A STUDY OF PROGRESSIVE OSCILLATORY WAVES IN WATER

TECHNICAL REPORT NO. 1

WAR DEPARTMENT — CORPS OF ENGINEERS, U. S. ARMY

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BEACH EROSION BOARD
OFFICE OF THE CHIEF OF ENGINEERS

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WAR DEPARTMENT,
OFFICE OF THE CHIEF OF ENGINEERS,
May 19, 1941.

Approved for publication.

By authority of the Secretary of War:

J. L. SCHLEY,
Major General,
Chief of Engineers.

WAR DEPARTMENT

OFFICE OF THE CHIEF OF ENGINEERS

BEACH EROSION BOARD

The Beach Erosion Board of the War Department was established by Section 2 of the River and Harbor Act approved July 3, 1930 (Public, 520, 71st Cong.), to cause investigations and studies to be made in cooperation with the appropriate agencies of various States on the Atlantic, Pacific, and Gulf coasts, and on the Great Lakes, and the Territories, with a view to devising effective means of preventing erosion of the shores of coastal and lake waters by waves and currents. The act approved June 26, 1936 (Public, 834, 74th Cong.), authorized the board to make investigations with a view to determining the most suitable method of beach protection and restoration of beaches in different localities; to advise the States, counties, municipalities, or individuals of the appropriate locations for recreational facilities; and to publish from time to time such useful data and information concerning the protection of the beaches as the board may deem to be of value to the people of the United States.

As of September 1940, the membership of the Beach Erosion Board is: Col. Jarvis J. Bain, C. E., senior member; Lt. Col. John F. Conklin, C. E.; Lt. Col. Charles H. Cunningham, C. E.; Maj. A. C. Lieber, jr., C. E., resident member; Dean Thorndike Saville, College of Engineering, New York University; Gen. Richard K. Hale, Director, Division of Waterways, Department of Public Works, Boston, Mass.; and Prof. Morrough P. O'Brien, University of California; 1st Lt. William C. Hall is recorder of the board.

TECHNICAL REPORTS

Technical Report No. 1—A Study of Progressive Oscillatory Waves in Water.

Technical Report No. 2—A Summary of the Theory of Oscillatory Waves (in preparation).

The authority for publication of this report was granted by an act for the improvement and protection of the beaches along the shores of the United States, Public, 834, 74th Congress, approved June 26, 1936.

This paper reports the results of the first of a series of experiments adopted by the board in 1939. It deals with the basic characteristics of oscillatory wave motion in water. The experimental work was under the direction of Lt. William C. Hall, C. E., and the report was prepared by Dr. Martin A. Mason, Chief of the Research Section, Beach Erosion Board.

A. C. Lieber, Jr.,
Major, Corps of Engineers,
Resident Member.

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LIST OF SYMBOLS

a=A function of the wave amplitude or height (feet).
$\alpha=\pi h/L$
C=The velocity of wave propagation (feet per second)
d=Still water depth (feet).
e=Base of natural logarithms.
g=Acceleration of gravity (feet per second per second).
h=Wave amplitude or height (feet).
h'=Depth below still water level (feet).
$k=2\pi/L$
L=Wave length (feet).
T=Wave period (seconds).
U=Velocity of mass transport (feet per second).
x=The horizontal semi-axis of a wave orbit (feet).
X=Horizontal axis of reference.
z=The vertical semi-axis of the wave orbit (feet).
Z=Vertical axis of reference.

SUBSCRIPTS

ave=Indicates an average value.
m=Indicates a measured value.
s=Indicates a value at the water surface.
t=Indicates a theoretical value.
z=Indicates a value at depth z.

DEFINITIONS

PROGRESSIVE OSCILLATORY WAVE—A periodic wave motion of finite amplitude advancing over a water surface without change of type; the length of wave is large compared to the wave height.

DEEP-WATER WAVE—A wave traveling in a depth of water greater than half the wave length. This definition, and that following, are used for purposes of classification, and do not denote a distinct physical separation of characteristics.

SHALLOW-WATER WAVE—A wave traveling in a depth of water less than half the wave length.

STILL WATER LEVEL—The level at which water would stand if wave action ceased.

WAVE LENGTH—The horizontal distance from crest to crest of the wave under discussion.

WAVE PERIOD—The time interval between consecutive similar phases of a wave motion.

WAVE HEIGHT—The vertical distance from top of wave crest to the bottom of the wave trough.

WAVE VELOCITY—The rate of advance of the wave crest relative to a fixed point.

ORBIT RADIUS—(Orbit Semi-axes)—The radius of the circle, or the semi-axes of the ellipse, which a water particle describes during the passage of an oscillatory wave.

MASS TRANSPORT—The translatory movement of a water particle induced by the passage of an oscillatory wave in deep water.

A STUDY OF PROGRESSIVE OSCILLATORY WAVES IN WATER

1. INTRODUCTION

Present knowledge of the motion of oscillatory waves in water consists almost wholly of theoretical studies made by mathematicians seeking either a solution of the classical hydrodynamical equations for an incompressible heavy liquid, which satisfies certain conditions dictated in part by logic and in part by experience, or a solution by geometrical methods based on observation of natural phenomena. Certain well-determined rigorous solutions of the problem, as posed, differing chiefly in their definition of the motion of the water particles involved have been obtained.

Gerstner's theory,[1] which was the first solution obtained, is based on geometrical considerations, and states that the wave surface is a trochoid, the paths of the water particles being circles. The diameters of these circles, or particle orbits, decrease with increasing depth below the surface according to an exponential law. The motion of the wave is rotational. The Gerstner solution rigorously satisfies the equations of motion and limiting conditions of the problem regardless of wave height.

Stokes criticized Gerstner's solution for requiring a rotation, since Lagrange had shown that the motions of liquids generated from rest under the influence of impulsive forces are necessarily irrotational; but did not otherwise question the exactness of the Gerstner solution.

This criticism led Stokes and others to search for a solution to the problem of surface waves, which would satisfy all the limiting conditions and be such that the eddy vector, or rotation, would everywhere be zero. Approximate solutions through a fifth approximation were obtained by Stokes, who failed, however, to prove the convergency of the series used in the approximations and consequently the exactness of his solution. Levi-Civita, following Stokes' work, succeeded in showing the series to be convergent and obtained a rigorous solution of the wave problem for the condition of infinite depth identical to that of Stokes.

In these latter solutions it is found that the particle orbits are not closed paths but open, indicating a current (defined as "mass transport") in the direction of wave propagation. Also, the velocity of propagation of the wave is dependent upon its height; whereas Gerstner's solution indicates independence from the wave height.

[1] References to all works cited will be found in appendix II, Bibliography.

There exist, therefore, two rigorous mathematical solutions of the wave problem for water of infinite depth: that of Gerstner (rotational, closed particle orbits, velocity of wave independent of height); and that of Stokes-Levi-Civita (irrotational, open particle orbits, velocity of wave dependent on wave height).

The validity of the solutions discussed is limited by the condition of infinite water depth. For the case of finite depth of water, in which we may consider the depth to be of an order of magnitude less than the wave length, the solution of the problem is much more difficult. Laplace, and later Airy, found a rigorous solution of the problem which is comparable to the solution of Gerstner for infinite depth. The Laplace-Airy solution defines the surface as an elliptic trochoid, the particle orbits as closed ellipses, the motion as rotational, and the wave velocity as independent of the wave height.

Stokes investigated this problem and arrived at a solution to a third approximation; the particle orbits are open, and the motion is irrotational. This solution is not, however, rigorous, and it remained for Struik, using the methods of Levi-Civita, to first find a rigorous solution of the problem of an irrotational wave in finite depths. Struik's solution is similar to Stokes' third approximation, and shows that the velocity of propagation depends on the wave height, the particle orbits are open ellipses, and the motion is irrotational.

The present status of oscillatory wave theory may be summarized, then, as follows: Rigorous solutions for infinite water depths; solutions of Gerstner and of Stokes-Levi-Civita (similar to those of Rayleigh); rigorous solutions for finite water depths; solutions of Laplace-Airy and of Stokes-Struik.

As noted by Favre, experimental confirmation of none of these theoretical solutions has been obtained. Observations of wave motion in the sea are very difficult to obtain with a high degree of accuracy and while laboratory studies may be made with facility and high accuracy, there is no confirmation that the waves studied are true counterparts of ocean waves. Observations of wave phenomena in nature are essential; but laboratory studies leading to the evaluation of existing theories may be invaluable as a guide to field observation programs and technique.

2. PURPOSE

It is the purpose of this study to seek in the laboratory experimental confirmation of the theories outlined above, by comparison of experimental and theoretical values of various wave characteristics. The experimental data were obtained with the equipment described in section 4, and are valid only for the case of uniform water depth. A short summary of previous experimental work will be found in appendix I.

3. PROCEDURE

The study was divided into separate consideration of deep-water and shallow-water waves, with the distinction between wave types made on the basis of the commonly employed criterion cited by Gaillard; deep-water waves are those propagated in depths greater than half the wave length, shallow-water waves are those propagated in depths less than half the wave length. It may be noted that while this division of wave types is arbitrary, it is sufficient for the purpose of this study.

Twelve deep-water and twenty shallow-water waves, covering the available range of wave characteristics were studied.

4. EQUIPMENT AND METHODS

The wave tank.—The concrete wave tank is 85 by 14 feet with a depth of 4 feet. One side is fitted with six glass windows each 24 by 40 inches. Each window has etched scales graduated to hundredths of a foot.

The wave generator.—The wave generator is a counter-weighted wooden plunger with the forward face set at 51° with the vertical. It is driven by a variable speed, three horsepower direct current motor. Wave heights are varied by setting eccentrics, adjustable by quarter-inch increments to 7½ inches, and plunger connecting arms, adjustable in 1½-inch increments.

Wave absorber.—A 6-inch layer of ¼ to ¾ inch commercial gravel placed at an angle of 12° with the horizontal is constructed in the end of the tank opposite the wave generator to serve as a wave absorber. A screen consisting of one thickness of number 16 screen wire laced to a thickness of ¼-inch wire mesh is stretched four inches above and parallel to the top of the gravel slope. A wave absorber of similar design but of greater slope is placed behind the plunger.

Wave height measurement.—Three sets of point and hook gages were used to determine still water levels and to measure wave heights. A point gage was adjusted to the top of the wave crests and a hook gage to the bottom of the troughs; the wave height being the distance measured between point and hook.

Wave length measurement.—Each of two point gages, one fixed and the other movable horizontally, were connected in an electrical circuit with a neon bulb and ground in the tank. The gages were adjusted to just touch the wave crests, the passage of a crest completing the electrical circuit, and causing a short flash of the neon bulb. Adjustment of the movable gage, to secure synchronous flashing of the neon bulbs, indicated that the points were set on a wave length or its multiple.

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Wave velocity measurement.—Wave velocity was measured by timing the passage of a wave crest over a measured baseline. Float-operated mercury switches in electrical circuits flashed neon bulbs on a panel to trace the progress of wave crests. A two-way switch and a key permitted selection of a wave crest for measurement. A solenoid in the electrical system automatically started and stopped a stop-watch and the wave crest closed the circuit at the beginning and ending of the course.

Photographs.—A Retina II–f2 camera and a 16 mm. Eastman Ciné Special motion picture camera comprised the photographic equipment.

Profiles against a 6-inch grid on the wall of the tank were photographed with both cameras.

Particle motion shown by injecting drops of a fluid mixture having the same specific gravity as water was photographed at the window at the center of the tank. Still exposures and motion pictures were made.

The fluid used was a mixture of xylol and butyl phthalate colored with zinc oxide. Injection was made with a glass tube having a small orifice at one end and a rubber bulb at the other. A scale on the tank window served for the measurement of orbit dimensions.

Wave period.—The wave period was computed from the average period of several trains of waves.

Measured wave length.—The movable point gage was adjusted to obtain synchronous flashes for an even multiple of wave length. The number of waves measured was noted by visual observation, and the average length of wave thus measured was recorded as the wave length.

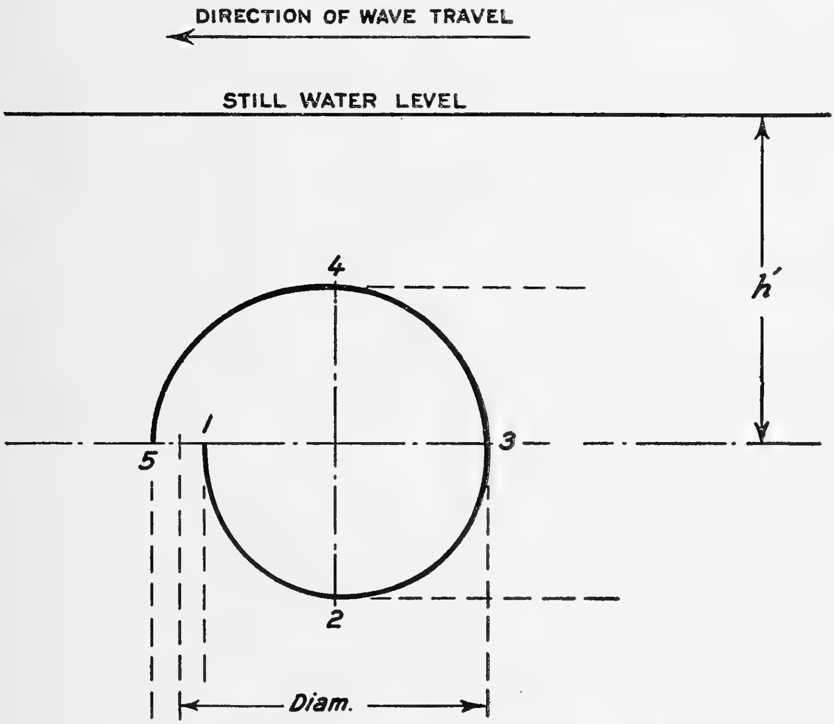
Measured wave velocity.—The wave travel was timed over a 30-foot course. The velocity in feet per second was obtained by dividing the length in feet by the time of travel in seconds.

