac{d\theta}{dt} = v\frac{d\theta}{ds}\frac{ds}{dt} = \frac{v^2}{r}.*$$

For an *example* we determine the tangential and normal components of the acceleration in the motion of the two preceding examples. Since $s = 2\,t^3$, $v = 6\,t^2$ and $dv/dt = 12\,t = a_t$; at $t = 2.4$ seconds, say, $a_t = 28.8$ feet per second per second. Also $a_n = v^2/r = 36\,t^4/20 = 1.8\,t^4$; at $t = 2.4$, $a_n = 59.7$ feet per second per second.

The (*resultant*) *acceleration* can be obtained from its axial or tangential and normal components. Thus

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{a_t^2 + a_n^2}.$$

* The following is an alternative proof: — Let AB (Fig. 272) be a portion of the path of the moving point P, and $O'A'$ and $O'B'$ represent the velocities of P when at A and B respectively. Then $A'B'$ represents the change in the velocity while P moves from A to B. Let

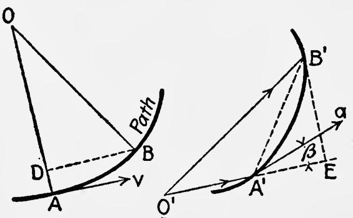


FIG. 272

v = the magnitude of the velocity at A, $v + \Delta v$ = that at B, $\Delta\theta$ = the angle between the normals (and the tangents) at A and B, and β the angle between the acceleration and the velocity at A. Then

$$a_t = a\cos\beta = \lim\,(A'B'/\Delta t)\lim\,(\cos B'A'E) =$$
$$\lim\,[(A'B'/\Delta t)\cos B'A'E] = \lim\,(A'E/\Delta t) = \lim\,[(O'B'\cos\Delta\theta - O'A')/\Delta t] =$$
$$\lim\,[v\,(\cos\Delta\theta - 1)/\Delta t + \Delta v\cos\Delta\theta/\Delta t] = \lim\,[v\,(\cos\Delta\theta - 1)/\Delta t] + \lim\,[\Delta v\cos\Delta\theta/\Delta t].$$

Now the first of these last two limits equals zero for it can be written

$$\lim\left[\frac{v\,(\cos\Delta\theta - 1)}{\Delta t}\frac{\Delta\theta}{\Delta\theta}\frac{\sin\Delta\theta}{\sin\Delta\theta}\right] = \lim\frac{v\,(\cos\Delta\theta - 1)}{\sin\Delta\theta}\lim\frac{\Delta\theta}{\Delta t}\lim\frac{\sin\Delta\theta}{\Delta\theta} =$$
$$\lim\,[v\,(\cot\Delta\theta - \csc\Delta\theta)]\,(d\theta/dt)\cdot 1 = 0\cdot(d\theta/dt)\cdot 1.$$

This final result = 0 because $d\theta/dt$ (rate at which θ changes) is not infinitely great. The second of the two limits mentioned equals $\lim\,(\cos\Delta\theta)\lim\,(\Delta v/\Delta t) = 1 \times dv/dt$. Hence $a_t = dv/dt$. Referring to the figure it will be seen that $a_n = a\sin\beta =$

$$\lim\frac{A'B'}{\Delta t}\lim\,(\sin B'A'E) = \lim\frac{A'B'\sin B'A'E}{\Delta t} = \lim\frac{B'E}{\Delta t} = \lim\frac{(v+\Delta v)\sin\Delta\theta}{\Delta t}.$$

Now $\sin\Delta\theta = BD/BO$, where O is the intersection of the normals at A and B; and if Δs stands for the arc AB then the last limit can be written

$$\lim\left[\frac{v+\Delta v}{\Delta t}\frac{BD}{BO}\frac{\Delta s}{\Delta s}\right] = \lim\frac{v+\Delta v}{BO}\lim\frac{\Delta s}{\Delta t}\lim\frac{BD}{\Delta s}.$$

The first of these three limits $= v/r$, the second $= v$, and the last $= 1$; hence $a_n = v^2/r$.

The angles which a makes with the x, y, and z axes are given respectively by

$$\cos^{-1}(a_x/a), \qquad \cos^{-1}(a_y/a), \qquad \text{and} \qquad \cos^{-1}(a_z/a).$$

The angle which a makes with the normal equals $\tan^{-1}(a_t/a_n)$. From $a = (a_t^2 + a_n^2)^{\frac{1}{2}}$ it appears that *a does not equal $a_t = dv/dt$ in general;* only when $a_n = 0$. *And a_n $(= v^2/r) = 0$ only when $v = 0$* or $r = \infty$, that is, where the moving point reverses direction of motion or where the radius of curvature is infinitely great.

Simple Harmonic Motion (see Art. 30). — The fact that the components of the velocity and acceleration — along any line — of a moving point P equal the velocity and acceleration of the projections of the point on that same line, enables one to get the formulas for velocity and acceleration in a *simple harmonic motion* very easily. Thus let P, Fig. 273, be a point describing the circle uniformly, and Q its projection on the horizontal diameter; then the motion of Q is a simple harmonic one (Art. 30). Let the amplitude of the s.h.m. (radius of the circle) = 2 feet, and the frequency of the s.h.m. (revolutions of P per unit time) = 100 vibrations per minute. Then the velocity of $P = 2\pi \times 2 \times 100 = 1260$ feet per minute = 21 feet per second, directed along the tangent at P as shown; and the acceleration of $P = 21^2 \div 2 = 220$ feet per second per second, directed along the radius PO. Now when PO makes an angle $\theta = 30°$ say, then the velocity of Q is $21 \sin 30° = 10.5$ feet per second; the acceleration of $Q = 220 \cos 30° = 180$ feet per second per second, directed toward O whether P is travelling clockwise or counter clockwise. Evidently the greatest velocity of Q obtains when Q is at O; that value equals 21 feet per second. The greatest acceleration of Q obtains when Q is at either end of its path; that value is 220 feet per second per second.

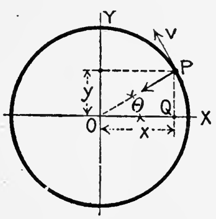


FIG. 273

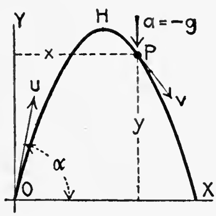


FIG. 274

General formulas for velocity and acceleration in simple harmonic motion can be as easily derived. Let r = amplitude, n = frequency. Then the velocity of $P = 2\pi rn$ and its acceleration $= 4\pi^2r^2n^2 \div r = 4\pi^2n^2r$. Hence velocity and acceleration of Q are respectively (see Fig. 273)

$$-2\pi rn \sin\theta \qquad \text{and} \qquad -4\pi^2n^2r\cos\theta.$$

§ 3. PROJECTILE WITHOUT AIR RESISTANCE. — Let u = the velocity of projection (initial velocity), and α = the angle of projection (angle between direction of projection and the horizontal), x and y = the coördinates of the projectile P (Fig. 274) at any time t after projection, v = the velocity of P, and

$a =$ the acceleration of P at the time t. The only force acting on the projectile during flight is gravity. Hence the acceleration of the projectile is vertically downwards at all times and equal to g (Art. 34), or $a_x = 0$ and $a_y = -g$. Since there is no x acceleration, the x velocity remains constant during the flight, and we find that value from the initial conditions (u, α) to be

$$v_x = u \cos \alpha. \tag{1}$$

The y velocity is decreased at all times by the y acceleration $-g$. In the interval t, that decrease is gt, and since the initial y velocity is $u \sin \alpha$, the y velocity at any time t is given by

$$v_y = u \sin \alpha - gt. \tag{2}$$

Since the x velocity remains constant, the x displacement in the interval t is given by

$$x = u \cos \alpha \cdot t. \tag{3}$$

The y velocity varies uniformly with the time; hence the average y velocity for the interval t is $\frac{1}{2}[(u \sin \alpha + (u \sin \alpha - gt)] = u \sin \alpha - \frac{1}{2} gt$. The y displacement for the interval equals the product of the average y velocity and the time or

$$y = u \sin \alpha \cdot t - \tfrac{1}{2} gt^2. \tag{4}$$

Foregoing results determine the velocity and position at any time t. They may be arrived at more directly by integrating the given equations

$$a_x = \frac{dv_x}{dt} = 0, \quad \text{and} \quad a_y = \frac{dv_y}{dt} = -g.$$

Thus integrating the first equation we find that $v_x = C_1$, where C_1 is a constant of integration whose value for reasons already stated is $u \cos \alpha$. Integrating the second equation we find that $v_y = -gt + C_2$ where C_2 is another constant of integration. From the initial conditions $v_y = u \sin \alpha$ when $t = 0$, and on substituting these values of v_y and t in the last equation we find that $C_2 = u \sin \alpha$; thus $v_y = -gt + u \sin \alpha$ as before. Now integrating $v_x = dx/dt = u \cos \alpha$, we get $x = u \cos \alpha \cdot t + C_3$. From initial conditions $x = 0$ when $t = 0$; therefore $v = 0 + C_3$, or $C_3 = 0$, and $x = u \cos \alpha \cdot t$ as before. Integrating $v_y = dy/dt = -gt + u \sin \alpha$, we get $y = -\frac{1}{2} gt^2 + u \sin \alpha \cdot t + C_4$. From initial conditions $y = 0$ when $t = 0$; therefore $0 = 0 + 0 + C_4$ or $C_4 = 0$, and $y = -\frac{1}{2} gt^2 + u \sin \alpha \cdot t$ as before.

The *trajectory* (path of the projectile) is a portion of a parabola as can be shown from the equation of the trajectory. To arrive at the equation we may combine equations (3) and (4) so as to eliminate t. Thus we find that

$$y\, 2u^2 \cos^2 \alpha = xu^2 \sin 2\alpha - gx^2. \tag{5}$$

Range and Greatest Height. — At the end X of the range, $y = 0$; hence the time of flight is given by $u \sin \alpha t - \frac{1}{2} gt^2 = 0$, or $t = (2u \sin \alpha)/g$. The range R equals the value of x in equation (3) when $t =$ the value just found; thus

$$R = (u^2 \sin 2\alpha) \div g.$$

R also equals the value of x in equation (5) when $y = 0$. The formula for R shows that the range is greatest — for a given velocity of projection — when $\alpha = 45°$. That greatest value is u^2/g.

At the highest point of the trajectory $v_y = 0$; hence the time of flight to that point is given by $u \sin \alpha - gt = 0$, or $t = (u \sin \alpha) \div g$. The height H of the trajectory equals the value of y in equation (4) when $t =$ the value just found; thus

$$H = \tfrac{1}{2} (u \sin \alpha)^2 \div g.$$

H also equals the value of y in equation (5) when $x = \frac{1}{2} R$.

34. Motion of the Center of Gravity of a Body

In Art. 31 we found that any rectilinear motion of a body depends in a very simple way upon the forces acting on the body. The relation between the motion of the center of gravity of a body (whether rigid or not) which has any sort of motion however complicated is also quite simply related to the forces exerted on the body as we shall see presently.

§ 1. A PARTICLE is a body so small that its dimensions are negligible in comparison with the range of its motion. In any motion of a particle no distinction need be made between the displacements (velocities or accelerations) of different points of the particle, for they are equal or practically so; and by displacement (velocity or acceleration) of the particle is meant the displacement (velocity or acceleration) of any point of the particle.

"Laws of Motion." — 1. *When no force is exerted upon a particle then it remains at rest or continues to move uniformly in a straight line.* 2. *When a single force is exerted upon a particle, then it is accelerated; the direction of the acceleration is the same as the direction of the force, and its magnitude is proportional to the force directly and to the mass of the particle inversely.* 3. *When one particle exerts a force upon another, then the latter exerts one on the former; and the two forces are equal, colinear, and opposite.*

These are essentially *Newton's Laws of Motion.* The form of statement here used differs however from that in which he announced them (1687). They are based on observation and experience. Newton was led to them through his study of the motions of heavenly bodies. No other moving bodies have been so accurately and extensively observed, and the agreement of the laws and those motions constitutes the best evidence of the correctness of the laws.

Law 3 has already been referred to (page 43, footnote). This law is doubted by some beginners in this subject. The doubt is sometimes expressed in this way: "When a horse pulls on a cart, then, if the cart pulls back on the horse an equal amount (as the law states), why is it that they generally move forward?" Close attention to the forces which act on the horse and on the cart should clear up this doubt. There are three forces exerted on the horse, — his weight (exerted by the earth), the pull of the cart, and the reaction exerted by the roadway on his hoofs. When the horizontal (forward) component of the reaction on his hoofs exceeds the pull back by the cart then the horse starts forward. There are three forces acting on the cart, — its weight (exerted by the earth), the pull of the horse, and the reaction of the roadway on the wheels. When the pull exceeds the horizontal (backward) reaction of the roadway then the cart starts forward. Or, the motion of horse and cart together may be explained like this: There are four forces acting upon the pair, — the weight of the horse, that of the cart, the reaction of the roadway on the horse, and that on the cart; the horse and cart start to move when the horizontal component of the reaction of the roadway on the horse exceeds that on the cart.

Law 2 is discussed at length in Art. 31 for the case of rectilinear motion, but is not referred to there as a "law." It covers curvilinear motion, as well as rectilinear, inasmuch as no reference to kind of motion is made in the law. We cannot give a real illustration of a particle moving under the action of a single force. But imagine a particle projected in some way, and then subjected to a single force inclined to the direction of projection; the particle would move in a curved path. (A ball in flight through the air is a near approach to our imagined illustration. This ball is acted upon by two forces, gravity and air resistance; but at moderate velocities the latter may be neglible in comparison with the former.) The law states that the direction of the acceleration of the particle agrees at each instant with the direction of the force, and that the magnitude of the acceleration is directly proportional to the force and inversely proportional to the mass of the particle ($a \propto F \div m$, where a = the acceleration, m = the mass of the particle, and F = the force acting upon it). It is shown in Art. 31, § 2, that the proportion $a \propto F/m$ can be written as an equation $F = Kma$ where K is a constant whose value depends on the units used for F, m, and a. Units may be chosen so that $K = 1$; such units are "systematic units"; for example, dyne (force), gram (mass), and centimeter per second per second (acceleration). We will continue to take $K = 1$ (as in Art. 31), thus implying the use of systematic units.

Law 1 is really included in law 2. For if there is no force acting on a particle during any particular interval of time, then the particle has no acceleration during the interval (according to law 2); and hence the velocity of the particle, whatever it may be, remains unchanged. Thus, if the velocity is zero at the beginning of the interval, then the velocity remains zero, that is the particle rests; if the velocity is not zero initially, then the velocity remains constant in magnitude and direction, that is the particle moves uniformly and in a straight line. This fact is important enough to warrant its statement in a separate law.

The word *inertia* is used in Mechanics to refer to the property of the matter involved more or less in laws 1 and 2. It refers to the fact that the natural state of a particle is rest or uniform rectilinear motion, that a particle is reluctant, as it were, to change that state, and responds only to an outside influence which we call force. We also express this fact by saying that matter is inert.

"Force of inertia" is a term which students sometimes use to express a notion, but generally in a vague way. For example, concerning the motion of a hockey puck projected without spin along the surface of smooth ice, it is stated sometimes that the puck is urged on by the (or its) force of inertia. This statement is at variance with the laws of motion. The only forces acting on the puck, after projection, are gravity and the reaction of the ice. There is no force urging the puck onward; it moves onward — for a time — because it was (forcibly) projected, and in spite of the retarding influence of the reaction of the ice. Were it not for this influence (friction), the puck would move across the entire field of ice at constant velocity, not because of any force urging it onward but because of no force to change its natural state (of uniform rectilinear motion).

For another illustration, imagine a yard stick mounted on a vertical axis, the wide sides of the stick being horizontal; also imagine a coin laid on the upper side and near the end of the stick remote from the axis, and that several pins are stuck about the coin to hold it in place when the yard stick is rotated. If the pins are not too strong and firm, then the stick may be rotated so rapidly that the pins will give way, and the coin will "fly off." Or, as some would say, the coin will be "thrown off by the force of inertia." Such statement is at variance with the laws of motion. The following is a description of the phenomenon in accordance with those laws. Before the stick is rotated, there are two forces acting on the coin, — its own weight (or gravity) and the reaction of the stick (upward and equal to the weight). When the stick is rotated, the coin is forced into an unnatural state (curvilinear motion) by some of the pins. We know from our experience and observation that the coin presses against the outer pins (remote from the axis) and that those pins press against the coin. Thus there is no force acting on the coin tending to throw it off the stick; on the contrary, the pins exert forces to hold it on. The coin eventually flies off — as the speed is increased — because the pressure of the coin against the pins gets large enough to make them give way; then the pins can no longer restrain the coin, and it takes on a natural state of motion. This motion is along the tangent to the (circular) path previously described by the coin at the point where the coin is supposed to have broken loose, and with velocity equal to that of the coin at failure. Of course this natural motion is short-lived, because after the coin has left the stick, it is subjected to a single unbalanced force (gravity) which interferes with the inclination — as it were — of the coin to move along the straight line mentioned (tangent).

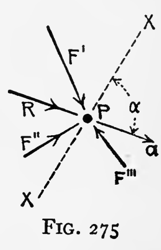


FIG. 275

When several forces act on a particle then the particle has a definite acceleration at each instant, which might of course equal zero under certain circumstances. Let F', F'', etc. (Fig. 275), be forces acting on the particle P, and $a =$ the acceleration, and $m =$ the mass of P. Obviously some single force R acting alone would give the particle that same acceleration. According to the second law R would have to act in the direction of the acceleration and equal ma. This force is the resultant of the forces F', F'', etc., which actually produce the acceleration.* Let α = the angle between the direction of the acceleration and any line, say the x axes of a coördinate frame. Then $R \cos \alpha = ma \cos \alpha$, or $R_x = ma_x$ where R_x and a_x denote the x components of R and a respectively. But R_x equals the algebraic sum of the x components of F', F'', etc. (Art. 4), and hence $\Sigma F_x = ma_x$.

§ 2. TWO OR MORE PARTICLES considered collectively are called a *system of particles*. We conceive a body (whether rigid or not) as consisting of particles, that is, as a system or collection of particles. Among the forces exerted upon any particle of a body some may be exerted by the particles of another body; such a force has been called an *external force* with reference to the body under consideration (Art. 10). A force exerted on a particle of a body by another particle of the same body is called an *internal force* with reference to the body. According to the third law of motion, if one particle of a body exerts a force upon another, then the second exerts a force upon the first; and these two forces are equal, colinear, and opposite. Hence, *a system of internal forces consists of pairs of equal, colinear, and opposite forces.*

Let Fig. 276 represent a body, not rigid necessarily, points 1, 2, 3, etc., being constituent particles of the body; let F_1, F_2, F_3, etc., be the external forces acting on the body, the other vectors (not lettered) being internal forces. Imagine the last equation of § 1 (which states that the algebraic sum of the components — along any line — of all the forces acting on a particle equals the product of the mass of the particle and the component of its acceleration along the line) written down for every particle of the body, and then imagine the left-hand members to be added and also the right-hand members; these sums are equal of course. To the first sum the internal forces contribute nothing, since those forces occur in certain pairs as already explained; hence the sum depends only on the external forces. We will denote the algebraic sum of their components along some line, say an axis of x, by ΣF_x as customarily. The second sum is $m'a_x' + m''a_x'' + \cdots$ where m', m'', etc., denote the masses of the particles and a_x', a_x'', etc., the x components of their accelerations respectively. A simple expression for this sum can be found as follows: — Let x', x'', etc., be the x-coördinates of the particles at any instant of the motion, and $\bar{x}$ = x-coördinate of their mass-center † at that instant; then

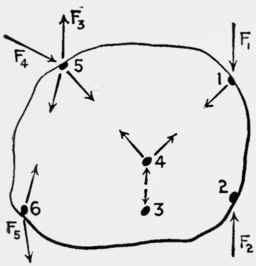


FIG. 276

$$m'x' + m''x'' + \cdots = \bar{x}\Sigma m.$$

* This force R is called resultant in accordance with the definition of the term in Art. 3, where first used. For if R were reversed, then acting alone it would give the particle an acceleration $- a$; and acting together with the forces F', F'', etc., the acceleration would be zero. R therefore is "equivalent" to F', F'', etc. All the relations between concurrent forces and their resultant developed in Statics hold here also for F', F'', etc., and R.

† *Mass-center* is another name for center of gravity. The former term seems more appro-

Differentiating with respect to time, we get

$$m'v_x' + m''v_x'' + \cdots = \bar{v}_x \Sigma m,$$

where v_x', v_x'', etc., are the x components of the velocity of the respective particles, and $\bar{v}_x$ is the x component of the velocity of the mass-center. Differentiating again we get

$$m'a_x' + m''a_x'' + \cdots = \bar{a}_x \Sigma m,$$

where $\bar{a}_x$ is the x component of the acceleration of the mass-center. If now we equate these simplified expressions for the sums mentioned we get

$$\Sigma F_x = M\bar{a}_x, \tag{1}$$

where M is written in place of Σm, the mass of the whole body, for simplicity.

Since ΣF_x does not include internal forces, $\bar{a}_x$ does not depend on those forces; that is to say, the acceleration of the mass-center of a system of particles does not depend at all upon internal forces.

Equation 1 is a mathematical form of an important principle which we will call *the principle of the motion of the mass-center.* It may be put into words as follows: *In any motion of a body (whether rigid or not) the algebraic sum of the components (along any line) of all the external forces equals the product of the mass of the body and the component of the acceleration of the mass-center along that line.* It is worth noting that equation (1) is just like the last equation of § 1 which relates to the motion of a particle. Hence, the motion of the mass-center of a body is the same as though the entire mass of the body were concentrated at the mass-center with all the external forces acting on the body applied to the dense point parallel to their actual lines of action. The use of systematic units (Art. 31) is presupposed; but if W/g be written in place of M (see Art. 31, § 2), where W is the weight of the body, then

$$\Sigma F_x = (W/g)\,\bar{a}_x \tag{2}$$

and any unit may be used for F and W, and any unit for g and $\bar{a}_x$.

Any number of equations like (1) or (2) can be written in a given case, one for each possible direction of resolving (x, y, z, u, etc.). Only three of these equations would be independent; the others would be superfluous. Thus we would have

$$\Sigma F_x = M\bar{a}_x, \qquad \Sigma F_y = M\bar{a}_y, \qquad \Sigma F_z = M\bar{a}_z.$$

When the mass-center describes a curve then it is usually more convenient to resolve along the tangent to the curve, the (principal) normal, and at right angles to the plane of the first two directions. The component of the acceleration in this last direction equals zero; calling the components in the first two

priate in the present discussion. Since masses of bodies are proportional to their weights (at the same place), we may substitute mass for weight in the formulas for the coördinates of the center of gravity (or mass-center) in Art. 21. (Mass-center is generally defined without reference to center of gravity, and then the identity of the two points is demonstrated.)

directions $\bar{a}_t$ and $\bar{a}_n$ respectively, we then have for our three equations of resolution

$$\Sigma F_t = M\bar{a}_t, \qquad \Sigma F_n = M\bar{a}_n, \qquad \Sigma F_3 = 0,$$

where ΣF_t, ΣF_n, and ΣF_3 stand for the algebraic sums of the components of all the external forces acting on the body, along the three lines of resolution just named.

Examples. — 1. Fig. 277 represents a flat car on which there is a body A weighing 4000 pounds. Suppose that the car is on a curve with no elevation of outer rail; that the speed of the car is increasing at 2 miles per hour per second. Required the reaction of the car on A at the instant when the velocity is 40 miles per hour, and where the radius of the curve is 1000 feet. There are two forces acting on A, its weight and the reaction of the car. For simplicity we imagine the reaction resolved into two components, normal pressure N (vertical) and friction F (horizontal); next we imagine F resolved into two components, along the tangent and the radius of the path of the center of mass of A, and we call them F_1 and F_2 respectively (see Fig. 277), where A is shown in plan and elevation. Therefore

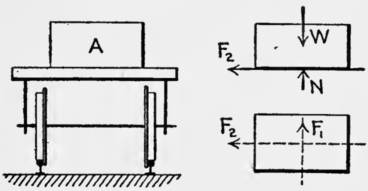


FIG. 277

$$\Sigma F_t = F_1 = M\bar{a}_t, \quad \Sigma F_n = F_2 = M\bar{a}_n, \quad \text{and} \quad \Sigma F_3 = N - 4000 = 0.$$

Now $\bar{a}_t = 2$ miles per hour per second $= 2.93$ feet per second per second. The velocity is 40 miles per hour or 58.7 feet per second, and therefore $\bar{a}_n = 58.7^2 \div 1000 = 3.44$ feet per second per second. The force-acceleration equations become $F_1 = (4000 \div 32.2)\ 2.93 = 364$ pounds, $F_2 = (4000 \div 32.2)\ 3.44 = 427$ pounds, and $N = 4000$ pounds. The reaction sought equals

$$\sqrt{(4000^2 + 364^2 + 427^2)} = 4039 \text{ pounds.}$$

We have assumed that A and the floor of the car are rough so as to furnish a frictional force large enough to hold A in place on the car; the necessary holding force $= \sqrt{(364^2 + 427^2)} = 561$ pounds.

2. A circular cylinder C (Fig. 278) is laid in a box which is mounted on a board as shown, and the whole thing is then rotated about a vertical axis AB. The weight of the cylinder is 30 pounds, $AC = 2$ feet, and the rate of rotation (constant) $= 60$ revolutions per minute. The pressures of the box on the cylinder are required. There are three forces acting on the cylinder, — its weight, the pressure P_1 of the bottom of the box, and a pressure P_2 exerted by one of the ends of the box. We assume that the cylinder rests against the lower end; the complete solution will determine whether the assumption is

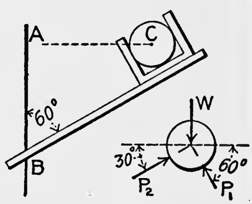


FIG. 278

correct. Because the rate of rotation is constant there are no pressures on the cylinder in the direction of motion (perpendicular to paper). The velocity of the mass-center = $2\pi 2 \times 60 = 754$ feet per minute = 12.5 feet per second; hence $a = 78$ feet per second per second, directed toward the axis of rotation. Now $\Sigma F_n = P_1 \sin 30° - P_2 \cos 30° = (30 \div 32.2)\, 78$, and $\Sigma F_3 = P_1 \cos 30° + P_2 \sin 30° - 30 = 0$. Solving them simultaneously for P_1 and P_2 we get $P_1 = 62.3$ and $P_2 = -48.0$ pounds. The negative sign means that we made a wrong assumption as to P_2; it acts downward and is exerted by the upper end of the box.

3. A *simple conical pendulum* consists of a "bob" suspended from a fixed point by a cord, arranged so the bob and cord can be rotated about a vertical through the fixed point. See Fig. 279 which represents such a pendulum by side and end views; AB is a forked vertical shaft; GG are guides fastened to the shaft, between which the bob may swing. When the shaft is rotated, the cord will deflect from the vertical. We now determine this deflection for any constant speed of rotation. Let l = length of cord, from point of suspension to the center of the bob; θ = angle of deflection; n = number of revolutions per unit time; W = weight of bob; T = tension in the cord. The bob is under the action of W, T, and the pressure P of one of the guides possibly; hence

$$\Sigma F_n = T \sin\theta = M\bar{a}_n; \quad \Sigma F_3 = T\cos\theta - W = 0; \quad \Sigma F_t = P = M\bar{a}_t.$$

When the speed is constant as here assumed, the deflection is constant, and the center of the bob describes a horizontal circle of radius $l \sin\theta$. The velocity of the center $= 2\pi l \sin\theta \cdot n$; hence $\bar{a}_n = 4\pi^2 n^2 l \sin\theta$, and $T \sin\theta = (W/g)\, 4\pi^2 n^2 l \sin\theta$. Solving this and $T\cos\theta = W$ simultaneously for θ we get $\cos\theta = g \div (4\pi^2 n^2 l)$. Also $T = W\, 4\pi^2 n^2 l \div g$; and since $\bar{a}_t = 0$, $P = 0$.

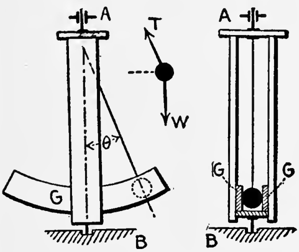


FIG. 279

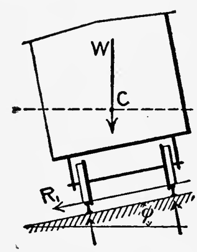


FIG. 280

Elevation of Outer Rail on Curves. — Fig. 280 represents a car "on a curve" in a railway track. We discuss certain features of the pressures of the car upon the track as the car runs around the curve. Imagine the rail pressure on each wheel resolved into three components, — one parallel to the ties (so-called flange pressure), one perpendicular to the track, and one parallel to the rails. Unless the curve is quite sharp, the forces of each of these three sets of components are parallel. We will suppose them parallel, and call the resultants of the three sets R_1, R_2, and R_3 respectively. Besides these three resultants there are acting on the car the weight W, the pull P_1 of the car ahead, and the pull P_2 of the car behind. Unless the curve is quite sharp P_1 and P_2 are practically parallel to the tangent to the curve under the middle of the car; we will assume them to be so. Then resolving along the normal (or radius of the curve), the vertical, and the tangent to the curve, we get

$$R_1 \cos \phi + R_2 \sin \phi = (W \div g)\, \bar{a}_n = (W \div g)\, v^2/r,$$

where v = velocity of the car and r = radius of the curve;

$$-R_1 \sin \phi + R_2 \cos \phi - W = 0; \quad \text{and} \quad P_1 - P_2 - R_3 = (W \div g)\, \bar{a}_t.$$

Solving the first and second simultaneously for R_1 and R_2 we get

$$R_1 = W\left(\frac{v^2}{gr} \cos \phi - \sin \phi\right), \quad \text{and} \quad R_2 = W\left(\frac{v^2}{gr} \sin \phi + \cos \phi\right).$$

It is obvious from the expression for R_1 that the resultant flange pressure may be equal to zero for certain values of v, r, and ϕ. It will be zero if $(v^2 \cos \phi) \div gr = \sin \phi$, or $\tan \phi = v^2/gr$.* When v, r, and ϕ are related in this way, then $W = R_2 \cos \phi$.

* This formula, or some modification of it, is used to determine the proper elevation of the outer rail on railroad curves, except as noted below. The following is a practical rule deduced from the formula: "The correct superelevation for any curve is equal to the middle ordinate of a chord [of the curve] whose length in feet is 1.6 times the speed of the train in miles per hour." On the Pennsylvania Railroad the rule is modified as follows: "No speed greater than 50 miles per hour should be assumed in determining the superelevation by the above method even though higher speed may be made. No superelevation exceeding 7 inches is permissible and none exceeding 6 inches should be used except at special locations on passenger tracks." The formula was deduced on the basis that resultant flange pressure should = zero. The same formula is arrived at by making ties of the track perpendicular to the resultant pressure between the floor of the car and any object resting upon it, or perpendicular to a plumb line suspended in the car.

CHAPTER IX

TRANSLATION AND ROTATION

35. Translation

A translation is such a motion of a rigid body that each straight line of the body remains fixed in direction; there is no turning about of any line of the body. The coupling or side rods of a locomotive (connecting the driving wheels on either side of the locomotive) have a translatory motion when the engine is running on a straight track. It should be noticed that our definition does not require rectilinear motion of each point of the moving body. But rectilinear translations are most common, and such translations have been quite fully discussed in Art. 31.

The motions of all points of a body in translation are alike. For, let A and B be any two points of the body, and A' and B' be the positions of those points in space at a certain instant and A'' and B'' their positions at a later instant. By definition of translation the lines $A'B'$ and $A''B''$ are parallel; and since the lines are equal in length the figure $A'B'\ B''A''$ is a parallelogram; and $A'A''$ and $B'B''$ (the displacements of A and B respectively) are equal and parallel. Since the displacements of all points of the moving body for any interval, long or short, are equal and parallel, the velocities of all points at any instant are alike, and hence also the accelerations. By displacement, velocity, and acceleration of a body having a motion of translation is meant the displacement, velocity, and acceleration respectively of any one of its points.

The general principle of Art. 34, relating to the motion of the mass-center of a body moving in any way, when applied to a translation, takes this form: the algebraic sum of the components — along any line — of the external forces acting on the body equals the product of the mass of the body and the component of the acceleration of the body along that line. This gives three independent "equations of motion," namely,

$$\Sigma F_x = Ma_x, \qquad \Sigma F_y = Ma_y, \qquad \Sigma F_z = Ma_z,$$

where x, y, and z denote three noncoplanar lines of resolution.

*The resultant of all the external forces acting on a body having a motion of translation is a single force; its line of action passes through the mass-center, the force is directed like the acceleration of the body, and its magnitude equals the product of the mass of the body and the acceleration.** Assuming that the resultant is a single force, most students will accede to the second statement in the foregoing

* The student is reminded that the resultant of a system of forces is a force, a couple, or a pair of noncoplanar forces (see Chapter I).

proposition, on the basis of their experience; for, they will say, if the resultant did not pass through the mass-center, the body would turn and not have a translatory notion. But it can be demonstrated as follows: Let Fig. 281 represent the body and points 1, 2, 3, etc., its constituent particles; the external forces acting on the body are not shown. Suppose that the acceleration is directed, say, toward the right, and let a = the magnitude of that acceleration, and m_1, m_2, m_3, etc. = the masses of the particles respectively. Then the resultants of all the forces acting on the several particles equal respectively m_1a, m_2a, m_3a, etc., all directed like the acceleration, as represented in the figure. Now this system of imaginary forces (resultants) is equivalent to all the real forces, external and internal, acting on the system of particles; and the resultant of the imaginary system and that of the real system are identical in magnitude, line of action, and sense. But the internal forces occur in pairs of equal, colinear, and opposite forces (Art. 34), and so constitute a balanced system and contribute nothing to the resultant of the real system. Hence, the resultant of the external system and that of the imaginary system are identical. We proceed now to ascertain the resultant from the latter system.

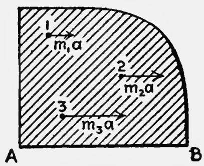


FIG. 281

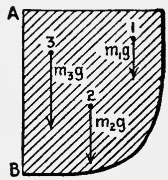


FIG. 282

The imaginary system (I) consists of parallel forces proportional to the masses of the particles, and the lines of action of the forces pass through the particles respectively. The system of earth-pulls (gravity, G) likewise consists of parallel forces proportional to the masses of the particles, and the lines of action of these pulls pass through the particles respectively. Hence systems I and G are very similar; and if we imagine the body turned so that the line AB (parallel to a) in Fig. 281 is vertical (Fig. 282) then systems I and G are still more alike. The difference is in the magnitudes of corresponding forces; the forces of I are respectively proportional to the forces of G. It follows that the line of action of the resultants of systems I and G coincide (in the body); but the resultant of system G passes through the mass-center of the body; and hence the resultant of system I (and the resultant of the external system) also passes through the mass-center. From Fig. 281 it is obvious that the resultant of the external system is a single force directed like the acceleration, and equals

$$m_1a + m_2a + \cdots = a\Sigma m = Ma.$$

The algebraic sums of the moments, or torque, of all the external forces about any line through the mass-center equals zero, for the resultant of those forces has no moment about such line. This principle gives three independent moment equations:

$$T_x = 0, \qquad T_y = 0, \qquad T_z = 0, \qquad (1)$$

where T_x, T_y, and T_z denote the moment-sums for three noncoplanar lines through the mass-center. Or we may take moments about any three lines and equate the torques of the external forces about that line to the moments of the resultant (Ma) about the same lines respectively.

Examples. — 1. A (Fig. 283) is a rectangular prism weighing 2000 pounds. The car is being started at 4 feet per second per second. Required the pressure of the car on the bottom of the prism. There are only two forces acting on the prism, — its own weight and the required pressure P. See the figure where P is shown resolved into two components (P_1 and P_2) at the base of the prism. The (unknown) distance from the point of application of P to the center of the base is denoted by x. $\Sigma F_y = P_1 - 2000 = Ma_y = 0$, or

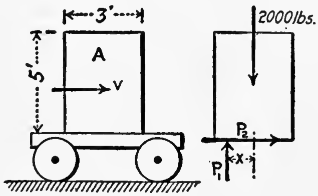


Fig. 283

$P_1 = 2000$; $\Sigma F_x = P_2 = (2000/32.2)\,4 = 248$. Hence $P = \sqrt{(2000^2 + 248^2)} = 2015$ pounds, and the inclination of P to the vertical $= \tan^{-1}(248/2000) = 8° 25'$. To determine x we take the torque, of the forces acting on the prism, about the horizontal line through the mass-center and perpendicular to the direction of motion and equate to zero. Thus $248 \times 2.5 - 2000\,x = 0$, or $x = 0.31$ feet $= 3.72$ inches. ($P_2 = 248$ pounds is friction, and the floor and prism must be rough enough to develop such a value, to prevent the slipping, here assumed not to occur. Thus the coefficient of friction must be not less than $248 \div 2000 = 0.124$ or about one-eighth. If the coefficient were less than one-eighth, the friction developed under the prism, say 200 pounds, could not give the prism an acceleration of 4 feet per second per second, only 3.22. Hence the prism would eventually be left behind. The prism is not "thrown off by the force of inertia" in such a case, as some would describe the phenomenon, but the car slips out from under the prism.)

2. C and C (Fig. 284) are two parallel cranks, their shafts being connected mechanically so that they rotate together with equal speeds and in the same direction. B is a bar pinned to the cranks. We discuss the forces acting on B when the mechanism is in motion. There are three such forces; the weight of B and the pressures of the pins on B. We will neglect the weight, or assume that the plane of the cranks is horizontal so that the bar lies upon the cranks and the supporting forces balance the weight. If the bar is uniform then it seems reasonable to

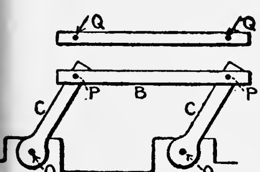


Fig. 284

assume that the pin pressures Q are parallel; if so they must be equal since the algebraic sum of their moments about the mass-center of B equals zero.

Moreover, the resultant of the pin pressures $= Q + Q = Ma$, where M = mass of the rod and a = its acceleration, and the pressures act in the direction of a. The acceleration of the bar is the same as that of the center of either pin P. If the cranks be made to turn uniformly, then the acceleration is in the direction PO and it equals v^2/r (Art. 32), where v = velocity of P and $r = PO$; hence $2Q = Mv^2/r = (W/g)(v^2/r)$, or $Q = \frac{1}{2}Wv^2/gr$.

3. Imagine a locomotive raised up off its track, and that steam is "turned on" so that the drivers are made to rotate at constant speed. If the connecting rod on one side be detached — the drivers being driven from the other side — then the side rod on the first side would be under the action of pin pressures just like those discussed in the preceding example. Each pressure equals $\frac{1}{2}Mv^2/r$, directed along its crank radius and toward the crank shaft. (The weight of the rod induces pressures equal to $\frac{1}{2}W$ upwards.)

When the locomotive is running on its track, then there is superimposed upon the motion of the side rod just discussed the forward (or backward) motion of the locomotive as a whole. The velocity of the side rod equals the vector sum of v and the velocity of the locomotive; and the acceleration of the rod equals the vector sum of the acceleration v^2/r and that of the locomotive. Now when the velocity of the locomotive is constant its acceleration is zero, and the acceleration of the side rod is still v^2/r and parallel to the cranks and directed as explained in example 1. Hence, even when the locomotive is running on a track, the pin pressures on the (lone) side rod are as when the locomotive is "jacked up" and running. Let V = speed of locomotive, R = radius of driving wheels; then $v = Vr/R$, and the pin pressures $= \frac{1}{2}(W/g)\,r\,V^2/R^2$ (weight of rod neglected). For example, let W = 275 pounds, r = 1 foot, R = 2.75 feet, and V = 60 miles per hour = 88 feet per second; then the pin pressures $= \frac{1}{2}(275/32.2) \times 1 \times (88 \div 2.75)^2 = 4425$ pounds.

Locomotive Side Rod. — We give here another solution of the side rod problem (see preceding examples). In Fig. 285 each pin pressure on the rod is represented by two components, horizontal and vertical. The vertical components are equal since the sum of the moments of all the forces acting on the rod (pressures and weight) about the center of gravity (at mid-length of the rod) equals zero; hence both vertical components are denoted by the same letter Y. The horizontal components are X_1 and X_2. Let a = the total, or absolute, acceleration of any and every point of the rod when the cranks make any angle θ with the downward vertical, and a_x and a_y = the horizontal and vertical components of a. Then

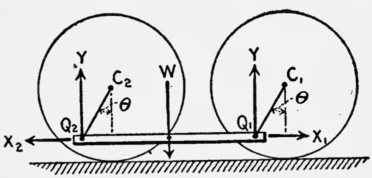


FIG. 285

$$X_1 - X_2 = Ma_x, \quad \text{and} \quad 2Y - W = Ma_y, \quad \text{or} \quad Y = \tfrac{1}{2}(W + Ma_y).$$

Presently we show how to find a_x and a_y for any position of the cranks. Then from the above we can determine $X_1 - X_2$ and Y. The values of X_1 and X_2 depend upon the load or pull on the locomotive, and how it is distributed among the driving wheels. But Y does not depend on the pull, only on W and a_y.

We now discuss the motion of one of the crank pins with the view of obtaining formulas for a_x and a_y. Let V = the velocity of the locomotive, A = its acceleration, R = radius of the driving wheels, and r = length of the cranks (CP, Fig. 286). It will be convenient to refer the motion of the crank-pin P to the coördinate axes shown; OY is the position occupied by the crank when P was in its lowest position. Let s be the distance of C from OY, and x and y the coördinates of P. Then

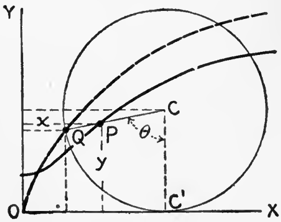


FIG. 286

$$s = R\theta,$$
$$x = s - r \sin \theta,$$
and
$$y = R - r \cos \theta$$

Now $a_x = d^2x/dt^2$ and $a_y = d^2y/dt^2$, and for use below $V = ds/dt = R\,d\theta/dt$, or $d\theta/dt = V/R$. Thus

$$\frac{dx}{dt} = \frac{ds}{dt} - r \cos \theta \cdot \frac{d\theta}{dt} = V - r \cos \theta \cdot \frac{V}{R} = V\left(1 - \frac{r}{R} \cos \theta\right);$$

$$a_x = \frac{dV}{dt}\left(1 - \frac{r}{R} \cos \theta\right) + V\frac{r}{R} \sin \theta \cdot \frac{d\theta}{dt} = A\left(1 - \frac{r}{R} \cos \theta\right) + \frac{V^2}{R^2} r \sin \theta;$$

$$\frac{dy}{dt} = r \sin \theta \cdot \frac{d\theta}{dt} = V\frac{r}{R} \sin \theta;$$

$$a_y = \frac{dV}{dt}\frac{r}{R} \sin \theta + V\frac{r}{R} \cos \theta \cdot \frac{d\theta}{dt} = A\frac{r}{R} \sin \theta + \frac{V^2}{R^2} r \cos \theta.$$

Thus it is seen that a_x and a_y depend on the velocity and acceleration of the locomotive. The largest values of a_x and a_y obtain at high speed, and then the A terms (in the expressions for a_x and a_y) are small and negligible compared to the V terms. So when we neglect these terms or when the acceleration of the locomotive is zero, then

$$a_x = (V/R)^2\, r \sin \theta, \quad \text{and} \quad a_y = (V/R)^2\, r \cos \theta.$$

When the rod is in its lowest position, $\theta = 0$, $a_x = 0$, $a_y = (V/R)^2 r$, $X_1 = X_2$, and $Y = \frac{1}{2} W + \frac{1}{2} (W/g)\,(V/R)^2 r;$ the forces Y act upward on the rod. When $\theta = 90°$, $a_x = (V/R)^2 r$, $a_y = 0$; the resultant of the two forces X acts toward the right and equals $(W/g)\,(V/R)^2 r$, and $Y = \frac{1}{2} W$. When the rod is in its highest position $\theta = 180°$, $a_x = 0$, $a_y = -(V/R)^2 r$; $X_1 = X_2$, and $Y = \frac{1}{2} W - \frac{1}{2} (W/g)\,(V/R)^2 r;$ for high speeds Y acts down on the rod. When $\theta = 270°$, $a_x = -(V/R)^2 r$, $a_y = 0$; the resultant of X_1 and X_2 acts toward the left and equals $(W/g)\,(V/R)^2 r$, and $Y = \frac{1}{2} W$.

36. Moment of Inertia and Radius of Gyration

§ 1. GENERAL PRINCIPLES, ETC. — Perhaps every student has observed that the effort required to start a body to rotating about a fixed axis seems to depend not only on the mass of the body but also on the remoteness of the material of the body from the axis of rotation. Fig. 287 represents a simple apparatus by means of which one can roughly "sense" this fact. It consists of a vertical shaft S to which a grooved pulley P and cross arm A are fastened rigidly, and a heavy body B which can be clamped on the cross arm. The pull or turning effort may be applied by means of a cord wrapped about the pulley. It is shown in the following article that this "rotational inertia" of a body is proportional to the "moment of inertia" of the body about the axis, and this article is devoted to a discussion of moment of inertia, as a preparation for the following article.*

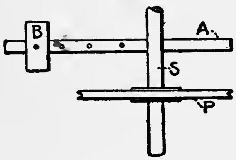


FIG. 287

The moment of inertia of a body with respect to a line is the sum of the products obtained by multiplying the mass of each particle of the body by the square of its distance from the line. Or, if I = moment of inertia, m_1, m_2, m_3, etc. = the masses of the particles, and r_1, r_2, r_3, etc., their distances respectively from the line or axis, then

$$I = m_1r_1^2 + m_2r_2^2 + \cdots = \Sigma mr^2;$$

or if the body is continuous, then

$$I = \int dM \cdot r^2, \tag{1}$$

where dM denotes the mass of any elementary portion and r its distance from the line about which moment of inertia is taken. The elementary portion must be chosen so that each point of it is equally distant from the line, there is doubt as to what distance to take for r.

It is plain from the foregoing formulas that a unit of moment of inertia depends upon the unit of mass and distance used. There is no single-word name for any unit of moment of inertia. Each unit is described by stating the units of mass and distance involved in it, and in accordance with the "make-up"

* Euler (1707–83) introduced the term "moment of inertia," and he explained its appropriateness (in his "Theoria Motus Corporum Solidorum," p. 167) somewhat as follows: The choice of the name, moment of inertia (Ger. tragheits-moment), is based on analogies in the equations of motion for translations and rotations. In a translation the acceleration is proportional directly to the "accelerating force" and inversely to the mass, or "inertia," of the moving body; and in a rotation the angular acceleration is proportional directly to the moment of the accelerating force and inversely to a quantity, Σmr^2, depending on the mass or inertia. This quantity, to complete a similarity, we may call "moment of inertia." Then we have for translations and rotations respectively,

linear acceleration = (force)/(inertia or mass); and
angular acceleration = (moment of force)/(moment of inertia).

of the unit. Thus, when the pound and the foot are used as units of mass and length respectively, then the unit of moment of inertia is called a pound-foot square; when the slug (about 32.2 pounds) and the foot are used, then the unit moment of inertia is called slug-foot square.*

The moment of inertia of any right prism — cross section of any form — with respect to any line parallel to the axis of the prism can be computed in a special way, preferred by some. Thus if we take as elementary portion a filament of the prism parallel to the axis, then $dM = (adA)\,\delta$ where $a =$ the altitude of the prism, $dA =$ the cross section of the filament, and $\delta =$ density; and

$$I = a\delta \int dA\, r^2. \tag{2}$$

This integral (extending over the area of the cross section) is called the moment of inertia of the cross section about the line specified (see appendix B).

Since a moment of inertia is one dimension in mass and two in length, it can be expressed as the product of a mass and a length squared; it is sometimes convenient to so express it. The *radius of gyration* of a body with respect to a line is such a length whose square multiplied by the mass of the body equals the moment of inertia of the body with respect to that line. That is, if k and I denote the radius of gyration and moment of inertia of the body with respect to any axis and $M =$ its mass, then

$$k^2M = I \quad \text{or} \quad k = \sqrt{I/M}. \tag{3}$$

The radius of gyration may be viewed as follows: If we imagine all the material of a body concentrated into a point so located that the moment of inertia of the material point about the line in question equals the moment of inertia of the body about that line, then the distance between the line and the point equals the radius of gyration of the body about that line. The material point is sometimes called the *center of gyration* of the body for the particular line.

To furnish still another view of radius of gyration we call attention to the fact that the square of the radius of gyration of a homogeneous body with respect to any line is the mean of the squares of the distances of all the equal elementary parts of the body from that line. For let r_1, r_2, etc., be the distances from the elements, dM, to the axis, and let n denote their number (infinite). Then the mean of the squares is

$$(r_1^2 + r_2^2 + \cdots)/n = (r_1^2\, dM + r_2^2\, dM + \cdots)/n\, dM = I/M = k^2.$$

Obviously the radius of gyration of a body with respect to a line is intermediate between the distances from the line to the nearest and most remote particle of the body. This fact will assist in estimating the radius of gyration of a body.

Examples. — 1. Required to show that the moment of inertia of a slender rod about a line through the center and inclined at an angle with the rod is $\frac{1}{12} Ml^2 \sin^2 \alpha$, where $M =$ mass, $l =$ length, and $\alpha =$ angle between the line and the axis of the rod. Let $a =$ the cross section of the rod, $\delta =$ the density, and

* For dimensions of a unit of moment of inertia, see Appendix A.

x = the distance of any elementary portion from the middle of the rod AB (Fig. 288). Then $dM = \delta\,(a\,dx)$, and the distance of the element from $CD = x \sin\alpha$. Hence

$$I = \int_{-\frac{1}{2}l}^{+\frac{1}{2}l} \delta a\,dx \cdot x^2 \sin^2\alpha = \delta a \sin^2\alpha \left[\frac{x^3}{3}\right]_{-\frac{1}{2}l}^{+\frac{1}{2}l} = \frac{\delta a \sin^2\alpha}{3}\,\frac{l^3}{4},$$

and this reduces to $\frac{1}{12} Ml^2 \sin^2\alpha$, since $\delta al = M$.

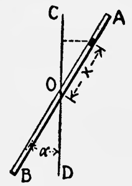


FIG. 288

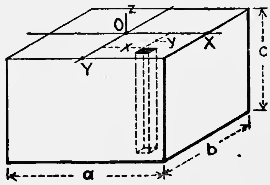


FIG. 289

2. Required to show that the moment of inertia of a right parallelopiped about a central axis parallel to an edge equals $\frac{1}{12} M\,(a^2 + b^2)$ where M = mass of the parallelopiped and a and b = the lengths of the edges which are perpendicular to that axis. See Fig. 289 where the z axis is the one to which this moment of inertia corresponds. We take for elementary portion a volume $dx\,dy\,dz$; its mass $= \delta\,(dx\,dy\,dz)$, and the square of its distance from the z axis $= x^2 + y^2$. Hence

$$I = \delta \int_{-a/2}^{+a/2} \int_{-b/2}^{+b/2} \int_0^c (x^2 + y^2)\,dx\,dy\,dz = \frac{\delta c}{12}\,(a^3 b + ab^3) = \text{etc.}$$

3. Required to show that the moment of inertia of a right circular cylinder with respect to its axis is $\frac{1}{2} Mr^2$, where M = its mass and r = the radius of its base. We use the special method for prisms (see equation 2) and choose polar coördinates (see Fig. 290); then $dA = \rho\,d\theta\,d\rho$ and $dM = \delta\,(a\rho\,d\theta\,d\rho)$; hence

$$I = a\delta \int dA\,\rho^2 = a\delta \int_0^r \int_0^{2\pi} \rho^3\,d\rho\,d\theta = \frac{a\delta r^4 2\pi}{4} = \text{etc.}$$

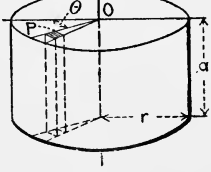


FIG. 290

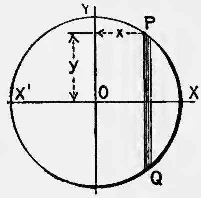


FIG. 291

4. Required to show that the moment of inertia of a sphere about a diameter is $\frac{2}{5} Mr^2$ where M = its mass and r = its radius. We might begin with equation (1) but we will use a special method, making use of the result found in example 3. We conceive the sphere made of laminas perpendicular to the diameter in question; determine the moment of inertia of each lamina; and then add the moments of inertia of the laminas. Let XX' (Fig. 291) be the diameter in

question, and PQ a section of one of the laminas; then the mass of the lamina is $\delta\,(\pi y^2\,dx)$. According to example 3 the moment of inertia of this lamina (cylinder) about its axis (XX') is $\frac{1}{2}\,\delta\,(\pi y^2\,dx)y^2$. Hence the moment of inertia of the sphere is

$$\int_{-r}^{+r} \tfrac{1}{2}\,\delta\,(\pi y^4\,dx) = \tfrac{1}{2}\,\pi\delta \int_{-r}^{+r} (r^2 - x^2)^2\,dx = \tfrac{8}{15}\,\delta\pi\,r^5 = \text{etc.}$$

§ 2. PARALLEL AXIS THEOREM; REDUCTION OR TRANSFORMATION FORMULAS. — There is a simple relation between the moments of inertia (and the radii of gyration) of a body with respect to parallel lines one of which passes through the mass-center of the body. By means of this relation we can simplify many calculations of moment of inertia, and avoid integrations (see examples following); it may be stated as follows:

The moment of inertia of a body with respect to any line equals its moment of inertia with respect to a parallel line passing through the mass-center plus the product of the mass of the body and the square of the distance between the lines. Or, if I = the first moment of inertia, $\bar{I}$ = the second (for the line through the mass-center), M = mass, and d = the distance between the parallel lines,

$$I = \bar{I} + Md^2. \tag{4}$$

Proof. — Let O (Fig. 292) be the mass-center, and P any other point of the body (not shown), LL the line about which the moment of inertia is I, and OZ a parallel line (through the mass-center) about which the moment of inertia is $\bar{I}$. Distance between these parallel lines is d. For convenience we take x and y axes through O, the former in the plane of the two parallel lines and the latter perpendicular to that plane. Let x, y, and z = the coördinates of P. The square of the distance of P from the z axis equals $x^2 + y^2$, hence $\bar{I} = \int dM\,(x^2 + y^2)$. The square of the distance of P from the line LL equals $(d - x)^2 + y^2$, hence

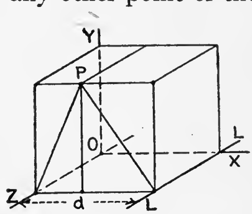


FIG. 292

$$I = \int [(d - x)^2 + y^2]\,dM = \int (x^2 + y^2)\,dM + d^2 \int dM - 2\,d \int x\,dM.$$

Now the first of the last three integrals = $\bar{I}$, and the second one = Md^2. If now we show that the third = 0, then formula (4) is proved. The third integral is proportional to the moment of the body with respect to the yz plane; but this plane contains the mass-center, and hence the moment equals zero (Arts. 21 and 23). Thus, if W = weight of the body,

$$\int x\,dM = \int x\,dW/g = (1/g)W\bar{x}.$$

If we divide both sides of equation (4) by M, we get $I/M = \bar{I}/M + d^2$, or

$$k^2 = \bar{k}^2 + d^2 \tag{5}$$

that is, *the square of the radius of gyration of a body with respect to any line equals the square of its radius of gyration with respect to a parallel line passing through the mass-center plus the square of the distance between the two lines.* According to (5) k is always greater than d; that is, the radius of gyration of a body with respect to a line is always greater than the distance from the line to the center of gravity of the body. But, if the dimensions of the cross sections of the body perpendicular to the line in question are small compared to d, then $\bar{k}/d$ is small compared to 1, and k equals d approximately (see example 2). In such a case the moment of inertia is approximately equal to Md^2.

Examples. — 1. Required the moment of inertia of a prism of cast iron (weighing 450 pounds per cubic foot) 6 inches $\times$ 9 inches $\times$ 3 feet with respect to one of the long edges. The block weighs 507 pounds. According to example 2, § 1, the moment of inertia of the block with respect to the line through the mass-center parallel to the long edge is 507 $(6^2 + 9^2) \div 12 = 4940$ pounds-inches². The square of the distance from a long edge to the mass-center = 29.25 inches²; hence the moment of inertia desired = $4940 + 507 \times 29.25$ = 19,760 pound-inches² = 4.27 slug-feet².

2. Required the radius of gyration of a round steel rod 1 inch in diameter with respect to a line 12 inches from the axis of the rod. According to example 3, § 1, the square of the radius of gyration of the rod with respect to its axis is $\frac{1}{2}\,0.5^2 = 0.125$ inches². According to equation (5) the radius of gyration desired $= \sqrt{(0.125 + 144)} = 12.01$, nearly the same as the distance from the line of reference to the mass-center of the rod.

3. It is required to show that the moment of inertia of a right circular cone with respect to a line through the apex and parallel to the base $= \frac{3}{20} M\,(r^2 + 4\,a^2)$ where M = mass of the cone, r = radius of its base, and a = its altitude.

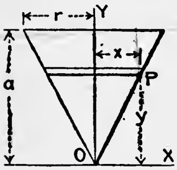


FIG. 293

We conceive the cone as made of laminas parallel to the base, find the moment of inertia of each lamina with respect to the specified line, and then add all the moments. For convenience we take the axis of the cone as the y-coördinate axis, and the line for which the moment of inertia is required as the x axis (Fig. 293). The moment of inertia of the lamina indicated about a diameter is $\frac{1}{4}\,dM \cdot x^2$ where dM = the mass of the lamina and x = its radius. Hence its moment of inertia about the x axis $= \frac{1}{4}\,dMx^2 + dMy^2$ (see equation $\overset{4}{3}$), and the moment of inertia of the entire cone $= \int (\frac{1}{4}\,dMx^2 + dMy^2)$, the limits being assigned so as to include all laminas. We choose to integrate with respect to y, and so must express dM and x in terms of y. From similar triangles in the figure $x/y = r/a$, or $x = ry/a$; obviously $dM = \delta\pi x^2\,dy = \delta\pi\,(r^2y^2/a^2)\,dy$ where δ = density. Hence

$$I = \int_0 \frac{\pi r^4 \delta y^4\,dy}{4\,a^4} + \int_0^a \frac{\pi r^2 \delta y^4\,dy}{a^2} = \frac{\pi r^4 \delta a}{20} + \frac{\pi r^2 \delta a^3}{5} = \text{etc.}$$

Composite Body. — By this term is meant a body which one naturally conceives as consisting of finite parts, for example, a flywheel which consists of a hub, several spokes, and a rim. The moment of inertia of such a body with respect to any line can be computed by adding the moments of inertia of all the component parts with respect to that same line. The radius of gyration of a composite body does not equal the sum of the radii of gyration of the component parts. It can be determined from equation (3), where I = moment of inertia of the whole body and M = its mass.

§ 3. RADIUS OF GYRATION OF SOME HOMOGENEOUS BODIES. — Let k = radius of gyration, a subscript with k referring to the axis with respect to which k is taken; thus k_x means radius of gyration with respect to the x axis. Also M = mass and δ = density.

Straight Slender Rod. — Let l = its length, α = angle between the rod and the axis. Then about an axis through the mass-center $k^2 = \frac{1}{12} l^2 \sin^2 \alpha$; about an axis through one end of the rod $k^2 = \frac{1}{3} l^2 \sin^2 \alpha$.

Slender Rod Bent into a Circular Arc (Fig. 294). — Let r = radius of the arc, then

$$k_x^2 = \tfrac{1}{2} r^2 [1 - (\sin \alpha \cos \alpha)/\alpha], \text{ and } k_y^2 = \tfrac{1}{2} r^2 [1 + (\sin \alpha \cos \alpha)/\alpha].$$

The divisor α must be expressed in radians (1 degree = 0.0175 radians). $k_z^2 = r^2$ (the z axis is through O and perpendicular to the plane of the arc).

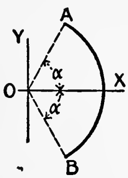


FIG. 294

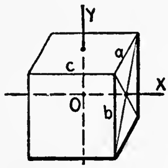


FIG. 295

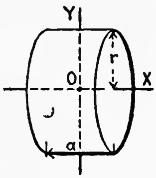


FIG. 296

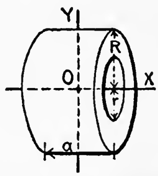


FIG. 297

Right Parallelopiped (Fig. 295). — The axis OX contains the mass-center, and is parallel to the edge c; $k_x^2 = \frac{1}{12}(a^2 + b^2)$.

Right Circular Cylinder (Fig. 296). — Both axes OX and OY contain the mass-center, r = radius and a = altitude; then

$$k_x^2 = \tfrac{1}{2} r^2; \quad k_y^2 = \tfrac{1}{12}(3 r^2 + a^2).$$

Hollow Right Circular Cylinder (Fig. 297). — Let R = outer radius, r = inner radius, and a = altitude; then

$$k_x^2 = \tfrac{1}{2}(R^2 + r^2); \quad k_y^2 = \tfrac{1}{4}(R^2 + r^2 + \tfrac{1}{3} a^2).$$

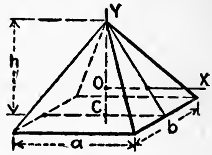


FIG. 298

Right Rectangular Pyramid (Fig. 298). — The x axis contains the mass-center and is parallel to the edge a; $M = \frac{1}{3} abh\delta$.

$$k_x^2 = \tfrac{1}{12}(b^2 + \tfrac{3}{4} h^2); \quad k_y^2 = \tfrac{1}{20}(a^2 + b^2).$$

Right Circular Cone (Fig. 299). — The x axis contains the mass-center, and is parallel to the base; $M = \frac{1}{3}\pi r^2 a\delta$.

$$k_x^2 = \tfrac{3}{20}(r^2 + \tfrac{1}{4}a^2); \quad k_y^2 = \tfrac{3}{10}r^2; \quad k_z^2 = \tfrac{3}{20}(r^2 + 4a^2).$$

Frustum of a Cone. — Let R = radius of larger base, r = radius of smaller base, and a = altitude. For the axis of the frustum

$$k^2 = \tfrac{3}{10}(R^5 - r^5) \div (R^3 - r^3); \quad I = \tfrac{1}{10}\pi h\delta (R^5 - r^5) \div (R - r).$$

Sphere. — Let r = radius. For a diameter

$$k^2 = \tfrac{2}{5}r^2; \quad I = \tfrac{8}{15}\pi r^5\delta.$$

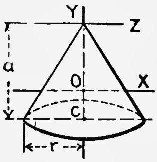


FIG. 299

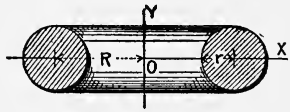


FIG. 300

Hollow Sphere. — Let R = outer and r = inner radius. For a diameter

$$k^2 = \tfrac{2}{5}(R^5 - r^5) \div (R^3 - r^3); \quad I = \tfrac{8}{15}\pi (R^5 - r^5)\,\delta.$$

Ellipsoid. — Let $2a$, $2b$, and $2c$ = length of axes. For the axis whose length = $2c$,

$$k^2 = \tfrac{1}{5}(a^2 + b^2); \quad I = \tfrac{4}{15}\pi abc\delta\,(a^2 + b^2).$$

Paraboloid Generated by Revolving a Parabola about its Axis. — Let r = radius of base and h = its height. For the axis of revolution

$$k^2 = \tfrac{1}{3}r^2; \qquad I = \tfrac{1}{6}\pi r^4 h\delta.$$

Ring (Fig. 300). — The x axis contains the mass-center and is parallel to the plane of the ring; the y axis is the axis of the ring.

$$k_x^2 = \tfrac{1}{2}R^2 + \tfrac{5}{8}r^2; \quad I_x = \pi^2 R r^2\delta\,(R^2 + \tfrac{5}{4}r^2).$$
$$k_y^2 = R^2 + \tfrac{3}{4}r^2; \quad I_y = 2\pi^2 R r^2\delta\,(R^2 + \tfrac{3}{4}r^2).$$

§ 4. EXPERIMENTAL DETERMINATION OF MOMENT OF INERTIA. — When the body is so irregular in shape that the moment of inertia desired cannot be computed easily, then an experimental method may be simpler. There are several experimental methods available.

By Gravity Pendulum. — This method is available if the body can be suspended and oscillated, like a pendulum, about an axis coinciding with or parallel to the line with respect to which the moment of inertia is desired. Let T = the time of one complete (to and fro) oscillation, c = distance from the mass-center to the axis of suspension, W = weight of the pendulum, g = acceleration due to gravity, k = radius of gyration, and I = moment of inertia about the axis of suspension; then

$$k = \frac{T}{2\pi}\sqrt{cg} \quad \text{and} \quad I = \frac{T^2 l W}{4\pi^2}. \qquad (1)$$

Above formulas are based on the formula for the time of oscillation or period of a pendulum $T = 2\pi \sqrt{k^2/cg}$ (see Art. 39). If the axis of suspension does not coincide with the line about which the moment of inertia is desired, then it remains to "transfer" I to that line (see § 2).

The desired moment of inertia can be determined without any time observation as follows: From the same axis about which the suspended body oscillates suspend a "mathematical pendulum," a very small bob with cord suspension (see Art. 39); adjust the length of the cord so that the periods (times of oscillation) of bob and body become equal; then

$$k = \sqrt{cl}, \text{ and } I = Wcl/g, \tag{2}$$

where l = the distance from the center of the bob to the axis of suspension and k, W, c, I have the same meaning as above. The foregoing result is based on the fact that k^2/c (for the pendulum) equals the length l of the mathematical pendulum (see Art. 39).

By Torsion Pendulum. — The torsion pendulum here referred to would consist of an elastic wire suspended in a vertical position, the lower end being fashioned or terminated in a disk so that objects, whose moments of inertia are to be determined, may be suspended on the wire and made to oscillate about its axis. Let t = the (observed) period (time of one oscillation) of the bare pendulum, t_1 = the (observed) period of the pendulum when it is loaded with a body A which is so regular in shape (as a cube or cylinder) that its moment of inertia about the axis of oscillation can be computed easily, and t_2 = the (observed) period of the pendulum when it is loaded with the body B whose moment of inertia is desired; further let I_1 = the (computed) moment of inertia of A and I_2 = the moment B about the axis of suspension. B should be suspended so that the axis of suspension coincides with or is parallel to the line (of B) about which the moment of inertia is desired. Then

$$I_2 = I_1 (t_2 - t) \div (t_1 - t). \tag{3}$$

This result is based on the fact that the square of the period of a torsion pendulum is proportional to the moment of inertia of the pendulum with respect to the axis of oscillation. Thus, if I = the moment of inertia of the bare pendulum, and C the proportionality factor, then

$$t^2 = CI, \; t_1^2 = C(I + I_1), \text{ and } t_2^2 = C(I + I_2).$$

These three equations may be combined so as to eliminate C and I and thus give equation (3).

If B cannot be suspended so as to make the axis of oscillation and the line (of B) about which the moment of inertia of B is desired coincident, then it remains to reduce, or transform, I_2 to that line (see § 20 of this article).

37. Rotation

§ 1. A rotation is such a motion of a rigid body that one line of the body or of the extension of the body remains fixed. The fixed line is the axis of the rotation. The motion of the flywheel of a stationary engine is one of rotation, and the axis of rotation is the axis of the shaft on which the wheel is mounted; the motion of an ordinary clock pendulum is one of rotation, and the axis of rotation is the horizontal line through the point of support and perpendicular to the axis of the pendulum. Obviously all points of a rotating body, except those on the axis if any, describe circles whose centers are in the axis and whose planes are perpendicular to the axis. The plane in which the mass-center of the body moves will be called the *plane of the rotation*, and the intersection of the axis of rotation and the plane of rotation will be called *center of rotation*. All points of the body on any line parallel to the axis move alike; hence the motion of the projection of the line on the plane of the motion represents that of all the points, and the motion of the body itself is represented by the motion of its projection.

