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ABSTRACT

A dynamic oligopoly model, having a market share attraction form, is formulated as a stochastic sequential game. Market shares evolve randomly over time according to dynamics that incorporate a general lag structure. The model incorporates "goodwill" as a measure of current and past effort decisions.

Qualitative properties of closed-loop, equilibrium strategies are developed by establishing a relationship between an equilibrium point of the dynamic, stochastic game and an associated static (one-period) game.

List of Symbols

Bold Roman Upper	Bold Roman Lower	Roman Upper	Roman Lower	Greek		Special	
$\mathbf{A}$	$\mathbf{a}$	A	a	α	alpha	∞	Infinity
$\mathbf{B}$	$\mathbf{e}$	B	b	β	beta	$\times$	Cartesian product
$\mathbf{E}$	$\mathbf{g}$	E	c	Λ	Lambda	$\equiv$	equivalent
$\mathbf{I}$	$\mathbf{h}$	H	d	δ	delta	$=$	equal
$\mathbf{J}$	$\mathbf{m}$	L	e	θ	theta	$\geq$	greater than or equal to
$\mathbf{K}$	$\mathbf{p}$	M	f	κ	kappa	$\leq$	less than or equal to
$\mathbf{L}$	$\mathbf{s}$	N	g	λ	lambda	$\ln$	natural log
$\mathbf{M}$	$\mathbf{w}$	Q	k	μ	mu	$\succeq$	"greater than or equal to" order
$\mathbf{W}$		R	m	ξ	xi	∂	round dee
$\mathbf{Y}$		S	q	π	pi	$>$	greater than
		T	r	Σ	Sigma	$<$	less than
		V	s	ϕ	phi	$\int$	integral
		X	t	Θ	Theta	$\cup$	union
			v			$\in$	belongs to
			w				
			y				

1. Introduction

Market share attraction models describe the manner in which a fixed market is partitioned among a number of competitors, given the effort allocations of each of the competitors. In these models, market shares have the form

$$\frac{\text{“my effective effort”}}{\text{“my effective effort”} + \text{“your effective allocation”}}.$$

Models that specify payoffs as a function of competitive effort allocations of this general form are prevalent in the economics, game theory, and operations research literature. See, for example, Schmalensee (1976), Shapley and Shubik (1977), Case (1979, Chapter 4), Ponssard (1981, Chapter 3), Shakun (1965), and Monahan (1987).

In this paper, we present a dynamic, stochastic oligopoly model in which payoffs have the market share attraction form. In any time period, the share of a fixed market accruing to one of $Q \geq 2$ competitors is a random function of current and past effort allocations of all of the Q competitors. Our focus in this paper is on the partitioning effects of competitive effort allocation rather than on the size of the market. The model's market share dynamics, i.e., the specification of how market shares evolve from period to period, incorporate a stochastic lag structure that has a rather general form. The market shares are based on a measure of "goodwill" that summarize current and past effort allocations; the competitors' discounted expected profits are their payoffs. The resulting model is a stochastic sequential game to which we apply the Nash equilibrium point solution concept.

Most of this paper's analysis is devoted to the determination of qualitative properties of an equilibrium point (**ep**) solution to a sequential game (**sg**) that has special structure. Even simple **sg**'s, such as the iterated prisoner's dilemma, possess extraordinarily many **ep**'s. This richness arises from strategies in which a player's action in a period depends on the past behavior of other players. There does not yet seem to be a generally effective framework for assessing the effects of information conditions on qualitative properties of **ep**'s of structured **sg**'s. See Albright and Winston (1979), Deshmukh and Winston (1978), Kirman and Sobel (1974), Mamer (1986), and Shubik and Sobel (1979) for examples of **sg**'s in which qualitative properties of selected **ep**'s *are* elicited. As in those papers, we study **ep**'s that depend on as little information as possible. In particular, we examine an **ep** that corresponds to a stationary Markov strategy (in the parlance of Markov decision processes). This relative simplicity leads to the analysis of a static (one-period) noncooperative game with a unique **ep** in the duopoly case.

While market share models are prevalent in the literature, the model we develop here has novel features. It is the first model to specify the *dynamic* evolution of market shares over time and to incorporate the direct and indirect effects of effort decisions made in the current period and earlier periods.

We are able to establish a number of interesting properties of equilibrium strategies even though the model specification is quite general. The class of strategies we admit includes open-loop, closed-loop, and reaction-function strategies. These properties of an **ep** are established by exploiting results in Sobel (1990a) regarding the relation between an **ep** of myopic-affine stochastic games and an **ep** of an associated static game. The analysis of the static game draws on results for one-period market share attraction models in Monahan (1987). (See Fershtman and Kamien (1987) for a comparison of the differences between open-loop and closed-loop strategies in a two-person differential game.)

The remainder of the paper is organized as follows. The dynamic market share model is specified in Section 2. Section 3 reduces the analysis of the dynamic game to the analysis of a static game. Several properties of an **ep** solution to the static, and hence to the dynamic game are developed in Section 4. Additional properties of the solution, especially those relating to the lag structure, are developed in Section 5. Concluding remarks are in Section 6.

2. The Model

Consider a single market with a sales potential of V units and Q competing firms. The effort expended by each of the Q competitors is specified by the Q-vector $\mathbf{g} = (g_1, \ldots, g_Q)$.

Let $\mu_q(\mathbf{g})$ be firm q's net profit. The market share attraction form specifies that

$$\mu_q(\mathbf{g}) = \frac{V M_q b_q g_q^\beta}{\sum_{j=1}^{Q} b_j g_j^\beta}, \tag{1}$$

where M_q is firm q's profit per unit sold and $q = 1, \ldots, Q$. The term in the numerator of (1) is the "attraction" of g_q, which is the effort expended by firm q. The denominator is the total attraction of effort of all Q competitors. The parameter b_q is the relative effectiveness of the effort expended by firm q. The attraction elasticity of effort is

$$\beta = \frac{d(b_q g_q^\beta)}{d g_q} \cdot \frac{g_q}{b_q g_q^\beta}.$$

We assume that all parameters are positive numbers and that $\beta < 1$.

Suppose that the Q firms compete in the fixed market over an infinite number of periods, starting with $t = 1$. Our model encompasses direct lagged effects of effort; that is, the effort expended over the last N periods directly influences the market share in the current period. The initial data thus include the levels of efforts expended in the N periods prior to $t = 1$. Let a_{qt} be firm q's effort in period t and let $\mathbf{a}_t$ be the column vector $(a_{1t}, \ldots, a_{Qt})'$, where prime denotes transpose. The initial data include the lagged direct effort allocations $\mathbf{a}_{-N+1}, \mathbf{a}_{-N+2}, \ldots, \mathbf{a}_0$.