Measured wave height.—Three combination hook and point gages were set at 12, 24, and 37 feet from the wave generator. In general, each point and hook was read to still water level before the beginning and after the end of the run. During each run, five readings were made. The average was taken for each gage, and the wave height recorded was the average for the three positions. Readings are to thousandths of a foot.

Wave profile.—The wave profile against the measured grid on the tank wall was scaled from the enlarged photograph of the wave. The ordinates were measured at the tenth points of the wave length and at additional points a twentieth and a fortieth of the length each side of the wave crest.

Orbit radius.—The measured value of the orbit radius was taken as one-fourth the sum of the horizontal and vertical diameters of the particle orbits obtained as described under "Photographs." The diameters were measured between vertical and horizontal tangents to the photographed curves. When the particle, because of mass transport or specific gravity effects, did not traverse a closed curve, the diameter was computed as shown on the curve, Figure 1.

It is assumed that the velocity due to mass transportation or differences in specific gravity were uniform over the period during which the orbit trace was obtained. Then the true diameter is the mean of the distance between the tangents at points 3–1 and 3–5.



ORBITAL MOTION

FIGURE 1.

The same reasoning is followed for the determination of the vertical diameters.

The depth of the orbit center below still water level is considered to be the depth of the mid point of the vertical diameter as determined above.

Mass transport.—The particle orbits were used to determine the existence and magnitude of mass transport. Referring to the figure of the preceding section, when a curve was open on the horizontal diameter, the amount by which it failed to close was considered as being due solely to mass transport. This distance (1–5 in the figure), in true magnitude, was divided by the wave period to obtain the velocity of mass transport. The direction of mass transport was determined by reference to the direction of travel of the wave, transport in the direction of wave travel being designated as positive.

Surface mass transport.—The existence and magnitude of surface mass transport was determined by a floating cork ball 1½ inches in diameter. The time for travel of the ball over a measured course of ten feet was observed and the velocity of surface mass transport obtained as $\frac{10}{T}$.

Percentage of wave height above still water level.—Measured values of the proportion of wave height above the still water level were obtained from wave profile photographs. The proportion is expressed as the ratio of the height of the wave crest above the still water level to the wave height. For each characteristic wave the percentage wave height above still water level was measured at three positions in the wave tank, 12, 24, and 37 feet from the wave generator.

5. DEEP-WATER WAVES

The basic data for the deep-water waves studied are given in table 1.

The ratio h/L is employed as a criterion of wave type—ratios greater than about 0.03 corresponding to storm waves—and as a parameter in the computation of wave characteristics by the Stokes-Levi-Civita equations.

TABLE 1.—*Basic data—Deep-water waves* $\left(\frac{d}{L}>0.5\right)$

1	2	3	4	5	6	7	8	9	10
Run	h_{ave}	T_m	C_m	L_m	d	d/L_m	h_{ave}/L_m	C_mT_m	$\frac{C_mT_m-L_m}{L_m}$
	Feet	*Sec.*	*F. p. s.*	*Feet*	*Feet*			*Feet*	*Percent*
1	0.276	0.850	4.39	3.89	3	0.771	0.0707	3.73	−4.4
2	.237	1.003	5.10	5.14	3	.584	.0460	5.17	−0.5
4	.230	0.860	4.40	3.83	2.5	.653	.0600	3.79	−1.0
5	.206	.985	5.00	4.87	2.5	.513	.0423	4.93	+1.2
13	.242	.848	4.42	3.79	3	.792	.0638	3.75	−1.1
14	.219	.969	4.89	4.76	3	.630	.0460	4.74	−0.4
17	.189	.966	4.81	4.75	2.5	.527	.0398	4.65	−2.1
18	.252	.847	4.23	3.57	2.5	.706	.0705	3.58	+0.3
19	.196	.842	4.31	3.55	3	.845	.0552	3.63	+2.2
20	.161	.996	5.07	5.02	3	.598	.0321	5.05	+0.6
22	.167	.843	4.33	3.60	2.5	.695	.0464	3.65	+1.4
23	.296	.988	5.05	4.92	2.5	.509	.0601	4.99	+1.4
Average									−0.20

The values listed in column 10, table 1, are of interest as showing the accuracy of measurement. For any type of wave motion the product of wave velocity and wave period is equal to the wave length.

This law is fundamental and any deviation therefrom may be considered as the result of errors in observation or measurement. In the present study the maximum deviation is of the order of 4 percent; the average error being 0.2 percent. In the evaluation of the data presented hereafter the accuracy of the test as indicated by column 10 should be considered.

The measured values of wave velocity and wave length, for each wave studied, are compared to the corresponding values computed from theory, in table 2. Columns 2 and 3 list the wave velocity and wave length as computed by the Gerstner (trochoidal) theory, where:

$$C_t = \frac{gT_m}{2\pi} \qquad (1)$$

$$L_t = C_t T_m \qquad (2)$$

T_m is the measured period which is assumed to have been correctly determined. Columns 4 and 5 list the percentage differences between measured and theoretical values. The averages given should be considered only in conjunction with similar averages; the comparisons for each test, when considered in company with values in column 10, table 1, giving a more accurate picture of the relation. Note, for example, Run 1, the accuracy of measurement is —4.4 percent, the comparison of measured and theoretical velocity is +0.7 percent. Inspection of the sign of the differences shows that if the measured velocity was 4.4 percent too small, then the value in column 4, table 2, when C_m is corrected by increasing it 4.4 percent, is of the order of +5.0 percent.

It will be noted that the agreement between the average measured values and the average theoretical values derived from Gerstner's formulae is of the order of 0.5 to 1.0 percent.

The theoretical values for wave velocity and wave length computed by the formulae of the Stokes-Levi-Civita theory where

$$C_t' = \frac{gT_m}{2\pi}\left(1 + \frac{\pi^2 h^2}{L^2} + \frac{\pi^4 h^4}{2L^4} - - - - - -\right) \qquad (3)$$

and $L'_t = C'_t T_m$

are given in columns 6 and 7.

Again comparing average measured and theoretical values as above, it is found that the agreement is of the order of 2.5 to 3.0 percent. Since this theory requires the existence of mass transport, i. e., a transport of water in the direction of wave travel, then in a closed tank (or any finite body of water) there must be a return flow of water counter to the direction of wave travel. It is assumed that this flow is uniform over the tank transverse cross-section, and reduction of the theoretical values by the velocity of the return flow has been made.

TABLE 2.—*Deep-water waves* $\left(\frac{d}{L}>0.5\right)$ *comparison of measured to theoretical velocity and length*

1	2	3	4	5	6	7	8	9
	Gerstner		$\frac{C_m-C_t}{C_t}$	$\frac{L_m-L_t}{L_t}$	Levi-Civita		$\frac{C_m-C'_t}{C_t}$	$\frac{L_m-L'_t}{L'_t}$
	C_t	L_t			C'_t	L'_t		
	F. p. s.	*Feet*	*Percent*	*Percent*	*F. p. s.*	*Feet*	*Percent*	*Percent*
1	4.36	3.71	+0.7	+4.9	4.39	3.75	0	+3.7
2	5.14	5.15	−0.8	−0.2	5.14	5.18	−0.8	−0.8
4	4.41	3.79	−0.2	+1.1	4.42	3.82	−0.5	+0.3
5	5.05	4.97	−1.0	−2.0	5.05	4.99	−0.0	−2.4
13	4.35	3.68	+1.6	+3.0	4.37	3.72	+1.1	+1.9
14	4.97	4.81	−1.6	−1.0	4.98	4.84	−1.8	−1.7
17	4.95	4.78	−2.8	−0.6	4.95	4.80	−2.8	−1.0
18	4.34	3.68	−2.8	−3.0	4.37	3.72	−3.2	−4.0
19	4.32	3.63	−0.2	−2.2	4.33	3.65	−0.5	−2.7
20	5.10	5.08	−0.6	−1.2	5.10	5.09	−0.6	−1.4
22	4.32	3.64	+0.2	−1.1	4.33	3.66	0	−1.6
23	5.06	5.00	−0.2	−1.2	5.08	5.04	−0.6	−2.4
Average			−0.85	−0.29			−0.9	−1.0

The measured values for the percentage of wave height above still water level are tabulated in table 3. The results are presented graphically in Figure 2, where the solid line represents the theoretical values derived from Gaillard's (trochoidal theory) formula:

$$\frac{Z_s}{h}=0.5+\frac{\pi h}{4L} \tag{4}$$

and the dotted line, the values derived from Levi-Civita's formula:

$$\frac{Z_s}{h}=0.5+\frac{\pi h}{4L}+\frac{\pi^3 h^3}{3L^3} \tag{5}$$

TABLE 3.—*Percentage of wave height above still water level—deep-water waves* $\left(\frac{d}{L}>0.5\right)$

Run	Position	$\frac{h}{L}$	Average wave height	Wave height above still water level	Percent above still water level	Run	Position	$\frac{h}{L}$	Average wave height	Wave height above still water level	Percent above still water level
1	1	0.0756	0.293	0.159	54.3	17	1	0.0392	0.186	0.094	50.6
1	2	.0666	.259	.156	60.2	17	2	.0400	.190	.098	51.6
1	3	.0717	.279	.153	54.8	17	3	.0404	.192	.106	55.1
2	1	.0483	.248	.144	58.1	18	1	.0762	.272	.159	58.5
2	2	.0471	.242	.130	53.8	18	2	.0698	.249	.142	57.0
2	3	.0428	.220	.124	56.3	18	3	.0661	.236	.144	61.0
4	1	.0611	.234	.130	55.6	19	1	.0569	.203	.120	58.9
4	2	.0582	.223	.122	54.8	19	2	.0552	.196	.106	54.1
4	3	.0608	.233	.135	57.9	19	3	.0541	.192	.104	54.2
5	1	.0425	.217	.116	53.5	20	1	.0293	.147	.076	51.7
5	2	.0425	.207	.108	52.2	20	2	.0321	.161	.085	52.8
5	3	.0400	.195	.106	54.3	20	3	.0351	.176	.096	54.6
13	1	.0697	.264	.161	61.0	22	1	.0486	.178	.096	53.9
13	2	.0594	.225	.132	58.7	22	2	.0472	.170	.090	53.0
13	3	.0622	.236	.126	53.3	22	3	.0431	.155	.086	55.5
14	1	.0450	.214	.109	50.9	23	1	.0620	.305	.179	58.7
14	2	.0475	.226	.110	48.7	23	2	.0602	.296	.169	57.1
14	3	.0458	.218	.120	55.0	23	3	.0583	.287	.155	54.0

NOTE.—Position 1 is 12 feet from wave generator. Position 2 is 24 feet from wave generator. Position 3 is 37 feet from wave generator.

The results of this comparison are not conclusive. For values of $\frac{h}{L} < \sim 0.05$ the measured values agree fairly well with the values required by either theory.

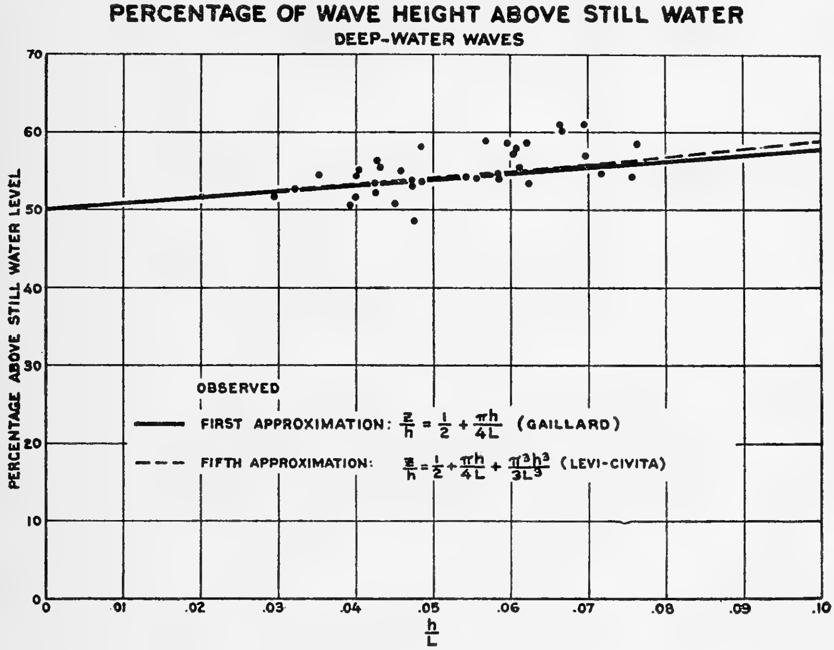


FIGURE 2.

