STRENGTH OF MATERIALS

A TEXT-BOOK

FOR

SECONDARY TECHNICAL SCHOOLS

BY

MANSFIELD MERRIMAN

Member of International Association for Testing Materials

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PREFACE TO THE FIRST EDITION

In the following pages the attempt is made to give a presentation of the subject of the strength of materials, beams, columns, and shafts, which may be understood by those not acquainted with the calculus. The degree of mathematical preparation required is merely that now given in high schools, and includes only arithmetic, algebra, geometry, and such a course in mechanics as is found in elementary works on physics. In particular the author has had in mind the students in the higher classes of manual training schools, and it has been his aim to present the subject in such an elementary manner that it may be readily comprehended by them and at the same time cover all the essential principles and methods.

As the title implies the book deals mainly with questions of strength, the subject of elastic deformations occupying a subordinate place. As the deductions of the deflections of beams are best made by the calculus they are not here attempted, but the results are stated so that the student may learn their uses; later, if he continues the study of engineering, his appreciation of the proofs that he will then read will be accompanied with true scientific interest.

All the rules for the investigation and design of common beams, including the subject of moment of inertia, are here presented by simple algebraic and geometric methods. As the mechanical ideas involved are by far the most difficult part of the subject, a special effort has been made to clearly present them, and to illustrate them by numerous practical numerical examples.

A chapter on the manufacture and general properties of materials is given, as also one on resilience and impact. Problems for students to solve are presented, and it should be strongly insisted upon that these should be thoroughly and completely worked out. It is indeed only by the solution of many numerical exercises that a good knowledge of the theory of the subject can be acquired.

NOTE TO THE SIXTH EDITION

In the fifth edition a new chapter was added on reinforced concrete, especially columns and beams. In this edition a new chapter on combined stresses is added, numerous changes have been made throughout, and many new problems introduced. The number of articles is increased from 72 to 91, the number of cuts from 48 to 54, and the number of problems from 140 to 230. For 45 of the new problems my thanks are due to Professor J. M. Jameson, of the Pratt Institute. It is hoped that the volume in its new form may advance the cause of sound technical education more effectively than before.

MANSFIELD MERRIMAN.

NEW YORK, JUNE, 1912.

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STRENGTH OF MATERIALS

Chapter 1

ELASTIC AND ULTIMATE STRENGTH

Art. 1. Direct Stresses

A 'stress' is an internal resistance which balances an exterior force. When a weight of 500 pounds is suspended by a rope, a stress of 500 pounds exists in every cross-section of the rope; or, if this rope is cut anywhere and the ends are connected by a spring balance, this will register 500 pounds. Stresses are measured in the same units as forces, namely, in pounds, tons, or kilograms.

A 'unit-stress' is the stress on a unit of area; this is expressed in pounds per square inch or in kilograms per square centimeter. Thus, when a bar of three square inches in cross-section is subject to a pull of 12 000 pounds, the unit-stress is 4000 pounds per square inch for the usual case when the total stress is uniformly distributed over the cross-section.

Three kinds of direct stress are produced by exterior forces which act on a body in such a way as to tend to change its shape; these are,

Tension, tending to pull apart, as in a rope.
Compression, tending to push together, as in a wall or column.
Shear, tending to cut across, as in punching a plate.

The forces which produce these kinds of stress may be called tensile, compressive, and shearing forces, while the stresses themselves are frequently called tensile, compressive, and shearing stresses.

A stress is always accompanied by a 'deformation' or change of shape of the body. As the applied force increases the deformation and the stress likewise increase, and if the force is large enough it finally overcomes the stress and the rupture of the body follows.

Tension and compression differ only in regard to direction (Fig. 1). A tensile stress in a bar occurs when two forces of equal intensity act upon its ends, each acting

Fig. 1

away from the end of the bar. In compression the direction of the forces is reversed, each acting toward the end of the bar. The tensile force produces a deformation called 'elongation' and the compressive force produces a deformation called 'shortening.' If P is the force in pounds, then the total stress in every section of the bar is equal to P.

Shear implies the action of two forces in parallel planes and very near together, like the forces in a pair of shears, from which analogy the name is derived. Thus, if a bar is laid upon two supports and two loads, each P pounds, are applied to it near the supports, there are hence produced near each support two parallel forces which tend to cut the bar across vertically (Fig. 2). In each of these sections the shearing stress is equal to P. The deformation caused by the shearing force P is a vertical sliding between the upward and downward forces; and the bar will be cut across if the external shear overcomes the internal stress.

In all cases of direct stress the total stress is supposed, unless otherwise stated, to be uniformly distributed over

Fig. 2

the area of the cross-section; this area will be called the 'section area.' Thus if A is the section area and S the unit-stress, then

$$P = AS, \qquad S = \frac{P}{A}, \qquad A = \frac{P}{S} \tag{1}$$

from which one of the quantities may be computed when the other two are given. For example, it is known that a wrought-iron bar will rupture under tension when the unit-stress S becomes 50 000 pounds per square inch; if the area A is $4\frac{1}{2}$ square inches, then the tensile force required to rupture the bar is $P = 4\frac{1}{2} \times 50\,000 = 225\,000$ pounds.

Prob. 1 *A*. A cast-iron bar which is to be subjected to a tension of 34 000 pounds is to be designed so that the unit-stress shall be 2500 pounds per square inch. What should be the section area in square inches? If the bar is round, what should be its diameter?

Prob. 1 *B*. If a cast-iron bar, $1\frac{1}{2} \times 2\frac{1}{4}$ inches in section area, breaks under a tension of 66 000 pounds, what tension will probably break a bar $1\frac{1}{4}$ inches in diameter?

Prob. 1 *C*. What should be the size of a round bar of structural steel to carry a tension of 200 000 pounds with a unit-stress of 15 000 pounds per square inch?

Art. 2. The Elastic Limit

When a tensile force is gradually applied to a bar it elongates, and up to a certain limit the elongation is proportional to the force. Thus, if a bar of wrought iron one square inch in section area and 100 inches long is subjected to a tension of 5000 pounds, it will be found to elongate 0.02 inches; if 10 000 pounds is applied, the elongation will be 0.04 inches; for 15 000 pounds it will be 0.06 inches, for 20 000 pounds 0.08 inches, for 25 000 pounds 0.10 inches. Thus far each addition of 5000 pounds has produced an elongation of 0.02 inches. But when the next 5000 pounds is added, making a total stress of 30 000 pounds, it will be found that the total elongation is about 0.13 or 0.14 inches, and hence the elongations are increasing more rapidly than the stresses.

The 'elastic limit' is defined to be that unit-stress at which the deformations begin to increase in a faster ratio than the stresses. In the above illustration this limit is about 25 000 pounds per square inch, and this indeed is the average value of the elastic limit for wrought iron. The term 'elastic strength' is perhaps more expressive than elastic limit, but the latter is the one in general use.

When the unit-stress in a bar is less than the elastic limit the bar returns, when the stress is removed, to its original length. When the unit-stress is greater than the elastic limit, the bar does not fully spring back, but there remains a so-called permanent set. In other words, the elastic properties of a bar are injured if it is stressed beyond the elastic limit. Hence it is a fundamental rule in designing engineering constructions that the unit-stresses in the members should never exceed the elastic limit of the material.

The following are average values of the elastic limits of the four materials most used in engineering construction under tensile and compressive stresses.

TABLE 1. ELASTIC LIMITS

Material	Pounds per Square Inch		Kilos per Sq. Centimeter	
	Tension	Compression	Tension	Compression
Timber	3 000	3 000	210	210
Cast Iron	6 000	20 000	420	1 400
Wrought Iron	25 000	25 000	1 750	1 750
Steel, structural	35 000	35 000	2 450	2 450

Values in English units must be carefully kept in mind by the student; and it may be noted that pounds per square inch multiplied by 0.07 will give kilos per square centimeter. But little is known concerning the elastic limit in shear; it is probably about three-fourths of the values above given for tension.

Prob. 2 *A*. A steel tie rod in a bridge is 1¼ inches in diameter. What load can be put on the rod if the unit-stress is not to exceed one-half the elastic limit?

Prob. 2 *B*. A square stick of timber is to carry a compressive load of 81 000 pounds. What should be its size in order that the unit-stress may be one-third of the elastic limit?

ART. 3. ULTIMATE STRENGTH

When a bar is under stress exceeding its elastic limit it is usually in an unsafe condition. As the stress is increased by the application of exterior forces the deformation rapidly increases, until finally the rupture of the bar occurs. By the term 'ultimate strength' is meant that unit-stress which occurs just before rupture, it being the highest unit-stress that the bar will bear.

The ultimate strengths of materials are from two to four times their elastic limits, but for some materials they are much greater in compression than in tension. The average values of the ultimate strengths will be given in subsequent articles.

The 'factor of safety' is a number which results by dividing the ultimate strength by the actual unit-stress that exists in a bar. For example, a stick of timber, 6×6 inches in section area, whose ultimate strength in tension is 10 000 pounds per square inch, is under a tensile stress of 32 400 pounds. The unit-stress then is $32\,400/36 = 900$ pounds per square inch, and the factor of safety is $10\,000/900 = 11$. The factor of safety was formerly much used in designing, but it is now considered the better plan to judge of the security of a body under stress by reference to its elastic limit. Thus in the above case, as the unit-stress is only one-third the elastic limit for timber, the degree of security may be regarded as sufficient.

Prob. 3 *A*. A bar of wrought iron $2\frac{1}{2}$ inches in diameter ruptures under a tension of 271 000 pounds. What is its ultimate strength in pounds per square inch?

Prob. 3 *B*. What should be the size of a round bar of structural steel to carry a tension of 125 000 pounds with a factor of safety of 5?

Prob. 3 *C*. What force is required to rupture in tension a cast-iron bar 8 inches in diameter, the ultimate tensile strength of cast iron being 20 000 pounds per square inch?

Art. 4. Tension

When a bar is tested under tension, it is done by loads which are gradually applied. The elongations increase proportionally to the stresses until the elastic limit is reached. After the unit-stress has exceeded the elastic limit the elongations increase more rapidly than the stresses, and a reduction in area of the cross-section of the bar often occurs. Finally the ultimate strength of the material is reached, and the bar tears apart.

A graphical illustration of these phenomena may be made by laying off the unit-stresses as ordinates and the elongations per unit of length as abscissas (Fig. 3). At

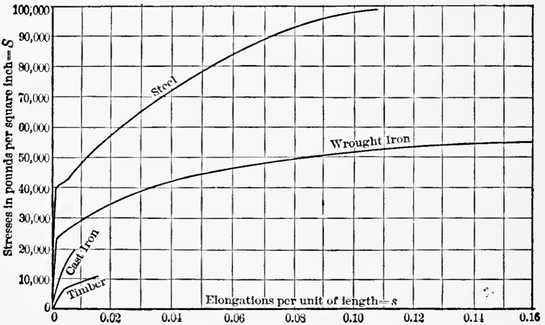

Fig. 3

various intervals, as the test progresses, the applied loads are observed and the resulting elongations are measured. The loads divided by the section area give the unit-stresses, while the total elongations divided by the length of the bar give the unit-elongations. On the plot a point is made for the intersection of each unit-stress with its corresponding unit-elongation, and a curve is drawn connecting the several points for each material. In this way curves are plotted showing the properties of each material. It is seen that each curve is a straight line from the origin 0 until the elastic limit is reached, showing that the elongations increase proportionally to the unit-stresses. At the elastic limit a sudden change in the curve is seen, and afterwards the elongation increases more rapidly than the stress. The end of the curve indicates the point of rupture. The curve for steel in the diagram is for a quality much stronger than structural steel, this being the kind mostly used in bridges and buildings.

The ultimate elongation is an index of the ductility of the material, and is hence generally recorded for wrought iron and steel; this is usually expressed as a percentage of the total length of the bar, or it is 100 times the unit-elongation. The following table gives mean values of the ultimate strengths and ultimate elongations for the principal materials used in tension.

TABLE 2. TENSILE STRENGTHS

Material	Ultimate Strength		Ultimate Elongation Per Cent.
	Pounds per Square Inch	Kilos per Sq.Centimeter	
Timber	10 000	700	1.5
Cast Iron	20 000	1 400	0.5
Wrought Iron	50 000	3 500	30
Steel, structural	65 000	4 500	25

All these values should be regarded as rough averages, since they are subject to much variation with different qualities of the material; for instance, poor timber may be as low as 6000, while strong timber may be as high as 20 000 pounds per square inch in ultimate tensile strength. The ultimate strengths given in the table should, however, be memorized by the student as a basis for future knowledge, and they will be used for all the examples and problems in this book, unless otherwise stated.

Prob. 4 *A*. What should be the diameter of a wrought-iron bar so as to carry a tension of 200 000 pounds with a factor of safety of 5? If the bar is cast iron, what should be its diameter?

Prob. 4 *B*. A bar of wrought iron one square inch in section area and one yard long weighs 10 pounds. Find the length of a vertical bar which ruptures under its own weight when hung at its upper end.

ART. 5. COMPRESSION

The phenomena of compression are similar to those of tension provided that the elastic limit is not exceeded, the shortening of the bar being proportional to the applied force. After the elastic limit is passed the shortening increases more rapidly than the stress. When the length of the specimen is less than about ten times its least thickness, failure usually occurs by an oblique splitting or shearing, as seen in Fig. 4. When the length is large compared with the thickness, failure usually occurs under a sidewise bending, so that this is not a case of simple compression. All the values given in the following table refer to the

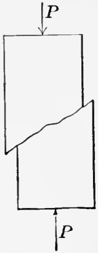

Fig. 4

short specimens; longer pieces are called 'columns' or 'struts,' and these will be discussed in Chap. 5.

The mean values of the ultimate compressive strengths of the principal materials are tabulated below. These are subject to much variation in different qualities of the materials, but it is necessary for the student to fix them in his mind as a preliminary basis for more extended knowledge. It is seen that timber is not quite as strong in compression as in tension, that cast iron is 4½ times as strong, that wrought iron and structural steel have the same ultimate strength in tension and compression.

Table 3. Compressive Strengths

Material	Ultimate Strength	
	Pounds per Square Inch	Kilos per Sq. Centimeter
Timber	8 000	560
Brick	3 000	210
Stone	6 000	420
Cast Iron	90 000	6 300
Wrought Iron	50 000	3 500
Steel, structural	65 000	4 500

The investigation of a body under compression is made by formula (1) of Art. 1. For example, if a stone block 8×12 inches in cross-section is subjected to a compression of 230 000 pounds, the unit-stress produced is $230\,000/96 = 2400$ pounds per square inch, and the factor of safety is $6000/2400 = 2\frac{1}{2}$; this is not sufficiently high for stone, as will be seen later.

Prob. 5 *A*. A brick $2 \times 4 \times 8$ inches weighs about $4\frac{1}{2}$ pounds. What will be the height of a pile of bricks so that the unit-stress on the lowest brick shall be one-half of its ultimate strength?

Prob. 5 *B*. A short cast-iron column is 12 inches in outside diameter and 10 inches in inside diameter. Compute its factor of safety when carrying a load of 165 000 pounds.

Art. 6. Shear

Shearing stresses exist when two forces acting like a pair of shears tend to cut a body between them. When a hole is punched in a plate, the ultimate shearing strength of the material must be overcome. If two thin bars are connected by a rivet and then are subjected to tension, the cross-section of the rivet between the plates is brought into shear. If a bolt is in tension, the forces acting on the head tend to shear or strip it off.

The following table gives the average ultimate shearing strength of different materials as determined by experi-

Table 4. Shearing Strengths

Material	Ultimate Strength	
	Pounds per Square Inch	Kilos per Sq. Centimeter
Timber, longitudinal	600	42
Timber, transverse	3 000	210
Cast Iron	20 000	1 400
Wrought Iron	40 000	2 800
Steel, structural	50 000	3 500

ment. For timber this is much smaller along the grain than across the grain; in the first direction it is called the longitudinal shearing strength, and in the second the transverse shearing strength. Rolled plates of wrought iron and steel, where the process of manufacture induces a fibrous structure, are also sheared more easily in the longitudinal than in the transverse direction.

Wooden specimens for tensile tests like that shown in Fig. 5 will fail by shearing off the ends if the length *ab* is not sufficiently great. For instance, suppose *ab* to be 6 inches, and the diameter of the central part to be 2 inches.

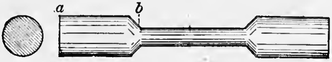

Fig. 5

The ends are grasped tightly by the machine and the cross-section of the central part thus brought under tensile stress. The force required to cause rupture by tension is

$$P = AS = 3.14 \times 1^2 \times 10\,000 = 31\,400 \text{ pounds.}$$

But the ends also tend to shear off along the surface of a cylinder whose diameter is 2 inches and whose length is *ab*; the force required to cause rupture by shearing on this surface is

$$P = AS = 3.14 \times 2 \times 6 \times 600 = 22\,600 \text{ pounds.}$$

and hence the specimen will fail by shearing off the ends. To prevent this the distance *ab* must be made longer than 6 inches.

Prob. 6 *A*. The beam in Fig. 2 is 3×4 inches in section-area, and P is 13 000 pounds. Compute the shearing unit-stress.

Prob. 6 *B*. A wrought-iron bolt $1\frac{1}{2}$ inches in diameter has a head $1\frac{1}{4}$ inches long. When a tension of 15 000 pounds is applied to the bolt, find the tensile unit-stress and the factor of safety for tension. Also find the unit-stress tending to shear off the head of the bolt, and the factor of safety against shear.

ART. 7. WORKING UNIT-STRESSES

When a body of cross-section A is under a stress P, the unit-stress S produced is found by dividing P by

A. By comparing this value of S with the ultimate strengths and elastic limits given in the preceding articles, the degree of security may be inferred. This process is called investigation. The student may not at first be able to form a good judgment with regard to the degree of security, this being a matter which involves some experience as well as acquaintance with engineering precedents and practice. As his knowledge increases, however, his ability to judge whether unit-stresses are or are not too great will constantly improve.

When a body is to be designed to stand a total stress P, the unit-stress S is first assumed in accordance with the rules of practice, and then the section area A is computed. Such assumed unit-stresses are often called working unit-stresses, meaning that these are the unit-stresses under which the material is to act or work. In selecting them, two fundamental rules are to be kept in mind:

1. They should be considerably less than the elastic limits.
2. They should be smaller for sudden stresses than for steady stresses.

The reason for the first requirement is given in Art. 2. The reason for the second requirement is that experience teaches that suddenly-applied loads and shocks are more injurious and produce higher unit-stresses than steady loads. Thus a bridge subject to the traffic of heavy trains must be designed with lower unit-stresses than a roof where the variable load consists only of snow and wind.

It will be best for the student to begin to form his engineering judgment by fixing in mind the following average values of the factors of safety to be used for different materials under different circumstances. The working unit-stress will then be found for any special

TABLE 5. FACTORS OF SAFETY

Material	For Steady Stress (Buildings)	For Varying Stress (Bridges)	For Shocks (Machines)
Timber	8	10	15
Brick and Stone	15	25	35
Cast Iron	6	15	20
Wrought Iron	4	6	10
Steel, structural	4	6	10

case by dividing the ultimate strengths by these factors of safety. For instance, a short timber strut in a bridge should have a working unit-stress of about $8000/10 = 800$ pounds per square inch.

It frequently happens that a designer works under specifications in which the unit-stresses to be used are definitely stated. The writer of the specifications must necessarily be an engineer of much experience and with a thorough knowledge of the best practice. It may be noted also that the particular qualities of timber or steel to be used will influence the selection of working unit-stresses, and in fact different members of a bridge truss are often designed with different unit-stresses. The two fundamental rules above stated are, however, the guiding ones in all cases.

Modern engineering is the art of economic construction. In numerous instances this will be secured by making all parts of a structure of equal strength, for if one part is stronger than another it has an excess of material which might have been saved.

Prob. 7 *A*. A wrought-iron rod is to be under a stress of 82 000 pounds. Find its diameter when it is to be used in a building, and also when it is to be used in a bridge.

Prob. 7 *B*. The total shear on each rivet of a lap-riveted joint is 2000 pounds. If the rivet is $\frac{5}{8}$ inches in diameter, find the factor of safety against shearing.

ART. 8. REVIEW PROBLEMS

The following problems may serve to test the student as to his knowledge of the preceding principles and methods. In solving problems it is very desirable that a neat and systematic method should be followed. The practice of making computations with a pencil on loose scraps of paper should be discontinued by every student who has followed it, and he should hereafter solve his problems in a special book, using pen and ink. Before beginning the solution, a diagram should be drawn whenever possible, for a diagram helps the student to understand the problem, and a problem thoroughly understood is really half solved. Before beginning the computation of a numerical problem, it is best to make a mental estimate of the answer, for thus the engineering judgement of the student will be developed. In Art. 54 will be found a few answers, but the student should never look there for an answer to a problem until he has completed its solution.

Prob. 8 *A*. During the tensile test of a steel bar $\frac{3}{4}$ inches in diameter the load at the elastic limit was found to be 17 600 pounds. What was the elastic limit in pounds per square inch?

Prob. 8 *B*. The pull on the piston-rod of a steam engine is 25 000 pounds. If the diameter of the rod is $2\frac{1}{4}$ inches, compute the factor of safety.

Prob. 8 *C*. During the tensile test of a cast-iron bar 1 inch in diameter rupture occurred under a load of 18 000 pounds. What was the tensile strength of the cast iron?

Prob. 8 *D*. In a test on a wooden specimen of shape shown in Fig. 5 a load of 1000 pounds was placed. Diameter of specimen was 1½ inches, and length of head was 6 inches. Find the tensile unit-stress and the shearing unit-stress.

Prob. 8 *E*. The piston of an engine is 12 inches in diameter, and the diameter of the piston rod is 2¼ inches. The maximum steam pressure is 120 pounds per sq. in. Find the tensile unit-stress on the rod and the factor of safety.

Prob. 8 *F*. A wrought-iron bolt 1½ inches in diameter has a head 1 inch long. Find the unit-stress tending to shear off the head when a tension of 3000 pounds is applied to the bolt.

Prob. 8 *G*. A pipe-rack in a shop is supported by four wrought-iron rods, each 1 inch in diameter. The total load supported by the rods is two long tons. Is the structure safe?

Prob. 8 *H*. A steel column is supported by a stone base. If the total load on the base is 30 000 pounds, and the factor of safety is to be 10, find the cross-section area of the stone.

Prob. 8 *J*. A steel tie rod in a roof truss is to sustain a load of 6000 pounds. Find the size of rod so that it shall have a factor of safety of 8.

Prob. 8 *K*. The total load on a vertical shaft hanger is 2000 pounds. The hanger is held in place by means of two 1-inch bolts. The length of head of each bolt is 1¼ inches. Find the factor of safety against shear and tension.

Prob. 8 *L*. A bridge carrying a total load of 320 000 pounds rests on two stone abutments. The bridge rests on four cast-iron bearing plates. Find the size of these plates so that the stone shall have a factor of safety of ten.

CHAPTER II

GENERAL PROPERTIES

ART. 9. AVERAGE WEIGHTS

The average weights of the six principal materials used in engineering constructions are given in the following table, together with their specific gravities. These are subject to more or less variation, according to the quality of the material. For instance, brick may weigh as low

TABLE 6. WEIGHT

Material	Pounds per Cubic Foot	Kilos per Cubic Meter	Specific Gravity
Timber	40	600	0.6
Brick	125	2 000	2.0
Stone	160	2 600	2.6
Cast Iron	450	7 200	7.2
Wrought Iron	480	7 700	7.7
Steel	490	7 800	7.8

as 100 or as high as 150 pounds per cubic foot, depending upon whether it is soft common quality or fine hard-pressed. Unless otherwise stated, the above values will be used in all the examples and problems in the following pages, and hence those in pounds per cubic foot must be carefully kept in the memory.

For computing the weights of bars, beams, and pieces of uniform section-area, the following approximate simple rules are much used by engineers:

A wrought-iron bar one square inch in section-area and one yard long weighs ten pounds.

Timber is one-twelfth the weight of wrought iron.

Brick is one-fourth the weight of wrought iron.

Stone is one-third the weight of wrought iron.

Cast iron is six percent lighter than wrought iron.

Steel is two percent heavier than wrought iron.

For example, if a bar of wrought iron be $1\frac{1}{2} \times 3$ inches in section and 22 feet long, its section-area is $4\frac{1}{2}$ square inches and its weight is $45 \times 7\frac{1}{3} = 330$ pounds. A steel bar of the same dimensions will weigh $330 + 0.02 \times 330 = 337$ pounds, and a cast-iron bar will weigh $330 - 0.06 \times 330 = 310$ pounds.

By reversing the above rules the section-areas are readily found when the weights per linear yard are given. Thus, if a stick of timber 15 feet long weighs 120 pounds, its weight per yard is 24 pounds and its section-area is $2.4 \times 12 = 28.8$ square inches.

Prob. 9 *A*. What is the weight of a stone block 12×18 inches and $4\frac{1}{2}$ feet long? How many square inches in the cross-section of a steel railroad rail which weighs 95 pounds per yard?

Prob. 9 *B*. If a cast-iron water pipe 12 feet long weighs 1000 pounds, what is its section-area? Find the diameter of a wrought-iron bar which is 24 feet long and weighs 1344 pounds.

ART. 10. TESTING MACHINES

The simplest method of testing is by tension, a specimen being used like that shown in Art. 6. The heads are either gripped in jaws, or they are provided with threads so that they may be screwed into nuts to which the forces are applied. The power may be furnished by a lever, a screw, or by hydraulic pressure, the last being the method in machines of high capacity. In these tests the elastic limit, ultimate strength, and the ultimate elongation are generally recorded, the latter being expressed as a percentage of the original length. For ductile materials the contraction of area of the fractured specimen is also noted, as this does not vary with the length of the specimen to the same extent as the ultimate elongation. In such tensile tests the load is applied gradually, and not suddenly or with impact.

The elastic limit is detected by taking a number of measurements of the elongation for different loads, and then noting when these begin to vary more rapidly than the stresses. For ductile materials the change is a sudden one, and it may be often noted by the drop of the scale beam of the machine.

Compressive tests are confined mainly to brick and stone, and are but little used for commercial tests of metals on account of the difficulty of securing a uniform distribution of pressure over the surfaces. Cement, which is always used in compression, is indeed usually tested by tension, this being found to be the cheaper and more satisfactory method.

The capacity of a testing machine is the number of pounds it can exert as tension or compression. A small machine for testing wire or cement need not have a capacity greater than 1000 or 2000 pounds. Machines of 50 000, 100 000, and 150 000 pounds for testing metals are common. The Watertown machine has a capacity of 800 000 pounds, and can test a small hair or a steel bar of 10 square inches section-area with equal precision. A list of the fourteen largest testing machines in the

United States is given in American Civil Engineers' Pocket Book (New York, 1912).

Fig. 6 shows an Olsen screw testing machine of 40 000 pounds capacity. The power is applied by hand by means of the crank on the left, and this causes the four vertical

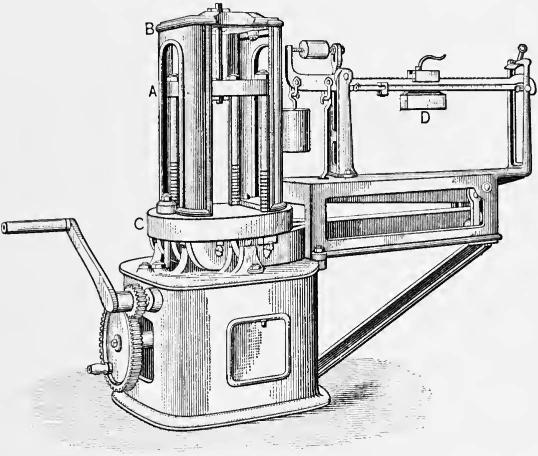

Fig. 6

screws to have a slow upward or downward movement. The upper ends of the screws are fastened to a table A, which hence partakes of the vertical motion. When a tensile test is to be made, one end of the specimen is gripped by jaws in the movable table A, and the other end by jaws in the upper fixed table B; in the figure a tensile specimen is seen in this position. The crank is then turned so as to cause the vertical screws and the movable table to descend, and thus a stress is brought upon the specimen. For a compressive test the specimen is placed between the lower fixed table C and the movable table A, the latter being caused to descend by turning the crank as before. The load applied to the specimen at any instant is weighed on the lever scale at the right by moving the weight D so that the scale arm will balance. Machines of greater capacity than 40 000 pounds are usually operated by power, which is transmitted from a motor to a pulley on the shaft of the machine.

Tests are also made by loading beams transversely and measuring the deflections, as well as finding the load required to produce rupture. The machine shown in Fig. 6 can be used for flexural tests of short beams by placing two supports on the lower fixed table, while for long beams a special attachment can be made to this table. In both cases the load is applied by lowering the movable table.

Prob. 10 *A*. What is the diameter of the largest bar of structural steel which can be tested in a machine of 100 000 pounds capacity?

Prob. 10 *B*. A steel eye-bar tested at Phœnixville was $10 \times 2\frac{5}{8}$ inches in size and 47 feet long. The length after rupture was 57.6 feet, and the area of the fractured cross-section was 13.0 square inches. Compute the percentage of ultimate elongation and the percentage of reduction of area.