We use the term "goodwill" to label the cumulative effect of effort. Let g_{qt} be firm q's goodwill in period t *after* effort is allocated in period t, and let $\mathbf{g}_t = (g_{1t}, \ldots, g_{Qt})'$. The model for the dynamics of goodwill is

$$\mathbf{g}_t = \mathbf{K}_{t-1}\mathbf{g}_{t-1} + \sum_{n=1}^{N} \mathbf{L}_n \mathbf{a}_{t-n+1}, \tag{2}$$

where $\mathbf{K}_t$ is a random $Q \times Q$ matrix. The vector $\mathbf{K}_0\mathbf{g}_0$ specifies the starting levels of goodwill and is included in the initial data. The model stipulates that goodwill levels at the end of period t depend stochastically on goodwill levels at the end of period $t-1$ and on efforts expended in the current and previous $N-1$ periods. Less general specifications of goodwill are prevalent in the literature; see, for example, Nerlove and Arrow (1961) and the references in Monahan (1983) and Heyman and Sobel (1984).

The $Q \times Q$ matrix $\mathbf{L}_n$ specifies the rate at which effort in period $t-n+1$ affects goodwill in period t, $n = 1, \ldots, N$ ($N \leq \infty$). When effort represents advertising expenditures, a number of papers support the argument that empirical advertising–sales phenomena often warrant an explicit lag structure ($N \geq 2$); see, e.g., Little (1979). However, the sensitivity of decisions to misspecification of the lag structure is a subject of continuing investigation [Bultez and Naert (1979) and Magat, et al. (1986)]. The models in these two papers can be interpreted as special cases of the monopoly version of (2). See Mann (1975) for examples of other advertising models that explicitly incorporate a lag structure.

Let δ_q be firm q's single-period discount factor ($0 < \delta_q < 1$). Until the Appendix, we assume that $\mathbf{L}_1$ and $\mathbf{L}_1 + \delta_q \mathbf{L}_2 + \ldots + \delta_q^{N-1} \mathbf{L}_N$ (for each q) are nonsingular matrices and $\mathbf{M} = (\mathbf{L}_1)^{-1}$ has nonnegative elements. Let λ_{qn} and ξ_{qt} be the qth diagonal elements in $\mathbf{L}_n$ and $\mathbf{K}_t$, respectively. We assume that $\mathbf{K}_1, \mathbf{K}_2, \ldots$ are independent and identically distributed but $\xi_{1t}, \ldots, \xi_{Qt}$ may be correlated for fixed t. Much of the empirical literature that estimates market shares with attraction models treats ξ_{qt} as a constant fraction, say θ_q, for all t; see, e.g., Naert and Weverbergh (1982). In these settings, $1 - \theta_q$ is the depreciation rate of firm q's goodwill; discrepancies between the data and the model are explained with "shocks" and "errors".

When $\mathbf{L}_n$ and $\mathbf{K}_t$ are diagonal matrices, (2) simplifies to

$$g_{qt} = \xi_{q,t-1} g_{q,t-1} + \sum_{n=1}^{N} \lambda_{qn} a_{q,t-n+1}. \tag{3}$$

Bultez and Naert (1979) (see, also, Magat, et al. (1980)) specify a model of goodwill which is a special case of (3) wherein $\xi_{q,t-1} = 0$ for all q and t and $\{\lambda_{qn}\}$ is a Pascal probability distribution.

We specify firm q's gross profit in period t as $\mu_q(\mathbf{g}_t)$. The net profit, denoted X_{qt}, is the gross profit less the cost associated with the level of effort; so $X_{qt} = \mu_q(\mathbf{g}_t) - a_{qt}$. Until the Appendix, we employ the market share attraction form (1) for $\mu_q(\cdot)$.

The present value of firm q's time stream of net profits is

$$\pi_q = \sum_{t=1}^{\infty} (\delta_q)^{t-1} X_{qt} = \sum_{t=1}^{\infty} (\delta_q)^{t-1} \left[\mu_q(\mathbf{g}_t) - a_{qt}\right]. \tag{4}$$

We assume that each firm selects its effort levels via a strategy, that is, a contingency plan that specifies nonnegative effort levels. The contingencies are the outcomes of stochastic elements in the model and the elapsed effort levels selected by the firm's competitors (and itself). The plan must be non-anticipative (i.e., it cannot depend on events that occur later in time), but it can depend arbitrarily (perhaps randomly) on the elapsed history of all of the firms' sequences of efforts and goodwills. Thus, "open-loop" decision rules and reaction functions are proper subsets of the set of all strategies.

We employ a Nash equilibrium point solution concept for the dynamic-oligopoly effort game with payoffs $E(\pi_q)$ that are expected values of present values. Let $v_q(d_1, \ldots, d_Q \mid h)$ denote $E(\pi_q)$ when $d_1, \ldots, d_Q$ are the strategies employed by firms $1, \ldots, Q$, respectively, and h specifies the initial conditions of the game $((a_{at}, g_{qt}, \mathbf{K}_t)$, for $t = -N+2, \ldots, 0)$. We say that $(d_1^*, \ldots, d_Q^*)$ is an **ep** with respect to H if

$$v_q(d_1^*, \ldots, d_q^*, \ldots, d_Q^* \mid h) \geq v_q(d_1^*, \ldots, d_{q-1}^*, d_q, d_{q+1}^*, \ldots, d_Q^* \mid h) \tag{5}$$

for all d_q, for all $q = 1, \ldots, Q$, and $h \in H$.

In the next section, we define the equilibrium point problem as an **sg** (sequential game) and pose questions regarding the structure of an **ep**. In order to answer the questions posed, we show how the **sg** is related to a static game.

3. The Market Share Attraction Game

Let $\mathbf{s}_t$, the *state of the sequential game at time t*, be the vector of goodwill levels *before* decisions in period t are made. Also, define g_{qt} as firm q's *action* in period t. Thus the vector of actions $\mathbf{g}_t = (g_{qt})$ is the vector of goodwill levels *after* decisions in period t are made but before profits are earned. From (3),

$$\mathbf{g}_t = \mathbf{s}_t + \mathbf{L}_1 \mathbf{a}_t \quad \text{and} \quad \mathbf{s}_{t+1} = \mathbf{K}_t \mathbf{g}_t + \sum_{n=2}^{N} \mathbf{L}_n \mathbf{a}_{t-n+2} \tag{6}$$

describe actions and states, respectively.