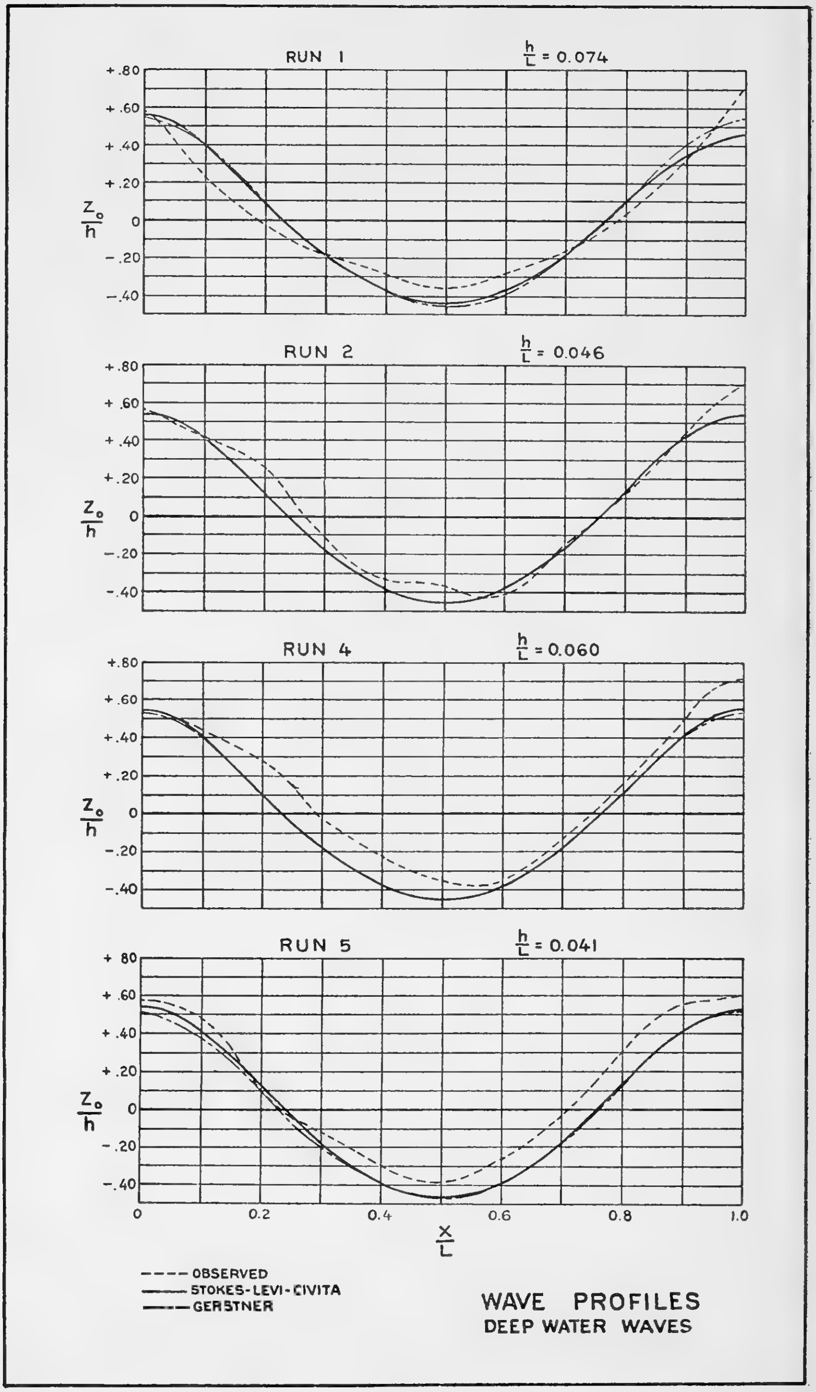
The measured wave profiles are shown in non-dimensional form on Figures 3 to 5. Plotted on the same graphs are the wave forms given by Gerstner's theory:

$$\begin{aligned} X_s &= R\theta - r \sin \theta \\ Z_s &= R - r \cos \theta \end{aligned} \tag{6}$$

and by Levi-Civita's theory:

$$\begin{aligned} Z_s = a \cos kX_s + \left(\frac{1}{2}a^2k + \frac{17}{24}a^4k^3\right) \cos 2kX_s + \\ + \left(\frac{3}{8}a^3k^2 + \frac{153}{128}a^5k^4\right) \cos 3kX_s - \frac{1}{3}a^4k^3 \cos 4kX_s + \\ + \frac{125}{384}a^5k^4 \cos 5kX_s \end{aligned} \tag{7}$$

where $$a = \frac{h}{2} - \frac{3}{64}k^2h^3 - \frac{584}{12288}k^4h^5$$



RUN 1
h/L = 0.074
RUN 2
h/L = 0.046
RUN 4
h/L = 0.060
RUN 5
h/L = 0.041
+.80
+.60
+.40
+.20
0
-.20
-.40
Z₀/h
0
0.2
0.4
0.6
0.8
1.0
X/L
OBSERVED
STOKES-LEVI-CIVITA
GERSTNER
WAVE PROFILES
DEEP WATER WAVES

FIGURE 3.

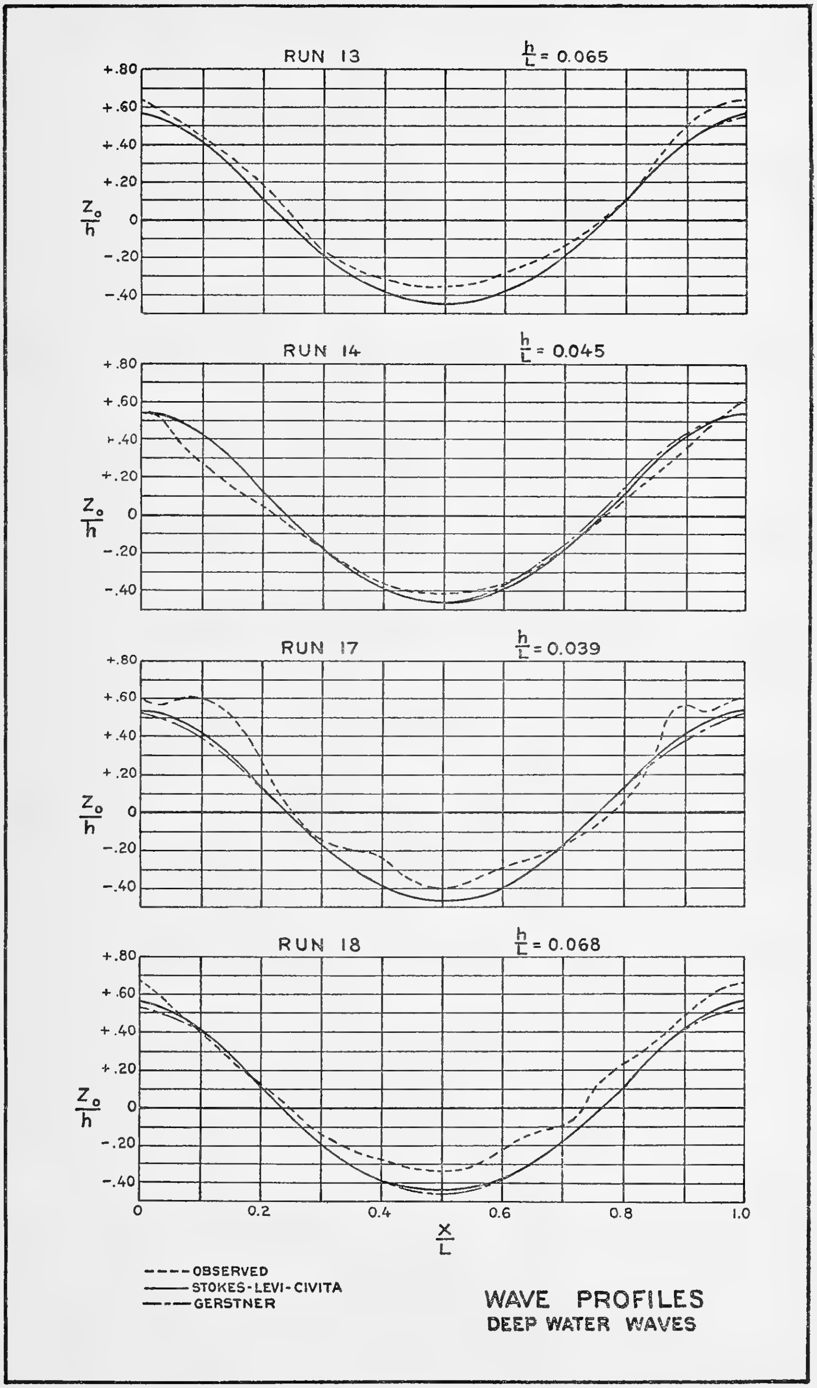

RUN 13
$\frac{h}{L}$ = 0.065
RUN 14
$\frac{h}{L}$ = 0.045
RUN 17
$\frac{h}{L}$ = 0.039
RUN 18
$\frac{h}{L}$ = 0.068
+.80
+.60
+.40
+.20
0
-.20
-.40
$\frac{Z_o}{h}$
0
0.2
0.4
0.6
0.8
1.0
$\frac{X}{L}$
OBSERVED
STOKES-LEVI-CIVITA
GERSTNER
WAVE PROFILES
DEEP WATER WAVES

FIGURE 4.

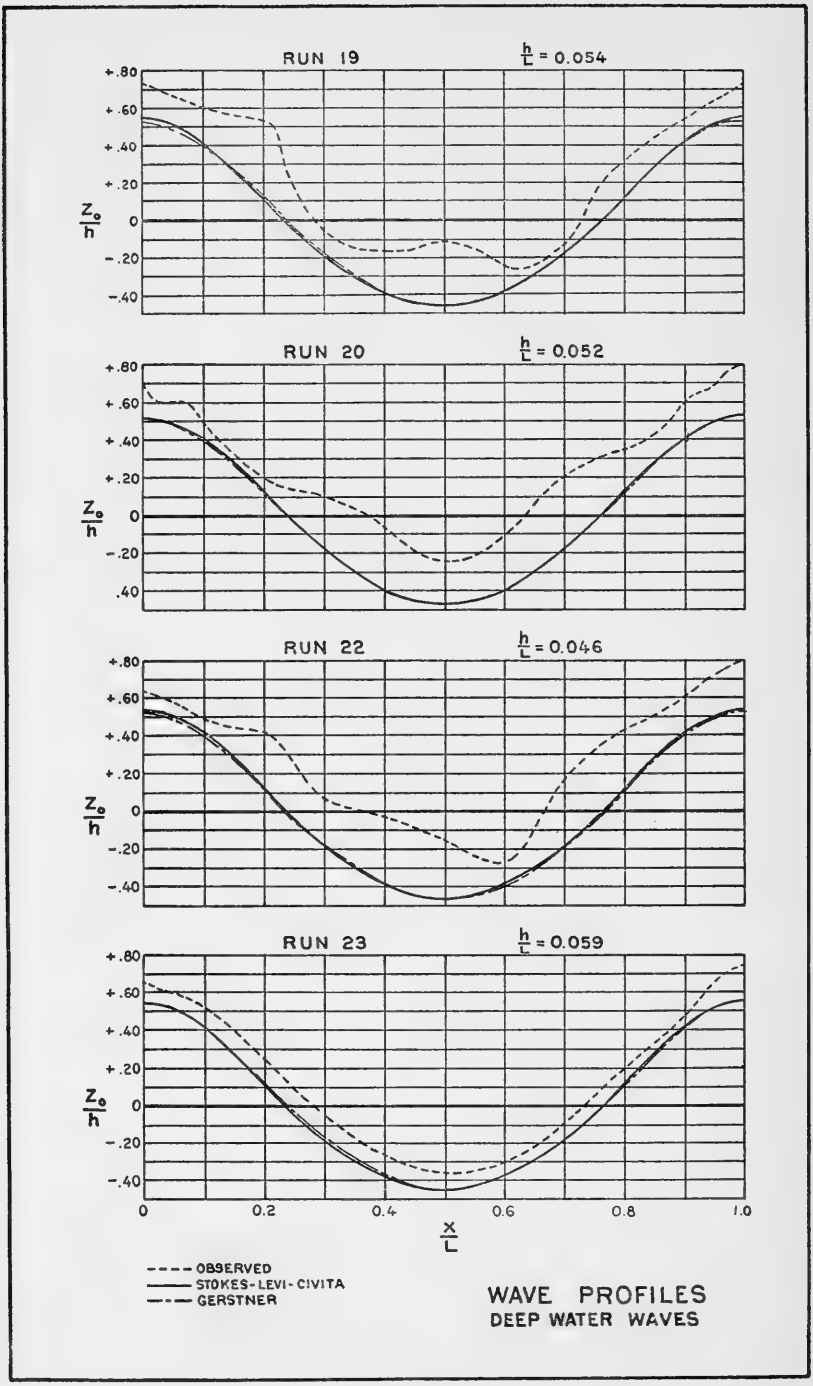

RUN 19
h/L = 0.054
RUN 20
h/L = 0.052
RUN 22
h/L = 0.046
RUN 23
h/L = 0.059
Z₀/h
+.80
+.60
+.40
+.20
0
-.20
-.40
0
0.2
0.4
0.6
0.8
1.0
x/L
OBSERVED
STOKES-LEVI-CIVITA
GERSTNER
WAVE PROFILES
DEEP WATER WAVES

FIGURE 5.

The actual wave profiles do not agree with either theoretical profile. The irregular profile of these waves is believed to be due to phenomena of reflection at the wave absorber, and to certain mechanical deficiencies in the wave generator. The elimination of the irregularities will require further study; the purposes of this investigation are not believed to require their elimination, though desirable.

