THE SUB-MECHANICS OF THE UNIVERSE

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THE SUB-MECHANICS OF THE UNIVERSE

BY

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PREFACE.

THIS memoir "On the Sub-Mechanics of the Universe" was communicated to the Royal Society on February 3, 1902, for publication in the *Philosophical Transactions*; it was read in abstract before the Society on February 13. It was under criticism by the referees of the Royal Society some five months. I was then informed by the Secretaries that it had been accepted for publication in full. At the same time the Secretaries asked me if I should be willing, on account of the size and character of the memoir, which seemed to demand a separate volume, to consent to what appeared to be an opportunity of making a substantial reduction in what would otherwise be the expense. The Cambridge University Press had already published two volumes of my Scientific Papers and were willing to share in the cost of publishing this as a separate volume to range with the other two, special copies being distributed by the Royal Society as in the case of the *Philosophical Transactions*. To this proposal I readily agreed.

OSBORNE REYNOLDS.

January 23, 1903.

ERRATUM.

p. 5, line 22: *for* 2 *read* q.

TABLE OF CONTENTS.

SECTION I.

Introduction.

SECTION II.

The General Equations of Motion of any Entity.

SECTION III.

The General Equations of Motion, in a purely Mechanical Medium, of Mass, Momentum, and Energy.

SECTION IV.

Equations of Continuity for Component Systems of Motion.

SECTION V.

The Mean and Relative Motions of a Medium.

SECTION VI.

Approximate Equations of Component Systems of Mean and Relative Motion.

SECTION VII.

The General Conditions for the Continuance of Component Systems of Mean and Relative Motion.

SECTION VIII.

The Conducting Properties of the Absolutely Rigid Granule, Ultimate-Atom or Primordian.

SECTION IX.

The Probable Ultimate Distribution of the Velocities of the Members of Granular Media as the Result of Encounters when there is no Mean Motion.

SECTION X.

Extension of the Kinetic Theory to include Rates of Conduction through the Grains when the Medium is in Ultimate Condition and under no Mean Strain.

SECTION XI.

The Redistribution of Angular Inequalities in the Relative System.

SECTION XII.

The Linear Dispersion of Mass and of the Momentum and Energy of Relative Motion by Convection and Conduction.

SECTION XIII.

The Exchanges between the Mean and Relative Systems.

SECTION XIV.

Conservation of Inequalities in the Mean Mass and their Motions about Local Centres.

SECTION XV.

THE DETERMINATION (1) OF THE RELATIVE QUANTITIES α'', λ'', σ AND G WHICH DEFINE THE CONDITION OF THE MEDIUM BY THE RESULTS OF EXPERIENCE: (2) THE GENERAL INTEGRATION OF THE EQUATIONS.

SECTION I.

INTRODUCTION.

1. By this research it is shown that there is one, and only one, conceivable purely mechanical system capable of accounting for all the physical evidence, as we know it, in the Universe.

The system is neither more nor less than an arrangement, of indefinite extent, of uniform spherical grains generally in normal piling so close that the grains cannot change their neighbours, although continually in relative motion with each other; the grains being of changeless shape and size; thus constituting, to a first approximation, an elastic medium with six axes of elasticity symmetrically placed.

The diameter of a grain, in C.G.S. units, is

$$5{\cdot}534 \times 10^{-18} = \sigma.$$

The mean relative velocities of the grains are

$$6{\cdot}777 \times 10 = \alpha''.$$

The mean path of the grains is

$$8{\cdot}612 \times 10^{-28} = \lambda.$$

These three quantities completely define the state of the medium in spaces where the piling is normal; they also define the mean density of the medium as compared with the density of water as

$$10^4 = 22\Omega.$$

The mean pressure in the medium, equal in all directions, is

$$1{\cdot}172 \times 10^{14} = p.$$

The coefficient of the transverse elasticity resulting from the gearing of the grains, where the piling is normal, is

$$9{\cdot}03 \times 10^{24} = n.$$

The rate of propagation of the transverse wave is

$$3{\cdot}004 \times 10^{10} = \tau \text{ or } \sqrt{n/\rho}.$$

The rate of propagation of the normal wave is

$$7{\cdot}161 \times 10^{10} = 2{\cdot}387 \times \tau.$$

The rate of degradation of the transverse waves, *i.e.* the dissipation resulting from the angular redistribution of the energy, or viscosity, is

$$5{\cdot}603 \times 10^{-16} = t_t$$

or such as would require fifty-six million years to reduce the total energy in the wave in the ratio $1/e^2$, or to one-eighth; thus accounting, by mechanical considerations, for the blackness of the sky on a clear dark night; while the degradation of the normal wave, *i.e.* the dissipation resulting from the linear redistribution of energy, is such that the initial energy would be reduced to one-eighth in the $(3{\cdot}923 \times 10^{-6})$th part of a second, or before it had traversed 2200 metres; and thus would account by mechanical considerations for the absence of any physical evidence of normal waves, except such evidence as might be obtained within some metres of the origin of the wave; as in the case of Röntgen rays.

2. In spaces in which there are local inequalities in the medium about local centres, owing to the absence or presence of a number of grains, in deficiency or excess of the number necessary to render the piling normal, such local inequalities are permanent; and are attended with inward or outward displacements and strains, as the case may be, extending indefinitely throughout the medium, causing dilatation equal everywhere to the strains but of opposite sign, *i.e.* dilatation equal to the volume of the grains, the presence or absence of which cause the inequality.

When the arrangement of the grains about the centres is that of a nucleus of grains in normal piling on which grains in the strained normal piling rest, the nucleus in normal piling cannot gear with the grains outside, in strained normal piling; so that there is a singular surface of misfit between the nucleus and the grains in strained normal piling.

Such singular surfaces are surfaces of weakness and may be surfaces of freedom or surfaces of limited stability with the neighbouring grains.

These singular surfaces, when their limited stability is overcome, are free to maintain their motion through the medium, by a process of propagation, in any direction; the number of grains entering the surface on the one side being exactly the same as the number leaving on the other side; so that when the inequalities are the result of the absence of grains they correspond to the molecules of matter.

If the singular surface of a negative inequality is propagating through a medium which is at rest, the grains forming the nucleus will have no motion, whatever may be the motion of the singular surface: but the strained normal piling, which surrounds the singular surface and moves by propagation with the singular surface, being of less density than the mean density of the medium, represents a displacement of the negative mass of the inequality, *i.e.* of the grains absent. And in whatever direction the singular surface is propagated the motion of the medium outside is such as represents equal and opposite momentum; as when a bubble is rising in water.

In exactly the same way, for inequalities resulting from an excess of grains, the momentum resulting from the displacement of the medium would be positive.

The principal stresses in the medium outside the singular surface of a negative inequality are to a first approximation two equal tangential pressures equal in all directions;

$$p_t = \tfrac{6}{5} p,$$

and a normal pressure $$p_r = \tfrac{3}{5} p,$$

the mean of these pressures being everywhere the mean pressure of the medium p equal in all directions.

Efforts, proportional to the inverse square of the distance, to cause two negative inequalities at finite distances to approach are the result of those components of the dilatation (taken to a first approximation only) which are caused by the variation of those components of the inward strain which cause curvature in the normal piling of the medium. The other components of the strain being parallel, distortions which satisfy the condition of geometrical similarity do not affect the effort. If the grains were indefinitely small there would be no effort. Thus the diameter of a grain is the parameter of the effort; and multiplying this diameter by the curvature of the medium and again by the mean pressure of the medium the product measures the intensity of the effort.

The dilatation diminishes as the centres of the negative inequalities approach, and work is done by the pressure in the medium, outside the singular surfaces, to bring the negative inequalities together.

The efforts to cause the negative inequalities to approach correspond, exactly, to gravitation, if matter represents negative mass.

Taking the mean density of the earth as $-5{\cdot}67$, as compared with water (-1),

the reciprocal of the density of the medium being	10^{-4},
the mean pressure of the medium	$1{\cdot}172 \times 10^{14}$,
σ the diameter of the grain	$5{\cdot}534 \times 10^{-18}$,
the mean radius of the earth	$6{\cdot}3709 \times 10^{8}$;

the effort to cause approach between the earth and a unit of matter on the surface (-1) is the product of these quantities multiplied by $4\pi/3$, or

$$p\sigma \times 10^{-4} \times \tfrac{4}{3}\pi \times 5\cdot67 \times 6\cdot3709 \times 10^{8} = 9\cdot81 \times 10^{2}.$$

The inversion is thus complete. Matter is an absence of mass, and the effort to bring the negative inequalities together is also an effort on the mass to recede. And since the actions are those of positive pressure there is no attraction involved; the efforts being the result of the *virtual* diminution of the pressure inwards.

3. If instead of the negative inequalities, as in the last article, the inequalities are positive, the efforts would be reversed, tending to separate the positive inequalities, and the analysis would be the same, except that the curvature would be negative. And it is important to notice that if such positive inequalities exist, the fact that they repel each other—*i.e.* they would tend to scatter through space—together with the evidence that the number of inequalities either positive or negative occupy an indefinitely small space as compared to the total volume of the medium, places any importance such positive inequalities might have on a footing of indefinitely less importance than that of the negative inequalities which are caused to accumulate by gravitation; and thus we have an explanation of the lack of evidence of any positive inequalities, even if such exist.

4. Besides the positive and negative inequalities there is another inequality which may be easily conceived, and—this is of fundamental importance—whatever may be the cause, it is possible to conceive that a number of grains may be removed from some position in the otherwise uniform medium, to another position. Thus instituting a complex inequality, as between two inequalities, one positive and the other negative; the number of grains in excess in the one being exactly the same as the number deficient in the other.

The complex inequalities differ fundamentally from the gravitating inequalities, inasmuch as the former involve an absolute displacement of mass while the latter have no effect on the mean position of the mass in the medium; and in respect of involving absolute displacement of mass the complex inequalities correspond with electricity.

Apart from the displacement of mass the complex inequalities differ from the gravitating inequalities. In the complex inequalities the parameter of the dilatation is not the diameter of a grain but one half the linear dimension of the volume occupied by the grains displaced, taken as spherical.

The effort to revert in the case of the complex inequality is the product of the pressure multiplied by the product of the volumes of the positive and negative inequalities and again by the parameter r_0. This is expressed when the positive and negative inequalities are at finite distance apart by

$$p\left(\frac{4\pi}{3}\right)^3 r_0^7 = -R,$$

R being essentially negative and the dimensions of the effort $(-R)$ are mlt^{-2} which express an effort to the displacement of mass.

The complex inequality which corresponds to the separation of the positive and negative inequalities is one displacement, not two. This fact admits of no question and might have been recognised long ago had it not been for the general assumption that positive electricity repels positive electricity, the fact being that the apparent repulsion of the positive electricities is the result of their respective efforts to approach their respective negative inequalities. By the assumption it became apparently possible to express the potential V, and the electricity q as rational quantities, when, as it now appears, the potential V and the electricity q are respectively $-(-e^2)^{\frac{1}{2}}\frac{1}{r}$ and $(-e^2)^{\frac{1}{2}}$, which are both irrational. Their product being the rational quantity

$$\frac{e^2}{r},$$

which, differentiated with respect to the distance, is

$$-\frac{e^2}{r^2} = R,$$

and the mechanical explanation of these is,

$$-\left\{-p\left(\frac{4\pi}{3}\right)^2 r_0^7\right\}^{\frac{1}{2}}\frac{1}{r} = V, \quad \left\{-p\left(\frac{4\pi}{3}\right)^2 r_0^7\right\}^{\frac{1}{2}} = 2,$$

and for the effort to revert, we have

$$p\left(\frac{4\pi}{3}\right)^2 \frac{r_0^7}{r^2} = -R.$$

Then for the electrostatic unit we have, since $r=1$, and $R=-1$,

$$p\left(\frac{4\pi}{3}\right)^2 r_0^7 = 1,$$

and from the known value of p the number of grains displaced through unit distance necessary to cause the unit effort is

$$1{\cdot}615 \times 10^{45},$$

and $r_0 = 6{\cdot}493 \times 10^{-3}$, from which we have the ratio of the effort to reinstate the normal piling, to the effort of gravitation, from the same number of grains absent in each inequality as are displaced in the complex inequality, the distances being the same,

$$1{\cdot}2 \times 10^{15},$$

so that the effort of attraction between two inequalities, the grains absent about each of which is the same as the grains displaced in instituting the complex inequality, is eighty-one thousand billions less than that of the electric effort.

5. Cohesion between the singular surfaces of the negative inequalities results from the terms which were not taken into account in the first approximation which correspond to gravitation. These secondary terms involve the inverse distance to the sixth power, and therefore have a very short range, and so correspond to efforts of cohesion of the singular surfaces as well as surface tensions having no effect unless the singular surfaces, or molecules, are within a distance very small compared with the diameter of the singular surface.

6. Transverse undulations in the medium, corresponding to the waves of light, are instituted by the disruptive reversion of the complex inequalities. The recoil sets up a vibration which is exhausted in initiating light.

7. Thus far the sketch of the results has included only those for which there exists sufficient evidence to admit of definite quantitative analysis. Nevertheless these quantitative results show that the granular medium, as already defined, accounts by purely mechanical considerations for the evidence, and affords the only purely mechanical explanation possible. If then the substructure of the universe is mechanical, all the evidence, not already adduced, is such as may be accounted for by an extension of the analysis, and this is found to be the case.

The results of the further analysis afford proof:—

Of the existence of coincidence between the periods of vibration of the molecules and the periods of the waves;

that dissociation of compound molecules proves the previous state to have been one of limited stability;

that the reassociation of compound molecules results from the reversion of complex molecules;

of the absorption of the energy of light by inequalities;

that negative inequalities affect the waves passing through;

that refraction is caused by the vibration of inequalities having the same periods as the waves;

that dispersion results from the greater number of coincidences as the waves get shorter;

that the polarization by reflection is caused only by that component of the transverse motion in the medium which is in the plane of incidence and results from the passage of the light from a space without, or with few, inequalities, through a surface into a space in which there are more inequalities;

that the metallic reflection results from the relative smallness of the dimensions of the molecules compared with the length of the wave and the closeness of their piling when the waves pass from a space without inequalities across the surface beyond which the inequalities are in closest order;

that the aberration of light results from the absence of any appreciable resistance to the motion of the medium when passing through matter.

8. It may be somewhat out of the usual course to describe the results of a research before any account has been given of the method by which these results have been obtained; but in this case the foregoing sketch of the purely mechanical explanation of the physical evidence in the universe by the granular medium has seemed the only introduction possible, and even so it is not with any idea that this introduction can afford any preliminary insight as to the methods by which these results have been obtained.

Certain steps, as it now appears, were taken for objects quite apart from any idea that they would be steps towards the mechanical solution of the problem of the universe.

The first of these steps was taken with the object of finding a mechanical explanation of the sudden change in the rate of flow of the gas in the tube of a boiler when the velocity reached a certain limit—perhaps this would be better described as a step towards a step*.

The second step was the discovery of the thermal transpiration of gas together with the analytical proof of the dimensional properties of matter†.

The third step was the discovery of the criterion of the two manners of motion of fluids‡.

* *Manchester Lit. and Phil. Soc.* 1874—5, p. 7.

† *Royal Soc. Phil. Trans.* 1879.

‡ *Royal Soc. Phil. Trans.* 1883.

And it was only on taking the fourth step, namely, the study of the action of sand, which revealed dilatancy as the ruling property of all granular media*, which directed attention to the possibility of a mechanical explanation of gravitation. In spite of the apparent possibility, all attempts to effect the necessary analysis failed at the time.

There was however a fifth step; the effecting of the analysis for viscous fluids, and the determination of the criterion†, which led to the recognition of the possibility of the analytical separation of the general motion of a fluid into mean varying motion, displacing momentum, and relative motion; and this suggested the possibility that the medium of space might be granular, the grains being in relative motion and at the same time being subject to varying mean motion. And this has proved to be the case. At the same time it became evident that it was not to be attacked by any method short of the general equations of a conservative system starting from the very first principles; and it is from such study that this purely mechanical account of the physical evidence has been obtained.

* *Phil. Mag.* 1885.

† *Royal Soc. Phil. Trans.* 1895, A.

SECTION II.

THE GENERAL EQUATIONS OF MOTION OF ANY ENTITY.

9. Axiom I. Any change whatsoever in the quantity of any entity within a closed surface can only be effected in one or other of two distinct ways:

(1) it may be effected by the production or destruction of the entity within the surface, or

(2) by the passage of the entity across the surface.

To express this general axiom in symbols I put;—Q for the quantity required to occupy unit volume, as an indefinitely small element of volume, δS, at any point within the surface is occupied. Q is thus the density of the entity at the point, and however it may vary from point to point is a single valued function of the position of the point:

$\Sigma(Q\delta S) = \iiint Q\,dx\,dy\,dz$ is put for the quantity within a space S enclosed by the surface s at the instant considered,

$\Sigma({}_oQ\delta S)$ is the quantity enclosed at a previous instant.

$\Sigma({}_pQ\delta S)$ is the quantity which has been produced within s during the interval, and

$\Sigma({}_cQ\delta S)$ is the quantity which has crossed the surface inwards during the interval.

Then
$$\Sigma(Q\delta S) = \Sigma({}_oQ\delta S) + \Sigma({}_pQ\delta S) + \Sigma({}_cQ\delta S)$$
is a complete expression for the Axiom.

Using $\delta[\;]$ to express any change effected in the time δt this may be written
$$\delta[\Sigma(Q\delta S)] = \delta[\Sigma({}_pQ\delta S)] + \delta[\Sigma({}_cQ\delta S)] \ldots\ldots\ldots\ldots\ldots (1).$$

And this equation (1) is the general equation of motion of any entity as founded on Axiom (I.).

10. *General equation of Continuity.*

Axiom II. When the entity considered is some particular form or mode of an entity which, like matter, momentum, or energy, can neither be produced or destroyed, any production or destruction of a particular form of the entity at a particular place and instant of time involves the destruction or production, at the same place and time, of an equal quantity of the same entity in some other form or mode.

To express this in symbols let Q refer to the general entity without distinction of form or mode and Q_1, Q_2, &c. respectively refer to the several particular forms or modes of the entity.

Then since

$$\delta [\Sigma ({}_pQ\delta S)] = 0,$$

$$\delta [\Sigma ({}_pQ_1\delta S)] = -\delta [\Sigma ({}_pQ_2\delta S + \&c.)] \quad\text{.................. (2)},$$

which is a general expression for the law of conservation, and is the general equation of continuity in terms of the several distinct actions of exchange between the different modes of the entity.

11. *Transformation of the Equations of Motion and continuity for a steady surface.*

Equations (1) and (2) hold however large or small the space S and the interval δt may be and whatever may be the motion of the surface s enclosing the space S; for the δ covers the Σ ( ).

If however the surface s be steady or fixed in space the δ may be covered by the Σ ( ) and the equations written

$$\Sigma [\delta (Q\delta S)] = \Sigma [\delta ({}_pQ\delta S)] + \Sigma [\delta ({}_cQ\delta S)] \quad\text{........... (3)},$$

$$\Sigma [\delta ({}_pQ_1\delta S)] = -\Sigma [\delta ({}_pQ_2\delta S + r)] \quad\text{........................(4)}.$$

Since these equations hold for indefinitely small spaces and indefinitely small intervals of time in the limit, when dx, dy, dz and dt are severally zero:—

$$\Sigma (Q\delta S) = Qdxdydz \quad\text{............................ (5)},$$

and

$$\Sigma [\delta (Q\delta S)] = \frac{d}{dt}(Q)\, dtdxdydz \quad\text{..................... (6)}.$$

In cases where Q is not a continuous function of t the meaning of such differential coefficients as that in the right member of equation (6) become unintelligible without further definition, and it seems desirable here to point out, once for all, in what sense they are used in this paper.

12. *Discontinuity.*

If Q is any function of xyz and t, which is single valued at every point of space at every instant, but which at a particular time t is discontinuous at a surface which is expressed by

$$\phi = \phi (x, y, z, t) = 0.$$

Where ϕ has positive values on one side of the surface and negative values on the other, then putting Q_1 for the continuously varying value of Q where ϕ is negative and Q_2 for Q where ϕ is positive, Q is at all times expressed by the limiting value of the function

$$F=\frac{Q_1+Q_2\,e^{n\phi}}{1+e^{n\phi}},$$

when n is infinite*.

For any finite value of n F is a continuous function of the variables, as are also the derivatives of F; and substituting F for Q, the limiting values, when n is infinite, of any functions derived from F by any mathematical process are taken as the values of the function expressed by the same mathematical process performed on Q†.

13. Having regard to the foregoing definition of the interpretation to be put upon the meaning of the differential coefficients in cases of discontinuity, the expressions obtained by equations (5) and (6) for the rates of convection into and production in such indefinitely small spaces may be treated as continuous functions of the coordinates.

Thus taking u, v, w for the component velocities of the entity, to which Q refers, passing a point x, y, z, relative to the surface of the elementary space $dxdydz$ at rest or in steady motion, since u, v, w are single valued at each point at any instant of time the convection into the space in the interval dt is expressed by

$$\left.\begin{array}{l} dt\,\dfrac{d}{dt}({}_cQ)\,dxdydz=-dt\left\{\dfrac{d}{dx}(uQ)+\dfrac{d}{dy}(vQ)+\dfrac{d}{dz}(wQ)\right\}dxdydz \\ \text{or at a point the rate of change by convection is} \\ \qquad \dfrac{d}{dt}{}_cQ=-\left\{\dfrac{d\,(uQ)}{dx}+\dfrac{d\,(vQ)}{dy}+\dfrac{d\,(wQ)}{dz}\right\} \end{array}\right\}\ \ldots\ldots(7),$$

* *Electricity and Magnetism*, Maxwell, § 8.

† *Electricity and Magnetism*, Maxwell § 8.

$$\frac{dF}{dt}=\frac{\dfrac{dQ_1}{dt}+\dfrac{dQ_2}{dt}e^{n\phi}}{1+e^{n\phi}}-\frac{n\,(Q_1-Q_2)\,e^{n\phi}}{(1+e^{n\phi})^2}\frac{d\phi}{dt}.$$

From which, taking n infinite, when ϕ is negative $\dfrac{dF}{dt}=\dfrac{dQ_1}{dt}$, when ϕ is positive $\dfrac{dF}{dt}=\dfrac{dQ_2}{dt}$ and when $\phi=0$

$$\frac{dF}{dt}=-\frac{ne^{n\phi}\dfrac{d\phi}{dt}(Q_1-Q_2)}{(1+e^{n\phi})^2},$$

which is infinite, but which, integrated, from ϕ negative to ϕ positive over an interval δt, indefinitely small, gives

$$\int\frac{dF}{dt}\,\delta t=Q_2-Q_1.$$

whence substituting in equations (1) and (2) for the indefinitely small element $dxdydz$ and the indefinitely small interval of time dt, these become:—

$$dt\frac{dQ}{dt}dxdydz = dt\left\{\frac{d}{dt}({}_pQ) - \frac{d}{dx}(uQ) - \frac{d}{dy}(vQ) - \frac{d}{dz}(wQ)\right\}dxdydz \ldots\ldots (8),$$

$$dt\frac{d}{dt}({}_pQ_1)\,dxdydz = -\,dt\left\{\frac{d}{dt}({}_pQ_2 + \&c.)\right\}dxdydz \;\ldots\ldots\ldots (9),$$

or at a point the rate of change is

$$\frac{d}{dt}(Q) = \frac{d}{dt}({}_pQ) - \left\{\frac{d}{dx}(uQ) + \frac{d}{dy}(vQ) + \frac{d}{dz}(wQ)\right\} \;\ldots\ldots\ldots (10),$$

$$\frac{d}{dt}({}_pQ_1) = -\frac{d}{dt}({}_pQ_2 + \&c.) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11).$$

Equation (10) expresses the rate of change in the density Q at a point in terms of the densities of the actions of production and convection at that point. While equation (11) expresses the relation which holds between the densities of the several actions of exchange between the different modes of Q.

14. *Moving Surface.*

In the equations (5) to (11) the surfaces of the element of space (δS or $dxdydz$) are steady, and in equations (3) and (4) the closed surface over which the summation is taken is also steady—the δ being covered by the Σ.

If, however, the motion of every point of the surface be taken into account it is possible to sum the results of equations (7), (8), (9) over the space enclosed by a surface in any manner of *continuous* motion.

Putting $\bar{u}$, $\bar{v}$, $\bar{w}$ for the component velocities of the surface at the point x, y, z, then the component motions of the entity represented by Q relative to the surface at this point are respectively

$$u - \bar{u}, \quad v - \bar{v}, \quad w - \bar{w},$$

and although $\bar{u}$, $\bar{v}$, $\bar{w}$ are only defined at the surface, since the motion of this surface is continuous, $\bar{u}$, $\bar{v}$, $\bar{w}$ may be taken as continuous function of x, y, z throughout the enclosed space. Then the rate of convection across the surface is expressed by

$$\frac{d}{dt}\Sigma({}_cQ\delta S) = -\iiint\left\{\frac{d}{dx}[(u-\bar{u})Q] + \frac{d}{dy}[(v-\bar{v})Q] + \frac{d}{dz}[(w-\bar{w})Q]\right\}dxdydz \ldots\ldots\ldots\ldots (12).$$

The instantaneous rate of production within the surface is not altered by the continuous motion of the surface. Therefore equation (1) becomes

$$\frac{d}{dt}\Sigma(Q\delta S) = \Sigma\left\{\delta S\frac{d}{dt}({}_pQ)\right\}$$
$$-\iiint\left\{\frac{d}{dx}[(u-\bar{u})Q]+\frac{d}{dy}[(v-\bar{v})Q]+\frac{d}{dz}[(w-\bar{w})Q]\right\}\ldots(13),$$

and integrating equation (10) over the surface, the rate of change in the space instantaneously enclosed as by a *fixed* surface is

$$\Sigma\left(dS\frac{dQ}{dt}\right) = \Sigma\left[dS\frac{d({}_pQ)}{dt}\right]$$
$$-\iiint\left\{\frac{d}{dx}(uQ)+\frac{d}{dy}(vQ)+\frac{d}{dz}(wQ)\right\}dx\,dy\,dz\ \ldots\ldots.(14);$$

whence substituting in equation (13) for

$$\Sigma\left\{\delta S\frac{d}{dt}({}_pQ)\right\}$$

from equation (14),

$$\frac{d}{dt}[\Sigma(QdS)] = \Sigma\left(dS\frac{dQ}{dt}\right)$$
$$+\iiint\left\{\frac{d}{dx}(\bar{u}Q)+\frac{d}{dy}(\bar{v}Q)+\frac{d}{dz}(\bar{w}Q)\right\}dx\,dy\,dz\ \ldots\ldots\ldots(15),$$

or as it may be written

$$\frac{d}{dt}[\Sigma(Q\delta S)] = \Sigma\left\{dS\left(\frac{d}{dt}+\bar{u}\frac{d}{dx}+\bar{v}\frac{d}{dy}+\bar{w}\frac{d}{dz}\right)Q\right.$$
$$\left.+Q\left(\frac{d\bar{u}}{dx}+\frac{d\bar{v}}{dy}+\frac{d\bar{w}}{dz}\right)\right\}\ \ldots\ldots\ldots\ldots(16).$$

SECTION III.

THE GENERAL EQUATIONS OF MOTION, IN A PURELY-MECHANICAL-MEDIUM, OF MASS, MOMENTUM AND ENERGY.

15. THESE equations are obtained by taking Q in equations (1) to (16) to refer successively to the density of *mass*, the density of the component, in a particular direction, of the momentum, and the density of the energy.

The forms of the equations so obtained, as well as the circumstances to which they are applicable, depend on the definition given, respectively, to the three entities.

If this definition is limited, strictly, to that afforded by the laws of motion as distinct from any physical or kinematical properties of *matter*, the equations will be the most general possible and applicable to all mechanical systems. In which case by introducing separately and step by step farther definition of the entities the effect of each such definition on the form of the equations and of the expressions for the resulting actions, to be obtained by integration of the equations, will be apparent; so that the individual effects of the several particular physical properties of matter may be analysed. While on the other hand if the definition is, in the first instance, such as that on which the equations of motion for fluids and elastic solids have been founded the equations so obtained will be essentially the same. And, although the significance of the several expressions in the equations as relating to accumulation, convection and production will be more clearly brought out they will afford no opportunity of analysing the several effects resulting from particular physical definition.

In this investigation the object sought, in the first instance, has been to render the equations the most general possible. Only introducing restrictive definition where the effect, of such definition, on the form of the expressions which enter into the equations and define the limiting circumstances to which the equations are applicable, becomes clearly defined.

16. A mechanical-system implies the existence, in the space occupied by the system, of an entity which possesses properties which distinguish the space so occupied from that which is unoccupied. If this entity includes everything that can occupy space, within the space occupied by the system, it is the mechanical-medium in which the system exists.

The sense in which mechanical-medium is here used is not that in which the term 'medium' or 'medium of space' is generally used in mechanical-philosophy, nor yet that for which "matter" is used. For although that which is recognised as matter is the only entity included in the equations of motion which has the property of occupying position in space, it is found necessary in order to account for experience to attribute to matter properties extending through spaces which are not occupied by matter, and to reconcile such extension with the absence of any mechanical properties as belonging to space itself it has been recognised that there exists in space some other entity, besides matter, which has the property of occupying position and is recognised in mechanical philsophy as the medium of space or the ether.

To the ether are attributed such mechanical properties, whatsoever these may be, as are necessary to account for the observed properties of matter which are not defined by implication in the laws of motion, as well as to account for all the properties extending outside the space occupied by the matter. This amounts to an admission that these physical or extended properties are not inherent in the matter nor yet in the ether, or in other words that they are not the properties of the entity which occupies position in space, but are the consequence of the mechanical actions and of the arrangement of the mechanical system of the Universe.

If then everything that occupies position in space is included by definition in the mechanical-medium, experience affords no reason for attributing to such medium inherent properties other than those required by the laws of motion and the law of conservation of energy, and so defined, the medium is here designated a *Purely-Mechanical-Medium.*

17. The properties of a *purely-mechanical-medium* necessitated by the laws of motion are

(1) The property of occupying definite position in space;

(2) The continuity or continuance in space and time;

(3) The property of definite capacity for momentum, *i.e.* definite mass;

(4) The property of receiving and communicating momentum in accordance with the laws of conservation of momentum and energy.

Since the mass of any particular portion of the medium measures the quantity of that portion of the medium and has identically the same position in space as that portion of the medium, this mass is identified with the particular portion of the medium. The density of the mass at every point in space is thus a measure of the density of the medium at every point; and the equations of motion and continuance in time and space of the mass are the equations of motion and continuance of the medium.

18. *The equations of continuity of mass.*

Putting $$\rho\,\delta S$$

for the capacity for momentum or *mass* in the indefinitely small space δS and substituting ρ for Q in equation (2) the equation for conservation of mass becomes

$$\delta\left[\Sigma\left({}_p\rho\,\delta S\right)\right] = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17);$$

and by equations (1) and (17) the equation of motion of mass becomes

$$\delta\left[\Sigma\left(\rho\,\delta S\right)\right] = \delta\left[\Sigma\left({}_c\rho\,\delta S\right)\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18).$$

Whence for the indefinitely small element of space $dxdydz$ and the indefinitely small interval of time dt it follows by equations (7) that

$$\frac{d\rho}{dt} + \frac{d\rho u}{dx} + \frac{d\rho v}{dy} + \frac{d\rho w}{dz} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19),$$

which is the general equation for density of *mass* or medium at a point.

19. *Position of mass.*

Taking x, y, z as defining the position of the indefinitely small steady space δs, and putting ρx, ρy, ρz successively for Q in equation (2), the equations for the conservation of the position of the mass become respectively

$$\Sigma\left[\delta\left\{{}_p(\rho x)\,\delta s\right\}\right] = 0, \quad \Sigma\left[\delta\left\{{}_p(\rho y)\,ds\right\}\right] = 0, \quad \Sigma\left[\delta\left\{{}_p(\rho z)\,ds\right\}\right] = 0 \ldots(20).$$

The equations for the rate of change of position of the mass within space over which the summation extends, become by equations (1) and (20)

$$\delta\left[\Sigma\left(\rho x\delta s\right)\right] = \delta\left[\Sigma\left\{{}_c(\rho x)\,\delta s\right\}\right], \text{ \&c., \&c.} \ldots\ldots\ldots\ldots\ldots\ldots(21).$$

Since x, y, z are not functions of the time, it follows by equation (19), if $\bar{x}$, $\bar{y}$, $\bar{z}$ define the position of the centre of gravity of the mass in the *steady* space over which the summation is taken, that

$$\frac{d\bar{x}}{dt} = \frac{\iiint (x - \bar{x})\left(\frac{d\rho u}{dx} + \frac{d\rho v}{dy} + \frac{d\rho w}{dz}\right) dxdydz}{\iiint \rho\, dxdydz}, \text{ \&c., \&c.} \ldots\ldots\ldots(22).$$

For in a fixed space,

$$\bar{x} = \frac{\Sigma\left(\rho x ds\right)}{\Sigma\left(\rho ds\right)} \quad \text{and} \quad \frac{d}{dt}\Sigma\left(\rho x ds\right) = -\iiint x\left(\frac{d\rho u}{dx} + \text{\&c.}\right) dxdydz;$$

$$\therefore \quad \frac{d}{dt}\left\{\bar{x}\Sigma\left(\rho ds\right)\right\} = \Sigma\left(x\frac{d\rho}{dt}ds\right).$$

Also $$\frac{d}{dt}\Sigma\left(\rho ds\right) = -\iiint\left(\frac{d\rho u}{dx} + \text{\&c.}\right) dxdydz.$$

For a space moving with the mass by (15)

$$\left.\begin{aligned}&\frac{d}{dt}\Sigma\left[(\rho x)\,\delta s\right]=\Sigma\left[\frac{d(\rho x)}{dt}\,\delta s\right]\\&\qquad+\Sigma\left[\left\{\frac{d}{dx}(\rho xu)+\frac{d}{dy}(\rho xv)+\frac{d}{dz}(\rho xw)\right\}\delta s\right]\end{aligned}\right\}\ldots(22\text{A}).$$

whence since x is not a function of t,

$$\Sigma\left(\rho\frac{d\bar{x}}{dt}\,\delta s\right)=\Sigma\,(\rho u\delta s),\ \&\text{c.},\ \&\text{c.}$$

20. Before proceeding to the consideration of *momentum* and *energy* it will be found convenient to express certain *general mathematical relations between the various expressions which enter into the equations for quantities into which* ρ *enters as a linear factor.*

When Q is put for ρq, where q is a factor which has only one value at each instant for each point in mass, but which value for the point in mass is a function of the time, then the derivatives of discontinuous functions having the meaning ascribed in Art. 12,

$$\frac{d\,({}_pQ)}{dt}=q\frac{d\,({}_p\rho)}{dt}+\rho\frac{d\,({}_pq)}{dt}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23).$$

And since by equation (17)

$$\left.\begin{aligned}&\frac{d\,({}_p\rho)}{dt}=0,\\&\frac{d\,({}_pQ)}{dt}=\rho\frac{d\,({}_pq)}{dt}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(24).$$

$$\left.\begin{aligned}\text{Also}\qquad&\frac{dQ}{dt}=q\frac{d\rho}{dt}+\rho\frac{dq}{dt},\\\text{and}\qquad&\frac{d\,({}_cQ)}{dt}=-\left\{\frac{d}{dx}(\rho uq)+\frac{d}{dy}(\rho vq)+\frac{d}{dz}(\rho wq)\right\}\\&\qquad=-q\left\{\frac{du\rho}{dx}+\&\text{c.}\right\}-\rho\left\{u\frac{dq}{dx}+\&\text{c.}\right\}\end{aligned}\right\}\ldots\ldots\ldots\ldots(25):$$

whence subtracting and having regard to equation (19)

$$\frac{dQ}{dt}-\frac{d\,({}_cQ)}{dt}=\rho\left\{\frac{dq}{dt}+u\frac{dq}{dx}+\&\text{c.}\right\},$$

therefore by equation (8) $\ldots\ldots\ldots\ldots\ldots\ldots(26).$

$$\frac{d\,({}_pQ)}{dt}=\rho\left\{\frac{dq}{dt}+u\frac{dq}{dx}+\&\text{c.}\right\}$$

Again, if $Q=\rho q=\rho q_1q_2$ and $Q_1=\rho q_1$, $Q_2=\rho q_2$, by equations (26),

$$\left.\begin{aligned}&\frac{d\,({}_pQ)}{dt}=\rho\left\{\frac{dq_1q_2}{dt}+u\frac{dq_1q_2}{dx}+\&\text{c.}\right\}\\&\quad=\rho\left[q_1\left\{\frac{dq_2}{dt}+u\frac{dq_2}{dx}+\&\text{c.}\right\}+q_2\left\{\frac{dq_1}{dt}+u\frac{dq_1}{dx}+\&\text{c.}\right\}\right]\end{aligned}\right\}\ldots(27),$$

and putting q_1 and q_2 respectively for q in equations (26) and substituting in the right member of the equations (27),

$$\left.\begin{aligned}\frac{dQ}{dt}-\frac{d\,({}_cQ)}{dt}&=q_1\left(\frac{dQ_2}{dt}-\frac{d\,({}_cQ_2)}{dt}\right)+q_2\left(\frac{dQ_1}{dt}-\frac{d\,({}_cQ_1)}{dt}\right)\\ \frac{d\,({}_pQ)}{dt}&=q_1\frac{d\,({}_pQ_2)}{dt}+q_2\frac{d\,({}_pQ_1)}{dt}\end{aligned}\right\}\text{............(28).}$$

21. In the equations (25) to (28) ρ is subject to the condition of conservation of mass, equations (17) and (19). If instead of ρ we take ρ'' as an abstraction of the density we obtain a corresponding but more general theorem, by putting

$$\frac{d\rho''}{dt}=-\left\{\frac{d\rho''u}{dx}+\frac{d\rho''v}{dy}+\frac{d\rho''w}{dz}\right\}+\frac{d\,({}_p\rho'')}{dt}\text{..............(23A),}$$

where the last term on the right expresses an arbitrary density; then

$$\frac{d\,({}_pQ)}{dt}=\rho''\frac{d\,({}_pq)}{dt}+q\frac{d\,({}_p\rho'')}{dt}\text{..............................(24A),}$$

$$\left.\begin{aligned}\frac{dQ}{dt}&=q\frac{d\rho''}{dt}+\rho''\frac{dq}{dt}\\ \frac{d\,({}_cQ)}{dt}&=-q\left(\frac{d\rho''u}{dx}+\&c.\right)-\rho''\left(u\frac{dq}{dx}+\&c.\right)\end{aligned}\right\}\text{...............(25A).}$$

Equating by (23A, 24A, 25A),

$$\left.\begin{aligned}\frac{dQ}{dt}-\frac{d\,({}_cQ)}{dt}&=q\frac{d\,({}_p\rho'')}{dt}+\rho''\left(\frac{dq}{dt}+u\frac{dq}{dx}+\&c.\right)\\ \frac{d\,({}_pQ)}{dt}&=q\frac{d\,({}_p\rho'')}{dt}+\rho''\left(\frac{dq}{dt}+u\frac{dq}{dx}+\&c.\right)\end{aligned}\right\}\text{............(26A).}$$

And putting $q=q_1q_2$ and $Q_1=\rho''Q_1$, $Q_2=\rho''Q_2$, we have

$$\left.\begin{aligned}q_1\left(\frac{dQ_2}{dt}-\frac{d\,({}_cQ_2)}{dt}\right)&=q_1q_2\frac{d\,({}_p\rho'')}{dt}+q_1\rho''\left(\frac{dq_2}{dt}+u\frac{dq_2}{dx}+\&c.\right)\\ q_2\left(\frac{dQ_1}{dt}-\frac{d\,({}_cQ_1)}{dt}\right)&=q_1q_2\frac{d\,({}_p\rho'')}{dt}+q_2\rho''\left(\frac{dq_1}{dt}+u\frac{dq_1}{dx}+\&c.\right)\\ \frac{dQ}{dt}-\frac{d\,({}_cQ)}{dt}&=q_1q_2\frac{d\,({}_p\rho'')}{dt}+\rho''\left(\frac{dq_1q_2}{dt}-u\frac{dq_1q_2}{dx}+\&c.\right)\end{aligned}\right\}\text{...(27A).}$$

From which it appears

$$\left.\begin{aligned}\frac{dQ}{dt}-\frac{d\,({}_cQ)}{dt}&=q_1\left(\frac{dQ_2}{dt}-\frac{d\,({}_cQ_2)}{dt}\right)+q_2\left(\frac{dQ_1}{dt}-\frac{d\,({}_cQ_1)}{dt}\right)-q_1q_2\frac{d\,({}_p\rho'')}{dt}\\ \frac{d\,({}_pQ)}{dt}&=q_1\frac{d\,({}_pQ_2)}{dt}+q_2\frac{d\,({}_pQ_1)}{dt}-q_1q_2\frac{d\,({}_p\rho'')}{dt}\end{aligned}\right\}\text{...(28A).}$$

22. *Momentum.*

The definition of momentum afforded or required by the laws of motion is, that the momentum in any particular direction is the product of the mass multiplied by the rate of displacement, in the particular direction, of the mass in which it resides. Since at each instant mass has position and capacity for momentum, and the rate of the displacement at the instant has magnitude and direction, momentum has *position, magnitude,* and *direction.*

Taking as before u, v, w to represent the component velocities of the mass passing a point at any instant, and ρ for the density of the mass at the same instant, the densities of the respective components of momentum are respectively

$$M_x = \rho u, \quad M_y = \rho v, \quad M_z = \rho w.$$

Substituting M_x for Q in equation (1) it becomes

$$\delta [\Sigma (M_x \delta S)] = \delta [\Sigma ({}_p M_x \delta S)] + \Sigma [({}_c M_x \delta S)], \text{ \&c., \&c.} \ldots\ldots (29).$$

By equation (2) substituting ${}_p M_x$ for ${}_p Q_1$,

$$\delta [\Sigma ({}_p M_x \delta S)] = - \delta [\Sigma ({}_p Q_2 \delta S + \text{\&c.})], \text{ \&c., \&c.} \ldots\ldots\ldots\ldots (30),$$

where $-\delta [\Sigma ({}_p Q_2 \delta S + \text{\&c.})]$ expresses the rate of destruction of momentum in direction x, in all other modes than that represented by $M_x \delta S$ within the space of S.

23. *Conduction of momentum by the mechanical medium.*

As $\Sigma (M_x \delta S)$ represents the sum of all the momentum in direction x within the space S, there is difficulty in realising how momentum in direction x can be produced or destroyed in any other mode. If, as in this research, $\rho \delta S$ is defined as including the total capacity for momentum within the indefinitely small space, δS, the production or destruction of momentum in direction x in any other mode than $M_x \delta S$, at a point within the space δS, requires that momentum should have entered the space without having been conveyed by the motion of the mass across the surrounding space. The difficulty thus presented naturally raises the question as to whether such production or destruction is necessarily implied in the laws of motion?—as to whether the entire exchanges of momentum cannot be accounted for as the result of convections by the moving mass?

That it is possible for momentum to be conveyed across a finite space by the mass within the space, and at the same time the momentum of the mass within the space to be zero, has long been recognised, and follows directly as a geometrical consequence of the fact that momentum possesses the property of being negative in exactly equal degree with that of being positive; just as does electricity; so that a stream of negative momentum in any direction, crossing a surface in a negative direction, has exactly the same geometrical significance as an equal stream of positive momentum crossing the same surface in a positive direction. The result being the convection by both streams of positive momentum in the positive direction and negative momentum in the negative direction at equal rates, while the sum of the momenta of the masses in the two streams taken together within the space is zero.

In such streams of momentum the action at a surface is, though purely kinematical, that of *exchange* of momentum between the spaces on the opposite sides of the surface, such exchange proceeding at a definite rate, which rate has a definite intensity at each point of the surface, and the *direction of the momentum exchanged is the direction of the motion of the mass* at each point. The condition that action and reaction are equal and opposite is thus completely satisfied—that is to say, not only is the action one of exchange of momentum, but it is also one of exchange of moment of momentum about every axis. Hence, where the boundary conditions of the medium admit of such opposite streams of momentum in different directions through the same space in the same interval of time, exchanges of momentum in any direction across any surface may be effected while the aggregate momentum is zero.

In this way, in the kinetic theory, the stresses in gases at any instant are completely accounted for, as the result of the convection of momentum conveyed by the molecules amongst which the motion is distributed uniformly in all directions. But even in the case of gas such convection does not account for the maintenance of the distribution of velocities amongst the molecules. This requires that the molecules should exchange momentum, and such exchange as appears by equation (13) cannot be accounted for as the result of kinetic convection by moving mass, but requires mechanical action between the molecules. In the kinetic theory, therefore, it is assumed that 'forces' exist between the molecules, when within certain distances of each other, either as the result of varying stresses in the matter, or as exerted through intervening space.

From these and like considerations it appears that, to whatever extent the transmission of momentum from one portion of space to another may be accounted for as the result of convection by moving mass, the communication of momentum from one portion of mass to another requires either that it be transmitted through space occupied by mass otherwise than as moving mass, or that it be destroyed in one place and produced in another.

Unless, therefore, it is assumed that, while mass has continuous existence in time and space, momentum can cease to exist in one place and, at the same time, come into existence, in the same quantity, at another place, that is unless we accept action at a distance, and thereby preclude all further definition and explanation, it is necessary that the purely-mechanical-medium, in addition to the properties of occupying position, and having capacity for momentum, should have the property of transmitting or conducting momentum through the space it occupies otherwise than by the convection consequent upon the motion of the mass; and, to completely satisfy the condition that the direction in which the exchange is effected is the direction of the momentum exchanged, it is necessary that the direction of conduction should everywhere be the same as, or the opposite to, that of the momentum conducted—that the conduction should be by streams, real or imaginary streams, of real or imaginary momentum in the same direction as that of the momentum, just as in the case of convection, except that in the latter case the streams and the momentum are real; so that if l, m, n refer to the direction in which h is measured, which is that of such a stream, of which p is the intensity, positive or negative, of the rate of exchange across a surface normal to h, the intensities of the rates of exchange of momentum, in direction h, across the surfaces yz, zx, xy are respectively pl, pm, pn, and the intensities of the rates of exchange of the components of momentum, in the direction of x, y, z, respectively, are

$$\begin{array}{lccc} \text{across } yz & pl^2, & plm, & pln, \\ zx & pml, & pm^2, & pmn, \\ xy & pnl, & pnm, & pn^2. \end{array}$$

This property of conducting momentum (on which all mechanical action depends), necessitated by the laws of motion as inherent in a purely-mechanical-medium, must be continuous in time and space if the medium is continuous in time and space. As possessed by the medium, therefore, the property differs from the property of strength or that of resisting stress possessed in various degrees by matter in respect to the limits to the strength, which limits depend on the physical condition of the matter and have no existence in the medium. This difference as regards limits, however, does not affect the correspondence, in character, between the property of conduction of momentum by the medium and the property of sustaining stress in matter.

The magnitude of stress being nothing more nor less than a measure of the intensity of the flux of the component of momentum, in the direction of the stress across the surface on which the stress acts, if the intensity of stress at a point on a surface is defined to be the intensity of the flux of momentum conducted, as distinct from that conveyed by the motion of the mass across the surface, the notation used for the expression of the stresses in matter becomes applicable for the expression of the components of momentum conducted, as distinct from that conveyed, in a purely-mechanical-medium. Thus

$$p_{xx},\ p_{yx},\ p_{zx},\ p_{xy},\ p_{yy},\ p_{zy},\ p_{xz},\ p_{yz},\ p_{zz},$$

the expressions, used by Rankine for the component intensities of the stress, in which the exchange of momentum is in the direction indicated by the second suffix and is across the surface perpendicular to the direction indicated by the first suffix, may be defined to express the intensities of the rates of conduction of the components of momentum in which the momentum is in the direction indicated by the second suffix and is conducted in the direction indicated by the first suffix.