Let $\mathbf{m}_q$ be the qth row of $\mathbf{M} = (\mathbf{L}_1)^{-1}$. From (6), $\mathbf{a}_t = \mathbf{M}(\mathbf{g}_t - \mathbf{s}_t)$ and

$$a_{qt} = \mathbf{m}_q(\mathbf{g}_t - \mathbf{s}_t). \tag{7}$$

So the level of effort expended by firm q in period t can be expressed in terms of period t's state and action vectors. It follows that tlre single-period rewards earned by firm q in period t can also be expressed solely in terms of the current state and action vectors; i.e,

$$X_{qt} = \mu_q(\mathbf{g}_t) - a_{qt} = \mu_q(\mathbf{g}_t) - \mathbf{m}_q(\mathbf{g}_t - \mathbf{s}_t). \tag{8}$$

The substitution of $\mathbf{a}_i = \mathbf{M}(\mathbf{g}_i - \mathbf{s}_i)$ for each i in the expression for $\mathbf{s}_{t+1}$ in (6) yields

$$\mathbf{s}_{t+1} = \mathbf{K}_t\mathbf{g}_t + \sum_{n=2}^{N} \mathbf{L}_n\mathbf{M}(\mathbf{g}_{t-n+2} - \mathbf{s}_{t-n+2}). \tag{9}$$

The equilibrium point problem is to find strategies $d_1^*, \ldots, d_Q^*$ for the Q firms that satisfy (5), where firm q's payoff is

$$E\left[\sum_{t=1}^{\infty}(\delta_q)^{t-1}X_{qt}\right], \tag{10}$$

and X_{qt} is given in (8), subject to the dynamics in (9). In light of (7), the choice of a new level of goodwill in period t, given the state of the process at the beginning of period t, corresponds to choosing a value for a_{qt}. We also require the sensible condition that effort levels be nonnegative, namely $\mathbf{m}_q(\mathbf{g}_t - \mathbf{s}_t) \geq 0$ for all q. We note that this inequality corresponds to $g_{qt} \geq s_{qt}$ for all q and t, where s_{qt} is the qth element of $\mathbf{s}_t$, only if $\mathbf{L}_1$ is a diagonal matrix with positive diagonal elements.

We discuss the existence of an **ep** in Section 3.4.

3.1 What Does The Equilibrium Look Like?

Consider two specifications of the game given in (10) in which all the parameters are identical with the exception of β, the attraction elasticity of effort in (1). In the absence of competition, we would expect that more effort will be expended in the setting when β

is higher, since the marginal benefit generated by the last unit of effort is higher in this case. With competition, however, the consequences of a higher elasticity of effort are not as obvious. Since the elasticity measure is common to all competitors and the size of the market is fixed, it may no longer be in each player's best interest to expend more effort in the setting exhibiting the higher elasticity. What determines which firms will expend more or less? We address such comparative statics issues in Sections 4 and 5. In Section 4, we show in some duopoly cases that it will be optimal to expend less. The requisite conditions for each case are specified in terms of the parameter values.

The effect of the attraction elasticity on equilibrium effort allocations is one of several results in Sections 4 and 5 that describe the influence of certain parameters on optimal allocations. There are few sequential games whose equilibrium points have been characterized qualitatively. However, we are able to conduct a fairly extensive parametric analysis by exploiting results in Sobel (1990a, 1990b) that establish conditions under which an **ep** of an **sg** coincides with an **ep** of an associated static game. Such **ep**'s are called *myopic*, since the repeated application of a solution to a one-period game is optimal in the multi-period game. The formal presentation of the general conditions, the equivalence result, and the demonstration that the conditions hold in the market share model are given in the appendix. In the next sub-section, we discuss the intuition behind the conditions and discuss their ramification in the market share attraction model.

3.2 An Equivalent Static Game

Here is the basic idea. Suppose that both the expected single-period rewards and the dynamical equation describing the evolution of the state from period to period are (a) additively separable in state and action and (b) linear in the state variable. Notice that (8) and (9) have this form. In (8), the single-period reward accruing to firm q is a non-linear function $[\mu(\mathbf{g}) - \mathbf{m}_q\mathbf{g}]$ of the "action" vector $\mathbf{g}$ plus a linear function of the state $(\mathbf{m}_q\mathbf{s}_t)$. In (9), we see that the dynamics of the decision process are linear and separable in both state and action. The conditions, therefore, correspond to requirements that the expected single-stage reward and the expected state next period be affine functions of the state vector. See Dirven and Vrieze (1986) for an advertising duopoly model with affine structure that differs from ours.

Under these conditions, it can be shown (see the Appendix for details) that there exists a function $H_q(\mathbf{g})$, specified below, such that

$$E\left[\sum_{t=1}^{\infty}(\delta_q)^{t-1}X_{qt}\right] = E\left[\sum_{n=1}^{\infty}(\delta_q)^{t-1}H_q(\mathbf{g}_t)\right] + \kappa_q \tag{11}$$

where κ_q is a constant that does not depend upon the strategies employed by the competitors. Let $\mathbf{e}_q$ be the qth unit row vector, $\mathbf{I}$ be the identity matrix,

$$\mathbf{J}_q = \left(\mathbf{L}_1 + \delta_q\mathbf{L}_2 + \ldots + \delta_q^{N-1}\mathbf{L}_N\right)^{-1},$$

and assume that $\mathbf{J}_q$ exists and $\mathbf{L}_1$ is nonsingular (but neither is necessarily a diagonal matrix). Then $H_q(\mathbf{g})$ has the form:

$$H_q(\mathbf{g}) = \mu_q(\mathbf{g}) - \mathbf{e}_q\mathbf{J}_q\left[\mathbf{I} - \delta_q E(\mathbf{K}_1)\right]\mathbf{g}. \tag{12}$$

Under the further assumption that $E(\mathbf{K}_1)$ and $\mathbf{L}_n$ for $n = 1, \ldots, N$ are diagonal matrices, (12) simplifies to

$$H_q(\mathbf{g}) = \mu_q(\mathbf{g}) - c_q g_q, \tag{13}$$

where

$$c_q = \frac{[1 - \delta_q E(\xi_{qt})]}{\sum_{n=1}^{N}(\delta_q)^{n-1}\lambda_{qn}}. \tag{14}$$

It is convenient to write $\mathbf{g}$ as $(g_1, \ldots, g_Q)$. For the remainder of this paper (until the Appendix), g_q refers to the qth component of $\mathbf{g}$ and not to the vector of actions in "period" q. We shall discuss the consequences of (11) and then examine the structure of $H_q(\cdot)$. Notice that $H_q(\cdot)$ does not depend upon the time period. Let Γ be the one-period noncooperative game amongst players $1, \ldots, Q$ in which player q chooses $g_q \geq 0$, $\mathbf{g} = (g_1, \ldots, g_Q)$, and player q's payoff is $H_q(\mathbf{g})$ defined by (12). Suppose $\mathbf{g}^* = (g_1^*, \ldots, g_Q^*)$ is an unrandomized Nash equilibrium point of Γ. It follows that an **ep** of the sequential game (1) is given by $g_{qt} = g_q^*$ for all t *if* such choices are feasible (i.e., $g_{qt}^* \geq s_{qt}$, where s_{qt} is the qth component of the state vector that satisfies the dynamics given in (9)). Therefore, setting $\mathbf{g}_t = \mathbf{g}^*$ for all t is an **ep** of the **sg** with respect to H, the set of initial conditions for which this strategy is feasible.