ART. 11. TIMBER

Good timber is of uniform color and texture, free from knots, sap wood, wind shakes, and decay. It should be well seasoned, which is best done by exposing it to the sun and wind for two or three years to dry out the sap. The heaviest timber is usually the strongest; also the darker the color and the closer the annular rings, the stronger and better it is, other things being equal. The strength of timber is always greatest in the direction of the grain, the sidewise resistance to tension or compression being scarcely one-fourth of the longitudinal.

The following table which gives average values of the ultimate strength of a few of the common kinds of timber will be useful for reference. These values have been determined from tests of small specimens carefully selected and dried. Large pieces of timber such as are

TABLE 7. STRENGTH OF TIMBER

Kind	Pounds per Cubic Foot	Pounds per Square Inch	
		Tensile Strength	Compressive Strength
Hemlock	25	8 000	5 000
White Pine	27	8 000	5 500
Chestnut	40	12 000	5 000
Red Oak	42	9 000	6 000
Yellow Pine	45	15 000	9 000
White Oak	48	12 000	8 000

actually used in engineering structures will probably have an ultimate strength of from fifty to eighty per cent of these values. Moreover, the figures are liable to a range of 25 per cent on account of variations in quality and condition arising from place of growth, time when cut, and method of seasoning. To cover these variations the factor of safety of 10 is not too high, even for steady stresses.

The shearing strength of timber is still more variable than the tensile or compressive resistance. White pine across the grain may be put at 2500 pounds per square inch, and along the grain at 500. Chestnut has 1500 and 600 respectively, yellow pine and oak perhaps 4000 and 600 respectively.

The elastic limit of timber is poorly defined. In precise tests on good specimens it is sometimes observed at about one-half the ultimate strength, but under ordinary conditions it is safer to put it at one-third. The ultimate elongation is small, usually being between 1 and 2 per cent.

Prob. 11 *A*. What should be the size of a short piece of yellow pine which is to carry a steady load of 80 000 pounds?

Prob. 11 *B*. If a piece of white cedar 2×2 inches in cross-section ruptures under a compression of 20 800 pounds, what is the size of a square section that will stand 25 000 pounds with a factor of safety of 10?

Art. 12. Brick

Brick is made of clay which consists mainly of silicate of alumina with compounds of lime, magnesia, and iron. The clay is prepared by cleaning it carefully from pebbles and sand, mixing it with about one-half its volume of water, and tempering it by hand stirring or in a pug mill. It is then moulded in rectangular boxes by hand or by special machines, and the green bricks are placed under open sheds to dry. These are piled in a kiln and heated for nearly two weeks until those nearest to the fuel assume a partially vitrified appearance.

Three qualities of brick are taken from the kiln; 'arch-brick' are those from around the arches where the fuel is burned, these are hard and often brittle; 'body-brick,' from the interior of the kiln, are of the best quality; 'soft brick,' from the exterior of the pile, are weak and only suitable for filling. Paving brick are burned in special kilns, often by natural gas or by oil, the rate of heating being such as to insure toughness and hardness.

The common size is $2 \times 4 \times 8\frac{1}{4}$ inches, and the average weight $4\frac{1}{2}$ pounds. A pressed brick, however, may weigh nearly $5\frac{1}{2}$ pounds. Good bricks should be of regular shape, have parallel and plane faces, with sharp angles and edges. They should be of uniform texture, and when struck a quick blow should give a sharp, metallic ring. The heavier the brick, other things being equal, the stronger and better it is.

Poor brick will absorb when dry from 20 to 30 per cent of its weight of water, ordinary qualities absorb from 10 to 20 per cent, while hard paving brick should not absorb more than 2 or 3 per cent. An absorption test is valuable in measuring the capacity of brick to resist the disintegrating action of frost, and as a rough general rule, the greater the amount of water absorbed the less is the strength and durability.

The crushing strength of brick is variable; while a mean value may be 3000 pounds per square inch, soft brick will scarcely stand 500, pressed brick may run to 10 000, and the best qualities of paving brick have given 15 000 pounds per square inch, or even more. Crushing tests are usually made on whole or half-bricks and are hence lacking in precision, since opposite surfaces are rarely truly parallel. Tensile and shearing tests of bricks are rarely made, and but little is known of their behavior under such stresses; the ultimate tensile strength may perhaps range from 50 to 500 pounds per square inch.

Prob. 12 *A*. Compute the unit-stress at the base of a brick wall 17 inches thick and 55 feet high. What is the factor of safety?

Prob. 12 *B*. A brick weighs 4.42 pounds when dry and 4.75 pounds after immersion for one day in water. What percentage of water has it absorbed? 7.8 %

Art. 13. Stone

Sandstone, as its name implies, is sand, usually quartzite, which has been consolidated under heat and pressure. It varies much in color, strength, and durability, but many varieties form most valuable building material. In general it is easy to cut and dress, but the variety known as Potsdam sandstone is very hard in some localities.

Limestone is formed by consolidated marine shells, and is of diverse quality. Marble is limestone which has been reworked by the forces of nature so as to expel the impurities, and leave a nearly pure carbonate of lime; it takes a high polish, is easily cut, and makes one of the most beautiful building stones.

Granite is a rock of aqueous origin metamorphosed under heat and pressure; its composition is quartz, feldspar, and mica, but in the variety called gneiss the mica is replaced by hornblende. It is fairly easy to work, usually strong and durable, and some varieties will take a high polish.

Trap, or basalt, is an igneous rock without cleavage. It is hard and tough, and less suitable for building constructions than other rocks, as large blocks cannot be readily obtained and cut to size. It has, however, a high strength, and is remarkable for durability.

The average weight of sandstone is about 150, of limestone 160, of granite 165, and of trap 175 pounds per cubic foot. The ultimate compressive strength of sandstone is about 5000, of limestone 7000, of granite 12 000, and of trap 16 000 pounds per square inch; these figures refer to small blocks, but the ultimate strength of large blocks is materially smaller.

The quality of a building stone cannot be safely inferred from tests of strength, as its durability depends largely upon its capacity to resist the action of the weather. Hence corrosion and freezing tests, impact tests, and observations of the behavior of stone under conditions of actual use are more important than the determination of crushing strength in a compression machine.

Prob. 13 *A*. Find the weight of a granite column 18 feet high and 18 inches in diameter.

Prob. 13 *B*. A stone pier 12×30 feet at the base, 8×24 feet at the top, and 16½ feet high is to be built at $6.37 per cubic yard. What is the total cost?

Art. 14. Cast Iron

Cast iron is a modern product, having been first made in England about the beginning of the fifteenth century. Ores of iron are melted in a blast furnace, producing pig iron. The pig iron is remelted in a cupola furnace and poured into moulds, thus forming castings. Beams, columns, pipes, braces, and blocks of every shape required in engineering structures are thus produced.

Pig iron is divided into two classes, foundry pig and forge pig, the former being used for castings and the latter for making wrought iron. Foundry pig has a dark-gray fracture, with large crystals and a metallic luster; forge pig has a light-gray or silver-white fracture, with small crystals. Foundry pig has a specific gravity of from 7.1

to 7.2, and it contains from 6 to 4 per cent of carbon; forge pig has a specific gravity of from 7.2 to 7.4, and it contains from 4 to 2 per cent of carbon. The higher the percentage of carbon the less is the specific gravity, and the easier it is to melt the pig. Besides the carbon there are present from 1 to 5 per cent of other impurities, such as silicon, manganese, and phosphorus.

The properties and strength of castings depend upon the quality of the ores and the method of their manufacture in both the blast and the cupola furnace. Cold blast pig produces stronger iron than the hot blast, but it is more expensive. Long continued fusion improves the quality of the product, as also do repeated meltings. The darkest grades of foundry pig make the smoothest castings, but they are apt to be brittle; the light-gray grades make tough castings, but they are apt to contain blow holes or imperfections.

The percentage of carbon in cast iron is a controlling factor which governs its strength, particularly that percentage which is chemically combined with the iron. As average values for the ultimate strength of cast iron, 20 000 and 90 000 pounds per square inch in tension and compression respectively are good figures. In any particular case, however, a variation of from 10 to 20 per cent from these values may be expected, owing to the great variation in quality. The elastic limit is poorly defined, there being no sudden increase in deformation, as in ductile materials.

The high compressive strength and cheapness of cast iron render it a valuable material for many purposes; but its brittleness, low tensile strength, and low ductility forbid its use in structures subject to variations of load or to shocks. Its ultimate elongation being scarcely one per cent, the work required to cause rupture in tension is small compared to that for wrought iron and steel, and hence as a structural material the use of cast iron must be confined entirely to cases of compression.

Prob. 14 *A*. A cast-iron bar weighing 31 pounds per linear yard is to be subjected to tension. How many pounds are required to rupture it?

Prob. 14 *B*. What must be the capacity of a testing machine to break a cast-iron block 2 inches square?

ART. 15. WROUGHT IRON

The ancient peoples of Europe and Asia were acquainted with wrought iron and steel to a limited extent. It is mentioned in Genesis, iv, 22, and in one of the oldest pyramids of Egypt a piece of iron has been found. It was produced, probably, by the action of a hot fire on very pure ore. The ancient Britons built bloomaries on the tops of high hills, a tunnel opening toward the north furnishing a draft for the fire, which caused the carbon and other impurities to be expelled from the ore, leaving behind nearly pure metallic iron.

Modern methods of manufacturing wrought iron are mainly by the use of forge pig (Art. 14), the one most extensively used being the puddling process. Here the forge pig is subjected to the oxidizing flame of a blast in a reverberatory furnace, where it is formed into pasty balls by the puddler. A ball taken from the furnace is run through a squeezer to expel the cinder and then rolled into a muck bar. The muck bars are cut, laid in piles, heated, and rolled, forming what is called merchant bar. If this is cut, piled, and rolled again, a better product, called best iron, is produced. A third rolling gives 'best best' iron, a superior quality, but high in price.

The product of the rolling-mill is bar iron, plate iron, shape iron, beams, and rails. Bar iron is round, square, and rectangular in section; plate iron is from ¼ to 1 inch thick, and of varying widths and lengths; shape iron includes angles, tees, channels, and other forms used in structural work; beams are I-shaped, and of the deck or rail form. Structural shapes and beams are, however, now almost entirely rolled in mild steel.

Wrought iron is metallic iron containing less than 0.25 per cent of carbon, and which has been manufactured without fusion. Its tensile and compressive strengths are closely equal, and range from 50 000 to 60 000 pounds per square inch. The elastic limit is well defined at about 25 000 pounds per square inch, and within that limit the law of proportionality of stress to deformation is strictly observed. It is tough and ductile, having an ultimate elongation of from 20 to 30 per cent. It is malleable, can be forged and welded, and has a high capacity to withstand the action of shocks. It cannot, however, be tempered so as to be used for cutting tools.

The cold-bend test for wrought iron is an important one for judging of general quality. A bar perhaps ¾×¾ inches and 15 inches long is bent when cold either by pressure or by blows of a hammer. Bridge iron should bend through an angle of 90 degrees to a curve whose radius is twice the thickness of the bar, without cracking. Rivet iron should bend through 180 degrees until the sides of the bar are in contact, without showing signs of fracture. Wrought iron that breaks under this test is lacking in both strength and ductility.

The process of manufacture, as well as the quality of the pig iron, influences the strength of wrought iron. The higher the percentage of carbon the greater is the strength. Best iron is 10 per cent stronger than ordinary merchant iron owing to the influence of the second rolling. Cold rolling causes a marked increase in elastic limit and ultimate strength, but a decrease in ductility or ultimate elongation. Annealing lowers the ultimate strength, but increases the elongation. Iron wire, owing to the process of drawing, has a high tensile strength, sometimes greater than 100 000 pounds per square inch.

Good wrought iron when broken by tension shows a fibrous structure. If, however, it is subject to shocks or to repeated stresses which exceed the elastic limit, the molecular structure becomes changed so that the fracture is more or less crystalline. The effect of a stress slightly exceeding the elastic limit is to cause a small permanent set, but the elastic limit will be found to be higher than before. This is decidedly injurious to the quality of the material on account of the accompanying change in structure, and hence it is a fundamental principle that the working unit-stresses should not exceed the elastic limit. For proper security indeed the allowable unit-stress should seldom be greater than one-half the elastic limit.

In a rough general way the quality of wrought iron may be estimated by the product of its tensile strength and ultimate elongation, this product being an approximate measure of the work required to produce rupture. Thus high tensile strength is not usually a good quality when accompanied by a low elongation.

Prob. 15 *A*. What should be the length of a wrought-iron bar, so that, when hung at its upper end, it will rupture there under the stress produced by its own weight?

Prob. 15 *B*. What is the section-area of a bar of wrought iron which weighs 10 pounds per linear foot.

Art. 16. Steel

Steel was originally produced directly from pure iron ore by the action of a hot fire, which did not remove the carbon to a sufficient extent to form wrought iron. The modern processes, however, involve the fusion of the ore, and the definition of the United States law is that "steel is iron produced by fusion by any process, and which is malleable." Chemically, steel is a compound of iron and carbon generally intermediate in composition between cast and wrought iron, but having a higher specific gravity than either. The following comparison points out the distinctive differences between the three kinds of iron:

	Per cent of Carbon	Spec. Grav.	Properties
Cast Iron......	5 to 2	7.2	Fusible, not malleable.
Steel..........	1.50 to 0.10	7.8	Fusible and malleable.
Wrought Iron..	0.30 to 0.05	7.7	Malleable, not fusible.

It should be observed that the percentage of carbon alone is not sufficient to distinguish steel from wrought iron; also, that the mean values of specific gravity stated are in each case subject to considerable variation.

The three principal methods of manufacture are the crucible process, the open-hearth process, and the Bessemer process. In the crucible process impure wrought iron or blister steel, with carbon and a flux, are fused in a sealed vessel to which air cannot obtain access; the best tool-steels are thus made. In the open-hearth process pig iron is melted in a Siemens furnace, wrought-iron scrap being added until the proper degree of carbonization is secured. In the Bessemer process pig iron is completely decarbonized in a converter by an air blast and then recarbonized to the proper degree by the addition of spiegeleisen. The metal from the open-hearth furnace or from the Bessemer converter is cast into ingots, which are rolled in mills to the required forms. The open-hearth process produces steel for guns, armor plates, machinery, shafts, and for structural purposes; the Bessemer process mainly produces steel for railroad rails.

The physical properties of steel depend both upon the method of manufacture and upon the chemical composition, the carbon having the controlling influence upon strength. Manganese promotes malleability and silicon increases the hardness, while phosphorus and sulphur tend to cause brittleness. The higher the percentage of carbon within reasonable limits the greater is the ultimate strength and the less the elongation.

A classification of steel according to the percentage of carbon and its physical properties of tempering and welding is as follows:

Extra hard,	1.00 to 0.60% C., takes high temper, but not weldable.
Hard,	0.70 to 0.40% C., temperable, welded with difficulty.
Medium,	0.50 to 0.20% C., poor temper, but weldable.
Mild,	0.40 to 0.05% C., not temperable, but easily welded.

It is seen that these classes overlap so that there are no distinct lines of demarcation. The extra-hard steels are used for tools, the hard steels for piston-rods and other parts of machines, the medium steels for rails, tires, and beams, and the mild or soft steels for rivets, plates, and other purposes.

The structural steel used in bridges and buildings has an ultimate tensile strength of from 60 000 to 70 000 pounds per square inch, with an elastic limit from 30 000 to 40 000 pounds per square inch. The hard and extra-hard steels are much higher in strength. By the use of nickel as an alloy steel has been made with an ultimate tensile strength of 277 000 and an elastic limit of over 100 000 pounds per square inch.

The compressive strength of steel is always higher than the tensile strength. The maximum value recorded for hardened steel is 392 000 pounds per square inch. The expense of commercial tests of compression is, however, so great that they are seldom made. The shearing strength is about three-fourths of the tensile strength.

Steel castings are extensively used for axle-boxes, cross-heads, and joints in structural work. They contain from 0.25 to 0.50 per cent of carbon, ranging in tensile strength from 60 000 to 100 000 pounds per square inch.

Steel has entirely supplanted wrought iron for railroad rails, and largely so for structural purposes. Its price being the same, its strength greater, its structure more homogeneous, the low and medium varieties are coming more and more into use as a satisfactory and reliable material for large classes of engineering constructions.

Prob. 16 *A*. If steel costs 3 cents per pound and nickel costs

35 cents per pound, what is the minimum cost of a pound of nickel steel which contains 3.25 per cent of nickel?

Prob. 16 *B*. A short steel piston-rod is to be designed to be used with a piston which is 20 inches in diameter and subject to a steam-pressure of 150 pounds per square inch. If the ultimate strength of the steel is 90 000 pounds per square inch, what should be its diameter, allowing a factor of safety of 15?

ART. 17. OTHER MATERIALS

Common mortar is composed of one part of lime to five parts of sand by measure. When six months old its tensile strength is from 15 to 30, and its compressive strength from 150 to 300 pounds per square inch. Its strength slowly increases with age, and it may be considerably increased by using a smaller proportion of sand.

Hydraulic mortar is composed of hydraulic cement and sand in varying proportions. The less the proportion of sand the greater is its strength. A common proportion is 3 parts sand to 1 of cement, the strength of this being about one-fourth of the neat cement. The natural cements are of lighter color, lower weight, and lesser strength than the Portland cement, but they are quicker in setting and cheaper in price. When one week old, neat natural cement has a tensile strength of about 125 and Portland cement about 300 pounds per square inch; when one year old the tensile strengths are about 300 and 500 pounds per square inch respectively. The compressive strength is from 8 to 10 times the tensile strength, and it increases more rapidly with age.

Concrete, composed of hydraulic mortar and broken stone, is an ancient material, having been extensively used by the Romans. It is mainly employed for foundations and monolithic structures, but in some cases large blocks have been made which are laid together like masonry. Like mortar, its strength increases with age. When six months old its mean compressive strength ranges from 1000 to 3000 pounds per square inch, and when one year old it is probably about fifty per cent greater.

Ropes are made of hemp, of manila, and of iron or steel wire with a hemp center. A hemp rope one inch in diameter has an ultimate strength of about 6000 pounds, and its safe working strength is about 800 pounds. A manila rope is slightly stronger. Iron and steel ropes one inch in diameter have ultimate strengths of about 36 000 and 50 000 pounds respectively, the safe working strengths being 6000 and 8000 pounds. As a fair rough rule, the strength of ropes may be said to increase as the squares of their diameters.

Aluminum is a silver-gray metal which is malleable and ductile and not liable to corrode. Its specific gravity is about 2.65, so that it weighs only 168 pounds per cubic foot. Its ultimate tensile strength is about 25 000 pounds per square inch, and its ultimate elongation is also low. Alloys of aluminum and copper have been made with a tensile strength and elongation exceeding those of wrought iron, but have not come into use as structural materials.

Prob. 17 *A* Ascertain the weight of lead and brass per cubic foot, and their ultimate tensile strengths.

Prob. 17 *B*. A bar of aluminum copper, $1\frac{1}{4} \times 1\frac{3}{8}$ inches in section, breaks under a tension of 42 800 pounds. What tension will probably break a bar of the same material which is $1\frac{7}{8} \times 2\frac{1}{4}$ inches in section?

ART. 18. REVIEW PROBLEMS

Prob. 18 *A*. Compute the weight of a cast-iron water-pipe 12 feet long, and 20 inches inside diameter, the thickness of the metal being 1 inch.

Prob. 18 *B*. A wrought-iron bar, 1 inch in diameter and 30 feet long, is hung at its upper end. What load applied at the lower end will stress it to the elastic limit?

Prob. 18 *C*. Find the weight of a wooden girder 10×12 inches in cross-section and 16 feet long.

Prob. 18 *D*. A cast-iron column is 20 feet long. Inside diameter of column is 8 and outside diameter is 10 inches. Find the weight of this column.

Prob. 18 *E*. During a tension test on a steel bar 0.8 inches in diameter the following data were obtained: Maximum load on specimen 30 000 pounds, load at elastic limit 23 000 pounds, original length of part tested, 8 inches, final length of part tested 10¼ inches, diameter at break 0.32 inches. Find the ultimate strength, elastic limit, percent elongation, and percent reduction of area.

Prob. 18 *F*. A circular brick stack is 30 feet high, its outside diameter being 8 feet and inside diameter 5 feet. Find total weight of stack. What is the unit-stress at the base of the stack?

Prob. 18 *G*. A cylinder head is held in place by six ⅝-inch studs. The diameter of the cylinder is 8 inches and the maximum steam pressure is 200 pounds per square inch. Find the unit tensile stress at root of threads if the diameter of stud at root of thread is 0.507 inches.

Prob. 18 *H*. A table with four legs carries a load of 1000 pounds uniformly spread over it. When the table weighs 85 pounds and each leg is 1¾ inch in diameter, what is the unit-stress in the legs?

Prob. 18 *J*. The pull upon a 1½-inch bolt passing through a yellow pine girder is 30 000 pounds. What size of washer must be used so that the unit-stress on the pine shall not exceed 1000 pounds per square inch?

CHAPTER 3

MOMENTS FOR BEAMS

ART. 19. THE PRINCIPLE OF MOMENTS

The moment of a force with respect to a point is a quantity which measures the tendency of the force to cause rotation about that point. The moment is the product of the force by the length of its lever-arm, the lever-arm being a line drawn from the point perpendicular to the direction of the force. Thus if P in Fig. 7 is any force and p the length of a perpendicular drawn to it

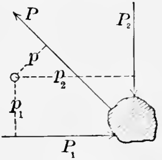

Fig. 7

from any point, the product Pp is the moment of the force with respect to that point. As P is in pounds and p is in feet or inches, the moment is a compound quantity which is called pound-feet or pound-inches.

The most important principle in mechanics is the principle of moments. This asserts that if any number of forces in the same plane be in equilibrium, the algebraic sum of their moments about any point in that plane is equal to zero. This principle results from the meaning

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of the word equilibrium, which implies that the body on which the forces act is at rest; and since it is at rest the forces taken collectively have no tendency to turn it around any point. All experience teaches that the principle of moments is, indeed, a law of nature whose truth is universal.

The point from which the lever-arms are measured is often called the 'center of moments.' Forces which tend to turn around this center in the direction of motion of the hands of a watch have positive moments, and those which tend to turn in the opposite direction have negative moments. Thus, in Fig. 7, the numerical values of Pp and P_1p_1 are negative, while that of P_2p_2 is positive. If the forces are in equilibrium, the sum $Pp+P_1p_1$ has the same numerical value as P_2p_2, or the algebraic sum of the three moments is zero; this will be the case wherever the center of moments is taken.

In all investigations regarding the strength of beams, the principle of moments is of constant application. A beam is a body held in equilibrium by the downward loads and the upward pressures of the supports. As the beam is at rest these forces are in equilibrium, and the algebraic sum of their moments is zero about any point in the plane. Moreover, by further use of the principle of moments the stresses in all parts of the beam due to the given loads may be determined.

Prob. 19 *A*. A lever is 5 feet long and the fulcrum is placed 3 inches from one end. What force will be required at the longer end to lift 1000 pounds at the shorter end?

Prob. 19 *B*. In the above figure, the three forces are in equilibrium, P_1 being 500 pounds, P_2 being 866 pounds, and the lever-arms being $p=1.5$ feet, $p_1=3.5$ feet, $p_2=5.1$ feet. Show that the force P is 4111 pounds.

Art. 20. Reactions of Supports

Let a simple beam resting on two supports near its ends be subject to a load P situated at 6 feet from the left support, and let the span be 24 feet (Fig. 8). Taking the center of moments at the right support, the lever-arm of R_1 is 24 feet, that of P is 18 feet, and that of R_2 is

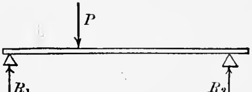

Fig. 8

0; then by the application of the principle of moments $R_1 \times 24 - P \times 18 = 0$, or $R_1 = \frac{3}{4}P$. Again, taking the center of moments at the left support the lever-arm of R_1 is 0, that of P is 6 feet, and that of R_2 is 24 feet; then likewise from the principle of moments $-R_2 \times 24 + P \times 6 = 0$, or $R_2 = \frac{1}{4}P$. The sum of these two reactions is equal to P, as should of course be the case.

The reactions caused by the weight of the beam itself may be found in a similar manner, the uniform load being supposed concentrated at its center of gravity in stating the equations of moments. Thus, if the weight of the beam be W, the two equations of moment are found to be $R_1 \times 24 - W \times 12 = 0$, and $-R_2 \times 24 + W \times 12 = 0$, from which $R_1 = \frac{1}{2}W$ and $R_2 = \frac{1}{2}W$.

The reactions due to both uniform and concentrated loads on a simple beam may also be computed in one operation. As an example, let there be a simple beam 12 feet long between the supports and weighing 35 pounds per linear foot, its total weight being 420 pounds (Fig. 9).

Let there be three loads of 300, 60, and 150 pounds, placed 3, 5, and 8 feet respectively from the left support. To find the left reaction R_1, the center of moments is taken

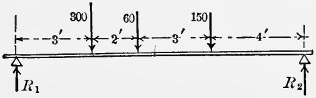

Fig. 9

at the right support and the weight of the beam regarded as concentrated at its middle; then the equation of moments is

$$R_1 \times 12 - 420 \times 6 - 300 \times 9 - 60 \times 7 - 150 \times 4 = 0$$

from which $R_1 = 520$ pounds. In like manner, to find R_2 the center of moments is taken at the left support; then

$$-R_2 \times 12 + 420 \times 6 + 300 \times 3 + 60 \times 5 + 150 \times 8 = 0$$

from which $R_2 = 410$ pounds. As a check the sum of R_1 and R_2 is found to be 930 pounds, which equals the weight of the beam and the three loads.

By means of the principle of moments other problems relating to reactions of beams may also be solved. For instance, if a simple beam 12 feet long weighs 30 pounds per linear foot and carries a load of 600 pounds, where should this load be put so that the left reaction may be twice as great as the right reaction? Here let x be the distance from the left support to the load; let R_1 be the left reaction and R_2 the right reaction. Then taking the centers of moments at the right and left support in succession there are found

$$R_1 = 180 + 50\ (12 - x), \quad R_2 = 180 + 50x$$

and placing R_1 equal to $2R_2$ there results $x = 2.8$ feet.

Prob. 20 *A*. A beam weighing 30 pounds per linear foot rests upon two supports 16 feet apart. A weight of 400 pounds is placed at 5 feet from the left end, and one of 600 pounds is placed at 8 feet from the right end. Find the reactions due to the total load.

Prob. 20 *B*. A wooden girder, 8×10 inches in section-area and 18 feet between supports, carries a uniformly distributed load of 500 pounds per linear foot for a distance of 8 feet from the left end. The remaining 10 feet carry a uniformly distributed load of 700 pounds per linear foot. Find the reactions at the supports.

Art. 21. Bending Moments

The 'bending moment' at any section of a beam is the algebraic sum of the moments of all the vertical forces on the left of that section. It is a measure of the tendency of those forces to cause rotation around that point. At the ends of a simple beam there are no bending moments, but at all other sections they exist, and the greater the bending moment the greater are the horizontal stresses in the beam, these stresses in fact being produced by the bending moment.

For example, let a beam 30 feet long have three loads of 100 pounds each, situated at distances of 8, 12, and 22 feet from the left support (Fig. 10). By the method of

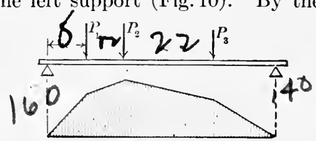

Fig. 10

the previous article the left reaction R_1 is 160 pounds and the right reaction R_2 is 140 pounds. For a section 4 feet from the left support the bending moment is $160 \times 4 = 640$ pound-feet, and for a section at 8 feet from the left support the bending moment is $160 \times 8 = 1280$ pound-feet. For a section 10 feet from the left support there are two vertical forces on the left of the section, 160 acting up and 100 acting down, so that the bending moment is $160 \times 10 - 100 \times 2 = 1400$ pound-feet. For a section at the middle of the beam the bending moment is $160 \times 15 - 100 \times 7 - 100 \times 3 = 1400$ pound-feet. For a section under the third load the bending moment is, in like manner, 1120 pound-feet, and for a section at 3 feet from the right support it is 420 pound-feet. The vertical ordinates underneath the beam represent the values of these bending moments, and the diagram thus formed shows how the bending moments vary throughout the length of the beam.

For a simple beam of span l and uniformly loaded with w pounds per linear unit, each reaction is $\frac{1}{2}wl$. For any section distant x from the left support (Fig. 11), the bending moment is $\frac{1}{2}wl \times x - wx \times \frac{1}{2}x$, where the lever-

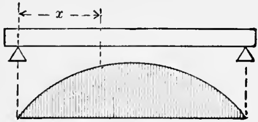

Fig. 11

arm of the reaction is x and the lever-arm of the load wx is $\frac{1}{2}x$. If w is 80 pounds per linear foot and l is 30 feet, the bending moment at any section is then $1200x - 40x^2$. For $x = 10$ feet, the bending moment is 8000 pound-feet; for $x = 15$ feet it is 9000 pound-feet; for $x = 20$ feet it is 8000 pound-feet, and so on. The diagram shows the distributions of moments throughout the beam, and it can be demonstrated that the curve joining the ends of the ordinates is the common parabola.

