A STUDY OF THE CIRCULAR-ARC BOW-GIRDER

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THE CIRCULAR-ARC BOW-GIRDER

BY

A. H. GIBSON

D.SC., ASSOC.MEM.INST.C.E., M.I.MECH.E.

Professor of Engineering in the University of St. Andrews, University College, Dundee

AND

E. G. RITCHIE

B.SC.

Assistant Lecturer in Engineering, University College, Dundee

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PREFACE

The problem of the state of equilibrium and of stress of the circular-arc bow-girder, *i.e.*, the girder forming a circular arc in plan such as is often used to support the balcony of a theatre, is one affording some difficulties of solution. These arise mainly from the fact that in addition to the bending moments and reactions involved in the case of the straight encastré girder, twisting moments are called into play at each section and at the ends of the bow-girder, and these moments affect very considerably the state of equilibrium of the girder.

The general problem was solved in a paper read before the Royal Society of Edinburgh by Professor Gibson in 1912, and the first portion of this book is based on the principles laid down in that paper. The solution in any particular case becomes easy if the end fixing moments and the reactions are known, and values of these have been calculated for the more important cases likely to occur in practice.

This investigation shows that the values of the various moments and reactions for a given loading depend on the relative values of the flexural rigidity, E I, and the torsional rigidity, C J, of the section. A knowledge of the geometrical properties of the section and of its material enable the former of these to be predetermined with some accuracy, but the authors have been unable to find any published data as to the values of the torsional rigidity for such commercial sections as are usual in structural engineering. With a view of obtaining such data experiments have been carried out by Mr. Ritchie on a number of commercial sections, and the result of this work forms the foundation for much of the second part of the book.

Chapter I. outlines the introductory theorems necessary for a thorough understanding of Chapter II., which deals with the equilibrium of the bow-girder. In Chapter III. the torsion of non-circular sections is considered, while Chapter IV. deals with the stresses involved by such torsion alone or combined with bending, and Chapter V. deals briefly with the general principles of design of a bow-girder exposed to both bending and twisting.

It is hoped that the treatment is sufficiently complete to enable any one familiar with the general principles of design of the ordinary straight plate-web or lattice girder to adapt these to any specific case of a bow-girder under uniform or concentrated loading.

In view of recent failures of structures in which straight beams exposed to some torsion have collapsed under seemingly inadequate loads, the data of Chapter III., emphasising as it does the extreme weakness of the commercial I, angle, or T section under torsion, should be of interest.

Appendices have been added, giving a list of integrals which will be useful to the reader working through the investigations of Chapter II., and also giving a table of the geometrical properties of some commercial sections.

A. H. G.
E. G. R.

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A STUDY OF THE CIRCULAR-ARC BOW-GIRDER

CHAPTER I

(1) Equilibrium of the Straight Girder.

If a girder straight in plan and horizontal when unloaded is exposed to a series of vertical loads, each section is subject to a bending moment M, whose magnitude varies from point to point. Under the influence of this moment the girder is bent, and, so long as the loads are not sufficient to produce stresses in excess of the elastic limit of the material, the radius of curvature R of the profile of the neutral axis at a point where the bending moment equals M is given by the relationship

$$\frac{1}{R} = \frac{M}{EI} \quad \ldots \ldots \ldots \quad (a)$$

where I is the moment of inertia of the section about a horizontal axis through the centroid of its area, and where E is the modulus of direct elasticity of the material. If y be the vertical displacement of the neutral axis at a point distant x from some datum point in the axis, it may readily be shown that

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$$

$$= \frac{d^2y}{dx^2} \text{ (approx.)}$$

so that, so long as the deflection of the beam is confined within practical limits,

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \quad \ldots \ldots \ldots \quad (b)$$

(2) Curvature, Slope, and Deflection.

From (b) it follows that if, at any one point, the girder is horizontal after loading, the slope $\frac{dy}{dx}$ at any other point at a distance l will be given by

$$\frac{dy}{dx} = \int_0^l \frac{M}{EI} \cdot dx \quad \ldots \ldots \ldots \quad (c)$$

and will therefore be represented to scale by the area of the $\frac{M}{EI}$ diagram between the two points, while if the slope at the first point is not zero, this area will measure the difference of slope at the two points. On integrating both sides of expression (c), the deflection y of the second point below the first is given by

$$y = \int_0^l \left(\frac{dy}{dx}\right) dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

$$= \int_0^l \int_0^l \frac{M}{EI} \, . \, dx.$$

In a given beam under load the slope changes from point to point, and the difference of slope at two points, a small distance δx apart, is given by $\frac{d}{dx}\left(\frac{dy}{dx}\right) \delta x$, or by $\left(\frac{d^2y}{dx^2}\right) \delta x$ or $\frac{M}{EI} \, . \, \delta x$, where M is the moment acting on the element included between the two sections. If the rest of the beam were to remain straight the deflection at a distance l from the element, due to the bending of the element under this moment, would be equal to

$$\frac{M}{EI} \, . \, \delta x \, . \, l$$

and if the slope at one end of the element were zero this would be the actual relative deflection at a distance l. Since every section of the beam is exposed to a bending moment, any element at a distance x from the point whose deflection is being considered contributes its quota $\frac{M}{EI} \, . \, x \, . \, \delta x$ to the resultant deflection, so that the actual deflection at the point l, relative to the point at which the slope is zero, is given by

$$\int_0^l \frac{M}{EI} \, . \, x \, . \, dx$$

$$\text{or by } A\bar{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

where A is the area of the $\frac{M}{EI}$ diagram included between the two points and $\bar{x}$ is the distance of its centroid from the point l.

If, instead of being zero, the slope at the point o is equal to i, the deflection at l, relative to this point is given by

$$i\,l + A\,\bar{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

Special Cases of Deflection.

In certain standard cases the maximum deflection is very readily calculated, and is as follows :—

	Deflection.
Beam of uniform section and of length l with single load W at centre .	$\frac{1}{48} \frac{Wl^3}{EI}$
Beam of uniform section uniformly loaded with w lbs. per foot run .	$\frac{5}{384} \frac{wl^4}{EI}$
Cantilever of uniform section with load W at end	$\frac{1}{3} \frac{Wl^3}{EI}$
Cantilever of uniform section uniformly loaded	$\frac{1}{8} \frac{wl^4}{EI}$

(3) Encastré and Continuous Beams.

A beam simply supported at its two ends has, everywhere, a curvature whose concavity is upwards. If, however, it is built in to supports at its ends, these supports prevent the beam adopting the slope natural to it when free, and a fixing moment is

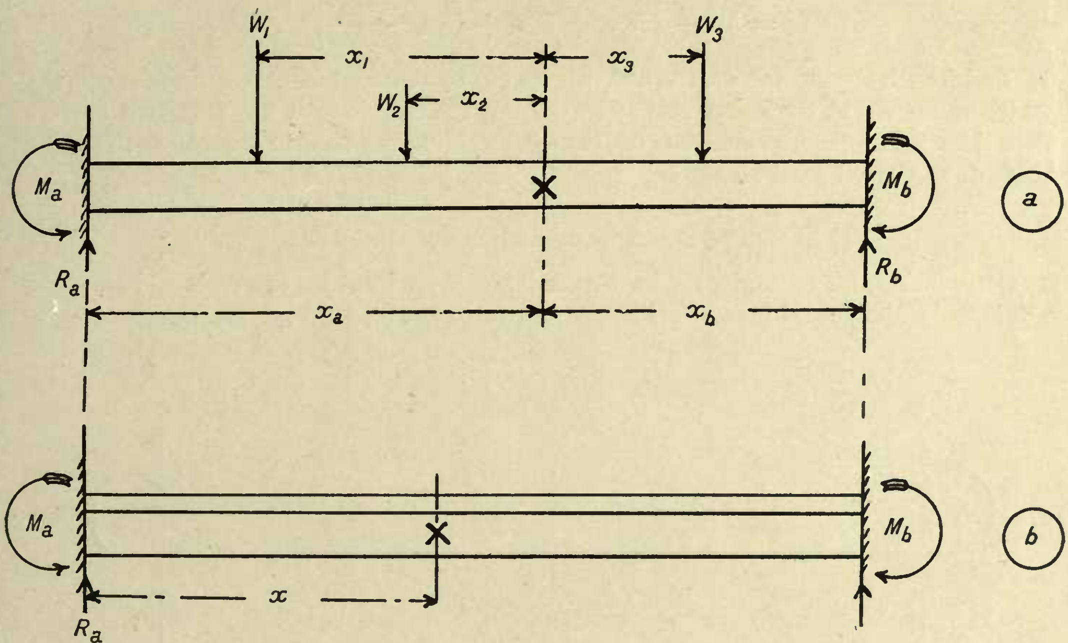


FIG. 1.

called into play at each support, these moments tending to make the beam concave downwards. The effect of the fixing moment is transmitted to every section of the beam, and at any such section as, say, X in the beam of Fig. 1A, for equilibrium

$$\begin{aligned} M_x &= M_a - (R_a x_a - W_1 x_1 - W_2 x_2) \qquad (g) \\ &= M_b - (R_b x_b - W_3 x_3) \end{aligned}$$

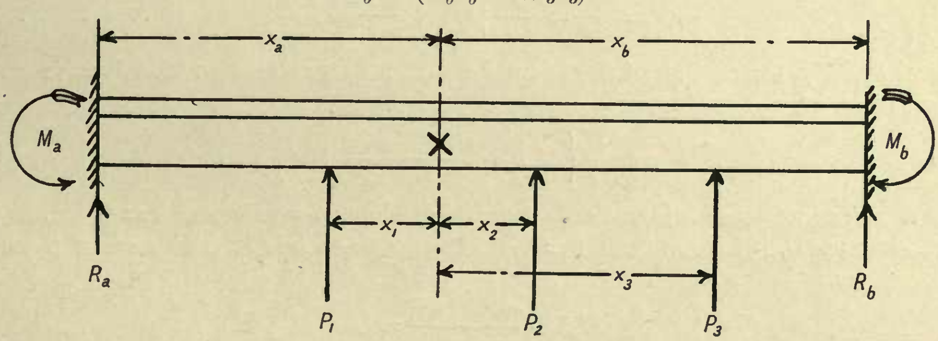


FIG. 2.

while at X in Fig. 1B, which represents a beam with uniform loading of magnitude w lbs. per foot run,

$$\begin{aligned} M_x &= M_a - \left(R_a x_a - \frac{w x_a^2}{2}\right) \qquad (h) \\ &= M_b - \left(R_b x_b - \frac{w x_b^2}{2}\right). \end{aligned}$$

If, as in Fig. 2, the beam has one or more intermediate supports, whose upward reactions are P_1, P_2, P_3, the moment at X, say between supports (1) and (2) is given by

$$M_x = M_a - \left\{R_a x_a + P_1 x_1 - \frac{w x_a^2}{2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (j)$$

$$= M_b - \left\{R_b x_b + P_2 x_2 + P_3 x_3 - \frac{w x_b^2}{2}\right\} \quad . \quad . \quad . \quad . \quad . \quad (k)$$

Under such circumstances, the magnitudes of the fixing moments M_a and M_b; of the reactions R_a, R_b, at the ends; and of P_1, P_2, P_3, the reactions at the supports, require to be determined before equations (g), (h), (j), or (k) can be used to determine the value of M_x at any given section.

(4) Encastré Beam with no Intermediate Supports.

Considering, for example, the case of a beam built in at its two ends and carrying a uniformly distributed load of w lbs. per foot run (Fig. 3),

$$M_x = M_a - R_a x + \frac{wx^2}{2}.$$

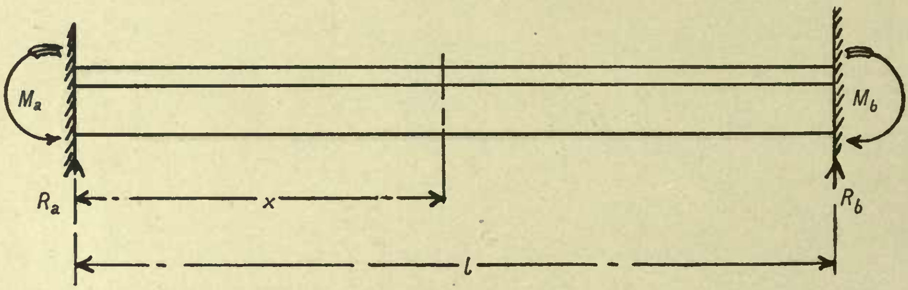


FIG. 3.

Since from symmetry $R_a = \frac{wl}{2}$, we have

$$\frac{M_x}{EI_x} = \frac{d^2y}{dx^2} = \frac{1}{EI_x}\left\{M_a + \frac{wx^2}{2} - \frac{wlx}{2}\right\} \quad . \quad . \quad . \quad . \quad . \quad (l)$$

If, for simplicity, the section of the beam be taken as constant so that $I_x =$ constant $=$ I, we have, on integrating,

$$\frac{dy}{dx} = \frac{1}{EI}\left\{M_a x + \frac{wx^3}{6} - \frac{wlx^2}{4} + A\right\} \quad . \quad . \quad . \quad . \quad . \quad (m)$$

where A is a constant of integration. Since the slope is zero where $x = 0$, *i.e.*, at the left-hand support, it follows that $A = 0$, and since the slope is also zero at B where $x = l$, we have

$$M_a l + \frac{wl^3}{6} - \frac{wl^3}{4} = 0$$

$$\therefore M_a = \frac{wl^2}{12}.$$

Substituting this value of M_a in (m) gives

$$\frac{dy}{dx} = \frac{w}{EI}\left\{\frac{l^2}{12} + \frac{x^3}{6} - \frac{lx^2}{4}\right\}$$

or, on integrating this,

$$y = \frac{w}{EI}\left\{\frac{l^2x}{12} + \frac{x^4}{24} - \frac{lx^3}{12} + B\right\}.$$

The constant B is determined from the fact that the deflection $y = 0$ when $x = 0$, so that $B = 0$.

(5) Encastré Beam with Intermediate Supports, or Continuous Beam on more than Two Supports.

Let A, B, C, (Fig. 4) represent three adjacent points of support on an encastré beam, or on a simple continuous beam with uniform loading w lbs. per foot run. To determine the moments M_a, M_b, and M_c, and the reactions R_a, R_b, R_c. Take the origin at A. Then between A and B,

$$M_x = M_a - R_a x + \frac{wx_2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (n)$$

$\therefore$ At B

$$M_b = M_a - R_a l_1 + \frac{wl_1^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (o)$$

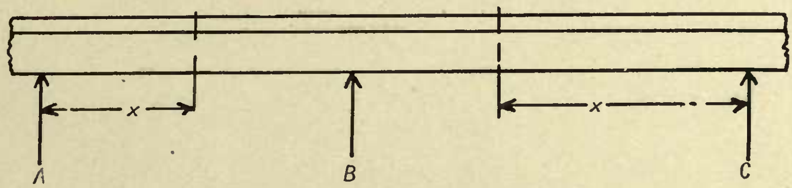


FIG. 4.

Similarly, working back from C to B,

$$M_b = M_c - R_c l_2 + \frac{wl_2^2}{2} \quad . \quad . \quad . \quad . \quad . \quad (p)$$

Writing (n) as

$$EI\frac{d^2y}{dx^2} = M_a - R_a x + \frac{wx^2}{2}$$

we get

$$EI\frac{dy}{dx} = M_a x - R_a\frac{x^2}{2} + \frac{wx^3}{6} + C \quad . \quad . \quad . \quad . \quad . \quad (q)$$

and

$$EI\,y = M_a\frac{x^2}{2} - R_a\frac{x^3}{6} + \frac{wx^4}{24} + Cx + D \quad . \quad . \quad . \quad (r)$$

Since $y = 0$ when $x = 0$, it follows that $D = 0$, while since $y = 0$ when $x = l$, we have

$$M_a\frac{l_1^2}{2} - R_a\frac{l_1^3}{6} + \frac{wl_1^4}{24} + Cl_1 = 0$$

$$\therefore C = -\left\{\frac{M_a l_1}{2} - \frac{R_a l_1^2}{6} + \frac{wl_1^3}{24}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (s)$$

From (q) the slope at B is given by

$$\left(\frac{dy}{dx}\right)_b = \frac{1}{EI}\left\{M_a l_1 - R_a\frac{l_1^2}{2} + \frac{wl_1^3}{6} + C\right\} \quad . \quad . \quad . \quad . \quad . \quad (t)$$

and on substituting from (s) this becomes

$$\left(\frac{dy}{dx}\right)_b = \frac{1}{EI}\left\{M_a\frac{l_1}{2} - R_a\frac{l_1^2}{3} + \frac{wl_1^3}{8}\right\}.$$

Similarly, taking C as origin and working back from C to B, we should get

$$-\left(\frac{dy}{dx}\right)_b = \frac{1}{EI}\left\{M_c\frac{l_2}{2} - R_c\frac{l_2^2}{3} + \frac{wl_2^3}{8}\right\}$$

the minus sign being taken before $\frac{dy}{dx}$, because x is now measured in the negative direction.

Equating these expressions for $\left(\frac{dy}{dx}\right)_b$, and eliminating terms containing R_a and R_c by substitution from equations (p) and (o), we get

$$(M_a + 2M_b)l_1 + (M_c + 2M_b)l_2 - \frac{w}{4}(l_1^3 + l_2^3) = 0 \quad . \quad . \quad . \quad (u)$$

the relationship commonly known as the equation of "three moments." With n points of support this theorem yields $n - 2$ equations, and the terminal conditions supply the additional two which are necessary before the n unknowns can be determined.

Taking, for example, the case of a beam resting on three equidistant supports and forming two spans each of length l, $M_a = M_c = 0$, and the foregoing equation reduces to

$$M_b = \frac{wl^2}{8}.$$

Also since
$$M_b = M_a - R_a l + \frac{wl^2}{2}$$

$$\therefore\ R_a = \frac{3}{8}wl$$

$$\therefore\ R_b = 2\,wl - 2\,R_a = \frac{5}{8}wl.$$

Again, taking the case of an encastré beam with a central support giving two spans, each of length l, from symmetry $M_a = M_c$, $l_1 = l_2 = l$, and equation (u) becomes

$$M_a + 2\,M_b - \frac{wl^2}{4} = 0 \quad . \quad . \quad . \quad . \quad . \quad (v)$$

From (o)
$$M_b = M_a - R_a l + \frac{wl^2}{2}$$

$$\therefore\ 3\,M\ \ - 2\,R_a l + \frac{3}{4}wl^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (w)$$

Since the slope at A where $x = 0$ is zero it follows from (q) and (s) that

$$\frac{M_a l}{2} - \frac{R_a l^2}{6} + \frac{wl^3}{24} = 0$$

$$\therefore\ 3M_a - R_a l + \frac{wl^2}{4} = 0 \quad . \quad . \quad . \quad . \quad . \quad (x)$$

and combining equations (w) and (x),

$$R_a = \frac{wl}{2}$$

$$\therefore\ R_b = 2\,wl - 2\,R_a = wl.$$

From (x)
$$3M_a = \frac{wl^2}{2} - \frac{wl^2}{4} = \frac{wl^2}{4}$$

$$\therefore\ M_a = \frac{wl^2}{12}$$

$\therefore$ from (v) $$M_b = \frac{wl^2}{8} - \frac{M_a}{2}$$
$$= \frac{wl^2}{12}.$$

(6) Encastré Beam with Uniform Loading—Effect of a Settlement of One Support.

Where the fixing moments M_a and M_b at the ends of an encastré beam of span l, or at any two intermediate supports of a continuous beam, are not equal, the moment due to these varies uniformly from M_a to M_b, and, at a point distant x from the end A, is equal to

$$M_a + \frac{x}{l}(M_b - M_a).$$

From equations (g) and (h) (p. 3), it is evident that in a loaded beam, fixed at the

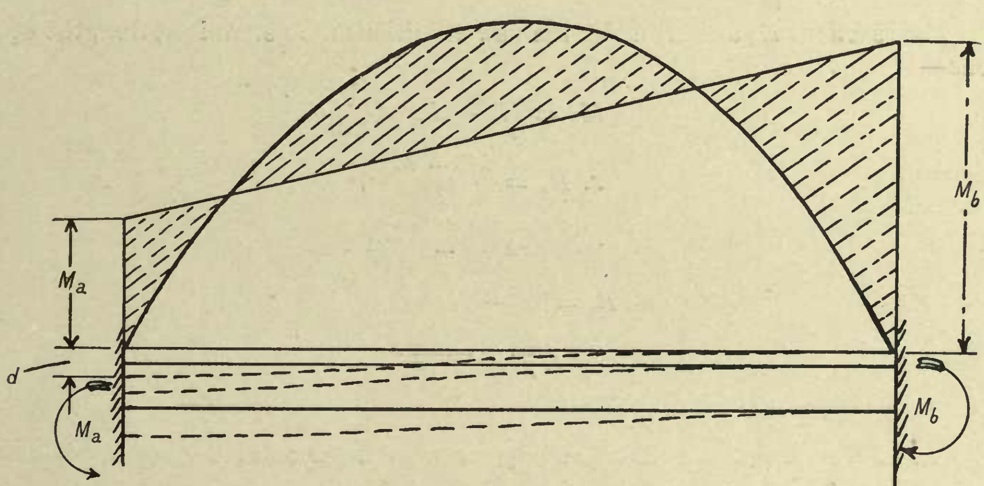


FIG. 5.

ends, the bending moment at any point is the difference between the bending moment which would be produced by the same loading on a beam simply supported at the ends, and that produced by the end moments, so that in the case of a uniformly loaded encastré beam with end moments M_a and M_b the diagram of effective bending moments is represented by the shaded area of Fig. 5.

If one of the supports A of such a beam sinks through a distance d, the ends remaining horizontal, the difference of slope of the ends is zero, and consequently from (c) (p. 1), the area of the effective B.M. diagram is zero.

$$\therefore \frac{M_a + M_b}{2} \cdot l - \frac{wl^2}{8} \cdot \frac{2}{3} l = 0$$

$$\therefore M_a + M_b = \frac{wl^2}{6}.$$

Again, since the relative deflection of the two ends is d, the moment of the effective B.M. diagram about A is equal to EId ((e) (p. 2)).

$$\therefore\ EId = M_b \frac{l}{2} \cdot \frac{2}{3} l + M_a \frac{l}{2} \cdot \frac{l}{3} - \frac{wl^2}{8} \cdot \frac{2}{3} l \cdot \frac{l}{2}$$

$$= (M_a + 2\ M_b) \frac{l^2}{6} - \frac{wl^4}{24}$$

$$\therefore\ M_a + 2\ M_b = \frac{6EId}{l^2} + \frac{wl^2}{4}$$

and

$$M_a + M_b = \frac{wl^2}{6}$$

$$\therefore\ M_a = -\frac{6EId}{l^2} + \frac{wl^2}{12}$$

$$M_b = \frac{6EId}{l^2} + \frac{wl^2}{12}$$

If the only moment is that induced by the settlement of the support, *i.e.*, $w = 0$, this $B.M. = \pm \frac{6EId}{l^2}$.

The reactions R_a and R_b under the new conditions are determined from the equations—

$$M_b = M_a - R_a l + \frac{wl^2}{2}$$

$$\therefore\ R_a = \frac{M_a - M_b}{l} + \frac{wl}{2}$$

$$= \frac{wl}{2} - \frac{12EId}{l^3}$$

$$\text{and } R_b = wl - R_a$$

$$= \frac{wl}{2} + \frac{12EId}{l^3}$$

(7) Beams with Unsymmetrical Loading, or with a Series of Concentrated Loads.

When the loading of an encastré or continuous beam is unsymmetrical or consists of a series of concentrated loads, a semi-graphical treatment based on the considerations outlined on p. 2 is preferable.

In Fig. 6, let A, B, C be three adjacent supports in a continuous beam, and let AGB, BHC represent the bending moment diagrams for such a loading on two simply supported spans AB and BC. Let $ADEB$ and $BEFC$ represent the fixing moment diagrams, and G_1', G_1, G_2', G_2 the positions of the centroids of the areas $ADEB$, AGB, $BEFC$, BHC.

Let the area $AGB = A_1$
" " $ADEB = A_1'$
" " $BHC = A_2$
" " $BEFC = A_2'$

Then, considering the span AB, taking A as origin, since the supports at A and B are at the same level

$$l_1 i_b = \frac{1}{EI} (A_1'\bar{x}_1' - A_1\bar{x}_1)$$

i_b being the slope at B.

Similarly for the span CB, taking C as origin, since the supports at C and B are at the same level

$$l_2 i_b = -\frac{1}{EI}(A_2'\bar{x}_2' - A_2\bar{x}_2)$$

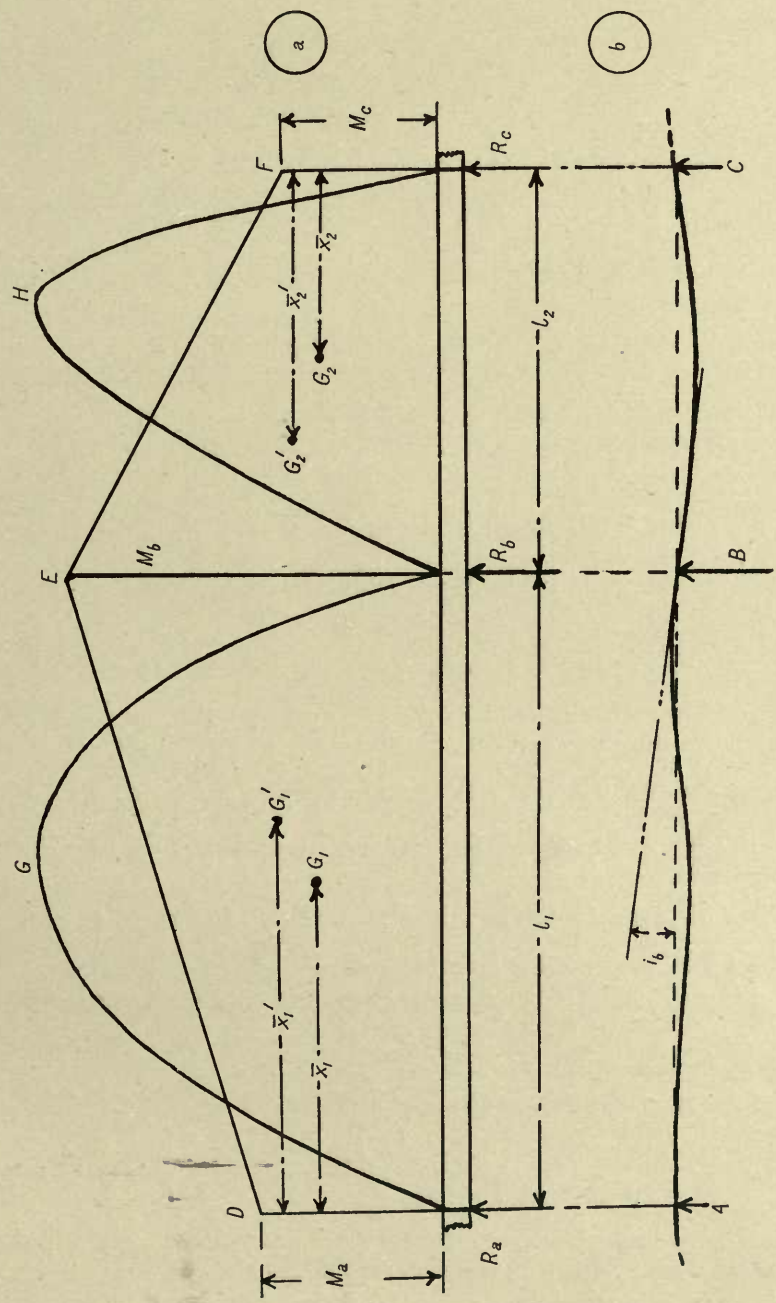


FIG. 6.

the negative sign being taken, since x is measured in opposite directions in the two cases.

Equating the two expressions for the slope at B gives

$$\frac{A_1'\bar{x}_1' - A_1\bar{x}_1}{l_1} = -\frac{A_2'\bar{x}_2' - A_2\bar{x}_2}{l_2} \quad . \quad . \quad . \quad . \quad . \quad (y)$$

Again, taking moments about A and C of the fixing moment diagrams on each span

$$A_1'\bar{x}_1' = \frac{l_1^2}{6}(M_a + 2M_b)$$

$$A_2'\bar{x}_2' = \frac{l_2^2}{6}(M_c + 2M_b)$$

and, on substituting these values, equation (y) becomes

$$M_a l_1 + 2M_b(l_1 + l_2) + M_c l_2 - 6\left\{\frac{A_1\bar{x}_1}{l_1} + \frac{A_2\bar{x}_2}{l_2}\right\} = 0 \quad . \quad . \quad (z)$$

This is the most general form of the equation of these moments and is applicable to any form of loading,

Writing $\quad A_1 = \frac{2}{3}\frac{wl_1^3}{8}$; $A_2 = \frac{2}{3}\frac{wl_2^3}{8}$; $\bar{x}_1 = \frac{l_1}{2}$; $\bar{x}_2 = \frac{l}{2}$

gives the equation for uniform loading, which is identical with (u), p. 6.

If some or all of the supports sink, B falling d_1 below A and d_2 below C, equation (z) becomes

$$M_a l_1 + 2M_b(l_1 + l_2) + M_c l_2 - 6\left\{\frac{A_1\bar{x}_1}{l_1} + \frac{A_2\bar{x}_2}{l_2}\right\}$$
$$+ 6EI\left(\frac{d_1}{l_1} + \frac{d_2}{l_2}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

(8) Resilience of a Girder Exposed to Bending.

If, under the action of a bending moment M, two originally parallel vertical sections of a beam, enclosing an element of length δx, become inclined to each other at an angle of δi, the work done by the moment in bending this element is equal to $M\frac{\delta i}{2}$. (This assumes that the moment is applied gradually, and, at any instant, is proportional to the curvature obtaining at that instant.)

$\therefore$ Whole work done in bending beam $= U = \frac{1}{2}\int M di$, where the integration is taken over the whole length of the beam.

But δi measures the difference of slope at the two ends of the element.

$$\therefore \delta i = \frac{\delta x}{R} = \frac{M}{EI}\cdot\delta x$$

So that if l be the length of the beam

$$U = \frac{1}{2}\int_0^l \frac{M_x^2}{EI}dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

This quantity is termed the resilience of the beam under the given loading, and is equal to the work done by the load or loads during the distortion of the beam. Thus, if a single load W be applied to the beam, causing its point of application to deflect through a distance y, the work done by it during its application is equal to $W\frac{y}{2}$.

$$\therefore W\frac{y}{2} = U = \frac{1}{2}\int_0^l \frac{M_x^2}{EI}\,dx$$

$$\therefore y = \frac{1}{W}\int_0^l \frac{M_x^2}{EI}\,dx$$

E.g., beam simply supported at the two ends. Single load W at a point C distant a from one end and b from the other end of the beam (Fig. 7).

$$\text{Here } R_a = W\left(\frac{b}{a+b}\right);\ R_b = W\left(\frac{a}{a+b}\right)$$

$$\text{Between } A \text{ and } C,\ M_x = \frac{Wbx}{(a+b)}$$

$$\therefore U_{(a-c)} = \frac{1}{2EI}\int_a^c \left(\frac{Wb}{a+b}\right)^2 x^2\,dx$$

$$= \frac{W^2b^2a^3}{6EI\,(a+b)^2}$$

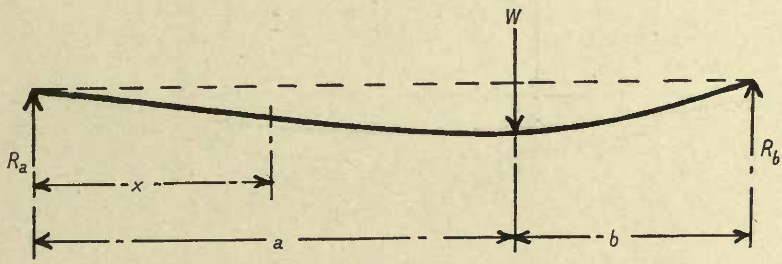


FIG. 7.