In the case where $E[\mathbf{K}_1]$ and $\mathbf{L}_n$, $n = 1, \ldots, N$ are diagonal, an equilibrium of the static game with payoffs given in (13) yields a dynamic equilibrium. Notice that the expression for c_q in (14) involves only parameters of the model and does not depend upon the decision variables of any of the Q firms. If c_q is interpreted as the "cost" per unit of goodwill chosen by firm q, then (13) represents the single-period "net reward" accruing to firm q if $\mathbf{g}$ is the level of goodwill is chosen by all of the competitors. When $\lambda_{q1}, \ldots, \lambda_{qN}$ is a probability distribution, as in Bultez and Naert (1979) and Magat, et al. (1986), then the denominator in (14) is a probability generating function with argument δ_q.

3.3 Repeatability

This subsection concerns the feasibility of each firm repeating its same level of goodwill in every period. The feasibility of $\mathbf{g}_t = \mathbf{g}^*$ for all t corresponds to $\mathbf{a}_t \geq 0$ for all t. From (6) and (7),

$$\mathbf{a}_t = \mathbf{M}\left[\mathbf{g}_t - \mathbf{K}_{t-1}\mathbf{g}_{t-1} - \sum_{n=2}^{N} \mathbf{L}_n \mathbf{a}_{t-n+1}\right]. \tag{15}$$

Notice that the sign of $\mathbf{a}_t$ depends on the signs of the differences of the components of the random vector in the bracketed term and on $\mathbf{a}_{t-1}, \ldots, \mathbf{a}_{t-N+1}$. It follows, therefore, that $\mathbf{a}_t \geq 0$ when $\mathbf{K}_t$ and $\mathbf{L}_n$ are not "too variable" and the lag structure is "moderate". In Case 4 below, we give an example that explicitly addresses these issues.

Although it seems difficult to obtain general sufficiency conditions for $\mathbf{a}_t \geq 0$ when $\mathbf{g}_t = \mathbf{g}^*$ (for all t), we can easily establish conditions for interesting special cases.

Case 1: No lagged direct effects.

If $N = 1$, the $\mathbf{K}_t$'s are diagonal matrices, and ξ_{qt} are random variables that take on values in $[0, 1)$, then (15) and nonnegativity of $\mathbf{M}$ imply $\mathbf{a}_t \geq 0$ for all t.

Case 2: $Q = 1$ (Monopoly version of the model).

If $\xi_{1t} \equiv \theta \in (0,1]$ for all t, then $\mathbf{g}_0 = \mathbf{g}_1 = \mathbf{g}^*$ and (2) imply $a_{1t} \geq 0$, where

$$a_{1t} = \frac{(1-\theta)g_1^*}{\sum_{n=1}^{N} \lambda_{1n}} \tag{16}$$

if $a_{1,-N+2}, \ldots, a_{1,0}$ equals the right side of (16). Hence, $a_{1t} \geq 0$ for all t if $\theta \leq 1$.

Case 3: (Deterministic dynamics with geometrically-distributed lags).

Suppose $\mathbf{K}_t \equiv \Theta = \text{diag}(\theta_q)$ and $\mathbf{L}_t = \mathbf{L}^t$ with $\mathbf{L} = \text{diag}(\lambda_q)$, $0 \leq \theta_q \leq 1$, and $0 < \lambda_q < 1$ for all q and t ($N = \infty$). Suppose further that $\mathbf{a}_{-N+2} = \ldots = \mathbf{a}_0 = 0$ and $\mathbf{g}_0 = \mathbf{g}^*$. Then (15) and $\mathbf{LM} = \mathbf{I}$ imply

$$\mathbf{a}_t = \mathbf{M}(\mathbf{I} - \Theta)\mathbf{g}^* - \sum_{n=1}^{t-1} \mathbf{L}^n \mathbf{a}_{t-n}$$

for $t = 1, 2, \ldots$. Therefore, $\mathbf{a}_1 = \mathbf{M}(\mathbf{I} - \Theta)\mathbf{g}^*$ and $\mathbf{a}_t = [\mathbf{M} - \mathbf{I}](\mathbf{I} - \Theta)\mathbf{g}^*$ for $t = 2, 3 \ldots$ and both vectors have nonnegative components.

Case 4: $N = 2$.

Let $\mathbf{E}$ be the $Q \times Q$ matrix that is all one's. We assume that $b\mathbf{E} \leq \mathbf{K}_1 < f\mathbf{E}$ (with probability one), $b \neq 1$, and $\mathbf{L}_2$ is a diagonal matrix. Then $\mathbf{L}_2\mathbf{a}_0 \leq \mathbf{M}(\mathbf{I} - \mathbf{K}_0)\mathbf{g}^*$ and $\lambda_{q2} \leq (1-f)/(1-b)$ for all q imply $a_{qt} \geq 0$ for all q and t, as we shall prove. These conditions can be interpreted as $\mathbf{a}_0$ not "too large" , ξ_{qt} not "too variable", and moderate lagged effects.

Let $\mathbf{u}_t = \mathbf{M}(\mathbf{I} - \mathbf{K}_t)\mathbf{g}^*$. From (15), $0 \leq \mathbf{a}_{t+1}$ if $\mathbf{L}_2\mathbf{a}_t \leq \mathbf{u}_t$; but $\mathbf{L}_2\mathbf{a}_t = \mathbf{L}_2\mathbf{M}(\mathbf{I} - \mathbf{K}_{t-1})\mathbf{g}^* \leq \mathbf{u}_t = \mathbf{M}(\mathbf{I} - \mathbf{K}_t)\mathbf{g}^*$, which corresponds to $0 \leq (\mathbf{I} - \mathbf{L}_2)\mathbf{M}\mathbf{g}^* + \mathbf{L}_2\mathbf{M}\mathbf{K}_{t-1}\mathbf{g}^* - \mathbf{M}\mathbf{K}_t\mathbf{g}^*$ whose right side is bounded below by $[(1-f)\mathbf{I} - (1-b)\mathbf{L}_2]\,\mathbf{M}\mathbf{g}^*$, which is nonnegative if $(1-f)/(1-b) \geq \lambda_{q2}$ for all q.

3.4 Existence of Equilibrium Points

Sufficient conditions for the existence of **ep**'s of discounted **sg**'s depend on the structure of the sets of states and actions, the players' single-period reward functions, and the dynamics (transition probabilities). An **ep** exists if there are only finitely many states and actions [Fink (1964)]. If the state space is a continuum, only approximation results are known ("ϵ–equilibria") [Whitt (1980) and Nowak (1985)].