When a beam is loaded both uniformly and with concentrated loads, the bending moments for all sections may be found in a similar manner. The maximum bending moment indicates the point where the beam is under the greatest horizontal stresses; this will usually be found near the middle of the beam and often under one of the concentrated loads. For simple beams resting on two supports at their ends all the bending moments are positive. It may further be noted that if the vertical forces on the right of the section be used, the same numerical values will be found for the bending moments.

Prob. 21 *A*. Two locomotive wheels, six feet apart and each weighing 20 000 pounds, roll over a beam of 27 feet span. Find the greatest reaction which can be caused by these wheels.

Prob. 21 *B*. A simple beam of 12 feet span weighs 60 pounds per linear foot, and has a load of 150 pounds at 8 feet from the left end. Compute the bending moments for sections distant 2, 4, 6, 8, 10 feet from the left support, and construct the diagram of bending moments.

Art. 22. Resisting Moments

Suppose a simple beam to be cut by an imaginary vertical plane MN and the portion on the right of that plane to be removed (Fig. 12). In order that the remaining part may be in equilibrium, forces must be applied to the section; in the figure horizontal forces are shown, and these represent the horizontal stresses in the section. The reaction and loads on the left of MN together with the stresses acting on that section constitute a system of forces in equilibrium. The algebraic sum of the moments of the reaction and loads with respect to the point D is the bending moment for the section MN, the value of which may be found by the methods of the last article. This bending moment tends to turn the beam in a clock-

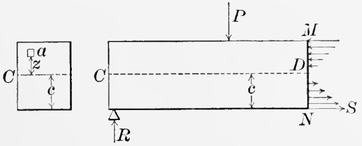

Fig. 12

wise direction about D, and it is balanced by the sum of the moments of the stresses acting on MN, which turn it in the opposite direction. Hence

Bending Moment = Resisting Moment.

It will now be shown how an expression for the second term of this equation can be obtained.

It is found by experiment that there is a certain line CD on the side of the beam which does not change in length under the bending, and hence there is no horizontal stress upon it. Below this neutral line the fibers in a simple beam are found to be elongated, and above it they are shortened; thus the stresses below the neutral line are tension and those above it are compression. A neutral line like CD also obtains in all longitudinal vertical sections of the beam. There is, in fact, a 'neutral surface' extending throughout the entire width of the beam, and the intersection of this neutral surface with any section area gives a line CC, which is called the 'neutral axis' of that section.

It is also found by experiment that the horizontal stresses in any section increase uniformly from the neutral axis to the top and bottom of the beam, provided the elastic limit of the material is not exceeded. Thus, if S is the horizontal unit-stress at the upper or lower side of the beam in Fig. 11, the unit-stress halfway between that side and the neutral axis is $\frac{1}{2}S$. Also, let c be the distance from the neutral axis to the upper or lower side of the beam, and z be any distance less than c, then the horizontal unit-stress at the distance z is $S(z/c)$.

'Resisting Moment' is the term used to denote the algebraic sum of the moments of all the horizontal stresses in a section with respect to its neutral axis. Let a (Fig. 12) be any small elementary area of the section at a distance z from the neutral axis. The unit-stress on this small area is $S(z/c)$ and hence the total stress on it is $aS(z/c)$. The moment of this stress with respect to the point D is $aS(z/c)$ multiplied by its lever-arm z, or $aS(z/c)z$. Hence

$$\text{Moment of stress on } a = (S/c)az^2$$

and the resisting moment is the algebraic sum of all these elementary moments for all possible values of z, or since S/c is a constant,

$$\text{Resisting Moment} = \frac{S}{c}(a_1z_1^2 + a_2z_2^2 + a_3\,z_3^2 + \cdots) = \frac{S}{c}\Sigma az^2$$

Here the notation Σaz^2 is used to denote the quantity $a_1z_1^2 + a_2z_2^2 + \ldots$. The letter Σ is not a factor, but a symbol which indicates the process of summation and it should be called 'summation of all values of.'

This quantity Σaz^2 is called the 'moment of inertia' of the cross-section of the beam, and it will be shown in

Art. 24 how its value is found. The moment of inertia is designated by I, and hence

$$\text{Resisting Moment} = \frac{SI}{c}$$

This expression for the resisting moment is applicable to all kinds of cross-sections. It is shown above that the bending moment for any section is equal to the resisting moment for that section; letting M be the value of the bending moment, found as in Art. 21, then

$$\frac{SI}{c} = M$$

is a general formula applicable to all kinds of beams. This formula will be constantly used in the next chapter.

The term 'moment of inertia' has no reference to inertia when applied to plane surfaces, as is here the case. It is merely a name for the quantity Σaz^2, and this quantity is found by multiplying each elementary area by the square of its distance from the given axis and taking the sum of the products. As z^2 is always positive whether z be positive or negative, Σaz^2 or the moment of inertia I is always positive. If all the elementary areas be taken as equal and n be their number, then $na = A$, the total area of the section. Hence $I = \Sigma az^2 = a\Sigma z^2 = A\,\Sigma z^2/n$. Designate the average of all the values of z^2 by r^2, then $\Sigma z^2/n = r^2$ and thus $I = Ar^2$ is another definition of moment of inertia. This is a constant which depends only on the size of the section area and its arrangement with respect to the neutral surface.

Prob. 22. In the above figure let MD and DN be each 6 inches and let the width of the beam CC be 8 inches. If the tensile unit-stress S on the bottom of the beam is 600 pounds per square inch, the compressive unit-stress on the top of the beam is also 600 pounds per square inch. Show that the total tensile stress is 14 400 pounds, and that the total compressive stress is also 14 400 pounds.

Art. 23. Centers of Gravity

The center of gravity of a plane surface is that point upon which a thin sheet of cardboard, having the same shape as the given surface, can be balanced when held in a horizontal position. In the investigation of beams its section area is the given surface, and it is required to know the distances from the top or bottom of the section to the center of gravity. The letter c will be used to denote these distances when they are equal, and the longest of these distances when they are unequal.

For a square, rectangle, or circle, whose depth is d, it is evident that $c=\frac{1}{2}d$. Also for a section of **I** shape, where the upper and lower flanges are equal in size, it is plain that $c=\frac{1}{2}d$.

For the **T** section, shown on the left of Fig. 13, the distance c is greater than $\frac{1}{2}d$, and its value is to be found by using the principle of moments. If the width of the horizontal flange is 4 inches and its thickness $1\frac{1}{4}$ inches, the area of the flange is 5 square inches; if the height of

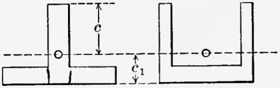

Fig. 13

the vertical web is 6 inches and its thickness 1 inch, the area of the web is 6 square inches. The total area of the cross-section is then 11 square inches. Now if this section is a thin sheet held in a horizontal plane, the weights of the two parts and the whole are represented by 5, 6, and 11. With respect to an axis at the end of the web the lever-arms of these weights are $6\frac{5}{8}$ inches, 3 inches, and c inches; the equation of moments then is

$$5\times6\tfrac{5}{8}+6\times3-11\times c=0$$

from which the value of c is found to be 4.65 inches. For the channel section, shown on the right of Fig. 13, the same method is to be followed as for the **T**.

The method of moments may thus be applied to areas as well as to forces. If a be any area and z the distance of its center of gravity from an axis, the product az is called the static moment of the area. The algebraic sum of the static moments of all parts of the figure is represented by Σaz, the summation of the values a_1z_1, a_2z_2, a_3z_3, etc. If A is the total section area, then

$$c=\frac{\Sigma az}{A} \tag{2}$$

is a general expression of the method of finding the distance c. If the axis is taken within the section, some of the z's are negative, and if the axis passes through the center of gravity of the section, then the quantity Σaz is zero.

When the cross-section is bounded by curved lines, as in a railroad rail, it is to be divided up into small rectangles and the value of a be found for each; the sum of all the a's is A, and then by the above method the value of c is computed. For the various rolled shapes found in the market the values of c are thus determined by the manufacturers and published for the information of engineers.

Triangular beams are never used, but it is often convenient to remember that for any triangle whose depth is d the value of c is $\frac{2}{3}d$.

For the angle section, shown in Fig. 14, the center of gravity usually lies without the section, and there are two values of c, called c_1 and c_2, to be determined. Let the thickness of each leg be $\frac{3}{4}$ inches, the length of the long leg be 6 and that of the short leg be 4 inches. The area of the long leg, including the lower corner, is $6\times\frac{3}{4}=4.5$ square inches, and its center of gravity is 3 inches below the axis AA and $3\frac{5}{8}$ inches to the right of axis BB.

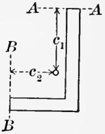

Fig. 14

The area of the short leg, excluding the corner, is $3\frac{1}{4}\times\frac{3}{4}=2.4375$ square inches, and its center of gravity is $5\frac{5}{8}$ inches below the axis AA and $1\frac{5}{8}$ inches to the right of the axis BB. Then, as the total area of the section is 6.9375 square inches, the equation of moments with respect to the axis AA is

$$6.9375\,c_1=4.5\times3+2.4375\times5.625,$$

and then $c_1=3.92$ inches. Also the equation of moments with respect to the axis BB is

$$6.9375\,c_2=4.5\times3.625+2.4375\times1.625,$$

from which $c_2=2.92$ inches.

Prob. 23 *A*. For Fig. 13 let $c=6$ and $c_1=3$ inches. If the unit stress S at the top of the web is 6400 pounds per square inch, what is the unit-stress S_1 on the lower side of the flange?

Prob. 23 *B*. A deck-beam used in buildings has a rectangular flange $4 \times \frac{3}{4}$ inches, a rectangular web $5 \times \frac{1}{2}$ inches, and an elliptical head which is 1 inch in depth and whose area is 1.6 square inches. Find the distance of the center of gravity from the top of the head.

ART. 24. MOMENTS OF INERTIA

The moment of inertia of a plane surface with respect to an axis is the sum of the products obtained by multiplying each elementary area by the square of its distance from that axis. In the discussion of beams the axis is always taken as passing through the center of gravity of the cross-section and parallel to the top and bottom lines of the cross-section. Let I be this moment of inertia as in Art. 22, its value is to be found by determining the quantity Σaz^2, the summation of all the values $a_1z_1^2$, $a_2z_2^2$, $a_3z_3^2$, etc.

To find I for a rectangle of breadth b and depth d, let CC be the axis through the center of gravity and parallel to b (Fig. 15). Let the elementary area a be a

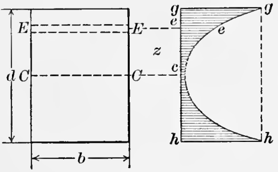

Fig. 15

small strip EE parallel to CC and at a distance z from it. Let a line gh be drawn parallel to the depth d of the rectangle, and normal to gh let lines be drawn equal to the squares of z; thus ee is the square of CE, and gg is the square of CG. Now the elementary product az^2 is the elementary area EE multiplied by the ordinate ee; hence Σaz^2 is represented by a solid standing on bd whose variable height is shown by the shaded area $ghhcg$. But the volume of this solid is the product of its length b and this shaded area. The curve ceg is a parabola because each line ee is the square of the corresponding altitude ce; accordingly the shaded area is one-third of $ghhg$. But gh is equal to d, and gg is equal to $(\frac{1}{2}d)^2$; thus the shaded area is represented by $\frac{1}{3}.d.\frac{1}{4}d^2$, or $\frac{1}{12}d^3$. Hence

$$I = b \times \tfrac{1}{12}d^3 = \tfrac{1}{12}bd^3$$

is the moment of inertia of a rectangle about an axis through its center of gravity and parallel to its base.

The moment of inertia is a compound quantity resulting by multiplying an area by the square of a distance; it thus contains the linear unit four times. If $b=3$ inches and $d=4$ inches, then $I=16$ inches^4, or the numerical unit of I is biquadratic inches.

Moments of inertia when referred to the same axis can be added or subtracted like any other qualities

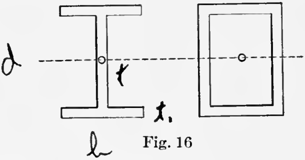

Fig. 16

which are of the same kind. Thus, let there be a hollow rectangular section whose outside depth and breadth are b and d and whose inside depth and breadth are b_1 and d_1, the thickness of the metal being the same throughout (Fig. 16). Then the moment of inertia of this section is found by subtracting the moment of inertia of the inner rectangle from that of the outer one, or

$$I = \tfrac{1}{12}bd^3 - \tfrac{1}{12}b_1d_1^3$$

is the moment of inertia for the rectangular section whose area is $bd - b_1d_1$.

For the common **I** beams whose flanges are equal the same method applies. Let b be the width of the flanges and d the total depth of the section shown on the right of Fig. 16; also let t be the thickness of the web and t_1 the thickness of the flanges. The moment of inertia of the area $(b-t)(d-2t_1)$ is then to be subtracted from the moment of inertia of the area bd, or

$$I = \tfrac{1}{12}bd^3 - \tfrac{1}{12}(b-t)(d-2t_1)^3$$

is the moment of inertia for the **I** section.

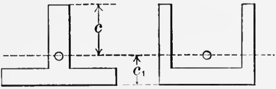

Fig. 17

For the **T** section the distance c from the end of the web to the axis through the center of gravity must first be computed by the method of the last article. Then c_1, the distance from the outside of the flange to the axis, is also known (Fig. 17). Let b be the breadth of the flange and t_1 its thickness, and let t be the thickness of the web. Then the moment of inertia of the area tc is one-half of that of a rectangle of depth $2c$, or $\tfrac{1}{2} \times \tfrac{1}{12}t(2c)^3$, which is $\tfrac{1}{3}tc^3$; also the moment of inertia of the area bc_1 is one-half of that of a rectangle of depth $2c_1$. Adding these together and subtracting the moment of inertia of the area $(b-t)(c_1-t_1)$, there results

$$I = \tfrac{1}{3}tc^3 + \tfrac{1}{3}bc_1^3 - \tfrac{1}{3}(b-t)(c_1-t_1)^3$$

which is the moment of inertia for the **T** section. The same formula applies to the ⊔ section if t is the thickness of the two webs.

The above formulas for **I** and **T** sections are correct for cast-iron beams where the corners are but little rounded. For wrought-iron and steel beams, however, the flanges are not usually of uniform thickness, and all the corners are rounded off by curves, so that the formulas are not strictly correct; for such shapes the numerical values of the moments of inertia for all the sections in the market are published by the manufacturers, so that it is not necessary for engineers to compute them. (See Art. 33.)

The moment of inertia of a circle with respect to its diameter as an axis is $I = \tfrac{1}{64}\pi d^4$ where d is the length of the diameter.

The moment of inertia of a plane surface with respect to any axis is equal to the moment of inertia with respect to a parallel axis through its center of gravity plus the area of the surface multiplied by the square of the distance between the two axes. Thus, let A be the area of a surface (Fig. 18), I its moment of inertia with respect to an axis CC through the center of gravity, and h the distance to another parallel axis DD, then the moment of inertia of the surface with respect to DD is $I_1 = I + Ah^2$. This principle is used to find the moment of inertia for compound and built-up sections, as illustrated in the following example.

For the section of a bridge-post, shown in Fig. 19, the area of each 12-inch channel and its moment of inertia with respect to an axis through its center of gravity are given as $A=7.35$ square inches and $I=144.0$ inches^4. For the plate, which is $16 \times \frac{3}{8}$ inches, the values are $A=6.00$ square inches and $I=0.007$ inches^4. From Art. 21

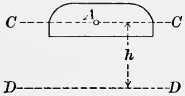

Fig. 18

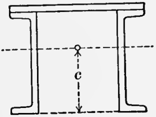

Fig 19

the distance c to the center of gravity of the compound section is found to be 7.99 inches. Hence for each channel $h=1.99$ inches, and for the plate $h=4.57$ inches. The moment of inertia for the entire built-up section is then

$$I_1=2(144.0+7.35\times 1.99^2)+0.007+6.00\times 4.57^2=471.4.$$

In this chapter the fundamental applications of moments to be used in discussing beams have been presented, and it is now possible to take up the subject and give the theory of equilibrium of beams clearly and logically, so that the student may undertake practical problems in the most satisfactory manner.

Prob. 24. A steel **I** beam weighing 80 pounds per linear foot is 24 inches deep, its flanges being 7 inches wide and $\frac{7}{8}$ inches mean thickness, while the web is 0.5 inches thick. The moment of inertia stated by the manufacturer is 2088 inches^4. Compute it by the formula here given.

Art. 25. Review Problems

Prob. 25 *A*. Three men carry a stick of timber, two taking hold at a common point and one at one of the ends. Where should be the common point so that each man may carry one-third of the weight?

Prob. 25 *B*. Compute the bending moments under each concentrated load for Fig. 9, taking the weight of the beam into account.

Prob. 25 *C*. The two bases of a trapezoid are 8 and 5 inches, and its height is 4 inches. Find the center of gravity.

Prob. 25 *D*. For a solid circular section the moment of inertia with respect to an axis through the center is $\frac{1}{64}\pi d^4$. Find the moment of inertia for a hollow circular section with outside diameter d_1 and inside diameter d_2.

Prob. 25 *E*. A simple beam of 16 feet span weighs 60 pounds per linear foot and has a concentrated load of 500 pounds at a distance of 4 feet from the left end. Compute the bending moments for several sections throughout the beam and construct the diagram of moments.

Prob. 25 *F*. Locate both gravity axes of a standard steel channel 8 inches deep, the average thickness of the web being 0.22 inches, average thickness of flange 0.40 inches, and width of flanges 2.26 inches.

Prob. 25 *G*. Compute the moments of inertia of the channel section with respect to each of the axes found in the last problem.

Prob. 25 *H*. Find the moment of inertia of a circle 3 inches in diameter. Also the moment of inertia of that circle with respect to another axis in the same plane, the shortest distance from the center of the circle to that axis being 5 inches.

Prob. 25 *J*. A timber cantilever 4×6 inches in section projects 6 feet out of a wall. What load must be put upon it so that the greatest shearing stress shall be 120 pounds per square inch?

Prob. 25 *K*. Show that the moment of inertia of a rectangle with respect to an axis passing through its base is $\frac{1}{3}bd^3$.

Chapter 4

CANTILEVER BEAMS AND SIMPLE BEAMS

Art. 26. Definitions and Principles

A simple beam is a bar resting upon supports at its ends, and is the kind most commonly in use. A cantilever beam is a bar resting on one support at the middle, or if a part of a beam projects out from a wall or beyond a support this part is called a cantilever beam. In a simple beam the lower part is under tension and the upper part under compression; in a cantilever beam the reverse is the case. Unless otherwise stated, all beams will be regarded as having the section area uniform throughout the entire length.

Since a beam is at rest the internal stresses in any section hold in equilibrium the external forces on each side of that section. Thus, if a beam be imagined to be cut apart

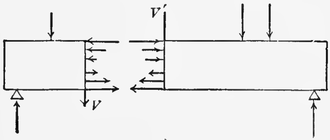

Fig. 20

and the two parts separated, as in Fig. 20, forces must necessarily be required to prevent the parts from falling. These internal forces or stresses may be resolved into horizontal and vertical components. The horizontal components are stresses of tension and compression, while the vertical components add together and form a stress V known as the resisting shear.

Each side of the beam is held in equilibrium by the vertical forces and stresses that act upon it. The vertical forces are the reaction and the loads, the stresses are the horizontal ones of tension and compression, and the vertical one of shear. The sum of all the horizontal tensile stresses must be equal to the sum of all the horizontal compressive stresses, or otherwise there would be longitudinal motion. The sum V of the vertical stresses must equal the algebraic sum of the reaction and loads, or otherwise there would be motion in an upward or downward direction. Lastly, the sum of the moments of the stresses in the section must equal the sum of the moments of the vertical forces, or otherwise there would be rotation. These statical principles apply to each part into which the beam is supposed to be divided.

It is found by experiment that the fibers on one side of the beam are elongated and on the other shortened, while between is a neutral surface, which is unchanged

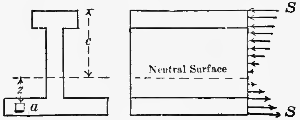

Fig. 21

in length. It is also found that the amount of elongation or shortening of any fiber is directly proportional to its distance from the neutral surface. Hence, if the elastic limit is not surpassed, the stresses are also proportional to their distances from the neutral surface (Fig. 21).

From the above it can be shown that the neutral axis passes through the center of gravity of the cross-section. For, if S be the unit-stress on the remotest fiber and c its distance from the neutral axis, then the unit-stress at the distance z is Sz/c, and the total stress on an elementary area a is aSz/c. The algebraic sum of all the horizontal stresses in the section then is $(S/c)\Sigma az$, where Σaz denotes the summation of all the values a_1z_1, a_2z_2, etc. From the above statical principles this sum must be zero, and it hence follows that Σaz must be zero; that is, the sum of the moments of the elementary areas is zero with respect to the neutral axis. Hence the neutral axis passes through the center of gravity, for the center of gravity is that point upon which the surface can be balanced, or it is that point for which $\Sigma az = 0$.

Prob. 26 *A*. An **I** beam which is 20 feet long weighs 700 pounds and the area of its cross-section is 10.29 square inches. What is the kind of material?

Prob. 26 *B*. Let $a_1 = 2$, $a_2 = 2.5$, $a_3 = 2.7$ square inches, and $z_1 = +3.5$, $z_2 = +1.5$, $z_3 = -2.6$ inches. Does the axis pass through the center of gravity of $a_1 + a_2 + a_3$?

ART. 27. RESISTANCE TO SHEARING

When a beam is short it sometimes fails by shearing in a vertical section near one of the supports. The force that produces this shearing is the resultant of all the vertical forces on one side of the section. Thus, in the simple beam of the first diagram (Fig. 22) this resultant is the reaction minus the weight of the beam between the reaction and the section; in the cantilever beam of the second diagram it is the loads and the weight of the beam on the left of the section.

'Vertical shear' is the name given to the algebraic sum of all the vertical forces on the left of the section under consideration. Thus in the first diagram of Fig. 22, if the reaction is 6000 pounds, the vertical shear V just at

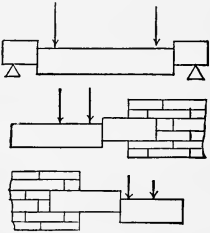

Fig. 22

the right of the support is 6000 pounds. If the beam weighs 100 pounds per linear foot, the vertical shear at a section one foot from the support and on the left of the single load is 5900 pounds. Again in the second diagram of Fig. 22, if the beam weighs 100 pounds per linear foot and if each concentrated load is 800 pounds, and the distance from the end to the section shown is 4 feet, the vertical shear in that section is 2000 pounds.

It is seen from these illustrations that in a simple beam the greatest vertical shear is at the supports, and that in a cantilever beam it is at the wall. Only these sections, then, need be investigated in a solid beam. For a simple beam of length l and carrying w pounds per linear unit, the greatest vertical shear is the reaction $\frac{1}{2}wl$. For a cantilever beam of length l, the greatest vertical shear due to uniform load is the total weight wl.

The vertical shear V produces in the cross-section an equal shearing stress. If A is the section area and S the shearing unit-stress acting over that area, then

$$V = AS, \qquad S = \frac{V}{A}, \qquad A = \frac{V}{S} \tag{3}$$

are the equations similar to (1) of Art. 1; these are used for the practical computations regarding shear in solid beams.

For example, consider a steel **I** beam weighing 250 pounds per yard and 12 feet long, over which roll three locomotive wheels 4 feet apart and each bearing 14 000 pounds (Fig. 23). The greatest shear will occur when

Fig. 23

one wheel is almost at the support as shown in the figure. By Art. 20 the reaction is found to be 28 500 pounds, and this is the greatest vertical shear V. By Art. 9 the area of the cross-section is found to be 24.5 square inches. Then the shearing unit-stress in the section is

$$S = \frac{28\,500}{24.5} = 1160 \text{ pounds per square inch}$$

which is a low working unit-stress for steel.

As a second example, consider a wooden cantilever beam which projects out from a bridge floor and supports a sidewalk. Let it be 6 inches wide, 8 inches deep, and 7 feet long, and let the maximum load that comes upon it be 7500 pounds. The vertical shear at the section where it begins to project is then 7590 pounds, or the load that it carries plus its own weight. As the section area is 48 square inches, the shearing unit-stress is a little less than 160 pounds per square inch. The factor of safety against shearing is hence about 19 (Art. 6), so that the security is ample.

It is indeed only in rare instances that solid beams of uniform cross-section are subject to dangerous stresses from shearing. Beams almost universally fail by tearing apart under the horizontal tensile stresses, and hence the following articles will be devoted entirely to the consideration of these bending stresses.

Prob. 27 *A*. A simple beam of cast iron is 3×3 inches in section and $5\frac{1}{2}$ feet long between supports. Besides its own weight, it is to carry a load of 4000 pounds at the middle and a load of 1000 pounds at $2\frac{1}{2}$ feet from the left end. Find the factor of safety against shearing.

Prob. 27 *B*. On a simple beam 12 feet long there are two loads, each 600 pounds, one at 3 feet from the left end, and one at 3 feet from the right end. Find the vertical shear due to these loads for a section near one of the supports, and also for any section between the loads.

Art. 28. Resistance to Bending

In Art. 27 it was shown that the resisting moment of the internal stresses in any section is equal to the bending moment of the external forces on each side of the section. Art. 21 explains how to find the bending moment, which hereafter will be designated by the letter M. In Art. 22 an expression for the resisting moment is derived. Therefore

$$\frac{SI}{c} = M \tag{4}$$

is the fundamental formula for the discussion of the flexure of beams, provided the elastic limit is not exceeded. Here S is the unit-stress of tension or compression on the top or bottom of the beam, c is the vertical distance of S from the center of gravity of the cross-section, and I is the moment of inertia of the cross-section. Art. 23 explains how to find c, and Art. 24 shows how I is determined. S is often called the 'fiber unit-stress,' meaning thereby the greatest horizontal unit-stress.

This formula shows that S varies directly with M, that is, the greatest tensile or compressive stress in the beam occurs at the section where M has its maximum value. For a simple beam under uniform load the bending moment M at any section distant x from the left support is, as shown in Art. 21,

$$M = \tfrac{1}{2}wl \,.\, x - wx \,.\, \tfrac{1}{2}x = \tfrac{1}{2}w(lx - x^2)$$

and if $x = \frac{1}{2}l$, this gives $M = \frac{1}{8}wl^2$ as the maximum bending moment; or if W be the total load wl, this may be written as $M = \frac{1}{8}Wl$. When concentrated loads are on a simple beam the maximum bending moment must usually be found by trial; it will generally be under one of those loads.

For a cantilever beam of length l the maximum bending moment always occurs at the wall (Fig. 24). For a uniform load of w per linear unit the bending moment at a section distant x from the end is the load wx into its lever-arm $\frac{1}{2}x$, and this is negative as it tends to produce rotation in a direction opposite to that of the hands of a watch (Art. 19). Thus, for any section $M = -\frac{1}{2}wx^2$, and when x becomes equal to l the maximum value is $M = -\frac{1}{2}wl^2$. The negative sign shows merely the direction in which rotation tends to occur, and when using formula (4) the value of M is to be inserted without sign. The diagram of bending moments for this case is a parabola, since M increases as the square of x.

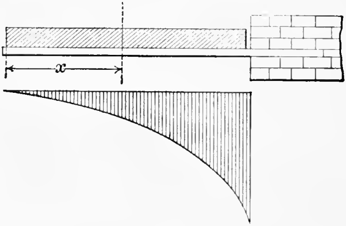

Fig. 24

For concentrated loads on a cantilever beam the bending moment M is $-P_1x$ until x passes beyond the second load; then $M=-P_1x-P_2(x-p)$ where p is the distance between the two loads. Thus the diagram of bending moments is composed of straight lines (Fig. 25),

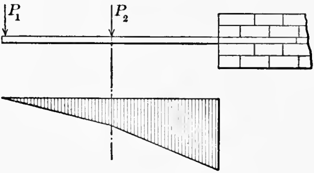

Fig. 25

the maximum M occurs when x becomes equal to l, and its value is $-P_1l-P_2(l-p)$.

It may be noted that the only difference in stating moment equations for a cantilever beam and for a simple beam lies in the fact that for the former there is no reaction at the left end. A cantilever beam is hence really simpler than a simple beam, as no reactions need be computed.

Prob. 28 *A*. A cantilever beam has a load of 800 pounds at its end, and is also uniformly loaded with 125 pounds per linear foot; its length is 5 feet. Compute the bending moments for five sections, one foot apart, and construct the diagram of bending moments.

Prob. 28 *B*. A simple beam weighing 60 pounds per linear foot is 13 feet in span and has a load of 1000 pounds at the middle. Compute the maximum bending moment.

ART. 29. SAFE LOADS FOR BEAMS

A safe load for a beam is one that produces a tensile or compressive unit-stress which is safe according to the principles set forth in Chapter 1. To find such a safe load for a given beam the safe value of S is to be assumed from those principles. Then in formula (4) the values of I and c are known. The maximum bending moment M is to be expressed in terms of the unknown load, and thus an equation is derived from which the load is found.