Whence, at any instant, the rates of conduction of the component of momentum from the outside into the indefinitely small steady element $dxdydz$ are respectively expressed by the left members of the equations (30 A),

$$\left.\begin{aligned} -\left\{\frac{dp_{xx}}{dx}+\frac{dp_{yx}}{dy}+\frac{dp_{zx}}{dz}\right\}dxdydz &= F_x dxdydz \\ -\left\{\frac{dp_{xy}}{dx}+\frac{dp_{yy}}{dy}+\frac{dp_{zy}}{dz}\right\}dxdydz &= F_y dxdydz \\ -\left\{\frac{dp_{xz}}{dx}+\frac{dp_{yz}}{dy}+\frac{dp_{zz}}{dz}\right\}dxdydz &= F_z dxdydz \end{aligned}\right\}\ldots\ldots\ldots\ldots(30\text{ A}),$$

F_x, F_y, F_z being merely contractions for the expressions in the left member.

24. Since, in order to satisfy the condition that action and reaction are equal, accumulation of momentum in the mode in which it is conducted is impossible, the expressions for the rate of conduction into the mass in the space $dxdydz$ must also express the rates at which momentum in the mode in which it is conducted, is produced in the mass in the space outside the element and destroyed within the element. Whence it follows that F_x, &c., respectively represent the rates at which the densities of the respective components of momentum, in other mode than that of M_x, &c., are destroyed within the element, and as these are the only rates at which momentum within the element is destroyed—F_x, &c. define the values of $({}_pQ_2 + \&c.)$ in equations (30), and the equations of continuity of the densities of the respective components of momentum in a purely-mechanical-medium become by equation (11)

$$\frac{d\,({}_pM_x)}{dt} = F_x,\ \&c.,\ \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31),$$

and substituting in equations (29) we have by equation (10)

$$\frac{d\,(M_x)}{dt} = F_x + \frac{d}{dt}\,({}_cM_x),\ \&c.,\ \&c. \ldots\ldots\ldots\ldots\ldots\ldots(32),$$

which are the equations of density of momentum in a purely-mechanical-medium expressed in terms of general symbols expressing the separate effects of the distinct actions of conduction and convection.

Substituting for F_x equations (30 A) and $d\,({}_cM_x)/dt$ from (7) we have the full detailed expressions for the equations of the densities of the components of momentum at a point

$$\frac{dM_x}{dt} = -\left\{\frac{d}{dx}(p_{xx} + \rho uu) + \frac{d}{dy}(p_{yx} + \rho uv) + \frac{d}{dz}(p_{zx} + \rho uw)\right\}, \text{ \&c., \&c.} \ldots (33).$$

The equations (32) and (33) are the equations of conservation of momentum in a purely-mechanical-medium, at a point, in which the first terms in the brackets on the right of (33) express the rates of change by conduction, and the second the rates of change by convection.

The integrals of the right members of these equations transform into surface integrals, and thus they express the condition that the change of momentum within any space S is solely the result of the passage of momentum across the surface of S.

25. *The conservation of the position of momentum.*

It appears from the previous article that the condition of conservation of momentum requires that action and reaction should be equal and opposite, but this is all; so far p_{xx}, p_{yx}, &c. may be independent of each other, and there is no indication that exchange must take place in the direction of the momentum exchanged. This is however expressed by the equations of conservation of the position of momentum.

Taking x, y, z and ρu, &c. as referring to a fixed point. Then multiplying each of the equations (33) by x, y, z, successively, we have

$$x\frac{dM_x}{dt} = -x\left\{\frac{d}{dx}(p_{xx} + \rho uu) + \text{\&c.}\right\}, \text{ \&c., \&c.} \ldots\ldots\ldots\ldots (34),$$

or transforming, since x, y, z are not functions of t,

$$\left.\begin{aligned}
\frac{d}{dt}(x\rho u) - p_{xx} - \rho uu &= -\left\{\frac{d}{dx}x\,(p_{xx} + \rho uu) + \text{\&c.}\right\} \\
\frac{d}{dt}(y\rho u) - p_{yx} - \rho uv &= -\left\{\frac{d}{dx}y\,(p_{xx} + \rho uu) + \text{\&c.}\right\} \\
\frac{d}{dt}(z\rho u) - p_{zx} - \rho uw &= -\left\{\frac{d}{dx}z\,(p_{xx} + \rho uu) + \text{\&c.}\right\}
\end{aligned}\right\} \ldots\ldots\ldots\ldots (35),$$

and corresponding equations, for $x\rho v$, &c. and $x\rho w$, &c.

The right members of these equations integrated over any space S represent surface integrals.

The integrals of p_{xx}, &c. on the left of the equations represent the respective rates of the displacement by conduction of the respective components of momentum within S, while those of ρuu, &c. represent the rates of displacement of momentum by connection within S.

Hence what these equations express is that the whole rate of displacement of momentum in S, less the internal rate of displacement, is equal to the rate of displacement of the momentum across the surface.

This, it appears, follows directly from the condition that action and reaction are equal—*i.e.* the equations of motion—and implies no relation between the components of conduction. Such conditions however follow from the further condition that the direction of exchange is the direction of the momentum exchanged.

26. *Conservation of moments of momentum.*

Subtracting equation (35) for $y\rho w$ from that for $y\rho v$,

$$\frac{d}{dt}(z\rho v - y\rho w) - (p_{zy} - p_{yz})$$
$$= -\left[\frac{d}{dx}\{z(p_{xy} + uv) - y(p_{xz} + uw)\} + \text{\&c.}\right] \quad\text{.......}(36);$$

whence in order that the rate of change in the moment of momentum about the axis of x may be expressed by a surface integral we have the condition, as previously obtained (Art. 23),

$$p_{zy} = p_{yz}, \text{ and similarly, that } p_{xz} = p_{zx} \text{ and } p_{yx} = p_{xy} \quad\text{...........}(36\text{A}).$$

27. *Boundary Surfaces.*

The conditions at the bounding surfaces of spaces continuously occupied by the medium may be of two kinds, according to whether the surface divides the medium from unoccupied space, or separates two continuous portions of the medium which are in contact at the surface.

Taking r, s, t for distances measured from a point in the surface in directions at right angles to each other, that in which r is measured being normal to the surface and l_r, m_r, n_r, l_s, m_s, n_s, l_t, m_t, n_t for the direction cosines of r, s, t respectively, then since $p_{xy} = p_{yx}$, &c., &c.,

$$\left.\begin{aligned}
p_{rr} &= p_{xx}l_r^2 + p_{yy}m_r^2 + p_{zz}n_r^2 + 2p_{yz}m_rn_r + 2p_{zx}n_rl_r + 2p_{xy}l_rm_r \\
p_{ss} &= p_{xx}l_s^2 + p_{yy}m_s^2 + p_{zz}n_s^2 + 2p_{yz}m_sn_s + 2p_{zx}n_sl_s + 2p_{xy}l_sm_s \\
p_{tt} &= p_{xx}l_t^2 + p_{yy}m_t^2 + p_{zz}n_t^2 + 2p_{yz}m_tn_t + 2p_{zx}n_tl_t + 2p_{xy}l_tm_t \\
p_{st} &= p_{xx}l_sl_t + p_{yy}m_sm_t + p_{zz}n_sn_t + p_{yz}(m_sn_t + n_sm_t) \\
&\qquad + p_{zx}(n_sl_t + l_sn_t) + p_{xy}(l_sm_t + m_sl_t) \\
p_{tr} &= p_{xx}l_tl_r + p_{yy}m_tm_r + p_{zz}n_tn_r + p_{yz}(m_tn_r + n_tm_r) \\
&\qquad + p_{zx}(n_tl_r + l_tn_r) + p_{xy}(l_tm_r + m_tl_r) \\
p_{rs} &= p_{xx}l_rl_s + p_{yy}m_rm_s + p_{zz}n_rn_s + p_{yz}(m_rn_s + n_rm_s) \\
&\qquad + p_{zx}(n_rl_s + l_rn_s) + p_{xy}(l_rm_s + m_rl_s)
\end{aligned}\right\}\text{...}(37).$$

Where the surface separates the medium from unoccupied space the stresses p_{rr} &c., are all zero at the surface, but where the surface divides two portions of the medium in contact, then the intensity of the flux across the surface at a point is the intensity of the rate at which such momentum is received by the one portion and lost by the other across the surface at the point, and by the foregoing notation p_{rr}, p_{rs}, p_{rt} respectively express the intensities of the rates of flux across the surface of the components of momentum in the direction in which r, s, t are respectively measured. These rates are the limiting values at the surface of the respective components of flux within the medium on either side of the surface in the directions in which r, s, t are measured, and are thus the limiting values, at the surface, of the expressions on the right side of the equations (1).

28. *Energy.*

Although the half of the *vis-viva* (that is half the rate of the displacement of the momentum, or half the product of the momentum multiplied by the rate of displacement of the mass) now called kinetic energy, has long been recognised as the general measure of the mechanical-effect of mechanical-action through space, the recognition of energy as a physical entity has resulted from the discovery of the reversibility of actions by which mechanical-action produces physical effects, and of the linear relations which exist between the physical measures of the physical effects so produced, and the kinetic energy which has been expended in producing them.

The discovery of these relations and the reversibility of the actions having led to the recognition of the existence in the Universe of physical entities which could be changed to and from the mechanical entity kinetic-energy, these physical entities, although not otherwise mechanically definable, have become recognised as modes of the general physical entity of which kinetic-energy is one mode and the only mode which is subject to strict mechanical definition; and hence followed the recognition of the law of conservation of energy.

Taking p_{xx}, &c. to have the significance ascribed to them in Art. 23, the intensities of the components of mechanical action—that is the intensities of the components of the flux of momentum, by conduction, from the negative to the positive side across a surface of which the direction of the normal is defined by l, m, n—are respectively expressed by

$$p_{xx}l + p_{yx}m + p_{zx}n, \text{ \&c., \&c.}$$

These are the expressions for the time-measures of the intensities of the components of mechanical action, in the directions of the perpendicular axes of reference, of the mass on the negative side of the surface, on the mass on the positive side of the surface, at a point in the surface.

Multiplying these time-measures respectively by u, v, w, the component velocities of the mass at the point, we obtain

$$u\,(p_{xx}l + p_{yx}m + p_{zx}n), \text{ \&c., \&c.,}$$

which are the corresponding space-measures of the respective components of the intensity of mechanical action at the point.

Adding these and multiplying by δs, the element of a closed surface, the integral over the surface is expressed by

$$\iint [(up_{xx} + vp_{xy} + wp_{xz})\,l + (up_{yx} + vp_{yy} + wp_{yz})\,m + (up_{zx} + vp_{zy} + wp_{zz})\,n]\,\delta S,$$

which is the space-measure of the mechanical action of the mass outside the closed surface on that within.

This (if there are no purely physical exchanges) is by the law of conservation of energy equal to the rate of change of energy in all its modes, within the surface—that is if there is no change by convection across the surface, which will be the case if the surface is everywhere moving with the mass.

The changes of energy may be partly in kinetic-energy and partly in other physical modes, according to the expression which is obtained by transforming the equations of momentum (33) by equation (26); multiplying respectively by u, v, w, integrating over the surface and adding, the equation becomes, when transformed by equation (15), taking $U = u$, &c., and assuming the actions continuous in space and time,

$$\left\{\begin{aligned} &\frac{1}{2}\frac{d}{dt}\iiint\{\rho\,(u^2+v^2+w^2)\}\,dx\,dy\,dz \\ &-\iiint\left\{\begin{aligned} &p_{xx}\frac{du}{dx} + p_{yx}\frac{du}{dy} + p_{zx}\frac{du}{dz} \\ &+ p_{xy}\frac{dv}{dx} + p_{yy}\frac{dv}{dy} + p_{zy}\frac{dv}{dz} \\ &+ p_{xz}\frac{dw}{dx} + p_{yz}\frac{dw}{dy} + p_{zz}\frac{dw}{dz} \end{aligned}\right\} dx\,dy\,dz \end{aligned}\right\}$$

$$= \iint \left\{\begin{aligned} &(up_{xx} + vp_{xy} + wp_{xz})\,l \\ &+ (up_{yx} + vp_{yy} + wp_{yz})\,m \\ &+ (up_{zx} + vp_{zy} + wp_{zz})\,n \end{aligned}\right\} \delta S. \quad . \; (38).$$

The right member is here the measure of mechanical action over the surface moving with the mass; so that the left member expresses the rate of change of energy, resulting from the mechanical action within the surface. The first term in the left member is the rate of change in kinetic energy, within the surface, and the second term expresses the rate of change of energy in other or physical modes within the surface as resulting from the mechanical action on the surface.

29. In a purely-mechanical-medium (including everything that has position in space and possessing *no physical properties other than are required by the laws of motion*) the kinetic-energy must include all the energy in the space over which the integration extends, hence as applied to such medium the second term on the left of equation (38) must be zero, however large or small the space over which the integration extends. Whence putting $2E=\rho(u^2+v^2+w^2)$ and transforming equation (38) by equation (15), the equation of energy for a fixed space becomes

$$\Sigma\left[\frac{dE}{dt}\,\delta S\right]=\iint[(up_{xx}+vp_{xy}+wp_{xz}+uE)\,l$$
$$+(up_{yx}+vp_{yy}+wp_{yz}+vE)\,m+(up_{zx}+vp_{zy}+wp_{zz}+wE)\,n]\,dS\ldots(39).$$

Whence since this holds whatsoever may be the size of the space enclosed, we have for the rate of change of the density of energy at a point, by differentiating the left member of equation (39) with respect to the limits

$$\frac{dE}{dt}=-\frac{d}{dx}(up_{xx}+vp_{xy}+wp_{xz})-\frac{d}{dy}(up_{yx}+vp_{yy}+wp_{yz})-\frac{d}{dz}(up_{zx}+vp_{zy}+wp_{zz})$$
$$-\frac{d(uE)}{dx}-\frac{d(vE)}{dy}-\frac{d(wE)}{dz}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(40).$$

30. In order to simplify the expressions N may be put for the rate at which density of the energy, in whatsoever mode, is produced by the mechanical action at any fixed point in space, and N_x, N_y, N_z for the densities of the energies which have been produced by the components in the directions in which x, y, z are measured respectively, so that

$$\left.\begin{aligned}&N=N_x+N_y+N_z.\\ \text{Then}\quad&\frac{dN_x}{dt}=-\left\{\frac{d}{dx}(up_{xx})+\frac{d}{dy}(up_{yx})+\frac{d}{dz}(up_{zx})\right\},\ \&c.,\ \&c.\end{aligned}\right\}\ldots\ldots(41),$$

$$\left.\begin{aligned}\text{and}\quad&\frac{dN}{dt}=\frac{dN_x}{dt}+\frac{dN_y}{dt}+\frac{dN_z}{dt}\\ &\Sigma\left\{\frac{dN}{dt}\,\delta S\right\}\\ &=\iint\{(up_{xx}+vp_{xy}+wp_{xz})l+(up_{yx}+vp_{yy}+wp_{yz})m+(up_{zx}+vp_{zy}+wp_{zz})n\}\,dS\end{aligned}\right\}\ldots(42).$$

Whence substituting in equation (40) it becomes

$$\frac{dE}{dt}=+\frac{dN}{dt}+\frac{d}{dt}({}_cE)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(43);$$

which may be obtained from (1) and (2) together with the condition that E is continuous—and is the equation for the density of energy—in terms of general symbols expressing the densities of the distinct actions of conduction and convection at a point.

31. *The condition of a purely-mechanical-medium.*

Equations (40) and (43) are the equations of continuity of energy in a purely-mechanical-medium in which the relation between the stresses and strains is continuously, that the second term in the left member of equation (38) is everywhere and continuously zero. Transposing the expression under the integral in the second term in the left member of equation (38) by (36A) and equating to zero we have

$$-\left\{p_{xx}\frac{du}{dx}+p_{yy}\frac{dv}{dy}+p_{zz}\frac{dw}{dz}+p_{yz}\left(\frac{dv}{dz}+\frac{dw}{dy}\right)+p_{zx}\left(\frac{dw}{dx}+\frac{du}{dz}\right)\right.$$
$$\left.+p_{xy}\left(\frac{du}{dy}+\frac{dv}{dx}\right)\right\}=0 \text{ ......} (44).$$

Then, for convenience, expressing equation (44) as $dR/dt=0$, equation (44) defines the action in the medium as being purely kinematical.

From the definition of p_{xx}, &c., &c. as components of intensity of a flux of momentum it follows geometrically that the value of the expression which forms the left member of equation (44) is independent of the direction in which the axes are taken. Hence, if i, j, k, are measured in the directions of the principal axes either of the rates of distortion or of the stresses at a point p and u, v, w are the components of the velocity in these directions, respectively, transforming to these axes we have by equation (44); since either ;—

$$\frac{dv}{d_k}+\frac{dw}{d_j}=0, \text{ \&c., \&c.; or } p_{jk}=0, \text{ \&c., \&c. ....} (45),$$

$$p_{ii}\frac{du}{d_i}+p_{jj}\frac{dv}{d_j}+p_{kk}\frac{dw}{d_k}=0 \text{........................} (46).$$

From these three conditions it appears that no energy is transformed in distorting the medium. And we have as the three possible conditions in a purely-mechanical-medium

$p_{ii}=p_{jj}=p_{kk}=0$; which is the condition of empty space .... (46A),

$p_{ii}=p_{jj}=p_{kk}$; and $\frac{du}{d_i}+\frac{dv}{d_j}+\frac{dw}{d_k}=0$; perfect fluid.

$\frac{du}{d_i}+\frac{dv}{d_j}+\frac{dw}{d_k}=0$; or $\frac{dw}{dy}+\frac{dv}{dz}=\frac{du}{dz}+\frac{dw}{dx}=\frac{dv}{dx}+\frac{du}{dy}=0$; perfect rigidity.

32. *The transformations of the directions of the energy, and angular redistribution.*

Kinetic energy has direction at every point, although not a vector, and the equations obtained by multiplying equations (33), respectively, by u, v, w are, respectively, the equations of energy in the directions of x, y, z.

For an element in a closed surface within the mass

$$\frac{1}{2}\frac{d}{dt}\iiint(\rho u^2)\,dxdydz - \iiint\left(p_{xx}\frac{du}{dx} + p_{yx}\frac{du}{dy} + p_{zx}\frac{du}{dz}\right)dxdydz$$

$$= -\iiint\left\{\frac{d}{dx}(p_{xx}u) + \frac{d}{dy}(p_{yx}u) + \frac{d}{dz}(p_{zx}u)\right\}dxdydz \quad \ldots(47),$$

&c., &c.

In these equations the members on the right represent work, in the directions x, y, z, respectively, done on the surface within which the integration extends. And as these efforts are all in the direction of x, y or z, respectively, they involve no change from one direction to another.

But the second terms on the left of each of the equations represent production of energy in the directions x, y, z respectively, at the expense of the energy in the other directions.

It is thus shown by condition (44)—which is that the sum of these terms, from the three equations, is zero—that, putting R_x, &c., &c. for the densities of the rates of angular dispersions at a point, from the directions x, y, z respectively, these are

$$\frac{dR_x}{dt} = -\left(p_{xx}\frac{du}{dx} + p_{yx}\frac{du}{dy} + p_{zx}\frac{du}{dz}\right), \text{ \&c., \&c.}$$

It is to be noticed that in a medium such that u, v, w do not represent the velocities of points in mass, R_x does not represent angular dispersion only, unless equations (44) are satisfied; and if not so satisfied dR_x/dt would represent the work done against the apparently physical actions in the medium, as well as the angular dispersion.

The analytical separation of this action is obtained by transforming the general equation, which becomes

$$\begin{aligned}\frac{dR}{dt} = &-\frac{1}{9}(p_{xx} + p_{yy} + p_{zz})\left(2\frac{du}{dx} - \frac{dv}{dy} - \frac{dw}{dz}\right) \\ &+ \frac{1}{2}\left\{p_{yx}\left(\frac{du}{dy} - \frac{dv}{dx}\right) + p_{zx}\left(\frac{du}{dz} - \frac{dw}{dx}\right)\right\} \\ &- \frac{1}{9}(p_{xx} + p_{yy} + p_{zz})\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) \\ &+ \left(p_{xx} - \frac{p_{xx} + p_{yy} + p_{zz}}{3}\right)\frac{du}{dx} + \frac{1}{2}\left\{p_{yx}\left(\frac{du}{dy} + \frac{dv}{dx}\right)\right. \\ &\left.+ p_{zx}\left(\frac{du}{dz} + \frac{dw}{dx}\right)\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(47\text{A}).\end{aligned}$$

From the member on the right of equation (47) it at once appears that the two first terms express angular dispersion only, while the second two terms express distortional motions only, which, by the conditions (45), are zero.

33. *The continuity of the position of energy.*

Kinetic energy has position; and hence, putting x, y, z for the point at which the density of energy is E, by equation (1)

$$\delta\,[\Sigma\,\{Ex\delta S\}] = \delta\,[\Sigma\,\{_c(Ex)\,\delta S\}] + \delta\,[\Sigma\,\{_p(Ex)\,\delta S\}], \text{ \&c., \&c. } \ldots(48),$$

in which x, y, z are not functions of time. And if $\bar{x}$, $\bar{y}$, $\bar{z}$ are put for the centre of energy, $\bar{u}$, $\bar{v}$, $\bar{w}$ for the component velocities of the surface, as in equations (12) to (16), Art. 14, we have at any instant,

$$\bar{x}\Sigma\,\{E\delta S\} = \Sigma\,\{Ex\delta S\}, \text{ \&c., \&c.}\ldots\ldots(49),$$

whence, differentiating with respect to time,

$$\frac{d\bar{x}}{dt}\,\Sigma\,\{E\delta S\} = -\,\bar{x}\,\frac{d}{dt}\,[\Sigma\,\{E\delta S\}] + \frac{d}{dt}\,[\Sigma\,\{Ex\delta S\}], \text{ \&c., \&c.}\ldots\ldots(50).$$

Then, by equation (15), these equations become

$$\begin{aligned}\frac{d\bar{x}}{dt}\,\Sigma\,\{E\delta S\} &= -\,\bar{x}\,\Sigma\left[\left\{\frac{dE}{dt} + \frac{dE\bar{u}}{dx} + \frac{dE\bar{v}}{dy} + \frac{dE\bar{w}}{dz}\right\}\delta S\right]\\ &\quad + \Sigma\left[\left\{x\,\frac{dE}{dt} + \frac{d\,(Ex\bar{u})}{dx} + \frac{d\,(Ex\bar{v})}{dy} + \frac{d\,(Ex\bar{w})}{dz}\right\}\delta S\right]\\ &= \Sigma\left\{(x - \bar{x})\left(\frac{dE}{dt} + \frac{d\,(E\bar{u})}{dx} + \frac{d\,(E\bar{v})}{dy} + \frac{d\,(E\bar{w})}{dz}\right)\delta S\right\}\\ &\quad + \Sigma\,(E\bar{u}\delta S), \text{ \&c., \&c.}\ldots\ldots(51).\end{aligned}$$

Whence, for a fixed surface, since $\bar{u} = \bar{v} = \bar{w} = 0$,

$$\frac{d\bar{x}}{dt} = \frac{\Sigma\left\{(x - \bar{x})\,\frac{dE}{dt}\,\delta S\right\}}{\Sigma\,(E\delta S)}, \text{ \&c., \&c.}\ldots\ldots(52).$$

For a surface moving everywhere with the mass so that $\bar{u} = u$, &c., equation (51) becomes

$$\frac{d\bar{x}}{dt} = \frac{\Sigma\left\{(x - \bar{x})\,\frac{d}{dt}\,(_pE)\,\delta S\right\} + \Sigma\,\{Eu\,\delta S\}, \text{ \&c., \&c.}}{\Sigma\,\{E\delta S\}}\ldots\ldots(53),$$

or, $$\frac{d}{dt}[\Sigma\,\{(Ex)\,\delta S\}] = \Sigma\left\{x\,\frac{d}{dt}\,(_pE)\,\delta S\right\} + \Sigma\,(Eu\delta S)\ \ldots\ldots(54),$$

where, as in equation (42), differentiating with respect to the limits

$$\frac{d}{dx}\,(_pE) = -\left\{\frac{d}{dx}\,(p_{xx}u + p_{xy}v + p_{xz}w) + \text{\&c.} + \text{\&c.}\ldots\right\}\ \ldots\ldots(55),$$

or $$= \frac{dN}{dt}\,.$$

34. *Discontinuity in the medium.*

It is to be noticed that the expressions in equations (37) to (55) are adapted to the cases in which the medium is continuous, so that for the complete expression of the actions where the medium is continuous within closed surfaces, only, it is necessary to express the conditions at the bounding surfaces by using the expressions in equations (37).

These complete expressions might very properly be introduced at this stage. But as the necessity for the definite use of these does not arise until a much later stage in this research, and then arises in a comparatively simple case which has already been much studied in some of its aspects, it is convenient to proceed as if the medium were continuous until this stage is reached. See equation (132), Section IX.

SECTION IV.

THE EQUATIONS OF CONTINUITY FOR COMPONENT SYSTEMS OF MOTION.

35 Component systems may be distinguished by definition of their component velocities or their density.

By a component system of motion distinguished by velocity is here understood a system of motion, howsoever defined, in which the velocity at any point is not necessarily the velocity of the mass at that point either in direction or magnitude.

Taking, as before, u, v, w, to express the components of the actual velocities of the mass at the point x, y, z and time t, and ρ for the density of the mass, and u'', v'', w'' as expressing the components, with respect to the same axes, of the velocity of a component system, there exist at each point the residual components

$$u' = u - u'', \quad v' = v - v'', \quad w' = w - w'' \text{ .... } (56).$$

The sums of these components $u'' + u'$, &c. satisfy the equations (33) Section III., and the following equation, for the resultant system, and if one of these systems is subject to any definition, actual or conditional, the equation for the resultant system becomes the equation for the residual system.

It is a very general method in mechanical analysis to separate the motion of the mass at each point into two component systems whenever the conditions are such that the independence of these systems is obvious. As, for instance, the motion of the mass at each point at any instant is considered as consisting of the motion of the centre of gravity of the whole mass at the instant together with another component system which is the motion at the point relative to the motion of the centre of gravity. But such instances have hitherto been considered as depending on special theorems, and do not appear to have suggested the study of the method which they involve as a general system of analysis apart from the existence of conditions which render the component systems completely independent.

It appears, however, that the manner in which the rates of increase of the momentum and kinetic energy of the one component system depend on convection by and transformations from the other may be subjected to general analytical expression, even when the definition is arbitrary and only conditional.

This is accomplished by equating the expressions for the rates of increase of u'', &c. at a point moving with the mass to arbitrary functions which, multiplied by ρ, express the rates at which density of momentum is transformed from the system $\rho u'$ into the system $\rho u''$ and represent the only rates of production of momentum in that system, so that the equations of motion of either of the component systems may then be obtained from equations (1) and (2) or (10) and (11) Section II. The equations so obtained will differ in form from the equations of the resultant system in five particulars.

(1) The equations for the component system will differ from that of the resultant system from the fact that u'', v'', w'' do not represent the whole causes of convection, which are u, v, w: so that the rate of increase of Q by convection is not

$$-\left\{\frac{d}{dx}(u''Q)+\frac{d}{dy}(v''Q)+\frac{d}{dz}(w''Q)\right\} \text{ but } -\left\{\frac{duQ}{dx}+\frac{dvQ}{dy}+\frac{dwQ}{dz}\right\}, \text{ \&c.,}$$

or

$$\frac{d}{dt}({}_cQ)=\frac{d}{dt}({}_{c''}Q)+\frac{d}{dt}({}_{c'}Q), \text{ \&c. } \qquad (57),$$

where the pre-suffix c'' indicates convections by u'' and c' indicates the convections by u', inwards across the bounding surface of the element.

(2) A difference in the form of the equations also results from the fact that $\rho u''$, $\rho v''$, $\rho w''$ are not the only modes in which densities of momentum in the directions x, y, z exist at a point in the medium. The rates of increase of density in the modes $\rho u''$, &c. by conduction, into the steady element of space $dxdydz$ are not the only increases other than by convection; since there are the further possibilities of exchanges of densities of momentum between the modes $\rho u''$, and $\rho u'$, &c. existing at the same point in the same mass.

That such abstract exchanges, without mechanical action, must result from the definition by which the component systems are distinguished is at once seen, for to this definition u'', v'', w'' are subject at each point and each instant. And therefore the rates of increase of u'', v'', w'', the defined components of acceleration of the moving mass, expressed by

$$\frac{du''}{dt}+u\frac{du''}{dx}+v\frac{du''}{dy}+w\frac{du''}{dz}, \text{ \&c. \&c.}$$

are subject to arbitrary definition independent of the actual accelerations of the mass. And

$$u''\frac{d\rho}{dt}+u''\left(\frac{d\rho u}{dx}+\frac{d\rho v}{dy}+\frac{d\rho w}{dz}\right)=0,\quad u'\left(\frac{d\rho}{dt}\right)+u'\left(\frac{d\rho u}{dx}+\text{\&c.}\right)=0.$$

Taking $$\frac{F_x''}{\rho''}, \text{ \&c., \&c.}$$

as arbitrary expressions for these defined rates of increase and multiplying by ρ we have as the equations of continuity for the components of momentum $\rho u''$, &c., &c. by equation (28) Section III.

$$\frac{d\rho u''}{dt} = \frac{d}{dt}({}_c\rho u'') + \rho\frac{d}{dt}({}_p u''), \text{ \&c., \&c. } \ldots\ldots\ldots\ldots\ldots (58),$$

and again by the equations for the resultant system

$$\frac{d}{dt}(\rho u'' + \rho u') = \frac{d}{dt}({}_c\rho u'' + {}_c\rho u') + F_x, \text{ \&c., \&c} \ldots\ldots\ldots\ldots (59).$$

Subtracting equation (58) we have for the other system

$$\frac{d\rho u'}{dt} = \frac{d}{dt}({}_c\rho u') + F_x - \rho\frac{d}{dt}({}_p u''), \text{ \&c., \&c} \ldots\ldots\ldots\ldots\ldots (60).$$

It thus appears that

$$\rho\frac{d}{dt}({}_p u''), \text{ \&c., \&c.,}$$

express rates of transformation of density of momentum from the component system $\rho u'$ to the system $\rho u''$, &c., &c., consequent on the geometrical conditions by which u'', v'', w'' are defined.

The arbitrary rates of increase of density of momentum represented by these transformations may be considered as variations either in an arbitrary system of stresses or an arbitrary system of convections to be determined by the actual definition.

(3) The equations of the component systems differ from that of the resultant system on account of the expression for the transformation of energy to and from each of the component systems in consequence of the definition to which they are subjected. The densities of each of these rates of transformation of energy are by equation (28), putting u'' for q_1, &c. respectively, the sums of the products of the densities of the component ratios of transformation of momentum to the particular component systems ($d_p\rho u''/dt$, &c.) respectively multiplied by the component velocity (u'', &c.) of the same system.

Thus expressing the density of energy so transformed at a point as $\rho_T(E'')$, &c., respectively, since there is no transformation of mass,

$$\left.\begin{aligned} \rho\frac{d_T}{dt}(E_x') &= -\left\{u'\frac{d({}_p\rho u'')}{dt} + v'\frac{d({}_p\rho v'')}{dt} + w'\frac{d({}_p\rho w'')}{dt}\right\} \\ \rho\frac{d_T}{dt}(E_x'') &= +\left\{u''\frac{d({}_p\rho u'')}{dt} + v''\frac{d({}_p\rho v'')}{dt} + w''\frac{d({}_p\rho w'')}{dt}\right\} \end{aligned}\right\} \ldots\ldots\ldots (61).$$

From these equations it will be seen, at once, that the sum of the transformations to the two component systems is not necessarily zero or that the transformation is not wholly between E'' and E'.

(4) The equations of energy for the component systems differ in form from that of the resultant system in consequence of the fact that the sum of the densities of the energies of the component systems, at a point, is not equal to the density of energy of the resultant system at that point, or that:

$$\left.\begin{aligned} &\tfrac{1}{2}\rho(u^2+v^2+w^2) \\ &\qquad = \tfrac{1}{2}\rho\,\{u''^2+v''^2+w''^2+u'^2+v'^2+w'^2+2(u''u'+v''v'+w''w')\} \\ \text{whence}\qquad &\rho(E-E''-E') = \rho(u''u'+v''v'+w''w') \end{aligned}\right\} \quad \ldots\ldots (32).$$

Whence it appears that the transformation of energy is not simply between the systems E'' and E', but also between each of these and the system $(u''u'+\&c.)$; so that besides the equations of energy of the component systems there is the equation of energy of the residual system to be considered.

The density of the rate of transformation to the residual system is by definition equal in value and opposite in sign to the sum of the rates of transformation to the energies of the component systems

$$\rho\left(\frac{d_T}{dt}(E'')+\frac{d_T}{dt}(E')\right).$$

Another expression for the transformation to the residual system is obtained by multiplying each of the rates of transformation of component of momentum to the component system, by the corresponding component of velocity of the other system and adding, as in equations (28).

The density of the rate of production into residual energy may be obtained in the same way by equation (28); then by equations (10) we obtain expressions for

$$\rho\frac{d_p}{dt}(E-E''-E') \text{ and } \frac{d}{dt}(\rho' u'').$$

(5) In the equation of motion for the resultant system of motion in a purely-mechanical-medium, $d(R)/dt$ the density of the rate at which energy is produced in other modes than E, is defined as zero; and hence the expression for this production disappears from the equation of energy. It does not however follow as a geometrical consequence that the expressions for $d(R'')/dt$ and $d(R-R'')/dt$, obtained from the equations of momentum by equation (28), are respectively zero. But it does follow that whatever these

values may be, they are pure abstractions resulting from the definition of the systems of motion, and are therefore transferences of such energy from the one system to the other. Therefore while it is necessary to retain these expressions in the equations of energy for the three systems, it is convenient to indicate that they express a transference by a pre-suffix T as $d\,({}_TR')/dt$.

36. *Component systems distinguished by distribution of mass.*

Taking, as before, ρ for the density of the mass at $xyzt$ and ρ'' for any defined density of mass at the same point, there exists the residual mass

$$\rho' = \rho - \rho'' \quad\quad (63).$$

The sum $\rho'' + \rho'$ satisfies equations (33) Section III. for the resultant system, also equations (58) and (60), Section IV., for the component systems distinguished by the distribution of velocity, and if ρ'' is subjected to any definition, actual or conditional, the equation for the resultant density defines the equation for residual density of mass.

The equations so obtained will differ in form from the equations for the resultant mass in one particular.

The fact that the integrals of ρ'' and ρ' do not, either of them, taken by themselves, represent the only mass included in the space over which the integrals extend, entails a difference in the form of the equations from that of the resultant system.

The rate of increase by convection of ρ'' is not necessarily the only rate of increase, since there are possibilities of exchanges between the densities ρ' and ρ'' at the same point.

That such exchanges must result from the definition is at once seen, for $d\rho''/dt$ is subject to these exchanges at each point at each instant, and therefore the defined rate of increase of the component density ρ'' at a point moving with the mass is subject to arbitrary definition independent of the rate of increase of the actual density.

Taking as in equations (24 A) Section III.

$$\frac{d_T\rho''}{dt} = \frac{d\rho''}{dt} + \frac{d\rho''u}{dx} + \frac{d\rho''v}{dy} + \frac{d\rho''w}{dz} \quad\quad (63\text{ A}),$$

as the arbitrary expression for this defined rate of increase, we have the equation of continuity for the component density

$$\frac{d\rho''}{dt} - \frac{d_c(\rho'')}{dt} = \frac{d_T(\rho'')}{dt} \quad\quad (64).$$

And by the equation for the resultant system

$$\left.\begin{aligned}\frac{d(\rho''+\rho')}{dt}-\frac{d_c(\rho''+\rho')}{dt}&=0\\ \frac{d\rho'}{dt}-\frac{d_c\rho'}{dt}=-\frac{d_T\rho''}{dt}&\end{aligned}\right\}\text{......................(65).}$$

Then, since by equation (24), $d_p(\rho u)/dt=\rho d_p u/dt$, substituting in equation (32), the equation at a point for the resultant system is

$$\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}=\frac{d_p u}{dt}\text{....................(66).}$$

Then multiplying by ρ'' and adding $u\dfrac{d\rho''}{dt}-u\dfrac{d_c\rho''}{dt}$ to the left member and the equivalent $ud_p\rho''/dt$ to the right member, we have for the equation of momentum of the defined density:

$$\left.\begin{aligned}\frac{d\rho''u}{dt}-\frac{d_c(\rho''u)}{dt}&=\rho''\frac{d_p u}{dt}+u\frac{d_T\rho''}{dt}\\ &=\frac{d_p(\rho''u)}{dt}\end{aligned}\right\}\text{...................(67),}$$

and in precisely the same manner

$$\left.\begin{aligned}\frac{d\rho'u}{dt}-\frac{d_c\rho'u}{dt}&=\rho'\frac{d_p(u)}{dt}-u\frac{d_T\rho'}{dt}\\ &=\frac{d_p(\rho'u)}{dt}\end{aligned}\right\}\text{...................(68).}$$

37. *Component systems of motion distinguished by density and velocity.*

Again substituting u'' and u' successively for u in equations (67) and (68) we have the four equations

$$\left.\begin{aligned}\frac{d\rho''u''}{dt}-\frac{d_c(\rho''u'')}{dt}=\frac{d_p(\rho''u'')}{dt}&=\rho''\frac{d_T u''}{dt}-u''\frac{d_{c'}\rho''}{dt}=d\left[{}_T\rho''u''\right]\\ \frac{d\rho'u''}{dt}-\frac{d_c(\rho'u'')}{dt}=\frac{d_p(\rho'u'')}{dt}&=\rho'\frac{d_T u''}{dt}+u''\frac{d_{c'}\rho''}{dt}\\ \frac{d\rho''u'}{dt}-\frac{d_c(\rho''u')}{dt}=\frac{d_p(\rho''u')}{dt}&=-\rho''\frac{d_T u''}{dt}-u'\frac{d_{c'}\rho''}{dt}+\frac{\rho''F_x-\rho F_x''}{\rho''}\\ \frac{d\rho'u'}{dt}-\frac{d_c(\rho'u')}{dt}=\frac{d_p(\rho'u')}{dt}&=-\rho'\frac{d_T u''}{dt}+u'\frac{d_{c'}\rho'}{dt}+\frac{\rho'F_x-\rho F_x''}{\rho''}\end{aligned}\right\}\text{......(69),}$$

together with corresponding equations for v'', w'', v', w'.

Adding the last three of equations (69) together, it appears that

$$\left.\begin{aligned}\frac{d(\rho u-\rho''u'')}{dt}-\frac{d_c(\rho u-\rho''u'')}{dt}&=\frac{d_p(\rho u-\rho''u'')}{dt}\\ &=F_x-\frac{d_T\rho''u''}{dt}\end{aligned}\right\}\text{.........(70),}$$

whence putting M_x'' for $\rho''u''$, M_x' for $\rho u - \rho''u''$, &c., &c., we have

$$\left.\begin{aligned} \frac{dM_x''}{dt} - \frac{d_cM_x''}{dt} = \frac{d_pM_x''}{dt} &= \rho''\frac{d_Tu''}{dt} + u''\frac{d_T\rho''}{dt} \\ \frac{dM_x'}{dt} - \frac{d_cM_x'}{dt} = \frac{d_pM_x'}{dt} &= F_x - \rho''\frac{d_Tu''}{dt} - u''\frac{d_T\rho''}{dt} \end{aligned}\right\} \quad \ldots\ldots\ldots (71).$$

It is to be noticed, however, that these last equations might be obtained by the simple definition of $(\rho u)''$, so that they do not express all the definition which results from the separate definition of ρ'', u''. The importance of this appears at once on proceeding to derive the corresponding equations of energy by multiplying the equations respectively by u'' and u', and transforming, which process since u'', v'' have defined values, gives definite results, whereas the mere definition of the product $(\rho u)''$ which leaves the definition of either factor incomplete would not admit of such derivation.

38. *Distribution of momentum in a component system.*

The condition imposed by the laws of motion, as the result of experience of physical actions,—that action and reaction are equal and opposite, and that the exchanges of momentum take place in the direction of the momentum exchanged,—will not of necessity be fulfilled by an arbitrarily defined component system. But should this not be so within all sensible spaces and times, the effects of one component system on the other will not accord with any physical action; so that for purposes of analysis the general expression for this condition in a component system is of the first importance.

It has already been shown that the first of the conditions requires that the integral rate of increase in each component of momentum, in a resultant system, shall be a surface integral, however small may be the limits (Section III, Art. 24). The same holds for a component system within defined limits; so that we must have, within such limits,

$$\iiiint \left\{\frac{d}{dt}\left[{}_c(M_x'')\right] + \frac{d}{dt}\left({}_pM_x''\right)\right\} dx\,dy\,dz\,dt$$
$$= \iiiint \left(\frac{dq_{xx}}{dx} + \frac{dq_{yx}}{dy} + \frac{dq_{zx}}{dz}\right) dx\,dy\,dz\,dt, \text{ \&c., \&c. } \ldots (72),$$

where so far q_{xx}, q_{yx}, &c. are arbitrary.

As in a resultant system it is necessary, in order to satisfy the second condition, that the integrals of the rates of increase of the moments of momentum should be surface integrals and that this may be the case within defined limits, it follows, as in Art. 26, that

$$\iiiint (q_{zy} - q_{yz})\, dx\,dy\,dz\,dt = 0, \text{ \&c., \&c. } \ldots\ldots\ldots\ldots\ldots\ldots (73),$$

which is the general condition to be satisfied by the component system $\rho u''$, &c. if the analysis is confined to physical properties.

If this condition is satisfied by the system $\rho''u''$, &c. it follows that since it is satisfied in the resultant system the same condition will be satisfied by the residual system $\rho u - \rho''u''$.

39. *The component equations of energy of the component systems as distinguished by density and velocity.*

Multiplying the first of equations (69) by u'' and transforming by equations (28 A), Section III., and putting $\rho''E_x''$ for $\rho''(u'')^2/2$, we have

$$\left.\begin{aligned}
&\frac{d(\rho''E_x'')}{dt} - \frac{d_c(\rho''E_x'')}{dt} = \frac{d_p(\rho''E_x'')}{dt} = u''\rho''\frac{d_T u''}{dt} + \frac{u''^2}{2}\frac{d_T\rho''}{dt} + \text{\&c.}\\
&\text{Also multiplying the third of equations (69) by } u' \text{ and transforming (28 A) we have}\\
&\frac{d\rho''\tilde{E}_x}{dt} - \frac{d_c(\rho''\tilde{E}_x)}{dt} = \frac{d_p(\rho''\tilde{E}_x)}{dt}\\
&\qquad = u''\left(\frac{\rho''F}{\rho} - \rho''\frac{d({}_Tu')}{dt}\right) + u'^2\frac{d({}_T\rho'')}{dt} + u'^2\frac{d({}_c\rho'')}{dt}.\\
&\text{Then multiplying the first by } u' \text{ and the third by } u'' \text{ and adding, \&c.}\\
&\frac{d(\rho''\overset{*}{E}_x)}{dt} - \frac{d_c(\rho''\overset{*}{E}_x)}{dt} = \frac{d_p(\rho''\overset{*}{E}_x)}{dt} = \frac{d_T\rho''u'u''}{dt} + \rho''u''\frac{\rho''F}{\rho} + \text{\&c.}\\
&\qquad = u'u''\frac{d_T(\rho'')}{dt} + \rho''(u' - u'')\frac{d_T u''}{dt} + \frac{\rho''u''F_c}{\rho} + \text{\&c.}\\
&\text{Again, multiplying the second by } u''\text{, \&c.}\\
&\frac{d\rho'E_x}{dt} - \frac{d_c(\rho'E_x'')}{dt} = \frac{d_p(\rho'E_x'')}{dt} = u''\rho'\frac{d_T u''}{dt} + \frac{u''^2}{2}\frac{d_T\rho'}{dt} + \text{\&c.}\\
&\text{Multiplying the fourth by } u'\text{, \&c.}\\
&\frac{d\rho'\tilde{E}_x}{dt} - \frac{d_c(\rho'\tilde{E}_x')}{dt} = \frac{d_p(\rho'\tilde{E}_x)}{dt}\\
&\qquad = u'\rho'\frac{d_T u'}{dt} + \frac{u'^2}{2}\frac{d_T\rho'}{dt} + u'\rho'F_x + \text{\&c.}\\
&\text{Then multiplying the second by } u' \text{ and the fourth by } u'' \text{ and adding, \&c.}\\
&\frac{d\rho'\overset{*}{E}_x}{dt} - \frac{d_c(\rho'\overset{*}{E}_x)}{dt} = \frac{d_p(\rho'\overset{*}{E}_x)}{dt} = \frac{d_T(\rho'u'u'')}{dt} - \frac{\rho'u''F}{\rho} + \text{\&c.}\\
&\qquad = u'u''\frac{d_T\rho''}{dt} + \rho'\frac{d_T(u' - u'')}{dt} + \frac{\rho'u''F_x}{\rho} + \text{\&c.}
\end{aligned}\right\} \quad \text{......(74).}$$

The first of these equations is the equation of the component system ρ'', u''.

Then adding together the several corresponding terms of the five equations following the first, we have

$$\frac{d(\rho E - \rho'' E'')}{dt} - \frac{d_c(\rho E - \rho'' E'')}{dt} = \frac{d_p(\rho E - \rho'' E'')}{dt} \quad \text{.........(75);}$$

for the energy of the system of momentum $\rho u - \rho'' u''$

$$\frac{d_p(\rho E - \rho'' E'')}{dt} = uF_x + vF_y + wF_z - \frac{d_T(\rho'' E'')}{dt} \quad \text{.................(76).}$$

40. *Generality of the equations for the component systems.*

As the actions which are respectively expressed by the several terms in the equations (68) to (72) $\left(\text{remembering } \frac{d_c(\;)}{dt} = \frac{d_{c'}(\;)}{dt} + \frac{d_{c''}(\;)}{dt}\right)$ are mechanically distinct, these equations are perfectly general and may be applied to the analysis of any resultant system of motion existing in a purely-mechanical-medium, into any two component systems which are *geometrically* distinguishable.

The motions in the two systems are not necessarily independent but the effects of the one on the other are generally expressed in the equations. Thus it may be that neither of the component systems is a conservative system, since one system may be subject to displacement of momentum by and may receive energy from the other system, although they both exist in a purely-mechanical-medium. And it thus appears that there may exist a non-conservative system of motion in a purely-mechanical-medium; that is to say, it appears that, so far as one abstract system of motion is concerned, a purely-mechanical-medium may be possessed of physical properties in consequence of the simultaneous existence of another system of motion. Thus where the only motion apparent to our senses is that of a component system, (the other component system being latent,) although this exists in a purely-mechanical-medium, the apparent system will not of necessity follow the laws of a conservative system, but is expressed by equations involving terms expressing the effects of the latent system on the apparent system, which apparent effects depend on certain physical properties in the medium. Such apparent physical properties however receive mechanical explanation when the complete motion of it is known; or, on the other hand, the experimental determination of these properties may serve to define the latent component motion so as to account, in the equations of the recognised system, for the terms expressing its effect; as for instance the *potential energy*.

41. *Further extension of the system of analysis.*

So far the complete expression of the equations of motion has been confined to the case of two component systems of motion. But by a precisely similar method either of the two component systems of motion may by further definitions be again abstracted into two or more component systems of motion which in virtue of the definition are geometrically distinguishable from each other and from the remaining component system.

If instead of taking u'', v'', w'' to express the defined components of the motion after the abstraction of the residual motion, we take

$$u'' + u''' + \&c., \quad v'' + v''' + \&c., \quad w'' + w''' + \&c.$$

and for ${}_{c}Q$ put ${}_{c'}Q + {}_{c''}Q + {}_{c'''}Q + \&c.$, for ${}_{T}M'$ put ${}_{P}M'' + {}_{P}M''' + \&c.$, and so on for the other functions, expressions are obtained for the equations of as many component systems of motion as are distinguishable by definition.

SECTION V.

THE MEAN AND RELATIVE MOTIONS OF A MEDIUM.

42. *Kinematical definition of mean motion and relative motion.*

By the mean motion of the medium is here understood an abstract component system of motion of which the mass and the components of the velocity respectively satisfy certain conditions as to distribution;—

(1) The condition of continuous velocity, that the mean component velocities are continuous functions of x, y, z and t, however discontinuous the mass may be, Art. 12.

(2) The condition of being mean velocities, that the quadruple integrals, with respect to the four variables, of the respective densities of the mean-components of the momentum (the components of the mean velocity multiplied by the density of the mass at each point) taken over spaces and times, the measures of which exceed certain defined limits, shall be the same as the corresponding integrals of respective components of the density of the resultant momentum.