In our model, the sets of states and actions are continua and the dynamics are not first-order Markov. We can dispose of the latter difficulty, namely that the expression for $\mathbf{s}_{t+1}$ in (6) involves actions taken in periods earlier than t, by suitably expanding the dimension of the state vector. However, the resulting model has a continuum of states (and actions) and the general theory yields only approximation results. We take an alternative approach and exploit (11).

Let $\mathbf{g}^*$ be an **ep** (unrandomized) of Γ. Then it follows from (11) and (5) that $\mathbf{g}_t = \mathbf{g}^*$ for all $t = 1, 2, \ldots$ is an **ep** of the **sg** with respect to the set of initial conditions such that $\mathbf{g}^*$ is repeatable, i.e., $P\{\mathbf{a}_t \geq 0 \text{ for all } t \,|\, \mathbf{g}_\tau = \mathbf{g}^* \text{ for all } \tau\} = 1$. Section 3.3 presents several instances of repeatability. It follows from Monahan (1987) that a sufficient condition for Γ to have a unique unrandomized equilibrium point $\mathbf{g}^*$ is $\beta < 1$ and $\mathbf{e}_q \mathbf{J}_q \left[\mathbf{I} - \delta_q E(\mathbf{K}_1)\right] \mathbf{g} > 0$ for each q. If $E(\mathbf{K}_1)$ and $\mathbf{L}_n$ for each n are diagonal matrices, the latter condition reduces to $c_q > 0$ for each q.

4. Properties of a Duopoly Equilibrium Point

It follows from Section 3 that the analysis of an **ep** for the **sg** with payoffs given by (10) can be reduced to the analysis of $\mathbf{g}^*$, an **ep** of the static game with payoffs given by either (12) or (13), depending upon the assumptions regarding the formation of goodwill. In this section, we restrict our attention to the duopoly game with $E[\mathbf{K}_1]$ and $\mathbf{L}_n$ being diagonal matrices, $n = 1, \ldots, N$.

The static game with payoffs given by (13) is a minor variation of the "linear budget" model in Monahan (1987). Drawing on the analysis in that paper, the following characteristics of an **ep** of the static game are readily established.

Lemma 1. *Let M_q, b_q, and $\beta < 1$ be the parameters given in (1) and let c_q be given by*

(14), $q = 1,2$. If $\mu(0,0) = 0$ and $VM_q > c_q$ for $q = 1,2$, then the unique **ep** *of the static game is given by*

$$g_1^* = \frac{T(r)}{c_1} \quad \text{and} \quad g_2^* = \frac{M_2 T(r)}{M_1 c_2}, \tag{17}$$

where

$$r \equiv \frac{c_1 M_2}{c_2 M_1} \quad \text{and} \quad T(r) = \frac{\beta V M_1 b_1 b_2 r^\beta}{[b_1 + b_2 r^\beta]^2}. \tag{18}$$

Several functions of the parameters of the model, discussed in detail in Monahan (1987), also play a significant role in establishing qualitative properties of an equilibrium point in the dynamic, stochastic setting of this paper. The first function is the *competitive advantage ratio*, defined as

$$R = \frac{b_1}{b_2} r^{-\beta}. \tag{19}$$

As we show below, the sign of $R - 1$ determines how various parameters influence the equilibrium goodwill levels. The condition that $R > 1$ corresponds to firm 1's total effectiveness of effort $(b_1(VM_1/c_1)^\beta)$ being greater than firm 2's total effectiveness of effort $(b_2(VM_2/c_2)^\beta)$.

The second function of the parameters of the model, established in Monahan (1987), is $\beta^*(r) = k/\ln(r)$ $(r \neq 1)$ where k is a solution to the hyperbolic equation

$$e^k = \frac{b_1(k+1)}{b_2(k-1)}.$$

When $r = 1$, firm 1's equilibrium effort allocation is an increasing function of β. When $r \neq 1$, firm 1's equilibrium allocation is an increasing (decreasing) function of β when $\beta < (>) \beta^*(r)$. Recall the hypothetical case in Section 3.1. Suppose β_1 and β_2 denote the effort elasticities in the two settings and $\beta_1 < \beta_2$. If $\beta_1 > \beta^*(r)$, then firm 1's equilibrium allocation is lower at $\beta = \beta_2$ then at $\beta = \beta_1$.

Both R and β^* appear in Table 1 which summarizes the structure of an equilibrium solution to the static game. Since these results are analogous to results given in Monahan (1987), the proofs are omitted. Qualitative characteristics of the equilibrium point can be deduced from the entries in this table. Result 7, for example, states that if the competitive advantage ratio is less than one ($R < 1$), then firm 1 is not as "strong" as firm 2 in the sense of total effectiveness of effort discussed above; so an increase in firm 2's profit per unit (M_2) is associated with a lower equilibrium level of firm 1's goodwill (i.e., g_1^* is lower). Results 7 and 8 specify how $\beta^*(r)$ determines the way in which the attraction elasticity effects optimal effort decisions. Results 10–13 indicate how $\beta^*(r)$ changes as either b_1 or b_2 increases.

Table 1 About Here

Much of the analysis in the next section hinges on the dependence of g_1^* on c_1 defined in (14). Note from (13) that c_1 can be viewed as the constant marginal cost of goodwill. It follows from (17) that the equilibrium level of firm q's goodwill diminishes as c_q increases and does so at a decreasing rate.

PROPOSITION 1. *Firm q's equilibrium goodwill level g_q^* is a decreasing convex function of c_q.*

If $\mathbf{g}_t = \mathbf{g}^*$ for each t, then $a_{qt}^* = \mathbf{m}_q(\mathbf{g}^* - \mathbf{s}_t)$ is the effort level expended by firm q in period t, when the state of the sequential game at the beginning of period t is $\mathbf{s}_t$ (cf. the expression in (7)). In the next subsection, we summarize the effects on a_{qt}^* of changes in some of the parameters.

4.1 Properties of Competitive Effort Levels

We have seen that a solution to the dynamic game has the following extremely simple structure: firm q simply "replenishes" its level of goodwill each period until it equals g_q^*. What does this simple strategy imply about optimal expenditures of effort? We show in this subsection, that the relative simplicity of the goodwill strategy being employed can induce complex expenditures of effort. The example below has a fractional tit-for-tat pattern of effort expenditures.

In a one-period deterministic duopoly attraction model, Monahan (1987) shows that the allocations of one competitor are proportional to the allocations of the other competitor. If one competitor increases effort, it is optimal for the other to do likewise. However, the competitive effort decisions in the dynamic attraction model can be quite different. In order to focus on the influence of the lag structure on effort decisions, we examine a special case of the general model given in Section 2.

4.1.1. $\mathbf{K}_t = \theta\mathbf{I}$ FOR ALL t.