The dimensions of the paths described by individual water particles are given in table 4. The theoretical values of the orbit radii are the same for the Gerstner and Stokes-Levi-Civita theories, their value being given by

$$R_t = ae^{-kh'} \tag{8}$$

where R_t=theoretical orbit radius at depth h′ below still water level. The last column of Table 4 lists the percentage differences between measured and theoretical values, the positive sign indicating that the measured value is larger than the theoretical. The data are presented graphically in Figures 6 and 7.

The comparison of measured and theoretical values of surface mass transport is shown in table 5. The theoretical values are given by the formula:

$$U_{t_s} = k^2a^2C - \frac{ka^2C}{2d} \tag{9}$$

derived by Stokes. It will be recalled that Gerstner's theory does not admit the existence of mass transport.

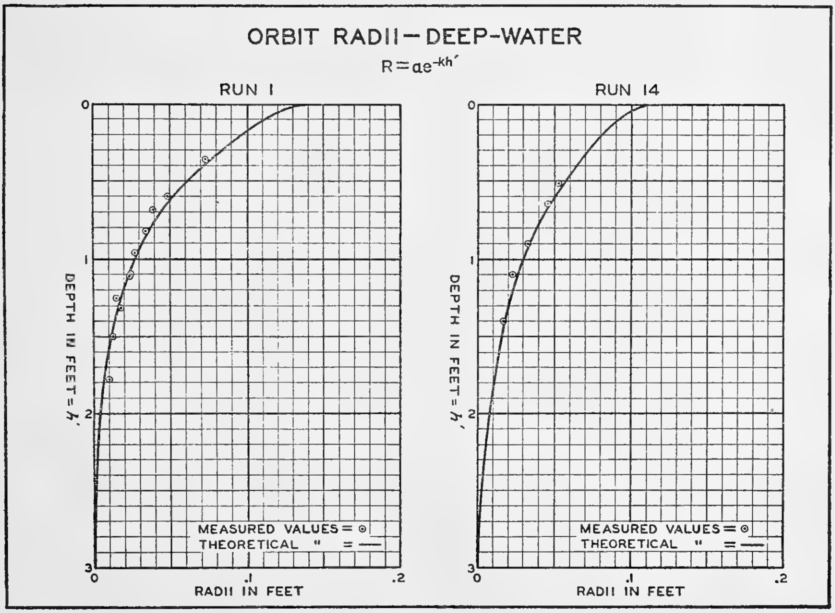


Figure 6.

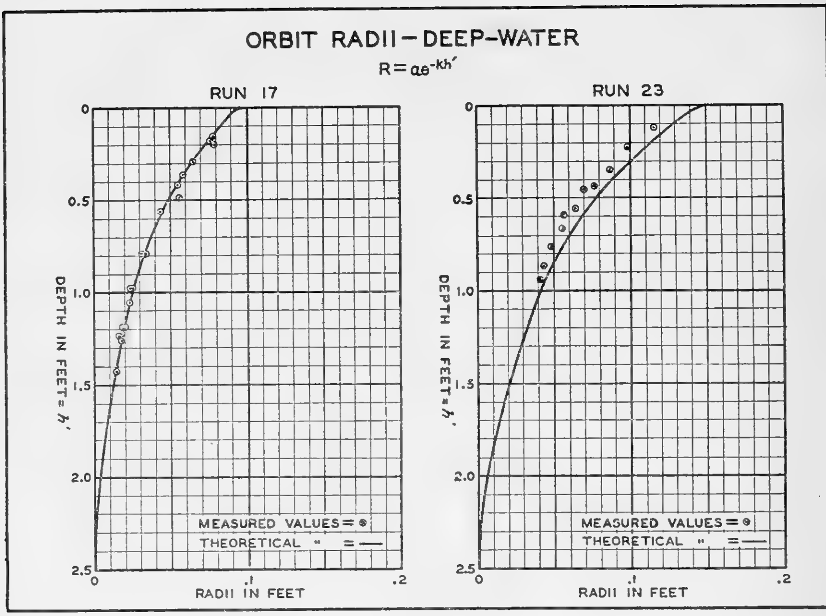


FIGURE 7.

TABLE 4.—*Orbit radii—Deep-water waves* $\left(\frac{d}{L}>0.5\right)$

Point	Depth	Measured			Theoretical radius R_t	$\frac{R_m-R_t}{R_t}$
		Horizontal diameter	Vertical diameter	Radius R_m		
1-1	0.69	0.080	0.078	0.0397	0.0454	−12.5
1-2	0.96	.055	.055	.0275	.0292	−5.7
1-3	1.11	.047	.047	.0235	.0226	−4.0
1-4	1.25	.028	.032	.0150	.0168	−10.7
1-5	1.50	.025	.025	.0125	.0125	0
1-6	0.357	.146	.146	.073	.0773	−5.6
1-7	0.600	.097	.099	.049	.0520	−5.8
1-8	0.820	.067	.068	.034	.0366	−7.1
1-9	1.10	.049	.049	.0245	.0232	+5.6
1-10	1.31	.033	.037	.0180	.0164	+9.7
1-11	1.78	.020	----------	.0100	.0077	+30.0
13-1	0.47	.106	.117	.0557	.0544	+2.4
13-2	0.62	----------	.077	.0385	.0438	−1.2
13-3	0.91	----------	.048	.024	.0263	−0.9
13-4	1.15	.037	.032	.0172	.0177	−2.8
14-1	0.51	.105	----------	.0525	.0561	−6.4
14-2	0.64	.093	----------	.0462	.0473	−2.3
14-3	0.90	.0655	----------	.0327	.0335	−2.4
14-4	1.10	.045	----------	.0225	.0258	−12.8
14-5	1.40	.035	----------	.0175	.0182	−3.8
17-1	0.18	.159	.150	.077	.0749	+2.8
17-2	0.365	.121	.115	.059	.0587	−0.5
17-3	0.415	.111	.107	.055	.0549	+0.2
17-4	0.56	.090	.085	.043	.0453	−5.1
17-5	0.79	.063	.060	.031	.0333	−6.9
17-6	1.05	.048	.045	.023	.0237	−2.9
17-7	1.19	.041	.039	.020	.0198	+1.0
17-8	1.26	.038	.034	.018	.0180	0
17-9	0.20	.163	.153	.079	.073	+8.2
17-10	0.49	.114	.110	.056	.0497	+12.7
17-11	0.79	.070	.063	.0332	.0335	−0.9

TABLE 4.—*Orbit radii—Deep-water waves* $\left(\frac{d}{L}>0.5\right)$ —Continued

Point	Depth	Measured			Theoretical radius R_t	$\frac{R_m-R_t}{R_t}$
		Horizontal diameter	Vertical diameter	Radius R_m		
17-12	0.98	0.053	0.052	0.0262	0.0261	+0.6
17-13	1.19	.039	.039	.0195	.0198	−1.6
17-14	1.42	.029	.024	.0142	.0145	−2.1
17-15	0.16	.163	.151	.0785	.0769	+2.1
17-16	0.29	.135	.126	.065	.0647	+0.4
17-17	0.79	.070	.064	.0335	.0336	−0.3
17-18	0.98	.053	.046	.0247	.0261	−5.4
17-19	1.23	.035	.033	.0170	.0187	−9.1
18-1	0.48	.118	.090	.0480	.0541	−11.3
18-2	0.70	.063	.056	.0297	.0368	−19.4
18-3	0.87	.042	.043	.0215	.0273	−21.3
18-4	0.96	.038	.041	.0195	.0232	−16.0
18-5	1.00	.035	.035	.0175	.0216	−18.6
19-1	0.29	.116	.120	.0590	.0602	−2.0
19-2	0.36	.100	.106	.0515	.0534	−3.5
19-3	0.55	.068	.074	.0355	.0388	−8.5
19-4	0.79	.048	.058	.0265	.0242	+9.5
19-5	0.89	.036	.046	.0205	.0203	+1.0
20-1	0.39	.090	.114	.0510	.0494	+3.2
20-2	0.49	.082	.104	.047	.0433	+8.5
20-3	0.84	.060	.078	.0345	.0290	+19.0
20-4	1.09	.036	.038	.0185	.0204	−9.3
22-1	0.61	.062	.068	.0325	.0287	+13.2
22-2	0.80	.046	.048	.0235	.0205	+14.6
22-3	1.05	.030	.030	.0150	.0133	+12.8
23-1	0.115	.237	.228	.116	.128	−9.4
23-2	0.217	.198	.199	.099	.112	−11.6
23-3	0.346	.174	.173	.087	.0950	−8.4
23-4	0.428	.152	.155	.077	.0856	−10.3
23-5	0.460	.142	.139	.070	.0823	−14.6
23-6	0.560	.128	.129	.064	.0708	−9.6
23-7	0.596	.115	.115	.057	.0691	−17.4
23-8	0.669	.114	.113	.056	.0629	−11.1
23-9	0.766	.097	.099	.049	.0573	−14.0
23-10	0.870	.088	.085	.043	.0486	−11.5
23-11	0.929	.081	.082	.041	.0451	−9.1
Average						−2.96

TABLE 5.—*Surface values—Mass transport*

Run	Average time	Base-line	Measured transport U_m	*d*	Theoretical transport U_t	$\frac{U_m-U_t}{U_t}$ percent
1	37.7	10	0.265	3	0.175	+51.4
2	63.9	10	.156	3	.117	+33.4
4	52.0	10	.192	2.5	.171	+12.3
5	102.8	10	.097	2.5	.087	+11.5
13	40.1	10	.249	3	.169	+47.4
14	66.3	10	.151	3	.093	+62.4
17	115.2	10	.087	2.5	.083	+4.6
18	64.4	10	.155	2.5	.167	−7.2
19	106.4	10	.094	3	.117	−16.2
20	341.8	10	.029	3	.044	−34.1
22	227.3	10	.044	2.5	.083	−47.0
23	105.8	10	.094	2.5	.158	−40.5
Average						+6.5

TABLE 6.—*Mass transport*

Point	Depth	Measured transport per period (feet)	U_m (ft./sec.)	d	U_t (ft./sec.)	U_m-U_t
1-1	0.69	0.0	0.0	3	+0.0008	−0.0008
1-2	0.96	−.010	−.0118	3	−.0126	+.0008
1-3	1.11	−.017	−.020	3	−.0163	−.0037
1-4	1.25	−.021	−.0248	3	−.0184	−.0064
1-5	1.50	−.019	−.0224	3	−.0205	−.0019
1-6	0.357	+.053	+.0624	3	+.0457	+.0167
1-7	0.60	+.039	+.0459	3	+.0089	+.0370
1-8	0.82	−.018	−.0212	3	−.0070	−.0142
1-9	1.10	−.019	−.0224	3	−.0161	−.0063
1-10	1.31	0	0	3	−.0191	+.0191
1-11	1.78	−.019	−.0224	3	−.0215	−.0009
13-1	0.47	+.0204	+.024	3	+.0195	+.0045
13-2	0.62	+.0076	+.009	3	+.0048	+.0042
13-3	0.91	−.0051	−.006	3	−.0090	+.003
13-4	1.15	0	0	3	−.0137	+.0137
13-5	1.78	−.0136	−.016	3	−.0171	+.0011
13-6	1.93	−.015	−.0172	3	−.0173	+.0001
13-7	1.61	−.011	−.0130	3	−.0168	+.0038
13-8	1.33	−.0053	−.0063	3	−.0154	+.0091
13-9	1.15	−.006	−.0071	3	−.0137	+.0066
14-1	1.60	−.0141	−.0146	3	−.0115	−.0031
14-2	1.65	−.0130	−.0134	3	−.0117	−.0017
14-3	1.25	−.021	−.0217	3	−.0092	−.0125
14-4	1.44	−.008	−.0083	3	−.0103	+.0020
14-5	0.99	+.0125	+.0129	3	−.0055	+.0184
14-6	1.14	−.0041	−.0042	3	−.0080	+.0038
14-7	0.65	+.006	+.0062	3	+.0055	+.0007
14-8	0.82	+.0016	+.0017	3	−.0012	+.0029
17-1	0.180	+.037	+.0383	2.5	+.0288	+.0095
17-2	0.315	+.0120	+.0124	2.5	+.0125	−.0001
17-3	0.415	+.0120	+.0124	2.5	+.0052	+.0072
17-4	0.560	0	0	2.5	−.0015	+.0015
17-5	0.79	0	0	2.5	−.0071	+.0071
17-6	1.05	−.0050	−.0052	2.5	−.0097	+.0045
17-7	1.19	−.0099	−.0103	2.5	−.0105	+.0002
17-8	1.26	−.0170	−.0176	2.5	−.0107	−.0069
17-9	1.18	+.001	+.001	2.5	−.0105	+.0115
17-10	0.95	−.003	−.003	2.5	−.0091	+.0061
17-11	0.65	+.024	+.025	2.5	−.0045	+.0295
17-12	0.69	+.018	+.019	2.5	−.0054	+.0244
18-1	1.30	−.0173	−.0204	2.5	−.0215	+.0011
18-2	1.14	−.0213	−.0252	2.5	−.0199	−.0053
18-3	1.02	−.0087	−.0103	2.5	−.0178	+.0075
18-4	0.86	−.0053	−.0063	2.5	−.0135	+.0072
18-5	.82	−.0093	−.0110	2.5	−.0120	+.0010
18-6	.71	−.0077	−.0091	2.5	−.0066	−.0025
18-7	.56	+.0107	+.0126	2.5	+.0053	+.0073
18-8	.50	+.0227	+.0268	2.5	+.0120	+.0148
18-9	.40	+.0170	+.0201	2.5	+.0269	−.0068
18-10	.27	+.0500	+.0591	2.5	+.0565	+.0026
19-1	.16	+.053	+.063	3	+.0629	+.0001
19-2	.68	−.017	−.0202	3	−.0006	−.0196
19-3	.86	−.0073	−.0086	3	−.0063	−.0023
19-4	.77	−.0077	−.0091	3	−.0038	−.0053
19-5	.91	−.006	−.0071	3	−.0070	−.0001
20-1	.92	−.001	−.001	3	−.0017	+.0007
20-2	.83	+.001	+.001	3	−.0004	+.0014
20-3	.70	−.0007	−.0007	3	+.0021	−.0028
20-4	.63	−.013	−.013	3	+.0037	−.0167
20-5	.52	0	0	3	+.0071	−.0071
20-6	.46	+.0027	+.0027	3	+.0094	−.0067
20-7	.38	+.0077	+.0077	3	+.0130	−.0053
20-8	.30	+.0027	+.0027	3	+.0173	−.0146
22-1	.17	+.023	+.0273	2.5	+.0398	−.0125
22-2	.22	+.021	+.0249	2.5	+.0316	−.0067
22-3	.30	+.017	+.0201	2.5	+.0218	−.0017
22-4	.39	+.0007	+.0008	2.5	+.0129	−.0121
22-5	.48	+.0013	+.0015	2.5	+.0068	−.0053
22-6	.68	−.0011	−.0013	2.5	−.002	+.0007
22-7	.77	−.005	−.0059	2.5	−.0042	−.0017
22-8	.78	−.015	−.0178	2.5	−.0044	−.0134
22.9	.86	−.0076	−.0090	2.5	−.0059	−.0031
22-10	1.03	0	0	2.5	−.0079	+.0079
22-11	1.09	0	0	2.5	−.0084	+.0084
23-1	0.15	+.029	+.0293	2.5	+.1038	−.0745
23-2	0.24	+.033	+.0334	2.5	+.0697	−.0363
23-3	0.44	−.0057	−.0058	2.5	+.0305	−.0363
23-4	0.55	−.0013	−.0013	2.5	−.0160	+.0147
23-5	0.82	−.0063	−.0064	2.5	−.0058	−.0006