Similarly between B and C

$$U_{(b-c)} = \frac{W^2b^3a^2}{6EI(a+b)^2}$$

$$\therefore U_{(a-b)} = \frac{W^2a^2b^2}{6EI(a+b)}$$

$$= W\frac{y}{2}$$

$$\therefore y = \frac{Wa^2b^2}{3EI\,(a+b)}$$

(9) Castigliano's Theorem.

Where more than a single load is applied, the problem is readily solved by an application of this theorem.

Suppose a structure, originally horizontal, to deflect through y_1 and y_2 at points P_1 and P_2 under the application of loads W_1 and W_2 (Fig. 8). Then assuming smooth supports, so that the work done by the end reactions is zero, we have—

$$U = \frac{W_1y_1}{2} + \frac{W_2y_2}{2}$$

Let W_1 be now increased to $(W_1 + \delta W_1)$, W_2 remaining constant, and let $y_1 + \delta y_1$, $y_2 + \delta y_2$, be the new deflections.

The additional work done $= \left\{W_1 + \frac{\delta W_1}{2}\right\} \delta y_1 + W_2 \delta y_2$

$$\therefore \delta U = W_1 \delta y_1 + W_2 \delta y_2 \quad . \quad . \quad . \quad . \quad . \quad (\gamma)$$

Now suppose the structure unstrained and gradually loaded with $(W_1 + \delta W_1)$ and W_2, these loads during application always maintaining towards each other the ratio of their final values. The final deflection must be the same as before, while the resilience is given by

$$\begin{aligned} U' &= \tfrac{1}{2}(W_1 + \delta W_1)(y_1 + \delta y_1) + \tfrac{1}{2} W_2 (y_2 + \delta y_2) \\ &= \tfrac{1}{2}\{(W_1 y_1 + W_2 y_2) + y_1 \delta W_1 + W_1 \delta y_1 + W_2 \delta y_2\} \\ &= \tfrac{1}{2}\{(W_1 y_1 + W_2 y_2) + y_1 \delta W_1 + \delta U\} \quad . \quad . \quad . \quad \text{from} \ . \ (\gamma) \\ &= \tfrac{1}{2}\{2U + \delta U + y_1 \delta W_1\} \end{aligned}$$

But $U' - U$ must equal δU.

$$\therefore \delta U = y_1 \delta W_1$$

$$\text{or} \ \frac{dU}{dW_1} = y_1$$

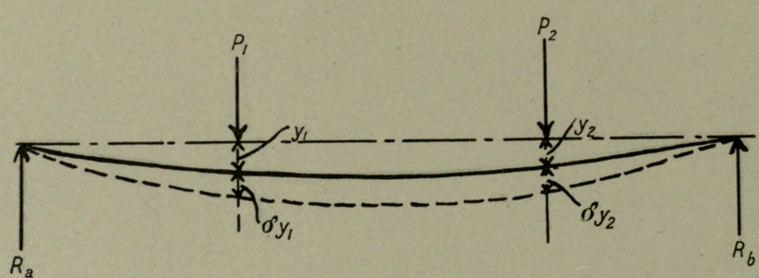


FIG. 8.

Similarly $\frac{dU}{dW_2} = y_2$, *ie.*, the derivative of U with respect to any one load equals the deflection of the point of application of that load.

(10) Resilience of a Beam Exposed to a Torque.

If a beam be exposed to a torque whose magnitude at a given point is T, successive plane sections suffer rotation about the longitudinal axis of the beam, and the relative rotation of two sections, distant δx apart, is equal to $\delta\theta$, where

$$\delta\theta = \frac{T}{CJ} \cdot \delta x$$

Here C is the modulus of transverse rigidity or the shear modulus of the material and J is the polar moment of inertia of the section, or its moment of inertia about an axis through its centroid perpendicular to its plane.[1]

The work done by the torque during this relative rotation is $T\frac{\delta\theta}{2} = \frac{T^2}{2CJ}\delta x$, so that over the whole length l of the beam the work done by the torque is given by

$$U = \tfrac{1}{2}\int_0^l \frac{T_x^2}{CJ} dx$$

[1] See Chapter III. for the effective value of J in any particular case.

Where a beam is exposed to both bending and twisting moments, its resilience is the sum of the works done by these moments, and this, by the principle of work, is equal to the work done by the applied loads during distortion.

(11) Deflection Produced by Shear Forces.

In addition to the deflections produced by the bending of a girder, there is some slight deflection due to the fact that each vertical layer is exposed to shear stress. In a straight beam, exposed only to bending and shear stresses, the deflection due to shear is always a small fraction of that due to pure bending, being greatest in a built up beam of I section in which the web is comparatively thin.[2]

In such a beam of normal proportions and span simply supported at the ends, the deflection due to shear is seldom more than 4 or 5 per cent. of that due to pure bending. In an encastré beam of this type the proportion may be as much as 20 or 25 per cent. In the type of bow-girder to which this treatise is particularly devoted, the deflection is mainly due to torsion, and moreover the proportion of the whole deflection due to torsion is greatest for those beam sections for which the shear deflection is greatest. Even in an extreme case, in a bow-girder the shear deflection does not amount to more than 4 or 5 per cent. of the whole, and will, in general, be much less than this. It has, in consequence, been neglected in the following discussion. Where, as in a large built-up bow-girder of I section with very slight curvature, it may be advisable to make allowance for the extra deflection, this may most easily and with sufficient accuracy be taken into account by using in the calculations a value of E about 20 per cent. less than the true value for the material.

[2] For a discussion of this point, see Morley's "Strength of Materials," p. 226, or any text-book on the same subject.

CHAPTER II

(12) The Circular-Arc Bow-Girder

A GIRDER built in to supports at one or both ends and forming an arc of a circle in plan, is subject, at each section, to both bending and twisting moments. At the supports, fixing moments of both kinds are called into play, and until these are known the resultant moment tending to cause rupture at any section is indeterminate. The following investigation is devoted to a consideration of the general state of elastic equilibrium of such a girder under various systems of loading.

The investigation is based on the theorem (p. 1) that in a straight beam, fixed horizontally at some point, the slope at any other point is given by the area of the $\frac{M}{EI}$ diagram between the two points. Where a girder is circular in plan and is subjected to both bending and twisting moments this theorem requires modification. Let M_θ and T_θ be the bending and twisting moments at a point P distant θ (in angular measure) from the support A (Fig. 9). Then a given slope at P in the direction of the tangent at this point produces a slope of $\cos(\theta_1 - \theta)$ times its magnitude at Q in the direction of the tangent at Q. Also an angular displacement γ at P, due to a torque between the support and this point, produces a slope $\gamma \sin(\theta_1 - \theta)$ at Q, in the direction of the tangent at Q.

It follows that if distances along the arc of the girder be represented by s, the resultant slope at Q, assuming the slope at the support to be zero, is given by

$$\left(\frac{dy}{ds}\right)_{\theta_1} = \int_0^{\text{arc } \theta_1} \frac{M_\theta}{EI_\theta} \cos(\theta_1 - \theta)\, ds + \int_0^{\text{arc } \theta_1} \frac{T_\theta}{CJ_\theta} \sin(\theta_1 - \theta)\, ds$$

Here I_θ and J_θ are the moments of inertia of the section at θ, about the axes of bending and of twisting.

Where the beam is of uniform section, this becomes

$$\left(\frac{dy}{ds}\right)_{\theta_1} = \frac{1}{EI} \int_0^{\text{arc } \theta_1} M_\theta \cos(\theta_1 - \theta)\, ds + \frac{1}{CJ} \int_0^{\text{arc } \theta_1} T_\theta \sin(\theta_1 - \theta)\, ds;$$

or, since, if r is the radius of the arc,

$$ds = r d\theta; \quad \frac{dy}{ds} = \frac{1}{r} \cdot \frac{dy}{d\theta};$$

$$\therefore \left(\frac{dy}{d\theta}\right)_{\theta_1} = \frac{r^2}{EI} \int_0^{\theta_1} M_\theta \cos(\theta_1 - \theta)\, d\theta + \frac{r^2}{CJ} \int_0^{\theta_1} T_\theta \sin(\theta_1 - \theta)\, d\theta.$$

(13) Circular-Arc Cantilever with Load W at Free End.

Let a (Fig. 9) be the angle subtended by the arc.

Then,

$$M_\theta = W \times CR = Wr \sin(a - \theta),$$
$$T_\theta = W \times RP = Wr\{1 - \cos(a - \theta)\};$$

$$\therefore \left(\frac{dy}{d\theta}\right)_{\theta_1} = \frac{Wr^3}{EI}\int_0^{\theta_1} \sin(a - \theta)\cos(\theta_1 - \theta)\,d\theta + \frac{Wr^3}{CJ}\int_0^{\theta_1}\{1 - \cos(a - \theta)\}\sin(\theta_1 - \theta)\,d\theta$$

On integrating[1] and simplifying, this becomes

$$\left(\frac{dy}{d\theta}\right)_{\theta_1} = \frac{Wr^3}{2EI}\left[\theta_1 \sin(a - \theta_1) + \sin\theta_1 \sin a\right] +$$
$$\frac{Wr^3}{2CJ}\left[2(1 - \cos\theta_1) + \theta_1 \sin(a - \theta_1) - \sin\theta_1 \sin a\right] \quad . \quad . \quad (1)$$

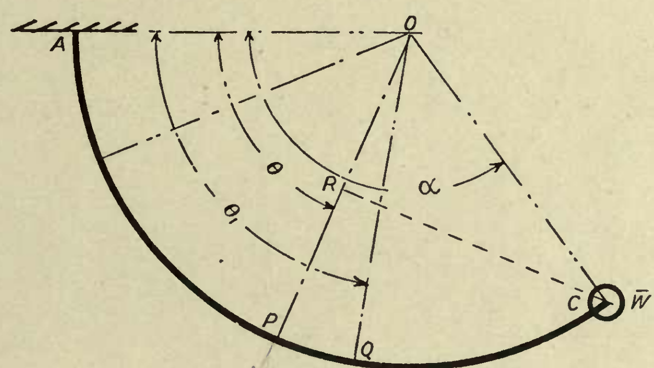


FIG. 9.

As θ_1 is any angle between o and a, on writing $\theta_1 = \theta$ in this expression and integrating between the limits θ_1 and o, we get the deflection at θ_1.

$$\therefore y_{(\theta_1)} = \frac{Wr^3}{2EI}\int_0^{\theta_1}\{\theta \sin(a - \theta) + \sin\theta \sin a\}\,d\theta +$$
$$\frac{Wr^3}{2CJ}\int_0^{\theta_1}\{2(1 - \cos\theta) + \theta \sin(a - \theta) - \sin\theta \sin a\}\,d\theta$$
$$= \frac{Wr^3}{2EI}\left[\theta_1 \cos(a - \theta_1) - \cos a \sin\theta_1\right] +$$
$$\frac{Wr^3}{2CJ}\left[\begin{array}{l}2(\theta_1 - \sin\theta_1) + \theta_1 \cos(a - \theta_1) + \\ \qquad \sin(a - \theta_1) + \sin a(\cos\theta_1 - 2)\end{array}\right] \quad . \quad . \quad (2)$$

At the free end $\theta_1 = a$, and we have

$$y_w = \frac{Wr^3}{2EI}\left[a - \cos a \sin a\right] + \frac{Wr^3}{2CJ}\left[3a - 4\sin a + \sin a \cos a\right] \quad . \quad (3)$$

As a check on the validity of the reasoning leading to these results the deflection

[1] For convenience in integrating this and other expressions occurring in the course of this investigation, a list of the necessary integrals is given in Appendix A.

at the weight may be calculated by equating the resilience of the beam to the work done during deflection. Taking, for convenience, the origin at the free end (Fig. 10),

$$M = Wr\sin\theta\,;\ T_\theta = Wr(1-\cos\theta)\,;$$

and, if l be the length of the beam, the resilience is given by—

$$\frac{1}{2EI}\int_0^l M_\theta^2 ds + \frac{1}{2CJ}\int_0^l T_\theta^2 ds = \frac{W^2r^3}{2EI}\int_0^a \sin^2\theta d\theta + \frac{W^2r^3}{2CJ}\int_0^a (1-2\cos\theta+\cos^2\theta)\,d\theta$$

$$= \frac{W^2r^3}{4}\left[\frac{a-\cos a\sin a}{EI} + \frac{3a-4\sin a+\sin a\cos a}{CJ}\right],$$

and, since this $= \frac{W}{2} \times$ deflection at weight,

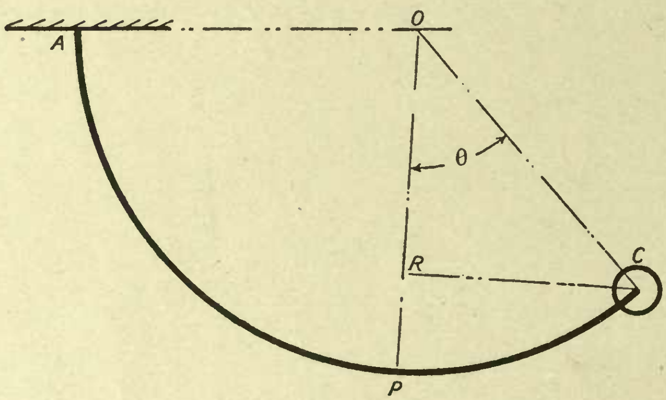


Fig. 10.

$$\therefore y_w = \frac{Wr^3}{2}\left[\frac{a-\cos a\sin a}{EI} + \frac{3a-4\sin a+\sin a\cos a}{CJ}\right]$$

which is identical with equation (3).

E.g., $a = \frac{\pi}{2} = 90°$,

$$y_w = \frac{Wr^3}{2}\left[\frac{\pi}{2EI} + \frac{1{\cdot}5\pi-4}{CJ}\right] = Wr^3\left[\frac{{\cdot}7854}{EI} + \frac{{\cdot}3562}{CJ}\right].$$

If $a = \frac{3\pi}{4} = 135°$,

$$y_w = \frac{Wr^3}{2}\left[\frac{3\pi+2}{4EI} + \frac{9\pi - \frac{16}{\sqrt{2}} - 2}{4CJ}\right] = Wr^3\left[\frac{1{\cdot}4281}{EI} + \frac{1{\cdot}8716}{CJ}\right].$$

(14) Circular-Arc Cantilever with Uniform Loading—w Lbs. per Unit Length.

Taking the origin at the free end, we have, at any point θ (Fig. 11)

$$M_\theta = \int_0^\theta wr^2\sin\phi d\phi = wr^2(1-\cos\theta),$$

$$T_\theta = \int_0^\theta wr^2(1-\cos\phi)\,d\phi = wr^2(\theta-\sin\theta),$$

as the moments produced at P by the loading on that portion of the beam between P and the free end.

$$\therefore \left(\frac{dy}{d\theta}\right)_{\theta_1} = \frac{wr^4}{EI}\int_{\theta_1}^{a}(1-\cos\theta)\cos(\theta-\theta_1)\,d\theta + \frac{wr^4}{CJ}\int_{\theta_1}^{a}(\theta-\sin\theta)\sin(\theta-\theta_1)d\theta,$$

where a is the total angle subtended by the beam. Integrating this expression and simplifying gives

$$\left(\frac{dy}{d\theta}\right)_{\theta_1} = \left[\begin{array}{l} \dfrac{wr^4}{EI}\left[\sin(a-\theta_1)-\cos\theta_1\left\{\dfrac{a-\theta_1}{2}+\dfrac{\sin 2a-\sin 2\theta_1}{4}\right\}-\dfrac{\sin\theta_1\sin^2 a-\sin^3\theta_1}{2}\right] \\ +\dfrac{wr^4}{CJ}\left[\begin{array}{l}\sin(a-\theta_1)-a\cos(a-\theta_1)+\theta_1-\cos\theta_1\left\{\dfrac{a-\theta_1}{2}-\dfrac{\sin 2a-\sin 2\theta_1}{4}\right\} \\ \qquad +\dfrac{\sin\theta_1\sin^2 a-\sin^3\theta_1}{2}\end{array}\right]\end{array}\right].$$

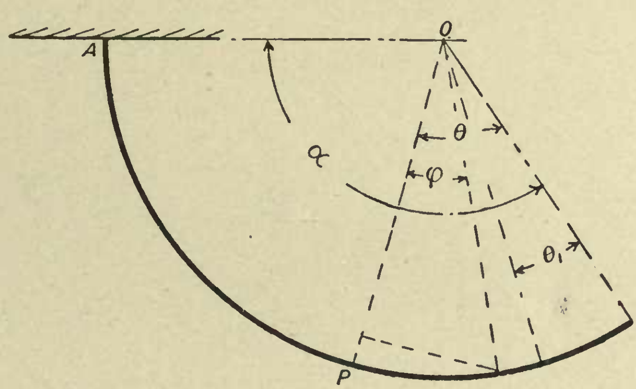


FIG. 11.

$$\therefore (y)_{\theta_1} = \left[\begin{array}{l} \dfrac{wr^4}{EI}\displaystyle\int_{\theta_1}^{a}\left[\sin(a-\theta)-\cos\theta\left\{\dfrac{a-\theta}{2}+\dfrac{\sin 2a-\sin 2\theta}{4}\right\}-\dfrac{\sin\theta\sin^2 a-\sin^3\theta}{2}\right]d\theta \\ +\dfrac{wr^4}{CJ}\displaystyle\int_{\theta_1}^{a}\left[\begin{array}{l}\sin(a-\theta)-a\cos(a-\theta)+\theta-\cos\theta\left\{\dfrac{a-\theta}{2}-\dfrac{\sin 2a-\sin 2\theta}{4}\right\} \\ \qquad +\dfrac{\sin\theta\sin^2 a-\sin^3\theta}{2}\end{array}\right]d\theta\end{array}\right]$$

$$= \left[\begin{array}{l} \dfrac{wr^4}{2EI}\left[\begin{array}{l}2-2\cos(a-\theta_1)+(a-\theta_1)\sin\theta_1+\cos a-\cos\theta_1+\dfrac{\sin\theta_1\sin 2a}{2} \\ \quad -\frac{1}{3}(\cos^3 a-\cos^3\theta_1)-\cos\theta_1\sin^2 a+\dfrac{a-\theta_1}{2}-\dfrac{\sin 2a-\sin 2\theta_1}{4}\end{array}\right] \\ +\dfrac{wr^4}{2CJ}\left[\begin{array}{l}2-2\cos(a-\theta_1)-2a\sin(a-\theta_1)+a^2-\theta_1^2+(a-\theta_1)\sin\theta_1 \\ \quad +\cos a-\cos\theta_1-\dfrac{\sin\theta_1\sin 2a}{2}+\frac{1}{3}(\cos^3 a-\cos^3\theta_1) \\ \quad +\cos\theta_1\sin^2 a-\dfrac{a-\theta_1}{2}+\dfrac{\sin 2a-\sin 2\theta_1}{4}\end{array}\right]\end{array}\right].$$

At the free end $\theta_1 = 0$, and the deflection becomes

$$y_o = \left| \begin{array}{l} \dfrac{wr^4}{2EI}\left[1 - \cos a - \tfrac{1}{3}(\cos^3 a - 1) - \sin^2 a + \dfrac{a}{2} + \dfrac{\sin 2a}{4}\right] \\ \qquad + \dfrac{wr^4}{2CJ}\left[1 - \cos a - 2a \sin a + a^2 + \tfrac{1}{3}(\cos^3 a - 1) + \sin^2 a - \dfrac{a}{2} + \dfrac{\sin 2a}{4}\right] \end{array} \right| ;$$

e.g., if $a = \frac{\pi}{2}$, the deflection at the free end becomes

$$y_o = wr^4 \left[\frac{\cdot 5594}{EI} + \frac{\cdot 1035}{CJ}\right].$$

(15) Circular-Arc Girder, Built in at Two Ends, with Single Load W.

Let the arc subtend an angle $(\pi - 2\phi)$, and let O (Fig. 12) be its centre; AB the line of supports; $AOW = a$; $BOW = \beta$; R_a and R_b the vertical reactions at A

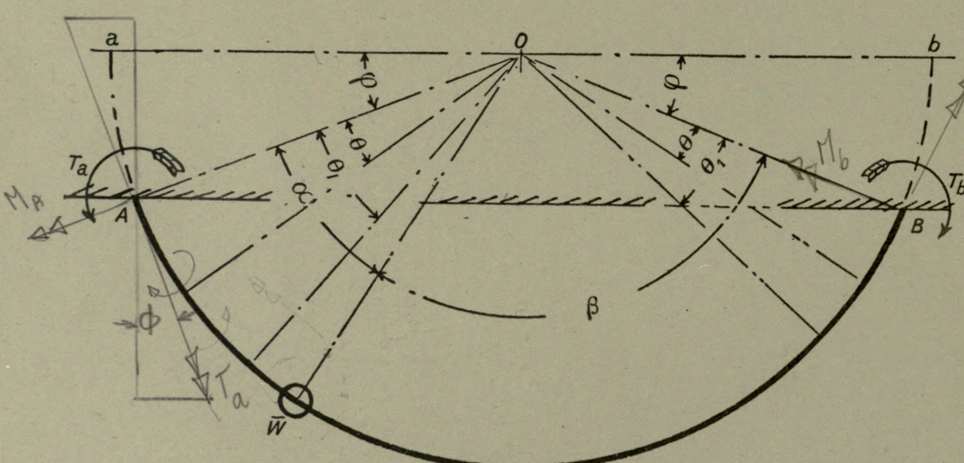


FIG. 12.

and B; M_a and M_b, T_a and T_b the bending and twisting moments at the supports A and B, the axes of these moments being respectively parallel to and perpendicular to OA and OB.

The bending and twisting moments at any point between A and W, distant θ from OA, are now given by

$$M_\theta = M_a \cos\theta - R_a r \sin\theta + T_a \sin\theta. \quad . \quad . \quad . \quad . \quad (4)$$
$$T_\theta = T_a \cos\theta + R_a r(1 - \cos\theta) - M_a \sin\theta \quad . \quad . \quad . \quad (5)$$

while the moments at a point between B and W, distant θ from OB, are given by similar expressions, with suffix b taking the place of suffix a.

Before these moments can be calculated for any particular case, the values of the six unknowns, M_a, M_b, T_a, T_b, R_a, R_b, are to be ascertained; and for this, six relationships between these unknowns are necessary.

Taking moments about B, of the forces and couples acting in a vertical plane we have, for equilibrium,

$$R_a(2r\cos\phi) - T_a\cos\phi - M_a\sin\phi - Wr\{\cos\phi + \cos(a + \phi)\} + T_b\cos\phi + M_b\sin\phi = 0$$

$$\therefore R_a = \frac{W}{2}\left\{1 + \frac{\cos(a+\phi)}{\cos\phi}\right\} + \frac{T_a - T_b}{2r} + \frac{M_a - M_b}{2r}\tan\phi \quad . \quad . \quad (6)$$

$$\text{while } R_b = \frac{W}{2}\left\{1 - \frac{\cos(a+\phi)}{\cos\phi}\right\} + \frac{T_b - T_a}{2r} + \frac{M_b - M_a}{2r}\tan\phi \quad . \quad . \quad (7)$$

$$= W - R_a \quad . \quad . \quad . \quad . \quad (7\text{A})$$

Again, taking moments about the line AB,

$$(M_a + M_b)\cos\phi - (T_a + T_b)\sin\phi = Wr\{\sin(a+\phi) - \sin\phi\} \quad . \quad . \quad (8)$$

while, equating the torques at the weight, as obtained by working from both ends of the girder,

$$T_a \cos a + R_a r(1 - \cos a) - M_a \sin a = -T_b \cos\beta - R_b r(1 - \cos\beta) + M_b \sin\beta \quad . \quad . \quad (9)$$

The other two necessary relationships are obtained by expressing the fact that both slope and deflection at the weight are the same, whether the latter is considered as being at one extremity of the arc AW, or of the arc BW.

The slope at any point θ_1 between A and W is given by

$$\left(\frac{dy}{d\theta}\right)_{\theta_1} = \frac{r^2}{EI}\int_0^{\theta_1} M_\theta \cos(\theta_1 - \theta)\,d\theta + \frac{r^2}{CJ}\int_0^{\theta_1} T_\theta \sin(\theta_1 - \theta)\,d\theta,$$

and, on substituting for M_θ and T_θ from (4) and (5) and integrating,

$$\left(\frac{dy}{d\theta}\right)_{\theta_1} = \left[\begin{aligned}&\frac{r^2}{2EI}\Big[M_a\{\theta_1\cos\theta_1 + \sin\theta_1\} - (R_a r - T_a)\,\theta_1 \sin\theta_1\Big]\\ &+\frac{r^2}{2CJ}\Big[(T_a - R_a r)\,\theta_1\sin\theta_1 + 2R_a r(1-\cos\theta_1) - M_a\{\sin\theta_1 - \theta_1\cos\theta_1\}\Big]\end{aligned}\right].$$

Similarly at any point between B and W, distant θ_1 from OB,

$$\left(\frac{dy}{d\theta}\right)_{\theta_1} = \left[\begin{aligned}&\frac{r^2}{2EI}\Big[M_b\{\theta_1\cos\theta_1 + \sin\theta_1\} - (R_b r - T_b)\,\theta_1 \sin\theta_1\Big]\\ &+\frac{r^2}{2CJ}\Big[(T_b - R_b r)\,\theta_1\sin\theta_1 + 2R_b r(1-\cos\theta_1) - M_b\{\sin\theta_1 - \theta_1\cos\theta_1\}\Big]\end{aligned}\right].$$

The slope at the weight is obtained by writing $\theta_1 = a$ in the first, or $\theta_1 = \beta$ in the second of these expressions, and is thus given by

$$\left(\frac{dy}{d\theta}\right)_w = \left[\begin{aligned}&\frac{r^2}{2EI}\Big[M_a\{a\cos a + \sin a\} - (R_a r - T_a)\,a \sin a\Big]\\ &+\frac{r^2}{2CJ}\Big[(T_a - R_a r)\,a\sin a + 2R_a r(1-\cos a) - M_a\{\sin a - a\cos a\}\Big]\end{aligned}\right] \quad (10)$$

or by

$$\left(\frac{dy}{d\theta}\right)_w = \left[\begin{aligned}&\frac{r^2}{2EI}\Big[M_b\{\beta\cos\beta + \sin\beta\} - (R_b r - T_b)\,\beta \sin\beta\Big]\\ &+\frac{r^2}{2CJ}\Big[(T_b - R_b r)\beta\sin\beta + 2R_b r(1-\cos\beta) - M_b\{\sin\beta - \beta\cos\beta\}\Big]\end{aligned}\right] \quad (11)$$

according as the point W is considered as forming part of span AW or of span BW.

On equating these two expressions, with the sign of the second changed since θ

is measured in opposite directions in the two sections, a further relationship between the unknowns is obtained.

Deflections.—Assuming the supports to be at the same level, integrating $\frac{dy}{d\theta}$ to obtain the deflection gives (between A and W)

$$y_{\theta_1} = \left[\begin{array}{l} \frac{r^2}{2EI}\int_0^{\theta_1}\Big[M_a\{\theta\cos\theta + \sin\theta\} - (R_a r - T_a)\,\theta\sin\theta\Big]\,d\theta \\ + \frac{r^2}{2CJ}\int_0^{\theta_1}\Big[(T_a - R_a r)\,\theta\sin\theta + 2R_a r(1-\cos\theta) - M_a\{\sin\theta - \theta\cos\theta\}\Big]\,d\theta \end{array} \right]$$

as the deflection at a point distant θ_1 from A. On integrating and simplifying, this becomes

$$y_{\theta_1} = \left[\begin{array}{l} \frac{r^2}{2EI}\Big[M_a\theta_1\sin\theta_1 - (R_a r - T_a)(\sin\theta_1 - \theta_1\cos\theta_1)\Big] \\ + \frac{r^2}{2CJ}\left[\begin{array}{l}(T_a - R_a r)(\sin\theta_1 - \theta_1\cos\theta_1) + 2R_a r(\theta_1 - \sin\theta_1) \\ \qquad + M_a(\theta_1\sin\theta_1 + 2\cos\theta_1 - 2)\end{array}\right] \end{array} \right] \quad . \quad . \quad (12)$$

Similarly for a point between B and W, distant θ_1 from B,

$$y_{\theta_1} = \left[\begin{array}{l} \frac{r^2}{2EI}\Big[M_b\theta_1\sin\theta_1 - (R_b r - T_b)(\sin\theta_1 - \theta_1\cos\theta_1)\Big] \\ + \frac{r^2}{2CJ}\left[\begin{array}{l}(T_b - R_b r)(\sin\theta_1 - \theta_1\cos\theta_1) + 2R_b r(\theta_1 - \sin\theta_1) \\ \qquad + M_b(\theta_1\sin\theta_1 + 2\cos\theta_1 - 2)\end{array}\right] \end{array} \right] \quad . \quad (13)$$

At the weight, θ_1 becomes a in (12) and β in (13) and these expressions give (A to W)

$$y_w = \left[\begin{array}{l} \frac{r^2}{2EI}\Big[M_a a\sin a - (R_a r - T_a)(\sin a - a\cos a)\Big] \\ + \frac{r^2}{2CJ}\left[\begin{array}{l}(T_a - R_a r)(\sin a - a\cos a) + 2R_a r(a - \sin a) \\ \qquad + M_a(a\sin a + 2\cos a - 2)\end{array}\right] \end{array} \right] \quad . \quad . \quad (14)$$

and (B to W)

$$y_w = \left[\begin{array}{l} \frac{r^2}{2EI}\Big[M_b\beta\sin\beta - (R_b r - T_b)(\sin\beta - \beta\cos\beta)\Big] \\ + \frac{r^2}{2CJ}\left[\begin{array}{l}(T_b - R_b r)(\sin\beta - \beta\cos\beta) + 2R_b r(\beta - \sin\beta) \\ \qquad + M_b(\beta\sin\beta + 2\cos\beta - 2)\end{array}\right] \end{array} \right] \quad . \quad . \quad (15)$$

On equating the identities (14) and (15) the final relationship is obtained, and from the six equations (6), (7), (8), (9), (10 = − 11), (14 = 15), the six unknown fixing moments and reactions may be determined in any particular case. These moments depend somewhat on the relative values of EI and of CJ, except where the load is in the middle of the span. An increase in the ratio $EI : CJ$ is accompanied by an increase in all the fixing moments. The effect on the values of M_a, of M_b, and of the end reactions, produced by a large variation in this ratio, is very small, especially when the angle a is large. The effect on the end torques is more pronounced, particularly for small values of a.