By *angular displacement* of a rotating body during any time interval is meant the angle described during that interval by any line of the body perpendicular to the axis of rotation. Obviously all such lines describe equal angles in the same interval, and we select a line which cuts the axis. Let the irregular outline (Fig. 301) represent a rotating body, the plane of rotation being that of the paper, and O the center of rotation. Let P be any point and θ the angle XOP, OX being any fixed line of reference. As customarily, θ is regarded as positive or negative according as OX when turned about O toward OP moves counter clockwise or clockwise. If θ_1 and θ_2 denote initial and final values of θ corresponding to any rotation, then the angular displacement $= \theta_2 - \theta_1$.

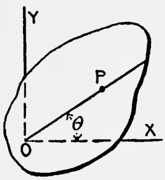


FIG. 301

The *angular velocity* of a rotating body is the time-rate at which its angular displacement occurs; or, otherwise stated, it is the time-rate at which any line of the body perpendicular to the axis describes angle. The time-rate at which OP describes angle, or the time-rate (of change) of θ is $d\theta/dt$ (see Art. 1, Note). Hence, if ω denotes angular velocity,

$$\omega = d\theta/dt. \tag{1}$$

Any angular displacement divided by the duration of that displacement gives the *average angular velocity* for that duration or interval of time. If the body is rotating uniformly (describing equal angles in all equal intervals of time), then the average velocity is also the actual velocity.

The formulas for angular velocity imply as *unit** the angular velocity of a body rotating uniformly and making a unit angular displacement in each unit time. There are several such units; thus, one revolution per minute, one degree per hour, one radian per second, etc. The last is the one usually used

* For dimensions of units of angular velocity and acceleration, see Appendix A.

herein. An angular velocity must be regarded as having *sign*, the same as that of $d\theta/dt$. Since $d\theta/dt$ is positive or negative according as θ increases or decreases algebraically, the angular velocity of a rotating body at any instant is positive or negative according as it is turning in the counter clockwise or clockwise direction at that time.

The *angular acceleration* of a rotating body is the time-rate (of change) of its angular velocity. If, as in the preceding, ω denotes the angular velocity, then the general expression for the time-rate of the angular velocity is $d\omega/dt$; hence if α denotes the angular acceleration,

$$\alpha = d\omega/dt = d^2\theta/dt^2. \tag{2}$$

The change in angular velocity which takes place during any interval of time divided by the length of the interval gives the *average angular acceleration* for that interval. If the velocity changes uniformly, then this average acceleration is also the actual acceleration.

The foregoing formulas imply as *unit** the angular acceleration of a body whose angular velocity is changing uniformly and so that unit angular velocity-change occurs in each unit time. One revolution per second per second, one radian per second per second, etc., are such units. An angular acceleration must be regarded as having *sign* — the same as that of $d\omega/dt$. Since $d\omega/dt$ is positive or negative according as ω increases or decreases algebraically, an angular acceleration is positive or negative according as the angular velocity is increasing or decreasing (algebraically).

There are simple *relations between the linear velocity v and linear acceleration a of any point P of a rotating body and the angular velocity and acceleration of the body.* Let r = the distance of P from the axis of rotation, s = distance travelled by P in any time from some fixed point in the path of P, and θ = the angle described by the radius to P in that same time. Then $s = r\theta$ if θ be expressed in radians; $ds/dt = r\, d\theta/dt$, or

$$v = r\omega.$$

Differentiating again, we find that $dv/dt = r\, d\omega/dt$, or

$$a_t = r\alpha; \quad \text{also} \quad a_n \,(= v^2/r) = r\omega^2.$$

Here a_t and a_n mean the tangential and normal components of the acceleration of P (Art. 34).

§ 2. EQUATION OF MOTION. — We have already called attention to the fact (Art. 36, footnote) that in the case of rotation the angular acceleration is proportional to the algebraic sum of the moments of all the external forces acting on the body directly and to the moment of inertia of the body inversely, both moments being about the axis of rotation. Or, if T_0 and I_0 be used to denote these moments, and α = the angular acceleration, then α is proportional to $(T_0 \div I_0)$; and, if systematic units (Art. 4) be used then

$$T_0 = I_0\alpha = Mk_0^2\alpha, \tag{3}$$

*** For dimensions of units of angular velocity and acceleration, see Appendix A.**

where M = mass of the body and k_0 = its radius of gyration about the axis of rotation. If W/g be written for M (Art. 4, § 2), then any unit of force (including W), any unit of length, and any unit of time may be used in (3).

The foregoing is called the equation of motion for a rotation; it may be derived from a consideration of the torque, about the axis of rotation, of all the forces acting on each particle of the body. Let P' (Fig. 302) represent a particle of the rotating body not shown, m' = its mass, and a' = its acceleration. Then the resultant of all the forces acting on $P' = m'a'$, and the tangential, normal, and axial components of this force are $m'a_t'$, $m'a_n'$, and 0 respectively. Similarly the tangential, normal, and axial components of the resultant of all the forces acting on the second particle P'' are $m''a_t''$, $m''a_n''$, and 0. All the radial or normal components are directed toward the axis of rotation, and all the tangential components clockwise or counter clockwise. Now the torque of all the forces acting on P' equals the torque of $m'a_t'$ and $m'a_n'$; this torque = $m'a_t'r'$. Simiarly the torque of all the forces acting on $P'' = m''a_t''r''$. Hence the torque of the forces acting on all the particles equals

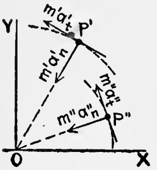


FIG. 302

$$m'a_t'r' + m''a_t''r'' + \cdot \cdot \cdot = m'r'\alpha r' + m''r''\alpha r'' + \cdot \cdot \cdot = \alpha \Sigma mr^2 = \alpha I_0.$$

Now the system of forces acting on all the particles consists of internal and external forces. The internal forces jointly have no torque since they consist of pairs of colinear, equal, and opposite forces. Hence, the torque of the external forces equals $I_0\alpha$.

Examples. — 1. Fig. 303 represents a circular disk of cast iron 4 inches thick and 3 feet in diameter. It is supported on a fixed horizontal shaft 3 inches in diameter. A cord is wrapped around the disk, and then a pull P = 100 pounds is applied to the cord as shown. What is the angular acceleration of the disk? The external forces acting on the disk and cord are the weight of the disk and cord P, and the reaction of the shaft. Only one of these, P, has a moment about the axis of rotation. We are assuming that the disk is homogeneous so that the center of gravity is in the axis of rotation, and that the shaft is frictionless. ΣM_0 of equation (3) is therefore $100 \times 1.5 = 150$ foot-pounds. Now the square of the radius of gyration of the disk about the axis of rotation is $\frac{1}{2}(1.5^2 + 0.125^2) = 1.133$ feet^2 (Art. 36). And since the weight of the disk is 1053 pounds, its moment of inertia about the axis of rotation is $(1053 \div 32.2)$ $1.133 = 37.0$ slug-feet^2. Hence the angular acceleration of the disk is $150 \div 37.0 = 4.0$ radians per second per second.

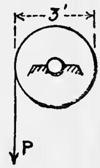


FIG. 303

2. Suppose that a turning force P in the preceding example is supplied not "by hand" but by means of a body suspended from the cord, and suppose that the body weighs 100 pounds. Obviously the system (disk and suspended body) moves with acceleration; hence the two forces acting on the body (gravity and the pull P of the cord) are not equal or balanced but have a resultant

downward (direction of the acceleration of the body). That resultant is $100 - P$, and it equals the product of the mass and acceleration of the body, or $100 - P = (100 \div 32.2) \times a$ where $a =$ the acceleration. The torque on the disk is $P \times 1.5$, and $1.5\,P = I\alpha = 37.0\,\alpha$. But $a =$ the tangential acceleration of any point on the rim of the disk $= 1.5 \times \alpha$, or $a = 1.5\,\alpha$. These three equations

$$100 - P = (100/32.2)\,a, \quad 1.5\,P = 37.0\,\alpha, \quad \text{and} \quad a = 1.5\,\alpha,$$

solved simultaneously give $\alpha = 3.41$ radians per second per second, less than in example 1 as was to be expected, because the pull P in this example is less than 100 pounds. The value of P as obtained from the foregoing equations is 8.41 pounds.

3. In Fig. 304 we take weight of $A = 64$ pounds, of $B = 96$ pounds, and of pulley $C = 144$ pounds; assume coefficient of friction under $B = \frac{1}{5}$ for sliding, axle friction zero; take diameter of pulley = 2 feet 6 inches, and the radius of gyration of the pulley about the axis of rotation = 10.6 inches. We show how to determine the acceleration of the system. Let $a =$ acceleration of A and B, and $\alpha =$ (angular) acceleration of the pulley. Obviously $a = 1.25\,\alpha$. Let us now consider the forces acting on each body A, B, and C. On A there are two, — gravity (64 pounds) and the pull of the cord P_1 (see Fig. 305). On

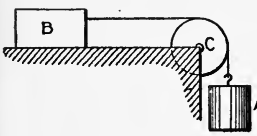


FIG. 304

FIG. 305

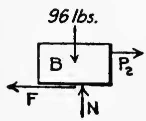


FIG. 306

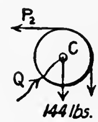


FIG. 307

B there are three, — gravity (96 pounds), the pull of the cord P_2, and the reaction of the supporting surface D (see Fig. 306 where this latter force is represented by two components N and F). On the pulley there are three forces, — gravity (144 pounds), the reaction Q of the axle, and the pressure of the cord. Since the mass of the cord is negligible, the tension at any point of the cord from A to the pulley is P_1, and at any point from B to the pulley it is P_2. Hence the pressure of the cord against the pulley equals the resultant of P_1 and P_2 (Fig. 307), and that pressure is equivalent to P_1 and P_2. Therefore the equation of motion becomes $(P_1 - P_2)\,1.25 = (144 \div 32)\,(10.6 \div 12)^2\alpha = 4.5 \times 0.778 \times \alpha = 3.5\,\alpha$. Since the acceleration of B is toward the right, the resultant force on it acts in that direction and equals $P_2 - F = P_2 - \frac{1}{5}\,N = P_2 - \frac{1}{5}\,96 = P_2 - 19.2$; and hence $P_2 - 19.2 = (96 \div 32)\,a = 3\,a$. Since the acceleration of A is downward the resultant force on A acts in that direction and equals $64 - P_1$; hence $64 - P_1 = (64 \div 32)\,a = 2\,a$. Now solving the three equations of motion,

$$(P_1 - P_2) \times 1.25 = 3.5\,x, \quad P_2 - 19.2 = 3\,a, \quad \text{and} \quad 64 - P_1 = 2\,a,$$

together with $a = 1.25\,\alpha$, we find that $a = 6.19$ feet per second per second, and $\alpha = 4.95$ radians per second per second. The equations also show that $P_1 = 51.62$ pounds, and $P_2 = 37.77$ pounds.

38. Axle Reactions

§ 1. Rotating bodies, such as machine parts, are commonly supported by shafts upon or with which the bodies rotate. In such a case, axle reaction means the force which the shaft exerts upon the rotating body. To determine such a force we make use of the principle of the motion of the mass-center. The principle states (Art. 24) that the algebraic sum of the components — along any line — of all the external forces acting on a body, moving in any way, equals the product of the mass of the body and the component of the acceleration of the mass-center along that line. In general, the principle furnishes three independent equations, one for each of three rectangular lines of resolution. If the mass-center of the (rotating) body does not lie in the axis of rotation then there are three lines of resolution which are generally more convenient to use than any others, and these we now describe. Let the circle (Fig. 308) be the path of the mass-center of a rotating body (not shown), O be the center of rotation (intersection of the axis of rotation and plane of the path of the mass-center), and C be the mass-center. Then the three convenient lines are the axis of rotation, the line OC, and a line perpendicular to the first two. The directions of these lines are called respectively axial, radial or normal (OC being a radius and normal of the circle), and tangential (the third line being parallel to the tangent at C). Now let $\Sigma F_t, \Sigma F_n$, and ΣF_a = the algebraic sums of the tangential, normal, and axial components of all the external forces acting on any rotating body; $\bar{a}_t$ and $\bar{a}_n$ = the tangential and normal components of the acceleration of its mass-center — the axial component of the acceleration equals zero; and M = the mass. Then

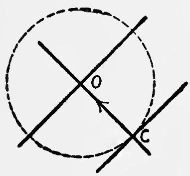


FIG. 308

$$\Sigma F_t = M\bar{a}_t, \quad \Sigma F_n = M\bar{a}_n, \quad \Sigma F_a = 0. \tag{1}$$

Systematic units (Art. 31) must be used in the foregoing. If W/g be substituted for M (Art. 31) then any unit may be used for force (including weight), any unit for length, and any unit for time.

Let $\bar{r}$ = radius of the circle described by the mass-center, $\bar{v}$ = velocity of the mass-center, α = angular acceleration, and ω = angular velocity of the rotating body at the instant under consideration; then (see Art. 37, §1)

$$a_t = \bar{r}\alpha, \quad \text{and} \quad \bar{a}_n = \bar{v}^2/\bar{r} = \bar{r}\omega^2,$$

and we may use these in equations (1).

If the mass-center of the rotating body is in the axis of rotation, then the acceleration of the mass-center is always zero, and the algebraic sum of the components of the external forces along any line equals zero.

Examples. — 1. AB (Fig. 309) is a bar of wrought iron 1.5 inches (perpendicular to paper) $\times$ 4 inches $\times$ 6 feet, suspended from a horizontal axis at A. Suppose that the bar is made to rotate and is then left to itself rotating under the influence of gravity, the axle reaction, and the initial velocity given to it. Suppose further that the initial velocity was such that when the bar gets into the position shown, the angular velocity is 60 revolutions per minute. Required the axle reaction in the position shown. The only forces acting on the body are its weight $W = 120$ pounds, and the axle reaction represented by two components R_1 and R_2. We neglect the axle friction; then the lines of actions of R_1 and R_2 cut the axis of rotation, and the equation of motion (Art. 37) becomes $W\,(2 \sin 35°) = I\alpha$. Now $I = (W/32.2)\,k^2 = (120/32.2)\,7.01$; hence $\alpha = 5.26$ radians per second per second, and $\bar{a}_t = 2 \times 5.26 = 10.52$ feet per second per second. The angular velocity, 60 revolutions per minute, equals 6.28 radians per second; hence $\bar{a}_n = 2 \times 6.28^2 = 78.8$ feet per second per second. Finally, equations (1) become

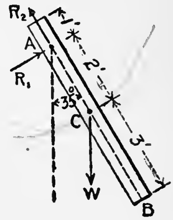


FIG. 309

$$120 \sin 35° - R_1 = (120/32.2)\,10.52 = 39.2, \text{ and}$$
$$R_2 - 120 \cos 35° = (120/32.2)\,78.8 = 294.$$

From the first $R_1 = 29.7$, and from the second $R_2 = 392$ pounds.

2. AB (Fig. 310) is a simple brake for retarding the motion of the drum C and suspended body W. Let $W = 2000$ pounds, weight of the drum $= 1800$ pounds, radius of gyration of drum about axis of rotation $= 2.5$ feet, coefficient of

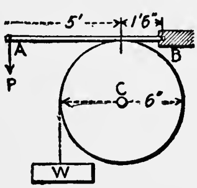


FIG. 310

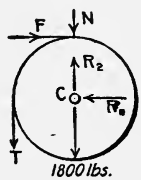


FIG. 311

friction "between" brake and drum $= 0.5$. Suppose that W is descending and the brake pull P is 1000 pounds. Required the axle reaction on the drum. Fig. 311 shows all the forces acting on the drum, — its own weight (1800 pounds), the brake pressure represented by two components N (normal pressure) and F (friction), the pull T of the rope, and the axle reaction represented by two components R_1 and R_2. From a consideration of the forces acting on the brake it is plain that $N = (1000 \times 6.5) \div 1.5 = 4333$ pounds; and hence $F = 0.5 \times 4333 = 2167$ pounds. Now in order to get T we write out the equations of motion of the drum and the suspended body. Since F is greater than the weight of the body the velocities of drum and body are being decreased; hence T is greater than W but less than F. If α = the acceleration of the drum and a = the acceleration of the suspended body then the equations of motion are

$$2167 \times 3 - T \times 3 = (1800/32.2)\, 2.5^2\alpha, \quad \text{and}$$
$$T - 2000 = (2000/32.2)\, a.$$

These equations and $a = 3\,\alpha$, solved simultaneously, give $T = 2103$ pounds. Since the acceleration of the mass-center of the drum equals zero,

$$2167 - R_1 = 0, \quad \text{or} \quad R_1 = 2167, \quad \text{and}$$
$$R_2 - 4333 - 1800 - 2103 = 0, \quad \text{or} \quad R_2 = 8236 \text{ pounds.}$$

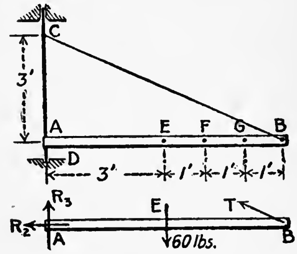
Therefore the axle reaction $= \sqrt{(2167^2 + 8237^2)} = 8500$ pounds inclined upwards and to the left at an angle of $14\frac{3}{4}$ degrees with the vertical.

§ 2. In some cases equations (1) do not suffice to determine the axle reaction, but in a certain common "symmetrical case" a simple principle furnishes the additional necessary equations. If the rotating body is symmetrical about a plane and the axis of rotation is perpendicular to that plane, then the resultant of all the external forces acting on the rotating body lies in the plane of symmetry (proof below). It follows that the algebraic sum of the moments of all the external forces about any line in that plane equals zero.

We prove that the resultant lies in the plane mentioned indirectly, by showing that the resultant of an equivalent (imaginary) system lies in that plane. This imaginary system of forces (first mentioned in Art. 35) consists of the resultants of all the forces acting on the several particles of the body; each resultant equals the product of the mass and acceleration of the corresponding particle, and the resultant and the acceleration agree in direction. Imagine the body to consist of elementary rods parallel to the axis of rotation, and each rod divided into elementary portions of equal length. Let m = mass of each elementary portion, and a' = the (common) acceleration of all the portions; then the resultant on each portion equals ma' and these resultants are all parallel and parallel to the plane of symmetry. Therefore the resultant R' of all the imaginary forces ma' is at mid-depth of the rod and lies in the plane of symmetry. Similarly the resultant R'' of all the imaginary forces ma'' on the elementary portions of the second rod lies in the plane of symmetry. And obviously the resultant of R', R'', etc., lies in the plane of symmetry.

Examples. — 1. AB (Fig. 312) is a slender rod, not drawn to scale, supported by a cord BC and a vertical shaft CD as shown; the rod weighs 60 pounds. A force $P = 30$ pounds is applied horizontally and perpendicular to AB at E, and makes the rod rotate about the shaft. Required the reaction of the shaft and the tension in the cord at the instant when the speed reaches 10 revolutions per minute. Four forces act on the rod, — its weight, P, the pull T of the cord, and the axle reaction. We will imagine this latter force resolved into three components, — R_1 (tangential, and not shown in Fig. 313), R_2 normal, and R_3 axial.

We need the values of the tangential and normal components of the acceleration of the mass-center for use in equations (1), and so we find α and ω first. The angular acceleration depends on the turning moment; the only force having such moment is the applied force P, and the moment is $30 \times 3 = 90$ foot-pounds. The moment of inertia of the rotating body about the axis of rotation is $\frac{1}{3}(60 \div 32.2) \times 6^2 = 22.35$ slug-feet² (see Art. 36). Hence the angular acceleration is $90 \div 22.35 = 4.03$ radians per second per second, and the tangential acceleration of the mass-center is $3 \times 4.03 = 12.09$ feet per second per second. The linear velocity of the mass-center is $3 \times 1.047 = 3.141$ feet per second (10 revolutions per minute = 1.047 radians per second); hence the normal acceleration of the mass-center is $3.141^2 \div 3 = 3.3$ feet per second per second. Therefore equations (1) become



FIGS. 312, 313

$$R_1 + 30 = (60 \div 32.2)\,12.09,$$
$$R_2 + T\cos 26^\circ\,34' = (60 \div 32.2)\,3.3, \quad \text{and}$$
$$R_3 - 60 + T\sin 26^\circ\,34' = 0.$$

From the first equation $R_1 = -7.5$; the negative sign means that R_1 acts opposite to P. The two remaining equations are not sufficient to determine R_2, R_3, and T. The additional necessary equation can be obtained by taking moments about the line through B say and perpendicular to the axis of rotation; thus

$$-R_3 \times 6 + 60 \times 3 = 0, \quad \text{or} \quad R_3 = 30 \text{ pounds.}$$

Substituting this value in the preceding equation, we find that $T = 67.1$, and hence $R_2 = -53.8$, the negative sign indicating that R_2 acts not as assumed but toward the right (on AB).

39. Pendulums

§ 1. GRAVITY PENDULUM. — By this term is meant the common pendulum, that is a body suspended on a horizontal axis so that it can be made to oscillate freely under the influence of gravity. A real pendulum is sometimes called a *compound* or *physical pendulum* to distinguish it from an imaginary one consisting of a mass-point or particle suspended by a massless cord; this latter is called a *simple* or *mathematical pendulum.* Let T = the period or time of one complete or double (to and fro) oscillation, k = the radius of gyration of the pendulum with respect to the axis of suspension, c = distance from the center of gravity of the pendulum to that axis, and 2β = the angle swept out by the pendulum in one single oscillation. Then, as will be shown presently, the period is given closely by

$$T = 2\pi\sqrt{k^2/cg}, \tag{1}$$

provided that β is small.* Since β does not appear in this formula the period of any pendulum is independent of β; that is all small oscillations of a pendulum have equal periods or, as we say, they are isochronous. When g is expressed in feet per second per second then k and c should be expressed in feet; T will be in seconds.

For the derivation of equation (1) let OG (Fig. 314) be a pendulum in any swinging position, O the center of suspension, G the center of gravity; let $W =$ the weight of the pendulum, $c = OG$, and θ the (varying) angle which OG makes with the vertical, regarded as positive when the pendulum is on the right side of the vertical, as shown. There are three forces acting on the pendulum, — gravity, the supporting force at the knife edge, and the pressure of the surrounding air. The moment of the first force about the axis of suspension is $Wc \sin \theta$; the moments of the other two forces we take as negligible. Hence the resultant torque on the pendulum in any position $= Wc \sin \theta$ practically. The angular acceleration $= d^2\theta/dt^2$ (see Art. 37); hence according to equation (3) of that article

$$Wc \sin \theta = -(W/g)\, k^2 d^2\theta/dt^2,$$

the negative sign being introduced because $\sin \theta$ and $d^2\theta/dt^2$ are always opposite in sign. It follows readily from the preceding equation that

$$d^2\theta/dt^2 = -(cg/k^2) \sin \theta = -A \sin \theta,$$

where A is an abbreviation for cg/k^2. We will assume that the greatest value of θ, that is β, is so small that $\sin \theta$ and θ are nearly equal; then as a good approximation we may substitute θ for $\sin \theta$, and have

$$d\theta^2/dt^2 = -A\theta.$$

To integrate this simply, let $u = d\theta/dt$; then $d^2\theta/dt^2 = du/dt = (du/d\theta)(d\theta/dt) = (du/d\theta)\, u$, and hence

$$(du/d\theta)\, u = -A\theta, \quad \text{or} \quad u\, du = -A\theta\, d\theta.$$

Now integrating and replacing u by $d\theta/dt$, we get

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = -A\frac{\theta}{2} + C_1,$$

where C_1 is a constant of integration. Remembering that $d\theta/dt =$ the angular velocity of the pendulum, we note that where $\theta = \beta$, there $d\theta/dt = 0$; therefore for these (simultaneous) values the preceding equation becomes $0 = -\frac{1}{2}A + C_1$, or $C_1 = \frac{1}{2}A$, and finally

$$d\theta/dt = \pm A\sqrt{\beta^2 - \theta^2}.$$

* The exact value of the period is given by

$$T = 2\pi\sqrt{k^2/cg}\left[1 + \left(\frac{1}{2}\right)^2 \sin^2\frac{\beta}{2} + \left(\frac{1}{2}\cdot\frac{3}{4}\right)^2 \sin^4\frac{\beta}{2} + \cdots\right].$$

If $\beta = 8$ degrees then the bracket above $= 1.00122$; and for smaller values of β the value of the bracket is still nearer unity. Hence the error in the approximate formula is less than one-eighth of one per cent if β does not exceed 8 degrees.

The positive sign is to be used when $d\theta/dt$ is positive; that is when the pendulum is swinging in the positive direction. Now let τ = the time required for the pendulum to swing out from its lowest to its highest position on the right, that is while θ changes from 0 to β. To get a value of this time we integrate the preceding equation as follows:

$$\sqrt{A}\int_0^{\tau} dt = +\int_0^{\beta}\frac{d\theta}{\sqrt{\beta^2-\theta^2}}, \quad \text{or} \quad \tau = \sqrt{\frac{1}{A}}\left[\sin^{-1}\frac{\theta}{\beta}\right]_0^{\beta} = \frac{\pi}{2}\sqrt{\frac{k^2}{cg}}.$$

Let τ' = the time required for a swing from the extreme right position to the lowest position, that is while θ changes from β to 0. To get this time we integrate as follows:

$$\sqrt{A}\int_0^{\tau'} dt = -\int_{\beta}^{0}\frac{d\theta}{\sqrt{\beta^2-\theta^2}}, \quad \text{or} \quad \tau' = -\sqrt{\frac{1}{A}}\left[\sin^{-1}\frac{\theta}{\beta}\right]_{\beta}^{0} = \frac{\pi}{2}\sqrt{\frac{k^2}{cg}}.$$

Hence τ and τ' are equal, as was to be expected. Finally, the time of one complete oscillation $= 4\tau = 2\pi\sqrt{k^2/cg}$, as was to be shown.

Let $\bar{k}$ = the radius of gyration of the pendulum with respect to an axis through its center of gravity and parallel to the axis of suspension; then $k^2 = \bar{k}^2 + c^2$ (see Art. 36), and hence

$$T = 2\pi\sqrt{\frac{\bar{k}^2+c^2}{gc}} = 2\pi\sqrt{\frac{c+\bar{k}^2/c}{g}} = 2\pi\sqrt{\frac{c}{g}\left(1+\frac{\bar{k}^2}{c}\right)}. \qquad (2)$$

Contrary to common belief, the period does not increase for all increases in c, the distance from the center of gravity to the axis of suspension. For examining the foregoing expression for T with reference to a variation in c, we find that

$$\frac{dT}{dc} = \frac{\pi(c^2-\bar{k}^2)}{c\sqrt{[gc(c^2+\bar{k}^2)]}}.$$

Now this is negative for all values of c less than $\bar{k}$, and positive for all values greater than k. Hence when c is less than $\bar{k}$ an increase in c decreases T; when c is greater than $\bar{k}$ an increase in c increases T. When $c = \bar{k}$, then $dT/dc = 0$, and T has its least value equal to $2\pi\sqrt{(2k/g)}$.

In the case of a simple pendulum of length l, the radius of gyration $k = l$ and also $c = l$; hence *the period of a simple pendulum* is given by

$$T = 2\pi\sqrt{l/g}.* \qquad (3)$$

A physical pendulum and a simple pendulum whose periods are equal are said to be equivalent. Periods are equal if $k^2/cg = l/g$, or $l = k^2/c$.

* Strictly speaking, a simple or mathematical pendulum can exist only in imagination, but a real pendulum made of a small bob suspended by means of a cord may be regarded as a simple pendulum in computing period. That is the period of the cord-bob pendulum $= 2\pi\sqrt{l/g}$ where l = the distance from the axis of suspension to the center of the bob. For $\bar{k}/c$ for the cord-bob pendulum is small compared to 1, and hence equation (3) gives $T = 2\pi\sqrt{c/g}$ practically.

Imagine the entire mass of a (real) pendulum concentrated into a point Q (Fig. 314), whose distance from the center of suspension O equals k^2/c. Then, as just shown, the period of such an (imaginary) simple or mathematical pendulum would be $2\pi\sqrt{(l/g)}$, where l is the length OQ or k^2/c; hence the period would be $2\pi\sqrt{(k^2/cg)}$, that is equal to the period of the real pendulum.

For this reason Q is sometimes called the *center of oscillation* of the pendulum. (It coincides with the center of percussion, see Art. 48.) The distance from the center of gravity to the center of oscillation is

$$GQ = \frac{k^2}{c} - c = \frac{\bar{k}^2 + c^2}{c} - c = \frac{\bar{k}^2}{c}.$$

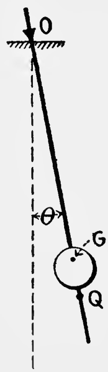


FIG. 314

FIG. 315

FIG. 316

"The centers of suspension and of oscillation of a pendulum are interchangeable," that is if a pendulum be suspended from Q (Fig. 315), then O becomes the center of oscillation. For, suppose that Q' is the center of oscillation corresponding to Q, then

$$GQ' = \frac{\bar{k}^2}{QG} = \frac{\bar{k}^2}{\bar{k}^2/c} = c;$$

hence Q' coincides with O. It follows from the property of interchangeability that the periods of a pendulum when suspended from O and from Q are equal.

The pendulum is our best device for accurately determining the acceleration due to gravity at any place. We have only to determine the period T and the length k^2/c of a pendulum at the place, and then compute g from the formula $T = 2\pi\sqrt{(k^2/cg)}$. But it is not easy to determine k^2/c directly. Captain Kater first (1818) made use of the property of interchangeability of centers of suspension and oscillation to make a pendulum whose length k^2/c could be determined accurately and readily. Fig. 316 represents a Kater pendulum in principle; O_1 and O_2 are two knife edges as shown at a known distance apart; W is a weight which can be slid along the rod and clamped where desired. The periods of oscillation for O_1 suspension and O_2 suspension would be different, but by shifting the weight and trying repeatedly, the periods can be made equal. When the periods are equal, then either knife edge is the axis of oscillation for the other as axis of suspension, and O_1O_2 (a known distance) is the length of the equivalent simple pendulum or the k^2/c of the formula. By means of a Kater pendulum the value of g for Washington was determined to be 980.100 centimeters per second per second. Values of g at many other places have been determined more simply — by comparing the periods of oscillation of a more ordinary pendulum at Washington and the other places. This comparison is based on the principle that the squares of the periods of oscillation of any pendulum at two different places are inversely proportional to the values of g at those places; hence if T_w and T = the periods at Washington and some other station and g = the acceleration at the latter place, then $g = 980.1\ (T_w/T)^2$.

§ 2. Torsion Pendulum. — This consists of a heavy bob suspended vertically by means of a light elastic wire, the wire being firmly embedded in the bob and in its support. Any horizontal couple applied to the bob will turn the bob and twist the wire. If the couple is not too large — so as not to stress the wire beyond its "elastic limit" — then the angular displacement of the bob will be proportional to the moment of the couple applied. That is, if C and C' = the moments of two couples applied successively and θ and θ' are the corresponding angular displacements produced by the couples, then $\theta/\theta' = C/C'$. Hence, the moment C required to produce any displacement is given by $C = (C'/\theta')\,\theta$. In any displaced position of the bob, the wire exerts a couple on the bob equal to the applied couple.

If the bob were released from any position of (moderate) angular displacement β, it would oscillate under the influence of the couple exerted by the wire. We will assume that this (varying) couple follows the law expressed above. Then the equation of motion (rotation) for the bob would be (see equation 3, Art. 37) $C = I\alpha$, where I = moment of inertia of the bob with respect to the axis of the wire and α = the (varying) angular acceleration. Since $\alpha = d^2\theta/dt^2$, and θ and $d^2\theta/dt^2$ are opposite in sign, the equation can be written

$$\frac{C'}{\theta'}\theta = -I\frac{d^2\theta}{dt^2} \quad \text{or} \quad \frac{d^2\theta}{dt^2} = -B\theta,$$

where B is an abbreviation for $(C'/\theta') \div I$. This last equation is just like the equation $d^2\theta/dt^2 = -A\theta$ of § 1, relating to the motion of a gravity pendulum except that B appears in one equation and A in the other. Hence the formulas in § 1 apply to this section if A be changed to B. Thus the time of one quarter complete oscillation of a torsion pendulum is

$$\tau = \sqrt{\frac{1}{B}}\left[\sin^{-1}\frac{\theta}{\beta}\right]_0^{\beta} = \frac{\pi}{2}\sqrt{\frac{I}{C'/\theta'}}.$$

The period (one complete to and fro oscillation) equals 4τ, or

$$T = 2\pi\sqrt{I \div (C'/\theta')}. \tag{1}$$

$I = Mk^2 = (W/g)\,k^2$ where W = weight and k = radius of gyration of the bob with respect to the axis of the wire. If, in a numerical case, W is taken in pounds and g in feet per second per second, k should be expressed in feet, C' in foot-pounds, and θ' (always) in radians; T will be in seconds. C'/θ' (the ratio of the moment of any twisting couple to the angle of twist produced) is a measure of the torsional stiffness of the wire, for that ratio is the moment required for twisting per radian of twist. Formula (1) shows that the stiffer the wire the less the period.

CHAPTER X

WORK, ENERGY, POWER

40. Work

§ 1. Definitions, Etc. — Work is a common word and has many meanings (see dictionary), but it is used in a single special sense in Mechanics. *Work* is said to be done upon a body by a force — also by the agent exerting the force — when the point of application of the force moves so that the force has a component along the path of the point of application. This component will be called the *working component* of the force; and the length of the path of the point of application the *distance through which the force acts.* The amount of work done by the force is taken as equal to the product of the working component and the distance through which the force acts.

The meaning of this measure of work done by a force is clear when the working component is constant. For example, suppose that the body represented in Fig. 317 is moved along the line AB by a number of forces, two of which (indicated) are constant in magnitude and in direction. During any portion of the motion, as from A to B, the work done by F_1 is $F_1(AB)$ and the work done by F_2 is $(F_2 \cos\theta)\, AB$. This expression when written $F\,(AB\cos\theta)$ means the product of the force and the component of the displacement along the line of action of the force, which is a "view" of amount of work done by a force sometimes more convenient than the other.

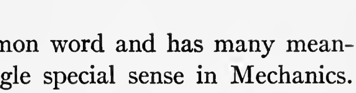


Fig. 317

When the working component is not constant in magnitude, then we may arrive at an expression for the work somewhat like this: — Let AB (Fig. 318) be the path of the point of application of one of the forces acting upon a body not shown, and P any point on the path. Let F = the force and ϕ = the angle between F and the tangent at P, and ds = the elementary portion of the path at P. Then the work done by F during the elementary displacement $= F\cos\phi \cdot ds$ or $F_t\,ds$ where F_t means working or tangential component of F; and the work done by F in the displacement from A to $B = \int F_t\, ds$, limits of integration to be assigned so as to include all elementary works $F_t\,ds$ in the motion from A to B. It is worth noting that if the force F acts normally to the path at all points, then $F_t = 0$ always, and the formula gives zero for the work done by F, as it should.

Fig. 318

The *unit work* is the work done by a force whose working component equals unit force acting through unit distance.* The unit of work depends upon the units used for force and distance; thus we have the foot-pound, centimeter-dyne, etc. The second unit is also called erg; and 10^7 ergs = 1 joule. The horse-power-hour and the watt-hour are larger units of work. They are the amounts of work done in one hour at the rates of one horse-power and one watt, respectively (see Art. 41); thus,

One horse-power-hour = 1,980,000 foot-pounds, and
One watt-hour = 3600 joules.

When the works done by several forces are under discussion, it may be convenient to give signs to their works according to this commonly used rule: When the working component acts in the direction of motion, the work of the force is regarded as positive; when opposite to the direction of motion then the work is regarded as negative. The formula $\int F \cos \phi \, ds$, with the lower and upper limits of integration to correspond to the initial and final positions A and B, respectively, observes this rule of signs for work, if s is measured positive in the direction of motion from some fixed origin to P, and ϕ is measured from the "positive tangent" around to the line of action of the force as shown in the figure. Forces which do positive work are sometimes called *efforts;* those which do negative work are called *resistances.*†

Work Diagram. — If values of F_t and s be plotted on two rectangular axes (Fig. 319) for all positions of the point of application of F, then the curve joining the consecutive plotted points might be called a "working force-space" (F_t-s) curve. The portion of the diagram "under the curve" (between the curve, the s-axis, and any two ordinates) is called the *work diagram* for the force F for the displacement corresponding to the bounding ordinates. The area of a work diagram represents the work done by the force during the displacement corresponding to the bounding ordinates. Proof: Let m = the force scale-number, and n = the space scale-number; that is, unit ordinate (inch) = m units of F_t (pounds) and unit abscissa (inch) = n units of s (feet). Also let A = area; then

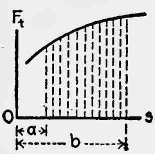


FIG. 319

$$A = \int_{x_1}^{x_2} y \, dx = \int_a^b \frac{F_t}{m} \frac{ds}{n} = \frac{1}{mn} \int_a^b F_t \, ds = \frac{\text{work}}{mn}.$$

Hence, $A\,(mn)$ = work; that is, A = work according to the scale number mn to be used for interpreting the area.

By *average working component* of F is meant a value of F_t which multiplied by the distance $s_2 - s_1$, or $b - a$, gives the work done by F. Obviously, that average working force is represented by the average ordinate to the curve of

* For dimensions of unit work, see Appendix A.

† The (negative) work done by a resistance on a body is often referred to as (positive) work done by the body against the resistance.

the work diagram. When that curve is straight, that is, when F_t varies uniformly with respect to s, then the average working component equals the mean of the initial and final values.

Fig. 320 is a fac-simile of a record made by the traction dynamometer (a spring balance essentially) in a certain train test. Abscissas represent distances travelled by the train, and ordinates represent "draw-bar pulls" (the pulls between the tender and first car of the train). Thus, the figure is a work diagram. To determine the area of such a diagram as this we first draw in an average curve "by eye," and then ascertain the area under this curve in any convenient way.

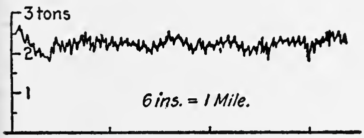


FIG. 320

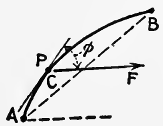


FIG. 321

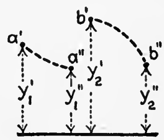


FIG. 322

§ 2. SOME SPECIAL CASES. — (i) The work done by a force which is constant in magnitude and direction equals the product of the force and the projection of the displacement of its point of application upon the line of action of the force. For, let F = the force, APB (Fig. 321) the path of its point of application, ϕ = the (variable) angle between F and the direction of the motion of the point of application P. Then the work done by F is

$$\int F \cos\phi \, ds = F \int ds \cos\phi,$$

where ds is an elementary portion of the path. Now $ds \cos\phi$ is the projection of the element ds upon F, or upon any line parallel to F, and $\int ds \cos\phi$ is the sum of the projections of all the elements of APB upon the line. But the sum of the projections = the projection of APB = the projection of the chord AB.

(ii) The work done by gravity upon a body in any motion equals the product of its weight and the vertical distance described by the center of gravity; the work is positive or negative according as the center of gravity has descended or ascended. Let w_1, w_2, etc., denote the weights of the particles of the body; y_1', y_2', etc., their distances above some datum plane — below which the body does not descend — at the beginning of motion; and y_1'', y_2'', etc., their distances above that plane at the end of the motion (see Fig. 322) where $a'a''$ is the path of the first particle, $b'b''$ that of the second, etc.). Also let W denote the weight of the body, and $\bar{y}'$ and $\bar{y}''$ the initial and final heights of its center of gravity above the plane. Then the works done by gravity on the particles, respectively, are $w_1(y_1'' - y_1')$, $w_2(y_2'' - y_2')$, etc., and the sum of these works can be written

$$(w_1y_1'' + w_2y_2'' + \ldots) - (w_1y_1' + w_2y_2' + \ldots).$$

The first term of this sum $= W\bar{y}''$, and the second $= W\bar{y}''$ (see Art. 21); hence the sum of the works done on all the particles equals

$$W\bar{y}'' - W\bar{y}' = W(\bar{y}'' - \bar{y}').$$

(iii) The algebraic sum of the works done by any number of forces having a common point of application during any displacement of that point equals the work done by their resultant during that displacement. For, let F', F'', F''', etc., = the forces, R = their resultant, and F_t', F_t'', F_t''', etc., and R_t = the components of the forces and of the resultant, respectively, along the tangent to the path of the point of application. Now $F_t' + F_t'' + F_t''' + \cdots = R_t$ (Art. 4). Hence $F_t'\,ds + F_t''\,ds + F_t'''\,ds + \cdots = R_t\,ds$, and

$$\int F_t'\,ds + \int F_t''\,ds + \int F_t'''\,ds + \cdots = \int R_t\,ds;$$

that is, the sum of the works done by the forces equals the work done by their resultant.

(iv) The work done by a pair of equal, colinear, and opposite forces in any displacement of their points of application equals

$$+\int_{r_1}^{r_2} F\,dr \quad \text{or} \quad -\int_{r_1}^{r_2} F\,dr$$

according as the two forces separate or draw the two points of application together; F = the common magnitude of the two forces — not constant necessarily —, r = the distance between their points of application, and r_1 and r_2 = initial and final values of r. Let A and B (Fig. 323) be the points of application of the two forces — acting on a body not shown — at any intermediate stage of the displacement, and suppose that the path of A is A_1A_2 and that of B is B_1B_2. Let x', y' be the coördinates of A, and x'', y'' those of B. (For simplicity in figure we have taken the paths of A and B as coplanar. The following proof could be extended to cover the case of any paths. The paths are not necessarily due to the forces F alone; but since we are concerned with the work done by these two forces only, no mention is made of any other forces concerned with the motion.) According to the preceding paragraph the work done by either force F in any displacement equals the sum of the works done by the x and y components of F in that same displacement. Hence in an elementary displacement ds the work of F on $A = (-F\cos\theta\,dx' - F\sin\theta\,dy')$, and the work of F on $B = (F\cos\theta\,dx'' + F\sin\theta\,dy'')$. The work done by both forces F in the elementary displacement is

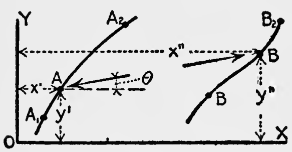


FIG. 323

$$F[\cos\theta\,(dx'' - dx') + \sin\theta\,(dy'' - dy')].$$

It will readily be seen from the figure that $(x'' - x')^2 + (y'' - y')^2 = r^2$; and, by differentiation, we find that

$$(x'' - x')(dx'' - dx') + (y'' - y')(dy'' - dy') = r\,dr.$$

Dividing by r and transforming we find that

$$\cos\theta\,(dx'' - dx') + \sin\theta\,(dy'' - dy') = dr.$$

Hence, the work done in the elementary displacement is $F\,dr$, and the work done in the displacement from A_1B_1 to A_2B_2 equals the integral of $F\,dr$ between the limits as stated. Obviously, changing the senses of the forces F (in the figure) so that they tend to make A and B approach, changes the sign of the total work done by the forces.

41. Energy

When the state or condition of a body is such that it can do work against forces applied to it, the body is said to possess *energy*. For example, a stretched spring can do work against forces applied to it if they are such that it may contract, and a body in motion can do work against an applied force which tends to stop it; the spring and the body, therefore, possess energy.

The amount of energy possessed by a body at any instant is the amount of work which it can do against applied forces while its state or condition changes from that of the instant to an assumed standard state or condition. The meaning of the standard condition is explained in subsequent articles. The *unit of energy* must, in accordance with the above, be the same as the unit of work. Thus we have the foot-pound, foot-ton, centimeter-dyne (or erg), the joule, horse-power-hour, watt-hour, etc. (see preceding article).*

§ 1. MECHANICAL ENERGY. — Energy is classified into kinds depending on the state or condition of the body, in virtue of which it has energy.

Kinetic Energy of a body is energy which the body has by virtue of its velocity. The amount of kinetic energy possessed by a particle at any instant is the work which it can do while the velocity changes from its value at that instant to some other value taken as a standard. It is customary to take zero velocity as the standard one; this being understood, then the amount of kinetic energy possessed by a particle is the work which the particle can do in "giving up all its velocity." The kinetic energy of a single particle whose mass and velocity are m and v, respectively, equals $\frac{1}{2}mv^2$. Proof: Let F_1, F_2, F_3, etc., be the forces which act on the particle P (Fig. 324) and eventually stop it; and let AB be the path, A the beginning (where velocity $= v_1$) and B the end where velocity $= 0$. Then we are to prove that the work done by the particle on the neighboring particles or bodies (which exert the forces F_1, F_2, F_3, etc.) equals $\frac{1}{2}mv_1^2$, during the motion.

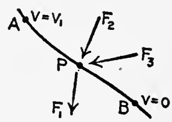


FIG. 324

Now, the work done by the forces F_1, F_2, F_3, etc., on the particle is

$$\int F_1\cos\phi_1\,ds + \int F_2\cos\phi_2\,ds + \cdots = \int (F_1\cos\phi_1 + F_2\cos\phi_2 + \cdots)\,ds,$$

* For dimensions of a unit of energy, see Appendix A.

where ϕ_1, ϕ_2, ϕ_3, etc., are the angles between F_1, F_2, F_3, etc., and the direction of motion (Art. 40). Since the particle P exerts forces on its neighbors, equal and opposite to F_1, F_2, etc., the work done by the particle on its neighbors is

$$-\int (F_1 \cos \phi_1 + F_2 \cos \phi_2 + \cdots)\, ds.$$

But $F_1 \cos \phi_1 + F_2 \cos \phi_2 + \ldots = ma_t = m\, dv/dt$, where a_t is the tangential component of the acceleration of the particle; hence the work done by P is

$$-\int_0^{s'} m\,(dv/dt)\, ds = -\int_{v_1}^{0} m\,(ds/dt)\, dv = -\int_{v_1}^{0} mv\, dv = \tfrac{1}{2}\, mv_1^2.$$

The kinetic energy of a body (a collection or system or particles) is the sum of the kinetic energies of the constituent particles of the body. We will now evaluate this sum for certain common cases, — namely, (i) translation, (ii) rotation, and (iii) combined translation and rotation.

(i) In translatory motion all particles of the moving body have at each instant equal velocities; hence, the sum of the kinetic energies of the particles is $\frac{1}{2}\, m_1v^2 + \frac{1}{2}\, m_2v^2 + \cdots = \frac{1}{2}\, v^2\, (\Sigma m)$, where m_1, m_2, etc., = the masses of the particles and v = their common velocity at the instant under consideration. Or, if M = the mass of the body and E = energy, then

$$E = \tfrac{1}{2}\, Mv^2 = \tfrac{1}{2}\, (W/g)\, v^2. \tag{1}$$

If 32.2 is written for g, then v should be expressed in feet per second. E will be in foot-pounds, foot-tons, etc., according as W is expressed in pounds, or tons, etc.

(ii) In a rotation about a fixed axis the velocity of any particle of the body equals the product of the angular velocity of the body, expressed in radians per unit time, and the distance from the particle to the axis of rotation (Art. 37). Hence, the sum of the kinetic energies of the particles of the body is

$$\tfrac{1}{2}\, m_1\, (r_1\, \omega)^2 + \tfrac{1}{2}\, m_2\, (r_2\omega)^2 + \cdots = \tfrac{1}{2}\, \omega^2\, \Sigma mr^2,$$

where ω = the angular velocity of the body at the instant under consideration, and r_1, r_2, etc., = the distances of the particles respectively from the axis of rotation. But Σmr^2 = the moment of inertia of the body about the axis of rotation; hence, the kinetic energy is given by

$$E = \tfrac{1}{2}\, I\omega^2 = \tfrac{1}{2}\, Mk^2\omega^2 = \tfrac{1}{2}\, (W/g)\, k^2\omega^2, \tag{2}$$

where I = the moment of inertia described, and k = the radius of gyration of the body about the axis of rotation. If 32.2 is written for g, then k should be expressed in feet and ω in radians per second ($\omega = 2\,\pi n$ where n = revolutions per second). Then E will be in foot-pounds, foot-tons, etc., according as W is expressed in pounds, tons, etc.

(iii) A body which has a combined plane translation and rotation (Art. 50), like a wheel rolling, has kinetic energy given by

$$E = \tfrac{1}{2}\, M\bar{v}^2 + \tfrac{1}{2}\, \bar{I}\omega^2 = \tfrac{1}{2}\, (W/g)\, \bar{v}^2 + \tfrac{1}{2}\, (W/g)\, \bar{k}^2\omega^2, \tag{3}$$

where M = mass of the body, $\bar{v}$ = velocity of the center of gravity, $\bar{I}$ = the moment of inertia of the body with respect to an axis through the center of gravity perpendicular to the plane of the motion, $\bar{k}$ = radius of gyration with respect to the same axis, and ω = the angular velocity of the motion. Proof of this formula is given in Art. 51. The portions $\frac{1}{2} M\bar{v}^2$ and $\frac{1}{2} \bar{I}\omega^2$ of the kinetic energy are sometimes called the translational and rotational components, respectively.

As an example of the use of the preceding formula we find the kinetic energy of a cylindrical disk, 6 feet in diameter and 400 pounds in weight, which is rolling so that the center has a velocity of 4 feet per second. $M = 400 \div 32.2 = 12.4$ slugs; the square of the radius of gyration of the disk is $\frac{1}{2}\, 3^2 = 4.5$ feet^2 (see Art. 36); and ω, the rate at which the wheel is turning, is $4 \div 3$ radians per second. Hence

$$E = \frac{1}{2}\, 12.4 \times 4^2 + \frac{1}{2}\, 12.4 \times \left[\frac{4}{3}\right]^2 = 148.8 \text{ foot-pounds.}$$

Potential Energy. — A body may possess energy which is not due to velocity. Thus two mutually attracting bodies can do work against forces applied to either or both if allowed to move so that they approach each other; and, as stated, a compressed or stretched spring can do work against applied forces if permitted to resume its natural length. The "change of condition or state" in the first case is a change in configuration, a change in the positions of the bodies relative to each other; and, in the second case, if we conceive of the spring as consisting of discrete particles, the change is also one in configuration (of the particles). Energy of a system of particles dependent on configuration of the system is called energy of configuration, and potential energy more commonly.

The amount of potential energy possessed by a system in any configuration is the work which it can do in passing from that configuration to any other taken as a standard, it being understood that no other change of condition takes place. The standard configuration may be chosen at pleasure, but it is convenient to so select it that in all other configurations considered the potential energy is positive.

A most common case of potential energy is that of the earth and an elevated body. In this case, standard configuration means one in which the body and earth are as near together as possible. Practically, it is necessary to regard the earth as fixed and the energy as resident in the elevated body. The amount of potential energy of an elevated body is just equal to the work which gravity would do upon the body during the descent into the standard or lowest position, and this work is given by Wh (see preceding article), where W = the weight of the body and h = the distance through which the center of gravity of the body can descend.

§ 2. OTHER FORMS OF ENERGY. — Kinetic energy and potential energy are often called *mechanical energy.* It is the opinion of some that all energy is mechanical, and some think that it is all kinetic. Whether either of these views be correct, it is practically necessary to recognize other forms. A mere enumeration of these with brief remarks is sufficient for the present purpose, since we shall deal mostly with energy known to be mechanical.

Thermal Energy. — A hot body may do work under favorable conditions; thus, if such a one is placed in a boiler containing water, the water will be heated and a part may be converted into steam which may drive a steam engine, that is, do work. By giving up its heat the hot body has done work, and, hence, by definition, it possessed energy in its heated state. Not only is this fact well known, but also the fact that a given quantity of heat represents a definite amount of energy. Thus, one British thermal unit (B.T.U.), which is the amount of heat required to raise the temperature of one pound of water one Fahrenheit degree, = 778 foot-pounds. And one (small) calorie, which is the amount of heat required to raise the temperature of one gram of water one Centigrade degree, $= 4.187 \times 10^7$ ergs (at 15 degrees). Based on the molecular hypothesis the common theory is that heat is due to the vibratory motion of molecules, that is, thermal energy is kinetic.

Chemical Energy. — Many substances combine chemically, and their combination gives evidence that they possessed energy. Thus, coal and oxygen combine and produce heat which, as we have seen, is a form of energy. We rightly say, therefore, that the coal and oxygen before combination possessed energy. Based on this molecular hypothesis the theory of chemical energy in cases where heat is generated in the chemical combination is that internal (molecular) forces of the substances do work during the combination, and, hence (see Art. 43), increase the kinetic energy of the molecules. According to this explanation the energy before combination is potential; and after, kinetic.

Electrical Energy. — If a charged storage battery be connected with a motor, work may be done by the latter. As the work is done, the electrical condition of the battery changes, and we therefore ascribe the energy to the battery. The energy is called electrical because it is due to a change of electrical condition. The nature of electrical energy is even less understood than that of thermal energy, and no commonly accepted explanation of it has yet been made.

42. Power

§ 1. In common parlance the word power has many meanings (see dictionary). Thus we hear of the power of a giant, power of example, power of the press, etc. And of things mechanical, we hear such expressions as a powerful derrick, a powerful cannon, a powerful pump, etc. On reflection we note that the adjective in these three expressions probably does not refer to the same feature of the derrick, cannon, and pump. A derrick is probably called powerful because it can lift a very heavy body, or exert a very great (lifting) *force*. A cannon is generally called powerful because it can project a heavy shot with great velocity, and we shall see presently such performance depends on the *energy* which the gun can impart to the shot. A pump is probably called powerful because it can elevate or transport a large quantity of liquid in a short space of time, or perform much *work per unit time.*

Use of the word power in the sense of force was very common in engineering literature at one time. Such usage is comparatively rare now, but not obsolete. Thus we read of the "tractive power of a locomotive" to denote pull in the bulletins of the American Locomotive Company. (But Goss in his *Locomotive Performance*, and Henderson in his *Locomotive Operation*, seem to prefer tractive force; and in *Locomotive Tests and Exhibits*, of the Pennsylvania Railroad System at the Louisiana Purchase Exposition, we find "tractive effort" to denote that pull.) The other two uses of the word power to denote (i) work or energy, and (ii) rate at which work is done or energy is transmitted or transformed are quite common. Thus in the same text-book we find: (i) "the actual power utilized is one-half the energy available," and (ii) "the power of the plant is about 470 horse-power" (258,500 foot-pounds per second, see below). And in another book there appear: (i) "the power of the rotating shaft could be converted into electrical energy," and (ii) "the power is here measured in kilowatts" (one kilowatt equals 10^{10} ergs per second, see below). It seems probable that this double usage of the word power in engineering literature will persist. In common with most authors, even those quoted above, we will define power in a single sense, namely, — as the rate at which work is done.

Units of Power * like units of work may be classed into gravitational, which vary slightly with locality, and absolute. Thus, the foot-pound per minute and the kilogram-meter per second are units of the first class; also the (practical)

$$\begin{aligned}\text{English and American horse-power} &= 550 \text{ foot-pounds per second}\\ &= 33{,}000 \text{ foot-pounds per minute,}\end{aligned}$$

and the

$$\begin{aligned}\text{Continental horse-power} &= 75 \text{ kilogram-meters per second}\\ &= 4500 \text{ kilogram-meters per minute.}\end{aligned}$$

The dyne-centimeter (or erg) per second is a unit of the second class; also the watt which is 10^7 ergs per second, and the (practical)

$$\text{kilowatt} = 1000 \text{ watts} = 10^{10} \text{ ergs per second.}$$

The Bureau of Standards has recently decided to adopt the English and American horse-power as the exact equivalent of 746 watts, thus making this horse-power an absolute unit. "Thus defined it is the rate of work expressed by 550 foot-pounds per second at 50° latitude and sea level, approximately the location of London, where the original experiments were made by James Watt to determine its value. The 'continental horsepower' is similarly most conveniently defined as 736 watts, equivalent to 75 kilogram-meters per second at latitude 52° 30′, or Berlin." †

* For dimensions of a unit of power see Appendix A.

† *Circular of the Bureau of Standards*, No. 34.

§ 2. MEASUREMENT OF POWER. — There is only one instrument in common use which measures power directly, the wattmeter. It measures electric power and reads in watts, hence the name wattmeter. Power other than electrical is generally measured indirectly by measuring the amount of work done or energy transmitted in a certain length of time; this work or energy divided by the time gives the average power for the period. And to measure the work or energy generally requires the measurement of a force; this force multiplied by the distance through which it acts (as explained later) gives the work or energy. Thus most appliances for ascertaining power measure force first of all, and so are properly called *dynamometers* (force-measurers). Dynamometers are of two kinds, — absorption and transmission. Those of the first kind absorb or waste the energy which they measure, and those of the second kind transmit the energy or nearly all of it. A great many dynamometers have been devised. Only one of each kind are here described.*

Prony Brake. — A simple form is shown in Fig. 325. *AA* are two bearing blocks which bear against the face of the pulley on the shaft of the motor or other machine whose power is to be measured; *BC* is the beam, one end of which is supported on a post *D* which rests on the platform of a weighing scale; *BB* are nuts by means of which the pressures between the pulley and the bearing blocks may be changed and consequently the frictional drag also when the pulley is turning. The drag on the brake tends to depress the end *C* when the pulley is rotating as indicated.

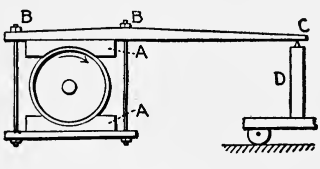


FIG. 325

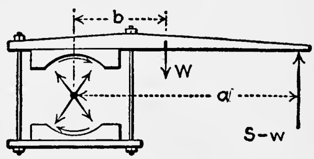


FIG. 326

Let S = the reading of the scale when the pulley is rotating at the desired speed, the brake then absorbing the energy which is to be measured; n = the revolutions of pulley per unit time; a = the horizontal distance from the support of C to the center of the shaft; and X = a correction explained below. Then the power equals

$$P = (S - X)\,2\pi a n. \tag{1}$$

If S and X are expressed in pounds, a in feet, and n in revolutions per minute, then

$$P = 0.000190\,(S - X)\,an \quad \text{horse-power.} \tag{2}$$

The meaning of X will appear from the following derivation of formula (1). Let F = the total frictional drag on the pulley while the energy to be measured

* For full descriptions of many others see Flathers' *Dynamometers* or Carpenter and Diederichs' *Experimental Engineering*.

is being absorbed and d = diameter of pulley. The work done on the pulley by this friction per revolution is $F\pi d$, and per unit time the work is $F\pi dn$. Now let W = weight of the brake, and w = weight of D; then it is plain from Fig. 326 that $\frac{1}{2} Fd + Wb = (S - w) a$, or

$$Fd = 2[S - (w + Wb/a)] a; \text{ and hence } P = [S - (w + Wb/a)] \, 2\pi an.$$

This last equation is like (1) except that X replaces $w + Wb/a$. Now obviously Wb/a is the pressure on the scale due to W; hence X is that portion of S due to W and w. X can be determined directly as follows: Loosen the screws BB and insert a small roller between the top of pulley and the upper block A, but without shifting C; then read the scale. That reading = X, for the pressure on the scale then = $w + Wb/a$.

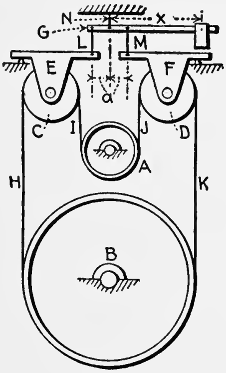


FIG. 327

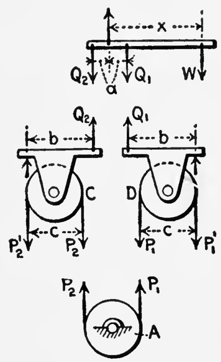


FIG. 328

Tatham Dynamometer. — This consists of four pulleys, A, B, C and D (Fig. 327), two levers E and F, a weighing beam G, and a belt $HIJK$. Pulleys A and B are mounted on the frame of the dynamometer; pulleys C and D are idlers and are mounted on the levers which, in turn, are supported on knife edges resting on the frame and by knife-edge links L and M suspended from the weighing beam, all as shown; the weighing beam is supported from the frame at N. The dimensions are such that the straight portions of the belt are vertical, and H and K are vertically below the knife-edge supports of the levers. The shafts of A and B extend backwards to connect with machines between which the energy to be measured is transmitted. In all cases, the connections to machines should be made so that the tension in I is greater than that in J (I "tight" and J "slack"); and, if possible, the machine whose power is to be determined should be connected to or be on the shaft of A. When the dynamometer is in operation, then L and M pull on the weighing beam; and, if the beam be balanced by the poise, then the scale-reading gives the difference in tensions of I and J, or $P_2 - P_1$ (see Fig. 328). Let S = the scale-reading, n = revolutions per unit time of pulley A, D = diameter of the pulley plus thickness of belt, and P = power; then

$$P = S\pi Dn.$$

For $S\pi D$ is the work done by the belt on A in one turn of A; and, hence, the work done per unit time is $S\pi Dn$.

Fig. 328 shows the forces acting on the various parts, and makes plain how the poise measures $P_2 - P_1$. Thus, from the right-hand lever $Q_1 = P_1c/b$; from the left-hand lever $Q_2 = P_2c/b$; and from the weighing beam $Wx = (Q_2 - Q_1)\,a = (P_2 - P_1)\,ac/b$. Hence, $P_2 - P_1 = (Wb/ac)\,x$. Now, Wb/ac is a constant, and so it is possible to graduate the scale beam (mark values of x on it), so that the readings will give the corresponding values of $P_2 - P_1$. (No mention has been made of the weights of the parts. These are counterbalanced by a balancing weight on the scale beam as in an ordinary platform scale.)

§ 3. INDICATOR; LOCOMOTIVE POWER. — To determine the work done in the cylinder of a steam or gas engine per stroke or per unit time, use is made of an instrument called an indicator. The indicator makes a diagram or "card" from which the intensity of the pressure on either side of the piston at any point of a stroke can be read. Fig. 329 represents, in principle, the original form of indicator as used by James Watt (1736–1819). A is a cylinder; B is a piston working against a coil spring C whose upper end is fixed; D is a pencil which presses against the card or paper E; F is a frame, movable right and left in suitable slides, for holding the paper or card. When the piston is moved the pencil simply makes a vertical line on the card; when the frame is moved the pencil makes a horizontal line. To take a diagram the cylinder of the instrument is connected with one end of the cylinder of the engine to be indicated, and the frame is connected to the cross-head of the engine with suitable reducing device so that the frame gets a motion just like that of the piston but greatly reduced. When the instrument is connected up, as just described, then the pencil describes a curve, something like $GHIJG$, the upper portion GHI being drawn during the forward stroke and the lower portion IJG during the return. The ordinates to the curve from the line of zero pressure K represent pressure per unit area in the cylinder, the scale of ordinates depending on the stiffness of the spring of course. The horizontal width of the diagram represents the stroke of the piston.

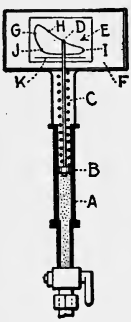


FIG. 329

Fig. 330 is a facsimile of an indicator card; the solid curve pertains to one end of the cylinder, and the dotted curve to the other end; AB is the line of zero pressure. The area $ACDEBA$ represents the work done on one side of the piston (per unit area) during the forward stroke, and the area $BEFCAB$ represents the work done on it during the return stroke. But the first work is positive, and the second negative; hence the work done on that side of the piston during both strokes is represented by the area enclosed by the curve *CDEFC*. Similarly, the area of the dotted curve represents the work done upon the other side of the piston (per unit area) during a to-and-fro stroke. The mean heights of these areas represent pressures per unit area which are called *mean effective pressures*, one for the head-end and one for the crank-end of the cylinder. Let p_1 = mean effective pressure for the head-end, p_2 = that for the crank-end, A = area of cross-section of the cylinder, A' = area of cross-section of the piston rod, and l = length of stroke. Then the work done by the steam in the head-end during two consecutive strokes $= p_1Al$; that done by the steam in the crank-end $= p_2(A - A')l$, and the total work done is the sum of these expressions.

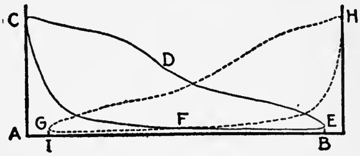


FIG. 330

The average of the mean effective pressures (p_1 and p_2) for the two ends of the cylinder is sometimes called *the* mean effective pressure (for the cylinder). Let p = this mean effective pressure (per unit area); a = the average of the areas of the two sides of the piston, or what amounts to the same thing, the area of the cross-section of the cylinder minus one-half the area of the cross-section of the piston rod; n = the number of strokes of the piston per unit time; and l = length of stroke, as before. Then, as will be shown presently, the work done on the piston per double stroke is $2\,pal$ closely; and, hence, the work done per unit time, or the power, is

$$P = plan. \tag{1}$$

If the customary units are used, namely, p in pounds per square inch, l in feet, a in square inches, and n in strokes per minute, then P, above, is in foot-pounds per minute; and

$$P = \frac{plan}{33{,}000} \text{ horse-powers.}$$

To justify $2pal$: — Let x be an area such that the product $2pxl$ gives the true work done on the piston in a double stroke; that is

$$2 \times \tfrac{1}{2}(p_1 + p_2)\,xl = p_1Al + p_2(A - A')\,l.$$

Then $x = A - \dfrac{p_2}{p_1 + p_2}A'$. Now $p_1 = p_2$ nearly, and so $\dfrac{p_2}{p_1 + p_2} = \tfrac{1}{2}$ very nearly. Therefore, as a close approximation $x = A - \tfrac{1}{2}A' = \tfrac{1}{2}(A + A - A')$, and this is the value assigned to a above. Hence, $2\,pal$ is a close approximation to the work done on the piston per double stroke.

For a *single-expansion, two-cylinder locomotive*, $P = 2\,plan$. Let s = the "piston speed," the actual distance which a piston describes in its cylinder per unit time; then $s = ln$ and

$$P = 2\,pas. \tag{2}$$

With customary units for p, a, and s (pounds per square inch, square inches, and feet per minute respectively) the foregoing formula gives P in foot-pounds per minute. Since the piston speed and the velocity of the locomotive are related, it is possible to express the indicated power of a locomotive in terms of its velocity. Thus let $v =$ the velocity of the locomotive, and $D =$ diameter of the driving wheels; then one turn of the drivers means a displacement of the locomotive equal to πD and a displacement of the piston relative to its cylinder equal to $2l$. Hence $v/s = \pi D/2l$, or $s = (2l/\pi D)v$. Substituting for s in the preceding formula for P, we find that

$$P = \frac{4pal}{\pi D} v = pd^2 \frac{l}{D} v, \tag{3}$$

where $d =$ diameter of the cylinder. (Strictly $d =$ the diameter of a circle whose area equals the area of the cross-section of the cylinder minus one-half the area of the cross-section of the piston rod.) With pounds per square inch for p, inches for d, l, and D, and feet per minute for v, the foregoing formula gives P in foot-pounds per minute. Both formulas for P show that the power of a locomotive is zero at starting, and would increase exactly with the velocity if the mean effective pressure were the same at all speeds.