For example, let a wooden cantilever beam be 2 inches wide, 3 inches deep, and 72 inches long, and let it be required to find what load P at the end will produce a unit-stress S of 800 pounds per square inch. Here the maximum value of M is $P\times72$. From Art. 23 the value of c is $1\frac{1}{2}$ inches and from Art. 24 the value of I is $4\frac{1}{2}$ inches^4. Then by (4) of Art. 28,

$$\frac{SI}{c}=\frac{800\times4.5}{1.5}=72P$$

from which P is found to be $33\frac{1}{3}$ pounds.

Again, let a simple beam of cast iron be 3 inches wide, 4 inches deep, and 36 inches long, and let it be required to find what uniform load will produce a unit-stress of 2000 pounds per square inch. Here let w be the uniform load per linear inch; the total load is wl, each reaction is $\frac{1}{2}wl$, and the maximum bending moment M is $\frac{1}{8}wl^2$. The value of c is 2 inches, and that of I is 16 inches^4. Then since l is 36 inches,

$$\frac{SI}{c} = \frac{2000 \times 16}{2} = 162w$$

from which $w = 98.8$ pounds per linear inch, and hence the total uniform safe load that can be put on the beam is about 3560 pounds.

The student should notice that in using formula (4) all lengths must be expressed in the same unit. If the length of a beam is given in feet it must be reduced to inches for use in the formula, because S, I, and c are expressed in terms of inches. Formula (4) cannot be used to find the load that will rupture a beam, except in the manner indicated in Art. 66.

Prob. 29 *A*. A steel **I** beam 7 inches deep and weighing 22 pounds per foot has for the moment of inertia of its cross-section 52.5 inches^4, and it is to be used as a simple beam with a span of 18 feet. What load P can it carry when the greatest unit-stress S is required to be 12 000 pounds per square inch?

Prob. 29 *B*. What safe uniformly distributed load can be carried by an oak beam, 4 inches wide and 6 inches deep, having a span of 16 feet, if the greatest unit-stress is not to exceed 1000 pounds per square inch?

Prob. 29 *C*. A wooden beam, 10×12 inches in section area, projects 6 feet from the wall of a building. What safe load can be suspended from the end of the beam so that there shall be a factor of safety of 10?

ART. 30. INVESTIGATION OF BEAMS

To investigate a beam acted upon by given loads the greatest unit-stress S produced by those loads is to be found from formula (4). From the given dimensions of the beam I and c are known, from the given loads the maximum value of M is to be found; then

$$S = \frac{Mc}{I}$$

is the equation for computing the value of S. Then by the rules of Chapter 1 the degree of security of the beam is to be inferred. As formula (4) is deduced under the laws of elasticity, it fails to give reliable values of S when the elastic limit is exceeded.

For example, consider a cast-iron ⊔ section which is used as a simple beam with a span of 6 feet, and upon which there is a total uniform load of 80 000 pounds. Let the total depth be 16 inches, the total width 12 inches, the thickness of the flange 2 inches, and the thickness of the webs 1 inch. By Art. 28 the greatest value of M is at the middle of the beam, this being $\frac{1}{8} \times 80\,000 \times 6 \times 12 = 720\,000$ pound-inches. By Art. 23 the value of c is found to be 10.7 inches. By Art. 24 the value of I is found to be 1292 inches^4. Then,

$$S = \frac{720\,000 \times 10.7}{1292} = 5960 \text{ pounds per square inch.}$$

This is the compressive unit-stress in the end of the web when the beam is placed in the ⊔ position, as is usually the case in buildings. On the base of the beam the tensile unit-stress is about half this value, since c_1 is about one-half of c. Thus under the compressive stress the beam has a factor of safety of about 15, and under the tensile stress it has a factor of safety of about 7. As the least factor of safety for cast iron should be 10, the beam has not the full degree of security required by the best practice.

Prob. 30 *A*. A piece of wooden scantling 2 inches square and 18 feet long is hung horizontally by a rope at each end and a student weighing 175 pounds stands upon it. Is it safe?

Prob. 30 *B*. A floor is supported by 3×8-inch wooden joists of 16 feet span spaced 18 inches apart. When this floor carries a total load of 200 pounds per square foot, what is the factor of safety of the joists?

Art. 31. Design of Beams

The design of a beam consists in determining its size when the loads and its length are given. The allowable working unit-stress S is first assumed according to the requirements of practice. From the given loads the maximum bending moment M is then computed. Thus in formula (4) everything is known except I and c, and

$$\frac{I}{c}=\frac{M}{S}$$

is an equation which must be satisfied by the dimensions to be selected.

For a rectangular beam of breadth b and depth d the value of c is $\frac{1}{2}d$, and the value of I is $\frac{1}{12}bd^3$. Thus the equation above becomes

$$bd^2=\frac{6M}{S}$$

and if either b or d be assumed the other can be computed. For example, let it be required to design a rectangular wooden beam for a total uniform load of 80 pounds, the beam to be used as a cantilever with a length of 6 feet, and the working value of S to be 800 pounds per square inch. Here the maximum value of M is $80 \times 3 = 240$ pound-feet $= 2880$ pound-inches. Thus $bd^2 = 21.6$ inches³. If b is taken as 1 inch, $d = 4.65$ inches; if b is 2 inches, $d = 3.29$ inches; if b is 3 inches, $d = 2.68$ inches. With due regard to sizes readily found in the market 3×3 inches are perhaps good proportions to adopt.

Prob. 31 *A*. A simple cast-iron beam of 14 feet span carries a load of 10 000 pounds at the middle. If its width is 4 inches, find its depth for a factor of safety of 10; also find its width for a depth of 12 inches.

Prob. 31 *B*. A yellow pine beam of 20 feet span is to carry a uniformly distributed load of 500 pounds per linear foot with a factor of safety of 9. The depth of the beam is to be 1½ times the breadth. Find the dimensions of the beam.

ART. 32. COMPARATIVE STRENGTHS

The strength of a beam is measured by the load it can carry with a given unit-stress S. Let it be required to investigate the relative strengths of the four following cases:

1st. A cantilever loaded at the end with W.
2d. A cantilever loaded uniformly with W.
3d. A simple beam loaded at the middle with W.
4th. A simple beam loaded uniformly with W.

Let l be the length in each case, and the cross-section be of breadth b and depth d. Then $c = \frac{1}{2}d$, and $I = \frac{1}{12}bd^3$. Then, from Art. 28, and formula (4),

$$\text{For 1st,} \quad M = Wl \quad \text{and } W = \frac{Sbd^2}{6l}$$

For 2d, $M = \frac{1}{2}Wl$ and $W = \frac{2Sbd^2}{6l}$

For 3d, $M = \frac{1}{4}Wl$ and $W = \frac{4Sbd^2}{6l}$

For 4th, $M = \frac{1}{8}Wl$ and $W = \frac{8Sbd^2}{6l}$

Hence the comparative strengths of the four cases are as the numbers 1, 2, 4, 8; that is, if four such beams are of equal size and length and of the same material, the second is twice as strong as the first, the third is four times as strong, and the last is eight times as strong as the first.

From these equations the following important laws regarding rectangular beams are derived:

The strength varies directly as the breadth and directly as the square of the depth.

The strength varies inversely as the length.

A beam is twice as strong under a distributed load as under an equal concentrated load.

The second and third of these laws apply also to beams having cross-sections of any shape.

The reason why rectangular beams are placed with the longest dimension vertical is now seen to be that the strength increases in a faster ratio with the depth than with the breadth. If the breadth is doubled the strength is doubled; if the depth is doubled the strength is four times as great as before.

A beam is said to be 'fixed' at its end when the end is fastened in a wall in such a manner that that end remains horizontal. The following are the maximum bending moments for such beams:

One end fixed, load W at middle, $M = \frac{3}{16}Wl$
One end fixed, uniform load W, $M = \frac{1}{8}Wl$
Both ends fixed, load W in middle, $M = \frac{1}{8}Wl$
Both ends fixed, uniform load W, $M = \frac{1}{12}Wl$

It is thus seen that beams fixed at their ends are stronger than simple beams similarly loaded, for under a given unit-stress S they will carry a greater load W. Moment diagrams for fixed beams are given in Art. 57

An 'overhanging beam' is shown in Fig. 26, the distance between the supports being l, one support being

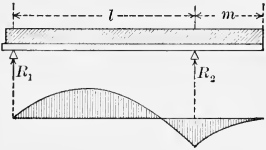

Fig. 26

at the left end and the other support being at a distance m from the right end. Under a uniform load of w per linear unit, the left reaction is $R_1 = \frac{1}{2}w(l - m^2/l)$, the maximum positive moment is $M = \frac{1}{2}R_1^2/w$ and the maximum negative moment is $M = \frac{1}{2}wm^2$. For instance, let $l = 10$ and $m = 6$ feet, and $w = 30$ pounds per linear foot; then the total weight of the beam is 480 pounds, the left reaction is 96 pounds, the maximum positive bending moment is 153.6, and the maximum negative bending moment is 540 pound-feet.

For a load P placed between the supports of the overhanging beam in Fig. 26, the reactions R_1 and R_2 are exactly the same as for a simple beam and the maximum moment is $\frac{1}{4}Pl$. For a load P placed at the end of the overhanging arm, the reactions are $R_1 = -Pm/l$ and $R_2 = +P(1-m/l)$, while the greatest bending moment is Pm.

Prob. 32 *A*. Show that a beam 3 inches wide, 6 inches deep, and 4 feet long is nine times as strong as a beam 2 inches wide, 4 inches deep, and $10\frac{2}{3}$ feet long.

Prob. 32 *B*. Compute, without using the above formulas, the reactions for an overhanging beam where the distance between the supports is 9 feet and the overhanging arm is 3 feet, the beam weighing 40 pounds per linear foot.

ART. 33. STEEL I BEAMS

Wrought-iron rolled beams have been much used in bridge and building construction, but now medium-steel beams are almost exclusively employed. The ultimate tensile strength of such steel will be taken as 65 000 pounds per square inch, and its elastic limit as 35 000 pounds per square inch, in the solution of examples and

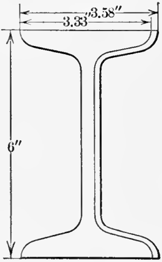

Fig. 27

problems hereafter given. These beams are manufactured in about thirteen different depths, and of each depth there are several different sizes or weights, so that designers have a large variety from which to select. In the following table only the heaviest and lightest sections of each

TABLE 8. STEEL I BEAMS

Depth Inches	Weight per Foot Pounds	Section Area A Sq. Inches	Moment of Inertia I Inches^4	Section Modulus $\frac{I}{c}$ Inches^3	Moment of Inertia I' Inches^4
24	100	29.4	2380	198	48.6
24	80	23.5	2088	174	42.9
20	75	22.1	1269	127	30.2
20	65	19.1	1170	117	27.9
18	70	20.6	921	102	24.6
18	55	15.9	796	88.4	21.2
15	55	15.9	511	68.1	17.1
15	42	12.5	442	58.9	14.6
12	35	10.3	228	38.0	10.1
12	31½	9.3	216	36.0	9.50
10	40	11.8	159	31.7	9.50
10	25	7.4	122	24.4	6.89
9	35	10.3	112	24.8	7.31
9	21	6.3	85.0	18.9	5.16
8	25¼	7.50	68.4	17.1	4.75
8	18	5.33	56.9	14.2	3.78
7	20	5.88	42.2	12.1	3.24
7	15	4.42	36.2	10.4	2.67
6	17¼	5.07	26.2	8.73	2.36
6	12¼	3.61	21.8	7.27	1.85
5	14¾	4.34	15.2	6.08	1.70
5	9¾	2.87	12.1	4.84	1.23
4	10½	3.09	7.1	3.55	1.01
4	7½	2.21	6.0	3.00	0.77
3	7½	2.21	2.9	1.93	0.60
3	5½	1.63	2.5	1.71	0.46

depth are given. The proportions of one of the sizes of the two 6-inch beams are shown in Fig. 27, the outer line on the right-hand side indicating the heavier section and the other one the lighter section.

In Table 8 the moments of inertia I in the fourth column are those about an axis through the centers of gravity and perpendicular to the web, and are those to be used in all beam computations. The values I' given in the last column are with respect to an axis through the center of gravity but parallel to the web; these are for use in the next chapter in the discussion of struts.

The quantity I/c is often called the 'section modulus' as it contains all the dimensions of the cross-section. The process of selecting an **I** section depends merely on finding a value of I/c which corresponds to the value of M/S, as shown in Art. 31; hence for convenience these values are tabulated in the fifth column of the table.

For example, an **I** beam in a floor is to have 20 feet span and to carry a uniform load of 13 500 pounds; what size is to be selected? The bending moment is $M = \frac{1}{8} \times 13\,500 \times 20 \times 12 = 405\,000$ pound-inches; and the working unit-stress S should be $\frac{1}{5} \times 65\,000$ pounds per square inch. Then from formula (4),

$$\frac{I}{c} = \frac{405\,000}{13\,000} = 31.2 \text{ inches}^3$$

and hence, from the table, the heavy 10-inch beam should be used.

Prob. 33 *A*. A heavy 15-inch steel **I** beam of simple span carries a uniform load of 42 net tons. Find its factor of safety if the span is 6 feet; also if the span is 9 feet.

Prob. 33 *B*. A steel **I** beam of 25 feet span is to carry a uniformly distributed load of 1000 pounds per linear foot. In addition there is a concentrated load of 6000 pounds at 10 feet from the left end. Find the proper size of **I** beam to be used.

ART. 34. BEAMS OF UNIFORM STRENGTH

The beams thus far discussed have been of uniform section throughout their entire length. As the bending moments are small near the ends of the beam the unit-stress S is there also small, and hence more material is used than is really needed. A beam of uniform strength is one so shaped that the unit-stress S is the same at all parts of the length.

For a cantilever beam loaded with P at the end, the bending moment at any distance x from the end is Px.

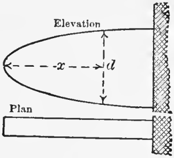

Fig. 28.

If the section is rectangular, formula (4) reduces to $\frac{1}{6}Sbd^2 = Px$, in which P and S are constant. If b is made the same throughout, then

$$d^2 = \frac{6Px}{Sb}$$

and therefore d^2 must vary directly as x. If $x = l$, the value of d is the depth d_1 at the wall, and accordingly $6P/Sb = d_1^2/l$; hence the equation becomes

$$d = d_1\sqrt{\frac{x}{l}}$$

Thus, if $x = \frac{1}{9}l$, $d = \frac{1}{3}d_1$; if $x = \frac{1}{4}l$, $d = \frac{1}{2}d_1$, and so on. As the squares of the depths vary with the distances from the end, the curve of the side of the beam should be the common parabola (Fig. 28).

For a rectangular cantilever beam uniformly loaded with w per linear unit the bending moment M is $\frac{1}{2}wx^2$ and

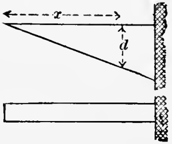

Fig. 29

formula (4) becomes $\frac{1}{6}Sbd^2 = \frac{1}{2}wx^2$. If the breadth is uniform throughout, then

$$d^2 = \frac{3wx^2}{Sb}$$

Here, if $x = l$, the value of d is the depth d_1 at the wall, and thus $3w/Sb = d_1P/l^2$. Accordingly,

$$d = d_1\frac{x}{l}$$

gives the depth for any value of x, and it shows that the elevation of the beam should be a triangle (Fig. 29).

The vertical shear near the end of the beam modifies slightly the form near the end. Thus for the first case above, if S' be the working shearing unit-stress there must be a section at the end whose area A is equal at least to P/S'.

Prob. 34. A simple beam of uniform strength is to be designed to carry a heavy load P at the middle. If d_1 be the depth at the middle, show that the depths at distances $0.1l$, $0.2l$, $0.3l$, and $0.4l$ should be $0.45d_1$, $0.63d_1$, $0.77d_1$, and $0.89d_1$.

ART. 35. REVIEW PROBLEMS

Prob. 35 *A*. Locate the neutral axis for a **T** section which is 3×3 inches and $\frac{3}{4}$ inches thick.

Prob. 35 *B*. A timber 4×6 inches in section projects 6 feet out of a wall. What load must be put upon it so that the greatest shearing stress shall be 120 pounds per square inch?

Prob. 35 *C*. A simple wooden beam, 8 inches wide, 9 inches deep, and 14 feet in span, carries two equal loads, one being 2.5 feet at the left and the other 2.5 feet at the right of the middle. Find these loads so that the factor of safety of the beam shall be 8.

Prob. 35 *D*. A simple wooden beam, 3 inches wide, 4 inches deep, and 16 feet span, has a load of 150 pounds at the middle. Compute its factor of safety.

Prob. 35 *E*. A simple beam of structural steel, $\frac{3}{4}$ inches deep and 16 feet span, is subject to a rolling load of 500 pounds. What must be its width in order that the factor of safety may be 6?

Prob. 35 *F*. Compare the strength of a joist, 3×8 inches, when laid with long side vertical with that when it is laid with short side vertical.

Prob. 35 *G*. Compare the strength of a light 9-inch steel **I** beam with that of a wooden beam 8 inches wide and 12 inches deep, the span being the same for both.

Prob. 35 *H*. A cast-iron cantilever beam is to be 4 feet long, 3 inches wide, and to carry a load of 15 000 pounds at the end. Find the proper depths for every foot of length, using 3000 pounds per square inch for the horizontal unit-stress and 4000 for the vertical shearing unit-stress.

Prob. 35 *J*. The wooden girders of a floor are 10×12 inches in cross-section, 25 feet span and 16 feet apart. The floor carries a load of 100 pounds per square foot. Find the maximum unit-stress at the middle of the girders.

Prob. 35 *K*. A steel pin, 8 inches long and 3 inches in diameter, is arranged like a simple beam to carry a load of 10 000 pounds at the middle. Find the maximum flexural unit-stress.

Chapter 5

COLUMNS OR STRUTS

Art. 36. General Principles

A bar under compression whose length is greater than about ten times its thickness is called a column or a strut. For shorter lengths the case is one of direct compression where the rules of Art. 5 apply. For the short specimen failure occurs by the shearing or splintering of the material. For the strut or column, however, failure generally occurs by a sidewise bending; this induces bending stresses, so that the phenomena of stress are more complex than in a beam.

Wooden and cast-iron columns are usually square or round, and are sometimes built hollow. Wrought-iron columns are made by riveting together channels, plates, and angle-irons. It is clear that a square or round section is preferable to a rectangular one, since then the tendency to bend is the same in all directions. For a rectangular section the bending will evidently occur in a plane parallel to the shorter side of the rectangle; thus in investigating such a column the depth d is this shorter side instead of the longer one, as in beams. When a single **I** beam is used as a column it tends to bend in a plane parallel to the flanges, and hence the moment of inertia to be used in this discussion is I', which is given in the last column of the table in Art. 33, the axis for this coinciding with the middle line of the web.

If a short prism whose section area is A is loaded with the weight P, the unit-stress is P/A, and this is uniformly distributed over the area A. For a column, however, this is not the case; while the mean unit-stress is still P/A, the unit-stress on the concave side, if bending occurs, may be very much greater than P/A. The longer the column the greater is this unit-stress on the concave side liable to become, and hence a long column cannot carry so large a load as a short one.

There are three ways of arranging the ends of columns (Fig. 30). Class (a) includes those with 'round ends' or

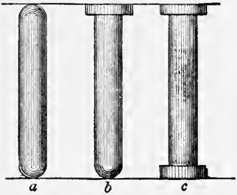

Fig. 30

those having their ends hinged on pins. Class (b) includes those with one end round and the other fixed; the piston-rod of a steam-engine is of this type. Class (c) includes those having fixed ends; these are used in bridge and building constructions. The figure here given is a symbolical representation, and is not intended to imply that the ends of the columns are necessarily enlarged in practice. It is found by experiment that class (c) is stronger than (b), and that (b) is stronger than (a).

Prob. 36. In a certain test wrought-iron tubes 2.37 inches in outer diameter and having a section area of 1.08 square inches were used. A tube 8 feet long failed under 24 800 pounds and a tube $3\frac{1}{2}$ feet long failed under 38 200 pounds. What load would be required to cause failure for a tube only 6 or 8 inches long?

ART. 37. RADIUS OF GYRATION

In the discussion of columns a quantity r, called 'radius of gyration of the cross-section,' is frequently used. It is defined to be that quantity whose square is equal to the least moment of inertia of the cross-section divided by the area of that cross-section, or

$$r^2 = \frac{I'}{A}$$

Thus, for a heavy 10-inch beam it is found by the table in Art. 33 that $r^2 = 9.50/11.8 = 0.805$ inches^2.

The values of r^2 for rectangles of least side d are readily obtained from the moments of inertia given in Art. 24;

For a solid rectangle, $r^2 = \frac{1}{12}d^2$

For a hollow rectangle, $r^2 = \dfrac{bd^3 - b_1d_1^3}{12(bd - b_1d_1)}$

For a solid square, $r^2 = \frac{1}{12}d^2$

For a hollow square, $r^2 = \frac{1}{12}(d^2 + d_1^2)$

The reason why the least moment of inertia is used for columns is that the bending tends to occur in a plane perpendicular to the axis about which the moment of inertia is the least. Thus, a rectangular strut bends in a plane parallel to the least side of its cross-section.

As circular cross-sections are frequently used for columns, the values of the moment of inertia for these will here be stated. Let d be the outer diameter and d_1 the inner diameter. Then

For a solid circle, $I = \frac{1}{64}\pi d^4$, $r^2 = \frac{1}{16}d^2$

For a hollow circle, $I = \frac{1}{64}\pi(d^4 - d_1^4)$, $r^2 = \frac{1}{16}(d^2 + d_1^2)$

Here the values of r^2 are found by dividing the first I by $\frac{1}{4}\pi d^2$ and the second I by $\frac{1}{4}\pi(d^2-d_1^2)$, these being the areas of the cross-sections.

From the last paragraph in Art. 24 it is seen that r^2 is the average of all the values of z^2 for the cross-section. There is, however, no way of finding this average r^2 except by first determining I and then dividing it by the area A.

Prob. 37 *A*. Compute the radius of gyration for a circular ring of 10 inches outer and 8 inches inner diameter.

Prob. 37 *B*. Find the least radius of gyration for a standard angle 4×4 inches and weighing 18.5 pounds per linear foot (see American Civil Engineers' Pocket Book, Sect. 4, Art. 48).

ART. 38. FORMULA FOR COLUMNS

Columns and struts generally fail under the stresses produced by combined compression and bending. The phenomena are so complex that no purely theoretical formula will fully represent all cases. The formula of Rankine is that which has the best rational basis, but this cannot here be fully developed as the laws of deflection have not yet been discussed.

Let P be the load on the vertical column, and let a horizontal plane ab cut it at the middle (Fig. 31). If A is the section area, the average compressive unit-stress P/A may be represented by the line cd. But in consequence of the bending this is increased to aq on the concave side and decreased to bq on the convex side. The triangles pdq and qdp represent the longitudinal bending stresses, as in beams. Let the maximum unit-stress aq be denoted by S. The part ap is equal to cd or to P/A. The part

pq is due to the bending and will be denoted by S_1. Hence the maximum unit-stress aq is given by

$$S=\frac{P}{A}+S_1$$

Now from the formula (4) established for cases of bending in Art. 30, the value of S_1 is Mc/I, where M is the bending

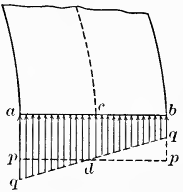

Fig. 31

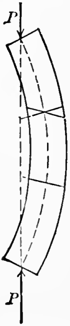

Fig. 32

moment of the external forces. Here the only external force is P, and its lever arm is the lateral deflection of the central line of the column. Let this lateral deflection be called f; then $M=Pf$, and accordingly,

$$S=\frac{P}{A}+\frac{Pcf}{I}$$

where c represents the distance ac in the figure.

Now let I be replaced by Ar^2, where r is the radius of gyration of the cross-section. Then the preceding equation becomes

$$S=\frac{P}{A}\left(1+\frac{cf}{r^2}\right)$$

which shows how the unit-stress S on the concave side increases with the lateral deflection f. By a discussion of the subject of deflection, such as is given in 'Mechanics of Materials,' it is shown that the value of f which is liable to occur increases as the square of the length of the beam, or cf may be made equal to ql^2, where q depends upon the kind of material and the arrangement of the ends (see Art. 56). The last equation may now be written,

$$\frac{P}{A}=\frac{S}{1+q\frac{l^2}{r^2}} \tag{5}$$

which is Rankine's formula for columns.

The values of q to be used in problems and examples in this book are given in the following table. These mean values have been derived by the consideration of numerous experiments on the rupture of columns and

TABLE 9. COLUMN CONSTANTS q

Material	Both Ends Fixed	One Fixed End and One Round End	Both Ends Round
Timber	$\frac{1}{3\,000}$	$\frac{2}{3\,000}$	$\frac{4}{3\,000}$
Cast Iron	$\frac{1}{5\,000}$	$\frac{2}{5\,000}$	$\frac{4}{5\,000}$
Wrought Iron	$\frac{1}{35\,000}$	$\frac{2}{35\,000}$	$\frac{4}{35\,000}$
Steel	$\frac{1}{25\,000}$	$\frac{2}{25\,000}$	$\frac{4}{25\,000}$

struts. It is seen that in all cases q is four times as large for round ends as for fixed ends, which results from the fact that a very long column with round ends has only one-fourth the strength of one with fixed ends.

Prob. 38 *A*. If $P/A=500$ pounds per square inch for a timber column with fixed ends, find from formula (5) the values of S when $l/r=0$, $l/r=50$, and $l/r=100$.

Prob. 38 *B*. When the length l becomes very small, show that formula (5) reduces to formula (1).

ART. 39. SAFE LOADS FOR COLUMNS

To find a safe load for a column of given size and material the working value of S is to be assumed by Art. 7. The value of r is determined by Art. 37, and q by the table in Art. 38. Then, from the formula (5),

$$P=\frac{AS}{1+q\frac{l^2}{r^2}}$$

which gives the safe load P for the column.

For example, let it be required to find the safe load for a timber strut 3×4 inches and 5 feet long, so that the greatest compressive unit-stress S may be 800 pounds per square inch. Here $b=4$ inches, $d=3$ inches, $r^2=\frac{1}{12}d^2=\frac{3}{4}$ inches², $l^2=3600$ inches², $l^2/r^2=4800$, $q=1/3000$, $ql^2/r^2=1.6$. Then

$$P=\frac{12\times800}{1+1.6}=3\,690 \text{ pounds}$$

which is the safe load for the strut. If the length is only about one foot, the safe load will be simply $P=12\times800=9600$ pounds. If the length is 12 feet, P will be found by the formula to be only 940 pounds. The influence of the length on the safe load is hence very great.

Prob. 39 *A*. A hollow cast-iron column to be used in a building is 6×6 inches outside dimensions and 5×5 inches inside dimensions, the length being 18 feet, and the ends fixed. Find its safe load.

Prob. 39 *B*. Find the safe load for the piston-rod of a steam engine, its diameter being 2 inches and its length 36 inches, when the allowable value of S is 5000 pounds per square inch.

ART. 40. INVESTIGATION OF COLUMNS

The investigation of a column under a given load consists in computing the unit-stress S from formula (5) and then comparing this with the ultimate strength and elastic limit of the material, having due regard to whether the stresses are steady, variable, or sudden (Art. 7). The value of S is

$$S=\frac{P}{A}\left(1+q\frac{l^2}{r^2}\right)$$

and the given data will include all the quantities in the second member.

For example, a wrought-iron tube used as a column with fixed ends carries a load of 38 000 pounds. Its outside diameter is 6.36 inches, its inside diameter 6.02 inches, and its length 18 feet. It is required to find the unit-stress S and the factor of safety. Here $P=38\,000$ pounds, $A=\frac{1}{4}\pi(6.36^2-6.02^2)=3.31$ square inches, $q=1/35\,000$, $l=18\times12=216$ inches, $r^2=\frac{1}{16}(6.36^2+6.02^2)=4.79$ inches^2. Then by the formula

$$S=\frac{38\,000}{3.31}\left(1+\frac{216\times216}{35\,000\times4.79}\right)$$

or $S=14\,700$ pounds per square inch. The factor of safety is thus about 4, which is a safe value if the column is used under steady stress, but too small if sudden stresses or shocks are liable to occur. If the length of this column is 36 feet, the unit-stress S will become about 25 000 pounds per square inch, so that its factor of safety will be only 2.2, a value far too low for proper security.

As a second example let a heavy 10-inch steel **I** beam which is 25 feet long be used as a strut in a bridge truss, the ends being hinged on pins. Let the compression on it be 5900 pounds. Here from the table in Art. 33 there is found $A = 11.8$ square inches and $I' = 9.50$ inches^4, whence $r^2 = 0.80$ inches^2; also $q = 4/25\,000$, $l = 300$ inches, $P = 5900$ pounds. Then, from the formula, S is found to be 9500 pounds per square inch, which is about one-third of the elastic limit of the material, and hence a safe value.