(3) The condition of momentum in space and time of the components of momentum of mean-velocities, that the integrals of the momentum of the mean velocities taken over the same limits as in (2) shall be respectively the same as in the resultant system.

(4) The condition of relative energy, that the quadruple integrals with respect to the four variables, taken over limits, of the products of the differences of the respective components of the actual, or resultant, and mean velocities, each multiplied by the density of the corresponding components of momentum of mean velocities, as defined in (2) shall be zero.

By the relative velocity of the medium is here understood the velocity which remains in the medium after the mean-velocity is abstracted from the resultant motion when this velocity satisfies certain conditions besides those entailed by the abstraction of the mean-velocity.

The conditions entailed by the abstraction of the momentum of mean-velocities are, besides the condition (4)—

(5) The condition of the momentum of relative-velocity, that the mean densities of the components of momentum of relative velocity are zero.

(6) The condition of distribution in space and time of the momentum of relative velocity, that, taken over the same limits as the mean velocity, the means of the products of the respective components of the momentum of the relative velocities multiplied by any one of the measures of the variables are all zero.

The further condition that must be satisfied by the velocity left after abstracting the mean motion in order that this may be relative-velocity is:

(7) The condition of position of energy of mean and relative velocities, that the mean values of the products of relative energies, as defined in (4), multiplied by measures of any one of the variables, shall be zero, or that the mean position of the energies of the mean-velocity, together with the energy of relative-velocity, shall be the mean position in time and space of energy of the resultant system.

By the mean density of mass is here understood an abstract system of mass which satisfies certain conditions as to distribution.

(8) The condition of continuous density, that the mean density is a continuous function of the variables.

(9) The condition of mean density, that the quadruple integrals with respect to the four variables of the mean-density taken over spaces and times which exceed certain defined limits shall be the same as the corresponding integrals of the actual density.

(10) The condition of distribution of mean-density, that mean position in time and space of the mean-mass shall be the same as the mean position of the resultant mass.

By the relative density of the medium is here understood the density (positive or negative) which remains in the medium after the mean-density has been abstracted, when this residual density satisfies certain conditions besides those entailed by the abstraction of the mean-density.

The conditions entailed by the abstraction of the relative density are:

(11) The condition of relative density, that the mean of the relative density is zero.

(12) The condition of distribution of relative mass, that the product of relative density multiplied by the measure of any one of the variables has no mean value when taken over the defined limits.

The further conditions which have to be satisfied by the relative density of mass are:

(13) The condition of momentum of relative mass, that the products of the components of mean velocity multiplied by the relative density of mass have no mean values over the defined limits.

(14) The condition of distribution of momentum of relative mass, that the products of the components of mean velocity multiplied by the relative density of mass and again by the measure of any one of the variables have no mean values over the defined limits.

(15) The condition of energy of relative mass, that the products of the squares of the components of mean velocity multiplied by the relative density have no mean values when taken over limits.

(16) The condition of position of energy of relative mass, that the products of the squares of the components of mean velocity multiplied by the relative density and again by the measure of any one of the variables have no mean values.

By the mean motion of the medium is here understood the product of the mean-velocity multiplied by the mean density, which is also the density of the mean momentum. And by the relative motion of the medium is understood the density of the resultant momentum less the mean momentum.

In the same way by the density of energy of mean-motion is understood the product of the square of mean-velocity multiplied by the mean-density of mass; and by the density of energy of relative motion is understood the density of energy of resultant motion less the density of energy of mean-motion.

43. *The independence of the mean and relative motions.*

It will be observed, that according to the foregoing definitions, in any resultant system which consists of component systems of mean- and relative-motion, satisfying all the conditions, all the motion which has any part in the mean momentum or in the mean-moments of momentum is, by integration, separated from the relative-motion in such a manner that the motion of each component system is subject to the laws of motion. Action and reaction being equal and opposite and the exchanges of momentum taking place in the direction of the momentum exchanged. And that the relative motion, separated out by integration, is confined to motions of linear and angular dispersion of momentum the effects of which on the mean-motion are such as correspond to the effect of observed physical properties of matter.

It also appears that all the conditions must be satisfied in the resultant motion in order that such separation may be effected.

44. Component systems of mean- and relative-motion are not a geometrical necessity of resultant motion. A very general process in Mechanical Analysis is to consider motion in a mechanical system for a definite interval of time as consisting, at each point of space at any instant of time, of component velocities which are the mean-component velocities of the whole mass over the whole time, together with components which are the differences between the actual components at the point and instant, and the mean-components. These systems respectively satisfy the conditions as to continuous and mean-velocity (1) and (2). Also the condition of relative-velocity (5), and that of relative-energy (4), but they do not satisfy the conditions as to distribution of mean-momentum or any of the other conditions; and hence are not mean and relative, except for particular classes of motion, in the sense in which these terms have been defined.

Such component systems of constant mean-motion in a defined space and time are a geometrical necessity in any resultant system. And, although I am not aware that it has been previously noticed, it appears that considering the number of geometrical conditions to be satisfied by the momentum of mean-velocity and of relative-velocity ((1), (2), (3), and as a consequence (5) and (6)), and the opportunities of satisfying them, the latter are sufficient for the former; so that every resultant system of motion existing in a defined space and time consists of two component systems which satisfy the conditions (1), (2), (3), (4), (5) and (6), although they do not, as a geometrical necessity, satisfy all the further conditions required for mean and relative motion as here defined.

45. THEOREM A.

Every resultant system of motion consists of a component system of mean motion which satisfies all the conditions of mean-velocity (1, 2, 3), *and the condition of relative energy* (4), *but not, of necessity, that of position of relative energy* (7); *together with another system which satisfies the conditions of relative velocity* (5) *and* (6), *but not of necessity* (7), *the condition of distribution of relative energy.*

Taking the mean-velocity at a point x, y, z at the time t within the defined limits, to be expressed by

$$u'' = A + (x - \bar{x}) A_x + (y - \bar{y}) A_y + (z - \bar{z}) A_z + (t - \bar{t}) A_t, \text{ \&c., \&c.} \ldots (77),$$

where the barred symbols refer to the mean-position of the mass within the limits, whether time or space, thus

$$\bar{x} = \frac{\iiiint x\rho\, dx\, dy\, dz\, dt}{\iiiint \rho\, dx\, dy\, dz\, dt}, \text{ \&c.} \ldots\ldots\ldots\ldots\ldots\ldots (78),$$

the limits being assumed; the conditions to be satisfied by the component velocity u'' are:

(1), (2), (5); that

$$\left.\begin{array}{ll} (3), (6) & \displaystyle\iiiint \rho\,(u - u'')\,dxdydzdt = 0, \\ & \displaystyle\iiiint x\rho\,(u - u'')\,dydxdzdt = 0, \text{ \&c., \&c., \&c.} \\ (4) & \displaystyle\iiiint \rho\,(u - u'')\,u''\,dxdydzdt = 0. \end{array}\right\} \quad\ldots\ldots\ldots\ldots (79).$$

The last of these conditions will be identically satisfied if the others are satisfied. Hence there are only five conditions to be satisfied, while in the expression for u'' there are five arbitrary constants, which are determined by putting

$$A = \frac{\iiiint(\rho u)\,dxdydzdt}{\iiiint(\rho)\,dxdydzdt} \quad\ldots\ldots\ldots\ldots (80),$$

then integrating the four equations of position and obtaining the values of A_x, A_y, A_z, A_t by elimination from the resulting equations. These values must be real since the A_x, &c. enter into the equations in the first degree only. The same reasoning applies to the component velocities v'' and w''; so that the first part of the theorem is proved.

To prove the second part all that is necessary is to observe that the condition (7) requires that

$$\iiiint x\rho\,(u - u'')\,u''\,dxdydzdt = 0 \quad\ldots\ldots\ldots\ldots (81),$$

when it is at once seen that this condition is not satisfied as a geometrical consequence of the definition of u'', since the terms involve products of the variables $x\,(y - \bar{y})\,\rho A_y$, &c., which do not necessarily vanish on integration: so that the second part of the theorem is proved.

46. Theorem B.

In a similar manner it appears that *every resultant system of mass consists of a component-system of mean-mass which satisfies all the conditions* (8), (9) *of mean density, and the conditions of relative density* (11) *and position of relative density* (12), *also the condition of momentum of relative mass* (13); *but does not satisfy, of necessity, the condition of distribution of momentum, of relative-mass, or of mean-mass* (10), (14), *nor the conditions of energy of relative mass,* (15) *and* (16).

Taking the mean-density of mass at x, y, z and t to be

$$\rho'' = D + (x - \bar{x})\,D_x + (y - y)\,D_y + (z - \bar{z})\,D_z + (t - \bar{t})\,D_t \ldots\ldots (82),$$

where, as before, the barred symbols refer to the mean position of mass between limits of time and space. And putting t_1, x_1, y_1, &c, as referring to the mean position in time and space, not of the mass, but of the time and space between limits. Since the mean value of ρ'' between limits is not the mean value at the centre of gravity or epoch, the conditions to be satisfied are:

$$\left.\begin{array}{l} (8),\ (9),\ (11) \\ \dfrac{\iiiint \rho\, dxdydzdt}{\iiiint dxdydzdt} = D + (x_1 - \bar{x}) D_x + (y_1 - \bar{y}) D_y + (z_1 - \bar{z}) D_z + (t_1 - \bar{t}) D_t \\ (10),\ (12) \\ \iiiint x(\rho - \rho'')\, dxdydzdt = 0, \text{ \&c., \&c., \&c.} \end{array}\right\} \ldots (83),$$

which five conditions determine D, D_x, D_y, D_z and D_t whatever may be the distribution of mass, so that putting $\rho' = \rho - \rho''$ the conditions (11) and (12),

$$\left.\begin{array}{l} \iiiint \rho'\, dxdydz = 0 \\ \iiiint x\rho'\, dxdydz = 0, \text{ \&c., \&c., \&c.} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (84)$$

are satisfied.

Again, since the constants A and D in the equations (77 and 83) for u'' and ρ'' are respectively the values of u'', ρ'', at the mean position of mass respectively, and the constants A_x, &c. and D_x, &c., are the differential coefficients of u'' and ρ'', respectively, the equations may be written

$$\left.\begin{array}{l} u'' = u'' + u', \text{ \&c., \&c.} \\ \rho'' = \rho'' + \rho', \text{ \&c., \&c.} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (85).$$

Then multiplying the corresponding members,

$$\rho u = \rho'' u'' + \rho' u'' + \rho u', \text{ \&c., \&c.} \ldots\ldots\ldots\ldots\ldots\ldots (86),$$

whence it appears, since the integrals of the last three terms on the right are by definition of necessity zero, that

$$\iiiint \rho u\, dxdydzdt = \iiiint \rho u''\, dxdydzdt \ldots\ldots\ldots\ldots (87),$$

so that condition (13) is of necessity satisfied, which concludes the proof of the first part of the theorem.

To prove the second part. Multiplying the equation respectively by x, &c., then, since the integrals of $x\rho u'$, &c. are zero while those of $x^2\rho'$ are not of necessity zero, and the expression of $x\rho u$, &c. includes the terms $x^2\rho' \dfrac{du''}{dx}$, &c., it appears that the product $\rho'' u''$ does not of necessity satisfy the condition of position of mean-momentum for every distribution of mass, which proves the second part of the theorem.

It has thus been proved that in order that a resultant-system of motion may satisfy the condition of consisting of a component system of mean-momentum which is a linear function of any one or more of the variables together with a component-system of relative-motion which satisfies all the conditions (1) to (15), the relative motion and the relative-mass must, whatever may be the mechanical cause, be subject to certain geometrical restrictions relative to the dimensions of the limits over which the mean motion is taken. With a view to studying the mechanical circumstances which cause such restrictions, where they are shown to exist by the existence of systems of mean and relative motion, it becomes important to generalise, as far as possible, the geometry of these restrictions.

47. *General conditions to be satisfied by relative-velocity and relative-density.*

The general condition to be satisfied by relative-velocity is that, in addition to the conditions which follow from the definition of mean-velocity, the integrals of the products of the density of relative component energy, $\rho u'' u'$, multiplied by the measure of any variable, are zero, or

$$\iiiint x\rho u'' u'\, dx\, dy\, dz\, dt = 0, \text{ \&c., \&c., \&c.} \qquad (88).$$

Hence as u'' is a linear function of the variables these conditions will be satisfied if $\rho u'$, multiplied by any variable, and again by the squares of any power of this variable, all vanish on integration with respect to all four variables, so that the general condition is at once seen to be that $\rho u'$, &c., the components of momentum of relative velocity, integrated between limits with respect to any two independent variables independent of the variable in which u'' varies, must have no mean value; and in the same way for v'', w'', since v'', w'' are not necessarily functions of the same one variable, in order to generally satisfy the conditions $\rho u'$, $\rho v'$, $\rho w'$ must vanish when integrated with respect to any two variables.

Again when the previous condition of relative velocity is satisfied, it appears that the general condition of position of mean-momentum,

$$\iiiint x\rho'' u''\, dx\, dy\, dz\, dt = \iiiint x\rho u\, dx\, dy\, dz\, dt, \text{ \&c., \&c.}$$

requires that the products $x^2\rho'$, &c. shall vanish when integrated between limits with respect to all four variables. Whence we have for the condition of relative mass—that the integrals of ρ' taken between limits with respect to any two independent variables which are independent of the variable in which u'' varies &c. must be zero.

If both the previous conditions are satisfied it appears that the conditions (15) and (16) will be satisfied for

$$\rho u - \rho'' u'' = \rho' u'' + \rho u' \qquad (89),$$

and since u'' is a linear function of the variables

$$(\rho u - \rho'' u'')\, u'' = \rho' u''^2 + \rho u' u'' \quad\ldots\ldots\ldots\ldots\ldots\ldots (90),$$

whence the integrals of both the terms on the right vanish by the previous conditions.

And further, the conditions

$$\iiiint x (\rho u - \rho'' u'') = 0, \text{ \&c., \&c., \&c.} \quad\ldots\ldots\ldots\ldots\ldots\ldots (91)$$

are satisfied; for by taking u'' constant in equation (77), by the definition of u'' we have one relation between four independent variables, so that there are three independent variables with respect to which u'' is constant. And in exactly the same way there are three independent variables with respect to which ρ'' is constant. Therefore u''^2 and ρ'' are each functions of one independent variable only. Hence in the expressions

$$x\rho' u''^2 + x\rho u' u'', \text{ \&c., \&c.},$$

since v'', w'' are not functions of the same variable as u'', $\rho' x$, &c. must vanish when integrated with respect to any two variables, or u'', v'', w'', must be constant. The factors of ρ' and $\rho u'$ are each functions of two independent variables only, and hence these terms vanish on integration between limits with respect to all four variables by the previous conditions of relative density and relative velocity.

Whence it appears that the general conditions, besides those which follow from the definitions of mean velocity and mean density, that must be satisfied by the momentum of relative motion and by relative density, are that these must have no mean values when integrated between limits with respect to any two independent variables independent of the variable with respect to which u'' varies, &c. And it is only resultant systems in which these conditions are satisfied that strictly consist of dynamical systems of mean- and relative-motion.

That these conditions can be strictly satisfied by any system within finite limits seems to be impossible; as for this it would require that, in a purely mechanical medium, there should be, in the same space and time, two masses moving in opposite directions, such that at each point the density of the momentum of the one was equal and opposite that of the other. It is however possible to conceive masses with equal and opposite momenta at any finite distance from each other, and in such cases the conditions may be conceived to be satisfied to any degree of approximation.

48. *Continuous states of mean- and relative-motion.*

The abstract systems of relative velocity and relative density as defined in the previous article must, as a geometrical necessity, be of an alternating character in respect of some of the variables, such that the respective means of the positive and negative masses of relative densities, and the positive and negative momentum of relative velocity, taken over the limits as to any two variables, balance. And as a consequence the distribution of such relative-masses and relative-velocities, whether regularly periodic, as in the case of waves of light or sound, or such as the so-called motions of agitation among the molecules of a gas, involves a geometrical scale of distribution defined by the dimensions of the variables over which the alternations balance.

Such scales of relative-density and velocity, clearly, define the inferior limits of the spaces and times over which the resultant system can consist of systems of mean- and relative-motion. But there is no necessity that the defined space and time over which the system of mean-motion extends should be confined to the dimensions of such scales. That is to say the defined space and time, over which the mean-system may be a linear function of the variables, may be in any degree larger than the minimum necessary for the satisfaction of the conditions of relative-density and relative-velocity, since these conditions will be satisfied for the whole space if they are continuously satisfied in every element of dimensions defined by these conditions.

49. Under such circumstances the expressions for the mean-motion admit of another interpretation, one which has already been discussed in a paper on "The Theory of Viscous Fluids*."

In this expression the mean-velocity at any point x, y, z, t is defined as the mean taken over an elementary space and time, of dimensions defined by the scales of the relative-velocity and density, so placed that the mean position of the mass within the element is defined by x, y, z, t.

Then, since by definition the relative-velocity and relative-density, as defined by integration over the whole space and time, have no mean value in the element, the mean velocity at x, y, z, t (the mean position of mass) obtained by integration over the element will be the same as that at the same point obtained by integration over the whole space and time, as in the first of equations (79); and since, by definition, not only the relative density, but also the variations of relative density, with respect to any variable, have no mean values in the element, the mean-density at the mean position x, y, z, t, obtained by integration over the element as in equations (87) will be the same as that obtained (as in the second equation (89)) by integration over the whole space and time.

* *Royal Soc. Phil. Trans.* 1894, pp. 123—164.

It thus appears that ρ'', u'', in equations (89) to (91) may be taken to represent the values of the mean-density and mean-velocity at x, y, z, t, as defined by integrations with respect to two variables over an element having dimensions defined by the scales of relative-velocity and relative-density, so placed that the mean position of the density in space and time is at x, y, z, t.

50. *The instruments for analysis of mean- and relative-motion.*

It further appears that, since in the method of Arts. 43 and 44 u'' may be taken to represent any entity, quantities consisting of the squares and products of u, u'', u', F/ρ may by the theorems of those articles be separated into mean- and relative-components which satisfy the conditions Art. 42, (1), (2), (3), (4), (5) and (6), respectively, the mean components being linear functions of the variables, and the relative components having no mean value when integrated with respect to any three independent variables over dimensions determined by the scales of relative-velocities and relative-density. And in the case of the quantities ρ', $\rho u'$, &c., subject to the further definition Art. 48, but only in the case where the relative components will have no mean values when integrated with respect to any two independent variables over the same scales. But in either case, if Q expresses the density of any function, integrating over definite limits about any point x, y, z, t as mean position of mass at that point we have

$$\left.\begin{array}{l} \dfrac{\iiiint Q\,dxdydzdt}{\iiiint dxdydzdt} = Q'', \\ \text{and} \\ \dfrac{\iiiint (Q - Q'')\,dxdydzdt}{\iiiint dxdydzdt} = 0, \\ \text{and putting } h \text{ and } k \text{ for any two variables,} \\ \iiiint h\,(Q - Q'')\,dxdydzdt = 0, \\ \iiiint hk\,(Q - Q'')\,dxdydzdt = 0. \end{array}\right\} \quad\text{..................(92).}$$

Equations (92) are thus the general instruments of mean and relative analysis.

51. *Approximate systems of mean- and relative-motion.*

The interpretation of the expressions for mean- and relative-motion considered in the last article is adapted to the consideration of systems in which the mean motion, taken over spaces and times which are defined by the scales of relative-density and relative-velocity, is everywhere approximately a linear function of the variables measured from the mean position and mean time. Thus if ρ'' and u'' are any continuous functions of the four variables x, y, z, t, taking $x_0 y_0 z_0 t_0$ as referring to a particular point and time, then at any other point x, y, z, t,

$$\left.\begin{aligned} \rho'' &= \rho_0'' + (x - x_0)\left(\frac{d\rho''}{dx}\right)_0 + \&\text{c.} + \frac{1}{2}(x - x_0)^2\left(\frac{d^2\rho''}{dx^2}\right)_0 + \&\text{c.} \\ u'' &= u_0'' + (x - x_0)\left(\frac{du''}{dx}\right)_0 + \&\text{c.} + \frac{1}{2}(x - x_0)^2\left(\frac{d^2u''}{dx^2}\right)_0 + \&\text{c.} \end{aligned}\right\} \quad \ldots\ldots.(93),$$

where the differential coefficients are all finite. Therefore as $(x - x_0)$, &c. approach zero all terms on the right except the first approximate to zero, and the terms of higher order which involve as factors multiples of the variables of degrees higher than the first become indefinitely small compared with the linear terms. It is therefore possible to conceive periodic or alternating functions of which the differential coefficients, continuous or discontinuous, are so much greater as to admit alternations to any finite number being included between such values of $(x - x_0)$, &c., as would leave the terms of the second and higher orders indefinitely small as compared with those of the first order, and those of the first indefinitely small as compared with the constant term. Therefore as long as ρ'' and u'' are finite and continuously varying functions of the variables it is always possible to conceive systems of relative-density and relative-motion which together with their differential coefficients satisfy the conditions of having approximately no mean values over the limits, and thus to any degree of approximation satisfy the conditions necessary to be relative-component systems to the mean system $\rho_0'' u_0'' + \&\text{c.}$ within the limits defined by the scale of relative motion.

The method of approximation therefore consists in obtaining

$$\rho'',\ u'',\ \rho''u'',\ \&\text{c.},\ \&\text{c.},$$

and the variations of these, Q'', when Q is any function of

$$\rho''u'',\ \rho'',\ \rho',\ \rho'u',$$

by integrating over the element taken about x, y, z, t, as the mean position, then using these quantities as determined for x, y, z, t, to express by expansion

$$\rho''u'',\ \&\text{c.},\ \&\text{c.},$$

for any other point within the limits of integration as in equation (93) so as to obtain the mean values of these terms in the equations by integration over the elements, neglecting the integrals of all terms which involve as factors functions of the increments of the variables of degrees higher than the first: and in this way may be obtained any necessary transformations of products of mean inequalities and rates of variation, as

$$u''d\rho''u'' = d\rho''u''^2 - u''\rho''du'',\ \&\text{c.}$$

It thus appears that the only motions neglected are those which are defined as small by the conditions, being of the second degree of the dimension of the scale of relative motion, while those retained may have any values at a point, and are, within the limits of approximation, linear functions of the variables; so that within the same limits ρ', $\rho u'$, &c., &c., satisfy by the special definition the conditions of having no mean values over the limits of any two variables; and generally Q' has no mean value over three independent variables.

As has already been pointed out the maintenance of such a system must depend on the distribution and constraints, and the process of analysis consists in assuming such a condition to exist at any instant, and then from the equations of motion ascertaining what circumstances, as to distribution and properties of conduction, the actions of convection and transformation by and to the relative-motion on the variations of the mean-motions will be to increase or to diminish these variations of the first and second orders.

52. *Relation between the scales of mean- and relative-motion.*

From the previous article it is clear that the absolute dimensions of the scale of mean-motion, as determined by the comparative values of the terms of higher orders as compared with those of the lower, do not enter into the degree of approximation to which the conditions of relative-mass and velocity are satisfied, except as compared with the scale of the relative-motion. But it does appear that the degree of approximation depends on the comparative values of these scales. And hence it is only under circumstances (whatever these may be) which maintain distributions of mass and velocity which admit of complete abstraction into two systems widely distinct as to relative scales, that systems of mean and relative motion can exist.

Thus, as we have previously pointed out, it is not sufficient that the relative motion, or one class of motions such as the motion of the molecules of a gas in equilibrium, should be subject to superior limits by the scale of distribution. It is equally necessary that the scale of variation of mean motions, such as the mean motions of a gas, should be subject to superior limits (whatever may be the cause) which prevent the scale of these mean-motions approaching that of the molecules. And it is the existence of circumstances which secure both these effects, which is indicated by resultant systems which satisfy the conditions of mean- and relative-motion as defined.

It has been already proved that the existence of component systems which satisfy the conditions of mean position of density and of relative energy, as well as those of mean-density and mean-position of momentum of mean-velocity, is not a geometrical necessity of the definition of mean-motion as is the existence of component systems which satisfy the latter conditions only. Were it not so there would be no point in the analysis, for then the existence of such component systems would reveal no special circumstances as to the geometrical distribution of the medium, or the motion in the medium, whereas it has now been shown that the existence in such systems of mean- and relative-motion, as indicated by the observed mean-motion and the apparent "physical" properties of the medium or matter, depends (if in a purely mechanical medium) upon circumstances which constrain the geometrical distribution of the motion of the medium. Thus the application of this method of analysis affords a general means of studying the conditions of the medium, either intermediate or fundamental, which would admit of such relative or latent motion as is necessary to account, as a mechanical consequence, for the apparently physical properties of matter and the medium of space.

SECTION VI.

THE APPROXIMATE EQUATIONS OF COMPONENT SYSTEMS OF MEAN- AND RELATIVE-MOTION.

53. THESE equations must conform to the general equations of component systems as expressed in the equations (61) to (76), Section IV.

Thus if in equations (69), (70), (71), together with equations (74), (75), (76), ρ'', u'' and $\rho' u'$ are at any time subject to the respective definitions for mean- and relative-motions, these suffice, for the instant, to determine the rates of transformation (as expressed by arbitrary functions) in terms of the several defined rates of convection and production.

Then these rates of transformation, as expressed in the defined symbols, having been substituted in the equations, these equations express the approximate rates of change of the mean and relative component systems at the instant.

These equations express, in terms of the so far defined mean and relative quantities, the initial approximate rates of change in the defined quantities and thus afford the means of studying whatever further conditions must hold in the distribution of the medium in order that these rates of change may tend to maintain or increase the degree of approximation to which the conditions of mean- and relative-motion are initially subject. This study of the further definition, however, must of necessity follow the complete expression of the initial equations, to which this section is devoted.

54. *Initial conditions.*

The initial conditions for approximate component systems of mean- and relative-motion, as defined in Arts. 50 and 51, Section V., define all mean quantities as continuous functions of the variables, such that within the limits over which the means are taken they are constant to a first approximation, whether they are the means of density, means of velocity, or means of component momentum; also the means of any products or derivatives of products, of velocity, or density, the means of any products of mean and relative quantities, while the products of the relative quantities, corresponding, multiplied by the density, are such that their means taken over the same limits are zero.

Thus if Q be any term expressing increase of density of mass, momentum, or of energy for the resultant system, or for either of the component systems at a point, x, y, z, t, at distance $\delta x, \delta y, \delta z, \delta t$,

$$\left.\begin{aligned} Q'' &= \frac{\iiiint Q\,dx\,dy\,dz\,dt}{\iiiint dx\,dy\,dz\,dt} - \delta x\frac{dQ''}{dx} + \&c.\\ Q' &= Q - Q'' \end{aligned}\right\}\qquad\text{.................(94),}$$

satisfy the conditions (1), (2), (3), (4), (5) and (6), Art. 42, of being respectively mean and relative, approximately,—that is to say Q'' is, approximately, a linear function of the variable, and Q' has approximately no mean value when integrated over any three independent variables.

Also if $\frac{dQ}{dx}$ is a derivative of any quantity

$$\left.\begin{aligned} \left(\frac{dQ}{dx}\right)'' &= \frac{d}{dx}(Q'')\\ \text{and}\qquad Q_1\frac{(dQ_2)''}{dx} &= \frac{d(Q_2''Q_1)}{dx} - \frac{Q_2''dQ_1}{dx} \end{aligned}\right\}\qquad\text{........................(95).}$$

55. *The rate of transformation, at a point, from mean-velocity, per unit of mass.*

From equation (58) or the first two of equations (69) transforming by equation (19),

$$\frac{du''}{dt} + u''\frac{du''}{dx} + v''\frac{du''}{dy} + w''\frac{du''}{dz}$$
$$+ u'\frac{du''}{dx} + v'\frac{du''}{dy} + w'\frac{du''}{dz} = \frac{d}{dt}({}_pu''),\ \&c.,\ \&c.\ \text{........(96).}$$

The first four terms in this are all mean accelerations, while the last three terms on the left are such that multiplied by ρ have no mean values—are entirely relative-accelerations—whence by definition it follows that since du''/dt is a mean-acceleration the right member must contain terms which exactly cancel the last three terms on the right, and that these form the only relative terms it can contain. These terms which represent the acceleration at a point per unit mass, due to convection of mean velocity by relative velocity, are the only transformation from mean velocity at a point.

Since after abstracting these terms the right member remains wholly mean, we have

$$\frac{d_pu''}{dt} = u'\frac{du''}{dx} + \&c. + \left(\frac{d_pu''}{dt}\right)''\ \text{..........................(97).}$$

56. *The rate of transformation at a point from relative velocity, per unit of mass.*

From equations (60), or the last two of equations (69),

$$\frac{d\rho u''}{dt} = \frac{d_c(\rho u')}{dt} + F_x - \rho\frac{d_pu''}{dt},\ \&c.,\ \&c.\ \text{...............(98).}$$

In this the term on the left is, by definition, such as has no mean value, hence taking a mean by equation (92), Section V.

$$\rho'' \left(\frac{d_p u''}{dt}\right)'' = \left\{\frac{d\,({}_c\rho u')}{dt} + F_x\right\}'' \ldots\ldots\ldots\ldots\ldots\ldots\ldots(99),$$

or dividing by ρ'' it appears that the transformation from relative-velocity to mean-velocity, at a point, is expressed by

$$\frac{1}{\rho''}\left\{\frac{d_c\,(\rho u')}{dt} + F_x\right\}'', \text{ \&c., \&c.};$$

that is the mean accelerations due to the mean convections of the relative-velocity by the relative-velocity, plus the mean acceleration due to conduction.

Substituting from equation (97) the expression $d_p u''/dt$ in equations (58) and (60), Section IV.

$$\left.\begin{aligned} \frac{d_p u''}{dt} &= u'\frac{du''}{dx} + v'\frac{du''}{dy} + w'\frac{du''}{dz} + \frac{1}{\rho''}\frac{d_c\,(\rho u')''}{dt} + \frac{F_x''}{\rho''}, \text{ \&c., \&c.} \\ \frac{d_p\,(u')}{dt} &= -u'\frac{du''}{dx} - v'\frac{du''}{dy} - w'\frac{du''}{dz} - \frac{1}{\rho''}\frac{d_c\,(\rho u')}{dt} + \frac{F_x''}{\rho''}, \text{ \&c., \&c.} \end{aligned}\right\} \ldots..(100).$$

57. *The rates of transformation of the energy of mean-velocity.*

As already pointed out, Art. 35, Section IV. equation (61), the rates of transformations of energies per unit of mass, of mean-velocity and relative-velocity, are respectively obtained by multiplying the rates of transformation of mean- and relative-velocity, u'' and u', &c., &c. respectively; thus

$$\left.\begin{aligned} \frac{1}{2}\frac{d_T(u'')^2}{dt} &= \frac{1}{2}u'\frac{d\,(u'')^2}{dx} + \text{\&c.} + \frac{u''}{\rho''}\left\{\frac{d_c\,(\rho u')}{dt} + F_x\right\}'', \text{ \&c., \&c.} \\ \frac{1}{2}\frac{d_T(u')^2}{dt} &= -\left\{u'u'\frac{du''}{dx} + u'v'\frac{du''}{dy} + u'w'\frac{du''}{dz}\right\} \\ &\qquad\qquad - \frac{u'}{\rho''}\left\{\frac{d_c\,(\rho u')}{dt} + F_x\right\}'', \text{ \&c., \&c.} \\ \frac{d_T\,(u''u')}{dt} &= -\frac{1}{2}\left\{\frac{d_T\,(u'')^2}{dt} + \frac{d_T\,(u')^2}{dt}\right\}, \text{ \&c., \&c.} \\ &= -\frac{1}{2}u'\frac{d\,(u'')^2}{dt} - \frac{(u''-u')}{\rho''}\left\{\frac{d_c\,(\rho u')}{dt} + F_x\right\}'' \\ &\qquad\qquad + u'u'\frac{du''}{dx} + \text{\&c., \&c.} \end{aligned}\right\} \ldots(101).$$

58. The expressions for the rates of transformation in equations (100) and (101) include all the rates of transformation of component velocities, and of the squares and products of the component velocities of the component systems of mean- and relative-velocities which enter as arbitrary functions into the equations (69) and (74). But as is pointed out in Art. 35, Section IV.

any one of these quantities, the rate of increase of which is expressed by one of the equations, may, by definition, be further abstracted into two component systems.

The component systems of the energies of the mean- and relative-velocity per unit mass may, therefore, be separately abstracted into mean and relative component systems. And the importance of this at once appears, since the process of analysis is solely between the mean and relative, and while $(u'')^2$ is mean and $(u''u')$ is relative, $(u')^2$, although positive, is not continuously distributed as a continuous function of the variables.

The rate of transformation from the mean rate of increase of energy of relative-velocities to relative-energy of relative velocity. Adding the second and fifth of the equations (74) as they stand, and substituting the expression for the transformation-function from the second of equations (101), we have

$$\frac{1}{2}\frac{d\rho\,(u')^2}{dt} = \frac{1}{2}\frac{d\,[{}_c(\rho u')^2 + u'F_x]}{dt} - \left\{\rho u'u'\frac{du''}{dx} + \&c.\right\} - \frac{\rho u'}{\rho''}\left\{\frac{d_c\,(\rho u')}{dt} + F_x\right\}'' \quad\ldots\ldots\ldots\ldots(102).$$

Then putting

$$(u')^2 = ((u')^2)'' + ((u')^2)' \quad\ldots\ldots\ldots\ldots(103),$$

where $((u')^2)''$ is obtained after the same manner as u''; putting $d\,({}_T((u')^2)'')/dt$ for the total rate of transformation, we have as in equations (97) and (98), substituting $((u')^2)''$ for u'' and the three last terms in equations (102) for F_x in equations (100), since the last term has no mean values,

$$\frac{1}{2}\frac{d_p\,((u')^2)''}{dt} = \frac{1}{2}\left\{u'\frac{d\,((u')^2)''}{dx} + \&c.\right\} + \frac{1}{2\rho''}\left\{\frac{d\,[d_c\rho\,((u')^2)']}{dt} + u'F_x\right\} - \frac{1}{\rho''}\left(\rho u'u'\frac{du''}{dx} + \&c.\right)'' \quad\ldots\ldots\ldots\ldots(104);$$

and

$$\frac{1}{2}\frac{d_p\,((u')^2)'}{dt} = -\frac{1}{2}\frac{d\,((u')^2)''}{dt} - \&c. \quad\ldots\ldots\ldots\ldots(104\text{ A}).$$

Then since

$$\frac{d_T u}{dt} = 0,$$

$$\left.\begin{aligned}
\frac{d_p\,(u^2)}{dt} &= \frac{d_T\,(u'')^2}{dt} + \frac{d_T((u')^2)''}{dt} + u'F_x,\\
\frac{1}{2}\frac{d_p\,[u^2 - (u^2)'']}{dt} &= \frac{1}{2}u'\frac{d\,((u'')^2 + (u'^2)'')}{dx} + \&c. - u'F_x{}'\\
&\quad + \frac{u''}{\rho''}\left\{\frac{1}{2}\frac{d_c\rho u'}{dt} + F_x\right\}\\
&\quad + \frac{1}{\rho''}\left\{\frac{1}{2}\frac{d_c\rho((u')^2)''}{dt} + u'F_x - \left(\rho u'u'\frac{du'}{dx} + \&c.\right)\right\}
\end{aligned}\right\}\ldots(105).$$

The expressions for the production of mean energy of relative motion which form the left members of equations (104) are not transformations from energy of mean motion only. They include the relative parts of the rates of convection and production of energy of relative motion which are being transformed to the system of relative energy. These rates of convection and production of relative-energy are expressed by the first two terms in the equations (104), while the last term expresses the only rates of transformation from energy of mean-motion.

Whence the only transformations from energy of the component mean motions are

$$-\rho\left\{u'u'\frac{du''}{dx}+v'u'\frac{du''}{dy}+w'u'\frac{du''}{dz}\right\},\ \text{\&c., \&c.}$$

59. *The rate of transformation from mean to relative energy.*

From equation (64), at a point,

$$\frac{d_T\rho''}{dt}=\frac{d\rho''}{dt}+\frac{d\rho''u''}{dx}+\frac{d\rho''v''}{dy}+\frac{d\rho''w''}{dz}+\frac{d\rho''u'}{dx}+\frac{d\rho''v'}{dy}+\frac{d\rho''w'}{dz}\ \text{.......}(106),$$

where the first four terms on the right are all mean, and the last three may be in part mean and in part relative. Hence the relative part of the convection of mean-density by the relative-velocity is the transformation to the relative density at a point, and this must form the only relative of the left member, and

$$\left.\begin{aligned}&\frac{d_T\rho''}{dt}=-\frac{d_{c'}\rho''}{dt}+\left(\frac{d_{c'}\rho''}{dt}\right)''+\left(\frac{d_T\rho''}{dt}\right)''.\\ &\text{Also from the last of equations (65)}\\ &-\frac{d_T\rho''}{dt}=\frac{d\rho'}{dt}+\frac{d\rho'u''}{dx}+\frac{d\rho'v''}{dy}+\frac{d\rho'w''}{dz}-\frac{d\,({}_{c'}\rho')}{dt}\end{aligned}\right\}\ \text{............}(107).$$

In the last of the equations (107) the first four terms on the right are relative, and therefore the mean rate of transformation is

$$\frac{d_T\rho''}{dt}=\frac{(d_{c'}\rho')''}{dt}\ \text{...............................}(108).$$

Then adding the mean and relative parts; since

$$\frac{(d_{c'}(\rho'))''}{dt}=-\frac{(d_{c'}(\rho''))}{dt},$$

and $$(\rho u'+\text{\&c.})''=0,$$

$$\frac{d_T\rho''}{dt}=-\frac{d_{c'}\rho''}{dt}\ \text{...............................}(109).$$

60. *The transformations for mean and relative momentum.*

We have $$\frac{d_T(\rho''u'')}{dt} = \rho''\frac{d_T u''}{dt} - u''\frac{d_{c'}\rho''}{dt} \quad\text{.......................(110).}$$

Then substituting from the first of equations (101) and (109), and transforming,

$$\frac{d_T(\rho''u'')}{dt} = -\frac{d_{c'}(\rho''u'')}{dt} + \left\{\frac{d_c(\rho''u')}{dt} + F_x\right\}'' + \&\text{c.} \quad\text{.........(111),}$$

and we have

$$\frac{d_T(\rho u - \rho''u'')}{dt} = \frac{d_{c'}(\rho''u'')}{dt} - \left\{\frac{d_c(\rho''u')}{dt} + F_x\right\} - \&\text{c.} \quad\text{..........(111 A).}$$

61. *The rates of transformation of mean-energy of the components of mean- and relative-velocity.*

From equations (74), (100) and (109) we have

$$\left.\begin{aligned} &\frac{1}{2}\frac{d[{}_T(\rho''(u'')^2)]}{dt} = -\frac{1}{2}\frac{d[{}_{c'}\rho''(u'')^2]}{dt} + \left\{u''\frac{(d_c\rho u')''}{dt} + F_x''\right\},\\ &\frac{1}{2}\frac{d[{}_T\rho''((u')^2)'']}{dt} = -\frac{1}{2}\frac{d[{}_{c'}(\rho''(u')^2)'']}{dt}\\ &\qquad + \left\{\frac{1}{2}\left[\frac{d_{c'}(\rho((u')^2)'}{dt}\right] + u'F_x\right\}'' - \left\{\rho u'u'\frac{du''}{dx} + \&\text{c.}\right\}'' \end{aligned}\right\} \quad\text{........(112).}$$

In the second of equations (112) it is the last term only that expresses transformation from energy of mean motion.

The last terms of equation (112) admit of different expression, by substituting for

$$\frac{d_c(\rho u')''}{dt}$$

its equivalent $$-\left\{\frac{d\rho u'u'}{dx} + \frac{d\rho v'u'}{dy} + \frac{d\rho w'u'}{dz}\right\}'',$$

or $$\left\{\frac{d\rho''(u'u')''}{dx} + \frac{d\rho''(v'u')''}{dy} + \frac{d\rho''(w'u')''}{dz}\right\},$$

and we have

$$u''\frac{(d_c\rho u')''}{dt} = -\left\{\frac{d(\rho''(u'u')''u'')}{dx} + \&\text{c.}\right\}\left\{\rho''(u'u')''\frac{du''}{dx} + \&\text{c.}\right\} \quad\text{...(113),}$$

also $$F_x = \frac{dp_{xx}}{dx} + \frac{dp_{yx}}{dy} + \frac{dp_{zx}}{dz};$$

so that by equation (95), F_x'' may be expressed by

$$\left\{\frac{dp''_{xx}}{dx} + \&c.\right\}.$$

Then we have

$$u''F_x'' = \frac{d(u''p''_{xx})}{dx} + \&c. - p''_{xx}\frac{du''}{dx} - \&c. \quad \text{...............(114)};$$

also
$$(u'F_x)'' = (uF_x - u''F_x)'' \quad \text{.....................................(115)},$$

and this may be expressed as

$$-\left\{\frac{d(up_{xx})''}{dx} + \&c.\right\} + \left\{\frac{d(u''p'')}{dx} + \&c.\right\}$$
$$+\left\{\left(p_{xx}\frac{du}{dx}\right)'' + \&c.\right\} - \left\{p''_{xx}\frac{du''}{dx} + \&c.\right\}.$$

Then substituting in the first of equations (112) we have for the rates of transformation to the energy of mean motion

$$\frac{1}{2}\frac{d[{}_T(\rho''(u'')^2)]}{dt} = -\frac{1}{2}\frac{d[{}_c(\rho''(u'')^2)]}{dt} - \left\{\frac{d[u''(\rho''(u'u')'')]}{dx} + \&c.\right\}$$
$$-\left\{\frac{d(u''p''_{xx})}{dx} + \&c.\right\} + \left\{[\rho''(u'u')'' + p''_{xx}]\frac{du''}{dx} + \&c.\right\} \text{...(116)},$$

and again substituting in the second of equations (112) we have for the rates of transformation to the energy of relative motion

$$\frac{1}{2}\frac{d[{}_T(\rho''((u')^2)'')]}{dt} = -\frac{1}{2}\frac{d[{}_{c'}\rho''((u')^2)'']}{dt}$$
$$+\frac{1}{2}\frac{d[{}_{c'}\rho((u')^2)']''}{dt} - \left\{\frac{d(up_{xx})''}{dx} + \&c.\right\}$$
$$+\left\{\left(p_{xx}\frac{du}{dx}\right)'' + \&c.\right\}$$
$$+\left\{\frac{d(u''p''_{xx})}{dx} + \&c.\right\}$$
$$-\left[\{\rho''(uu')'' + p''_{xx}\}\frac{du''}{dx} + \&c.\right] \text{............(117)}.$$

The purpose of this transformation is easily seen on adding the equations. The two last terms in each equation cancel, showing that they represent a transformation between the rate of increase of the mean-energies of relative- and mean-velocities; while changing the sign of the right members of the resulting equation, which then represent the rate of transformation to the energy of residual motion, or of relative energy, these become

$$\left.\begin{aligned} \frac{1}{2}\frac{d_T[\rho u^2 - \rho''(u^2)'']}{dt} &= \frac{1}{2}\frac{d[{}_{c'}\rho''(u^2)]''}{dt} \\ &\quad - \frac{1}{2}\frac{d[{}_{c'}(\rho(u')^2)']''}{dt} + \left\{\frac{d[u''(\rho''(u'u')'')]}{dx} + \&c.\right\} \\ &\quad + \left\{\frac{d(up_{xx})''}{dx} + \&c.\right\} - \left\{p''_{xx}\frac{du}{dx} + \&c.\right\}'', \ \&c., \&c. \end{aligned}\right\} \ldots(118);$$

and these are the exact forms in which the rate of transformation to relative-energy, obtained by substituting u^2, $(u^2)''$, $(u^2)'$, uF for u, u'', u', F respectively in equation (111) for relative momentum, is expressed.

In a purely mechanical medium the last terms in these equations (118) represent the mean rate of angular dispersion both of mean and relative motion of energy, as explained in Art. 32, Section III., while the integrals of the remaining terms are all surface integrals. It is thus seen that the rates of exchange between mean-energy and relative-energy are purely conservative within the limits of the approximation.

On the other hand, the integral rates of exchange by transformation between mean-energy of mean-motion and mean-energy of relative-motion as expressed by the integrals of the last terms of equations (116), (117) are not surface integrals, nor are these rates confined to angular dispersion; so that they express exchanges at each point which are not expressed by a surface integral, and thus appear to represent those actions of the relative-motion on the mean-motion the study of which is the object of the investigation. But this is found on closer examination not to be the case.

62. *The expressions for transformations of energy from mean to relative motion.*

The expressions $\rho''(u'u')''\dfrac{du''}{dx} + \&c.$, which occur in the last terms of equations (116) and (117), are simply transformation terms expressing the mean effect of the convections of relative-momentum by relative motion on the energy of mean motion, and this is the most general and most important transformation.

The other transformations are the results of conduction. These are included in the expressions

$$\left\{p''_{xx}\frac{du''}{dx} + \&c.\right\}, \quad \left\{p_{xx}\frac{du}{dx} + \&c.\right\}'',$$

as they occur in equations (116) and (117), but they are not explicitly expressed by these. The first of these expressions includes the rate at which the energy of the component of mean-motion is being increased by angular dispersion from the energy of the other components of mean-motion, as well as the rate at which the energy of the component of mean-motion is being increased by transformation from the energy of the corresponding component of relative-motion. The second of these expressions includes both the rates at which energy of the component of mean-motion and the energy of the component of relative-motion are increasing, by angular dispersion, at the expense of the other components in their respective systems,—together with the rate at which energy of the component of the resultant system is being increased by transformation from energy in some other mode—which latter rate does not exist if u, v, w are the motions of points in mass.

In the expressions

$$\left(p''_{xx}\frac{du''}{dx} + \&c.\right), \ \&c., \ \&c.,$$

and

$$\left(p_{xx}\frac{du}{dx} + \&c.\right)'', \ \&c., \ \&c.,$$

the analysis necessary to separate out the expressions for the separate actions in either system is furnished by equations (47 A), Section III., the symbols for the mean and the relative motions being substituted for those of the resultant system.