The mean effective pressure depends upon the boiler pressure obviously, and on the cut-off and piston speed.* The American Locomotive Company has adopted the line $ABCD$ (Fig. 331), as expressing the variation of mean effective pressure with change of piston speed, for the manner of running (cut-off, etc.) which engine men usually employ. Thus, for all speeds up to 250 feet per minute, the mean effective pressure is taken at 85 per cent of the boiler pressure; at 500 feet per minute, it is taken at about 65 per cent, etc. Let $p_0 =$ boiler pressure and $K =$ ratio of mean effective to boiler pressure, which may be called speed coefficient for convenience; so that $p = Kp_0$. Then the formula for indicated power of the locomotive can be written

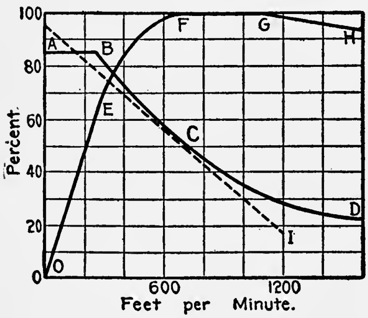


FIG. 331.

$$P = 2paKs. \tag{4}$$

Thus, for a given boiler pressure the power varies as Ks. The line $DEFGH$ (Fig. 331) is a graph of the preceding equation, the maximum value of P being called 100 per cent. It appears, then, that for the American Locomotive Company speed coefficients, the power increases uniformly up to a piston speed of 250 feet per minute, then less rapidly up to a maximum value at about 700 feet per minute, then remains nearly constant up to about 1000 feet per minute, and then diminishes.

* See Fig. 42 in Goss' *High Pressures in Locomotive Service*, which shows clearly how the mean effective pressure varied in a test made by him.

43. Principles of Work and Energy

§ 1. PRINCIPLE OF WORK AND KINETIC ENERGY. — In any displacement of a single particle the forces acting upon it, if any, do more or less work; and, in general, the velocity of the particle is changed, and, hence, the kinetic energy also. There is a simple relation between the total work done upon the particle by all the forces acting upon it in the displacement and the change in the kinetic energy as we will now show. Let P (Fig. 332) be the particle; m = its mass; OAB be its path (not a plane curve necessarily); v_1 = its velocity at A, and v_2 = its velocity at B; R = the resultant of all the forces acting on P; and R_t = the component of R along the tangent to the path at P. Then the work done by all the forces during an elementary displacement ds is $R_t\,ds$. But $R_t = ma = m\,dv/dt$, where a = tangential component of the acceleration of P. Hence the work done on P in the displacement ds is $m\,(dv/dt)\,ds = m\,(ds/dt)\,dv = mv\,dv$; and the work done in the total displacement AB is

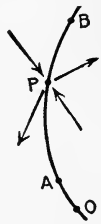


FIG. 332

$$\int_{v_1}^{v_2} mv\,dv = \tfrac{1}{2}\,mv_2^2 - \tfrac{1}{2}\,mv_1^2.$$

Now $\frac{1}{2}\,mv_2^2$ is the kinetic energy of the particle at B, and $\frac{1}{2}\,mv_1^2$ is its kinetic energy at A; hence $\frac{1}{2}\,mv_2^2 - \frac{1}{2}\,mv_1^2$ is the increment in the kinetic energy of P. Thus we have the simple relation, — in any displacement of a particle, the work done by all the forces acting upon it equals the increment in the kinetic energy of the particle. If the total work done is positive then the increment in the kinetic energy is positive also, and there is a real gain and increase in velocity; if the total work done upon the particle is negative, then the increment in the kinetic energy is negative and there is a loss and decrease in velocity.

Let P_1, P_2, P_3, etc., be the particles of any body (not rigid necessarily). In any displacement of the body,

work done by forces acting upon P_1	=	increment in kinetic energy of P_1,
" " " " " " P_2	=	" " " " " P_2,
" " " " " " P_3	=	" " " " " P_3,
etc.	=	etc.

Adding we get total work done on all particles = sum of increments in their kinetic energies = increment in kinetic energy of the system. That is, *in any displacement of any body the total work done upon it by all the external and internal forces acting upon it equals the increment in the kinetic energy of the body.*

In a displacement of a rigid body the total work done by the internal forces equals zero. Proof: — Consider any internal force exerted, say, on P_1 by P_2; P_1 exerts an equal, opposite, and colinear force on P_2. Since the body is rigid the distance between the points of application (P_1 and P_2) of these two forces does not change, and hence (Art. 40) the total work done by these two forces equals zero. But all the internal forces occur in such pairs; hence, the total work done by all the internal forces equals zero, as stated. Thus we have the principle, — *in any displacement of a rigid body the total work done upon the body by the external forces acting upon it equals the increment in the kinetic energy of the body.*

From these principles it follows that the rate at which work is done upon a body equals the rate at which it gains kinetic energy. But the rate at which work is done is power; so we may state that *the combined power of all the forces doing work on a body at any instant equals the rate at which it is gaining kinetic energy then.*

The foregoing principles written out mathematically would take the form: work done = increment in kinetic energy. Since work is of the form force $\times$ distance or space, we may state that the "space-effect" of force is kinetic energy. (The "time-effect" of force is momentum, see Art. 45.) The foregoing principles are especially well adapted for ascertaining the change in velocity — velocity-square, rather — when it is possible to compute the total work done on the body under consideration for the space in which the change takes place. By their means we may ascertain also something about the forces or displacement which accompany any given change in the kinetic energy of a body. We illustrate by means of some

Examples. — 1. A (Fig. 333) is a body weighing 400 pounds. It is dragged along a rough horizontal plane B by a force P, inclined as shown; $P = 80$ pounds. The coefficient of friction is about 1/10. What is the velocity acquired from rest in the first 10 feet? In the first 20 feet? The normal pressure between A and $B = 400 - 80 \sin 20° = 372.6$ pounds; hence, the friction = 37.3 pounds. Now we know all the forces acting on A. Gravity (400 pounds) does no work on A; the work done by P during a displacement of 10 feet $= (80 \cos 20°) \times 10 = 752$ foot-pounds; the reaction of B on A does a work $= -37.3 \times 10 = 373$ foot-pounds. Hence, the total work done on $A = 752 - 373 = 379$ foot-pounds; and this is also the amount of the gain in the kinetic energy of A during 10 feet of displacement. Let $v_1 =$ the velocity (in feet per second) at the end of the first 10 feet; then the kinetic energy of A at the end of the first 10 feet $= \frac{1}{2}(400/32.2)\, v^2 = 6.21\, v_1^2$ foot-pounds. Hence $6.21\, v_1^2 = 379$, or $v_1 = 7.81$ feet per second. Let $v_2 =$ the velocity of A at the end of the first 20 feet; then the kinetic energy of $A = 6.21\, v_2^2$. Since the work done on A during the first 20 feet = 758 foot-pounds, $6.21\, v_2^2 = 758$, or $v_2 = 11.0$ feet per second.

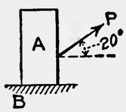


FIG. 333

Such a problem can be solved also by first finding the acceleration. Thus, since the resultant force acting on $A = 80 \cos 20° - 37.3 = 37.9$ pounds, the acceleration $= 37.9 \div (400/32.2) = 3.05$ feet per second per second. The time for describing the first 10 feet = the velocity acquired $\div$ the acceleration $= v_1/3.05 = 0.328\, v_1$. The distance = the average velocity $\times$ the time; that is, $10 = \frac{1}{2} v_1 \times 0.328\, v_1$, or $v_1 = 7.81$ feet per second as before. Obviously, the first method is more direct than the second.

2. A piece of timber $12'' \times 12'' \times 16'$ is suspended by means of two parallel ropes as shown in position $A'B'$ (Fig. 334). The ropes are 10 feet long and the timber weighs 800 pounds. It is raised into the position AB, two feet above $A'B'$, and then allowed to swing. What are its kinetic energy and velocity when it reaches its lowest position? The forces acting on the timber during its descent are gravity, the pulls of the ropes, and air pressure. We neglect the last. At each instant the pulls are normal to the direction of the displacement of their respective points of application; therefore the pulls do no work. The work done by gravity during the descent = $800 \times 2 = 1600$ foot-pounds. Since this is the total work done on the timber, the kinetic energy of the timber in its lowest position = 1600 foot-pounds. Now the timber has a motion of translation — no turning —, and therefore at each instant all points of the timber have identical velocities (Art. 35). Hence, if v = the velocity in the lowest position, then

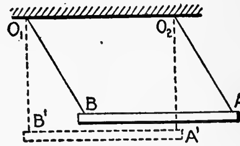


FIG. 334

$$\tfrac{1}{2}(800/32.2)\, v^2 = 1600, \quad \text{or} \quad v = 16 \text{ feet per second.}$$

3. A certain flywheel and its shaft weigh 400 pounds; the radius of gyration of both with respect to the axis of rotation = 10 inches. The wheel is set to rotating at 100 revolutions per minute, and is then left to itself, coming to rest under the influence of axle friction and air resistance after making 84 turns. Required, the average torque of the resistances. The moment of inertia of the wheel and shaft, about the axis, $= (400/32.2)\,(10/12)^2 = 8.64$ slug-feet². The angular velocity, 100 revolutions per minute, $= 2\pi\, 100/60 = 10.47$ radians per second. Hence, the kinetic energy of this wheel and shaft, when released, $= \frac{1}{2}\, 8.64 \times 10.47^2 = 474$ foot-pounds. Besides the forces mentioned above, gravity and the normal pressure of the bearings act on the wheel and shaft, but these do no work during the stoppage. Let M = average torque of the resistances in foot-pounds; then the work done by them during the stoppage is $-M\, 2\pi\, 84 = -528\, M$ foot-pounds. This equals the gain in the kinetic energy of the wheel; that is, $528\, M = 474$, or $M = 0.90$ foot-pounds.

4. A (Fig. 335) is a sheave supported on a smooth horizontal shaft. A is 3 feet in diameter, and its radius of gyration with respect to the axis of rotation = 9 inches. The weights of A, B, and C are 100, 200, and 300 pounds, respectively. The system is released and allowed to move under the influence of gravity and the resistances brought into action. Required the velocity of the suspended weights when they have moved through 10 feet. The system moves under the action of the following external forces, — gravity, axle reaction, air resistance, and the internal reactions between sheave and rope and the fibers of the rope. If the rope is quite flexible then the forces

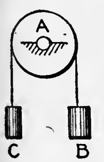


FIG. 335

in the rope do little work; this will be neglected. If the rope does not slip on the sheave, then no work is done by the reaction between rope and sheave. Thus, little or no work is done by the internal forces. The work done by air resistance is small unless the speeds of the moving bodies get high; it will be neglected. The work done by the frictional component of the axle reaction per turn is $M 2\pi$, where M is the frictional moment which we will assume has been found to be 10 inch-pounds. In the displacement under consideration, 10 feet for B and C, the wheel makes $10/3\pi$ turns. Hence, the total work done by friction $= (10 \times 2\pi)(10/3\pi) = 66.7$ inch-pounds $= 5.6$ foot-pounds. Gravity does no work on A; on B and C its work $= 300 \times 10 - 200 \times 10 = 1000$ foot-pounds. We neglect its work on the rope as small. Hence, the total work done on the system $= 1000 - 5.6 = 994.4$ foot-pounds. Now let $v =$ the required velocity in feet per second; then the angular velocity of the wheel $= v \div 1.5 = 0.667\,v$ radians per second. The kinetic energy of the system equals

$$\frac{1}{2}\frac{300}{32.2}v^2 + \frac{1}{2}\frac{200}{32.2}v^2 + \frac{1}{2}I\,(0.667\,v)^2,$$

where $I =$ moment of inertia of the sheave. Now $I = (100/32.2) \times (9/12)^2 = 1.75$ slug-feet^2. Hence, the kinetic energy of the system $= 8.16\,v^2$ foot-pounds. Thus the work-energy equation is $994.4 = 8.16\,v^2$; hence $v = 11$ feet per second.

5. A certain pair of car wheels with their axle weigh 2000 pounds. Their diameter is 33 inches and the radius of gyration of wheels and axle is 9 inches. They are rolled along a level track until their speed is 60 revolutions per minute, and are then left under the influence of the rolling resistance of the track, coming to rest after rolling a distance of 1000 feet. (Data not from an actual experiment.) Required, the average rolling resistance. When released, the angular velocity of the wheels = one revolution per second = 6.28 radians per second, and the linear velocity of their centers $= \pi\,33/12 = 8.64$ feet per second. Hence, the kinetic energy =

$$\frac{1}{2}\frac{2000}{32.2} \times 8.64^2 + \frac{1}{2}\frac{2000}{32.2}(9/12)^2 \times 6.28^2 = 3010 \text{ foot-pounds.}$$

This is also the value of the work done by the rolling resistance, air resistance neglected. Hence, the rolling resistance is equivalent to a constant pull-back of $3010/1000 = 3$ pounds.

§ 2. MOVING TRAINS. — We will now apply the principles of work and energy to some train problems. First, we briefly consider the forces directly concerned with the motion of a train consisting of engine, tender, and cars. For convenience we regard the train as consisting of two parts, namely, the locomotive (engine and tender) and the cars; notation as in Art. 42, § 3.

Locomotive. — For simplicity we regard the locomotive as being driven by an imaginary (forward) force F equivalent to the steam pressures. To be equivalent the work done by F per unit time (or power of F) must equal the indicated power of the locomotive, or Fv must equal $pd^2(l/D)\,v$; hence,

$$F = p\,d^2l/D.$$

This force F we will call the *cylinder effort* of the locomotive. The resistances to motion experienced by a locomotive running alone on a straight and level track may be put into three groups: — (i) Those which arise through its action as a machine, consisting of friction in the working mechanism (valves and gear, cross-head, piston, crank pins, and journals of driving-wheels); (ii) those which arise through its action as a vehicle, like the resistances experienced by the cars (see below); (iii) the air resistance. For convenience we may regard all the resistances in each group lumped, as it were, into a single resistance acting backward on the locomotive. We call them machine resistance, vehicle resistance, and frontal resistance, respectively; and we designate them by R_m, R_v, and R_f. The sum of these three is called locomotive resistance, and will be denoted by R_l. Thus, we regard a moving locomotive as under the action of the following forces (see Fig. 336): gravity, the support-

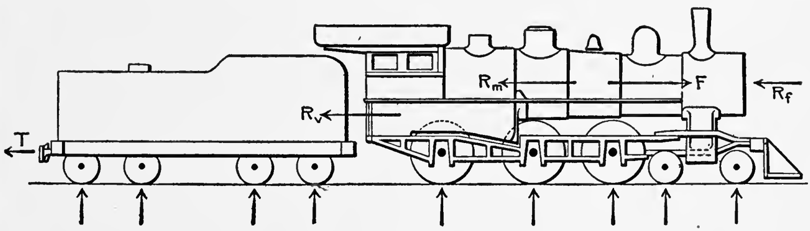


FIG. 336

ing forces of the track (having no components along the rails), the draw-bar pull T, the locomotive resistance $(R_m + R_v + R_f)$, and the moving force $pl\,d^2/D$. The actual external forces are shown in Fig. 337.

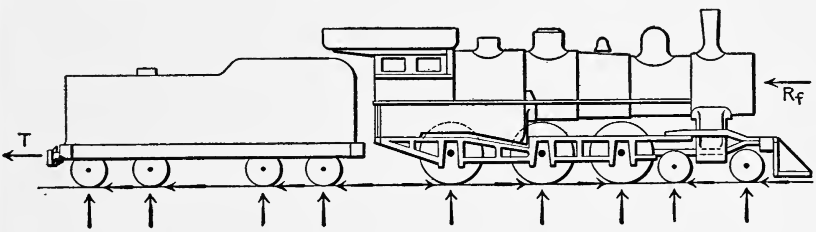


FIG. 337

If the velocity of the locomotive is constant and the track is straight and level, then for any run of length L the work-energy equation is

$$[(p\,d^2l/D) - T - R_l]\,L = 0; \quad \text{hence} \quad T = (p\,d^2l/D) - R_l.$$

If the velocity is changing, then the power equation is

$$\left(\frac{p\,d^2l}{D} - T - R_l\right)v = \frac{d}{dt}\left(\frac{1}{2}Mv^2\right) = Mva,$$

where v = velocity of locomotive and a = its acceleration. Hence

$$T = (p\,d^2l/D) - R_l - Ma.$$

If the locomotive is running on a grade then the grade resistance R_g must be included in an obvious way.

According to the American Locomotive Company (*Bulletin*, No. 1001), the resistances in pounds are as follows:

R_f = 0.24 V^2, where V is velocity in miles per hour;
R_m = 22.2 × weight on drivers, in tons; and
R_v = the same as for cars (see further on).

The Cars. — The cars are urged forward by only one force, the pull of the tender on the first car; this is called *draw-bar pull.* The cars are retarded by several forces, namely: The rolling resistance of the rails upon the treads of the car wheels; the journal friction at the axles of the wheels; the air resistance; and miscellaneous forces, due to oscillation and concussion. The "laws" of these separate resistances are known only in a very general way. Because of lack of knowledge of these separate items of resistance, and, for convenience, it is customary to "lump" them into a single equivalent resistance, called *train resistance.* Thus we may imagine trains to be without actual track, journal, air, etc., resistance, but subjected to this equivalent force, conceived as a single pull backward on the train. A train of cars, then, may be regarded as moving under the action of four forces, namely, the draw-bar pull, the train resistance, gravity, and a supporting force exerted by the track, having no components along the rails.

Many *experiments* have been made to determine train resistance, special "dynamometer cars" (equipped with instruments for measuring and recording speed of train, draw-bar pull, steam pressure, wind velocity and direction, etc.) being used for that purpose now-a-days. The methods for determining train resistance are very simple in principle. One method is this: — the locomotive drags the cars along a straight, level track at a constant speed; the draw-bar pull and the speed are measured. Then the (total) train resistance for that speed equals the draw-bar pull. But level stretches of track are not always convenient of access, and constant speeds are not easily maintained. For an experiment on a grade let H = the ascent or descent of the center of gravity of the train during the experiment, L = the length of the run, W = weight of cars, T = average draw-bar pull, R = average train resistance. Then the grade resistance is $R_g = \pm WH/L$, according as the train is ascending or descending the grade, and the work-energy equation is

$$(T - R_t - R_g)\,L = E,$$

where E is the gain in kinetic energy of the cars during the run, to be regarded as negative if there is a loss of kinetic energy. Hence

$$R_t = T - R_g - E/L.$$

This gives average train resistance for the speeds of the run, or, perhaps, the train resistance for the average speed of the run. Another method is based on the power equation (the rate at which work is done on the cars equals the rate at which they gain kinetic energy); this is

$$(T - R_t - R_g)\, v = \frac{d}{dt}\left(\frac{1}{2} M v^2\right) = Mva,$$

where M = mass of cars, v = velocity, and a = acceleration. Hence

$$R_t = T - R_g - Ma.$$

If the train is being retarded then a should be regarded as negative. There are many practical difficulties in carrying out experiments as suggested; discussion of these is not appropriate here.*

Obviously, train resistance depends upon many conditions, as state of track and rolling-stock, weather and wind, and velocity of train. It is practically impossible to express the influence of all these conditions in a formula for train resistance. For a long time a favorite formula was the so-called

Engineering News formula, $r = 2 + \frac{1}{4} V$,

where r = train resistance in pounds per ton (weight of cars), and V = velocity of train in miles per hour. Recent experiments have shown very clearly that train resistance (per ton) depends very much on the loading of the cars, being much less for heavily loaded cars than for empties, and not so much on velocity as formerly believed. The American Locomotive Company in Bulletin No. 1001 states that "The best data available shows that the resistance varies from about 2.5 to 3 pounds for 72-ton cars to 6 to 8 pounds for 20-ton cars" (see Fig. 338); and "for speeds from 5 to 10 up to 30 to 35 miles per hour the resistance is practically constant." Schmidt, in the bulletin already mentioned, gives formulas for train resistance (per ton) for trains consisting of cars of different average weights; also the following as an approximation

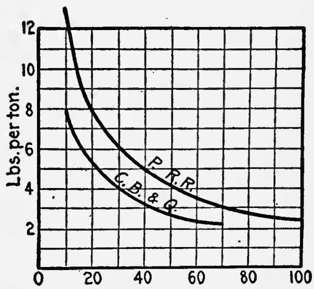


FIG. 338

$$r = \frac{V + 39.6 - 0.031\, w}{4.08 + 0.152\, w},$$

where r = train resistance in pounds per ton, V = velocity in miles per hour, and w = average weight of cars in tons.

* See Schmidt's *Freight Train Resistance*, University of Illinois Bulletin No. 43.

Example. — 1. A certain locomotive (engine and tender) weighs 178.5 tons, 106 tons on the drivers. There are two cylinders, 23 inches (diameter) $\times$ 32 inches (stroke); the drivers are 63 inches in diameter; and boiler pressure is 200 pounds per square inch. Required the maximum draw-bar pull which this locomotive can exert on a level track at 20 miles per hour. The cylinder effort is

$$Kp_0 d^2 l/D = (K\,200 \times 23^2 \times 32) \div 63 = K\,53{,}800 \text{ pounds.}$$

Now the piston speed $s = 2\,vl/\pi D$ (see preceding article) $= (2 \times 20 \times 32) \div (\pi \times 63) = 6.465$ miles per hour $= 569$ feet per minute. The speed coefficient (see Fig. 331) is about 0.60; hence the cylinder effort is $0.60 \times 53{,}800 = 32{,}300$ pounds. The frontal resistance $= 0.24 \times 20^2 = 96$ pounds; the machine resistance $= 22.2 \times 106 = 2350$ pounds; the vehicle resistance is about 4 pounds per ton or $4 \times (178.5 - 106) = 290$. Hence, the total locomotive resistance is about 2740 pounds, and the maximum draw-bar pull $= 32{,}300 - 2740 = 29{,}560$ pounds about.

2. A freight train consists of 30 cars, average weight with load being 60 tons. What is the "resistance" of this train at 20 miles per hour? According to P. R. R. (Fig. 338), the resistance is about 3.5 pounds per ton or 6300 pounds total; according to C. B. & Q., it is about 2.5 pounds per ton or 4500 pounds total. According to Schmidt's formula, it is about 4.4 pounds per ton or 7920 pounds total.

3. The locomotive (example 1) pulls the train (example 2) along a straight track. Required to show graphically how the cylinder effort and the various resistances vary with the velocity, assuming laws of resistances, etc., as in the preceding examples. As in example 1 the cylinder effort $F = K\,53{,}800$; the piston speed $s = 2\,vl/\pi D = 28.45\,v$ feet per minute where v is velocity of locomotive in miles per hour. Thus we have

$v =$	0	5	10	15	20	25	30	35	40
$s =$	0	142	284	426	569	710	853	995	1138
$K =$	0.85	.85	.825	.714	.604	.493	.417	.353	.306
$F =$	22.9	22.9	22.2	19.2	16.2	13.2	11.2	9.5	8.2

the value of K having been taken from the curve $ABCD$ (Fig. 331). We next compute the values of the cylinder effort from $F = K\,26.9$ (tons) for the velocities v. Plotting the results we get curve 1 (Fig. 339). As in example 1 we take machine resistance as 2350 pounds, and vehicle resistance (for locomotive) as 290 pounds at all velocities. The frontal resistance (given by $R_f = 0.24\,v^2$) varies. At $v = 0$, $R_f = 0$; and at $v = 35$ miles per hour, $R_f = 385$ about pounds. Inasmuch as R_f is small compared with the other elements of locomotive resistance, we will take a mean value of R_f, say 200 pounds, and regard it as correct for all velocities. Then the locomotive resistance $R_l = 2350 + 290 + 200 = 2840$ pounds. Plotting this we get the straight line number 2. Taking train resistance at $3\frac{1}{2}$ pounds per ton, as in example 2, gives $R_t = 6300$ pounds, assumed to be independent of velocity for the range from 5 to 35 miles per hour. This plotted gives line number 3.

The ordinate between lines 1 and 3 at any velocity represents the net or resultant driving or accelerating force on the train at that velocity. Thus, at 20 miles per hour that ordinate scales about 23,500 pounds.

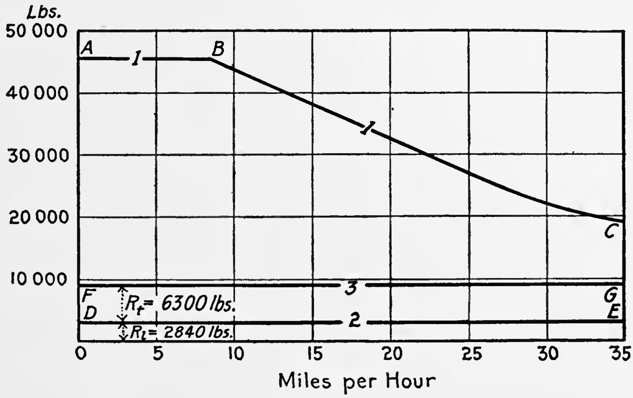


FIG. 339

4. Referring to the train of the preceding example: — Required to show how its acceleration varies with the velocity. Under the preceding example it was explained that any ordinate between lines 1 and 3 represents net driving force on the train. Hence the acceleration at any velocity = such ordinate (to scale) divided by the mass of the train. Thus at 20 miles per hour, say, the ordinate represents about 23,500 pounds, and the acceleration = 23,500 ÷ (3,957,000/32.2) = 0.1914 feet per second per second. In this way the accelerations at other velocities might be computed, and then the curve of accelerations determined. This curve would resemble curve 1; indeed the accelerations are proportional to the ordinates from line 3 to line 1.

44. Efficiency; Hoists

§ 1. EFFICIENCY OF MACHINES. — Among the machines and appliances used in the industries there are some whose function is the conversion or transmission of energy. For example, — an electric dynamo which converts mechanical into electrical energy; a steam engine which converts energy of steam into (kinetic) energy of its flywheel; a line-shaft which transmits energy from one place in a shop to one or more other places; etc. In this article, "machine" means the kind of machine or appliance just described. The amount of energy supplied to a machine in any interval of time, for conversion or transmission, is called the *input* for that time; the amount of energy converted into the desired form or transmitted to the desired place is called the *output*. Experience has shown that output is always less than input; that is a machine does not convert or transmit the entire input. The difference between output and input, for the same interval of time of course, is called lost energy or *loss* simply. By *efficiency*, in this connection, is meant the ratio of output to input; that is if e = efficiency, then

$$e = output \div input.$$

Most machines are designed for a definite rate of working or for a certain load called its full load. Then we speak of the efficiency of a machine at full load, half-load, quarter over-load, etc., these efficiencies being different generally. The two following tables are given to furnish some notion of the efficiencies of the more common machines.*

FULL-LOAD EFFICIENCY OF

	Per cent
Hydraulic turbines	60–85
impulse wheels	75–85
Steam boilers	50–75
engines	5–20
turbines	5–20
Gas and oil engines	16–30
Electric dynamos	80–92
motors	75–90
transformers	50–95

EFFICIENCY OF SOME MACHINE ELEMENTS*

	Per cent
Common bearing, singly	96–98
Common bearing, long lines of shafting	95
Roller bearings	98
Ball bearings	99
Spur gear cast teeth, including bearings	93
Spear gear cut teeth, including bearings	96
Bevel gear cast teeth, including bearings	92
Bevel gear cut teeth, including bearings	95
Belting	96–98
Pin-connected chains, as used on bicycles	95–97
High-grade transmission chains	97–99

* From Kimball and Barr's *Elements of Machine Design.*

The efficiency of a combination of machines, A, B, C, etc., A transmitting to B, B to C, etc., is the product of the efficiencies of the individual machines. For, let e_1, e_2, e_3, etc. = the efficiencies of the separate machines A, B, C, etc., and e = the efficiency of the group. Then if E = the input for A, the output of $A = e_1E$ = the input for B; the output of $B = e_2e_1E$ = the input for C; the output of $C = e_3e_2e_1E$ = the input for D; etc. Hence, the output of the last machine ÷ the input of the first $= (e_1e_2e_3 \ldots)E \div E = e_1e_2e_3 \ldots$, or

$$e = e_1 \cdot e_2 \cdot e_3 \ldots .$$

For example, if a dynamo is run by a steam engine, then the efficiency of the combination or set = the product of their separate efficiencies, say $0.20 \times 0.90 = 0.18$ or 18 per cent.

§ 2. HOISTING APPLIANCES, ETC. — There are certain rather simple appliances by means of which a given force can overcome a relatively large resistance; as, for example, the lever, the wedge, the screw, the differential pulley. By *mechanical advantage* of such appliances is meant the ratio

* For detailed information see Mead's *Water Power Engineering*, from which most values in the first table were taken; Gebhardt's *Steam Power Plant Engineering;* and Franklin and Esty's *Elements of Electrical Engineering.*

of the resistance to the driving force or effort. In most cases this ratio is constant during operation; also, all equal displacements of the point of application of the effort result in equal displacements of the point of application of the resistance, and these displacements take place along the lines of action of the effort and resistance respectively. We will generally use a and b to denote these displacements, and F and R to denote effort and resistance respectively. Then the works done by effort and resistance are respectively Fa and $-Rb$.

Although we do not regard these appliances under discussion as devices for transmitting energy they may be so regarded. The input is the work done by the effort; the output is that done against the (useful) resistance; and the loss is the work done against friction. Hence, efficiency equals the ratio of the useful work done by the appliance to the input, meaning by useful work that done against the resistance (not including friction). Or, in terms of our symbols,

$$e = Rb \div Fa. \tag{1}$$

Let F_0 = the effort which would be required to overcome the resistance R if the machine were frictionless; then $F_0a = Rb$. Substituting in (1) we find that efficiency is given also by

$$e = F_0 \div F. \tag{2}$$

Let R_0 = the resistance which F could overcome if the machine were frictionless; then $Fa = R_0b$. Substituting in (1) we find that efficiency is given also by

$$e = R \div R_0. \tag{3)*}$$

Most of the appliances now under discussion can be operated backward as well as directly. For example, the lever, the wedge, the screw, etc., can be used to lower a heavy body as well as to raise it. Some of these appliances, which can be run either way, will run backward without direct assistance when loaded; that is the load will overcome the internal friction. Such appliances are said to *overhaul*. Some will not run backward unassisted; that is the load cannot overcome the internal friction. Such appliances are said to *self-lock*. An appliance will overhaul or self-lock according as its (direct)† efficiency is greater or less than one-half, if the works done in overcoming friction in a forward and in an equal backward motion are equal

* Formulas (2) and (3) hold even when the displacements a and b are not along the lines of action of F and R. In such a case let f and r respectively = the components of a and b along F and R. Then $e = Rr \div Ff$, $F_0f = Rr$, $Ff = R_0r$, etc.

† When a machine is run backwards it is said to have *reversed efficiency*, by (considerable) extension of the definition of efficiency. In such case the load (on the hoist, for example) is the effort, and the applied force is regarded as the useful resistance. In case the machine overhauls so that the applied force (P say) must assist the load, then by considerable stretch of imagination $-P$ is regarded as the useful resistance; the computed (reversed) efficiency is negative.

(usual case). Proof: — As before, let F = the effort, R = the (useful) resistance, a and b = corresponding displacements of F and R, and w = the work done against friction, all in forward motion of the appliance. Then $Fa = Rb + w$. Now if the efficiency (forward motion) is greater than one-half, then more than one-half of the work Fa is expended usefully (against R); that is Rb is greater than w, and hence R could overcome the friction in backward motion. If the efficiency (forward motion) is less than one-half, then less one-half the work Fa is done against R; that is Rb is less than w, and hence R could not overcome friction unassisted in backward motion.

Inclined Plane. — In Art. 20 there is a discussion relating to a body resting on an inclined plane, in two certain limiting cases of equilibrium; namely, — the body is on the point of ascending and descending. Three formulas are derived there for the force P which would just start the body up, just permit it to descend, and cause it to descend. These formulas were derived from "conditions of equilibrium." Now when the body on the plane is moving up or down with constant velocity, then the forces acting on it are balanced; that is the conditions of equilibrium hold. Therefore, the results arrived at in Art. 20 hold even when such motion obtains but the coefficient of friction and angle of friction of the formulas must be regarded as kinetic coefficient and angle respectively (see Art. 45) which are strictly analogous to the static coefficient and angle respectively (see Art. 19). These latter will be denoted by μ_0 and ϕ_0 respectively in this article. We will not borrow all following formulas from Art. 19 as just explained, but will derive some independently.

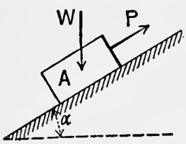


FIG. 340

When the pull P (Fig. 340) is directed along the plane, then the normal pressure = $W \cos \alpha$ and the kinetic friction = $\mu W \cos \alpha$. For any displacement a along the plane, the distance through which gravity works is $a \sin \alpha$.

(i) When the body A is dragged up the plane, the effort = P, the useful resistance = W; and for displacement a at constant speed (or initial and final velocities = 0), $Pa = Wa \sin \alpha + \mu W \cos \alpha \cdot a$, or

$$\frac{W}{P} = \frac{1}{\sin \alpha + \mu \cos \alpha} = \frac{\cos \phi}{\sin (\alpha + \phi)}.$$

The efficiency is the ratio of the work $Wa \sin \alpha$ to the work Pa, or

$$e = \sin \alpha / (\sin \alpha + \mu \cos \alpha) = (\sin \alpha \cos \phi) / \sin (\alpha + \phi).$$

(ii) When the plane is quite steep, then the body A would slip down unless prevented from so doing by a suitable force. It is shown in Art. 19 that such slipping would occur when the inclination is greater than the angle of repose or static angle of friction ($\alpha > \phi_0$). If the body is permitted to slide down against an applied force P up the plane, then W is the effort and $Wa \sin \alpha = Pa + \mu W \cos \alpha \cdot a$, and

$$P = W (\sin \alpha - \mu \cos \alpha) = W \sin (\alpha - \phi) / \cos \phi.$$

In this case efficiency is the ratio of the work of the resistance P to the work of the effort W; that is

$$e = (\sin\alpha - \mu\cos\alpha)/\sin\alpha = \sin(\alpha - \phi)/\sin\alpha\cos\phi.$$

(iii) When the inclination of the plane is less than the angle of repose ($\alpha < \phi_0$) then the body would not slip down the plane under the action of gravity alone. If the body is assisted down by a force P along the plane then all the work of gravity and P is expended against friction; thus $Wa\sin\alpha + Pa = \mu W\cos\alpha \cdot a$, or

$$P = W(\mu\cos\alpha - \sin\alpha) = W\sin(\phi - \alpha)/\cos\phi.$$

In this case the only resistance is the friction, and yet $-P$ is sometimes regarded as useful resistance and the (reversed) efficiency then as $-Pa \div Wa\sin\alpha$, or

$$e = -(\mu\cos\alpha - \sin\alpha)/\sin\alpha = -\sin(\phi - \alpha)/\sin\alpha\cos\phi.$$

Wedge. — In Fig. 341 M is a bearing block sustaining a load W, N is the wedge and A, B, and C are guides. Let α = angle of the wedge, and ϕ = the angle of (kinetic) friction for all rubbing surfaces. As just explained under inclined plane we may borrow formulas from Art. 19 for the present purpose. Thus when P raises W, the mechanical advantage is given by

$$\frac{W}{P} = \frac{1}{\tan(2\phi + \alpha)}$$

If there were no friction ($\phi = 0$), the effort P_0 required to overcome W would be $W\tan\alpha$; hence the efficiency of the actual wedge is given by (see equation 2)

$$e = \tan\alpha/\tan(2\phi + \alpha).$$

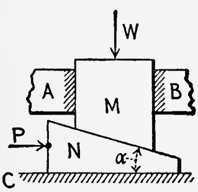


FIG. 341

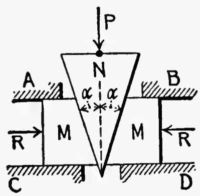


FIG. 342

As explained in Art. 19 the wedge would not slip out (under the action of the load W when $P = 0$) if the wedge angle is less than the double (static) angle of friction ($\alpha < 2\phi_0$). In such a case the force P required to withdraw the wedge is given by

$$P = W\tan(2\phi_0 - \alpha).$$

Fig. 342 represents another arrangement of a wedge; N is the wedge and M and M are bearing blocks against which resistances R are exerted. Borrow-ing from Art. 19 we find that when the force P depresses the wedge, the mechanical advantage is given by

$$\frac{R}{P} = \frac{\cot(\phi + \alpha) - \tan\phi}{2}.$$

If there were no friction ($\phi = o$), the effort P_0 required to depress the wedge against the resistances R would $= 2\,R/\cot\alpha$; hence the efficiency of the actual wedge arrangement is

$$e = [\cot(\phi + \alpha) - \tan\phi] \div \cot\alpha.$$

As explained in Art. 19 forces R could not push the wedge out even when $P = 0$ if the wedge angle is less than double the angle of friction ($\alpha < \phi_o$); that is such a wedge self-locks. The force P required to withdraw the wedge is given by

$$P = 2\,R \div [\cot(\phi_0 - \alpha) + \tan\phi_0].$$

Screw and Lever. — Fig. 343 represents a square thread screw in which the only friction occurs at the nut N. Let α = the pitch angle, h = pitch, r = mean radius of the thread; then $\tan\alpha = h/2\pi r$. Also let l = length of the lever. (i) When the load W is being overcome then the mechanical advantage is (see Art. 19)

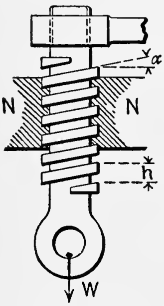


FIG. 343

$$\frac{W}{P} = \frac{l}{r}\,\frac{1}{\tan(\alpha + \phi)}.$$

If the screw were frictionless the force P_0 would $= W(r/l)\tan\alpha$; hence the efficiency is

$$e = \tan\alpha/\tan(\alpha + \phi).$$

It is shown in Art. 19 that the screw self-locks if the pitch angle is less than the static angle of friction ($\alpha < \phi_0$). To start the screw in the direction of W in such a case requires a moment given by

$$Pl = Wr\tan(\phi_0 - \alpha).$$

Mr. Albert Kingsbury has made numerous experiments on screw-jacks to determine their coefficient of friction.* "The conclusions which the results seem to warrant are: That for metallic screws in good condition, turning at extremely slow speeds, under any pressure up to 14,000 pounds per square inch of bearing surface, and freely lubricated before application of pressure, the following coefficients may be used."

Lubricant	Minimum	Maximum	Mean
Lard oil	0.09	0.25	0.11
Heavy machinery oil (mineral)	0.11	0.19	0.143
Heavy machinery oil and graphite (equal volumes)	0.03	0.15	0.07

* *Trans. Am. Soc. M. E.*, 1896, Vol. 17, p. 96.

Pulley. — Fig. 344 represents a simple pulley with part of a rope or chain upon it. Let S = tension in leading or off side of the rope, and T = tension in the following or on side. Then evidently S is greater than T, for S overcomes not only T but also the friction at the pin and the "rigidity" of the rope. The resistance due to pin friction = the product of the coefficient of axle friction (see Art. 45) and the pressure on the pin; this pressure = $S + T$. Hence, if f = coefficient and r = radius of the pin, the work done against friction per revolution of the pulley = $2\pi r(S + T)$. The work done in bending or unbending (inelastic) rope over the pulley is proportional to the amount of rope so bent per revolution (that is $2\pi R$), and it seems to be proportional also to the tension, to the area of the cross section of the rope and inversely proportional to the radius R. Thus the work of bending = $C2\pi RT\,d^2/R = C2\pi Td^2$, where d = diameter of the rope and C is an experimental coefficient depending on the kind of rope and perhaps other elements. Likewise the work of unbending (at off side) = $C2\pi Sd^2$. Now, if we equate the work done by the effort S to the work done against rigidity, the resistance T, and the axle friction, and then simplify the resulting equation, we get

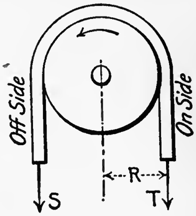


FIG. 344

$$S\left(1 - f\frac{r}{R} - C\frac{d^2}{R}\right) = T\left(1 + f\frac{r}{R} + C\frac{d^2}{R}\right).$$

This equation can be written in the following approximately correct form, —

$$S = \left(1 + 2f\frac{r}{R} + 2C\frac{d^2}{R}\right)T = KT,$$

where K is an abbreviation for $1 + 2fr/R + 2Cd^2/R$. According to experiments by Eytelwein, C equals about 0.23 when d and R are expressed in inches. The American Bridge Company made some experiments to determine C and K for such pulleys and rope as are in common use in tackle for construction work, and found that C depends not only on kind of rope, as expected, but also on the size of rope. The following table is taken from their report.*

Dimensions and coefficients	Hemp				Wire
Diameter of rope, d	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$\frac{3}{4}$
Center pin to center rope, R	$3\frac{7}{8}$	$4\frac{9}{16}$	$5\frac{3}{8}$	6	$7\frac{3}{8}$
Diameter of pin, $2r$	$\frac{7}{8}$	1	$1\frac{1}{8}$	$1\frac{3}{8}$	$2\frac{1}{2}$
Coefficient C	0.23	0.20	0.19	0.17	0.9
Coefficient K	1.20	1.21	1.23	1.24	1.16

These values of C and K are higher than those usually employed. C = 0.08 to 0.22 is advised for hemp rope,† and "K = 1.06 to 1.07 may be considered maximum practical values since generally K = 1.02 to 1.04."‡

* *Trans. Am. Soc. C. E.*, 1903, Vol. 51, p. 161; also *Eng. Rec.*, 1903, Vol. 48, p. 307.

† Hütte, *Taschenbuch* (Twentieth Edition), Vol. 1, p. 247.

‡ Bottcher-Tolhausen, *Cranes*, p. 15.

Tackle. — In the fixed pulley (Fig. 345) when lifting, the efficiency $= Wa/Pa = T/S = 1/K$. When lowering, W is the effort and P the resistance; hence efficiency $= Pa/Wa = T/S = 1/K$. In the movable pulley (Fig. 346) when lifting, $W = P + T = P\,(1 + K)/K$. For any displacement of $P = a$ say, the displacement of $W = \frac{1}{2}\,a$; hence efficiency $= (W\,\frac{1}{2}\,a) \div Pa = (1 + K)/2\,K$. When lowering, $W = P + S = P\,(1 + K)$; and hence efficiency $= Pa \div (W\,\frac{1}{2}\,a) = 2/(1 + K)$.

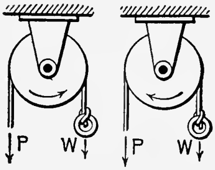


FIG. 345

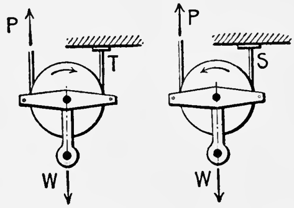


FIG. 346

In a similar way we can determine the efficiency (and mechanical advantage) of any combination of pulleys in terms of K. For example, consider the tackle represented in Fig. 347. There are two separate pulleys in each block A and B. The pulleys in a block are generally alike in size but are here represented unlike for clearness. Let P = the applied pull and W = load, P_1, P_2, P_3, and P_4 = the tensions as indicated in Fig. 348. When lifting, $P_1 = P/K$, $P_2 = P_1/K = P/K^2$, $P_3 = P_2/K = P/K^3$, and $P_4 = P_3/K = P/K^4$. Since $W = P_1 + P_2 + P_3 + P_4$ we have also

$$W = \frac{P}{K} + \frac{P}{K^2} + \frac{P}{K^3} + \frac{P}{K^4} = \frac{P}{K^4}\,(K^3 + K^2 + K + 1).$$

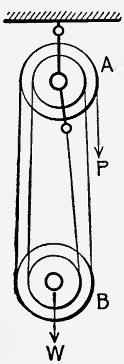


FIG. 347

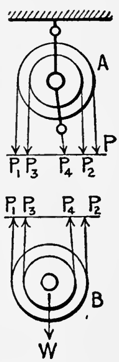


FIG. 348

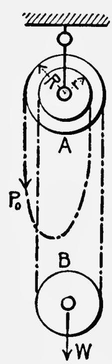


FIG. 349

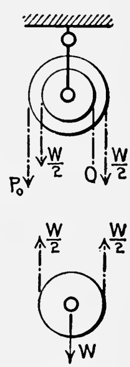


FIG. 350

If there were no lost work ($K = 1$), then all the tensions would equal P, and the load W_0 would be $4\,P$. Hence the efficiency equals $W/4\,P$ (see equation 3), or

$$e = (K^3 + K^2 + K + 1)/4\,K^4.$$

Special (Chain) Hoists. — Fig. 349 represents a Weston differential hoist. The upper block contains two pulleys differing slightly in diameter; they are fastened together. The lower block contains only one pulley. The pulley grooves have pockets into which the links of the chain fit; thus slipping of the chain is prevented. The chain is endless and is reeved as shown. If there were no lost work, then the tension in each portion of the chain to block B would equal one-half the load (Fig. 350), and the pulls on the block A would be as indicated in the figure. Now if R and r = the distances from the center of the pin in block A to the axis of the chain as indicated then moments about the axis of the pin give

$$P_0R + \tfrac{1}{2}Wr = \tfrac{1}{2}Wr, \text{ or } W = P_0 2R/(R - r);$$

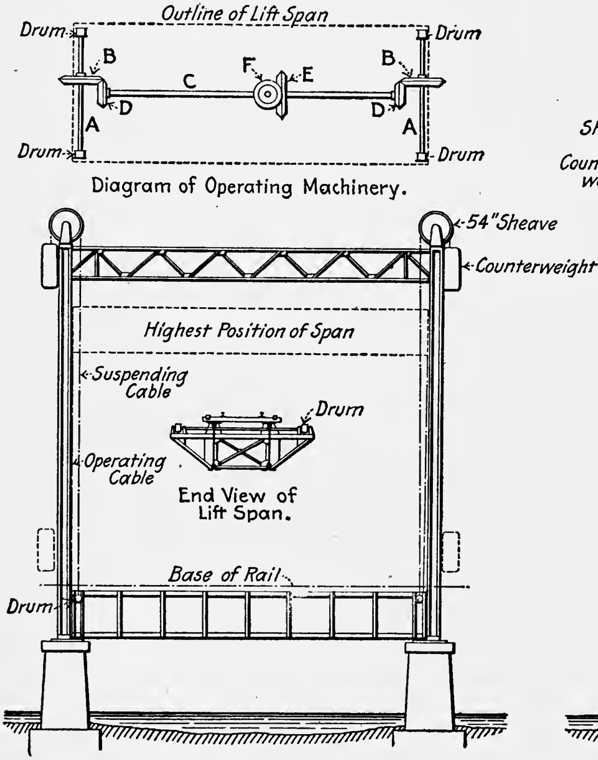
the ratio, $W/P_0 = 2R/(R - r)$ may be made very large by making $R - r$ small. The mechanical advantage is

$$\frac{W}{P} = \frac{W}{P_0}e = \frac{2R}{R - r}e,$$

where P = the actual force required to raise W and $e \doteq$ efficiency. These hoists are made of various capacities up to $W = 3$ tons; their efficiencies are relatively low, from about 25 to 40 per cent according to the manufacturers' lists. In the so-called Duplex and Triplex hoists the upper blocks are screw-geared and spur-geared respectively. At full load the efficiency of these hoists vary from about 30 to 40 and from 70 to 80 per cent.

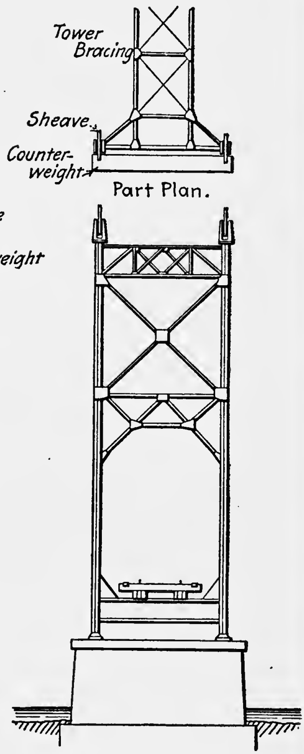
Example. — We will now show how to apply some of the preceding principles and formulas in a computation relating to the operating machinery of the vertical lift bridge represented in Figs. 351 and 352. The lift span when down in place rests on two piers. When up it is balanced by two counterweights as shown. Each counterweight is suspended by means of two pairs of one-inch cable; each pair of cables extends upwards from the counterweight, over a sheave and downward to a point of attachment on the lift span. At each corner of the lift span there is a spirally grooved drum carrying two one-half-inch cables. Each cable has one end attached to its drum; the other end of the up-haul cable is attached to a point vertically above at the top of the tower, and the other end of the down-haul cable is similarly attached at the base of the tower. As the drum is revolved, one cable is wound upon it and the other is paid out. The two drums at either end of the span are mounted upon a single cross-shaft A, which carries a bevel gear B. The gears BB mesh with bevel pinions DD mounted on the longitudinal shaft C which also carries a bevel gear E. E meshes with a bevel pinion F on a vertical shaft which carries a capstan head. This capstan head takes a horizontal lever by means of which a man operates the mechanism. To lift the bridge he rotates the capstan headed shaft in the proper direction and drives the drums; they wind the up-haul cable upon themselves and pay out the down-haul cable as already described. This winding up necessitates upward motion of the bridge.

The length of the lever (radius of circle in which the man walks as he operates) is 6 feet. The pinions F and D are alike; each is 6.86 inches in diameter and has 21 teeth. The spur wheels E and B are also alike; each is 16.87 inches in diameter and has 53 teeth. The drums are 18 inches, the sheaves 54 inches, and the sheave shafts are $3\frac{1}{2}$ inches in diameter. The lift span weighs 68,000 pounds and each counterweight weighs one-half that amount. Thus the span would be perfectly balanced, if the mechanism were frictionless and the cables without stiffness and weight, and no effort would be required to operate the bridge.



Side Elevation.

FIG. 351



Cross Section.

FIG. 352

In the following computation the weight of cables is neglected. Then the tension in each counterweight cable on the counterweight side of the sheaves is one-fourth of 34,000 pounds or 8500 pounds. When the span is being lifted, the tension on the other or following side of the sheave is less than 8500 pounds. Call that tension T; then

$$8500 = KT, \text{ or } T = 8500 \div K$$

(see under "pulley" above). We will take $K = 1.06$; then $T = 8020$ pounds, and hence the lift on the span due to counterweights $= 8 \times 8020 = 64{,}160$ pounds. This leaves $68{,}000 - 64{,}160 = 3840$ pounds to be furnished by the four up-haul cables, or 960 pounds apiece.

Let a and b = respectively any corresponding displacements of the effort at the hand lever and the resistance 960 pounds at each drum; then

$$\frac{a}{b} = \frac{6 \times 12}{3.43} \times \frac{53}{21} \times \frac{16.87}{18} = 50.$$

Hence, if the mechanism were frictionless the effort P_0 required to produce 960 pounds tension in one rope $= 960 \div 50 = 19.2$ pounds; and the effort P required to produce that tension by means of the actual mechanism $= 19.2 \div e$, where e = the efficiency of that part of the mechanism which transmits from P to the resistance 960 pounds. The efficiency of each pair of gears and necessary bearings we take as 0.95; the efficiency of a drum about $1 \div 1.03 = 0.97$; hence $e = 0.95 \times 0.95 \times 0.97 = 0.875$. Therefore $P = 19.2 \div 0.875 = 22$ pounds, and the effort (at the lever) required to develop a tension of 960 pounds at the four drums $= 4 \times 22 = 88$ pounds.

The computation can also be made as follows: — We regard the total force Q exerted at the hand lever and the force of gravity on the counterweights as two efforts which overcome the (useful) resistance (gravity on the lift span) and the wasteful resistances in the entire mechanism. For any rise b of the lift-span the counterweights descend an equal distance and the hand-lever effort works through a distance $50\,b$; and since the efficiency of a sheave $= 1 \div 1.06 = 0.944$, we have

$$Q \times 50\,b \times 0.875 + 2 \times 34{,}000 \times b \times 0.944 = 68{,}000 \times b, \text{ or } Q = 87 \text{ pounds.}$$

45. Kinetic Friction

§ 1. KINETIC FRICTION, or Friction of Motion, is the friction between two bodies when sliding actually occurs. The *coefficient of kinetic friction* for two bodies is the ratio of the kinetic friction to the corresponding normal pressure between them. The *angle of kinetic friction* is the angle between the normal pressure and the total pressure (resultant of the normal pressure and the kinetic friction). One of the so-called laws of friction states that the kinetic coefficient is less than the static coefficient (Art. 19), and implies that there is a sudden or abrupt change in the values of the coefficients. Experiments by Jenkin and Ewing* on the kinetic coefficients at speeds as low as 0.0002 foot per second (about $\frac{3}{4}$ foot per hour) lead them to conclude that "it is highly probable that the kinetic coefficient gradually increases when the velocity becomes extremely small, so as to pass without discontinuity into the static coefficient." Experiments by Kimball† also indicate that there is no abrupt

* *Phil. Trans. Roy. Soc.*, 1877, Vol. 167, Part 2.

† *Am. Jour. Sci.*, 1877, Vol. 13, p. 353.

change from static to kinetic coefficient. Moreover, they show that the kinetic coefficient may be greater than the static. Galton and Westinghouse experiments* indicate that the coefficient for dry surfaces probably decreases progressively from the value of the static coefficient as the velocity increases. See the following table of

COEFFICIENTS OF FRICTION AT VARIOUS SPEEDS

Cast-iron Brake Shoes on Steel-tired Wheels

Velocity		Coefficients			Number of tests
Miles per hour	Feet per second	Maximum	Minimum	Mean	
0+	0+	....	....	0.330	...
10	14.5	0.281	0.161	.242	54
20	29	.240	.133	.192	69
30	44	.196	.098	.164	94
40	59	.194	.088	.140	70
50	73	.153	.050	.116	55
60	88	.123	.058	.074	12

The foregoing coefficients are based on observations taken very soon after application of the brakes. The wide variation from minimum to maximum value at any velocity was due in part to the different intensities of pressure employed at that velocity; in general the coefficient decreases with increase in intensity until seizing occurs.

Continual rubbing of dry surfaces abraids them and decreases the coefficient of friction. This effect is clearly shown in the following table, also taken from the Galton and Westinghouse experiments.

COEFFICIENT OF FRICTION AS AFFECTED BY TIME OF RUBBING

Cast-iron Brake Shoes on Steel-tired Wheels

Miles per hour	Time after applying brakes				
	0+	5 seconds	10 seconds	15 seconds	20 seconds
20	0.182	0.152	0.133	0.116	0.099
27	.171	.130	.119	.081	.072
37	.152	.096	.083	.069	.....
47	.132	.080	.070	.....	.....
60	.072	.063	.058	.....	.....

The discrepancies between the two foregoing tables are due in part to the fact that the values for time 0 + in the second table are based on comparatively few experiments.

The following table gives coefficients for several different materials; it shows also influence of intensity of pressure, velocity and water lubrication.

* *Proc. Inst. Mech. Engrs.*, 1879, p. 170.

COEFFICIENTS OF FRICTION *

Various Brake-shoe Materials on Steel-tired Wheels

Materials	Pressure, pounds per square inch	Velocity, miles per hour		Lubrication
		7	15	
Cast iron	10	0.43	0.37	none
Cast iron	40	.36	.30	none
Oak	10	.60	.55	none
Oak	40	.43	.40	none
Poplar	10	...	.72	none
Poplar	40	...	.53	none
Cast iron	20	.32	.28	water
Cast iron	80	.30	.26	water
Oak	40	.037	.032	water
Oak	120	.073	.055	water
Poplar	40	.041	.038	water
Poplar	120	.070	.053	water

* From Experiments by Ernest Wilson, *Engr. News*, 1909, Vol. 62, p. 736.

COEFFICIENTS OF KINETIC FRICTION (Rough Averages)

Compiled by Rankine from Experiments by Morin and others

Wood on wood, dry	0.25–0.50	Leather on oak	0.27–0.38
soapy	.2	Leather on metals, dry	.56
Metals on oak, dry	.5– .6	wet	.36
wet	.24– .26	greasy	.23
soapy	.2	oily	.15
Metals on elm, dry	.2 – .25	Metals on metals, dry	.15–0.2
Hemp on oak, dry	.53	wet	.3?
wet	.33		

The following table gives coefficients for eight ship launchings and hence for similar cases. The lubricant used seems to have been mainly tallow.

COEFFICIENTS FROM LAUNCHING DATA.*

Load, tons per square foot	Coefficient of friction	Load, tons per square foot	Coefficient of friction
0.50	0.0532	1.62	0.0370
0.79	.0487	2.05	.0324
1.03	.0400	3.56	.0257
1.29	.0407	4.50	.0217

* From Peabody's *Naval Architecture*.

§ 2. PIVOT AND JOURNAL FRICTION. — *Pivots.* — Let W = load, μ = coefficient of friction. (i) In the case of a flat pivot (Fig. 353) the average pressure per unit area of contact is $W/\pi R^2$. On any element of area dA the normal pressure = $(W/\pi R^2)\, dA$ (supposing that the total pressure is uniformly distributed), and the frictional resistance on the element = $\mu(W/\pi R^2)dA$. The moment of this resistance about the axis of the shaft $= \mu(W/\pi R^2)dA\cdot\rho$. We take $dA = \rho d\theta\cdot d\rho$; then the total resisting moment $=$

$$\int_0^{2\pi}\int_0^r \mu\frac{W}{\pi R^2}d\theta\rho^2\,d\rho = \mu W\frac{2}{3}R.$$

Thus the actual resistance may be regarded as a single force $= \mu W$ with an arm $= \frac{2}{3}R$; and, for example, the work done against friction per revolution or the power lost may be computed simply on that basis. Thus the work done per revolution $= \frac{4}{3}\pi\mu WR$, and the power lost $= \frac{4}{3}\pi\mu WRn$ where $n =$ number of revolutions per unit time.

(ii) In a similar way we might determine the resisting (frictional) moment in a collar bearing pivot (Fig. 354). We would find the moment to be

$$\tfrac{2}{3}\mu W(R^3-r^3)\div(R^2-r^2).$$

Hence we may regard the resistance as a single force $= \mu W$ with an arm $\frac{2}{3}(R^3-r^3)\div(R^2-r^2)$.

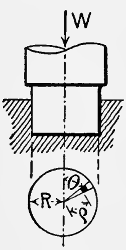


FIG. 353

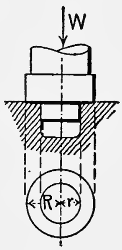


FIG. 354

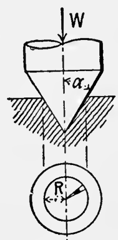


FIG. 355

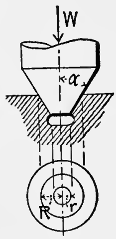


FIG. 356

(iii) In the conical pivot (Fig. 355), the total normal pressure, and hence the friction too, is increased by wedge action. Let $p =$ the intensity of normal pressure at any point of the contact, regarded as constant. Then the normal pressure on an elementary area $dA = pdA$. Since the friction has no vertical component, the vertical component of the normal pressures on all the elementary areas $= W$; that is,

$$\int p\,dA\cdot\sin\alpha = W = pA\sin\alpha,\ \text{or}\ p = \frac{W}{A\sin\alpha}.$$

But $A\sin\alpha =$ the horizontal projection of the actual surface of contact. Hence the intensity of the normal pressure is independent of α, the pivot angle. For Fig. 355, $p = W/\pi R^2$; hence the normal pressure on the elementary area dA is $(W/\pi R^2)dA$ and the frictional resistance $= \mu(W/\pi R^2)dA$. The moment of this resistance about the axis of the shaft $= \mu(W/\pi R^2)dA\cdot\rho$, and the entire resisting moment $=$ the integral of this expression. For simplicity in integration, imagine dA to be of such shape that its horizontal projection equals $\rho d\theta\cdot d\rho$ (see Fig. 355). Then $\sin\alpha\cdot dA = \rho d\theta\cdot d\rho$, and the resisting moment $=$

$$\int_0^{2\pi}\int_0^R\frac{\mu W\,d\theta\rho^2\,d\rho}{\pi R^2\sin\alpha} = \frac{\mu W}{\sin\alpha}\,\frac{2}{3}R.$$

Hence we may regard the resistance as a single force $\mu W/\sin\alpha$ with an arm $\frac{2}{3}R$.

(iv) In a similar way we may compute the resisting (frictional) moment in the case of a frustrated conical pivot (Fig. 356). We would find that the resisting moment =

$$\frac{\mu W}{\sin\alpha}\frac{2}{3}\frac{R^3 - r^3}{R^2 - r^2}.$$

Hence we may regard the friction as a single force $= \mu W/\sin\alpha$ with an arm $\frac{2}{3}(R^3 - r^3)/(R^2 - r^2)$.

Journal Friction. — We do not attempt to compute the normal pressure and frictional resistance at each point of a journal bearing and then the resisting moment as in the case of pivots. So-called *coefficients of journal friction* have been determined from direct experiments on journal friction. This coefficient is the ratio of the frictional resistance to the pressure between journal and the bearing. Thus in a certain experiment there were 20 babbitt bearings sustaining a $2\frac{7}{16}$-inch shaft; the load per bearing was 2,000 pounds, and it was found that 1124 watts were required to run the shaft at 350 revolutions per minute. All the power was used to overcome the journal friction. Since 1124 watts = 49,600 foot-pounds per minute and 350 revolutions per minute corresponds to a (shaft) surface velocity of 223 feet per minute, the total frictional resistance $= 49{,}600 \div 223 = 222$ pounds or 11.1 pounds per bearing. Hence the coefficient of journal friction in this particular instance was $11.1 \div 2000 = 0.0055$.

The pressure between a journal and its bearing is not uniformly distributed over the surface of contact. By nominal intensity of pressure ("pressure" for brevity) is meant the whole pressure divided by the product of the length and diameter of the bearing. Thus in the experiment just mentioned, the length of each bearing was $9\frac{2}{3}\frac{1}{2}$ inches; hence the nominal intensity was $2000 \div (2\frac{7}{16} \times 9\frac{2}{3}\frac{1}{2}) = 90$ pounds per square inch.

It has been found from numerous experiments that coefficients of journal friction depend on (i) the method of lubrication, (ii) the lubricant, (iii) its temperature, (iv) the velocity of rubbing, and (v) intensity of pressure on the bearing.

(i) Tower* and Goodman† report the following relative coefficients as showing effect of the method of applying the lubricant:

Method	Tower	Goodman
Bath	1.00	1.00
Saturated pad	....	1.32
Ordinary pad	6.48	2.21
Siphon	7.06	4.20

* *Proc. Inst. Mech. Engrs.*, 1883.

† *Mechanics Applied to Engineering*, 1896.

(ii) The following table (according to Tower) indicates how the coefficient depends on the lubricant. Numbers are relative.

Sperm oil	Rape oil	Mineral oil	Lard oil	Mineral grease
1.00	1.06	1.29	1.35	2.17

The journal was steel; gun metal brass embracing somewhat less than one-half the circumference of the journal; speed 300 revolutions per minute; nominal loads from 100 to 310 pounds per square inch; oil temperature 90° F.; bath lubrication. Tower states also: "the numbers represent the relative thickness or body of the various oils, and also in their order, though perhaps not in their numerical proportions, their relative weight-carrying power. Thus sperm oil, which has the highest lubricating power, has the least weight-carrying power; and though the best oil for light loads would be inferior to the thicker oils if heavy pressures or high temperatures were to be encountered. (iii) The coefficient decreases with increased temperature (see Figs. 357 and 359). But if the temperature gets so high as to lower the viscosity greatly, then the lubricant gets squeezed out and the coefficient increases. (iv) In general the coefficient increases with increase of speed (see Figs. 358 and 360 and accompanying explanations). But at the lower speeds the coefficient may decrease with increase of velocity (see Fig. 358). (v) The coefficient decreases with increase of intensity of pressure (see Figs. 358 and 361). But the intensity may become so great that the lubricant gets squeezed out, and then the coefficient increases and seizing occurs.

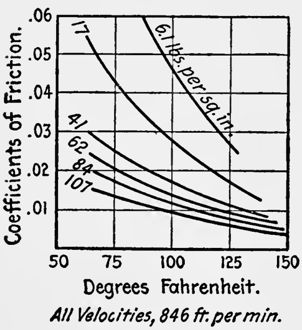


FIG. 357

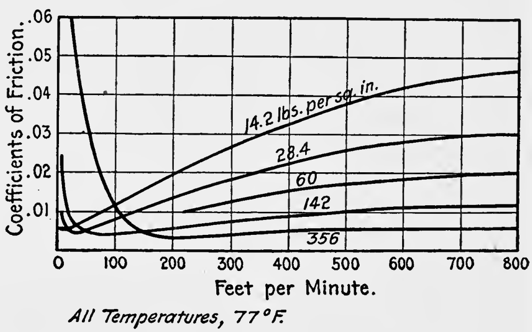


FIG. 358

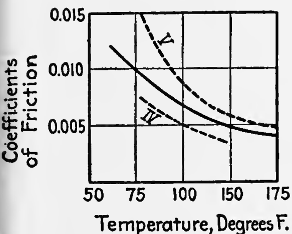
Figs. 357 and 358 are after Stribeck.* Gas-motor oil and ring oiler were used in the experiments. Figs. 359, 360, and 361 are after Lasche.† They show respectively how the coefficient depends on temperature, velocity, and

* *Z. d V. d. I.*, 1902, Vol. 46, p. 1341.
† *Z. d V. d. I.*, 1902, Vol. 46, p. 1881.

pressure. The lubrication was forced; journal and bearing combinations as follows:

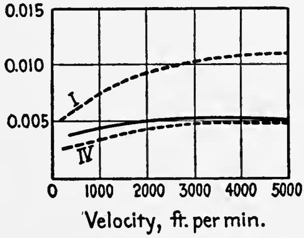
Number	I	II	III	IV	V
Journal	steel	nickel steel	nickel steel	nickel steel	wrought iron
Bearing	white metal	white metal	....	bronze	white metal

The heavy line in each figure represents the average law for the five combinations, and the other two curves relate to the two combinations departing most from the average result.



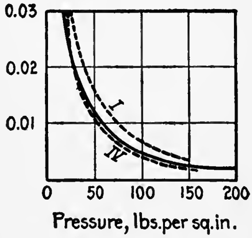
All Pressures, 92.5 lbs. per sq. in.
All Velocities, 197 ft. per min.

FIG. 359



All Pressures, 92.5 lbs. per sq. in.
All Temperatures, 112° F.

FIG. 360



All Temperatures, 112° F.
All Velocities, 197 ft. per min.

FIG. 361

Roller and Ball Bearings generally develop less resistance to turning than ordinary bearings. For descriptions of roller and ball bearings and information on their coefficients of resistance, students are referred to works on machine design; see also Kent's Pocket Book, or the American Civil Engineer's Pocket Book for coefficients of resistance. But to furnish some notion of relative values the two following tables are given. In the experiments from which the first was compiled* the speed was 560 revolutions per minute, and the loads were from 113 to 456 pounds. In the ball bearings† the balls were $\frac{7}{8}$ inch in diameter and they ran in grooves, cross sections of which were arcs of circles whose radius was equal to two-thirds the diameter of the balls; the circle of the centers of the balls was 4 inches in diameter.

COEFFICIENTS OF JOURNAL FRICTION

Different Bearings

Diameter of journal	Flexible rollers	Solid rollers	Babbitt bearing
$1\frac{1}{16}$	0.018	0.022	0.043
$2\frac{3}{16}$	.014	.....	.082
$2\frac{7}{16}$	.032	.021	.096
$2\frac{15}{16}$	.025	.027	.107

Ball Bearings

Load in pounds	Revolutions per minute		
	65	385	780
840	0.0033	0.0035	0.0037
2,420	.0017	.0018	.0019
4,500	.0015	.0015	.0015
10,800	......	......	.0011

* Benjamin, *Machinery, N. Y.*, 1905, Vol. 25, p. 000.
† Stribeck, *Z. d V. d. I.*, 1901, Vol. 45, p. 121.

CHAPTER XI

MOMENTUM AND IMPULSE

46. Linear Momentum and Impulse

§ 1. (LINEAR) MOMENTUM. — By momentum of a moving particle is meant the product of its mass and velocity. We regard momentum as having direction, namely, that of velocity; thus, momentum is a vector quantity. By momentum of a collection of particles is meant the vector-sum of the momentums of the particles. For example, let m' and m'' = the masses of two particles (Fig. 362), v' and v'' = the velocities of the particles at a certain instant, and suppose that $AB = m'v'$ and $BC = m''v''$ according to some convenient scale; then AC represents the momentum of the two particles.