Prob. 40 *A*. A pine stick 3×3 inches and 12 feet long is used in a building as a column with fixed ends. Find its factor of safety under a load of 3000 pounds. If its length is only one foot, what is the factor of safety?

Prob. 40 *B*. A rectangular wooden column, 12×12 inches in outside dimensions and 9×9 inches in inside dimensions, is 14 feet long. Compute the unit-stress S when the load P is 10 000 pounds and the ends are fixed.

ART. 41. DESIGN OF COLUMNS

When the length of a column is given and the load to be carried by it, the design consists in selecting the proper material and then finding the dimensions so that the unit-stress S in formula (5) may have the proper value. This is often done by trial, dimensions being assumed and inserted in (5), and if these do not fit, changes are made in them until a satisfactory agreement is found. For example, let it be required to find the size of a square wooden column with fixed ends and 24 feet long to carry a load of 100 000 pounds with a unit-stress S of 800 pounds per square inch. If the column is very short the area A should be $100\,000/800 = 125$ square inches, and the side of the square about 11 inches. The column

24 feet long must be larger than this; assume it 16 inches. Then, from the formula of the last article find the value of S; this being a little larger than 800 shows that 16 inches is too small. Again, trying 17 inches, S is found to be a little smaller than 800. Hence 16½ inches is an approximate solution of the problem.

Equations can be derived, however, for finding the size of solid square and round columns by placing for A and r^2 in formula (5) their values in terms of the side or diameter d. Thus for a solid square column

$$d^4 - \frac{P}{S}d^2 = \frac{P}{S}.12ql^2$$

and for a solid round column

$$d^4 - \frac{4P}{\pi S}d^2 = \frac{4P}{\pi S}.16ql^2$$

As an example, take the data of the last paragraph, where $P=100\,000$, $S=800$, $q=1/3000$, and $l=24\times12$. Inserting these in the first equation it becomes $d^4-125d^2=41\,472$, and solving, there is found $d^2=275.5$, whence $d=16.6$ inches is the side of the square column.

For hollow square and round columns equations can be derived in a similar way for finding the inner side or diameter d_1 when the outer side or diameter d is given. Thus for a hollow square column

$$d_1^4 + \frac{P}{S}d_1^2 = d^4 - \frac{P}{S}(d^2 + 12ql^2)$$

and for a hollow round column

$$d_1^4 + \frac{4P}{\pi S}d_1^2 = d^4 - \frac{4P}{\pi S}(d^2 + 16ql^2)$$

For example, let it be required to find the inner diameter

d_1 for a cast-iron hollow round column with fixed ends, which is 18 feet long and 10 inches outer diameter, and which is to carry a steady load of 240 000 pounds. Here the working value of S is 15 000 pounds per square inch, and $q=1/5000$. Then the last equation gives $d_1^2=60.7$ whence $d_1=7.8$ inches for the inner diameter.

Prob. 41 *A*. Find what steel **I** beam 12 feet long may be used as a column to carry a load of 100 000 pounds, taking the working value of S at 12 000 pounds per square inch.

Prob. 41 *B*. A hollow square column of wood with fixed ends and 14 feet long has outside dimensions of 12×12 inches and carries a load of 9450 pounds. Find the inside dimensions so that S shall be 900 pounds per square inch.

ART. 42. ECCENTRIC LOADS

Thus far it has been supposed that the load is so applied to the end of a column that its line of action coincides

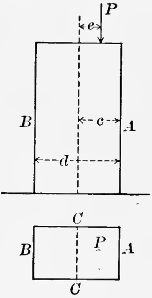

Fig. 33

with the axis of the column. In many instances, however, this is not the case. Let Fig. 33 represent a short block where the load P is applied at a distance e from the axis passing through the center of gravity of the cross-section. The distribution of the internal compressive unit-stresses in every section is then not uniform. The mean unit-stress on the area A is P/A, but this is increased on the side nearest P and decreased on the opposite side by the unit-stress due to the flexure. Let CC be the neutral axis of the cross-section and c the distance to the side, let I be the moment of inertia and r the radius of gyration of the cross-section with respect to the axis CC; let S' be the flexural unit-stress at the side of the column. Then from the flexure formula (4), $S'=Mc/I$. But the bending moment M is Pe, hence $S'=Pec/I=Pec/Ar^2$. Adding this to the mean unit-stress P/A, there results

$$S=\frac{P}{A}\left(1+\frac{ce}{r^2}\right) \qquad (5)$$

which is the compressive unit-stress on that side of the column nearest P. On the other side of the column the unit-stress is found by changing the $+$ sign to $-$.

A small eccentricity e causes the unit-stress S to deviate much from the mean value P/A. For a rectangular section $r^2=\frac{1}{12}d^2$ and $c=\frac{1}{2}d$, so that

For side A of the prism, $S_1=\frac{P}{A}\left(1+6\frac{e}{d}\right)$

For side B of the prism, $S_2=\frac{P}{A}\left(1-6\frac{e}{d}\right)$

When $e=\frac{1}{6}d$, then $S_1=2P/A$ which is double the mean value, while $S_2=0$. When $e=\frac{1}{3}d$, then $S_1=3P/A$ and $S_2=-P/A$; hence the side B is under tension instead of compression. It is thus seen that, in placing loads on a column, eccentricity of application should be avoided.

The above formula (5) applies to a short column or to one in which l/r does not exceed 10. For longer columns it is customary to add the quantity ce/r^2 to the denominator in Rankine's formula, which thus becomes

$$\frac{P}{A}=\frac{S}{1+q\frac{l^2}{r^2}+\frac{ce}{r^2}} \qquad (5)'$$

This formula may be used for finding the safe load on a column having an eccentric load, for investigating an existing column, or for designing a section for a proposed column.

Prob. 42 *A*. Using formula (5)′ find the safe load for the data given in Prob. 39 *A*, taking the eccentricity of the load as ¾ inches.

Prob. 42 *B*. Using formula (5)′ find the factor of safety for the data given in Prob. 40 *A*, taking the eccentricity of the load as ⅜ inches.

ART. 43. THE STRAIGHT-LINE FORMULA

Another formula for columns is that called the straight-line formula, because the relation between P/A and l/r is the same as that between y and x in the equation of a straight line. This formula is

$$\frac{P}{A}=S-C\frac{l}{r}$$

in which S is the unit-stress on the concave side of the column and C is a quantity which varies with the material and the condition of the ends. For columns with fixed ends which are used in buildings under steady loads the following are used in cases of design:

For cast iron, $P/A=10\,000-40(l/r)$
For wrought iron, $P/A=12\,000-60(l/r)$
For structural steel, $P/A=16\,000-70(l/r)$

These formulas only apply when P is in pounds, A in square inches, and when the value of l/r is less than 120. They do not have the same degree of reliability as Rankine's formula, since they are wholly empirical. When specifications require that they should be used, this must be done, but otherwise Rankine's formula (5) should be employed.

For example, find the safe load for a hollow cast-iron column 6×6 inches in outside dimensions and 5×5 inches in inside dimensions, the length being 18 feet and the ends fixed. Here $A=11$ square inches, $r^2=\frac{1}{12}(36+25)=5.08$ whence $r=2.252$ inches, $l/r=95.9$, and then from the formula $P=67\,800$ pounds. In this solution no use is made of the unit-stress S on the concave side of the column. By Rankine's formula, using $S=15\,000$ pounds per square inch, there is found $P=58\,100$ pounds, which is a more reliable value.

Again, let it be required to find the diameter of a solid cast-iron strut 6 feet long to safely carry a steady load of 64 000 pounds. Here for a very short strut, where $l=0$, the area required is $A=64\,000/10\,000=6.4$ square inches, which corresponds to $d=2.85$ and $r=0.71$ inches. Assume then $d=4$ inches, whence $A=12.57$ square inches, $r=1$ inch, and $l/r=72$; inserting these in the formula there is found $P=89\,000$ pounds which, being greater than the given value, shows that 4 inches is too large a diameter. Assume again that $d=3.5$ inches, whence $A=9.62$ square inches, $r=0.875$ and $l/r=84.6$; inserting these in the formula, there is found $P=63\,600$ pounds, which is very close to the given value, so that $d=3.5$ inches is a satisfactory solution of the problem by the straight-line formula.

Prob. 43. A column of structural steel has the dimensions stated in Art. 25 for Fig. 19. What steady load can it carry according to the straight-line formula?

ART. 44. REVIEW PROBLEMS

Prob. 44 *A*. Find the safe steady load for a hollow short cast-iron column which is 12 inches in outside and 9 inches in inside diameter.

Prob. 44 *B*. Given $q = 1/5000$ and $S = 9000$ pounds per square inch for a cast-iron column. Plot a curve for formula (5), taking values of l/r as abscissas and values of P/A as ordinates.

Prob. 44 *C*. Determine the safe load for a fixed-ended timber column 3×4 inches in section and 10 feet long, so that the greatest compressive unit-stress may be 800 pounds per square inch.

Prob. 44 *D*. A cylindrical wrought-iron column with fixed ends is 12 feet long, 6.36 inches in outside diameter, 6.02 inches in inside diameter, and carries a load of 49 000 pounds. Find its factor of safety.

Prob. 44 *E*. Compute the size of a square timber column with fixed ends to carry a load of 100 000 pounds with a factor of safety of 10, its length being 12 feet.

Prob. 44 *F*. A beam 25 feet long carries a uniform load of 3000 pounds per linear foot, and is supported at its ends by two round cast-iron columns 15 feet long. The columns have fixed ends and are 6 inches in outer diameter. Find the inner diameter of the columns so that the unit-stress S is 10 000 pounds per square inch.

Prob. 44 *G*. A 10-inch standard **I** beam weighing 30 pounds per linear foot is used as one of the compression members in a small bridge. The column is fixed-ended and is 20 feet long. Will the column be safe for a load of 50 000 pounds?

Prob. 44 *H*. The piston-rod of an engine is circular in shape and its stroke is three feet. The maximum load upon the piston is 20 000 pounds. Find the proper diameter for the rod, using S as 6000 pounds per square inch.

CHAPTER 6

THE TORSION OF SHAFTS

ART. 45. PHENOMENA OF TORSION

Torsion is that kind of stress which occurs when external forces tend to twist a body round an axis. A shaft which transmits power is twisted by the forces applied to the pulleys, and thus all its cross-sections are brought into stress. This stress is a kind of shearing but the forces acting in different parts of a section are not parallel.

Let one end of a horizontal bar be rigidly fixed, and to the free end let a lever be attached at right angles

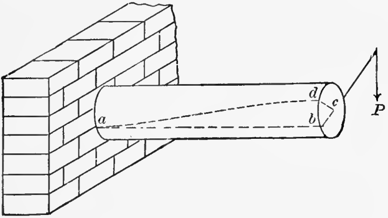

Fig. 34

to its axis (Fig. 34). A weight P hung at the end of this lever will twist the shaft so that a line ab which originally was horizontal will assume a spiral form ad, while the radial line cb will move to the position cd. It has been shown by experiments that, if the material is not stressed beyond its elastic limit, the angles bcd and bad are proportional to the applied weight P, and that on the removal of this weight the lines cd and ad will return to their original positions. If the elastic limit is exceeded this proportionality does not hold, and if the stress is made great enough the bar will be ruptured.

Let p be the lever-arm of P with respect to the axis c. Then experience also shows that the amount of twist is proportional to p. The product Pp is the moment of P with respect to the axis, and it is called the 'twisting moment.' If there are several forces P_1, P_2, etc., acting on the shaft with lever-arms p_1, p_2, etc., the total twisting moment Pp is the algebraic sum of the separate moments P_1p_1, P_2p_2, etc., those being positive which tend to turn in the direction of the hands of a watch, and those negative which turn in the opposite direction.

For example, let the three lever-arms be applied to a bar at the points B, C, and D, whose distances from A

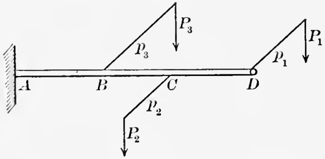

Fig. 35

are 5, 8, and 12 feet. Let the forces in Fig. 35 be $P_1 = 30$ pounds, $P_2 = 60$ pounds, and $P_3 = 100$ pounds, their lever-arms being $p_1 = 2.5$ feet, $p_2 = 2.0$ feet, and $p_3 = 3.5$ feet. Then for all sections between D and C the twisting moment is $+30 \times 2.5 = +75$ pound-feet; for all sections between C and B the twisting moment is $+30 \times 2.5 -$

$60 \times 2.0 = -45$ pound-feet; and for all sections between B and A the twisting moment is $+30 \times 2.5 - 60 \times 2.0 + 100 \times 3.5 = +305$ pound-feet. Thus the tendency to twist between B and C is in the opposite direction to that in the other parts of the bar.

Prob. 45 *A*. If a force of 600 pounds acting at 5 inches from the axis twists the end of a shaft 30 degrees, what force acting at 12 inches from the axis will twist it 60 degrees?

Prob. 45 *B*. It is found by experiment that the angle *bcd* in Fig. 34 is proportional to the length of the bar when P and p are constant. If the angle *bcd* is 6° 35′ for a shaft 9.4 feet long, what will this angle be for a shaft 13.5 feet long?

ART. 46. POLAR MOMENTS OF INERTIA

In the discussion of shafts the moments of inertia of cross-sections are required with respect to a point at the center of the shaft and not with respect to an axis in the same plane, as in beams and columns. The 'polar moment of inertia' of a surface is defined as the sum of the products obtained by multiplying each elementary area by the square of its distance from the center of the surface. Thus if a be any elementary area and x its distance from the center the quantity Σax^2 is the polar moment of inertia, Σ being the symbol of summation, which denotes that all the values $a_1x_1^2$, $a_2x_2^2$, etc., are to added until the entire surface is covered.

In Fig. 36 let a be any elementary area and z its distance from an axis AB passing through the center of gravity of the section; then Σaz^2, or the summation of all the values of az^2, is the moment of inertia with respect to the axis AB (Art. 24). Also, if y is the distance from a to an axis CD which is normal to AB, then Σay^2 is the moment of inertia with respect to the axis CD. But since $z^2+y^2=x^2$, the product Σax^2 is equal to $\Sigma az^2+\Sigma ay^2$; that is, the polar moment of inertia is the sum of the

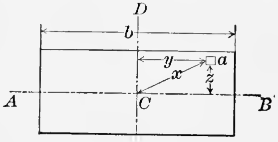

Fig. 36

moments of inertia taken with respect to any two rectangular axes.

The polar moment of inertia is represented by J. By the aid of the above principle its value is readily found from the values of I given in Arts. 24 and 37. Let d be the diameter of a circle; then,

$$\text{For a solid circle,} \qquad J=\tfrac{1}{32}\pi d^4$$

Also, in the case of a hollow section, let d be the outer and d_1 be the inner diameter; then,

$$\text{For a hollow circle,} \qquad J=\tfrac{1}{32}\pi(d^4-d_1^4)$$

The circular sections are most frequently used for shafts, and the discussions of this chapter apply mainly to such shafts. The theory of the torsion of square and rectangular bars is very complicated and cannot be given here.

Prob. 46 A. Show that the polar moment of inertia for the hollow circular section is $\tfrac{1}{8}A(d^2+d_1^2)$, where A is the section area.

Prob. 46 B. Show that the polar moment of inertia for a square section area is $\tfrac{1}{6}d^4$.

ART. 47. FORMULA FOR TORSION

If two cross-sections are taken in a shaft very near together, each section tends to twist with respect to the other, and shearing stresses are found to exist in all parts of the section. These stresses are zero at the center and greatest at the boundary of a circular section, and they act everywhere perpendicular to the lever-arms drawn to them from the center. If the elastic limit is not exceeded it is found that the stresses are proportional to their lever-arms.

Let P be the force acting with the lever-arm p which produces the twisting moment Pp (Fig. 37). This must

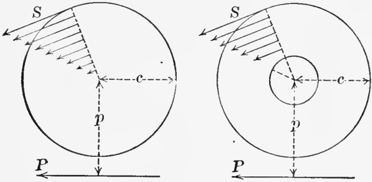

Fig. 37

be equal to the resisting moment of the internal stresses. Let S be the shearing unit-stress at the remotest part of the section whose distance from the center is c. Then the stress at a distance $\frac{1}{2}c$ from the center is $\frac{1}{2}S$, and the stress at a distance x from the center is Sx/c. The total stress on an elementary area a at a distance x from the center is then aSx/c, and the moment of this stress with respect to the center is $(S/c)ax^2$. The resisting moment is the sum of all the values of S/cax^2, or, since S and c are constants, this sum is $(S/c)\Sigma ax^2$. But, as seen in the last article, the quantity Σax^2 is the polar moment of inertia J. Accordingly the resisting moment of the internal shearing stresses is SJ/c, and, equating this to the twisting moment Pp, there results

$$\frac{SJ}{c}=Pp \tag{6}$$

which is the fundamental formula for the torsion of shafts with circular cross-sections.

This formula is analogous with formula (4) for beams, and is used in a similar manner to investigate and design shafts. The unit-stress S is here always a shearing stress, and its working values are to be determined by applying factors of safety to the ultimate shearing strengths given in Art. 6. Shafts which transmit power are subject to variable loads, and often to shocks, and hence their values of S should be taken low. Formula (6) is subject to the same limitation as formula (4), namely, it is only true when the unit-stress S is less than the elastic limit of the material (see Art. 66).

For example, the twisting moment Pp being 20 000 pound-inches, it is required to find the shearing stress produced by it in a circular shaft 4 inches in diameter. Here $c=2$ inches, $J=25.13$ inches^4, and then by (6) the value of S is found to be 1590 pounds per square inch. If the shaft is of wood this is too low a value of S, it being about one-half of the ultimate shearing strength; if it is of wrought iron or steel there is a high factor of safety.

Prob. 47 *A*. A round steel shaft is subject to a twisting moment of 2500 pound-inches. What should be its diameter so that the greatest shear S may be 6000 pounds per square inch?

Prob. 47 *B*. A pulley 36 inches in diameter is placed on a 2-inch wrought-iron shaft, and the effective pull of the belt on the pulley is 500 pounds. What is the factor of safety of the shaft?

ART. 48. SHAFTS TO TRANSMIT POWER

'Work' is the product of a force by the distance through which it is exerted. Thus, if a weight of 10 pounds is lifted vertically a distance of 5 feet there are performed 50 foot-pounds of work. If this weight is moved horizontally, however, the force required depends only on frictional and other resistances; if these require a force of 3 pounds and this be exerted through a distance of 5 feet, then 15 foot-pounds of work are performed.

'Power' is work performed in a given time. The unit of power is the 'horse-power,' which is defined as 33 000 foot-pounds of work performed in one minute. Thus, if 99 000 foot-pounds of work are performed in one minute, the power exerted is 3 horse-powers; if 99 000 foot-pounds of work are performed in two minutes, the power exerted is $1\frac{1}{2}$ horse-powers.

Power from a motor is usually transmitted to a shaft by belts, and the shaft then transmits the power to the places where the work is to be performed. In doing this the shaft is brought under stress. Let H be the power transmitted through a belt to a pulley. Let P be the tangential force in pounds brought by the belt on the circumference of the pulley, and let p be the radius of the pulley in inches. Let n be the number of revolutions made by the shaft and pulley in one minute. In one revolution a force of P pounds acts through $2\pi p$ inches, and the work of $P \times 2\pi p$ pound-inches, or $\frac{1}{6}\pi Pp$ pound-feet, is performed. In one minute the work performed is

$\frac{1}{6} n\pi Pp$ pound-feet. The number of horse-powers exerted is found by dividing this work by 33 000, or

$$H = \frac{n\pi Pp}{198\ 000}$$

The twisting moment Pp may now be replaced by the resisting moment SJ/c, and hence

$$\frac{SJ}{c} = \frac{198\ 000H}{n\pi} \qquad (7)$$

which is the formula for the discussion of round shafts that are used to transmit power.

For such a shaft c is equal to one-half of the outer diameter, whether its section be solid or hollow. The unit-stress S is here always that for shearing, and in selecting its safe value a high factor of safety is to be used, as the shaft is subject to variable stresses. It is noticed that S varies inversely with n, that is, for a given power transmitted the slower the speed the greater is the stress in the shaft.

Prob. 48 *A*. If a round shaft one inch in diameter transmits one horse-power at 100 revolutions per minute, show that the shearing stress produced is about 3200 pounds per square inch.

Prob. 48 *B*. A steel shaft making 300 revolutions per minute is 3 inches in diameter. What horse-power is being transmitted when the shearing unit-stress is 6000 pounds per square inch?

ART. 49. SOLID SHAFTS

For round solid shafts of diameter d, the polar moment of inertia is $\frac{1}{32}\pi d^4$, the value of c is $\frac{1}{2}d$, and formula (7) then reduces to

$$Sd^3 = 321\ 000\frac{H}{n},$$

in which d must be taken in inches and S in pounds per square inch. From this formula S may be found for a given shaft which transmits power, or d may be computed when it is required to design a shaft for that purpose.

For example, let it be required to find the factor of safety of a round solid shaft of wrought iron, 2½ inches in diameter, when transmitting 25 horse-power at 100 revolutions per minute. Here $d=2.5$ inches, $H=25$, $n=100$, and the formula gives

$$S=\frac{321\,000\times 25}{2.5^3\times 100}=5\,140 \text{ pounds per square inch}$$

so that the factor of safety is about 10; this is a high value for a shaft not subject to shocks.

As an example of design, let it be required to find the diameter of a wrought-iron shaft when transmitting 90 horse-power at 250 revolutions per minute. Here the factor of safety will be taken at 8, or the allowable unit-stress S at 7000 pounds per square inch. Then, from the formula,

$$d^3=\frac{321\,000\times 90}{7000\times 250}=16.5$$

and hence the diameter d should be 2 ⅝ inches.

The above formulas do not apply to square shafts; in these the greatest stress is not at the corners but along the middle of the sides. It is shown in Mechanics of Materials (tenth edition) that the formula for a solid square shaft of side d is

$$Sd^3=283\,600\frac{H}{n}$$

For example, let it be required to find how many horse-powers are transmitted by a wooden shaft 12 inches square when it makes 25 revolutions per minute and S is 200 pounds per square inch. Here all quantities in the formula are known except H, and the solution gives $H=305$ horse-powers.

Prob. 49 *A*. Find the horse-power that can be transmitted by a solid round steel shaft of $6\frac{1}{4}$ inches diameter when making 150 revolutions per minute, S being 7500 pounds per square inch.

Prob. 49 *B*. Compare the horse-powers per pound weight of shaft which can be transmitted by a round shaft of diameter d and a square shaft of side d.

Art. 50. Hollow Shafts

Hollow forged steel shafts are now much used for ocean steamers, as their strength is greater than solid shafts of the same area of cross-section. If d is the outside and d_1 the inside diameter, the value of J is $\frac{1}{32}\pi(d_4-d_1^4)$ and c is $\frac{1}{2}d$. These inserted in (7) give

$$S\frac{d^4-d_1^4}{d}=321\,000\frac{H}{n}$$

which is the formula for investigation and discussion.

For example, a nickel steel shaft of 17 inches outside diameter is to transmit 16 000 horse-powers at 50 revolutions per minute; what should be the inside diameter so that the unit-stress S may be 25 000 pounds per square inch? Here everything is given except d_1, and from the equation its value is found to be 11 inches nearly. The area of the cross-section of this shaft will be about 132 square inches, and its weight per linear foot about 449 pounds.

Prob. 50 *A*. If a hollow shaft has the same area of cross-section as a solid one, and if the inside diameter of the hollow shaft is one-half of the outside diameter, prove that the hollow shaft is 44 percent stronger than the solid one.

Prob. 50 *B*. The tail shaft of a marine engine is 15 inches outside and 10 inches inside diameter. What horse-power is being transmitted when the shaft is making 300 revolutions per minute and the unit-stress S is 8000 pounds per square inch?

ART. 51. REVIEW PROBLEMS

Prob. 51 *A*. If a force of 80 pounds, acting at 18 inches from the axis, twists the end of a shaft through 15 degrees, what force will produce the same result when acting at 4 feet from the axis?

Prob. 51 *B*. Compute the polar moment of inertia for a hollow shaft with outside diameter 18 inches and inside diameter 10 inches.

Prob. 51 *C*. Compute the shearing unit-stress for the shaft of the last problem when it is subject to a twisting moment of 2500 pound-inches.

Prob. 51 *D*. Find the horse-power that can be transmitted by a wrought-iron shaft 3 inches in diameter when making 50 revolutions per minute, the value of S being 6000 pounds per square inch.

Prob. 51 *E*. Find the diameter of a solid wrought-iron shaft to transmit 90 horse-power at 250 revolutions per minute, the value of S being 7000 pounds per square inch.

Prob. 51 *F*. Find the ratio of the strength of a hollow shaft to that of a solid one, the section areas being equal, and the outside diameter of the hollow section being three times as great as the inside diameter.

Prob. 51 *G*. The crank of an engine is 9 inches long and the maximum tangential thrust brought upon it by the connecting-rod is 5000 pounds. Find the diameter of a steel shaft to stand the above twisting moment when the allowable stress S is 6000 pounds per square inch.

Prob. 51 *H*. What horse-power will be transmitted by a hollow shaft of 8 inches outside and 5 inches inside diameter when running at 300 revolutions per minute, the value of S being 7000 pounds per square inch? Find the diameter of a solid steel shaft to transmit the same horse-power with the same speed and unit-stress.

CHAPTER 7

ELASTIC DEFORMATIONS

ART. 52. THE MODULUS OF ELASTICITY

It was explained in Chapter 1 that, when a bar is subject to stresses produced by gradually applied forces, the elongations increase proportionately with the stresses, if the elastic limit is not exceeded. This law of elasticity enables the elongations of bars and the deflections of beams to be computed, provided none of the stresses exceeds the elastic limit of the material.

The 'modulus of elasticity' in tension is the ratio of the unit-stress to the unit-elongation. Thus, if a bar one inch long and one square inch in cross-section is under the stress S an elongation s is produced, and

$$E = \frac{S}{s} \tag{8}$$

is the modulus of elasticity. If the bar has a section area A which is acted on by the pull P, then the unit-stress S is P/A; if the bar has the length l, an elongation e is produced and the unit-elongation s is given by e/l.

For compression E is the ratio of the unit-stress to the unit-shortening accompanying that stress, and in general E is the ratio of the unit-stress to the unit-deformation. Since s is an abstract number, E is expressed in the same unit as S, that is, in pounds per square inch or kilos per square centimeter.

Within the elastic limit S increases at the same rate as s, and thus E is a constant; beyond the elastic limit there is no proper modulus of elasticity. For different materials under the same unit-stress S, the value of E increases as s decreases; thus E is a measure of the stiffness of materials. Formula (8) also gives

$$s=\frac{S}{E}$$

which is the change of a unit of length of a bar under a given unit-stress S.

The values of the moduluses of elasticity for tension and compression are practically the same, and their mean values for the different materials are given in the following table. For shear the moduluses of elasticity are about

TABLE 10. MODULUSES OF ELASTICITY

Material	Pounds per Square Inch	Kilos per Sq. Centimeter
Timber	1 500 000	105 000
Cast Iron	15 000 000	1 050 000
Wrought Iron	25 000 000	1 750 000
Steel	30 000 000	2 100 000

one-third of those stated in the table. These values show that, within the elastic limit, steel is the stiffest of the four materials, it being 20 percent stiffer than wrought iron, twice as stiff as cast iron, and twenty times as stiff as timber. In other words, a given stress less than the elastic limit will elongate a timber bar twenty times as much as a steel bar, a cast-iron bar twice as much, and a wrought-iron bar 20 per cent more.

Prob. 52 *A*. A bar one inch square and 2 inches long elongates 0.0004 inches under a tension of 5000 pounds. Compute the modulus of elasticity.

Prob. 52 *B*. When a steel bar 30 feet long was subjected to a tensile unit-stress of 12 000 pounds per square inch, it elongated about 0.143 inches. Compute the modulus of elasticity of steel.

ART. 53. ELONGATION UNDER TENSION

Let a bar whose section area is A and whose length is l be under the tension P, and let e be the elongation produced. The unit-stress S is P/A and the unit-elongation s is e/l. Then the modulus of elasticity E is

$$E=\frac{S}{s}=\frac{Pl}{Ae}$$

and hence, if P/A be less than the elastic limit,

$$e=\frac{Pl}{AE}$$

is the elastic elongation of the bar due to the applied tension P.

For example, let it be required to find the elongation of a wrought-iron bar 30 feet long when stressed up to its elastic limit. Here $P/A=25\,000$ pounds per square inch, $E=25\,000\,000$ pounds per square inch, and $l=360$ inches. Then from the formula, $e=0.36$ inches. This is the elastic elongation; the ultimate elongation will be about 72 inches. In all cases, as seen from the figure in Art. 4, the elastic elongations are very small compared with the ultimate elongations.

Prob. 53 *A*. A steel eye-bar 30 feet long is $1\frac{1}{2}\times 6$ inches in size. How much does it elongate under a pull of 90 000 pounds?

Prob. 53 *B*. What is the ultimate elongation of a steel bar 1 inch square and 10 feet long?