In order to facilitate the application of the results of this analysis, and to make

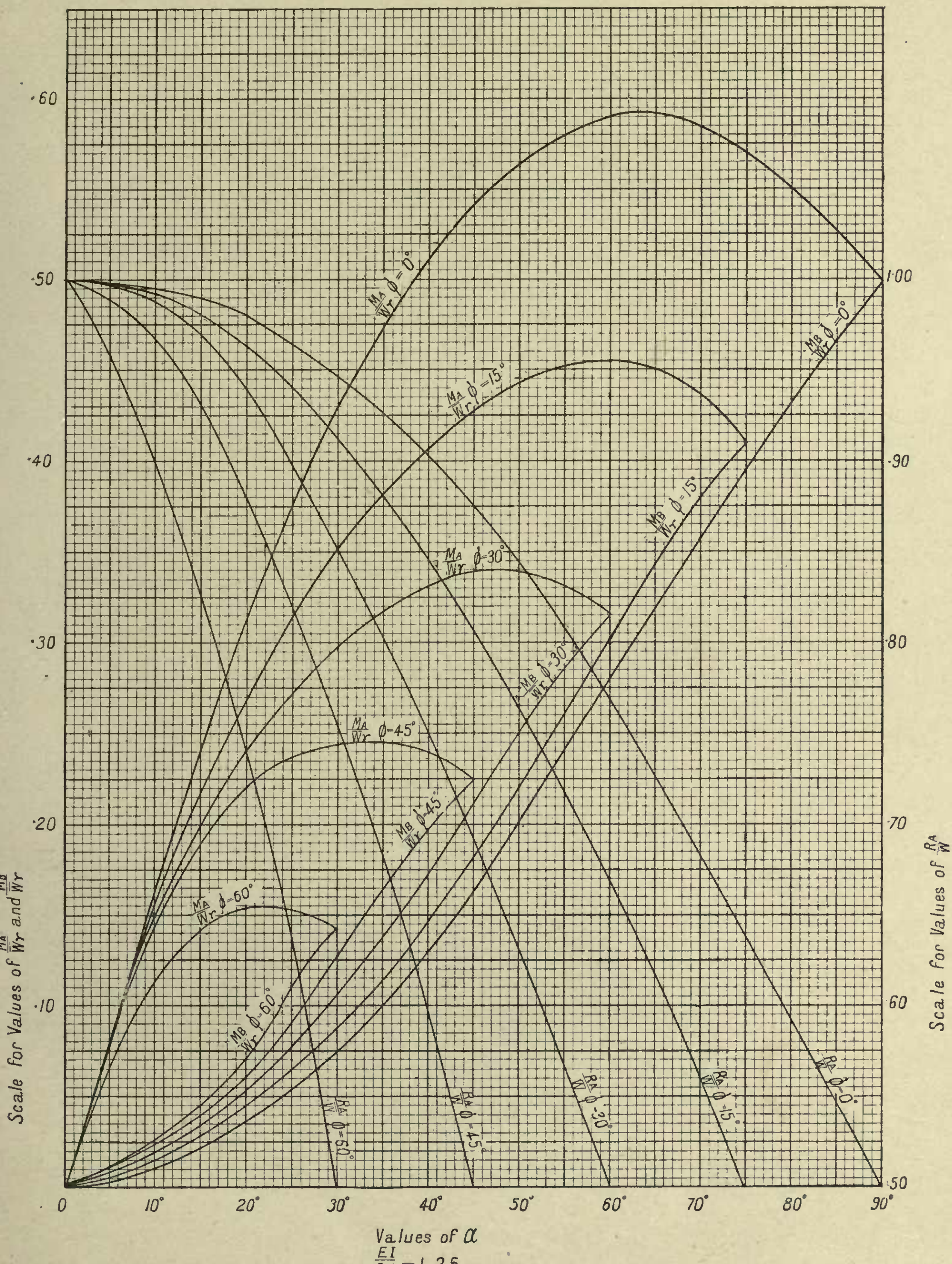


FIG. 13.—Values of M_A, M_B and R_A for a bow girder built in at both ends, subtending an angle $180° - 2\phi$, and carrying a load W at a point distant α from end A.

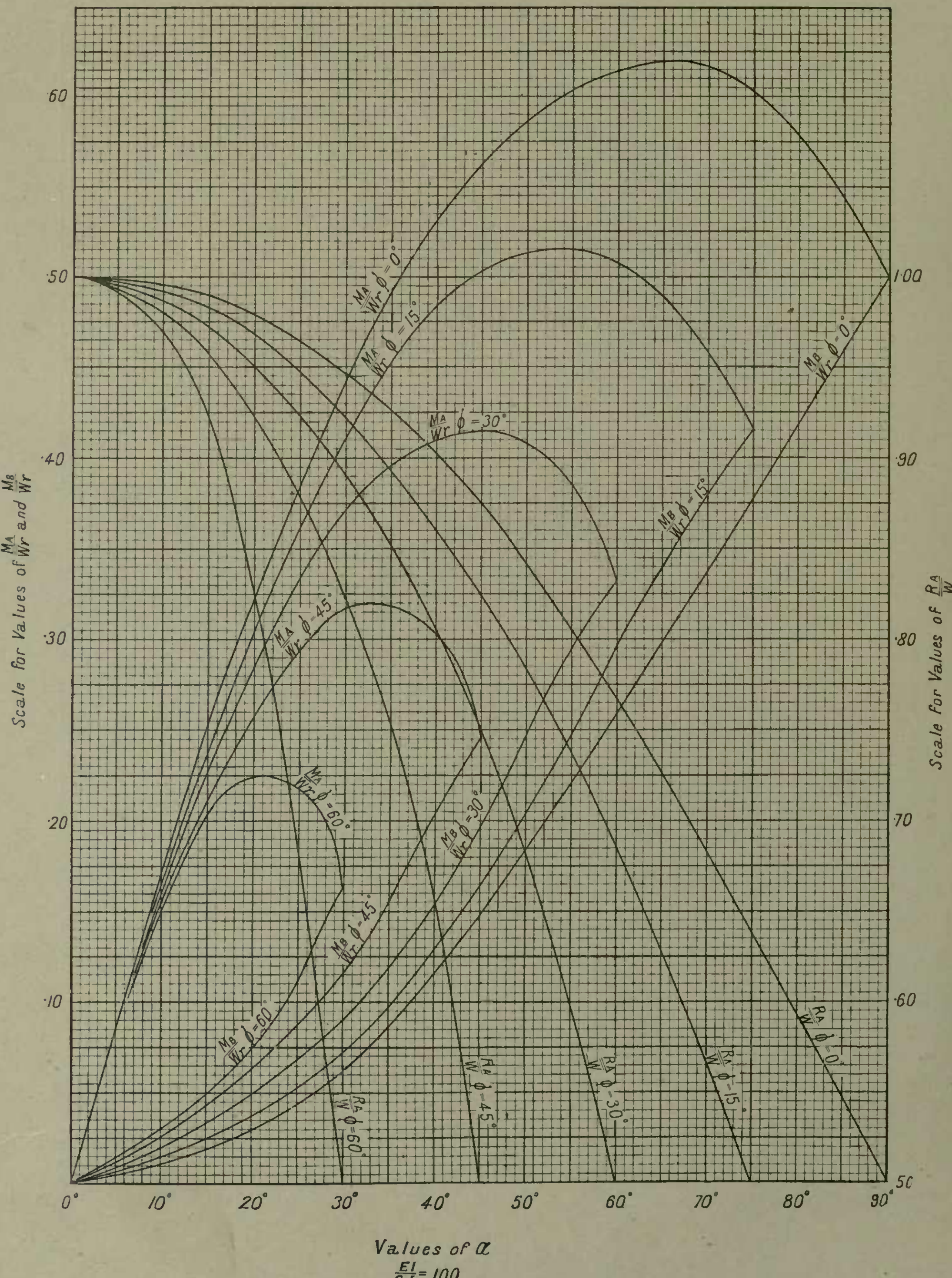


FIG. 14.—Values of M_A, M_B and R_A for a bow girder built in at both ends, subtending an arc $(180° - 2\phi)$, and with a single weight W distant α from the end A.

it more useful in practice, the foregoing equations have been solved for a series of values of a and of ϕ, and the values of the end moments and reactions have been calculated for a series of values of $EI : CJ$. Owing to the comparatively small effect of this ratio on the end bending moments and reactions, values of these have

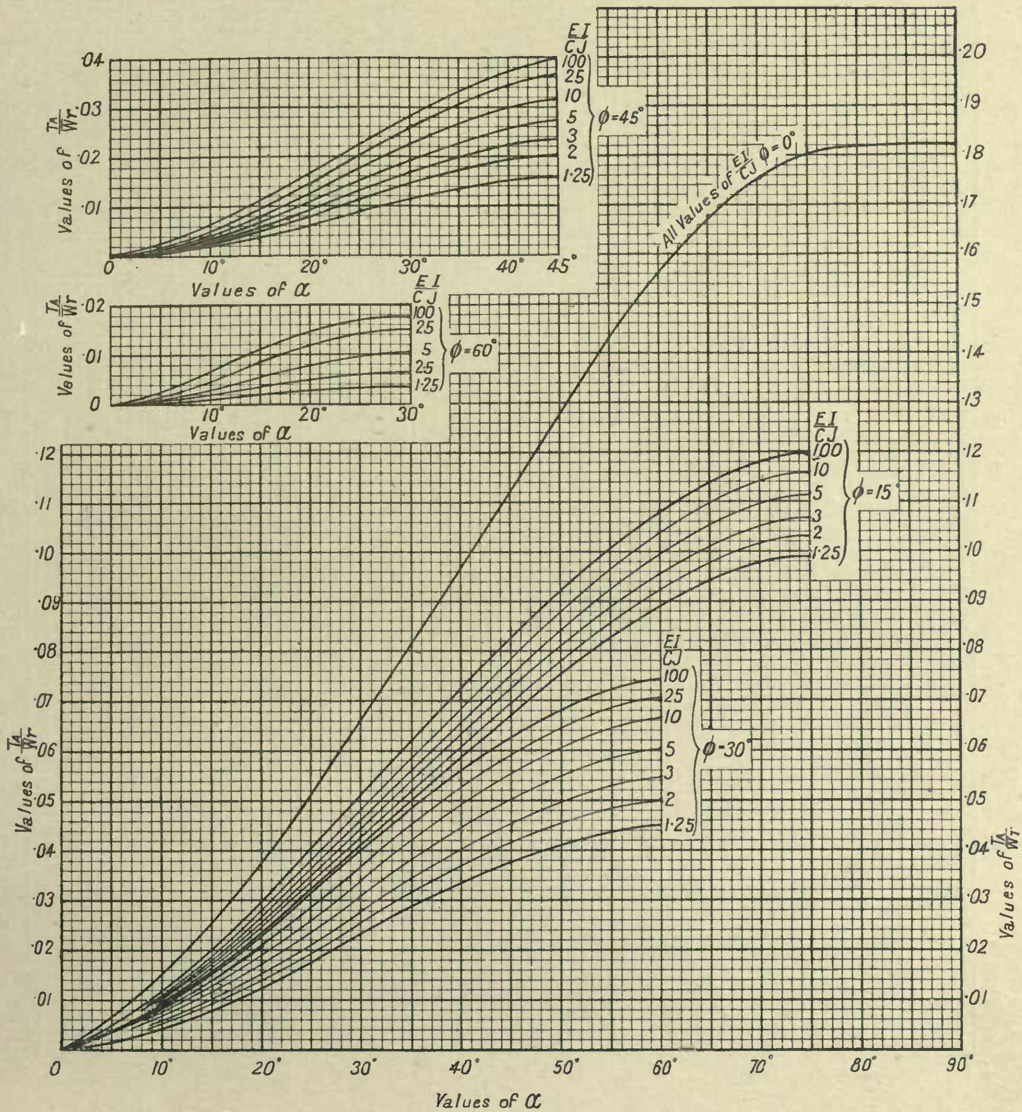


FIG. 15.—Values of $T_A \div \overline{W}r$ for a bow girder built in at both ends, subtending an angle $180° - 2\phi$, and carrying a single load $\overline{W}$ at a distance a from the end A.

only been calculated for the extreme cases likely to be found in practice—viz., for $EI : CJ = 1·25$ (its approximate value is a solid circular section) and for $EI : CJ = 100$. These results are plotted as curves in Figs. 13 and 14, and for intermediate values of the ratio the moments and reactions may be obtained with a sufficient degree of accuracy by interpolation from these curves.

Owing to the relatively greater variation in the end torques, values of these for a series of values of $EI : CJ$ have been calculated, and are plotted in Figs. 15 and 16. By substitution from these values in equations (4), (5), (12), and (13), the values of

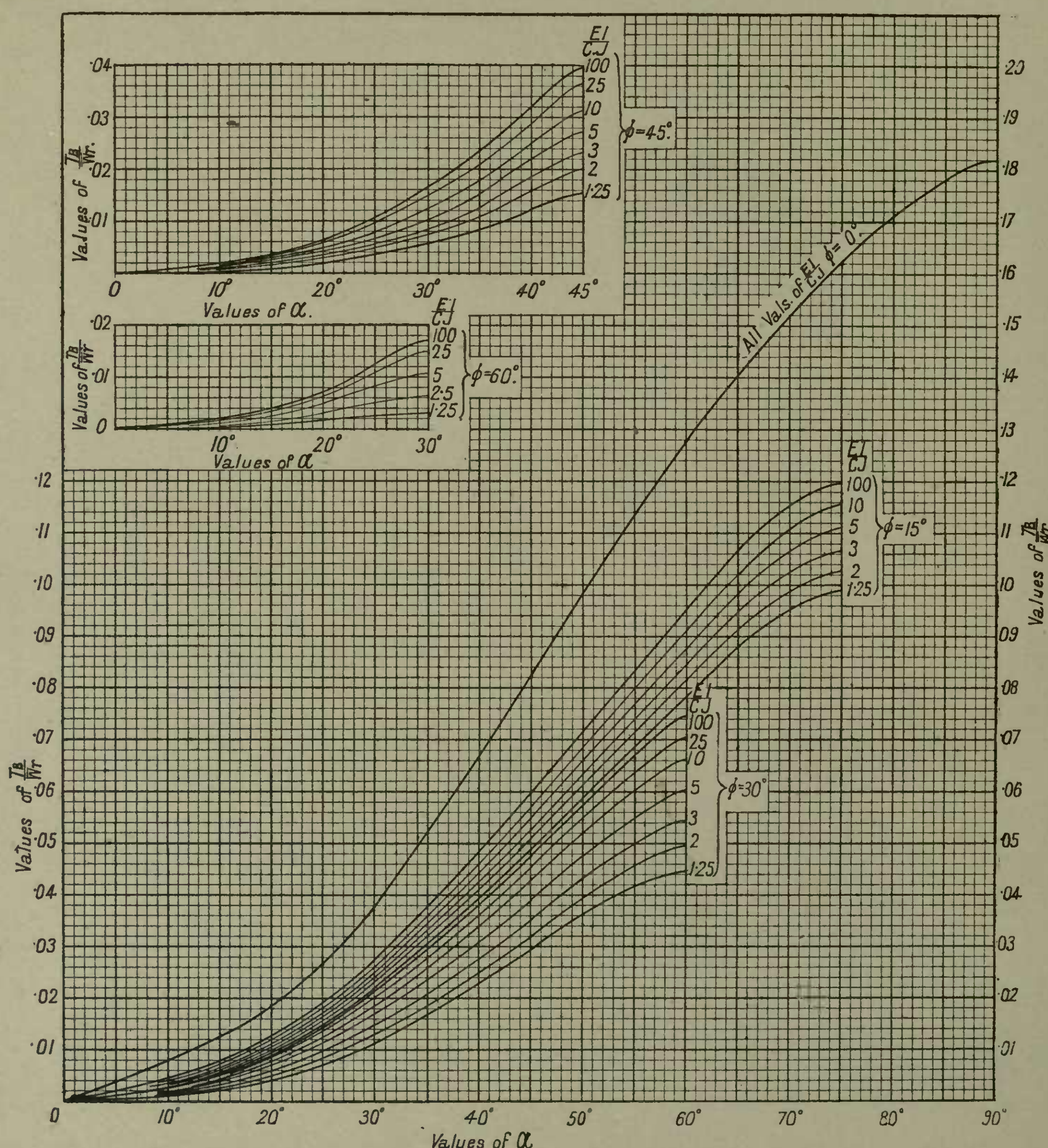


Fig. 16.—Values of $T_B \div Wr$ for a bow girder built in at both ends, subtending an angle $180° - 2\phi$, and with a single load W at a distance a from the end A.

the bending and twisting moments, and of the deflections at any point of the girder, may be obtained.

Special Cases.

Semicircular Bow-Girder with Single Load W in any Position.—Here $a + \beta = 180°$; $\phi = 0$; and the foregoing equations simplify. The values of the various constants for such a girder have been calculated for the case where $EI = 1{\cdot}25\,CJ$, and are given in Table I.

TABLE I.

a	0° 180°	15° 165	30° 150	45° 135	60° 120	75° 105	90°
$\frac{R_a}{W}$	1·00	·990	·940	·870	·764	·640	·500
$\frac{R_b}{W}$	0·0	·0104	·060	·131	·236	·361	·500
$\frac{M_a}{Wr}$	0·0	·239	·428	·542	·590	·571	·500
$\frac{M_b}{Wr}$	0·0	·0200	·0725	·165	·276	·395	·500
$\frac{T_a}{Wr}$	0·0	·0251	·0662	·115	·155	·181	·182
$\frac{T_b}{Wr}$	0·0	·0118	·0382	·082	·128	·161	·182

In the particular case where $a = 90° = 1{\cdot}5708$ radians (*i.e.*, weight at centre of span) from symmetry

$$\begin{cases} R_a = R_b = {\cdot}5\,W \\ M_a = M_b = {\cdot}5\,Wr \\ T_a = T_b \end{cases}$$

From (10) the value of $\frac{dy}{d\theta}$ at the weight $\left(a = \frac{\pi}{2}\right)$ is given by

$$\frac{r^2}{2EI}\left\{Wr\left(1-\frac{\pi}{2}\right)+\pi T_a\right\} - \frac{r^2}{2CJ}\left\{Wr\left(1-\frac{\pi}{2}-2\right)-\pi T_a\right\} = \left\{\frac{r^2}{2EI}-\frac{r^2}{2CJ}\right\}\left\{Wr\left(1-\frac{\pi}{2}\right)+\pi T_a\right\}.$$

From symmetry this equals zero ;

$$\therefore\ T_a = Wr\left(\frac{\pi-2}{2\pi}\right) = {\cdot}182\,Wr,$$

and in this case both M_a and T_a are independent of the relative values of EI and CJ. The curves of Fig. 17 and 18 show respectively the values of the bending and twisting

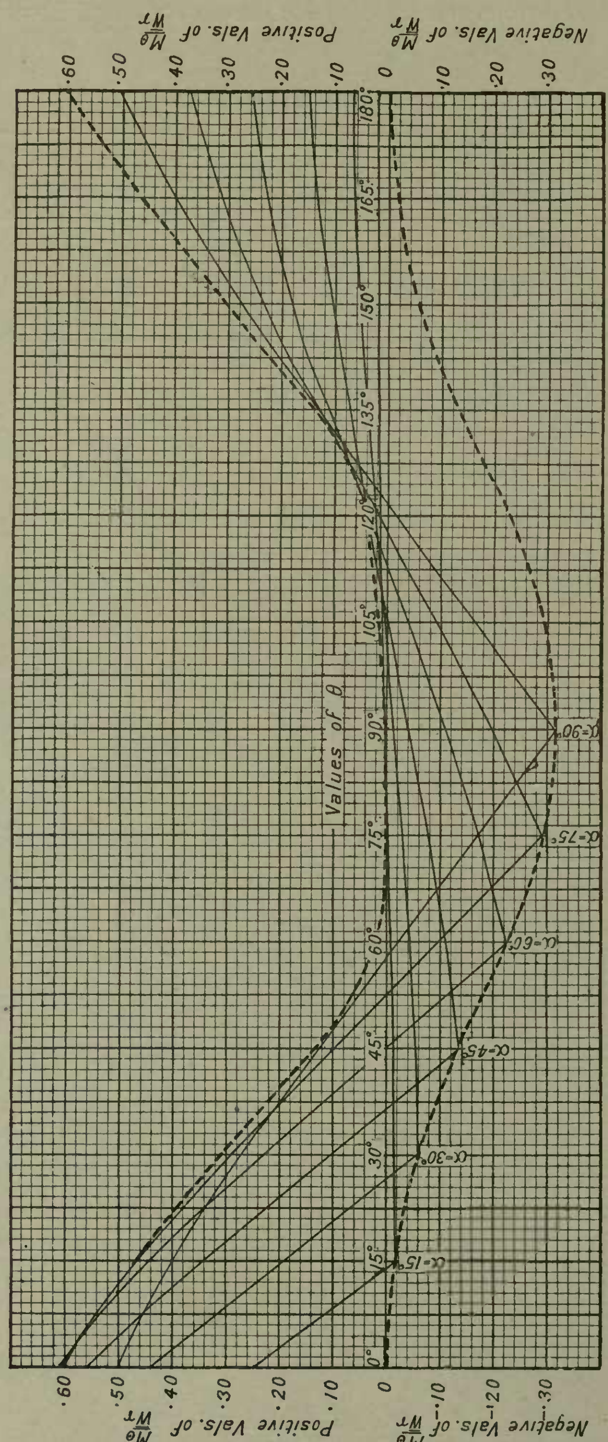


FIG. 17.—Bending-moment diagram for a semicircular-arc girder with concentrated load W in various positions. Bending-moment diagram for rolling load W indicated by dotted lines.

moments at each section of a semicircular girder due to a single load W at any distance a (degrees) from one end, and ordinates of the envelopes to these curves—shown in

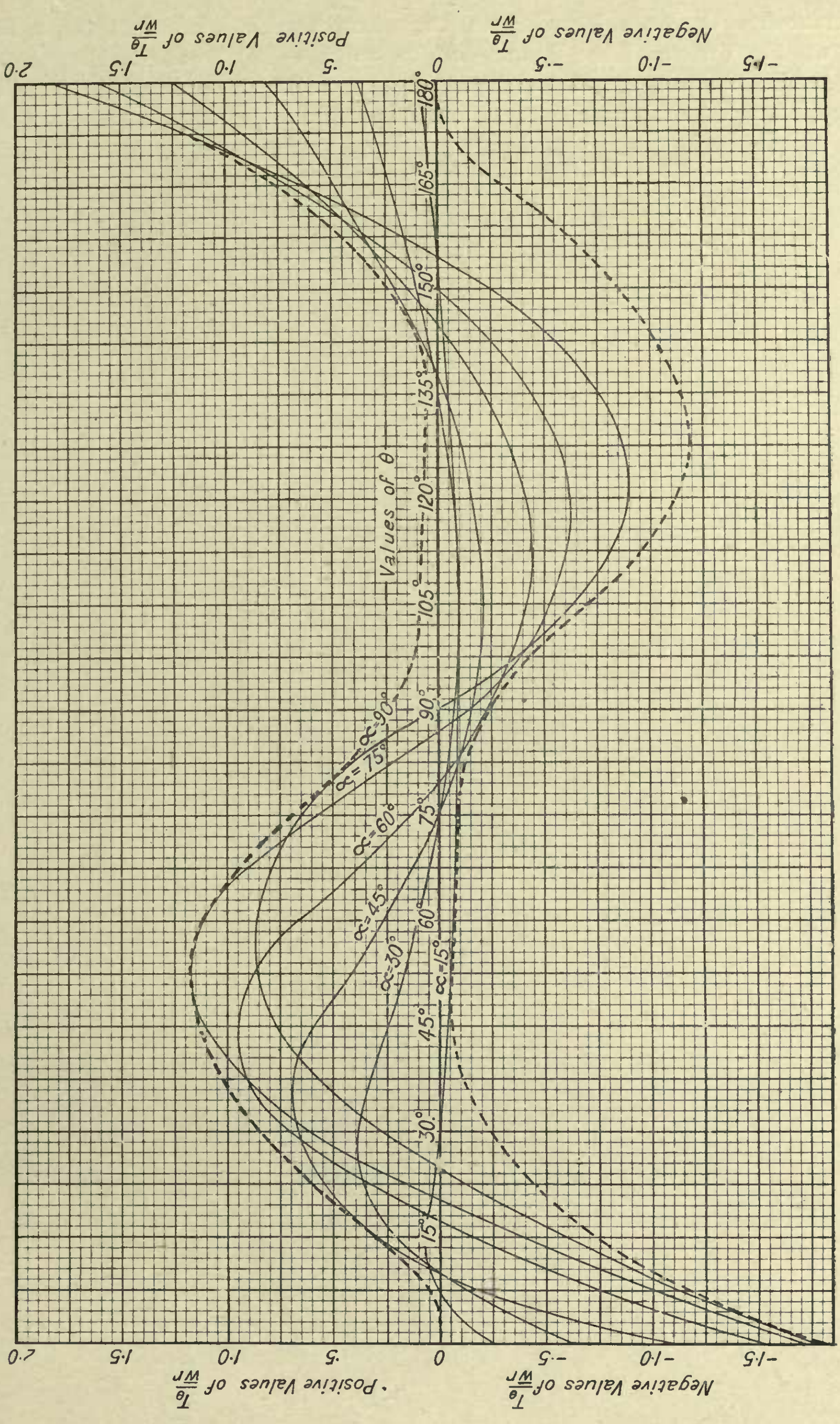


FIG. 18.—Twisting-moment diagram for semicircular-arc girder with concentrated load $\overline{W}$ in various positions. Twisting-moment diagram for rolling load $\overline{W}$ indicated by dotted lines.

dotted lines—give the maximum positive or negative moments produced at any point by a concentrated rolling load of this magnitude.

Circular-Arc Girder, subtending an Angle less than 180°, and carrying a Single Weight at the Centre of the Span.—Let $2a = (\pi - 2\phi)$ be the angle subtended (Fig. 12). The moment of the weight about $AB = Wr\,(1 - \sin\phi)$, and as, from symmetry, $M_a = M_b$; $T_a = T_b$; equation (8) becomes

$$M_a \cos\phi - T_a \sin\phi = \frac{Wr}{2}(1 - \sin\phi)$$

or

$$M_a = \frac{Wr}{2\cos\phi}(1 - \sin\phi) + T_a \tan\phi,$$

also

$$R_a = R_b = \frac{W}{2}$$

On substituting these values of M_a and R_a, equation (10) becomes

$$\left(\frac{dy}{d\theta}\right)_w = \left|\begin{array}{l} \dfrac{Wr^3}{2EI}\left[\left(\dfrac{1-\sin\phi}{2\cos\phi} + \dfrac{T_a}{Wr}\tan\phi\right)\{a\cos a + \sin a\} - \left(\dfrac{1}{2} - \dfrac{T_a}{Wr}\right) a\cos\phi\right] \\ + \dfrac{Wr^3}{2CJ}\left[\left(\dfrac{T_a}{Wr} - \dfrac{1}{2}\right) a\cos\phi + 1 - \sin\phi - \left(\dfrac{1-\sin\phi}{2\cos\phi} + \dfrac{T_a}{Wr}\tan\phi\right)\{\sin a - a\cos a\}\right] \end{array}\right|.$$

From symmetry this equals zero, and, on substituting for a and ϕ and equating to zero, the value of T_a is obtained. Except in a semicircular girder ($\phi = 0$), this value depends on the ratio of $EI : CJ$. The following values have been calculated for the case in which this ratio equals 1·25.

$\phi°$	0°	15°	30°	45°	60°
$\dfrac{M_a}{Wr}$	·50	·410	·314	·223	·140
$\dfrac{T_a}{Wr}$	·182	·099	·045	·0157	·0032

Knowing M_a and T_a, the deflection at the weight may be obtained by substituting these values in equation (14), p. 20.

(16) Circular-Arc Bow-Girder, Built in at both Ends, with Uniform Loading—w lbs. per Unit Length.