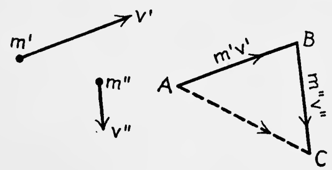


FIG. 362

Since the component of the vector AC along any line equals the algebraic sum of the components of the vectors AB and BC along that line, it follows that the component of the momentum of a pair of particles along any line equals the algebraic sum of the components of their momentums along that line. Obviously, this proposition can be extended to a collection of any number of particles. A simple expression for this component can be arrived at as follows: Let m', m'', m''', etc. = the masses of the particles; v', v'', etc. = their velocities; and v'_x, v''_x, etc. = the components of these velocities along any line x. Then the component of the momentum of the collection along this line = $m'v'_x + m''v''_x + \ldots$ Now if x', x'', etc. = the x coördinates of the moving particles, and $\bar{x}$ = the x coördinate of the mass-center, all at the same instant, then $m'x' + m''x'' + \ldots = \bar{x}\Sigma m$ (Art. 34); and differentiating with respect to t, we get $m'dx'/dt + m''dx''/dt + \ldots = (d\bar{x}/dt)\Sigma m$, or

$$m'v'_x + m''v''_x + \ldots\ldots = \bar{v}_x\Sigma m = M\bar{v}_x,$$

where $M = \Sigma m$ = the mass of the collection. That is, the x component of the momentum of the collection of particles equals the product of the mass of the collection and the x component of the velocity of the mass-center. Hence, the component momentum is just the same as though all the material of the body were concentrated at the mass-center.

In the case of a body having a motion of translation, all the particles have at any instant velocities which are equal in magnitude and the same in direction (Art. 35). Hence the momentums of the particles are parallel, and their vector sum $= m'v + m''v + \quad . \quad . \quad . \quad = v\Sigma m = Mv$, where $v =$ their common velocity and $M =$ the mass of the body.

The definition of momentum implies that the unit of momentum equals the momentum of a body of unit mass moving with unit velocity. The magnitude of the unit, therefore, depends on the units of mass and velocity used. No single word has been generally accepted for any unit of momentum. The dimensional formula for momentum is $F'T'$ (see appendix A), that is, a unit momentum is one dimension in force and one in time. Hence, any unit of momentum may be and commonly is called by names of the units of force and time used. Thus the unit of momentum in the C.G.S. system is called the dyne-second; in the "engineers' system," the pound (force) -second, etc.

In Art. 34 it is explained that the acceleration of the mass-center of any collection of particles does not depend at all on the forces which the particles exert upon each other but on the external forces; also that the algebraic sum of the components of the external forces along any line equals the product of the mass of the system and the component of the acceleration of the mass-center along that line, that is,

$$F'_x + F''_x + \quad . \quad . \quad . = M\bar{a}_x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where F'_x, F''_x, etc., are the components of the external forces along a line x, and $\bar{a}_x$ is the x component of the acceleration of the mass-center. Now $\bar{a}_x$ equals the rate at which the x component of the velocity of the mass-center changes, that is, $\bar{a}_x = d\bar{v}_x/dt$, where $\bar{v}_x$ is the x component of the velocity of the mass-center; hence, $M\bar{a}_x = M\,d\bar{v}_x/dt = d(M\bar{v}_x)/dt$; and finally

$$F'_x + F''_x + \quad . \quad . \quad . \quad = d(M\bar{v}_x)/dt. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But Mv_x is the x component of the momentum of the system, and $d(Mv_x)/dt$ is the rate at which that component changes; hence the *algebraic sum of the components of the external forces along any line x equals the rate at which the x component of the linear momentum changes.*

The principle just arrived at (equation 2) was derived from the law of motion of the mass-center (equation 1), and it is essentially an alternative form of the law. But practically the former seems to apply more simply in certain cases as the following examples show.

Fig. 363 represents a jet of water impinging against a flat plate. Required the pressure of the jet upon the plate. Let $W =$ the weight of water impinging per unit time, $v =$ the velocity of the water in the jet, and $\alpha =$ the angle between the jet and the plate as indicated. We suppose that the water does not rebound from the plate with any considerable velocity; then the momentum of the water after striking has no component normal to the plate. The momentum of an amount of water equal to W before striking is $(W/g)v$, and the component of that momentum along the normal to the plate $= (W/g)v \sin\alpha$; hence the change in the (normal) component momentum is $(W/g)v \sin\alpha$. This change takes place in unit time; therefore, it is the rate at which momentum along the normal is changed, and also the value of the normal pressure of the plate against the jet. The jet exerts an equal (normal) pressure against the plate. If the plate is rough, then the water also exerts a frictional force on the plate.

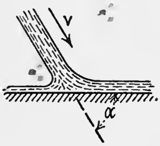


FIG. 363

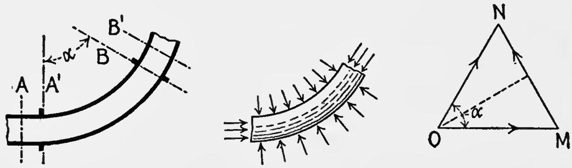


FIG. 364

For another example, we will determine the pressure on a bend in a pipe by water flowing through it at constant velocity. Let W = the weight of the water flowing past any section of the pipe per unit time; v = velocity of the water, assumed to be the same at all points of inlet and outlet cross sections of the bend; and α = the angle of the bend (Fig. 364). Also let Δt = the time required for the body of water AB to move into the position $A'B'$. The momentum of the body of water at the beginning of the interval = that of AA' + that of $A'B$; its momentum at the end of the interval = that of $A'B$ + that of BB'. Hence the change in the momentum of the body of water in the time Δt = momentum of AA' − momentum of BB'. These momentums respectively are in the direction AA' and BB'; each equals $(W\Delta t/g)v$. Hence the change of momentum under consideration is represented by the vector MN where OM and ON represent the two momentums just mentioned. But $MN = 2(OM) \sin \frac{1}{2}\alpha$; hence the change = $2(W\Delta t/g)v \sin \frac{1}{2}\alpha$, and the rate at which the change occurs = $2(W/g)v \sin \frac{1}{2}\alpha$. The direction of this rate is MN; it bisects the angle α. This rate of change of momentum is maintained by the forces acting on the body of water in $A'B$. Those forces consist of gravity G, the pressures P_1 and P_2 (of the water) on the front and rear faces of the body, and the pressure P of the bend upon it. Their resultant $R = 2(W/g)v \sin \frac{1}{2}\alpha$, and R bisects α. If R, G, P_1 and P_2 are known then P can be determined. For it is such a force which compounded with G, P_1 and P_2 gives R. The pressure of the water on the bend = $-P$.

For another example, we take the jet propeller of a ship. This consists essentially of a pump which takes in water from the sea and ejects it from nozzles toward the rear (to propel the ship forward). Let W = weight of water so ejected per unit time, v = velocity of the ship, and V = velocity of the ejected water relative to the ship. The absolute velocity of the jet (relative to the sea) = $V - v$. Hence the amount of momentum produced by the pumping plant (pump, pipes, etc.) per unit time = $(W/g)(V - v)$. The direction of this is horizontal and backward; hence the plant exerts a force on the body of water within the passages at any instant equal to $(W/g)(V - v)$; the water exerts an equal force forward on the passages.

If the algebraic sum of the components — along any line — of the external forces acting on a body equals zero, then the rate of change of the component momentum (along that line) equals zero; hence, if the sum remains zero for any interval of time, the component momentum remains constant. This is known as the principle of *conservation of linear momentum.* It follows that if there are no external forces acting on the body, its linear momentum remains constant. The grand illustration of this principle is furnished by the solar system. Even the nearest stars exert no appreciable attractions on the solar system, and so the members of the system move under the action of their mutual attractions only. Accordingly, the component of the momentum of the system along any line does not change; the linear momentum is, therefore, constant in amount and direction. It follows that the mass-center of the system moves uniformly, and in a straight line.

§ 2. (LINEAR) IMPULSE. — If the magnitude and direction of a force are constant for any interval of time, then the product of the magnitude of the force and the interval is called the impulse of the force for that interval. If the magnitude varies, then the impulse for any interval equals the sum of the impulses for all the elementary periods of time which make up the interval; that is impulse =

$$\lim\,[F'\Delta t + F''\Delta t + \ . \ . \ . \ . \ . \ ] = \int F\,dt,$$

where F = the varying force. If the direction of the force varies, we regard the impulse for any elementary portion of time as a vector quantity having the direction of the force, and then in principle we add (vectorially) the elementary impulses for all the portions of time which make up the interval. That is to say, we integrate $F\,dt$ vectorially, arriving at a definite vector quantity.

Units of impulse depend on the units of force and time used.* There is no current single-word name for any unit of impulse. Each unit is named by the names of the units of force and time involved in it. Thus, in the C.G.S. system the unit of impulse is the dyne-second; in the "engineers' system" the unit of impulse is the pound (force) -second.

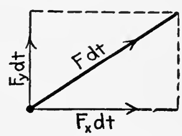


FIG. 365

It is evident (Fig. 365) that the elementary inpulse $F\,dt$ is the resultant of the impulses of the x and y components of F (or x, y, and z components, if preferred). Hence the x, y, and z components respectively of the impulse of F equal the impulses of the components of the force F. If we integrate equation (2) over any interval $t_2 - t_1$, say, we get

$$\int_{t_1}^{t_2} F'_x\,dt + \int_{t_1}^{t_2} F''_x\,dt + \ \ . \ . \ . \ \ = \tfrac{1}{2}\,M\bar{v}_2 - \tfrac{1}{2}\,M\bar{v}_1 = \Delta\,(M\bar{v}), \qquad (3)$$

in which $\bar{v}_1$ and $\bar{v}_2$ = the velocity of the center of gravity of the system at times t_1 and t_2 respectively. Equation (3) can be put into the following

* See appendix A.

principle of (linear) impulse and momentum. The algebraic sum of the components — along any line — of the impulses of the external forces acting on any system of particles equals the increment in the component of the momentum of the system along that same line, the sum and the increment referring to any interval of time.

The principle of impulse and momentum answers such questions as, — how much velocity in a given time? or how much time to produce a certain velocity? For example, it is required to ascertain how much time is required to give a velocity of 10 feet per second to a certain body by sliding it along a horizontal rail by means of a constant push of 20 pounds, the body weighing 100 pounds and the frictional resistance of the rail being 8 pounds. The external forces acting on the body are gravity, the push, and the reaction of the rail, the horizontal and vertical components of which are friction and the "normal pressure." Only the impulses of the push and friction have components along the line of motion; hence

$$20\,t - 8\,t = (100/32.2)\,40,$$

where t = the required time. Therefore $t = 10.3$ seconds. Solution of such a problem by earlier methods of this book would be as follows: Let a = the acceleration; then $a = (20 - 8) \div (100/32.2) = 3.86$ feet per second per second. Hence $t = 40 \div 3.86 = 10.3$ seconds.

47. Impact or Collision

§ 1. BLOW. — Momentum of a blow, energy of a blow, and especially force of a blow are terms generally used more or less vaguely. But when one of the two colliding bodies is fixed, then the first two terms are taken to mean the momentum and the kinetic energy respectively of the moving body just before the impact, perfectly definite quantities. If the motion is one of translation, these are $Mv = (W/g)v$ and $\frac{1}{2}\,Mv^2 = \frac{1}{2}\,(W/g)v^2$ respectively. If in a numerical case we write $g = 32.2$ (feet per second per second), v should be expressed in feet per second; W may be expressed in any force unit. If the pound is used, then the momentum is in pound-seconds and the energy in foot-pounds.

Force of a blow means the pressure which two colliding bodies exert upon each other. The pressure changes during the collision. Analysis of this variation is beyond the scope of this book. We will deal only with average values of the force of a blow. In the first place, it should be noted that there are two average values of the force of a given blow, — a space-average and a time-average. We explain the distinction by means of an example, but we choose the simpler case of a varying horizontal pull dragging a body along a smooth horizontal surface instead of a blow. Let us suppose first that the pull varies uniformly with respect to time, from a zero value to 40 pounds in 20 seconds (see Fig. 366). Then the time-average is represented by the average ordinate to the line which shows how the force varies with respect to the time; hence it is 20 pounds. We wish to find now how the force varies with respect to distance. Let P = the value of the pull at any time t after starting; then the law of force is $P = 2\,t$. Also let M = mass of the body; a and v respectively = the acceleration and velocity at any time t, and s = the displacement up to that time. Then

$$a = \frac{P}{M} = \frac{2}{M}t, \quad v = \frac{1}{M}t^2, \quad \text{and} \quad s = \frac{1}{3\,M}t^3.$$

The total displacement (s_1) in the 20 seconds = $(1/3\,M)$ 8000. It follows from the last equation that

$$t = (3\,Ms)^{\frac{1}{3}}; \quad \text{hence } P = 2\,(3\,Ms)^{\frac{1}{3}}.$$

This equation determines the graph shown in Fig. 367, from which it is apparent that the space-average force is more than 20 pounds, or the time-aver-

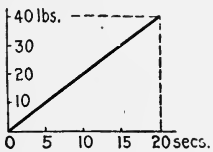


FIG. 366

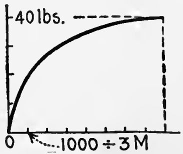


FIG. 367

age. The space-average equals the area under the curve divided by the base of the area. The area is

$$\int_0^{s_1} P\,ds = \int_0^{s_1} 2\,(3\,Ms)^{\frac{1}{3}}\;ds = 1.5\,(3\,M)^{\frac{1}{3}}\,s_1^{\frac{4}{3}};$$

hence the space-average $= 1.5\,(3\,M)^{\frac{1}{3}}\,s_1^{\frac{1}{3}} = 30$ pounds.

It will be observed that the space-average is that constant force whose work equals the work done by the (real) varying force (see Art. 40). Likewise the time-average is that constant force whose impulse equals the impulse of the (real) varying force. Hence the space-average equals the quotient of the work done by the force (equal to the kinetic energy produced by the force) and the distance through which the force acted; and the time-average equals the quotient of the impulse of the force (equal to the momentum produced by the force) and the duration of the impulse.

Crushers or crusher gages are small cylinders of copper or lead — one inch diameter and one inch high are common proportions — used in a way described presently to determine the energy and force of a blow. Fig. 368 shows several energy-compression curves for $1\frac{1}{2}$- by $1\frac{1}{2}$-inch lead cylinders.* Curve A-B is a so-called static curve, and was obtained by crushing a lead cylinder in an ordinary testing machine at the (slow) speed indicated. The amount of compressing force and the amount of the compression were observed at frequent stages during the test, and from these observations the amount of work

* *American Machinist*, Vol. 33, Part 1, p. 436 (1910).

done on the cylinder up to each stage was computed. Amounts of compression and corresponding amounts of work were plotted to determine the curve. Curve C is a static curve but for a higher speed. D is a so-called dynamic curve. It was obtained from drop or impact tests in which each crusher was subjected to a blow from a "hammer" dropped upon it. The hammer

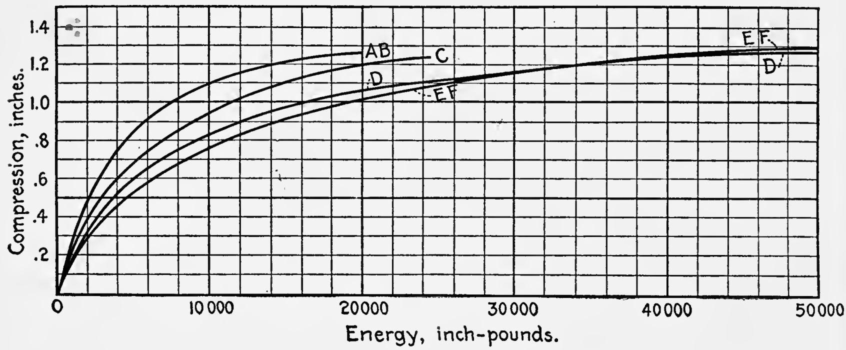


FIG. 368

weighed 1330 pounds, and the maximum drop used was 38 inches. For each test the amount of compression c was observed and the amount of work 1330 $(h + c)$ was computed, where h = height of drop to the cylinder; and this compression and work were plotted for one point on the curve. Curve E-F is a dynamic curve obtained from tests in which the hammers were lighter and the drops higher than for D.

Crushers are used as follows to determine the energy of a blow, as of a steam hammer for example. The crusher is placed on the anvil and subjected to the blow. Then the amount of the compression of the crusher is measured, and the corresponding energy is read off from the appropriate compression-energy curve (previously determined from tests on crushers like the one used). The space-average force of such a blow equals the quotient of the energy of the blow and the compression unless c/h is not a small fraction; in that case the space-average $= W(h + c) \div c$.

A very skillful use of the crusher was made by Lieut. B. W. Dunn to determine not only the average but the actual value of the force of a blow at any instant during the impact.* He devised apparatus which made a photographic record (space-time graph, Art. 29) of the motion of a hammer during the impact. From that graph he deduced the velocity-time, and from this the retardation-time graph; then the force of the blow F at any instant from $F = W + (W/g)\, a$, where W = weight of hammer and a = retardation at the instant. The order of measurements involved in this apparatus is indicated by these circumstances of one test: amount of compression of crusher about

* *Journal of the Franklin Institute*, Vol. 144, p. 321 (1897).

$\frac{1}{6}$ inch and time of impact about $\frac{3}{1000}$ second; the weight of hammer was 33 pounds and the drop 15 inches. For the copper crushers used the maximum pressure occurred just before the end of the compression, and its value was slightly less than twice the space-average.

§ 2. MOTION AFTER COLLISION. — In this section we discuss the changes of motion of one or both colliding bodies due to the collision in certain comparatively simple cases. In most cases of collision the pressures which the colliding bodies exert on each other are enormous compared with other forces acting on the bodies. For example, the space-average pressure between two billiard balls colliding with velocity of 8 feet per second is about 1300 pounds. Therefore in discussing changes of motion of the bodies during collision we may neglect the other (ordinary) forces acting on the bodies, gravity for example; that is we regard the two bodies jointly as under the action of no external forces. Hence, according to the principle of conservation (Art. 46), the momentum of the two bodies jointly is not changed by the impact.

If the center of gravity of two bodies about to collide are moving along the same straight line, then the collision or impact is called *direct;* if otherwise, *oblique.* If the pressures which two colliding bodies exert upon each other during impact are directed along the line joining their centers of gravity, then the impact is called *central;* if otherwise, *eccentric.* These are the kinds of impact called simple, above.

Direct Central Impact. — We assume that the bodies have motions of translation before impact. Since the impact is supposed to be central, the pressure (of impact) on each body acts through the center of gravity of that body and does not turn it. Hence the motion of each body after collision is one of translation. Let A and B be the two bodies,

M_1 and M_2 = their masses,
u_1 and u_2 = their velocities just before impact,
and v_1 and v_2 = their velocities just after impact respectively.

We regard these velocities as having sign; velocity in one direction (along the line of motion) being positive, and that in the other being negative. Then the momentum of the two bodies before impact $= M_1u_1 + M_2u_2$, and after impact it $= M_1v_1 + M_2v_2$. Since the momentums before and after impact are equal, we have

$$M_1v_1 + M_2v_2 = M_1u_1 + M_2u_2. \tag{1}$$

The foregoing expressions are correct whether A and B are moving in the same or opposite directions before or after the impact. Thus, if both are moving toward the right before impact, at 8 and 10 feet per second say, their momentum is $8\ M_1 + 10\ M_2$; but if A is moving toward the right and B toward the left, their momentum is $8\ M_1 - 10\ M_2$.

It has been learned experimentally that when two spheres A and B collide directly and centrally the velocity of separation is always less than and opposite to the velocity of approach, and the ratio of these two velocities seems to depend only on the material of the two spheres. The ratio of the velocity of separation to that of approach (signs disregarded) is called *coefficient of restitution;* it is generally denoted by e. The following are approximate values of e for a few materials,

glass $\frac{15}{16}$, ivory $\frac{8}{9}$, steel and cork $\frac{5}{9}$, wood about $\frac{1}{2}$, clay and putty 0.

Now the velocity of approach equals $u_1 - u_2$ (or $u_2 - u_1$), — the first with reference to A (regarded as fixed) and the second with reference to B (regarded as fixed) —, and the velocity of separation is $v_1 - v_2$ (or $v_2 - v_1$). Since these velocities are opposite in direction, we have

$$-(v_1 - v_2)/(u_1 - u_2) = e, \quad \text{or} \quad -(v_1 - v_2) = e\,(u_1 - u_2). \tag{2}$$

Equations (1) and (2) solved simultaneously for the final velocities v_1 and v_2 give

$$v_1 = u_1 - (1 + e)\frac{M_2}{M_1 + M_2}(u_1 - u_2); \; v_2 = u_2 - (1 + e)\frac{M_1}{M_1 + M_2}(u_2 - u_1). \tag{3}$$

If one of the colliding bodies is fixed, say B, then $u_2 = 0$, and M_2 is the mass of B and its supports, infinitely great. Thus we have $v_1 = - eu_1$.

Oblique Central Impact. — We assume as before that the bodies A and B have a motion of translation before impact; then the pressure on each during the impact acts through the center of gravity and produces no turning. Let U_1 and U_2 = the velocities of A and B before impact; V_1 and V_2 their velocities after impact; u_1 and u_2 = the components of U_1 and U_2 along the line of impact pressure (joining the centers of gravity of A and B when in contact); v_1 and v_2 = the components of V_1 and V_2 along that line; and w_1 and w_2 = the components of U_1 and U_2 at right angles to that line. See Fig. 369 which represents one of several possible ways of oblique collision.

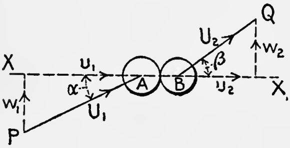


FIG. 369

Since the impact pressure on either body has no component transversely to the line of pressure XX, the component of the momentum of either body at right angles to XX is not changed. Hence the transverse component of the velocity of either body is not changed by the impact. The longitudinal components are changed as in direct impact, and v_1 and v_2 are given by equations (3). The final velocities V_1 and V_2, therefore, are determined, V_1 by its components v_1 and w_1, and V_2 by its components v_2 and w_2.

Loss of Energy in Impact. — Let L = the loss of kinetic energy; then

$$L = (\tfrac{1}{2}\,M_1U_1^2 + \tfrac{1}{2}\,M_2U_2^2) - (\tfrac{1}{2}\,M_1V_1^2 + \tfrac{1}{2}\,M_2V_2^2).$$

Now $U_1^2 = u_1^2 + w_1^2$, $U_2^2 = u_2^2 + w_2^2$, $V_1^2 = v_1^2 + w_1^2$, and $V_2^2 = v_2^2 + w_2^2$; hence

$$L = \tfrac{1}{2}\,M_1(u_1^2 - v_1^2) + \tfrac{1}{2}\,M_2(u_2^2 - v_2^2).$$

Substituting for v_1 and v_2 their values from equation (3) and simplifying we get

$$L = \tfrac{1}{2}\,(1 - e^2)\,\frac{M_1M_2}{M_1 + M_2}\,(u_1 - u_2)^2.$$

For perfectly elastic bodies ($e = 1$), $L = 0$. For other bodies $(1 - e^2)$ is not zero but a positive quantity; and since $(u_1 - u_2)$ is not zero, L is always a finite positive quantity. That is, in every collision of bodies not perfectly elastic there is loss of kinetic energy. If the bodies are without elasticity ($e = 0$), the loss $= \frac{1}{2} [(M_1M_2)/(M_1 + M_2)] (u_1 - u_2)^2$.

The foregoing is essentially Newton's analysis of impact. Several more recent analyses have been made independent of any coefficient of restitution but taking into account the vibrations set up in the colliding bodies. On account of the difficulties of the problem they include only impact of spheres and cylinders end on. Explanation of these analyses fall beyond the scope of this book.*

48. Angular Momentum and Impulse

§ 1. ANGULAR MOMENTUM. — The linear momentum of a moving particle is a vector quantity, as explained in Art. 46; the magnitude of the momentum is mv (where m = mass of the particle and v = its velocity), and the direction is that of the velocity. We go farther now and assign position to the momentum and to the momentum-vector. The position, or position-line, of the momentum of a moving particle is the line through the particle in the direction of the velocity. Thus the linear momentum of a particle is a "localized" vector quantity, — like a concentrated force, which has magnitude, direction and a definite position, or line of action as it is more commonly called.

We apply the term *moment of momentum* to a product which is analogous to the product which we call moment of a force about a line. Thus the moment of momentum of a moving particle about a line (or *angular momentum* as it is also called) is the product of the component of the momentum perpendicular to the line — the other component being parallel to it — and the distance from the line to the perpendicular component. (Compare definition of moment of a force about a line, Art. 8.) For example, let O (Fig. 370) be the position of the moving particle at a given instant, OC the direction of its velocity, and $OABC$ a parallelogram whose sides are parallel and perpendicular to the line LL', an axis of moments. (QQ is a plane perpendicular to LL' represented to make the figure more plain.) Then according to some scale OC represents the momentum mv, and OA and OB represent components of mv perpendicular and parallel to LL' respectively. The angular momentum of the particle about LL' is $OA \times PL$. It follows from the definition of

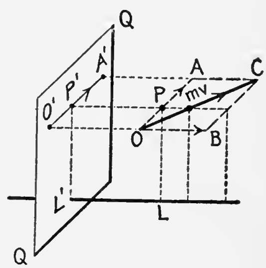


FIG. 370

* See Love's *Theory of Elasticity*, Vol. 2; *Nature*, Vol. 88, p. 531 (1912) for an instructive paper by Prof. Hopkinson, on "The Pressure of a Blow"; also *Journal of the Franklin Institute*, Vol. 172, p. 22 (1911) for an account of some determinations of the time of impact of metal spheres.

the term, that the angular momentum of a particle about a line parallel to its momentum is zero; and about a line perpendicular to its momentum it is equal to the product of the momentum and the distance from the line to the particle.

There is another method for computing the angular momentum of a moving particle about a line which is more simple generally than that described in the definition of angular momentum. It is as follows: we resolve the momentum into any three rectangular components, one of which is parallel to the axis of moments — then the other two are perpendicular to the axis —, and add the moments of the two perpendicular components about the line; the sum equals the angular momentum of the particle. Proof: Imagine the momentum OC (Fig. 370) resolved first into two rectangular components OA and OB as before, and then OA into any two rectangular components perpendicular to LL'. These last two are not shown in the figure but their relations to OA and the axis LL' are shown in projection on the plane QQ in Fig. 371. The moment of the component OM about LL' is $O'M \times L'm$

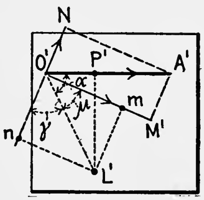


FIG. 371

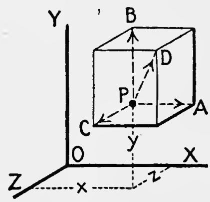


FIG. 372

$= O'M \times O'L' \sin \mu = O'M \sin \mu \times O'L'$. The moment of the component $O'N$ is $O'N \times L'n = O'N \times O'L' \sin \nu = O'N \sin \nu \times O'L'$. Hence the sum of the moments $= (O'M \sin \mu + O'N \sin \nu)\, O'L' = O'A' \sin \alpha \times O'L' = O'A' \times O'L' \sin \alpha = O'A' \times L'P'$ which is the angular momentum of the particle as defined.

By angular momentum of any collection of particles (body) about a line is meant the algebraic sum of the angular momentums of the particles about that line. In the case of a rigid body rotating about a fixed axis, the angular momentum of the body about the axis of rotation can be computed quite easily. Thus let m_1, m_2, etc., = the masses of the particles of the body; r_1, r_2, etc., = the distances of the particles respectively from the axis of rotation; and ω = the angular velocity of the body. Then the linear velocities of the particles are respectively $r_1\omega$, $r_2\omega$, etc. (Art. 37), and their linear momentums are $m_1r_1\omega$, $m_2r_2\omega$, etc. These momentums are perpendicular to the axis of moments; hence the angular momentums are $m_1r_1\omega r_1$, $m_2r_2\omega r_2$, etc. And since these are of the same sign, the angular momentum of the body is $m_1r_1^2\omega + m_2r_2^2\omega + \ldots = \omega\Sigma mr^2 = \omega I$, where I = the moment of inertia of the body about the axis of rotation (Art. 36).

A general formula for the angular momentum of a body about a line can be arrived at as follows: Let P (Fig. 372) be one of the particles of the body, OX the line about which to compute the angular momentum, and PD = the velocity of P. Let $OXYZ$ be a set of fixed coördinate axes; x, y, and z = the (varying) coördinates of P; m = mass of P; v = velocity of P; v_x, v_y, and v_z = the axial components of v (represented by PA, PB, and PC respectively). Then to some scale, PD represents the momentum mv of the particle, and PA, PB, and PC represent the axial components of the momentum; these equal mv_x, mv_y, and mv_z respectively. Hence the angular momentum of P about OZ is $mv_yx - mv_xy$, and the angular momentum of the entire body is

$$\Sigma\,(mv_yx - mv_xy).$$

We will now ascertain how the angular momentum of a body about any line depends on the forces concerned in the motion. Let P, Fig. 373, be one of the particles of a body, OX a fixed line about which the angular momentum in taken, R = the resultant of all the forces acting on this particle, v = its velocity, and a = its acceleration. Further, let the coördinates of P at any particular instant under consideration be x, y, and z referred to axes one of which is the line OX; R_x, R_y, and R_z = the axial components of R; v_x, v_y, and v_z = the axial components of v; a_x, a_y, and a_z = the axial components of a; and T_z = the torque of all the forces acting on P about the z axis. Then $T_z = R_yx - R_xy$ (Art. 8); and since $R_x = ma_x$ and $R_y = ma_y$ (Art. 34),

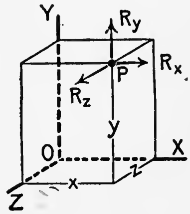


FIG. 373

$$T_z = ma_yx - ma_xy.$$

Now imagine one equation like the last written down for each particle of the body. The sum of the left-hand members equals the sum of the right-hand members of course. To the first sum the internal forces (exerted by the particles upon each other) contribute nothing because these internal forces occur in pairs, the forces of each being colinear, equal, and opposite, and so the moments of such two forces cancel. Therefore, the first sum is also the torque of the external forces about the z axis. Thus, we have where

$$\Sigma T_z = \Sigma\,(ma_yx - ma_xy), \tag{1}$$

where ΣT_z = the torque of all the external forces, acting on the body, about the z axis. The second sum, $\Sigma\,(ma_yx - ma_xy)$, equals the rate at which the angular momentum of the body about the z axis is changing. We prove this by differentiating the expression for angular momentum about the z axis, $\Sigma\,(mv_yx - mv_xy)$, with respect to the time; thus

$$\frac{d}{dt}\Sigma\,(mv_yx - mv_xy) = \Sigma\left[m\left(\frac{dv_y}{dt}x - v_y\frac{dx}{dt}\right) - m\left(\frac{dv_x}{dt}y - v_x\frac{dy}{dt}\right)\right].$$

Now $dv_y/dt = a_y$, $dx/dt = v_x$, $dv_x/dt = a_x$, and $dy/dt = v_y$; and substitution of these equivalents of the four derivatives in the long equation gives

$$\frac{d}{dt}\Sigma\,(mv_yx - mv_xy) = \Sigma\,(ma_yx - ma_xy),$$

which was to be proved. Thus finally we have the important principle that *the torque of the external forces, acting on any body, about any line equals the rate at which the angular momentum of the body about that line is changing*, or

$$\Sigma T_z = dh_z/dt, \tag{2}$$

where the line in question is called z, and h_z = the angular momentum of the body about that line.

For an example we will apply the foregoing principle to determine the torque of the water flowing through the water motor (Barker Mill) represented in Fig. 374. Essentially, the motor consists of a horizontal cylinder AB, mounted on a vertical pivot C, and an inlet D connected by a water-tight sleeve joint to a feed pipe E. On opposite sides of the cylinder and near its ends there are orifices or nozzles through which the water escapes horizontally. The water turns the motor in the opposite direction. Let W = the weight of water escaping per unit time, v = the velocity of escape relative to the orifices, and ω = the angular velocity of the motor. The amount of water which escapes in a short interval of time Δt is $W\Delta t$; and, since the absolute velocity of escape $= v - r\omega$ (Art. 53), the angular momentum of this water about the axis of rotation is $(W\Delta t/g)\ (v - r\omega)\,r$. Hence the rate at which the motor gives angular momentum to the water is

$$(W/g)\ (v - r\omega)r,$$

and this equals the torque of the motor on the water; also the torque of the water on the motor.

If the torque — about any line — of the external forces acting on a body equals zero, then the rate of change of the angular momentum of the body about that line equals zero; hence, if the torque remains zero for any interval of time, then the angular momentum remains constant. This is known as the principle of the *conservation of angular momentum.* It can be well illustrated by means of the apparatus on which the man (Fig. 375) is standing. It consists of a metal plate A supported on balls in suitable circular races in A and B so that A can be rotated about the line C with very little friction resistance; B is fixed. Imagine that a man has mounted the plate A and holds a balancing pole as shown, all being at rest; then the angular momentum of the man-plate-pole system about CC equals zero. Now suppose that the man exerts himself in any way, to move the pole about for example, but touches nothing except A and the pole. The only external forces acting on the system are gravity, reactions of the balls on A, and the air pressure. The first has no torque about C; the other two very little and are negligible here. Hence there is no external torque about C, and the angular momentum of the system about C equals zero always. This is strikingly illustrated if the man, without moving his feet on the plate, trys to rotate the pole (over his head as shown) about C. In doing so, he and A begin to rotate in the opposite direction. If I and I' = the moments of inertia of man (and A) and the pole respectively about C, and ω and ω' = their angular velocities at any instant, then the principle requires that the angular momentums $I\omega$ and $I\omega'$ shall be equal (and opposite). Or, imagine the man-plate-pole system is given an angular velocity by external means (the man holding the rod as shown, say), and then left to itself. If now the man should change the pole into a vertical position before him, he would reduce the moment of inertia of the system (about C) very materially; and since the angular momentum must remain constant, the angular velocity of the system would increase accordingly.

The grand illustration of the principle of conservation of angular momentum is furnished by the solar system. The system moves under the influence of no external forces; hence the angular momentum of the system about any line remains constant. The angular momentum about a certain line through the mass-center of the system is greater than that about any other such line. The line is known as the invariable axis of the system — a plane perpendicular to it as the invariable plane — and "is the nearest approach to an absolutely fixed direction yet known."

Center of Percussion. — Fig. 376 represents a body OC suspended like a pendulum; O is the center of suspension, and C is the center of gravity or mass-center of the body. Let R = the reaction of the axle supporting the pendulum,

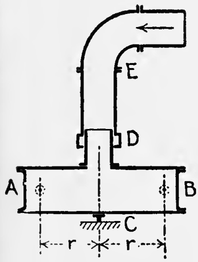


FIG. 374

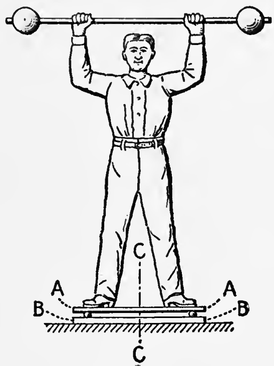


FIG. 375

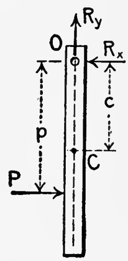


FIG. 376

and P = the time average force of a blow applied as shown. In general, R would not be vertical during the blow; so let R_x and R_y = the horizontal and vertical components of the time-average of R during the blow. The value of R_x depends not only on the force of the blow P but also on the arm of the blow with respect to the axis of suspension. It will be shown presently that if the arm has a certain value, then R_x equals zero. The point Q in OC (extended) and in the line of action of a blow applied as just explained so that there is no component axle reaction parallel to the blow, is called the center of percussion of the body for the particular axis of suspension. (Q is the point that was called center of oscillation in Art. 39.) The distance of the center of percussion from the axis of suspension equals

$$q = k^2/c = c + \bar{k}^2/c,$$

where k = the radius of gyration of the pendulum about the axis of suspension, c = the distance from the center of gravity, to that axis, and $\bar{k}$ = the radius of gyration about a line through the mass-center and parallel to the axis of suspension.

To develop the expression for q given above let M = the mass of the body, p = the arm of P about the axis of suspension, ω = the angular velocity of the body produced by the blow, and Δt = the duration of the blow. By the end of the blow the velocity of C will be $c\omega$, and practically horizontal; hence, according to Art. 46,

$$P - R_x = Mc\omega/\Delta t.$$

The only force which has a torque about O during the blow is P; hence

$$Pp = Mk^2\omega/\Delta t.$$

These two equations solved simultaneously for R_x give $R_x = P(1 - cp/k^2)$; therefore, if $p = k^2/c$, $R_x = 0$ which was to be shown.

Every American boy has batted a baseball a few times in such a way that the bat "stung" his hands; and he soon learned that such stinging is a result of impact near his hands or quite near the big end of the bat; in fact, quite remote from the center of percussion of the bat (with reference to the particular axis of rotation about which the bat was being swung at the instant of impact). Such a blow also results in rapid vibrations of the material of the bat which cause the sting. Large pendulums are used in certain impact testing machines for striking a blow. To avoid the impulsive reaction at the suspension and vibrations in the pendulum, they are always so arranged that the line of action of the blow passes through the center of percussion of the pendulum.

§ 2. ANGULAR IMPULSE. — If the line of action of a force is fixed in position then the angular impulse of that force for any interval about any line is the moment of the impulse of the force for the interval about that line. The moment of an impulse is computed just like moment of a force (Art. 8) or angular momentum; that is, we resolve the impulse into two components, one parallel and one perpendicular to the line and then we take the product of the perpendicular component and the distance from it to the line. If the line of action of the force changes then the angular impulse of the force about any line for any interval is the algebraic sum of the angular impulses for all the elementary portions of time which comprise the interval. Thus let F = the force, $t'' - t'$ = the interval, θ = the angle between the line of action F

and the line, and p = the perpendicular distance between the two lines. Then the angular impulse is

$$\int_{t'}^{t''} F\,dt \cdot \sin\theta \cdot p = \int_{t'}^{t''} F \sin\theta \cdot p\,dt.$$

Since $F \sin\theta \cdot p$ = the torque of the force about the line in question, the angular impulse of the force may also be regarded as the time-integral of the torque of the force. Hence, if T = the torque of the force about the line at any instant then the angular momentum for the interval equals

$$\int_{t'}^{t''} T\,dt.$$

Now let us integrate equation (2) over any interval $t'' - t'$ say; then

$$\int_{t'}^{t''} \Sigma T_z\,dt, \text{ or } \Sigma \int_{t'}^{t''} T_2\,dt, = h_z'' - h_z' = \Delta h_z, \tag{3}$$

in which h_z' and h_z'' denote the angular momentums of the body about the z axis at the times t' and t'' respectively. Equation (3) can be put into the following *principle of angular impulse and momentum:* The sum of the angular impulses of all the external forces acting on a body about any line equals the increment in the angular momentum of the body about that line.

49. Gyrostat

§ 1. GENERAL DESCRIPTION. — The words gyroscope and gyrostat are generally used synonymously but sometimes a distinction is made, as follows: A gyrostat consists of a wheel and axle, both being symmetrical to the axis of the axle, and mounted so that they may be rotated about that axis; a gyroscope consists of a gyrostat mounted in a frame which can be rotated. Fig. 377 represents a common form of gyroscope; the gyrostat (wheel W and axle AA') is supported by a ring R which can be rotated about the axis BB'; the axle BB' is supported by the forked pillar F which can be rotated about the axis CC'. Thus the wheel can be rotated about its center into any desired position. The gyroscope seems to have been designed for illustrating principles of composition of rotations (Art. 54). In 1852 Foucault (French physicist) made an interesting application of the instrument; by its means he practically made visible the rotation of the earth. More recently the gyroscope has been made use of in several connections, — to steer a torpedo, to serve as a substitute, unaffected by the iron of the ship, for the ordinary (magnetic) mariner's compass, to stabilize a mono-rail car, and to steady a ship in a rough sea; it has been proposed also to stabilize flying machines by means of a gyroscope.

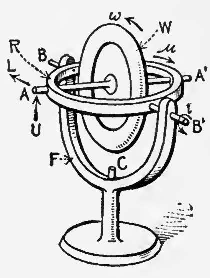


FIG. 377

When its wheel is spinning, a gyroscope possesses properties which seem peculiar to students as yet uninformed in the matter, inasmuch as it does not always respond as expected to efforts made to change its motion or position. For example, if a gyroscope like that represented in Fig. 377, well made and practically frictionless at all bearings and pivots, be grasped by the pillar and then moved about in any way, the axle of the wheel remains fixed in direction in spite of any attempt to alter it. The (gimbal) method of support makes it impossible to exert any resultant torque on the gyrostat (by way of the pillar) about any line through the center; and hence, as will be proved later, the direction of the axle cannot be thus changed. It is this property of permanence of direction of the spin-axis of a gimbal-supported gyrostat which is made use of in the self-steering torpedo.

For another example, consider the effect of a torque applied directly to the gyrostat. A vertical force, say, applied at A would turn the gyrostat when not spinning about the axis B. But when spinning, that force U would rotate the spin-axis about the axis C, the direction of rotation depending upon the direction of spin. When the gyrostat is spinning in the direction indicated by the arrow ω, then such force U would rotate the spin-axis about C in the direction indicated by the arrow μ. Again, a horizontal force applied at A, say, would turn the gyrostat when not spinning about the axis C. But when spinning, such force L would rotate the spin-axis about BB'; and in the direction indicated by the arrow λ if the spin is as indicated. This behavior of a spinning gyrostat under the action of torque is exhibited more strikingly by a gyroscope represented plainly in Fig. 378. The wheel may be spun on the axle A; the gyrostat and its frame may be rotated about the axis BB'; and all may be rotated about the axis CC'. W is the weight which can be clamped on the stem A' to balance or unbalance the frame with respect to the axis BB'. Now imagine W clamped so that the frame (with W and the gyrostat) is unbalanced.

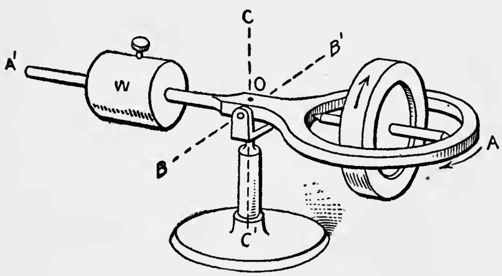


FIG. 378

Then if the gyrostat is set spinning and the frame be released in the position shown, say, the frame will not rotate about BB' but about CC'. The direction of this rotation depends on the direction of spin and on the direction of the torque of gravity about BB'. If, for example, W is clamped quite near BB' so that the torque of gravity is clockwise as seen from B and the spin is as indicated, then A rotates toward B. This rotation persists except in so far as it is interfered with by friction at the pivots, and air resistance. We might recite still other peculiar performances of a gyrostat but the foregoing suffice for our purpose. Professor Perry's book on "Spinning Tops" would be found interesting in this connection.

Any such rotation of the axis of a spinning gyrostat is called a *precessional motion* or precession of the axis or of the gyrostat; the axis and the gyrostat are said to precess. We will call precession *normal* or *oblique* according as the axis precesses about a line perpendicular or inclined to the axis. It may not be clear from the foregoing examples of precession how to predict the direction of precession that would result by applying a given torque to a gyrostat with a given spin. The following is a simple rule for predicting; it is based on the dynamics of the whole matter as will be seen later: "When forces act upon a spinning body tending to cause rotation about any other axis than the spinning axis, the spinning axis sets itself in better agreement with the new (other) axis of rotation; perfect agreement would mean perfect parallelism, the direction of rotation being the same." (From "Spinning Tops".) Or, what amounts to the same thing, the precession is such as to turn the spin-vector* toward the couple or torque-vector.

The following is an incomplete proof of the foregoing rule. Further explanation is given in the next section and in Art. 56. Fig. 379 represents a gyrostat pivoted at O so that it can be rotated freely about that point; we suppose the center of gravity of the gyrostat to be at O. Imagine that the gyrostat is at rest, not spinning, in the position shown, and that a downward force is applied to the axle on the left-hand side of O and

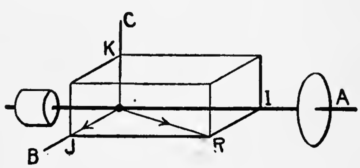


Fig. 379

downward. The torque makes the gyrostat rotate about the axis OB, that is the torque produces angular momentum about that axis. The amount of angular momentum produced is proportional to the torque and to the duration of its action (see Art. 48). This angular momentum may be represented by a vector on OB, the length of the vector representing the amount of the angular momentum and the arrow-head pointing so as to agree with the direction of rotation, according to the usual convention, that is, forward in this case. Now imagine that the axis of the gyrostat is at rest in the position shown but the wheel spinning, say, counter-clockwise when viewed from the right. The angular momentum of the spinning gyrostat about its axis† would be represented by a vector on OA pointing in the direction OA; let OI be that vector.

* A *spin-vector* is a vector on the axis of spin, its arrow-head pointing to the place from which the spin appears counter-clockwise; or — what amounts to the same thing — the arrow-head points in the direction along which the axis would advance if it were a right-hand screw turning in a fixed nut. The length of the vector — immaterial in the present connection — represents the angular velocity of spin to some convenient scale. Likewise the couple-vector (see Art. 8) is a vector perpendicular to the plane of the couple pointing to the place from which the rotation, which the couple tends to produce, would appear counter-clockwise; or — what amounts to the same thing — the arrow-head points in the direction along which the vector would advance if it were a right-handed screw turned by the couple in a fixed nut.

† This angular momentum is greater than that for any other line, and hence may be regarded as the total or resultant angular momentum of the gyrostat (see Art. 55).

Now suppose that the torque already described comes into action, and let OJ represent the angular momentum which it would produce in a short interval of time. This angular momentum added to the original angular momentum gives OR as the resultant angular momentum of the gyrostat at the end of the interval. It seems, therefore, that the spin-axis would coincide with OR at the end of the interval; indeed, that axis does approach OR, that is the spin-axis turns toward the torque-axis as stated in the rule which we undertook to prove.

The approach just mentioned is not a direct one; the gyrostat yields slightly to the torque just as though there were no spin; that is the wheel rises (in this instance) slightly. This is only the first (small) swing of a rapid oscillation of the spin-axis — nutation as it is called — which accompanies the (more prominent) precession of the spin-axis toward the torque-axis. The (unavoidable) friction at the pivot O rapidly damps this oscillation so that the oscillation generally escapes notice. The mentioned rise of the spin-axis may be explained as follows: In the approach of that axis toward OR the gyrostat rotates about OC, due to which it acquires angular momentum about OC, clockwise when viewed from above; but since there is no torque about OC, the gyrostat can acquire no (resultant) angular momentum about that line (see Art. 48 on conservation); hence the spin-axis rises so that at each instant the component along OC of the angular momentum due to spin just equals the angular momentum due to the rotation about OC.

There is another item of gyrostat behavior worth noting here. Suppose that the gyrostat shown in Fig. 378 to be precessing as already explained. If the precession be hurried, say by means of a horizontal push applied at A', the center of gravity of the frame (with gyrostat and weight) rises; if the precession be retarded, the center of gravity descends. This behavior is in accordance with the rule for predicting precession. In the first case we have a torque about CC'; the torque vector is in the direction OC'; the spin-vector is in the direction OA'; and in accordance with the rule OA' turns toward OC', that is the center of gravity rises. In the second case we have a torque about CC' but the torque-vector is OC; and the spin-vector OA' turns toward that vector, that is the center of gravity descends. Thus we may state as another rule: Hurry a precession, the gyrostat rises or opposes the torque which causes the precession; retard a precession, the gyrostat falls, or yields to the torque which causes the precession.

Self-steering Torpedo. — The gyroscope of such a torpedo is linked to appropriate valves of a compressed air engine in such a way that any turning of the spin-axis toward either side of the torpedo causes the engine to turn the (vertical) rudder of the torpedo in the opposite direction. Prior to projection of a torpedo, the gimbals are locked so as to hold the spin-axis of the gyrostat parallel (or inclined at any desired angle) to the axis of the torpedo. During the discharge of the torpedo, the gyrostat is automatically set spinning and the gimbals are unlocked. During the flight, the spin-axis continues to point in its original direction; any deviation of the torpedo from its intended course changes the inclination of the spin-axis relative to the torpedo; simultaneously the gyroscope actuates the rudder as explained, and the torpedo is deflected back toward its proper direction. Like a common pendulum swinging to its lowest position, the torpedo swings beyond a mean direction, and is then swung back again by the rudder. And this oscillation is kept up during the flight so that the actual path of the torpedo is a zigzag, about two feet wide. A gyrostat (wheel and axle) weighing 2 pounds and rotating at 2500 revolutions per minute has been made to serve the purpose just described.

Gyro-compass. — For our purpose we may regard a gyro-compass as consisting essentially of a gyrostat (wheel and axle), the axle supported in a ring or case, and the ring suspended from above. See *A*, Fig. 380. Such a compass, when the gyrostat is spinning, sets its spin-axis into the plane of the meridian at the place where the compass happens to be. Imagine such a compass to be set up at the equator with its spin-axis pointing east and west, and suppose that the direction of spin is counter-clockwise when viewed from the west. The rotating earth carries the gyrostat eastward; the spin-axis would remain parallel to its original position if the gyrostat were supported in frictionless gimbals, and would in time be positioned as shown at *B*. Now consider the gyrostat as shown at *B*, supported not in gimbals but suspended from above as in the gyrocompass. The supporting force (above) and the force of gravity would have a torque counter-clockwise as viewed from the north; thus the torque vector would point toward the reader. The spin-vector points to the right; hence the torque would turn the end of the spin-axis marked *n* from the west toward the north.

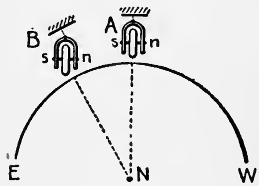


Fig. 380

Of course the action is not precisely as outlined above, that is the spin-axis does not remain parallel to its original position for a time and then yield to the influence of the torque mentioned. The action is really continuous; the slightest rotation of the compass with the earth from the position *A* induces the gravity torque, and the spin-axis begins to turn toward the meridian as described.

Though the restraint of the support (fine wire in the Sperry and mercury float in the Anschütz compass) is very small, the gravity torque is so small that the turning of the spin-axis into the meridian is very slow. Like a magnetic compass the gyro-compass swings beyond the meridian from a deflected position and oscillates for a time. In the Anschütz type the period of a free oscillation is about 1 hour and 20 minutes. Special damping arrangements reduce the oscillations to zero (from a deflected position of 40 degrees) in about one and one-half hours. The spin is maintained electrically, at about 20,000 revolutions per minute.

Mono-rail Car. — A car on a single rail can be rendered stable even if the center of gravity of the car is above the rail by means of a suitable gyroscope apparatus. Fig. 381 represents the germ of one type of such apparatus. AA' is the spin-axis, L is a lever rigidly fastened to the axle BB' by means of which the gyrostat can be made to precess about BB'. Imagine the car to be standing or travelling in an upright position, the gyrostat spinning, and a man standing on the car so that he may grasp and operate the lever. Now suppose that the car is tilted, as by a wind against either side. The car exerts tilting forces on the gyrostat axle at B and B', the torque-vector of which is parallel

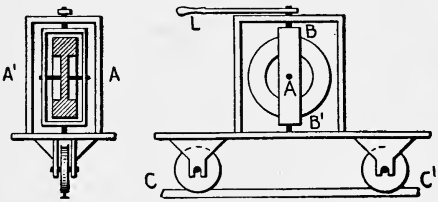


FIG. 381

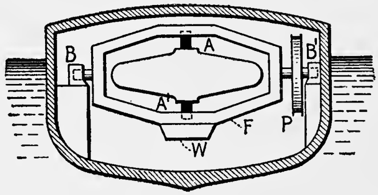


FIG. 382

to the rail; hence (see the stated rule) the spin-axis begins to set itself parallel to the rail, that is it precesses about BB'. The axle BB' exerts (righting) reactions on the car but if the man will hurry the precession, the (heavy, rapidly spinning) gyrostat will rise against the tilting forces and carry the car back with it toward the vertical position. It is conceivable that a skillful operator could put the car back into its vertical position in one swing, but in general he would swing the car beyond the vertical, then back again and after a few oscillations, into its vertical position.

Gyro-stabilizers as now built automatically perform the function of the man of the preceding explanation, and they include two gyrostats, spinning in opposite directions, to enable the car to run on a curve. The gyrostat wheels of a certain Brennan mono-rail car (40 feet long and weighing 22 tons) are $3\frac{1}{2}$ feet in diameter; each weighs $\frac{3}{4}$ tons, and spins at 3000 revolutions per minute (in a vacuum to avoid air friction). Such a car has taken curves of 105 feet radius at a speed of 7 miles per hour without appreciable disturbance of the level of the car floor. The spin is maintained by electric means; in fact each gyro-wheel is made the armature of a motor and this is driven by a generator on the car.

Schlick Gyro-stabilizer for Reducing the Rolling of a Ship. — This is represented in Fig. 382. The gyrostat is mounted in a rigid frame F which is supported in bearings B and B' fixed on the ship. Thus the wheel can be spun about AA' and the axle AA' can precess about BB'. P is a brake pulley by means of which this precession can be controlled. Explanation of the steadying action of this device is beyond the scope of this article. Such a stabilizer has been tried out in a ship 110 feet long, 12 feet wide and of 58 tons displacement. The gyro-wheel weighed 1100 pounds, was 1 meter in diameter, and was spun at 1600 revolutions per minute. In still water the ship would settle down from a heel of 20 degrees to one of $\frac{1}{2}$ degree in about 20 single oscillations; the period was about $4\frac{1}{8}$ seconds. The stabilizer produced the same extinction in less than three oscillations of 6 seconds period. (See London *Engineering*, Vol. 83, p. 448 (1907)).

§ 2. RATE OF NORMAL PRECESSION; DETERMINATION OF FORCES. — In the preceding section, we discussed the effect of a torque on a spinning gyrostat in a qualitative way; we will now discuss the matter quantitatively. Let I = the moment of inertia of the gyrostat about the axis of spin and ω = the angular velocity of spin; then $I\omega$ = the angular momentum of the gyrostat about that axis (Art. 48). If T = the applied torque, the angular momentum produced by it in the element of time dt is $T\,dt$, and the angular approach of the spin-axis toward the torque-axis in that time is IOR (Fig. 379) $= \tan^{-1}(T\,dt/I\omega) = T\,dt/I\omega$. The rate at which this angle is described, that is the angular velocity of precession — generally denoted by Ω — is

$$\Omega = (IOR)/dt = T/I\omega.$$

If the torque is applied so that its vector is always perpendicular to the axis of spin OA, then there is no torque about OA and hence ω is constant; if also the magnitude of the torque is constant, then it follows from the preceding formula that Ω is constant. That is, in the case assumed, the velocities of spin and precession are constant. The case is quite analogous to that of a moving particle subjected to a constant force whose line of action is always perpendicular to the direction of motion and in a given plane. Such a force does not change the magnitude of the velocity but continually changes the direction of it; indeed, the particle describes a circle with constant speed (Art. 34). Let P (Fig. 383) be the particle, m = its mass, v = its velocity, F = the force, PQ be the path and r = the radius of the circle. The linear momentum = mv; the angle POQ through which the vector mv is turned in any time t is vt/r. Since $r = mv^2/F$ (see Art. 34), the angle $= tF/mv$. Hence the rate at which F turns the linear momentum vector is F/mv, a result strictly analogous with $T/I\omega$, the rate at which the torque T turns the angular momentum vector $I\omega$. The result can be arrived at, independently of Art. 34, in a way to bring out the analogy still more. We may regard F constant in direction for an element of time dt. During that time it produces an amount of momentum, in its own direction PQ, equal to $F\,dt$. Let PJ represent this momentum and PI the initial momentum mv. At the end of the interval the (resultant) momentum is represented by PR. Hence the change in the direction of the momentum is $IPR = (F\,dt) \div (mv)$, and the rate at which the change occurs is the change divided by dt, that is F/mv.

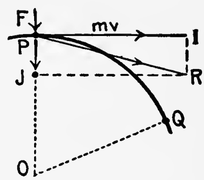


FIG. 383

The Forces Acting on a Gyrostat Precessing Normally at Constant Speed. — We will now determine certain conditions which the forces in such a case always fulfill. Incidentally, we give an alternative derivation of the formula $\Omega = T/I\omega$. We take the gyrostat represented by two projections in Fig. 384. AN is the axis of spin, the perpendicular to the paper at O is the axis of precession, and Q is the mass-center of the gyrostat. The assumed directions of spin and precession are indicated by the curved arrows ω and Ω respectively.

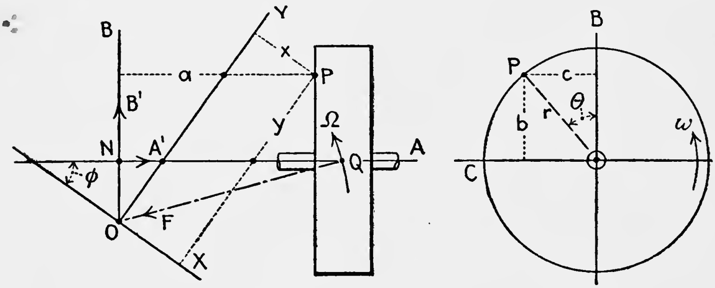


FIG. 384

For the investigation we shall use two sets of coördinate axes, one fixed and one moving. The fixed set is OX, OY, and OZ, the latter not shown; OZ is taken coincident with the precession-axis, and OX and OY in the plane in which the spin-axis moves. The moving set consists of NA, NB, and NC; NA is the spin-axis (as already stated), NB is the common perpendicular to the axes of spin and precession, and NC is perpendicular to NA and NB. Let I' = the moment of inertia of the gyrostat about the axis NC, e = the distance (ON) between the axes of spin and precession, ϕ = the (varying) angle which the spin-axis makes with OX, P be any particle of the gyrostat, m = its mass, r = its distance from the axis of spin, θ = the (changing) angle BNP, a, b, and c = the coördinates of P with respect to the moving axes, and x, y, and z = its coördinates with respect to the fixed axes.

It follows from the trigonometric relations in the figure that

$$x = a\cos\phi - (b+e)\sin\phi = a\cos\phi - r\cos\theta\sin\phi - e\sin\phi,$$
$$y = a\sin\phi + (b+e)\cos\phi = a\sin\phi + r\cos\phi\cos\theta + e\cos\phi,$$
and
$$z = c = b\sin\theta.$$

Differentiating these expressions with respect to time (and noting that a, r, and e are constants, and that $d\theta/dt = \omega$ and $d\phi/dt = \Omega$), we get the following values of the x, y, and z components of the velocity of P:

$$v_x = (c\omega - a\Omega)\sin\phi - (b+e)\,\Omega\cos\phi,$$
$$v_y = (a\Omega - c\omega)\cos\phi - (b+e)\,\Omega\sin\phi,$$
and
$$v_z = b\omega.$$

The angular momentums of P about the axes OX, OY, and OZ respectively are (see Art. 48)

$$m(v_z y - v_y z), \quad m(v_x z - v_z x), \quad \text{and} \quad m(v_y x - v_x y).$$

If we substitute in these expressions for v_x, v_y, and v_z their values as just deduced, then sum up for all the particles of the gyrostat, we arrive at the following simple expressions for the angular momentums of the gyrostat about the x, y, and z axes respectively:*

$$h_x = I\omega \cos\phi, \quad h_y = I\omega \sin\phi, \quad \text{and} \quad h_z = I'\Omega.$$

Differentiating these expressions for h_x, h_y, and h_z, with respect to time (and remembering that ω and Ω are assumed to be constant), we find that the rates at which the angular momentums change are

$$dh_x/dt = -I\omega\Omega \sin\phi, \quad dh_y/dt = I\omega\Omega \cos\phi, \quad \text{and} \quad dh_z/dt = 0.$$

(These rates are also the values of the torques, of all the external forces acting on the gyrostat, about the x, y, and z axes repectively; see Art. 48.) Now consider the instant, or position of the gyrostat, when the spin-axis NA is parallel to the x axis. Then $\phi = 0$, and the rates respectively $= 0$, $I\omega\Omega$, and 0; hence the external forces have no torques about the axes of spin and precession but a torque equal to $I\omega\Omega$ about the common perpendicular to those axes. Or, if T_s, T_p, and T respectively denote the torques about the axes of spin and precession and their common normal, then

$$T = I\omega\Omega, \text{ and } T_s = T_p = 0. \tag{1}$$

Let M = mass of the gyrostat, and r = the distance from its mass-center to the precession axis. Since the mass-center describes a circle with constant speed its acceleration at any instant is directed from the mass-center toward the center of the circle and equals $r\Omega^2$ (Art. 32). Hence, according to the principle of the motion of the mass-center (Art. 34), the sum of the components of all the forces (acting on the gyrostat) along radius of the circle (through the mass-center perpendicular to the precession-axis) equals $Mr\Omega^2$, and the sum of the components at right angles to this line equals zero. Or, if R_r, R_p, and R_3 respectively be used to denote the sums of the components along the radius of the circle, along the precession-axis, and along the common perpendicular to those two lines,

$$R_r = Mr\Omega^2 = (W/g)\, 4\, r\pi^2 N^2, \text{ and } R_p = R_3 = 0, \tag{2}$$

where N = the number of precessional revolutions per unit time.

Examples. — (1) Fig. 385 represents a side and end view of the armature of the motor of an electric locomotive. The armature shaft is parallel to the ties of the track. We will discuss the forces acting on the armature when the locomotive is rounding a curve. Inasmuch as we are not now concerned with the driving of the locomotive by this motor we will assume that the armature is spinning but under no load, the locomotive being driven around the curve by another locomotive. And for simplicity, we assume that there is no elevation of the outer rail, so that the precession of the armature is normal; that is,

* In reducing the summations, the student should note that

$$\Sigma m(b^2 + c^2) = I, \quad \Sigma m(a^2 + b^2) = I', \text{ and}$$
$$\Sigma mbc = \Sigma mca = \Sigma mab = \Sigma mb = \Sigma mc = 0.$$

we take the angle between the axis of spin and the (vertical) precession-axis to be 90 degrees. We take the weight of the armature = 8000 pounds, its radius of gyration = 15 inches, its speed = 750 revolutions per minute,

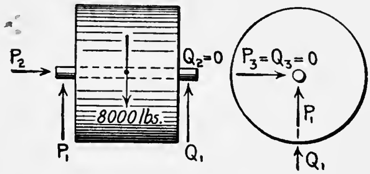


FIG. 385

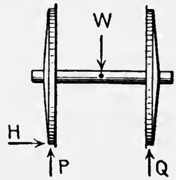


FIG. 386

distance between centers of bearings = 4 feet, the radius of the curve = 2000 feet, and the speed of the car = 30 miles per hour (= 44 feet per second). Then

$$I = (8000/32.2)\ (15/12)^2 = 388 \text{ slug-feet,}^2$$
$$\omega = 750 \times 2\,\pi/60 = 78.54 \text{ radians per second,}$$
and
$$\Omega = 44/2000 = 0.022 \text{ radians per second.}$$

The forces acting on the armature are gravity and the reactions P and Q of the bearings on the armature shaft. We neglect axle friction and imagine each reaction resolved into three components, vertical, parallel to the armature, shaft, and parallel to the rails. We distinguish these components by the subscripts 1, 2, and 3, respectively (see the figure). If the center of the curve is on the right, then evidently the armature presses outward against the bearing P and hence $Q_2 = 0$. Since the sum of the component forces along the rails = 0, P_3 and Q_3 must be equal and opposite, or else equal zero. Since the torque about the axis of precession must = 0, P_3 and $Q_3 = 0$. According to equation (2),

$$P_2 = (8000/32.2)\ (44^2/2000) = 240 \text{ pounds.}$$

The torque of all the forces acting on the gyrostat about the common perpendicular to the spin and precession axes equals

$$I\omega\Omega = 388 \times 785.4 \times 0.022 = 6700 \text{ foot-pounds.}$$

If the direction of spin is the same as the direction of rotation of the car wheels, then the torque is clockwise seen from the rear; hence

$$P_1(2000 + 2) + Q_1(2000 - 2) - 8000 \times 2000 = 6700.$$

We have also $P_1 + Q_1 = 8000$; hence, solving these two equations simultaneously, we find

$$P_1 = \frac{8000}{2} + \frac{6700}{4} = 5675, \text{ and } Q_1 = \frac{8000}{2} - \frac{6700}{4} = 2325 \text{ pounds.}$$

If the armature were not spinning ($\omega = 0$), or the car were running on a straight track ($\Omega = 0$) then $I\omega\Omega$ would equal zero, and hence the reactions P_1 and Q_1 would equal 4000 pounds. Thus the effect of the spin and precession is to increase one reaction and decrease the other by $6700 \div 4 = 1375$ pounds. This increase and decrease are called the gyrostatic couple or gyrostatic effect. The force P_2 does not depend on the spin of the armature, only on the radius of the curve and the velocity of the car. It is often described as the centrifugal effect.

(2) Fig. 386 represents a pair of car wheels which we assume to be rounding a curve. We will determine the forces acting on them. We assume that the wheels are "coned" so that there is true rolling; even if there were slipping — because of the excess length of the outer over the inner rail — our results would be practically correct. We neglect the tilt of the track and so regard the precession as normal. Let W = the weight of the wheels (including their axle), M = their mass, k = the radius of gyration of wheels, r = their radius, V = velocity of the center of gravity, R = the radius of the curve, and f = gage of the track. Further let P and Q = the vertical components of the pressure of the outer and inner rails on the wheels; H = the transverse component of the pressure of the outer rail. Besides these there are components along the rails with which we are not concerned. According to the first of equations (1),

$$P\,(R + \tfrac{1}{2}f) + Q\,(R - \tfrac{1}{2}f) - WR - Hr = Mk^2V^2/Rr;$$

and according to the first two of equations (2)

$$H = MV^2/R \text{ and } P + Q = W.$$

Solving these three simultaneously for P and Q we get

$$P = \frac{W}{2} + \frac{MV^2r}{Rf} + \frac{Mk^2V^2}{Rrf},$$

$$Q = \frac{W}{2} - \frac{MV^2r}{Rf} - \frac{Mk^2V^2}{Rrf}.$$

The first terms in these two expressions are due to gravity. The second terms are due to centrifugal action; they have the same values as if the wheels were skidding, that is, they do not depend on the spin of the wheels. The third terms are due to gyrostatic action; the components of P and Q which they stand for constitute the so-called gyrostatic couple.

§ 3. GYROSTATIC REACTION. — In general, any system of forces can be compounded into a single force acting through any desired point and a couple (Art. 9). Let us imagine all the forces acting on a gyrostat which is precessing normally to be compounded into a force acting through the mass-center of the gyrostat and a couple. Let these be denoted by F and C respectively. As will be proved presently,

$$F = Mr\Omega^2 = (W/g)r\,4\pi^2N^2, \tag{1}$$

where M = mass of gyrostat, W = its weight, r = distance from its center of gravity to the precession-axis, Ω = angular velocity of precession, and N = number of revolutions per unit time about the precession-axis. F acts along the perpendicular from the mass-center to the precession-axis, and in that direction (see Fig. 384),

$$C = I\omega\Omega = (W/g)k^2 4\pi^2 nN, \tag{2}$$

where I = the moment of inertia of the gyrostat about the axis of spin, k = its radius of gyration about the same axis, ω = the angular velocity of the spin, and n = number of revolutions of spin per unit time. The plane of the couple is perpendicular to the common perpendicular to the axes of spin and precession. The sense of the couple may be described as follows: Imagine a vector laid off on the axis of spin to represent the direction of the spin; then the vector representing the couple at any instant is parallel to the position which the spin-vector will occupy at the end of a quarter of the precession period (time required for one turn about the axis of precession). See Fig. 384; NA' is the spin-vector, and NB' is the couple-vector. That F and C have values and other characteristics as stated follows from the fact that such a force and couple together satisfy all the conditions imposed by equations (1) and (2) of § 2 on the system of forces acting on the gyrostat.