TABLE 6.—*Mass transport*—Continued

Point	Depth	Measured transport per period (feet)	U_m (ft./sec.)	d	U_t (ft./sec.)	$U_m - U_t$
23-6	0.76	−0.0020	−0.0020	2.5	−0.0022	+0.0002
23-7	.89	−.0010	−.0010	2.5	−.0096	+.0086
23-8	1.12	−.0050	−.0051	2.5	−.0179	+.0128
23-9	0.115	+.028	+.0282	2.5	+.1062	−.0780
23-10	.217	+.012	+.0121	2.5	+.0750	−.0629
23-11	.346	+.020	+.0203	2.5	+.0465	−.0262
23-12	.428	+.009	+.0091	2.5	+.0323	−.0232
23-13	.460	+.014	+.0142	2.5	+.0274	−.0132
23-14	.560	−.010	−.0101	2.5	+.0148	−.0249
23-15	.596	+.011	+.0111	2.5	+.0112	−.0001
23-16	.669	.000	.000	2.5	+.0044	−.0044
23-17	.766	+.005	+.051	2.5	−.0026	+.0536
23-18	.870	+.012	+.0121	2.5	−.0086	+.0207
23-19	.929	+.006	+.0061	2.5	−.0113	+.0174
Average difference						−0.0021

Table 6 lists comparative values of mass transport at various depths for each of the waves studied. The measured values were obtained as described in section 4. The theoretical values are given by Stokes' theory as:

$$U_t = k^2a^2Ce^{-2kh'} - \frac{ka^2C}{2d} \tag{10}$$

Since mass transport in a closed tank is modified by a return flow which results in zero and negative measured values of mass transport in the bottom section of the tank, percentage comparisons are not feasible.

The data are presented graphically in Figures 8 and 9.

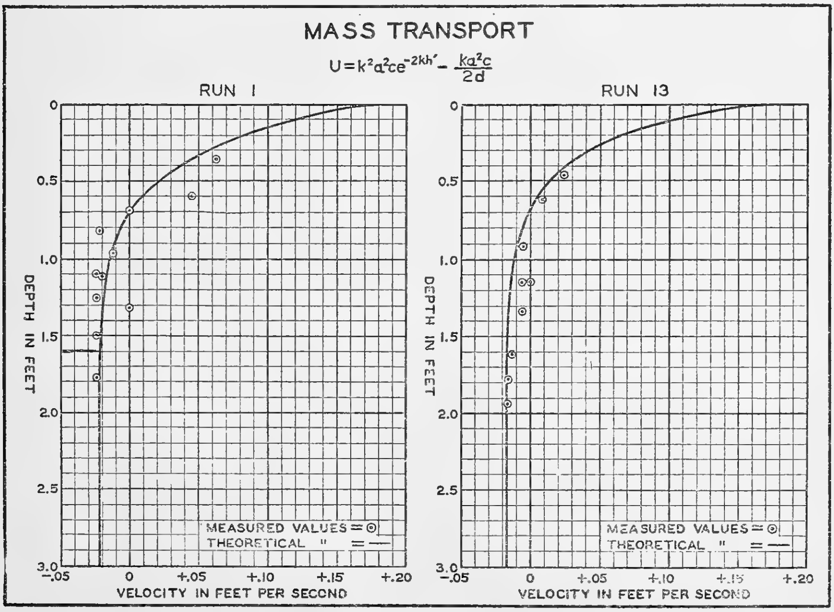


FIGURE 8.

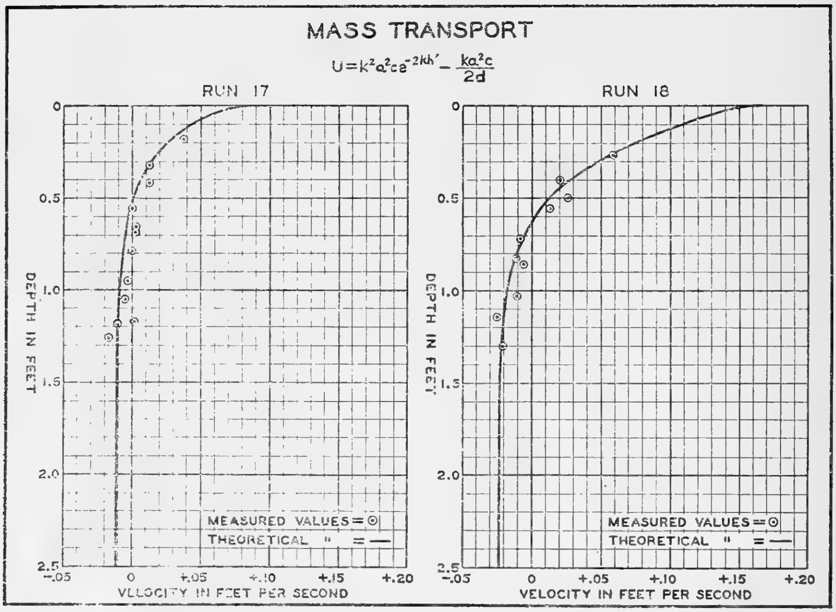


FIGURE 9.

6. SHALLOW-WATER WAVES

The basic data for the shallow-water waves studied are given in table 7.

TABLE 7.—*Basic data—shallow-water waves* $\left(\frac{d}{L}<0.5\right)$

1 Run	2 h_{ave} ft.	3 T_m sec.	4 C_m ft./sec.	5 L_m ft.	6 d ft.	7 $\frac{d}{L_m}$	8 $\frac{h_{ave}}{L_m}$	9 b_s/a_s	10 $\frac{C_mT_m}{\text{ft.}}$	11 $\frac{C_mT_m-L_m}{L_m}$ percent
3	0.182	1.220	6.25	7.52	3	0.399	0.0242	0.987	7.62	+1.3
6	.154	1.205	6.05	6.97	2.5	.359	.0221	.980	7.29	+4.3
7	.159	1.432	6.08	8.97	2	.223	.0177	.886	8.71	−3.1
8	.213	1.196	5.62	6.81	2	.294	.0313	.952	6.72	+1.3
9	.280	0.987	4.97	4.89	2	.409	.0573	.988	4.91	+0.4
10	.243	1.436	6.20	9.15	2	.219	.0265	.880	8.90	−2.7
11	.246	1.231	5.77	7.24	2	.276	.0340	.940	7.10	−1.9
12	.213	1.003	4.84	5.08	2	.393	.0419	.986	4.86	−4.3
15	.256	1.200	5.78	7.20	3	.417	.0356	.989	6.94	−3.6
16	.232	1.188	5.56	6.96	2.5	.359	.0333	.980	6.61	−5.0
21	.310	1.232	6.10	7.64	3	.393	.0406	.986	7.51	−1.3
24	.252	1.251	6.15	7.65	2.5	.327	.0329	.969	7.69	+0.5
25	.157	0.990	4.95	4.83	2	.414	.0325	.989	4.90	+1.4
26	.251	1.222	5.79	7.12	2	.281	.0353	.944	7.08	−0.6
27	.233	1.496	6.51	9.53	2	.210	.0245	.866	9.73	+2.1
28	.168	1.496	5.90	8.77	1.5	.171	.0192	.772	8.82	+0.6
29	.242	1.468	5.77	8.62	1.5	.174	.0281	.800	8.47	−1.7
30	.145	1.911	6.49	12.11	1.5	.124	.0119	.654	12.40	+2.4
31	.089	2.080	5.37	11.09	1.0	.090	.0080	.512	11.16	+0.6
32	.179	1.506	5.09	7.86	1.0	.127	.0228	.665	7.66	−2.5

The ratio b_s/a_s column 9, is the ratio of the vertical to horizontal amplitude of motion of the water particles in the wave surface.

The remarks previously made concerning the accuracy of measurement are applicable also to these shallow-water wave tests.

The measured values of wave velocity and wave length, for each wave studied, are compared to the corresponding values computed from theory in table 8. Columns 2 and 3 list the wave velocity and wave length as computed by the Laplace-Airy (trochoidal) theory, where:

$$C_t = \frac{gT_m}{2\pi} \tanh \frac{2\pi d}{L} \tag{11}$$

and:

$$L_t = C_t T_m$$

T_m is the measured period which is assumed to have been correctly determined. Columns 4 and 5 list the percentage differences between measured and theoretical values. It will be noted that the agreement between the measured values and the theoretical values derived from the Laplace-Airy formulae is not as close as for the deep-water waves, varying as much as about 6.5 percent.

TABLE 8.—*Shallow-water waves* $\left(\frac{d}{L}<0.5\right)$—*comparison of measured to theoretical velocity and length*

1	2	3	4	5	6	7	8	9
Run	Gerstner		$\frac{C_m-C_t}{C_t}$	$\frac{L_m-L_t}{L_t}$	Stokes-Struik		$\frac{C_m-C'_t}{C'_t}$	$\frac{L_m-L'_t}{L'_t}$
	C_t ft./sec.	L_t ft.	percent	percent	C'_t ft./sec.	L'_t ft.	percent	percent
3	6.16	7.53	+1.5	−0.1	6.19	7.56	+1.0	−0.5
6	6.04	7.29	+0.6	−4.1	6.06	7.31	−0.2	−4.7
7	6.49	9.30	−6.3	−3.5	6.51	9.34	−6.6	−4.0
8	5.81	6.96	−3.3	−2.2	5.87	7.04	−4.3	−3.3
9	4.97	4.93	0	−0.8	5.14	5.10	−3.3	−4.1
10	6.45	9.29	−3.9	−1.5	6.51	9.38	−4.8	−2.5
11	5.91	7.30	−2.4	−0.8	5.99	7.40	−3.7	−2.2
12	5.05	5.09	−3.8	−0.2	5.13	5.17	−5.7	−1.7
15	6.06	7.30	−4.6	−1.4	6.14	7.39	−5.9	−2.6
16	5.95	7.09	−6.5	−1.8	6.00	7.15	−7.3	−2.7
21	6.19	7.56	−1.4	+1.0	6.30	7.79	−3.2	−1.9
24	6.19	7.77	−0.6	−1.5	6.25	7.84	−1.6	−2.4
25	5.01	4.97	−1.2	−2.8	5.06	5.02	−2.2	−3.8
26	5.94	7.28	−2.5	−2.2	5.97	7.32	−3.0	−2.7
27	6.61	9.91	−1.5	−3.8	6.68	10.02	−2.5	−4.9
28	6.05	9.06	−2.5	−3.2	6.10	9.14	−3.3	−3.0
29	5.98	8.81	−3.5	−2.1	6.09	8.97	−5.3	−3.9
30	6.39	12.22	+1.5	−0.9	6.41	12.27	+1.2	−1.3
31	5.44	11.35	−1.3	−2.3	5.50	11.46	−2.4	−3.2
32	5.11	7.71	−0.4	+2.0	5.10	7.73	−0.2	+1.7
Average			−2.1	−1.6			−3.2	−2.7

The theoretical values for wave velocity and wave length computed by the formulae of the Stokes-Struik theory, where

$$C_t' = \frac{gT_m}{2\pi} \frac{e^{\frac{2\pi d}{L}} - e^{-\frac{2\pi d}{L}}}{e^{\frac{2\pi d}{L}} + e^{-\frac{2\pi d}{L}}} \left[1 + \frac{e^{\frac{8\pi d}{L}} + e^{-\frac{8\pi d}{L}} + 2\left(e^{\frac{4\pi d}{L}} + e^{-\frac{4\pi d}{L}}\right) + 12}{\left(e^{\frac{2\pi d}{L}} - e^{-\frac{2\pi d}{L}}\right)^4} \frac{\pi^2 h^2}{L^2} \right] \tag{12}$$

and:

$$L'_t = C'_t T_m$$

are given in columns 6 and 7.