The special case considered here is a deterministic version of the general model with a one-period lag. We assume $\mathbf{L}_n = 0$, $n > 2$ and $\mathbf{K}_t = \theta\mathbf{I}$ for all t, where θ is a positive scalar. In this case, $\mathbf{g}_t = \mathbf{g}^*$ for all t and (15) imply

$$\mathbf{a}_t^* = \mathbf{L}_1^{-1}\left[\mathbf{g}^*(1-\theta) - \mathbf{L}_2\mathbf{a}_{t-1}^*\right], \tag{20}$$

where $\mathbf{g}^*$ is the vector of equilibrium goodwill levels.

Intertemporal variation of competitive effort decisions is implicit in (20). For example, suppose that $\mathbf{L}_1 = \text{diag}(\lambda_{q1})$ with $\lambda_{q1} > 0$, and $\mathbf{L}_2 = [\alpha_{ij}]$, $i, j = 1, 2$. From (20),

$$a_{qt}^* = \left[(1-\theta)g_q^* - \alpha_{q1} - a_{q,t-1}^*\lambda_{q1} - \alpha_{q2}a_{3-q,t-1}^*\right]/\lambda_{q1} \tag{21}$$

for $q = 1, 2$. Competitor q's equilibrium level of expenditure of effort in period t depends upon its own equilibrium goodwill level and the levels of effort expended in period $t-1$ by itself and its competitors.

Example.

The following example illustrates the period-to-period variability of effort allocations that can accompany a policy that maintains a constant stock of goodwill. The example is also interesting because the two competitors follow a "tit-for-tat" strategy discussed, for example, in Kalai and Stanford (1985).

In Figure 1, we graph (21) as a function of t. The parameter values are:

$$(1-\theta)g_1^*/\lambda_{11} = 110 \qquad \alpha_{11}/\lambda_{11} = 0.4 \qquad \alpha_{12}/\lambda_{11} = 0.005$$

$$(1-\theta)g_2^*/\lambda_{21} = 90 \qquad \alpha_{21}/\lambda_{21} = 0.4 \qquad \alpha_{22}/\lambda_{21} = 0.005$$

$$a_{1,0}^* = 0 \qquad a_{2,0}^* = 85$$

Each competitor employs a "pulsing" expenditure policy–in one period, one competitor expends relatively more than the other and the pattern is reversed in the following period. While these cycles repeat, the optimal effort levels dampen over time. In effect, we have an example where a fractional tit-for-tat strategy is optimal in the larger class of closed-loop strategies. Note that the pulsing effort allocations arise even though each competitor is applying a "myopic" equilibrium strategy of maintaining goodwill levels at $\mathbf{g}^*$.

Insert Figure 1 About Here

The pulsing phenomenon can persist even in the presence of random goodwill effects. For example, we have simulated equilibrium effort allocations in a model without direct lagged effects ($\mathbf{L}_n = 0$ for all $n \geq 2$) and with ξ_{qt} is uniformly distributed between 0.0 and 0.5 (Case 1 in Section 3.3). We observed pulsing expenditure patterns over a wide range of parameter values.

5. Other Properties of an Equilibrium

This section presents additional results describing the qualitative dependence of equilibrium efforts on model parameters.

Proposition 2. *If $0 \leq E(\xi_{q1}) \leq 1$ and $\lambda_{qn} \geq 0$ for all q and n, then the goodwill level g_q^* is an increasing function of the discount factor δ_q.*

Proof: Differentiating (14) with respect to δ_q establishes that c_q is a nonincreasing function of δ_q. The result then follows from Proposition 1. ∎

The influence of the discount factor on equilibrium goodwill is not unexpected. A decrease in the discount factor diminishes the valuation of future returns generated by current goodwill. Current period payoffs are concave in goodwill, and hence both effort and goodwill decline as the discount factor decreases.

The remaining results in this section concern the effect of the lag structure on equilibrium allocations. Let $\Lambda_q = (\lambda_{q1}, \ldots, \lambda_{qN})$ and $\Lambda'_q = (\lambda'_{q1}, \ldots, \lambda'_{qN})$ be different specifications of firm q's diagonal elements in the matrices $\mathbf{L}_1, \ldots, \mathbf{L}_N$ in the dynamical equation (2) for goodwill. We write $\Lambda_q \succeq \Lambda'_q$ if

$$L \equiv \sum_{n=1}^{N} \delta_q^{n-1} \lambda_{qn} \geq L' \equiv \sum_{n=1}^{N} \delta_q^{n-1} \lambda'_{qn}. \tag{22}$$

If $\Lambda_q \succeq \Lambda'_q$, current effort allocations have less impact in the "primed" model than they do in the "unprimed" model. In this sense, the lag structure in the "primed" model is weaker than in the "unprimed" model.

As in Proposition 2, firm q's equilibrium level of goodwill is lower if the impact of future events on current profits is reduced. Let firm q's goodwill level with the "primed" lag structure be g'_q.

PROPOSITION 3. *If* $\Lambda_q \succeq \Lambda'_q$, *then* $g_q^* \geq g_q'^*$.

PROOF: The hypothesis is that $L \geq L'$, so that

$$c_q = \frac{1 - \delta_q E(\xi_{q1})}{L} \leq \frac{1 - \delta_q E(\xi_{q1})}{L'} = c'_q.$$

Therefore, Proposition 1 implies that $g_q^*(c_q) \geq g_q'^*(c_q)$. ∎

Bultez and Naert (1979) use Lydia Pinkham data to estimate the parameters of a Pascal-distributed lag structure. The next result summarizes the influence of the Pascal parameters on equilibrium effort levels.

Suppose λ_{qn} is Pascal-distributed with parameters x and y; i.e.,

$$\lambda_{qn} = (1 - x)^y \binom{y + n - 2}{n - 1} x^{n-1}, \quad n = 1, 2, \ldots \text{ and } y = 0, 1, \ldots. \tag{23}$$

COROLLARY 4. *If $\{\lambda_{qn}\}$ satisfies (23), then g_q^* increases (decreases) as y (x) increases.*

PROOF: Using (23), it can be shown that

$$L \equiv \sum_{n=1}^{N} (\delta_q)^{n-1} \lambda_{qn} = \sum_{n=1}^{N} (\delta_q)^{n-1} (1-x)^y x^n \binom{y+n-1}{n},$$

which simplifies to $L = \{(1-x)/(1-\delta x)\}^y$. Therefore, $\partial L/\partial x < 0$ and $\partial L/\partial y > 0$, which (from Proposition 3) establishes the result. ■

The interpretation of these results is simplified by noting a relationship developed in the proof of Proposition 3. There we show that c_q, hence g_q^*, is an increasing function of Λ. We established Corollary 4 by showing that Λ increases (decreases) with y (x). The interpretation, therefore, is consistent with the discussion following Proposition 2.