Let $\pi - 2\phi$ be the angle subtended by the arc (Fig. 12). The total load $= wr\,(\pi - 2\phi)$ lbs.

$$\therefore R_a = R_b = wr\left(\frac{\pi}{2} - \phi\right).$$

The centre of gravity of the load is at a distance from the line of supports given by

$$r\left\{\frac{\sin\left(\frac{\pi}{2} - \phi\right)}{\frac{\pi}{2} - \phi} - \sin\phi\right\} = 2r\left\{\frac{\cos\phi - \left(\frac{\pi}{2} - \phi\right)\sin\phi}{\pi - 2\phi}\right\} \quad . \quad . \quad (16)$$

Let M_a, M_b, T_a, T_b, have the same meanings as before. Then, from symmetry, $M_a = M_b$, $T_a = T_b$; and, on taking moments about the line AB

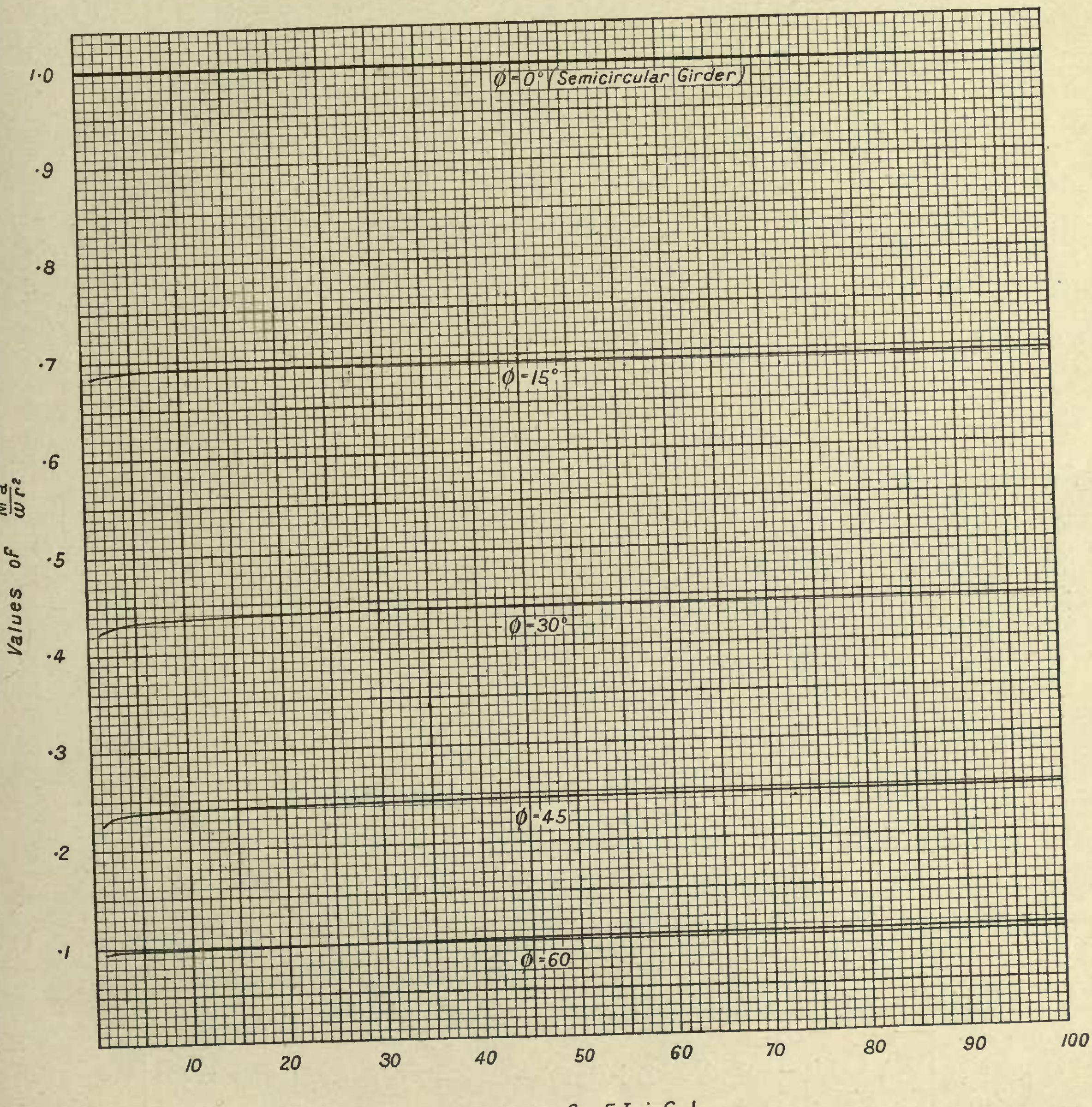


FIG. 19.—Values of M^a for a girder with uniform loading, subtending an angle $180° - 2\phi$.

$$2M_a \cos\phi - 2T_a \sin\phi = 2wr^2 \left\{\cos\phi - \left(\frac{\pi}{2} - \phi\right)\sin\phi\right\}$$

$$\therefore M_a = wr^2 \left\{1 - \left(\frac{\pi}{2} - \phi - \frac{T_a}{wr^2}\right)\tan\phi\right\} \quad . \quad . \quad (17)$$

Taking the origin of ϕ at the supports,

$$M_\theta = M_a \cos\theta - R_a r \sin\theta + T_a \sin\theta + wr^2(1 - \cos\theta)^{1}$$
$$= (M_a - wr^2)\cos\theta - (R_a r - T_a)\sin\theta + wr^2 . \quad . \quad . \quad . \quad (18)$$
$$T_\theta = T_a \cos\theta + R_a r(1 - \cos\theta) - M_a \sin\theta - wr^2(\theta - \sin\theta)^{1}$$
$$= (T_a - R_a r)\cos\theta - (M_a - wr^2)\sin\theta + R_a r - wr^2\theta \quad . \quad . \quad (19)$$

If the girder is fixed horizontally at the ends,

$$\left(\frac{dy}{d\theta}\right)_{\theta_1} = \frac{r^2}{EI}\int_0^{\theta_1} M_\theta \cos(\theta_1 - \theta)d\theta + \frac{r^2}{CJ}\int_0^{\theta_1} T_a \sin(\theta_1 - \theta)d\theta$$

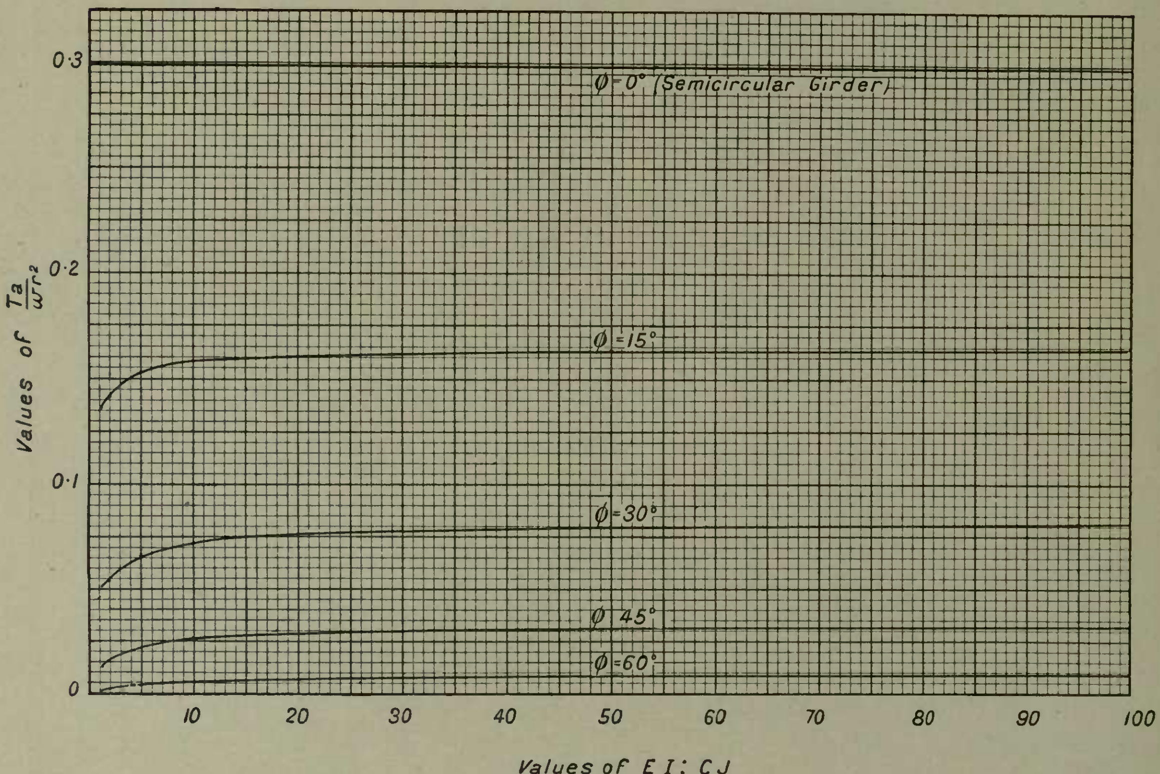


FIG. 20.—Values of T_a for a girder with uniform loading subtending an angle $180° - 2\phi$.

and, on substituting for M_θ and T_θ from (18) and (19), this gives

$$\left(\frac{dy}{d\theta}\right)_{\theta_1} = \left|\begin{array}{l} \dfrac{r^2}{2EI}\left[(M_a - wr^2)\{\theta_1\cos\theta_1 + \sin\theta_1\} - (R_a r - T_a)\theta_1\sin\theta_1 + 2wr^2\sin\theta_1\right] \\ + \dfrac{r^2}{2CJ}\left[\begin{array}{r}(T_a - R_a r)\theta_1\sin\theta_1 - (M_a - wr^2)\{\sin\theta_1 - \theta_1\cos\theta_1\} \\ + 2R_a r(1 - \cos\theta_1) - 2wr^2(\theta_1 - \sin\theta_1)\end{array}\right]\end{array}\right| \quad (20)$$

Writing θ for θ_1 in this expression, and integrating between the limits θ_1 and 0 we have

[1] The last terms, representing the moments due to the portion of the load between A and θ, being obtained as at the beginning of § (4).

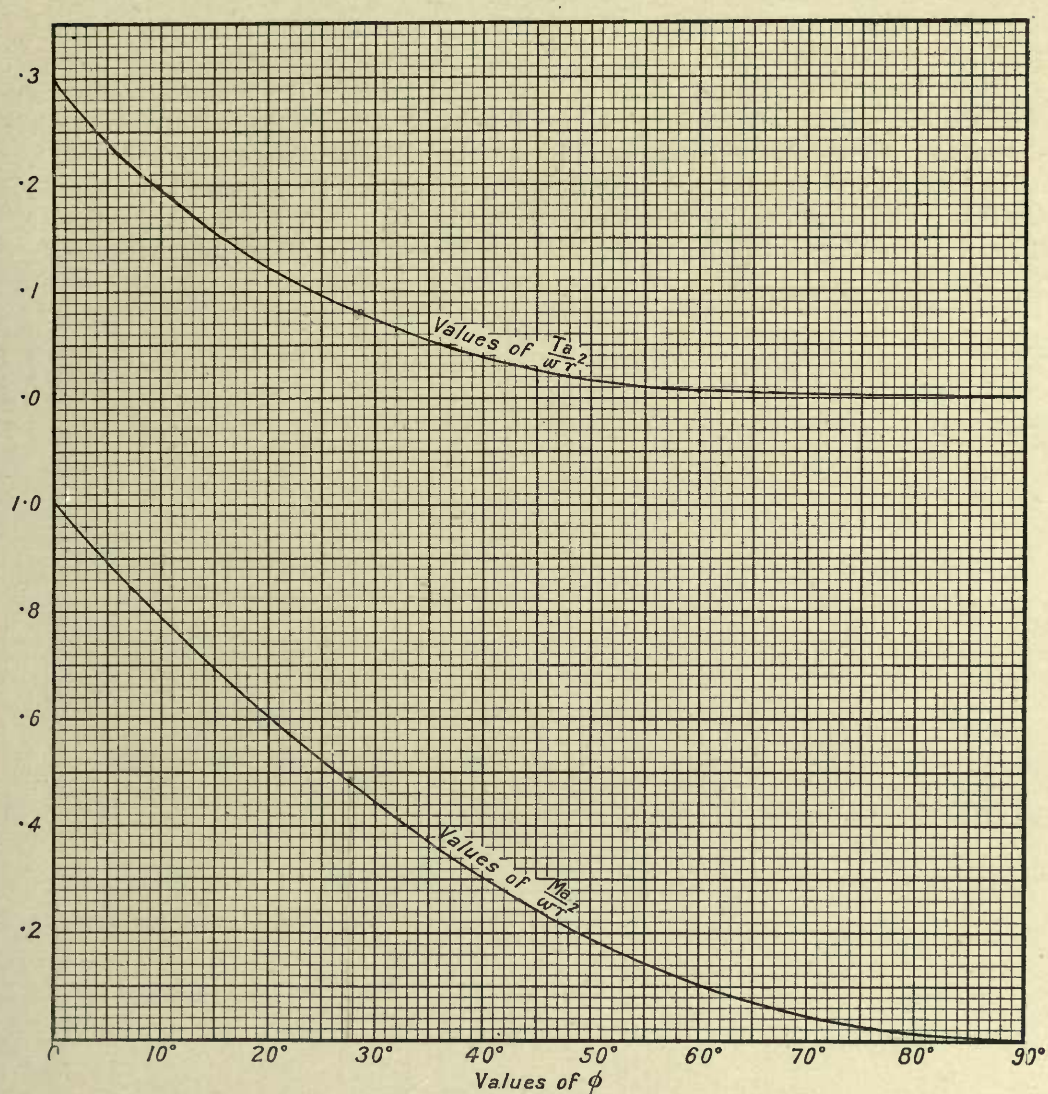


FIG. 21.—Values of M_a and of T_a in a girder with uniform loading, subtending an angle $[180° - 2\phi]$. $EI : CJ = 10$.

$$y_{\theta_1} = \left[\begin{array}{l} \frac{r^2}{2EI}\left[(M_a - wr^2)\theta_1 \sin\theta_1 - (R_a r - T_a)(\sin\theta_1 - \theta_1\cos\theta_1) - 2wr^2(\cos\theta_1 - 1)\right] \\ + \frac{r^2}{2CJ}\left[\begin{array}{l}(T_a - R_a r)(\sin\theta_1 - \theta_1\cos\theta_1) + \{(M_a - wr^2)(\theta_1\sin\theta_1 + 2\cos\theta_1 - 2)\} \\ \qquad + 2R_a r(\theta_1\sin\theta_1) - 2wr^2\left(\frac{\theta_1^2}{2} + \cos\theta_1 - 1\right)\end{array}\right]\end{array}\right] \quad (21)$$

From symmetry $\frac{dy}{d\theta}$ is zero at the centre of the span where $\theta_1 = \frac{\pi}{2} - \phi$, and by substituting this value for θ_1 in (20), and by also substituting for M_a its value $wr^2\left\{1 - \left(\frac{\pi}{2} - \phi - \frac{T_a}{wr^2}\right)\tan\phi\right\}$ and equating to zero, the value of T_a may be obtained, after which the values of M_θ and T_θ for any point on the girder may be obtained by substitution in (18) or (19).

The values of M_a, T_a, M_θ, T_θ have been calculated from the foregoing equations for one-half of a uniformly loaded girder for a series of values of ϕ, and of θ for each value of ϕ. These values depend slightly on the relative value of EI and of CJ, and in Figs. 19 and 20 values of M_a and of T_a are plotted for a series of values of $EI : CJ$. Fig. 21 shows the variation of M_a and of T_a with ϕ, for a given value of $EI : CJ$. The curves of this figure are calculated for the case where this ratio equals

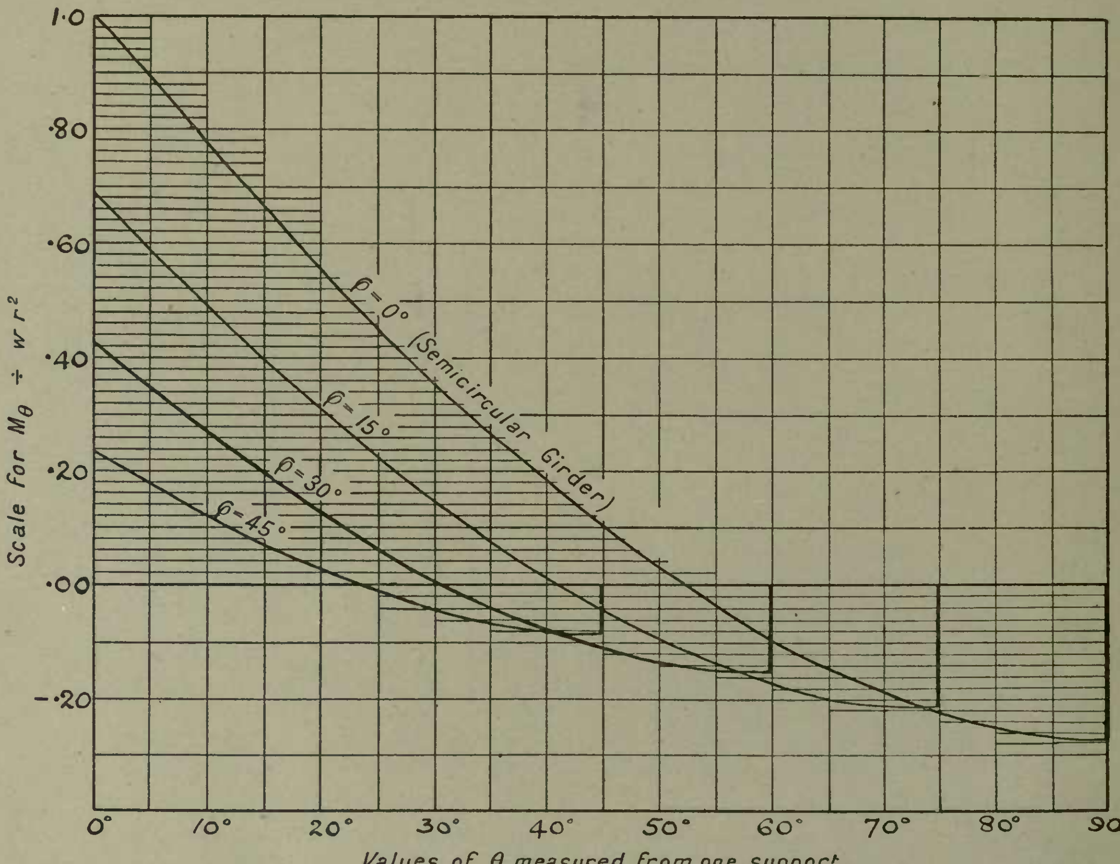


FIG. 22.—Bending-moment diagrams for one-half of a uniformly loaded circular-arc, subtending an angle of $[180° - 2\phi]$.

10, and for purposes of design these values may be taken as sensibly accurate for any likely values of the ratio.

Figs. 22 and 23 show respectively the bending moment M_θ, and the twisting moment T_θ at each point of a uniformly loaded bow girder subtending an arc $180-2\phi$ degrees.

Special Case.

Semicircular Girder with uniform Load.—Here $\phi = 0$, and we have :—

$$M_a = M_b = wr^2 : R_a = R_b = \frac{\pi}{2} \cdot wr :$$

$$\left(\frac{dy}{d\theta}\right)_{\theta_1} = \left[\begin{array}{l} \frac{r^2}{2EI}\left[(T_a - R_a r)\theta_1 \sin\theta_1 + 2wr^2 \sin\theta_1\right] \\ \quad + \frac{r^2}{CJ}\left[(T_a - R_a r)\theta_1 \sin\theta_1 + 2R_a r(1-\cos\theta_1) - 2wr^2(\theta_1 - \sin\theta_1)\right] \end{array}\right] \quad (20')$$

$$y_{\theta_1} = \left[\begin{array}{l} \frac{r^2}{2EI}\left[(T_a - R_a r)(\sin\theta_1 - \theta_1\cos\theta_1) - 2wr^2(\cos\theta_1 - 1)\right] \\ \quad + \frac{r^2}{2CJ}\left[\begin{array}{l}(T_a - R_a r)(\sin\theta_1 - \theta_1\cos\theta_1) + 2R_a r(\theta_1 - \sin\theta_1) \\ \qquad - 2wr^2\left(\frac{\theta_1^2}{2} + \cos\theta_1 - 1\right)\end{array}\right] \end{array}\right] \quad . \quad (21')$$

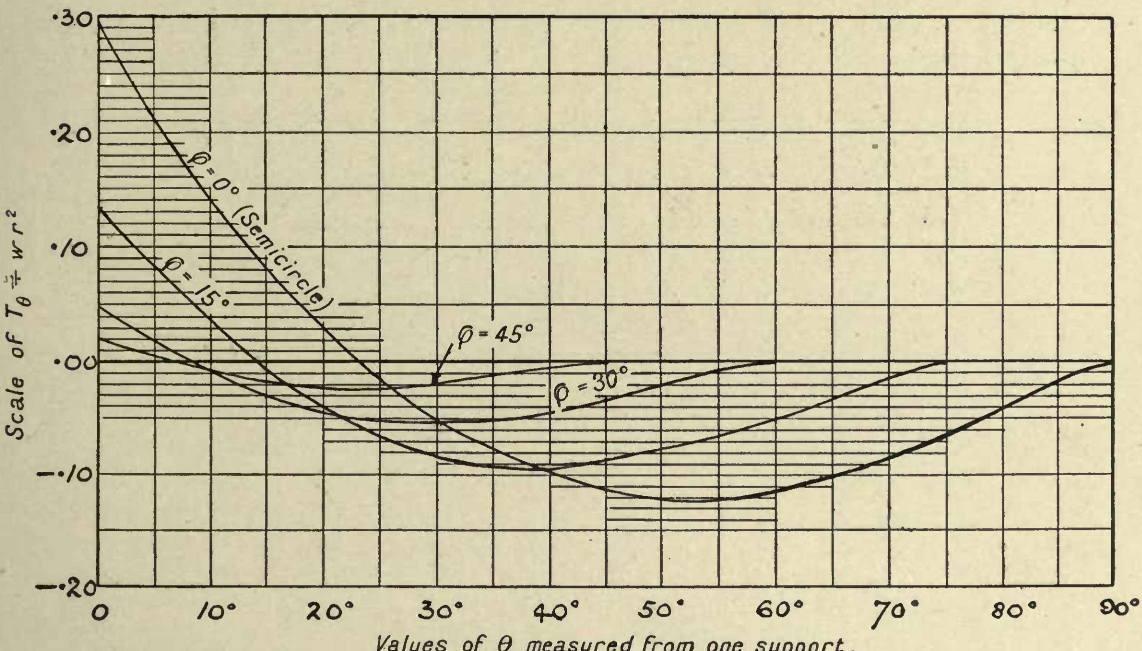


FIG. 23.—Twisting-moment diagrams for one-half of a uniformly loaded circular-arc girder, subtending an angle $[180° - 2\phi]$.

Substituting for M_a and R_a in (20′), writing $\frac{\pi}{2}$ for θ, and equating to zero, gives

$$T_a = wr^2 \times \frac{2}{\pi}\left(\frac{\pi^2}{4} - 2\right) = \cdot 298 wr^2,$$

and on substituting in (18) and (19)

$$M_\theta = wr^2(1 - 1\cdot 2728 \sin\theta),$$
$$T_\theta = wr^2(1\cdot 5708 - 1\cdot 2728 \cos\theta - \theta).$$

This makes $M_\theta = 0$ when $\sin\theta = \frac{1}{1\cdot 2728} = \cdot 7850$; *i.e.*, when $\theta = 51°43'$, and makes $T_\theta = 0$ when $\theta = 22°40'$, and again when $\theta = 90°$. M_θ is a maximum when $\frac{dM_\theta}{d\theta} = 0$; *i.e.* when $\cos\theta = 0$, and therefore at the supports. T_θ is a maximum when

$\frac{dT_\theta}{d\theta} = 0$, *i.e.* when $\sin\theta = \cdot 7850$, or when $\theta = 51°43'$.

Writing $\theta_1 = \frac{\pi}{2}$ in (21′) and substituting for T_a and R_a, the deflection at the centre is given by

$$y_{(\text{centre})} = wr^4\left[\frac{\cdot 7272}{2EI} + \frac{\cdot 053}{2CJ}\right].$$

(17) Circular-Arc Bow-Girder, Subtending an Angle $(180-2\phi)°$, Built in at the Ends and Carrying a Uniformly Loaded Platform.

Let w lb. per unit area be the load on the platform whose area will be $\frac{r^2}{2}\left\{\pi - 2\phi - \sin 2\phi\right\}$. Imagine the latter to be divided into a series of strips parallel to AB, each of these strips transmitting its load to the girder at its ends. The length of the particular strip resting on the girder at points distant θ from A and B,

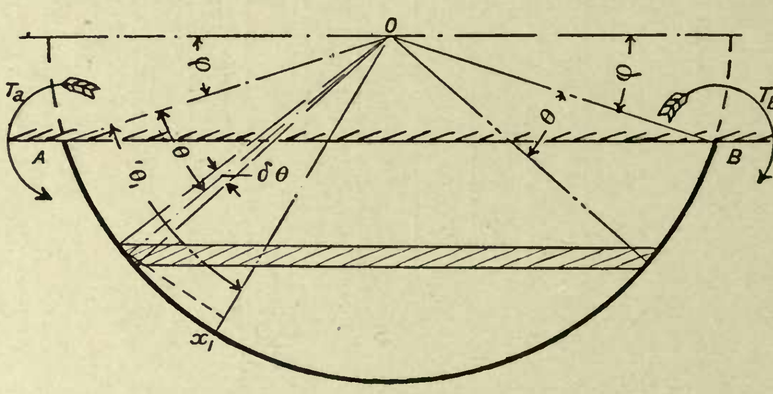


FIG. 24.

is $2r\cos(\theta+\phi)$ (Fig. 24). If this strip covers a length $\delta s = r\delta\theta$ of the girder, its width is $r\delta\theta\cos(\theta+\phi)$, and the load on it is $2wr^2\cos^2(\theta+\phi)\delta\theta$.

Its moment about $AB = 2wr^3\cos^2(\theta+\phi)\{\sin(\theta+\phi) - \sin\phi\}\delta\theta$,

$$\therefore \text{Moment of whole load, about } AB = 2wr^3\int_0^{\frac{\pi}{2}-\phi}\cos^2(\theta+\phi)\{\sin(\theta+\phi) - \sin\phi\}\delta\theta$$

$$= 2wr^3\left\{\frac{\cos^3\phi}{3} - \frac{\sin\phi}{4}(\pi - 2\phi - \sin 2\phi)\right\}$$

Since, from symmetry, $M_a = M_b$; $T_a = T_b$; it follows that

$$M_a\cos\phi - T_a\sin\phi = wr^3\left\{\frac{\cos^3\phi}{3} - \frac{\sin\phi}{4}(\pi - 2\phi - \sin 2\phi)\right\}$$

$$\therefore M_a = wr^3\left\{\frac{\cos^2\phi}{3} - \frac{\tan\phi}{4}(\pi - 2\phi - \sin 2\phi)\right\} + T_a\tan\phi.$$

Again, since the total load is

$$2wr^2\int_0^{\frac{\pi}{2}-\phi}\cos^2(\theta+\phi)d\theta$$

$$= \frac{wr^2}{2}\left\{\pi - 2\phi - \sin 2\phi\right\}$$

$$\therefore R_a = R_b = \frac{wr^2}{4}\left\{\pi - 2\phi - \sin 2\phi\right\}.$$

The bending and twisting moments at a point x_1 distant θ_1 from OA are given by

$$M_{\theta_1} = M_a \cos\theta_1 - (R_a r - T_a)\sin\theta_1 + \int_0^{\theta_1} wr^3 \cos^2(\theta+\phi)\sin(\theta_1-\theta)\,d\theta.$$

$$T_{\theta_1} = (T_a - R_a r)\cos\theta - M_a \sin\theta + R_a r - \int_0^{\theta_1} wr^3\cos^2(\theta+\phi)\{1-\cos(\theta_1-\theta)\}d\theta,$$

the last term in each case representing the moment, bending or twisting, about the point x_1 (Fig. 24), of the load between A and x_1.

On integrating these terms and writing θ for θ_1, the general expressions for M_θ and T_θ become

$$M_\theta = M_a\cos\theta - (R_a r - T_a)\sin\theta + \frac{wr^3}{3}\left[\begin{array}{r}(\cos\theta-1)\{\cos\theta-\sin^2\phi+\sin 2\phi\sin\theta\}\\ +2\sin^2\theta\cos^2\phi\end{array}\right] \quad (22)$$

$$\begin{aligned}T_\theta = (T_a - R_a r)\cos\theta \\ - M_a\sin\theta + R_a r - wr^3\end{aligned}\left[\begin{array}{r}\dfrac{\theta}{2}+\sin\theta\left\{\dfrac{\cos2\phi\cos\theta}{2}+\dfrac{2}{3}(1-\cos\theta)-\cos^2\phi\right\}\\ -\dfrac{\sin2\phi}{6}(1-\cos\theta)^2\end{array}\right] \quad (23)$$

As before, if the girder be fixed horizontally at the ends

$$\left(\frac{dy}{d\theta}\right)_{\theta_1} = \frac{r^2}{EI}\int_0^{\theta_1} M_\theta\cos(\theta_1-\theta)d\theta + \frac{r^2}{CJ}\int_0^{\theta_1} T_\theta\sin(\theta_1-\theta)d\theta,$$

and, on substituting the foregoing values of M_θ and T_θ and integrating, this gives the value of $\frac{dy}{d\theta}$ at any point θ_1. Thus

$$\left(\frac{dy}{d\theta}\right)_{\theta_1} = \left[\begin{array}{l} \dfrac{r^2}{2EI}\left[\begin{array}{l} M_a(\theta_1\cos\theta_1+\sin\theta_1) - (R_a r - T_a)\theta_1\sin\theta_1 \\ +\dfrac{wr^3}{3}\left\{\begin{array}{r}\sin\theta_1\left(\dfrac{7}{3}-\sin^2\phi\{3+\theta_1\}\right)-\theta_1\cos\theta_1(1+\sin^2\phi)\\ -\dfrac{2}{3}\sin2(\theta_1+\phi)\end{array}\right\}\end{array}\right] \\ +\dfrac{r^2}{2CJ}\left[\begin{array}{l}(T_a-R_a r)\theta_1\sin\theta_1 - M_a(\sin\theta_1-\theta_1\cos\theta_1)+2R_a r(1-\cos\theta_1) \\ -wr^3\left\{\begin{array}{r}\theta_1-\sin\theta_1\left(\dfrac{7}{9}+\cos^2\phi-\dfrac{\cos2\phi}{3}+\dfrac{\sin2\phi}{6}+\dfrac{\cos\theta_1\cos2\phi}{3}\right.\\ \left.-\dfrac{4}{9}\cos\theta_1\right)\\ +\theta_1\cos\theta_1\left(\cos^2\phi+\dfrac{\sin2\phi}{6}-\dfrac{2}{3}\right)+\dfrac{\sin2\phi}{9}(3\theta_1\sin\theta_1\\ -\sin^2\theta_1+\cos\theta_1-1)\end{array}\right.\end{array}\right]\end{array}\right] \quad (24)$$

From symmetry the slope is zero at the centre of the beam where $\theta_1 = \frac{\pi}{2} - \phi$, and, on substituting this value for θ_1 in (24), and also substituting the values of M_a and R_a as given on pp. 34 and 35, and equating to zero, the value of T_a may be obtained.

E.g., Semicircular Girder ($\phi = 0$).

In this case, on putting $\phi = 0$ in (24)

$$\therefore \left(\frac{dy}{d\theta}\right)\theta_1 = \left[\begin{array}{l} \dfrac{r^2}{2EI}\left[\begin{array}{l} M_a\{\theta_1 \cos\theta_1 + \sin\theta_1\} - (R_a r - T_a)\theta_1 \sin\theta_1 \\ \quad + \dfrac{wr^2}{3}\left\{\dfrac{7}{3}\sin\theta_1 - \cos\theta_1\left(\dfrac{4}{3}\sin\theta_1 + \theta_1\right)\right\} \end{array}\right] \\ + \dfrac{r^2}{2CJ}\left[\begin{array}{l} (T_a - R_a r)\theta_1 \sin\theta_1 - M_a\{\sin\theta_1 - \theta_1\cos\theta_1\} \\ \quad + 2R_a r(1-\cos\theta_1) - wr^3\left\{\theta_1 - \sin\theta_1\left(\dfrac{13}{9} - \dfrac{1}{9}\cos\theta_1\right) \right. \\ \qquad\qquad \left. + \dfrac{1}{3}\theta_1\cos\theta_1\right\} \end{array}\right] \end{array}\right] \tag{24'}$$

At the centre, where $\theta_1 = \frac{\pi}{2}$, the slope is zero, and $M_a = \frac{wr^3}{3}$; $R_a = wr^2 \cdot \frac{\pi}{4}$.

$$\therefore \frac{wr^5}{2EI}\left[\frac{1}{3} - \left(\frac{\pi}{4} - \frac{T_a}{wr^3}\right)\frac{\pi}{2} + \frac{7}{9}\right] + \frac{wr^5}{2CJ}\left[\left(\frac{T_a}{wr^3} - \frac{\pi}{4}\right)\frac{\pi}{2} - \frac{1}{3} + \frac{13}{9}\right] = 0$$

$$\therefore T_a = \left(\frac{\pi}{4} - \frac{20}{9\pi}\right) wr^3 = \cdot 078 wr^3.$$

It follows that, on substituting in (22) and (23)

$$M_\theta = wr^3\left\{\frac{1+\sin^2\theta}{3} - \cdot 7074 \sin\theta\right\}$$

$$T_\theta = wr^3\left\{\frac{\sin\theta\cos\theta}{6} + \frac{\pi}{4} - \cdot 7074\cos\theta - \frac{\theta}{2}\right\}.$$

The deflection at any point θ_1 is obtained by writing $\theta_1 = \theta$ in (24′) and integrating between the limits θ_1 and 0. Thus,

$$y_{\theta_1} = \left[\begin{array}{l} \dfrac{r^2}{2EI}\left[\begin{array}{l} M_a\theta_1 \sin\theta_1 - (R_a r - T_a)(\sin\theta_1 - \theta_1\cos\theta_1) \\ \quad + \dfrac{wr^3}{9}\left\{10 - 10\cos\theta_1 - 2\sin^2\theta_1 - 3\theta_1\sin\theta_1\right\} \end{array}\right] \\ + \dfrac{r^2}{2CJ}\left[\begin{array}{l} (T_a - R_a r)(\sin\theta_1 - \theta_1\cos\theta_1) + 2R_a r(\theta_1 - \sin\theta_1) \\ \quad + M_a(\theta_1\sin\theta_1 + 2\cos\theta_1 - 2) \\ \quad - \dfrac{wr^3}{9}\left\{\dfrac{9\theta_1^2}{2} + 16\cos\theta_1 - 16 + \dfrac{\sin^2\theta_1}{2} + 3\theta_1\sin\theta_1\right\} \end{array}\right] \end{array}\right] \tag{25}$$

At the centre, where $\theta_1 = \frac{\pi}{2}$

$$y_{\text{centre}} = \left[\begin{array}{l} \dfrac{r^2}{2EI}\left[M_a\dfrac{\pi}{2} - (R_a r - T_a) + \cdot 3653 wr^3\right] \\ + \dfrac{r^2}{2CJ}\left[(T_a - R_a r) + R_a r(\pi - 2) + M_a\left(\dfrac{\pi}{2} - 2\right) - \cdot 0350 wr^3\right] \end{array}\right]$$

$$= \frac{wr^5}{2}\left[\frac{\cdot 1815}{EI} + \frac{\cdot 0121}{CJ}\right].$$

(18) Girder with Unsymmetrical Loading.

Where the loading of a girder does not admit of being represented by a simple trigonometrical expression, or where the girder is not of uniform cross section throughout its length, a solution is most readily obtained by dividing the load, including the dead load due to the girder itself, into a series of comparatively short lengths, and by calculating the moments due to each of these portions of the load separately, by an application of the reasoning and results of § (15). In practice a first approximation would be obtained by assuming a likely value for the cross section and weights at each point, and by then applying these results. A second approximation would then be made taking into account the weight of the girder calculated from the sections found necessary by the first approximation, and this would in the majority of cases give results sufficiently near for all practical purposes.