The gyrostat exerts reactions on the bodies which exert forces upon it equal and opposite to those forces respectively. Hence those reactions are equivalent to $-F$ and $-C$, where $-F$ and $-C$ denote a force and a couple respectively equal and opposite to F and C. Now $-F$ is independent of the spin or precession (see equation 1) but C depends on both. Hence $-C$ is called the gyrostatic (part of the) reaction.

In the examples of the preceding section we determined the forces acting on certain gyrostats, and it is easy to pick out the gyrostatic reactions. Thus, in example (1) the armature shaft exerts downward forces of 5675 and 2325 pounds on its left- and right-hand bearings as seen from the rear. As already pointed out each of these pressures is the resultant of two components, thus:

$$4000 + 1675 \text{ and } 4000 - 1675,$$

the second component being the gyrostat reaction; they are the couple denoted by $-C$. In example (2) the car wheels exert downward pressures equal to P and Q. The third components of these reactions constitute the gyrostatic reaction of the wheels.

A side (paddle) wheel steam boat sustains gyrostatic reactions in certain circumstances. When such a boat is turning, the (pair of) paddle wheels and shaft exert a gyrostatic couple on the boat which makes the boat heel. When the boat is, say, travelling forward and turning to starboard, the couple heels the boat to port. Likewise a screw-propelled ship sustains a gyrostatic couple when she is turning; it is due to the precession of the screw and shaft (and turbine too if so equipped). The couple depresses the bow or stern depending on the direction of turning of the ship and sense of rotation of the screw. It has been suggested that the gyrostatic reactions to which (comparatively frail) torpedo-boat destroyers are subject may over-tax their strength. The fact is, these reactions are quite insignificant compared to other straining actions which such boats withstand (see J. and J. G. Gray's "Treatise on Dynamics," page 531).

A flying machine is subjected to a gyrostatic reaction of its propeller, shaft and engine when turning or when describing any curved path. When turning, the reaction tends to raise or depress the front of the machine, depending on the circumstances. Propellers being right-hand screws (turning clockwise when viewed from the rear), the front is raised (unless prevented by the air man) when he turns to the right. When he makes a dive the couple tends to advance the side of the machine on the right-hand side of the air man. The flight of a machine fitted with two screws which rotate in opposite directions is not thus interfered with by gyrostatic reactions. Each propeller exerts a couple on the machine but the two couples are always opposite. It has been suggested that gyrostatic reactions of propellers and motors may have been the cause of some flying-machine accidents. However, a well-built machine can safely withstand such reactions even under conditions of legitimate quick driving and turning. Thus, for a dive or turn at the rate of one revolution in 20 seconds, it has been ascertained* that a 100-horse-power Gnome motor — speed not stated, but probably about 1200 revolutions per minute — exerts a gyrostatic couple of 140 foot-pounds; and the (suitable) propeller, a couple of 184 foot-pounds. The forces involved in the couples come upon the flying machine at the supports of the engine and the propeller shaft.†

* M. O'Gorman in *The Aeronautical Journal* for April, 1913.

† For a full discussion of the subject of this article, consult Crabtree's *Spinning Tops and Gyroscopic Motion.*

CHAPTER XII

TWO DIMENSIONAL (PLANE) MOTION

50. Kinematics of Plane Motion

§ 1. Plane motion is a motion in which every point of the moving body remains at a constant distance from a fixed plane. Each point of the body moves in a plane; that is, its motion is uniplanar. By *plane of the motion* is meant the plane in which the mass-center of the body moves. The wheels of a locomotive running on a straight track have plane motion; also a book which is slid about in any way on the top of a table. A translation (Art. 35) may or may not be a plane motion; a rotation about a fixed axis (Art. 37) is always a plane motion.

In a plane motion all points of the moving body which lie on a perpendicular to the plane of the motion move alike, and the motion of the projection of this line on the plane of the motion correctly represents the motion of all the points. So also the motion of the projection of the moving body upon the plane of the motion correctly represents the motion of the body itself. Thus we have a plane figure (the projection just mentioned) moving in a plane representing a plane motion of a body; and since the motion of the plane figure is uniplanar, the motion of the body is called uniplanar. Hereafter, we will sometimes refer to the projection of the body as the body itself.

By *angular displacement* of a body whose motion is plane is meant (as in rotation) the angle described by any line of the body which is in the plane of the motion. Obviously all such lines describe equal angles in the same interval of time. As in rotations also, displacements are regarded as positive or negative according as they are due to counter-clockwise or clockwise turning of the body. Let the irregular outline (Fig. 387) represent the projection of the moving body on the plane of the motion, AB a fixed line of the projection, and OX a fixed reference line; also let θ denote the angle XOA, it being regarded as positive or negative according as OX, when turned about O toward AB, turns counter-clockwise or clockwise. If θ_1 and θ_2 denote initial and final values of θ corresponding to any motion of the body, then the angular displacement $= \theta_2 - \theta_1 = \Delta\,\theta$.

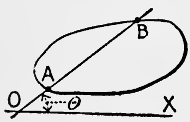


FIG. 387

If a body has a plane motion, its *angular velocity* is the time-rate at which its displacement occurs, and its *angular acceleration* is the time-rate at which its angular velocity changes. These definitions are precisely similar to those of the angular velocity and acceleration of a rotation about a fixed axis

(Art. 37); hence the expressions, units, and rules of signs given in that article hold also for any plane motion. The expressions are

$$\omega = d\theta/dt \quad \text{and} \quad \alpha = d\omega/dt = d^2\theta/dt^2,$$

ω and α denoting angular velocity and acceleration of the moving body respectively.

§ 2. Any uniplanar displacement of a body can be accomplished by means of a translation of the body followed by a rotation, or vice versa. Thus let $A_1B_1C_1$ (Fig. 388) be one position of a body ABC, and $A_2B_2C_2$ a subsequent position. By means of a translation the body can be displaced so that one of its points is put into its final position; thus a translation to $A_2b'C'$ puts A_2

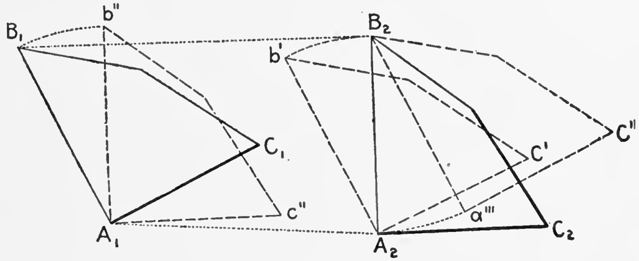


FIG. 388

into its final position. Then a rotation of the body about A_2 puts the body into its final position. Or, by means of a rotation we can put the body into an intermediate position $A_1b''c''$ so that each line in it will be parallel to its final position (in $A_2B_2C_2$); and then the body may be put into its final position by a translation. Obviously, the translation and rotation might be performed simultaneously.

The point (or axis) of the body about which we imagine the rotation to occur is called *a base point* (or *base axis*). Fig. 388 also represents a displacement from $A_1B_1C_1$ to $A_2B_2C_2$, accomplished with B as base point. A translation puts the body into the position $B_2a'''C''$, and a suitable rotation about B_2 puts it into the final position $B_2A_2C_2$. It is clear that the amount of the translation component depends on the base point; thus A_1A_2 is the translation for A as base point, while B_1B_2 is the translation for B as base point. But the amount of the rotation component does not depend on the base point; thus the rotation equals the angle $b'A_2B_2$ for A as base point, and it equals the angle $a'''B_2A_2$ for B as base point.

The successive small displacements of ABC from $A_1B_1C_1$ to $A'B'C'$, $A''B''C''$, etc., to $A_2B_2C_2$ (Fig. 389) already mentioned (and which altogether approximate to a continuous motion of ABC in which all points of the body move along smooth curves), can each be made by a small simultaneous translation and rotation. And if we take some one point as base point for all these small displacements then we may regard the motion as a continuous combined or simultaneous translation and rotation, the translation being like the motion of the base point and the rotation being about that point. In accordance with this view, the velocity of any point of the moving body at any particular instant consists of two components, one corresponding to the translation and one to the rotation. Thus let A (Fig. 390) be the chosen base point, v' = the velocity of A for the position of the body shown, and ω = the angular velocity

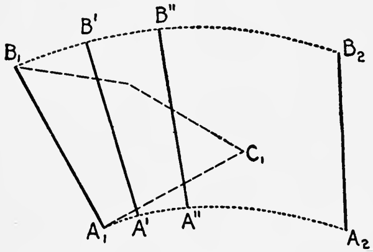


FIG. 389

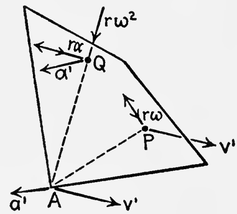


FIG. 390

of the body at the instant under consideration. Then the first component of the velocity of any point P equals v' and is directed like v; the second component equals $r\omega$ ($r = AP$) and is directed at right angles to AP, the sense depending on the sense of ω (clockwise or counter-clockwise). Also the acceleration of any point consists of two components, one corresponding to the translation component of the motion and one to the rotation. Thus let a' be the acceleration of the base point, and α = the angular acceleration of the body. Then the first component of the acceleration of any point Q equals a' and is

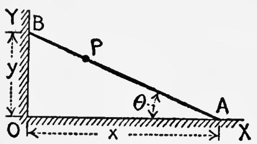


FIG. 391

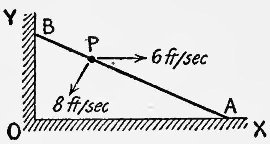


FIG. 392

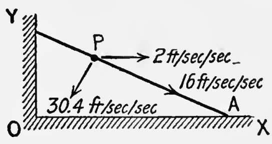


FIG. 393

directed like a'; the second component we describe by means of two components, as in a rotation about a fixed axis (see Art. 37), one of which (the normal component) is directed along QA and the other (the tangential component) is at right angles to QA. The normal component equals $r\omega^2$ ($r = AQ$) and is always directed from Q to A, toward the base point or center of the rotational component; the tangential component equals $r\alpha$, and obviously its sense depends on the sense of the angular acceleration.

For a numerical example let us consider the motion of the bar AB (Fig. 391) the ends of which slide along the lines OA and OB. Let the length of the bar = 6 feet, and the velocity and acceleration of A = 6 feet per second and 2 feet per second per second respectively (both toward the right) when $\theta = 30$ degrees. Required the velocity and acceleration of P, 4 feet from A. It is plain from the figure that $6 \cos \theta = x$; hence,

$$-6 \sin \theta \, d\theta/dt = dx/dt, \quad \text{or} \quad -6 \sin \theta \cdot \omega = v, \tag{1}$$

where ω = the angular velocity of the bar and v = velocity of A at any instant. Differentiating the last equation with respect to time we get

$$-6\,(\omega \cos \theta \cdot d\theta/dt + \sin \theta \cdot d\omega/dt) = dv/dt, \text{ or}$$

$$-6\,(\omega^2 \cos \theta + \alpha \sin \theta) = a, \tag{2}$$

where α = the angular acceleration of the bar and a = the acceleration of A at any instant. Now when $\theta = 30°$, (1) gives $\omega = -2$ radians per second, and (2) gives $\alpha = -7.6$ radians per second per second. The negative signs mean that ω and α are counter-clockwise, clockwise having been taken as positive for θ. Finally, the velocity components of P are $v = 6$, and $4 \times \omega = -8$ feet per second as shown in Fig. 392; the acceleration components of P are $a = 2$, $4 \times \alpha = -30.4$, and $4 \times \omega^2 = 16$ feet per second per second as shown in Fig. 393.

§ 3. Any uniplanar displacement of a body can be accomplished by means of a single rotation. Thus consider the displacement of ABC from the position $A_1B_1C_1$ to $A_2B_2C_2$ (Fig. 394). The point A can be brought from A_1 to A_2 by means

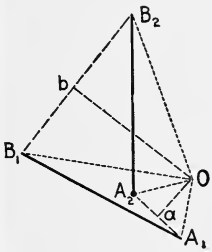


FIG. 394

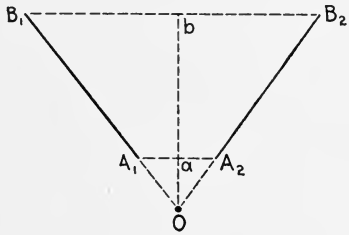


FIG. 395

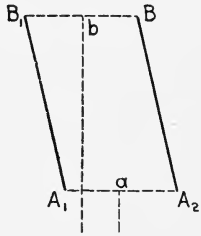


FIG. 396

of a rotation of AB about any point on the perpendicular bisector aO (of A_1A_2); and B can be brought from B_1 to B_2 by means of a single rotation of AB about any point on the perpendicular bisector bO (of B_1B_2). If the intersection of the bisectors is taken for the center of rotation of both A and B, then the amounts of the rotations (angles A_1OA_2 and B_1OB_2) are equal; hence, the line AB (and body ABC) can be displaced from one position to any other (uniplanar displacement) by means of a single rotation as stated.

In case the two bisectors coincide (Fig. 395), then the angles B_1 and B_2 are equal and hence the lines A_1B_1 and A_2B_2 extended intersect on the bisector ab extended; this extension is the center of rotation C which would displace AB from A_1B_1 to A_2B_2. In case the bisectors are parallel (Fig. 396) the center of rotation is "at infinity," and the displacement is a translation; thus a uniplanar translation may be regarded as a rotation about a center of infinity.

The actual continuous motion of AB from one position A_1B_1 to another A_2B_2 (in which A and B describe smooth curves) can be closely duplicated by a succession of rotations of AB from A_1B_1 (Fig. 389) into successive intermediate positions $A'B'$, $A''B''$, etc., until A_2B_2 is reached. Each small rotation is made about a definite center O', O'', etc. (not shown). The closer these intermediate positions are taken (and the more numerous and closer the centers of rotation O', O'', etc.) the more nearly do the successive rotations reproduce the actual continuous motion. "In the limit," the actual motion is reproduced by the rotations, the centers of rotation forming a continuous line. Thus we may regard any uniplanar motion of a body as consisting of a continuous rotation about a center which, in general, is continuously moving. The position of the center O about which the moving body is rotating at any instant is called the *instantaneous center* of the motion for the particular instant or position (of the body) under consideration, and the line through that center and perpendicular to the plane of the motion is called the *instantaneous axis* of the motion for that instant.

In general, the instantaneous center moves about in the body and in space. Its path in the body is called body centrode; its path in space the space centrode. Thus, in the case of a wheel rolling on a plane, the instantaneous center at any instant is the point of contact between the wheel and plane; the successive instantaneous centers on the wheel trace or mark out the circumference and this line is the body centrode; the successive instantaneous centers in space trace or mark out the track and this line is the space centrode. It can be shown that any plane motion may be regarded as a rolling of the body centrode on the space centrode.

Now in a rotation about a fixed axis the velocities of all points of the body are proportional to the distances of the points from the axis of rotation, and the velocities are respectively normal to the perpendiculars from the points to the axis (Art. 37); the velocity of any particular point is given by $v = r\omega$, where $v =$ the velocity of the point, $r =$ the distance of the point from the axis, and $\omega =$ the angular velocity of the body. So too, in the case of a uniplanar motion, the velocities of all points of the body at any particular instant are proportional to the distances of the points from the instantaneous axis (corresponding to that instant); the velocities are respectively normal to the perpendiculars from the points to the instantaneous axis; and the velocity v of any particular point is given by $v = r\omega$, where $r =$ the distance from the point to the axis and $\omega =$ the angular velocity of the body.

By means of the foregoing velocity relations, we can locate the instantaneous center for any given position of the moving body if the directions of the velocities of two of its points are given; and then if the value of one velocity is given we can compute the angular velocity of the body and the velocity of any other point. For an example we will consider the connecting rod of an engine (BC,

Fig. 397), in the position shown, the speed being 100 revolutions per minute. Since the velocity of the point B of the rod is along the tangent to the crank-pin circle at B, the instantaneous center of the connecting rod is on the normal to the tangent at B, that is on AB or its extension; and since the velocity of the point C of the rod is along AC, the instantaneous center is on the normal to AC. Hence the instantaneous center is at the intersection O. Now velocity of $B = 2\pi \times AB$ (to scale) $\times 100 = 2000$ feet per minute; hence, the angular velocity of the rod $= 2000 \div OB$ (to scale) $= 185$ radians per minute. The velocity of $C = OC$ (to scale) $\times 185 = 1110$ feet per minute.

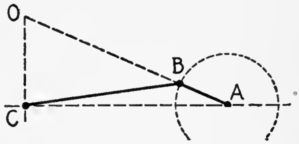


FIG. 397

51. Kinetics of Plane Motion

§ 1. GENERAL PRINCIPLES. — From the principle of the motion of the mass-center (Art. 34) we may write at once

$$\Sigma F_x = M\bar{a}_x, \quad \Sigma F_y = M\bar{a}_y, \quad \text{and} \quad \Sigma F_z = 0; \tag{1}$$

where ΣF_x, ΣF_y, and ΣF_z = the algebraic sums of the components of the external forces acting on the body along three rectangular lines, the third one being at right angles to the plane of the motion, $\bar{a}_x$ and $\bar{a}_y$ respectively = the x and y components of the acceleration of the mass-center, and M = the mass of the body. In addition to the above, we have another simple relation (established later),

$$\bar{T} = \bar{I}\alpha = M\bar{k}^2\alpha \tag{2}$$

where $\bar{T}$ denotes the torque of all the external forces about the line through the mass-center and perpendicular to the plane of the motion, $\bar{I}$ = the moment of inertia of the body about the line just mentioned, $\bar{k}$ = the radius of gyration of the body about that line, and α = the angular acceleration of the moving body. Systematic units (Art. 31) must be used in equations (1) and (2). But we may substitute W/g for M (where W = the weight of the body and g = the acceleration due to gravity) and then use any convenient units for force (and weight), length, and time.

To derive equation (2), let Fig. 398 represent the moving body, C be the mass-center, $\bar{a}$ = the acceleration of C, ω and α = the angular velocity and acceleration respectively of the body. Further, let P_1, P_2, etc., be particles of the body; m_1, m_2, etc., = their masses; r_1, r_2, etc., = their distances from the line through C and perpendicular to the plane of the motion; and R_1, R_2, etc., = the resultants respectively of all the forces acting on P_1, P_2, etc. We will regard the motion as consisting of a translation like the motion of C and a rotation about the "base axis" through C. Then the acceleration of P_1 can be regarded as consisting of three components, $\bar{a}$, $r_1\alpha$, and $r_1\omega^2$ as indicated; likewise the acceleration of P_2 can be regarded as consisting of three components, $\bar{a}$, $r_2\alpha$, $r_2\omega^2$; etc. Therefore, the resultant R_1 consists of three components $m_1\bar{a}$, $m_1r_1\alpha$, and $m_1r_1\omega^2$ directed like the corresponding accelerations; similarly, the resultant R_2 consists of three components $m_2\bar{a}$, $m_2r_2\alpha$, and $m_2r_2\omega^2$ directed like the corresponding accelerations; etc. Now the torque of all the forces acting on P_1 = the torque of the (three) components of R_1; similarly, the torque of all the forces acting on P_2 = the torque of the (three) components of R_2; etc. Hence, the torque of all the forces acting on all the

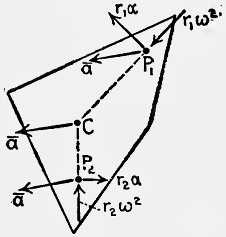


FIG. 398

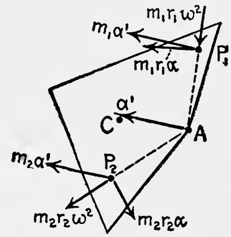


FIG. 399

particles (external and internal forces acting on the body) = the torque of the components (as $m\bar{a}$, $mr\alpha$, and $mr\omega^2$) of all the resultants R_1, R_2, etc. Since the internal forces occur in pairs of equal, opposite, and colinear forces, they contribute nothing collectively to the first torque just mentioned. It is plain from the figure that the normal components $m_1r_1\omega^2$, $m_2r_2\omega^2$, etc., have no torque about the (base) axis. Since the resultant of the components $m_1\bar{a}$, $m_2\bar{a}$, etc., passes through the mass-center (Art. 35), they have no torque about the axis. The torque of the remaining set of components is

$$m_1r_1\alpha r_1 + m_2r_2\alpha r_2 + \ldots = \alpha\Sigma mr^2 = \alpha\bar{I}$$

(see Art. 36). Hence, we have $\bar{T} = \bar{I}\alpha$, or equation (2).*

To show how to "apply" equations (1) and (2) we will discuss the rolling of a homogeneous cylinder on an inclined plane. We take the weight of the

* For some cases it may be desirable to know the value of the torque of the external forces about an axis (perpendicular to the plane of the motion) not containing the mass-center. The value of such torque may be arrived at just as above except that we take the line in question as base axis. That is, we regard the motion as consisting of a translation like the motion of the line and a rotation about that line. Thus let A (Fig. 399) be the base point (intersection of that line and the plane of the motion), and a' = the acceleration of A. Then the components of R_1, R_2, etc., are as represented. As before, the normal or radial components have no torque about the (base) axis; the torque of the tangential components is $I\alpha$, where I = moment of inertia about the axis (through A), but in this case the torque of the (translation) components m_1a', m_2a', etc., is not zero. The resultant of these components = Ma' (where M = mass of the body), and its line of action passes through the mass-center in the direction of a'; hence the torque of the components equals the product of Ma' and its arm with reference to A which let us call p. Therefore the torque of the external forces equals the sum or difference of $I\alpha$ and $Ma'p$ according as these two torques are alike or opposite in sense.

cylinder = 200 pounds, the diameter of its bases = 3 feet, and the inclination of the plane 25 degrees. Further, we assume that the cylinder and plane do not distort each other, so that there is only line-contact between them and no "rolling resistance" (Art. 52); also that the surfaces in contact are sufficiently rough to prevent slipping so that the rolling is perfect. There are only two external forces acting on the rolling cylinder, its own weight and

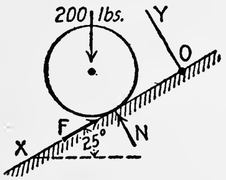


FIG. 400

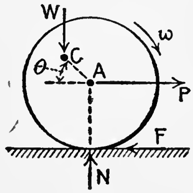


FIG. 401

the reaction of the plane, but the latter is represented by two components, N and F, in Fig. 400. Since the mass-center moves in a line parallel to the incline, $\bar{a}_x = \bar{a}$, and $\bar{a}_y = 0$; hence equations (1) become

$$200 \sin 25^\circ - F = (200 \div 32.2)\, \bar{a},$$
$$N - 200 \cos 25^\circ = 0, \text{ and } 0 = 0.$$

The second equation shows that $N = 181$ pounds. The first equation contains two unknowns (F and $\bar{a}$) and does not furnish the value of either of them; so we resort to equation (2). Since $\bar{k}^2 = \frac{1}{2}\ 1.5^2 = 1.125$ (see Art. 36), equation (2) becomes

$$F \times 1.5 = (200 \div 32.2)\, 1.125 \times \alpha.$$

Now we have two equations but three unknowns, and so we need an additional equation; this is given by the (simple) relation between $\bar{a}$ and α. Since there is no slipping, the displacement s of the mass-center in any interval of time and the angular displacement θ of the cylinder for that interval are related thus: $s = 1.5\,\theta$ (θ in radians and s in feet); hence $d^2s/dt^2 = 1.5\, d^2\theta/dt^2$, or $\bar{a} = 1.5\,\alpha$. Substituting $1.5\,\alpha$ for $\bar{a}$ in the first equation and then solving simultaneously with the fourth, we find that $\alpha = 6.05$ radians per second per second ($\bar{a} = 9.07$ feet per second per second) and $F = 28.2$ pounds.

For another example we will discuss the forces acting on a rolling wheel whose center of gravity is not in the axis of the wheel, and we suppose that the speed of rolling is maintained uniform by a suitable horizontal force P (Fig. 401). Let W = weight of the wheel, r = its radius, and c = the distance from its center A to the center of gravity C; further let θ = the angle between AC and the horizontal in the position of the wheel under consideration. There are three forces acting on the wheel, P, W, and the reaction of the roadway (represented for convenience by two components N and F). Equations (1) become

$$P - F = (W/g)\bar{a}_x \quad \text{and} \quad N = W - (W/g)\bar{a}_y.$$

Since the angular velocity is constant, $\alpha = 0$, and equation (2) becomes

$$F(r + c\sin\theta) - Nc\cos\theta - Pc\sin\theta = 0.$$

These equations contain five unknowns (P, F, N, $\bar{a}_x$, and $\bar{a}_y$), and so we need other equations. Obviously the relations between $\bar{a}_x$, $\bar{a}_y$, and θ furnish the additional equations. To determine these let us regard the rolling as consisting of a translation with A as base point and a rotation about A. Then since A moves uniformly, the acceleration of the translational component $= 0$; and there being no angular acceleration, the acceleration of the rotational component of the motion of C is wholly radial (along CA) and equals $c\omega^2$. Hence $\bar{a}$ equals $c\omega^2$ and is directed from C to A; and

$$\bar{a}_x = c\omega^2\cos\theta, \quad \text{and} \quad \bar{a}_y = - c\omega^2\sin\theta.$$

Substituting these values of $\bar{a}_x$ and $\bar{a}_y$ in the first two equations, and solving them simultaneously with the third we find that

$$P = W\left(\frac{c}{r} + \frac{c\omega^2}{g}\right)\cos\theta; \qquad F = W\frac{c}{r}\cos\theta; \qquad N = W\left(1 - \frac{c\omega^2}{g}\sin\theta\right).$$

For ω we may write $2\pi n$, where $n =$ the number of turns of the wheel per unit time.

It follows from the foregoing results that P and F are always opposite; that P and F act as shown whenever the center of gravity C is on the left of the vertical through the center A (θ between $-90°$ and $+90°$); that P and F act opposite to the directions indicated in the figure when C is on the right of the vertical through A; that N always acts upward unless $c\omega^2\sin\theta$ is greater than g; that the greatest value of N obtains when C is vertically below A ($\theta = -90°$) and then $N = W(1 + c\omega^2/g)$. This excess $Wc\omega^2/g$ over W in the value of N is called "hammer blow" in locomotive parlance, but the hammer blow of a locomotive driving wheel depends also upon the side rods attached to the wheel (see Art. 35).

Equations (1) contain no term depending on the rotation of the body about the mass-center; therefore, they show that the motion of the mass-center is entirely independent of the rotation about that point. And as already pointed out (Art. 34), the acceleration of the mass-center is the same as though the entire body were concentrated at the mass-center and all the external forces were applied at that point parallel to their actual lines of action. Equation (2) contains no term depending on the motion of the mass-center; therefore, the rotation of the body about the mass-center is independent of any motion of the mass-center itself. And on comparing equations (2) with the equation of motion for rotations about fixed axes (Art. 37), it becomes plain that the external forces produce rotation about a free (moving) axis through the mass-center as though that axis were fixed. Thus we have complete *independence of translation and rotation* effects of the external forces. This independence holds only for the mass-center.

To illustrate we will apply the principle of independence to explain *center of percussion;* Art. 48 includes an explanation based on other principles. Let AB (Fig. 402) be a prismatic bar lying on a horizontal surface, and C its center of gravity. Now imagine the bar to be struck a blow in the line F. The only other forces acting on the bar are gravity and the supporting force of the surface; these produce no appreciable effect on the motion during the blow. The motion produced, therefore, consists of a translation as though the blow F acted through the mass-center, and a rotation about the mass-center as though the mass-center were fixed. Any point beyond C gets a velocity toward the right due to the translation, and a velocity toward the left due to the rotation. For some particular point these two velocities are equal and opposite, and hence if the bar were pivoted there, the pivot would feel no pressure from the bar during the blow. For such a point, G is the center of percussion. Let us now find where this pivot point is. For that purpose let M = mass of the bar, $\bar{k}$ = its radius of gyration about the line through C perpendicular to the supporting surface, f = the arm of the blow F about the mass-center, R be the pivot point, r = its distance from C, a = the average acceleration of the mass-center, α = the average angular acceleration of the body during the blow, and Δt = the duration of the blow. The velocities of R due to the translation and rotation respectively equal $a\Delta t$ and $r\alpha\Delta t$. Now

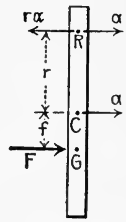


FIG. 402

$$a = F/M \quad \text{and} \quad \alpha = Ff/M\bar{k}^2;$$

therefore, for the pivot point we have

$$(F/M)\,\Delta t = r(Ff/M\bar{k}^2)\,\Delta t, \quad \text{or} \quad fr = \bar{k}^2.$$

That is, $r = \bar{k}^2/f$. For a given pivot the distance of the center of percussion from the center of gravity is $f = \bar{k}^2/r$, which agrees with the result reached in Art. 48.

Kinetic Energy of a Body with Plane Motion.—Let M = the mass of the body, W = its weight, $\bar{I}$ = its moment of inertia about a line through the mass-center perpendicular to the plane of the motion, $\bar{k}$ = its radius of gyration about the same line, $\bar{v}$ = the velocity of the mass-center, and ω = the angular velocity of the body. Then the kinetic energy of the body equals

$$\tfrac{1}{2}M\bar{v}^2 + \tfrac{1}{2}\bar{I}\omega^2 = \tfrac{1}{2}(W/g)\bar{v}^2 + \tfrac{1}{2}(W/g)\bar{k}^2\omega^2. \qquad (1)$$

The latter is the more convenient form generally for use in a numerical case. If g is taken as 32.2 (feet per second per second), then the foot and second should be adhered to as units of length and time; ω should be expressed in radians per unit time. If W be expressed in pounds, tons, etc., then the result will be in foot-pounds, foot-tons, etc.

The first term of (1) equals the kinetic energy which the body would have if its motion were one of translation with velocity equal to $\bar{v}$; and the second term equals the kinetic energy which it would have if its motion were one of rotation about a fixed axis through the mass-center and perpendicular to the plane of the motion. Hence the kinetic energy of a body with any plane motion may be regarded as consisting of two parts; they are called translational and rotational.

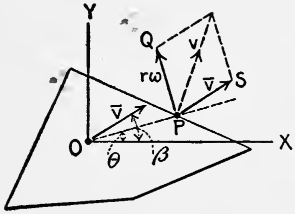


FIG. 403

The following is a derivation of the preceding formula after the view that a plane motion is a combined translation and a rotation (Art. 50, § 2). Let Fig. 403 represent the moving body, C its mass-center, and P any other point of the body. Also let r = the distance of P from the line through C perpendicular to the plane of the motion, and v = the velocity of P. Then v is the resultant of $\bar{v}$ and $r\omega$ as indicated. The angle $QPS = 90 - (\beta - \theta)$, where β and θ are the angles which $\bar{v}$ and OP respectively make with the x axis. Therefore

$$v^2 = \bar{v}^2 + r^2\omega^2 - 2\,\bar{v}r\omega \sin(\beta - \theta),$$

and the kinetic energy of the entire body ($\Sigma \frac{1}{2} mv^2$) equals

$$\tfrac{1}{2}\,\bar{v}^2\Sigma m + \tfrac{1}{2}\,\omega^2\Sigma mr^2 - 2\,\bar{v}\omega\,(\sin\beta\Sigma mr\cos\theta + \cos\beta\,\Sigma mr\sin\theta).$$

Now $r\cos\theta$ and $r\sin\theta$, respectively, equal the x and y coördinate of P. Hence $\Sigma mr\cos\theta = \Sigma mx = \bar{x}\Sigma m$ (see page 158), $\bar{x}$ denoting the x coördinate of the mass-center; and since $\bar{x} = 0$, $\Sigma mr\cos\theta = 0$. Similarly, $\Sigma mr\sin\theta = 0$. Hence the foregoing expression for the kinetic energy reduces to

$$\tfrac{1}{2}\,\bar{v}^2\Sigma m + \tfrac{1}{2}\,\omega^2\Sigma mr^2 = \tfrac{1}{2}\,M\bar{v}^2 + \tfrac{1}{2}\,I\omega^2.$$

The following is a derivation based on the view that any plane motion consists of a succession of instantaneous rotations (Art. 50, § 3). Let I = the moment of inertia of the body about that line which is the (instantaneous) axis of rotation at the instant in question, d = the distance from that axis to the mass-center, ρ = the distance of any point P of the body from the axis, v = velocity of P (as before), and ω = angular velocity of the body. Then $v = \rho\omega$, and the kinetic energy of the body is

$$\Sigma\tfrac{1}{2}\,mv^2 = \tfrac{1}{2}\,\omega^2\Sigma m\rho^2 = \tfrac{1}{2}\,I\omega^2. \qquad (2)$$

This is a much simpler expression than (1) but not so convenient to use generally, because I refers to an axis not fixed in the body. It remains to reduce (2) to (1). According to the parallel axis theorem (Art. 36, § 2), $I = \bar{I} + Md^2$; hence

$$\tfrac{1}{2}\,I\omega^2 = \tfrac{1}{2}\,\bar{I}\omega^2 + \tfrac{1}{2}\,M(d\cdot\omega)^2 = \tfrac{1}{2}\,\bar{I}\omega^2 + \tfrac{1}{2}\,M\bar{v}^2.$$

For an example we will compute the kinetic energy of a solid cylinder rolling on a plane surface. Let W = weight of cylinder, D = its diameter, and n = number of turns of the cylinder per unit time. Then $M = W/g$, $\bar{v} = \pi Dn$, $\bar{I} = \frac{1}{8}(W/g)D^2$ (see Art. 36), and $\omega = 2\pi n$. Hence the kinetic energy of the cylinder equals

$$\tfrac{1}{2}(W/g)\pi^2D^2n^2 + \tfrac{1}{4}(W/g)\pi^2D^2n^2.$$

Thus it appears that two-thirds of the energy is translational and one-third is rotational.

§ 2. DYNAMICS OF A SIMPLE MOVING VEHICLE. — Let W = weight of the body of the vehicle and its load, if any; w = the weight of each wheel (including one-half of the axle if the wheels are rigidly mounted on their axles); k = radius of gyration of wheel (with one-half of axle in case mentioned); r = radius of wheel; n = number of wheels; and v = velocity of the vehicle. The kinetic energy of each wheel is

$$\tfrac{1}{2}(w/g)v^2 + \tfrac{1}{2}(w/g)k^2(v/r)^2 = \tfrac{1}{2}(w/g)(1 + k^2/r^2)v^2.$$

Hence the kinetic energy of the entire vehicle is

$$\frac{1}{2}\left[\frac{W}{g} + \frac{nw}{g}\left(1 + \frac{k^2}{r^2}\right)\right]v^2.$$

Comparing this expression with that for the kinetic energy of a body with a motion of translation, we see that the motion of the entire vehicle may be regarded as one of translation provided that the weight of the vehicle is taken equal to $W + nw(1 + k^2/r^2)$. For modern freight cars r = 16.5 inches and k = 9.5 inches (about); hence $k^2/r^2 = 0.35$. Therefore the "effective inertia" of the wheels when rolling is about one-third greater than when at rest or skidding.

Height of Draw Bar. — Fig. 404 represents a vehicle, as a railroad car, being dragged on a level track by a pull P. The other external forces acting on the

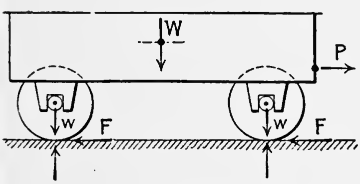


FIG. 404

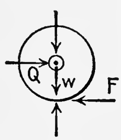


FIG. 405

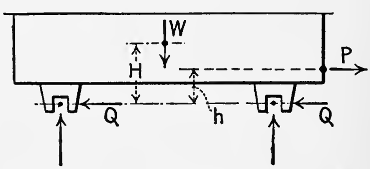


FIG. 406

car are gravity $(W + nw)$ and the reactions of the rails on the wheels (each represented by its horizontal and vertical components). In Fig. 405 there are represented all the external forces acting on one wheel, in Fig. 406 those acting on the car body. The pressures between axles and bearings are represented by their horizontal and vertical components; axle friction is disregarded. Let a = the acceleration of the car; then the angular acceleration of the wheels $= a/r$. Consideration of the forces on the wheel, equation (2), § 1, shows that

$$Fr = \frac{w}{g} k^2 \frac{a}{r}, \quad \text{or} \quad F = \frac{k^2}{r^2} a.$$

We have also (according to equations 1)

$$Q - F = \frac{w}{g} a, \quad \text{or} \quad Q = \frac{w}{g}\left(1 + \frac{k^2}{r^2}\right) a.$$

Consideration of the forces acting on the car body shows that $P - nQ = (W/g)a$, or

$$P = \left[\frac{W}{g} + \frac{nw}{g}\left(1 + \frac{k^2}{r^2}\right)\right] a.$$

When applied high up on the car, P tends to raise the rear end, decreasing the rear vertical axle pressures and increasing the forward vertical axle pressures. When applied low, P produces the opposite effect. Obviously, when applied in some certain line, P has no such effect on the vertical axle pressures. We will now locate that line; let $h =$ its height above the plane of the axes of the axles, and $H =$ the height of the center of gravity of the car body and its load above that plane. When the car is at rest (P and $Q = 0$), the (vertical) pressures of the axles on the car body take on certain values. If, when P (and nQ) act on the car body, their resultant acts through the center of gravity, then those forces do not tend to rotate the car body and do not affect vertical pressures of or on the axles already mentioned. Thus, to provide against extra loading or unloading of axles by P (draw-bar effect), the moments of P and nQ about the transverse horizontal line through the center of gravity of the car body (and load) should balance. That is, we should have $Ph = QH$, or

$$h = \frac{H}{1 + (nw/W)(1 + k^2/r^2)}.$$

52. Rolling Resistance

§ 1. ROLLERS. — In the present connection a roller is taken to differ from a wheel (of a vehicle) in that the latter sustains its load indirectly through its axle, while the former has no axle but takes its load directly. When a roller (or wheel) is made to roll, it experiences more or less resistance from the track (or roadway) upon which it rolls. Obviously the amount of this resistance depends in large part on the nature of the surfaces in contact and on the amount of the pressure between them. In the case of an inelastic roadway (A, Fig. 407) the roller leaves a rut, and there is a continual expenditure of energy in thus (permanently) deforming the track as well as against friction due to actual rubbing between roller and track. In the case of an elastic roadway (B, Fig. 407) also, there is rubbing between the roller and the deforming and recovering positions (OA and OB) and consequently friction loss.* In any case there is expenditure of energy against the (internal) friction in portions of the roller and track which are deforming or recovering.†

Let R = the resultant reaction of the track on the roller. Obviously the point of application of R is on the surface (or arc) of contact between wheel and roadway; and it will be shown presently that this point is in front of the vertical diameter of the roller, the roadway supposed to be horizontal. The distance from this point to the diameter is called the *coefficient of rolling resistance;* we will denote it by c, and express numerical values of the coefficient in inches. Obviously the coefficient of rolling resistance depends on the nature of the wheel and roadway, and is greater for yielding surfaces than for rigid ones. It would seem that the coefficient depends on the load but in certain cases at least the coefficient is not influenced much by it. The coefficient is claimed to be independent of the radius of the roller; also that it varies as the square root of the radius. The precise way in which the coefficient varies with the conditions named has not been established. Below we give some of the meager experimental data relating to the matter.

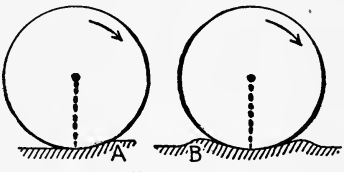


FIG. 407

Coulomb seems to have made the first experiments to determine coefficients of rolling resistance. The following are his results for

LIGNUM VITÆ ROLLERS ON OAK "PIECES"

Load.	Diameter = 2.18 ins.	Diameter = 6.55 ins.
220.5 lbs.	c=0.0174 ins.	c=0.0196 ins.
1102.7	.0205	.0197
2205.	.0196	.0196

For the circumstances of these experiments, it appears that the coefficient does not vary much with the load or the diameter of the roller. For elm rollers 6.55 and 13.11 inches in diameter on oak pieces, Coulomb found c = 0.0327 inches.‡

The following also are quoted from Morin's Mechanics:

OAK ROLLERS (diameter = 7.87 inches) ON POPLAR STRIPS

Width of strips.	Load.	Coefficient c.
0.97 ins.	409 lbs.	0.00637 ins.
3.94	400	.00287

* See *Phil. Trans. Roy. Soc.*, Vol. 166, Part 1 (1876), for experiments on rollers rolling on a rubber roadway, by Prof. Osborne Reynolds.

† See *Jour. and Trans. Soc. of Engrs.* (London), Vol. 3, p. 180 (1912) for an analysis of this element of rolling resistance, by Prof. Herbert Chatley.

‡ From English translation by Joseph Bennett of Morin's *Mechanics*, p. 339.

In these experiments, increasing the length of bearing from 0.97 to 2.94 (about triple) more than halved the coefficient. Thus it appears that the coefficient depends on the loading per unit length of contact between roller and roadway. But the coefficient probably does not decrease indefinitely with increase of length of contact.

For some conditions the coefficient seems to vary as the square root of the radius of the roller, that is

$$c = \phi \sqrt{r},$$

where ϕ is another coefficient and r = radius of the roller. Dupuit gives the following average values:

Wood on wood	$\phi = 0.0069$
Iron on moist wood	.0063
Iron on iron	.0044
Wheel on macadam	.19

For the conditions of his experiments,* Prof. C. L. Crandall takes the coefficient of rolling resistance as proportional to the square root of the radius, that is $c = \phi \sqrt{r}$. Roller plates used were $1\frac{1}{2}$ inches thick; rollers 1, 2, 3 and 4 inches in diameter, all $1\frac{1}{2}$ inches long except the first whose length was 1 inch. Plates and rollers were used as they came from the plane and lathe; were not polished or filed. Loads varied from 350 to 2500 pounds per linear inch in contact. The coefficient did not seem to vary much with load; with materials it varied as follows:

Cast iron	$\phi = 0.0063$
Wrought iron	.0120
Steel	.0073

These values refer to cast-iron plates; for wrought-iron plates they should be increased about 13 per cent, and for steel plates they should be decreased by that amount.

Fig. 408 represents in principle the device used by Coulomb to determine the coefficient of rolling resistance. W = weight of roller, W_1 and W_2 = weights of suspended bodies as shown. By adjusting the difference between W_1 and W_2 the roller was made to roll quite uniformly. When rolling at constant speed, the reaction R of the track on the roller is vertical, and $R = W + W_1 + W_2$. Also there is no resultant torque on the roller; hence the moment of R must be counter-clockwise (in this illustration), and so the point of application of R is in front of the vertical diameter of the roller (as stated). It follows that $(W_2 - W_1)\, r' = Rc = (W + W_1 + W_2)\, c$; or

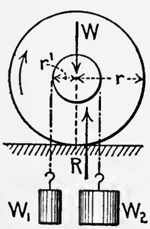


FIG. 408

$$c = r'\,(W_2 - W_1)/(W + W_1 + W_2),$$

from which c can be computed easily.

* *Trans. Am. Soc. C.E.*, Vol. 32, p. 99 (1894).

Fig. 409 represents in principle the device used by Crandall. There were two rollers under load (and a third one to preserve stability only), and three plates as shown. The lower plate was supported on the weighing table of a testing machine; load was applied on the upper plate; and then the middle plate was subjected to a force P sufficient to start the plate. Thus the middle plate was subjected to the reactions of the two main rollers, inclined as shown.

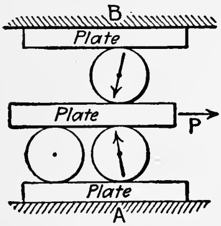


FIG. 409

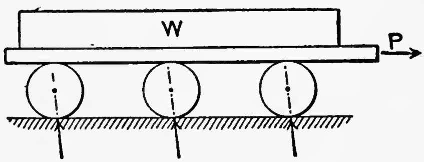


FIG. 410

Let R = these reactions (nearly equal), and θ = their inclination to the vertical. Then, evidently, $P = 2\,R \sin\theta = 2\,Rc/r$, and $R\cos\theta = W$ or $R = W$ nearly; hence

$$P = 2\,Wc/r \quad \text{and} \quad c = \tfrac{1}{2}\,Pr/W.$$

Rollers are generally used for moving a heavy load as shown in Fig. 410. Let r = radius of rollers, c = their coefficient of rolling resistance (assumed same for top and bottom contacts), R_1, R_2, etc., = the reactions of the rollers, θ = their inclinations to the vertical, W = load, and P = the pull required to move the load. Then since θ is small, $(R_1 + R_2 + \quad . \quad . \quad . \quad) = W$ (nearly); and since $\sin\theta = c/r$, $P = (R_1 + R_2 + \quad . \quad . \quad . \quad)\,c/r$. Hence

$$P = Wc/r.$$

§ 2. ROLLING WHEEL. — The general nature of rolling resistance in the case of a wheel is like that against a roller. A rolling wheel of a vehicle experiences axle friction as well as rolling resistance,* and few experiments have been made to determine them separately. For cast-iron wheels 20 inches in diameter on cast-iron rails Weisbach and Rittinger, respectively, found for the coefficient of rolling resistance c = 0.0183 and 0.0193 inches.† For an iron railroad wheel 39.4 inches in diameter, Pambour gives c = 0.0196 to 0.0216 inches.

Fig. 411 represents a simple case of rolling wheel. The velocity is supposed to be constant, so maintained by the force P as shown. Practically it would be difficult to so apply P without friction, but inasmuch as we are interested just now in rolling resistance, this axle friction (due to P) will be disregarded. W = the weight of the wheel, including its load if any; the axle friction due to the load will be disregarded too. R = the reaction of the roadway. Evi-

* See Baker's *Roads and Pavements* for full information on total resistance to traction of vehicles (due to rolling resistance and axle friction).

† Coxe's translation of Weisbach's *Mechanics*.

dently R has a backward component (overcome by P); and since there is no angular acceleration, the torque on the wheel = 0, and hence R acts through the center of the wheel. Therefore, R cuts the rim of the wheel in front of the

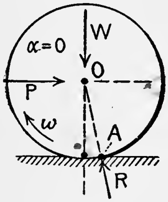


FIG. 411

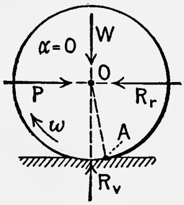


FIG. 412

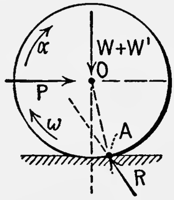


FIG. 413

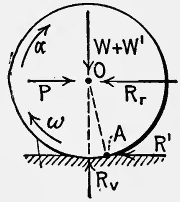


FIG. 414

lower end of the vertical diameter, and of course within the arc of contact of wheel and roadway. The vertical component of $R = W$. Since the vertical distance between A and O equals r (nearly)

$$Pr = Wc, \quad \text{or} \quad P = Wc/r.$$

This gives the force P (applied as shown) required to maintain constant speed to overcome rolling resistance. If we imagine R resolved into horizontal and vertical components, we get a good view of the mechanics of the case. See Fig. 412, in which R_v is the vertical component of R, and R_r is the horizontal component.

Fig. 413 represents a wheel rolling toward the right with increasing velocity, the acceleration being produced by the force P. Obviously the point of application of the reaction R of the roadway is in front of the vertical diameter of the wheel as indicated. Since the angular acceleration is clockwise, the resultant torque on the wheel about the axis through its center is in that direction too. Therefore, R acts somewhat as shown. For convenience imagine R resolved into two components at A, R' horizontal and the other along the radius AO; then imagine this latter component resolved into a horizontal component R_r and a vertical one R_v (see Fig. 414). Let W = weight of wheel and W' = weight of its load, if any; then

$$R_v = W + W', \quad \text{and } R_r = R_v \tan\theta = (W + W')c/r, \text{ nearly.}$$

Let a = the acceleration of the center of the wheel, k = the radius of gyration of the wheel with respect to its axis; then

$$P - R_r - R' = (W/g)a, \quad \text{and } R'r = (W/g)k^2a/r.$$

These two equations (solved simultaneously) show that the acceleration produced by a given force P is

$$a = \frac{P - (W + W')c/r}{M\,(1 + k^2/r^2)};$$

and that the force P required to produce a given acceleration is

$$P = (W + W')\frac{c}{r} + M\left(1 + \frac{k^2}{r^2}\right)a,$$

where M = mass of the wheel = W/g.

53. Relative Motion

§ 1. Motion Relative to a Point. — We can specify position of a point only by means of a set of reference axes or some other equivalent base described or implied in the specification. Thus when we say that Chicago is $10\frac{1}{2}$ degrees west and 3 degrees north of Washington — the cities regarded as points — we are really specifying the position of the former city with reference to the parallel and the meridian through Washington. But we say briefly that the specification is relative to Washington. So too when we say that a moving ship A is 40 miles east and 50 miles north of another ship B at a certain instant, we are specifying position of A by means of the parallel and the meridian through B at the instant in question; but we say that the specification is relative to B, the coördinate axes being understood. Being small compared to the distances mentioned (40 and 50 miles), the ships were regarded as mere points. If, however, the ships were at close quarters, then to describe the position of A relative to B we would specify the position of at least two points in A (bow and stern for example) relative to axes fixed in B, as indicated in Fig. 415, say. Even if B were turning about, we would still use those axes to specify subsequent positions of A relative to B. For the present we will deal with position (and motion) of points (or bodies regarded as mere points) relative to another base point — not body — and it should be understood that the coördinate axes, though moving with the base point, remain fixed in direction.

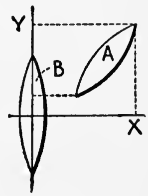


Fig. 415

Let the points 0, 1, 2, 3, etc. (Fig. 416), on the lines *aa* and *bb* be the positions

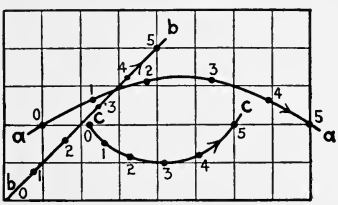


Fig. 416

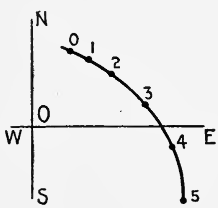


Fig. 417

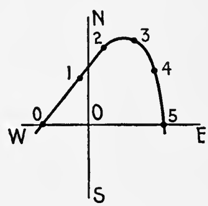


Fig. 418

(relative to a lighthouse), say, of two ships A and B, at 12, 1, 2, . . . o'clock of a certain day; then the lines are the paths (relative to the lighthouse) of the ships. From these paths we have made the following tabulation of coördinates of the positions of A relative to B. These coördinates if plotted on rectangular axes, representing a parallel and a meridian, determine the path of A relative to B (Fig. 417).

Time (hours)	0	1	2	3	4	5
East (degrees)	10	15	21	30	37	40
North (degrees)	20	19	16	10	2	−6

Taking points 0, 1, 2, etc., on the line *cc* (Fig. 416) as the positions (relative to the lighthouse) of a third ship *C* at the hours mentioned, we have the following tabulation of the coördinates of the positions of *A* relative to *C* from which the path of *A* relative to *C* (Fig. 418) was constructed. Thus it is clear that in general the path of a moving point depends on the point of reference or base point.

Time (hours)	0	1	2	3	4	5
East (degrees)	−12	−3	4	12	18	20
North (degrees)	0	12	20	23	22	15

For another illustration, imagine a table (Fig. 419), several balls (for bearings) on the table, a drawing board resting on the balls, another set of balls on the board, and a sheet of plate glass resting on these latter balls. Imagine

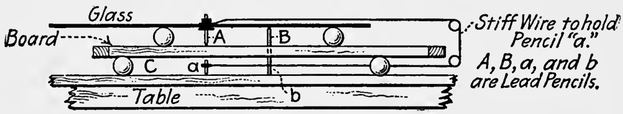


FIG. 419

also a pencil fastened to the glass so that its point *A* presses against the board, and another pencil fastened to the board so that its point *B* presses against the glass. When either board or glass or both are rolled about on the balls, the pencils (if suitable and properly mounted) trace lines on the board and glass. If the edges of the board and glass (or coördinates axes through *A* and *B*) are kept in fixed directions, then *A* traces its path relative to *B*, and *B* traces its path relative to *A*.

By velocity of a point relative to another point is meant the rate at which the first point is traversing its path relative to the second point at the instant in question. We regard this velocity as having direction, that of the tangent to the (relative) path at the point corresponding to the instant in question. By acceleration of a point relative to another point is meant the rate at which the velocity of the first point relative to the second is changing at the instant in question.

Motion of Two Points Relative to Each Other. — The velocities and accelerations of two points relative to each other are equal and opposite. This proposition may be illustrated by comparing the path of ship *A*, of the foregoing illustration, relative to *B* (Fig. 416) with the path of *B* relative to *A* (Fig. 420), constructed from the following tabulation of the positions of *B* relative to *A* at the hours 12, 1, 2, etc. Thus for the hour from 2 to 3 o'clock,

Time (hours)	0	1	2	3	4	5
East (degrees)	−10	−15	−21	−30	−37	−40
North (degrees)	−20	−19	−16	−10	−2	6

for example, the displacement of *A* relative to *B* is represented by vector 2-3 of Fig. 417, and the displacement of *B* relative to *A* is represented by vector

2-3 of Fig. 420. Apparently these vectors are equal and parallel (also opposite); and it seems that such displacement vectors would be equal, parallel, and opposite for any interval of time. If this be true, then it follows that the rates at which these displacements occur (the relative velocities) are equal and opposite at each instant; and if the velocities are always equal and opposite then their rates of change (the relative accelerations) are also equal and opposite at each instant.

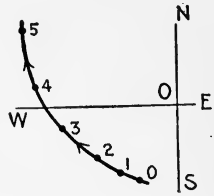


FIG. 420

To prove that displacements such as mentioned in the preceding illustration are equal and opposite, we will use the glass-board illustration. Suppose that the pencils A and B are attached at the middle points of the glass and board respectively, and that at a certain instant glass and board are in the positions shown at (1) in Fig. 421, and at a later instant in positions shown at (2); the table is not shown. A_1 and B_1 and A_2 and B_2 are the corresponding positions of the pencil points. During this displacement, A will have traced some such line as $A'A_2$ and B the line $B'B_2$. A_1A' is equal and parallel to B_1B_2; hence $A_1B_1B_2A'$ is a parallelogram, and $A'B_2$ and A_1B_1 are equal and parallel. B_1B' is equal and parallel to A_1A_2; hence $B_1A_1A_2B'$ is a parallelogram, and $B'A_2$ and B_1A_1 are equal and parallel. It follows that $A'B_2B'A_2$ is a parallelogram, and so $A'A_2$ and $B'B_2$ are equal and parallel. That is, the displacement of A relative to B (chord $A'A_2$) is equal and parallel to the displacement of B relative to A (chord $B'B_2$). Obviously the senses of the displacements are opposite.

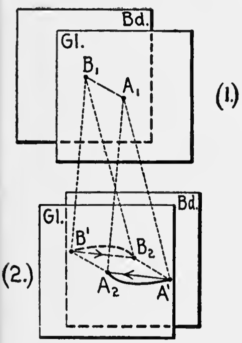


FIG. 421

Motions of Two Points Relative to a Third Point. — For convenience we regard the third point as fixed, and call velocities and accelerations relative to that point as absolute. To illustrate this case we will modify the glass-board apparatus as follows: Imagine another pencil a rigidly fastened to the glass plate so it presses against the table as shown directly under A, and B extended downward so that its lower end b presses on the table. Then when the glass and board are moved about without turning, a and b draw the paths of A and B relative to any (third) point as C on the table; and as already stated, A and B draw their paths relative to each other.

In this case two problems arise: (a) Given the velocity (or acceleration) of a point relative to a second point, and the absolute velocity (or acceleration) of the second; required the absolute velocity (or acceleration) of the first point. (b) Given the absolute velocities (or accelerations) of two points; required the velocity (or acceleration) of either of the two points relative to the other.

(a) To do this problem we merely need to add (vectorially), or compound, the velocity (or acceleration) of the first point relative to the second and the absolute velocity (or acceleration) of the second; the sum is the desired quantity. To justify this solution we first show that the (vector) sum of the displacement of the first point relative to the second and the absolute displacement of the second point equals the absolute displacement of the first, all displacements being taken for any interval of time. It will follow that the relative and absolute velocities (and accelerations) are related as above stated. Referring to our glass-board-table device, let A, B, and C be the three points respectively. Let (1) and (2), Fig. 421, be the positions of glass and board at the beginning and end of any interval, as before. Then $A'A_2$ is the displacement of A relative to B as explained; B_1B_2 is the absolute displacement of B; and A_1A_2 is the absolute displacement of A. As already shown, the quadrilaterals in the figure are parallelograms; hence the vector sum of $A'A_2$ and B_1B_2 equals A_1A_2.

(*b*) Let A and B be the first two points and C the third, and the velocity (or acceleration) of A relative to B the desired quantity. According to (*a*), the absolute velocity (or acceleration) of A = the vector sum of the velocity (or acceleration) of A relative to B and the absolute velocity (or acceleration) of B. Therefore the (desired) velocity (or acceleration) is such a velocity (or acceleration) which when added vectorially to the absolute velocity (or acceleration) of B = the absolute velocity (or acceleration) of A. For example let v_a and v_b (Fig. 422) be the absolute velocities (or accelerations) of A and B; then if OM and ON be drawn to represent v_a and v_b respectively, NM will represent the velocity (or acceleration) of A relative to B.

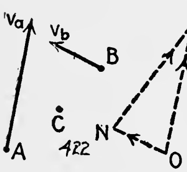


FIG. 422

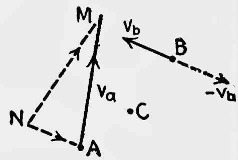


FIG. 423

The problem can be solved also on the basis of the principle that if we add equal velocities (or accelerations) to the absolute velocities (or accelerations) of the two points we do not change the velocities (or accelerations) of either of the points relative to the other. Thus, taking the preceding example, we will add to v_a and v_b a velocity equal and opposite to v_b (Fig. 423); then the new $v_b = 0$ and the new $v_a = NM$. Since now B is at rest relative to C, the new velocity of A relative to C is also the velocity of A relative to B.

§ 2. MOTION OF A POINT RELATIVE TO A BODY. — As explained in § 1, we specify the positions of a moving point relative to another moving point by means of reference axes of fixed directions through the second point, but its positions relative to a moving body by means of reference axes fixed in the body. See illustrations of the ships. Then the path of a point relative to a body is the line through the successive positions of the point relative to the body. Thus, to illustrate, consider again the glass-board-table apparatus (Fig. 419). When both the glass and board are rolled about in any way, the pencil A traces a line on the board, and that line is the path of A relative to the board.

By velocity of a point relative to a moving body is meant the rate at which the point traverses its path relative to the body at the instant in question. By acceleration of a point relative to a moving body is meant the rate at which the velocity of the point relative to the body is changing at the instant in question.

When a point P is moving relative to a moving body B then the absolute velocity of P equals the vector sum of its relative velocity and the absolute velocity of that point of B with which P coincides at the instant in question. For simplicity of proof we take the pencil A of the glass-board-table apparatus as the moving point P and the board as the moving body B. Since P and B have plane motion, the proof is not general. Let Bd_1 (Fig. 424) be the position of B at

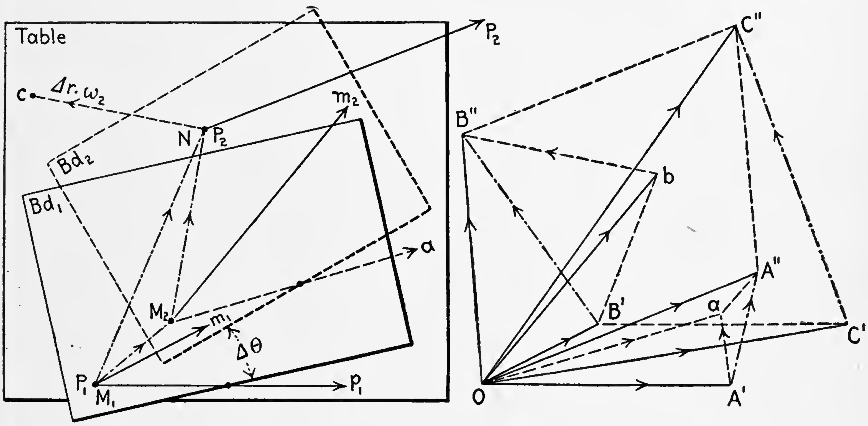


FIG. 424

a particular time t_1, and Bd_2 the position of B at a later time t_2; also P_1 and P_2 respectively, the positions of P at those times. Let M be the point of B with which P coincides at time t_1. At time t_1, M is at M_1 (under P_1); and at time t_2, M is at M_2. Then for the interval $t_2 - t_1$ the absolute displacement of P is P_1P_2; the relative displacement of P is M_2P_2; and the absolute displacement of M is M_1M_2. Obviously $P_1P_2 = M_2P_2 + M_1M_2$ (vectorially). Since this relation holds for any interval, the rates at which these displacements occur (velocities) are related in the same way; that is, the absolute velocity of P = its relative velocity + the velocity of M.

When a point P is moving relative to a moving body B then the absolute acceleration of P equals the vector sum of three accelerations, namely — the relative acceleration of P, the absolute acceleration of that point of B with which P coincides at the instant in question, and a so-called complimentary acceleration. The complimentary acceleration equals twice the product of the relative velocity of P and the angular velocity of B at the instant in question; its direction is the same as that of the linear velocity of p where Pp is a vector representing the relative velocity of P due to the angular velocity of B.

For simplicity again we restrict the proof to plane motions. Let P_1p_1 (Fig. 424) $=$ the relative velocity of P at the time t_1, and $M_1m_1 =$ the absolute velocity of M at that instant. The vector sum of these two velocities equals the absolute velocity of P at the time t_1. Making OA' and OB' to represent these velocities respectively, we get the diagonal OC' to represent the absolute velocity of P at the time t_1. Let N be the point of the board with which P coincides at the time t_2; N is under P_2 then. The velocity of N (at time t_2) equals the vector sum of the velocity of M_2 and the velocity of N "about" M_2. Now the velocity of N about M_2 equals the product of M_2N and the angular velocity of the board (at time t_2), or $\Delta r \times \omega_2$, where $\Delta r = M_2N$ and $\omega_2 =$ the angular velocity. The direction of this velocity $\Delta r \cdot \omega_2$ is perpendicular to M_2N as indicated (assuming that ω_2 is counter-clockwise). OB and bB'' are equal and parallel to M_2m_2 and $\Delta r \cdot \omega_2$ respectively; hence OB'' is the velocity of N at time t_2. Now let P_2p_2 $(= OA'')$ be the relative velocity of P at time t_2. Then the diagonal OC'' of the parallelogram on OA'' and OB'' is the absolute velocity of P at time t_2. Therefore $C'C''$ is the increment in the absolute velocity of P for the interval $t_2 - t_1$. It follows readily from the geometry of the figure that

$$C'C'' = A'A'' + B'B'', \tag{1}$$

vectorial addition being understood here and in the following.

Now let $M_2a = P_1p_1$ and the angle between these vectors equal the angular displacement $\Delta\theta$ of the board during the interval $t_2 - t_1$. Then the increment in the relative velocity of P for that interval equals the difference between the vectors M_2a and P_2p_2. Oa is equal and parallel to M_2a; hence aA'' is that difference. Therefore

$$A'A'' = A'a + \Delta v_r = 2\, v_r \sin \tfrac{1}{2}\, \Delta\theta + \Delta v_r,$$

where v_r means relative velocity of P at time t_1. Since Ob is equal and parallel to M_2m_2 (velocity of M at time t_2), $B'b$ is the increment in the velocity of M during the interval $t_2 - t_1$; and since $bB'' = \Delta r \cdot \omega_2$,

$$B'B'' = \Delta v_m + \Delta r \cdot \omega_2,$$

where v_m means velocity of M. Substituting the foregoing values of $A'A''$ and $B'B''$ in equation (1), we get

$$C'C'' = \Delta v_r + \Delta v_m + 2\, v_r \sin \tfrac{1}{2}\, \Delta\theta + \Delta r \cdot \omega_2. \tag{2}$$

Now let $\Delta t = t_2 - t_1$, and t_2 approach t_1; then we get

$$\lim \frac{C'C''}{\Delta t} = \lim \frac{\Delta v_r}{\Delta t} + \lim \frac{\Delta v_m}{\Delta t} + v_r \lim \frac{\Delta\theta}{\Delta t} + \lim \frac{\Delta r}{\Delta t}\,\omega_2.$$

The left-hand member is the absolute acceleration of P; the first term of the right-hand member is the relative acceleration of P; the second term is the acceleration of M. Lim $(\Delta\theta/\Delta t) = \omega_1$, the angular velocity of the board at time t_1; hence the third term $= v_r\omega_1$. Lim $(\Delta r/\Delta t)\omega_2 =$ lim $(\Delta r/\Delta t) \times$ lim ω_2 $= v_r\omega_1$. Hence the third and fourth terms are equal in magnitude, and if their directions — they are vectors — are parallel, then their sum $= 2\, v_r\omega_1$. The direction of the third term is the limiting direction of $A'a$, perpendicular to OA' or v_r obviously. The direction of the fourth term is the limiting direction of bB'' or Nc. Now Nc is always (as t_2 approaches t_1) perpendicular to M_2P_2; and since M_2P_2 is the relative displacement of P, the limiting direction of M_2P_2 is P_1p_1 (or v_r). Hence the limiting direction of Nc is perpendicular to v_r. Thus the sum of the last two terms $= 2\, v_r\omega_1$, and it has the direction mentioned, perpendicular to v_r. And this sum is the so-called complimentary acceleration; it is called also acceleration of Coriolos after him who first discovered the relation between the accelerations under discussion.

CHAPTER XIII

THREE DIMENSIONAL (SOLID) MOTION

54. Body with a Fixed Point, Kinematics of

§ 1. Spherical Motion means motion of a rigid body with only one point of the body fixed. Each point of the body, excepting the fixed one, moves on the surface of a sphere, whence the name spherical motion.

Any spherical displacement of a body can be accomplished by means of a rotation about some line of the body passing through the fixed point, and fixed in space. Proof: — Evidently, we may describe any position of the body by describing the positions of two of its points, not in line with the fixed point. Let A and B denote two such points, equally distant from the fixed point O; then during any motion of the body, A and B move on the surface of the same sphere. Let OA_1B_1 be one position of the body, and OA_2B_2 another. Then we are to prove that the points A and B could be brought from A_1B_1 to A_2B_2 by means of a single rotation about some fixed line through O. Let the lines A_1B_1 (Fig. 425) and A_2B_2 be arcs of great circles of the sphere mentioned; these arcs are equal since A and B are points of a rigid body. The lines A_1A_2 and B_1B_2 are arcs of great circles; M and N bisect these arcs; MR and NR are great circles perpendicular to A_1A_2 and B_1B_2 respectively.

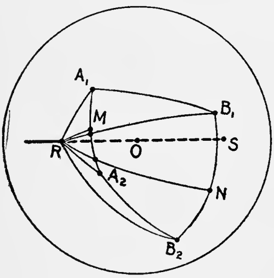


Fig. 425

In general two such great circles do not coincide but intersect at two points, R and S. The diameter ROS is the axis, rotation about which would produce the given displacement, proven presently. Let A_1R, A_2R, B_1R, and B_2R be arcs of great circles. Since A_1A_2R and B_1B_2R are isosceles triangles, $A_1R = A_2R$ and $B_1R = B_2R$; and, as already stated, $A_1B_1 = A_2B_2$. Hence the triangles RA_1B_1 and RA_2B_2 are equal, and the angle $A_1RB_1 = A_2RB_2$. Finally,

$$A_1RA_2 = A_1RB_2 + A_2RB_2 = A_1RB_2 + A_1RB_1 = B_1RB_2.$$

Hence a rotation of the great circles A_1R and B_1R about RS of an amount equal to the angle A_1RA_2 would displace A from A_1 to A_2 and B from B_1 to B_2.