TABLE 9.—*Percentage of wave height above still water level, shallow-water waves* $\left(\frac{d}{L}<0.5\right)$

Run	Position	$\frac{h}{L}$	Average wave height	Wave height above still water level	Percent wave height above still water level
3	1	0.0250	0.188	0.101	53.7
3	2	.0246	.185	.098	53.0
3	3	.0217	.163	.088	54.0
6	1	.0229	.160	.086	53.7
6	2	.0221	.154	.082	53.2
6	3	.0221	.154	.076	49.4
7	1	.0191	.171	.088	51.5
7	2	.0175	.157	.083	52.9
7	3	.0167	.150	.081	54.0
8	1	.0331	.225	.118	52.4
8	2	.0308	.210	.113	53.8
8	3	.0301	.205	.118	57.7
9	1	.0593	.290	.169	58.3
9	2	.0568	.278	.165	59.3
9	3	.0556	.272	.157	57.8
10	1	.0275	.252	.131	52.0
10	2	.0262	.240	.125	52.1
10	3	.0258	.236	.129	54.7
11	1	.0363	.263	.142	54.0
11	2	.0334	.242	.136	56.1
11	3	.0320	.232	.132	56.9
12	1	.0431	.219	.108	49.3
12	2	.0439	.223	.124	55.6
12	3	.0392	.199	.102	51.2
15	1	.0368	.265	.131	49.4
15	2	.0357	.257	.132	51.3
15	3	.0342	.246	.127	51.7
16	1	.0358	.249	.125	50.2
16	2	.0332	.231	.123	53.2
16	3	.0339	.236	.144	51.2
21	1	.0412	.315	.170	53.9
21	2	.0397	.303	.155	51.2
21	3	.0406	.310	.168	54.2
24	1	.0329	.252	.137	54.4
24	2	.0328	.251	.131	52.2
24	3	.0328	.251	.144	57.4
25	1	.0338	.163	.090	55.2
25	2	.0331	.160	.084	52.5
25	3	.0313	.151	.078	51.7
26	1	.0367	.261	.149	56.9
26	2	.0355	.253	.134	53.0
26	3	.0337	.240	.133	55.4
27	1	.0237	.226	.133	58.8
27	2	.0236	.225	.124	55.1
27	3	.0255	.243	.132	54.3
28	1	.0191	.167	.084	50.3
28	2	.0173	.152	.076	50.0
28	3	.0226	.198	.093	47.0
29	1	.0315	.272	.151	55.5
29	2	.0289	.249	.124	49.8
29	3	.0238	.205	.110	53.6
30	1	.0101	.123	.062	50.4
30	2	.0115	.139	.088	63.3
30	3	.0147	.178	.110	61.8
31	1	.0081	.090	.059	65.5
31	2	.0073	.081	.052	64.2
31	3	.0084	.093	.060	64.5
32	1	.0262	.206	.111	53.9
32	2	.0214	.168	.085	50.6
32	3	.0209	.164	.101	61.6

Again comparing measured and theoretical values as above, it is found that the agreement is on the whole good though individual comparisons show differences amounting to about 4.5 percent.

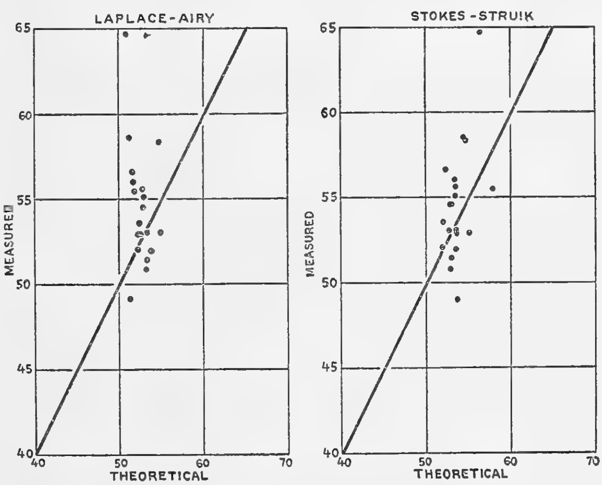
The measured values for percentage of wave height above still water level are tabulated in table 9. The results are presented graphically in Figure 10. The theoretical values are derived from the

Laplace-Airy formula, similar to that of the Gerstner theory (Eq. 4), and the Stokes-Struik formula:

$$\frac{Z_s}{h}=0.5+\frac{\pi h}{4L}\left[\frac{\left(e^{\frac{4\pi d}{L}}-e^{-\frac{4\pi d}{L}}\right)\left(e^{\frac{4\pi d}{L}}+e^{-\frac{4\pi d}{L}}+4\right)}{\left(e^{\frac{2\pi d}{L}}-e^{-\frac{2\pi d}{L}}\right)^4}\right] \tag{13}$$

With few exceptions the measured percentage of wave height above still water level is several percent higher than that indicated by either the Laplace-Airy or Stokes-Struik theories; agreeing most nearly with the former theory.

RUN	MEASURED	THEORETICAL	
		AIRY	STOKES
3	53.6	52.5	52.0
6	52.1	52.1	51.9
7	56.6	51.7	52.2
8	54.6	52.9	53.0
9	58.4	54.7	54.7
10	52.9	52.6	53.6
11	55.6	52.9	53.4
12	52.0	53.9	53.5
15	50.8	53.1	52.9
16	51.5	53.2	52.9
21	53.1	54.8	53.4
24	54.6	53.0	52.9
25	53.1	53.2	52.7
26	55.1	53.0	53.5
27	56.1	51.8	53.3
28	49.1	51.3	53.6
29	52.9	52.2	55.1
30	58.5	51.2	54.4
31	64.7	50.9	56.4
32	56.4	51.9	57.8



PERCENTAGE WAVE HEIGHT ABOVE STILL WATER LEVEL
SHALLOW-WATER WAVES

FIGURE 10.

Some of the measured wave profiles are shown in nondimensional form on Figures 11 and 12. Plotted on the same graphs are the wave forms given by the Laplace-Airy theory:

$$\begin{aligned} X_s&=R\theta-x\sin\theta \\ Z_s&=z\cos\theta \end{aligned} \tag{14}$$

and by the Stokes-Struik theory:

$$Z_s=a\cos kX_s-\frac{\left(e^{\frac{2\pi d}{L}}+e^{-\frac{2\pi d}{L}}\right)\left(e^{\frac{4\pi d}{L}}+e^{-\frac{4\pi d}{L}}+4\right)}{\left(e^{\frac{2\pi d}{L}}-e^{-\frac{2\pi d}{L}}\right)^4}ka^2\cos 2kX_s+ \tag{15}$$

$$5\frac{\left(e^{\frac{12\pi d}{L}}+e^{-\frac{12\pi d}{L}}\right)+14\left(e^{\frac{8\pi d}{L}}+e^{-\frac{8\pi d}{L}}\right)+19\left(e^{\frac{4\pi d}{L}}+e^{-\frac{4\pi d}{L}}\right)+32}{\left(e^{\frac{2\pi d}{L}}-e^{-\frac{2\pi d}{L}}\right)^6}\frac{k^2a^3}{4}\cos 3kX_s$$

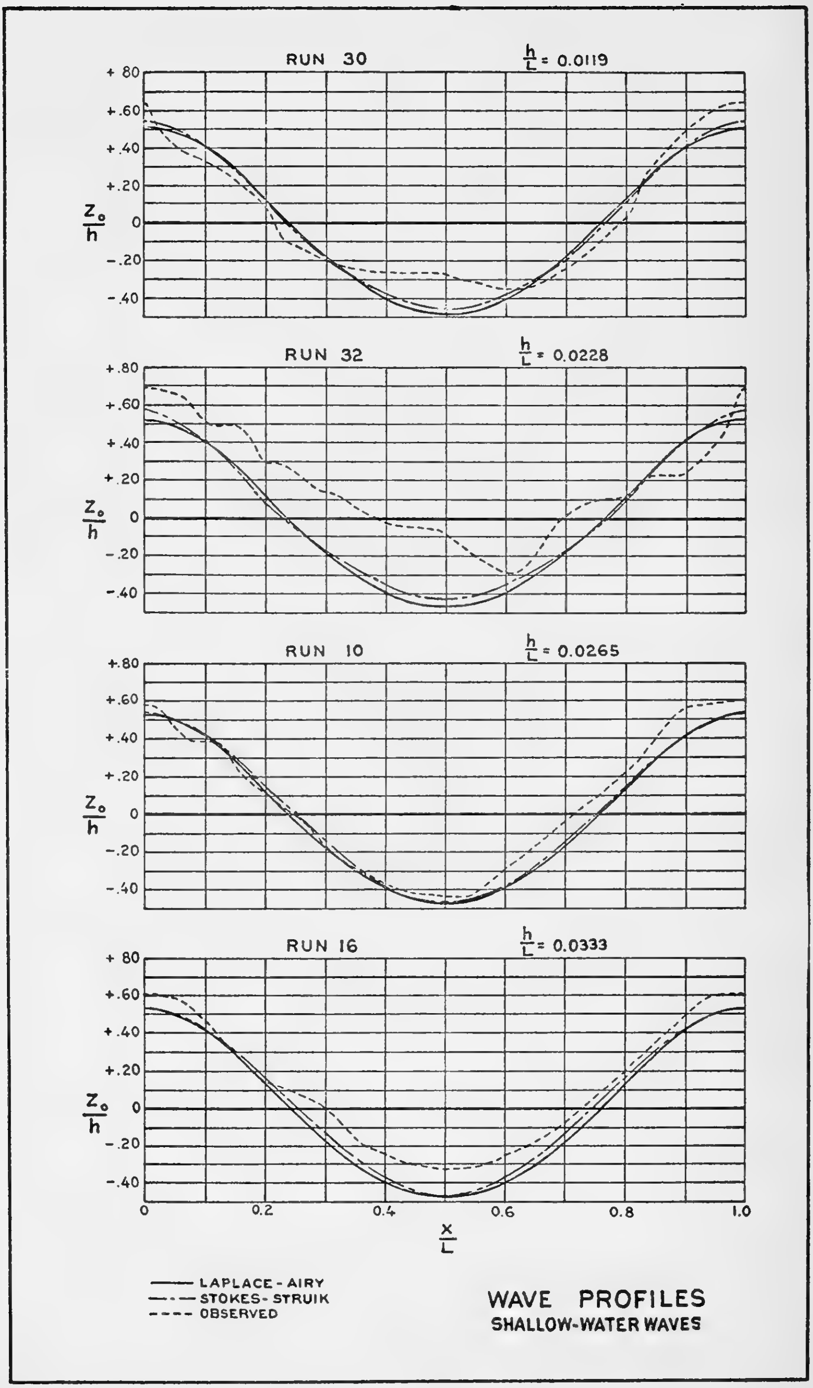

RUN 30
h/L = 0.0119
RUN 32
h/L = 0.0228
RUN 10
h/L = 0.0265
RUN 16
h/L = 0.0333
+.80
+.60
+.40
+.20
0
-.20
-.40
Zo/h
0
0.2
0.4
0.6
0.8
1.0
x/L
LAPLACE-AIRY
STOKES-STRUIK
OBSERVED
WAVE PROFILES
SHALLOW-WATER WAVES

FIGURE 11.

Where $$a=\frac{h}{2}-\frac{k^2h^3}{32}f(e^{kd})$$

The actual wave profiles do not agree with either theoretical profile.

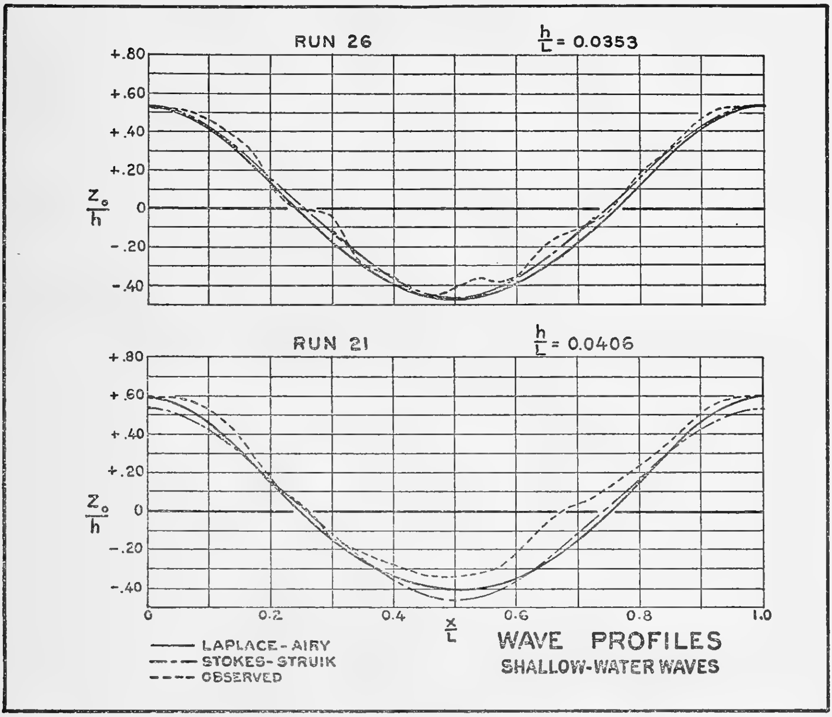


FIGURE 12.