Putting $p = \dfrac{p_{xx} + p_{yy} + p_{zz}}{3}$, the first two terms in these equations (47 A) which express the rates of angular dispersion in the directions of x, y, z respectively on the square of the components of the mean and the resultant system, become respectively

$$-\left[\frac{1}{3}p''\left(2\frac{du''}{dx} - \frac{dv''}{dy} - \frac{dw''}{dz}\right)\right.$$
$$\left. + \frac{1}{2}\left\{p''_{yx}\left(\frac{du''}{dy} - \frac{dv''}{dx}\right) + p''_{zx}\left(\frac{du''}{dz} - \frac{dw''}{dx}\right)\right\}\right], \ \&c., \ \&c.,$$

$$-\left[\frac{1}{3}p\left(2\frac{du}{dx} - \frac{dv}{dy} - \frac{dw}{dz}\right) + \frac{1}{2}\left\{p_{yx}\left(\frac{du}{dy} - \frac{dv}{dx}\right) + p_{zx}\left(\frac{du}{dz} - \frac{dw}{dx}\right)\right\}\right], \ \&c., \ \&c.$$

The corresponding expressions for the rate of increase of the resilience are

$$-\left[\frac{1}{3}p''\left(\frac{du''}{dx} + \frac{dv''}{dy} + \frac{dw''}{dz}\right) + (p''_{xx} - p'')\frac{du''}{dx}\right.$$
$$\left. + \frac{1}{2}\left\{p''_{yx}\left(\frac{du''}{dy} + \frac{dv''}{dx}\right) + p''_{zx}\left(\frac{du''}{dz} + \frac{dw''}{dx}\right)\right\}\right], \ \&c., \ \&c.,$$

$$-\left[\frac{1}{3}p\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) + (p_{xx} - p)\frac{du}{dx}\right.$$
$$\left. + \frac{1}{2}\left\{p_{yx}\left(\frac{du}{dy} + \frac{dv}{dx}\right) + p_{zx}\left(\frac{du}{dz} + \frac{dw}{dx}\right)\right\}\right], \ \&c.. \ \&c.$$

Substituting these for $\left(p''_{xx}\frac{du''}{dx}+\&c.\right)$ and $\left(p_{xx}\frac{du}{dx}+\&c.\right)''$ as they enter into equations (116) and (117), these equations become

$$\frac{1}{2}\frac{d_T[\rho''(u'')^2]}{dt}=-\frac{1}{2}\frac{d_{c'}[\rho''(u'')^2]}{dt}-\left\{\frac{d_{c''}[\rho(u'u')'']}{dx}+\&c.\right\}-\left\{\frac{d\,u''p''_{xx}}{dx}+\&c.\right\}$$
$$+\left\{\frac{1}{3}p''\left(2\frac{du''}{dx}-\frac{dv''}{dy}-\frac{dw''}{dz}\right)+\frac{1}{2}\left\{p''_{yx}\left(\frac{du''}{dy}-\frac{dv''}{dx}\right)+p''_{zx}\left(\frac{du''}{dz}-\frac{dw''}{dx}\right)\right\}\right.$$
$$+\left[\frac{1}{3}p''\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right)+(p''_{xx}-p'')\frac{du''}{dx}\right.$$
$$\left.+\frac{1}{2}\left\{p''_{yx}\left(\frac{du''}{dy}+\frac{dv''}{dx}\right)+p''_{zx}\left(\frac{du''}{dz}+\frac{dw''}{dx}\right)\right\}\right]$$
$$+\left[\left\{\rho''(u'u')''\frac{du''}{dx}\right\}+\&c.\right],\ \&c.,\ \&c.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(116\text{A}).$$

$$\frac{1}{2}\frac{d[{}_T\rho''(u'u')'']}{dt}=-\frac{1}{2}\frac{d[{}_{c'}\rho''(u'u')'']}{dt}+\frac{1}{2}\frac{d[{}_{c'}\rho(u'u')']''}{dt}-\left\{\frac{d(u'p_{xx})''}{dx}+\&c.\right\}$$
$$-\left[\frac{p''}{3}\left(2\frac{du'}{dx}-\frac{dv'}{dy}-\frac{dw'}{dz}\right)''+\frac{1}{2}\left\{p_{yx}\left(\frac{du'}{dy}-\frac{dv'}{dx}\right)''-p_{zx}\left(\frac{du'}{dz}-\frac{dw'}{dx}\right)''\right\}\right]$$
$$-\left[\frac{p''}{3}\left(\frac{du'}{dx}+\frac{dv'}{dy}+\frac{dw'}{dz}\right)''+\left\{(p_{xx}-p)\frac{du'}{dx}\right\}''\right.$$
$$\left.-\frac{1}{2}\left\{p''_{yx}\left(\frac{du'}{dy}+\frac{dv'}{dx}\right)''+p''_{zx}\left(\frac{du'}{dz}+\frac{dw'}{dx}\right)''\right\}\right]$$
$$-\left[\left\{\rho''(u'u')''\frac{du''}{dx}\right\}+\&c.+\&c.\right],\ \&c.,\ \&c.\ldots\ldots\ldots\ldots\ldots\ldots\ldots(117\text{A}).$$

In these equations the first three terms in the members on the right express rates of linear redistribution of the energy of components of motion of the respective systems, while the fourth terms express, respectively, rates of energy received from the other components of the same system by angular dispersion, and the fifth and the last terms express the direct exchanges between the two systems, of mean density of energy, by transformation.

This last statement however is only true when, as in the case of the resultant system, in a purely mechanical medium, there is no resilience in the resultant system, for the fifth term in the last equation expresses rates of decrease of the resilience in the resultant system less that of the abstract resilience in the mean-system; so that, if the former is not zero, this term, besides the exchange by transformation, expresses the total rate of increase of the resilience of the resultant system.

In a granular medium when u, v, w are the component velocities at points in mass, and there is no resilience in the resultant system, the sum of the resilience of the mean and relative systems is zero, and the fourth term in equation (117) has the identical value, under opposite sign, as the fourth term in equation (116), which expresses rate of decrease of abstract resilience in the mean system.

The first term in the brackets represents the angular dispersion by distortion under mean strains, equal in all directions, and the second represents the rates of angular dispersion by rotational motion of the mass.

63. *The equations for the rates of change of density of mean- and relative-mass.*

By equations (64) and (109) we have for mean density

$$\frac{d\rho''}{dt} = \frac{d_{c''}\rho''}{dt} \qquad \text{(119)},$$

and by equations (65) and (109) we have for the equation of relative mass

$$\frac{d\rho'}{dt} = \frac{d\,({}_{c''}\rho')}{dt} + \frac{d\,({}_{c'}\rho)}{dt} \qquad \text{(119A)}.$$

64. *The equation for mean momentum.*

By equation (58) and the first of equations (100) we have for the equation of mean momentum

$$\frac{d\rho''u''}{dt} = \frac{d_{c''}\rho''u''}{dt} - \left\{\frac{d\,(p_{xx} + \rho''\,(u'u')'')}{dx} + \text{\&c., \&c.}\right\} \qquad \text{(120)},$$

and by equations (60) and the second of equations (100) we have the equation of relative momentum

$$\frac{d(\rho u - \rho''u'')}{dt} = \frac{d_c(\rho u)}{dt} - \frac{d_{c''}(\rho''u'')}{dt} - d\,({}_c\rho u')'' - \left[\frac{dp'}{dx} + \text{\&c.}\right] \qquad \text{(121)}.$$

65. *The equations for the rate of change of the density of mean-energy of the components of mean-motion and of the mean-energy of the components of relative-velocity.*

Substituting for the transformation function in the first of equations (74) from equation (116), the equation for mean density of energy of mean motion becomes

$$\frac{1}{2}\frac{d\,[\rho''\,(u'')^2]}{dt} = \frac{1}{2}\frac{d\,[{}_{c''}\,(\rho''\,(u'')^2)]}{dt} - \left\{\frac{d[u''\,(\rho''(u'u')'' + p_{xx}'')]}{dx} + \text{\&c.} + \text{\&c.}\right\}$$
$$+ \left\{(\rho''\,(u'u')'' + p_{xx})\frac{du''}{dx} + \text{\&c.} + \text{\&c.}\right\},\ \text{\&c., \&c.} \qquad \text{(122)},$$

and the equations for the mean density of energies of relative-velocity become

$$\frac{1}{2}\frac{d\,[\rho''((u')^2)'']}{dt} = +\frac{1}{2}\frac{d\,[_{c''}(\rho''((u')^2)'')]}{dt} + \frac{1}{2}\frac{d\,[_{c'}(\rho((u')^2)')]''}{dt} - \left\{\frac{d\,(up_{xx})''}{dx} + \&c.\right\}$$
$$+\left\{\left(p_{xx}\frac{du}{dx}\right)'' + \&c.\right\}$$
$$+\left\{\frac{d\,(u''p'')}{dx} + \&c.\right\}$$
$$-\left\{[\rho''(u'u')'' + p_{xx}'']\frac{du''}{dx} + \&c.\right\}, \ \&c., \ \&c. \ \ldots\ldots\ldots(123).$$

66. *The equation for density of relative-energy.*

Proceeding in the same manner as in equations (74) and substituting the rate of transformation to relative-energy equation (118), the equation for relative-energy of component velocities becomes

$$\frac{1}{2}\frac{d\,[\rho u^2 - \rho''(u^2)'']}{dt} = \frac{1}{2}\frac{d\,[_{c''}(\rho u^2 - \rho''(u^2)'')]}{dt} + \frac{1}{2}\frac{d\,[_{c'}(\rho(u^2)'')]}{dt}$$
$$+\frac{1}{2}\frac{d\,[_{c'}(\rho(u^2)')]}{dt} + \frac{1}{2}\frac{d\,[_{c'}\rho((u')^2)']''}{dt}$$
$$+\left\{\frac{d\,[u''\rho''(u'u')'']}{dx} + \&c.\right\}$$
$$-\left\{\frac{d\,[up_{xx}]'}{dx} + \&c.\right\} + \left\{\left(p_{xx}\frac{du}{dx}\right)' + \&c.\right\},$$
$$\&c., \ \&c. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(124).$$

67. *Complete equations.*

$$\frac{1}{2}\frac{d\,[\rho''((u'')^2 + (v'')^2 + (w'')^2)]}{dt} = \frac{1}{2}\frac{d\,[_{c''}\{\rho''((u'')^2 + (v'')^2 + (w'')^2)\}]}{dt}$$
$$-\left\{\begin{array}{l} +\left\{\dfrac{d\,[u''(\rho''(u'u')'' + p_{xx}'')]}{dx} + \&c.\right\} \\ +\left\{\dfrac{d\,[v''(\rho''(v'u')'' + p_{xy}'')]}{dx} + \&c.\right\} \\ +\left\{\dfrac{d\,[w''(\rho''(w'u')'' + p_{xz}'')]}{dx} + \&c.\right\} \end{array}\right\}$$
$$+\left\{\begin{array}{l} +\left\{(\rho''(u'u')'' + p_{xx}'')\dfrac{du''}{dx} + \&c.\right\} \\ +\left\{(\rho''(v'u')'' + p_{xy}'')\dfrac{dv''}{dy} + \&c.\right\} \\ +\left\{(\rho''(w'u')'' + p_{xz}'')\dfrac{dw''}{dz} + \&c.\right\} \end{array}\right\} \ \ldots\ldots(125).$$

$$\frac{1}{2}\frac{d\left[\rho''((u')^2+(v')^2+(w')^2)''\right]}{dt}=\frac{1}{2}\frac{d\left[{}_{c''}(\rho''((u')^2+(v')^2+(w')^2)'')\right]}{dt}$$

$$+\frac{1}{2}\frac{d\left[{}_{c'}(\rho\,((u')^2+(v')^2+(w')^2)')\right]''}{dt}$$

$$-\left\{\begin{array}{l}+\left\{\dfrac{d\left[(up_{xx})''-u''p_{xx}''\right]}{dx}+\&\text{c.}\right\}\\ +\left\{\dfrac{d\left[(vp_{yx})''-v''p_{yx}''\right]}{dx}+\&\text{c.}\right\}\\ +\left\{\dfrac{d\left[(wp_{zx})''-w''p_{zx}''\right]}{dx}+\&\text{c.}\right\}\end{array}\right\}+\left\{\begin{array}{l}+\left\{\left(p_{xx}\dfrac{du}{dx}\right)''+\&\text{c.}\right\}\\ +\left\{\left(p_{yx}\dfrac{dv}{dx}\right)''+\&\text{c.}\right\}\\ +\left\{\left(p_{zx}\dfrac{dw}{dx}\right)''+\&\text{c.}\right\}\end{array}\right\}$$

$$-\left\{\begin{array}{l}+\left\{(\rho''\,(u'u')''+p_{xx}'')\dfrac{du''}{dx}+\&\text{c.}\right\}\\ +\left\{(\rho''\,(v'u')''+p_{yx}'')\dfrac{dv''}{dy}+\&\text{c.}\right\}\\ +\left\{(\rho''(w'u')''+p_{zx}'')\dfrac{dw''}{dz}+\&\text{c.}\right\}\end{array}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(126).$$

$$\frac{1}{2}\frac{d\left[\rho\,(u^2+v^2+w^2)-\rho''\,((u^2)''+(v^2)''+(w^2)'')\right]}{dt}$$

$$=\frac{1}{2}\frac{d\left[{}_{c''}(\rho\,(u^2+v^2+w^2))-\rho''\,((u^2)''+(v^2)''+(w^2)'')\right]}{dt}$$

$$+\frac{1}{2}\frac{d\left[{}_{c'}(\rho\,(u^2+v^2+w^2)'')\right]}{dt}$$

$$+\frac{1}{2}\frac{d\left[{}_{c'}(\rho\,(u^2+v^2+w^2)')-\{{}_{c'}(\rho\,(u^2+v^2+w^2))\}''\right]}{dt}$$

$$+\left\{\begin{array}{l}+\left\{\dfrac{d\left[u''\rho''\,(u'u')''\right]}{dx}+\&\text{c.}\right\}\\ +\left\{\dfrac{d\left[v''\rho''\,(v'u')''\right]}{dx}+\&\text{c.}\right\}\\ +\left\{\dfrac{d\left[w''\rho''(w'u')''\right]}{dx}+\&\text{c.}\right\}\end{array}\right\}$$

$$-\left\{\begin{array}{l}+\left\{\dfrac{d\left[up_{xx}\right]'}{dx}+\&\text{c.}\right\}\\ +\left\{\dfrac{d\left[vp_{yx}\right]'}{dx}+\&\text{c.}\right\}\\ +\left\{\dfrac{d\left[wp_{zx}\right]'}{dx}+\&\text{c.}\right\}\end{array}\right\}+\left\{\begin{array}{l}+\left\{\left(p_{xx}\dfrac{du}{dx}\right)'+\&\text{c.}\right\}\\ +\left\{\left(p_{yx}\dfrac{dv}{dx}\right)'+\&\text{c.}\right\}\\ +\left\{\left(p_{zx}\dfrac{dw}{dx}\right)'+\&\text{c.}\right\}\end{array}\right\}\quad\ldots\ldots\ldots(127).$$

The equations (119) to (127) are the equations for mean and relative component systems of any resultant system in which the conditions are satisfied, irrespective of the medium being a purely mechanical medium; that is to say, irrespective of whether or not in the resultant system (ρ, u, v, w, p_{xx}, &c.) are related to the actual, mechanical-medium, or represent the densities, motions and stresses of a component system of mean-motion of the resultant system.

It has already been pointed out (Art. 52) that the absolute scale of the variations of the mean motion has no part in determining the degree of approximation, but only the relative magnitude as compared with the scale of variations of the relative motion. So that any component of mean-motion may be a resultant system if the conditions exist which ensure its satisfying the conditions of mean and relative motion. There is however this difference according to whether the unqualified symbols refer to the purely mechanical medium or not. If they do refer to the mechanical medium, then the last terms in equation (124) and the last but two in (123) represent angular dispersion of energy only, and the last term in equation (127) and the last but one in (126) are zero; if not, they represent changes of energy.

SECTION VII.

THE GENERAL CONDITIONS FOR THE CONTINUANCE OF COMPONENT SYSTEMS OF MEAN- AND RELATIVE-MOTION.

68. THE general conditions for the existence of mean-, and relative-motion, as defined in Art. 47, Section V., are that the components of momentum of relative-velocity, as well as the relative density, must respectively be such that their integrals with respect to any two independent variables, taken over limits defined by the scale of relative-motion, have no mean values.

By equation (1), Section II., it follows that for the continuance of such states the respective rates of increment of these quantities by all causes, convection and production, must satisfy the same conditions. Therefore as the necessary and sufficient conditions we have, that

$$\int_0^t \frac{d(\rho u')}{dt}\,dt, \quad \int_0^t \frac{d(\rho v')}{dt}\,dt, \quad \int_0^t \frac{d(\rho w')}{dt}\,dt, \quad \int_0^t \frac{d\rho'}{dt}\,dt,$$

where the limit t may have any value, when integrated between the limits, as initially defined by the relative scales, with respect to any two independent variables shall be zero within the limits of approximation.

The satisfaction of these conditions does not follow as a geometrical consequence of the initial condition.

The rate of change in the density of relative-momentum is a consequence of the space rates of the variation of the convections and conductions existing at the instant. And initially the mean- and relative-motions are subject to definition, from which, as a geometrical consequence, their variations, in space, are also subject to definition, which although less complete has been already fully defined, Art. 45, Section V.

It therefore follows that the general conditions to which the initial rates of increase, by convections and conductions, are subjected, are defined. And this at once appears on considering the equations of motion for the momentum of relative-velocity, which are obtained by substituting in equations (98) the expressions for the rates of transformation from equations (100), Section VI.

$$\frac{d(\rho u')}{dt}\,dt = -\left\{\frac{d}{dx}(\rho u'u') + \frac{d}{dy}(\rho v'u') + \frac{d}{dz}(\rho w'u')\right\}dt$$

$$-\rho\left(u'\frac{du''}{dx} + v'\frac{dv''}{dy} + w'\frac{dw''}{dz}\right)dt$$

$$+\frac{\rho}{\rho''}\left\{\frac{d}{dx}(\rho u'u') + \frac{d}{dy}(\rho v'u') + \frac{d}{dz}(\rho w'u')\right\}''dt$$

$$-\frac{\rho'}{\rho''}F_x''\,\delta t + F'\,\delta t, \text{ \&c., \&c. } \qquad \text{(128).}$$

In these equations, according to the method of approximation, all the terms in the member on the right are such as have no mean values when integrated over any three variables, as a geometrical consequence of the definition.

It therefore appears that it does not follow as a geometrical consequence that

$$\frac{d(\rho u')}{dt}, \text{ \&c., \&c.,}$$

should satisfy the condition of having no mean values when integrated with respect to any two variables, to the same degree of approximation as do the initial values of $\rho u'$, $\rho v'$, $\rho w'$. And this applies to both rates of increment by convection and rates of increment by relative accelerations.

If, then, this condition is to be continuously satisfied it must be as the result of some redistributing effects of the actions of conduction on the convections. For the rates of increase by convection are a geometrical consequence of the initial motions which are subject to the definition as to scale and relative-motion; while on the other hand, the rates of increase by conduction depend on the conducting properties of the medium, as well as on the distribution of the medium in space and time.

69. The fourth property of mass, necessitated by the laws of motion, is that of exchanging momentum with other mass, Art. 17, Section II., and it now appears that this is the fundamental property on which the existence of systems of mean- and relative-motion depends.

For if there were no conduction, that is, if mass were completely penetrable by mass; so that two continuous masses could pass through each other without affecting each other's motion; then the only rates of increase would be those by convection, each point of mass preserving its course with no interruption, with constant velocity, and there could be no redistribution. Hence :—

Certain properties of conduction are necessary for the maintenance of systems of approximately mean- and relative-motion.

70. Notwithstanding the extremely abstract reasoning on which the foregoing conclusion is based it is definite. And it appears possible to carry this reasoning further and so obtain conclusive evidence as to what the general properties of conduction and the general distributions of the medium must be for the maintenance of the mean- and relative-systems, when the resultant system is purely mechanical.

71. The general laws of conduction of momentum by a purely mechanical medium, as defined by the laws of motion, have already been deduced (Section III. Art. 24), and the effects of conduction in displacing momentum and in angular dispersion of *vis viva* have been proved (Section III. Arts. 31—2), and also the effect of conduction on the resilience, if any. However, since there is no resilience in a purely mechanical medium, it at once follows that the medium must be perfectly free to change its shape without changing its volume, or it must consist of mass or masses, whether infinite, finite, or indefinitely small, each of which absolutely maintains its shape and volume; that is to say, each of which is a perfect conductor of momentum.

Thus the class of media in which the general conducting properties satisfy, as a resultant system, the condition of being a purely mechanical system is not large; being confined to

(1) The "perfect fluid";

(2) The perfect solid;

(3) Perfect discontinuous solids;

(4) Perfect discontinuous solids with perfect fluid within their interstices.

This class of media all satisfy the conditions for purely mechanical media as resultant systems. But it does not follow, as a geometrical necessity, that they all satisfy the conditions of consisting of mean and relative component systems.

For although any medium which satisfies the conditions of consisting of component systems of mean and relative motion must of necessity satisfy the conditions as a resultant system, the converse of this is not a necessity.

It therefore remains to obtain from the previous definition the further limitations imposed, as a geometrical necessity, by the conditions of consisting of component systems of approximately mean- and relative-motion.

72. *Evidence as to the properties of conduction for component systems.*

(1) From the equations (128) it appears, as already pointed out, that in order that

$$\int_0^t \frac{d(\rho u')}{dt} dt, \text{ \&c., \&c.,}$$

may satisfy the condition of having no mean values, when integrated between the limits of the scale, in time and space, of relative motion, over any two independent variables to any defined degree of approximation, the time integrals of the members on the right must satisfy the same condition.

Whence it follows that the condition for the maintenance of the inequalities steady requires that the rate of increment, as expressed by all terms on the right, in each of the equations (128), shall be such as has absolutely no mean value when integrated over limits, with respect to any two independent variables.

This condition, although it applies only in a somewhat particular case, is such as must be satisfied for the maintenance of mean and relative systems to be general, and hence any evidence that may be derived from it must be perfectly general.

To apprehend the importance of this evidence we have only to consider, what has already been pointed out, that the first four terms in the right members in each of the equations (128) require, as a geometrical necessity, integration between limits over three independent variables in order that they may have no mean values. Whence it follows that in order to maintain the inequalities steady the fifth term, which expresses relative rates of increment of momentum by conduction, must be such when integrated, over limits, with respect to any two variables, as will exactly cancel the integrals of the other four terms when they are taken over the same limits with respect to the same two variables.

Thus we have for a particular case, which however must occur in all general systems consisting of component systems of mean- and relative-motion, an inexorable condition as to the necessary properties of conduction.

It will be readily granted that the satisfaction of this condition involves the absolute dependence of the functions F_x', &c., on the condition of the medium and its relative-motion.

(2) Evidence as to the necessary properties of the medium is also obtained from the condition that the inequalities must be maintained small.

The satisfaction of the condition of equality between the rates of opposite actions resulting from transformation, convection, and conduction, does not define the magnitudes of the inequalities which may be maintained, but only the fact that they remain steady.

It therefore appears that the definition of the relative values of the inequalities which are maintained depends on a balance of rates of institution and decrement. And in order that such a balance should institute itself and remain steady, it is necessary that the state of the medium shall be such that integrals of F_x', &c., taken over limits with respect to any two independent variables, shall be such functions of the inequalities that they increase with the inequalities and are of opposite sign, whereby the inequalities are subject to logarithmic rates of decrement.

Then, whatever might be the rates of institution of inequalities resulting from all the other actions, the inequalities would increase, increasing the rates of decrement by conduction until these balanced the rates of increment, that is until the other actions were cancelled by the actions expressed by F_x', &c., after which the inequalities would remain steady as long as the rate of institution remained steady.

(3) Evidence as to the necessary properties is also obtained from the conditions that define the scales of relative motion.

Where mean motion is everywhere uniform this condition requires that the scale of relative velocities and relative mass shall approximate to some finite scale at which it will remain as long as the mean motion is everywhere uniform. This does not follow as a geometrical necessity of the initial definition, for if constraining limits were absent from the mass, the actions which insure the logarithmic rates of decrement would continue to diminish the scale indefinitely; hence inferior limits of relative-mass and relative-motion define the properties of the medium as regards limiting constraints.

73. This evidence, together with the definitions of mean-velocity and mass, suffices to differentiate the four general states of media, which, as resultant systems, satisfy the conditions of being purely mechanical, from those which also satisfy the conditions of consisting of component systems of approximately mean and relative motion.

Since continuous mass cannot pass through continuous mass without exchanging momentum, the reciprocal actions between the masses in relative motion will be to cause continual diversions of the paths of points in mass.

And by definition of relative motion, if there is no mean motion, the mean component momentum in any positive direction is exactly equal to the mean of the negative momentum in the same direction. Therefore the mean rate of increase of component momentum in the positive direction, by the components of the reciprocal relative accelerations, is exactly equal to the mean rate of increase by the component reciprocal accelerations of the component momentum in the negative direction. The mean motions being uniform, the reciprocal accelerations have no effect on energy of relative motion in all three independent directions. Whence the effects of the component reciprocal accelerations are rates of change in the positive and negative component momenta, in one direction, with the positive and negative momenta in other directions. Such exchanges of positive and negative momenta from one direction to another are possible only when the component accelerations of relative motion are, not resultant accelerations, but, are the means of the components of resultant reciprocal accelerations with various degrees of divergence from the direction of the previous motion.

And it is thus shown that any angular redistribution of positive and negative components of momenta, or, which is the same thing, of the *vis viva* of the component velocities, results solely from the impenetrability of the medium.

74. From the foregoing reasoning it might be inferred that the impenetrability of mass together with the definition of relative motion must secure logarithmic rates of decrement of all inequalities provided that the medium were sufficiently mobile. That this is not the case is however at once seen from the theory of a "perfect fluid."

(*a*) For in such media every point in mass is in complete normal constraint by the surrounding medium, with lateral freedom. So that, while no point can move without affecting the motion of every other point in some degree, there is no lateral action. Thus the continuous finite accelerations do not cause finite diversions of the paths of points in mass from the previous directions at any point of their courses, but cause finite curvature of these paths. And thus the paths of adjacent points are ultimately parallel. There being no finite lateral deviation, there is no lateral exchange of momentum in the direction of motion at any point.

Whence such lateral exchange of momentum being necessary in order that there may be general rates of logarithmic decrement of inequalities, it follows that in a perfect fluid there cannot exist logarithmic rates of decrement of all inequalities of relative motion.

It thus appears, since, as has already been pointed out, general logarithmic rates of decrement of all angular inequalities are necessary for the maintenance of approximate systems of mean and relative motion, that a perfect fluid, although satisfying the condition of a purely mechanical medium as a resultant system, cannot satisfy, generally, the condition of consisting of component systems of approximately mean and relative motion.

(*b*) A perfect continuous solid, that is a continuous mass which conducts momentum perfectly, whether direct or lateral, can only move as one piece, and therefore cannot consist of component systems of mean and relative motion.

(*c*) It thus appears that of the class of media that satisfy the conditions of a purely mechanical medium, neither the perfect fluid nor the perfect solid satisfies the condition of consisting of component systems of approximately mean and relative motion. And as these are the only two continuous media in the class we have the conclusion: that no continuous medium can satisfy the condition of consisting of component systems of mean and relative motion.

(*d*) If then the conditions for mean and relative systems are to be satisfied it can only be by discontinuous media.

These all include perfectly conducting parts and are capable of separation into two classes according to whether or not these parts are or are not in such constraint with each other that each part is in complete constraint with the neighbouring parts; lateral as well as normal.

(*e*) In media in which the perfectly conducting parts are each in complete lateral as well as normal constraint with their neighbours, there can be no logarithmic rates of decrement. Whence, as in the case of a perfect fluid, such discontinuous media cannot generally consist of component systems of approximately mean and relative motion.

It thus appears that no purely mechanical medium can satisfy the condition of consisting of approximate systems of mean and relative motion unless it includes discontinuous perfectly conducting parts, each of which has certain degrees of freedom with its neighbours.

(*f*) If, therefore, it could be shown that, as in the other purely mechanical media, these discontinuous media, with degrees of freedom, do not admit of logarithmic rates of decrement of the inequalities of relative motion, it would follow that component systems of approximately mean and relative motion are impossible.

As it is, however, it can be shown that these discontinuous media, with or without perfect fluid occupying the interstices, as long as the perfectly conducting parts have any degrees of freedom with their neighbours, do admit of, and not only admit of, but entail, logarithmic rates of decrement of all inequalities of relative-momentum.

This will be fully proved in the following sections. But it is sufficient at this stage to show how this comes about.

(*g*) The actions between perfectly conducting masses are instantaneous finite exchanges of momentum in the direction of the common normal to the surfaces at contact. The direction of this normal has no necessary connection with the direction of the relative motion of the masses before contact; therefore the direction of relative motion after contact has no necessary connection with the direction before contact. And thus the actions will be to render the path of the centre of each mass a rectilinear polygon in space, with angles which may be anything from 0 to π according to the freedoms.

Such action entails that mean component, positive or negative, acceleration of the relative motion in any direction is not a resultant acceleration, but the mean of the component resultant impulses in all directions, thus securing continued angular redistribution in direction and magnitude of the relative momentum of each of the perfectly conducting masses; so that any mean inequality in the relative motion is subjected to rates of decrement proportional to the inequality, and to the mean of the positive or negative components of relative velocity, divided by the scale of relative motion—to a logarithmic rate of decrement.

(*h*) The evidence furnished by the necessity of the maintenance of the scales of relative mass and relative motion has not been drawn upon in the foregoing reasoning, and therefore may now be brought forward as confirming the conclusion already arrived at; that the only media that satisfy the conditions of mean and relative component systems are those which involve discontinuous perfectly conducting parts, since such media are the only media in which limits to the scales of relative mass and relative motion are of necessity maintained.

75. Having thus arrived, for reasons shown, at the conclusions that the only purely mechanical media which can consist of component systems of approximately mean- and relative-motion are those which consist of perfectly conducting members which have certain degrees of independent movement, and that such media of necessity satisfy the condition of securing logarithmic rates of decrement of all mean inequalities in the positive or negative components of relative-momentum in every direction, the further analysis may be confined to this class of media only.

It is still a class of media and not a single medium.

Such media may be distinguished according as the interstices between the grains are occupied by perfect fluid or are empty of mass. But this is by no means the only distinction. For the perfectly conducting members may have any shapes, and hence may include any possible kinematical arrangement or trains of mechanism, provided that there is always a certain amount of freedom or backlash, as it is called in mechanism; or they may consist of parts of any similar shape but of different sizes or of parts the same in size and shape, as for instance, spheres of equal size and mass. Nor is this all, for the relative extent of the freedom as compared with the size of the members may introduce fundamental distinctions in the properties of media consisting of similar members.

76. This last source of distinction, arising from the relative extent of the freedoms as compared with the dimensions of the grains, being perfectly general however the media may otherwise be distinguished, is a subject for general treatment, the outlines of which may with advantage be drawn at this stage from the evidence, already adduced, as to the conducting properties of the media consisting of component systems of approximately mean- and relative-motion.

In this preliminary discussion of the effect of the extent of the freedoms, relative to the dimensions of the perfectly conducting members, the latter may be considered as being spherical grains of equal size and mass.

In the first place it must be noticed that, so far, in this section, no account has been taken of any transformation of mass or of the displacement of momentum by conduction, so that the logarithmic rates of decrement by accelerations refer only to changes in the direction of the *vis viva*, leaving out of account the fact that there is displacement of momentum by conduction at each encounter, and, thus, the reasoning, so far, does not touch on the possibility of redistribution of inequalities of rates of conduction of component momenta.

It has, however, been shown that, owing to the fact that the directions of the normals at contact are independent of the directions of relative motion before contact, in a granular medium, there must exist rates of redistribution of all mean angular inequalities in *vis viva* of the components of relative motion, whatever may be the inequalities in rates of conduction of momentum in different directions.

Thus far, then, for anything that has been shown in the previous reasoning, the actions which determine the rates of displacement of momentum by conduction may be independent of any effect of the independence of the direction of the normals at contact, and the direction of the relative motion of the grains before contact, which, as shown, secures angular dispersion of the momentum of relative motion.

77. In the simple case of uniform spherical grains, which may be conceived to be smooth, without rotation, whatever may be the relative paths of the grains as compared with their diameters, if the state of the relative-motion is without angular inequalities, since this state is maintained by the continual finite exchanges of momentum lateral to their paths, the mean component of the aggregate momentum in an interval of time, determined by the time scale of relative motion, must be the same in all directions, as also must be the aggregate component paths traversed in a positive direction, and also those traversed in a negative direction.

But it in nowise follows as a necessity of complete angular dispersion of components of momentum, within the limits of relative motion, that the mean length of the component paths traversed in one direction shall be the same as the mean of those in another direction.

The clear apprehension of this fact is of extreme importance, when we come to consider the rates of displacement by conduction of momentum; this is easily seen:—

If each grain traverses the same aggregate, positive and negative, component paths in the same time, but their mean component paths in one direction differ from those in another, since the paths are limited by encounters, and the displacement, by conduction, of momentum in the direction of the component is the mean of the product of the diameter of the grain multiplied by the component of the relative momentum; then, if the mean component conductions are the same in all directions, the number of the conductions in any direction must be inversely proportional to the component mean path in that direction. And thus the rate of displacement of momentum in any direction must be inversely proportional to the mean component path in any direction.

78. In order to secure that the rates of displacement of the momentum shall be approximately equal in all directions, it is not sufficient that there should be logarithmic rates of decrement of the mean inequalities of the relative components of momentum, positive or negative, but requires in addition that there should be logarithmic rates of decrement of mean inequalities in the mean component paths of the grains.

The length of the path of a grain in any direction depends only on the positions of the surrounding grains; and if the mean distance between the grains is such that the probable length will carry its centre through several surfaces set out by the centres of these other grains, then, since all possible arrangements of the grains would be probable, all directions of the normal at encounter would be equally probable, whatever might be the directions of the paths. And hence continual encounters would lead to such distribution of the grains that the probable length of the path would be equal in all directions; and, so, there would be logarithmic rates of decrement of inequalities in the lengths of the mean paths in different directions.

78 A. Evidence of the necessity of such logarithmic rates of decrement of inequalities in the arrangement of the mass is furnished by the equations of relative-mass; in a manner similar to that furnished by the equations of relative-motion as to the necessity of logarithmic decrement of the inequalities of *vis viva*.

This at once appears from the equations of relative-mass (119), which may be expressed:

$$\frac{d(\rho')}{dt} = -\left\{\frac{d(\rho' u)}{dx} + \&c.\right\} - \left\{\frac{d(\rho u')}{dx} + \&c.\right\}.$$

In this equation, according to the limits of approximation, the terms in the right member are such as have no mean values when integrated over the defined limits with respect to three independent variables.

Therefore it does not follow as a geometrical consequence of the definition of relative mass that

$$\frac{d\rho'}{dt}$$

should satisfy the condition of having no mean value, when integrated over definite limits with respect to any two independent variables, to the same degree of approximation as do the initial values of ρ'; and this applies both to the rates by convection and the rates by transformation.

If then the conditions are to be continuously satisfied, it must be as the result of the redistributing actions on the rates of convection by the mean-velocity, which alone institutes inequalities.

78 B. Inequalities in the integrals of relative mass, over defined limits, with respect to any two independent variables, correspond to inequalities in the *products* and *moments* of relative mass. And it thus appears that these inequalities have no connection with inequalities in the mean-mass, which is a mean over all four variables.

Therefore these inequalities are inequalities in the symmetry or angular arrangement of the relative mass.

This significance of the inequalities becomes apparent on multiplying both members of the equation of relative mass by the square of any variable, as x^2, or by the product of two variables, as yz, and taking the mean over all four variables; as

$$x^2 \frac{d(\rho')}{dt} = -x^2 \left\{\frac{d(\rho' u'')}{dx} + \&c.\right\} - x^2 \left\{\frac{d(\rho u')}{dx} + \&c.\right\} \ldots\ldots (128\text{ A}).$$

Then if $x^2\rho'$ integrated over all four variables satisfies the conditions to any degree of approximation, the maintenance of the same degree of approximation requires that

$$x^2 \frac{d\rho'}{dt}$$

should satisfy the identical conditions to the same degree of approximation.

Hence we have the necessity, in order to maintain the inequalities steady, that, whatever may be the rate of institution, resulting from distortional mean motions, as expressed by the first term in the right member, the rate of rearrangement resulting from the transformation expressed by the second term must be such as exactly counteracts the rate of institution.

78 C. It thus appears, as in the case of Art. 72, that this condition of equality between the rates of institution and rearrangement can be satisfied only when the rate of rearrangement, as expressed by the second term, depends on, and is proportional to, the inequality instituted.

78 D. From this evidence it appears that the logarithmic rate of decrement of inequalities in the mean arrangement of the grains, which has been shown (Art. 78A) to follow as the result of diffusion in granular media, is a necessity for the maintenance of systems of mean and relative motion.

And thus it appears that granular media *may* satisfy the condition of consisting of component-systems which are mean and relative in respect of conductions as well as convections.

78 E. It also appears, and perhaps this is of greater analytical importance, that the two rates of logarithmic decrement, that of inequalities of *vis viva*, and that of rearrangement of mean inequalities in the symmetry of the mean arrangement of the grains, which also secures the redistribution of angular inequalities in the rates of component conduction of momentum, are in a measure independent and are analytically distinct.

79. The inequalities in the mean symmetrical arrangement of the mass, although, being the most remote, they have presented the greatest difficulties to recognition and analytical separation, are of primary importance and distinguish between classes of granular media. It has been shown that logarithmic decrement of these inequalities results from diffusion among the grains.

79 A. It does not, however, follow that such logarithmic rates of decrement would exist when the grains were in such close order that no grain could break through the closed surface which might be drawn through the centres of its immediate neighbours. For then, whatever might be the order of arrangement of the grains, notwithstanding the existence of a certain extent of freedom, it could undergo no change.

If in this last case the general state of the medium were such that the mean freedoms of each grain were equal in all directions, so that there were no inequalities in the mean component paths in different directions, the relative-motion would be in a state of mean equilibrium without inequalities and the rates of displacement, by conduction, would be equal in all directions.

But if, from the last condition, the medium were subjected to a mean distortional strain, however small, the mean component paths of the grains would no longer be equal in all directions; and the rates of displacement of the momentum, by conduction, would be no longer equal in all directions, but would be such as tended to reinstitute the former condition; that is to say, the rearrangement of the grains within the limits of freedom would be such as to balance, not the external mean stresses by which the strains were brought about, but the stresses necessary to maintain the strain steady. And thus the logarithmic decrement would not be to a state in which the mean paths were equal in all directions, but to a state in which the inequalities in the mean paths were such as to maintain the necessary inequalities in the rates of displacement, by conduction, to secure equilibrium under the external stresses.

80. It thus appears that, while the effect of relative accelerations to redistribute all mean inequalities, in the angular distribution of relative *vis viva*, is independent of any symmetry in the mean arrangement of the grains, and, hence, of mean angular inequalities in the mean component paths of the grains, and is therefore subject to no limits. Whatever the relative freedoms of the grains may be, the angular redistribution of inequalities in the mean component paths depends solely on the rate of redistribution of the mean inequalities in the symmetry of the arrangement of the grains and is subject to limits depending on the relative lengths of the mean component paths of the grains, taken in all directions, as compared with the diameters of the grains.

81. It also appears that the definite limit, at which redistribution of the lengths of the mean paths ceases, is that state of relative freedoms which does not permit of the passage of the centre of any grain across the triangular plane surface set out by the centres of any three grains which are neighbours.

This definite limiting condition obviously corresponds to that at which all diffusion of the grains amongst each other ceases.

82. It thus appears that there is a fundamental difference in media, otherwise similar, according to whether or not the freedoms are within or without this limit.

This difference amounts to discontinuity in the media, for within the limit there will be no rearrangement of the grains however long a time may elapse or whatever the state of strain may be. While outside the limit, in however small a degree, any state of mean strain must ultimately be relaxed however long the time.

83. The time taken for such relaxation will in some way be a function of the degree in which the freedoms are without the limit of no diffusion which will range from infinity to zero, so that there are continuous degradations in the properties of the media according to the degree in which the freedoms exceed the fundamental limit.

84. The independence of the redistribution of relative *vis viva* on this fundamental limit to redistribution of the arrangement of mass in media consisting of perfectly hard spheres, or of masses of any rigid shapes, does not appear to have formed a subject of study by those who have developed the kinetic theory of gases; so that however complete this development may be with respect to limited classes of granular media which have formed the subjects of this study, the methods employed can have been applicable only to those classes of media in which the extent of the relative freedoms has, in a large degree, been outside the fundamental limit of no diffusion.

85. It seems important that the limitation imposed, by the methods of analysis hitherto used in the kinetic theory, on the class of media to which that theory applies, should be distinctly pointed out here, before proceeding to the further analysis of the general theory. Otherwise confusion might arise in the mind of any reader acquainted with the conclusions already accepted as resulting from the kinetic theory, as to the reason why, after having arrived at the general conclusion that the only media which can consist of component systems of mean and relative motion belong to the class of granular media with some degree of freedom, which is also the class of media to which the kinetic theory has been applied, any further analysis should not simply follow the lines of the kinetic theory as hitherto developed?

This question having been anticipated by the answer which is given in the previous paragraph, in which it is shown that the general class of granular media is subject to fundamental differentiation according as the ratio of the mean paths of the grains to the dimensions of the grains is within certain limits; and that hitherto the method of the kinetic theory has not been such as to take account of these limits, and is thus only applicable to media in which the relative paths are large as compared with the linear dimensions of the grains*.

86. Besides the fundamental limit of no diffusion there is also another fundamental limit, which appears as soon as a finite relation between the paths and the linear dimensions of the grains is contemplated. This limit is that to which the medium approaches as the paths of the grains approach zero.

If the granular medium is in a steady condition, then if the relative *vis viva* is finite there will be some extent of freedom. But for any given *vis viva* the mean paths will depend on the rates of conduction or *vice versa*. Thus it is possible that the relative mean paths may be indefinitely small as compared with the diameters of the grains, and the rates of conduction indefinitely large.

87. It has been shown Art. 74 (a) that a granular medium, in which the grains are in such arrangement that each grain is in complete constraint by its neighbours, cannot consist of mean and relative systems of motion. While from the previous paragraph it appears that granular media in which there is finite relative-energy may approach within any approximation of the condition of complete constraint with their neighbours.

88. The conclusion, as stated at the end of the last paragraph, has a fundamental significance. It clears the way to the recognition of the definite geometrical distinction between the effects of redistribution in media, otherwise similar, in which the mean paths are respectively within and without the fundamental limit of no diffusion.

* *Phil. Mag.* 1860, Vol. XIX. p. 19, Vol. XX. p. 21.

When there is no relative motion and each grain is in complete constraint with its neighbours, if there is no mean motion, it follows, at once, that the directions of the normals, at the points of contact, to the surfaces of the grains, whatever these directions may be, are undergoing no change—are fixed in space.

If then, as shown in the last paragraph, granular media in which there is *vis viva* of relative-motion may approach indefinitely to the condition of complete constraint, it follows that in such media, when the mean paths are indefinitely small compared with the diameters of the grains, the directions of the normals at points of contact approximate indefinitely to certain definite directions fixed in space, that is, as long as there is no mean-motion. Thus we have the definite geometrical distinction, that as long as the mean paths are within the fundamental limit of no diffusion, and there is no mean-motion, the normals to the surfaces at encounters are within certain angles of directions fixed in space; while if the mean paths are without these limits, in however small a degree, the normals continually change their directions so that, if sufficient time is allowed, all directions are equally probable.

89. While within the fundamental limit any one grain can only have contacts with a strictly limited number of other grains, in the case of

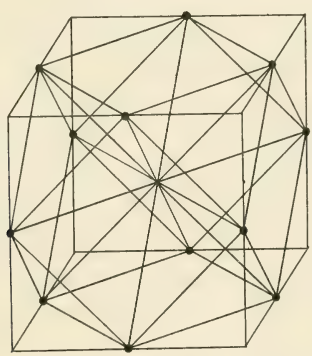


Fig. 1.

uniform spherical grains, in regular symmetrical piling, the number of grains any grain can come in contact with is twelve, so that if there is no strain in the medium and the mean paths are indefinitely small, as compared with the diameter, there are twelve fixed normals in which this grain can have contact with other grains. The twelve normals radiate from the centre of the grain, and when the grains are in the regular formation each normal is in the same line with an opposite normal so that there are six fixed axes symmetrically situated in which encounters take place. And as the resultant accelerations are in the directions of the normals at encounter, these six directions of the normals are six axes of conduction of momentum.

These axes pass through the twelve middle points in the edges of a cube circumscribing each grain, if there are no mean strains in the medium, and are thus symmetrically placed with respect to the three principal axes of the cube. This is shown in Fig. 1, p. 83.

If, then, the rates of conduction across surfaces perpendicular to these six axes are equal, the momentum conducted being in the direction of the axes, the grains will, of necessity, be in mean equilibrium.

This state of equilibrium in no way depends on the mean density of the relative *vis viva* of the grains. Therefore, in the limit, as the mean paths of the grains become indefinitely small, as compared with their diameters, as regards the direction of the rates of conduction, whatever the relative *vis viva* may be, the state will be the same.

Thus, if there is no relative motion, but the grains are under stress, equal in all directions, by rates of conduction resulting from actions at the boundaries of the medium, the rates and directions of the resultant actions would be the same as if the rates of conduction resulted from the exchanges of momentum of relative-motion.

90. This limiting similarity between the states of media, one of which, having no system of relative motion, is purely kinematical, and cannot satisfy the conditions of consisting of mean and relative systems of motion, while the other, essentially, satisfies these conditions, has a fundamental significance, although (except by the recognition that in the one case the conduction results from mean actions at the boundaries of the medium, while in the other the conductions are between the moving grains) this significance in no way appears as long as there are no mean strains in the media.

If these media are subject to any indefinitely small distortional strains the discontinuity between them, as classes of media, appears.

In the case of kinematical media without mean strain, the stresses being equal in all directions and finite, no strain will result from indefinitely small stresses, nor will any strain result until the mean distortional stresses arrive at the same order as the mean stress equal in all directions. Thus if p

represents the stress, equal in all directions, and $p_{xx} - p$ is the normal stress imposed in the direction in which x is measured, the stress in the direction at right angles remaining equal to p (and not affected by the strain), there will be no strain until p_{xx} is greater than $2p$. Whence it follows that any distortional strain is attended by an increase of mean volume occupied by the medium equal to the contraction in the direction in which x is measured, since there is no work spent in resilience, or in accelerations of relative *vis viva*. Thus the kinematical medium has absolute stability up to certain limits*.

91. On the other hand, the granular medium with relative motion, however small may be the mean paths, when subject to no distortional strain, and to indefinitely small distortional stresses, yields in proportion to the stress so that such stress is equal to the strain multiplied by a coefficient which is constant if the terms involving the square and higher powers of the strain are neglected; and this medium has the character of a perfectly elastic solid for indefinitely small strains. It has therefore no finite absolute stability, and no dilatation as long as the squares of the strains are indefinitely small. As the strains increase, however, dilatation ensues, as expressed by the terms involving the squares and higher powers of the strains.

Thus, although for small strains the two media are fundamentally different, as the strains become larger the conditions of the two classes of media approximate towards similarity, as regards the relation between stresses and strains; and thus the door opened to mechanical analysis by the recognition and analytical study of the property of dilatancy, as belonging to all media consisting of rigid discontinuous members, is not closed to the analysis of systems of mean and relative motion. So far from this being the case, the recognition of the coexistence of relative motion, by easing off the condition of absolute stability, belonging to the purely kinematical system, supplying as it were kinetic cushions at the corners, has removed difficulties which otherwise rendered analysis impossible.

92. The primary conclusion arrived at in this section, that the only media which, as purely mechanical resultant systems, can consist of component systems of mean and relative motion, are those which consist of discontinuous perfectly conducting members with some degree of freedom, while limiting, as already pointed out, the scope of the subsequent analysis necessary for the definite expression of the several rates of action resulting from convections in such media, also indicates the methods by which this analysis may be accomplished.

* *Phil. Mag.* Dec. 1885, "On the Dilatancy of Media composed of Rigid Particles in Contact."

Given the mean actions across the boundaries of any portion of the medium, the mean action of the grains enclosed is, at any instant, a mean function of the generalised ordinates which define the shapes, positions and dimensions of the members, the intervals of freedom, number of grains in unit volume, their velocities and their directions of motion.

Thus the method of analysis is to express the several probable mean rates of action, resulting from convection and conduction, in terms of the mean *vis viva* of relative velocity, the mean component-paths and mean paths, their number, mean-mass, and any other generalised mean ordinates that the shapes of the grains may entail. Then these expressions may be substituted in the members on the right of the equations, Section VI., since these include general expressions for the several actions.

The method thus indicated constitutes a general extension, or completion, of the method employed in the kinetic theory of gases.

SECTION VIII.

THE CONDUCTING PROPERTIES OF THE ABSOLUTELY RIGID GRANULE, ULTIMATE-ATOM OR PRIMORDIAN.

93. ALTHOUGH the absolutely rigid atom is as old as any conception in physical philosophy, the properties attributed to it are outside any experience derived from the properties of matter. In this respect, the perfect atom is in the same position, though in a different way, as that other physical conception—the perfect fluid. Both of these conceptions represent conditions to which matter, in one or other of its modes, apparently approximates, but to which, the results of all researches show, it can never attain, although this experience shows that there is still something beyond.

The analysis of the properties of conducting momentum, which must belong to the perfect atom considered as of uniform finite density, is obtained from the principle of conduction defined in Art. 72, Section VII.; from which it appears that it must conduct in all directions at an infinite rate, or that it must be capable of sustaining stress of infinite intensity, tension, compression or shearing; while it is shown that the property of conducting negative momentum in a positive direction or *vice versâ* requires that the momentum and the conduction shall be imaginary.

In the case of matter (rigid bodies) these imaginary stresses and rates of conduction are held to imply rates of actual conduction, round the outside of the bodies, in the medium of the ether. A conclusion confirmed in the case of matter by the existence of limits to the intensities of these stresses. Such outside conduction is at variance with the conception of fundamental atoms outside of which there is no conducting medium and which atoms do not possess the properties of changing their shapes or of separating into parts.