Imagine any actual continuous spherical motion of a body, in which the two points A and B of the body are displaced from A_1 to A_2 and B_1 to B_2 respectively. Let A', A'', etc., be several intermediate positions of A, and let B', B'', etc., be corresponding intermediate positions of B. As already shown, the displacements of AB from A_1B_1 to $A'B'$, from $A'B'$ to $A''B''$, from $A''B''$

to $A'''B'''$, etc., might be accomplished by single rotations about definite fixed lines $R'OS'$, $R''OS''$, $R'''OS'''$, etc. If a large number of intermediate positions $A'B'$, $A''B''$, etc., be assumed, and if the successive rotations be accomplished in times equal to the times required for the actual displacements in the continuous motion, then the succession of rotations would closely resemble the actual continuous motion. The more numerous the intermediate positions, and the more numerous the succession of single rotations, the more closely would the succession resemble the actual motion. "In the limit," the succession would reproduce the actual motion; hence we may regard any spherical motion of a body as consisting of a continuous rotation about a line through the fixed point, the line continually shifting about in the body and in space. The line about which the body is rotating at any instant is the *instantaneous axis* (of rotation) at that instant.

At any particular instant of a spherical motion, the body is rotating about the instantaneous axis at a definite rate; this rate is called the angular velocity of the body at that instant. We will, generally, denote magnitude of angular velocity by ω. In a rotation about a fixed axis, the (linear) velocity of any point of the body equals the product of the angular velocity and the perpendicular distance (or radius) from the point to the axis; and the direction of the linear velocity is perpendicular to the plane of the radius and the axis. So too in a spherical motion, the linear velocity of any point of the body at any instant equals the product of the angular velocity at that instant and the radius (perpendicular from the point to the instantaneous axis for that instant); the direction of that velocity is perpendicular to the plane of the radius and the axis.

Any angular velocity ω may be represented by means of a vector laid off on the corresponding instantaneous axis; the length of the vector is made equal to ω according to some convenient scale, and the sense of the vector indicates the direction of the rotation according to some convention. We will always associate direction with angular velocity in the way just described; that is, we regard angular velocity as a vector quantity. In a spherical motion, angular velocity changes in direction continuously; it may or may not change in amount too. In any case, the rate at which the (vector) angular velocity is changing at any instant is called the *angular acceleration* at that instant. (See page 148 for note on rate of change of a vector quantity.) This rate or acceleration has a definite amount and direction at each instant, and hence is a vector quantity too. We will use α to denote the magnitude of an angular acceleration.

§ 2. COMPOSITION AND RESOLUTION OF ANGULAR VELOCITIES. — Imagine a body P to be rotating about a line l, fixed in a body A; and that A is rotating about a line l_2, intersecting l_1 and fixed in a body B (Fig. 426). For convenience we call the motion of P relative to B its absolute motion, and we regard this absolute motion as a resultant motion consisting of the (component) rotations about l_1 and l_2. We will show presently that the absolute motion of P is spherical, and that the angular velocity of that motion equals the vector sum of the angular velocity of P relative to A and that of A relative to B.

That the absolute motion is spherical will be conceded as almost self-evident; for the point O (of P) does not move at all, being a point of B, and O appears to be the only point of P which is fixed. But more on this matter later.

Let Oa and Ob (Fig. 427) be the two lines l_1 and l_2 at the instant in question, and let the angular velocities (of rotation) about those lines be ω_1 and ω_2 respectively, and in the directions indicated. Let C, not shown, be any point of the body P in the plane of the lines l_1 and l_2 (or paper). If C is taken above

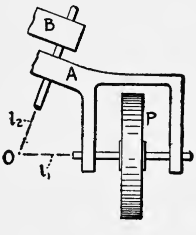


FIG. 426

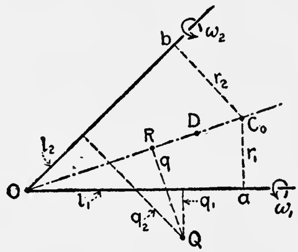


FIG. 427

Oa, then the rotation ω_1 alone brings C up out of the paper; if below, then ω_1 depresses C. If C is above Ob, then ω_2 alone brings C up out of the paper; if below, ω_2 depresses C. Hence if C is in either acute angle between l_1 and l_2, the two rotations give C displacements in opposite directions. Let C_0 be such a point C, and so chosen too that the two displacements of C_0 in an element of time dt would be equal. If r_1 and r_2 = the distances of C_0 from l_1 and l_2 respectively, these displacements = $r_1\omega_1\,dt$ and $r_2\omega_2\,dt$. Hence, $r_1\omega_1 = r_2\omega_2$, or $OC_0 \sin\alpha \cdot \omega_1 = OC_0 \sin\beta \cdot \omega_2$; and

$$\sin\alpha \cdot \omega_1 = \sin\beta \cdot \omega_2, \quad \text{or } \sin\alpha/\sin\beta = \omega_2/\omega_1. \tag{1}$$

Let D be any other point on OC_0; then its displacements due to ω_1 and ω_2 in the time dt are respectively $(OD \sin\alpha)\,\omega_1\,dt$ and $(OD \sin\beta)\omega_2\,dt$. But these are equal, since $\sin\alpha \cdot \omega_1 = \sin\beta \cdot \omega_2$; hence all points on OC_0 have zero velocity at the instant in question. Evidently, there are no other points in the body P whose velocity is zero at the instant; hence the state of motion of P is a rotation about OC_0, a line fixed by equation (1).

Let ω = the angular velocity of the rotation of P (about OC_0); Q (Fig. 427) be any point of P in the plane of the paper; q, q_1, and q_2 = distances of Q from OC_0, l_1, and l_2 respectively. Then the displacements of Q due to ω_1 and ω_2 are respectively $q_1\omega_1\,dt$ and $q_2\omega_2\,dt$. These displacements for Q as chosen are in the same direction; hence the total or resultant displacement $= (q_1\omega_1 + q_2\omega_2)\,dt$, and the linear velocity of Q (displacement per unit time) $= q_1\omega_1 + q_2\omega_2$. Now the angular velocity of the body P equals the linear velocity of its point Q divided by the distance of Q from the (instantaneous) axis of rotation OC_0, or

$$\omega = (q_1\omega_1 + q_2\omega_2) \div q.$$

It follows from the trigonometry of the figure that

$$q_1 = q\cos\alpha + (OR)\sin\alpha, \quad \text{and } q_2 = q\cos\beta - (OR)\sin\beta.$$

Substituting these values of q_1 and q_2 in the expression for ω and noting equation (1), we arrive at

$$\omega = \omega_1\cos\alpha + \omega_2\cos\beta. \tag{2}$$

Equations (1) and (2) respectively enable us to determine the axis (OC_0) of the resultant of two angular velocities ω_1 and ω_2 and the amount of the resultant angular velocity ω.

By means of equations (1) and (2) we can show that ω is the vector-sum of ω_1 and ω_2. Let OM and ON (Fig. 428), on l_1 and l_2 (Figs. 426 and 427), represent ω_1 and ω_2, and let $OMNR$ be a parallelogram. Then

$$\omega_1 \sin MOR = \omega_2 \sin NOR.$$

Comparing this with equation (1), we see that $MOR = \alpha$ and $NOR = \beta$; hence the parallelogram construction gives a diagonal which coincides with

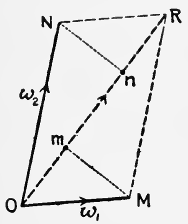


FIG. 428

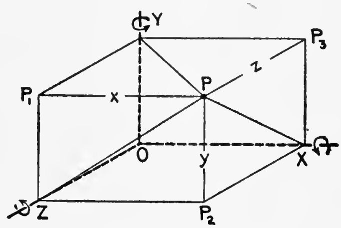


FIG. 429

the instantaneous axis of the absolute motion. It will be readily seen from the parallelogram that

$$OR = \omega_1\cos\alpha + \omega_2\cos\beta;$$

hence the length of the diagonal gives the magnitude of the angular velocity (see equation 2).

Reverting now to the proposition that the absolute motion of P is spherical, we note that the axis of instantaneous rotation OC_0 (Fig. 427) is always in the plane of l_1 and l_2 and hence is not fixed. Therefore there is only one fixed point of P, the intersection of l_1 and l_2.

Obviously, the foregoing analysis could be extended to a case of simultaneous rotations about three or more concurrent axes l_1, l_2, l_3, etc. Hence, in any case, the resultant motion is spherical, and the resultant angular velocity is given by the vector-sum of the component angular velocities. Conversely, the angular velocity of any spherical motion can be resolved into any number of concurrent components, and the vector-sum of the components is equal to the given velocity.

§ 3. VELOCITY OF ANY POINT OF THE MOVING BODY. — Let P (Fig. 429) be any point of a moving body (not shown), fixed at O; x, y, and z the (changing) coördinates of P with reference to fixed axes OX, OY and OZ; ω_x, ω_y, and ω_z = the components of the angular velocity of the body with respect to those axes; v = the linear velocity of P; and v_x, v_y, and v_z = the components of v along those axes. Then as will be proved presently

$$v_x = z\omega_y - y\omega_z, \quad v_y = x\omega_z - z\omega_x, \quad v_z = y\omega_x - x\omega_y. \tag{3}$$

If the body were rotating about the x axis only, then P would be describing a circle about X, and the velocity of P would be $XP \times \omega_x$. This velocity has no x component, and it is plain from the figure that the y and z components of that velocity respectively are $-z\omega_x$ and $y\omega_x$. These component velocities of P due to angular velocity ω_x are scheduled below; also the component velocities due to angular velocities ω_y and ω_z. It is plain from the schedule that the total component velocities due to the three angular velocities are as given by equations (3).

Rotation about	OX produces	$v_x = 0$,	$v_y = -z\omega_x$,	and	$v_z = y\omega_x$.
	OY produces	$v_x = z\omega_y$,	$v_y = 0$,	and	$v_z = -x\omega_y$.
	OZ produces	$v_x = -y\omega_z$,	$v_y = x\omega_z$,	and	$v_z = 0$.

55. Body with a Fixed Point, Kinetics of a

§ 1. ANGULAR MOMENTUM. — Certain definitions and notions set forth in Chapter XI will be recalled now, and then we will develop anew the subject of angular momentum with special reference to the kind of motion now under consideration. The momentum of a moving particle at any instant is the product of the mass of the particle and its velocity at that instant (Art. 46). The momentum is regarded as having direction — that of the velocity; also we regard it as having position — that of the line through the particle and in the direction of the velocity. Hence momentum of a particle is a localized vector quantity. The (localized) vector representing the momentum of a particle we call the momentum-vector of the particle. By angular momentum of a (moving) particle with respect to or about a line is meant the moment of the momentum about that line; that is the product of that component of the momentum which is perpendicular to the line — the other component being parallel to it — and the distance between the line and the perpendicular component (Art. 48). (Compare definition of moment of a force about a line, Art. 8.)

It will be of assistance now to represent angular momentum about any line by a vector coinciding with the line, the length and sense of the vector to represent the magnitude and sense of the angular momentum. Thus we regard angular momentum as a localized vector quantity. A moving particle has angular momentum about every conceivable line — except a line which cuts or is parallel to the momentum-vector of the particle; but in the case of a particle of a moving body with a fixed point — and such particles are assumed from now on — the angular momentum of the particle about a line through the fixed point normal to the plane of that point and the momentum-vector of the particle is of prime importance, because that angular momentum is greater than the angular momentum about any other line through the point. Hence we may call it *the* angular momentum of the particle. The angular momentum of a particle of a body with a fixed point equals the product of the momentum of the particle and the perpendicular distance from the fixed point to the momentum-vector, or mvp where m = mass of the particle, v = its velocity at the instant under consideration, and p = the distance just mentioned.

We now show that the (rectangular) component along any line through the fixed point of the body, of the vector representing the angular momentum of any particle represents the angular momentum of the particle about that line. Let P (Fig. 430) be the particle (of mass m); PQ the momentum-vector

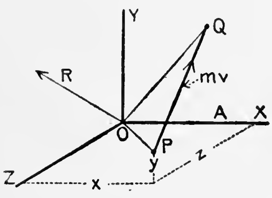


FIG. 430

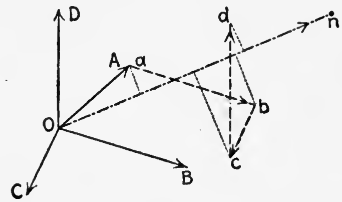


FIG. 431

(= mv) of P; O the fixed point of the body (not shown); OR the angular momentum-vector; OA any line through O; and h_a the component of OR along that line. Also let OA be the x axis of a set of coördinate axes with origin at O; x, y, and z = the coördinates of P; and α, β, and γ = the direction angles (with respect to those axes) of v. Then

$$h_a = mvp \cos (AOR) = mvp (\cos \gamma \cdot y - \cos \beta \cdot z)/p = m (v_z y - v_y z).$$

But $m(v_z y - v_y z)$ is the angular momentum of P about OA (see Art. 48); hence the component of OR along OA does represent the angular momentum of P about OA as stated.

If we should add (or compound) the vectors representing the (resultant) angular momentums of the several particles of a body, we would arrive at a vector of definite magnitude, position, and sense. Thus, suppose that OA, OB, OC, etc. (Fig. 431) are the vectors; then the closing side On of the vector-polygon $OAbcd$ . . . (not plane) is the vector-sum. Evidently this vector does not depend in any way on any coördinate system. We now prove that this vector represents the angular momentum of the body about the line On. According to definition, the angular momentum of the body about that line is the algebraic sum of the momentums of all the particles about that line (see

Art. 48). The angular momentum of the first particle about On = the projection of Oa on On; that of the second particle = the projection of ab on On; etc. Hence, the algebraic sum of the projections equals the algebraic sum of the angular momentums of the particles, that is, the angular momentum of the body about On. This particular angular momentum of the body we call *the* (or resultant) angular momentum of the body; we will denote it by h.

Let h_x, h_y, and h_z respectively equal the components of h along any three rectangular axes through the fixed point of the moving body; the axes need not be fixed in space. Then obviously h_x, h_y, and h_z respectively equal the algebraic sums of the angular momentums of the particles of the body about the x, y, and z axes; that is

$$h_x = \Sigma m(v_z y - v_y z), \quad h_y = \Sigma m(v_x z - v_z x), \quad h_z = \Sigma m(v_y x - v_x y). \tag{1}$$

These expressions for the component angular momentums can be transformed into the following (involving angular and not linear velocities):

$$\begin{aligned} h_x &= + I_x\omega_x - J_z\omega_y - J_y\omega_z, \\ h_y &= - J_z\omega_x + I_y\omega_y - J_x\omega_y, \\ h_z &= - J_y\omega_x - J_x\omega_y + I_z\omega_z. \end{aligned} \tag{2}$$

I_x, I_y, and I_z = the moments of inertia of the body about the x, y, and z axes respectively; or symbolically,

$$I_x = \Sigma m(y^2 + z^2), \quad I_y = \Sigma m(z^2 + x^2), \quad I_z = \Sigma m(x^2 + y^2).$$

J_x, J_y, and J_z respectively = the products of inertia of the body with respect to the two coördinate planes intersecting in the x, y, and z axes (Art. 57); or symbolically,

$$J_x = \Sigma myz, \quad J_y = \Sigma mzx, \quad J_z = \Sigma mxy.$$

Symbols ω_x, ω_y, and ω_z denote the axial components of the angular velocity of the body. Equations (2) may be deduced from equations (1) by substituting for v_x, v_y, and v_z their values from equations (3) of Art. 54, and then simplifying. If the coördinate axes x, y, and z are principal axes of the body at the fixed point (Art. 57), then J_x, J_y, and J_z = 0; and

$$h_x = I_x\omega_x, \quad h_y = I_y\omega_y, \quad \text{and } h_z = I_z\omega_z.$$

§ 2. RATE OF CHANGE OF ANGULAR MOMENTUM. — Let O be the fixed point of a moving body; OX, OY, and OZ a set of fixed axes; and OA, OB, and OC another (rectangular) set rotating about O, but not necessarily fixed in the body. Let ω = the angular velocity of the body, and θ = the angular velocity of the rigid frame consisting of the axes OA, OB, and OC. (If these axes are fixed in the body, θ and ω are equal.) Also let θ_1, θ_2, and θ_3 = the components of θ about the axes OA, OB, and OC; and h_1, h_2, and h_3 = the angular momentums of the body about the axes OA, OB, and OC respectively. Then, as will be proved presently, the rates at which the angular momentums about the (moving) axes OA, OB, and OC respectively are changing at any instant are given by

$$\frac{dh_1}{dt} - h_2\theta_3 + h_3\theta_2, \quad \frac{dh_2}{dt} - h_3\theta_1 + h_1\theta_3, \quad \frac{dh_3}{dt} - h_1\theta_2 + h_2\theta_1,$$

where the derivatives, the h's and the θ's all pertain to the instant in question. Furthermore, since the torque of the forces acting on a body about any line equals the rate at which the angular momentum of the body about that line is changing, we have

$$\begin{aligned} T_1 &= (dh_1/dt) - h_2\theta_3 + h_3\theta_2, \\ T_2 &= (dh_2/dt) - h_3\theta_1 + h_1\theta_3, \\ T_3 &= (dh_3/dt) - h_1\theta_2 + h_2\theta_1, \end{aligned} \tag{3}$$

where T_1, T_2, and T_3 are the torques about the axes OA, OB, and OC. Torques and rates pertain to the same instant of course.

Evidently, the angular momentums h_1, h_2, and h_3 do not depend on the x, y, and z axes; hence the rates of change of the angular momentums do not depend on those axes. We may choose any positional relation between the two sets of axes for deducing expressions for the rates. For simplicity, we choose them coincident as shown in Fig. 432. Let the vectors OA, OB, and OC represent h_1, h_2, and h_3 respectively. Suppose first that the frame $OABC$ is not rotating ($\theta = 0$). Then the rate of change of angular momentum about OA is due wholly to growth in h_1; growth in h_2 and h_3 would not affect the rate; the rate would be dh_1/dt. Now consider the effect of the angular velocity of the frame $OABC$ on the rate of change of angular momentum about OA. The component velocity θ_1 turns h_2 and h_3 (in the plane YOZ), and hence does not affect the rate in question. The component velocity θ_2 turns h_1 and h_3 (in the plane ZOX); the turning of h_1 does not affect the rate but the turning of h_3 contributes $h_3\theta_2$. The component velocity θ_3 turns h_1 and h_2 (in the plane XOY); the turning of h_1 does not affect the rate but the turning of h_2 contributes $-h_2\theta_3$.* Finally the total rate is $(dh_1/dt) - h_2\theta_3 + h_3\theta_2$, as was to be shown. In a similar way, the stated values of the other two rates could be arrived at.

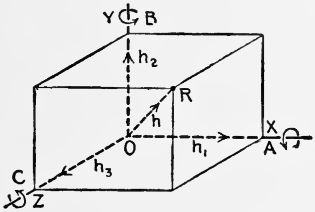


FIG. 432

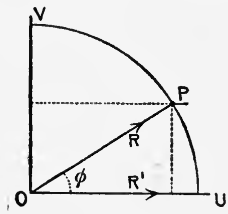


FIG. 433

* Let OP (Fig. 433) be a vector R of constant magnitude rotating about O, in the plane of the paper, with angular velocity ω $(= d\phi/dt)$. Also let R' = the component of R along the fixed line OU; then $R' = R\cos\phi$. The rate at which R' changes is dR'/dt, or $-R\sin\phi \cdot d\phi/dt = -R\sin\phi\cdot\omega$. Now when R coincides with OU, $\phi = 0$, and the rate of change of $R' = 0$. When R is perpendicular to OU, $\phi = \frac{1}{2}\pi$, and the rate of change of $R' = -R\omega$. If ω is positive (rotation counter clockwise), then $-R\omega$ is negative; and this means that R' is decreasing (obvious from the figure). If ω is negative (rotation clockwise), then $-R\omega$ is positive; and this means that R' is increasing (obvious from the figure).

§ 3. KINETIC ENERGY. — As in § 3 of Art. 54, let ω_x, ω_y, and ω_z = the axial components of the angular velocity of the moving body at any particular instant; x, y, and z = the coördinates of some particle P of the body then; and v_x, v_y, and v_z = the axial components of the velocity of P. If m = the mass of the particle, then its kinetic energy at the instant in question is

$$\tfrac{1}{2} m (v_x^2 + v_y^2 + v_z^2).$$

Now if we substitute for v_x, v_y and v_z their values from equations (3) of Art. 54, we arrive at a new expression for the kinetic energy of P; and if we sum up such expressions for all the particles of the body, we find that the kinetic energy of the body is

$$\tfrac{1}{2} I_x\omega_x^2 + \tfrac{1}{2} I_y\omega_y^2 + \tfrac{1}{2} I_z\omega_z^2 - J_x\omega_y\omega_z - J_y\omega_z\omega_x - J_z\omega_x\omega_y,$$

where I_x, I_y, and I_z are the moments of inertia of the moving body about the x, y, and z axes respectively, and J_x, J_y, and J_z are the products of inertia of the body with respect to the pairs of coördinate planes intersecting in the x, y, and z axes respectively (Art. 57) all at the instant in question. That is

$$J_x = \Sigma myz, \quad J_y = \Sigma mzx, \quad J_z = \Sigma mxy.$$

The products of inertia may be zero; then the kinetic energy equals

$$\tfrac{1}{2} I_x\omega_x^2 + \tfrac{1}{2} I_y\omega_y^2 + \tfrac{1}{2} I_z\omega_z^2.$$

56. Gyrostat

§ 1. STEADY OBLIQUE PRECESSION. — Let OC (Fig. 434) be the spin-axis of a gyrostat, OZ a fixed axis about which OC is rotating or precessing at a steady rate, θ = the constant angle ZOC, ω = the angular velocity of spin, and Ω = the angular velocity of precession. Let OX and OY be fixed axes, perpendicular to each other and to OZ; OA an axis on the plane of ZOC and perpendicular to OC; and OB perpendicular to OA and OC. OA and OB are moving axes, rotating with OC about OZ.

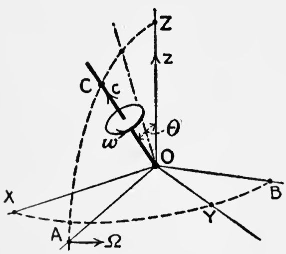


FIG. 434

The motion of the gyrostat consists of the component rotations ω and Ω about OC and OZ respectively. The resultant of those components is a rotation about the diagonal of the parallelogram on the vectors Oc and Oz representing ω and Ω (Art. 54), and the angular velocity of that resultant rotation is represented by that diagonal. Hence, the components of the angular velocity of the gyrostat along OA, OB, and OC are respectively

$$-\Omega \sin\theta, \quad 0, \quad \text{and} \quad \omega + \Omega\cos\theta = n,$$

n being an abbreviation for $\omega + \Omega\cos\theta$.

It may be well to note the distinction between the velocity of spin ω and n. The spin velocity is the angular velocity of the gyrostat relative to the moving frame $OABC$; it is the product of 2π and the number of times per unit time which a point on the gyrostat pierces the plane ZOC. The angular velocity n is the component of the absolute angular velocity of the gyrostat along the fixed line with which OC happens to coincide at the instant in question. Since OA, OB, and OC are principal axes of the gyrostat (Art. 57), the angular momentums of the gyrostat about these axes are respectively

$$-A\Omega\sin\theta, \quad 0, \quad \text{and} \quad C(\omega + \Omega\cos\theta) = Cn,$$

where A and C denote the moments of inertia of the gyrostat about OA and OC respectively.

Since the entire frame of axes $OABC$ is rotating about OZ with angular velocity Ω, the components of that velocity along OA, OB, and OC are respectively,

$$-\Omega\sin\theta, \quad 0, \quad \text{and} \quad \Omega\cos\theta.$$

Substituting now in equation (3) of Art. 55, we get as the required values of the torques of the external forces about the axes OA, OB, and OC respectively

$$T_1 = 0,$$
$$T_2 = Cn\Omega\sin\theta - A\Omega^2\sin\theta\cos\theta,$$
and
$$T_3 = 0.$$

Therefore for steady spin and precession, there must be no torque about any line in the plane of the spin and precession axes (T_1 and $T_3 = 0$) but a torque equal to

$$Cn\Omega\sin\theta - A\Omega^2\sin\theta\cos\theta$$

about a line perpendicular to those axes.

Let us now consider whether a gyrostat may precess steadily under the influence of gravity and the pivot reaction only. Let W = the weight of the gyrostat, and h = the distance of its center of gravity from the pivot; then the torques of gravity about OA, OB, and OC are respectively 0, $Wh\sin\theta$, and 0. We assume that the pivot is so well made that the torques of the reaction about the lines mentioned equal zero practically. Hence the gyrostat is not subjected to any torque about OA and OC but a torque of $Wh\sin\theta$ about OB. If now the quantities W, h, θ, etc., be given such values that

$$Wh\sin\theta = Cn\Omega\sin\theta - A\Omega^2\sin\theta\cos\theta,$$

then all the conditions for steady precession will be satisfied. Evidently such values can be assigned, in general. Indeed if we solve the preceding equation for Ω, we get

$$\Omega = \frac{Cn \pm \sqrt{(C^2n^2 - 4AWh\cos\theta)}}{2A\cos\theta},$$

from which it is plain that in general there are two possible velocities of precession for a given gyrostat, spin ω, and inclination θ. But if $C^2n^2 = 4AWh\cos\theta$, then there is only one value of Ω; and if $C^2n^2 < 4AWh\cos\theta$, then Ω is imaginary, and the gyrostat will not precess under the conditions imposed.

A gyrostat whose center of gravity is at the pivot will precess steadily for certain conditions of impressed spin, precession, and obliquity. For, suppose that spin, precession, and obliquity are so arranged that $Cn = A\,\Omega\cos\theta$; then $T_3 = 0$, that is no torque is required to maintain the precession. Hence the gyrostat, with center of gravity at the pivot, would continue to precess.

Gyrostat in a Case. — The foregoing analysis must be modified for a spinning gyrostat in a frame or case which does not spin but merely precesses with the axis of spin. Let A and C be moments of inertia of the spinning part as before, and A' and C' the corresponding moments of inertia of the case. Then the angular momentums of the case about the axes OA, OB, and OC (Fig. 434) are respectively

$$-A'\Omega\sin\theta, \quad 0, \quad \text{and} \quad C'\Omega\cos\theta.$$

These may be added to the earlier expressions for corresponding momentums of the spinning part to arrive at values of the angular momentums of the entire gyroscope. Then substituting in equation (3) of Art. 55 as before, we find that the necessary torques about OA, OB, and OC for steady precession are respectively

$$T_1 = 0,$$
$$T_2 = (Cn + C'\Omega\cos\theta)\,\Omega\sin\theta - (A + A')\,\Omega^2\sin\theta\cos\theta,$$

and $$T_3 = 0,$$

where $n = \omega + \Omega\cos\theta$ as before.

§ 2. UNSTEADY OBLIQUE PRECESSION. — Imagine a gyrostat to have been started spinning in some way, and then released and left to itself on a frictionless pivot under the action of the pivot reaction and gravity. The subsequent motion will now be investigated.

Let ω_0 = the angular velocity of spin, and θ_0 = the angle between the axis of spin and the vertical at release. Let Fig. 434 represent the gyrostat at some instant after its release; ω = the velocity of spin (velocity of the gyrostat relative to the plane ZOC); Ω = the velocity of turning of the plane ZOC (which we will continue to call velocity of precession); and θ = the angle ZOC. We do not assume ω, Ω, and θ to be constants.

At the instant of release, when there is not yet any motion of the axes $OABC$ the total angular velocity of the gyrostat is ω_0. At a later instant the angular velocity of the gyrostat is the resultant of its velocity of spin ω (relative to the frame $OABC$) and the angular velocity of the frame. Now this latter velocity has the following components along OA, OB, and OC respectively,

$$\Omega\sin\theta, \quad \dot{\theta},\text{*} \quad \text{and } \Omega\cos\theta;$$

hence the (resultant) angular velocity of the gyrostat has the following components along OA, OB, and OC respectively,

$$-\Omega\sin\theta, \quad \dot{\theta}, \quad \text{and } \omega + \Omega\cos\theta = n,$$

* According to this (*fluxional*) notation, a symbol with a dot over it means the time rate of the quantity represented by the symbol; thus $\dot{\theta}$ means $d\theta/dt$, $\dot{s}$ means ds/dt, etc.

where n is an abbreviation for $\omega + \cos\theta$ as in § 1. And the angular momentums about those same lines are

$$- A\,\Omega \sin\theta, \quad B\dot{\theta}, \quad \text{and} \quad Cn.$$

According to equation (3), Art. 55, the rate at which the angular momentum about OC is changing is

$$C\dot{n} + A\,\Omega \sin\theta \cdot \dot{\theta} - B\dot{\theta} \sin\theta;$$

but this rate equals zero since there is no torque about OC. And because $A = B$, $C\dot{n} = 0$; hence n is constant, and therefore always equals its initial value, that is,

$$n = 0.$$

This does not mean that the spin velocity, ω, is constant.

Since there is no torque about the (fixed) axis OZ, the angular momentum about that line remains constant; thus that angular momentum at any instant equals its initial value, or

$$A\,\Omega \sin\theta \times \sin\theta + C\omega_0 \cos\theta = C\omega_0 \cos\theta_0.$$

This equation shows that

$$\Omega = C\omega_0(\cos\theta_0 - \cos\theta) \div A \sin^2\theta, \tag{1}$$

from which one may compute the velocity of the plane ZOC, or the velocity of precession.

Investigation of the (nutational) motion of the spin-axis in the (azimuthal) plane ZOC can be made simplest by means of the principle of work and kinetic energy (Art. 43). From the instant of release of the gyrostat to any subsequent instant, gravity does an amount of work on the gyrostat equal to the product of the weight and the vertical descent of the center of gravity. If, as in § 1, W = the weight, and h = the distance from the pivot to the center of gravity, then the work done by gravity is

$$Wh(\cos\theta_0 - \cos\theta).$$

The initial kinetic energy of the gyrostat is $\frac{1}{2} C\omega_0^2$, and its kinetic energy at a later instant is

$$\tfrac{1}{2} A\,\Omega^2 \sin^2\theta + \tfrac{1}{2} B\dot{\theta}^2 + \tfrac{1}{2} C\omega_0^2.$$

Now the change in kinetic energy is due to the work done by gravity; hence, since $A = B$,

$$\tfrac{1}{2} A(\Omega^2 \sin^2\theta + \dot{\theta}^2) = Wh\,(\cos\theta_0 - \cos\theta) \tag{2}$$

From this equation and (1), it is possible to compute the angular velocity of nutation for any value of θ. Thus if we eliminate Ω between equations (1) and (2), we get

$$\dot{\theta}^2 = \frac{2\,Wh}{A}(\cos\theta_0 - \cos\theta) - \frac{C^2\omega_0^2\,(\cos\theta_0 - \cos\theta)^2}{A^2 \sin^2\theta}. \tag{3}$$

Now the angular velocity $\dot{\theta}$ must be real, and hence the right-hand member of (3) cannot be negative; it may equal zero or be positive. The right-hand member is zero (and then $\dot{\theta} = 0$), when $\theta = \theta_0$; also when θ equals θ_1, where

$$\cos\theta_1 = \lambda - \sqrt{1 - 2\lambda\cos\theta_0 + \lambda^2},$$

λ being an abbreviation for $C^2\omega_0^2/2\,WhA$. Any value of θ between θ_0 and θ_1 makes the right-hand member of (3) positive, and gives two equal values of $\dot{\theta}$ of opposite sign. Hence the spin-axis oscillates in the azimuthal plane, θ varying between θ_0 and θ_1.

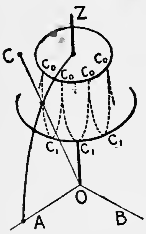


FIG. 435

Whenever θ is greater than θ_0, Ω is positive (see equation 1); but when $\theta = \theta_0$, then $\Omega = 0$. Hence the azimuthal plane rotates always in one direction but its velocity is zero every time when the center of gravity of the gyrostat gets into its highest position ($\theta = \theta_0$).

In Fig. 435 the curve $C_0C_1C_0C_1$ represents the path of a point on the spin-axis of the gyrostat; C_0C_0 are the highest and C_1C_1 the lowest positions reached by the point; $ZOC_0 = \theta_0$ and $ZOC_1 = \theta_1$.

57. Principal Moments of Inertia and Axes

§ 1. MOMENT OF INERTIA AND RADIUS OF GYRATION of a body with respect to (or about) a line are defined in Art. 36. It is shown there, among other things, that the moments of inertia and radiuses of gyration of a body with respect to parallel lines are very simply related (§ 2); and of such moments of inertia and radiuses of gyration, the one about the line through the mass-center is least. We will now examine the moments of inertia of a body about all lines through any point of it. It will be shown that in general there is one line about which the moment of inertia is maximum, and a second line, perpendicular to the first, about which the moment of inertia is a minimum. These two lines and the one perpendicular to their plane at the point in question are called the *principal axes* at the point, and the moments of inertia about those lines are the *principal moments of inertia* at the point. These axes are important dynamically (see § 2).

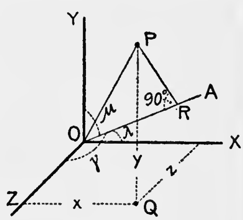


FIG. 436

Let P (Fig. 436) be a point of a body (not otherwise shown); O any other point, not in the body necessarily; and OA any line through O. Let λ, μ, and ν = the direction cosines of OA with respect to any coördinates axes with origin at O, m = mass of P, r = distance of P from OA, and I = the moment of inertia of the body about OA. According to definition, $I = \Sigma mr^2$. Now $r^2 = (OP)^2 - (OR)^2$. OP is a diagonal of a parallelopiped of which lines x, y, and z are three intersecting edges; hence $OP^2 = x^2 + y^2 + z^2$. OQ is one side of the closed (gauche) polygon $OZQPRO$; and since any side of a closed polygon equals the algebraic sum of the projections of the other sides upon it, $OR = \lambda x + \mu y + \nu z + 0$. Hence,

$$r^2 = x^2 + y^2 + z^2 + (\lambda x + \mu y + \nu z)^2.$$

Expanding this expression for r^2 and arranging terms we would find that

$$I = \Sigma m[\lambda^2(y^2 + z^2) + \mu^2(z^2 + x^2) + \nu^2(x^2 + y^2) - 2\,\mu\nu yz - 2\,\nu\lambda zx - 2\,\lambda\mu xy].$$

In this (space) summation λ, μ, and ν are constants; hence

$$I = \lambda^2\Sigma m(y^2 + z^2) + \cdots\cdot - 2\,\mu\nu\Sigma myz - \cdots\cdot$$

Now $y^2 + z^2$, $z^2 + x^2$, and $x^2 + y^2$ respectively = the squares of the distances of P from the x, y, and z axes; hence if A, B, and C = the moments of inertia of the body with respect to the x, y, and z axes, we have

$$A = \Sigma m(y^2 + z^2), \quad B = \Sigma m(z^2 + x^2), \quad \text{and} \quad C = \Sigma m(x^2 + y^2).$$

The remaining summations in the foregoing expression for I are the so-called *products of inertia* of the body with respect to the two coördinate planes intersecting in the x, y, and z axes respectively. Let D, E, and F respectively denote these products of inertia, that is

$$D = \Sigma myz, \quad E = \Sigma mzx, \quad \text{and} \quad F = \Sigma mxy.$$

Then we have

$$I = \lambda^2 A + \mu^2 B + \nu^2 C - 2\,\mu\nu D - 2\,\nu\lambda E - 2\,\lambda\mu F. \tag{1}$$

If we know the moments of inertia (A, B, and C) of a body about each one of a set of coördinate axes, and the products of inertia (D, E, and F) with respect to each pair of the coördinate planes, then by means of formula (1) we can find the moment of inertia I of the body about any line through the origin of coördinates. And by means of formula (4) of Art. 36, we can transfer this I to any parallel axis desired. Thus the two formulas enable one to "transfer" from the coördinate axes to any line whatsoever.

Imagine a length OS laid off on OA (Fig. 436) so that OS, which we will call ρ, is inversely proportional to the radius of gyration of the body about OA. That is, if k = the radius of gyration and K a factor of proportionality, then $\rho = K/k$. Such points S for all lines OA would lie on the surface of an ellipsoid (proved presently) called the *momental ellipsoid* of the body for the selected point O. Let X, Y, and Z = the coördinates of S; they equal $\rho\lambda$, $\rho\mu$, and $\rho\nu$ respectively. Then equation (1) multiplied by ρ^2 reduces to

$$AX^2 + BY^2 + CZ^2 - 2\,DZY - 2\,EYX - 2\,FXY = K^2M, \tag{2}$$

where M = the mass of the body. This is the equation of an ellipsoid with center at O (see any standard work on Analytic Geometry).

In general, the axes of an ellipsoid are unequal in length. Hence, the radius of gyration (and the moment of inertia) about the shortest axis of the momental ellipsoid is greater than the radius of gyration (and moment of inertia) about any other line through the center of the ellipsoid, and the moment of inertia about the longest axis is less than that about any other line through the center. Thus we have shown that there are two lines at right angles to each other through any point of a body (or of its extension) about which the moments of inertia of the body are maximum and minimum. The momental ellipsoid might of course be one of revolution in a special case, or even a sphere.

If two of the products of inertia equal zero, say E and F, then the equation of the momental ellipsoid is

$$AX^2 + BY^2 + CZ^2 - 2\,DYZ = K^2M,$$

which shows that the ellipsoid is symmetrical with respect to the yz plane. Hence the x axis coincides with one of the axes of the ellipsoid, that is with one of the principal axes of the body at the point O. If the three products of inertia equal zero, then the ellipsoid is symmetrical with respect to the three coördinate planes, and hence each coördinate axis is a principal axis at the origin. Then if I_1, I_2, and I_3 denote the principal moments of inertia, formula (1) becomes

$$I = \lambda^2 I_1 + \mu^2 I_2 + \nu^2 I_3.$$

Symmetrical Bodies.—If a homogeneous body is symmetrical with respect to a plane, then any line perpendicular to the plane is a principal axis at the point where it pierces the plane. For, take such line as the x axis, and the y and z axes in the plane. Then for every particle of the body whose coördinates are a, b, and c, there is another one whose coördinates are $-a$, b, and c; hence Σmzx and $\Sigma mxy = 0$, and therefore as explained the x axis is a principal axis at the origin of coördinates.

If a homogeneous body has two planes of symmetry at right angles to each other, then their intersection is a principal axis at every point of that line. For if the two planes be taken as coördinate planes and any plane perpendicular to them as the third coördinate plane, then it is obvious that the three products of inertia equal zero; hence the intersection of the planes of symmetry (one of the coördinate axes) is a principal axis at the origin of coördinates (taken at any point on the intersection).

§ 2. FREE AXES. — The axes of principal moments of inertia at the mass-center of a rigid body are called free axes of the body because they possess a certain property which may be described as follows: If the body could be set to rotating about any one of these axes and then left to itself entirely free from all external forces, even gravity, it would continue to rotate about that axis. To demonstrate this, we will imagine this axis to be a shaft resting in bearings, and then show that the bearings would exert no pressures whatever on the shaft. It will follow that such bearings are not necessary to hold the shaft in position. Let B_1B_2 (Fig. 437) be the axis, B_1 and B_2 the bearings, O the mass-center of the body, and ω = the angular velocity. Also let O be the origin of the axes x, y, and z as shown, P and Q = the reactions of the bearings on the shaft; and P_x, P_y, and P_z = the axial components of P, and Q_x, Q_y, and Q_z = those of Q. Evidently, P_z or $Q_z = 0$. Since the mass-center is at rest, the sums of the x, y, and z components of all the forces (P and Q) acting on the body equal zero. Hence

$P_z = Q_z = 0$; P_x and Q_x are opposite; also P_y and Q_y.

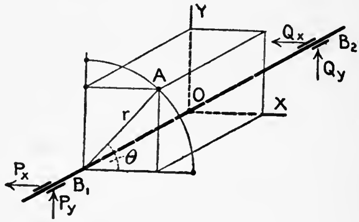


FIG. 437

That is, the external forces consist of two couples, P_x and Q_x, and P_y and Q_y. Let A be one of the particles of the body; m = its mass; r = its distance from the axis; θ = the (varying) angle which r makes with xz plane (as shown); x, y, and z = the coördinates of A; v_x, v_y, and v_z = the axial components of the velocity of A. Since $v_z = 0$, the angular momentums of A about the x and y axes are respectively

$$-mv_yz = -mv\cos\theta \cdot z = -mr\omega\cos\theta \cdot z = -\omega mxz,$$

and $$mv_xz = -mv\sin\theta \cdot z = -mr\omega\sin\theta \cdot z = -\omega myz.$$

Hence the angular momentums of the entire body about the x and y axes respectively are

$$-\omega\Sigma mxz \quad \text{and} \quad -\omega\Sigma myz.$$

Since the z axis is a principal axis, these summations (or products of inertia) equal zero; that is the angular momentums about the x and y axes equal zero, at all times. It follows that there is no torque about either axis at any time; hence there are no such couples, P_xQ_x and P_yQ_y.

If the axis of rotation B_1B_2 is not a principal axis, then the angular momentums $-\omega\Sigma mxz$ and $-\omega\Sigma myz$ are not always zero nor are they constant in value during a revolution; hence the torques of the couples P_xQ_x and P_yQ_y are not always zero. Such couples can be sensed roughly by supporting an irregular shaped body by means of one's hands as bearings and then making it rotate. Of course one would feel the dead weight of the body but also pulls and pushes due to the tendency or effort of the axis of rotation to get away. If a regular body were selected, one would be apt to rotate it about a principal axis, and so miss the effect just described.

In general a body has three free axes, the axes of greatest, least, and mean moments of inertia — axes through the mass-center being understood. There is an interesting difference among these axes, namely, rotation about the axis of greatest or least moment of inertia is stable, while rotation about the axis of mean moment is unstable. That is, if the body were set rotating about either of the first two axes, then any slight deviation of the axis of rotation from such principal axis would not be followed by continually increasing deviations; but if the initial rotation be about the axis of mean moment then slight deviation is followed by still greater change. Explanation of these properties would take us too far afield.

58. Any Motion of a Rigid Body; Summary of Dynamics

§ 1. ANY MOTION OF A RIGID BODY. — A rigid body can be displaced from one position A into another position B by means of a translation followed by a rotation. For, it is obvious that a translation can be selected so as to move any chosen point O of the body from its original position (in A) into its final position (in B). From this intermediate position, the body can be put into its final position by means of a rotation (of suitable amount) about a (certain) fixed axis through the final position of O (see Art. 54). The displacement might be effected in the reverse order, that is a rotation followed by a translation. For, a rotation about a fixed line through the "base point" O could be made so as to put the two lines OP and OQ, P and Q being two points of the body not in line with O, parallel to their final positions (in B); and a suitable translation would put those lines (and the body) into their final positions.

Evidently, the rotation and the translation could be made simultaneously. Therefore any actual motion of a body from one position into another may be regarded as a succession of infinitesimal simultaneous translations and rotations. All the translations may refer to the same base point, but in general the successive rotations do not occur about the same line of the body. Thus we may regard any motion of a rigid body as consisting of a translation (in which each point of the body moves just like the base point), combined with a rotation about a line through the base point, the line shifting about in the body, generally. There is not only such (kinematic) independence of translation and rotation as just explained, but if we take the center of gravity as base point, then there is also independence of translation and rotation dynamically. That is to say, we may ascertain the translation (or motion of the center of gravity) quite independently of the rotation; and the rotation about the center of gravity quite independently of the translation. We proceed to demonstrate this independence.

As already shown in Art. 34, the acceleration of the mass-center of a body (even if not rigid) may be determined as though all the material of the body were concentrated at the mass-center, and all the external forces were applied at that (dense) point. To see that the rotation about the mass-center is independent of the translation, let us apply the principle that the torque of all the external forces about any line equals the rate at which the angular momentum about that line is changing (Art. 48), taking the line through the mass-center. Let O (Fig. 438) be the mass-center, and P any other point of the moving body; a, b, and c = the (changing) coördinates of O with respect to fixed axes QX, QY, and QZ; x, y, and z = the coördinates of P with respect to the same axes, and x', y', and z' = the coördinates of P with respect to a

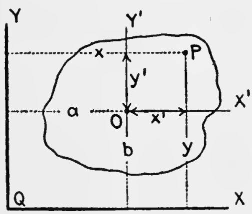


FIG. 438

parallel set of axes, the origin being at O. Furthermore, let m = mass of P, and v_x, v_y, and v_z = the x, y, and z components of the velocity of P; then the angular momentum of P about the z' axis say (through O and parallel to the z axis) $= mv_yx' - mv_xy'$, and the angular momentum of the entire body about that line equals

$$\Sigma m(v_yx' - v_xy').$$

Now $x = x' + a$, and $y = y' + b$; hence $v_x = dx'/dt + da/dt$ and $v_y = dy'/dt + db/dt$. Therefore, the angular momentum =

$$\Sigma m(x'\,dy'/dt + x'\,db/dt - y'\,dx'/dt - y'da/dt) =$$
$$\Sigma mx'\,dy'/dt + (db/dt)\,\Sigma mx' - \Sigma my'\,dx'/dt - (da/dt)\Sigma my'.$$

Since $\Sigma mx'$ and $\Sigma my' = 0$, the second and fourth terms equal zero. Hence the angular momentum equals

$$\Sigma m(x'\,dy'/dt - y'\,dx'/dt).$$

Now this expression does not depend on the motion of the center of gravity at all; moreover, it is just like the expression for the angular momentum of a body rotating about a fixed point (see Art. 55).

§ 2. SUMMARY; MOTION OF A RIGID BODY. — The following summary is based upon the order of development of dynamics followed in this book; it may therefore be regarded as a brief review.

Motion of Translation (rectilinear or otherwise). — The resultant of all the external forces acting on the body is a single force. Its line of action passes through the mass-center, and hence the external forces have no torque about any line through that point. The resultant and the acceleration of the moving body have the same direction; hence the algebraic sum of the components of the external forces at right angles to the acceleration equals zero. The acceleration is proportional to the resultant directly and to the mass of the body inversely; or $a = R/M$ (where a = acceleration, R = resultant, and M = mass) if systematic units be used. The (linear) momentum of the body $= Mv = (W/g)v$, where v = velocity, W = weight, and g = acceleration due to gravity. The kinetic energy of the body $= \frac{1}{2} Mv^2 = \frac{1}{2} (W/g)\, v^2$.

Rotation about a Fixed Axis. — The torque of all the external forces about the axis of rotation and the angular acceleration of the rotating body are alike in sense. The angular acceleration is proportional to the torque directly and to the moment of inertia of the body (with respect to the axis of rotation) inversely; or $\alpha = T/I$ (where α = angular acceleration, T = torque, and I = moment of inertia), if systematic units be used. (It is generally convenient to take $I = Mk^2 = (W/g)k^2$ where k = radius of gyration of the body about the axis of rotation. We have also

$$\Sigma F_n = M\bar{a}_n, \quad \Sigma F_t = M\bar{a}_t, \quad \text{and}\ \Sigma F_3 = 0.$$

The summations mean the algebraic sums of the components of the external forces — including the axle reaction if any — along three certain lines, namely, — (1) the perpendicular to the axis of rotation through the mass-center, (2) the perpendicular to the plane of the line just mentioned and the axis of rotation, and (3) the axis of rotation. Symbols $\bar{a}_n$ and $\bar{a}_t$ denote the components of the acceleration of the mass-center along the first two lines respectively; $\bar{a}_n = \bar{r}\omega^2$ and $\bar{a}_t = \bar{r}\alpha$, where $\bar{r}$ = the distance from the mass-center to the axis of rotation, ω = the angular velocity, and α = the angular acceleration of the body. If the angular velocity is constant then $\Sigma F_t = 0$; if the mass-center is in the axis of rotation, then the three summations equal zero.

The angular momentum of the body $= I\omega = Mk^2\omega = (W/g)k^2\omega$; its kinetic energy $= \frac{1}{2} I\omega^2 = \frac{1}{2} Mk^2\omega^2 = \frac{1}{2} (W/g)k^2\omega^2$.

Uniplanar Motion. — It may be regarded as a combined translation and rotation. Motion of the mass-center is given by

$$\Sigma F_x = M\bar{a}_x \quad \text{and} \quad \Sigma F_y = M\bar{a}_y,$$

where ΣF_x and ΣF_y respectively mean the algebraic sums of the components of the external forces along axes x and y in the plane of the motion; and $\bar{a}_x$ and $\bar{a}_y$ = the x and y components of the acceleration of the mass-center. The rotation of the body about the mass-center is given by $T = \bar{I}\alpha$, where T = the torque of the external forces about the perpendicular to the plane of the motion through the mass-center, $\bar{I}$ = the moment of inertia of the body about that line, and α = the angular acceleration of the body. The kinetic energy of the body is given by $\frac{1}{2} M\bar{v}^2 + \frac{1}{2} \bar{I}\omega^2$, where $\bar{v}$ = the velocity of the mass-center and ω = the angular velocity of the body at the instant in question.

Rotation about a Fixed Point. — The principle of motion of the mass-center (§ 3) furnishes three independent equations of motion of the mass-center like

$$\Sigma F_x = M\bar{a}_x,$$

where the symbols have meanings already explained. The axes x, y, and z must not be parallel. The principle of torque and angular momentum furnishes three independent equations like

$$T_x = dh_x/dt,$$

where T_x is the torque of the forces acting on the body about any line x, and h_x is the angular momentum of the body about the same line. That line and the other two may be taken at pleasure; generally lines through the fixed point are simplest.

Any (Solid) Motion. — Motion of the mass-center and rotation about the mass-center are independent (§ 1). We may treat these motions separately; the first as the motion of a particle whose mass equals that of the body under the action of forces like those acting on the body; the second as though the body were fixed at the mass-center.

§ 3. SUMMARY; MOTION OF ANY SYSTEM OF PARTICLES, SOLID OR FLUID. — We will call the system a body but without implying that it is rigid except as noted.

Principle of Motion of Mass-Center. — The motion of the mass-center does not depend at all on the internal forces; it moves just as if all the material of the moving body were concentrated at the mass-center and all the external forces were applied to that (dense) point in their actual directions of course. Thus the component of the acceleration of the mass-center along any line is proportional to the algebraic sum of the components of the external forces along that line directly and to the mass of the body inversely. Or, if systematic units are used $\bar{a}_x = \Sigma F_x \div M$, where $\bar{a}_x$ and ΣF_x = the mentioned component acceleration and algebraic sum respectively and M = mass. The foregoing equation is generally written

$$\Sigma F_x = M\bar{a}_x.$$

Principle of Force and Momentum. — The algebraic sum of the components — along any line — of the external forces acting on the body at any instant equals the rate at which the component (along the same line) of the momentum of the body is changing then. Or

$$\Sigma F_x = \frac{d}{dt} M\bar{v}_x,$$

where $\bar{v}_x$ = the component of the velocity of the mass-center along the line called x. $M\bar{v}_x$ = the x component of the linear momentum of the body.

Principle of Impulse and Momentum. — The algebraic sum of the impulses of the components — along any line — of all the external forces for any interval equals the increment in the component of the momentum of the body along that line for that interval. Or

$$\Sigma \int_{t'}^{t''} F_x \, dt = M\bar{v}_x'' - M\bar{v}_x'.$$

Principle of Torque. — The torque of all the external forces, acting on any body, equals the torque of the resultants of all the forces acting on the particles of the body, all torques being taken about any line. An expression for this latter torque was deduced in Art. 48, the line about which torque was taken being called a z coördinate axis; it is $\Sigma m\,(a_y x - a_x y)$, where m = the mass of any particle, x and y = the coördinates of the particle, and a_x and a_y = the x and y components of its acceleration all at the instant in question. Thus the principle gives

$$T_z = \Sigma m(a_y x - a_x y),$$

where T_z means the torque of all the external forces about the z axis.

Principle of Torque and Angular Momentum. — The torque of the external forces acting on the moving body about any line equals the rate at which the angular momentum of the body about that line is changing (Art. 48). Or

$$T_z = \frac{d}{dt} \Sigma(mv_y x - mv_x y),$$

the torque and angular momentum being taken about a z axis of a coördinate frame. If, in a given case, there is no torque during an interval then the angular momentum about that line remains constant; this is the principle of conservation of angular momentum.

Principle of Angular Impulse and Momentum. — The angular impulse of all the external forces — about any line — for any interval equals the increment in the angular momentum of the body about that line and for that interval. Or

$$\int_{t'}^{t''} T_z \, dt = \Delta\Sigma\, (mv_y x - mv_x y).$$

Principle of Work and Kinetic Energy. — The total work done upon a body by all the external and internal forces during any displacement of the body equals the increment in the kinetic energy of the body during the interval. If the total work is positive then there is a real gain; if negative, then there is loss. If the body is a rigid one, the internal forces do no work; the total work done upon the body by the external forces equals the increment in its kinetic energy.

Principle of Conservation of Energy. — If a body is isolated so that it is beyond the influence of other bodies, then during any change of condition of the body, the amount of its energy remains constant. There may be a transfer of energy from one part of the body to another, but the total gain or loss in one part is exactly equivalent to the loss or gain in the remainder.

D'Alemberts' Principle, not heretofore discussed. — The resultant of all the forces acting on any particle of a body is called the *effective force* for that particle. Its magnitude equals the product of the mass and acceleration of the particle; its direction is the same as that of the acceleration. The group of effective forces for all the particles of a body is called the effective system (of forces) for the body. It should be noted that these forces are fictitious or imaginary, equivalent respectively to the actual forces acting upon the particles. The principle may be stated in two forms: — (a) The external system of forces and the effective system are equivalent, and (b) the external system and the reversed effective system jointly balance, or are in equilibrium.

APPENDIX A

THEORY OF DIMENSIONS OF UNITS

§ 1. Dimensions of Units. — The magnitude of a quantity is expressed by stating how many times larger it is than a standard quantity of the same kind and naming the standard. Thus, we say that a certain distance is 10 miles, meaning that the distance is 10 times as great as the standard distance, the mile. The number expressing the relation between the magnitude of the quantity and the standard (the number 10 in the illustration) is called the *numeric* (or numerical value) of the quantity, and the standard is called the *unit.*

A unit for measuring any kind of quantity may be selected arbitrarily, but it must of course be a quantity of the same kind as the quantity to be measured. Thus, as unit of velocity we might select the velocity of light, as unit of area the area of one face of a silver dollar, etc. Many units in use are arbitrarily chosen, that is without reference to another unit (for example, the bushel and the degree); but generally it is convenient practically to define them with reference to each other. All mechanical and nearly all physical quantities can be defined in terms of three arbitrarily selected units, not dependent on any other units. These are called *fundamental units,* and the others, defined with reference to them, *derived units.* It is customary in works on theoretical mechanics and physics to choose as fundamental the units of

length, mass, and *time;*

but it is sometimes more convenient to take as fundamental the units of

length, force, and *time.*

We give an analysis of derived units with reference to each of these sets of fundamentals, and two tables in which the absolute units are referred to the first set of fundamentals and the gravitational units to the second set. But either set might serve as fundamentals for all absolute and gravitational units.

A statement of the way in which a derived unit depends on the fundamental units involved in it is called a statement of the dimensions of the unit. For example,

$$\frac{\text{one square yard}}{\text{one square foot}} = \frac{(\text{one yard, or three feet})^2}{(\text{one foot})^2} = 9.$$

Thus, a unit of area depends only on the unit of length used, and the unit of area varies as the square of the unit of length. This relation is expressed in the form of a "dimensional equation" as follows:

$$(\text{unit area}) = (\text{unit length})^2,$$

and briefly a unit area is said to be "two dimensions in length." Similarly, a unit volume is said to be three dimensions in length. We proceed to determine "dimensional formulas" for the units of several of the quantities of mechanics. The student should be able to determine formulas (see subsequent tables) for the others.

Velocity. — According to the definition of velocity (Art. 28), a unit velocity is directly proportional to the unit length and inversely to the unit time; hence if $\mathbf{V}$, $\mathbf{L}$, and $\mathbf{T}$ denote units of velocity, length, and time respectively, the dimensional equation is

$$\mathbf{V} = \mathbf{L}/\mathbf{T} = \mathbf{LT}^{-1},$$

and a unit velocity is one dimension in length and minus one in time.

Acceleration. — According to the definition of acceleration (Art. 28), a unit acceleration is proportional directly to the unit velocity and inversely to the unit time; hence if $\mathbf{A}$ denotes unit acceleration, the dimensional equation is

$$\mathbf{A} = \mathbf{V}/\mathbf{T} = \mathbf{L}/\mathbf{T}^2 = \mathbf{LT}^{-2},$$

and a unit acceleration is one dimension in length and minus two in time.

Angular Velocity. — According to the definition of angular velocity (Art. 37), a unit angular velocity is proportional directly to the unit angle and inversely to the unit time; hence if $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$ denote units of angular velocity and angle respectively, the dimensional equation is

$$\boldsymbol{\omega} = \boldsymbol{\theta}/\mathbf{T} \quad \text{or} \quad \boldsymbol{\omega} = \mathbf{T}^{-1},$$

since units of angle (degree, radian, etc.) are independent of the fundamental units. A unit angular velocity is therefore minus one dimension in time.

Angular Acceleration. — According to the definition of angular velocity (Art. 37), a unit angular acceleration is proportional directly to the unit angular velocity and inversely to the unit time; hence if $\boldsymbol{\alpha}$ denotes unit angular acceleration, the dimensional equation is

$$\boldsymbol{\alpha} = \boldsymbol{\omega}/\mathbf{T} = \mathbf{T}^{-2},$$

and a unit angular acceleration is minus two dimensions in time.

Force. — In accordance with the equation of motion of a particle (Art. 31), $R = ma$, or

$$\text{"force} = \text{mass} \times \text{acceleration;"}$$

that is, the unit force is directly proportional to the units of mass and acceleration. Hence if $\mathbf{F}$ and $\mathbf{M}$ denote units of force and mass respectively, the dimensional equation is

$$\mathbf{F} = \mathbf{MA} = \mathbf{LMT}^{-2},$$

ABSOLUTE SYSTEMS

Names of Quantities.	Dimensional Formulas.	Names of Units.	
		C.G.S.	F.P.S.
Length	L	centimeter (cm)	foot (ft)
Mass	M	gram (gr)	pound (lb)
Time	T	second (sec)	second (sec)
Velocity	LT^{-1}	cm/sec ("kine")	ft/sec
Acceleration	LT^{-2}	cm/sec² ("spoud")	ft/sec²
Angular velocity	T^{-1}	rad/sec	rad/sec
Angular acceleration	T^{-2}	rad/sec²	rad/sec²
Force	LMT^{-2}	dyne	poundal (pdl)
Weight	LMT^{-2}	dyne	pdl
Moment of mass	LM	gr-cm	lb-ft
Moment of inertia (body)	L^2M	gr-cm	lb-ft
Moment of force	L^2MT^{-2}	cm-dyne	ft-pdl
Work	L^2MT^{-2}	cm-dyne ("erg")	ft-pdl
Energy	L^2MT^{-2}	cm-dyne ("erg")	ft-pdl
Power	L^2MT^{-3}	erg/sec	ft-pdl/sec
Impulse	LMT^{-1}	dyne-sec ("bole")	pdl-sec
Momentum	LMT^{-1}	dyne-sec ("bole")	pdl-sec
Density	$L^{-3}M$	gr/cm³	lb/ft³
Specific weight	$L^{-2}MT^{-1}$	dyne/cm³	pdl/ft³
Moment of area	L^3	cm³	ft³
Moment of inertia (area)	L^4	cm⁴	ft⁴
Stress	LMT^{-2}	dyne	pdl
Stress intensity	$L^{-1}MT^{-2}$	dyne/cm²	pdl/ft²

and a unit force is one dimension in length, one in mass, and minus two in time.

Mass. — If we regard length, force, and time as fundamental units, then the last equation written as follows is the dimensional equation for a unit mass:

$$\mathbf{M} = \mathbf{FT}^2/\mathbf{L} = \mathbf{L}^{-1}\mathbf{FT}^2,$$

and a unit mass is minus one dimension in length, one in force, and two in time.

Work. — According to the definition of work (Art. 40), the unit of work is directly proportional to the units of force and length; hence if $\mathbf{W}$ denotes unit work, the dimensional equation is

$$\mathbf{W} = \mathbf{LF} = \mathbf{L}^2\mathbf{MT}^{-2},$$

and a unit work is one dimension in length, one in force, or two in length, one in mass, and minus two in time.

Power. — According to the definition of power (Art. 42), a unit of power is proportional directly to the unit work and inversely to the unit time; hence if $\mathbf{P}$ denotes unit of power, the dimensional equation is

$$\mathbf{P} = \mathbf{W}/\mathbf{T} = \mathbf{LFT}^{-1} = \mathbf{L}^2\mathbf{MT}^{-3},$$

and a unit power is one dimension in length and force and minus one in time, or two in length, one in mass, and minus three in time.

GRAVITATION SYSTEMS

Names of Quantities.	Dimensional Formulas.	Names of Units.	
		F.P. (force) S.	M.K. (force) S.
Length	L	foot (ft)	meter (m)
Force	F	pound (lb)	kilogram (kg)
Time	T	second (sec)	second (sec)
Velocity	LT^{-1}	ft/sec	m/sec
Acceleration	LT^{-2}	ft/sec²	m/sec²
Angular velocity	T^{-1}	rad/sec	rad/sec
Angular acceleration	T^{-2}	rad/sec²	rad/sec²
Mass	$L^{-1}FT^2$	"geepound" (glb)	"geekilogram" (gkg)
Weight	F	lb	kg
Moment of mass	FT^2	glb-ft	gkg-m
Moment of inertia	LFT^2	glb-ft²	gkg-m²
Moment of force	LF	ft-lb	kg-m
Work	LF	ft-lb	kg-m
Energy	LF	ft-lb	kg-m
Power	LFT^{-2}	ft-lb/sec	kg-m/sec
Impulse	FT	lb-sec	kg-sec
Momentum	FT	lb-sec	kg-sec
Density	L^2FT^2	glb/ft³	gkg/m³
Specific weight	$L^{-3}F$	lb/ft³	kg/m³
Moment of area	L^3	ft³	m³
Moment of inertia	L^4	ft⁴	m⁴
Stress	F	lb	kg
Stress intensity	$L^{-2}F$	lb/ft²	kg/m²

§ 2. Applications of the Theory of Dimensions. — A knowledge of the theory of dimensions is probably of most value to the beginner as a help to a clear understanding of the different mechanical quantities and the relations between them. The theory is useful practically in other ways, three of which we mention.

(1) *As a test of the accuracy of equations between mechanical quantities.* — Such an equation if *rationally* and *correctly* deduced must be homogeneous, that is the terms in it must be the same in kind. To ascertain whether terms are the same in kind we write the dimensional form of the equation, reduce the terms to their simplest forms and compare; if they are alike, the terms are the same in kind. To illustrate let us consider equation (4), Art. 25,

$$T = \frac{1}{2} wa \left(1 + \frac{a^2}{16 f^2}\right)^{\frac{1}{2}},$$

where T denotes tension (or force), w weight (or force) per unit length, and a and f lengths. Using **L, M,** and **T,** the dimensional form of the equation is

$$\mathbf{LMT^{-2}} = \frac{\mathbf{LMT^{-2}}}{\mathbf{L}} \quad \mathbf{L}\left(\frac{\mathbf{L^2}}{\mathbf{L^2}}\right)^{\frac{1}{2}}.$$

Since the right-hand member reduces to $\mathbf{LMT^{-2}}$, the two members are alike in kind, as they should be. The coefficient ½ and the term 1 were omitted from the dimensional equation because they are independent of **L, M,** and **T.** Using **L, F,** and **T,** the dimensional form of the equation is

$$\mathbf{F} = \frac{\mathbf{F}}{\mathbf{L}}\ \mathbf{L}\left(\frac{\mathbf{L}^2}{\mathbf{L}^2}\right)^{\frac{1}{2}},$$

which is simpler than the first form. Indeed dimensional equations based on **L, F,** and **T** are generally the simpler in the case of formulas with which engineers have to deal, particularly if mass does not appear in the formula.

Showing that an equation is homogeneous does not prove that it is correct, but that it may be correct; showing that an equation is non-homogeneous shows it to be incorrect. Since abstract numbers do not appear in the dimensional form of an equation, the test for homogeneity does not disclose errors in numerical coefficients and terms, nor errors in signs.

(2) *To express a magnitude in different units.* — Obviously the numerical value of a given quantity changes inversely as the magnitude of the unit used; thus a certain distance may be expressed as

10 mi., 17,600 yds., and 52,800 ft.,

and plainly the numerics are respectively as 1, 1760, and 5280, while the corresponding units are as 5280, 1760, and 1.

Let q_1 be the known numerical value of a quantity when expressed in the unit Q_1, and q_2 the numeric (to be found) of the same quantity expressed in the unit Q_2; then

$$q_1/q_2 = Q_2/Q_1, \quad \text{or} \quad q_1 = q_2Q_2/Q_1.$$

The ratio Q_1/Q_2 can be easily computed by substituting for Q_1 and Q_2 their *equivalents* in terms of fundamental units; thus if a, b, and c are the dimensions of Q_1 (and Q_2),

$$Q_1 = k_1\,(\mathbf{L}_1{}^a\mathbf{M}_1{}^b\mathbf{T}_1{}^c) \quad \text{and} \quad Q_2 = k_2(\mathbf{L}_2{}^a\mathbf{M}_2{}^b\mathbf{T}_2{}^c),$$

where $\mathbf{L}_1$, $\mathbf{M}_1$, and $\mathbf{T}_1$ are the *particular* fundamentals for Q_1; $\mathbf{L}_2$, $\mathbf{M}_2$, and $\mathbf{T}_2$ those for Q_2; and k_1 and k_2 numerical coefficients (very often unity). Finally,

$$q_1 = q_2\frac{k_1}{k_2}\left(\frac{\mathbf{L}_1}{\mathbf{L}_2}\right)^a\left(\frac{\mathbf{M}_1}{\mathbf{M}_2}\right)^b\left(\frac{\mathbf{T}_1}{\mathbf{T}_2}\right)^c.$$

As an example, let us determine how many watts in 10 horse-power. Since Q_1 (horse-power) = 550 ft-lb-sec^{-1}, and Q_2 (watt) = 10^7 ergs per sec. = 10^7 cm-dyne-sec^{-1},

$$q_1 = 10\frac{550}{10^7}\frac{\text{ft}}{\text{cm}}\frac{\text{lb}}{\text{dyne}}\frac{\text{sec}^{-1}}{\text{sec}^{-1}} = 10\frac{550}{10^7}(30.48)\,(4.45 \times 10^5)\,(1) = 7640.$$

(3) *To ascertain the unit of the result of a numerical calculation.* — Substitute for the quantities the names of the units in which they are expressed, and then repeat the calculation, treating the names as though they were

algebraic quantities. The reduced answer is the name of the unit of the numerical answer. Thus in the formula for the elongation of a rod due to a pull at each end, Pl/AE (wherein P denotes pull, l length of the rod, A area of cross-section, and E Young's modulus for the material), suppose that $P =$ 10,000 lbs., $l =$ 50 in., $A =$ 0.5 in^2, $E =$ 30,000,000 lbs/in^2; the calculations for elongation and name of unit are

$$\frac{10{,}000 \times 50}{0.5 \times 30{,}000{,}000} = 0.33, \quad \text{and} \quad \frac{\text{lbs} \times \text{in}}{\text{in}^2 \times \text{lbs/in}^2} = \frac{\text{lbs} \times \text{in} \times \text{in}^2}{\text{in}^2 \times \text{lbs}} = \text{in.}$$

APPENDIX B

MOMENT OF INERTIA AND RADIUS OF GYRATION OF PLANE AREAS *

§ 1. Elements of the Subject. — In the subject of Strength of Materials, students of engineering meet certain quantities, formulas for which are like

$$\int dA \cdot x^2,$$

where dA denotes elementary area and x the distance of dA from some line, and the integration is to be extended over some finite area as the cross section of a beam, column, etc. In Chapter V of this book there are integrals like

$$\int dA \cdot x, \quad \text{or} \quad dA_1 \cdot x_1 + dA_2 \cdot x_2 + \cdots$$

Since each of the terms in this summation is the product of an elementary area and its distance from some line (the y coördinate axis), each term (and their sum) has been called "moment of area;" this name is in line with "moment of force" which is a similar product. Likewise, since

$$\int dA \cdot x^2 = (dA_1 \cdot x)x_1 + (dA \cdot x_2)x_2 + \cdots,$$

and since each term of the summation may be regarded as the moment of a moment, the terms (and more particularly their sum) are called "second moments of area." Thus these names for the integrals are quite appropriate. But the names are not in general use; others not so appropriate are more common. The first moment is generally called statical moment, and the second is generally called moment of inertia. This latter name came into use because the integral named is like a certain other integral, $\int dM \cdot r^2$ (Art. 36) which has been previously called (with some reason) moment of inertia. Students should recognize at once that an area has no inertia, and hence in the ordinary sense of the words, no moment of inertia. There is therefore no physical meaning whatsoever attached to moment of inertia of an area. Nevertheless, the term is so firmly established that we will follow the common usage. Thus,

Moment of inertia of an area with respect to or about a straight line (or axis) is the sum of the products obtained by multiplying each elementary

* Writers on Strength of Materials usually refer to works on Mechanics for a discussion of this subject, and for that reason this appendix is included herein.

part of the area by the square of its distance from the line. Engineers have occasion to compute moments of inertia of plane areas only, and about a line which is either in the plane or perpendicular to it. The moment of inertia of an area about a line perpendicular to the plane of the area is called a polar moment of inertia, and the line a polar axis.