The relation between the measured and theoretical average semi-axes of the paths described by individual water particles are given in table 10. The theoretical values of the orbit radii are the same for the Laplace-Airy and Stokes-Struik theories, their values being given by

$$\text{horizontal semi-axis:} = x_t = a\frac{\cosh\frac{2\pi}{L}(d-h')}{\sinh\frac{2\pi d}{L}} \qquad (16)$$

$$\text{vertical semi-axis:} = z_t = a\frac{\sinh\frac{2\pi}{L}(d-h')}{\sinh\frac{2\pi d}{L}}$$

where x_t and z_t=theoretical orbit sem-iaxis at depth h' below still water. The positive sign indicates that the measured value is larger than the theoretical. Some of the representative data are presented graphically in Figure 13.

TABLE 10.—*Average orbit semi-axes, shallow-water waves* $\left(\frac{d}{L}<0.5\right)$

Run	$\frac{d}{L}$	Average Horizontal $\frac{X_m-X_t}{X_t}$	Average Vertical $\frac{Z_m-Z_t}{Z_t}$	Run	$\frac{d}{L}$	Average Horizontal $\frac{X_m-X_t}{X_t}$	Average Vertical $\frac{Z_m-Z_t}{Z_t}$
31	0.090	−10.7	+32.8	24	0.327	−1.1	−3.4
30	.124	−8.7	+8.8	6	.359	−10.3	−34.4
32	.127	−3.3	+13.5	16	{ .359	−1.7	+3.7
28	.171	−16.4	−27.8		{ .359	−3.2	+8.9
29	.174	−0.6	−7.2	12	.393	−7.6	−1.9
27	.210	+1.0	−7.1	21	.393	−6.3	−5.6
10	.219	+1.7	+1.7	9	.409	−3.8	−5.1
7	.223	−17.5	−23.6	25	.414	+2.7	+1.8
11	.276	−7.2	−5.7	15	.417	+1.3	−0.6
26	.281	+2.0	+1.7				

X_m=measured horizontal semi-axis
X_t=theoretical horizontal semi-axis.
Z_m=measured vertical semi-axis.
Z_t=theoretical vertical semi-axis.

TABLE 11.—*Mass transport, shallow-water waves* $\left(\frac{d}{L}<0.5\right)$

Point	Depth feet	*d* feet	U_t ft./sec.	U_m ft./sec.	U_m-U_t ft./sec.
21-1	0.016	3	+0.077	+0.040	−0.037
2	.138		+.062	−.002	−.064
3	.106		+.064	+.002	−.062
4	.278		+.045	+.003	−.042
5	.408		+.031	+.009	−.022
6	.689		+.012	−.015	−.027
7	.495		+.023	+.002	−.021
8	.672		+.013	.000	−.013
9	.876		+.003	−.018	−.021
10	1.089		−.004	+.008	+.012
11	1.184		−.006	+.012	+.018
24-1	0.190	2.5	+.033	+.005	−.028
2	.170		+.035	+.022	−.013
3	.242		+.029	+.013	−.016
4	.407		+.018	−.014	−.032
5	.625		+.008	+.002	−.006
6	.485		+.014	+.006	−.008
7	.851		+.001	−.008	−.009
8	1.120		−.006	+.007	+.001
9	1.014		−.004	+.014	+.018
25-1	0.146	2.0	+.027	+.009	−.018
2	.266		+.018	+.009	−.009
3	.357		+.013	−.006	−.019
4	.489		+.007	+.014	+.007
5	.436		+.009	−.003	−.012
6	.635		+.002	.000	−.002
7	.743		−.001	+.004	+.005
8	1.018		−.004	+.003	+.007
26-1	0.035	2.0	+.049	−.005	−.054
2	.221		+.029	−.011	−.040
3	.414		+.015	.000	−.015
4	.567		+.006	−.002	−.008
5	.754		−.002	+.002	+.004
6	.825		−.004	+.007	+.011
7	.560		+.007	−.005	−.012
8	1.190		−.012	+.002	+.014
27-1	0.030	2.0	+.026	+.003	−.023
2	.107		+.022	+.012	−.010
3	.184		+.018	+.017	−.001
4	.319		+.013	+.031	+.018
5	.357		+.012	.000	−.012
6	.474		+.008	.000	−.008
7	.568		+.005	−.004	−.009
8	.877		−.002	+.005	+.007
9	1.113		−.006	+.008	+.014
10	1.101		−.005	−.015	−.010

Table 11.—*Mass transport, shallow-water waves* $\left(\frac{d}{L}<0.5\right)$—Continued

Point	Depth feet	*d* feet	U_t ft./sec.	U_m ft./sec.	U_m-U_t ft./sec.
28-1	0.021	1.5	+0.014	+0.036	+0.022
2	.102		+.011	+.025	+.014
3	.209		+.008	+.044	+.036
4	.285		+.006	+.035	+.029
5	.417		+.003	+.031	+.028
6	.491		+.002	+.021	+.019
7	.550		+.001	+.025	+.024
8	.577		+.000	+.007	+.007
9	.695		.000	+.003	+.003
10	.841		−.003	+.000	+.003
29-1	.031	1.5	+.029	−.005	−.034
2	.135		+.021	+.024	+.003
3	.229		+.015	+.012	−.003
4	.345		+.009	+.016	+.007
5	.538		+.001	−.008	−.009
6	.655		−.003	+.038	+.041
7	.910		−.009	+.029	+.038
8	1.055		−.011	+.015	+.026
9	1.037		−.011	+.025	+.036
30-1	0.028	1.5	+.066	+.045	+.039
2	.177		+.004	+.040	+.036
3	.320		+.002	+.047	+.045
4	.426		+.002	+.035	+.033
5	.529		+.001	+.024	+.023
6	.585		+.000	+.022	+.022
7	.722		+.000	+.003	+.003
8	.838		−.001	−.003	−.002
9	.870		−.001	+.010	+.011
31-1	.029	1.0	+.002	−.007	−.009
2	.170		+.001	+.019	+.018
3	.229		+.001	−.003	−.004
4	.343		+.000	−.012	−.012
5	.552		−.001	+.003	+.004
32-1	.093	1.0	+.012	−.005	−.017
2	.112		+.011	.000	−.011
3	.248		+.005	−.007	−.012
4	.248		+.005	−.011	−.016
5	.363		+.001	.000	−.001
6	.443		−.001	−.006	−.005
7	.559		+.004	−.003	+.001

The comparison of measured and theoretical values of mass transport, at various depths below the water surface, is shown for several representative waves in table 11. As in the case of deep-water waves, the trochoidal theory does not admit the existence of mass transport; the theoretical value given by the Stokes-Struik theory is:

$$U_t=k^2a^2C\frac{e^{\frac{4\pi}{L}(d-h')}+e^{\frac{-4\pi}{L}(d-h')}}{\left(e^{\frac{2\pi d}{L}}-e^{\frac{-2\pi d}{L}}\right)^2}-\frac{h^2}{d}\frac{\sqrt{\frac{\pi g}{32L}}}{\sqrt{\tanh\frac{2\pi d}{L}}} \tag{17}$$

The data are presented graphically on Figure 14.

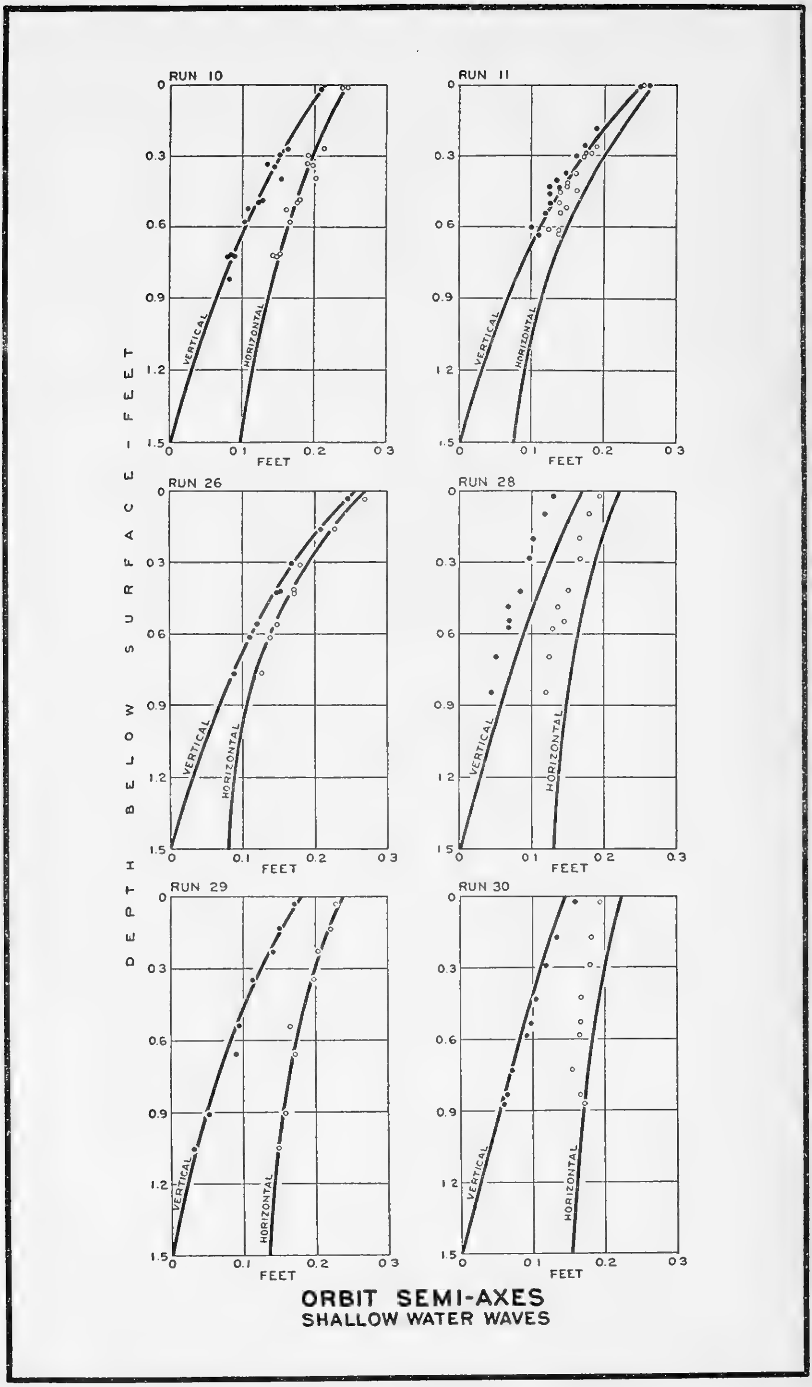

RUN 10
RUN 11
RUN 26
RUN 28
RUN 29
RUN 30
VERTICAL
HORIZONTAL
DEPTH BELOW SURFACE - FEET
FEET
ORBIT SEMI-AXES
SHALLOW WATER WAVES

FIGURE 13.

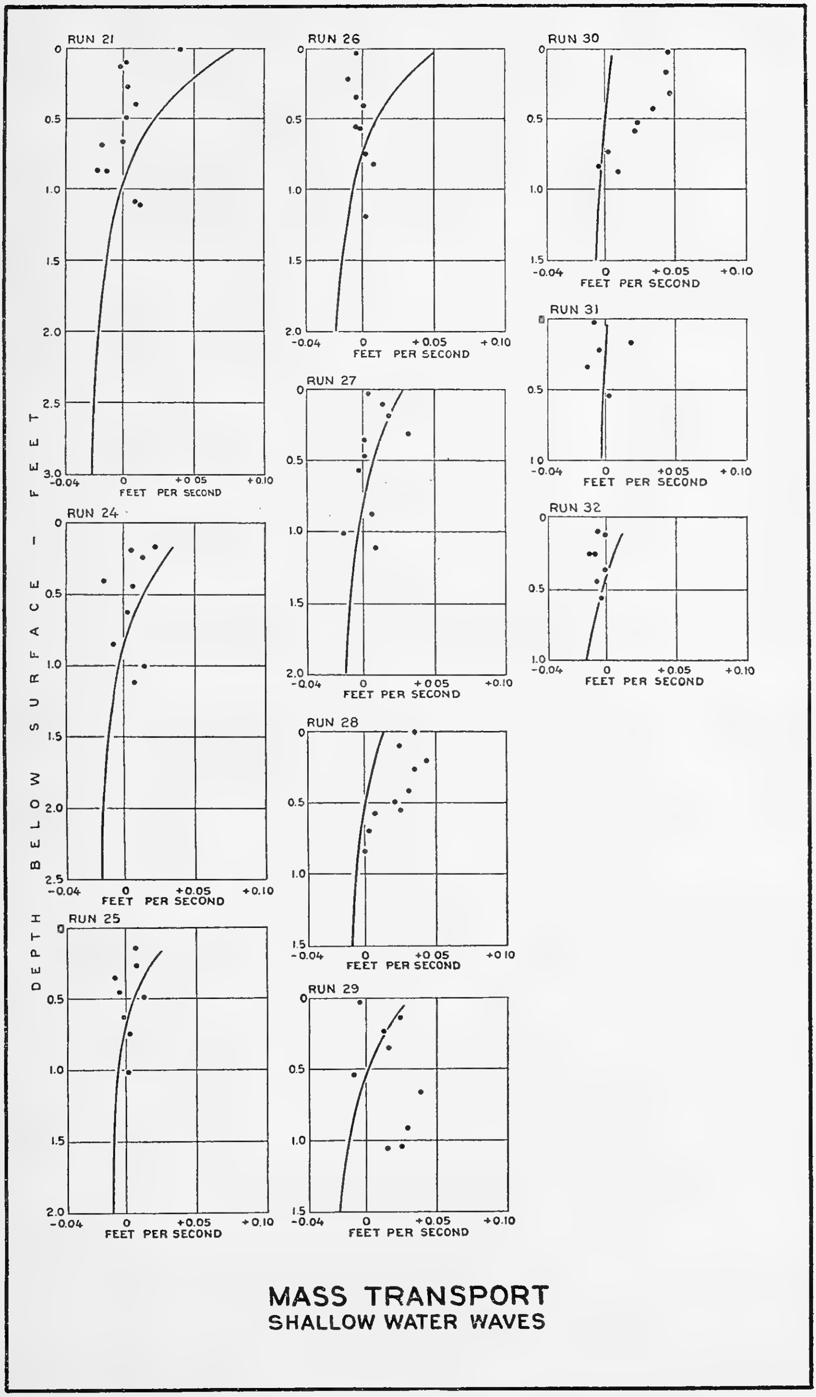

RUN 21
RUN 24
RUN 25
RUN 26
RUN 27
RUN 28
RUN 29
RUN 30
RUN 31
RUN 32
DEPTH BELOW SURFACE — FEET
FEET PER SECOND
MASS TRANSPORT
SHALLOW WATER WAVES

FIGURE 14.