(19) Bow-Girder Built in at the Ends and Resting on Intermediate Supports.

Assuming all the supports to be at the same level, the reactions of the intermediate supports may be most readily obtained by expressing the fact that the upward deflections at these supports caused by their reactions, are equal to the downward deflections produced at the same points by the loading.

(20) Girder with Uniform Loading and Central Support.

Let P be the reaction of this support. Let $180 - 2\phi$, or 2α, be the angle subtended by the arc of the girder.

The upward deflection at the centre due to the reaction is given by equation (14), in which $W = P$, and in which M_α and T_α have the values given by the curves of Figs. 13—16, for the corresponding value of α or $(90^\circ - \phi)$. The downward deflection at the centre due to the load is obtained by substituting α for θ_1, and by substituting the corresponding values of M_α and T_α as given by the curves in Figs. 19—21, in equation (21).

E.g., $\alpha = 90^\circ$; $\phi = 0$ (semicircular girder).

The upward deflection at centre

$$= \frac{Pr^3}{2EI}\left[\frac{\pi}{4} - (\cdot 500 - \cdot 182)\right] + \frac{Pr^3}{2CJ}\left[(\cdot 182 - \cdot 500) + \frac{\pi}{2} - 1 + \frac{\pi}{4}\left(\frac{\pi}{2} - 2\right)\right]$$

$$= Pr^3\left[\frac{\cdot 4674}{2EI} + \frac{\cdot 0382}{2CJ}\right].$$

The downward deflection at the centre, due to the loading

$$= wr^4\left[\frac{\cdot 7272}{2EI} + \frac{\cdot 053}{2CJ}\right]$$

and on equating these

$$P = wr\left[\frac{\cdot 7272CJ + \cdot 053EI}{\cdot 4674CJ + \cdot 0382EI}\right].$$

The value of this depends slightly on the ratio of EI to CJ. Taking this ratio as 1·25, gives

$$P = wr\left\{\frac{\cdot 7928}{\cdot 5147}\right\} = 1\cdot 54wr.$$

Again,

$$R_a + R_b + P = \pi w r$$

$$\therefore\ R_a = R_b = \frac{wr}{2}\left\{\pi - 1{\cdot}54\right\}$$
$$= {\cdot}801wr.$$

Also

$$M_a + M_b = 2wr^2 - Pr$$
$$= {\cdot}46wr^2$$
$$\therefore\ M_a = M_b = {\cdot}23wr^2.$$

The value of T_a is the difference between the values produced by the load and by

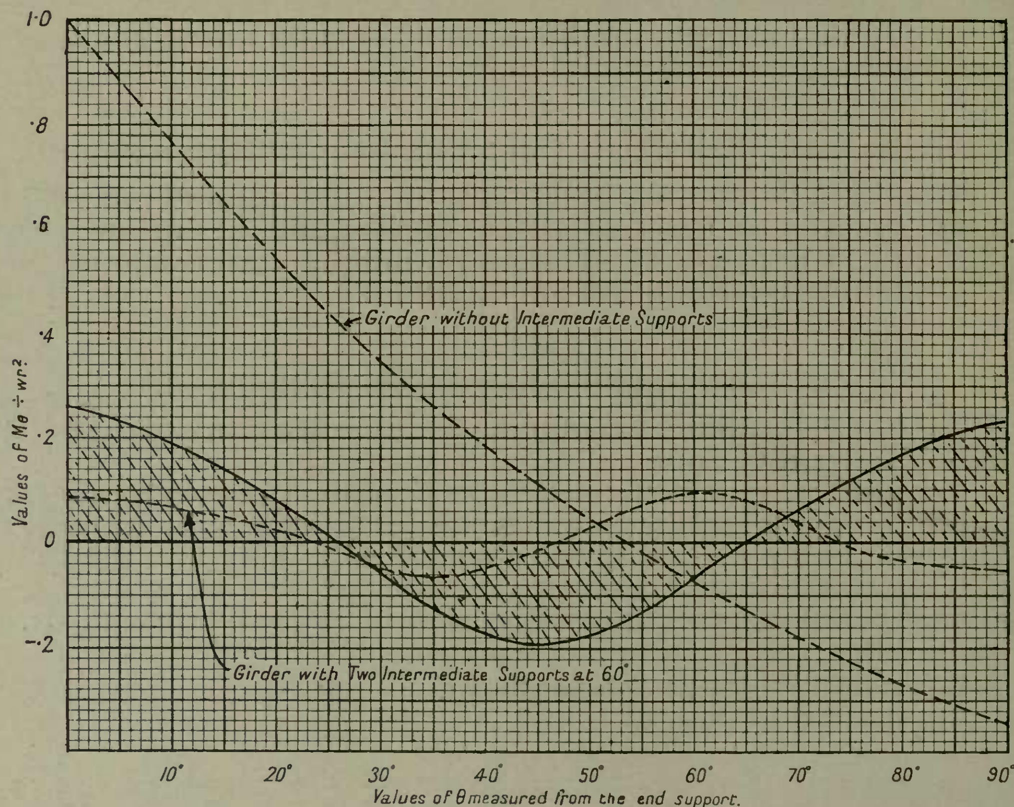


FIG. 25.—Bending moments in a uniformly loaded circular-arc built in at the ends and having a central support. (Full-line curve.)

the upward reaction P. The first of these is ${\cdot}298wr^2$ (Fig. 20); the second is ${\cdot}182Pr$ (Fig. 16).

$$\therefore\ T_a = \{{\cdot}298 - ({\cdot}182 \times 1{\cdot}54)\}\, wr^2$$
$$= {\cdot}018wr^2.$$

This value may be obtained alternatively by substituting the foregoing values of M_a and of R_a in equation (20) with $\theta_1 = \frac{\pi}{2}$, and by equating to zero.

The values of M_θ and of T_θ at any point between the end and the support and distant θ from the end then become, on substituting in equations (18) and (19)

$$M_\theta = wr^2\{1 - {\cdot}77\cos\theta - {\cdot}783\sin\theta\},$$
$$T_\theta = wr^2\{{\cdot}801 - {\cdot}783\cos\theta + {\cdot}77\sin\theta - \theta\}.$$

If $EI : CJ = 10$ the values of the end moments and reactions become $P = 1{\cdot}47\ wr$; $R_a = R_b = {\cdot}835wr$; $M_a = M_b = {\cdot}265wr^2$; $T_a = T_b = {\cdot}030wr^2$, and equations (18) and (19) become

$$M_\theta = wr^2\{1 - {\cdot}735 \cos\theta - {\cdot}805 \sin\theta\}$$
$$T_\theta = wr^2\{{\cdot}835 - {\cdot}805 \cos\theta + {\cdot}735 \sin\theta - \theta\}.$$

Figs. 25 and 26 show the bending and twisting moments at each section of one-

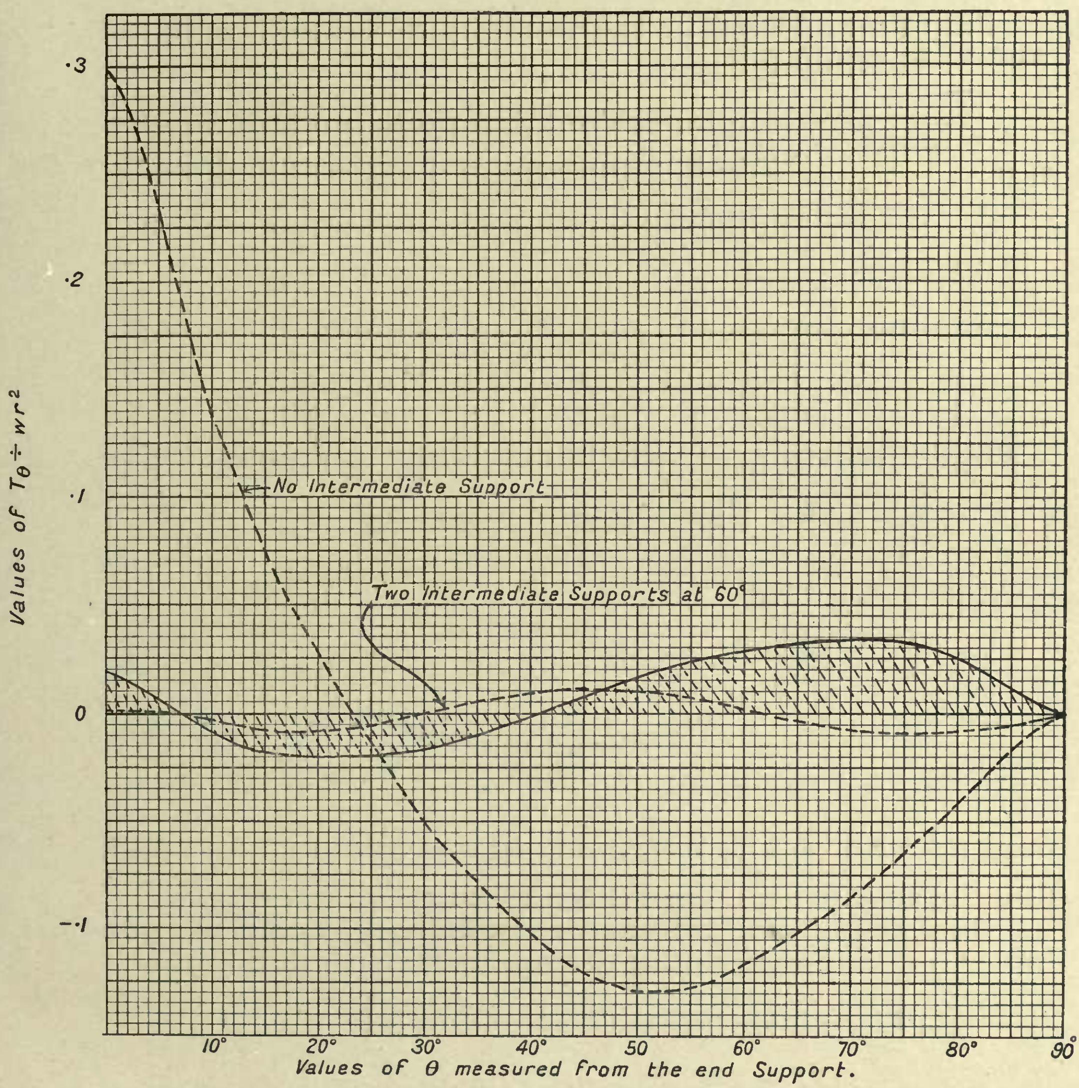


Fig. 26.—Twisting moments in a uniformly loaded circular-arc built in at the ends and having a central support. (Full-line curve.)

half of such a girder with a central support and with $EI \div CJ = 1{\cdot}25$, while for comparison the moments with the same loading but without the central support are shown by the dotted line curves on the same diagrams.

Where the girder subtends an angle less than 180°, the problem may be solved in an exactly similar manner by making use of the requisite relationships from the foregoing curves.

(21) Circular-Arc Girder, built in at the Ends, with Uniform Loading, and with two Symmetrical Intermediate Supports.

Let the angle subtended by the girder be $(180-2\phi)^\circ$, and let the supports (at C and D, Fig. 27) be distant γ from each end. Let the upward reaction at each support $= P$. Let M_a'', T_a'', R_a'' represent such end conditions at A as would be produced by these two reactions alone, and let M_a', T_a', R_a' represent such end conditions as would be produced by the load alone, with supports removed.

Under these conditions the downward deflection at C and D due to the loading would be, by equation (21)

$$y_\gamma = \left[\begin{array}{l} \frac{r^2}{2EI}\Big[(M_a'-wr^2)\gamma\sin\gamma-(R_a'r-T_a')(\sin\gamma-\gamma\cos\gamma)-2wr^2(\cos\gamma-1)\Big] \\ +\frac{r^2}{2CJ}\left[\begin{array}{l}(T_a'-R_a'r)(\sin\gamma-\gamma\cos\gamma)+(M_a'-wr^2)\{\gamma\sin\gamma+ \\ 2\cos\gamma-2\}+2R_a'r(\gamma-\sin\gamma)-2wr^2(\frac{\gamma^2}{2}+\cos\gamma-1)\end{array}\right]\end{array}\right] \quad (26)$$

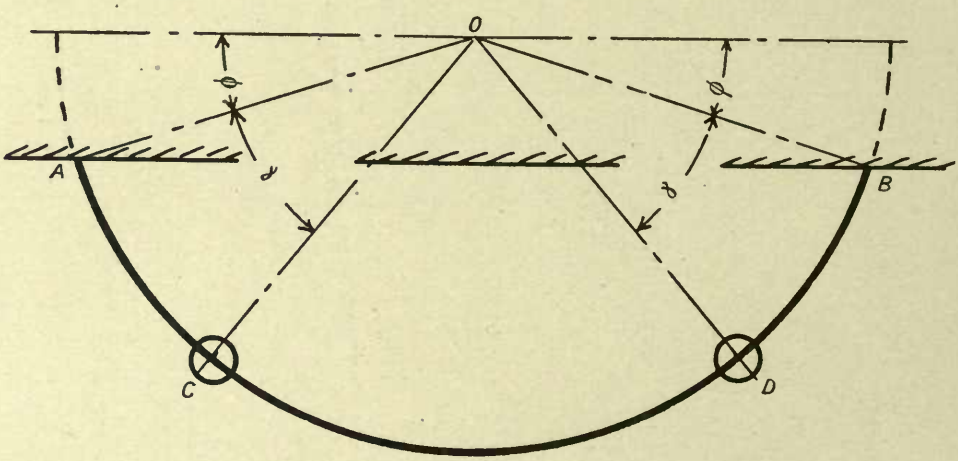


Fig. 27.

where $R_a' = wr\left(\frac{\pi}{2}-\phi\right)$, and M_a' and T_a' for the particular value of ϕ obtaining in the girder, are given by the curves of Figs. 19—21.

The upward deflection at C and, from symmetry, at D, due to the two upward forces P is obtained by substituting γ for θ_1 in equation (12), which becomes

$$y_\gamma = \left[\begin{array}{l} \frac{r^2}{2EI}\Big[M_a''\gamma\sin\gamma-(R_a''r-T_a'')(\sin\gamma-\gamma\cos\gamma)\Big] \\ +\frac{r^2}{2CJ}\left[\begin{array}{c}(T_a''-R_a''r)(\sin\gamma-\gamma\cos\gamma)+2R_a''r(\gamma-\sin\gamma) \\ +M_a''(\gamma\sin\gamma+2\cos\gamma-2)\end{array}\right]\end{array}\right] \quad . \quad . \quad (27)$$

The values of M_a'', R_a'', T_a'' for use in this expression are the sum of the corresponding values produced by each of the two forces P acting at points distant γ from A and from B, and may evidently be obtained by adding the values of M_a and M_b, R_a and R_b, T_a and T_b, as obtained from the curves of Figs. 13—16 for a girder having the correct value of ϕ, and having the force P at γ from A.

On substituting these values, each of which is given in terms of P, in equation

(27) and equating to (26), the resultant expression contains P as the only unknown and enables this to be calculated.

E.g., Semicircular girder with uniform loading and with two piers at 60° from the ends of the span ($\phi = 0$; $\gamma = 60°$).

From Figs. 19 and 20 the values of M_a', and T_a' for substitution in equation (26) are $M_a' = wr^2$; $T_a' = \cdot 298wr^2$; while $R_a' = 1{\cdot}5708wr$, and, on substituting, the downward deflection at the supports ($\gamma = 60°$) is given by

$$y_{60°} = wr^4\left[\frac{\cdot 564}{2EI} + \frac{\cdot 037}{2CJ}\right].$$

The values of M_a'', T_a'', and R_a'' for substitution in (27) are, from Figs. 13, 14, 15 and 16

$$M_a'' = (M_a + M_b)_{\phi=0,\,\gamma=60°} = (\cdot 588 + \cdot 278)Pr = \cdot 866Pr.$$
$$T_a'' = (T_a + T_b)_{\phi=0,\,\gamma=60°} = (\cdot 156 + \cdot 127)Pr = \cdot 283Pr.$$
$$R_a'' = P,$$

and, on making these substitutions,

$$y_{60°} = Pr^3\left[\frac{\cdot 539}{2EI} + \frac{\cdot 035}{2CJ}\right].$$

Equating these two expressions for $y_{60°}$ gives

$$P = wr\left[\frac{\cdot 564CJ + \cdot 037EI}{\cdot 539CJ + \cdot 035EI}\right],$$

and taking $EI = 1{\cdot}25CJ$, this makes $P = 1{\cdot}05wr$.

The reactions at A and B are then given by

$$R_b = R_a = R_a' - R_a'' = wr\left(\frac{\pi}{2} - 1{\cdot}05\right) = \cdot 521wr.$$

while the moments M_a and M_b are given by

$$M_b = M_a = M_a' - M_a'' = wr^2(1 - \cdot 866 \times 1{\cdot}05) = \cdot 091wr^2.$$

The torques T_a and T_b are given by

$$T_b = T_a = T_a' - T_a'' = wr^2\{\cdot 298 - \cdot 283 \times 1{\cdot}05\} = \cdot 001wr^2.$$

The state of affairs at any point on the girder is thus given by the relations (equations (18) and (19)):—Between A and C—

$$M_\theta = M_a \cos\theta - (R_a r - T_a)\sin\theta + wr^2(1 - \cos\theta)$$
$$= wr^2\{1 - \cdot 909\cos\theta - \cdot 520\sin\theta\}$$
$$T_\theta = (T_a - R_a r)\cos\theta + R_a r - M_a\sin\theta - wr^2(\theta - \sin\theta)$$
$$= wr^2\{\cdot 521 - \cdot 520\cos\theta + \cdot 909\sin\theta - \theta\}$$

Between C and the centre (θ being measured from OA)—

$$M_\theta = M_a\cos\theta - (R_a r - T_a)\sin\theta + wr^2(1 - \cos\theta) - Pr\sin(\theta - 60°)$$
$$= wr^2\{1 - 1{\cdot}045\sin\theta\}$$
$$T_\theta = (T_a - R_a r)\cos\theta + R_a r - M_a\sin\theta - wr^2(\theta - \sin\theta) + Pr\{1 - \cos(\theta - 60°)\}$$
$$= wr^2\{1{\cdot}571 - 1{\cdot}045\cos\theta - \theta\}.$$

Fig. 28 shows the bending and twisting moment diagrams for such a girder, while for purposes of comparison these have also been drawn as dotted line curves on Figs. 25 and 26. From these it appears that the maximum values of the moments with and without supports have the following ratios, the bending and twisting moments for the span without intermediate supports being taken as unity.

	Number of Intermediate Supports.		
	none	one at centre	two at 60°
Maximum bending moment .	1·0	·26	·09
Maximum twisting moment .	1·0	·11	·035

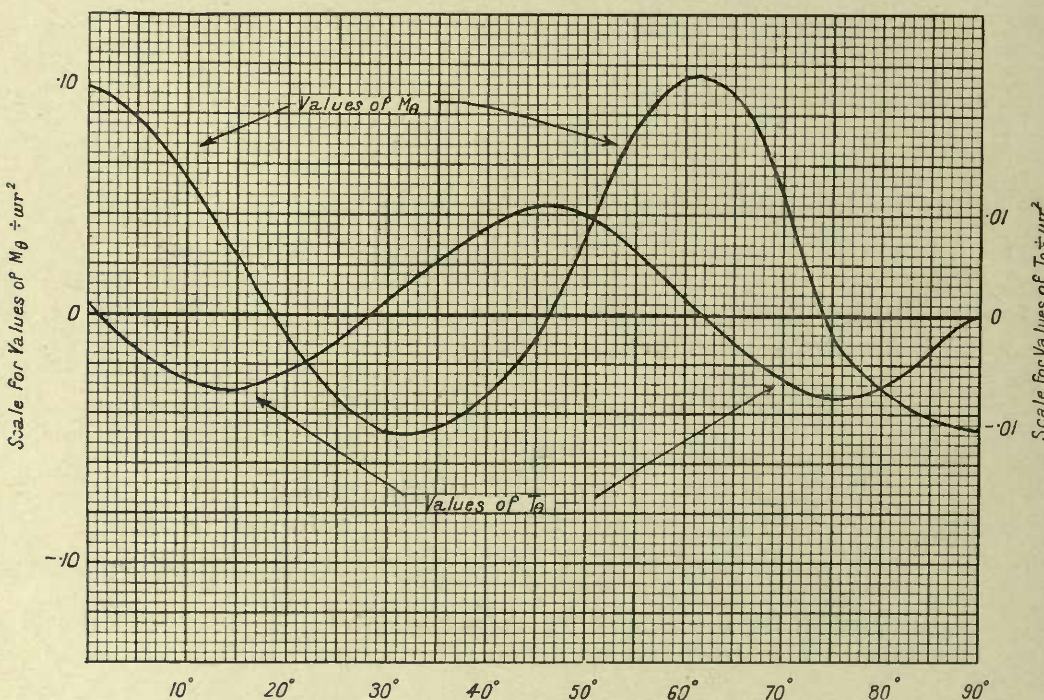


FIG. 28.—Diagrams of bending and twisting moments for uniformly loaded semicircular girder, with two intermediate supports, distant 60° from each end.

The following table shows how the fixing moments and reactions vary with the ratio of $EI : CJ$ in the foregoing example.

$\frac{EI}{CJ}$	$\frac{P}{wr}$	$\frac{R_a}{wr}$	$\frac{M_a}{wr^2}$	$\frac{T_a}{wr^2}$
1·25	1·05	·521	·091	·001
100	1·06	·511	·081	—·002

From these figures it appears that a considerable change in this ratio has very little effect on the magnitude of these moments.

Semicircular Girder with uniform Loading and with two Piers at 45° from Ends of Span.

In this case, the end constants and pier reactions for $EI = 1{\cdot}25\ CJ$ become

$$P = 1{\cdot}460wr \quad ; \quad M_a = -\ {\cdot}031wr^2;$$
$$R_a = R_b = {\cdot}111wr \quad ; \quad T_a = {\cdot}010wr^2.$$

As before, between A and C

$$M_\theta = M_a \cos\theta - (R_a r - T_a)\sin\theta + wr^2(1 - \cos\theta),$$
$$T_\theta = (T_a - R_a r)\cos\theta + R_a r - M_a \sin\theta - wr^2(\theta - \sin\theta),$$

while between C and the centre

$$M_\theta = M_a \cos\theta - (R_a r - T_a)\sin\theta - Pr\sin(\theta - 45^\circ) + wr^2(1 - \cos\theta),$$
$$T_\theta = (T_a - R_a r)\cos\theta + R_a r - M_a\sin\theta + Pr\left\{1 - \cos(\theta - 45^\circ)\right\} - wr^2(\theta - \sin\theta).$$

(22) Semicircular Girder, built in at the Ends, with Uniform Loading, and with three Intermediate Supports.

Let the supports be arranged symmetrically, P_1 and P_2 being the reactions at the outer supports and Q that at the central support. These reactions may be obtained by expressing the facts (1) that the downward deflection at the centre due to the loading is equal to the sum of the upward deflections at the centre due to the forces P_1, P_2, and Q, in their respective positions; and (2) that the downward deflection at P_1 due to the loading is equal to the upward deflection at this point due to forces P_1, P_2, and Q; thus if, for example, P_1 and P_2 are each at 45° from the ends, we have—

Downward deflection at Q due to loading

$$= wr^4\left\{\frac{{\cdot}7272}{2EI} + \frac{{\cdot}053}{2CJ}\right\}.$$

Downward deflection at P_1 or P_2 due to loading

$$= wr^4\left\{\frac{{\cdot}3928}{2EI} + \frac{{\cdot}0213}{2CJ}\right\},$$

these values being obtained from equation (21′) by substituting the values of θ, viz., 90° and 45°, and of M_a and T_a from Figs. 19 and 20.

Again, the upward deflection at Q due to force Q

$$= Qr^3\left[\frac{{\cdot}4674}{2EI} + \frac{{\cdot}0382}{2CJ}\right] \text{ from (14) and Figs. 13 and 14,}$$

and the upward deflection at Q due to the two forces P_1 and $P_2(=P)$

$$= 2Pr^3\left[\frac{{\cdot}2110}{2EI} + \frac{{\cdot}0594}{2CJ}\right] \text{ from (13) and Figs. 13 and 14.}$$

Also the upward deflection at P_1 due to force P_1

$$= Pr^3\left[\frac{{\cdot}1865}{2EI} + \frac{{\cdot}0055}{2CJ}\right] \text{ from (14) and Figs. 13 and 14,}$$

while the upward deflection at P_1 due to P_2

$$= Pr^3\left\{\frac{{\cdot}0845}{2EI} + \frac{{\cdot}0085}{2CJ}\right\} \text{ from (13) and Figs. 13 and 14,}$$

and the upward deflection at P_1 due to force Q

$$= Qr^3\left\{\frac{\cdot 2297}{2EI}+\frac{\cdot 015}{2CJ}\right\} \text{ from (13) and Figs. 13 and 14.}$$

Collecting and equating deflections at the same points gives

$$wr(\cdot 7272CJ + \cdot 053EI) = Q(\cdot 4674CJ + \cdot 0382EI) + P(\cdot 4220CJ + \cdot 0594EI),$$
$$wr(\cdot 3928CJ + \cdot 0213EI) = Q(\cdot 2297CJ + \cdot 015EI) + P(\cdot 2710CJ + \cdot 0140EI),$$

where $P = P_1 = P_2$.

If $EI = 1\cdot 25CJ$, the solution of this gives

$$Q = \cdot 74wr; \quad P = \cdot 83wr.$$

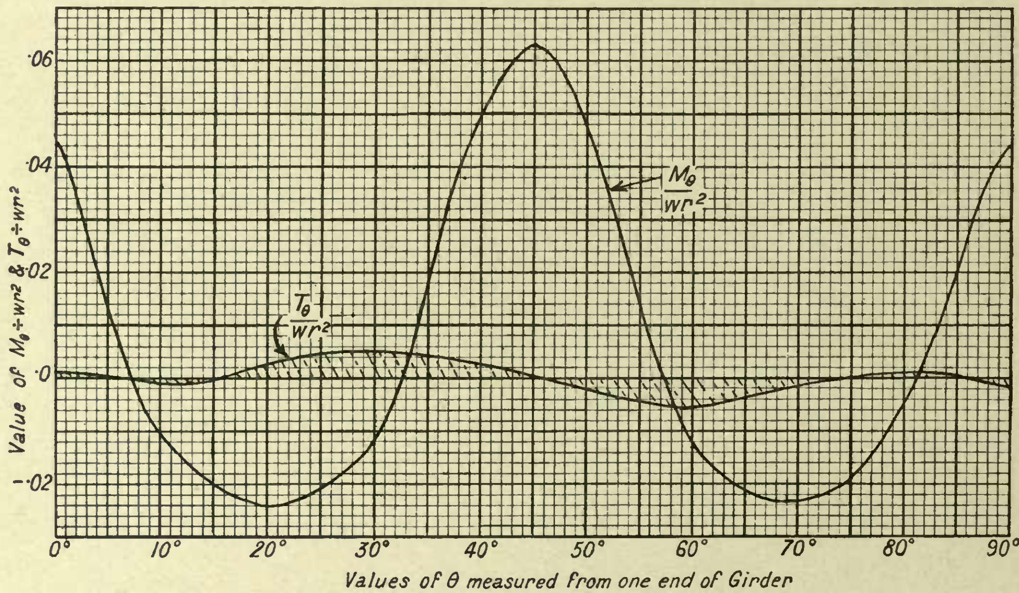


FIG. 29.—Bending and twisting moment diagrams for one-half of a uniformly loaded semicircular girder with three intermediate supports at 45°.

From this

$$R_a = R_b = \tfrac{1}{2}\{\pi wr - Q - 2P\}$$
$$= \cdot 37wr.$$

Also

$$M_a + M_b = 2wr^2 - 2Pr\sin 45° - Qr$$
$$= \cdot 088wr^2$$
$$\therefore\ M_a = M_b = \cdot 044wr^2$$

while T_a (from Figs. 15 and 16)

$$= \cdot 298wr^2 - \cdot 112Pr - \cdot 182Qr - \cdot 083Pr$$
$$= (\cdot 298 - \cdot 297)wr^2$$
$$= \cdot 0010\,wr^2.$$

Fig. 29 shows the bending and twisting moment diagrams for one-half of this girder, and a comparison of these diagrams with those of Figs. 25 and 26 indicates to what extent the maximum moments are reduced by the addition of the third support.

(23) Effect of Depression of Supports.

Where a bow-girder is used to support the circle of a theatre, intermediate supports are often provided by cantilevers built into the rear walls of the theatre. If erected so that under no load the ends of these are level with the end supports of the bow girder, their deflection under load will reduce the supporting pressure to a value below that obtaining with rigid supports, will increase the end reactions, and, generally speaking, will increase the average bending and twisting moment over the whole girder.

If P be the end load on a given cantilever, its deflection at the free end is proportional to P and is equal to kP where k depends on the dimensions of the cantilever. For example, if of uniform section, of moment of inertia I', and of length l, $k = \dfrac{l^3}{3EI'}$.

The actual deflection under load of the bow-girder at this point is thus kP, and if y would be its deflection with the support removed, the upward deflection due to the upward force P is equal to $y - kP$.

Expressing y in terms of the load on the girder, and expressing the upward deflection due to P in terms of P as in §§ 20, 21, and 22, and equating this to $y - kP$, the pressure P on the support is obtained in terms of the load as in the examples of the preceding articles.

E.g., **Semicircular Girder with Uniform Loading, built in at the Ends and Supported at the Centre by the End of a Cantilever.**

Deflection at support, with support removed . . . . . $= wr^4\left[\dfrac{\cdot 7272}{2EI} + \dfrac{\cdot 053}{2CJ}\right]$ . . . p. 34

Actual deflection at support . . $= kP$

Upward deflection at centre due to force P . . . . . . $= Pr^3\left[\dfrac{\cdot 4674}{2EI} + \dfrac{\cdot 0382}{2CJ}\right]$ . . . p. 37

$$\therefore\ Pr^3\left[\frac{\cdot 4674}{2EI} + \frac{\cdot 0382}{2CJ}\right] + kP = wr^4\left[\frac{\cdot 7272}{2EI} + \frac{\cdot 053}{2CJ}\right]$$

$$\therefore\ P = wr\left[\frac{\cdot 7272 + \cdot 053\dfrac{EI}{CJ}}{\cdot 4674 + \cdot 0382\dfrac{EI}{CJ} + \dfrac{2kEI}{r^3}}\right].$$

Thus, for example, if the cantilever be of uniform section, of moment of inertia I' and of length l, so that $k = \dfrac{l^3}{3EI'}$, this becomes

$$P = wr\left[\frac{\cdot 7272 + \cdot 053\dfrac{EI}{CJ}}{\cdot 4674 + \cdot 0382\dfrac{EI}{CJ} + \dfrac{2}{3}\dfrac{l^3}{r^3}\cdot\dfrac{I}{I'}}\right].$$

The following table shows how, in the case where $l = r$ and $I = I'$, the yielding of this support would modify the end moments and reactions as compared with those experienced with a rigid support or with a cantilever so erected and designed as to deflect under load to the level of the end supports. These figures apply to the case where $EI = 10\,CJ$.