The almost universal symbol for moment of inertia is I. A subscript on the symbol indicates the axis to which the moment of inertia refers; thus I_x means moment of inertia about the x axis. Using ρ to stand for distance of the elementary area dA from the axis we have a general formula

$$I = \int dA \cdot \rho^2 = dA_1 \cdot \rho_1^2 + dA_2 \cdot \rho_2^2 + \cdots \qquad (1)$$

In using this formula care must be taken to select the elementary areas so that all parts of each are equally distant from the axis. If this is not done, then the distance ρ is uncertain. This caution is illustrated in the first example following.

Each term in the preceding series is the product of four lengths; hence a moment of inertia of an area is four "dimensions" in length. The numerical value of a moment of inertia of an area is usually computed with the inch as unit length, and the corresponding unit moment of inertia is called a "biquadratic inch," abbreviated in.^{4}

Examples. — 1. It is required to ascertain the value of the moment of inertia of a rectangle whose breadth and height are b and h respectively, about

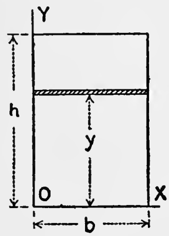


FIG. 1

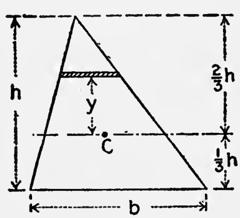


FIG. 2

the base. If we choose for dA a strip parallel to the base (see Fig. 1), then all parts of dA are equally distant from the base. Let y = that distance; then $dA = b\,dy$, and

$$I_x = \int_0^h (b\,dy)y^2 = \frac{b}{3}\Big[y^3\Big]_0^h = \tfrac{1}{3}\,bh^3.$$

If we take a vertical strip for elementary area, then $dA = h\,dx$. Now ρ, the distance of dA from the base, is ambiguous; one would naturally take it to be $\frac{1}{2}\,h$. Trying $\frac{1}{2}\,h$ we get

$$I_x = \int_0^b (h\,dx)\,(\tfrac{1}{2}\,h)^2 = \tfrac{1}{4}\,h^3[x]_0^b = \tfrac{1}{4}\,bh^3,$$

which differs from the first result and is wrong.

2. It is required to ascertain the value of the moment of inertia of a triangle whose base and height are b and h respectively, about a line through its centroid parallel to the base. For (any) elementary area we take a strip as shown in Fig. 2; let y = its distance from the axis, and u = its length. Then $dA = u\,dy = (b/h)\,(\frac{2}{3}h - y)dy$, and hence

$$I_x = \frac{b}{h}\int_{-\frac{1}{3}h}^{+\frac{2}{3}h} (\tfrac{2}{3}h - y)\, y^2\, dy = \tfrac{1}{36}\, bh^3.$$

3. It is required to ascertain the value of the moment of inertia of a circle whose radius is r about a polar axis through its center. Here it is practically necessary to take an elementary area of the second order, as $dx\,dy$ or $\rho\, d\theta \cdot d\rho$ in polar coördinates. We choose the latter (see Fig. 3). Then

$$I_z = \int_0^r \int_0^{2\pi} (\rho\, d\theta \cdot d\rho)\rho^2 = \tfrac{1}{2}\pi r^4.$$

Radius of Gyration. — Since any moment of inertia of an area is four "dimensions" in length, it can be expressed as the product of the area and a

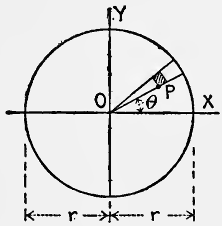


FIG. 3

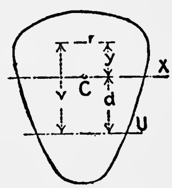


FIG. 4

length squared. It is sometimes convenient to so express it. The length is called the radius of gyration of the area about the line to which the moment of inertia refers; thus if k and I denote radius of gyration and moment of inertia of an area A about the same line, then

$$k^2A = I, \quad \text{or} \quad k = \sqrt{I/A}. \tag{2}$$

This length k was called radius of gyration because of the analogy between it and another length which had been previously called radius of gyration. This other length is defined by the equation $k^2M = I$, where I is the moment of inertia of a body of mass M about some line (Art. 36). For this length, the term radius of gyration is more or less appropriate, but for the first, defined by equation (2), the term is not appropriate, except through the analogy.

It is worth noting that the square of the radius of gyration of an area with respect to any line is the mean of the squares of the distances of all the equal elementary parts of the area from that line. For let ρ_1, ρ_2, etc., be the distances from the elements (dA) to the line, and let n denote their number (infinite); then the mean of the squares is

$$(\rho_1^2 + \rho_2^2 + \rho_3^2 + \ \ldots\ )/n = (\rho_1^2\, dA + \rho_2^2\, dA + \ldots\ )/n\, dA = I/A = k^2.$$

Parallel Axes Theorem. — There is a very simple relation between the moments of inertia (and the radiuses of gyration) of an area with respect to parallel axes, one of which passes through the centroid of the area. Thus the moment of inertia of an area with respect to any line exceeds the moment of inertia with respect to a parallel line through the centroid by an amount equal to the product of the area and the square of the distance between the lines. Or if I and $\bar{I}$ denote the moments of inertia respectively, A the area, and d the distance between the lines, then

$$I = \bar{I} + Ad^2. \tag{3}$$

Proofs of the Theorem. — (1) When the two axes are in the plane area. — Let C (Fig. 4) be the centroid of the area, U and X the two parallel lines or axes, and v and y = the ordinates of the elementary area dA from those lines respectively. Then

$$I = \int dA \cdot v^2 = \int dA (y + d)^2 = \int dA \cdot y^2 + 2\,d \int dA \cdot y + d^2 \int dA.$$

Now $\int dA \cdot y^2 = \bar{I}$; $\int dA \cdot y = 0$, shown presently; and $\int dA = A$; hence, $I = \bar{I} + Ad^2$. $dA \cdot y$ is the statical (or first) moment of dA about CX, and the sum of all such terms as $dA \cdot y$ is the statical moment of the area about CX. Since this line contains the centroid, the statical moment equals zero (Art. 22). (2) When the two lines are perpendicular to the area. — Let O and C (Fig. 5) be the points where the two parallel lines pierce the area, C being the centroid of the area. We take OC for an x coördinate axis, and the y axis as shown. Let x and y = the coördinates of dA; then since $d = OC$, the square of the distance of dA from O is $(x + d)^2 + y^2$. Hence the moment of inertia of the area with respect to the parallel line through O is

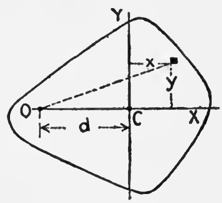


FIG. 5

$$\int dA\,[(x + d)^2 + y^2] = \int dA\,(x^2 + y^2) + 2\,d \int dA \cdot x + d^2 \int dA.$$

Now $x^2 + y^2$ equals the square of the distance from dA to C; hence

$$\int dA (x^2 + y^2) = \bar{I};\ \text{and as already shown} \int dA \cdot x = 0,\ \text{and} \int dA = A.$$

Therefore $I = I + A \cdot d^2$.

If we divide both sides of equation (3) by A we get $I/A = \bar{I}/A + d^2$, or

$$k^2 = \bar{k}^2 + d^2, \tag{4}$$

where k and $\bar{k}$ respectively denote the radiuses of gyration of the area with respect to any line and a parallel line through the centroid and d is the distance between the lines. According to this equation k is always greater than

d; that is, the radius of gyration of an area with respect to a line is always greater than the distance from the line to the centroid of the area. But if the line is outside the area so that d is great compared with the greatest dimension of the area in the direction of d, then $\bar{k}/d$ is small compared to 1 and k equals d approximately. In such a case, the moment of inertia equals Ad^2 approximately.

The parallel axes theorems enable one to simplify many calculations on moment of inertia and often to avoid integrations. Thus having found by integration in example 1 preceding (or otherwise) that the moment of inertia of a rectangle with respect to its base equals $\frac{1}{3}bh^3$ (where b and h are base and height respectively), we can write at once that the moment of inertia with respect to the median parallel to the base is

$$\tfrac{1}{3}bh^3 - bh\left(\tfrac{1}{2}h\right)^2 = \tfrac{1}{12}bh^3.$$

With respect to a line parallel to the base at quarter or three quarters height, the moment of inertia is

$$\tfrac{1}{12}bh^3 + (bh)\left(\tfrac{1}{4}h\right)^2 = \tfrac{7}{48}bh^3.$$

In steel structural design it is often necessary to compute the moment of inertia or radius of gyration of the cross section of a beam or column which is to be "built up" of so-called "structural shapes," about some line of the section. Fig. 6 represents the section of a built-up column consisting of a web plate W, two side plates S, and four Z bars. Manufacturers of such shapes publish "hand books" which include detailed information about the shape sections, — dimensions, area, position of centroid, moments of inertia and radiuses of gyration about several lines through the centroid, etc. Thus for the Z section, $6 \times 3\frac{1}{2} \times \frac{3}{4}$ inches, it is given that its area = 8.63 inches², its moments of inertia respectively about horizontal and vertical axes through its centroid (Fig. 6) = 42.12 and 15.44 inches⁴.

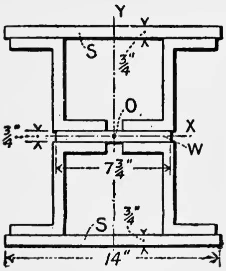


FIG. 6

For another example of the use of the parallel axes theorem, we will compute the moment of inertia of the built-up section represented in Fig. 6 about the x axis. The moment of inertia of the web-plate section (7.75×0.75 inches) is

$$\tfrac{1}{12}\, 7.75 \times 0.75^3 = 0.27 \text{ inches}^4.$$

The moment of inertia of the two side plate sections (14×0.75 inches) about the x axis is

$$2\left[\tfrac{1}{12}\, 14 \times 0.75^3 + (14 \times 0.75)\, 6.75^2\right] = 959.0 \text{ inches}^4,$$

6.75 inches being the distance from the centroid of either rectangle to the x axis. The moment of inertia of the four Z sections about the x axis is

$$4\left[42.12 + (8.63 \times 3.375^2)\right] = 561.7 \text{ inches}^4,$$

3.375 inches being the distance from the centroid of a Z section from the x axis. Hence the moment of inertia of the entire section about the x axis is

$$0.3 + 959.0 + 561.7 = 1521.0 \text{ inches}^4.$$

While the moment of inertia of a composite area with respect to a line can be found by adding the moments of inertia of the component parts about that line, the radius of gyration of the area cannot be found in that way. To find the radius of gyration in such a case, find the moment of inertia first, and then use $k = (I/A)^{\frac{1}{2}}$. For example, let it be required to find the radius of gyration of the cross section of two $6 \times 4 \times 1$ inch angles, placed as shown in Fig. 7 about the line XX through their centroid. We find in a hand book

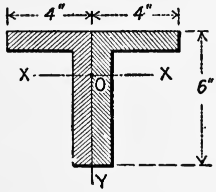


FIG. 7

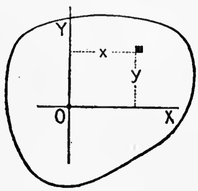


FIG. 8

that the radius of gyration of a single angle about the line XX is 1.85 inches, and that the area of one section is 9 inches². Hence the moment of inertia of the pair about $XX = 2\ (9 \times 1.85^2)$, and the radius of gyration of the pair is

$$\sqrt{\frac{2\ (9 \times 1.85^2)}{2 \times 9}} = 1.85 \text{ inches.}$$

Three Rectangular Axes Theorem. — The moment of inertia of an area with respect to any polar axis (perpendicular to the area) equals the sum of the moments of inertia of the area with respect to any two rectangular axes which intersect the polar axes and lie in the area. If the rectangular axes and the polar axis be regarded as x, y, and z coördinates axes respectively, then the theorem can be written

$$I_z = I_x + I_y. \tag{5}$$

To prove this theorem let x and y = the coördinates of the element dA (Fig. 8). Then the distance of dA from the z axis is $(x^2 + y^2)^{\frac{1}{2}}$, and hence

$$I_z = \int dA\,(x^2 + y^2) = \int dA \cdot x^2 + \int dA \cdot y^2 = I_y + I_x.$$

If equation (5) be divided by A, we get at once

$$k_z^2 = k_x^2 + k_y^2, \tag{6}$$

where k_x, k_y, and k_z denote the radiuses of gyration with respect to the x, y, and z axes respectively.

Graphical Determination of the Moment of Inertia of a Plane Area. — If the area is quite irregular in shape so that it cannot be divided into simple parts whose moments of inertia are known, then the method now to be explained may be resorted to for finding the moment of inertia of the irregular area about any line in its plane. This method is merely graphical integration. Let the area at the left in Fig. 9 be the irregular area and XX the line

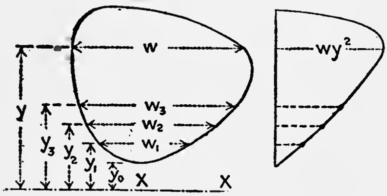


FIG. 9

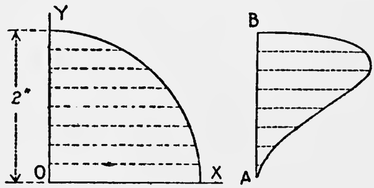


FIG. 10

about which the moment of inertia of the area is required. Let w = the width of the area, parallel to XX, at any point of the figure; and y = the distance of the point or width from XX. Then

$$I = \int (w\,dy)\,y^2 = \int (wy^2)\,dy = \int w'\,dy,$$

where w' is merely an abbreviation for wy^2. Now suppose we multiply several widths w by the square of the corresponding distances y, lay off the products wy^2 to any convenient scale from a perpendicular to XX as shown, and then draw a smooth curve through the ends of the ordinates or distances wy^2 or w'. The area between this smooth curve and the perpendicular equals

$$\int w'\,dy, \text{ and hence it represents } I.$$

Evidently, the modified area, as we may call it, must be interpreted according to some scale as we will explain in connection with

w	y	wy^2	w	y	wy^2
2.00	0	0	1.54	1.25	2.40
1.96	0.25	0.122	1.30	1.50	2.92
1.91	0.50	0.477	0.93	1.75	2.85
1.83	0.75	1.03	0	2.00	0
1.71	1.00	1.71			

A numerical example. — Instead of an irregular figure, we take a regular one so that we can compute its moment of inertia by an exact method also, and thus check the graphical method. We will compute the moment of inertia of a circular quadrant (Fig. 10) of 2-inch radius about one of the straight sides. We have taken nine widths w, see adjoining table. At the right-hand of the figure we have laid off the products wy^2, or w', to the scale 1 inch = 5 inches³.

The new area is 0.25 inches². Since the scale of the quadrant is 1 inch = 2.5 inches, the scale of the new area is 1 square inch = $5 \times 2.5 = 12.5$ inches⁴. Hence, the construction gives $0.25 \times 12.5 = 3.12$ inches⁴ as the moment of inertia desired. The exact formula ($\frac{1}{16}\pi r^4$) gives 3.142 inches⁴.

§ 2. FORMULAS FOR MOMENT OF INERTIA AND RADIUS OF GYRATION FOR SOME SPECIAL CASES. — In the following, I and k are symbols for moment of inertia and radius of gyration respectively. Only a few formulas for k are stated; in any case k can be computed from $\sqrt{I/\text{area}}$.

Rectangle. — Let b = base and h = altitude. About a line through the center parallel to b, $I = \frac{1}{12}\, bh^3$. About a line through the center parallel to h, $I = \frac{1}{12}\, hb^3$. About the base b, $I = \frac{1}{3}\, bh^3$. About the side h, $I = \frac{1}{3}\, hb^3$. About a diagonal, $I = \frac{1}{6}\, b^3h^3/(b^2 + h^2)$. About a line through the center perpendicular to the rectangle, $I = \frac{1}{12}\,(bh^3 + hb^3)$.

Square. — Make $b = h$ in foregoing. The moment of inertia for all axes in the plane of the square and passing through the center is $\frac{1}{12}\, h^4$, where h is the length of one side of the square.

Hollow Rectangle. — Let B and b = outer and inner breadths, and H and h = outer and inner heights. About an axis parallel to B and b and passing through the center, $I = \frac{1}{12}\,(BH^3 - bh^3)$.

Triangle. — Let b = base and h = altitude. About the base, $I = \frac{1}{12}\, bh^3$. About a line through the centroid, and parallel to the base, $I = \frac{1}{36}\, bh^3$. About a line through the vertex and parallel to the base, $I = \frac{1}{4}\, bh^3$.

Regular Polygon. — Let A = area, R = radius of circumscribed circle, r = radius of inscribed circle, and s = length of a side. About any axis through the center and in the plane of the polygon, $I = \frac{1}{24}\, A(6\, R^2 - s^2) = \frac{1}{48}\, A(12\, r^2 + s^2)$. About a line perpendicular to the plane of the polygon passing through the center, I = double the preceding I.

Trapezoid. — Let B = long base, b = short base, and h = altitude. About the long base, $I = \frac{1}{12}\,(B + 3\, b)h^3$. About the short base, $I = \frac{1}{12}\,(3\, B + b)h^3$. About a line through the centroid and parallel to the bases,

$$I = \tfrac{1}{36}\,(B^2 + 4\, Bb + b^2)\, h^3/(B + b).$$

Circle. — Let d = diameter and r = radius. About a diameter, $I = \frac{1}{64}\,\pi d^4 = \frac{1}{4}\,\pi r^4$; $k = \frac{1}{4}\, d = \frac{1}{2}\, r$. About a line through the center and perpendicular to the circle, $I = \frac{1}{32}\,\pi d^4 = \frac{1}{2}\,\pi r^4$; $k = \sqrt{\frac{1}{8}}\, d = \sqrt{\frac{1}{2}}\, r$.

Semicircle. — Let d = diameter and r = radius. About the bounding diameter or about the line of symmetry, $I = \frac{1}{128}\,\pi d^4 = \frac{1}{8}\,\pi r^4$. About a line through the centroid and parallel to the bounding diameter,

$$I = (9\,\pi^2 - 64)\, d^4/1152\,\pi = 0.00686\, d^4 = 0.110\, r^4.$$

Hollow Circle. — Let D and d = outer and inner diameters, and R and r = outer and inner radiuses. About a diameter, $I = \frac{1}{64}\,\pi\,(D^4 - d^4) = \frac{1}{4}\,(R^4 - r^4)$; $k = \frac{1}{4}\,(D^2 + d^2)^{\frac{1}{2}} = \frac{1}{2}\,(R^2 + r^2)^{\frac{1}{2}}$. About a line through the center and normal to the circle, $I = \frac{1}{32}\,\pi\,(D^4 - d^4) = \frac{1}{2}\,\pi\,(R^4 - r^4)$; $k = \sqrt{\frac{1}{8}}\,(D^2 + d^2)^{\frac{1}{2}} = \sqrt{\frac{1}{2}}\,(R^2 + r^2)^{\frac{1}{2}}$.

Circular Segment. — Let A = area of the segment, r = radius of the arc, and 2α = the angle subtended at the center by the arc. About the line of symmetry of the segment,

$$I = \tfrac{1}{4} Ar^2 [1 - \tfrac{2}{3} (\sin^3 \alpha \cos \alpha)/(\alpha - \sin \alpha \cos \alpha)].$$

About the diameter of the circle which is parallel to the straight side of the segment,

$$I = \tfrac{1}{4} Ar^2 [1 + \tfrac{2}{3} (2 \sin^3 \alpha \cos \alpha)/(\alpha - \sin \alpha \cos \alpha)].$$

Circular Sector. — Let A = area of the sector, r = radius of the arc, and 2α = the angle subtended at the center by the arc. About the line of symmetry of the sector,

$$I = \tfrac{1}{4} Ar^2 (1 - \sin \alpha \cos \alpha/\alpha).$$

About a line through the center perpendicular to the line of symmetry and in the plane of the sector,

$$I = \tfrac{1}{4} Ar^2 (1 + \sin \alpha \cos \alpha/\alpha).$$

About a line through the center of the arc and perpendicular to the plane of the sector, $I = \tfrac{1}{2} Ar^2$.

Parabolic Segment bounded by an arc of a parabola and a chord which is perpendicular to the axis of the parabola. Let a = distance from the vertex to the chord and b = length of the chord. About the axis of the parabola $I = \tfrac{4}{15} ab^3$. About the tangent at the vertex of the parabola, $I = \tfrac{4}{7} ba^3$.

Ellipse. — Let $2a$ and $2b$ = lengths of the axes of the ellipse. About the $2a$ axis, $I = \tfrac{1}{4} \pi ab^3$. About the $2b$ axis, $I = \tfrac{1}{4} \pi ba^3$. About a line through the center and perpendicular to the ellipse, $I = \tfrac{1}{4} \pi ab(a^2 + b^2)$.

§ 3. PRODUCT OF INERTIA AND PRINCIPAL AXES. — Preparatory to another matter, we will now discuss briefly a quantity called product of inertia of a plane area with respect to two rectangular coördinate axes in the plane. By this term is meant the sum of all the products obtained by multiplying each elementary area by its coördinates. Thus if dA_1, dA_2, etc., denote (second order) elements of the area, and (x_1y_1), (x_2y_2), etc., denote their coördinates respectively, then the product of inertia is $dA_1x_1y_1 + dA_2x_2y_2 + \ldots$, or

$$J_{xy} = \int dA\,xy, \tag{1}$$

J_{xy} being the symbol which we shall use for product of inertia with respect to axes x and y. It is plain from the definition and expression that a unit product of inertia is four "dimensions" in length. Like moments of inertia we will express products of inertia in biquadratic inches.

Unlike a moment of inertia, a product of inertia may be zero or negative. For example, the product of inertia of the rectangle (Fig. 11) with respect to the axes OX and OY is zero, which may be shown as follows: for every elementary area whose coördinates are (a, b), there is one whose coördinates are $(a, -b)$, and hence the product of inertia of the pair is $dA \cdot ab - dA \cdot ab = 0$; therefore the product of inertia of the entire area is zero.

The product of inertia of the rectangle with respect to the axes $O'X'$ and $O'Y'$ is negative; for one of the coördinates of each element is negative and the other is positive, and hence the product of inertia of each element is negative. Even for different pairs of axes with the same origin the product of inertia of an area may be positive, zero, or negative. Thus, the product of inertia of the triangle (Fig. 11) for the axes shown is zero. If the axes be turned clockwise about O slightly, then the product of inertia is negative; and if turned counter clockwise slightly, then it is positive. As will be shown presently there is always one pair of axes through each point of an area with respect to which the product of inertia is zero, and this pair is of prime importance in certain particulars. If the area has a line of symmetry, then some of the pairs of axes for which the product of inertia of the area is zero can be identified easily; indeed for such an area, the product of inertia is zero with respect to the axis of symmetry and any line (in the area) perpendicular to that axis. For if we think of the elementary areas grouped into pairs symmetrical with respect to the axis of symmetry, then we see that the product of inertia of each pair — and hence that of the entire area — equals zero.

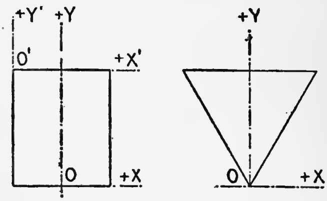


FIG. 11

Parallel Axes Theorem for Products of Inertia. — There is a simple relation between the products of inertia of an area with respect to two parallel sets of coördinate axes, the origin of one set being at the centroid of the area. It is expressed by

$$J = \bar{J} + A\bar{x}\bar{y}, \tag{2}$$

where $\bar{J}$ = the product of inertia about the axis through the centroid, J = the product about the other pair, A = the area, and $(\bar{x}\bar{y})$ = the coördinates of the centroid with respect to the second set of axes. To deduce equation (2), let C (Fig. 12) be the centroid of the area, O any other point, CU and CV one set of axes, OX and OY another parallel set, (u, v) and (x, y) = the coördinates of any elementary area dA with respect to these sets of axes respectively. Then $x = u + \bar{x}$ and $y = v + \bar{y}$; also

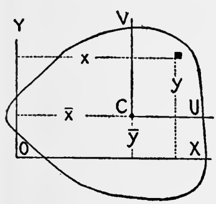


FIG. 12

$$J = \int dA(u + \bar{x})(v + \bar{y}) = \int dA \cdot uv + \bar{x}\bar{y}\int dA + \bar{x}\int v\,dA + \bar{y}\int u\,dA.$$

Now $\int dA \cdot uv = \bar{J}$; $\bar{x}\bar{y}\int dA = A\bar{x}\bar{y}$; $\int v\,dA$ and $\int u\,dA$ = the statical moments of A about CU and CV respectively, and these moments equal zero since these lines contain the centroid (Art. 22). Therefore $J = \bar{J} + A\bar{x}\bar{y}$.

We will now illustrate by determining the product of inertia of the angle section ($7 \times 3\frac{1}{2} \times 1$) shown in Fig. 13 with respect to the axes CX and CY. Imagine the section divided into two rectangles as shown; their areas are $3.5 \times 1 = 3.5$ inches², and $6 \times 1 = 6$ inches². The coördinates of the centroids of these areas with respect to the axes CX and CY are $(0.79, -2.21)$ and $(-0.46, 1.29)$ respectively. Now the product of inertia of each rectangle with respect to axes through its centroid parallel to CX and CY is zero; therefore according to the parallel axes theorem, the product of inertia of the entire section about CX and CY is

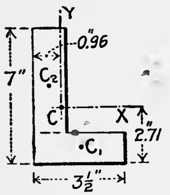


FIG. 13

$$[0 + 3.5\,(0.79)\,(-2.21)] + [0 + 6\,(-0.46)\,(1.29)]$$
$$= -6.11 - 3.56 = -9.67 \text{ inches}^4.$$

Inclined Axis Theorem for Moment of Inertia. — Let OX and OY (Fig. 15) be any two rectangular axes in the area and OU and OV another pair, XOU being any angle α. It is plain from the figure that

$$v = y \cos \alpha - x \sin \alpha, \quad \text{and} \quad u = y \sin \alpha + x \cos \alpha.$$

If these values for u and v be substituted in $I_u = \int dA \cdot v^2$, it will be found on simplifying that

$$I_u = I_x \cos^2 \alpha + I_y \sin^2 \alpha - J_{xy} \sin^2 2\,\alpha \;\; . \; . \; . \qquad (3)$$

With this equation it is possible to find the moment of inertia of an area with respect to an axis through any point in the plane, if the moments and the product of inertia of the area with respect to two rectangular axes through the point are known.

Obviously, the moment of inertia of an area with respect to different lines through the same point are unequal in general. We will show presently that, generally, there is one line for which the moment of inertia is greater, and a line for which the moment of inertia is smaller than for any other line through the point; also that these two lines are at right angles to each other. They are called the *principal axes* of the area for the particular point; and corresponding to those axes we speak of the *principal moments of inertia* and *principal radiuses of gyration* of the area for the point. The condition for a maximum or minimum value of I_u is that $dI_u/d\alpha = 0$. Now from (3),

$$dI/d\alpha = -2\,I_x \sin \alpha \cos \alpha + 2\,I_y \sin \alpha \cos \alpha - 2\,J_{xy} \cos 2\,\alpha.$$

Let us denote by α' the value of α which makes this zero. Then we have

$$-I_x \sin 2\,\alpha' + I_y \sin 2\,\alpha' - 2\,J_{xy} \cos 2\,\alpha' = 0, \text{ or}$$

$$\tan 2\,\alpha' = \frac{2\,J_{xy}}{I_y - I_x}. \qquad (4)$$

In general, this equation gives two values of $2\,\alpha'$ differing by 180 degrees; hence two values of α' differing by 90 degrees. These two values of α' fix two lines (u axes) which are the principal axes for the point under consideration. If $J_{xy} = 0$ and $I_y = I_x$, then equation (4) is ambiguous. For such case, equation (3) shows that $I_u = I_x = I_y$; that is, I_u does not depend on the inclination, and there is no maximum or minimum value for any axis through the point.

The condition expressed by equation (4), for locating the principal axes, can be stated somewhat differently. Referring to Fig. 14, it will be seen that

$$J_{uv} = \int dA \cdot uv = \tfrac{1}{2}(I_x - I_y)\sin 2\alpha + J_{xy}\cos 2\alpha. \tag{5}$$

Apparently this may equal zero for certain values of α; indeed if we equate it to zero we will arrive at equation (4). Hence, *the principal axes are such a pair for which the product of inertia is zero.*

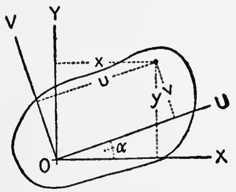


FIG. 14

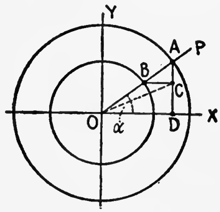


FIG. 15

For an example, we will locate the principal axes of the section shown in Fig. 13, for the point C, it being given that $I_x = 45.37$ and $I_y = 7.53$ inches⁴. In the preceding example, it is shown that $J_{xy} = -9.67$ inches⁴. Therefore according to equation (4),

$$\tan 2\alpha' = 2(-9.67)/(7.53 - 45.32) = 0.5118; \text{ hence}$$
$$2\alpha' = 27^\circ 6' \text{ or } 207^\circ 6', \quad \text{and} \quad \alpha' = 13^\circ 33' \text{ or } 103^\circ 33'.$$

Substituting these two values successively in equation (3), we find as the principal moments of inertia

$$I_1 = 47.70, \quad \text{and} \quad I_2 = 5.20 \text{ inches}^4.$$

There is a simple graphical construction for the radius of gyration of an area about any line through a given point, if the principal axes and radiuses of gyration of the area for that point are known. Let O (Fig. 15) be the point (area not shown), OP the line, OX and OY the principal axes, and k_x and k_y the principal radiuses of gyration respectively. We draw two circles with centers at O and radiuses equal to k_x and k_y; and we call the intersection of these circles with OP, A and B respectively. We draw lines through A and B parallel to OY and OX respectively and call their intersection C. Then OC equals the desired radius of gyration (about OP). For when the axes x and y of equation (3) are principal axes, $J_{xy} = 0$ and

$$I_u = I_x \cos^2\alpha + I_y \sin^2\alpha, \quad \text{or} \quad k_u^2 = k_x^2\cos^2\alpha + k_y^2\sin^2\alpha;$$

but OD (Fig. 15) $= k_x \cos\alpha$ and $CD = k_y \sin\alpha$, and hence $(OD)^2 + (CD)^2$, which equals $(OC)^2$, $= k_x^2 \cos^2\alpha + k_y^2 \sin^2\alpha = k_u^2$, or $OC = k_u$.

§ 4. INERTIA CURVES. — By means of a certain (*inertia*) *circle*, we can locate principal axes, find principal moments of inertia, etc., — in short, do graphically what we did algebraically in the preceding section. We will now show how to draw and use this circle; proof of the method is supplied later.

Let the shaded portion of Fig. 16 be the area under consideration. To

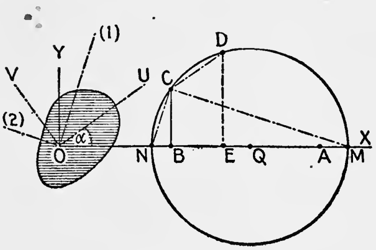


FIG. 16

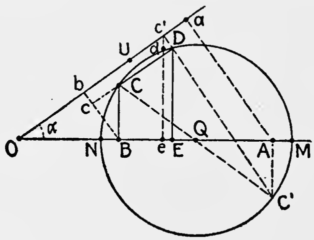


FIG. 17

draw the required circle we must know (as in § 3 to apply equations 3 and 4) the moments of inertia of the area about two rectangular axes through the point under consideration and the product of inertia about those axes; we will suppose these quantities (I_x, I_y, and J_{xy}) to have been determined. First we lay off OA and OB to represent I_x and I_y respectively, according to some convenient scale; draw BC from B parallel to OY, and make $BC = J_{xy}$ (requiring that BC be drawn in the positive or negative y direction according as J_{xy} is positive or negative); we bisect AB in Q, and then draw the circle with center at Q and radius equal to QC. This is the inertia circle of the area for the axes OX and OY. If we letter the intersections of the circle with OX say M and N, then the principal axes for O are parallel to CM and CN, and the corresponding (principal) moments of inertia are equal to OM and ON, according to the scale used. To find the moment of inertia of the area about any line through O as OU: — draw a secant through C parallel to OU, and mark its intersection with the circle D; from D draw a line parallel to the y axis and mark its intersection with the x axis E; then OE equals the desired moment of inertia I_u. Incidentally we may note that ED represents J_{uv}, the product of inertia of the area with respect to OU and OV.

We will prove first that the construction for I_u is correct. Equation (3) can be written

$$I_u = (I_x \cos\alpha - J_{xy} \sin\alpha)\cos\alpha + (I_y \sin\alpha - J_{xy}\cos\alpha)\sin\alpha,$$

and this form suggests the proof. Since (Fig. 17) $I_x = OA$, $I_y = OB$, and $J_{xy} = BC = AC'$,

$$\begin{aligned} I_u &= (OA\cos\alpha - AC'\sin\alpha)\cos\alpha + (OB\sin\alpha - BC\cos\alpha)\sin\alpha \\ &= (Oa - ac')\cos\alpha + (Bb - Bc)\sin\alpha \\ &= Oc'\cos\alpha + (bc = c'D)\sin\alpha = Oe + (dD = eE) = OE. \end{aligned}$$

If we imagine OU to turn about O clockwise, say, CD (drawn parallel to OU) turns about C; and OE (and therefore J_u) increases. The greatest value of I_u (larger principal moment of inertia) obtains when E is at M; then D is at M, and the corresponding principal axis is parallel to CM. If we imagine OU to turn counter clockwise, OD turns and OE (or I_u) gets smaller. The smallest value of I_u (the lesser principal moment of inertia) obtains when E is at N; then D is at N and the corresponding principal axis is parallel to CN.

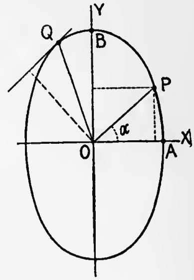


FIG. 18

Inertia Ellipse. — Let OX and OY (Fig. 18) be the principal axes of an area (not shown) for the point O, k_1 and k_2 respectively = the radiuses of gyration of the area with respect to those axes, and $OA = k_2$ and $OB = k_1$; then the ellipse AB is called the inertia ellipse of the area for the point O. Let r = any radius as OP, k = the radius of gyration of the area about OP, and p = the perpendicular from O to either tangent parallel to OP; then, as will be shown presently,

$$k = k_1k_2/r = p. \tag{1}$$

Since the coördinates of P are $r\cos\alpha$ and $r\sin\alpha$, the equation of the ellipse can be written

$$\frac{r^2\cos^2\alpha}{k_2^2} + \frac{r^2\sin^2\alpha}{k_1^2} = 1, \quad \text{or} \quad r^2 = \frac{k_1^2k_2^2}{k_1^2\cos^2\alpha + k_2^2\sin^2\alpha}.$$

It follows from equation (3), § 3, that $k^2 = k_1^2\cos^2\alpha + k_2^2\sin^2\alpha$; hence $r^2 = k_1^2k_2^2/k^2$, or $k = k_1k_2/r$. One of the well-known properties of the ellipse is that the product of any radius and the perpendicular from the center to either tangent parallel to that radius is constant; that is $rp = k_1k_2$; hence $k_1k_2/r = p$.

PROBLEMS

The number in Parentheses following a problem number refers to the article which pertains to that problem.

1–(3). Compound the 80 and 110 lb. forces (Fig. 1) by means of the parallelogram law. (To describe the line of action of the resultant, note where it cuts edges of the square board. Use scales of about 4 ins. and 40 lbs. to the inch.)*

2–(3). Compound the 50 and 60 lb. forces (Fig. 1) by means of the triangle law. (Make the vector diagram separate from the space diagram, and use standard notation.)

3–(3). Compound the 60 and 70 lb. forces (Fig. 1) algebraically. (Specify the direction of the resultant by means of the angles between it and the two given forces.)

4–(3). Compound the 50 and 90 lb. forces (Fig. 1).

5–(3). Resolve the 40 lb. force (Fig. 1) into two components, one parallel to the 70 lb. force and one vertical, by a graphical method.†

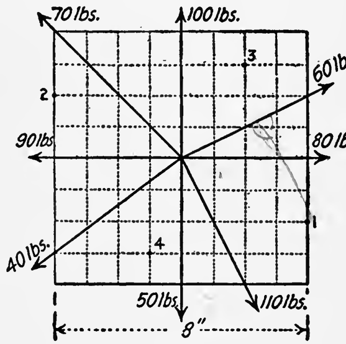


Fig. 1

6–(3). Resolve the 100 lb. force (Fig. 1) into two components, one of which acts in the lower edge of the square and the other through the upper right-hand corner.

–(3). Resolve the 110 lb. force (Fig. 1) into horizontal and vertical components.

7–(3). Draw a square and letter the corners A, B, C, and D consecutively. Imagine a force of 100 lbs. to act in AB and in the direction AB. Resolve it into components acting in the other three sides.

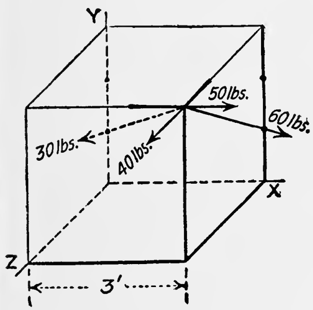


Fig. 2

9–(4). Compound the 40, 50, 60, and 70 lb. forces (Fig. 1) graphically. (Do not draw the force polygon in the space diagram; use standard notation.)

10–(4). Compound the 70, 90, 100, and 110 lb. forces (Fig. 1) algebraically. (Specify the direction of the resultant by means of the angle between it and the horizontal.)

11–(4). Compound the four forces of the cube in Fig. 2.

12–(5). Compute the moment of the 60 lb. force (Fig. 1) about point 1.

* Complete composition requires that the magnitude, line of action, and sense of the resultant be determined.

† Complete resolution requires that the magnitude, line of action, and sense of each component be determined.

13–(5). Compute the moment of the 40 lb. force (Fig. 1) about point 2, making use of the principle of moments.

14. A certain chimney is 150 ft. high and weighs 137,500 lbs. Suppose that it is subjected to a horizontal wind pressure of 54,000 lbs., uniformly distributed along its height. Determine where the line of action of the resultant of the weight and pressure cuts the ground.

15. Fig. 3 represents the cross section of a masonry dam. It weighs 150 lbs/ft³ and the water pressure against it is 112,500 lbs. per foot length of dam. The resultant pressure acts at right angles to the face of the dam and 20 ft. above its base. The center of gravity of the cross section is 11.46 ft. from the face of the dam and 24 ft. above the base. Find where the resultant of the weight and the pressure cuts the base.

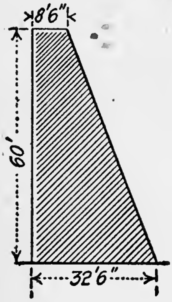


FIG. 3

16–(5). Imagine a clockwise couple of 2 ft-lbs. to act on the square board of Fig. 1. Then compound the couple and the 40 lb. force.

17–(5). Fig. 4 represents a 3 ft. pulley on the end of a shaft; the pulley is subjected to a pull of 100 lbs. applied tangentially as shown. Resolve the force into a force acting through the center of the pulley and a couple.

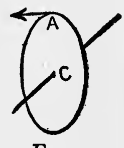


FIG. 4

18–(6). Compound the four forces (wind pressures) represented in Fig. 5. (Be prepared to give the inclination of the resultant and the point where the line of action cuts the floor.)

19–(6). Fig. 6 represents one-half of an arch and certain loads applied to it. $P_1 = 4000$, $P_2 = 5000$, $P_3 = 6000$, and $P_4 = 10,000$

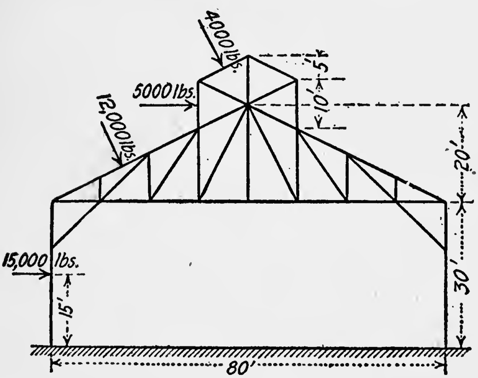


FIG. 5

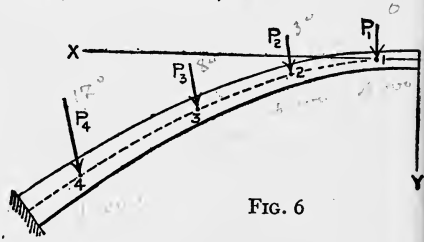


FIG. 6

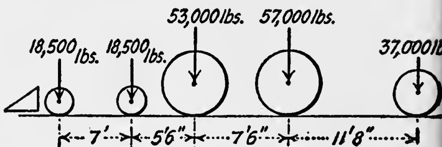


FIG. 7

lbs.; their inclinations are 0°, 3°, 8°, and 12° respectively; the coördinates of points 1, 2, 3, and 4 are (1.6, 0.1), (4.9, 0.7), (8.4, 2.1), and (12.8, 4.8), all in feet. Compound the four load by the second method. (Specify the line of action of the resultant by means of the angle between it and the x axis and the intercept on that axis.)

20–(7). Determine the resultant of the locomotive wheel-loads (Fig. 7).

21–(7). Determine the resultant of the loads described in Prob. 19 algebraically.

22–(8). Compute the moments of each of the forces represented in Fig. 2 about the x, y, and z axes.

23–(8). Determine the resultant of the three couples acting on the 4 ft. cube represented in Fig. 8. (Specify the plane of the resultant by means of the angles which a normal to the plane makes with the edges of the cube.)

24–(9). Determine the resultant of all except the 300 lb. forces (Fig. 8).

25–(10). State what you can about the resultant in the following cases:

(a) A system of coplanar concurrent forces for which $\Sigma F_y = 0$; for which $M\Sigma_a = 0$; for which the force polygon closes.

(b) A system of noncoplanar concurrent forces for which $\Sigma F_x = 0$.

(c) A system of coplanar parallel forces for which $\Sigma F_x = 0$; for which $M\Sigma_a = 0$; for which the force polygon closes.

(d) A system of coplanar nonconcurrent nonparallel forces for which $\Sigma F_x = 0$; for which $\Sigma F_x = \Sigma F_y = 0$; for which $\Sigma M_a = \Sigma M_b = 0$.

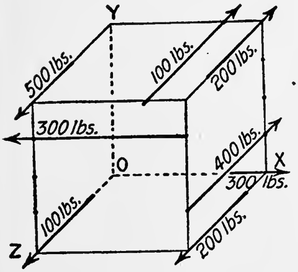


FIG. 8.

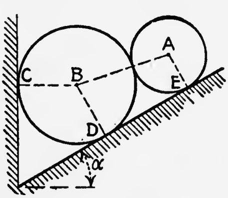


FIG. 9

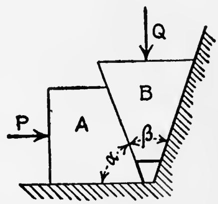


FIG. 10

26–(11). A and B (Fig. 9) are two smooth cylinders supported by two planes as shown. A weighs 200 lbs. and B 100 lbs.; the diameter of A is 6 ft. and of B 10 ft.; $\alpha = 30°$. Determine the pressures on the planes and that between the cylinders.

27–(11). Fig. 10 represents two wedges; $\alpha = 70°$ and $\beta = 40°$. A push P of 1000 lbs. can sustain what load Q if all rubbing surfaces are smooth?

28–(11). The chains AB and AC (Fig. 11) are 5 ft. long. When BC = 8 ft. and the suspended load W = 2 tons, what is the tension on each chain? If the safe pull for each chain is 3 tons, how large may the spread BC be?

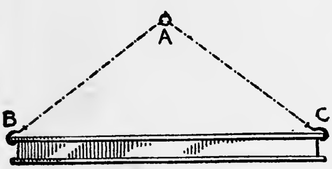


FIG. 11

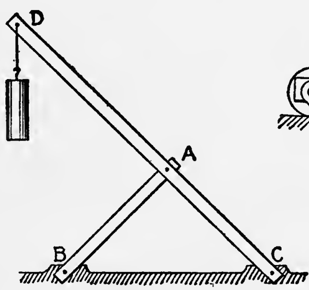


FIG. 12

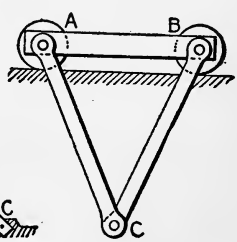


FIG. 13

29–(11). Two bars AB and CD (Fig. 12) are connected by a pin at A and to a floor by pins B and C. BC = 8 ft., AB = AC = 5 ft., and AD = 8 ft. A weight of 100 lbs. is suspended from D. Determine the pin pressures at A, B, and C.

30–(11). A carrier is arranged as shown in Fig. 13. The bar AB connecting the axles of the wheels is 24 ins. long. The bars AC and CB are each 30 ins. long. There is a load of 1200 lbs. at C. Determine the compression in AB and the tension in AC and BC.

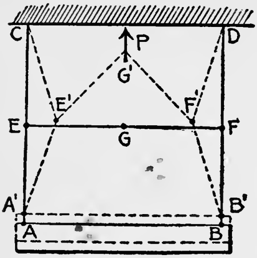


FIG. 14

31–(11). AB (Fig. 14) is a bar suspended from a ceiling by means of vertical ropes AC and BD. The middle points E and F are connected by another rope. $AB = AC = BD = 8$ ft. A vertical force P is applied at the middle G, deflects the ropes as shown by the dotted lines, and raises the bar. How large must P be to support the bar (weighing 1000 lbs.) 6 ins. above its original position?

32–(11). The cylinder of the steam engine (Fig. 15) is 10 ins. in diameter, the crank AB is 5 ins. long, and the connecting rod BC is 15 ins. long. Assume the engine to be stalled in the position shown, $\theta = 60$ degrees, and the steam pressure 150 lbs/in^2. Determine the push on the connecting rod BC and the pressure against the cross-head guide D.

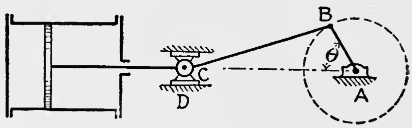


FIG. 15

33–(11). The bell-crank ABC (Fig. 16) is pinned to a wall at A; a cylinder G is suspended by means of a cord from D as shown; $BD = 4$ ins. The cylinder weighs 80 lbs. and is smooth. Determine all the forces which act upon the bell-crank.

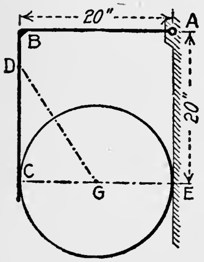


FIG. 16

34–(11). Fig. 17 represents a riveting machine operated by compressed air. It consists of a rigid frame F on which the air cylinder C is mounted; P is the piston; AB is the piston rod pinned to the piston at A so that the rod can be rotated somewhat about A inside of the (hollow) piston; the toggle link BD is pinned to the frame at D; the toggle link BE is pinned to the plunger Q (movable in a vertical guide on the frame) at E; HH are the rivet dies between which the rivet is squeezed. $AB = 19$ ins.; $BD = 13$ ins.; $BE = 10$ ins.; the diameter of the cylinder is 10 ins. Assume the air pressure to be 100 lbs/in^2 and then determine the pressure at the pins D and E, the pressure against the guide, and the pressure on the rivet. (To "lay out" this mechanism begin at D, then fix A, then B, and then E.) Solve the problem when A is advanced 2 ins. from the position shown.

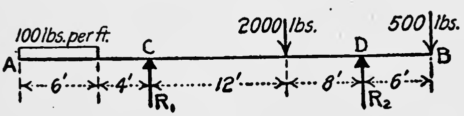


FIG. 18

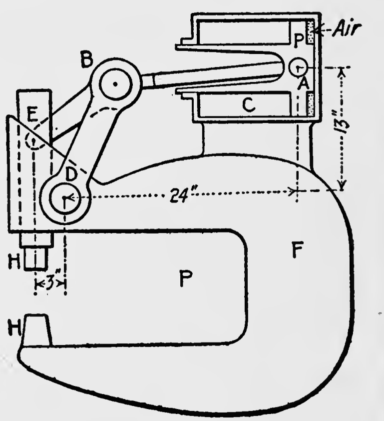


FIG. 17

35–(12). The beam AB (Fig. 18) is supported at C and D, and it sustains three loads as shown. The beam weighs 50 lbs. per lineal feet. Determine each supporting force, or reaction.

36–(12). Fig. 19 represents a shutter dam; AB is the shutter, and CD and CE are braces. The shutter and the braces are pinned together at C; the shutter rests against an inclined stop at A; brace CD is pinned to a bed plate at D; brace CE rests against a bed socket at E. The shutter is 4 ft. wide and its length $AB = 12$ ft. The water pressure is 16,000 lbs., and its "center" is at F, 4 ft. from A. Determine the reactions at D and E due to the water pressure.

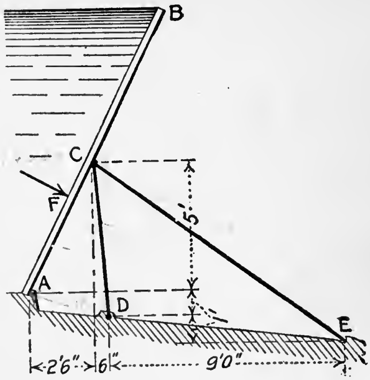


FIG. 19

37–(13). Fig. 20 represents a truss supported by a shelf B on a wall and a horizontal tie A; $AB = 9$ ft. and $BC = 12$ ft. Determine the reactions at A and B due to the loads.

38–(13). AB (Fig. 21) is a beam supported by a rod CD and a pin at A; $AB = 9$ ft. $AC = 3$ ft., $AD = 8$ ft., and $AE = 5$ ft. The beam weighs 400 lbs. and the load, $= W$ 1000 lbs. Determine the pull at C and the pressure at A.

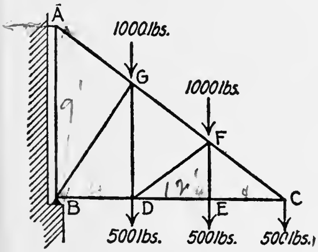


FIG. 20

39–(13). The crane represented in Fig. 22 is supported by two floors as shown. E is a hole in the upper floor and F is a cylindrical socket in the lower floor. The crane weighs 5 tons and its center of gravity is 2 ft. to the left of the axis of the post. Determine the pressures on the floors when the load W is 5000 lbs.

40–(13). A and B (Fig. 23) are two horizontal pegs in a wall; they are 3 and 6 ft. above the floor respectively, and the horizontal distance between them is 4 ft. A smooth straight bar CD, 15 ft. long and weighing 200 lbs., is placed under A and over B with its lower end on the floor, but is not sprung into that position. Determine all the pressures on the bar, due to its own weight.

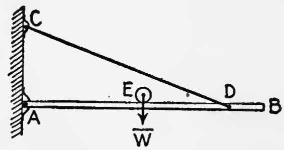


FIG. 21

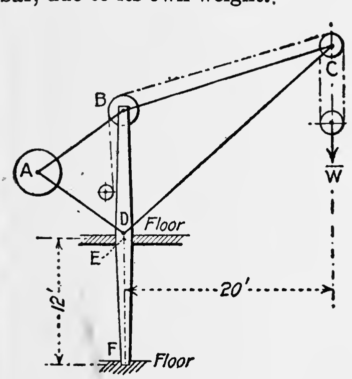


FIG. 22

41. AB (Fig. 24) is a bar 12 ft. long fastened to the floor at A by a pin and it rests at C on a smooth cylinder 4 ft. in diameter. The center of the cylinder is 6 ft. to the right of A and is connected by a horizontal cord to the bar at D. A weight of 100 lbs. is hung on the free end of the bar. What is the pressure between the bar and the cylinder; between the cylinder and the floor; what is the tension in the cord; and what is the pressure exerted by the nip on the bar A? Consider the cylinder and the bar as weightless.

42. AB (Fig. 25) is a bar 20 ins. long, and weighs 10 lbs. It rests on a peg C and against a

smooth wall at A, as shown. What vertical force applied at B will preserve the equilibrium of the bar?

43. If the weight of the bar in Prob. 42 is 12 lbs. and a load weighing 4 lbs. is suspended at B, at what angle must the bar be placed to insure equilibrium?

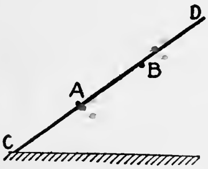


FIG. 23

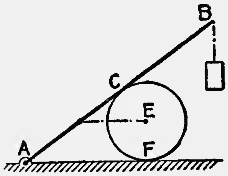


FIG. 24.

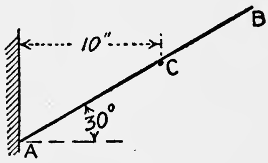


FIG. 25

44–(14). Figs. 26 and 27 are two outline views of a steam shovel; the former represents a dumping and the latter a digging position. A is the "A-frame," B the boom, and D the dipper. The pin P (axis perpendicular to the paper) is seated on the upper half of a "fifth wheel" which permits swinging of the boom about the vertical axis PQ. Two engines are mounted on the boom, — the main engine which operates the hoisting drum, and the thrusting engine which operates the pinion meshing with a rack on the bottom of the dipper handle.

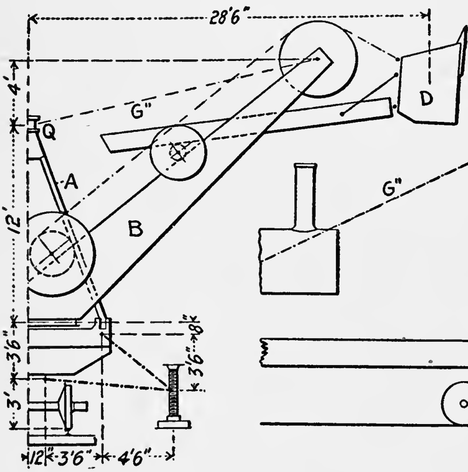


FIG. 26

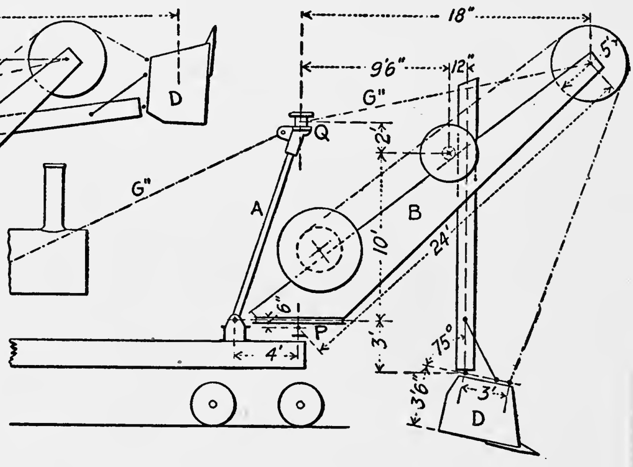


FIG. 27

Many of the parts of a shovel are most severely stressed when the dipper is encountering an unyielding obstruction in the bank. We indicate how some of these stresses may be determined and then ask the student to determine others.

The actual resistance of the bank against the dipper cannot be determined with certainty because the line of action of the resistance is generally unknown. It doubtless depends largely on the direction in which the cutting edge of the dipper tends to move in the bank, determined mainly by the pull of the hoisting rope and the thrust on the dipper handle. Some designers assume that the line of action of the resistance for the digging position shown in Fig. 27 is about along the bottom of the dipper. Making this assumption and analyzing the system of forces acting on the dipper and its handle (resistance of the bank, hoisting pull, weight of dipper and handle, and thrust on the handle) we find that the resistance is about 20,000 lbs. We might proceed now and determine the pressures developed at various points in the structure and mechanism on account of this bank resistance. For instance, analysis of all the external forces acting on the boom, dipper and handle, main and thrusting engines (resistance of bank, pull of front guys G', pin pressure at base of boom, and weight of parts under consideration) shows that the pull of the guys is about 22,000 lbs.

The student should now determine the stress in each leg of the A-frame and that in each back guy G'. (These guys are fastened to the car at points 9 ft. apart. $22\frac{1}{2}$ ft. from the base of the A-frame and on the same level with that base.)

45-(14). Suppose that the shovel is digging as shown in Fig. 27, but with the boom at right angles to the track as shown in Fig. 26. The pull of the front guys is 22,000 lbs. as in the preceding problem. Determine the stress in each leg of the A-frame, and the stress in each back guy.

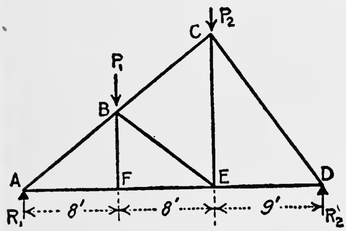


FIG. 28

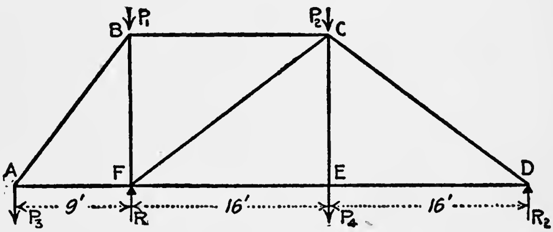


FIG. 29

46-(15). The truss represented in Fig. 28 is supported at A and D; $CE = 12$ ft., $P_1 = 1000$ lbs. and $P_2 = 2000$ lbs. Determine the amount and kind of stress in each member.

47-(15). The truss represented in Fig. 29 is supported at F and D; $BF = CE = 12$ ft., $P_1 = P_2 = 2000$ lbs., and $P_3 = P_4 = 1000$ lbs. Determine the amount and kind of stress in each member.

48-(15). The truss represented in Fig 30 is supported at A and E; each load $P = 1000$ lbs. Determine the amount and kind of stress in each member.

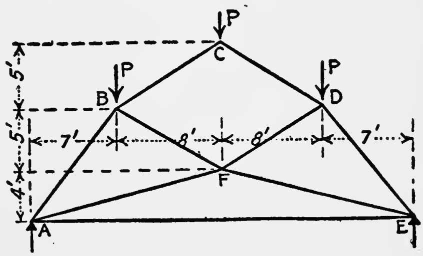


FIG. 30

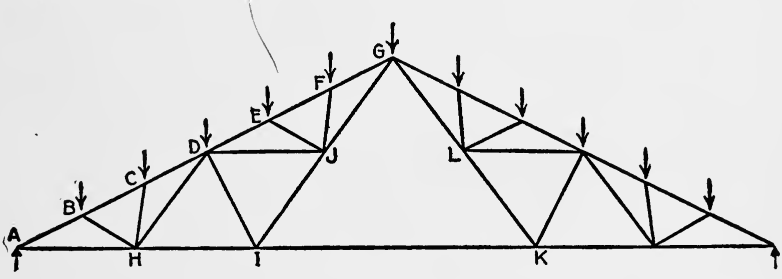


FIG. 31

49-(15). The truss represented in Fig. 31 is supported at each end; span = 80 ft.

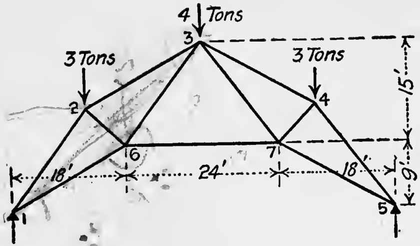


FIG. 32

and rise = 20 ft.; consecutive points on AG are equidistant; DI is perpendicular to AG; H bisects AI, and J bisects GI; each load = 1000 lbs. Determine the amount and kind of stress in IK, and be prepared to describe how to determine the stress in every other member.

50–(16). Solve Prob. 46 graphically.

51–(16). The truss represented in Fig. 32 is supported at each end. The points 1, 2, 3, 6 and the points 3, 4, 5, 7 are at the vertices of parallelograms. Draw a stress diagram for the truss loaded as shown, and make a record of the stresses in the members.

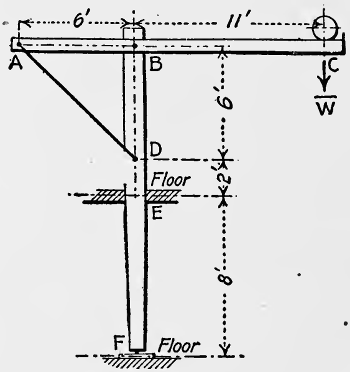


FIG. 33

52–(16). Solve Prob. 47 graphically.

53–(17). Fig. 33 represents a crane consisting of three members, a boom AC, a brace AD, and a post BF. The crane is supported at E and F by two floors. The load W = 5 tons. Determine all forces acting on each member.

54–(17). The crane represented in Fig. 34 consists of a post AB, a boom CD, and braces DE and FG. The crane is supported by sockets at A and B as shown. The boom passes freely through a smooth slot in the post at H so that any reaction existing there will be vertical. The counterweight at D is $\frac{1}{2}$ ton, the load W is $\frac{1}{2}$ ton, and the latter is 21 ft. from the axis of the post. Determine all the forces which act upon each member.

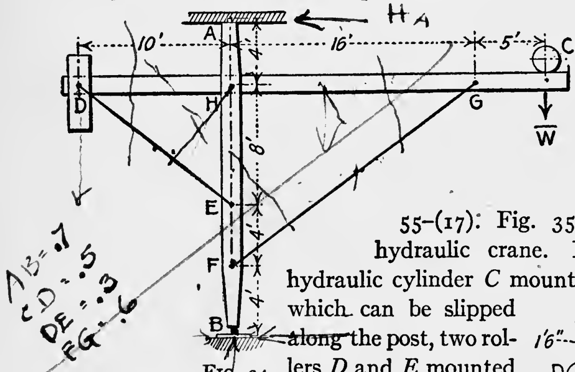


FIG. 34

55–(17). Fig. 35 represents a certain type of hydraulic crane. It consists of a post AB, an hydraulic cylinder C mounted on the post, a large sleeve S which can be slipped along the post, two rollers D and E mounted on the sleeve, a boom EF, and a tie rod FG. When water (under pressure) is admitted to the cylinder, the pistons are pushed upward; the upper one bears against the sleeve, and rolls the entire part $DEFG$ up along the post. Let the load W = 2 tons and suppose that it is 10 ft. out from the axis of the post; then determine all the forces which act upon each pin (D, E, and G).

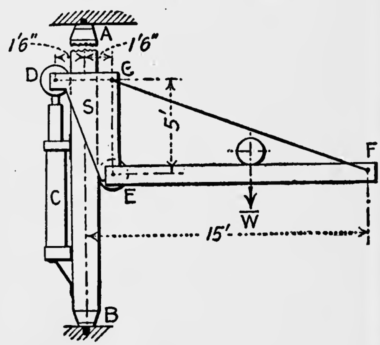


FIG. 35

56–(17). Solve Prob. 54 but take into account the weights of the members of the crane as follows: post $AB = 0.7$ ton, boom $CD = 0.5$ ton, brace $DE = 0.3$ ton, and brace $FG = 0.6$ ton. Middle of boom is 5 ft. 6 ins. from axis of post.

57–(17). Solve Prob. 53 but take into account the weights of the members which are as follows: post $BF = 0.5$ ton, brace $AD = 0.2$ ton, and boom $AC = 0.7$ ton. The boom is 18 ft. long; its center of gravity is 2 ft. 6 ins. from B.

58–(18). Fig. 36 represents a crane supported by a foot-step bearing at B and a collar-bearing at C. B can furnish horizontal and vertical support, and C can furnish

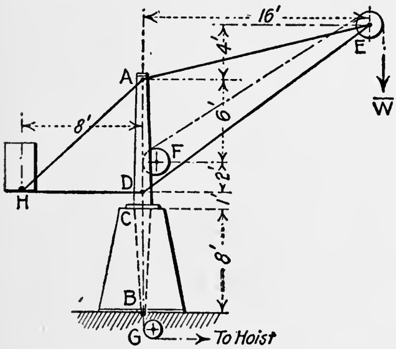


FIG. 36

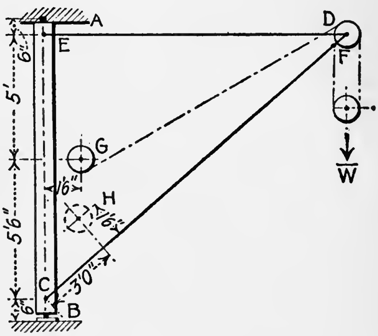


FIG. 37

horizontal support only. The pulleys E and F are 1 ft. in diameter; the hoisting cable enters the post at F, descends through the post, over pulley G, and to the hoist as shown. The counter-weight H is 2 tons and the load 4 tons. Determine all the forces which act upon each member.

59–(18). The crane represented in Fig. 37 consists of a post AB, a boom CD, and a tie rod DE. The pulley at D and the winding drum at G are 1 ft. in diameter. The load W is 1 ton. $DE = 12$ ft. Determine all the forces which act on each member.

60–(18). Imagine the winding drum (Prob. 59) to be mounted in bearings at H (supported by the brace CD) instead of at G. Then solve.

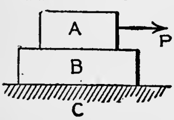


FIG. 38

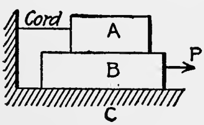


FIG. 39

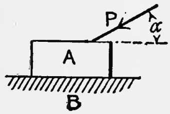


FIG. 40

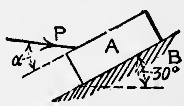


FIG. 41

61–(19). A (Fig. 38) weighs 100 lbs. and B 200 lbs. A, B, and C are very rough. Make separate sketches of A and B and represent all the forces which act on each body when $P = 20$ lbs. (not large enough to produce any slipping).