6. SUMMARY

We formulated a stochastic, dynamic oligopoly model in which intra-period gross profits have a market share attraction form. Market shares in the model evolve randomly and respond to previous effort levels via a general lag structure. The model utilizes goodwill as a measure of the market consequences of current and past effort decisions. We observed that the model's dynamics and single-period rewards possess an affine structure that yields a relatively simple expression for each firm's expected discounted expected profits. This simple expression led to a single-period noncooperative game whose equilibrium points may identify equilibrium points of the dynamic oligopoly model. The key requirement is the condition that it must be feasible for every for each firm to repeat its portion of the single-period game's equilibrium point. Several examples were given where this repeatability condition is satisfied. Further, we present an example where repetition of the equilibrium point induces continual fluctuation of effort levels.

We exploited the market share attraction form further by identifying an equilibrium point of the one-period game with aesthetic appeal. Finally, we converted this equilibrium point to an equilibrium point of the dynamic oligopoly model and deduced the qualitative dependence of equilibrium effort levels on various model parameters.

APPENDIX

Suppose that a discrete-time stochastic optimization process is observed and controlled denumerably many times by the composite actions of players $1, \ldots, Q$. Let states (i.e., values taken by the stochastic process) be j-dimensional column vectors with real components, let S be the state space, and let $A_{\mathbf{s}q}$ be player q's nonempty set of feasible actions if the state is $\mathbf{s}' = (s_1, \ldots, s_j)$, where prime denotes transpose. The state and player q's action in period t are $\mathbf{s}_t$ and g_{qt}, respectively. Let $\mathbf{g}_t = (g_{1t}, \ldots, g_{Qt})$. It is expedient to permit the rewards and dynamics in a period to depend on earlier states and actions; so the pertinent history in period t is $\mathbf{h}_t = (\mathbf{s}_{-B+2}, \mathbf{g}_{-B+2}, \ldots, \mathbf{s}_1, \mathbf{g}_1, \ldots, \mathbf{s}_{t-1}, \mathbf{g}_{t-1}, \mathbf{s}_t)$ for an appropriate integer B.

Player q's reward in period t is the random variable X_{qt} and player q's single-period discount factor is δ_q, $0 < \delta_q < 1$. Let $A = \times_{q=1}^{Q} \cup_{\mathbf{s} \in S} A_{\mathbf{s}q}$.

For $b = 1, \ldots, B$, let $\mathbf{w}_b(\cdot)$ be a column j-vector-valued function on A and $\mathbf{W}_b$ be a $j \times j$ matrix, and for each player q, let $y_{qb}(\cdot)$ be a real-valued function on A and $\mathbf{Y}_{qb}$ be a row j-vector.

Suppose for all $t = 1, 2, \ldots$

CONDITION I:

$$E\left(\mathbf{s}_{t+1} \middle| \mathbf{h}_t, \mathbf{g}_t\right) = \sum_{b=1}^{B} \left[\mathbf{w}_b(\mathbf{g}_{t-b+1}) + \mathbf{W}_b \mathbf{s}_{t-b+1}\right] \tag{A1}$$

CONDITION II:

$$E\left(X_{qt} \middle| \mathbf{h}_t, \mathbf{g}_t\right) = \sum_{b=1}^{B} \left[y_{qb}(\mathbf{g}_{t-b+1}) + \mathbf{Y}_{qb} \mathbf{s}_{t-b+1}\right]. \tag{A2}$$

Let $\mathbf{C}_q = \sum_{i=1}^{B} (\delta_q)^i \mathbf{W}_i$. The following result is established in Sobel (1990a).

PROPOSITION 1. *If $\mathbf{I} - \mathbf{C}_q$ is nonsingular, Conditions I and II imply that (11) is valid for every control strategy and initial history for which the expectation on the left side exists* *and is finite. Moreover, the number κ_q depends on h_1 but not on the strategy, and*

$$H_q(\mathbf{g}) = \sum_{b=1}^{B} (\delta_q)^{b-1} \left[y_{qb}(\mathbf{g}) + \mathbf{Y}_{qb} \left[\mathbf{I} - \mathbf{C}_q\right]^{-1} \sum_{k=1}^{B} (\delta_q)^{k-1} \mathbf{w}_k(\mathbf{g}) \right]. \tag{A3}$$

We assume that $\mathbf{I} - \mathbf{C}_q$ is nonsingular and now show that Conditions I and II are satisfied by the market share attraction model, whose single-stage reward and dynamics are specified by (8) and (9), respectively. We see that (8) satisfies Condition II with $y_{qb}(\cdot) \equiv 0$ and $\mathbf{Y}_{qb} = 0$ if $b \geq 2$, $y_{q1}(\mathbf{g}_t) = \mu_q(\mathbf{g}_t) - \mathbf{m}_q \mathbf{g}_t$, and $\mathbf{Y}_{q1} = \mathbf{m}_q$. Likewise, (9) satisfies Condition I with $B = N - 1$, $\mathbf{w}_1(\mathbf{g}) = [E(\mathbf{K}_1) + \mathbf{L}_2\mathbf{M}]\,\mathbf{g}$, $\mathbf{W}_1 = -\mathbf{L}_2\mathbf{M}$, $\mathbf{w}_n(\mathbf{g}) = \mathbf{L}_{n+1}\mathbf{M}\mathbf{g}$ and $\mathbf{W}_n = -\mathbf{L}_{n+1}\mathbf{M}$ for $n > 1$.

Now we prove that the substitution of these elements in (A3) leads to (12). Substitution into (A3) yields

$$\begin{aligned} H_q(\mathbf{g}) &= \mu(\mathbf{g}) - \mathbf{m}_q \mathbf{g} \\ &\quad + \delta_q \mathbf{m}_q \left[\mathbf{I} - \delta_q(-\mathbf{L}_2\mathbf{M}) - \sum_{n=2}^{N-1} (\delta_q)^n \mathbf{L}_{n+1}\mathbf{M} \right]^{-1} \cdot \\ &\quad \cdot \left[[E(\mathbf{K}_1) + \mathbf{L}_2\mathbf{M}]\mathbf{g} + \sum_{n=2}^{N-1} (\delta_q)^{n-1} \mathbf{L}_{n+1}\mathbf{M}\mathbf{g} \right] \end{aligned} \tag{A4}$$

Note that

$$\left[\mathbf{I} + \sum_{n=1}^{N} (\delta_q)^n \mathbf{L}_{n+1}\mathbf{L}_1^{-1} \right]^{-1} = \left[[\mathbf{I}\mathbf{L}_1 + \sum_{n=1}^{N} (\delta_q)^n \mathbf{L}_{n+1}\mathbf{L}_1^{-1}\mathbf{L}_1] \cdot \mathbf{L}_1^{-1} \right]^{-1}. \tag{A5}$$