	$\frac{P}{wr}$	$\frac{R_a}{wr}$	$\frac{M_a}{wr^2}$	$\frac{T_a}{wr^2}$
Rigid support . . .	1·47	·835	·265	·030
Elastic support . . .	·828	1·157	·586	·147
No central support . .	—	1·571	1·00	·298

It will be noted that, due to yielding of the support, the end moments (and approximately also the average moments) in the girder are increased roughly in the

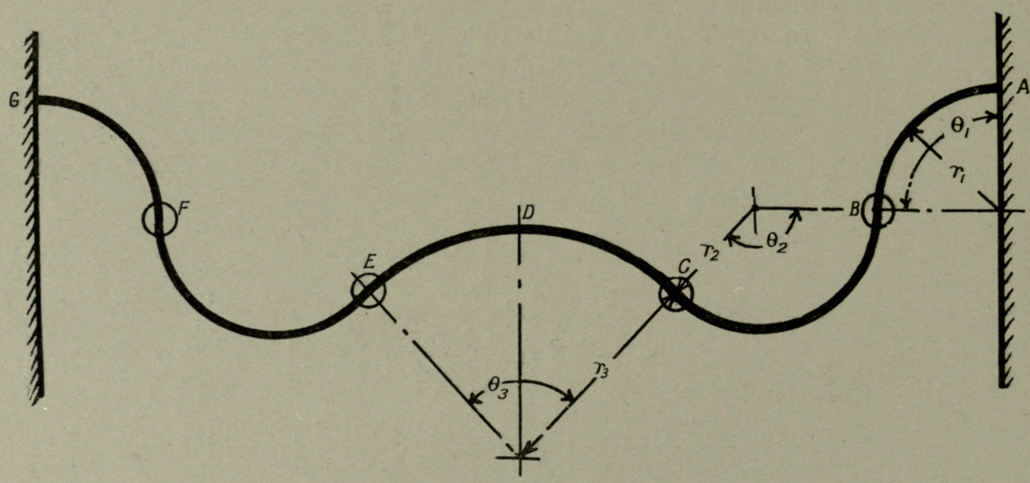


Fig. 30.

proportion in which the value of P is diminished. Owing to the comparative shortness of the cantilever it will generally be more economical to design this so as to take the load P corresponding to a rigid support, and to erect this with sufficient camber to allow its deflection under this load to bring it down to the level of the end supports. The same reasoning in general holds however many intermediate supports may be used.

(24) Compound Bow-Girder.

The state of equilibrium of a compound bow-girder of the type illustrated in Fig. 30, may be obtained by an application of the methods used for the simpler forms. In the case shown, with intermediate supports at the points of inflexion B, C, E and F, the whole of the reactions and the end moments M_a, M_g, T_a, T_g, are unknown. From symmetry, however, with uniform loading $M_g = M_a$; $T_g = T_a$; $R_f = R_b$; $R_e = R_c$; $R_g = R_a$, so that in effect the only unknowns are M_a, T_a, R_a, R_b, R_c.

Knowing the radii r_1, r_2, r_3, and the angles θ_1, θ_2, θ_3, the total load on the girder, and the position of its centre of gravity, may readily be obtained as in art. 16, p. 28.

Calling wl the load, let $\bar{x}$ be the distance of its centre of gravity from the line joining AG, and let x_1 and x_2 be the distances of supports B and C from this line.

Then taking moments about AG gives

$$T_a = \frac{wl\bar{x}}{2} - (R_a x_2 + R_b x_1).$$

Again
$$\frac{wl}{2} = R_a + R_b + R_c,$$

so that if R_b and R_c are known, R_a and T_a may be deduced from these equations. This leaves in effect three unknowns, M_a, R_b and R_c, and in order to determine these, three further equations are necessary.

These are to be obtained as follows:—

(1) *Span AB.*—Write down the expressions for the slope and deflection at B in terms of R_a, M_a, and T_a. These are the same as equations (20) and (21), pp. 30 and 31, with r_1 taking the place of r. Equating the deflection at B to zero gives the first of the required relationships.

(2) Determine values of M_b and T_b from equations (18) and (19), p. 30, in terms of R_a, M_a, and T_a.

(3) *Span BC.*—Obtain the slope and deflection at C in terms of M_b, T_b, R_a and R_b, and of the slope at B. Equating the deflection at C to zero gives the second of the required relationships.

(4) From equations (18) and (19) determine M_c and T_c.

(5) *Span CD.*—Obtain the slope at D and equate to zero. This gives the third relationship.

(25) Shear Force at a given Section.

The vertical shear force at any section of a bow-girder is the same as would be experienced at the corresponding section of a straight girder subject to the same loading and to the same reactions. Thus, between an end support—reaction R_a—and the first concentrated load W_1, the shear force is constant, except for the weight of the girder itself, and equal to R_a. Between this load and a second load W_2, the reaction is $R_a - W_1$.

In the case of a uniformly loaded girder, carrying w lbs. per foot run, the shear force at a distance x, measured along the arc, from the support A is $R_a - wx$ for all points between the end and any intermediate support. If there be an intermediate support at a distance x_1 from the end A, and if its reaction be P_1, the shear force at a point distant x from A, between this intermediate support and any third support, is given by

$$R_a + P_1 - wx$$

and so on.

(26) Experimental Verification of Formulæ.

In order to verify the formulæ of this chapter by experiment, measurements of deflection have been made by the authors on a series of bow-girders fixed at one or both ends and loaded either by single concentrated loads or by a uniform load. Some of these girders were of circular section, others of angle section. Values of EI and of CJ were obtained by deflection and torsion experiments on straight lengths of the same sections, and these values have been adopted in the calculations.

The following are the results of the experiments:—

TABLE II.

Series.	Type of section.	Conditions.	Angle subtended by arc.	Deflection (ins.).	
				Measured.	Calculated.
a	Circular	Circular arc cantilever with weight at free end	90° 135°	1·469 4·475	1·475 4·475
b	,,	Ditto with uniform loading	90°	·510	·502
c	,,	Semicircular bow girder fixed at ends with single load at a from one end—deflection at weight	$a = 30°$,, 45° ,, 60° ,, 90°	·043 ·117 ·202 ·307	·043 ·115 ·204 ·307
d	,,	Circular arc girder with single weight at centre	120°	·075	·074
e	,,	Ditto with uniform loading	180°	·310	·306
f	Angle	Circular arc girder with weight at centre	90° 180°	·011 ·124	·012 ·116
g	Angle	Semicircular bow girder with single load at 45° from one end	Deflection at weight	·036	·032
			Deflection at centre	·068	·072

From these figures it appears that there is a very close agreement between experimental and calculated values in every case.

(27) Non-circular Sections.

The foregoing formulæ are of general application to a beam of any section of which the EI and CJ are known. The former of these products is usually known or can be determined by calculation with a close degree of approximation for any commercial section. While the geometrical polar moment of inertia J of any section may also be calculated, the product of this J and the shear modulus C of the material does not, however, give the effective value of CJ for use in these formulæ, except in the case of circular sections. The reason for this and the question of the effective value of J for non-circular sections is considered in some detail in the following chapter.

CHAPTER III

(28) The Torsional Rigidity of Non-Circular Sections.

On the assumptions that the displacement of every point in a section under torsion is proportional to its distance from the centroid of the section, and that a section originally plane remains plane after straining, the angle of twist of a straight member of length l is given by

$$\theta = \frac{Tl}{CJ} \quad . \quad . \quad (28)$$

where J is the polar moment of inertia of the section, as deduced from its geometrical properties.

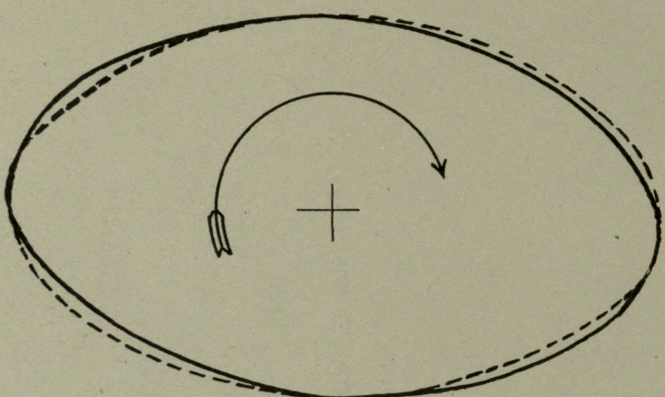


Fig. 31.

If the section is circular, these assumptions are fully justified by experiment so long as the stresses involved are within the elastic limit of the material.

But this is not the case for any but a circular section. In any other section radial lines originally straight do not remain straight after straining, and sections originally plane become warped under strain. For example, Fig. 31 shows the shape assumed by each section of an elliptical shaft, and Fig. 32 indicates the deformation of a square section under strain. The net result of this is that a given torque produces a greater angular displacement than is indicated by formula (28), and the angle of twist is given by

$$\theta = \frac{Tl}{CkJ} = \frac{Tl}{CJ'} \quad . \quad . \quad (28\text{A})$$

where J' is the effective polar moment of inertia of the section.

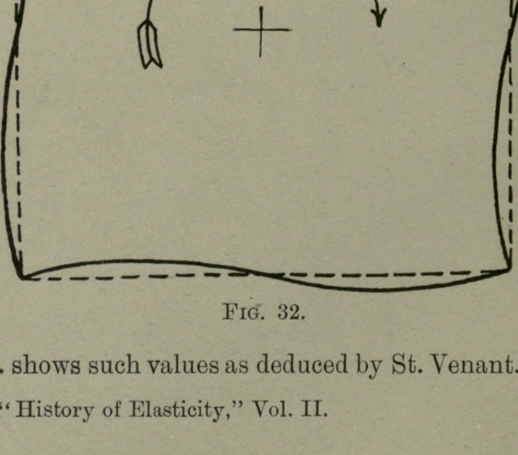


Fig. 32.

In a few simple cases, where the profiles of the section are the graphical representations of definite mathematical functions, values of J' may be deduced from considerations of strain, and Table III. shows such values as deduced by St. Venant.[1]

[1] Todhunter and Pearson, "History of Elasticity," Vol. II.

TABLE III.

Type of Section.	Remarks.	Effective value of $J\ (=J')$.
Solid ellipse . . .	Major axis — $2a$ Minor axis — $2b$	$\dfrac{\pi a^3 b^3}{a^2+b^2}$
Hollow ellipse . . .	Major axes, $2a$ and $2a_1$ Minor axes, $2b$ and $2b_1$	$\dfrac{\pi a^3 b^3}{a^2+b^2}\left[1-\left(\dfrac{a_1}{a}\right)^4\right]$
Square	Side $= s$	$\cdot 14 s^4$
Rectangle	Lengths of sides, b and d	$\dfrac{db^3}{3}\left\{1-\dfrac{\cdot 6b}{d}\left(1-\dfrac{b^4}{12d^4}\right)\right\}$
Any symmetrical section, including rectangles, in which the ratio of outside dimensions in any two directions in a cross-section is not very great	$A=$ area of section $J=$ geometrical polar moment of inertia	$\dfrac{A^4}{40J}$

It becomes apparent from St. Venant's investigation that there is always greatest distortion at that part of the section of a shaft or beam under torque, where the surface is nearest the axis. The distortion, and hence the intensity of stress, becomes very great at the apex of any re-entrant angle, becoming infinite where the apex of this angle

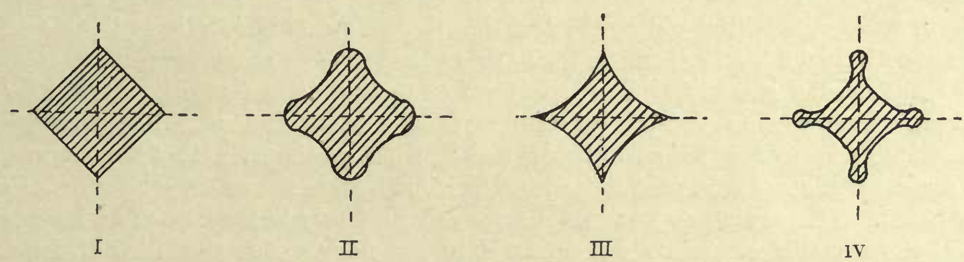


FIG. 33.

coincides with the centroid of the section. On the other hand, the distortion and stress in the neighborhood of projecting points is very small, so that while such projecting areas at a distance from the axis add largely to the magnitude of the polar moment of inertia, their effect on the tortional resistance of the section is usually inconsiderable. Thus such sections as are usual in I, or channel beams, and which offer a very efficient distribution of material to resist simple flexure, are relatively inefficient to resist torsion, and their inefficiency becomes more pronounced as the distance of their main members from the centroid of the section is increased.

As having an interesting bearing on these points the results of investigations on the following sections may be cited. These are (Fig. 33)

(1) Square section.
(2) Ditto with slightly concave sides, and round corners.
(3) Ditto ditto ditto and acute corners.
(4) Star-shaped section with four rounded points.

Writing $\theta = \frac{TL}{CJ'}$, where J', the effective moment of inertia of the section, equals kJ, St. Venant showed that the values of k for these sections were :—

Section	1	2	3	4
k	·843	·819	·778	·537

The concavity in section 3 was about $\frac{1}{20}$ of the length of the side, and this small degree of concavity reduces the value of k by approximately 8 per cent. As shown by the value of k for section 2, this concavity has more influence in diminishing the torsional stiffness of a beam, for the same moment of inertia, than the rounding of the corners has in increasing it. The large effect of a greater degree of concavity, accompanied by the massing of material in projecting points of the section, is well marked in section 4. As compared with a circular section of the same cross-sectional area and weight, these sections offer only ·891, ·867, ·828 and ·674 times respectively the resistance to torsion, notwithstanding the fact that the moments of inertia of their section are respectively 1·05, 1·06, 1·07, and 1·25 times that of the circular section.

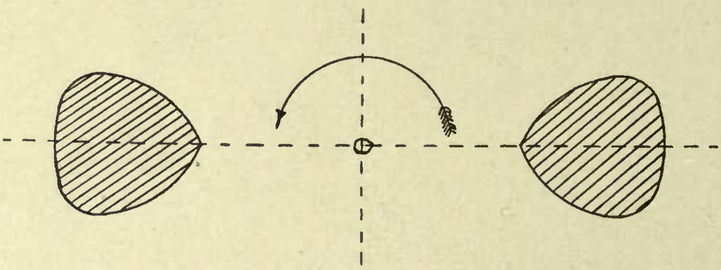


Fig. 34.

St. Venant's investigation of the form of section shown in Fig. 34 is also of interest. This section consists of two isolated masses of material symmetrically situated with respect to the axis of twist; and on the assumption that this represents the section of a beam subjected to torque, the investigation shows that the value of k is only ·0185. This section approximates more or less closely to the case of an I beam in which the material is mainly concentrated in the flanges, the thickness of the web being small. Comparison between this value for k, and the values obtained by experiment on I sections (see Table V.), is instructive. It is evident that a structural member consisting of two flat bars connected by a lattice bracing must of necessity be excessively weak in torsion.

For complex sections, and indeed for the great majority of commercial sections, the difficulties involved in a mathematical investigation of the value of J' are insuperable, and such values can only be determined from torsion experiments.

(29) Experimental Investigation of Torsional Rigidity of Commercial Sections.

Such experiments have been carried out by one of the authors and are described in the following pages. In all, twenty-one beam sections were tested. The details and dimensions of these are given in Table IV. With the exception of the solid circular and rectangular sections, and the welded tubes, which were of wrought iron, all were of mild steel.

TABLE IV.

No.	Section.	Dimensions.					Moments of Inertia (ins. units).		
		Width.	Depth.	Thickness of Flange.	Thickness of Web.	Area. sq. ins.	I_x.	I_y.	J.
1	I	5·01″	8·02″	·605″	·30″	8·02	91·10	13·10	104·20
2	do.	3·01″	3·00″	·325″	·200″	2·43	3·70	1·20	4·90
3	do.	1·75″	4·78″	·324″	·190″	1·91	6·70	0·26	6·96
4	do.	1·66″	3·16″	·23″	·17″	1·222	1·92	·177	2·097
5	do.	·99″	1·95″	·165″	·22″	·6825	·336	·0281	·3641
6	do.	·76″	1·50″	·165″	·14″	·4141	·1328	·0124	·1452
7	Channel	·97″	2·00″	·227″	·22″	·7825	·413	·0618	·4748
8	Angle	1·175″	1·175″	·250″	—	·5245	·0615	·0615	·1230
9	do.	1·00″	1·00″	·185″	—	·3363	·0275	·0275	·0550
10	Tee	1·58″	1·58″	·231″	·21″	·650	·1450	·0739	·2189
11	do.	·99″	·99″	·135″	·145″	·2573	·0236	·0108	·0344
12	Solid Rectangular	·87″	1·96″	—	—	1·70	·5460	·1075	·6535
13	do.	·51″	1·62″	—	—	·827	·1810	·0180	·1990
14	Solid Square	·96″	·96″	—	—	·920	·0702	·0702	·1404
15	Hollow Rectangular	·872″	1·432″	× ·0360″ thick		·151	·0479	·0223	·0702
16	Hollow Square	1·500″	1·500″	× ·0502″ ,,		·296	·1035	·1035	·2070
17	Solid Circular	1·01″ dia.				·801	·0510	·0510	·1020
18	do.	·876″ dia.				·601	·0288	·0288	·0576
19	Hollow Circular (Welded)	O.S. dia. 1·305″ I.S. dia. 1·05″				·473	·0826	·0826	·1652
20	Hollow Circular (Solid-drawn)	O.S. dia. 1·005″ I.S. dia. ·923″				·124	·0144	·0144	·0288
21	Hollow Oval (Solid-drawn)	·862″ × 1·74″ × ·045″ thick				·1788	·0515	·0173	·0688

The method of carrying out the torsion tests was as follows.—The beam under test was mounted between the centres of a six-foot lathe, centre-pops being made on the ends of the beam at the centre of gravity of the section, to receive the lathe centres. To one end of the beam was clamped a lever from which was suspended a hanger fitted with a knife edge, and carrying the load. Two pointers, each three feet long, could be clamped to the beam at any desired position. These pointers moved over scales, clamped to the bed-plate, and graduated in degrees and minutes. Readings were taken to the nearest minute. The other end of the beam was clamped to the head of the lathe, the gear being locked to prevent rotation.

On the addition of each increment of load, scale readings were taken at both pointers. In order to eliminate the effect of friction at the centres, the torque lever was elevated slightly, and allowed to decend slowly, depressed slightly, and allowed to rise slowly, the angle of mean position being noted. Observations were made for both loading and unloading, and the mean angle of twist per unit of load so obtained. The value of the product of C and J' was then found from the formula.

$$CJ' = \frac{Tl}{\theta}$$

where the symbols have the significance already ascribed to them.

In each case the experiment was repeated over a span of about half the original span. In no case did the two values of CJ' so obtained differ by more than 3 per cent.

The values of the product of E and I were also determined by supporting the beam on two massive knife-edges firmly bolted to the bed-plate. Load was applied to a hanger fitted with a hardened point, suspended from the middle point of the beam. Deflections were measured by means of a micrometer microscope sighted on to a silk fibre fixed to the beam. These deflections were observed to the nearest ·001 inch. Readings were taken for both loading and unloading, and the mean deflection per unit load calculated. The value of EI was then found from the relationship

$$EI = \frac{Wl^3}{48\delta}.$$

In order to obtain the values of the two moduli E and C, specimens were cut from the thickest part of each section, turned down to a diameter of about ·18 inch, and cut to a length of about 9 inches. The values of C were then found by means of a small torsion meter, and the values of E determined by supporting the specimens on knife-edges and applying a load at the middle of the span. The values of the constants so found have been tabulated in Table V., which also shows the results of the torsion and bending experiments on the beams.

The Bending Tests show that in general the experimental and theoretical values of $E\,I$ agree closely. In the few cases where a fairly large discrepancy exists between them, it is probably due mainly to the fact that the section was not perfectly uniform throughout the length of the beam. These figures indicate roughly the discrepancy that might be expected from calculations based on the ordinary suppositions that a beam is of uniform section throughout, and is perfectly straight from end to end.

One point of considerable interest is brought out in the above tests. It will be observed that in the case of the I, channel, and other sections, the values of E obtained are not equal for both axes of bending. In the case of the large I section, for instance, the observed values of E when the web is vertical and when the web is horizontal are respectively $30{\cdot}7 \times 10^6$ and $26{\cdot}4 \times 10^6$ in.-lb. units. In the former case, the web provides 14·5 per cent. and in the latter case only ·64 per cent. of the total moment of inertia. Generally speaking, therefore, the modulus of elasticity of the metal in the flanges is less than that of the metal in the web; this want of uniformity being undoubtedly produced in the process of rolling. This is confirmed by the results of experiments by Prof. E. Mar-

TABLE V.

Number.	Section.	Axis of Flexure.	Bending Experiments. Observed Value of $EI \div 10^6$.	Bending Experiments. Observed Value of $E \div 10^6$.	Bending Experiments. Value of $E \div 10^6$ from Small Round Specimen.	Bending Experiments. Theoretical Value of $EI \div 10^6$.	Bending Experiments. Ratio $\frac{\text{Theoretical } EI}{\text{Observed } EI}$.	Torsion Experiments. Observed Value of $CJ \div 10^6$.	Torsion Experiments. Value of $C \div 10^6$ from Small Round Specimen.	Torsion Experiments. Theoretical Value of $CJ \div 10^6$.	Torsion Experiments. Ratio $\frac{\text{Observed } CJ}{\text{Theoretical } CJ}$.	Observed Value of $\frac{EI}{CJ}$.	Theoretical Value of $\frac{EI}{CJ}$	Ratio $\frac{\text{Observed Value } \frac{EI}{CJ}}{\text{Theoretical Value } \frac{EI}{CJ}}$
1	I		—	—	—	2730·0	—	12·52	—	1252·0	·010	218·5	—	—
			—	—		390·5	—					31·15	—	—
2	do.		—	—	—	110·8	—	1·190	—	58·89	·0202	93·10	—	—
			—	—		36·0	—					30·30	—	—
3	do.		—	—	—	200·9	—	·699	—	83·50	·0084	287·0	—	—
			—	—		7·80	—					11·18	—	—
4	do.		59·0	30·70	30·0	57·8	·979	·267	11·32	23·60	·01129	220·8	2·445	90·7
			4·65	26·40		5·31	1·141					17·41	·225	77·5
5	do.		10·48	31·20	30·4	10·21	·976	·106	11·30	4·110	·0259	98·90	2·485	39·85
			·781	27·80		·853	1·092					7·35	·2075	35·47
6	do.		4·025	30·30	31·2	4·12	1·022	·0585	11·61	1·691	·0347	68·90	2·445	28·12
			·343	27·60		·386	1·127					5·86	·229	25·55
7	Channel		12·10	29·40	32·3	13·34	1·100	·1618	11·61	5·51	·0294	74·91	2·415	31·00
			1·661	26·90		1·990	1·195					10·29	·361	28·45
			1·661	26·90		1·990	1·195					10 29	·361	28·45
8	Angle		1·659	27·00	29·6	1·825	1·102	·1405	10·95	1·342	·1046	11·80	1·360	8·67
			1·748	28·50		1·825	1·045					12·43	1·360	9·15

9	do.		·713	26·00	28·7	·790	1·109	·0501	11·00	·605	·0830	14·22	1·308	10·87
			·739	26·90		·790	1·071					14·72	1·308	11·22
10	Tee		3·885	26·90	31·0	4·500	1·160	·1706	11·50	2·508	·0680	22·75	1·798	12·62
			3·940	27·22		4·500	1·142					23·08	1·798	12·86
			2·210	30·01		2·290	1·032					13·00	·911	14·24
11	do.		·599	25·32	30·0	·710	1·185	·0205	11·41	·391	·0524	29·20	1·818	16·10
			·590	25·00		·710	1·201					28·80	1·811	15·82
			·323	29·90		·324	1·002					15·78	·827	19·10
12	Solid Rectangular		15·50	28·41	28·0	15·26	·985	3·680	12·10	7·911	·4645	4·21	1·925	2·192
			3·02	28·00		3·00	·990					·822	·379	2·170
13	do.		4·79	26·52	27·0	4·90	1·021	·632	11·00	2·194	·2900	7·56	2·24	3·38
			·474	26·31		·490	1·033					·750	·224	3·36
14	Solid Square		1·905	27·10	29·8	2·095	1·098	1·370	11·35	1·595	·8591	1·390	1·310	1·06
15	Hollow Rectangular		1·270	26·60	26·85 Assumed	1·280	1·008	·531	11·00 Assumed	·772	·6880	2·390	1·656	1·441
			·601	27·10		·595	·989					1·133	·770	1·472
16	Hollow Square.		2·560	24·90	26·0	2·690	1·046	1·975	11·00 Assumed	2·28	·8670	1·301	1·180	1·102
17	Solid Circular		1·380	27·12		1·380	1·000	1·133	11·10	1·133	1·00	1·221	1·221	1·000
18	do.		·808	28·14		·808	1·000	·657	11·42	·657	1·00	1·232	1·232	1·000
19	Hollow Circular (Welded)		2·230	26·05		2·230	1·000	1·979	11·99	1·979	1·00	1·130	1·130	1·000
20	Hollow Circular (Solid-drawn)		·417	28·90		·417	1·000	·346	11·99	·346	1·00	1·202	1·202	1·000
21	Hollow Oval (Solid-drawn)		1·325	25·7	25·3 Assumed	1·302	·985	·495	11·00 Assumed	·755	·6560	2·66	1·73	1·540
			·425	24·92		·431	1·010					·859	·571	1·501

[To face p. 54.

burg,[1] in which tension test pieces were cut from the flange, web, and root, of several I beam sections. Tests on these specimens showed a considerable variation in E at different points in a section, and indicated generally a lower value of E for the flanges than for the web. The minimum value of E was invariably obtained from the test piece cut from the junction of web and flange. In the authors' experiments the channel section was tested with the web both in tension and in compression, and it is interesting to note that the flexural strength is the same in each case. In the angle sections also, the flexural rigidity is sensibly the same whether the flange is in tension or compression.

The Torsion Tests afford substantial confirmation of St. Venant's deductions as to the inefficiency of material in the neighbourhood of projecting points and of sharp corners in a beam section. The extreme weakness of all commercial sections is apparent from the figures given in column 12 of Table V. The inefficiency of I and channel sections is especially remarkable, while tee and angle sections are little better.

The hollow circular section is the most efficient of all for withstanding torsion. It is, however, inefficient when exposed to bending, and is for many reasons ill adapted for girder work. Next in order of efficiency comes the box section. So long as the ratio of depth to breadth is moderate, this is equally well adapted for resisting either torsion or bending, and would appear to afford the most economical distribution of material when both are to be resisted.

Solid and Hollow Rectangular Sections.—Reference to Table V. shows that k is sensibly the same for a hollow as for a solid square section, having a value ·86 in the latter and ·87 in the former case. The theoretical value of k deduced from St. Venant's formula for a solid square section is ·84 which is in close agreement with the experimental value.

The agreement between calculated and experimental results in the case of the solid rectangular sections is equally close. Thus for section 12 (Table V.), depth $\div$ breadth $= 2{\cdot}25$, St. Venant's formula gives $k = {\cdot}47$ against the measured value ·46, while for section 13, depth $\div$ breadth $= 3{\cdot}18$, the theoretical and measured values of k are each equal to ·29. For the hollow rectangular section No. 15 (depth $\div$ breadth $= 1{\cdot}64$), the experimental value of k is ·69, while St. Venant's value for a solid section with the same ratio of breadth to depth is ·68.

It thus appears that the value of k for a hollow rectangular section is sensibly the same as that of a solid section of the same overall dimensions; depends only on the ratio of breadth to depth and not on the thickness of the walls; and that the value is practically identical with St. Venant's theoretical value for the corresponding solid rectangle.

Values of k for such sections, having different values of the ratio, breadth $\div$ depth, are given in Table VI., while Table VII. shows how the effective value of J varies with this ratio in such sections having the same area or weight per foot run. It will be noted that while both k and J' diminish with an increase in the ratio, the relative diminution of J' is not nearly so great as that of k. The relative diminution of J' is approximately the same for hollow as for solid sections with the same overall dimensions.

Owing to the inefficiency of the material in the corners and at the ends of the flanges of a typical commercial box section (Fig. 35) under torsion, the value of J or of J' for such a section should be computed not on the whole area but on the portion included by the rectangle *abcd*.

[1] *Engineering News*, Vol. 62, 1909, p. 168.

TABLE VI.

Ratio. $\frac{\text{Greater Side, } 2c}{\text{Lesser Side, } 2b.}$	Value of k in $CJ' = k\frac{TL}{\theta}$.	Ratio. $\frac{\text{Greater Side, } 2c}{\text{Lesser Side, } 2b.}$	Value of k in $CJ' = k\frac{TL}{\theta}$.
1·0	·841	5·0	·135
1·5	·721	5·5	·113
2·0	·550	6·0	·096
2·5	·413	7·0	·073
3·0	·316	8·0	·057
3·5	·247	9·0	·045
4·0	·198	10·0	·037
4·5	·161	20·0	·010

TABLE VII.—EFFECTIVE VALUES OF J FOR RECTANGULAR SECTIONS HAVING THE SAME CROSS-SECTIONAL AREA.

Ratio $\frac{2c}{2b}$.	$2c$.	$2b$.	Theoret. J.	k.	Effective J or J'.
1	1·0	1·0	·166	·841	·140
2	1·416	·708	·209	·550	·115
4	2·00	·500	·354	·198	·070
6	2·448	·408	·511	·096	·049
10	3·160	·316	·917	·037	·034

Since an increase in depth renders a section more efficient to resist bending, the most effective value of this ratio when both torsion and bending are to be resisted, depends on the relative values of the two moments. With zero bending moment the section should be square. With zero torque, experience shows that the ratio of breadth to depth should be between 3·5 and 5·0 for best results. With both torsion and bending the most economical ratio will usually lie somewhere between 2·0 and 3·5, its value increasing as the ratio of bending moment to twisting moment increases.

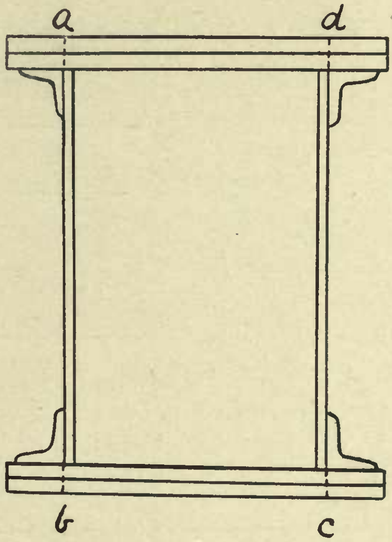


Fig. 35.

I Sections.—A comparison of the results of the torsion tests on I sections Nos. 1 to 6, Table V., indicates that the ratio of actual to calculated value of J diminishes with an increase in the size of the section. The penultimate column in Table VIII. gives the values of k for these sections. The value of J' in inch units is given with a fair degree of accuracy by the relationship

$$J' = \frac{A^2}{60} \quad . \quad . \quad . \quad (29)$$

where A is the area of the section in square inches. The last column of this table shows values of $A^2 \div 60$, while experimental values of J' are given in column 6.

Table VIII.

Section Number Table V.	Approximate Dimensions.	$\frac{2c}{2b}$	Area "A"	J	J'	k	$\frac{A^2}{60}$
1	$8'' \times 5''$	1·60	8·02	104·0	1·04	·010	1·07
2	$4\frac{3}{4}'' \times 1\frac{3}{4}''$	2·73	1·90	6·96	·058	·0083	·060
3	$3'' \times 3''$	1·00	2·43	4·90	·099	·0202	·098
4	$3'' \times 1\frac{1}{2}''$	1·90	1·22	2·10	·024	·0114	·025
5	$2'' \times 1''$	1·97	·682	·364	·0094	·0260	·0078
6	$1\frac{1}{2}'' \times \frac{3}{4}''$	1·97	·414	·145	·005	·0344	·0029

From these figures it appears that for sections 1 to 4 the formula gives results which are accurate within about 3 per cent. These are all commercial sections. The agreement is not so close for section 5, and is unsatisfactory for section 6. These two are not commercial sections, and the relative thickness of web and of flanges is much greater than in commercial sections, especially in section 6, in which the discrepancy is most pronounced. Probably for all normal commercial I sections expression (29) will give results sufficiently accurate for purposes of design.