It becomes clear therefore that any fundamental atom must be considered as something outside—of another order than—material bodies, the properties of which are not to be considered as a consequence of the laws of motion and conservation of energy in the medium but as the prime cause of these laws.

94. If, for the sake of simplicity, the medium consist of closed spherical surfaces of equal radii $\sigma/2$ with the same internal constitution—anything or nothing—and the interstices between them are unoccupied; these surfaces having the property of maintaining their motions, uniform in direction and magnitude, across the intervals, and that of instantly reversing the components of their relative velocities in the directions to the surfaces at contact on encounter without having changed their shapes; such a medium, however far it might go to satisfy the kinematical conditions necessary for the physical properties of matter, would of necessity entail the laws of motion and the conservation of energy; and would thus constitute a purely mechanical medium in which the results would be the same whatever might be the constitution of the space within the surfaces.

The mean density in such a medium would be measured by the number (N) of closed surfaces divided by the space occupied. And the density within the surfaces would be the reciprocal of the volume enclosed ($\pi\sigma^3/6$).

Since each of the grains represents the same mass, this mass becomes the standard of mass; and being common to all the grains, is of no analytical importance.

In the same way σ, the diameter of the grains, becomes the standard of scale in the medium; and being the same for all the grains has no analytical importance.

It is, therefore, important and convenient, as adapting the notation to any arbitrary system of units, to define the mass of a grain in terms of the dimensions of the grains in the arbitrary units.

The most definite and convenient definition appears to be that which makes the mean density of the medium, when the grains are piled in their closest order, a maximum, that is when each grain has contact with twelve neighbours at the same time. In this way the mass of a grain is expressed by

$$\frac{\sigma^3}{\sqrt{2}},$$

where σ is the diameter of a grain expressed in arbitrary units.

Then if ρ'' expresses the mean density of the medium

$$\rho'' = \frac{N\sigma^3}{\sqrt{2}} \qquad \text{(129).}$$

And thus ρ'' becomes unity when the grains are in closest order.

SECTION IX.

THE PROBABLE ULTIMATE DISTRIBUTION OF VELOCITIES OF THE MEMBERS OF GRANULAR MEDIA AS THE RESULT OF ENCOUNTERS, WHEN THERE IS NO MEAN MOTION.

95. *Maxwell's Theory.*

Since the only action between elastic hard particles, as considered by Maxwell, is that of exchanging each other's relative motion in the direction of contact at the instant of contact, and the action of the grains, as defined in Section VIII., is identically the same, notwithstanding that it is not ascribed to elasticity, Maxwell's* proof of the law of probable distribution of velocities to which the action between the particles tends, applies equally to the grains. This law of Maxwell's is perfectly general and independent of all circumstances as to shape and size of the particles, and the extent of their freedoms, as long as there is freedom in all directions, and there is no distortional mean motion.

According to this law the mean of the energy, taken over limits of space, such as define the scale of the relative velocity of the motion in each degree of freedom, is the same for each and every degree of freedom, and is constant when equilibrium has been established. From this it follows that the time-mean of the energy of motion in each degree of freedom is the same, and is equal to the space-mean.

In the case of all the grains being similar and equal the mean component velocities positive or negative are the same, whether taken with respect to time, or to space. And when the grains differ the mean component velocities are inversely as the square roots of the masses.

This law of distribution, to which the relative-velocities, in any granular medium, tend when the mean motion ceases, being general requires no further exposition here.

* *Phil. Mag.* 1860, Part I., pp. 20—23, Props. I, II, III, IV.

In following up the consequences of the law, to which the mean component *vis viva* tends, on the mean distribution of the spheres, Maxwell, it appears, has tacitly introduced an assumption which, although legitimate in cases in which the diameters of the spheres are negligible as compared with the mean-paths of the spheres between encounter, has completely obscured the fact that the mean arrangement of the grains does not depend solely on fulfilment of the law of distribution of the *vis viva*; but also depends on the hindrance which the surrounding grains may offer to the enclosed grain in changing its neighbours.

When the grains are small compared with spaces separating them this hindrance becomes negligibly small. And, further, whatever effect it might have is entirely dependent on the conduction through the grains; so that the neglect of the displacement of momentum by conduction renders any account of such mutual constraints which the grains may impose on each other futile.

It now appears, however, that taking account of the conditions, we have in these a class of actions which, however insignificant they may be when the density is small, entirely dominate all other actions when the density approaches maximum density. And it thus becomes evident that the failure of the kinetic theory, as applied to gases, to apply to the liquid and solid states of matter is owing to this tacit assumption that the distribution of the mass depends only on the action which secures that the distribution of *vis viva* shall approach that of uniform angular dispersion as the medium approaches a state of equilibrium.

It will thus be seen, that accepting Maxwell's law of probable distribution of *vis viva*, it still remains necessary for the purpose of definite analysis, to define the limits of its consequences on the probable arrangement of the grains, *i.e.* of mass.

96. *Maxwell's law of probable distribution of vis viva is independent of equality in the lengths of the mean paths.*

This is founded on the demonstration (1) that when two elastic spheres, having relative-velocities in any particular direction, undergo chance encounter, all directions of subsequent relative-motion are equally probable, and (2) the demonstration that whatever may be the shape of the elastic bodies the same law holds, as to the linear velocity, and is further extended to their rotational motions. As consideration here is confined to the case of smooth spheres it is sufficient to take into account the first case only.

The most general expression of this law for uniform grains is, taking x, y, z to represent the component velocities of grains in the directions x, y, z respectively, and N for the number of grains in unit space, the numbers of grains which have component velocities which, respectively, lie between $x + \delta x$, $y + \delta y$, $z + \delta z$, are

$$\delta N = \frac{N}{\alpha^3 (\pi)^{\frac{3}{2}}} e^{-\frac{\{x^2+y^2+z^2\}}{\alpha^2}} \delta x\, \delta y\, \delta z \quad \text{.................(130).}$$

From this definite expression of the law it will be seen that it is confined to direction only and would apply equally to cases where in some directions the grains were making short paths and in others long paths, as well as to that in which the mean paths are equal in all directions. Q. E. D.

97. *The distribution of mean and relative velocities of pairs of grains.*

In Proposition V. of the same paper Maxwell extended the law of probable distribution of *vis viva* to the distribution of the relative *vis viva* of all pairs of grains. He does not seem, however, to have further extended it to that of the mean motions of the pairs; which is remarkable as it appears to follow directly from his method and would have saved him much subsequent trouble.

These extensions do not in the least involve the arrangement of the grains. It is however convenient to introduce the demonstration of the law of distribution of the mean-velocities here, for the purpose of reference, and it is simpler to demonstrate both at the same time.

Taking $\bar{x}$, $\bar{y}$, $\bar{z}$ as the components of the mean-velocity of a pair of grains and x', y', z' as the relative components of the same pair, and x_1, y_1, z_1, x_2, y_2, z_2 as the components of the individual motions, we have

$$x_1 = \bar{x} + x', \quad y_1 = \bar{y} + y', \quad z_1 = \bar{z} + z',$$
$$x_2 = \bar{x} - x', \quad y_2 = \bar{y} - y', \quad z_2 = \bar{z} - z'.$$

Then for the numbers of grains for which x_1 is between x_1 and $x_1 + \delta x_1$, y_1 between y_1 and $y_1 + \delta y_1$, z_1 between z_1 and $z_1 + \delta z_1$, and x_2 is between x_2 and $x_2 + \delta x_2$, &c., &c.

$$\left.\begin{aligned} n_1 &= \frac{N_1}{\alpha^3(\pi)^{\frac{3}{2}}} e^{-\left\{\frac{(\bar{x}+x')^2}{\alpha^2}+\frac{(\bar{y}+y')^2}{\alpha^2}+\frac{(\bar{z}+z')^2}{\alpha^2}\right\}} d\bar{x}\, d\bar{y}\, d\bar{z} \\ n_2 &= \frac{N_2}{\alpha^3(\pi)^{\frac{3}{2}}} e^{-\left\{\frac{(\bar{x}-x')^2}{\alpha^2}+\frac{(\bar{y}-y')^2}{\alpha^2}+\frac{(\bar{z}-z')^2}{\alpha^2}\right\}} dx'\, dy'\, dz' \end{aligned}\right\} \quad \text{.........(131).}$$

The first of these equations expresses the probable number of grains having mean-velocities between $\bar{x}$ and $\bar{x} + \delta\bar{x}$, &c., &c., for any particular value of x', the relative-velocity, &c., &c.

And the second equation in the same way expresses the number of grains having relative-velocities between x' and $x' + \delta x'$, &c., &c., for any value of $\bar{x}$, &c., &c. Whence the probability of the double event is expressed by the product

$$n_1 n_2 = \frac{N_1 N_2}{\alpha^6 \pi^3} e^{-\frac{1}{\alpha^2}\{(\bar{x}+x')^2+(\bar{x}-x')^2+(\bar{y}+y')^2+(\bar{y}-y')^2+(\bar{z}+z')^2+(\bar{z}-z')^2\}} d\bar{x}\, dx'\, d\bar{y}\, dy'\, d\bar{z}\, dz'$$

$$\text{.........(132).}$$

Then if $\bar{r} = \bar{x}^2 + \bar{y}^2 + \bar{z}^2$ and $r' = x'^2 + y'^2 + z'^2$, the number of pairs having mean-velocities between $\bar{r}$ and $r + \delta r$ and relative velocities between r' and $r + \delta r$ is

$$n_1 n_2 = \frac{N_1 N_2}{\alpha^6 \pi^3} e^{-2(\bar{r}^2 + r'^2)}\, d\bar{x}\, d\bar{y}\, d\bar{z}\, dx'\, dy'\, dz' \quad \text{...........(133)}.$$

These admit of integration either with respect to $\bar{x}$, $\bar{y}$, $\bar{z}$, or x', y', z'.

Thus integrating $\bar{x}$, $\bar{y}$, $\bar{z}$ from $\bar{x} = -\infty$ to $\bar{x} = \infty$ we find

$$\frac{N_1 N_2}{(\sqrt{2}\alpha)^3 (\pi)^{\frac{3}{2}}} e^{-\frac{r'^2}{2\alpha^2}} dx'\, dy'\, dz' \quad \text{.....................(134)}$$

for the whole number of pairs whose components of relative velocities are between x' and $x' + \delta x'$, y' and $y' + \delta y'$, z' and $z' + \delta z'$. And integrating for r' instead of $\bar{r}$ we find

$$\frac{N_1 N_2}{(\alpha/\sqrt{2})^3 (\pi)^{\frac{3}{2}}} e^{-\frac{2r^2}{\alpha^2}} d\bar{x}\, d\bar{y}\, d\bar{z} \quad \text{........................(135)}$$

for the number of pairs whose mean components of velocity are between $\bar{x}$ and $\bar{x} + \delta\bar{x}'$, &c., &c.

These may be expressed in a more convenient form by substituting $-r^3 d\cos\theta\, d\phi$ for dx, dy, dz.

And applying this to the three expressions for the number—

of grains having velocities between r and $r + \delta r$,

of pairs having relative-velocities between $\sqrt{2}r$ and $\sqrt{2}\,(r + \delta r)$,

of pairs having mean velocities between $r/\sqrt{2}$ and $(r + \delta r)/\sqrt{2}$,

since N is the number of grains in unit volume and $N(N-1)$ is the number of pairs of grains,

$$\frac{N4\,(r)^2}{\alpha^3 \sqrt{\pi}} e^{-\frac{r^2}{\alpha^2}} \delta r = n_1 \quad \text{...........................(136)},$$

$$\frac{(N-1)\, N4\, (\sqrt{2}r)^2}{(\sqrt{2}\,\alpha)^3 \sqrt{\pi}} e^{-\frac{(\sqrt{2}r)^2}{2\alpha^2}} \sqrt{2}\,\delta r = (N-1)\, n_1 \quad \text{.........(137)},$$

$$\frac{(N-1)\, N4\, (r/\sqrt{2})^2}{(\alpha/\sqrt{2})^3 \sqrt{\pi}} e^{-\frac{(r/\sqrt{2})^2}{\alpha^2/2}} \delta r/\sqrt{2} = (N-1)\, n_1 \quad \text{.........(138)}.$$

Q. E. D.

The first and second of these laws of angular distribution of *vis viva* are the same as those given by Maxwell; and the third, that for the distribution of the mean *vis viva* of pairs of grains, leads to the same results as Maxwell arrived at in a different manner. Together they constitute the principal means of giving definite quantitative expression to the results of the analysis of the actions in a granular medium. And it is important to notice that they are derived from the probable independence of the preceding and antecedent directions of the relative velocities of a pair of grains before and after encounter under conditions in which the mean density and constitution of the medium remain unaltered.

In Proposition VI. Maxwell has shown the rates at which the several members of the medium exchange *vis viva*, using arbitrary constants. And in his Proposition VII. he proceeds to the demonstration of the probable length of the path of a grain in terms of N, the number of grains in unit volume, s the diameter of a grain, and v the velocity. He has first shown that if r is the relative velocity of a particle with respect to N particles in unit volume, this particle will approach within the distance s of $N\pi rs^2$ particles in a unit of time.

Thus in Propositions VIII. and IX. he determines the number of pairs moving according to the laws expressed in equations (137) and (138) which will undergo encounters in a unit of time, and in Proposition X. determines the mean path of a particle to be

$$l = \frac{1}{N\sqrt{2}\pi s^2}.$$

In this result there are two things to be noticed.

In the first place the πs^2 in the denominator represents the area of the target exposed to the centre of a spherical grain by another grain in the direction of their relative motion; while the $\sqrt{2}$ is merely the ratio of the mean relative velocity of the pair to the mean velocity of either grain, equations (136), (137). It is thus seen that, although the dimensions of the grain are, perforce, taken into account as determining the probability of an encounter, no account is taken of the third dimension of the grain in diminishing the actual distance the centres of the grains would travel between encounters. Hence Maxwell's mean path l can only be an approximation when his s is small with respect to his l.

The second point to be noticed in Maxwell's deduction of the mean path is that he has tacitly assumed l to be the same in all directions. And has thus assumed not only that the density is constant, which is assumed in the determination of his laws of distribution of *vis viva*, but also that the arrangements of the particles must be such that the mean chance of encounter is equal in all directions, a condition which does not enter into the laws of distribution of *vis viva*, and consequently limits the application of this mean path to conditions of the medium such that all directions afford equal chance of encounter. A condition which is obviously approximated to as the actual density becomes small compared with the maximum density, when each particle is in continuous contact with twelve neighbours.

98. In pointing out the limits to the application of Maxwell's analysis of the action in a medium of hard elastic spheres, my chief object has been to direct attention to those extensions and modifications which are necessary to render the analysis general, and thus to present a clear idea as to how far Maxwell's method may be applied. At the same time it seemed very desirable to show clearly, that in extending the analysis to include conditions of the medium to which Maxwell had not applied his method, there is nothing at variance with the results he had obtained under the condition to which his application of this method extended.

Maxwell's laws of the probable distribution of *vis viva*, and mass, extended to include the mean *vis viva* of pairs of grains, are, as already pointed out, perfectly general.

But it is necessary to obtain expressions in terms of the quantities which define the relative motions of the medium for the *rates* at which the actions of conduction through the grains displace momenta and *vis viva* of relative motion, which expressions shall, if possible, be as general as the law of distribution of *vis viva*.

In the media considered by Maxwell the distances between the grains are assumed to be large compared with the dimensions of the grains. Whereas in the general theory it is fundamental that cases should be considered in which the distances between the centres of the grains, which are neighbours, approach indefinitely near to the linear dimensions of the grains.

Such consideration involves methods of analysis by which the several effects of the action between the grains may be defined whatever may be the relation between σ the diameters of the grains and λ their mean path.

In the first instance the consideration of these rates is confined to states of the media in which, whatever may be the density as compared with the possible density, the arrangements of the grains, however varying, are such that the mean actions in every direction are similar and equal; the medium being everywhere in mean equilibrium. And afterwards to proceed to the effects of inequalities both angular and linear.

SECTION X.

EXTENSION OF THE KINETIC THEORY TO INCLUDE PROBABLE RATES OF CONDUCTION THROUGH THE GRAINS, WHEN THE MEDIUM IS IN ULTIMATE CONDITION AND IS UNDER NO MEAN STRAIN.

99. THE mean rates of convection and conduction of momentum, expressed in equations (120) by p_{xx}, p_{yx}, &c., and $\rho''(u'u')''$, $\rho''(v'u')''$, &c., admit of expression as

$$p + p_{xx} - p,\ p_{yx},\ \text{\&c.};\ \tfrac{1}{3}\rho''(v'v')'' + \rho''(u'u')'' - \tfrac{1}{3}\rho''(v'v')'',\ \rho''(v'u')'',\ \text{\&c.},$$

where $\quad p = \frac{1}{3}(p_{xx} + p_{yy} + p_{zz}),\ \rho''(v'v')'' = \rho''(u'u' + v'v' + w'w')''$

and in this case p and $\frac{1}{3}\rho''(v'v')''$ represent the mean action, equal in all directions, while $p_{xx} - p$, $\rho''(u'u')'' - \frac{1}{3}\rho''(v'v')''$ &c., p_{yx}, &c. and $\rho''(v'u')''$ represent inequalities.

In this first extension of the kinetic theory the object is to express the actions indicated by p and $\rho''(v'v')''$ only, assuming that the inequalities are zero, in terms of the quantities which define the condition of the medium.

100. *To determine the mean path of a grain.*

The mean path of a grain expressed by λ is the distance traversed by its centre between encounters, which is not the component in the direction of its motion, of its distance between the points at which the two actual contacts, which limit the path, have occurred, although it approximates to this as λ/σ becomes large.

Maxwell has shown that neglecting σ/λ the mean path of a grain and the relative path of a pair of grains are expressed by

$$\lambda = \frac{1}{\sqrt{2}\pi\sigma^2 N} \text{ and } \sqrt{2}\lambda = \frac{1}{\pi\sigma^2 N} \quad\ldots\ldots\ldots\ldots\ldots\ldots(139)$$

respectively, while both of these are obtained from

$$\sqrt{2}\pi\lambda\sigma^2 = \frac{1}{N} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(140),$$

where N expresses the number of grains in unit volume; so that either member represents the mean volume maintained free from other grains by the kinetic action of each grain.

In this estimate however no account is taken of the striking distance, of the centres of the pair of grains, from the plane, normal to their relative paths before contact, through the point of contact, so that the centres of both grains are assumed to be in this plane at the instant of contact.

When λ/σ is large we have all positions of the projection, in the direction of relative motion of the striking grains, over the disc $\pi\sigma^2/4$, equally probable, and then the probable mean relative striking distance in the direction of relative motion is

$$\frac{2}{3}\sigma.$$

This is a relative distance and the corresponding actual extension of their actual paths is, by equations (136) and (137),

$$\frac{\sqrt{2}}{3}\sigma.$$

101. The assumption that all positions of the projection, in the direction of relative motion, of the striking grains are equally probable over the disc area $\pi\sigma^2/4$ is obviously legitimate when λ is large compared with σ, and hence these estimates of the probable mean striking distance when λ/σ is large are precisely on the same footing as Maxwell's estimate of the mean path neglecting σ/λ. But there does not seem to be the same ground for this assumption when σ/λ is large; while, on the other hand, there is evidence, as pointed out in Section VII. (Arts. 88 and 89), that, when the grains are close, the normals at encounter fall into line (approximately) with the direction of a finite number of axes, fixed in space, not more than six.

In this article the arrangement of the grains is assumed to be similar in all directions; so that, whatever may be the law of distribution of the projections of encounters on the disc-area, the probability will be equal in all directions at equal distances from the centre of the disc.

Therefore taking θ, as before, for the angular distance from the axis of the disc at which the normal at encounter meets the hemisphere of unit radius, the law of radial distribution on the disc may be expressed by a function of $\cos\theta$, which function will depend only on the ratio σ/λ. Thus as a general expression for the probable mean striking distance we have

$$\frac{2\pi\sigma\int_0^{\frac{\pi}{2}}\cos\theta\,(1+A_1\cos\theta+\&c.)\sin\theta\,d\sin\theta}{2\pi\int_0^{\frac{\pi}{2}}(1+A_1\cos\theta)\sin\theta\,d\sin\theta}=\frac{2}{3}\sigma\,.f\left(\frac{\sigma}{\lambda}\right)\ \ldots\ (141),$$

in which A_1 &c. are functions of σ/λ only; and as the law of radial distribution of the striking distance is perfectly general we have in the right member a perfectly general expression for the mean relative striking distance of a pair of grains in the direction of their relative motion. And dividing this by $\sqrt{2}$ we have for the mean probable actual striking distance of a grain

$$\frac{\sqrt{2}}{3}\sigma f\left(\frac{\sigma}{\lambda}\right).$$

Thus as a general expression for the mean path of a grain we have

$$\left.\begin{array}{l} \lambda = \dfrac{1}{\sqrt{2}}\left\{\dfrac{1}{\pi\sigma^2 N} - \dfrac{2}{3}\sigma f\left(\dfrac{\sigma}{\lambda}\right)\right\}, \\ \text{and for the volume maintained by a grain} \\ \sqrt{2}\pi\left\{\lambda\sigma^2 + \dfrac{\sqrt{2}}{3}\sigma^3 f\left(\dfrac{\sigma}{\lambda}\right)\right\} = \dfrac{1}{N}. \end{array}\right\} \quad \text{..............(142).}$$

102. *Further definition of* $f(\sigma/\lambda)$.

Since the foregoing expression for the volume from which a grain excludes other grains applies to all conditions of the medium it must include the case in which λ is indefinitely small; in which case, if the medium is in uniform condition with three perpendicular axes of similar arrangement, the unique condition is that in which the volume maintained by each grain approximates to $\sigma^3/\sqrt{2}$, as explained in Section IX., each grain being in contact with 12 neighbours. In this case N approximates to $\sqrt{2}/\sigma^3$ which is the reciprocal of the volume maintained by the grain, which thus approximates to the volume of the spherical grain multiplied by $6/\sqrt{2}\pi$. Substituting this for the right member of the second equation (142) we have for the limit when σ/λ is large

$$f\left(\frac{\sigma}{\lambda}\right) = \frac{1}{4}\frac{6}{\sqrt{2}\pi} \quad \text{..........................(143).}$$

Then, again, if λ/σ is large the value to which $f(\sigma/\lambda)$ approximates is unity. Whence for an expression satisfying all cases in a uniform medium with three axes of similar arrangement it appears that we may take

$$\left.\begin{array}{l} f\left(\dfrac{\sigma}{\lambda}\right) = 1 - a^2 e^{-b^2\frac{\lambda}{\sigma}} \\ \text{where}\quad a^2 = 1 - 6/4\sqrt{2}\pi \quad \text{and } b^2 \text{ is arbitrary} \end{array}\right\} \text{........................(144).}$$

It is convenient however to render the expression for this function a little more general, since in a granular medium although generally in uniform condition, with three axes of similar arrangement, there may exist localities where the arrangements vary about local centres; the medium being still in equilibrium and λ/σ being small. Under such conditions the limits of variation are defined by the fact that equilibrium requires that each grain shall be in approximate contact with at least four grains. And it seems that these may be included by substituting $1-G/4$, where G has the value $6/\sqrt{2}\pi$ when the medium is in uniform condition, and values ranging to the limit $18/4\sqrt{2}\pi$ when the medium is in varying condition, as about centres of disarrangement, instead of $6/4\sqrt{2}\,\pi$ in a^2. Then

$$f\left(\frac{\sigma}{\lambda}\right)=1-\left(1-\frac{G}{4}\right)e^{-b^2\frac{\lambda}{\sigma}} \quad\text{.....................}(145).$$

By definition (Section IX.) $\rho=N\sigma^3/\sqrt{2}$, and by the second equation (142)

$$\frac{\sigma}{\lambda}=\frac{2\pi\rho}{1-2\pi\frac{\sqrt{2}}{3}f\left(\frac{\sigma}{\lambda}\right)\rho} \quad\text{.....................}(146).$$

103. In order to render the expressions for the mean relative-path of a pair of grains and the mean path of a grain, taking account of the three dimensions of the grains, general and complete, use has been made, equation (139) in Art. 100, of the ratio $(1/\sqrt{2})$ of the mean path of the grain to the mean relative-path of a pair of grains as determined by Maxwell for conditions in which the third dimension is negligible.

The legitimacy of this assumption therefore remains to be proved. But before proceeding to the proof of this proposition the proofs of two other geometrical propositions are desirable, as they depend directly on the law of distribution of the component-striking distance over the area of the normal disc.

104. The first of these propositions is:

When a pair of grains having any particular relative velocity ($\sqrt{2}\,V_1'$), all directions being equally probable, undergo chance encounter, the probable mean product of the displacement of momentum, in the direction of the normal at encounter, by conduction, multiplied by the component of $\sqrt{2}\,V_1'$ in the direction of the normal is

$$\tfrac{2}{3}\sqrt{2}\,V_1'\sigma f\left(\frac{\sigma}{\lambda}\right).$$

To prove this, let χ be the acute angle between two diameters drawn through the centre of a sphere of unit radius in the directions of the normal at contact and that of the relative motion before contact, and let ω be any small area on the surface of the sphere taken so that its mean position is at the point in which the diameter in the direction of the normal meets the surface of the sphere.

Then by the law of probability of the striking distance it follows that, at a chance encounter, the probability of the normal meeting the surface in ω is

$$\frac{\omega\cos\chi\,(1-A_1\cos\chi+\&c.)}{\pi},$$

or multiplying this probability by the product of the normal component of the relative velocity $\sqrt{2}\, V_1' \cos\chi$, and again by σ, the normal displacement, integrating over the hemisphere for all values of χ, and dividing this integral by the integral of the probability of an encounter on ω for all values of χ over the hemisphere, we have for the probable mean product of the normal component of relative velocity multiplied by the displacement

$$\frac{2\pi \int_0^{\frac{\pi}{2}} \frac{\omega}{\pi} \sqrt{2}\, \sigma V_1' \cos\chi\, (1 - A_1 \cos\chi + \&c.) \sin\chi\, d \sin\chi}{2\pi \int_0^{\frac{\pi}{2}} \frac{\omega}{\pi} \cos\chi\, (1 - A_1 \cos\chi + \&c.) \sin\chi\, d\chi} = \frac{\sqrt{2}}{3} \sigma 2 V_1' f\left(\frac{\sigma}{\lambda}\right) \ldots (147).$$

Q. E. D.

105. The second of the two geometrical propositions is:

The probable mean component conduction of component momentum in any fixed direction at a single collision is

$$\frac{2}{3} \frac{\sigma^3}{\sqrt{2}} \left[\sqrt{2}\, \sigma \frac{V_1'}{3} f\left(\frac{\sigma}{\lambda}\right)\right].$$

To prove this we have to multiply the mean product of normal displacement multiplied by the component of the relative velocity by $(\sigma^3/\sqrt{2})$ the mass of a grain; thus obtaining the expression for the mean displacement, in the direction of the normal at encounter, of momentum at a single encounter, as

$$\frac{\sigma^3}{\sqrt{2}} \left[\frac{2}{3} \sigma \sqrt{2}\, V f\left(\frac{\sigma}{\lambda}\right)\right].$$

Then, taking θ as the angle which the direction of the normal makes with any fixed direction, say that in which χ is measured, and resolving the normal displacement σ and the mean normal component of V in the direction of χ, multiplying by $\sin\theta d\theta$, integrating over the sphere and dividing by 4π,

$$\frac{\frac{\sigma^3}{\sqrt{2}} 2\pi \frac{2}{3} \sigma \sqrt{2}\, V_1' f\left(\frac{\sigma}{\lambda}\right) \int_0^{\frac{\pi}{2}} \cos^2\theta \sin\theta\, d\theta}{4\pi} = \frac{2}{3} \frac{\sigma^3}{\sqrt{2}} \frac{\sqrt{2}}{3} \sigma V_1' f\left(\frac{\sigma}{\lambda}\right) \ldots (148).$$

Q. E. D.

This expression for the probable mean-component conduction at a single encounter is one of the factors of the rate of component conduction by pairs of grains having particular relative velocity $\sqrt{2}\, V_1'$, the other factor being the number of collisions that take place between such pairs in unit space in unit time.

This second factor involves the discussion of the ratio of the mean path to that of the relative path of a pair of grains.

106. *The number of collisions between pairs of grains, having particular relative velocities, in unit of time, in unit space.*

Taking N for the number of grains in unit space and substituting V_1' for r in the equations (136), (137), (138), Section IX., we have for the numbers of grains having velocities between V_1' and $V_1' + \delta V_1'$

$$\frac{N 4 (V_1')^2 e^{-\frac{(V_1')^2}{\alpha^2}}}{\alpha^3 \sqrt{\pi}} d V_1' = n_1 \quad \text{......................(149)},$$

for the number of pairs of grains having relative-velocities between $\sqrt{2}\, V_1'$ and $\sqrt{2}\,(V_1' + \delta V_1')$

$$\frac{N(N-1)\, 4 (\sqrt{2}\, V_1')^2}{(\sqrt{2}\, \alpha)^3 \sqrt{\pi}} e^{-\frac{(\sqrt{2}\, V_1')^2}{2\alpha^2}} \sqrt{2}\, d V_1' = (N-1)\, n_1 \quad \text{......(150)},$$

and for the number of pairs of grains having mean-velocities between $V_1'/\sqrt{2}$ and $(V_1' + \delta V_1')/\sqrt{2}$,

$$\frac{N(N-1)\, 4 (V_1'/\sqrt{2})^2}{(\alpha/\sqrt{2})^3 \sqrt{\pi}} e^{-\frac{(V_1'/\sqrt{2})^2}{\alpha^2/\sqrt{2}}} d V_1'/\sqrt{2} = (N-1)\, n_1 \quad \text{......(151)}.$$

107. From the equations of distribution of velocities, relative-velocities, and mean-velocities amongst the grains and pairs of grains in unit volume, it follows that the proportion of the N grains having velocities between V_1' and $V_1' + \delta V_1'$ is the same as the proportion of the $N(N-1)$ pairs of grains having relative-velocities between $\sqrt{2}\, V_1'$ and $\sqrt{2}\,(V_1' + \delta V_1')$ as well as the proportion of $N(N-1)$ pairs having mean-velocities between $V_1'/\sqrt{2}$ and $(V_1' + \delta V_1')/\sqrt{2}$, since for every one of the grains having velocities between V_1' and $V_1' + \delta V_1'$ there are $(N-1)$ pairs of grains having relative-velocities between $\sqrt{2}\, V_1'$ and $\sqrt{2}\,(V_1' + \delta V_1')$ and $(N-1)$ pairs having mean-velocities between $V_1'/\sqrt{2}$ and $(V_1' + \delta V_1')/\sqrt{2}$.

Multiplying the equations (136), (137), (138) respectively by V_1', $\sqrt{2}\, V_1'$, and $V_1'/\sqrt{2}$ respectively, and integrating from $V_1' = 0$ to $V_1' = \infty$, we have for the mean velocity of grains, the mean relative-velocity of pairs of grains, and the mean mean-velocity of pairs of grains,

$$(V_1')'' = \frac{2\alpha}{\sqrt{\pi}}, \qquad \sqrt{2}\,(V_1')'' = \frac{2\sqrt{2}\, \alpha}{\sqrt{\pi}} \quad \text{and} \quad (V_1')''/\sqrt{2} = \frac{\sqrt{2}\, \alpha}{\sqrt{\pi}} \text{...(152)}.$$

And as the grains are of equal mass the relative velocity of each grain in a pair is half the relative velocity of the pair; so that the mean relative velocity of each grain in the pairs is

$$\frac{(V_1')''}{\sqrt{2}} = \frac{\sqrt{2}\, \alpha}{\sqrt{\pi}} \quad \text{..............................(153)}.$$

108. *To find the mean path of the grains, taking $\sqrt{2}\lambda$ for the mean path of the pairs.*

Each grain has at any instant $N-1$ relative paths with the $N-1$ other grains in unit volume, and $N-1$ relative velocities, so that the N grains have in all $N(N-1)$ relative paths and $N(N-1)$ relative velocities.

A change in the actual velocity of any one grain causes a change in the relative velocity of each of the $N-1$ pairs of which it is a member. And as at an encounter between the members of a pair two grains change their actual velocities, there are $2(N-1)$ changes at each collision in the $N(N-1)$ relative velocities of the pairs in unit volume. The mean relative path of a pair of grains between changes being by definition $\sqrt{2}\lambda$, the mean relative path of a grain is $\lambda/\sqrt{2}$. And considering a particular pair of grains, their paths and velocities relative to each other, though continually changing, are always parallel and equal, so that the distances relative to each other traversed by each of the grains in unit of time have a mean value $(V_1')''/\sqrt{2}$, and the mean number of changes of relative path and velocity in unit of time is

$$\frac{(V')''/\sqrt{2}}{\lambda/\sqrt{2}}=\frac{(V')''}{\lambda}.$$

Whence the number of changes in all the relative paths of all the grains is $N(N-1)(V')''/\lambda$; and since there are $2(N-1)$ changes for each collision the number of collisions in unit volume in unit time is

$$\frac{N}{2}\frac{(V')''}{\lambda}.$$

Having thus found the number of collisions between the N grains in unit volume in unit of time, since there are two grains engaged in each collision the total number of encounters made by all the individual grains in a unit of volume in a unit of time is twice the number of collisions: that is

$$\frac{N(V')''}{\lambda}.$$

Therefore the mean number of paths traversed by each grain in unit time is

$$\frac{(V')''}{\lambda}.$$

Then since $(V')''$ is the mean distance traversed by a grain in unit time, dividing by the number of encounters the mean path is

$$\frac{V''}{V''}\lambda=\lambda \qquad\qquad (154).$$

Therefore if $\sqrt{2}\,\lambda$ is the mean relative path of pairs of grains, λ is the mean path of a grain. It also appears that the mean number of collisions in unit of time in unit volume is

$$\frac{N}{2}\frac{(V')''}{\lambda} = \frac{N}{\lambda}\cdot\frac{\alpha}{\sqrt{\pi}} \qquad \text{(155).}$$

And the mean number of grains a grain encounters in unit time is

$$\frac{(V')''}{\lambda} = \frac{2\alpha}{\sqrt{\pi}\,.\,\lambda} \qquad \text{(156).}$$

109. *The mean path of a pair of grains.*

This follows directly from the last proposition. For as the number of mean paths of pairs of grains is identical with the number of relative paths of pairs, and the mean velocities of pairs is one-half their relative velocities, the mean paths of the pairs must be one-half the mean relative path of the pairs, that is, must be equal to the mean relative path of each grain of the pair, or

$$\frac{\lambda}{\sqrt{2}}.$$

110. *The number of collisions of pairs of grains having relative velocities between $\sqrt{2}\,V_1'$ and $\sqrt{2}\,(V_1'+dV_1')$.*

Since the mean relative distance traversed between changes by a pair of grains irrespective of relative velocity is $\sqrt{2}\,\lambda$, the mean time of a pair of grains having relative velocity $\sqrt{2}\,V_1'$ in traversing their mean path ($\sqrt{2}\,\lambda$) is λ/V_1'.

Then since the number of pairs of grains in unit volume having relative velocities between $\sqrt{2}\,V_1'$ and $\sqrt{2}\,(V_1'+dV_1')$ is $N(N-1)$, and each of these pairs changes V_1'/λ times in unit time, the total number of changes of these pairs in unit of time is

$$n\,(N-1)\,\frac{V_1'}{\lambda}.$$

And since there are $2\,(N-1)$ changes for each collision, we have for the numbers of collisions of the $n\,(N-1)$ pairs of grains in unit of time, equation (148),

$$\frac{n_1 V_1'}{2\lambda} = \frac{N}{2}\,\frac{V_1'}{\lambda}\,\frac{4\,(V_1')^2}{\alpha^3\sqrt{\pi}}\,e^{-\frac{(V_1')^2}{\alpha^2}}\,.\,dV_1' \qquad \text{(157).}$$

The integral of this from $V_1'=0$ to $V_1'=\infty$ gives the number of collisions of the N grains in unit time.

111. *The mean rate of conduction of component momentum in the direction of the momentum conducted.* Cases 1 and 2.

Multiplying the probable mean component conduction from a mean collision of a pair with relative velocity $\sqrt{2}\ V_1'$, equation (136), by the number of collisions in a unit of time, equation (157), and integrating V_1' between the limits $V_1' = 0$ to $V_1' = \infty$ we have for the mean rate of conduction

$$\rho \cdot \frac{\sqrt{2}}{3} \frac{\sigma}{\lambda} f\left(\frac{\sigma}{\lambda}\right) \frac{4\,(V_1')^4}{\alpha^3 \sqrt{\pi}} \cdot e^{-\frac{(V_1')^2}{\alpha}} \cdot dV' \int_0^{\frac{\pi}{2}} \frac{1}{6}\, d\,(-\cos^3\theta) = \rho \cdot \frac{\sqrt{2}}{3} \frac{\sigma}{\lambda} \cdot f\left(\frac{\sigma}{\lambda}\right) \frac{(V'V')''}{3} \quad \text{.........(158)},$$

whence since $(V'V')'' = 3\,(U'U')''$

$$\rho \cdot \frac{\sqrt{2}}{3} \frac{\sigma}{\lambda} f\left(\frac{\sigma}{\lambda}\right) (U'U')'' = p_{xx}'', \text{ \&c., \&c.} \quad \text{............(159)}.$$

112. The left members of equation (159) express in terms of the quantities which define the relative motion of the medium, the mean normal stresses, or the mean rates of conduction of momentum, in the direction of the momentum conducted. And besides these there are the mean tangential stresses, or rates of conduction in directions at right angles to the direction of the momentum conducted.

These rates are obtained by substituting in equation (158), for $\cos^3\theta$, &c., &c., $\cos\theta \sin\theta \cos\phi$, &c., which when integrated over the surface of a hemisphere are zero, if all directions of relative motion are equally probable, but have values in a medium with linear inequalities when the axes of reference are other than the principal axes of the inequalities.

It is therefore necessary to obtain their integral values over the several groups of pairs having relative velocities in directions in which the sign of the component displacement is the same as that of the component of normal velocity, as

$$\frac{\frac{2}{3} \frac{\sigma}{\lambda} f\left(\frac{\sigma}{\lambda}\right) \sqrt{2}\ V_1' \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \cos\phi \sin\theta\, d\theta\, d\phi}{\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta\, d\phi} = \frac{4}{3} \frac{\sigma}{\lambda} f\left(\frac{\sigma}{\lambda}\right) \frac{\sqrt{2}\ V_1'}{3\pi} \quad \text{...(160)},$$

which multiplied by the mass and the number of collisions and taking the mean is

$$\frac{4}{12} \rho \frac{\sigma}{\sqrt{2}\,\lambda} f\left(\frac{\sigma}{\lambda}\right) \frac{(V'V')''}{3\pi}, \ -\text{\&c.}, \ +\text{\&c.}, \ -\text{\&c.} \quad \text{......(161)},$$

so that to each of these groups of pairs there is a corresponding group for which the normal components of mean-relative motions are of opposite sign, the mean taken over the two groups or over the whole unit sphere is zero; so that in a medium without linear inequalities

$$p_{xy}'' = 0, \text{ \&c., \&c.} \quad \text{..........................(162)}.$$

113. *The mean rate of convection of components of momentum in the direction x by grains having velocities V_1', for which all directions are equally probable, is expressed by*

$$\frac{2\pi\rho\int_0^\pi \lambda V_1' \frac{V_1'}{\lambda}\cos^2\theta\sin\theta\, d\theta}{2\pi\rho\int_0^\pi \sin\theta d\theta} = \frac{V_1'^2}{3} \quad\text{..............(163)},$$

which becomes, taking Maxwell's expression for the mean value of v^2 from 0 to ∞, $(\alpha^2 . \frac{3}{2})$, when multiplied by the product of the mass into the number of grains,

$$\frac{1}{3} N \frac{\sigma^3}{\sqrt{2}} (V_1' V_1')'' = \rho \frac{\alpha^2}{2} \quad\text{........................(164)}.$$

And for the mean rate of momentum conveyed in the direction of the momentum

$$\rho'' \frac{\alpha^2}{2} = \rho' (U'U')'', \text{ \&c., \&c.} \quad\text{.....................(165)}.$$

For the lateral convections of momentum the expression is

$$\frac{\frac{1}{8}\rho\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \lambda \frac{(V_1'V_1')''}{\lambda} \cos\theta \sin^2\theta d\theta \sin\phi d\phi}{\frac{1}{8}\rho\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin\theta d\theta d\phi} = \frac{\alpha^2}{\pi}, - \text{\&c.}, + \text{\&c.}, - \text{\&c.} \quad\text{.......(166)},$$

where the integration extends, as in the case of lateral conduction, over groups of grains of which the directions are such that $\cos\theta$, $\sin\theta$, $\cos\phi$, &c. have the same signs, positive or negative. The groups in which the corresponding signs are opposite have integrals with the opposite signs negative or positive, so that for the complete integrals

$$\rho'' (V'W')'' = 0, \text{ \&c., \&c.} \quad\text{.....................(167)}.$$

114. *The total rates of displacement of mean-momentum in a uniform medium.*

Adding the expressions for the rates of conduction and convection in the respective members of equations (159) and (165), also (162) and (167), we obtain for the whole rates of displacement of the components of momentum

$$\left.\begin{aligned} p_{xx}'' + \rho''(U'U')'' &= \rho''\left\{1 + \frac{2}{3}\frac{\sigma}{\sqrt{2}\lambda} f\left(\frac{\sigma}{\lambda}\right)\right\}(U'U')'', \text{ \&c. \&c.} \\ p_{xy}'' + \rho''(U'V')'' &= 0, \text{ \&c. \&c.} \end{aligned}\right\} \quad\text{...(168)}.$$

115. *The number of collisions which occur between pairs of grains having mean velocities between $V_1'/\sqrt{2}$ and $(V_1'+dV_1')/\sqrt{2}$.*

Since the mean distance traversed between changes of a pair of grains, irrespective of mean velocity, is $\lambda\sqrt{2}$, the mean time of a pair of grains having mean velocity $V_1'/\sqrt{2}$ in traversing their mean path is λ/V. And since the number of pairs of grains in unit volume having mean velocities between $V_1'/\sqrt{2}$ and $(V_1'+dV_1')/\sqrt{2}$ is $n(n-1)$, and each of these pairs changes V/λ times in a unit of time, the total number of changes of these mean paths is

$$n(N-1)\frac{V_1'}{\lambda}.$$

And since there are $2(N-1)$ changes for each collision the number of collisions of the $n(n-1)$ pairs of grains in unit volume in unit time is

$$\frac{n}{2}\cdot\frac{V_1'}{\lambda}=\frac{N}{2}\frac{V_1'}{\lambda}\frac{4V_1^2}{\alpha^3\sqrt{\pi}}e^{-\frac{(V_1')^2}{\alpha^2}}.dV \qquad \text{.................(169)},$$

which integrated gives the total number of collisions $\alpha/\sqrt{\pi}.\lambda$.

116. *The mean velocities of pairs having relative velocities $\sqrt{2}V_1'$ and $V_1'/\sqrt{2}$.*

Since the time of existence of a pair between changes, whatever the mean and relative velocity, is the time of existence of both the mean and relative velocities between changes, and the mean ratio of the mean and relative paths between changes is that of $1/\sqrt{2}$ to $\sqrt{2}$ or 1 to 2, it follows that the mean ratio of the mean and relative velocities is 1 to 2. And hence the mean velocity of all pairs having relative velocities between $\sqrt{2}\,V_1'$ and $\sqrt{2}(V_1'+dV_1')$ is between $V_1'/\sqrt{2}$ and $(V_1'+dV_1')/\sqrt{2}$. Q. E. D.

117. *All directions of mean velocity of a pair are equally probable whatever the direction of the mean velocity.*

This follows directly from the expression for the number of pairs having particular mean and relative velocities

$$-N_1(N_2-1)\left\{\frac{r_1^2}{\left(\frac{\alpha}{\sqrt{2}}\right)^3.\pi^{\frac{3}{2}}}.e^{-\frac{2r_1^2}{\alpha^2}}.dr_1d(\cos\theta_1).d\phi_1\right.$$
$$\left.\times\frac{r_2^2}{\left(\frac{\alpha}{\sqrt{2}}\right)^3.\pi^{\frac{3}{2}}}.e^{-\frac{2r_2^2}{\alpha^2}}.dr_2d(\cos\theta_2)d\phi_2\right\},$$

r_1 being the mean velocity, r_2 the relative velocity and $\theta_1\phi_1$, $\theta_2\phi_2$ having reference to the angular positions of r_1 and r_2.

For, taking $r_1\delta r_1$ and $r_2\delta r_2$ constant, and ascribing any particular values to $\theta_2\phi_2$ and $\delta\theta_2\delta\phi_2$, the number of pairs, having a mean velocity V_1 in directions such that, referred to the centre of a sphere of unit radius, they meet the spherical surface element $d\cos\theta_1 d\phi_1$, is to the total number which meet the sphere as $d\cos\theta_1 d\phi_1$ is to 4π. Q. E. D.