62–(19). A (Fig. 39) weighs 100 lbs. and B 200 lbs. For A and B, $\mu = \frac{1}{2}$; for B and C, $\mu = \frac{1}{3}$. How large must P be to cause slipping?

63–(19). A (Fig. 40) weighs 100 lbs.; the surfaces in contact are very rough; $P = 50$ lbs., and $\alpha = 20°$. Determine the friction F and the normal pressure N.

64–(19). A (Fig. 40) weighs 100 lbs.; $\alpha = 20°$, and $\mu = 0.6$. How large must P be to start A? How large is F when slipping impends?

65–(19). A (Fig. 40) weighs 100 lbs., $\alpha = 40°$, $\mu = 0.6$, $P = 200$ lbs. Does P move A?

66–(20). Same as Prob. 63 but refer to Fig. 41.

67–(20). Same as Prob. 64 but refer to Fig. 41.

68–(20). Same as Prob. 65 but refer to Fig. 41.

69–(20). Fig. 42 represents a double-wedge device for raising and lowering a heavy load W.* The device consists of wedges A and B and bearing blocks C and D; W = 200,000 lbs. The coefficient of friction is 0.5. How large are the required pushes P to raise the load? How large are the required pulls to lower the load? (First consider C and determine the forces acting upon it.)

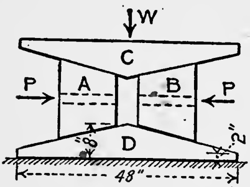


FIG. 42

70–(20). Fig. 43 represents, somewhat conventionalized, an adjusting device used in making the closure (insertion of the last few members) of a large cantilever bridge (Beaver River).† The mechanical elements are a double wedge W, a screw S, and a lever L. The accessories are a head piece H, two struts A, and two wedge-blocks B; they are pin-connected as shown. C and C are two portions of the bridge member to be connected; they are under compression P and pin-bear against the compression blocks. The nut, which bears against the head piece, can be turned by means of the lever, and the screw and wedge raised or lowered. Raising the wedge separates the wedge blocks and parts C and C. Determine the necessary moment (of force) on the lever for raising the wedge against pressures P = 1,235,000 lbs., assuming that the struts A are vertical and the following data: mean diameter of screw = $4\frac{1}{2}$ ins.; pitch of screw = $\frac{1}{4}$ in.; bevel of wedge (each side) = 1 in 10; mean radius of nut where it bears on the head piece = 9 ins.; coefficient of friction for all rubbing surfaces = $\frac{1}{4}$. (Consider first a wedge-block, and determine all the forces which act upon it.)

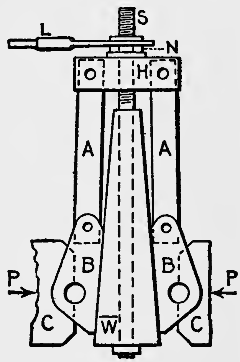


FIG. 43

71–(20). Fig. 44 represents a screw toggle used in the erection of a steel arch (Niagara Falls and Clifton Bridge).‡ It consists of four multiple links pinned together as shown, a right-and-left screw S with nuts N, and a lever L. The toggle is supported by the anchor rod R and brace B. The "pulling end" of the toggle was connected to the arch under construction, supplying the supporting force P. Assume mean diameter of screw = 2 ins., pitch = $\frac{1}{5}$ in., coefficient of friction = 0.3; also that now the diagonal MM = 16 ft., 4 ins., and the diagonal NN = 4 ft., 4 ins. Determine the couple on the lever which will shorten NN; which will lengthen NN.

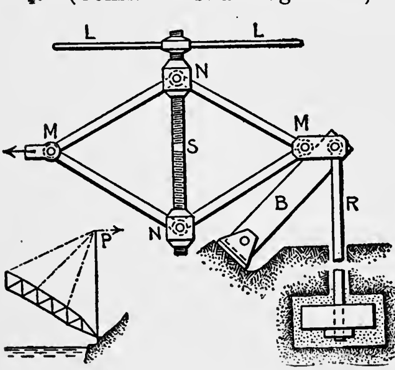


FIG. 44

72–(20). Solve Prob. 34, taking into account the friction at all rubbing surfaces (pins, piston, and guide). Pins A and E are 2 ins. and pins B and D are 3 ins. in

* *Engineering News*, July 15, 1911. † *Engineering Record*, June 10, 1911.

‡ Skinner's *Details of Bridge Construction.*

diameter. The coefficient of friction is $\frac{1}{4}$. (Solve graphically and make drawing of riveter full size or larger.)

73–(20). Fig. 45 represents a band-brake. The diameter of the wheel is 1 ft., 8 ins., the angle of lap = 255°, P = 60 lbs., and the coefficient of friction is $\frac{1}{5}$; the wheel is turning clockwise. Compute the frictional moment and the pull on the pins A and B. Solve for the case when the wheel is turning in the other direction.

74–(21). Fig. 46 represents a crank-arm for a shaft, by plan and elevation — dotted lines to be disregarded. Locate the center of gravity of the arm.

75–(21). Solve Prob. 74 but change width at thin end as shown by dotted lines. (See Obelisk, Art. 24.)

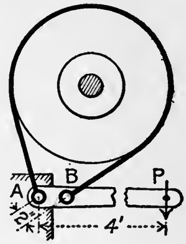


FIG. 45

76–(21). Fig. 47 represents a connecting rod for a steam engine by plan and elevation. The rod is $1\frac{1}{2}$ ins. thick except as noted. Determine the distances of the center of gravity from the center of each hole.

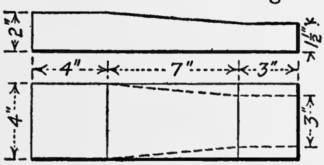


FIG. 46

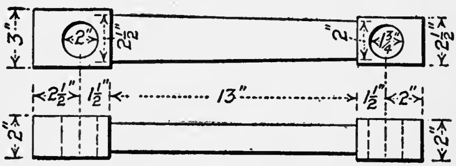


FIG. 47

77–(21). Fig. 48 represents a thin plate into which holes were punched at A and B, and the pieces glued on at C and D respectively. Area of hole A = 4 ins.^{2}; that of B = 2 ins^2. Locate the center of gravity of the modified plate.

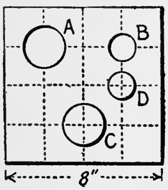


FIG. 48

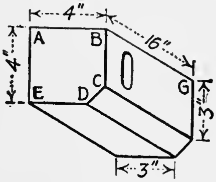


FIG. 49

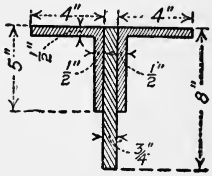


FIG. 50

78–(21). Fig. 49 represents a five-sided prism made from a rectangular prism by bevelling one edge as shown. There is a cylindrical hole 2 ins. in diameter in the piece, its axis being parallel to the line AB, 4 ins. from the face $ABCDE$, and $1\frac{1}{2}$ in. below the face ABG. Find the center of gravity of the modified prism.

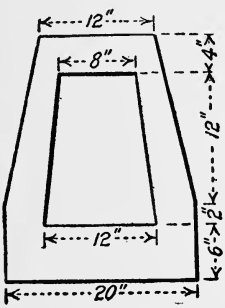


FIG. 51

79–(22). Fig. 50 is the cross section of a steel beam "built-up" of two angles $5 \times 4 \times \frac{1}{2}$ ins. and a plate $8 \times \frac{3}{4}$ in. The centroid of each angle is 1.57 ins. from the back of the shorter leg. Determine the position of the centroid of the entire section.

80–(22). Fig. 51 represents the cross section of a machine part. Determine the position of the centroid of the cross section.

81–(23). Prove that the distance of the centroid of a triangle from its base equals one-third the altitude.

82–(23). Prove that the distance from the centroid to the base of a paraboloid of revolution formed by revolving a parabola about its axis equals one-third the altitude a (see Fig. 168, page 94).

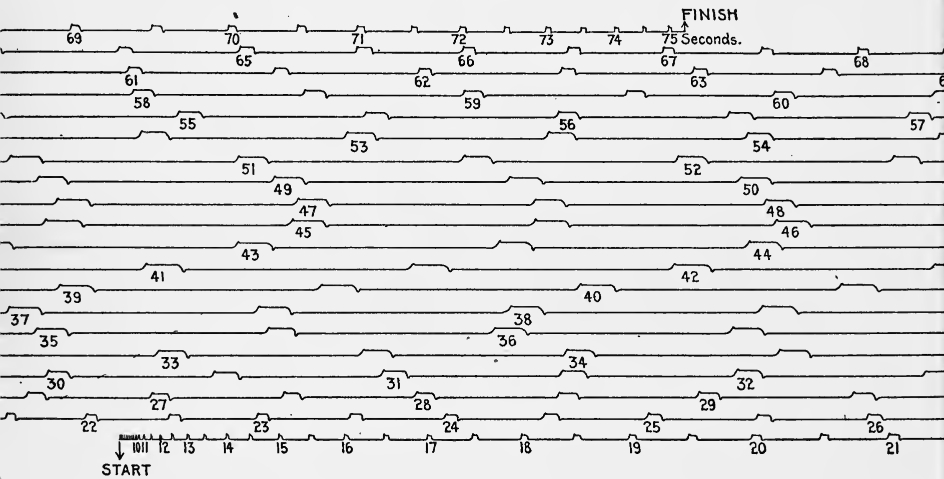
83–(24). The cross section of a certain cylinder is elliptical; the axes of the ellipse $= 2a$ and $2b$, and the length of the cylinder $= l$. Show that the radius of gyration of the cylinder with respect to the $2b$ axis of the middle section $= (\frac{1}{4}a^2 + \frac{1}{12}l^2)^{\frac{1}{2}}$.

84–(25). A cord is supported at two points on the same level 30 ft. apart, and its lowest point is 8 ft. below the level of the supports. If the load is 20 lbs. per horizontal ft., what are the tensions at the supports and at the lowest point?

85–(26). A cable is to be suspended between two points at the same level 200 ft. apart; the sag is to be 80 feet. Determine the length of the cable.

86–(27). A rope 100 feet long is suspended from two points A and B at the same level 60 ft. apart. A body weighing 1000 lbs. is suspended from a point C x ft. distant from A. Determine the tension in AC when $x = 10$, 20, 30, 40, 50, 60, 70, 80, and 90 ft. Make a graph showing how the tension varies with x.

87–(28). Fig. 52 is a chronographic record of the launching of the U.S.S. California (*Transactions of Naval Architects and Marine Engineers*, Vol. 12). Determine



1 Revolution of Chronograph Cylinder to 40ft. Travel of Ship
(Time between Notches = $\frac{1}{2}$ Second)

FIG. 52

the velocity of the ship at the twentieth second in the following three ways: first, from the average velocities for at least four intervals after the instant; second, from the average velocities for at least four intervals before the instant; third, from the average velocities for the half-seconds immediately before and after the instant.

88–(28). The following velocities (feet per second) were computed from the chronographic record (Fig. 52) by taking the mean of the average velocities for the half-seconds immediately preceding and following the instants or times listed below.

$t =$	15	16	17	18	19	20	21	22	23	24
$v =$	2.50	3.00	3.55	4.20	4.80	5.45	6.10	6.75	7.45	8.15

Compute the acceleration for $t = 16$ secs.

89–(28). Reduce a sprint of 100 yds. in 10 secs. to miles per hour. Compare the retardation of a train at 4 mi/hr/sec with the retardation of gravity on a ball thrown vertically upward.

90–(28). A point P moves in a straight line so that $s = 2t^3 - 5t^2$, where s (in feet) equals the distance of P from a fixed origin in the path at any time t (in minutes). Determine the velocity and acceleration when $t = 1$ min.; when $t = 2$ mins. Interpret the negative signs.

91–(28). A certain point P of a mechanism is made to move in a straight line by means of a crank in such a way that $s = 3 \cos 2\theta$, where $s =$ the distance of P from a fixed origin in the path of P and $\theta =$ the angle which the crank makes with a fixed line of reference. The crank rotates uniformly at 100 rev/min. Determine position, velocity, and acceleration of P when $\theta = 60°$. Interpret signs of the results.

92–(28). In a certain "gunnery experiment" the shot was fired through screens placed 150 ft. apart. The times (in seconds) of piercing were observed with the following results:

screen	1	2	3	4	5	6	7
time	0	0.0666	0.1343	0.2031	0.2729	0.3439	0.4161

Determine the velocity at the fourth screen.

93–(28). A point P moves in a straight line so that $a = 4 - 2t$, where a is in feet per minute per minute and t in minutes. When $t = 0$, $v = 0$ and $s = 0$. Determine general formulas for v and s. What are v and s when $t = 4$? when $t = 5$?

94–(28). A certain electric train can get up full speed of 24 mi/hr in a distance of 150 ft., and can stop from full speed in a distance of 100 ft. What is the shortest time in minutes in which the train can make a run between two stations 650 ft. apart, the train starting from one station and coming to full stop at the other? (Assume that the starting and stopping are accomplished uniformly with respect to time.)

95–(28). A certain train can be retarded at a rate of 4 mi/hr/sec by braking. Determine the times (in seconds) and the distances (in feet) in which the train can be stopped from 10, 20, 30, and 40 mi/hr. (Assume that the retardation is the same at all speeds.)

96–(29). Draw the distance-time and velocity-time graphs for the interval from 15 to 24 secs. of the launching mentioned in Prob. 87, and determine the velocity and acceleration at the twentieth second from the graphs.

97–(29). Fig. 53 shows the acceleration-time graph for a certain rectilinear motion. When $t = 0$, v and $s = 0$. Construct the v-t and s-t graphs.

98–(29). Make a sketch of the velocity-time graph for the train-run described in Prob. 94, calling the lengths of the three periods t_1, t_2, and t_3 respectively. Then use the principle that "area under the curve" represents distance travelled to find values of t_1, t_2, and t_3, and finally the time for the entire run.

99–(30). The period of a certain simple harmonic motion is 8 secs., and the amplitude is 6 ins. What is the maximum velocity? the maximum acceleration? For the motion from one extreme point in the path to the center, what is the average velocity? the average acceleration?

100–(30). Four particles, Q_1, Q_2, Q_3, and Q_4, are describing simple harmonic motions in AB (Fig. 54); the period of each motion is 8 secs. At a certain instant the four

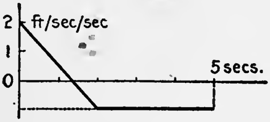


Fig. 53

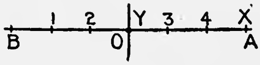


Fig. 54

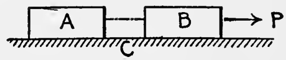


Fig. 55

particles are at points 1, 2, 3, and 4 respectively; Q_1 and Q_3 are moving toward the right and Q_2 and Q_4 are moving toward the left. Write out the expressions for the x coördinates of the moving points t secs. after the instant mentioned. ($AB = 12$ ins., and is divided into sixths by the points.)

101–(31). A (Fig. 55) weighs 200 lbs., B weighs 100 lbs.; the coefficient of friction "under" A is $\frac{1}{5}$, that under B is $\frac{1}{4}$; $P = 300$ lbs. Determine the acceleration of A and B, and the tension in the rope connecting them.

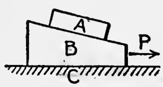


Fig. 56

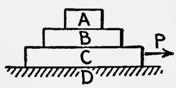


Fig. 57

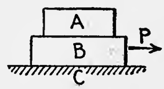


Fig. 58

102–(31). Suppose that the supporting surface in the preceding problem is not horizontal but inclined at 30 degrees to the horizontal. Then solve.

103–(31). A (Fig. 56) weighs 50 lbs. and B weighs 100 lbs.; the pull P gives A and B an acceleration of 2 ft/sec/sec. Determine the magnitude and direction (referred to the horizontal) of the pressure between A and B.

104–(31). Two bodies are connected somewhat as two cars, and are placed on a plane inclined at 30 degrees to the horizontal. The lower one weighs 600 lbs. and is smooth, that is, there is no resistance to its sliding on the plane. The upper one weighs 1000 lbs., and the coefficient friction under it is $\frac{1}{10}$. With what acceleration will the bodies slide down when released? Will there be tension or pressure at the connection? What is its value?

105–(31). The weights of A, B, and C (Fig. 57) are 50, 100, and 200 lbs. respectively. Contacts between A, B, and C are very rough; between C and D very smooth; $P = 100$ lbs. Determine the forces which the bodies exert upon each other. Sketch each body separately, showing the forces acting on it.

106–(31). A (Fig. 58) weighs 100 lbs., and B weighs 200 lbs. The coefficient of kinetic friction under B is $\frac{1}{5}$; the coefficient of static friction under A is $\frac{1}{10}$. When $P = 75$ lbs., will A slip? How great is the friction under A? How large a force P would just make A slip?

107–(31). A (Fig. 56) weighs 50 lbs., and B weighs 100 lbs. C is perfectly smooth; the coefficient of static friction "between" A and B is $\frac{1}{5}$; the angle between the top of B and the horizontal is 25 degrees. How great may P be without making A slip on B?

108-(31). A (Fig. 59) weighs 100 lbs., and B weighs 50 lbs. The coefficient of friction under A is $\frac{1}{5}$. Neglect the inertia of the pulley and the friction at its axle, and find the acceleration of A and B, and the tension in the cord. (The resultant of the three forces acting on A is $T-20$, where $T =$ tension; and the resultant of the two forces acting on B is $50 - T$. Now write the equations of motion, $R = (W/g)a$, for A and B, and solve them simultaneously for a and T.)

109-(31). Show that the acceleration of the suspended bodies and the tension in the cord of the Atwood machine (Fig. 60) are respectively

$$a = g\,\frac{(W_2 - W_1)}{W_1 + W_2} \quad \text{and} \quad T = \frac{(2\,W_1W_2)}{W_1 + W_2}$$

when the inertia of the pulley and the axle friction are negligible.

110-(31). Fig. 61 represents a simple engine, without connecting rod. Stroke = 18 ins., speed = 150 r.p.m. Piston and rod weigh 120 lbs. When $x = 3$ ins., steam pressure = 2000 lbs. Determine the pressure of crank pin P on the piston rod. When the piston is advanced 6 ins. beyond the position shown ($x = -3$ ins.), the steam pressure is still 2000 lbs. Determine the pressure of the pin on the piston rod for this position.

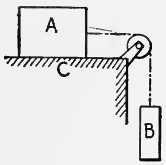


FIG. 59

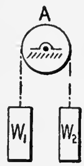


FIG. 60

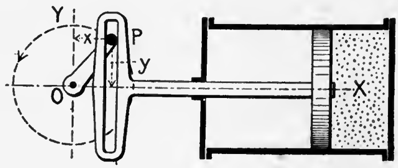


FIG. 61

111-(31). Suppose that Fig. 61 represents an air compressor, steam being changed to air and the crank turning clockwise. Determine the pressures of the crank pin for the two positions mentioned in the preceding problem ($x = +3$ ins. and $x = -3$ ins.).

112-(31). Fig. 62 represents, in principle, a certain "throw" testing machine for subjecting a metal specimen to rapid changes of direct stress (tension and compression). S is the specimen, firmly screwed into two bosses M and N. W is a weight firmly fastened to the lower boss. The parts named can be oscillated in the vertical guides G by means of an ordinary crank-connecting rod mechanism (OP-PC). When the machine is not running, the specimen is subjected to a tension equal to the weight of N and W. When the machine is running, the stress on the specimen changes continuously.* Let $OP = \frac{1}{2}$ in., $PC = 9$ ins., weight of N and $W = 25$ lbs., and speed of crank = 2000 rev/min. Determine the stress on the specimen at each end of a stroke or oscillation, and at the middle of the stroke.

FIG. 62

113-(31). Take data except speed as in preceding problem. Determine the speed

* For detailed description see *Phil. Trans. Roy. Soc.*, Ser. A, Vol. 199 (1902).

which would make the stress on the specimen equal to zero at the upper end of the stroke. What would the stress be at the lower end at that speed?

114–(32). A point P starts at A (Fig. 63), and moves in the circle as indicated traversing distance s so that $s = 2\,t^2$, where t is time after starting in seconds and s is in feet; radius $OA = 20$ ft. Draw the hodograph for the first 3 secs. Then determine the average accelerations for the intervals 1 to 3, 1.5 to 3, 2 to 3, 2.5 to 3. Next determine the magnitude and direction of the acceleration when $t = 3$ from these average accelerations.

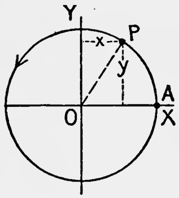


FIG. 63

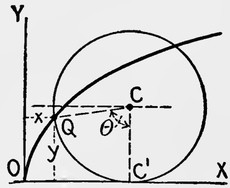


FIG. 64

115–(33). Determine v_x, v_y, v, a_x, a_y, a_t, a_n, and a for the motion described in Prob. 114 and $t = 3$ secs.

116–(33). The point Q (Fig. 64) on the rim of a wheel rolling in a straight line describes a curve known as cycloid. Let v' = velocity of the center of the wheel C, a' = the acceleration of C, and R = radius of the wheel. Find formulas for the x and y components of the velocity and acceleration of Q when in the position shown. (Let s = the abscissa of C, and x and y = the coördinates of Q. Then $x = s - R \sin\theta$, and $y = R\,(1 - \cos\theta)$; also $s = R\theta$.)

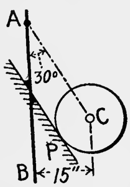


FIG. 65

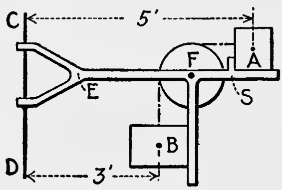


FIG. 66

117–(33). A point Q describes a simple harmonic motion; the frequency = 100 (to and fro) oscillations per minute and the amplitude = 3 ft. Determine the average accelerations of Q for the following distances traversed: first 6 ins. from one end of its path; second 6 ins.; third 6 ins.; and first 18 ins.

118–(33). A 16-inch gun can give a projectile weighing 2400 lbs., a muzzle velocity of 1465 mi/hr, and at an elevation of 15 degrees can throw it approximately 9 mi. Compute the range for the velocity and elevation stated neglecting air resistance, and compare with the actual range.

119–(34). A cylinder C (Fig. 65) is suspended by a cord and rests against a smooth inclined plane P as shown. The cylinder weighs 20 lbs.; its diameter is one foot. The plane is rotated at 30 rev/min about the vertical axis AB. Determine the tension in the cord and the pressure against the plane.

120–(34). CD (Fig. 66) is a vertical axis about which E can be rotated. A is a body resting on E, and B is suspended by means of a cord fastened to A as shown. A weighs 10 lbs. and B weighs 20 lbs. Suppose that E makes 30 rev/min; then compute the pressure at the stop S. The centers of A and B are 5 and 3 ft. from CD respectively. (Neglect friction under A, at B and the pulley axle.)

121–(34). Suppose that A and B in Prob. 120 are rough, the coefficients of static friction being $\frac{1}{4}$ for each. What rate of rotation would lift B?

122–(34). T (Fig. 67) is a horizontal whirling table. A and B are spheres connected by an elastic cord, the tension in which is 30 lbs. when the table is at rest. A weighs 10 lbs. and B weighs 40 lbs. What are the pressures of the stops S' and S'' against the spheres when the table is rotated about CD at 20 rev/min?

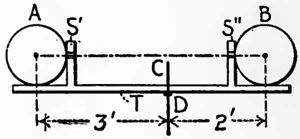


FIG. 67

123–(35). Suppose that the floor of the car and A (Fig. 283, Art. 35) are very rough so that A will not slip on the car; then ascertain how great an acceleration of the car would result in tipping of A.

124–(35). Suppose that the coefficient of friction in Prob. 123 is $\frac{1}{3}$. If the applied push on the car is gradually increased, thus increasing the acceleration gradually, will A slip or tip eventually?

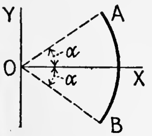


FIG. 68

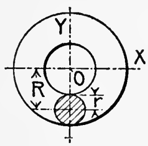


FIG. 69

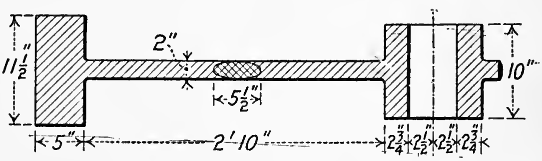


FIG. 70

125–(35). The Scotch cross-head (Fig. 61) described in Prob. 110 presses against the stuffing box and on the cylinder by reason of the weight of the cross-head and the pressure of the crank-pin on it. Suppose that the center of gravity of the cross-head is 15 ins. from the center of the slot, the center of the piston is 24 ins. from the same point, and the center of the stuffing box is 13 ins. from O. Determine the pressures mentioned when the circumstances are as in Prob. 110 (steam pressure = 2000 lbs., etc.).

126–(36). Show that the moment of inertia of the slender wire AB (Fig. 68) about the x-axis is $\frac{1}{2}\, Mr^2[1 - (\sin\alpha\cos\alpha)/\alpha]$, where M = mass of the wire.

127–(36). Show that the moment of inertia of a right circular cone about its axis is $\frac{3}{10}\, Mr^2$, where M = the mass of the cone and r = the radius of its base.

128–(36). Show that the moment of inertia of the ring or torus (Fig. 69) about the z-axis is $M\,(R^2 + \frac{3}{4}\,r^2)$, where M = the mass of the ring.

129–(36). The length of a homogeneous right elliptic prism is l, and the semi-axes of its cross section are a and b. Prove that the radius of gyration of the prism with respect to a line through its center of gravity parallel to the axis b is $(\frac{1}{4}\,a^2 + \frac{1}{12}\,l^2)^{\frac{1}{2}}$.

130–(36). Fig. 70 is a section of a cast-iron flywheel; there are six spokes. The cross section of each spoke is elliptical, the axes of the ellipse being 2 inches and $5\frac{1}{2}$ ins. long. Compute the moments of inertia of rim, spokes, and hub with respect to the axis of the wheel; also the radius of gyration of the wheel about that axis.

131–(37). In order to produce a tension of 100 lbs. in the cord of Ex. 2, Art. 37, how heavy must the suspended body be?

132–(37). A, B, and C (Fig. 71) weigh 100 lbs., 30 lbs., and 34.4 lbs., respectively. The diameter of C = 2 ft. 3 ins., and the radius of gyration of C about the axis of rotation = 1 ft.; ϕ = 30 deg. Friction under A, when the system is moving, = 10 lbs. Determine the acceleration of A, B, and C, and the tensions, the system having started without initial velocity. (Neglect axle friction.)

133–(37). A, B, and C (Fig. 72) weigh 50 lbs., 100 lbs., and 150 lbs. respectively. C is a solid disk of cast iron 16 ins. in diameter. Determine the acceleration of A, B, C, and also the pulls of the cord on A and B. (Neglect axle friction.)

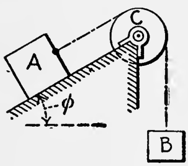


FIG. 71

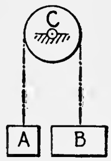


FIG. 72

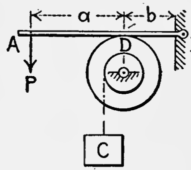


FIG. 73

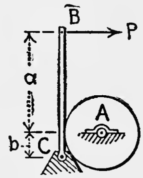


FIG. 74

134–(37). AB (Fig. 73) is a brake for regulating the descent of the suspended body C. C weighs 1000 lbs., the drum 2000 lbs., the diameter of the drum = 12 ft., that of the brake wheel = 14 ft., a = 4 ft., b = 6 ins., and the radius of gyration of the entire rotating system about the axis of rotation = 4 ft. When P = 100 lbs. and the coefficient of brake friction is $\frac{1}{4}$, what is the acceleration of C? (Neglect axle friction.)

135–(37). The wheel A (Fig. 74) is a solid cylinder weighing 1000 lbs. and its diameter is 8 ft. It is desired to arrange a brake BC as shown, by means of which the speed of the wheel may be reduced from 100 rev/min to zero in 10 secs. The coefficient of friction at $D = \frac{1}{4}$; the available pull P is 100 lbs. Determine the ratio a/b. (Neglect axle friction.)

136–(38). Determine the magnitude and direction of the axle reactions in Probs. 131 and 132; in Probs. 133 and 134.

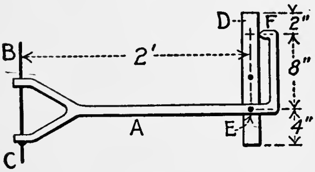


FIG. 75

137–(38). In Fig. 74, a = 6 ft. and b = 6 ins.; the wheel weighs 400 lbs. The coefficient of brake friction $= \frac{1}{2}$. When the wheel is turning clockwise, a push P of 120 lbs. is applied. Determine the axle reaction.

138–(38). A (Fig. 75) is a rigid piece which can be rotated about the vertical axis BC. D is a vertical bar pinned to A at E, and rests against A at F; the bar is 14 ins. long and weighs 20 lbs. The speed of rotation is 100 rev/min. Determine the pressures on D.

139–(38). In Ex. 1 of Art. 38, § 2, take P as applied at F and solve; then as applied at G and solve.

140–(39). Compute the length to the nearest hundredth inch of the simple seconds pendulum for your locality.

141–(40). The body C (Fig. 76) weighs 50 lbs. It is dragged up the plane by P (= 40 lbs.) and Q (= 20 lbs.). The frictional resistance is 5 lbs.; $\alpha = 30°$. Compute the work done on C by each force acting on it while C is moved from A to B, a distance of 15 ft.

142–(40). ABC (Fig. 77) is a smooth rail in the form of a vertical semicircle of 4 ft. radius. D is a body, weighing 50 lbs., which can be made to slide along the rail. P is a force of 150 lbs. always inclined 30 deg. to the horizontal; Q is a force of 40 lbs. always directed along the tangent. Compute the work done on D by all the forces acting on it while D is moved from A to B.

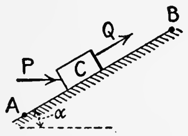


Fig. 76

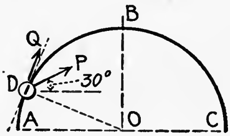


Fig. 77

143–(40). Solve the preceding problem on the supposition that P is always directed toward B.

144–(40). In order to retard the motion of a launching ship, ropes were fastened to it and to points on the shore, so that the ship broke many of the ropes as it progressed. In order to estimate the retarding effect of each rope broken, tension tests were made on samples of the rope (7-in. manilla). Fig. 78 shows the average tension-stretch curve for these tests. The average strength of the samples was about

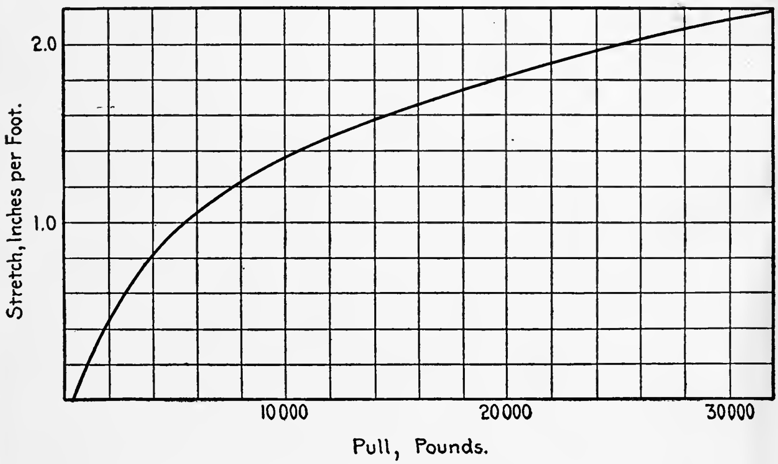


Fig. 78

32,500 lbs. It was assumed that the efficiency of the knots used would be about 80 per cent, and therefore that the ropes would fail at about 26,000 lbs. On the basis of this assumption and the curve, it was estimated that each rope (20 ft. long) would do 60,000 ft-lbs. of work on the ship before breaking. Can you check this estimate? (Data taken from *Trans. Soc. Nav. Archts. and Mar. Engrs.*, 1903, p. 295.)

145–(41). Show that the rotational part of the kinetic energy of a rolling sphere is two-sevenths of its total kinetic energy.

146–(41). A certain freight car with its load weighs 60 tons. Each pair of wheels with its axle weighs 1800 lbs., and the radius of gyration of a pair and axle with respect to the axis of the axle is 0.81 ft.; the diameter of the wheels is 33 ins. Determine the ratio of the rotational part of the kinetic energy of the moving car (and load) to the translational part.

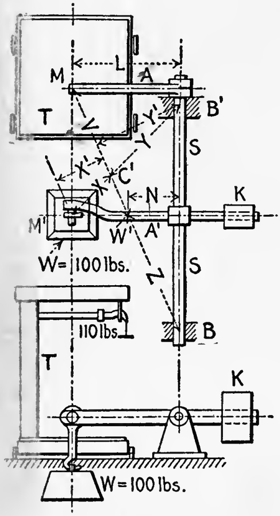


FIG. 79

147–(42). In the *American Machinist* for Dec. 2, 1909, there appears a communication in which an alleged "fault in brake dynamometers" is pointed out and explained. The writer states that on several occasions he got ridiculous results with a Prony brake. The "enigma" became clear to him when he encountered a "paradox" in his experimental work, described by him as follows:

In Fig. 79 S represents a shaft, mounted in two bearings BB', carrying two levers, arms AA', each exactly 50 inches long from center of shaft to the fulcrums MM', respectively, and firmly keyed to the shaft. At K is represented a counterweight which balanced the two lever arms and brought the center of gravity about the center line of the shaft S; T represents a platform scale and W represents a weight, which weighed 100 pounds when placed on the scales T. When W of 100-pounds weight was hung on the fulcrum N', the scales just balanced at 110 pounds. At first the paradox almost paralyzed the brain, but on closer examination the mystery was easily solved, as follows: Considering A and A' to be firmly keyed to the shaft S, then the two arms and shaft S become practically one solid mass. Therefore, when any weight W is placed on the fulcrum M' of the lever A', the whole mass will tend to rotate about a line passing through the points of support M and B, with a moment of W times the lever arm X'. The shaft S at the point B' will be fetched forcibly up against the top of the box or bearing cap of bearing B', which will resist the rotation of the mass about MB, with a balancing moment equal to WX', or a reaction on the bearing cap equal to

$$\frac{WX'}{Y'}, \text{ or } \frac{WX}{Y}.$$

Now, it is evident that the resultant of these two forces is a downward vertical force C at the point C' equal to $W + (WX)/Y$ which load is distributed between the points of support B and M inversely proportional to their respective distances from the point C'. Hence the load on the scales T will be represented by

$$\frac{ZC}{V+Z}, \text{ or } \frac{Z}{V+Z}\left(W + \frac{WX}{Y}\right).$$

Hence a weight of 100 pounds on the fulcrum M will produce a load on the scales T equal to

$$\frac{Z}{V+Z}\left(100 + \frac{100\,X}{Y}\right)$$

instead of 100 pounds as generally believed. The above condition obtains, more or less, in the vast majorities of dynamometers, and is sometimes so exaggerated as to make the results positively ridiculous. In the case of a motor test let W represent the tangential pull on the armature, an equal upward pull on the opposite side of the shaft might tend to balance the error, or it might tend to make matters worse, depending upon the position of the other points in the diagram, but wherever W may fall the results will be most erroneous. For instance, suppose that W happens to fall on the line MB at W', then it is evident that the weight exerted on the scales T will be equal to

$$\frac{WZ}{V+Z}, \text{ instead of } \frac{NW}{L}$$

as generally accepted.

Show that the writer is mistaken in his assertion that

$$\frac{Z}{V+Z}\left(100+\frac{100\,X}{Y}\right) \text{ does not } = 100, \text{ and that } \frac{WZ}{V+Z} \text{ does not } = \frac{NW}{L};$$

and, hence, that his explanation of the "enigma" does not explain.

148–(42). Fig. 80 represents Durand's dynamometer. A, B, C, and D are sprocket wheels of equal diameter; A and B are mounted on a beam XYT which is carried by the well-known Emery steel-plate support or knife-edge at E. The knife-edge rests on the standard R. Sprocket wheels C and D are mounted on R. The bars SS are fastened rigidly to the beam, and engage loosely with a pin on R, thus limiting rotation of the beam. The sprocket chain passes over A, under D, over B, under C, and up to A. The shafts for C and D are extended forward and back; and on these extensions pulleys may be mounted, or universal joint couplings may be attached, for the receipt and delivery of power. (For detailed description see *American Machinist* for June 20, 1907.)

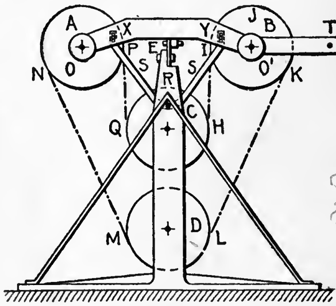


FIG. 80

$OEO'T$ is horizontal; PQ and IH are vertical; MN and KL are inclined at an angle of 27 deg. with the vertical; $OE = O'E = 12$ ins.; and $ET = 24$ ins. Suppose that an electric motor on the shaft of C turns counter-clockwise at 100 rev/min, and transmits to a machine on the shaft of D, and that a weight of 40 lbs. at T keeps the beam XY balanced. What is the power of the motor?

149–(42). Assume that the law of mean effective pressure and piston speed is represented by the dotted line in Fig. 331 of the text, so that

$$p = p_0\,[0.95 - (7\,s + 11{,}000)],$$

where p = mean effective pressure, p_0 = boiler pressure, and s = piston speed in feet per minute. Then derive a formula for indicated locomotive power. Find piston speed at which power is maximum. Also graph your formula in the figure, calling the maximum power 100 per cent.

150–(42). Let D = diameter of the driving wheels of a locomotive in inches; l = stroke in inches; d = diameter of the cylinder in inches; p_0 = boiler pressure in pounds per square inch; and V = velocity of the locomotive in miles per hour. Assume that the mean effective pressure varies as described in the preceding problem. Derive a formula for the indicated power of the locomotive in horse powers for any velocity V.

151–(43). A certain body weighs 400 lbs., and is dragged along a rough horizontal plane by a force of 80 lbs. The force is inclined 20 deg. upward from the horizontal; the coefficient of friction between the body and plane is about $\frac{1}{10}$.

At a certain point in the motion, the velocity of A is 5 ft/sec. What is the velocity of A 10 ft. beyond the point?

152–(43). For the purposes of comparing the "running qualities" of certain freight car trucks, they were tested substantially as follows: Each one was made to roll down a steep incline to give it "initial velocity," and then it passed onto a moderate upgrade; the velocity was measured at two points on the upgrade; then the loss of kinetic energy was computed. These losses furnished a comparison. The upgrade was 0.38 per cent, and the points at which velocities were measured were 257.2 ft. apart. One of these trucks (four-wheeled) weighed 18,150 lbs.; each pair of wheels and axle 1800 lbs. The diameter of wheels was 33 ins.; the radius of gyration of a pair and axle was 0.81 ft. In one test the velocities at the two points were 14.95 and 11.05 ft/sec. Determine the average "truck resistance," a single imaginary force equivalent to actual resistance experienced by the truck. (Experiments by Prof. L. E. Endsley for American Steel Foundries.)

153–(43). The suspended body C (Fig. 81) weighs 10 lbs. The coefficient of friction under the brake is $\frac{1}{5}$; $r_1 = 4\frac{1}{2}$ ins., $r_2 = 6$ ins., $a = 2$ ft., and $b = 1$ ft. C is allowed to descend 6 ft., thus turning the wheel, and then the brake is put on, with $P = 20$ lbs. How much farther will C descend? (Neglect axle friction.)

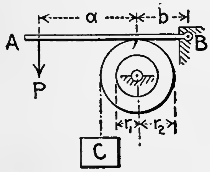


FIG. 81

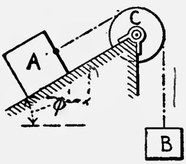


FIG. 82

154–(43). A, B, and C respectively (Fig. 82) weigh 100, 30, and 64.5 lbs. The diameter of C = 30 ins., and its radius of gyration about the axis of rotation = 1 ft.; $\phi = 30$ deg. The friction under A = 10 lbs. Determine the velocity of the system when A has moved through 10 ft. from rest.

155–(43). Copy Fig. 339 (pertaining to Exs. 2 and 3, Art. 43, § 2) using scale 1 in. = 10,000 lbs. and 5 mi/hr. (*a*) Make a graph in your copy which will show how the accelerating force is apportioned between the locomotive and the cars. What is there in your finished figure which represents draw-bar pull? (*b*) Modify your figure for the case of the train when on an upgrade of 0.5 per cent.

156–(43). Make graphs showing how the total train resistance in pounds varies with the velocity in miles per hour according to Schmidt's formula and the *Engineering News* formula for the train described in Ex. 2, Art. 43, § 2.

157–(43). Make a new figure (as for problem 155) assuming that the train resistance varies according to Schmidt's formula. First assume level track; then modify the diagram for the case of an upgrade of 0.5 per cent.

158–(43). Referring to the preceding problem with train on upgrade: (*a*) Make a graph showing how the acceleration changes with velocity. (*b*) Find the time required for the velocity to change from 10 to 20 mi/hr. (See § 3, Art. 28.)

159–(43). Make a graph showing how the velocity of the train of the preceding problem (on the upgrade) changes with the time (in seconds) during the run mentioned.

160–(43). Make a graph showing how the distance covered by the train of the preceding problem (on an upgrade) changes with the time.

161–(44). Fig. 83 represents in outline a certain small vertical-lift bridge. The lifting span is counterweighted as shown. At the center of the span there is a cross-shaft having on each end a drum long enough to provide for two up-haul and two down-haul cables. From each drum the cables are led to deflecting sheaves at each

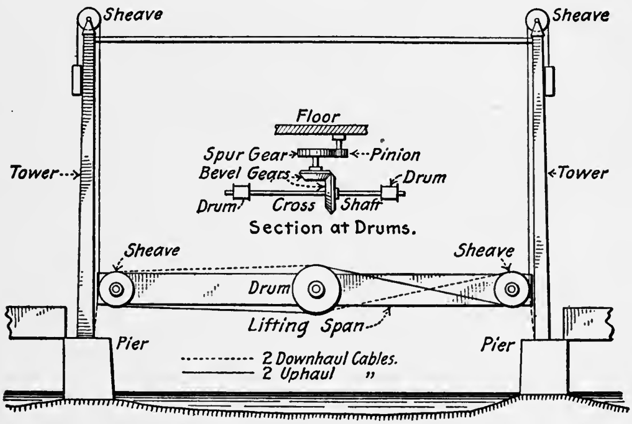


FIG. 83

end of the span, beyond which they are led to attachments at the top and bottom of the tower, as shown. The cross-shaft is driven through a pair of bevel gears by a vertical shaft connected by a single set of spur gearing to a second vertical shaft to the capstan head of which the operating lever is fitted when the span is to be raised or lowered. For full description see *Engineering News* for July 18, 1912.

The dimensions, etc., are as follows, but those marked "assumed" are missing in the published description: — Weight of lifting span = 58,000 lbs., and of each counterweight 29,000 lbs.; length of operating lever = 6 ft. (assumed); number of teeth in pinion 17, in spur gear 68, and in each bevel gear 30; diameter of drums = 18 ins. (assumed); diameter of (four) deflecting sheaves = 12 ins., and diameter of shafts for same = 1 in. (assumed); diameter of (four) counterweight sheaves = 54 ins. (assumed); and diameter of shafts for same = $3\frac{1}{2}$ ins. (assumed); diameter of (eight) counterweight cables = 1 in.; and of other cables $\frac{3}{8}$ in. Determine the necessary effort (force) at the end of the operating lever required to lift the span; to lower it.

162–(44). Fig. 84 represents the arrangement of tackle, engines, etc., used for moving a large building (three stories, 120 × 142 ft., weighing about 8000 tons). Pulls were applied at six points on the rear of the building as shown. The four blocks under and the three immediately in front of the building are single (one sheave or pulley in each); *A* and *B* are single, *C* and *D* double, and *E* and *F* triple. The pulling cable from each engine extends to *A*, and is reeved through *A* and *B*, ending at *A*; a second cable is fastened to *A* and reeved through *C* and *D*, ending at *C*; a third cable is fastened to *C* and reeved through *E* and *F*, ending at *E*. Blocks *E* are merely hooked to the three blocks immediately in front of the building; blocks *B*, *D* and *F* are held in place by cables fastened to deadmen (buried logs or the like). The runs of cable from *A* to *C* and from *C* to *E* are really parallel to the main runs;

they are shown inclined to avoid confusion of lines. The pull of each engine was about one ton. (For fuller description see *Engineering Record* for Nov. 22, 1913.) Assume K to be 1.15, and compute the total pull exerted on the building; also the pull exerted on each deadman cable. Which one or ones of all the cables is subjected to the greatest pull?

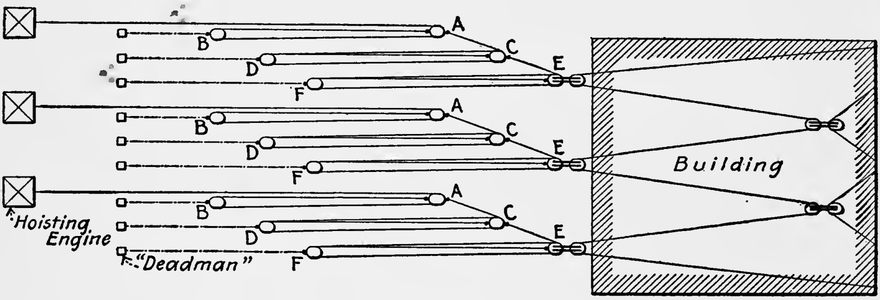


FIG. 84

163-(44). The building (preceding problem) was moved forward 40 ft. for each setting of the equipment. How far did blocks A, C, and E travel for each setting? How much cable was wound on the drum of each engine?

164-(45). Fig. 85 represents the mechanism for operating a small bascule bridge of a single draw span. The train of gears, A, B, C, D, and E rests on the (fixed) ap-

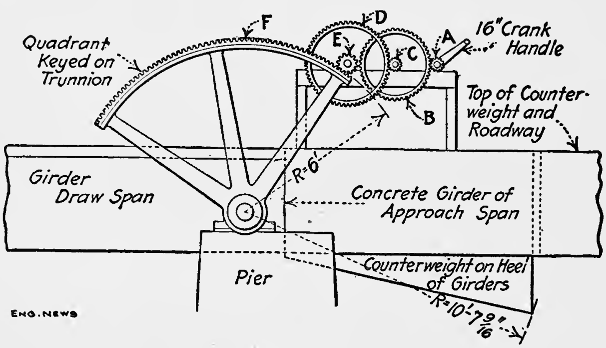


FIG. 85

proach span. The quadrant and the draw span are keyed to the same trunnion, supported on the pier shown. When the hand crank is turned counter clockwise (in the view shown), the quadrant rotates clockwise, and the free end of the draw span lifts. The total weight of the draw span and counterweight is 115,000 lbs., and the center of gravity of that (moving) part of the bridge is in the axis of the trunnion. The trunnion is 7 ins. in diameter. The following description of the gear train is sufficient for our purpose:

Gear	A	B	C	D	E	Quadrant
Number of teeth	13	94	15	122	11	57

(For fuller description see *Engineering News* for July 24, 1913, or the paper by the designer, Prof. L. E. Moore, in *Engineering and Contracting* for Aug. 13, 1913.) Determine how large a force applied to the crank handle at right angles to the crank is required to raise the draw span.

165–(45). Fig. 86 represents in plan certain elements of the downstream (miter) gate of the lock at the Keokuk Dam. Each leaf of the gate is hung on hinges somewhat like an ordinary door; but the lower hinge is a hemispherical pivot or pintle

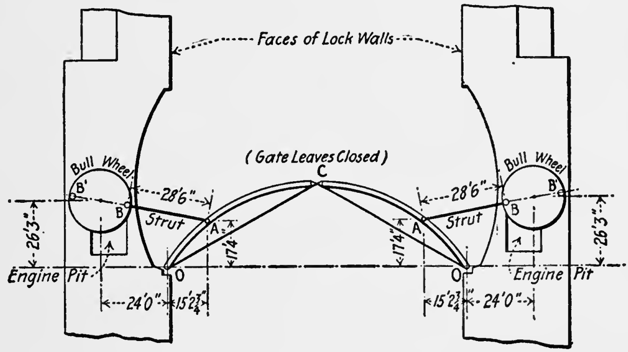


Fig. 86

and it takes up all the direct weight of the leaf, the upper hinge taking up only horizontal pull. Each leaf is opened and closed by means of an operating strut AB, pinned to the top of the leaf and to the rim of a horizontal bull-wheel; each wheel is driven by an engine through a train of gears. Each leaf weighs 463,000 lbs.; the distance from its center of gravity to the (vertical) axis of its hinges is 31 ft. 8 ins.; the distance between the hinges is 48 ft.; the diameter of the upper hinge pin is 12 ins.; the radius of the pintle is 9 ins. Assume coefficients of friction for pin and pintle to be 0.05 and 0.15 respectively. Determine the reactions at the hinge pin and pintle due to the weight of the leaf, and the moment of the frictional resistance to swinging, about the axis of the hinges.

166–(45). Fig. 87 represents certain details of the operating mechanism for the lock gate described in the preceding problem. It will be noticed that when the gate is wide open, the axis of the operating strut is over the center of the bull-wheel. The dimensions, proportions, etc., are such that a turn of the wheel through 180° from the position shown closes the leaf; and then the center of the wheel is again in line with the axis of the strut. From a large drawing, we have scaled the arms of the thrust of the strut with respect to the axes of rotation of the leaf and bull-wheel for thirteen positions of the strut, corresponding respectively to the open position of the wheel, 15° turn, 30° turn, etc. (see adjoining table). Compute the torque required on the bull-wheel for overcoming the frictional resistance at the hinges for each of the thirteen positions, neglecting the frictional resistance at the pins of the strut, at the center pin of the wheel and at the rollers under the rim of the wheel. Make a curve which shows how this torque varies with θ. What does the area under the curve represent?

Position	θ, degrees	a', feet	a'', feet
Open 1	0	19.9	0.0
2	15	18.8	1.8
3	30	18.2	3.8
4	45	18.4	5.8
5	60	19.2	7.8
6	75	20.3	9.5
7	90	21.5	10.7
8	105	22.5	11.2
9	120	23.0	10.7
10	135	22.9	9.4
11	150	22.3	7.0
12	165	21.3	3.7
Shut 13	180	19.9	0.0

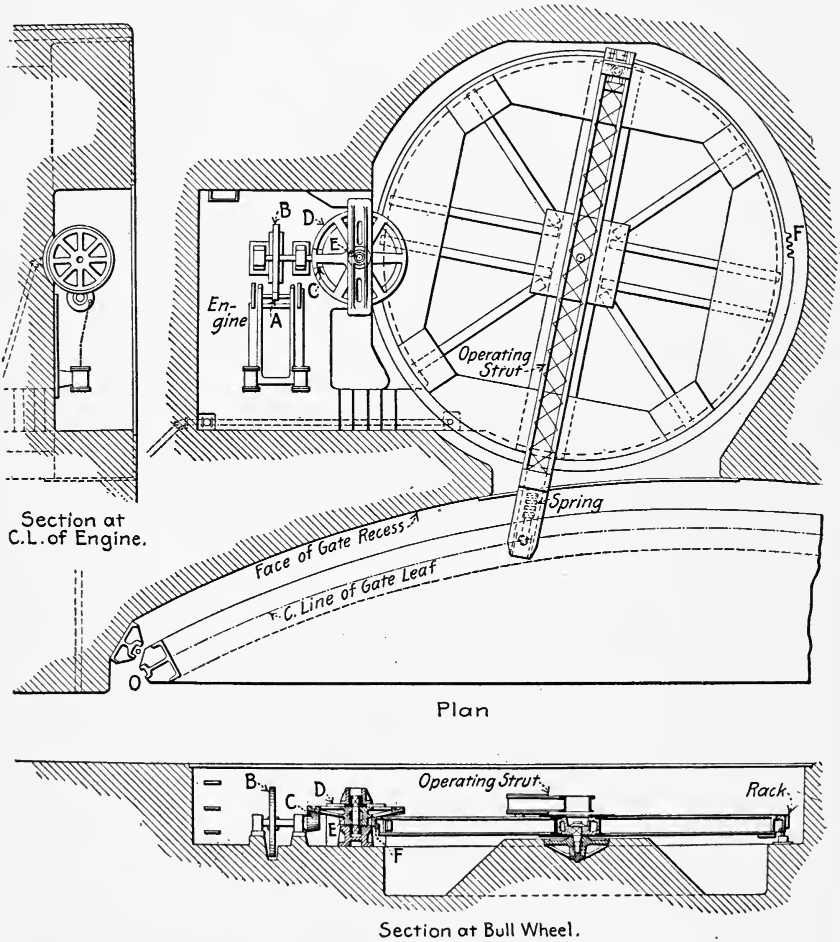


FIG. 87

167–(45). The shaft of the engine, which operates the gate described in the preceding problems, carries a pinion A (Fig. 87); A drives a spur wheel B on a second shaft which also carries a bevel pinion C; C drives a bevel spur wheel D on a vertical shaft which also carries a pinion E; E gears with the rack F and thus drives the bull-wheel. The numbers of teeth on the pinions, wheels, etc., are as follows:

A	B	C	D	E	F
18	126	20	90	16	150

the last for one-half the circumference, more not required. Neglecting all friction loss in the operating mechanism, compute the torque required at the engine shaft for each of the thirteen positions of the gate mentioned in the preceding example, and make a graph which shows how this torque varies with θ. Recompute, but allow for friction loss by means of (estimated) efficiency of the gear train. What is the total amount of work done at the engine shaft in closing one gate?

168–(45). The engine (preceding problems) is run at 370 rev/min. Compute the rate (in horse-power) at which the engine works, at the engine shaft, when closing a leaf at each of the thirteen positions mentioned. Show by means of a graph how the power varies with θ and the time. (The author is indebted to Mr. B. H. Parsons, Mechanical Engineer of the Mississippi River Power Company, for the data of these problems relating to the Keokuk Lock.)

169–(46). Water is flowing through a certain 6-in. pipe at a velocity of 4 ft/sec. Compute the resultant pressure of the water against a right-angle bend in the pipe. (Assume that the water pressure is the same at both ends of the bend, and equals 100 lbs/in².)

170–(46). Actually, the water pressure (referring to the preceding problem) is greater at the inlet end of the bend. Assume that the pressures are 104 and 100 lbs/in²; then solve.

171–(46). A certain three and one-half inch hose is conducting water at a velocity of 20 ft/sec. There is a circular bend of 180° in the hose; the radius of the bend is 8 ft. Assume water pressure at both ends of the bend to be 100 lbs/in². Determine the resultant water pressure on the bend. How much pressure (tending to straighten the hose) is there per inch of bend.

172–(46). Water is projected into a smooth channel with borders so that the magnitude of the velocity of the stream is not changed, only its direction. Determine the pressure of the stream against the channel.

173–(47). A body whose mass = M is dragged along a smooth horizontal plane by a force which varies uniformly with the displacement, the force being zero when the displacement = 0 and 40 lbs. when the displacement = 10 ft. Determine the time-average value of the force.

174–(47). Fig. 88 is a part copy of a figure from a report on certain tests of an hydraulic (railway) buffer by Mr. Carl Schwartz, published in the *Journal of the American Society of Mechanical Engineers* for June, 1913. An abstract of the report is printed in *Engineering News* for Sept. 11, 1913. The buffer consists essentially of a cylinder 22 ins. in diameter, and a piston; the working stroke is 11 ft. The buffer is firmly anchored at the stopping point, with the piston rod in the line of approach of the buffer of the car or locomotive to be stopped. The cylinder is grooved so as to allow water to pass by the piston during a stop.

The curve marked "speed" shows how the speed of the locomotive, in this instance, varied during the 12 secs. preceding impact, and also during the impact. Thus the speed was about 5.6 mi/hr at the beginning of the test; it increased to about 7.3 in $8\frac{1}{2}$ secs; then it decreased uniformly up to the instant of impact after which it decreased much more rapidly. The curve marked pressure shows how the hydraulic pressure behind the piston varied during the impact. Thus the initial pressure on each side of the piston was about 45 lbs/in²; after the instant of impact the pressure shot up to a maximum of 925 lbs/in², and then decreased to about 80. The entire travel of the piston in this case was 3 ft. (not indicated in the figure). The locomotive weighed 100 tons.

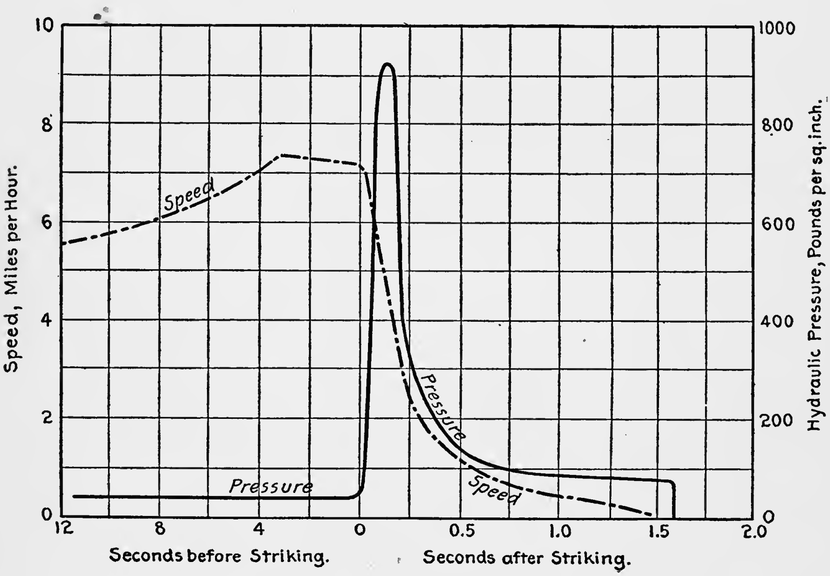


FIG. 88

Compute the time-average and the space-average force which stopped the locomotive, neglecting the effect of the so-called train resistance. Estimate the train resistance from the retardation of the locomotive just before the impact, and then recompute the averages just mentioned. Measure the area under the pressure curve and interpret it. Does the shape of the curve suggest any improvement in the buffer?

175–(48). The power of an operating hydraulic turbine equals the product of the angular velocity of the turbine and the rate at which angular momentum (about the axis of rotation) of the flowing water is changed in its passage through the turbine. Prove.

176–(48). A certain homogeneous prism is $2 \times 6 \times 36$ ins. in dimensions. It is mounted so that it can oscillate like a common pendulum about either of two axes of suspension. Both axes contain the center of one small face of the prism; one axis is parallel to the 2-in. edges, and the other is parallel to the 6-in. edges. Locate the center of percussion for each of these axes.

177–(49). Describe the gyrostatic reaction which a screw-propelled ship sustains when pitching (in a rough sea).

178–(49). In the *General Electric Review*, Vol. IX, pages 117 and 118, there appears the following: "The spin of a precessing body increases the centrifugal force about the axis of precession. Take the case of a wheel spinning about a horizontal axis supported at one end which is precessing about a vertical axis through the point of support. The total centrifugal force is

$$\frac{WV^2}{gR}\left(1 + \frac{k^2p^2}{2r^2}\right)$$

which equals the ordinary centrifugal force WV^2/gR plus the additional centrifugal force due to spin (gyroscopic centrifugal force) $(WV^2k^2p^2)/(gR2r^2)$. W = weight of the gyroscope, k = its radius of gyration, R = the radius of the circle of precession, r = the radius of the spinning wheel, V = the linear velocity of the precession, v = the peripheral velocity of the wheel, and p = the ratio v/V." Presumably, R means the radius of the circle described by the mass-center of the wheel. Ascertain in your own way whether any force, appropriately called centrifugal force, has the value above stated in the case in question.

179–(49). On page 144 of the journal mentioned in the preceding problem there appears this statement. "The total vertical force on the outside rail [car wheels running around a curve] due to gyroscopic action will therefore be $(3\,WV^2k^2) \div (2\,gRrx)$." W = the weight of a pair of wheels and axle (presumably), k = radius of gyration of the pair and axle (about their axis), r = the radius of the wheels, R = radius of the curve, x = gage of the track, and V = the velocity of the car. Can you prove the statement?

180–(50). A wheel 6 ft. in diameter rolls on a straight track. At a certain instant the velocity and acceleration of its center are 10 ft/sec. and 4 ft/sec/sec. Determine the acceleration of the lowest point of the wheel at the instant in question.

181–(51). When a slender body, such as a pole, chimney, etc., is tipped over from an upright position, the motion is one of rotation about the point of contact of the body and the surface which supports the body until slip occurs at the contact or the lower end lifts from the surface. Assume that the slender body is hinged to the supporting surface so that it cannot slip or lift, and then determine the vertical and horizontal components (V and H) of the supporting force for various positions of the tipping body. Draw curves showing how V and H vary with the angular displacement of the pole from the vertical. How could you ascertain whether slip or lift would occur first?

182–(51). Referring to the preceding problem, assume that the pole is supported on the ground, and that slip cannot occur during tipping. The lower end of the pole will lift when a certain degree of tip is reached; afterwards the pole moves under the influence of gravity only. Until the pole strikes ground, it rotates with the angular velocity which it had at the instant when the contact was broken, and the center of gravity moves in a parabolic path due to its initial velocity (when the contact was broken) and action of gravity. Determine the distance from the (original) point of support of the pole to where it first strikes the ground.

183–(52). In Fig. 410, the load W = 18,000 lbs.; the diameter of the rollers = 15 ins.; the coefficient of rolling resistance "under" the rollers = 0.020, that "over" the rollers = 0.025. How large a force P is required to move the load? Determine the two forces which act upon a roller supposing that the load is distributed equally among the rollers.

184–(52). Referring to Prob. 162: The rollers used were 3 ins. in diameter; about 2000 were used. They were of steel 2 ft. long and rolled between steel plates above and below. Assume that your computed result in Prob. 162 is the value of the pull actually exerted on the building when moving on a level stretch. Then compute the average coefficient of rolling resistance.

185–(53). Two men A and B are walking at a speed of 4 mi/hr along east and west and north and south paths respectively. Compute the velocity of A relative to B when A is walking northward and B eastward; when A is walking northward and B westward.

186–(53). The disk (Fig. 89) is 4 ft. in diameter and is rotating uniformly about O at one rev/sec. A point P is moving uniformly along the diameter AB from A toward B at a speed of 4 ft/sec. Determine the absolute velocity of P when midway between A and O; when midway between O and B.

187–(53). Suppose that P (see preceding problem) is moving from C toward A; the angle $\phi = 150°$, and when P reaches A its speed is 6 ft/sec (along CA). What is the absolute velocity of P then?

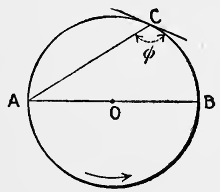


FIG. 89

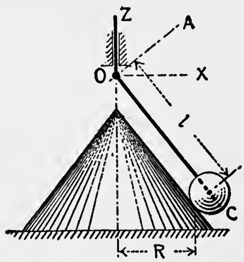


FIG. 90

188–(53). A certain square is 6 × 6 ft., and its corners are lettered A, B, C, and D in succession around the perimeter. The square is rotating uniformly about a line through A perpendicular to its plane at one rev/sec; a point P is moving uniformly along CD and in that direction at 6 ft/sec. Determine the absolute velocity and acceleration of P when it reaches the mid position between C and D.

189–(54). The sphere (Fig. 90) is suspended from the end of a vertical shaft OZ by means of the rod OC extending into and rigidly fastened to the sphere. The shaft and the rod are connected by a Hooke's (flexible) joint. When the shaft is rotated it exerts a torque on the rod which in turn makes the sphere roll around on the cone. Assume that the sphere is 2 ft. in diameter, $R = 4$ ft., $l = 8$ ft., and that the shaft makes 150 rev/min. Determine the angular velocity of the sphere, and the x, y, and z components of that velocity.

190–(55). Referring to the preceding problem, suppose that the sphere is cast iron (weighing 450 lbs/ft^3). Then compute the angular momentum of the sphere and determine the rate at which the angular momentum is changing.

191–(55). Suppose that there is no "rolling resistance" (Art. 52) between sphere and cone. Then determine the following: normal pressure and friction between cone and sphere; the torque which the shaft must exert on the rod; and the x, y, and z components of the supporting force at O.

192–(56). Fig. 91 represents in principle the Griffin Mill for grinding cement. The cross piece of the (upright) frame supports the upper (vertical) shaft S by means of a thrust ball bearing. The large pulley P is rigidly fastened to the shaft. The pulley hub HH is extended downward and is restrained laterally by the guides GG, thus virtually forming an extension of the shaft. The "roll" is rigidly fastened to the "roll shaft" and both are suspended on a cylindrical seat on the inside of the hub of the pulley as shown. Thus the roll and its shaft can oscillate like a common pendulum about a perpendicular to the paper at O. K is a cross head rigidly fastened to the roll shaft but slipping in vertical guides on the hub when the roll and its shaft oscillate like a common pendulum. The "die" is a hard metal ring between which and the roll the grinding of the cement takes place as explained presently. When the mill is idle, the roll shaft hangs in a vertical position; if the pulley be rotated the guides in the hub exert a torque on the cross head, and the roll shaft is made to rotate in the vertical position with the pulley. When it is desired to start the mill for grinding, the roll is first pulled outward "with an iron hook," and then the power is turned on at the pulley. The roll shaft rotates with the pulley; promptly, the roll begins and continues to roll on the die (ring), a great pressure being developed between roll and die. Material to be ground is fed into the mill so that some is caught between the roll and the die and then pulverized. Suitable paddles on the lower side of the roll continually toss the material which collects in the recess of the base; eventually it is caught between roll and die.

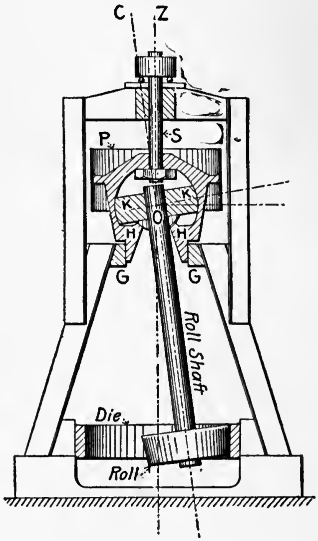


FIG. 91

It will be noted that the roll and its shaft constitute a large gyrostat. We now propose the problem of determining the pressure between the roll and the ring when the mill is operating. The makers (Bradley Pulverizer Co.) state it to be about 15,000 lbs. for their giant size when run at a pulley speed of 165 to 170 rev/min. The following data, approximated in some cases, was taken from drawings furnished by the makers of the mill. The die is 40 ins. in diameter (inside), 8 ins. high; from the plane of its top to the point of suspension O is 5 ft. $4\frac{1}{2}$ ins. The roll weighs 880 lbs.; its larger diameter is 24 ins. The roll shaft weighs 600 lbs.; its length over all is 6 ft. $9\frac{1}{2}$ ins.; its point of suspension O is 6 ins. from the upper end; its diameter varies from $5\frac{1}{8}$ ins. at the cross head to $6\frac{1}{2}$ ins. at the roll but the ends in the cross head and roll are tapered. For simplicity, make the following approximations: roll-shaft uniform diameter is $5\frac{3}{4}$ ins., smaller diameter of roll = 22 ins., and its thickness is 8 ins. As a further close approximation for locating center of gravity and determining required moments of inertia, assume that the roll is a cylinder 23 ins. in diameter and 8 ins. thick (with $5\frac{3}{4}$ ins. hole for the roll shaft).

193–(57). A certain right cone with a circular base is homogeneous; the diameter of its base is 4 ft.; the altitude is 6 ft.; and half the apex angle is 20°. Determine the radius of gyration of the cone with respect to an element of its curved surface.

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