7. DISCUSSION

Previous to any discussion of the results obtained in this study certain remarks should be made concerning their limitations.

Primarily, it should be noted that the generated wave requires a certain distance of travel after its generation to reach a stable form. It was noted during the course of the experiments that the measured values of wave velocity, wave length, and wave height, changed with time and position in the tank. For example, an average decrease in wave height of 0.011 foot was noted over a space of 12 feet immediately preceding the center observation window of the wave tank; while over the succeeding 13 feet the average decrease was only of the order of 0.001 foot. Further observations indicated that at a distance of 2 feet from the plunger, the shape of the particle orbits was essentially as indicated by theory, but that the forward travel of the water particles was small and practically constant from surface to bottom.

In the generation of the wave it seems reasonable to assume that since the water is not displaced according to the laws governing the particle motions then a certain number of cycles, or oscillations, of the water surface will be required after displacement of the water by the generating plunger, before the water particles take on the more or less stable movement indicated by both theory and observation. The variation of wave height should be a sufficient criterion as to the attainment of a stable form by the generated wave. Observations of many waves indicate that in the present wave tank and with the present method of generation, a distance equivalent to about 6 to 10 wave lengths is required for the assumption of a stable form by the wave. These experiments were by no means definitive and the results cited must be regarded as merely indicative of a condition, rather than as quantitative values accurately defining the condition.

The distance from the wave generator to the section of the wave tank selected for making the observations for this study is about 24 feet. It follows from the above discussion that insofar as this effect is concerned the most accurate results are those for waves whose lengths were in the neighborhood of 4 feet or less.

It is believed that the magnitude of the error resulting from observation of the wave characteristics before a stable form is attained is, in general, small. However, for the determination of mass transport, which is a secondary effect, the error introduced by this condition may be appreciable. The quantitative determinations of mass transport reported herein therefore may not be the true magnitudes of the effect.

The ratio $\frac{d}{L}$ has been adopted in wave experimentation as a criterion to differentiate deep- and shallow-water waves. The reasoning leading to the adoption is based on the fact that above a certain value (about 0.5) of the ratio, the value of the term tanh $\frac{2\pi d}{L}$ appearing in the Gerstner and Laplace-Airy equations of wave motion approaches very closely to unity. It should, however, be noted that the distinction of deep- or shallow-water waves made on this basis is arbitrary and, as far as the Stokes' theories are concerned, may have practically no significance. In fact, Stokes defines the deep-water wave as one for which "the depth of the fluid is very great compared with the length of a wave", when we may without sensible error suppose the depth to be infinite.

In reality, as an examination of the formulae in Section 6 will show, the ratio $\frac{d}{L}$ is effective as a parameter in Stokes' theory until large values of the ratio are reached. If the comparison of shallow-water wave theory with experimental results shows Stokes' theory to be most nearly applicable to the waves studied, it would appear, therefore, that the present arbitrary distinction between deep- and shallow-water waves should be revised to be more in accord with the principles on which Stokes makes the distinction.

Considering first the results of section 5, the comparison of wave lengths and wave velocities (all comparisons are understood to be between measured and theoretical values, unless specifically stated otherwise) indicates agreement of experiment with both of the theories studied within the limits of experimental error, the closer agreement being with the Gerstner theory.

The comparison of the proportions of wave height above still water level indicates agreement with both theories within the limits of the experimental error, for values of $\frac{h}{L}$ less than about 0.05. Above this value the measured percentages tend to be higher than those predicted by either theory.

It should be noted in this connection, and also with respect to the wave profiles, that the experimental values are susceptible to considerable error by reason of a phenomenon of reflection from the wave absorber. The first waves generated from water at rest had smooth, flowing profiles. As soon as these first waves reached the absorber it was noticed that small, short waves appeared to be superimposed on the generated waves, until after a short time the initial smooth wave surface became a surface of more or less irregular outline. This condition is shown clearly in figures 3 to 5. Furthermore the fact that the generated waves had not attained a stable form before measurement probably had an appreciable effect on the shape measurements.

With respect to the wave profiles there is little evidence to show agreement with either of the theories investigated.

As noted previously, Gerstner's theory does not provide for mass transport of the water in the direction of wave travel, the paths of the water particles being, in the case of deep water, circles. According to the Stokes-Levi-Civita theory the paths are open; but are basically circles modified by the effect of mass transport. The magnitude of the motion is the same in either case, except as modified by mass transport. The results listed in tables 5 to 7 show that the curves are not closed, but open, indicating that mass transport does exist. The measured values of the orbit radii are generally in fair agreement with theory, with a tendency for the experimental values to be smaller than the theoretical being shown.

The results of the measurements of mass transport are not conclusive except insofar as they confirm the existence of mass transport. Quantitative verification of the theory could not be made with the equipment now available. The importance of mass transport in natural beach processes should not be underestimated. It is, for example, a possible explanation of rip tides, since on any extended ocean coast there may exist a localized seaward current; i. e. a rip tide, in compensation for the shoreward current of mass transport.

For the case of shallow-water waves (section 6) the comparison of wave length and wave velocity indicates agreement within the limits of experimental error with both of the theories considered, the closer agreement being with the Stokes-Struik theory.

The comparison of proportion of wave height above still water level is inconclusive, verification of either theory within the limits of experimental error being indicated.

The wave profiles show noticeably better agreement with theory than do the deep-water wave results. Within the limits of error agreement with either theory is indicated.

The comparison of orbit semi-axis values shows, as for the deep-water waves, a tendency toward smaller measured values than are indicated by theory. However, the existence of mass transport is clearly shown. The magnitude of the mass transport effect could not be quantitatively verified.

8. CONCLUSIONS

1. All of the characteristics of the oscillatory wave in deep or shallow water required by the irrotational wave theories have been reproduced in the wave tank, with the exception of the wave profile.

2. It has been shown that mass transport does exist, as indicated by the theories of Stokes, but the theoretical magnitudes have not been quantitatively verified.

3. Since mass transport has been shown to exist, it follows that the Gerstner and Laplace-Airy wave theories, which do not admit its existence, are not applicable to the waves studied.

4. The magnitude of the difference in value of the wave characteristics, other than mass transport, defined by the rotational or irrotational theories is smaller than the experimental error associated with the tests described herein.

5. Certain experience in wave experimentation has been gained by this essentially exploratory study which leads to the following general statements.

a. Prior to further experimentation for the purpose of verifying wave theory, studies should be made of the generation of irrotational waves by mechanical means, and of the necessary length of wave travel after generation required for the wave to achieve a stable, or permanent, form.

b. The techniques of wave measurement in the laboratory should be improved, a maximum error in experimentation of 0.25 percent being desirable when comparisons similar to those of this study are to be made.

c. Provision should be made in future experiments to determine directly the existence or non-existence of rotation in the wave motion by study of the particle motion.

Appendix I

PREVIOUS INVESTIGATIONS RELATIVE TO THE VERIFICATION OF WAVE THEORY

This appendix lists and discusses briefly the available data regarding the verification of wave characteristics predicted by theory. The complete references noted are listed in appendix II.

This study does not include wave generation, pressures, breaking, or movement of materials, hence these subjects are not discussed.

Primary elements.—(Velocity, length, period)—Gaillard (1904) found the measured velocity to be 3 percent greater for 85 observations than the Gerstner theoretical velocity for deep water. In shallow water he found the measured velocity to average 0.2 percent greater than the Laplace-Airy theoretical velocity for 84 observations at St. Augustine, Fla., and 631 observations on Lake Superior.

Thorade (1931) lists the following table summarizing a large number of comparisons of measured velocity with velocity computed from length and period by Gerstner's equations for a large number of observations.

Author	Year	Ocean	$\frac{C}{C_L}$	Percent	$\frac{C}{C_T}$	Percent	Remarks
Paris	1871	All oceans	96	±2	96	±3	5 groups, totaling about 4,000 observations.
de Benaze	1874	Atlantic	96	±2			25 groups, swell.
			109	±3			14 groups, wind waves.
H. M. S. "Gazelle"	1874–76	All oceans	100	±2	101	±5	4 groups, violent winds.
Abercromby	1888	Pacific	106	±3	114	±7	6 groups, violent winds.
Schott	1893	Atlantic and Indian.	99	±2	99	±3	7 groups, wind waves.
			104	±1	109	±3	8 groups, swells.
Gassenmayr	1896	Atlantic	90	±2	83	±4	20 groups, wind waves.
			99	±7	96	±3	8 groups, swells.
Average			100	±1	100	±2	

In 1929, at Long Branch and Seaside Heights, N. J., the Beach Erosion Board found that for 206 observations in shallow water the measured velocity averaged 1.5 percent greater than the Laplace-Airy theoretical velocity.

A large number of studies have indicated that oscillatory waves generated in tanks follow the theoretical relationships for velocity, length, and period. DeCaligny (1843) and Hagen (1861) were pioneers in this field. Larras, Stucky and Bonnard, Bagnold and others have used this relationship as a check in connection with their experiments. Mitchim (1939) found that for 28 deep-water runs his measured velocities were 1 percent too large according to Gerstner's equation and 0.3 percent too small for the third approximation of Rayleigh.

Wave profile.—Wave profiles have been studied both in nature and in the wave tank. Gaillard's work contains four profiles photographed in the Duluth Canal and compared to the theoretical shapes. Cornish noted trochoid-like shapes in his observations. Kohlschutter and Schumacher on the "Meteor" investigations (1928) and Weinblum, Schnadel, and Block on the "San Francisco" expedition photographed waves corresponding closely to trochoids.

In wave tanks, Meyer-Peter, Larras, Waters, and others, have checked their wave shapes against the theory with good agreement. R. D. Meyer noted that in shallow water the wave front was steeper and the back slope flatter than the trochoid.

Dimensionless ratios.—The height-length ratio of waves has received considerable attention. Paris found that the ratio in deep water $\frac{L}{h}=39$ corresponded to a light sea, 21 to a rough sea, and 19 to a heavy sea. Schott's observations verified these findings. Gaillard found that the $\frac{L}{h}$ ratio varied from 9.1 to 15.0 for 235 observations in shallow water; Cornish observed ratios as low as 13 during storms.

Larras used the length-height ratio as a check on his work on vertical jetties, keeping the ratio between 20 and 11 for shallow-water conditions. Mitchim noted this ratio in applying the third approximation of Rayleigh to his results, using the limits of 12 and 26 for his studies in deep water.

The depth-length ratio has been generally accepted as the criterion of deep- and shallow-water waves. Almost all investigators have used this check in determining the applicable formula.

The percentage of wave height above still water level is of particular significance in studies of wave pressure against structures. Gaillard made a total of 834 observations at St. Augustine and Duluth (1890–1902), and the Beach Erosion Board made 365 observations at Long Branch, N. J., in 1929–30.

In laboratories, the following studies are known to this office: Larras, France, 1937; K. C. Reynolds, Massachusetts Institute of Technology, 1937; F. K. Klauck, Massachusetts Institute of Technology, 1937; C. H. Waters, University of California, 1938; and the Beach Erosion Board, 1939, made observations on the percentage of wave height above still water level in conjunction with other studies. All these observations, both in nature and in the wave tank, made in shallow water under a variety of bottom conditions, showed close general agreement with the Laplace-Airy theory.

Orbit radii.—Aimé (1839) observed upright ellipses off the coast of Algeria. De Caligny (1843) and Hagen (1861) found that water particles in shallow water traveled in paths resembling but distinctly different from the ordinary ellipse.

Mitchim at the University of California in 1939 made 114 observations of deep-water waves and found an average difference of 8.2 percent between laboratory wave and theory.

Mass transport.—Mitchim in his 1939 study found a qualitative verification of the theory of mass transport for 142 observations in the University of California laboratory wave tank.

APPENDIX II

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