118. *The probable component of mean velocity of a pair having relative velocity $r_2 = \sqrt{2}\, V_1$ in the direction of the normal at encounter.*

Since $r_1 = r_2/2$ and $r_2 = \sqrt{2}\, V_1'$, $r_1 = V_1'/\sqrt{2}$. In all directions the probable component value is

$$\pm \frac{V_1'}{2\sqrt{2}}.$$

119. *The probable mean transmission of vis viva at an encounter in the direction of the normal.*

When two equal spheres encounter, the displacement of energy by conduction of momentum is the product of the displacement σ multiplied by twice the product of the components of the mean velocity and relative velocity of a pair in the direction of the normal. Therefore since the probable component of mean velocity in the direction of the normal (last article) is $V_1'/2\sqrt{2}$, and the probable component of the relative velocity as obtained by dividing out the σ in equation (147) is $2\sqrt{2}\,.f(\sigma/\lambda)\,.V_1/3$, the probable displacement of *vis viva* in the direction of the normal is

$$\pm \frac{2\rho}{N}\frac{\sigma}{3}f\left(\frac{\sigma}{\lambda}\right)V_1^2 = \pm \frac{2\rho}{N}\frac{\sigma}{3}f\left(\frac{\sigma}{\lambda}\right)(l^2+m^2+n^2)\,V_1^2 \ldots\ldots\ldots (170).$$

If l, m, n are the directions of the normal referred to fixed axes, the component displacements of the *vis viva* of components parallel to the axes are

$$\pm \{l^3 + lm^2 + ln^2\}\frac{\sigma}{3}f\left(\frac{\sigma}{\lambda}\right), \text{ \&c., \&c.}$$

120. *The mean distance through which the actual vis viva of a pair of grains having relative velocities between $\sqrt{2}\, V_1'$ and $\sqrt{2}\,(V_1' + \delta V_1')$ is displaced at a mean collision.*

Since the mean velocities of pairs of grains having relative velocity $\sqrt{2}\, V_1'$ is $V_1'/\sqrt{2}$ and the actual *vis viva* of such a pair is

$$2\left(r_1^2 + \frac{r_2^2}{4}\right) = 4\,(V_1/\sqrt{2})^2 = 2V_1,$$

we have for the displacement of the total *vis viva* of a pair of grains

$$\pm \frac{\sigma}{3}f\left(\frac{\sigma}{\lambda}\right).$$

And since the displacement of *vis viva* by convection by a grain having velocities between V_1' and $V_1' + \delta V_1'$ between encounters is λV_1^2 and there are, in unit time, twice as many mean paths traversed as there are collisions, the relative rates of displacement of *vis viva* by convection and conduction are as λ to $\sigma . f(\sigma/\lambda)/3$, and the displacement of *vis viva* on encounter is in cases (1) and (2)

$$\lambda + \frac{\sigma}{3} f\left(\frac{\sigma}{\lambda}\right).$$

It thus appears that, while, as has already been shown, the range of mass or any mean quantity carried by mass is λ, and the range of relative velocity or momentum is

$$\lambda \left\{1 + \frac{\sqrt{2}}{3} \frac{\sigma}{\lambda} f\left(\frac{\sigma}{\lambda}\right)\right\},$$

the range of *vis viva* is

$$\lambda \left\{1 + \frac{1}{3} \frac{\sigma}{\lambda} f\left(\frac{\sigma}{\lambda}\right)\right\}.$$

121. *The probable mean component displacement of vis viva at a mean collision by conduction.*

Multiplying the mean normal conduction of *vis viva* at a collision of a pair of grains having relative velocity $\sqrt{2}\, V_1'$ by $\cos\theta . \sin\theta . d\theta . 2\pi$ and integrating from $\theta = 0$ to $\theta = \pi/2$ and dividing by 2π we get

$$\pm \int_0^{\frac{\pi}{2}} \frac{2\rho}{N} \cdot \frac{\sigma}{3} \cdot f\left(\frac{\sigma}{\lambda}\right) \frac{d . \cos^2\theta}{2} = \mp \frac{\rho}{N} \frac{\sigma}{3} f\left(\frac{\sigma}{\lambda}\right) \text{............(171)}.$$

122. *The probable mean component displacement of vis viva by convection between encounters by a grain having velocities between V_1' and $V_1' + dV_1'$.*

Multiplying the product of the *vis viva* of the grain V_1^2 into the probable displacement (λ) by $\cos\theta . \sin\theta . d\theta$, dividing by 2π and integrating from $\theta = 0$ to $\theta = \pi/2$, the rate of the mean probable convection is

$$\frac{\rho\lambda}{N} V_1^2 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\frac{d \sin^2\theta}{2} . d\phi}{2\pi} = \frac{\rho}{N} \frac{\lambda V_1^2}{2}, \text{ \&c., \&c. ....(172)}.$$

123. *The mean component flux of vis viva.*

Since there are two mean paths traversed for each collision, adding twice the mean component displacement by convection for one path to the mean displacement by conduction at an encounter and multiplying by $n_1 V_1/2\lambda$, the expression for the mean flux by grains having directions such that $\cos\theta$ and $\cos\phi$ are positive, and for pairs of grains for the mean velocity of which $\cos\theta$ and $\cos\phi$ are positive, is

$$\frac{\rho}{N}\left\{1+\frac{\sigma}{3\lambda}\cdot f\left(\frac{\sigma}{\lambda}\right)\right\}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} d\left(\frac{\sin^2\theta}{2}\right).\, d\phi\, .\,(V_1')^3\, .\, \frac{n}{4\pi}$$

$$=\frac{1}{8}\frac{n}{N}\cdot\rho\left\{1+\frac{\sigma}{3\lambda} f\left(\frac{\sigma}{\lambda}\right)\right\}\frac{V_1^3}{2}\,\text{.........}(173).$$

124. *The mean component flux of component vis viva.*

The flux of the components of *vis viva* may be separated for direct action by substituting $\cos^2\theta\, .\, \frac{d\sin^2\theta}{2}$ for $\frac{d\sin^2\theta}{2}$ in the last equation and integrating:

$$\left.\begin{aligned}&-\frac{\rho}{N}\left(1+\frac{\sigma}{3\lambda}\right) f\left(\frac{\sigma}{\lambda}\right)\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{d\cos^2\theta}{4}\, .\, d\phi\, .\, V_1'^3\,\frac{n}{4\pi}\\ &\qquad\qquad=\frac{\rho}{8}\cdot\frac{n}{N}\left(1+\frac{\sigma}{3\lambda}\right) f\left(\frac{\sigma}{\lambda}\right)\cdot\frac{V_1^3}{4}\\ &\text{and for lateral action by substituting } \sin^2\theta\, .\, \cos^2\phi\,\frac{d\sin^2\theta}{2}:\\ &\frac{\rho}{N}\left(1+\frac{\sigma}{3\lambda}\right) f\left(\frac{\sigma}{\lambda}\right)\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{d\sin^4\theta}{4}\left(\frac{1+\cos 2\phi}{2}\right).\, V_1^3\, .\, d\phi\, .\, \frac{n}{4\pi}\\ &\qquad\qquad=\frac{\rho}{8}\cdot\frac{n}{N}\left(1+\frac{\sigma}{3\lambda}\right) f\left(\frac{\sigma}{\lambda}\right)\frac{V_1^3}{8}\end{aligned}\right\}\text{......}(174).$$

125. *The component of flux of mass in a uniform medium.*

Since mass is not subject to conduction, and the probability of a grain having velocity V_1' is n_1/N while the probable mean path is λ and the number of collisions in unit space and time between the grains having velocities between V_1' and $(V_1'+\delta V_1')$ is

$$n_1\, .\, \frac{V_1'}{\lambda},$$

the component in direction of x of a grain of which the direction is defined by $\sin\theta\, .\, d\theta\, .\, d\phi$ is $\lambda\cos\theta$, and multiplying by the number of mean paths traversed by each of such grains in a unit of time we have

$$\frac{\lambda\cos\theta n\, .\, \frac{V_1'}{\lambda}\sin\theta\, .\, d\theta\, .\, d\phi}{4\pi}=\frac{n}{4\pi}\, .\, V_1'^2\,\frac{d\sin^2\theta\, .\, d\theta\, .\, d\phi}{2}\,\text{......}(175).$$

Then integrating from $\theta=0$ to $\theta=\pi/2$ and $\phi=0$ to $\phi=\pi/2$ and from $V_1'=0$ to $V_1'=\infty$

$$\int_0^\infty\frac{n_1 V_1}{4}=\frac{N}{4}\cdot\frac{2\alpha}{\sqrt{\pi}}\,\text{..........................}(176),$$

and taking account of the mass of a grain

$$\rho \cdot \frac{\alpha}{\sqrt{\pi}}$$

is the flux of mass by the grains for which $\cos\theta$ is positive, &c., &c.

126. The extension of the kinetic theory has thus been carried as far as to include the expression of the rate of flux of momentum, *vis viva*, and mass, by conduction, as well as by convection, in the ultimate state of the medium without mean strain. Q. E. D.

It is to be noticed that the analysis effected in this section does not complete the extensions which are desirable, and possible, as these include the extension for the expression of the rates of conduction as well as convection, when the medium is subject to mean uniformly varying conditions though still in equilibrium.

These form the subject of Section XII. so that their consideration may follow the consideration of the logarithmic rates of redistribution of angular inequalities resulting from the varying condition of the medium on which they depend.

SECTION XI.

REDISTRIBUTION OF ANGULAR INEQUALITIES IN THE RELATIVE SYSTEM.

127. WHEN a granular medium, however uniform and symmetrical its mean initial condition, passes from a state of equilibrium and mean rest into a state in which there are mean rates of strain, there follow, as a consequence, rates of establishment of inequalities in the mean distribution in the relative system, which are expressed by the rates of transformation from mean to relative motion, as in the last term in equations (116) and (117) and in (116 A) and (117 A).

The general analysis of the effects of the mean motion on the relative motion for granular media comes later in the research*; and it is sufficient here to have pointed out the general source of such inequalities, as in this section we are not concerned with the source except in as far as it may be an assistance in realizing the general distinction between the two classes of inequalities. Thus the inequalities which are called into existence by rates of strain partake of the characteristics of the rates of strain.

Local volumetric rates of strain, which cause the density to vary from point to point, institute what will here be called *linear inequalities*, while uniform distortional rates of strain institute what will here be called *angular inequalities*.

The inequalities so instituted, owing to the activity of the relative-motion, are subjected to rates of redistribution proportional to their magnitudes, and it is the determination of these rates in terms of the constants which define the condition of the medium that constitutes the purpose of this section and the next.

These two rates of redistribution, like the volumetric and distortional strains, are analytically distinguishable as belonging to different classes of mean actions.

The rates of angular redistribution have the characteristics of production at a point. Their integrals are not surface integrals, and they are included in the expression for angular redistribution in the fourth term, equation (117 A).

* Section XIII.

The rates of linear redistribution, on the other hand, have the characteristics of a flux. Their integrals are surface integrals, and they are included in the expressions for the linear rates of distribution in the second and third terms, equation (117 A).

It thus appears that these rates require separate treatment, and as the analysis for the linear rate depends, to some extent, on the angular rate, the angular rate is taken first as the subject for this section, and the linear for the subject of the next, Section XII.

128. *Logarithmic rates of angular redistribution by conduction through the grains as well as by convection by the grains.*

The necessity of logarithmic rates of angular redistribution in the mean angular inequalities in the *vis viva* of relative-motion, and of inequalities in the symmetry of the mean arrangement of the grains, for the maintenance of approximately mean- and relative-motion has already been proved in Section VII.; and the actions on which these rates depend have undergone considerable *qualitative* analysis (to use a chemical expression) in the same section. What is necessary, therefore, in this section is the application of the definite, or quantitative, analysis for the definition of these rates.

The first step in this direction is the definite consideration, in the concrete, of the instantaneous effects of encounters between hard spherical grains of equal mass and dimensions.

For this purpose use is here made of the conceptions and the method given by Rankine in his paper "On the Outlines of the Science of Energetics*," a remarkable paper, which seems to have received but little notice.

129. In a purely mechanical medium, since any variation of any component-velocity of a point in mass can only result from some action of exchange of density of energy with other points in mass, there are always masses engaged in such an exchange. Considering these to include all the mass through which the exchange extends (as between some particular portion of the medium and all the rest) the sum of the energies of the components of motion, in any particular direction—that of x—immediately before the exchange is the active accident, or the "effort," of the component energy to vary itself, by conversion into some other mode, which, in a purely mechanical system, considered as a resultant system, can only be energy of component motion in some directions y and z at right angles to x.

The energy so converted into directions y and z is called the "passive accident." And in the same way the sum of the energies in the directions y and z, antecedent to the action, is the active accident or the effort of these energies to vary the energy in the direction x.

* *Proc. of the Phil. Soc. Glasgow*, Vol. III. No. 1; Rankine's *Scientific Papers*, p. 209.

It is at once apparent that the result of such accident is, taking account of the dimensions of the grains, to produce three instantaneous effects, while, if the dimensions of the grains are neglected as being small (as has been the case in the kinetic theory), only one of these effects is recognised as the result of the exchanges of energy on the instant. And although this one effect has been taken into account in the kinetic theory its position in that theory has not been generally defined, nor has it been made the subject of separate expression in the equations.

The first, and hitherto the only, published mention it has received as a specific effect occurs in Arts. 20 and 21 of my paper "On the Theory of Viscous Fluids*," where reference is made to the "angular redistribution of relative-mean motion."

It was not however till some time afterwards that I was able to distinguish, geometrically, the circumstances on which the existence of angular redistribution of relative motion depend, and obtain separate expressions for their effect.

It is included in those terms in equations (47 A), Section III. of this research, which are not surface integrals, although not specifically expressed, being associated with the resilience-effects in these equations for a resultant system; the specific expressions for the separate effects for a resultant system are however effected in equations (47 A).

The instantaneous action of which this angular redistribution is the effect turns out to be the only instantaneous action on the energy of the *relative* motions of the mass or densities of masses engaged other than the effects on resilience; so that, when the masses engaged are two equal hard spheres, angular dispersion of the energy of their relative velocities, that is, of their velocities relative to their mean position, is the only instantaneous effect on this relative energy. This theorem may be easily proved.

130. When two hard spheres encounter, their relative-velocities are in the same direction, and their momenta, relative to axes moving with their mean-velocity, are equal and opposite. Suppose the axis of x to be the direction of relative motion. Then at encounter the grains exchange components of momenta in directions of the line of centres, and thus the relative component momentum of each sphere in the direction of the line of centres is reversed; so that if the line of centres does not coincide in direction with the lines of relative motion, the instantaneous effect (1) of conduction is exchange of energy of component motion from the direction x to those of y and z at right angles to x. This is angular redistribution of the energies of component motion, and is the only change of the energies of the relative motions, measured from the moving axes. For as the relative

* *Royal Soc. Phil. Trans.*, Vol. 186 (1895) A, pp. 146—7.

momenta in direction of the line of centres of the respective grains are reversed at the instant there is no change in the position of their energies; so that at the instant there is no linear displacement of the energy of the relative motions. Q. E. D.

131. The other fundamental effects of the action between the grains—those which have been neglected in the kinetic theory—are (2) the displacement of momentum which results when two spheres encounter, having components of actual momentum (referred to fixed axes), in the direction of the line of centres, which differ in magnitude, causing the instant displacement of the difference of the component momenta, in the direction of the line of centres, through a distance σ, or the sum of the radii of the spheres. And (3) the instantaneous exchange of actual component energies in the direction of the normal.

This linear redistribution of momenta by conduction and the consequent linear displacement of their energy, relative to fixed axes, when there is mean motion, are the complement of the angular redistribution of energy, the three effects being the total instantaneous effect of the encounter, which admit of analytical separation, as long as there is no resilience.

132. The concrete effects of encounters between the grains must be considered as belonging to the resultant system in which there is no resilience. For when the effects come to be analytically separated by integration into effects on the mean and relative systems respectively, if there are rates of strain in the mean system there will be, perforce, abstract complementary resilience-effects in both systems.

It therefore appears that, if the mean effects of encounters are to be considered as belonging to the relative system, it is necessary to assume that the mean-motion is not undergoing strain, or that any rates of strain are indefinitely small. Then since the relative motions are the only motions, the following theorem requires no further demonstration.

133. If the directions, velocities and positions of the grains, constituting a granular medium, be considered, at any instant, as a complex accident, at the instant an encounter occurs, between any pair of grains, the three instantaneous effects, already discussed, will constitute an instantaneous finite variation in the complex accident, which variation will continue the same finite change, from the condition that would have existed, had the pairs passed through each other without effect, no matter what other variations might have taken place. Also, the subsequent effects resulting from the first encounter will remain unchanged. And thus, the integral effect of an encounter, at a time subsequent to the encounter, is its instantaneous effect added to all effects which ensue as a consequence of the encounter. In a granular medium, since each encounter involves two grains, the number of changes would increase as the sum of the series in geometrical progression with the factor 2; so that in a time ten times as long as the average time between two encounters, by the same grains, the number of effects resulting from a single encounter would be on the average 8000.

Thus taking account of the three analytically distinct instantaneous effects, in a time ten times as long as the average life of a path, the effects of an encounter would entail, on the average, 8000 changes in the directions of paths of grains, 8000 linear shunts of component momenta through the distance σ in different directions, and 8000 shunts of the difference of the *vis viva* of the normal velocities through σ in the direction of the normals.

Assuming, then, that in these changes, or variations of the complex accident, each has its effect in removing a portion of any mean inequality, which portion is proportional to the mean inequality, some idea may be gathered of the predominance of the effect of these changes in bringing about and maintaining the mean condition of the medium to which the changes tend.

134. In order to form definite estimates, in terms of the quantities, or mean constants, which define the condition of the medium, of the rates of decrement of inequalities from the condition to which the variations tend, as well as to find expressions for the resulting condition of the medium, it seems, in the first place, necessary to define, somewhat precisely, what are the immediate after-effects which follow, severally, from the three instantaneous effects which have been analytically distinguished. For such definition the following general theorems may be proved.

THEOREM. *The only effect which follows the instantaneous effects of an encounter, until there occurs another in which one of the grains is engaged, is the linear change in position of mass, energy, and momentum, which results from the instantaneous change in the direction of vis viva.*

The proof of this theorem follows, at once, from the analytical definition of the three effects and their continued existence.

For the instantaneous effect of linear displacement of the component momenta by conduction through the distance σ in the direction of the common normal remains unaltered and hence produces no further effect till the next encounter.

And exactly in the same way the instantaneous exchange of the energy or *vis viva* of the components of the velocity of the grains, in the direction of the normal, remains unchanged until the next encounter. Therefore it follows that the instantaneous changes in the direction and velocity (which is obtained for each grain by superimposing on its actual velocity, before contact, the normal component of the relative velocity of the pair, measured in the direction opposite to the normal component of the velocity of the grain before contact) represent the actual changes in the directions and velocities of the respective grains, whence, as these effects are to institute rates of linear displacement of mass, momentum and energy by convection, these are the only changes, and they are the after-effects of the instantaneous change in the direction of *vis viva*. Q. E. D.

135. From the theorem in Art. 134 it follows, as a corollary, that:—

The instantaneous, and after-effects of an encounter (before the next encounter of either of the grains) are confined absolutely to normal displacements of mass, and of normal components of momentum and energy; so that they have no effect whatsoever on the positions of mass, momentum or energy as measured in directions at right angles to the normal.

Therefore whatever may be the directions and velocities of pairs of grains before encounters, if the normals at encounter are all parallel to one axis, there is no lateral redistribution as the result of the encounters, whatsoever may be the extent of the normal redistributions.

136. From the principle stated in the corollary, Art. 135, that the redistributions resulting from encounters are confined to the directions of the normals at encounter, the following theorem may be proved.

THEOREM. *In a granular medium, in its ultimate state, without angular inequalities in the vis viva, &c., &c., the rates of angular redistribution of the vis viva will be equal in all directions, and equal to the rate of redistribution in the directions of the normals, if the directions of the normals are such that all the lines, drawn from a point, parallel to the directions of the normals, meet the surface of a sphere, about the point, of unit radius, in points which are symmetrically distributed over the surface of the sphere.*

For in granular media, without angular inequalities, if λ/σ is large, all directions are equally probable for the normals of encounters, in which the changes in normal *vis viva* are equal; so that the probable rates of redistribution of inequalities are equal in all directions.

And in media in which σ/λ is small, as has been shown (Section VII. Art. 89), the directions of the normals will be arranged about n axes symmetrically placed; $n = 4$ being the smallest number of mean normals that admits of symmetrical arrangement; and $n = 12$ the largest number, and the number in the ordinary piling. These mean normals being parallel to six axes, so that the probable arrangement in each group, of the directions of the normals, at encounters, in which the changes of normal *vis viva* are equal, will be similar about the axes; and it has to be shown that the rates of distribution will be the same in all directions.

This proof follows from the principle of the resolution of stresses or component *vis viva*.

If the angles between any line OA drawn through a point O, and the lines drawn through the point O, in the directions of the normals, are respectively θ_1, θ_2, &c., then the sum of the products of $p_1 \cos^2 \theta_1$, $p_2 \cos^2 \theta_2$, &c. is the rate of redistribution in the direction OA, and is the same for all directions if the directions of the normals are symmetrical. Q. E. D.

137. The theorem in Art. 136 includes the redistribution of the actual *vis viva* between the grains, as this results from the same exchanges in directions of the same normals as determine the directions of *vis viva*; and, further, includes the redistribution of the limited displacement of normal momentum by conduction. Q. E. D.

138. When the mean condition is such that there are more normals in any one direction than in those at right angles, the rates of redistribution will be greater in that direction in which there are most normals. But, as regards the *vis viva*, as long as the distribution of the normals is such that the normal redistribution is in no direction zero, there will be rates of redistribution which, though not equal in all directions, all tend to bring about an equal distribution of *vis viva* in all directions, and also tend to bring about the normal distribution of the actual *vis viva* of the grains.

As long as the inequalities in the symmetry of the directions of the normals are small, the effect on the rates of redistribution will be very small, that is, on the rate of redistribution of *vis viva*, and on the actual distribution of velocities of the grains, whatever may be the state of the medium as regards the ratio σ/λ.

Thus for the component *vis viva* and actual *vis viva* there is a continuous law of rate of redistribution and only one even when σ/λ becomes indefinitely large, so that the directions of the normals approximate to steady axes which only change their position on account of mean strain in the medium.

139. The redistribution of rates of limited conduction of momentum, or the limited displacement of normal momentum, is primarily dependent on the rates of redistribution of the directions of the normals. And the redistribution of the normals is primarily dependent on the redistribution of the positions of mass, which again has a primary dependence on diversion of the paths, as the after-effect of the instantaneous angular redistribution of *vis viva*, but this dependence on the divergence of the path is essentially limited by the value of σ/λ.

If this is small—that is if the freedoms are great—then, after an encounter, it is a matter of chance, like the length of the path of a grain, in what direction the normal at the next encounter will be, all angles being equally probable, and consequently the redistribution of the normals is determined by this probability.

But when the condition of the medium is such that σ/λ is large the greatest possible distance a grain can travel before the next encounter may be much less than σ, and this in any direction, in which case the possible direction of the normal is limited by a conical surface, which may be of angle zero, in the limit.

Then the rate of redistribution of the normals varies with the angle of this cone. Thus, as σ/λ approximates to ∞, the directions of the normals approximate to fixed axes according to the arrangement of the grains; in which case there is a redistribution of the rates of conduction of momentum or of the conduction of energy.

And here it may be noticed, that before the grains become virtually close, a limit is reached at which change of neighbours, or diffusion of the grains, ceases, and as soon as that limit is reached the mean position of the grain is constant, except for mean strains, and then the normals group round mean axes which only move with the mean strains of the medium.

Thus the displacements of normal momentum and energy depend on the arrangement of the grains apart from the mean freedoms, and the redistribution of the conduction depends on the redistribution of inequalities in the symmetry of the arrangement of the grains, so that, although both the angular redistribution of the *vis viva* and rearrangement of inequalities in the symmetry of the mean arrangement of the grains, are included in the fourth term of equation (117 A), expressing angular redistribution, they have not been analytically separated, in the terms, as depending on angular dispersion of *vis viva* and rearrangement of the inequalities in the symmetry of the mean arrangement of the grains.

The analytical separation of the abstract actions on which the two effects of angular redistribution respectively depend, effected by the demonstration of the foregoing theorems, renders it possible to deal with the two rates separately and so to obtain analytical definition of the respective rates in terms of the constants which define the state of the medium.

140. *The analytical definition of the rates of angular redistribution of inequalities in the directions of vis viva of relative motion.*

As these actions do not appear to have been the subjects of previous consideration it is necessary to demonstrate two preliminary propositions before considering the mean effects.

141. *The energy of component motion in any direction cannot by its own effort increase the energy of component motion in this direction.*

This proposition might be taken as self-evident; but it may be definitely proved. In the case of spherical grains the proof is simplified, and particularly if the relative-motion is such that the only inequalities are in the energies of motion in different directions—unequal angular dispersion.

Taking the axes of reference fixed, l, m, n and l', m', n', and l'', m'', n'' as the direction cosines of the normal at the point of contact and of two other directions at right angles, also $u_1, v_1, w_1, u_2, v_2, w_2$ for the antecedent velocities of the two grains, and $U_1, V_1, W_1, U_2, V_2, W_2$, for the subsequent velocities, it follows as a direct result of the exchange of the components of motion in the direction of the normal that at a single encounter,

$$\left.\begin{aligned} U_1^2 + U_2^2 - u_1^2 - u_2^2 = &- 2(m^2+n^2)\,l^2(u_2-u_1)^2 \\ &+ 2l^2\{m^2(v_2-v_1)^2 + n^2(w_2-w_1)^2\} \\ &+ 4l^2mn(v_2-v_1)(w_2-w_1) \\ &+ 2(2l-1)\{lm(u_2-u_1)(v_2-v_1) \\ &\qquad + nl(w_2-w_1)(u_2-u_1)\} \end{aligned}\right\} \text{\&c., \&c.} \ldots (177).$$

Then, since for any two spheres with particular relative motion, $u_2 - u_1$, $v_2 - v_1$, $w_2 - w_1$, the probability of their normal, at the point of contact, having a direction within any small area, $\sin\theta d\theta d\phi$, on a sphere of unit radius, having its centre at the centre of one of the spheres, assuming all angles of relative motion after encounter equally probable, is:

$$\frac{\sin\theta d\theta d\phi \cos\chi}{\pi},$$

where χ is the angle between two radii, one meeting the surface of the unit sphere in the direction of the point of contact, and the other in the direction of the relative motion, drawn so that χ is an acute angle, so that χ is always between zero and $\pi/2$.

142. *The active and passive accidents.*

In considering the action resulting from conduction of momentum of two spheres at a single encounter, the problem is greatly simplified by taking the direction of one of the axes of reference to be that of the relative motion of the spheres; while, as will be seen, it does not lose in generality.

Taking χ to be measured in the direction of the relative motion, $v_2 - v_1$, $w_2 - w_1$ are each zero, and putting

$$\tfrac{1}{2}(u_1+u_2)^2 + \tfrac{1}{2}(u_2-u_1)^2 \text{ for } u_1^2 + u_2^2, \text{ \&c., \&c.}$$

in equation (177) we have

$$\left.\begin{aligned} U_1^2 + U_2^2 - \tfrac{1}{2}(u_1+u_2)^2 - \tfrac{1}{2}(u_2-u_1)^2 &= -2(m^2+n^2)\,l^2(u_2-u_1)^2 + 0 + 0 \\ V_1^2 + V_2^2 - \tfrac{1}{2}(v_1+v_2)^2 - \qquad 0 \qquad &= -0 + 0 + 2m^2l^2(u_1-u_2)^2 \\ W_1^2 + W_2^2 - \tfrac{1}{2}(w_1+w_2)^2 - \qquad 0 \qquad &= -0 + 2n^2l^2(u_2-u_1)^2 + 0 \end{aligned}\right\} \ldots (178),$$

in which the ciphers represent the values of the terms having factors $(v_2 - v_1)$ and $(w_2 - w_1)$.

Multiplying these equations by the factor of probable positions of the normal and integrating over the sphere of unit radius, since $\cos\chi$ is positive and equal to $\pm \cos\theta = \pm l$, the equations become on transposing the last terms in the left members

$$\left.\begin{aligned} U_1^2 + U_2^2 - \tfrac{1}{2}(u_1+u_2)^2 &= \left(\frac{u_1-u_2}{2}\right)^2 - \tfrac{1}{3}(u_2-u_1)^2 + 0 + 0 \\ V_1^2 + V_2^2 - \tfrac{1}{2}(v_1+v_2)^2 &= \quad 0 \qquad - 0 + 0 + \tfrac{1}{6}(u_2-u_1)^2 \\ W_1^2 + W_2^2 - \tfrac{1}{2}(w_1+w_2)^2 &= \quad 0 \qquad - 0 + \tfrac{1}{6}(u_2-u_1)^2 + 0 \end{aligned}\right\} \quad \ldots\ldots(179),$$

where, since the square of the relative motion, $(u_2 - u_1)^2$, is double the sum of the squares of differences between the actual component motions and the mean component motions,

$$\left.\begin{aligned} \left(u_2 - \frac{u_2+u_1}{2}\right)^2 + \left(u_1 - \frac{u_1+u_2}{2}\right)^2 &= \frac{1}{2}(u_2-u_1)^2 \\ U_1^2 + U_2^2 - \frac{1}{2}(u_1+u_2)^2 &= \frac{1}{3}\left\{\left(u_2 - \frac{u_1+u_2}{2}\right)^2 + \left(u_1 - \frac{u_1+u_2}{2}\right)^2\right\} \\ V_1^2 + V_2^2 - \frac{1}{2}(v_1+v_2)^2 &= \frac{1}{3}\left\{\left(u_2 - \frac{u_1+u_2}{2}\right)^2 + \left(u_1 - \frac{u_1+u_2}{2}\right)^2\right\} \\ W_1^2 + W_2^2 - \frac{1}{2}(w_1+w_2)^2 &= \frac{1}{3}\left\{\left(u_2 - \frac{u_1+u_2}{2}\right)^2 + \left(u_1 - \frac{u_1+u_2}{2}\right)^2\right\} \end{aligned}\right\} \ldots(180).$$

The left members of equations (178) express, respectively, the effects, both active and passive, of the accidents on the energies of the components of motion in the directions of x, y, z respectively.

The first terms in the right members, *which are all negative*, or zero, express the effects of the active accidents on the energies in these directions respectively, while the last two terms, which are positive, or zero, express the effects of the passive accidents in these directions. Q.E.D.

143. The active accidents are work spent by the efforts produced by $u_2 - u_1$, $v_2 - v_1$, $w_2 - w_1$, respectively, in other directions than those of x, y, z respectively. Thus the effort in the direction of the normal caused by $u_2 - u_1$ is $2l(u_2 - u_1)$ and the component of the relative velocity $u_2 - u_1$ in the direction of the normal is $l(u_2 - u_1)$; so that the total result of this effort is $-2l^2(u_2-u_1)^2$, work spent by energy in direction of x. Of this $2l^4(u_2-u_1)^2$ is work returned to the energy in direction of x; so that the portion of the energy in the direction x expended in (passive accidents) changing the energy in directions of y and z is $-2(l^2-1)l^2(u_2-u_1)^2$, and the passive accidents in the directions of y and z are $2l^2m^2(u_2-u_1)^2$, $2l^2n^2(u_2-u_1)^2$ respectively.

144. *The angular dispersion of relative motion.*

The equations (180) show that considering the chance encounter between two grains, whatever their relative-motion before encounter, all directions of the subsequent relative-motion are equally probable. So that any angular inequality in their relative-motion is virtually extinguished after a single encounter; although if the pair have any mean-motion, whatever it may be, the inequality in this remains as before encounter. Q. E. D.

145. *The mean angular inequalities.*

Before we can pass from dispersion of the component relative-velocities of a pair of grains to that of the mean-inequalities of all the grains the demonstrations of several propositions become necessary.

For reasons, which will appear, we have here to consider only such mean angular inequalities as are introduced in the relative motion of the medium while the mean system is undergoing mean rates of strain.

These inequalities, as Maxwell has shown, for a medium consisting of equal hard spheres, are expressed by, taking N for the number of grains in unit volume,

$$\surd N = \frac{Ne^{-\left\{\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}\right\}}}{\alpha\beta\gamma\,(\pi)^{\frac{3}{2}}}\,dxdydz\,^* \quad\text{..................(181)},$$

where α^2, β^2, γ^2 are double the mean of squares of the respective component velocities.

Since the differences between α^2, β^2, γ^2 and the mean $(\alpha^2+\beta^2+\gamma^2)/3$ are always small compared with their mean it becomes more convenient to alter the notation and, taking α^2 as expressing the mean of $\alpha^2+\beta^2+\gamma^2$, to take $\alpha(1+a)$, $\alpha(1+b)$, $\alpha(1+c)$ respectively for Maxwell's α, β, γ; a, b, c are then small fractions of unity such that their squares may be neglected and for the mean squares we have

$$\alpha^2(1+2a), \quad \alpha^2(1+2b), \quad \alpha^2(1+2c),$$

and the inequalities are $2a\alpha^2$, $2b\alpha^2$, $2c\alpha^2$; $2a$, $2b$, $2c$ being the coefficients of inequality from the mean of the mean squares of the respective components.

It is to be noticed that in equation (136) the axes of reference are the principal axes of the space variations of the mean motions of the medium—the principal axes of distortional mean motions—and also of the inequalities.

146. *The angular inequalities in the mean relative motions of pairs of grains have the same coefficients of inequality as the mean actual motions.*

Integrating equation (181) with respect to y and z from $-\infty$ to $+\infty$

$$\surd N = \frac{Ne^{-\frac{x^2}{\alpha^2}(1-2a)}}{\alpha(1+a)\surd\pi}\,dx \quad\text{.........................(182)}.$$

Then, after Maxwell, taking x_1 as a particular component of velocity in direction of x, the number of grains which have component velocities between x_1 and $x_1+\delta x_1$ is

$$\frac{N}{\alpha(1+a)\surd\pi}\,e^{-\frac{x_1^2}{\alpha^2}(1-2a)}\,dx.$$

* *Phil. Trans. Royal Soc.*, 1866, p. 64.

And again taking $x_2 = x_1 + x'$ the number of grains between $x_1 + x'$ and $x_1 + x' + \delta x'$ is

$$\left(\frac{N}{\alpha(1+a)\sqrt{\pi}} e^{-\frac{(x_1+x')^2}{\alpha^2}(1-2a)}\right) dx.$$

Then the number of pairs of grains which satisfy both these conditions is

$$\frac{NN}{\alpha^2(1+2a)\sqrt{\pi}} e^{-\frac{(\sqrt{2}(1-a))^2}{\alpha^2}\left\{\left(x_1+\frac{x'}{2}\right)^2+\frac{x'^2}{4}\right\}} dx_1 dx'.$$

Then, since $x_1 + \frac{x'}{2}$ may have any value from $-\infty$ to $+\infty$ for any value of x', integrating for x_1 between these limits for any particular value of x', the number of pairs which have component relative-velocities, in direction x, between x' and $x' + \delta x'$ is:

$$\frac{N^2}{\sqrt{2}\,\alpha(1+a)\sqrt{\pi}} e^{-\frac{x'^2}{2\alpha^2}(1-2a)} dx'.$$

In exactly the same way it is shown that the numbers of component relative-velocities between y' and $y' + \delta y'$ and between z' and $z' + \delta z'$ are respectively

$$\frac{N^2}{\sqrt{2}\,\alpha(1+b)\sqrt{\pi}} e^{-\frac{y'^2}{2\alpha^2}(1-2b)} dy',$$

$$\frac{N^2}{\sqrt{2}\,\alpha(1+c)\sqrt{\pi}} e^{-\frac{z'^2}{2\alpha^2}(1-2c)} dz'.$$

Multiplying these expressions by x'^2, y'^2, z'^2 respectively and integrating from $-\infty$ to $+\infty$, and dividing by N^2, we have for the mean-squares of the respective components, in the directions x, y, z

$$2\alpha^2(1+2a), \quad 2\alpha^2(1+2b), \quad 2\alpha^2(1+2c),$$

which have precisely the same coefficient of angular inequalities as the mean squares of the components of the actual velocities obtained from equations (181)

$$\alpha^2(1+2a), \quad \alpha^2(1+2b), \quad \alpha^2(1+2c). \qquad \text{Q. E. D.}$$

147. *The mean squares of the components of relative-motion of all pairs are double the mean squares of the components of actual motion.*

In the last paragraph of the last article it has been shown that the mean squares of the components of relative-motion of all pairs including the inequalities are double the mean squares of the components of the actual motion, so that no further demonstration is necessary.

148. *The rate of angular redistribution of the mean inequalities in the actual motion is the same as the rate of redistribution of the angular inequalities in the relative motion of all pairs.*

This follows at once from the inequality of the coefficients of inequalities which has already been proved.

149. *The rate of angular dispersion of the mean inequalities in vis viva.*

It has been shown, equations (180), that the angular inequality in the squares of the relative velocities of any pair of grains is virtually extinguished at a single encounter. From this it follows that the virtual inequality in the motion of any grain exists only from the time of the institution of the inequality to the time of its next encounter.

This time is expressed by

$$\frac{\lambda_1}{V_1},$$

V_1 being the actual velocity of the grain, and λ_1 the distance traversed before encounter.

This distance λ_1 may be anything from 0 to ∞. But it is proved by Maxwell to be independent of V_1 and to have a probable mean value, neglecting σ as compared with λ, of

$$\lambda = \frac{1}{\sqrt{2}\,\pi\sigma^2 N} \quad\text{.............................(183).}$$

Taking σ into account, as will be shown, the probable value of λ becomes

$$\lambda = \frac{1}{\sqrt{2}\,\pi\sigma^2 N} - \sqrt{\frac{2}{3}}\,\sigma f\left(\frac{\sigma}{\lambda}\right) \quad\text{....................(184).}$$

The probable path being λ, the probable time of any grain with velocity V_1 is

$$\frac{\lambda}{V_1}.$$

It thus appears that, although the mean relative distance traversed between encounters by pairs of grains having the same relative velocities V_1 is independent of $\overline{V}_1$, the mean time between encounters varies inversely as $\overline{V}_1$.

In order therefore to obtain the probable mean time of existence of inequalities in the angular distribution of the *vis viva*, it is not sufficient to find the probable value of the mean time $\frac{\lambda}{V_1}$, for all values of V_1, since this would only be the probable mean time between encounters during which the inequalities in the mean velocity are sustained.

150. *The mean time of mean inequalities of vis viva.*

The direction of motion of each grain is the direction of its path; so that if l, m, n are the direction-cosines of the motion, the probable times of the continuance of the components of motion in directions x, y, z are

$$\frac{\lambda l}{V_1 l},\quad \frac{\lambda m}{V_1 m},\quad \frac{\lambda n}{V_1 n},$$

and since the chance of a collision in a unit of time is V_1/λ the probability of continued existence is

$$e^{-\frac{V_1}{\lambda}t},$$

and the probability of continuing for a time

$$t = \frac{n_1\lambda}{V_1}$$

is

$$e^{-n_1}.$$

Whence it follows that, taking account of all the pairs of grains at different relative velocities, but moving nearly in the same directions, the times for which their continuance is equally probable are

$$t_1 = \frac{n_1\lambda l}{V_1 l}, \quad t_2 = \frac{n_2\lambda l}{V_2 l}, \text{ \&c. } \ldots\ldots\ldots\ldots\ldots\ldots\ldots(185),$$

so that, multiplying $V_1^2l^2$, $V_2^2l^2$, &c. respectively by t_1, t_2, &c., and adding, the sum will be equal to

$$\Sigma\{n_1\lambda l^2(V_1 + V_2 + \text{\&c.})\}, = \frac{n_1}{2}\,\overline{V}\lambda l,$$

and similarly for the other two components.

And putting $\overline{V}$ and $\overline{V^2}$ respectively for the mean values of V and V^2, the mean time of equal probability for the continued existence of $\overline{V^2}$ is obtained by dividing the product by $\overline{V^2}$: $\dfrac{n_1\lambda l^2\overline{V}}{\overline{V^2}l^2}$, and for the other components

$$\frac{n_1\lambda m^2\overline{V}}{\overline{V^2}m^2}, \quad \frac{n_1\lambda n^2\overline{V}}{\overline{V^2}n^2}.$$

These mean times, it will be noticed, are independent of the directions of the groups, being all expressed by

$$\bar{t} = \frac{n_1\lambda\overline{V}}{\overline{V^2}}, \text{ where the probable continuance is } e^{-n_1} = e^{-\frac{\overline{V^2}}{\lambda\overline{V}}\bar{t}} \ldots\ldots(186).$$

Differentiating this expression with respect to t,

$$\frac{dn_1}{dt} = \frac{\overline{V^2}}{\lambda\overline{V}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(187).$$

From equation (181) the mean values of u^2, v^2, w^2 are found to be

$$\frac{\alpha^2}{2}(1 + 2a), \quad \frac{\alpha^2}{2}(1 + 2b), \quad \frac{\alpha^2}{2}(1 + 2c).$$

In these α^2 is constant, and $a + b + c = 0$, and the inequalities are

$$\frac{\alpha^2}{2}(1 + 2a) - \frac{\alpha^2}{2} = 2a\frac{\alpha^2}{2}, \text{ \&c., \&c. } \ldots\ldots\ldots\ldots\ldots\ldots(188).$$

Then by equation (187) the probability of continued existence is expressed by

$$2a\frac{\alpha^2}{2} = -2a_1\frac{\alpha^2}{2}e^{-\left(\frac{\frac{3\alpha^2}{2}}{\frac{2\alpha\lambda}{\sqrt{\pi}}}\right)t}.$$

Whence if $n_1 = 0$,

$$\frac{\alpha^2}{2}\frac{d(2a)}{dt} = -6a_1\frac{\alpha^2}{2}\frac{\sqrt{\pi}}{2\alpha\lambda}, \text{ \&c., \&c.} \quad\ldots\ldots\ldots\ldots\ldots\ldots(189),$$

or

$$\frac{d(2a)}{dt} = -2a\left(3\frac{\alpha\pi}{4\lambda}\right), \text{ \&c., \&c.} \quad \text{Q.E.F.}$$

151. Translated into the notation adopted in this research for the expression of the velocities of the component system of relative motion, we have for the mean inequalities referred to their principal axes,

$$\rho''\left[(u'u')'' - \tfrac{1}{3}(u'u' + v'v' + w'w')''\right], \text{ \&c., \&c.} \quad\ldots\ldots\ldots(190),$$

and for the rates of dispersion with reference to the same axes we have, putting $\partial_2/\partial_2 t$ in place of d/dt to distinguish these as rates of angular dispersion,

$$\rho''\frac{\partial_2}{\partial_2 t}\left[(u'u')'' - \tfrac{1}{3}(u'u' + v'v' + w'w')''\right] = \frac{3}{4}\frac{\pi}{\lambda}\alpha\rho''\left[(u'u')'' - \frac{(u'u' + v'v' + w'w')''}{3}\right],$$
$$\text{\&c., \&c.,} \quad\ldots\ldots(191),$$

where $2\alpha/\sqrt{\pi}$ is the time-mean of the velocities of a grain, and λ is the measure of the scale of the system of relative motion. (N.B. These rates are independent of σ.)

As already pointed out, Art. 146, the expressions in equations (189) and (190) for the inequalities are with reference to their principal axes only; so that in order to obtain expressions that shall apply for any axes it is necessary to effect the transformation from the principal axes, at a point, to fixed axes.

152. *Rates of angular dispersion referred to axes which are not necessarily principal axes of rates of distortion.*

Taking $l_1m_1n_1$, $l_2m_2n_2$, $l_3m_3n_3$ to be respectively the direction cosines of the principal axes with reference to any rectangular system of fixed axes,

$$a',\ b',\ c',\ f',\ g',\ h'$$

to be the mean values of u'^2, v'^2, w'^2, $v'w'$, $w'u'$, $u'v'$ (u', &c., as before, representing the relative velocities referred to the principal axes 1, 2, 3), and let a, b, c, f, g, h, be their corresponding mean values when referred to the fixed axes of x, y, z.

Then

$$\left.\begin{aligned}
&f' = g' = h' = 0\\
&a = l_1^2a' + l_2^2b' + l_3^2c'\\
&b = m_1^2a' + m_2^2b' + m_3^2c'\\
&c = n_1^2a' + n_2^2b' + n_3^2c'\\
&f = m_1n_1a' + m_2n_2b' + m_3n_3c'\\
&g = n_1l_1a' + n_2l_2b' + n_3l_3c'\\
&h = l_1m_1a' + l_2m_2b' + l_3m_3c'
\end{aligned}\right\} \quad \text{..................(192).}$$

From these, adding the second, third, and fourth,

$$a + b + c = a' + b' + c' \quad \text{........................(193).}$$

Also since the principal axes do not change their position in consequence of the dispersion of the inequalities

$$\left.\begin{aligned}
&\frac{\partial_2 (a)}{\partial_2 t} = l_1^2 \frac{\partial_2 (a')}{\partial_2 t} + l_2^2 \frac{\partial_2 (b')}{\partial_2 t} + l_3^2 \frac{\partial_2 (c')}{\partial_2 t}, \text{ \&c., \&c.}\\
&\frac{\partial_2 (f)}{\partial_2 t} = m_1n_1 \frac{\partial_2 (a')}{\partial_2 t} + m_2n_2 \frac{\partial_2 (b')}{\partial_2 t} + m_3n_3 \frac{\partial_2 (c')}{\partial_2 t}, \text{ \&c., \&c.}
\end{aligned}\right\} \text{.....(194).}$$

Then substituting from equations (190) for $\partial_2 a'/\partial_2 t$, &c., in (194), and remembering that $l_1u' + l_2v' + l_3w'$, when referred to the principal axes is the same as u' referred to the fixed axes, we have by equation (193), for the rates of dispersion, referred to any axes,

$$\left.\begin{aligned}
&\rho'' \frac{\partial_2}{\partial_2 t} [(u'u')'' - \tfrac{1}{3}(u'u' + v'v' + w'w')'']\\
&\qquad\qquad = -\frac{3}{4} \rho'' \frac{\sqrt{\pi}}{\lambda} \alpha [(u'u')'' - \tfrac{1}{3}(u'u' + v'v' + w'w')''], \text{ \&c., \&c.}\\
&\rho'' \frac{\partial_2}{\partial_2 t} [(v'u')''] = \frac{3}{4} \rho'' \frac{\sqrt{\pi}}{\lambda} \alpha (v'u')'', \text{ \&c. \&c.}\\
&\rho'' \frac{\partial_2}{\partial_2 t} [(w'u')''] = \frac{3}{4} \rho'' \frac{\sqrt{\pi}}{\lambda} \alpha (w'u')'', \text{ \&c., \&c.}
\end{aligned}\right\} \text{...(195).}$$

153. *The analytical definition of the rates of angular redistribution of inequalities in rates of conduction through the grains.*

As already proved, Arts. 78 C and 79, Section VII., and the theorem Art. 136 in this section, the angular inequalities in the rates of conduction are the result of unsymmetrical arrangement of the grains. And as, according to the definitions of mean- and relative-mass, Art. 47, the mean-mass is independent of the arrangement, since the number of grains within the scale of relative-mass is not affected by the arrangement, the inequalities in the rates of conduction are the result of unsymmetrical arrangement of the relative-mass.

It has also been shown, Art. 77, Section VII., that angular inequalities in the mean conduction result from angular inequalities in the lengths of the mean paths of the grains, and it has been further pointed out that angular inequalities in the lengths of the mean paths are the result of the distortion rates of mean strain. And the number of paths traversed being inversely proportional to their lengths, there are more mean paths traversed in directions in which the relative paths are shortest.

It thus appears that, although the rates of conduction are not of the same dimensions as the mean paths or the position of relative-mass, the rates of angular redistribution of the angular inequalities are the same.

154. *The rate of angular redistribution of mean inequalities in the position of the relative-mass in terms of the quantities which define the state of the medium.*

When, owing to the rates of distortional or rotational strain in the mean-motion of a granular medium, there are instantaneous inequalities in the symmetry of the arrangement of the grains, there will be inequalities in the lengths of the mean component paths; and, the number of component paths traversed being inversely proportional to their lengths, there will be more relative paths traversed in the directions in which they are shortest.

Then, since after each encounter all directions of relative paths are equally probable, after each encounter any inequality which may be attributed to any pair of grains is virtually extinguished. And, as shown in Art. 150, the probability for the continued existence for a time

$$t_1 = n_1 \frac{\lambda}{V_1} \text{ is } e^{-n_1} \quad \text{.............................(196).}$$

From this it follows, as in equation (185),

$$t_1 = n_1 \frac{\lambda_1 l_1}{V_1 l_1}, \quad t_2 = n_2 \frac{\lambda_2 l_2}{V_2 l_2}, \text{ \&c., \&c. ....(197),}$$

in which expressions the direction cosines l_1, m_1, n_1, &c. are nearly constant and n_1, the index of probability, is constant.

Therefore taking the products $(t_1 \overline{V}_1 + \text{\&c.})$ and dividing the mean product by $\overline{V}$—the mean velocity—the mean time of existence of the inequality is found to be

$$\bar{t} = n_1 \frac{\overline{\lambda}}{\overline{V}} \quad \text{..................................(198),}$$

and the mean probability of continued existence is

$$e^{-n_1} = e^{-\frac{\overline{V}}{\overline{\lambda}}\bar{t}} \quad \text{.............................(199),}$$

which when the inequalities are small becomes

$$e^{-\frac{2}{\sqrt{\pi}}\frac{\alpha}{\lambda}\bar{t}}.$$

If, then, we take a, f, &c., the angular inequalities in the positions of relative mass, we have for the relative rates of angular dispersion,

$$\frac{\partial_2(a)}{\partial_2(t)} = -\frac{2}{\sqrt{\pi}}\frac{\alpha}{\lambda}a, \quad \frac{\partial_2(f)}{\partial_2(t)} = -\frac{2}{\sqrt{\pi}}\frac{\alpha}{\lambda}f \text{ ....(200).}$$

It will be observed that the logarithmic rate of decrement of inequalities in relative mass differs somewhat from that of the *vis viva.* This is a consequence of the difference in the mean time of probable existence of V and of V^2.

155. *The limits to the dispersion of angular inequalities in mean mass.*

The numerical coefficient is the only respect in which the rate of angular redistribution of mass differs from that of *vis viva* as long as λ/σ is large. But as the density becomes large, unlike the redistribution of *vis viva*, the redistribution of relative mass depends on two circumstances, the inequalities being small in both cases.

Inequalities in *vis viva* are not subject to any limits imposed by the neighbouring grains and consequently all directions of motion are equally probable, however close the grains may be, and whatever may be the arrangement of the grains.

On the other hand the possibility of angular rearrangement of the grains turns on the possibility of a grain passing through the triangular surface set out by the centres of three of its neighbouring grains ; and this possibility is closed at some density less than that of maximum density. The density at which this closure is effected is that at which diffusion ceases and the state of permanent distortional elasticity commences. Before this density is reached the diffusion becomes slower and slower as the density increases ; so that in a granular medium of which the mean condition is uniform, but which is steadily contracting, the chance of a grain finding a clear way between three of its neighbours diminishes, and each grain dwells longer and longer in the same mean position in the medium, until all chance ceases and its mean position is definitely defined, notwithstanding that it has still a certain range of freedom. For the general consideration of the rate of rearrangement of mass it is necessary to take account of the probability of a grain returning after encounters to the formation before encounter, and this presents great difficulties. But it will be sufficient to point out here that owing to the instantaneous action at encounter, no more than two grains are ever in contact at the same time, so that there is no chance of combination of the grains, and that the mean position of two grains is not altered at encounter while the relative motions are reversed.