Angle, Tee, and Channel Sections.—An examination of the results of the tests on the angle, tee, and channel sections of Table V., shows that the value of k varies widely with the type of section. The value of J' is given within about 2 per cent. in every case by the relationship

$$J' = \frac{A^{2 \cdot 3}}{m} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where m varies with the type of section. Values of k and of m are given in Table. IX.

TABLE IX.

Section.	Mean value of k.	m.
Channel .	·025	40
Tee . .	·06	25
Angle .	·09	18

Compound Girder.—Experiments were also carried out on a compound girder of the type shown in Fig. 36. This consists of two 8″ × 4″ commercial I sections, distant 10·3 inches centre to centre, and tied together at intervals of 2′ 6″ by plates across the bottom flanges. The value of J for this combination is 370 inch units; the value of J' is 2·05 inch units; and the value of k is ·0055. Calling A the total area of both sections,

$$J' = \frac{A^2}{110}$$

as compared with the value $\frac{A^2}{60}$ for a single girder of the same total weight per foot run as the combined girder.

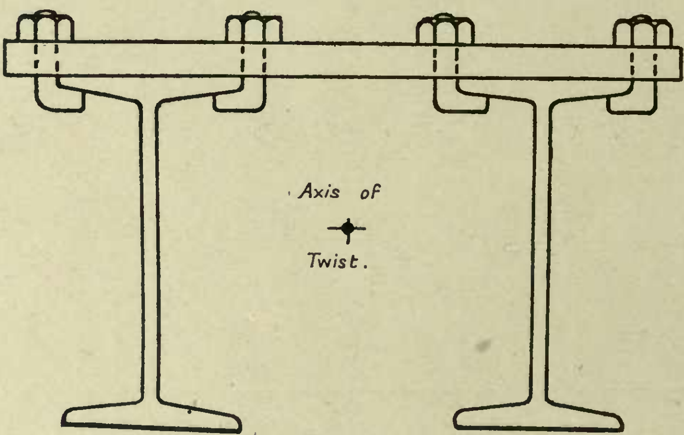


FIG. 36.

Tests on Hollow Box Sections filled in with Concrete.—Since in a hollow box section torsion is accompanied by distortion of the webs and flanges (Fig. 46) it was anticipated that by filling the interior of such a section with concrete this relative distortion might be reduced to some extent, and the section be stiffened in consequence. To test this point the hollow sections Nos. 15 and 16, Table IV., were filled with cement grout and, after setting for four weeks, were again tested in torsion. The effect of this is, however, not great. *E.g.*, with section (15), J' without filling was ·0483, and with filling ·0508, while in section (16) J' was increased from ·1645 to ·1941 by the filling.

CHAPTER IV

Magnitude of Shear Stresses in a Beam under Torsion

(30) Beam of Circular Section.

In a beam of circular section the shear produced by torsion is everywhere circumferential, and varies directly as the distance from the axis of twist. Thus if f be the magnitude of this shear at a radius r, and f_s its magnitude at the surface where the radius is a, we have

$$f = f_s \cdot \frac{r}{a}.$$

The moment of the shear on an elementary concentric ring of radius r and of radial width δr will therefore be

$$\frac{2\pi r^2 \cdot f_s \cdot \delta r}{a},$$

and on integrating this expression over the whole section of the beam and equating the result to the external torque T, we have

$$f_s = \frac{Ta}{J} \qquad \qquad (31)$$

Here f_s is the maximum circumferential shear in the section. This formula is applicable to both solid and hollow circular sections.

(31) Sections other than Circular.

In a non-circular section under torsion the assumptions that the shear at any point is perpendicular to the radius at that point and is proportional to its distance from the axis of twist, are no longer true. It has been shown both by St. Venant and by Bach [1] that the maximum transverse shear stress in any non-circular section under torque occurs at that point on the surface which is nearest to the axis of twist; that the stress is great in the neighbourhood of re-entrant angles and zero in the neighbourhood of projecting corners.

Expressions for the maximum shear in the case of a few of the simpler sections such as the ellipse and the rectangle have been deduced by St. Venant, and are given on p. 72. Autenreith [2] assumes that the stress at a given point P (Fig. 37) on the boundary of any solid or hollow section bounded by a continuous curve convex outwards, is given by

$$f_s = \frac{2T}{Ar} \qquad \qquad (32)$$

where T is the torque, A the area of the section, and r is the length of the perpendicular from the centroid of the section on to the tangent at P. The maximum shear stress will thus occur where r is a minimum, *i.e.*, at the end of the minor axis of the section, and the minimum surface shear at the end of the major axis.

On the same assumptions the surface shear in a hollow section having a continuous

[1] "Elastizität und Festizkeit."

[2] *Zeitschrift des Vereines deutscher Ingenieure*, 1901, p. 1099

boundary, in which the ratio of inner to outer radius is sensibly constant and equal to γ for all radii, is given by

$$f_s = \frac{2T}{A(1+\gamma^2)r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

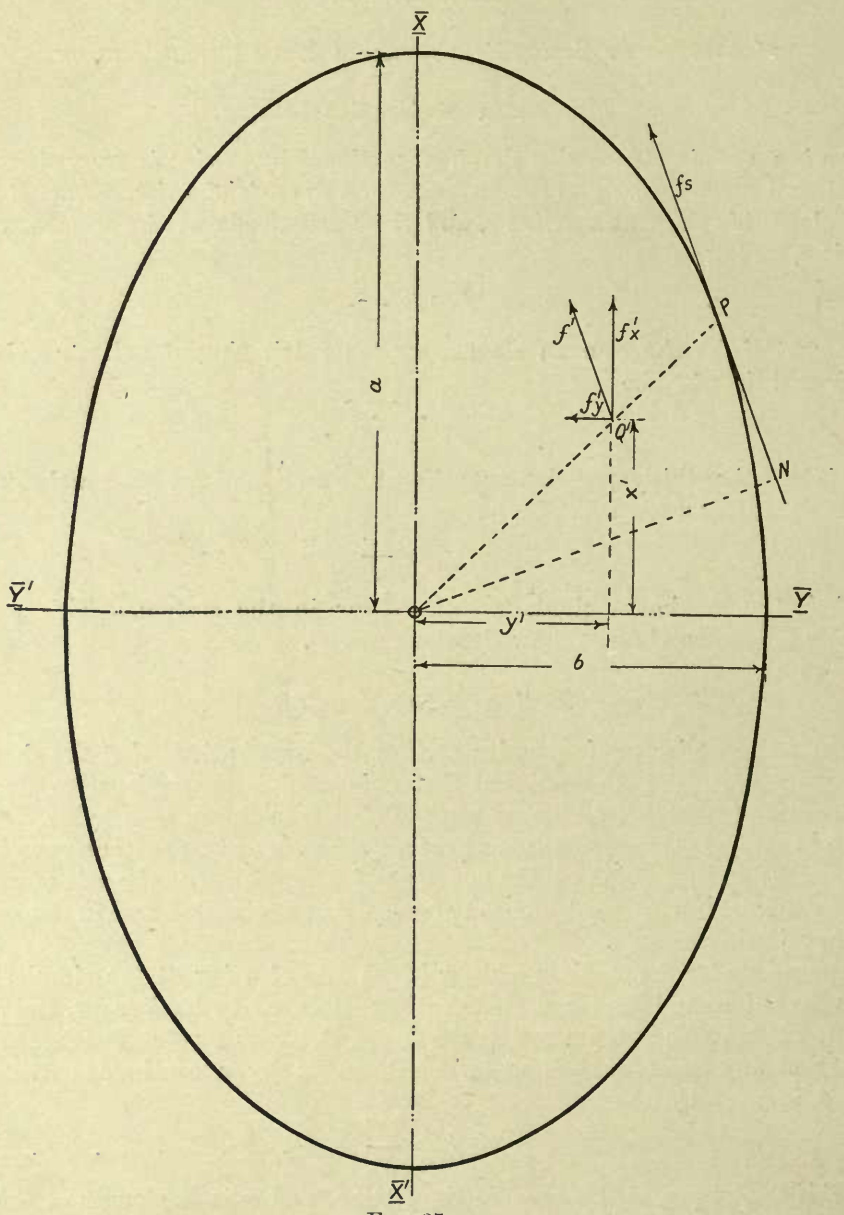


FIG. 37.

(32) Solid and Hollow Elliptical Sections.

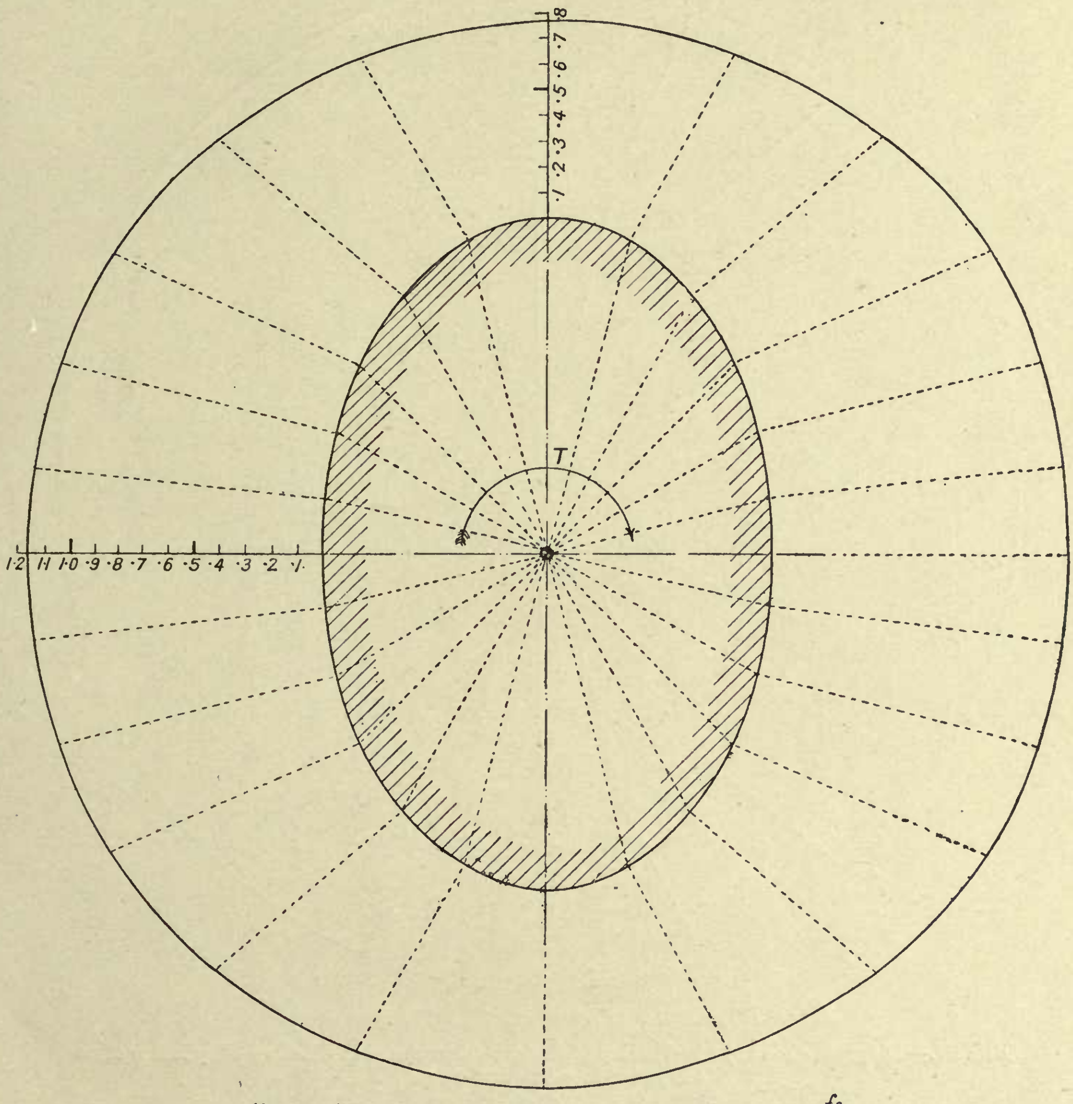
For a solid or hollow elliptical section, having semi-major and minor axes a and b, the value of r at any point P whose co-ordinates are xy (Fig. 37) is given by

$$r = \frac{a^2b^2}{\sqrt{a^4y^2 + b^4x^2}} = \frac{ab^2}{\sqrt{y^2(a^2-b^2)+b^4}} \quad . \quad . \quad . \quad . \quad . \quad (34)$$

In a hollow section having a and b as the semi-major and semi-minor axes of its external surface, the area of section is

$$\pi\left[ab - a_1b_1\right], \text{ and since } \frac{a_1}{a} = \frac{b_1}{b} = \gamma$$

$$\therefore\ A = \pi ab\left\{1 - \gamma^2\right\}$$



Note:– Intercepts of Normals Give Values of $\frac{f_s}{T}$

FIG. 38.—Diagram showing distribution of surface shear stress in a solid elliptical section subjected to a twisting moment.

Thus in the general case

$$f_s = \frac{2T}{\pi ab\left[1 - \gamma^2\right]\left[1 + \gamma^2\right]r}$$

$$= \frac{2T\sqrt{y^2(a^2 - b^2) + b^4}}{\pi a^2b^3\left[1 - \gamma^4\right]} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

and for a solid elliptical section ($\gamma = 0$) this becomes

$$f_s = \frac{2T\sqrt{y^2(a^2 - b^2) + b^4}}{\pi a^2 b^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

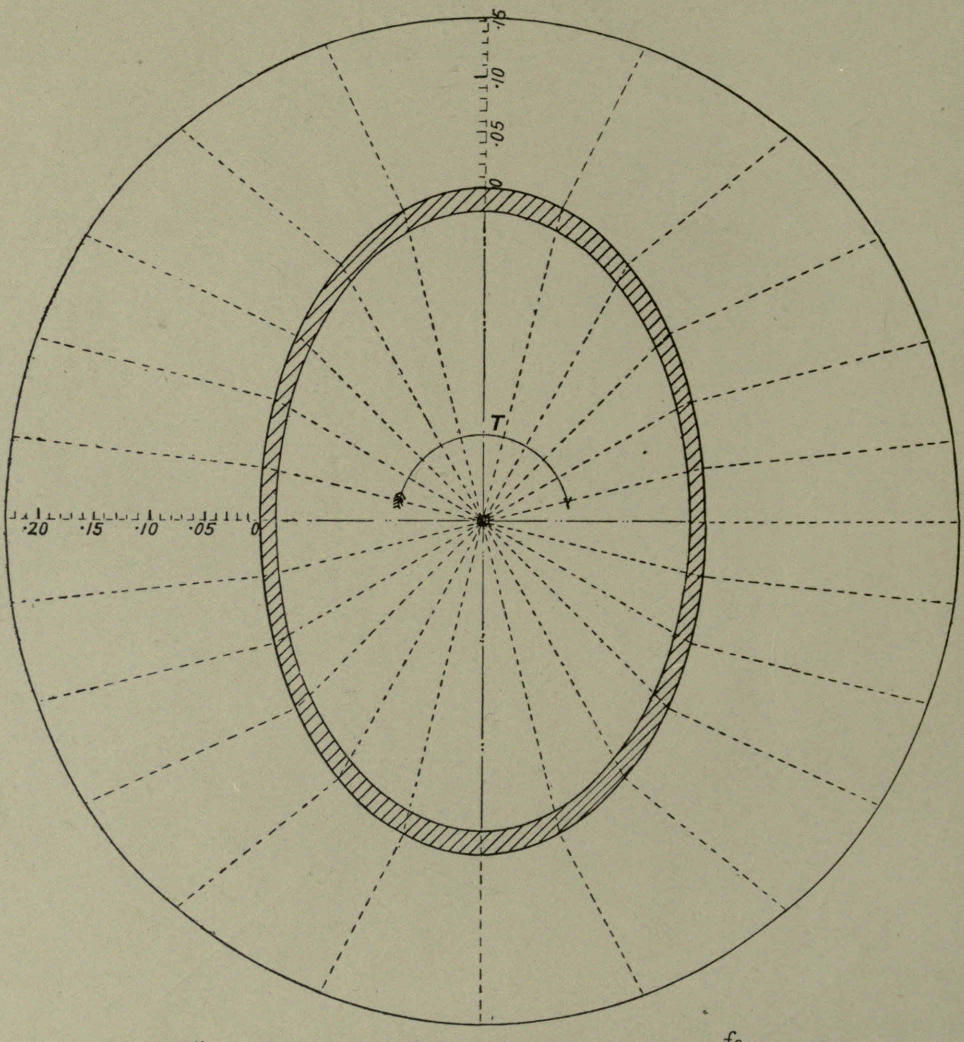


FIG. 39.—Diagram showing distribution of surface shear stress in a hollow elliptical section subjected to a twisting moment.

The maximum shear occurs at the end of the minor axis where $y = b$, and is given by

$$f_{(\text{max.})} = \frac{2T}{\pi a b^2 (1 - \gamma^4)} = \frac{2Tb}{\pi(ab^3 - a_1 b_1^3)} \quad . \quad . \quad . \quad . \quad . \quad (37)$$

which agrees with St. Venant's result.

The minimum stress on the periphery is given by

$$f_{(\text{min.})} = \frac{2T}{\pi a^2 b (1 - \gamma^4)} = \frac{2Ta}{\pi[a^3 b - a_1^3 b_1]}.$$

Where $a = b = r$, each of these expressions reduces to

$$f = \frac{2Tr}{\pi[r^4 - r_1^4]} \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

the expression for the shear at the periphery of a hollow circular section.

Figs. 38 and 39 show respectively the distribution of surface shear in a solid and a hollow elliptical section, in each of which $a : b = 1{\cdot}5$, while $\gamma = {\cdot}934$. These are subject to the same torque and have the same cross sectional area. The magnitude of the stress is indicated by the normal to the surface, intercepted between the surface and the curve. In this case the maximum stress in the solid section is 5 times as great as in the hollow section.

In a solid circular section of the same area the maximum stress is ·82 times that in the solid elliptical section, while in a hollow circular section having the same thickness and the same area as the hollow elliptical section, the maximum stress is ·76 times that in the latter section.

While the assumptions made in deducing the foregoing formulae give results in close agreement with experiment if the boundary is a continuous curved line, they fail to do so if the section has a discontinuous boundary. In the latter case the researches of Bach indicate a state of zero stress at projecting points, and, in an extreme case would postulate zero stress at the corners of a polygonal section no matter how closely this approximates to a circle. To obviate this difficulty Autenreith assumes that the stress at such a corner depends upon the included angle, being zero for a right angle, and that, at any point in the surface of such a section in which this angle is not less than 90°, it is given by

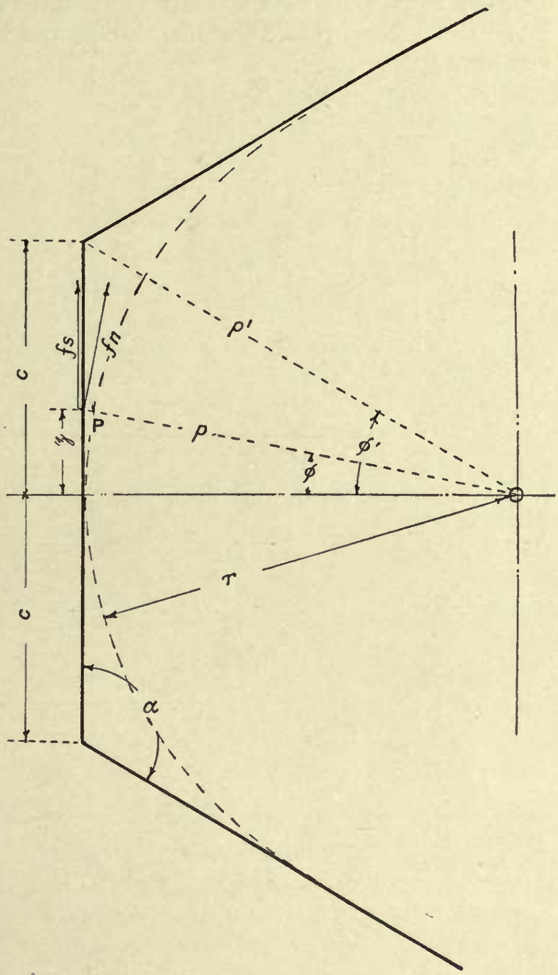


FIG. 40.

$$f_s = \frac{2\beta}{r}\left[1 - \left(\frac{z}{c}\right)^2 \sin a\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

where f_s is the circumferential shear stress; r the length of the perpendicular from the centroid to the corresponding side of the polygon; β a constant; z the distance from the mid point of the side to the point at which the stress is required; c half the length of the side; and a is the included angle (Fig. 40).

When $a = 180°$, *i.e.*, for a circular section, this makes $f_s =$ constant. When $a = 90$, *i.e.*, for a square or rectangular section, f_s becomes zero when $z = c$ (at corner), and attains a maximum value when $z = 0$, *i.e.*, at the centre of the side. In these two extreme cases the formula thus agrees with the results of experiment. Assuming that at any point in the interior of the section the component of the shear stress normal to the radius vector is proportional to the distance from the centroid, an expression may be obtained for the moment of the shear on any element, and on integrating this over the whole section and equating to the torque the value of the constant β may be obtained.

This is given by

$$2\beta = \frac{36T}{A[18 - 4\sin a]} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

where A is the area of the section.

Since $\rho \sin \phi = z$ (Fig. 40) equation 39 becomes

$$f_s = \frac{36T}{rA[18 - 4\sin a]}\left\{1 - \left(\frac{\rho \sin \phi}{c}\right)^2 \sin a\right\} \quad . \quad . \quad . \quad . \quad (41)$$

For a hollow polygonal section in which the ratio of inner and outer radii vectores is sensibly constant and equal to γ, this formula becomes

$$f_s = \frac{36T}{rA[18(1+\gamma^2) - 4\sin a(1+\gamma^2+\gamma^4)]}\left\{1 - \left(\frac{\rho \sin \phi}{c}\right)^2 \sin a\right\} \quad . \quad (42)$$

In each case the maximum shear occurs at the middle of the side of the polygon where $\phi = 0$, and is given by $\frac{\Omega T}{Ar}$, where, for a solid section,

$$\Omega = \frac{18}{9 - 2\sin a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

and, for a hollow section,

$$\Omega = \frac{18}{9(1+\gamma^2) - 2\sin a(1+\gamma^2+\gamma^4)} \quad . \quad . \quad . \quad . \quad (44)$$

(33) Rectangular Sections—Box Sections.

In a solid rectangular section (Fig. 41), whose longer side is $2c$ and shorter side $2b$, r for the shorter side is c, and for the longer side is b. Also $\sin a = 1$, so that, for the longer side equation 41 becomes

$$f_s = \frac{18T}{7Ab}\left[1 - \left(\frac{\rho \sin \phi}{c}\right)^2\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

and for the shorter side

$$f_s = \frac{18T}{7Ac}\left[1 - \left(\frac{\rho \sin \phi}{b}\right)^2\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (45\text{A})$$

Thus the maximum stress in the longer side (at its mid point, where $\phi = 0$) is given by

$$f_{(\text{max.})} = 2{\cdot}57\,\frac{T}{Ab} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

and the maximum stress in the shorter side by

$$f_{(\text{max.})} = 2{\cdot}57\,\frac{T}{Ac} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46\text{A})$$

At the corners in each case $f_s = 0$.

In the case of a hollow rectangular or box section in which γ is sensibly constant equation (42) applies. The shear at any point in the longer side is given by

$$f_s = \frac{18T}{Ab[7(1+\gamma^2) - 2\gamma^4]}\left\{1 - \left(\frac{\rho \sin \phi}{c}\right)^2\right\} \quad . \quad . \quad . \quad . \quad (47)$$

from which

$$f_{(\text{max.})} = \frac{18T}{Ab[7(1+\gamma^2) - 2\gamma^4]} \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

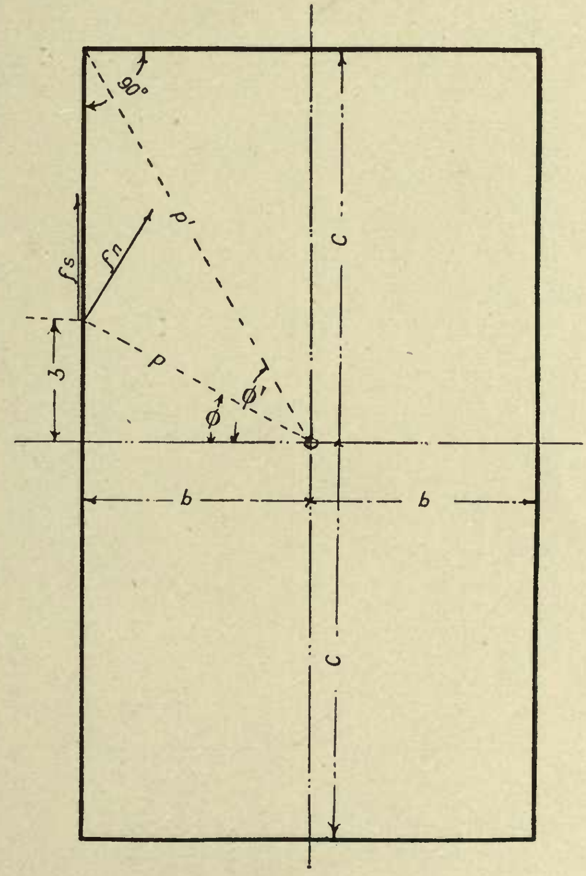


FIG. 41.

while for the shorter side

$$f_s = \frac{18T}{Ac[7(1+\gamma^2) - 2\gamma^4]}\left\{1 - \left(\frac{\rho \sin \phi}{b}\right)^2\right\} \quad . \quad . \quad . \quad . \quad (49)$$

and

$$f_{(\text{max.})} = \frac{18T}{Ac[7(1+\gamma^2) - 2\gamma^4]} \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

From equations (45) and (47) it appears that the curves of stress distribution in a rectangular section are parabolic.

Figs. 42 and 43 show such curves drawn respectively for a solid and a hollow rectangular section having the same ratio 1·5, of depth to breadth, and the same cross sectional area and weight per foot run. In the hollow section the ratio of inside to outside dimensions, or γ, is ·975. From these curves it appears that the maximum stress in the box section is about 19 per cent. of that in the solid section.

Comparing diagrams 39 and 43, it appears that the ellipitical section is the more efficient in that the maximum stress is only 72% of that in the box section. In the

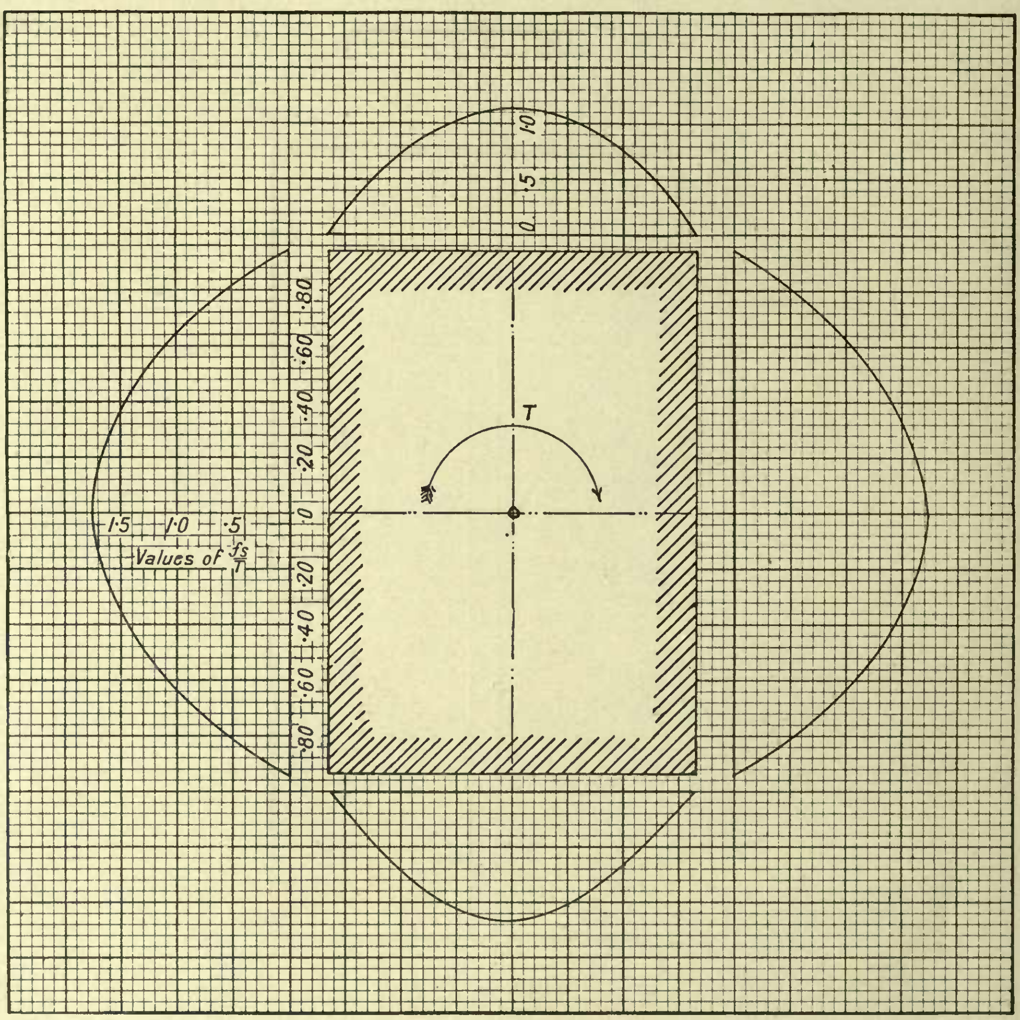


Fig. 42.—Diagram showing variation in surface shear stress in a solid rectangular section subjected to a twisting moment.