Art. 54. Shortening under Compression

If a bar of cross-section A and length l be under the compression P it shortens the amount e. For a short bar where no lateral deflection can occur the unit-stress S is uniform over the cross-section, and the shortening follows the same law as does the elongation in tension, and hence

$$e=\frac{Pl}{AE}$$

Here, as before, the unit-stress P/A must not exceed the elastic limit of the material.

For example, let a cast-iron bar one inch in diameter and 5 inches long be under a compression of 30 000 pounds. Here $P=30\ 000$ pounds, $A=0.785$ square inches, $l=5$ inches, and $E=15\ 000\ 000$ pounds per square inch. Then from the formula, $e=0.0128$ inches; but this result is of no value, because the unit-stress P/A is nearly double the elastic limit of cast iron. If, however, P is given as 3000 pounds, then the formula properly applies, and e is found to be 0.0013 inches.

Prob. 54 *A*. A wrought-iron bar 18 inches long weighs 24 pounds. How much will it shorten under a compression of 7250 pounds?

Prob. 54 *B*. The piston-rod of a steam engine is 4 inches in diameter and 30 inches long, while the piston is 24 inches in diameter. What is the change in length of the piston-rod when the steam pressure is 200 pounds per square inch?

Art. 55. Deflection of Cantilever Beams

The best method of deriving formulas for the deflections of beams is by the help of the calculus. These methods are given in higher works on the subject; see for instance Merriman's Mechanics of Materials. The formulas will be stated here without proof, and be accompanied by illustrations showing their value and importance.

When a load P is at the end of a cantilever beam whose length is l (Fig. 38), a deflection of that end results, which

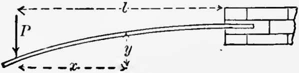

Fig. 38

will be designated by f. This deflection will evidently be the greater the greater the load and the longer the length of the beam. The formula for it is

$$f=\frac{Pl^3}{3EI}$$

in which E is the modulus of elasticity of the material (Art. 52) and I is the moment of inertia of the cross-section (Art. 24). The ordinate y at any distance x from the free end is given by $y=\frac{1}{2}f(3n-n^2)$ in which n represents x/l.

Fig. 39

When a uniform load is on the beam let this be called W (Fig. 39). Then the deflection is

$$f=\frac{Wl^3}{8EI}$$

It is thus seen that the deflection varies as the cube of the length of the beam, so that if the length is doubled the deflection will be eight times as great. It is also seen that a uniform load produces only three-eighths as much deflection as a single load at the end. The ordinate y at any distance x from the free end is given by $y = \frac{1}{3}f\ (4n - n^4)$ where n represents x/l.

For example, let it be required to compute the deflection of a cast-iron cantilever 2×2 inches and 6 feet long, due to a load of 100 pounds at the end. Here $P = 100$ pounds, $l = 72$ inches, $E = 15\,000\,000$ pounds per square inch, and $I = \frac{1}{12}2^4 = 1\frac{1}{3}$ inches^4. Then from the formula, $f = 0.622$ inches, which is the deflection at the end.

For a rectangular cross-section of breadth b and depth d the value of I is $\frac{1}{12}bd^3$. Thus the deflections of rectangular beams vary inversely as b and d^3. As stiffness is the reverse of deflection, it is seen that the stiffness of a beam is directly as its breadth, directly as the cube of its depth, and inversely as the cube of its length. The laws of stiffness are hence quite different from those of strength.

Prob. 55 *A*. A steel **I** beam 8 inches deep and 6 feet long is used as a cantilever to carry a uniform load of 240 000 pounds. What will be its deflection?

Prob. 55 *B*. A cantilever beam 6.3 feet long and loaded at the end has a deflection of 0.48 inches at that end. What is the deflection of a point half-way between that end and the wall?

ART. 56. DEFLECTION OF SIMPLE BEAMS

When a simple beam of span l has a load P at the middle (Fig. 40), each reaction is $\frac{1}{2}P$. If this beam is imagined to be inverted it will be seen to be equivalent to two cantilevers of length $\frac{1}{2}l$, each having the load $\frac{1}{2}P$ at the end. Hence in the first formula for the deflection of a cantilever, given in Art. 55, if l is replaced by $\frac{1}{2}l$ and P by $\frac{1}{2}P$, it becomes

$$f=\frac{Pl^3}{48EI}$$

which gives the deflection of a simple beam due to a load at the middle.

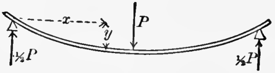

Fig. 40

When a simple beam is loaded with w per linear unit the total load wl is represented by W. The deflection at the middle due to this load is

$$f=\frac{5Wl^3}{384EI}$$

which is only five-eighths of the deflection caused by the same load at the middle.

The formulas of this and the preceding article are only valid when the greatest horizontal stress S produced by the load is less than the elastic limit. These formulas can be expressed in terms of S by substituting the values of P and W from the formula (4) of Art. 28. Thus for the simple beam with load at the middle $\frac{1}{4}Pl=SI/c$, and for the uniform load $\frac{1}{8}Wl=SI/c$. Hence

For the single load P, $\quad f=\frac{Sl^2}{12Ec}$

For the uniform load W, $\quad f=\frac{5Sl^2}{48Ec}$

which show that the deflections of beams under the same unit-stresses increase directly as the squares of their lengths.

Prob. 56 *A*. In order to find the modulus of elasticity of oak, a bar 2×2 inches, and 6 feet long, was loaded at the middle with 50 pounds, and then with 100 pounds, the corresponding deflections being found to be 0.16 and 0.31 inches. Compute the modulus of elasticity E.

Prob. 56 *B*. Compute the deflection of a steel **I** beam 6 inches deep and 16 feet long when it is loaded so that the flexural unit-stress at the middle equals the elastic limit of the material.

ART. 57. RESTRAINED BEAMS

A beam is said to be restrained at one end when that end is horizontally fixed in a wall and the other end rests on a support (Fig. 41). In this case the reaction of the support is less than for a simple beam. For a uniform load

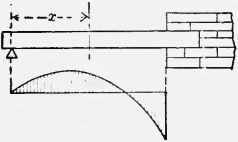

Fig. 41

of w per linear unit over the span l it is proved in Mechanics of Materials that the reaction at the support is $\frac{3}{8}wl$, provided the elastic limit is not exceeded. The bending moment at any section distant x from the support then is $\frac{3}{8}wlx - \frac{1}{2}wx^2$, and this shows that when $x = \frac{3}{4}l$ there is no bending moment; when $x = \frac{3}{8}l$ the greatest positive bending moment is $\frac{9}{128}wl^2$, and when $x = l$ the greatest negative bending moment is $\frac{1}{8}wl^2$; the distribution of bending moments being as shown in the figure. Also, the maximum deflection is

$$f = \frac{wl^4}{185EI} = \frac{Wl^3}{185EI}$$

which occurs when x has the value $0.4215l$.

For a beam fixed at both ends and uniformly loaded (Fig. 42) there is a negative bending moment $\frac{1}{12}wl^2$ at

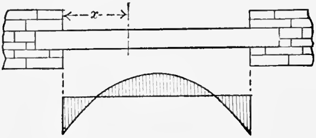

Fig. 42

each wall and a positive bending moment $\frac{1}{24}wl^2$ at the middle; also the deflection at the middle is

$$f=\frac{Wl^3}{384EI}$$

in which W is the total uniform load wl.

It is seen that in these restrained beams the lower side is partly in tension and partly in compression, since a positive bending moment indicates the former and a negative one the latter (Art. 30). For a simple beam the greatest bending moment is $\frac{1}{8}Wl$, for a beam fixed at both ends the greatest bending moment is $\frac{1}{12}Wl$; hence if both be the same size the restrained beam will carry the greater load, or if both carry the same load the restrained beam may be of smaller size than the simple one. Thus if beams can be fixed horizontally at their ends the construction may be more economical.

Prob. 57 *A*. When a beam is fixed at one end and supported at the other, the reaction of the supported end due to a load P at the middle is $\frac{5}{16}P$. Show that there is a positive bending moment $\frac{5}{32}Pl$ under the load, and a negative bending moment $\frac{3}{16}Pl$ at the wall. Also draw the diagram of bending moments.

Prob. 57 *B*. Show that the deflection of a simple beam is five times as great as that of a beam fixed at both ends, both beams being uniformly loaded.

Art. 58. Twist in Shafts

When a shaft of length l transmits H horse-powers at a speed of n revolutions per minute, one end of the shaft is twisted with respect to the other through an angle of D degrees, this being the angle bcd in Fig. 34. When the elastic limit is not exceeded, this angle is, for a round shaft,

$$D = 3\,610\,000\frac{Hl}{nFJ}$$

in which J is the polar moment of inertia of the section (Art. 46) and F is the modulus of elasticity for shear (Art. 52). Here l must be taken in inches, J in inches^4, and F in pounds per square inch.

For example, let a steel shaft 125 feet long, 17 inches outside diameter, and 11 inches inside diameter transmit 16 000 horse-powers at 50 revolutions per minute. Here $H = 16\,000$ horse-powers, $l = 1500$ inches, $n = 50$, $F = 10\,000\,000$ pounds per square inch, $J = 6765$ inches^4. Then from the formula $D = 25.3$ degrees, which is the angle through which a point on one end is twisted relative to the corresponding point on the other end. If this shaft revolves with a speed of only 25 revolutions per minute while doing the same work, its angle of twist will be twice as great and the stresses in it also twice as great as before. The formula also shows that the angle of twist varies directly as the length of the shaft.

The above formula may also be written

$$D = 56.5\frac{Ppl}{FJ}$$

from which D may be computed when the twisting moment Pp is given.

Prob. 58 *A*. A solid steel shaft 125 feet long and 16 inches in diameter transmits 8000 horse-powers at a speed of 25 revolutions per minute. Compute the angle of twist.

Prob. 58 *B*. A solid steel shaft 8 feet long and 2 inches in diameter is subject to the twisting moment brought by a belt on a pulley of 30 inches diameter, the effective pull of the belt being 200 pounds. Compute the angle of twist.

ART. 59. REVIEW PROBLEMS

Prob. 59 *A*. Show, for timber and wrought-iron bars stressed to their elastic limits, that the change of length of the former is double that of the latter.

Prob. 59 *B*. Compute the tensile force required to stretch a bar of structural steel, $1\frac{3}{4} \times 9\frac{3}{4}$ inches in section area and 23 feet $3\frac{1}{4}$ inches long, so that its length may become 23 feet $3\frac{7}{8}$ inches.

Prob. 59 *C*. Show that the modulus of elasticity is the unit-stress which would stretch a bar to double its original length, provided this could be done without impairing the elasticity of the material.

Prob. 59 *D*. What unit-stress will shorten a block of cast iron 0.4 percent of its length?

Prob. 59 *E*. A cast-iron bar, 2 inches wide, 4 inches deep and 6 feet long, was balanced upon a support and a weight of 4000 pounds hung at each end, when the deflection of each end was found to be 0.401 inches. Compute the modulus of elasticity.

Prob. 59 *F*. Compute the elastic deflection of a light steel 10-inch beam of 30 feet span, due to its own weight, when resting on supports at the ends.

Prob. 59 *G*. Compute for the beam of the last problem the deflection when the beam is fixed at both ends.

Prob. 59 *H*. A wrought-iron shaft, 5 feet long and 2 inches in diameter, is twisted through an angle of 7 degrees when transmitting 4 horse-powers at 120 revolutions per minute. Compute the shearing modulus of elasticity.

CHAPTER 8

MISCELLANEOUS APPLICATIONS

ART. 60. WATER AND STEAM PIPES

The pressure of water or steam in a pipe is exerted in every direction and tends to tear the pipe apart longitudinally. This external force is resisted by the internal tensile stresses which act in the walls of the pipe normal to the radii. If p is the pressure per square inch exerted by the water or steam, D the diameter, and l the length of the pipe, the total pressure P exerted on any diametral plane is $p \cdot lD$. If t is the thickness of the pipe and S the tensile unit-stress, the total resisting stress will be $S \cdot 2lt$, if the thickness is not large compared with the diameter. Hence

$$plD = 2Slt \quad \text{or} \quad pD = 2St$$

is the formula for discussing water or steam pipes.

Water pipes are made of cast iron or wrought iron, the former being more common, while for steam the latter is preferable. A water pipe liable to the shock of water-ram should have a high factor of safety, and in steam pipes the factors should also be high. The formula above deduced shows that the thickness of a pipe must increase with its diameter, as also with the internal pressure to which it is to be subjected.

For example, let it be required to find the proper thickness for a wrought-iron steam pipe of 18 inches diameter to resist a steam-pressure of 250 pounds per square inch. With a factor of safety of 10 the working unit-stress is 5500 pounds per square inch. Then, from the formula,

$$t=\frac{pD}{2S}=\frac{250\times18}{2\times5500}=0.45 \text{ inches}$$

so that a thickness of ½ inch would probably be selected.

The transverse resistance of a pipe is double the longitudinal, or $pD=4St$ applies to the case of transverse resistance. This equation also applies to hollow spheres under internal pressure.

Prob. 60 *A*. Find the factor of safety of a cast-iron water pipe 12 inches in diameter and ⅝ inches thick under a pressure of 130 pounds per square inch.

Prob. 60 *B*. What internal pressure per square inch will burst a cast-iron water pipe of 24 inches diameter and ¾ inches thickness?

ART. 61. RIVETED LAP JOINTS

When two plates are joined together by rivets and the plates then subjected to tension, there is brought a shear upon the rivets which tends to cut them off. A riveted lap joint is one where one plate simply laps over the other.

For a lap joint with a single row of rivets (Fig. 43), let P be the tensile stress which is transmitted from one plate to another by means of one rivet; let a be the pitch of the rivets, or the distance from the center of one rivet to the center of the next; let d be the diameter of a rivet and t the thickness of the plate. The plate tends to tear apart on the section area $(a-d)t$, while the rivet tends to shear off on the section area $\frac{1}{4}\pi d^2$. Accordingly, if S_t

and S_s are the unit-stresses for tension and shear, then

$$P = t(a-d)S_t \quad \text{and} \quad P = \tfrac{1}{4}\pi d^2 S_s$$

are formulas for the discussion of this case.

For example, a steel water pipe 30 inches in diameter has a longitudinal rivet seam with one row of rivets,

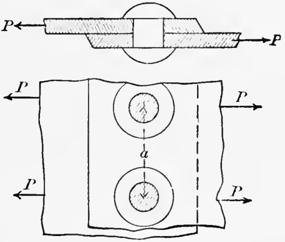

Fig. 43

the diameter of the rivets being ¾ inches, their pitch 2 inches, and the thickness of the plate ½ inches. If the interior water-pressure is 130 pounds per square inch, what are the unit-stresses in tension and shear? Here the total pressure on a diametral plane of length equal to the pitch is $130 \times 2 \times 30 = 7800$ pounds. Then for tension on the plate

$$S_t = \frac{3900}{\frac{1}{2}(2-\frac{3}{4})} = 6240 \text{ pounds per square inch}$$

and for shear on the rivet

$$S_s = \frac{3900}{0.785 \times \frac{9}{16}} = 8860 \text{ pounds per square inch}$$

Here the factors of safety are about 10 for the plates and about 6 for the rivets, so that the joint may be regarded as a satisfactory one.

The thickness required for a boiler or pipe having a longitudinal lap joint like that shown in Fig. 43 is found from the equation

$$2t(a-d)S_t = paD$$

where D is the diameter and p the inner unit-pressure. For example, let $D=30$, $a=2$, and $d=\frac{3}{4}$ inches, while $p=130$ pounds per square inch. Then, for $S_t=6000$ pounds per square inch, the thickness is $t=130\times2\times30/2\times1\frac{1}{4}\times60\,000=0.52$ inches.

When two rows of rivets are used, these are staggered, so that the rivets in one row come opposite the middle

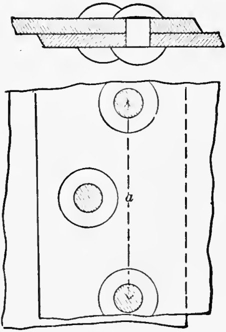

Fig. 44

of the pitches in the other row (Fig. 44). Here the tension P is distributed over two rivet sections instead of one, and

$$P=t(a-d)S_t \qquad P=\tfrac{1}{2}\pi d^2S_s$$

are the formulas for investigation. Thus for the data of the above example, if there are two rows of rivets,

S_t is 12 500 pounds per square inch, and $S_s = 8900$ pounds per square inch, which are good working values for mild steel if the pipe is not subjected to the shock of water-ram.

The working unit-stress for shear should be about three-fourths of that for tension, or $S_s = \frac{3}{4}S_t$. Equating the above values of P under this condition gives a joint where the security of the plates in tension is the same as that of the rivets in shear; thus

$$a = d + 0.59\frac{d^2}{t} \qquad\qquad a = d + 1.18\frac{d^2}{t}$$

the first being for single lap riveting and the second for double lap riveting. These are approximate rules for finding the pitch when the thickness of plates and diameter of rivets are given.

Prob. 61 *A*. A steel water pipe 30 inches in diameter has rivets $\frac{3}{4}$ inches in diameter and plates $\frac{1}{2}$ inches thick. If double riveting is used, what should be the pitch of the rivets?

Prob. 61 *B*. Compute the factor of safety of a steel boiler 5 feet in diameter and $\frac{5}{16}$ inches thick when it is subject to a steam pressure of 300 pounds per square inch, there being longitudinal lap joints having rivets of $\frac{3}{4}$ inches diameter with $2\frac{1}{8}$ inches pitch.

ART. 62. RIVETED BUTT JOINTS

When two plates butt together cover plates are used on one or both sides; if the covers are on both sides each

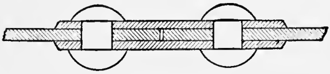

Fig. 45

is one-half the thickness of the main plate (Fig. 45). The shear on each rivet is here divided between the upper and the lower cross-section, this being called a case of double shear. Thus if P is the tension which is transmitted through one rivet, d the diameter of a rivet, a the pitch, and t the thickness of the main plate,

$$P=t(a-d)S_t \qquad\qquad P=2\,.\,\tfrac{1}{4}\pi d^2 S_s$$

which are the same as for two rows of lap riveting.

The 'efficiency' of a riveted joint is the ratio of the strength of the joint to that of the solid plate. If the joint is designed so as to be of equal strength in tension and shear this efficiency is

$$\frac{t(a-d)S_t}{taS_t}=1-\frac{d}{a}$$

Thus if the rivets are $\frac{3}{4}$ inches in diameter and the pitch is 2 inches the efficiency is $1-\frac{3}{8}=0.625$, that is, the riveted joint has only 62.5 percent of the strength of the solid plate. Single lap riveting has usually an efficiency of about 60 percent, while double lap riveting and common butt riveting have from 70 to 75 percent. By using three or more rows of rivets efficiencies of over 80 percent can be secured.

When a joint is not of equal strength in tension and shear there are two efficiencies, one being the ratio of the tensile strength of the joint to that of the solid plate, and the other the ratio of the shearing strength to that of the solid plate. The least of these is the true efficiency.

Prob. 62 *A*. A butt joint with two cover-plates has the main plate $\frac{1}{2}$ inches thick, the rivets $\frac{7}{8}$ inches in diameter, and the pitch of the rivets $2\frac{7}{8}$ inches. Compute the efficiency.

Prob. 62 *B*. Show that the efficiency of a butt joint, based on the shear of rivets, is double that of a lap joint.

ART. 63. STRESSES DUE TO TEMPERATURE

A bar which is free to move elongates when the temperature rises and shortens when it falls. But if the bar is under stress, or is fixed so that it cannot elongate or contract, the change in temperature produces a certain unit-stress. This unit-stress is that which would cause a change in length equal to that produced in the free bar by the change in temperature.

The coefficient of linear expansion is the elongation of a bar of length unity under a rise of temperature of one degree. For the Fahrenheit degree the average values of the coefficients of expansion are as follows:

For brick and stone,	$C=0.0000050$
For cast iron,	$C=0.0000062$
For wrought iron,	$C=0.0000067$
For steel,	$C=0.0000065$

Thus a free bar of cast iron 1000 inches long will elongate 0.0062 inches for a rise of one degree, and 0.62 inches for a rise of 100 degrees.

The elongation of a bar of length unity for a change of t degrees is hence $s=Ct$. But (Art. 52) the unit-stress due to the unit-elongation s is $S=sE$, where E is the modulus of elasticity. Therefore

$$S=CtE$$

is the unit-stress produced by a change of t degrees on a bar which is fixed. If the temperature rises S is compression, if the temperature falls S is tension.

For example, consider a wrought-iron rod which is used to tie together two walls of a building and which is screwed up to a stress of 10 000 pounds per square inch. If the temperature falls 50 degrees there is produced a tensile unit-stress,

$$S = 0.0000067 \times 50 \times 25\,000\,000 = 8400$$

and hence the total stress in the rod is 18 400 pounds per square inch. If the temperature rises 50 degrees the stress in the bar is reduced to 1600 pounds per square inch. In all cases the unit-stresses due to temperature are independent of the length and section area of the bar.

Prob. 63 *A*. A cast-iron bar 6 feet long and 4×4 inches in section is confined between two immovable walls. What pressure is brought on the walls by a rise of 40 degrees in temperature?

Prob. 63 *B*. When steel railroad rails are improperly laid with their ends close together at a temperature of 40°, what compressive unit-stress occurs when the temperature rises to 80°?

ART. 64. SHRINKAGE OF HOOPS

A hoop or tire is frequently turned with the inner diameter slightly less than that of the cylinder or wheel upon which it is to be placed. The hoop is then expanded by heat and placed upon the cylinder, and upon cooling it is held firmly in position by the radial stress produced. This radial stress causes tension in the hoop.

Let D be the diameter of the cylinder upon which the hoop is to be shrunk and d the interior diameter to which the hoop is turned. If the thickness of the hoop is small, D will be unchanged by the shrinkage and d will be increased to D. The unit-elongation of the hoop is then $s = (D-d)/d$, and hence the unit-stress produced is

$$S = sE = \frac{D-d}{d}E$$

where E is the modulus of elasticity of the material.

A common rule in steel-hoop shrinkage is to make $D-d$ equal to $\frac{1}{1500}d$; that is, the cylinder is turned so that its diameter is $\frac{1}{1500}$th greater than the inner diameter of the hoop. Accordingly the tangential unit-stress which occurs in the hoop after shrinkage is $30\,000\,000/1500 = 20\,000$ pounds per square inch.

When the hoop is thick the above rule is not correct, for a part of the stress produced by the shrinkage causes the diameter of the cylinder to be decreased. The rules for this case are complex ones, and cannot be developed in an elementary text-book; they will be found in Chapter XVI of Mechanics of Materials.

Prob. 64. Upon a cylinder 18 inches in diameter a thin wrought-iron hoop is to be placed. The hoop is turned to an inner diameter of 17.98 inches and then shrunk on. Compute the tensile unit-stress in the hoop.

ART. 65. SHAFT COUPLINGS

Let a shaft be in two parts which are connected by a flange coupling. (Fig. 46). A shows the end view

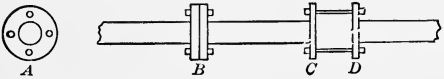

Fig. 46

and B the side view of the coupling. The flanges of the coupling are connected by bolts which are brought into shearing stress in transmitting the torsion from one part of the shaft to the other.

Let the shaft be solid and of diameter D, let there be n bolts of diameter d, and let h be the distance from the center of the shaft to the center of a bolt. If D and d are assumed, as also the distance h, then, as shown in Art. 124 of Mechanics of Materials, the formula

$$n=\frac{(d+2h)D^3}{(d^2+8h^2)d^2}$$

gives the number of bolts which is required in order that the strength of the bolts may be the same as the strength of the shaft. Since d is usually much smaller than h it may be neglected within the parentheses; then the above formula becomes $n=D^3/4hd^2$ which is a simpler expression generally used in practice.

For example, let $D=8$ inches, $d=1$ inch, and $h=12$ inches, then the second formula gives $n=10.7$, so that 11 bolts should be used. If $D=8$ inches, $d=1\frac{1}{4}$ inches, and $h=12$ inches, the formula gives $n=6.8$ so that seven bolts should be used.

The case shown at CD in the above figure is one that should never occur in practice, because here the bolts are subject to a bending stress as well as to the shearing stresses due to the torsion. It is clear that this bending stress will increase with the length between the flanges, and that the bolts should be greater in diameter than for the case of pure shearing.

Prob. 65. A solid steel shaft 16 inches in diameter transmits 16 000 horse-powers at 50 revolutions per minute. Design a flange coupling for this shaft.

ART. 66. RUPTURE OF BEAMS AND SHAFTS

The formulas (4) and (6) deduced in Arts. 28 and 47 for the discussion of beams and shafts are only valid when the unit-stress S is less than the elastic limit of the material. Formula (4) can, however, be used for the rupture of beams, if S be taken as a certain quantity intermediate between the ultimate compressive and tensile strengths of the material. This quantity, which is called the 'modulus of rupture,' has been determined by breaking beams under transverse loads and then computing S from the formula. In like manner formula (6) can be used for the rupture of round shafts if the modulus of rupture, as found by experiment, is used instead of the ultimate shearing strength.

The following table gives average values of the modulus of rupture as determined by testing beams and columns

TABLE 11. MODULUSES OF RUPTURE

Material	For Beams		For Round Shafts	
	Pounds per Square Inch	Kilos per Square Centimeter	Pounds per Square Inch	Kilos per Square Centimeter
Timber	9 000	630	2 000	140
Stone	2 000	140		
Cast Iron	35 000	2 450	25 000	1 750
Wrought Iron			40 000	2 800
Steel, hard	80 000	5 600	70 000	4 900

to destruction. Wrought iron and mild steel have no proper modulus of rupture when used as beams, since they continually bend and do not break until the ultimate load is reached. By the use of these values formulas (4) and (6) may be applied to the solution of numerous problems relating to the rupture of beams and shafts.

For example, let it be required to find the size of a square cast-iron simple beam of 6 feet span that will rupture under its own weight. Let x be the side of the square. The weight of a cast-iron bar x square inches in section area and one yard long is $9.4x$ pounds; thus the weight of the beam is $18.8x$ pounds. The bending moment is $\frac{1}{8}Wl$ or $169.2x$ pound-inches. The value of c is $\frac{1}{2}x$ and that of I is $\frac{1}{12}x^4$. Then, by formula (4),

$$\frac{35\,000 \times x^4}{6x} = 169.2x$$

and the solution of this gives $x = 0.17$ inches.

It is to be noted, when formulas (4) and (6) are used for cases of rupture, that they are entirely empirical and have no rational basis.

Prob. 66 *A*. What load P applied at the middle of a cast-iron bar, 18 inches long and 1 inch square, will cause its rupture?

Prob. 66 *B*. What force P, acting at 24 inches from the axis of a steel shaft 1.4 inches in diameter, will cause failure by torsion?

ART. 67. REVIEW PROBLEMS

Prob. 67 *A*. What internal pressure per square inch will burst a cast-iron sphere 24 inches in diameter and $\frac{3}{4}$ inches thick?

Prob. 67 *B*. A wrought-iron boiler, 63 inches in diameter and $\frac{11}{16}$ inches thick, carries a steam pressure of 180 pounds per square inch. Find the factor of safety of the metal when the efficiency of the longitudinal riveted joint is 87 percent.

Prob. 67 *C*. Draw a figure for a double-riveted butt joint and deduce formulas for the same. Find the efficiencies for plate and rivets when plate thickness is $\frac{1}{2}$ inch, pitch of rivets is $3\frac{1}{4}$ inches, and diameter of rivets is $\frac{15}{16}$ inches.

Prob. 67 *D*. Find the radial unit-pressure between the rim and tire of a locomotive driving-wheel when the shrinkage is $\frac{1}{1500}$, the diameter of the tire being 60 inches and its thickness $\frac{3}{4}$ inches.

Prob. 67 *E*. A solid shaft 6 inches in diameter is coupled by bolts $1\frac{1}{4}$ inches in diameter with their centers 5 inches from the axis. How many bolts are necessary?

CHAPTER 9

REINFORCED CONCRETE

ART. 68. CONCRETE AND STEEL

Columns and beams are made by ramming concrete into wooden forms or boxes which surround their sides, these being removed after the concrete has hardened. Steel rods are often placed in the forms and the concrete rammed around them, and this combination is often called 'reinforced concrete.' The object of inserting the steel is to make a safer and stronger construction than is possible with concrete alone, and to do this at a lower cost than is possible when only steel is used. Since 1895 there has been a great development in this kind of construction, steel rods being now extensively used for the reinforcement not only of columns and beams but also in walls, sewers, and arches.

The kind of concrete generally used for this purpose is made of Portland cement, sand, and broken stone, an excellent grade being of the proportions 1 cement, 2 sand, 4 stone by measure, and a lower grade being of the proportions 1 cement, 3 sand, 6 stone; these two grades are frequently called '1 : 2 : 4 concrete' and '1 : 3 : 6 concrete'; the former is stronger and better than the latter, but its cost is higher. The average weight of concrete is about 150 pounds per cubic foot, or about the same as that of sandstone.