Since $(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ for nonsingular matrices $\mathbf{A}$ and $\mathbf{B}$, (A5) simplifies to $\mathbf{L}_1\mathbf{J}_q$ and (A4) becomes

$$H_q(\mathbf{g}) = \mu(\mathbf{g}) - \mathbf{m}_q\mathbf{g} + \delta_q \mathbf{m}_q \mathbf{L}_1 \mathbf{J}_q [E(\mathbf{K}_1) + \sum_{n=2}^{N} (\delta_n)^{n-2} \mathbf{L}_n \mathbf{M}]\mathbf{g}. \tag{A6}$$

But $\sum_{n=2}^{N} (\delta_q)^{n-2}\mathbf{L}_n = (1/\delta_q)\left[(\mathbf{J}_q)^{-1} - \mathbf{L}_1\right]$. Therefore, (A6) becomes (after substituting $\mathbf{m}_q\mathbf{L}_1 = \mathbf{e}_q$)

$$\begin{aligned} H_q(\mathbf{g}) &= \mu(\mathbf{g}) - \mathbf{m}_q\mathbf{g} + \delta_q \mathbf{m}_q \mathbf{L}_1 \mathbf{J}_q \left[E(\mathbf{K}_1) + \left[(\mathbf{J}_q)^{-1} - \mathbf{L}_1\right] \frac{1}{\delta_q} \mathbf{M} \right] \mathbf{g} \\ &= \mu(\mathbf{g}) - \mathbf{m}_q\mathbf{g} + \delta_q \mathbf{e}_q \mathbf{J}_q E(\mathbf{K}_1)\mathbf{g} + \delta_q \mathbf{e}_q \left[\mathbf{I} - \mathbf{J}_q\mathbf{L}_1\right] \frac{1}{\delta_q} \mathbf{M}\mathbf{g} \\ &= \mu(\mathbf{g}) - \mathbf{e}_q \mathbf{J}_q [\mathbf{I} - \delta_q E(\mathbf{K}_1)]\mathbf{g}, \end{aligned}$$

which is (12).

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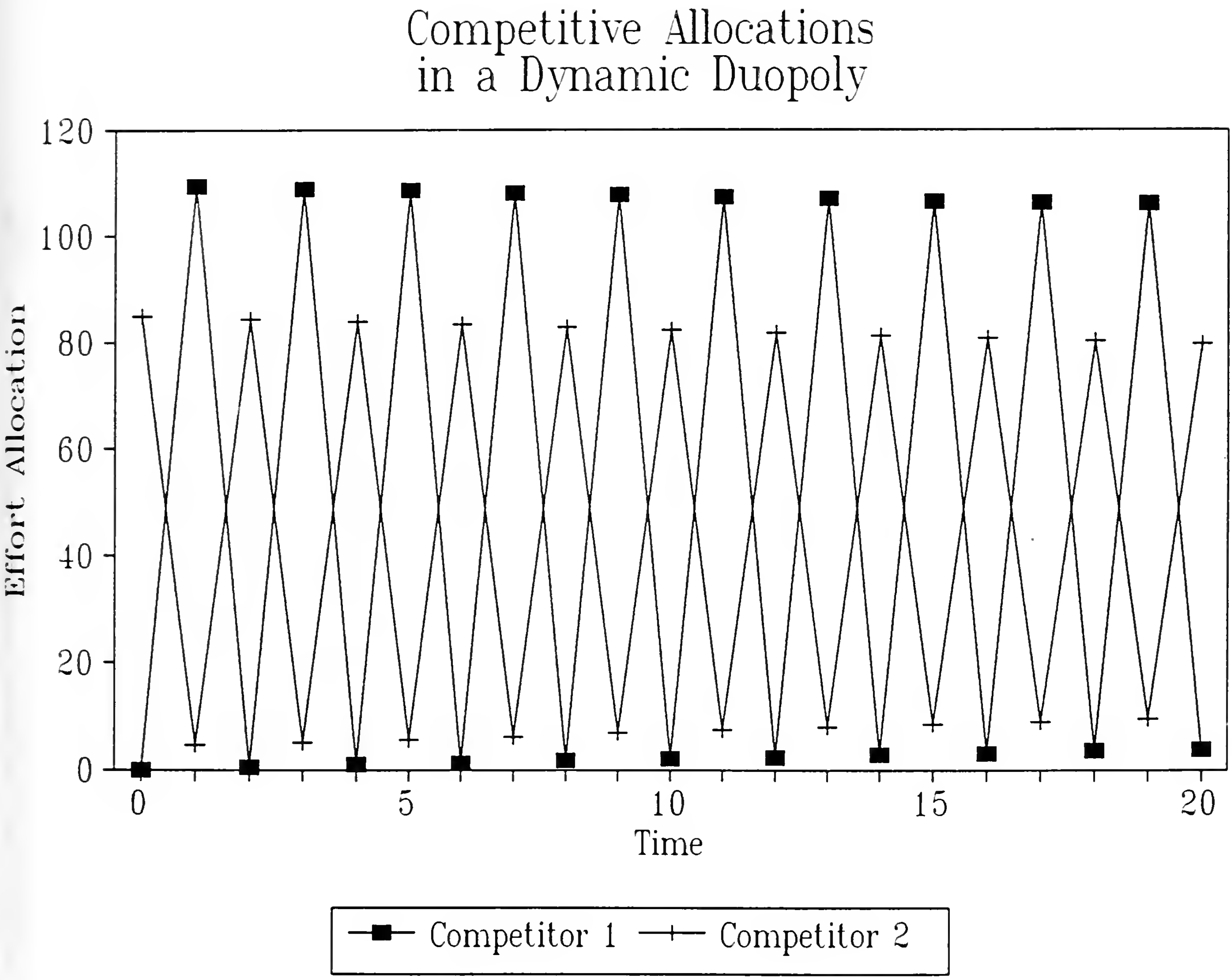
Result Number	Variable	Assumption	Conclusion Variable	Direction†
1	b_1	$R > 1$	g_1^*	+
2	b_1	$R < 1$	g_1^*	−
3	b_2	$R > 1$	g_1^*	−
4	b_2	$R < 1$	g_1^*	+
5	M_1	− − −	g_1^*	+
6	M_2	$R > 1$	g_1^*	+
7	M_2	$R < 1$	g_1^*	−
8	β	$\beta > \beta^*$	g_1^*	+
9	β	$\beta < \beta^*$	g_1^*	−
10	b_1	$r < 1$	β^*	−
11	b_1	$r > 1$	β^*	+
12	b_2	$r < 1$	β^*	+
13	b_2	$r > 1$	β^*	−

†Increasing and decreasing are denoted "+" and "−", respectively.

Table 1. Qualitative Properties of an Equilibrium


Competitive Allocations
in a Dynamic Duopoly
Effort Allocation
0
20
40
60
80
100
120
0
5
10
15
20
Time
Competitor 1
Competitor 2