Ratio $\frac{\text{depth}}{\text{breadth}} = 1·5$; Area of section $= 2·4$.

ordinary box section used in practice the value of γ will not in general be the same for the top and bottom flanges as for the webs, nor can it be the same for different points on web or flange since these are of uniform thickness. From the following table, which shows calculated values of Ω in the formula

$$f_{(\max.)} = \frac{\Omega T}{Ar} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

for a hollow box section 4 ft. square and with different thicknesses of metal, it appears, however, that a given percentage variation in γ only produces about one-half the same

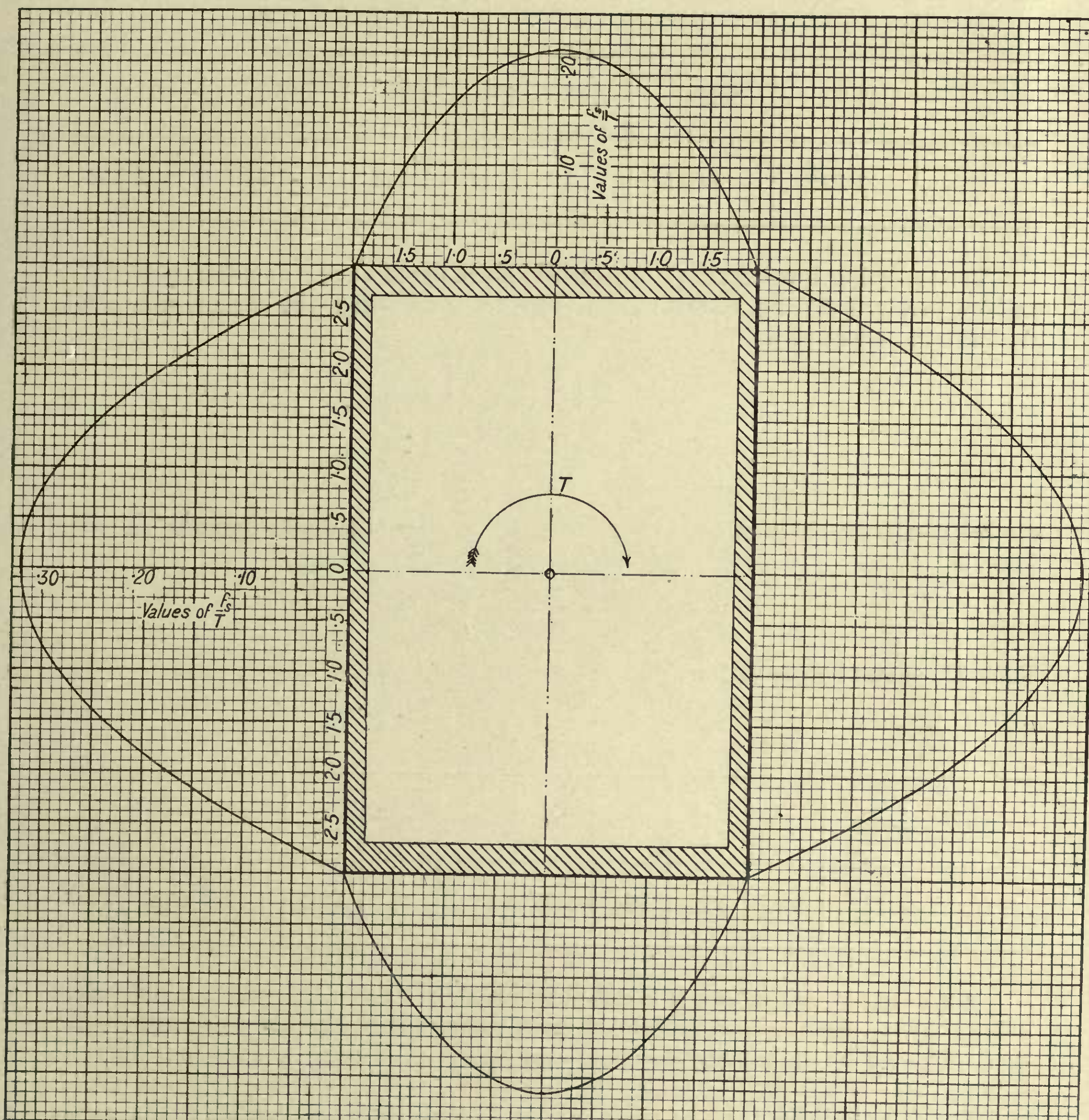


FIG. 43.—Diagram showing variation in surface shear stress in a rectangular box-section subjected to a twisting moment.

$$\text{Ratio } \frac{\text{depth}}{\text{breadth}} = 1{\cdot}5.$$

$$\gamma = {\cdot}975.$$

$$\text{Area of section} = 2{\cdot}4.$$

percentage variation in Ω. In practice the mean of the values of γ measured at the mid points of the two sides will give results within a few per cent. of the truth.

Thickness of Metal.	$\frac{1}{4}''$	$\frac{1}{2}''$	$\frac{3}{4}''$	$1''$	$1\frac{1}{4}''$	$1\frac{1}{2}''$
γ	·989	·978	·968	·958	·947	·937
A	47·75	95·0	141·7	188·0	233·7	279·0
Ω	1·510	1·517	1·532	1·538	1·542	1·548

The foregoing investigations of Autenreith are based upon a consideration of the stresses involved during torsion. St. Venant, considering the strains produced, obtained the expression

$$f_{(\text{max.})} = \left[\frac{15c + 9b}{40c^2b^2}\right] T$$
$$= \frac{\Omega T}{Ab}$$

for the maximum shear stress in a rectangular section of sides $2c$ and $2b$. In this formula

$$\Omega = 1{\cdot}5 + 0{\cdot}9\,\frac{b}{c}.$$

Table IX. shows how Ω varies with the ratio of depth to breadth.

Table IX.

Ratio $\frac{c}{b}$.	1	2	3	4	5	6	7	8	9	10
Ω (St. Venant) .	2·40	1·95	1·80	1·72	1·68	1·65	1·63	1·62	1·60	1·59

According to Autenreith Ω is independent of the ratio $c \div b$, and has a constant value 2·57, so that stresses calculated from Autenreith's formula are greater than those obtained by St. Venant, the difference becoming more pronounced as this ratio is increased.

Bach's experiments on the whole appear to show that Autenreith's values are in closer accord with the result of experiment, and for purposes of design these may be adopted with some confidence. The calculated stresses, if they err at all, will do so on the side of safety.

(34) I Sections.

Little definite is known as to the magnitude and distribution of stress in a member of I section under torque, except that the stress is greatest at the mid point of the web and is zero at the extremities of the flanges. Since the stress is always large in the neighbourhood of a re-entrant angle, it is probable that it will be large at the junction of web and flange, particularly where the radius of the fillet at this point is small. As to this point, however, no definite information is available.

From experiments on I sections made of lead Bach found that rupture always occurred at that point on the web nearest to the centroid of the section, and deduced the expression

$$f_{(\text{max.})} = 4{\cdot}5\,\frac{T}{At} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

where A is the total area of the section and t is the thickness of the web.

Some confirmation of this formula has been obtained by the authors. Thus considering I section No. 1 (Table IV.), the effective value of J' for the whole section is 1·04, while J' for the web if isolated from the rest of the section would be approximately ·086. Adopting these values, the web may be expected to take approximately $\frac{\cdot 086}{1\cdot 04} = \cdot 082$ of the total torque, and from formula (46), p. 64, the maximum stress in the web would then be equal to

$$\frac{2{\cdot}57 \times \cdot 082\, T}{Ab}$$

where b is the half thickness of the web, or $\frac{t}{2}$.

On making this substitution the formula becomes

$$f_{(\text{max.})} = \frac{4{\cdot}2\, T}{At}$$

which is in fair agreement with Bach's expression for the same stress.

Although the stress at other parts of the section is indeterminate, experiment shows that if the web is made stiff enough to withstand this stress the remainder of the section is amply strong.

(35) Horizontal Shear in a Beam Subject to Torsion.

Whatever be the magnitude of the transverse shear stress due to torsion at a point in a vertical section of a horizontal beam, this shear will be accompanied by an equal shear stress on the horizontal plane passing through the same point. In a beam of box section in which the depth exceeds the breadth, or in a beam of I section, the magnitude of this shear on horizontal layers is a maximum at the neutral axis.

(36) Resultant Shear on Horizontal and Vertical Sections of a Beam Exposed to Torsion or Bending.

The resultant shear at any point in a horizontal or vertical section of a beam is the algebraic sum of the shears due respectively to bending and to torsion. The shear stress due to torsion has already been discussed. The shear stress due to bending, or to the application of the vertical loads and reactions which produce bending, varies from point to point in a section.

If q denotes the intensity of shear due to this vertical loading at a point distant

z_1 from the axis of bending, and if the breadth of the section at this point be y_1, this shear stress is given by [1]

$$q = \frac{F}{Iy_1}\int_{z_1}^{z_2} yzdz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

where F is the shear force at the section in question, and z_2 is the distance of the outer fibres of the section from the neutral axis.

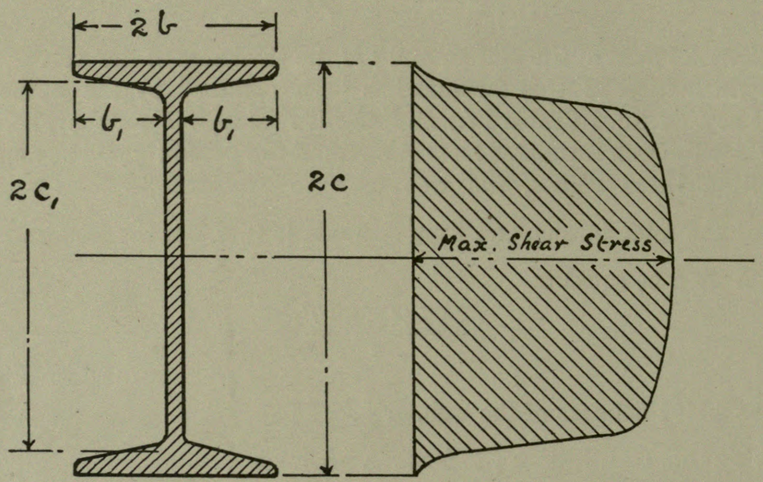


FIG. 44.

In a rectangular section of breadth $2b$ and depth $2c$, $y = y_1 = 2b$ is constant, while $y_2 = c$, and expression (53) becomes

$$q = \frac{F}{I}\int_{1}^{c} zdz$$

$$= \frac{F}{2I}[c^2 - z_1^2]$$

$$= \frac{3}{8}\frac{F}{bc^3}[c^2 - z_1^2] \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

This distribution of shear over the section is parabolic. The maximum value occurs at the neutral axis where $z_1 = 0$, and is equal to $\frac{3}{8}\frac{F}{bc}$ or $\frac{3}{2}\frac{F}{A}$, or to 1·5 times the mean shear over the section. The minimum value, zero, occurs at the outer extremity of the section where $z_1 = c$.

(37) I and Box Sections.

In the case of an I or rectangular box section the breadth is constant over the web and is suddenly increased at the flanges. As a result of this the magnitude of the shear stress in the flanges is much less than that in the web. The distribution of this stress is indicated in Fig. 44. In an average section the intensity of stress in the

[1] Morley, "Strength of Materials," Chapter V.

web does not change greatly, and the usual assumption that the web carries the whole vertical shear force with uniform distribution gives stresses which are in fair agreement with, and usually slightly higher than those actually attained.

In a hollow box section formed by the rectangles $2b$, $2c$, and $2b_1$, $2c_1$, or in the corresponding I girder (Fig. 44), in the flange at a height z_1 from the neutral axis.

$$q = \frac{F}{2I}\left\{c^2 - z_1^2\right\}$$

$$= \frac{3}{8}\frac{F}{[bc^3 - b_1c_1^3]}\left\{c^2 - z_1^2\right\} \quad . \quad . \quad . \quad . \quad (55)$$

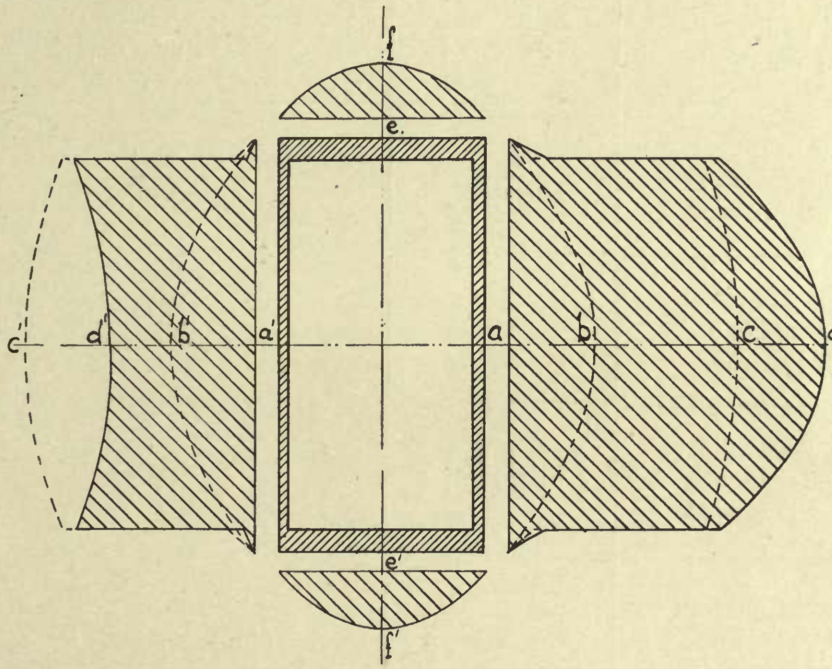


FIG. 45.

while in the web at a height z_1,

$$q = \frac{3}{8}\frac{F}{[bc^3 - b_1c_1^3]}\left\{\frac{b(c^2 - c_1^2)}{b - b_1} + c_1^2 - z_1^2\right\} \quad . \quad . \quad . \quad . \quad (56)$$

and, at the neutral axis,

$$q_{(\max.)} = \frac{3}{8}\frac{F}{[bc^3 - b_1c_1^3]}\left\{\frac{bc^2 - b_1c_1^2}{b - b_1}\right\} \quad . \quad . \quad . \quad . \quad (57)$$

It should be noted that whereas the shear on a vertical section produced by the vertical loading acts in the same direction at all points in the section, that due to torsion acts in opposite directions at opposite ends of a diameter. It follows that the shear stresses due to bending and torsion act in the same direction in one of the webs of a box girder, and in opposite directions in the other, and that under such combined moments one web will be much more heavily stressed than the other.

The nature of the resultant shear stress distribution over the vertical section of such a girder is indicated by the curves of Fig. 45.

Table X.

Type of Section.	Maximum surface shear stress.	
	St. Venant.	Autenrieth.
Solid Circular Radius r	$\frac{2T}{\pi r^3}$	$\frac{2T}{\pi \gamma^3}$
Hollow Circular radii r_1 and r_2	$\frac{2T\gamma_1}{\pi[r_1^4 - r_2^4]}$	$\frac{2T\gamma_1}{\pi[r_1^4 - r_2^4]}$
Solid Elliptical Major Axis $= 2c$ Minor „ $= 2b$	$\frac{2T}{\pi cb^2}$	$\frac{2T}{\pi cb^2}$
Hollow Elliptical formed by $[2c\ 2b]$ $[2c_0\ 2b_0]$	—	$\frac{2Tb}{\pi[cb^3 - c_0 b_0^3]}$
Solid Rectangular Long Side $= 2c$ Short Side $= 2b$	$\left[\frac{15c + 9b}{40c^2b^2}\right] T$	$\cdot 643 \frac{T}{cb^2}$
Hollow Rectangular $\gamma =$ const. Short side $= 2b$	—	$\frac{18}{[7\{1 + \gamma^2\} - 2\gamma^4]} \cdot \frac{T}{Ab}$
Any Polygonal Section Rad. of Inscribed Circle $= r$ Included Angle $= a$	—	$\frac{18}{[9\{1+\gamma^2\} - 2\sin\alpha[1+\gamma^2+\gamma^4]]} \cdot \frac{T}{Ar}$
I Web Thickness $= t$	—	$4\cdot 5 \frac{T}{A}$

CHAPTER V

(38) General Principles of Design of the Bow-Girder.

From the data of Chapters III. and IV., it appears that where a beam is exposed to any appreciable torsion, the box section is from every point of view the most suitable, and, for beams of considerable span, or carrying heavy loads, is the only practicable section. For comparatively small spans; for spans in which the radius of curvature is large and the angle sub-tended by the arc between successive supports is small, or for moderate loads, the I section may be permissible, but in general its use is to be deprecated wherever combined torsion and bending is anticipated.

In any case, where not barred by other considerations, intermediate supports are, as shown by the results of the investigations in Chapter II., of the greatest value in reducing the applied moments, and especially the twisting moment at a given section.

In a box section exposed to twisting and bending, a general consideration of the problem indicates that most economical results are to be obtained where the ratio of depth to breadth has a value somewhere between 2·0 and 3·5, the former value applying to encastré beams without intermediate supports and subtending an angle in the neighbourhood of 180°, and the latter for beams adequately supported at intermediate points or subtending angles not exceeding 45°. The following may be taken as affording a first approximation to the relative dimensions of such a girder designed for heavy duty:—

	Angle subtended by arc between supports.					
	180°	150°	120°	90°	60°	30°
$\frac{\text{depth}}{\text{breadth}}$	2·0	2·25	2·5	2·75	3·0	3·25

Having assumed a suitable section for the girder, the tensile and compressive stresses due to the bending moment, and the shear stresses due to the vertical loading, are to be determined for each section of the girder, as in the case of a straight girder, the value of the bending moment being obtained from the data of Chapter II. The value of the twisting moment at each section having been calculated in the same way, the shear stress due to this may be determined by an application of the results of Chapter III., and this shear stress is to be added to the shear stress due to the vertical loading, to give the actual shear at a given point in the section. In the box or I section both components of shear have their maximum value at the neutral axis. The shear in the flanges of such a girder, due to the vertical loading, is sensibly zero. That due to torsion is in general also small, and where the flanges are of adequate thickness to withstand the direct stresses due to bending there is little question as to their ability to take care of the additional small stress due to torsion. Having obtained the resultant shear in the webs, these should be designed by the ordinary rule applicable to the web of a straight plate-web girder subject to the same stress.[1]

Under torsion such a girder tends to buckle as shown by the dotted lines of Fig. 46, and particular attention should be paid to stiffening the webs against this action. Under normal circumstances this may be accomplished by the use of angle or tee stiffeners, between flanges, reinforced if necessary, where the torsion is greatest, by the addition of a cover-plate to the web.

The pitch of the stiffeners should, strictly speaking, diminish as the torsion increases. Where torsion is large the pitch should not exceed the depth of the girder, for girders less than 2 feet 6 inches deep, and should not exceed about one half the depth for a girder 6 feet deep.

Special attention should be paid to the design of the riveting at the junction of web and flange, since this has not only to withstand a shear of magnitude equal to that of the vertical shear at this point, but has also to resist the tendency to relative distortion indicated in Fig. 46. This latter effect also involves the use of somewhat heavier angle sections than are usual in the straight girder.

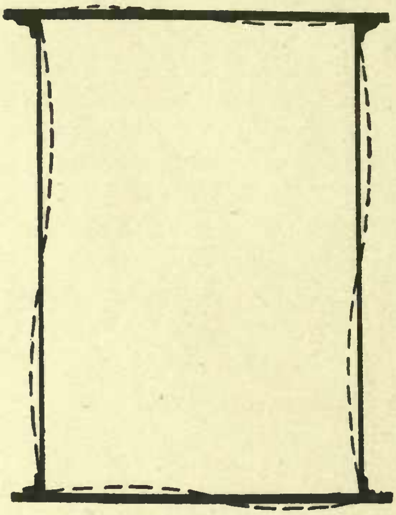


Fig. 46.

Where joints in the web plates are necessary these should be placed where the sum of torsional and load shear is a minimum.

As an example the preliminary design of a bow girder of uniform section of 30 feet radius, built in at the ends and subtending an angle of 120°, and carrying a uniform load of 2 tons per foot run, may be considered. The values of M_θ and T_θ for such a girder having $EI : CJ = 1{\cdot}25$, are given by the curves of Figs. 22 and 23, ϕ being 30°. From these curves it appears that M_θ has its maximum value ($\cdot 42\, wr^2$) at the support, while at this point $T = \cdot 048\, wr^2$. The maximum value of T_θ ($\cdot 052\, wr^2$) occurs at approximately 30° from the support, but since at this point M_θ is zero, and since the vertical shear force is only $wr \left\{\frac{\pi}{2} - \phi - \frac{\pi}{6}\right\}$ as against $wr \left\{\frac{\pi}{2} - \phi\right\}$ at the support, the latter will be the point of maximum resultant stress.

Preliminary investigation indicates that a box girder 5 feet deep and 2 feet wide, with flanges $1\frac{1}{2}$ inches thick and webs $\frac{1}{2}$ inch thick will be somewhere near the required section. For such a section $I = 104 \times 10^3$ (inches)^4 units; while $J = 110 \times 10^3$ units. From Table VI., k for the given ratio of depth to breadth is $\cdot 413$, so that $J' = 45{\cdot}5 \times 10^3$ (inches)^4 units. Assuming $E = 30 \times 10^6$ lbs. per square inch and $C = 12 \times 10^6$ lbs. per square inch, the effective value of $EI : CJ$ becomes $5{\cdot}73$.

From Figs. 19 and 20 it appears that the values of the end moments M_a and T_a for this value of the ratio when $\phi = 30$, are $M_a = \cdot 435 wr^2$ and $T_a = \cdot 067\, wr^2$.

The effective load per foot run, including the weight of the girder, is approximately $2{\cdot}2$ tons, so that the moments become

[1] See "The Design of Plate Girders and Columns," Lilley, or any similar work.

$$M = \cdot 435 \times 2\cdot 2 \times 900 = 880 \text{ ft. tons}$$
$$T = \cdot 067 \times 2\cdot 2 \times 900 = 133 \text{ ft. tons}$$

while the shear force $F = 2\cdot 2 \times 30 \times \frac{\pi}{2} \times \frac{120}{180} = 69$ tons.

Flanges.—Adopting a working stress of 6 tons per square inch in tension and compression, and assuming an effective depth of 57 inches, we have

$$6 \times a_f \times \frac{57}{24} = 880$$
$$\therefore\ a_f = 61\cdot 8 \text{ square inches}$$

where a_f is the flange area.

Assuming this to include $\frac{1}{8}$ the area of the webs ($= \frac{1}{8} \times 57 = 7$ square inches approx.) the required area of flange plates and angles is 54·8 square inches. This might be made up of

2 plates, $\frac{3}{4}'' \times 33''$	$= 49\cdot 5$	square inches
2 angles $6\frac{1}{2}'' \times 4\frac{1}{2}'' \times \cdot 55''$	$= 11\cdot 5$	,,
Total	61·0	

From this is to be deducted the area corresponding to two rivets, and assuming these to require 1-inch holes, this will be approximately 5 square inches, leaving an effective area of 56·0 square inches, or slightly more than is required.

Webs.—Calling a_w the area of the two webs, the maximum shear stress due to vertical loading $= \frac{69}{a_w}$ tons square inches. The maximum shear stress due to torque $= \frac{1\cdot 54T}{Ab}$ (p. 65, equation 48), where A is the effective area of the section to resist torsion and b is the breadth across the webs. Allowing $\frac{1}{2}$ inch between the edges of angles and of flange plates, $2b$ becomes equal to $33 - 10 = 23$ inches, while

$$A = (a_w + \text{area of a } 23'' \text{ width of flanges})$$
$$= a_w + \frac{99 \times 23}{33}$$
$$= a_w + 69 \text{ square inches}$$

The resultant shear stress in vertical and horizontal planes at the neutral axis is then given by

$$\frac{69}{a_w} + \frac{1\cdot 54 \times 133 \times 24}{(a_w + 69) \times 23}$$

Equating this to the working shear stress, say 3 tons per square inch, and simplifying gives

$$a_w^2 - 25\cdot 3a_w - 1587 = 0,$$

from which $a_w = 54\cdot 4$ square inches.

If t be the thickness of the web plates this makes

$$2t \times 57 = 54\cdot 4$$
$$\therefore\ t = \cdot 477 \text{ inch}$$

or, say, $\frac{1}{2}$ inch.

Rivets.—Assuming the centre line of the riveting at the junction of webs and flanges to be 3 inches from the edge of the web, or at a distance 25·5 inches from the neutral axis, the shear stress at this point due to the vertical loading is, by equation (56), p. 71, equal to 0·90 ton per square inch of web section.

The shear stress at the same point, due to torsion, is, by (47), p. 65, equal to

$$\frac{1{\cdot}54T}{Ab}\left\{1-\left(\frac{z_1}{c}\right)^2\right\}$$

where $\frac{z_1}{c}=\frac{25{\cdot}5}{30}$

so that this stress equals $\cdot2775 \times \frac{1{\cdot}54T}{Ab}$

$$=\frac{\cdot2775 \times 1{\cdot}54 \times 133 \times 24}{(57+69) \times 23}$$

$$= \cdot47 \text{ ton per square inch}$$

The resultant horizontal or vertical shear at this point is therefore $\cdot90 + \cdot47 = 1{\cdot}37$ tons per square inch.

Considering one of the web plates, the horizontal shear force corresponding to the shear stress over a horizontal length p inches is

$$1{\cdot}37\, pt \text{ tons}$$
$$= \cdot685\, p \text{ tons}$$

Then if p be the pitch of the rivets and R the safe working resistance to shear of one rivet

$$p=\frac{R}{\cdot685} \text{ inch}$$

Adopting a working stress of 5 tons per square inch for rivets in shear, and using $\frac{7}{8}$-inch rivets (area ·602 square inch), gives

$$p=\frac{5 \times \cdot602}{\cdot685}=4{\cdot}4 \text{ inches.}$$

To allow for the stress on the rivets due to the tendency to distortion indicated in Fig. 46, the pitch would be reduced to about 4 inches, or alternatively two rows of rivets with a correspondingly greater pitch would be used.

Stiffeners.—Considering the web as a column whose effective length is $\sqrt{2}$ times the distance between adjacent stiffeners the allowable mean shear stress depends on the ratio of this length l to the least radius of gyration "r" of the plate. For a $\frac{1}{2}$-inch plate $r\left(=\frac{t}{\sqrt{12}}\right)=\cdot144$ and $l \div r = 6{\cdot}92l$. In the case in question the mean stress in the web is approximately $(3 + 1{\cdot}4) \div 2 = 2{\cdot}2$ tons, and for this stress Moncrieff[1] has shown that the maximum permissible value of $l \div r$ is about 265. This makes $l = 265 \div 6{\cdot}92 = 38{\cdot}3$ inches, in which case the distance between the stiffeners would be $38{\cdot}3 \div \sqrt{2} = 27$ inches. As the shear diminishes, this distance is to be increased to suit, up to a maximum of about 3 feet 6 inches.

Over the end bearings the stiffeners should be designed as columns of sufficient strength to transmit the total load. Intermediate stiffeners would be about $4'' + 3\frac{1}{2}'' + \frac{3}{8}''$ angles.

For a more detailed examination of this point and of details of design the reader is advised to consult any modern work on the design of girders.

[1] J. M. Moncrieff, *Trans. Am. Soc. C. E.*, Vol. XLV., 1901. See also "Structural Engineering," Husband & Harby, Longmans & Co., p. 154.

APPENDIX A

The following list of integrals will be found of service in solving the various problems involved in the circular-arc bow-girder.

$$\int \theta \cos \theta \, d\theta = \theta \sin \theta + \cos \theta ;$$

$$\int \theta \sin \theta \, d\theta = \sin \theta - \theta \cos \theta$$

$$\int \cos^2 \theta \, d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} ;$$

$$\int \sin^2 \theta \, d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4}$$

$$\int \cos^3 \theta \, d\theta = \sin \theta - \frac{\sin^3 \theta}{3} ;$$

$$\int \sin^3 \theta \, d\theta = - \frac{\cos \theta}{3} (\sin^2 \theta - 2)$$

$$\int_0^{\theta_1} \sin (\theta_1 - \theta) \, d\theta = 1 - \cos \theta_1 ;$$

$$\int_0^{\theta_1} \cos (\theta_1 - \theta) \, d\theta = \sin \theta_1$$

$$\int_0^{\theta_1} \theta \sin (\theta_1 - \theta) \, d\theta = \theta_1 - \sin \theta_1 ;$$

$$\int_0^{\theta_1} \cos \theta \cos (\theta_1 - \theta) \, d\theta = \frac{\theta_1}{2} \cos \theta_1 + \frac{\sin \theta_1}{2}$$

$$\int_0^{\theta_1} \theta \cos (\theta_1 - \theta) \, d\theta = 1 - \cos \theta_1 ;$$

$$\int_0^{\theta_1} \sin \theta \cos (\theta_1 - \theta) \, d\theta = \frac{\theta \sin \theta_1}{2}$$

$$\int_0^{\theta_1} \cos \theta \sin (\theta_1 - \theta) \, d\theta = \frac{\theta_1 \sin \theta_1}{2} ;$$

$$\int_0^{\theta_1} \sin \theta \sin (\theta_1 - \theta) d\theta = \frac{\sin \theta_1}{2} - \frac{\theta_1}{2} \cos \theta_1$$

$$\int_0^{\theta_1} \cos^2 \theta \sin (\theta_1 - \theta) d\theta = \frac{1}{3} (1 + \sin^2 \theta_1 - \cos \theta_1) ;$$

$$\int_0^{\theta_1} \cos^2 \theta (\theta_1 - \theta) d\theta = \frac{\sin \theta_1}{3} (2 \cos \theta_1 + 1)$$

$$\int_0^{\theta_1} \sin^2 \theta \sin (\theta_1 - \theta) \, d\theta = \frac{(1 - \cos \theta_1)^2}{3} ;$$

$$\int_0^{\theta_1} \sin^2 \theta \cos (\theta_1 - \theta) d\theta = \frac{2}{3} \sin \theta_1 (1 - \cos \theta_1)$$

$$\int_0^{\theta_1} \cos^3 \theta \sin (\theta_1 - \theta) \, d\theta = \frac{\sin \theta_1}{4} \left(\frac{1}{4} \sin 2\theta_1 + \frac{3}{2} \theta_1 \right)$$

$$\int_0^{\theta_1} \cos^3 \theta \cos (\theta_1 - \theta) \, d\theta = \frac{3}{16} \cos \theta_1 (\sin 2\theta_1 + 2\theta_1) + \frac{\sin \theta_1}{4}$$

$$\int_0^{\theta_1} \sin^3 \theta \sin (\theta_1 - \theta) \, d\theta = \frac{1}{4} \left(\sin \theta_1 + \frac{\sin \theta_1 \cos^2 \theta_1}{2} - \frac{3}{2} \theta_1 \cos \theta_1 \right)$$

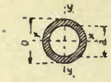
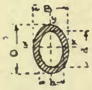
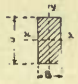
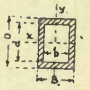
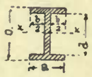
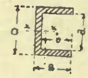
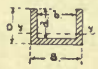
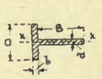
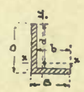
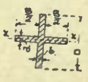
$$\int_0^{\theta_1} \sin^3 \theta \cos (\theta_1 - \theta) \, d\theta = \frac{3}{16} \sin \theta_1 (2\theta_1 - \sin 2\theta_1)$$

$$\int_0^{\theta_1} \sin 2\theta \sin (\theta_1 - \theta) \, d\theta = \frac{2}{3} \sin \theta_1 (1 - \cos \theta_1) ;$$

$$\int_0^{\theta_1} \sin 2\theta \cos (\theta_1 - \theta) \, d\theta = - \frac{2}{3} \cos 2\theta_1$$

APPENDIX B.

Moments of Inertia of Various Sections.

Section.	Moments of Inertia.	
	I_x	I_y
	$\frac{\pi D^4}{64}$	$\frac{\pi D^4}{64}$
	$\frac{\pi}{64}\left[D^4 - d^4\right]$	$\frac{\pi}{64}\left[D^4 - d^4\right]$
	$\frac{\pi B D^3}{64}$	$\frac{\pi D B^3}{64}$
	$\frac{\pi}{64}\left[BD^3 - bd^3\right]$	$\frac{\pi}{64}\left[DB^3 - db^3\right]$
	$\frac{BD^3}{12}$	$\frac{DB^3}{12}$
	$\frac{1}{12}\left[BD^3 - bd^3\right]$	$\frac{1}{12}\left[DB^3 - db^3\right]$
	$\frac{1}{12}\left[BD^3 - bd^3\right]$	—
	—	$\frac{1}{12}\left[bD^3 + Bd^3\right]$
	$\frac{1}{12}\left[BD^3 - bd^3\right]$	—
	—	$\frac{[BD^2 - bd^2]^2 - 4BDbd\,[D - d]^2}{12\,[BD - bd]}$
	$\frac{1}{12}\left[bD^3 + Bd^3\right]$	—
	—	$\frac{[BD^2 - bd^2]^2 - 4BDbd\,[D - d]^2}{12\,[BD - bd]}$
	$\frac{[BD^2 - bd^2]^2 - 4BDbd\,[D - d]^2}{12\,[BD - bd]}$	$\frac{[DB^2 - db^2]^2 - 4BDbd\,[B - b]^2}{12\,[BD - bd]}$
	$\frac{1}{12}\left[bD^3 + Bd^3\right]$	$\frac{1}{12}\left[bD^3 + Bd^3\right]$

$$J = I_x + I_y.$$

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