The strength of concrete increases with its age, reaching nearly the highest value by the end of the first year. Its ultimate compressive strength is much higher than the ultimate tensile strength, and the following are average values of these in pounds per square inch for concrete one year old:

	Compressive Strength	Tensile Strength
For 1 : 2 : 4 concrete	3500	300
For 1 : 3 : 6 concrete	2500	200

These figures, like all those given in this chapter, refer only to concrete made with Portland cement. The ultimate shearing strength of concrete is from 800 to 1000 pounds per square inch.

It is seen from these values of the ultimate strengths that concrete is not well adapted to resist tension, and under tensile stresses it is almost impracticable to use concrete, unless it be strengthened by reinforcing rods of metal.

Concrete suffers a greater change of shape under a given applied unit-stress than steel, or the stiffness of steel is much greater than that of concrete. Mean values of the modulus of elasticity for the two grades of concrete are in pounds per square inch:

For 1 : 2 : 4 concrete	$E=3\,000\,000$
For 1 : 3 : 6 concrete	$E=2\,000\,000$

For steel the mean value of E is 30 000 000 pounds per square inch (Art. 52), and hence it is seen that concrete suffers an elastic deformation ten or fifteen times as great as that of steel when subjected to the same stress per square inch.

The elastic limit of concrete is not well defined, but as a rough average it may be taken in compression at about one-sixth and in tension at about one-fifth of the ultimate strength. The allowable working unit-stress for concrete under compression is generally taken as about one-seventh of the ultimate strength, that is, about 500 pounds per square inch for 1 : 2 : 4 concrete and about 350 pounds per square inch for 1 : 3 : 6 concrete.

The rods or bars used for reinforcement are generally of structural steel having an ultimate strength of about 60 000 and an elastic limit of about 35 000 pounds per square inch. These may be the round, square, and rectangular shapes such as are everywhere found in the market, and various special patented forms are also widely used. The Ransome rods are square bars which have been twisted so that the corner lines are spirals; the Thacher and Johnson bars are rolled so as to have protuberances or swellings at intervals along the length; these forms are claimed to possess special advantages in preventing the rods from slipping in the concrete. The Kahn bar has projecting fins which are intended to prevent beams from shearing, and there are also several other forms which are advertised and used. Some of these are claimed to be advantageous in having an elastic limit much higher than that of structural steel.

Prob. 68*A*. Consult the advertising columns of the engineering journals and obtain pictures of several kinds of reinforcing bars for beams.

Prob. 68*B*. Compute the amount of shortening of a 1 : 2 : 4 concrete column, 12 inches square and 9 feet long, under a load of 60 000 pounds.

ART. 69. COMPOUND BARS

A bar which is under tension is usually called a rod, while one under compression is called a strut. The rods and struts considered in the previous chapters may be called simple ones, as each has been only of one material; a compound rod or strut, however, is one formed of two or more kinds of materials. For example, a steel rope with a hempen center, or a concrete column with a steel bar within it parallel to the axis, are instances of compound tension and compression members.

When a rod of two materials is subject to a longitudinal tension P, part of this is resisted by one material and part by the other. Let A_1 be the section area of one material and A_2 that of the other, and let S_1 be the unit-stress over the area A_1, and S_2 the unit-stress over the area A_2. Then A_1S_1 and A_2S_2 are the total stresses on the two sections, and hence, since the resisting stresses must equal the applied tension,

$$P = A_1S_1 + A_2S_2 \tag{9}$$

is a necessary equation of equilibrium. When P, A_1, and A_2 are given, the values of S_1 and S_2 cannot be determined from this equation, and hence a second condition between them must be derived.

This second condition is established from the fact that the elongation of the two parts of the bar is the same. Let E_1 be the modulus of elasticity of the first material and E_2 that of the second. Then, if the elastic limit of neither material is exceeded, the elongation of a unit of length of the first material is S_1/E_1 (Art. 52) and that of the second material is S_2/E_2. Since these must be equal,

$$\frac{S_1}{E_1}=\frac{S_2}{E_2} \qquad \text{or} \qquad \frac{S_1}{S_2}=\frac{E_1}{E_2}$$

which shows that the unit-stresses in the two materials are proportional to their moduluses of elasticity. If E_2 is 10 times as great as E_1, then S_2 must be 10 times as great as S_1.

By help of the above formulas the values of S_1 and S_2 due to a load P on a compound rod of two materials may be easily found. The above reasoning applies also to compression if the length of the strut is not more than about 10 times its thickness, so that lateral flexure does not change the uniform distribution of the stresses. For example, consider a wooden bar having wrought-iron straps fastened along two opposite sides; here E_1 for the timber is 1 500 000, while E_2 for the wrought iron is 25 000 000 pounds per square inch, so that E_1/E_2 is 0.06, and hence the unit-stress S_1 in the timber is equal to $0.06S_2$, so that, if S_2 is 5000 pounds per square inch, S_1 will be 300 pounds per square inch. The formula $S_1/S_2=E_1/E_2$ cannot, however, be used when S_1 exceeds the elastic limit of the timber or when S_2 exceeds the elastic limit of the wrought iron.

The determination of the values of S_1 and S_2 for this compound rod is now easily made when A_1, A_2, and P are given. Let A_1 for the timber be 36 square inches and A_2 for the wrought iron be 4 square inches. Let the load P be 60 000 pounds. Since $S_1=0.06S_2$, formula (9) gives

$$60\,000=36\times0.06S_2+4S_2$$

from which $S_2=9740$ pounds per square inch for the wrought iron, and hence $S_1 = 580$ pounds per square inch for the timber. It is here seen that the total stress which comes on the wrought iron is 4×9740 or about 39 000 pounds, while that on the timber is about 21 000 pounds. The metal hence carries the greater part of the load, and it does this largely by virtue of its greater stiffness. In general, the higher the value of E for a material in a compound bar, the greater is the part of the load carried by it.

Prob. 69. A short timber strut, 8×8 inches in section, has four steel plates fastened to its sides, each being $6 \times \frac{1}{2}$ inches in size, and it carries a load of 180 000 pounds. Compute the compressive unit-stresses in the two materials.

ART. 70. REINFORCED COLUMNS

Concrete columns are generally square or round. Fig. 47 shows a square form having four steel rods im-

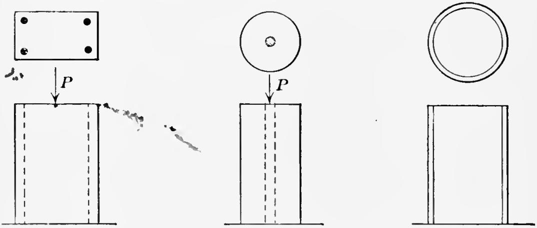

Fig. 47 Fig. 48 Fig. 49

bedded in it near the corners. Fig. 48 shows a round column having a single rod through its axis. Fig. 49 shows a round column formed by a hollow metal cylinder which is filled with concrete. The length of these columns will be considered short compared with the thickness, so that no tendency to lateral flexure exists.

The investigation of a reinforced column consists in determining the compressive unit-stresses due to the given load P, and comparing them with the allowable unit-stresses. Let A_1 be the section area of the concrete and A_2 that of the metal, and let E_1 and E_2 be the corresponding moduluses of elasticity. Let n represent the known ratio E_2/E_1; then, as shown in the last article, the value of S_2/S_1 must also be n, provided the elastic limit of neither material is exceeded. Inserting for S_2 its value nS_1 in the first formula of Art. 69 and solving, there results

$$S_1 = \frac{P}{A_1 + nA_2} \qquad (9)'$$

from which the unit-stress in the concrete may be computed, and then the unit-stress in the metal is found from $S_2 = nS_1$.

As an example of investigation, take a reinforced column of 1 : 3 : 6 concrete which is 14 inches square and has four steel rods, each 3/8 inches in diameter, parallel with its length near the corners, as in Fig. 47, while the load P is 71 000 pounds. Here the section area of the four rods is $A_2 = 0.442$ square inches and that of the concrete is $A_1 = 196 - 0.44 = 195.56$ square inches. Then, since n is 15 (Art. 68), formula (9)′ gives $S_1 = 71\,000/202.2 = 351$ pounds per square inch for the concrete, and hence $S_2 = 5270$ pounds per square inch for the steel. These are safe working stresses, that for the steel being very low.

When a reinforced column is to be designed, the load P and the unit-stress S_1 for the concrete are known or assumed, while n is to be taken as 10 or 15, depending on the kind of concrete (Art. 68). Then the above investigation shows that the section areas A_1 and A_2 of the two materials are to be determined so as to satisfy the equation

$$A_1+nA_2=\frac{P}{S_1}$$

In order to do this, one section area is usually assumed and the other can then be computed. Evidently many different sets of values of A_1 and A_2 can be found, and the one to be used will generally be determined by convenience and local conditions. For example, let the column in Fig. 48 be of 1 : 2 : 4 concrete for which S_1 is to be 500 pounds per square inch, and it be required that the diameter shall be 6 inches, while the load P is 20 000 pounds. Let it be required to find the diameter d of the single steel rod. Here $A_2=\frac{1}{4}\pi d^2$, and $A_1=\frac{1}{4}\pi(36-d^2)$, and then

$$\tfrac{1}{4}\pi(36-d^2+10d^2)=20\,000/500$$

from which d is found to be 1.28 inches, so that a rod $1\frac{5}{16}$ inches in diameter should be used.

For a column like Fig. 49, cast iron is often used for the outer casing, and since E is 15 000 000 pounds per square inch for cast iron, the value to be used for n will be about $7\frac{1}{2}$ for 1 : 3 : 6 concrete, which is the grade that would be most likely to be used for a column of this kind. The safe load P which such a column can carry may then be found by formula (9), in which $S_1=350$ and $S_2=7\frac{1}{2}\times 350=2620$ pounds per square inch. The investigation or design of such a column may be made by the methods above explained.

Prob. 70. A column like Fig. 49 is to be made as described in the last paragraph. The diameter of the concrete is 2 feet 6 inches, and the load to be carried is 500 000 pounds. Compute the thickness of the cast-iron casing.

ART. 71. BEAMS WITH SYMMETRIC REINFORCEMENT

Although beams of timber are sometimes reinforced by fastening metal plates upon the sides, the most common example of reinforcement is that of a concrete beam in which the steel is imbedded. Concrete beams are usually rectangular and Fig. 50 represents a section of one with no reinforcing rods, this being called a plain concrete beam; such a beam is not well adapted for carrying heavy loads on account of the low tensile strength of the concrete (Art. 68). Fig. 51 gives a section of a

Fig. 50

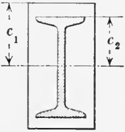

Fig. 51

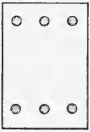

Fig. 52

concrete beam in which a steel I beam is imbedded; although this form is sometimes used, the steel part is generally made sufficiently large to carry the given loads, the office of the concrete being to protect the steel from fire. Fig. 52 is a form in which steel rods are imbedded near both the top and the bottom of the beam, and symmetrically arranged with respect to the neutral axis of the concrete section.

The discussion of a plain concrete beam is made by the help of formula (4) and the methods explained in Arts. 28–31. For example, let the beam be 8 inches wide, 10 inches deep, 6 feet in span and let it be required to compute the total uniform load which it can carry when the concrete is stressed to 100 pounds per square inch on the tensile side. The bending moment for the total load W is $\frac{1}{8}W\times72$ in inch-pounds (Art. 21). The section modulus I/c is, from Arts. 23 and 24, found to be $\frac{1}{6}\times8\times10^2=$ 133.3 inches³. Accordingly, the flexure formula (4) gives

$$M=S(I/c), \qquad \text{or} \qquad 9W=100\times133.3$$

from which $W=1480$ pounds is the total uniform load which the beam can carry. Since this beam weighs about 500 pounds, an additional uniform load of about 980 pounds will stress the concrete on the tensile side to 100 pounds per square inch.

For cases where the imbedded steel is placed symmetrically with respect to the concrete section as in Figs. 51 and 52, the flexure formula (4) may be modified so as to take the two different materials into account. Let S_1 be the unit-stress in the concrete on the remotest fiber at the distance c_1 from the neutral axis, S_2 the unit-stress in the steel at the distance c_2, and I_1 and I_2 the moments of inertia of the concrete and steel sections. M being the maximum bending moment carried by the beam, this must be equal to the sum of the resisting moments of the two sections (Art. 26). Therefore, for compound beams,

$$M=\frac{S_1I_1}{c_1}+\frac{S_2I_2}{c_2}$$

Now, for the beam as for the column, the change of length of any line drawn on the sides parallel with the length must be the same for both concrete and steel. At the distance unity from the neutral axis the unit-stress in the concrete is S_1/c_1 and the change in a unit of length is then S_1/c_1E_1, if E_1 is the modulus of elasticity of the concrete. Similarly for the steel the change in a unit of length at the distance unity from the neutral axis is S_2/c_2E_2. Hence

$$\frac{S_1}{c_1E_1}=\frac{S_2}{c_2E_2}, \qquad \text{whence} \qquad S_2=\frac{c_2}{c_1}nS_1$$

in which n represents the ratio E_2/E_1, the value of which is 10 or 15 (Art. 68). Also inserting this value of S_2 in the above expression for M, there results

$$M=\frac{S_1}{c_1}(I_1+nI_2) \qquad \text{or} \qquad S_1=\frac{Mc_1}{I_1+nI_2} \tag{10}$$

which are formulas for the design and investigation of reinforced concrete beams.

As an example, let it be required to find the total uniform load W which a beam like Fig. 52 can carry, S_1 being 100 pounds per square inch and n being 10. Let the width be 8 inches, the depth 10 inches, the span 6 feet, and each of the six steel rods be ½ inch in diameter and have its center 3 inches from the neutral axis. The value of M in pound-inches is $\frac{1}{8}W\times72$, or $9W$. The section area of each rod is 0.196 square inches, so that the moment of inertia I_2 is $6\times0.196\times3^2=10.6$ inches^4. The moment of inertia of the 8×10-inch rectangle is $\frac{1}{12}\times8\times10^3=666.7$ inches^4, and hence $I_1=666.7-10.6=656$ inches^4. Then, since S_1/c_1 is 100/5, the first formula in (10) gives $9W=20(656+106)$, from which $W=1720$ pounds, which is 13 percent more than a plain concrete beam of the same dimensions can carry.

For the above case the unit-stress S_2 in the steel rod is quite small. If the rods are at the upper and lower surfaces of the beam, the above formula shows that S_2 is nS_1, but since the rods are only 3 inches from the neutral axis $S_2 = 3/5 \times 10 \times 100 = 600$ pounds per square inch, while the steel may safely bear twenty-five times as great a unit-stress. The full safe strength of the steel rods cannot indeed be developed unless the tensile strength of the concrete is entirely overcome. A symmetric arrangement of the rods, like that in Fig. 52, is not economical and is rarely used for beams. The above discussion indicates, however, the principles involved, and the formulas will be modified in the next article so as to apply to the usual cases of unsymmetric reinforcement.

Prob. 71. A beam like Fig. 52 is 10 inches wide, 12 inches deep, 14 feet in span, and has eight steel rods each $3/8$ inches in diameter, four being 1 inch below the top and four being 1 inch above the bottom. Compute the unit-stresses for the 1 : 2 : 4 concrete and for the steel when the beam is loaded with 200 pounds per linear foot besides its own weight.

ART. 72. UNSYMMETRIC REINFORCEMENT

For the reasons stated in the last article, reinforced concrete beams are generally built with imbedded rods only on the tensile side, as shown in Fig. 53. Let S_1 be the compressive unit-stress on the upper surface of the beam, T_1 the tensile unit-stress on the lower surface, and T_2 the tensile unit-stress in the steel. Let n, as before, represent the ratio E_2/E_1 found by dividing the modulus of elasticity of the steel by that of the concrete. Let b be the breadth and d the depth of the rectangular section, A the section area of the steel rods, the centers of which are at the distance h below the middle of the beam. Let M be the bending moment of the loads for the given section, and let it be required to find the values of S_1, T_1, and T_2 due to M. The following formulas are demonstrated in Mechanics of Materials (eleventh edition).

Case I—When the loads on the beam are light, so that the unit-stress on the tensile side does not exceed about one-half of the tensile strength of the concrete, the distribution of stresses in the vertical section is that shown in

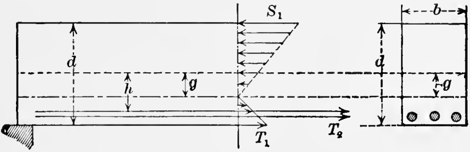

Fig. 53

Fig. 53. The neutral surface in this case lies at a certain distance g below the middle of the beam, and the value of g may be computed from

$$g = \frac{h}{1 + \dfrac{bd}{nA}}$$

Then the unit-stresses for the concrete are

$$S_1 = \frac{d + 2g}{d^2 + 12gh} \cdot \frac{6M}{bd} \qquad T_1 = \frac{d - 2g}{d^2 + 12gh} \cdot \frac{6M}{bd}$$

while that for the steel is

$$T_2 = \frac{2n(h - g)}{d^2 + 12gh} \cdot \frac{6M}{bd}$$

and from these formulas the beam may be investigated, provided the load does not produce a value of T_1 greater than about 100 pounds per square inch.

For example, let $b = 10$ and $d = 12$ inches, $h = 4$ inches, $A = 2.4$ square inches, and $n = 15$ for 1:3:6 concrete

(Art. 68). Then $g = 0.923$ inches is the distance of the neutral surface below the middle of the beam. Now let the span of the beam be 8 feet and the uniform load upon it be 2400 pounds; the bending moment at the middle is $M = \frac{1}{8}Wl = 28\,800$ inch-pounds. The compressive unit-stress on the upper surface of the concrete is then found to be $S_1 = 106$ pounds per square inch, and the tensile unit-stress on the lower surface is $T_1 = 78$ pounds per square inch, while the tensile unit-stress in the steel is $T_2 = 1\,110$ pounds per square inch. Both S_1 and T_2 are quite small, but T is about one-third of the ultimate tensile strength.

Case II—When a heavy load is applied to a reinforced concrete beam the tensile resistance of the concrete is first overcome, vertical cracks extending upward from the lower side, and thus a greater stress is thrown upon the steel. The theoretic analysis for this case is a difficult one unless it is assumed that the concrete below the neutral surface

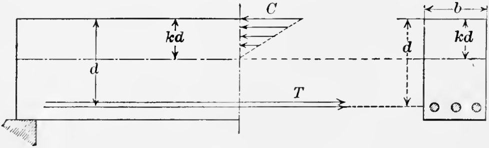

Fig. 54

exerts no material resistance, and this assumption is the one usually made. Fig. 54 shows the distribution of stresses, the neutral surface being usually above the middle of the beam. Let b be the breadth of the beam, d the distance of the centers of the reinforcing rods below the top, and kd the distance of the neutral surface below the top, k being a number less than unity. Also let the ratio A/bd be called p. First, compute k from

$$k = -np + \sqrt{2np + (np)^2}$$

and then the unit-stresses are

$$C = \frac{2M}{k(1 - \frac{1}{3}k)bd^2} \qquad T = \frac{M}{(1 - \frac{1}{3}k)Ad} \tag{11}$$

the first being for compression on the concrete and the second for tension on the steel. As an example, let $b = 12$ and $d = 4.5$ inches, $A = 0.6$ square inches, $l = 60$ inches, the uniform load be 1800 pounds, and $n = 10$. Then $k = 0.373$ which locates the depth of the neutral surface below the top of the beam. From the given load and span, M is found to be 13,500 inch-pounds, and then from formula (11) the compressive stress on the upper surface of the concrete is $C = 340$ pounds per square inch, and the tensile stress in the steel rods is $T = 5710$ pounds per square inch. Both of these are lower than maximum allowable values.

The formulas of Case II are those usually required for the investigation of a reinforced concrete beam unless it is so lightly loaded that the unit-stress T_1, found by the formula of Case I, is less than about 100 pounds per square inch.

Prob. 72. Let a reinforced beam of 1:2:4 concrete be 24 inches wide, 5 inches deep, and 6 feet span, with 1.2 square inches of steel at 2 inches below the middle; compute the unit-stresses S_1, T_1, T_2, due to the light uniform load of 1125 pounds. Also compute the unit-stresses C and T due to the heavy uniform load of 4500 pounds.

ART. 73. DESIGN OF BEAMS

When a reinforced concrete beam is to be designed the allowable unit-stresses for concrete and steel are given or assumed, as also the span and width, and it is then required to compute the depth of the beam and the section area of the steel. When this is done according to the formulas applicable to Fig. 53, the steel is stressed but slightly; if T_1 is taken as 100 pounds per square inch for the concrete, the stress T_2 for the steel will be less than 1500 pounds per square inch. It is found impossible to economically design a beam on this theory and have unit-stresses prevail that are satisfactory, this being due to the low tensile strength of the concrete. Nothing remains to be done, therefore, but to allow the concrete to crack on the tensile side and thus to stress the steel higher in tension than is otherwise possible. The formulas (11) given above for the distribution of stresses shown in Fig. 54 may be transformed so as to be applicable to cases of design.

The quantities usually given in designing are the allowable compressive unit-stress C on the concrete, the allowable tensile unit-stress T on the steel, the ratio $E_2/E_1=n$, the bending moment M, and the breadth b of the rectangular beam. It is required to find the depth d of the beam and the section area A of the reinforcing steel. Let the ratio T/C be designated by t, then for the case of Fig. 54,

$$d=\frac{n+t}{\sqrt{2n^2+3nt}}\sqrt{\frac{6M}{bC}}, \qquad A=\frac{n}{2t(n+t)}bd \qquad (11)'$$

are the formulas for computing d and A. The unit-stress C should be taken as high as allowable by the specifications; T should not be higher than the highest allowable value, but it may be taken lower than this value if economy in cost is promoted.

For example, a rectangular beam of 1 : 3 : 6 concrete is to have a span of 14 feet, a breadth of 20 inches, and is to carry a uniform load of 300 pounds per square foot, including its own weight. It is required to find the depth of the beam and the section area of the reinforcing rods so that the unit-stresses C and T shall be 350 and 14 000 pounds per square inch respectively. Here $n=15$, $t=T/C=40$, and $b=20$ inches. The total load on the beam is $300\times14\times20/12=7000$ pounds, and the bending moment is $M=\frac{1}{8}\times7000\times14\times12=147\ 000$ inch-pounds. Inserting these values in the first of the above formulas, there is found $d=13.0$ inches. Then $bd=20\times13=260$ square inches, and from the second formula $A=0.90$ square inches.

For the above case the section area of the steel is about one-third of one percent of the section area bd of the concrete. Higher percentages of steel are frequently used, from 0.60 to 1.25 percent being common values, but these are probably not economical except for high-class concrete and low-priced steel. The depth d computed by the above formula is that from the top of the beam to the centers of the reinforcing rods. The actual depth of the beam is, however, greater than d by 1 or $1\frac{1}{2}$ inches, the extra thickness of concrete serving to protect the steel from corrosion and from the effects of fire.

Prob. 73 *A*. For the above numerical data except that for T, compute the depth d and the section A, taking the value of T as 12 000 pounds per square inch. Also taking the value of T as 9000 pounds per square inch. If steel costs 50 times as much as concrete, per cubic unit, which of the three beams is the cheapest?

Prob. 73 *B*. Design two reinforced concrete beams 12 inches wide for a span of 12 feet 6 inches and a uniform load of 1500 pounds per linear foot, one being 1 : 2 : 4 concrete and the other 1 : 3 : 6 concrete.

ART. 74. GENERAL DISCUSSIONS

The formulas and methods above presented for reinforced concrete beams are valid when the unit-stresses in the concrete are proportional to their distances from the neutral surface, and this is the case only when the changes of length are proportional to the unit-stresses. Concrete is a material in which this proportionality does not exist for compressive stresses higher than 500 pounds per square inch, so that it cannot be expected that the formulas will apply to cases of rupture. Formulas for rupture have been deduced by Hatt and others in which the unit-stresses are taken as varying according to a parabolic law with their distances from the neutral surface, and such formulas are sometimes used for designing beams by applying proper factors of safety.

The phenomena of failure of reinforced concrete beams have been fully ascertained by the experiments made by Talbot in 1905. The beams were tested by applying two concentrated loads at the third points of the span, and the deflections at the middle were measured for several increments of loading, as also horizontal changes of length near the top and bottom. Under light loads the tensile resistance of the concrete was plainly apparent; when the tensile unit-stress in the concrete approached the ultimate strength, the neutral surface rose and the stress in the steel increased. A little later fine vertical cracks appeared on the tensile side, while the tensile stresses in the steel and the compressive stresses in the concrete increased faster than the increments of the load. The last stage was a rapid increase in the deformations, and rupture generally occurred by the crushing of the concrete on the upper surface, the steel being then stressed beyond its elastic limit.

In some cases, especially in short beams, failure is observed to occur by an oblique shearing near the supports, and this may be prevented by inclined or vertical reinforcing rods. In designing a beam, however, it is usually not necessary to make computations for this shearing stress because the dimensions obtained by the flexure formulas are for a beam which is to carry only about one-fifth or one-sixth of the load that causes rupture.

The above discussions give only an introduction to the subject of reinforced concrete. Many questions in regard to beams remain yet to be settled, but it is believed that the methods above given are fundamental and not liable to essential change. It seems likely that the use of combined concrete and steel, not only for beams and columns, but for arches, walls, piers, dams, and aqueducts, is to increase rapidly and become a most important feature in engineering construction.

Prob. 74. A plain concrete beam, 12 inches wide, 13½ inches deep, and 14 feet span, broke under two single loads, each of 1300 pounds and placed at the third points of the span. Compute the modulus of rupture by the common flexure formula. If this beam has one percent of steel placed 1½ inches above the lower surface, compute the values of C and T.

ART. 75. REVIEW PROBLEMS

Prob. 75 *A*. Find the safe load for a short column of 1 : 2 : 4 concrete which is 2×3 feet in section area.

Prob. 75 *B*. A short timber column, 6×8 inches in section, has two steel plates, each ⅜×8 inches, bolted to the 8-inch sides. Compute the load P when the timber is stressed to 750 pounds per square inch.

Prob. 75 *C*. A pier of 1 : 3 : 6 concrete, 6 feet in diameter, is surrounded by a cast-iron casing 1.15 inches thick. What part of the total load is carried by the concrete?

Prob. 75 *D*. For a 1 : 2 : 4 concrete beam like Fig. 52 let $b=10$ and $d=12$ inches, and the six steel rods be 5 inches from the neutral surface. Find the size of the rods when the beam is 13 feet in span and carries a total uniform load of 5000 pounds.

Prob. 75 *E*. For the dimensions in Prob. 72, what uniform load will probably cause the concrete to begin to fail in tension?

Prob. 75 *F*. Design a beam of 1 : 2 : 4 concrete with three reinforcing rods, the width to be 8 inches and the span 6 feet, so that it will safely carry a total uniform load of 2500 pounds.

Prob. 75 *G*. Consult a paper by Sewall and the accompanying discussions in Vol. 56 of Transactions of American Society of Civil Engineers, and ascertain different opinions as to what should be the comparative cost of concrete and steel in order to produce the most economical reinforced beam.

Prob. 75 *H*. If a force of 235 pounds, acting at the end of a lever 17.5 inches long, twists the end of a shaft of 6.5 feet length through an angle of 14° 45′, what force acting at the end of a lever 9.75 inches long will cause a twist of 28° 31′ when the length of the shaft is 10.62 feet?

Prob. 75 *K*. What center load will rupture a wooden beam 1 inch square and 12 inches in span?

Prob. 75 *L*. A wrought-iron simple beam is 2×2 inches in section. What must be its length so that it will rupture under its own weight?

CHAPTER 10

COMBINED STRESSES

ART. 76. COMPRESSION AND FLEXURE

When a simple beam is subject to compressive forces at its ends the compressive stress due to this force increases the deflection, and hence also the compressive stress on the upper side of the beam. Let M be the bending moment at the middle of the beam, found as in Art. 21, c the distance of the neutral surface of the beam from the compression side (Art. 23), and I the moment of inertia of the section area (Art. 24); then the unit-stress due to the simple flexure is Mc/I (Art. 30). Let A be the section area of the beam and P the compressive load acting on the ends; then P/A is the unit-stress due to the direct compression (Art. 1). Roughly and approximately, then, the total compressive unit-stress on the upper side of the beam is $Mc/I+P/A$.

A more extended discussion will show (as in Mechanics of Materials, tenth edition, p. 256) that the following formula obtains for the case where the simple beam is loaded at the middle:

$$S=\frac{P}{A}+\frac{Mc}{I}\Big/\left(1-\frac{Pl^2}{12EI}\right)$$

where E is the modulus of elasticity of the material (Art. 52). For a uniform load on the beam the number 12 is to be replaced by 9.6.

For example, let a simple wooden beam 16 feet long, 10 inches wide, and 9 inches deep be under an axial compression of 40 000 pounds, while at the same time it carries a total uniform load of 2000 pounds. Here $M = \frac{1}{8}Wl = 48\,000$ pound-inches, $c = 4.5$ inches, $I = \frac{1}{12}bd^3 = 607.5$ inches^4, $l = 96$ inches, $P = 40\,000$ pounds, $E = 1\,500\,000$ pounds per square inch, and $A = 90$ square inches. Inserting these values in the formula, there is found $S = 444 + 428 = 872$ pounds per square inch as the flexural unit-stress at the middle of the beam due to the combined compression and flexure.