In the next section it will appear that the linear dispersion of *vis viva* of grains is very slow as the angular dispersion is very great, so that any chance activity of a grain of an exceptional character is immediately dispersed amongst its neighbours and brought back to the mean.

When therefore the density is such that λ/σ is very small and the density is nearly the maximum, *i.e.* when G is nearly $6/\sqrt{2}\pi$, there is no rearrangement of the grains, and this will hold good as G increases provided that the extent of the medium for which the value of G is large is very small.

Thus we have two states of the medium in which the rates of rearrangement are defined, and between these a gap in which the definition is difficult.

Fortunately this difficulty is confined to a very small portion of the total range of density, being that between the density at which diffusion ceases and that at which diffusion becomes easy.

This gap covers a region of which the higher limit of ρ is slightly less than $1/\sqrt{2}$, when the distribution is uniform, and is equal to $1/3$ at irregular points and surfaces; λ/σ being small in both cases.

For values of ρ above these limits there is no diffusion and consequently no redistribution in the arrangement of mass, while for values of ρ below these limits the change in rate of redistribution is very rapid at first, then gradually settling down to the same relative rate as that of redistribution of *vis viva*.

If then we take as before $a = \partial_1(a)/\partial_1(t)$, &c. to represent the small angular inequalities instituted by the distortion in the mean system during the time $\partial_2(t)$; the rates of redistribution to which these are subjected will approximate to that to which the *vis viva* is subjected as ρ approximates to zero. Thus the law of redistribution has an asymptote

$$\frac{\partial_2(a)}{\partial_2(t)} = -\frac{2}{\sqrt{\pi}\,\lambda}\,\alpha a \quad\text{..........................(201).}$$

Then if we take $f(G)$ as expressing a coefficient by which the upper limit of ρ must be multiplied to bring it to unity

$$\left.\begin{aligned} \frac{\partial_2(a)}{\partial_2(t)} &= -\frac{2}{\sqrt{\pi}\,\lambda}\,\alpha a\,\frac{1-f(G)\rho}{1+e^{-\infty}}\{1-f(G)\rho\}, \text{ \&c., \&c.} \\ \frac{\partial_2(f)}{\partial_2(t)} &= -\frac{2}{\sqrt{\pi}\,\lambda}\,\alpha f\,\frac{1-f(G)\rho}{1+e^{-\infty}}\{1-f(G)\rho\}, \text{ \&c., \&c.} \end{aligned}\right\} \quad\text{.....(202)}$$

are expressions which give the rates of redistribution correctly except, perhaps, in the immediate region of the higher limit.

156. *The rates of probable redistribution of angular inequalities in the rates of conduction.*

Any angular inequalities in the rates of conduction result, solely, from angular inequalities in the distribution of mass, but the coefficients of the rates of redistribution are not the same for rates of redistribution of mass as for the redistribution of conduction.

The mean time of continued existence of the path of a grain

$$\bar{t} = \frac{n_1 \bar{\lambda}}{\bar{V}} \qquad \text{.................................(203)},$$

is not the mean time for the continued existence of the product of the mean path multiplied by the *vis viva*. If however the mean time for the mean path be multiplied by the factor

$$\text{we have} \qquad \left.\begin{aligned} \frac{\bar{V^2}}{\bar{V}^2} &= \frac{8}{3\pi} \\ \frac{\bar{V^2}}{\bar{V}^2}\bar{t} &= \frac{n_1 \lambda \bar{V}}{\bar{V}^2} \end{aligned}\right\} \qquad \text{............................(204)},$$

which is the same coefficient as for the time of continued existence of *vis viva*.

To obtain the expressions for the probable relative rates of angular redistribution of angular inequalities in the rates of conduction corresponding to the rates of angular redistribution of angular inequalities in the distribution of mass, we have to multiply the relative rates of redistribution of mass by the factor

$$\frac{3\pi}{8}.$$

Then substituting the actual inequalities in the angular rates of conduction

$$(p_{xx}'' - p''), \quad p_{yx}'', \quad p_{zx}'', \text{ \&c.,}$$

for a, f, &c., the expressions for the rates of redistribution of these inequalities of conduction are

$$\left.\begin{aligned} \frac{\partial_2}{\partial_2 t}(p_{xx}'' - p'') &= -\tfrac{3}{4}\sqrt{\pi}\,\frac{\alpha}{\lambda}\,\frac{1 - f(G)\rho}{1 + e^{-\infty(1-f(G)\rho)}}(p_{xx}'' - p''), \text{ \&c., \&c.} \\ \frac{\partial_2}{\partial_2 t}(p_{yx}'') &= -\tfrac{3}{4}\sqrt{\pi}\,\frac{\alpha}{\lambda}\,\frac{1 - f(G)\rho}{1 + e^{-\infty(1-f(G)\rho)}}\,p_{yx}'', \qquad \text{\&c., \&c.} \\ \frac{\partial_2}{\partial_2 t}(p_{zx}'') &= -\tfrac{3}{4}\sqrt{\pi}\,\frac{\alpha}{\lambda}\,\frac{1 - f(G)\rho}{1 + e^{-\infty(1-f(G)\rho)}}\,p_{zx}'', \qquad \text{\&c., \&c.} \end{aligned}\right\} \text{...(205).}$$

In these equations (204) for the rates of angular dispersion of the distortional inequality, and the two rotational inequalities in conduction, as well as in the corresponding equations for the rates of angular dispersion of the corresponding inequalities in the *vis viva* of relative motion (195), the analysis for each inequality has been effected separately in terms of the quantities which define the state of the medium.

These six rates of dispersion for each of the components in directions x, y, and z added together constitute the rate of increase of the energy of the component of relative motion received from the other components of the same system. And thus it appears that the expressions for these six rates of redistribution are the analytical equivalent, in terms of the quantities which define the condition of the medium, of the fourth term in the equation (117 A); which may be expressed as

$$-\left[p'\left\{\frac{du'}{dx}-\frac{1}{3}\left(\frac{du'}{dx}+\frac{dv'}{dy}+\frac{dw'}{dz}\right)\right\}+\frac{1}{2}\left\{p'_{yx}\left(\frac{du'}{dy}-\frac{dv'}{dx}\right)\right.\right.$$
$$\left.\left.+p'_{zx}\left(\frac{du'}{dz}-\frac{dw'}{dx}\right)\right\}\right]\text{\&c., \&c.}$$

Q. E. F.

SECTION XII.

THE LINEAR DISPERSION OF MASS AND OF THE MOMENTUM AND ENERGY OF RELATIVE-MOTION, BY CONVECTION AND CONDUCTION.

157. THESE actions are expressed by the second, third and fifth terms in equations (123), or more concisely by the second and third terms in (117 A),

$$\frac{1}{2}\left\{\frac{d}{dx}\left[(\rho u'u' + p_{xx})'u'\right] + \&c.\right\}'', \ \&c., \ \&c.$$

It has been shown that the actions of the component mean and relative stresses on the space-variations of the relative velocities $(p'\,du'/dx + \&c.)''$ are confined to the resilience and the angular dispersion of the energy of the components of relative-motion at the points where the inequalities of angular distribution exist; and therefore do not account for any linear redistribution from point to point.

Linear redistribution requires the conveyance or transmission of energy, &c. from one space to another, and the integrals of these actions must be surface integrals.

These actions of linear redistribution are again such that their effects can be studied only by considering the causes which determine the rates at which energy, &c., is carried and conducted across a plane from opposite sides. The relative-velocities at which the grains arrive at a plane, or which come in collision with a grain intersected by the plane, are not determined by any action at the plane, but by the antecedent actions.

As far as these actions of redistribution depend on the convections, that is, neglecting the dimensions of the molecules, they have been taken into account in the kinetic theory of gases.

Clausius was the first to obtain the true explanation* on the supposition that the mean distance between the molecules was so great, compared with their dimensions, that the latter might be neglected. In this method he takes

* *Pogg. Ann.* 1860.

account of the principle, that after a collision the mean velocity of the pair is the same as before, and of the consequence, that the molecules crossing a plane surface, perpendicular to the directions in which the inequality varies, from opposite sides, must have mean velocities such that their sum, in the direction of the downward slope of the inequality, is equal to V, the mean velocity of the encountering molecules, the same as if they arrived at the plane from uniform gas in motion with this mean velocity, V_1; the uniform gas being discontinuous at the surface in respect of density and velocity, but continuous in respect of mean *vis viva*; the density and the mean relative-velocity on either side of the plane surface being that of the varying gas at a distance proportional to the mean path of a molecule.

Maxwell by a law of force (which he had arrived at from his experiments on viscosity* as the fifth power of the distance) obtained a numerically different, but otherwise, essentially, the same law.

In a communication—"On the dimensional properties of matter in the gaseous state†"—I have fully discussed this action, of the linear redistribution by the convections; confirming and extending Clausius' explanation.

In that paper, by making use of the arbitrary constant s for the *mean-range*, or distance from the plane at which the molecules crossing the plane receive their characteristics as those of a uniform gas in motion with the mean velocity, V, of the molecules which cross in unit of time, the assumption that this distance is proportional to the mean path is avoided, and this is important where the mean path (λ) is of the same order as the dimensions, σ, of the molecule or grain.

In these analyses account has not been taken of any effects of conduction: so that, neither Clausius' nor Maxwell's, nor yet my own previous method is directly applicable for the determination of the rates of linear dispersion of linear inequalities in a medium in which σ and λ are of the same order, or in which λ/σ is small.

It thus appears that to render the analysis general these methods must be extended by taking account of the expressions (159), (162), (165), for the rates of flux by conduction of momentum, as well as of *vis viva* in terms of λ and σ; so as to obtain expressions for the *mean-ranges* of *mass*, momentum, and *vis viva*, as determined by conduction as well as by convection.

158. The analysis, to be general, must take account of all possible variations in the arrangement of the grains.

But in the first instance it is obviously expedient to restrict the arrangement of the grains, to be considered, to those which have three axes, at right angles, of similar arrangement, as in the octahedral formation; in which cases,

* "On the Dynamical Theory of Gases," *Phil. Trans. Royal Soc.*, p. 49, 1866.

† *Phil. Trans. Royal Soc.*, 1879, Part II.

whatever may be the formation, equilibrium is secured when the internal arrangement of the medium is uniform along each of the three axes; and the external actions on the medium over planes which are perpendicular to the axes are also uniform.

159. *Mean-ranges.*

Having obtained expressions for the rates of flux of mass, momentum, and *vis viva*, respectively, by conduction as well as by convection, for any group of grains in any direction, in a uniform medium, it remains to analyse these expressions so as to obtain the component mean-ranges of mass, momentum, and *vis viva*.

It is to be noticed that mass and *vis viva* are scalar, while momentum or velocity is vector; and that this fact gives the mean-ranges of momentum and velocity a different significance from those of mass, and *vis viva* or energy.

The mean-range of convection by grains in the direction of their actual motion, whatever they may convey, is λ. And the mean-range of conduction, at encounters between pairs of grains in the direction of the normal, whatever is conducted, is σ.

160. *The component mean-ranges.*

The respective component mean-ranges of conduction and convection are obtained by multiplying the components of the rate of flux by convection, in the direction of the elementary group, by the component of λ in that direction, and the component rate of flux by conduction, in the direction of the elementary group, by the component of σ in that direction, respectively, integrating for the general group and dividing by the integral flux for the same group.

The component mean-range of mass.

As mass is not conductible the mean-range of conduction is zero. The component mean-range—that of convection—is then obtained from equation (175) as

$$\frac{\int_0^{2\pi}\int_0^{2\pi} \lambda n V_1' \, d\left(-\frac{\cos^3\theta}{3}\right) d\phi}{\int_0^{\pi}\int_0^{2\pi} n V_1' d\left(\frac{\sin^2\theta}{2}\right) d\phi} = \tfrac{2}{3}\lambda \qquad (206).$$

161. *The component mean-range of momentum or component velocity.*

In equations (158) and (163) if the factors for convection and conduction under the signs of integration are multiplied respectively by $\lambda\cos\theta$ and $\sigma\cos\theta$, and integrated with respect to θ from $\theta=0$ to $\theta=\pi/2$, $\phi=0$ to $\phi=\pi/2$ and divided by the respective integrals of the flux, between the same limits, the component ranges of momentum in the direction of the momentum, by convection and conduction, respectively, are found to be

$$\tfrac{3}{4}\lambda \text{ and } \tfrac{3}{4}\sigma.$$

And performing the same operation on equations (160) and (166), the component mean-ranges of momentum at right angles to the direction of the momentum, by convection and conduction, respectively, are

$$\tfrac{3}{8}\lambda \text{ and } \tfrac{3}{8}\sigma.$$

162. *The mean-range of vis viva.*

Multiplying the convections and conductions, under the signs of integration, in the three equations (172), (171), (174) respectively by $\lambda\cos\theta$ and $\sigma\cos\phi$ and dividing by the respective integral rates of flux, the respective mean-ranges are found to be, for convection and conduction,

For actual energy	$\tfrac{2}{3}\lambda$	and	$\tfrac{2}{3}\sigma$,	coefficient	$\tfrac{2}{3}$.
Direct displacement	$\tfrac{4}{5}\lambda$	,,	$\tfrac{4}{5}\sigma$,	,,	$\tfrac{26}{35}$.
Lateral ,,	$\tfrac{8}{15}\lambda$	,,	$\tfrac{8}{15}\sigma$,	,,	$\tfrac{24}{35}$.

The mean-ranges of momentum and *vis viva*, inasmuch as they are expressed in terms of λ and σ, are general when λ has the value expressed in equation (146).

It should be noticed that while the mean-range of the grains in an elementary group is λ, the mean path from centre to centre, owing to conduction, the mean-range of the velocities and the squares of the velocities are respectively extended to

$$\lambda + \frac{\sqrt{2}}{3}\sigma f\left(\frac{\sigma}{\lambda}\right) \text{ and } \lambda + \frac{\sigma}{3} f\left(\frac{\sigma}{\lambda}\right);$$

that is to say the velocity of the grain is not determined by the mean condition at the centre of the grain at which it last undergoes encounter, but at a position further back; and this becomes of fundamental importance when λ/σ is small.

163. *The mean characteristics of the state of the medium.*

The mean quantities which define the state of a (spherical) granular medium in uniform condition are

(1) $\sigma^3/\sqrt{2}$, the mass of a grain,

(2) the constants in the expression $f\left(\frac{\sigma}{\lambda}\right)$, Art. 102,

(3) u'', v'', w'', the mean velocities of the medium,

(4) N, the number of grains in unit volume,

(5) α, where $3x^2/\sqrt{2} = (V_1'V')''$.

Of these five mean characteristics (1) and (2) stand in different position from the rest, (1) being constant in time and (2) depending on the ultimate arrangement of the grains, and the consideration of these may be deferred.

The mean characteristics (3), (4) and (5) all enter into the definition of the state of a medium in uniform condition.

164. *Characteristic velocities, densities and mean-velocities of the grains.*

From equation (136) it appears that, referred to axes moving with the mean motion of the medium (u'', &c.), the number of grains having velocities between V_1' and $V_1' + \delta V_1'$ in directions which referred to the centre meet the surface of a sphere of unit radius in the small element $d(\cos\theta)\, d(\phi)$, is

$$\frac{n}{4\pi} d\cos\theta\, d\phi = \frac{N}{(\pi)^{\frac{3}{2}}} \left(\frac{V_1'}{\alpha}\right)^2 e^{-\left(\frac{V_1'}{\alpha}\right)^2} d\left(\frac{V_1'}{\alpha}\right) d\theta\,.\,d\phi \text{ ....(207).}$$

Dividing by N

$$\frac{n}{4\pi N} d\cos\theta\, d\phi = \frac{1}{(\pi)^{\frac{3}{2}}} \left(\frac{V_1'}{\alpha}\right)^2 e^{-\left(\frac{V_1'}{\alpha}\right)^2} d\left(\frac{V_1'}{\alpha}\right) d\theta\,.\,d\phi \text{ ....(208).}$$

If then in one state of the medium α has the value α_1, and in another state has the value $\alpha_2 = \alpha_1 (1 + \partial\alpha_1/\alpha_1)$, the characteristic velocities, for which

$$\frac{n_1}{N_1} = \frac{n_2}{N_2} \text{ ....(209),}$$

will be V_1' and $V_2' = V_1'(1 + \partial\alpha_1/\alpha_1)$.

The inequality between the characteristics is:

$$V_1' \frac{\partial\alpha_1}{\alpha_1}.$$

In the same way for the characteristic densities if the numbers of grains in the two states are N_1 and $N_2 = N_1\left(1 + \frac{\partial N_1}{N_1}\right)$ the characteristic numbers of the two states are

$$n_1 \text{ and } n_1\left(1 + \frac{\partial N_1}{N_1}\right),$$

with the inequality $n_1 \dfrac{\partial N_1}{N_1}$.

And if u'' and $u''(1 + \partial u''/u'')$ are mean component velocities in the two states the characteristics are

$$u_1'' \text{ and } u_2'' = u_1''\left(1 + \frac{\partial u_1''}{u_1''}\right) \text{ ....(210).}$$

165. *Characteristic rates of flux when the axes are fixed.*

Putting $l = \cos\theta$, $m = \sin\theta\cos\phi$, $n = \sin\theta\sin\phi$ for the direction cosines of the normal at contact of a pair of grains referred to axes moving with the mean motion of the medium, in the directions of x, y, z, and remembering that the range of convection is λ while that of conduction is σ, that for momentum the rates of the fluxes are $\sqrt{2}\sigma f\left(\frac{\sigma}{\lambda}\right)\Big/3\lambda$ and for *vis viva* $\sigma f\left(\frac{\sigma}{\lambda}\right)\Big/3\lambda$, and putting $\partial({}_cQ)_{xx}$, &c., and $\partial({}_pQ)_{xx}$ for the respective rates of convection and conduction of an elementary group in direction defined by $-d(\cos\theta)\,d\phi$, with respect to fixed axes; for the flux of mass we have by equation (175)

$$\partial\,({}_cQ_1)_{xx} = \rho\left(u'' + \frac{\lambda V_1'}{\lambda}\cos\theta\right)\frac{n_1}{N_1}\,d\left(\frac{-\cos\theta}{4\pi}\right)d\phi,\ \&c.,\ \&c.\ \ldots(211).$$

And by the last Art.

$$\partial\,({}_cQ_2)_{xx} = \partial\,({}_cQ_1)_{xx} + \left(\delta(\alpha)\frac{d}{d\alpha} + \delta(u'')\frac{d}{du''} + \delta(N)\frac{d}{dN}\right)\partial\,({}_cQ_1)_{xx},$$

whence the inequality of flux is

$$\partial\,({}_cQ_2)_{xx} - \partial\,({}_cQ_1)_{xx} = \left(\delta(\alpha)\frac{d}{d\alpha} + \delta(u'')\frac{d}{du''} + \delta(N)\frac{d}{dN}\right)\partial({}_cQ_1)_{xx}\ldots(212).$$

Equation (212) is general and Q may represent mass, momentum or *vis viva*.

166. *Rates of convection and conduction of momentum by an elementary group.*

Substituting the mean-rate of flux of momentum by convection, and noticing that the component mean-path is increased from $\lambda\cos\theta$ to $\lambda(u'' + V_1'\cos\theta)/V_1'$ while the conduction is not altered by the mean-motion—omitting the square of the mean-motion and dividing out the λ, we have:—

For direct action referred to fixed axes

$$\left.\begin{array}{l}\partial\,({}_cQ_1)_{xx} + \partial\,({}_pQ_1)_{xx} = \rho\left\{(u'' + V_1'\cos\theta)^2 + \dfrac{\sqrt{2}}{3}\dfrac{\sigma}{\lambda}f\left(\dfrac{\sigma}{\lambda}\right)V_1'^2\cos^2\theta\right\}\dfrac{n_1}{N}d(-\cos\theta)\,d\phi \\ \qquad\&c. \qquad\qquad\qquad\qquad \&c. \\ \qquad\&c. \qquad\qquad\qquad\qquad \&c.\end{array}\right\}$$

$$\ldots\ldots\ldots\ (213).$$

For lateral action

$$\partial\,({}_cQ_1)_{yx} + \partial\,({}_pQ_1)_{yx} = \rho\left\{(u'' + V_1'\cos\theta)(v'' + V_1'\sin\theta\cos\phi)\right.$$

$$\left. + \frac{\sqrt{2}}{3}\frac{\sigma}{\lambda}f\left(\frac{\sigma}{\lambda}\right)V_1'\cos\theta\sin\theta\cos\phi\right\}\frac{n_1}{N}d(-\cos\theta)\,d\phi\ldots\ldots(214).$$

167. *For the rate of displacement of vis viva by an elementary group referred to fixed axes.*

Taking, as before, $\lambda(u'' + V_1'\cos\theta)/u'$ for λ and omitting, for the sake of simplicity, all quantities of the second order, such as u''^2/λ and $\lambda\sigma^2$, we have for the direct rate of displacement

$$\left.\begin{array}{l}\partial\,(Q_1)_{xx} = \rho\left\{(u'' + V_1'\cos\theta)(u'' + V_1'\cos\theta)^2\right. \\ \quad + \left[\dfrac{\sqrt{2}\sigma}{3\lambda}f\left(\dfrac{\sigma}{\lambda}\right)V_1'^2(u''\cos\theta + v''\sin\theta\cos\phi + w''\sin\theta\sin\phi)\cos^3\theta\right. \\ \quad \left.\left. + \dfrac{\sigma}{3\lambda}f\left(\dfrac{\sigma}{\lambda}\right)V_1'^2\cos^3\theta\right]\right\}\dfrac{n_1}{N}\dfrac{d(-\cos\theta)\,d\phi}{4\pi}\end{array}\right\}\ldots(215).$$

The first term within the brackets on the right, which is the convection term, becomes, omitting the terms of second order,

$$3u'' V_1'^2 \cos^2\theta + V_1'^3 \cos^3\theta.$$

One part of the first of these two terms expresses the rate of displacement of mean *vis viva* by u''; while the remainder of this term expresses the displacement of the inequality of *vis viva* $(2u'' V_1' \cos^2\theta)$ by V_1'.

The second of the two terms, which changes sign with $\cos\theta$, expresses the displacement $(V_1'^2 \cos^2\theta)$ by $V_1' \cos\theta$.

The second term within the brackets expresses the displacement resulting from conduction on the mean normal velocity, and this does not change sign with $\cos\theta$.

For the lateral action

$$\left.\begin{aligned} \partial(Q_1)_{yx} = \rho \Bigg\{ & (u'' + V_1' \cos\theta)(v'' + V_1' \sin\theta\cos\phi)^2 \\ & + \left[\frac{\sqrt{2}\sigma}{3} f\left(\frac{\sigma}{\lambda}\right) V_1'^2 (u''\cos\theta + v''\sin\theta\cos\phi + w''\sin\theta\sin\phi)\cos\theta\sin^2\alpha\cos^2\phi \right. \\ & \left. + \frac{\sigma}{3\lambda} f\left(\frac{\sigma}{\lambda}\right) V_1'^2 \cos\theta\sin^2\theta\cos^2\phi \right]\Bigg\} \frac{n_1}{N}\, d(-\cos\theta)\, d\phi \end{aligned}\right\} \quad \text{.........(216).}$$

168. *The inequalities in the mean rates of flux of mass, momentum and vis viva resulting from space variations in the mean characteristics in a medium of equal spherical grains.*

When the mean state of the medium *varies continuously* from point to point, so that $(\lambda/N)(dN/dx)$,

$$\left\{\lambda + \sqrt{2}\sigma f\left(\frac{\sigma}{\lambda}\right) \Big/ 3\right\} \frac{d\alpha}{\alpha} \Big/ dx, \quad \left\{\left(\lambda + \sqrt{2}\sigma f\left(\frac{\sigma}{\lambda}\right) \Big/ 3\right) \Big/ \alpha\right\} du''/dx,$$

and $(\lambda/\alpha)\, d\alpha/\alpha dt$ are of the first order of small quantities, the mean characteristics N, α, u'', &c., obtained by integrating over a unit of volume, taking account of the motion in all directions, are taken as the mean characteristics at the centre P of the unit element.

Then it follows that if PQ represents a distance r of the order $\lambda + \sigma$, having a direction defined by l, m, n, the characteristics at Q will, to the first order of small quantities, be, putting I for any one of the characteristics,

$$I_Q = I_P + r\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right) I_P \quad \text{..............(217).}$$

If, then, r is the range of I, whether it is λ, $\sqrt{2}\sigma f\left(\frac{\sigma}{\lambda}\right)\Big/3$ or $\sigma f\left(\frac{\sigma}{\lambda}\right)\Big/3$, as the case may be, and it be assumed that the group of grains arriving at P, from the direction of Q, *arrive* as from a *uniform* medium having characteristics which are the mean characteristics at Q, the inequalities in the mean rates of flux at p would be obtained by substituting

$$I_Q - I_P = r\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right) I_P \text{ ....(218)}$$

for $\partial(I)$ and integrating $\iint \partial(I) \sin\theta \, d\theta \, d\phi$ for the partial groups.

There is however nothing in the definition of the mean characteristics, at a point, in a *varying* medium, as stated above, to warrant the assumption that the grains arriving from the direction Q will arrive at P with the mean characteristics of the medium at Q.

The mean characteristics are the means of all the groups at Q, whereas the grains arriving at P from Q must, unless PQ is at right angles to the direction in which the medium varies, differ from the mean at Q taken in all directions; and therefore cannot have the mean characteristics at Q. It is necessary therefore to obtain further evidence before we can determine what are the characteristics of the elementary groups in different directions, which evidence is found in the conditions of equilibrium of the varying medium.

169. *The conditions between the variations in the mean characteristics, α, u'', &c., N or ρ, in order that a medium, in which σ and the constants in $f\left(\frac{\sigma}{\lambda}\right)$ are constant, may be in steady condition with respect to all the characteristics.*

The condition of equilibrium of a medium in mean uniform condition requires that u'', α and N should each be constant for all positions and all directions; so that in a medium in which any one of these mean characteristics varies, the rest being constant, the equilibrium would be disturbed. But it does not follow that equilibrium would be impossible if two or more of the mean characteristics vary.

For the case where σ/λ is small these general conditions have been already determined, in the study of the conduction of heat by Clausius*, and more generally, in the study of the dimensional properties of matter in the gaseous state†. In the latter instance, this was accomplished by the recognition that if the mean characteristics, u'', α, N, of flux by a mean group of molecules arriving at P were the mean characteristics of the medium at Q, PQ being the range of the characteristics, the three conditions

* *Pogg. Ann.*, Jan. 1862; *Phil. Mag.*, June 1862.

† *Phil. Trans. Royal Soc.*, 1897, Part II. pp. 786–803.

of steady density, steady momentum and steady *vis viva*, could not be satisfied; whereas if the characteristics, α and N, of the flux arriving at P from Q were the characteristics at Q, while instead of the characteristics u'', v'', w'' at Q arbitrary functions of x, y, z (U, V, W) are taken for the mean velocities of the arriving group, all the conditions could be satisfied; and the values of U, V, W be determined in terms of u'', v'', w'', α and N.

This method may be applied for the determination of the conditions between the mean characteristics, U, α, N and u'', when $\frac{\sigma}{\lambda}$ is large as when small, now that the expressions for the mean rates of flux and mean ranges, resulting from conduction, have been determined, as well as those resulting from convection, in *a uniform medium*.

170. *The equation for the mean flux.*

Substituting U for u'', &c. in the expressions for the characteristic rates of flux by an elementary group ( ), remembering that λ is the range of convection and σ the range of conduction, that

$$\left.\begin{array}{ll} \left.\begin{array}{l} \partial N = \lambda\left(l\dfrac{dN}{dx} + m\dfrac{dN}{dy} + n\dfrac{dN}{dz}\right), \\ \partial\alpha = \lambda\left(l\dfrac{d\alpha}{dx} + m\dfrac{d\alpha}{dy} + n\dfrac{d\alpha}{dz}\right), \end{array}\right\} & \text{For convection} \\ \partial\alpha = \sigma\left(l\dfrac{d}{dx} + m\dfrac{d}{dy} + n\dfrac{d}{dz}\right)\alpha, & \text{For conduction} \end{array}\right\} \quad \ldots\ldots(219),$$

in the expression for the inequality of the flux, and integrating from $\theta = 0$ to $\theta = \pi$ and from $\phi = 0$ to $\phi = 2\pi$, the equation for the mean flux is obtained to a first approximation.

For the flux of mass.

From equations (176), the equation for the flux of mass in direction of x is:—

$$\rho u'' = \rho U - \frac{2}{3}\frac{\lambda}{\sqrt{\pi}}\left(\alpha\frac{d\rho}{dx} + \rho\frac{d\alpha}{dx}\right), \text{ \&c., \&c.} \quad \ldots\ldots\ldots\ldots(220).$$

Equation (220) has reference to fixed axes, for moving axes the equations become

$$0 = \rho''(U - u'') - \frac{2}{3}\frac{\lambda}{\sqrt{\pi}}\left\{\alpha\frac{d\rho''}{dx} + \rho''\frac{d\alpha}{dx}\right\}, \text{ \&c., \&c.} \quad \ldots\ldots(221).$$

These equations define the values U, V, W in terms of the characteristics (u'', α, ρ or N), the mean characteristics at the point.

For the rates of flux of momentum to a first approximation.

From the first of equations (213) the rates of direct flux of momentum become, to a first approximation, assuming λ *to be the same in all directions,*

$$\left.\begin{aligned}
&\rho''(u'u')''+p''_{xx}=\left\{1+\frac{\sqrt{2}}{3}\frac{\sigma}{\lambda}f\left(\frac{\sigma}{\lambda}\right)\right\}\cdot\\
&\qquad\left\{\rho''\frac{\alpha^2}{2}+\frac{4}{3}\frac{\lambda}{\sqrt{\pi}}\left[\alpha\frac{d}{dx}[\rho(U-u'')]+\rho(U+u'')\frac{d\alpha}{dx}\right]\right\},\ \&c., \&c.\\
&\textit{For lateral flux.}\\
&\text{From the second of (213) the equations become}\\
&\rho''(u'v')''+p''_{xy}=\left\{1+\frac{\sqrt{2}}{3}\frac{\sigma}{\lambda}f\left(\frac{\sigma}{\lambda}\right)\right\}\frac{2}{3}\frac{\lambda}{\sqrt{\pi}}\left[\alpha\left\{\frac{d}{dx}[\rho''(V-v'')]\right.\right.\\
&\qquad\left.\left.+\frac{d}{dy}[\rho''(U-u'')]\right\}+\rho''(V-v'')\frac{d\alpha}{dx}+\rho''(U-u'')\frac{d\alpha}{dy}\right]
\end{aligned}\right\}\ldots(222).$$

For the rates of flux of vis viva to a first approximation.

From equations (215) the equations for the rate of flux of direct *vis viva* become

$$\left.\begin{aligned}
&\{\rho''(u'u'+p_{xx})u'\}''=\frac{3}{2}\left\{1+\frac{\sqrt{2}\sigma}{15\lambda}f\left(\frac{\sigma}{\lambda}\right)\right\}(U-u'')\rho''\alpha^2\\
&\qquad\qquad-\frac{4}{5}\left\{\lambda+\frac{\sigma^2}{3\lambda}f\left(\frac{\sigma}{\lambda}\right)\right\}\left[\alpha^3\frac{d\rho''}{dx}+\rho''\frac{d\alpha^3}{dx}\right]\\
&\textit{For lateral flux.}\\
&\text{From equation (216)}\\
&[\rho''(u'v'+p_{xy})v']''=\frac{1}{2}\left\{1+\frac{\sqrt{2}\sigma^2}{15\lambda}f\left(\frac{\sigma}{\lambda}\right)\right\}(U-u'')\rho\alpha^2\\
&\qquad\qquad-\frac{4}{15}\left\{\lambda+\frac{\sigma^2}{3\lambda}f\left(\frac{\sigma}{\lambda}\right)\right\}\left[\alpha^3\frac{d\rho}{dx}+\rho\frac{d\alpha^3}{dx}\right]
\end{aligned}\right\}\ldots(223).$$

The values of $U-u''$, &c., as defined in equations (221), are small quantities of the first order. Hence as these quantities, and their space variations, enter into the rates of momentum as factors of the small distances λ and σ only, the terms into which they enter are all of the second order of small quantities, as compared with p, and may therefore be neglected as being within the limits of approximation. Omitting these terms from equations (222), the rates of flux of momentum to the first order of small quantities are by convection:

$$\left.\begin{aligned}
&\rho''(u'u')''=\rho''\frac{\alpha^2}{2},\ \&c., \&c.,\\
&\rho''(u'v')''=0,\ \&c., \&c.,\\
&\text{and by conduction, equation (159),}\\
&p''_{xx}=\frac{\sqrt{2}}{3}\frac{\sigma}{\lambda}f\left(\frac{\sigma}{\lambda}\right)\rho''\frac{\alpha^2}{2},\ \&c., \&c.,\\
&p''_{xy}=0,\ \&c., \&c.
\end{aligned}\right\}\ldots\ldots\ldots\ldots(224).$$

The total rates of flux of momentum being

$$\left.\begin{aligned} p''_{xx} + \rho''(u'u')'' &= \left\{1 + \frac{\sqrt{2}}{3}\frac{\sigma}{\lambda} f\left(\frac{\sigma}{\lambda}\right)\right\}\rho\frac{\alpha^2}{2} \\ p''_{xy} + \rho''(u'v')'' &= 0, \text{ \&c., \&c.} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(225).$$

Substituting in equations (223) the values of $U - u''$, &c., as obtained from equations (221), the rates of flux of the *vis viva* of the component motions become by transformation:

$$\left.\begin{aligned} (\rho''u'u'u')'' &= \frac{\lambda}{15}\frac{\alpha}{\sqrt{\pi}}\left(3\alpha^2\frac{d\rho}{dx} - \frac{21}{2}\rho\frac{d\alpha^2}{dx}\right) \\ (p_{xx}u')'' &= -\frac{\sigma}{15}\frac{f\left(\frac{\sigma}{\lambda}\right)\alpha}{\sqrt{\pi}}\left\{\left(\frac{4\sigma}{\lambda} - \sqrt{2}\right)\alpha^2\frac{d\rho''}{dx}\right. \\ &\qquad\qquad \left. + \left(\frac{6\sigma}{\lambda} - \frac{1}{\sqrt{2}}\right)\rho''\frac{d\alpha^2}{dx}\right\} \\ \left.\begin{aligned}(\rho u'v'v')'' &= \tfrac{1}{3}(\rho u'u'u')'' \\ (p_{xy}v')'' &= \tfrac{1}{3}(p_{xx}u')''\end{aligned}\right\} &\text{ by equation (223)} \end{aligned}\right\} \text{, \&c., \&c. } \ldots(226).$$

And for the rate of flux of the total *vis viva*

$$\left\{\begin{aligned} &\rho u'u'u' + p_{xx}u' \\ &+ \rho u'v'v' + p_{xy}v' \\ &+ \rho u'w'w' + p_{xz}w' \end{aligned}\right\}'' = \frac{1}{9}\frac{\alpha}{\sqrt{\pi}}\left\{\begin{aligned} &\lambda\left(3\alpha^2\frac{d\rho}{dx} - \frac{21}{2}\rho\frac{d\alpha^2}{dx}\right) \\ &-\sigma f\left(\frac{\sigma}{\lambda}\right)\left(\frac{4\sigma}{\lambda} - \sqrt{2}\right)\alpha^2 d\rho - \left(\frac{6\sigma}{\lambda} - \frac{1}{\sqrt{2}}\right)\rho\frac{d\alpha^2}{dx} \end{aligned}\right\}$$

$$\text{\&c., \&c. } \ldots\ldots\ldots(227).$$

The equations (221) to (227) as they stand are perfectly general.

So far however these equations satisfy the conditions of steady density and steady *vis viva*, only, on the supposition that the conditions of mean-mass are satisfied. And these conditions explicitly involve the space variations of λ; as is at once seen from equations (225).

171. *The conditions of equilibrium of mass referred to axes moving with the mean motion of the medium.*

Differentiating equations (225) with respect to x, y, z, respectively, and transforming, the general conditions of the equilibrium of mass may be expressed as

$$\frac{d\rho}{dx} = \rho\left\{\frac{\sqrt{2}\sigma f\left(\frac{\sigma}{\lambda}\right) - \sqrt{2}\lambda a^2b^2e^{-\frac{b^2\lambda}{\sigma}}}{3\lambda + \sqrt{2}\sigma f\left(\frac{\sigma}{\lambda}\right)}\frac{1}{\lambda}\frac{d\lambda}{dx} - \frac{1}{\alpha^2}\frac{d\alpha^2}{dx}\right\}\text{, \&c., \&c.} \ldots(228),$$

and from equation (146), differentiating and transforming,

$$-\frac{d\rho}{dx} = \rho\frac{3\lambda + \lambda\sqrt{2}\,a^2b^2e^{-\frac{b^2\lambda}{\sigma}}}{3\lambda + \sqrt{2}\,\sigma f\left(\frac{\sigma}{\lambda}\right)}\frac{1}{\lambda}\frac{d\lambda}{dx}\text{, \&c., \&c.} \ldots\ldots\ldots\ldots(229).$$

Adding the equations (228) and (229) the condition of equilibrium is

$$0 = \rho\left(\alpha^2 \frac{d\lambda}{dx} - \lambda \frac{d\alpha^2}{dx}\right), \text{ \&c., \&c. } \ldots\ldots\ldots\ldots\ldots\ldots(230).$$

The rates of flux of vis viva when the medium is in equilibrium.

Substituting in the first and second of equations (226), (227) respectively from equation (228) the respective rates of direct flux by convection and conduction are expressed as:

$$\left.\begin{aligned}
\rho''(u'u'u')'' &= -\frac{\alpha}{15\sqrt{\pi}}\left\{6\lambda^2 \frac{3+\sqrt{2}\,a^2b^2e^{-\frac{b^2\lambda}{\sigma}}}{3\lambda+\sqrt{2}\,\sigma f\left(\frac{\sigma}{\lambda}\right)} + 21\lambda\right\}\rho\frac{1}{2}\frac{d\alpha^2}{dx}, \text{ \&c., \&c.} \\
(p_{xx}u')'' &= -\frac{\alpha}{15\sqrt{\pi}}\left\{-2(4\sigma-\sqrt{2}\lambda)\,\sigma f\left(\frac{\sigma}{\lambda}\right)\frac{3+\sqrt{2}\,a^2b^2e^{-\frac{b^2\lambda}{\sigma}}}{3\lambda+\sqrt{2}\,\sigma f\left(\frac{\sigma}{\lambda}\right)}\right. \\
&\qquad \left. + \left(\frac{12\sigma}{\lambda}-\sqrt{2}\right)\sigma f\left(\frac{\sigma}{\lambda}\right)\right\}\rho\frac{1}{2}\frac{d\alpha^2}{dx}, \text{ \&c., \&c.}
\end{aligned}\right\} \ldots(231),$$

the respective rates of lateral convection and conduction being one-third of the corresponding direct rates.

Adding the respective members of the equations (231) the expression for the total rate of direct flux of *vis viva* by convection and conduction is:

$$\begin{aligned}
[(\rho''u'u' + p_{xx})\,u']'' = -\frac{\alpha}{15\sqrt{\pi}}&\left\{\left(6\lambda^2 - 2(4\sigma-\sqrt{2}\lambda)\sigma f\left(\frac{\sigma}{\lambda}\right)\right)\frac{3+\sqrt{2}a^2b^2e^{-\frac{b^2\lambda}{\sigma}}}{3\lambda+\sqrt{2}\sigma f\left(\frac{\sigma}{\lambda}\right)}\right. \\
&\left. + 21\lambda + \left(\frac{12\sigma}{\lambda}-\sqrt{2}\right)\sigma f\left(\frac{\sigma}{\lambda}\right)\right\}\rho\frac{1}{2}\frac{d\alpha^2}{dx}, \text{ \&c., \&c. } \ldots\ldots(232).
\end{aligned}$$

Then, since the rates of flux of lateral *vis viva* are each one-third of the normal rate, the total rate becomes

$$\begin{aligned}
\left\{\begin{aligned}&(\rho''u'u' + p_{xx})\,u' \\ &+(\rho''u'v' + p_{xy})\,v' \\ &+(\rho''u'w' + p_{xz})\,w'\end{aligned}\right\}'' = \frac{\alpha}{9\sqrt{\pi}}&\left\{\left(6\lambda^2 + 2(4\sigma-\sqrt{2}\lambda)\sigma f\left(\frac{\sigma}{\lambda}\right)\right)\frac{3+\sqrt{2}a^2b^2e^{-\frac{b^2\lambda}{\sigma}}}{3\lambda+\sqrt{2}\sigma f\left(\frac{\sigma}{\lambda}\right)}\right. \\
&\left. + 21\lambda + \left(\frac{12\sigma}{\lambda}-\sqrt{2}\right)\sigma f\left(\frac{\sigma}{\lambda}\right)\right\}\rho\frac{1}{2}\frac{d\alpha^2}{dx}, \text{ \&c., \&c. } \ldots\ldots(233).
\end{aligned}$$

The equations from (221) to (233) are perfectly general to a first approximation of the inequalities, the axes moving with the mean motion of the medium, the medium being in steady condition, and the arrangement such that a^2 and b^2 are constant.

172. *The coefficients of the component rates of flux of* $(\sigma^3 . \alpha^2/2\sqrt{2})$ *the mean component vis viva of the grain.*

By equation (129 B) Section VIII.

$$\rho = N \frac{\sigma^3}{\sqrt{2}}.$$

Substituting this in equations (233), dividing by N and putting $(C_2^2 + D_2^2)$ for the product of the first two factors of the member on the right, these members take the form:

$$(C_2^2 + D_2^2) \frac{\sigma^3}{\sqrt{2}} \frac{d}{dx} \left(\frac{\alpha^2}{2}\right),$$

as expressing the relative rates of flux of the *vis viva* of the grains across surfaces moving with the mean motion of the medium.

These rates expressed by the space rates of variation of the *vis viva* of the grains multiplied by the coefficient $(C_2^2 + D_2^2)$ express the rate of flux under the condition of steady motion.

But as long as the scales of the variation of α^2 are sufficiently large, as compared with the squares of the scale of the relative mass and the mean paths, to come within the limits of approximation for the maintenance of mean and relative systems, the rates at each point will be approximately the same as under the conditions of equilibrium.

Then if the inequalities of mean motion are so small that the inequalities instituted in N, λ, and α may be neglected as compared with N, λ, α, *i.e.* if the scales of mean motion are sufficiently large and the inequalities sufficiently small, the coefficients C_2^2 and D_2^2, which are respectively the coefficients for convection and conduction, may be taken as constants within the limit of approximation.

173. *The rate of dispersion of linear inequalities in the vis viva of the grains.*

Putting

$$\frac{1}{N}\frac{\partial_3}{\partial_3 t} \breve{I}_{xx} = \frac{1}{N}\frac{d}{dx}\left[(p_{xx} + \rho u'u')\, u' - (p_{xy} + \rho u'v')\, v' - (p_{xz} + \rho u'w')\, w'\right] \ldots (234),$$

we have

$$\left.\begin{aligned} \frac{1}{N}\frac{\partial_3}{\partial_3 t} \breve{I}_{xx} &= -(C_2^2 + D_2^2)\frac{\sigma^3}{\sqrt{2}}\frac{d^2}{dx^2}\left(\frac{\alpha^2}{2}\right) \\ \frac{1}{N}\frac{\partial_3}{\partial_3 t} \breve{I}_{xy} &= -(C_2^2 + D_2^2)\frac{\sigma^3}{\sqrt{2}}\frac{d^2}{dy^2}\left(\frac{\alpha^2}{2}\right) \\ \frac{1}{N}\frac{\partial_3}{\partial_3 t} \breve{I}_{xz} &= -(C_2^2 + D_2^2)\frac{\sigma^3}{\sqrt{2}}\frac{d^2}{dz^2}\left(\frac{\alpha^2}{2}\right) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (235).$$

Thus although not vectors the component rates of redistribution depend severally on the component inequalities, and admit of separate expressions which when added together give the expression

$$\frac{1}{N}\frac{\partial_3}{\partial_3 t}\breve{I} = (C_2^2 + D_2^2)\frac{\sigma^2}{\sqrt{2}}\nabla^2\left(\frac{\alpha^2}{2}\right) \quad \text{..................(236)}.$$

And multiplying by N

$$\frac{\partial_3 \breve{I}}{\partial_3 t} = (C_2^2 + D_2^2)\,\rho\,.\,\nabla^2\left(\frac{\alpha^2}{2}\right).$$

174. The expressions for the coefficients C_2^2 and D_2^2 involve the arbitrary constant b^2, so that the general expression cannot be completely interpreted until b^2 is defined. But the terms which depend upon b are very small except for states of the medium in which λ is greater than $\sigma/10$ or less than 10σ; so that outside these limits the coefficients are independent of b^2 within the limits of approximation.

Then, outside these limits, the expressions for C_2^2 and D_2^2, as appears from equation (233), when σ/λ is small, are, within the limits of approximation,

$$\left.\begin{aligned} C_2^2 &= \frac{3\lambda\alpha}{\sqrt{\pi}} \\ D_2^2 &= 0 \end{aligned}\right\} \quad \text{..............................(237)}.$$

And when σ/λ is large

$$\left.\begin{aligned} C_2^2 &= 0 \\ D_2^2 &= \frac{4}{3}\frac{\sigma^2}{\sqrt{\pi}}\frac{\alpha}{\lambda}\left(\frac{G}{4}\right) \end{aligned}\right\} \quad \text{...........................(238)}.$$

And these values become infinite in the limit.

175. *Summary and conclusions as to the rates of redistribution by relative motion.*

The equations (202) express, in terms of the quantities which define the relative motion of the medium, the rates of angular rearrangement of the relative-mass, by institution of relative motion, corresponding to the last term in equations (119) Section VI.

Equations (235) Section XII. express the linear redistribution of inequalities in *vis viva* of relative motion by the actions of convection and conduction corresponding to the second and third terms of equations (117 A) Section VI.

Equations (195) and (205) express the respective rates of angular redistribution of angular inequalities in the *vis viva* of relative motion, resulting from convections and conductions respectively, corresponding to the fourth term in equations (117 A).

The second term in the equations (119) Section VI. is the only term in the equations of mass which does not become zero when ρ'' is constant in time and space. Therefore equations (202) express the only redistributive actions on mass, equation (204), resulting from relative motion. These redistributions of relative-mass are essentially positive dispersions of unsymmetrical arrangement, at rates which are proportional to the inequalities in the arrangement of the mass. But subject to the same limit as the permanent diffusion, as λ/σ becomes small.

Thus the action of relative-motion on the mass is that of positive dispersion of all inequalities.

The second, third and fourth terms in equations (117 A) are the only terms in the equation which depend on relative motion only; that is, are the only terms in these equations that do not necessarily vanish when the *vis viva* of mean motion is constant.

Therefore the equations (195) and (204), Section XI., express the only redistributive actions on the *vis viva* resulting from relative motion.

From these equations it appears that all these actions are essentially dispersive of inequalities, at rates proportional to the inequalities multiplied by coefficients depending on the characteristics of the medium; the only limit being that imposed by the nearness of the grains, which is the same limit as that of permanent diffusion as expressed in equation (205).

It thus appears that to a first approximation the action of the relative motion on relative mass and relative *vis viva* is essentially that of positive dispersion of inequalities; in which the rates of linear dispersion, and of angular dispersion of *vis viva*, by convection, are subject to no limit, while those of angular rearrangement of mass and of angular dispersion of *vis viva* by conduction are subject to a finite limit as the grains become closer.

The generalization of the dispersive actions.

The numerical coefficients of the several rates of redistribution expressed in the equations (202), (195), (205) relate to a medium consisting of uniform spherical grains. But if, for these numerical coefficients, arbitrary constants are substituted, these equations become general, that is to say, they include all discontinuous media in which the separate members do not alter their shape or size.