Prob. 76. A simple wooden beam, 10 inches wide and 4 feet long, carries a uniform load of 500 pounds per linear foot and is subjected to a longitudinal compression of 40 000 pounds. Find the depth of the beam so that the maximum compressive unit-stress may be 800 pounds per square inch.

ART. 77. TENSION AND FLEXURE

Let a beam be subject to flexure by transverse loads and also to a tension in the direction of its length. The effect of the tension is to decrease the deflection of the beam and hence also the flexural unit-stress. The formula of the last article can be applied to this case by a change in one of the signs. Thus for the case of a single load at the middle of the simple beam,

$$S = \frac{P}{A} + \frac{Mc}{I} \Big/ \left(1 + \frac{Pl^2}{12EI}\right)$$

and for a uniform load on the beam the number 12 is to be changed to 9.6.

For example, take a steel eye bar, 18 feet long, 1 inch wide, and 8 inches deep, which is under a longitudinal tension of 80 000 pounds. The weight of this beam is

490 pounds and $M = 12\,230$ pound-inches. Also $c = 4$ inches, $I = 42.67$ inches^4, $E = 30\,000\,000$ pounds per square inch, $P = 80\,000$ pounds, and $A = 8$ square inches. Then the formula gives $S = 10\,000 + 950 = 10\,950$ pounds per square inch due to the combined flexure and tension. By using the approximate method noted in the first paragraph of the last article, there is found $S = 10\,000 + 1240 = 11\,240$ pounds per square inch.

Prob. 77. A light steel **I** beam, 4 inches deep, and 10 feet long, has a load of 650 pounds at the middle and is under the longitudinal tension of 20 000 pounds. Compute the flexural unit-stress due to the combined loads.

Art. 78. Shear and Axial Stress

Let a bar having the section area A be subjected to the longitudinal tension P, and at the same time to a shear V at right angles to its length. The axial unit-stress on the section area is P/A which will be designated by S, and the shearing unit-stress is V/A which will be denoted by S_s. These two direct stresses combine to produce tensile, compressive, and shearing unit-stresses in other directions. The following formulas, demonstrated in Mechanics of Materials, Art. 105 (tenth edition), give the greatest of these internal unit stresses:

Max. shearing unit-stress, $S_s' = \sqrt{S_s^2 + \frac{1}{4}S^2}$

Max. tensile unit-stress, $S_1 = S_s' + \frac{1}{2}S$

Max. compression unit-stress, $S_2 = S_s' - \frac{1}{2}S$

When P is compression, then the second of these formulas gives the maximum compressive unit-stress and the third gives the maximum tensile unit-stress.

For example, take a bolt one inch in diameter which is subject to a longitudinal tension of 5000 pounds and at the same time to a cross shear of 3000 pounds. Here $S=6366$ and $S_s=3820$ pounds per square inch. Then the first formula gives $S_s'=4970$ pounds per square inch for the maximum shearing unit-stress, $S_1=8155$ pounds per square inch for the resultant maximum tensile unit-stress, and $S_2=1790$ pounds per square inch for the resultant maximum compressive unit-stress which act in the bolt in directions different from the applied unit-stresses.

Prob. 78. A short bolt ¾ inch in diameter is subjected to a longitudinal compression of 2000 pounds and at the same time to a cross shear of 3000 pounds. Find the maximum compressive, tensile, and shearing unit-stresses which exist in the bolt.

ART. 79. COMPRESSION AND TORSION

Compression and torsion are combined when a loaded vertical shaft rests in a step at its foot. Here there is a compressive unit-stress S due to the weight of the shaft and its loads, a torsional unit-stress S_s due to the transmitted horse-power (Art. 48). These combine to produce the resultant unit-stresses S_s', S_1, and S_2 which may be computed by the formulas of the last article.

To find the diameter of a vertical solid shaft for this case the following formula may be used:

$$\pi d^3 S_s' = \sqrt{(16\,Pp)^2+(2Wd)^2}$$

in which d = diameter, S_s' = the working shearing unit-stress, Pp = the twisting moment computed by Art. 48, W = the weight of the shaft and its loads. Assumed values of d are to be inserted in the formula until one is found that satisfies it. This formula also serves to compute S_s' directly when d is given.

A vertical shaft is sometimes so arranged that its weight and loads are supported near the top on a series of circular disks, sometimes called a thrust bearing. The shaft is thus brought into tension instead of compression, and this is a better arrangement because there is then no liability to flexure. The above formula applies also to this case.

Prob. 79. A vertical shaft, weighing with its loads 6000 pounds, is subjected to a twisting moment by a force of 300 pounds acting at a distance of 48 inches from its center. If the shaft is structural steel, 4 feet long and 2 inches in diameter, find its factor of safety.

Art. 80. Flexure and Torsion

This case occurs when a horizontal shaft for the transmission of power is loaded transversely with weights. Let S be the flexural unit-stress computed from (4) and S_s the torsional shearing unit-stress found from Art. 49. Then by the last article the resultant maximum unit-stresses are

$$\text{Max. shearing unit-stress,} \quad S_s' = \sqrt{S_s^2 + \tfrac{1}{4} S^2}$$

$$\text{Max. flexural unit-stress,} \quad S' = \frac{1}{2} S + \sqrt{S_s^2 + \tfrac{1}{4} S^2}$$

These can be used to find the greatest internal unit-stress, and then the factors of safety of the material are known.

It is thus seen that the actual maximum unit-stresses in a shaft due to combined flexure and torsion are much higher than those derived from the formulas for flexure or torsion alone. In determining the diameter of a shaft it is hence necessary to take S_s' as the allowable shearing unit-stress and S' as the allowable tensile or compressive unit-stress. For a round shaft of diameter d, the value of

S_s is Ppc/J where Pp is the twisting moment (Art. 45) or $S_s = 16Pp/\pi d^3$, while the value of S is Mc/I (Art. 30) or $S = 32M/\pi d^3$. Inserting these in the above formulas they reduce to

$$\pi d^3 S_s' = 16\sqrt{(Pp)^2 + M^2}$$
$$\pi d^3 S' = 16(M + \sqrt{(Pp)^2 + M^2})$$

the first being for the resultant shearing and the second for the resultant tension or compression. Since the allowable S_s' is smaller than the allowable S', it often happens that the first formula will give a larger diameter than the second.

As an example, find the diameter of a horizontal steel shaft to transmit 90 horse-powers at 250 revolutions per minute, when the distance between bearings is 8 feet and there is a load of 480 pounds at the middle, the allowable unit-stress being $S_s' = 5000$ and $S' = 7000$ pounds per square inch. Here the twisting moment is $Pp = 63\,030 \times 90/250 = 22\,690$ pound-inches, and the bending moment is $M = 480 \times 96/8 = 5760$ pound-inches. Then using the first formula the diameter d is found to be 2.9 inches, while from the second formula it is 2.8 inches. Hence the shaft should be about 3 inches in diameter.

Prob. 80. A horizontal steel shaft of 17 inches outer and 11 inches inner diameter is to transmit 16 000 horse-powers at 50 revolutions per minute, the distance between bearings being 18 feet. Taking into account the flexure due to the weight of the shaft, compute the maximum unit-stresses.

ART. 81. TENSION AND COMPRESSION

When a tensile unit-stress S_1 exists in a body acting in a certain direction and a second tensile unit-stress S_2 is applied in the same direction, then the resultant unit-stress is S_1+S_2. When S_1 is tensile and S_2 compressive then the resultant unit-stress is S_1-S_2; if S_1 is greater than S_2 then S_1-S_2 is tensile, but if S_1 is less than S_2 then it is compressive.

When a tensile unit-stress S_1 exists in a body acting in a certain direction and a second tensile unit-stress S_2 is applied in a direction at right angles to S_1, then the true resultant unit-stress T_1 is $S_1-\lambda S_2$ where λ is the 'factor of lateral contraction.' The mean value of λ for wrought iron and steel is about $\frac{1}{3}$ while for cast iron it is about $\frac{1}{4}$. Thus if a tensile unit-stress of 5000 pounds per square inch acts in a certain direction on a steel bar and another tensile unit-stress of 6000 pounds per square inch acts at right angles to the first, then the true resultant tensile unit-stress in the original direction is $T_1=5000-\frac{1}{3}(6000)$ $=3000$ pounds per square inch.

Let a body be subject to tensile forces acting in three rectangular directions, as for instance upon the faces of a cube. Let S_1, S_2, S_3 be the tensile unit-stresses in the three directions. Then the true unit-stresses acting in these three directions are

$$T_1=S_1-\lambda S_2-\lambda S_3$$
$$T_2=S_2-\lambda S_3-\lambda S_1$$
$$T_3=S_3-\lambda S_1-\lambda S_2$$

If any unit-stress S is compression it is to be taken as negative in the formulas, then the true unit-stresses are tensile or compressive according as their numerical values are positive or negative.

As an example, let a steel bar 2 feet long and 3×2 inches in section area be subject to a tension of 60 000 pounds in the direction of its length, and to a compression of 432 000 pounds upon the two flat sides. Here $S_1 = 60\,000/6 = 10\,000$ pounds per square inch, $S_2 = -432\,000/72 = -6000$ pounds per square inch, and $S_3 = 0$. Then, taking λ as $\frac{1}{3}$, the true internal unit-stresses are $T_1 = +12\,000$, $T_2 = -9330$, $T_3 = -1330$ pounds per square inch. For this case the true tensile unit-stress T_1 is 20 percent greater than S_1 and the true compressive unit-stress T_2 is more than 50 percent greater than S_2.

Prob. 81. A common brick, $2\frac{1}{2} \times 4 \times 8\frac{1}{4}$ inches, is subject to a compression of 3200 pounds upon its top and bottom faces, 500 pounds upon its sides, and 60 pounds upon its ends. Taking λ as $\frac{1}{5}$, compute the true internal unit-stresses in the three directions.

ART. 82. REVIEW PROBLEMS

Prob. 82 *A*. An I beam, 12 inches deep and weighing 35 pounds per foot, acts as a simple beam with a span of 30 feet. Compute the flexural unit-stress at the middle due to its own weight.

Prob. 82 *B*. When an axial compression of 60 000 pounds acts on the beam of the last problem, find the unit-stress due to the combined compression and flexure.

Prob. 82 *C*. A horizontal eye bar, $1\frac{1}{4} \times 9$ inches in section, is under a tension of 120 000 pounds. Find the tensile unit-stress at the middle due to the combined tension and flexure.

Prob. 82 *D*. Compute the greatest tensile, compressive, and shearing unit-stresses due to the combination of a direct tension of 24 000 pounds with a cross-shear of 7500 pounds, both acting on a bar $1\frac{3}{4}$ inches in diameter.

Prob. 82 *E*. A vertical steel shaft weighs with its loads 64 000 pounds and transmits 1200 horse-powers at 75 revolutions per minute. What should be the diameter?

Prob. 82 *F*. A cast-iron ball is subjected in every direction to a uniform hydrostatic pressure of 625 pounds per square inch. What is the actual true compressive unit-stress which exists at every point within the ball?

Chapter 11

RESILIENCE OF MATERIALS

Art. 83. Fundamental Ideas

When a force of uniform intensity P is exerted through the distance e the work performed is measured by the product Pe. When a bar is tested in a machine, however, the force gradually and uniformly increases from 0 up to the value P and produces the elongation e; here the work performed is $\frac{1}{2}Pe$, because the average value of the uniformly increasing force is $\frac{1}{2}P$. In the first place the work may be represented by a rectangle of height P and base e; in the second case the work may be represented by a triangle of height P and base e; and the area of the triangle is one-half that of the rectangle.

As the external force increases from 0 up to P the internal stress in the bar increases gradually and uniformly from 0 up to S. The internal work of these stresses is called the 'resilience' of the bar. As the internal work equals the external work $\frac{1}{2}Pe$, this quantity is a measure of the resilience.

Strength is the capacity of a body to resist force; stiffness is the capacity of a body to resist deformation; resilience is the capacity of a body to resist work. The higher the resilience of a material the greater is its capacity to resist the work of external forces.

Elastic resilience is that internal work which has been performed when the internal stress reaches the elastic limit. Ultimate resilience is that internal work which has been performed when the body is ruptured. Ultimate strength is usually from two to three times the elastic strength; ultimate elongation is always much greater than elastic elongation; and ultimate resilience is very much larger than elastic resilience.

Resilience, like work, is expressed in foot-pounds, or inch-pounds, usually in the latter unit. Thus, if a bar is subject to a stress which gradually and uniformly increases from 0 up to 5000 pounds and is accompanied by an elongation of 0.5 inches, the resilience is 1250 inch-pounds.

Prob. 83. A wrought-iron bar weighing 30 pounds per linear foot is subject to a stress of 5000 pounds per square inch which is accompanied by an elongation of 0.25 inches. What is the resilience in inch-pounds?

Art. 84. Elastic Resilience of Bars

Let a bar of length l and section area A be under a tension P, which produces a unit-stress S equal to the elastic limit of the material and an elongation e. The elastic resilience of the bar is then equal to $\frac{1}{2}Pe$. Now $P=SA$, and by Art. 53 the elastic elongation is $e=Pl/AE=Sl/E$; hence letting K represent the elastic resilience, the product $\frac{1}{2}Pe$ becomes

$$K=\frac{S^2}{2E}Al \qquad (12)$$

or the elastic resilience of a bar is proportional to its section area and to its length, that is, to its volume.

When the bar has a section of one square inch and a length of one inch, then Al is one cubic inch, and the elastic resilience is

$$k = \frac{S^2}{2E}$$

in which S is the elastic limit of the material.

This quantity k is called the modulus of resilience, since for any given material it is a constant. For bars under tension the average values of S are given in Art. 2, and those of E in Art. 52. Using these constants, the 'modulus of resilience' k has the following values for tension:

For timber,	$k = 3$ inch-pounds
For cast iron,	$k = 1$ inch-pound
For wrought iron,	$k = 12$ inch-pounds
For hard steel,	$k = 42$ inch-pounds

These figures show that the capacity of steel to resist work within the elastic limit is the greatest of the four materials, and that of cast iron the least.

For a bar of any size the elastic resilience is found by multiplying its volume by the modulus of resilience k. Thus, a bar of timber whose volume is 50 cubic inches has an elastic resilience of about 150 inch-pounds, that is, the external work required to stress it up to the elastic limit is 150 inch-pounds. The particular shape of the bar is unimportant; it may be 5 inches in section area and 10 inches long, or 2 inches in section area and 25 inches long, or any other dimensions which give a volume of 50 cubic inches.

The above formula (12) also gives the work required to produce any unit-stress S which is less than the elastic limit. For example, let it be required to find the work needed to stress a bar of wrought iron up to 12 500 pounds per square inch, the diameter of the bar being 2 inches and its length 18 feet. Here $S = 12\,500$ pounds per square inch, $E = 25\,000\,000$ pounds per square inch, $A = 3.14$ square inches, and $l = 216$ inches. Then

$$K = \frac{12\,500^2 \times 3.14 \times 216}{2 \times 25\,000\,000} = 2\,120 \text{ inch-pounds.}$$

If the bar is required to undergo this stress 250 times per minute, the work required in one minute is $250 \times 2120 = 530\,000$ inch-pounds $= 44\,200$ foot-pounds. The power expended in stressing the bar is hence $44\,200/33\,000 = 1.34$ horse-powers.

When a bar is under a unit-stress S_1 and this is increased by additional exterior loads to S_2, the resilience due to these loads is

$$K = (S_2^2 - S_1^2)\frac{Al}{2E}$$

provided S_2 be not greater than the elastic limit.

Prob. 84. A bar of steel 10 feet long and weighing 490 pounds is stressed in one second from 4000 up to 9000 pounds per square inch. What work and what horse-power are expended in doing this?

ART. 85. ELASTIC RESILIENCE OF BEAMS

When a simple beam of span l is brought into stress by a load P applied gradually and uniformly at the middle, the deflection f results and the work $\frac{1}{2}Pf$ is performed. This work equals the resilience of the beam. The value of f in terms of the horizontal unit-stress is given in Art. 56, and the value of P in terms of the unit-stress S is given by (4) of Art. 28. Accordingly the product $\frac{1}{2}Pf$ has the value

$$K = \frac{S^2}{2E} \cdot \frac{r^2}{3c^2} \cdot Al$$

in which I, the moment of inertia of the section, has been replaced by its equivalent Ar^2, where A is the section area and r its least radius of gyration (Art. 37).

For a simple beam under a full uniform load the elastic resilience is given by

$$K = \frac{S^2}{2E} \cdot \frac{Sr^2}{15c^2} \cdot Al$$

which is 1⅗ times that of the simple beam with a single load at the middle.

These expressions show that the elastic resilience of beams of similar cross-sections is proportional to their volumes. For rectangular sections where the depth is d, the value of c is $\frac{1}{2}d$ and that of r^2 is $\frac{1}{12}d^2$; thus r^2/c^2 is $\frac{1}{3}$. Hence a rectangular bar under tensile stress has nine times the resilience of a rectangular beam loaded at the middle and 5⅝ times that of the same beam under a full uniform load.

Prob. 85. What horse-power is required to stress in one second a heavy 20-inch steel **I** beam of 24 feet span from 500 up to 8000 pounds per square inch, this being done by a load at the middle?

ART. 86. ULTIMATE RESILIENCE

The ultimate resilience of a body is equal to the external work required to produce rupture. The ultimate resilience greatly surpasses the elastic resilience, it being for wrought iron and steel sometimes five hundred times as large. It is not possible, however, to establish a formula by which the ultimate resilience can be computed, because the law of increase of the deformations beyond the elastic limit is unknown.

If a diagram is made showing the increase of elongation with stress, as in the figure of Art. 4, the abscissas indicating the elongations and the ordinates the stresses, then the area included between the curve and the axis of elongations represents the ultimate resilience for one cubic inch of the material. The total ultimate resilience is then found by multiplying this area by the volume of the specimen in cubic inches.

In Art. 15 it was remarked that the product of the ultimate strength and ultimate elongation is an index of the quality of wrought iron and steel. This is so because it is a rough measure of the ultimate resilience or resistance to external work. A measure which more closely fits the area given by a stress-diagram is

$$k = \tfrac{1}{3}s(S_e + 2S_t)$$

in which S_e is the elastic limit, S_t the ultimate tensile strength, and s the ultimate unit-elongation. For example, take a wrought-iron specimen where $S_e = 25\,000$ and $S_t = 50\,000$ pounds per square inch, while $s = 30$ percent $= 0.30$; then $k = 12\,500$ inch-pounds is the ultimate resilience for one cubic inch of the material.

Prob. 86. Show from the values given in Arts. 2 and 4 that the average ultimate resilience of timber in tension is about 50 percent greater than that of cast iron.

ART. 87. SUDDEN LOADS

When a tension is gradually applied to a bar it increases from 0 up to its final value, while the elongation increases from 0 to e and the unit-stress increases from 0 to S. A 'sudden load' is one which has the same intensity from the beginning to the end of the elongation; this elongation being produced, the bar springs back, carrying the load with it, and a series of oscillations results, until finally the bar comes to rest with the elongation e. The temporary elongation produced is greater than e, and hence also the temporary stresses produced are greater than S.

Let P be a suddenly applied load and y the temporary elongation produced by it; the external work performed during its application is Py. Now let Q be the internal stress corresponding to the elongation y; this increases gradually and uniformly from 0 up to Q, and hence its resilience or internal work is $\frac{1}{2}Qy$. But, since internal work must equal external work,

$$\tfrac{1}{2}Qy = Py, \qquad \text{or} \qquad Q = 2P$$

that is, the sudden load P produces a temporary internal stress equal to $2P$.

Now after the oscillations have ceased, the bar comes to rest under the steady load P and has the elongation e. If the elastic limit of the material has not been exceeded, corresponding elongations are proportional to their stresses; thus

$$\frac{y}{e} = \frac{Q}{P} = 2, \qquad \text{or} \qquad y = 2e$$

that is, the sudden load produces a temporary elongation double that caused by the same load when gradually applied.

If A is the section area of the bar the unit-stress S under the gradual load is P/A, and the temporary unit-stress produced under the sudden load is $2P/A$ or $2S$. The unit-stresses temporarily produced by sudden loads are hence double those caused by steady loads. It is for this reason that factors of safety are taken higher for variable loads then for steady ones.

Prob. 87. A simple beam of wrought iron, 2×2 inches and 18 inches long, is to be loaded with 3000 pounds at the middle. Show that the beam will be unsafe if this be applied suddenly.

ART. 88. STRESSES DUE TO IMPACT

Impact is said to be produced in a bar or beam when a load falls upon it from a certain height. The temporary stresses and deformations in such a case are greater than for sudden loads, and may often prove very injurious to the material. If the elastic limit is not exceeded, it is possible to deduce an expression showing the laws that govern the stresses produced by the impact. This will here be done only for the case of impact on the end of a bar.

When the load P falls from the height h upon the end of a bar and produces the momentary elongation y, the work performed is $P(h+y)$. The stress in the bar increases gradually and uniformly from 0 up to the value Q, so that the resilience or internal work is $\frac{1}{2}Qy$. Hence there results

$$\tfrac{1}{2}Qy = P(h+y)$$

Also, if e is the elongation due to the static load P, the law of proportionality of elongation to stress gives

$$\frac{y}{e} = \frac{Q}{P}$$

By solving these equations the values of Q and y are

$$Q = P\left(1+\sqrt{\frac{2h}{e}+1}\right)$$

$$y = e\left(1+\sqrt{\frac{2h}{e}+1}\right)$$

which give the temporary stress and elongation produced by the impact.

If $h=0$ these formulas reduce to $Q=2P$ and $y=2e$, as found in the last article for sudden loads. If $h=4e$ they become $Q=4P$ and $y=4e$; if $h=12e$ they give $Q=6P$ and $y=6e$. Since e is a small quantity for any bar it follows that a load P dropping from a moderate height upon the end of a bar may produce great temporary stresses and elongations. If these stresses exceed the elastic limit they cause molecular changes which result in brittleness and render the material unsafe.

The above expressions for Q and y are not exact, as the resistance against motion due to the inertia of the material has not been taken into account. In Chapters XIII and XIV of Mechanics of Materials (tenth edition) the subject is discussed far more completely than has been possible here.

Prob. 88. In an experiment upon a spring a steady weight of 15 ounces on the end produced an elongation of 0.4 inches. What temporary elongation would be produced when the same weight is dropped upon the end of the spring from a height of 7 inches?

ART. 89. REPEATED STRESSES

Ultimate strength is usually understood to be that steady unit-stress which causes the rupture of a bar in one application. Experience and experiment teach, however, that rupture may be caused by a unit-stress less than the ultimate strength when that unit-stress is applied to a bar a large number of times in succession. For example, Wöhler showed that a bar of wrought iron could be broken in tension by 800 applications of 52 800 pounds per square inch and by 10 140 000 applications of 35 000 pounds per square inch, the range of stress in each application being from 0 up to the designated value.

It has also been shown by Wöhler and others that the greater the range of stress, the less is the unit-stress required to rupture it with a large number of applications. Also that when the range of unit-stress is from 0 up to the elastic limit, rupture occurs only after an enormous number of applications.

Let P_1 be the least and P_2 the greatest tensile stresses which occur in a bar under repeated stress and let n be the ratio P_1/P_2. Let S_u be the ultimate strength and S_e the elastic limit of the material. Then Weyrauch's formula for the unit-stress which ruptures the bar after an enormous number of repetitions is

$$S = S_e\left(1 + \frac{S_u - S_e}{S_e} n\right)$$

For structural steel, using the mean values given in Tables 1 and 2, this becomes $S = 35\,000(1 + \tfrac{6}{7}n)$ and for wrought iron $S = 25\,000(1 + \tfrac{1}{2}n)$. For example, let a bar of structural steel range in tension from 80 000 to 160 000 pounds; then $n = \tfrac{1}{2}$ and $S = 47\,500$ pounds per square inch is the tensile unit-stress which will rupture it after an enormous number of applications, although the ultimate strength observed in one application is 65 000 pounds per square inch. The above formula also applies to the case where P_1 and P_2 are both compressive stresses.

When P_1 and P_2 are stresses of different kinds, one being tension and the other compression, let n be the ratio P_1/P_2 without respect to sign. Then Weyrauch's formula for the unit-stress which produces rupture after an enormous number of applications is

$$S = S_e(1 - \tfrac{1}{2}n)$$

For structural steel this becomes $S = 35\,000(1 - \tfrac{1}{2}n)$ and for wrought iron $S = 25\,000(1 - \tfrac{1}{2}n)$. For example, if the forces in a bar of structural steel range from 80 000 pounds compression to 160 000 pounds tension, then $n = \tfrac{1}{2}$ and $S = 26\,200$ pounds per square inch is the compressive unit-stress which will rupture the bar after an enormous number of applications.

The above formulas are sometimes used to find working unit-stresses for designing bars, such a factor of safety being used that the value of S shall be less than one-half of S_e.

Prob. 89. A wrought-iron bar is subject to axial forces which range from 320 000 to 400 000 pounds. Compute the rupturing unit-stress S after an enormous number of repetitions, first when both forces are tension, second when the smaller one is compression and the larger one is tension.

ART. 90. REVIEW PROBLEMS

Prob. 90 *A*. How many foot-pounds of work are required to stress a steel piston-rod, 3 inches in diameter and 4 feet long, from 0 up to 16 000 pounds per square inch?

Prob. 90 *B*. What horse-power is required to stress the rod of the last problem 120 times in one minute?

Prob. 90 *C*. Compute the horse-power required to deflect, 59 times per second, a wrought-iron cantilever beam, $2 \times 3 \times 72$ inches, so that at each deflection the unit-stress S shall range from 0 to 9000 pounds per square inch.

Prob. 90 *D*. What work is required to rupture by tension a wrought-iron bar which weighs 325 pounds?

Prob. 90 *E*. Discuss the case where a sudden load P is applied to a bar which is already under the static load P_1. What is the maximum stress?

Prob. 90 *F*. What is the height from which a weight must fall upon the end of a bar in order to produce a deformation equal to three times the static deformation?

Prob. 90 *G*. Consult Test of Metals, published annually by the U. S. Ordnance Department, and describe some of the endurance tests made by Howard on rotating shafts.

Prob. 90 *H*. A bar of wrought iron is to be subjected to repeated stresses ranging from 16 000 pounds tension to 80 000 pounds tension. What should be its diameter using a safety factor of 4 in the above formula for *S*?

Prob. 90 *J*. Solve the last problem, taking the smaller stress as compression and the larger one as tension.

ART. 91. ANSWERS TO PROBLEMS

Below are given answers to a few of the problems stated in the preceding pages, the number of the problem being placed in parentheses. However satisfactory it may be to a student to know the true result of a solution, let him remember that after commencement day answers to problems will never be given.

(1 *A*) $4\frac{5}{32}$ inches.
(3 *C*) About 1 000 000 pounds.
(4 *A*) $4\frac{1}{2}$ inches for wrought-iron bar.
(5 *A*) 1780 feet.
(6 *B*) Factors are 6 and 16.
(7 *A*) $3\frac{3}{8}$ inches for second case.
(10 *B*) Reduction of area = 50.5 percent.
(13 *B*) $1058.84.
(14 *A*) 66 000 pounds.
(16 *A*) 4.04 cents.
(16 *B*) $3\frac{1}{8}$ inches.
(17 *B*) 105 000 pounds.
(18 *A*) 1200 pounds.
(23 *B*) 4.04 inches.
(24) 2097 inches^4.
(25 *J*) 2840 pounds.
(29 *A*) 3130 pounds.
(31 *A*) 18 inches.
(33 *A*) 5.9 and 3.9.
(35 *D*) Nearly 8.
(35 *E*) 24 inches.
(35 *F*) $2\frac{2}{3}$ to 1.
(36) 54 000 pounds.

(39 *A*) 58 000 pounds.
(44 *D*) 3⅓.
(44 *E*) 13½ inches.
(45 *A*) 500 pounds.
(47 *A*) 1.3 inches.
(51 *A*) 30 pounds.
(51 *E*) 2⅝ inches.
(51 *F*) 144 to 100.
(53 *A*) 0.12 inches.
(54 *A*) 0.0011 inches.
(59 *F*) 14 500 000 pounds per square inch.
(63 *A*) 59 500 pounds.
(66 *B*) 1500 pounds.
(67 *D*) 500 pounds per square inch.
(74) $S_r = 200$ pounds per square inch.
(75 *B*) 126 000 pounds.
(78) $S_1 = 9420$ pounds per square inch.
(79) Nearly 6.
(83) 5625 inch-pounds.
(88) 2.8 inches.
(90 *F*) $h = 1.5\,e$.
(90 *H*) $d = 3.7$ inches.

Prob. 91. The horizontal flexural unit-stress on the upper surface of a railroad rail is 9500 pounds per square inch and the vertical compressive unit-stress due to the weight of the driving-wheel is 17 500 pounds per square inch. Compute the true resultant unit-stresses in the horizontal and vertical directions.

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