Whence the conclusion follows, that discontinuous, purely mechanical media satisfy the condition for the maintenance of the state of relative motion.

SECTION XIII.

THE EXCHANGES BETWEEN THE MEAN- AND RELATIVE-SYSTEMS.

176. It has been shown (Sections XI. and XII.) that the effect of the relative motion is to disperse all inequalities in the mean *vis viva* of relative motion and in the arrangement of the mean-mass; the rates and the limits of these actions having been expressed in terms of the quantities which define the relative motion.

It remains therefore (1) to effect such analysis of the terms in the equations which express the effect of inequalities, in the mean-system, in instituting inequalities in the relative-system, as is necessary to define the actions they express, in terms similar to those in which the rates of redistribution are expressed; and (2), by combining the effects of the respective actions of institution and redistribution, to arrive at expressions for the resultant inequalities which may be maintained.

The only terms, which remain to be considered in the members on the right, of the equations of component *vis viva* of mean- and relative-motion (123) after transferring the first term on the right, which is the convection term:

$$\frac{1}{2}\frac{d}{dt}\left[{}_{c''}\rho\,(u')^2\right], \text{ \&c., \&c.,}$$

to the left member, are those terms which are concisely expressed as the fifth and sixth terms in equations (117 A).

Therefore these terms are the only terms which express exchanges of *vis viva* between the two systems taken as a whole. And since these terms do not become surface integrals they express the exchange, at points, of *vis viva* from the mean-system to the relative-system. And further, these terms are transformation terms solely; so that they each express, under the opposite sign, the exact rates of exchange as the corresponding terms in the equations (116 A). Thus the fifth term in equations (117 A) expresses the rate at which *vis viva* is received by the relative-system from the mean-system on account of the diminution of the abstract resilience in that system, while the sixth term in (117 A) expresses the rate of exchange of kinetic energy necessary in order to satisfy the condition of no energy in the residual system, the expressions under opposite signs being identical in the two systems.

177. *The initiation of inequalities in the state of the medium.*

Since the terms in (117 A) express the only actual rates of exchange of energy between the two systems, and the effects of the relative-system are purely dispersive, it at once appears that in a medium, in a state of general equilibrium, inequalities can be initiated only by acceleration of mean-motion, and whatever the state of the medium may be, all initiation of inequalities springs from acceleration of mean-motion as the prime cause. This being so, any rate of change which may result by transformation from inequalities in the mean-motion will be expressed as:

$$\frac{d_1(\;)}{d_1 t} \text{ or } \frac{\partial_1(\;)}{\partial_1 t},$$

according to whether or not the rate of convection $d_{c''}(\;)/dt$ is or is not included in the action.

In this way the joint actions of institution and redistribution are expressed as

$$\frac{d_1(\;)}{d_1 t} + \frac{d_2(\;)}{d_2 t}.$$

178. As presenting by far the greatest difficulty, and thus entailing the most discussion, the rates of institution of angular inequalities in the rates of conduction through the grains demand first consideration. These rates, it would seem, have not hitherto been the subject of analytical treatment; and although the expressions for these rates of institution are clearly distinguishable, now that the conductions are separated from the convections, the interpretation of these terms presents difficulties owing, partly, to the novelty of the conceptions involved.

It appears that the analysis of these conductions constitutes the kinetic theory of the abstract elastic properties in the mean-system of a granular medium, that is to say, properties of distortional elasticity.

The terms which express the rates of increase of abstract resilience in the mean-system are included in the last term but one in the right members of equations (116 A).

In a purely mechanical medium there is no resilience in the resultant system, so that these terms in the mean-system have their identical counterpart under the opposite sign in the corresponding equations of the relative-system. But that which has rendered this subject obscure, is that the counterpart is under different expressions.

This is owing to the generality of the equations, which are not confined to a purely mechanical medium. However, on changing the signs of the terms in (116 A) we have the interpretation of the corresponding terms in (117 A). These terms,

$$\left[\frac{p''}{3}\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right)+(p''_{xx}-p'')\frac{du''}{dx}\right.$$
$$\left.+\frac{1}{2}\left\{p''_{yx}\left(\frac{du''}{dy}+\frac{dv''}{dx}\right)+p''_{zx}\left(\frac{du''}{dz}+\frac{dw''}{dx}\right)\right\}\right], \text{ \&c., \&c.,}$$

represent the rate at which kinetic energy in directions x, &c. is being abstracted from the relative-motion to supply the abstract mean resilience, depending on conduction, to the mean-system of motion. This is obvious, as regards the first of the terms within the brackets, for the components in directions x, y and z. But as these represent uniform expansion multiplied by uniform pressure, both the expansion and pressure being equal in all directions, it introduces no angular inequalities in the relative *vis viva*. It is however these terms, or more strictly, the three corresponding terms for the directions x, y and z taken together, that, owing to their simplicity, reveal the *modus operandi* by which the conduction through grains, of changeless shape or volume, can affect the work done in contracting the space in which they exist.

It is not the conductions that are the active agents. But these conductions are a passive necessity of the space occupied by the grains; and thus measure the contraction of the freedom of the grains, owing to their volume. Whence, it is at once realized that the amount of increase of kinetic energy, which would result from a contraction of the entire space occupied, would not be the same as it would be if the grains, while conserving their mass, ceased to occupy volume. For in the latter case, taking V the velocity of the grains and ρ for the density, and supposing the action were what is called "isothermal," the velocity V remaining constant, the rate of displacement of momentum would not be $\rho V^2/3$, as it would be if the volumes of the grains were zero.

Neither would this stress vary with ρ but with $\rho\{1+\phi(\rho)\}$ where $\phi(\rho)$ represents virtual contraction of the space free to the motions of the centres of the grains. Thus the variation of the kinetic energy caused by a mean volumetric strain in the medium is increased by the proportion of the volume occupied by the grains to the exclusion of other grains. It is thus seen that it is this excess of work in any mean strain, resulting from the virtual space from which the grains shut each other out, that is measured by the conductions. These effects have been fully expressed in equations (158) and (159), Section X., and are easily realized in the case of volumetric strain. But it is quite a different matter to realize how a purely distortional strain, which neither affects the volume of the space nor the volume of the grains, can produce a virtual alteration of freedom open to the grains or inequalities in rates of conduction; and hence the importance of the evidence derived from the consideration of the volumetric strain in the interpretation of the results of distortional strains as expressed in the three last terms within the brackets. From these it appears at once that the action which determines the character of any effect there may be is rate of distortion, which also determines the rate of action, while the subject acted upon is the component of conduction induced by the distortional strain. In the first of these distortional terms, for instance,

$$\frac{1}{3}(p''_{xx} - p'')\frac{du''}{dx},$$

we see that all actions on the mean rates of conduction, expressed by p'', equal in all directions, are expressly excluded. The recognition of this is important as it shows the independence of the actions, in so far that if the distortional strain does not induce any change in the rate of conduction there is no effect. This raises the question: what is it that determines whether or not these distortional strains shall have any effect? And the answer to this is furnished from the experience derived from the volumetric strain. If the mean distortional strain, by altering the relative positions of the grains from what they would have been without the distortional strains, so alters the mean extent of freedom in the directions of the principal axes of the rates of strain, there will be effects, otherwise not. "Limiting the freedoms" is only an expression for altering the probable mean paths, and as a distortional strain consists essentially of strains in directions at right angles, such that one of these strains is of opposite sign and equal to the sum of the others, the action of a distortional strain is not to alter the mean density, nor if σ/λ is small the mean paths of the grains, taken in all directions, but to institute inequalities, increasing the mean paths in the directions in which the strain is positive, and decreasing them in those directions in which it is negative.

It becomes plain, therefore, (1) that no matter what the mass or number of grains may be, if the volumes are such that the space they occupy is negligible compared with the space through which they are dispersed, the effect of distortional strains on the conductions must also be negligible.

And (2) that any effect the distortional strains may produce on account of the size of the grains depends on the change in the angular arrangement of the grains, as measured by the angular inequalities in the mean paths, that may be instituted.

And from these two conclusions it appears definitely that the abstract exchanges of *vis viva*, from the mean system to the relative system, in consequence of distortional strain in the former, and the space occupied by the grains in the latter, depend solely on the angular arrangements, as they are here called, of the grains.

This general and definite conclusion brings into view, for the first time, the fundamental place which the conditions to be satisfied by the relative mass, as set forth in Section V., as resulting from first principles, occupy in the exchanges between the two systems.

It also calls our attention to the fact, pointed out in the preamble to Section IX., that the tacit assumption in the kinetic theory of gases, that the redistribution of *vis viva* entailed the redistribution of mass, has limited the application of this theory to circumstances in which the conductions are negligibly small, and reveals the necessity, for the general theory, of a study of the law of redistribution of mass resulting from the dispersion of mass as a subsequent effect of encounters, and as being in some respects independent of, and of equal importance with, Maxwell's law of redistribution of *vis viva*.

Although in such studies of the kinetic theory as I have seen I have not found any reference to the existence of such a law or the necessity of its study, in a recent reference to the celebrated paper by Sir George G. Stokes, "On the Equilibrium of Elastic Solids," I was much relieved to find that, in his discussion of Poisson's theory of elasticity, he expresses the opinion that it is important to take into account the possible effects of new relative positions which the molecules may take up, in which I recognise a reference to what I have called the angular distribution of the grains.

179. *The probable rates of institution of inequalities in the mean angular distribution of mass.*

When the condition of the granular medium is such that the probable mean path of a grain is the same in all directions—that is, when the mean of the paths of all the grains moving approximately in one direction is the same, whatever direction this may be—there are no angular inequalities in the arrangement of the grains. And when the means of the paths of grains moving approximately in the same directions are different for different directions, these differences serve to measure the inequalities in the angular arrangement of the grains.

And in exactly the same way the angular inequalities in the number of encounters between pairs of grains having relative-mean paths approximately in the same direction serve (and are rather more convenient) to measure the angular inequalities in the mass.

Such relative angular inequalities are instituted solely by distortional motion in the mean system. And the rate of distortion is one of the factors of the product which represents the rate of institution of the relative inequality; the other factor being the ratio of the average normal conduction of momentum at an average encounter of a pair of grains, divided by twice the average convection by a grain in the direction of its path.

By equation (147) the normal conduction at a mean collision is

$$\frac{2}{3}\sqrt{2}\,(V')''\,\sigma f\left(\frac{\sigma}{\lambda}\right),$$

and by equations (155) and (156), there are two mean paths traversed for each collision, and the mean displacement of momentum, by the convection of a grain between encounters, is $\lambda\,\bar{V}'$.

Therefore the ratio of the corresponding normal conductions and normal convections is

$$\frac{2}{3}\,\frac{\sqrt{2}\,(V')''\,\sigma}{2\,(V')''\,\lambda}\,f\left(\frac{\sigma}{\lambda}\right)=\frac{\sqrt{2}}{3}\,\frac{\sigma}{\lambda}\,f\left(\frac{\sigma}{\lambda}\right)\;\ldots\ldots\ldots\ldots\ldots\ldots(239).$$

And the rates of institution of relative angular inequalities in the arrangement of the mass are represented by

$$\frac{\partial_1 a'}{\partial_1 t}=\frac{\sqrt{2}}{3}\,\frac{\sigma}{\lambda}\,f\left(\frac{\sigma}{\lambda}\right)\left\{2\,\frac{du''}{dx}-\frac{2}{3}\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right)\right\},\ \text{\&c., \&c.*}\ldots(240).$$

This is, only, when u'', v'', w'' are referred to the principal axes of the rates of distortion. And da'/dt, db'/dt, dc'/dt, represent the relative rates of increase of the mean paths of pairs of grains having relative motion in the directions of x, y, and z respectively. The rates of relative increase of pairs of grains, having directions of motion other than the directions of the principal axes, are obtained from those in the directions of the principal axes as in the ellipsoid of strain.

Besides expressing the inequalities in the angular distribution of mass and in the mean relative paths, da', &c., express the rates of increase of the inequalities in the numbers of encounters between pairs of grains having relative velocities in the directions of the principal axes. But they do not, without further resolution, properly represent the rates of increase of the inequalities in the rates of conduction in the directions of the principal axes; since the directions of encounter, that is, the normals at encounter, may depart by anything short of a right angle from the direction of the relative motion of a pair.

Before proceeding to consider the relative-inequalities in the rates of conduction, however, it seems desirable to call attention to the distinction between *rates of strain* and *strains*.

It will be noticed, after what has already been said as to the difference between the effects of volumetric strains and distortional strains, that in what follows, the expressions da'/dt, &c. are used to express the rates of increase of relative-inequalities resulting from rates of distortion, while

* N.B. The a', b', c', in this article have no relation to (a, b, c) as used in equations (181) &c. for inequalities of *vis viva*.

these expressions are equally applicable to the rates of volumetric strain. Thus the expressions,

$$\frac{\sqrt{2}}{3}\frac{\sigma}{\lambda}f\left(\frac{\sigma}{\lambda}\right)\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right),$$

and

$$\frac{\sqrt{2}}{3}\frac{\sigma}{\lambda}f\left(\frac{\sigma}{\lambda}\right)\left[2\frac{du}{dx}-\frac{2}{3}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\right],$$

express, respectively, the rate of relative increase of λ, the mean path, in all directions, and the rate of increase of the inequality in the mean value of the mean paths of the pairs of grains having motion in the direction of x only. This at first may appear paradoxical; but the explanation becomes clear when it is remembered that a rate of strain does not represent a strain, however small.

For a finite rate of strain to cause a strain it must exist for a finite time. And to convert the expression for a rate of strain into the expression for a strain it must be multiplied by the expression for a time; recognising this, the difference between the effects of volumetric strains and distortional strains is at once seen. In the uniform volumetric strain the effects on the path of every pair of grains, whatever the direction of the paths, are the same; whereas in the distortional strain, if the strain in direction of one of the principal axes is positive, the sum of the strains in the other two axes is equal and negative, and thus they neutralise each other except for such effects as result from rearrangement of the grains.

Noticing this, it is seen that the *rates of strain* in the directions of the principal axes on the pairs of grains with relative motion only, in one or other of these axes, are perfectly independent. And, assuming that there are no initial inequalities, these independent rates express the initial rates of increase of the initial inequalities in the mean relative paths, with relative-motion in the directions of the principal axes of rates of distortion. And, as long as the relative inequalities are very small, this independence will be approximately maintained.

Taking δt as an indefinitely small increment of time and multiplying both members of equations (146) by this time we have, putting $a' = da'\delta t/dt$, as a first approximation to the effects of the rates of institution,

$$a'=\frac{\sqrt{2}}{3}\frac{\sigma}{\lambda}f\left(\frac{\sigma}{\lambda}\right)\left\{2\frac{du''}{dx}-\frac{2}{3}\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right)\right\}\delta t, \text{ \&c., \&c.} \quad \ldots(241),$$

or since λ is not affected by the distortional strains we may put for the actual rates

$$\lambda(1+a')=\lambda\left[1+\frac{\sqrt{2}}{3}\frac{\sigma}{\lambda}f\left(\frac{\sigma}{\lambda}\right)\left\{2\frac{du''}{dx}-\frac{2}{3}\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right)\right\}\delta t\right], \text{ \&c., \&c.} \quad \ldots\ldots\ldots(242),$$

which express the increase in the mean paths of pairs of grains having relative velocities in the directions of the principal axes.

Then since the numbers of encounters between such pairs are inversely as the increase of the paths, we have, equating the reciprocals of both members,

$$1 - a' = 1 - \frac{\sqrt{2}}{3}\frac{\sigma}{\lambda} f\left(\frac{\sigma}{\lambda}\right)\left\{2\frac{du''}{dx} - \frac{2}{3}\left(\frac{du''}{dx} + \frac{dv''}{dy} + \frac{dw''}{dz}\right)\right\}\delta t \quad \text{...(243).}$$

From which we have for the rate of relative increase of encounters the numbers of pairs with relative motion in the directions x, y, z,

$$-a' = -\frac{\sqrt{2}}{3}\frac{\sigma}{\lambda} f\left(\frac{\sigma}{\lambda}\right)\left\{2\frac{du''}{dx} - \frac{2}{3}\left(\frac{du''}{dx} + \frac{dv''}{dy} + \frac{dw''}{dz}\right)\right\}\delta t \quad \text{......(244).}$$

Having thus obtained to a first approximation expressions for the effect of rates of institution of inequalities in the pairs of grains having relative motion in the directions of the principal axes, we may proceed as in Art. 149 to find, to a like approximation, the effect of these inequalities in the numbers of encounters on the normal conductions in the directions of the principal axes of distortion.

180. *The initiation of angular inequalities in the distribution of the probable rates of conduction resulting from angular redistribution of the mass.*

Taking x', y', z' as measured in the directions of the principal axes of the distortional strains, and $-a'$, $-b'$, $-c'$ respectively for the relative inequalities in numbers of encounters between pairs of grains having relative velocities in the directions of x', y', z' respectively, where $a' + b' + c' = 0$, we have for the probable relative inequality in the number of encounters of pairs of grains having relative motion in the directions defined by l', m', n' referred to the principal axes,

$$-(l'^2a' + m'^2b' + n'^2c'), \text{ since } f' = g' = h' = 0.$$

Then, taking l_1, m_1, n_1 as the direction cosines of the principal axis measured in direction x', with respect to any arbitrary system of axes measured in directions of x, y, z; l_2, m_2, n_2 and l_3, m_3, n_3 being the direction cosines of the principal axes of y' and z' respectively referred to the arbitrary system, the inequalities in encounters between pairs in directions x, y, z respectively are expressed by

$$-(l_1^2a' + l_2^2b' + l_3^2c'), \text{ \&c., \&c.} \quad \text{.....................(245)}$$

respectively. Then using $-a_1$, $-b_1$, $-c_1$ to express these inequalities, we may also take, in the usual way, f, g, h, the probable tangential inequalities,

$$f = \frac{1}{2}\left(\frac{dv}{dz} + \frac{dw}{dy}\right), \text{ \&c., \&c.} \quad \text{..................(246),}$$

$$-(m_1n_1a' + m_2n_2b' + m_3n_3c'), \text{ \&c., \&c.}$$

Then to find the inequality in the number of encounters having normals in the directions of the axes of x, y, z, respectively, resulting from encounters between pairs of grains in all directions, we must express the probable number of pairs having relative velocities in a direction defined by l, m, n referred to the directions of x, y, z; such an expression is

$$a_1 = l^2a + m^2b + n^2c + 2mnf + 2nlg + 2lmh \quad \text{............(247).}$$

Then the angular distances of the direction of a_1 from this line to the axes of x, y, z respectively are defined by l, m, n respectively; and the probability of the normal at encounter being in the direction of x is la_1, in the direction of y is ma_1, and in the direction of z is na_1. These are the inequalities in the numbers of encounters of which the directions of the normals are in the directions x, y, z, respectively, resulting from encounters between pairs having relative motion defined by l, m, n. Then integrating $-a_1 l$, $-a_1 m$, $-a_1 n$ over hemispheres having axes in the directions of x, y, z, respectively, we obtain, respectively, on dividing by π the mean inequalities in the probability of encounters having normals in the directions of the axes x, y, z. Thus putting $l = \cos\theta$, $m = \sin\theta\sin\phi$, $n = \sin\theta\cos\phi$,

$$-2\pi\int_0^{\frac{\pi}{2}}\left\{\frac{d\cos^4\theta}{4\pi}a + \frac{1}{2\pi}\left(\frac{d\cos^2\theta}{2} + \frac{1}{4}\frac{d\cos^4\theta}{4}\right)(b+c)\right\}$$
$$= \frac{a}{2} + \frac{1}{4}(b+c) \quad \text{.........(248).}$$

181. The mean relative inequalities in normal conduction are obtained after the manner in which equation (148) is obtained, by resolving the components of mean normal conduction in the directions of x, y, z respectively, and multiplying them by the expressions for a, b, c, &c. equations (247).

Then, since $a + b + c = 0$, we have for the probable inequalities respectively $a/4$, $b/4$, $c/4$.

Our object however is not to obtain the inequalities in the probable number of encounters, but the inequalities in the mean normal conduction in the directions of the principal axes.

The mean relative inequality of normal conduction is obtained by the same method as in Art. 104. This requires that for the direction of x, la_1 must be multiplied by $\frac{2\sigma}{3}\sqrt{2}f\left(\frac{\sigma}{\lambda}\right)V_1 l$, and then integrated. Thus

$$\frac{2}{3}\sqrt{2}V\frac{\sigma}{\lambda}f\left(\frac{\sigma}{\lambda}\right)\pi\int_0^{\frac{\pi}{2}}\left\{\frac{d\cos^5\theta}{5\pi}a + \frac{1}{2\pi}\left(\frac{d\cos^3\theta}{3} + \frac{d\cos^5\theta}{5}\right)(b+c)\right\}, \text{ \&c., \&c.} \quad \text{.........(249),}$$

reduce to

$$\left.\begin{aligned}
-\sqrt{2}V_1\sigma f\left(\frac{\sigma}{\lambda}\right)\left(\frac{2}{5}a + \frac{2}{15}(b+c)\right) &= -\frac{2}{3}\frac{4}{15}\sqrt{2}V_1\sigma a \\
-\sqrt{2}V_1\sigma f\left(\frac{\sigma}{\lambda}\right)\left(\frac{2}{5}b + \frac{2}{15}(c+a)\right) &= -\frac{2}{3}\frac{4}{15}\sqrt{2}V_1\sigma b \\
-\sqrt{2}V_1\sigma f\left(\frac{\sigma}{\lambda}\right)\left(\frac{2}{5}c + \frac{2}{15}(a+b)\right) &= -\frac{2}{3}\frac{4}{15}\sqrt{2}V_1\sigma c
\end{aligned}\right\} \quad \text{......(250).}$$

These are the inequalities in the probable normal conductions in the directions of the axes of x, y, z respectively, and it remains to find the inequalities in the probable conductions in the directions of the principal axes.

The probable inequalities in the conductions resulting from an encounter, having the normals in the direction of x, are obtained by substituting the expressions for a, b, c in the preceding equations, then resolving the normal components of V_1 and σ, in the members on the right of these equations, in the directions of x', y', z' respectively, integrating over a sphere of unit radius and dividing by 4π. Thus since $a' + b' + c' = 0$,

$$\left.\begin{aligned} &\frac{4}{15}\frac{2}{3}2\pi\sqrt{2}\,\sigma V_1 f\left(\frac{\sigma}{\lambda}\right)\left\{\int_0^\pi l_1^2(l_1^2a' + l_2^2b' + l_3^2c')\frac{dl_1}{4\pi}\right\} \\ &\quad = -\frac{2}{15}\frac{2}{3}\sqrt{2}\,\sigma V_1 f\left(\frac{\sigma}{\lambda}\right)\left\{\frac{6}{5}a' + \frac{6}{15}(b' + c')\right\} \\ &\quad = -0\cdot32 f\left(\frac{\sigma}{\lambda}\right)\frac{2\sqrt{2}\,\sigma V_1 a'}{9} \end{aligned}\right\}, \text{ \&c., \&c.} \ldots\ldots(251).$$

It will be observed that these expressions are for inequalities of the probable component of conduction in the directions of the principal axes, taking into account the relative inequalities in probable normal conduction in all directions; and that they do not express rates of conduction corresponding to the expressions in equations (158) and (159), but if multiplied by $\sigma^3/\sqrt{2}$ the mass of a grain, they express inequalities of conduction corresponding to the conductions expressed in equation (148).

To obtain the expressions for the inequalities in the rates of the relative component conductions in the directions of the principal axes of distortion, the expressions for the corresponding component conductions must be multiplied severally by the number of encounters each grain undergoes in unit time, and by the number of grains in unit space, as expressed by the integral of equation (157).

Comparing the expressions thus obtained with the rates of conduction, equation (158), it is at once seen that the inequalities in the probable rate of component conduction in the directions of the principal axes of distortion are, remembering that a expresses $\partial_1(a')\,\partial_1 t/\partial_1 t$, &c.,

$$0\cdot32\,\frac{\sqrt{2}}{3}\frac{\sigma}{\lambda}\rho f\left(\frac{\sigma}{\lambda}\right)\frac{\alpha^2}{2}\frac{\partial_1}{\partial_1 t}(a')\,\partial_1 t = \frac{\partial_1}{\partial_1 t}(p''_{x'x'} - p'')\,\partial_1 t, \text{ \&c., \&c.} \ldots.(252).$$

Then although the significance of the a' and a, &c., used to express relative inequalities in mean paths have no relation to the a' and a, &c., used to express inequalities in the *vis viva*, in equations (192—194) they are of similar significance and admit of similar transformation, whence it follows that by a process strictly corresponding to that followed in Art. 152, these rates of conduction transformed to any system of rectangular fixed axes x, y, z,

$$\left.\begin{aligned} \frac{\partial_1(a)}{\partial_1 t}\delta_1 t &= \left\{l_1^2\frac{\partial_1(a')}{\partial_1 t} + l_2^2\frac{\partial_1(b')}{\partial_1 t} + l_3^2\frac{\partial_1(c')}{\partial_1 t}\right\}\delta_1 t, \text{ \&c.} \\ \frac{\partial_1(f)}{\partial_1 t}\delta_1 t &= \left\{m_1n_1\frac{\partial_1(a')}{\partial_1 t} + m_2n_2\frac{\partial_1(b')}{\partial_1 t} + m_3n_3\frac{\partial_1(c')}{\partial_1 t}\right\}\delta_1 t, \text{ \&c.} \end{aligned}\right\} \ldots\ldots\ldots(253),$$

then dividing by δt and substituting the values of $\dfrac{\partial_1(a')}{\partial_1 t}$, &c. from equations (146)

$$\left.\begin{aligned}\frac{\partial_1(a)}{\partial_1 t}&=-\frac{\sqrt{2}}{3}\frac{\sigma}{\lambda}f\left(\frac{\sigma}{\lambda}\right)\left\{2\frac{du''}{dx}-\frac{2}{3}\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right)\right\}, \text{ \&c.}\\ \frac{\partial_1(f)}{\partial_1 t}&=-\frac{\sqrt{2}}{3}\frac{\sigma}{\lambda}f\left(\frac{\sigma}{\lambda}\right)\left(\frac{dv''}{dz}+\frac{dw''}{dy}\right)\cdot\frac{1}{2}, \text{ \&c., \&c.}\end{aligned}\right\}\text{ ....(254).}$$

To convert these into rates of institution of inequalities in the probable rates of conduction they must be multiplied by the constant coefficient of the $\partial_1(a')/\partial_1 t$ in equations (252) which by equations (159) may be expressed as: $0{\cdot}32p''$; the coefficients of the right members of equations (254) may also be expressed by $2p''/\rho\alpha^2$. Therefore

$$\left.\begin{aligned}&-\frac{0{\cdot}32p''^2}{\rho\frac{\alpha^2}{2}}\left\{2\frac{du''}{dx}-\frac{2}{3}\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right)\right\}=\frac{\partial_1}{\partial_1 t}(p''_{xx}-p''), \text{ \&c., \&c.}\\ &-\frac{0{\cdot}32p''^2}{\rho\frac{\alpha^2}{2}}\,2\left\{\frac{dv''}{dz}+\frac{dw''}{dy}\right\}=\frac{1}{2}\frac{\partial_1}{\partial_1 t}(p''_{yz}), \text{ \&c., \&c.}\end{aligned}\right\}\text{ .....(255)}$$

express the initial rates of increase of probable angular inequalities in the rates of conduction, resulting from distortional rates of strain in the mean-system, which are expressed in the last term but one of equations (117 A).

The rates of increase of conduction resulting from rates of change of density.

By equations (239) the relative rates of increase of p'' are the products of the relative rates of change of density multiplied by the ratio of the rate of conduction to the rate of convection; the last factor is

$$\frac{\sqrt{2}}{3}\frac{\sigma}{\lambda}f\left(\frac{\sigma}{\lambda}\right)=\frac{p''}{\rho\frac{\alpha^2}{2}}.$$

Thus for the relative rate of increase of p''

$$\frac{1}{p''}\frac{\partial_1(p'')}{\partial_1 t}=-\frac{p''}{\rho\frac{\alpha^2}{2}}\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right)$$

the actual rate of increase being ....(256).

$$\frac{\partial_1(p'')}{\partial_1 t}=-\frac{p''^2}{\rho\frac{\alpha^2}{2}}\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right)$$

182. *The transformation of vis viva or kinetic stress.*

This as expressed in the last term of equations (117 A) and multiplied by 2 so as to express the rate of increase of *vis viva* (not energy), is

$$2\rho''\left\{(u'u')''\frac{du''}{dx}+(v'u')''\frac{du''}{dy}+(w'u')''\frac{du''}{dz}\right\}, \text{ \&c., \&c.}$$

If the axes are principal axes of rates of distortion and the medium is in uniform condition the last two terms within the brackets are zero. Then taking a', b', c' for the relative inequalities, which are initially zero, we have for the rates of increase

$$\rho\frac{\alpha^2}{2}\frac{\partial_1 a'}{\partial_1 t}=\rho''(u'u')''\left\{2\frac{du''}{dx}-\frac{2}{3}\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right)\right\}, \text{ \&c., \&c.} \ldots(257).$$

Putting $l_1m_1n_1$, $l_2m_2n_2$, $l_3m_3n_3$ for the direction cosines of the principal axes referred to any system of rectangular axes and taking a, b, c, f, g, h as expressing the inequalities when referred to other fixed axes, by the well-known theorem

$$\left.\begin{aligned}
a &= l_1{}^2a' + l_2{}^2b' + l_3{}^2c'\\
b &= m_1{}^2a' + m_2{}^2b' + m_3{}^2c'\\
c &= n_1{}^2a' + n_2{}^2b' + n_3{}^2c'\\
f &= m_1n_1a' + m_2n_2b' + m_3n_3c'\\
&\text{\&c.} \qquad \text{\&c.}
\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(258).$$

where

$$a+b+c=a'+b'+c'$$

Then

$$\frac{\partial_1 a}{\partial_1 t}=l_1{}^2\frac{da'}{dt}+l_2{}^2\frac{db'}{dt}+l_3{}^2\frac{dc'}{dt} \ldots\ldots\ldots\ldots\ldots\ldots(259),$$

and substituting for the values of da'/dt, &c., from (257)

$$\rho\frac{\alpha^2}{2}\cdot\frac{\partial_1 a}{\partial_1 t}=\rho''\alpha^2\left\{\frac{du''}{dx}-\frac{1}{3}\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right)\right\}, \text{ \&c., \&c.} \ldots\ldots(260),$$

$$\rho\frac{\alpha^2}{2}\cdot\frac{\partial_1 (f)}{\partial_1 t}=2\rho\frac{\alpha^2}{4}\left(\frac{dv''}{dz}+\frac{dw''}{dy}\right), \text{ \&c., \&c.} \ldots\ldots\ldots\ldots(261).$$

Then putting $\alpha^2/2$ for $\dfrac{(u'u'+v'v'+w'w')''^2}{3}$,

$$\frac{\partial_1}{\partial_1 t}\left[\rho''\frac{\alpha^2}{2}\right]=-\rho''\frac{3\alpha^2}{2}\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right) \ldots\ldots\ldots\ldots(262),$$

$$\left.\begin{aligned}
\frac{\partial_1}{\partial_1 t}\left[\rho''(u'u')''-\frac{\alpha^2}{2}\right]&=-2\rho''\frac{\alpha^2}{2}\left\{\frac{du''}{dx}-\frac{1}{3}\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right)\right\}, \text{ \&c., \&c.}\\
\frac{\partial_1}{\partial_1 t}\left[\rho''(v'w')''\right]&=-2\rho''\frac{\alpha^2}{2}\cdot\frac{1}{2}\left(\frac{dv''}{dz}+\frac{dw''}{dy}\right), \text{ \&c., \&c.}
\end{aligned}\right\} (263).$$

These equations express the initial rates of increase of angular inequalities in the rates of convection resulting from distortional rates of strain in the mean system, which are expressed in the last terms of equations (117 A).

183. *The institution of linear inequalities in the rates of flux of vis viva of relative motion by convection and conduction.*

Thus far the analysis for the rates of institution of inequalities in the *vis viva* and rates of conduction has been confined to the effects of uniform rates of strain in the mean-motion extending throughout the medium, whether distortional, rotational, or volumetric. When however the rates of mean volumetric strain are other than uniform, as long as the parameters of such motion are large as compared with the parameters which define the spaces over which the means of the relative mass and relative-momentum are approximately zero, the analysis of the effects resulting from small variations in the rates of strain in the mean-motions, in instituting linear dispersive inequalities in the mean *vis viva*, $\rho(\alpha^2)''/2$, of relative-motion, follows as a second approximation on that which has preceded.

In Section V. equation (93), it is shown that provided the relative motion and relative mass are subjected to such redistribution as to maintain the scales, over which they must be integrated, small compared with the corresponding scales of the mean-motion, the conditions for mean- and relative-systems will be approximately satisfied.

The expressions for the rates of institution of linear dispersive inequalities by convection and by conduction are given by equations (261) and the last of equations (256)

$$\left.\begin{aligned} \frac{\partial_1}{\partial_1 t}\left(\rho\frac{\alpha^2}{2}\right) &= -\frac{2}{3}\rho\frac{\alpha^2}{2}\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right) \\ \frac{\partial_1}{\partial_1 t}(p'') &= -\frac{2}{3}\frac{p''^2}{\frac{\alpha^2}{2}}\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right) \end{aligned}\right\}\dots\dots\dots\dots\dots(264).$$

184. *The institution of inequalities in the mean motion.*

In the case of a space within which there are no inequalities, in either system, the institution of inequalities in the mean system within the space must be the result of some mean inequalities in the mean state of the medium outside the space—of some action across the boundaries; since in an infinite medium, including all the mass, all actions must be between one portion of the medium and another.

For the sake of analysis however it is legitimate to consider the mean actions on the boundaries of any space, as determined by the scale of mean-motions, as arbitrary. And it is important to notice that such mean actions on the mean motion are the only actions that it is legitimate to treat as arbitrary; since, as has been shown in the last article, the institution of inequalities in the relative motion results solely from the action of the mean motion.

Arbitrary accelerations may be finite or infinite and by assuming the accelerations infinite we are enabled to institute finite inequalities in the mean motion in an indefinitely short time, and this without instituting any inequalities in the relative motion, as the instantaneous result of the institution of the inequalities in the mean motion; whence, it appears, that we may, for the purpose of analysis, start with a medium without any inequalities in the mean mass, relative mass, or relative motion, but with arbitrary inequalities in the mean-motion. With such an initial start we have, from equations (120) Section VI.,

$$\rho \frac{\partial_1 u''}{\partial_1 t} = 0, \text{ \&c., \&c.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (265).$$

185. *The redistribution of inequalities in the mean-motion.*

The effect of the instantaneous institution of inequalities in the mean motion is an instantaneous finite acceleration to the institution of inequalities in the relative motion as expressed in equations (255) to (263) as the result of transformation; the action including both the convections and conductions. This acceleration of the inequalities, in *vis viva* of relative motion, including conduction, is also an acceleration to the institution of the space-rates of variation of these inequalities, and these space-rates of variation of the inequalities of relative motion are transformed back as accelerations of the mean motion.

Thus, although $\partial_1 u''/\partial_1 t = 0$, the institution of du''/dx, say, has instituted an acceleration to the institution of inequalities, the space variations of which react as accelerations on the mean-motion. That these reactions are dispersive, of inequalities in the mean motion, follows definitely from the sequence of the rates of action already defined.

To prove this we may consider the acceleration of any one of the inequalities, instituted by the mean motion, as to its rate of reaction, on the inequalities of position of the mean-momentum, by itself—independently of other inequalities. Considering the effect of acceleration of the inequality

$$\rho'' (u'v')'' + p''_{xy}$$

on the acceleration of the rate of increase of mean-momentum, it appears, at once, from the equations (120) that the reaction resulting from this inequality affects both u'' and v''. These effects may be considered separately. But from equations (255) to (263) it appears that the rate of institution of the inequality $\rho'' (u'v')'' + p''_{xy}$ depends on the mean inequalities

$$\frac{du''}{dy} + \frac{dv''}{dx};$$

so that if du''/dy is zero there will still be reaction unless dv''/dx is also zero.

From equations (255) to (263) the rate of institution of the inequality is

$$\frac{\partial_1}{\partial_1 t}(\rho''(u'v')''+p''_{xy}) = -\left(\rho''\frac{\alpha^2}{2}+\frac{0\cdot64}{2\rho''\alpha^2}p''^2\right)\left(\frac{du''}{dy}+\frac{dv''}{dx}\right) \quad \ldots(266).$$

Then changing the sign and differentiating with respect to y we have for the rate of increase of reaction from this inequality,

$$\rho''\frac{\partial_1^2}{\partial_1 t^2}(u'') = \left(\rho''\frac{\alpha^2}{2}+\frac{0\cdot64}{2\rho''\alpha^2}p''^2\right)\left(\frac{d^2u''}{dy^2}+\frac{d^2v''}{dydx}\right) \quad \ldots\ldots\ldots(267).$$

Differentiating this last equation with respect to y the acceleration of the rate of increase of the inequality in the mean motion is

$$\rho''\frac{\partial_1^2}{\partial_1 t^2}\left(\frac{du''}{dy}\right) = \left(\rho''\frac{\alpha^2}{2}+\frac{0\cdot64}{2\rho''\alpha^2}p''^2\right)\frac{d^2}{dy^2}\left(\frac{du''}{dy}+\frac{dv''}{dx}\right) \quad \ldots\ldots(268).$$

This equation expresses the partial effect of the inequality $\rho''(u'v')''+p''_{xy}$ on du''/dy. And proceeding in a similar manner we have for the other partial effect on dv''/dx

$$\rho''\frac{\partial_1^2}{\partial_1 t^2}\left(\frac{dv''}{dx}\right) = \left(\rho''\frac{\alpha^2}{2}+\frac{0\cdot64}{2\rho''\alpha^2}p''^2\right)\frac{d^2}{dx^2}\left(\frac{du''}{dy}+\frac{dv''}{dx}\right) \quad \ldots\ldots\ldots(269).$$

Then adding, the total effect becomes

$$\rho''\frac{\partial_1^2}{\partial_1 t^2}\left(\frac{du''}{dy}+\frac{dv''}{dx}\right) = \left(\rho''\frac{\alpha^2}{2}+\frac{0\cdot64}{2\rho''\alpha^2}p''^2\right)\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\left(\frac{du''}{dy}+\frac{dv''}{dx}\right) \ldots(270).$$

It is at once seen that this equation represents a positive *acceleration* to dispersion of the inequality in the mean motion, $du''/dy+dv''/dx$, as the result of the rate of institution of the inequality $\rho''(u'v')''+p''_{xy}$.

In a similar manner it may be shown that the effects of the five distortional inequalities, in the rates of convection and conduction, are accelerations to the dispersion of the five remaining inequalities in the rates of increase of mean motion. These, together with rates of dispersion of the volumetric inequalities, admit of expression in a general form.

186. *The inequalities in the component of mean motion.*

$$-\left\{\frac{du''}{dx}-\frac{1}{3}\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right)\right\}, \quad -\frac{\dfrac{du''}{dy}+\dfrac{dv''}{dx}}{2},$$

$$-\frac{\left(\dfrac{du''}{dz}+\dfrac{dw''}{dx}\right)}{2}, \quad -\frac{1}{3}\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right), \text{ \&c., \&c.,}$$

admit of expression after the manner of expression of component stresses by simply substituting I''_{xx} for p''_{xx}, &c., &c., and we may further simplify the expressions by putting I''_v for $(I''_{xx}+I''_{yy}+I''_{zz})/3$.

In the same way we may take $\check{I}_{xx}$ for $\{\rho''(u'u')''+p''_{xx}\}$. In this way we have for the three typical expressions of accelerations to rates of increase in inequalities of mean motion

$$\left.\begin{aligned}
\rho\frac{\partial_1^2}{\partial_1 t^2}(I''_{xx}-I''_v) &= \left(\rho''\alpha^2+\frac{0\cdot 64}{\rho''\alpha^2}p''^2\right)\frac{d^2}{dx^2}(I''_{xx}-I''_v)\\
\rho\frac{\partial_1^2}{\partial_1 t^2}(I''_{xy}+I''_{yx}) &= \left(\rho''\frac{\alpha^2}{2}+\frac{0\cdot 64}{2\rho''\alpha^2}p''^2\right)\left(\frac{d^2}{dy^2}+\frac{d^2}{dx^2}\right)(I''_{xy}+I''_{yx})\\
\rho\frac{\partial_1^2}{\partial_1 t^2}(I''_v) &= \left(\rho''\frac{\alpha^2}{3}+\frac{4}{3\rho''\alpha^2}p''^2\right)\left\{\frac{d^2}{dx^2}(I''_{xx})+\frac{d^2}{dy^2}(I''_{yy})+\frac{d^2}{dz^2}(I''_{zz})\right\}
\end{aligned}\right\} \quad \text{............(271).}$$

Each of these types, it will be observed, expresses acceleration to the dispersion of the inequality of the mean motion.

Whence it appears that the instantaneous institution of inequalities in mean-motion is also an instantaneous institution of accelerations to the dispersion of the inequalities in the mean motion. Q. E. D.

It will be observed that since by definition the mean relative components taken over the scale of relative motion are all zero, there can be no change in the mean momenta as the result of exchanges between the two systems. And hence the action of dispersion can be, only, changes of the position of the momentum from one place to another.

187. In the consideration of the equations for momentum the question of dissipation of energy of mean-motion to that of relative-motion does not arise. But, as an acceleration to dispersion of inequalities of the mean-motion is an acceleration to decrease the component momentum where it is greater and increase it where it is less, so that there is no change in the integral momentum of mean motion, it follows, as a necessary consequence, the acceleration to dispersion of momentum entails an acceleration to dissipation of energy of mean-motion to that of relative-motion. The expression for these initial accelerations to dissipation of energy may be obtained in various ways, one of which is involved in the proof of the following theorem:

The initial rates of institution of inequalities as expressed in equations (255) *to* (263), *for convections and conductions, are essentially accelerations to mean rates of increase of the vis viva of relative-motion as well as to the redistribution of inequalities in the mean system.*

The terms which express exchanges of energy by transformation from the mean system to the relative system, which are the only exchanges between the systems, are the last of the terms in each of the equations (116 A). Then putting $\rho''\partial_1(u'u')/\partial_1(t)$, &c., &c., as the initial effects of the instantaneous institutions of inequalities in the mean motion on the relative motion, we have

$$\rho''\frac{\partial_1}{\partial_1 t}(u'u')'' = -\left\{\begin{array}{l} \frac{1}{3}\left[\rho''\frac{u'u'+v'v'+w'w'}{3}+p\right]''\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right) \\ +\left[\rho''u'u'-\frac{u'u'+v'v'+w'w'}{3}+p_{xx}-p\right]''\frac{du''}{dx} \\ +\frac{1}{2}\left\{[\rho''v'u'+p_{yx}]''\left(\frac{du''}{dy}+\frac{dv''}{dx}\right)\right. \\ \left.+[\rho''w'u'-p_{zx}]''\left(\frac{du''}{dz}+\frac{dw''}{dx}\right)\right\} \end{array}\right\}\ldots(272),$$

and two corresponding expressions for the other components.

By equation (265) $\partial_1 u''/\partial_1 t$, &c., &c. as well as all inequalities of relative motion are initially zero; so that, initially, both members are zero. Then performing the operation $\partial_1/\partial_1 t$ on both members and observing that by equation (265) this operation has no effect on the mean inequalities,

$$\frac{\partial_1}{\partial_1 t}\rho''\frac{\partial_1}{\partial_1 t}(u'u')'' = -\left\{\begin{array}{l} \frac{1}{3}\frac{\partial_1}{\partial_1 t}\left[\rho''\frac{\alpha^2}{2}+p''\right]\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right) \\ +\frac{\partial_1}{\partial_1 t}\left[\rho''\left(u'u'-\frac{\alpha^2}{2}\right)+p_{xx}-p\right]''\frac{du''}{dx} \\ +\frac{\partial_1}{\partial_1 t}[\rho''v'u'+p_{yx}]''\left(\frac{du''}{dy}+\frac{dv''}{dx}\right) \\ +\frac{\partial_1}{\partial_1 t}[\rho''w'u'+p_{zx}]''\left(\frac{du''}{dz}+\frac{dw''}{dx}\right) \end{array}\right\}\ldots\ldots(273),$$

and two corresponding equations for the other components.

These three equations taken together express in terms of the differential coefficients the rates of institution of inequalities of the relative motion, expressions for which in terms of the mean motion are given in equations (255) to (263); and substituting these expressions for the differential coefficients in each of the three equations, and adding the corresponding members, we have for the total initial rate of acceleration of the rate of increase of relative energy

$$\rho''\frac{\partial_1^2}{\partial_1 t^2}\left(\frac{3}{2}\alpha^2\right) = \left(\rho''\alpha^2+\frac{0{\cdot}64}{\rho''\alpha^2}p''^2\right)\left\{\left(\frac{du''}{dx}\right)^2+\left(\frac{dv''}{dy}\right)^2+\left(\frac{dw''}{dz}\right)^2\right\}$$
$$+\frac{1}{2}\left\{\left(\frac{du''}{dy}+\frac{dv''}{dx}\right)^2+\left(\frac{dv''}{dz}+\frac{dw''}{dy}\right)^2+\left(\frac{dw''}{dx}+\frac{du''}{dz}\right)\right\}\ldots\ldots(274).$$

The member on the right is essentially positive while the left member expresses the acceleration of the mean rate of the *vis viva*. Q. E. D.

188. The first term on the right, equation (274), expresses the acceleration of the rate of mean-energy of relative motion resulting from the inequalities of the *direct* space variations of the mean motion, including both volumetric and distortional effects, while the second term expresses the acceleration of the rate of mean-energy in consequence of the tangential space variations of mean-motion.

These accelerations are all positive, tending to produce a dispersive condition of relative-motion.

The tendency, thus proved, of the effect of transformation from energy of mean-velocity to energy of relative-velocity, at each point, so to direct the signs of inequalities in relative *vis viva* as to cause dispersion of both energy of mean and energy of relative-velocity, and to render the effect of transformation, of mean-motion to energy of relative-motion, positive, is quite independent of all other actions or effects; and, although not hitherto analytically separated in the theory of mechanics, is now seen to be one of the most general kinematical principles—the prime principle which underlies those effects which have long been recognised from experience and generalised as the law of universal dissipation of energy.

The analytical separation of this principle does not immediately explain universal dissipation. It accounts for the initial acceleration to the dispersive condition, but it does not, alone, account for irreversibility of the dissipation.

The proof of this at once follows from equations (271), the general solution of which is

$$I'' = f\left(\mp\sqrt{\rho''\alpha^2 + \frac{0{\cdot}64}{\rho''\alpha^2}p''^2}\right)(t-y) \qquad \text{..............(275)},$$

which expresses two reciprocal inequalities of mean motion proceeding in opposite directions uniformly at velocities

$$\pm\sqrt{\rho''\alpha^2 + \frac{0{\cdot}64}{\rho''\alpha^2}p''^2}.$$

If then u'' be everywhere reversed, the direction and the rate of propagation of the reversed inequality remaining the same, will bring the state of the relative motion back to the initial condition. And this applies to all inequalities, so that if there were no other action than that of transformation including its effects on the mean and relative inequalities, these effects would be perfectly reversible.

189. *The conservation of the dispersive condition depends on the rates of redistribution of the relative motion.*

By equations (271) and (274) it appears that as long as the inequalities of relative-motion are zero while the inequalities in the mean motion are finite the signs of the acceleration to the dispersive condition are always positive. Therefore if these inequalities remain small as compared with the energy of relative motion, while the signs of the inequalities of the mean-motion are not changed, a dispersive condition is secured. From which it follows that any cause which maintains these inequalities small, compared with the relative energy, will render the dispersion irreversible by reversing the mean motion, no matter how great the acceleration to the dispersive condition arising from the prime tendency to the dispersive condition.

Such actions exist in the angular and the linear dispersions, of the angular and linear inequalities of *vis viva* of relative motion, and rates of conduction through the grains, equations (195) and (205), Section XI., and (236), Section XII.

From equation (266) it appears that the instantaneous reversal of the mean motion has no effect (instantaneous) on the relative motion; so that this is not simultaneously reversed. And thus it is not the resultant motion that is subject to reversal, but only the abstract mean motion, while the abstract relative motion continues as before to redistribute the reversed mean motion.

This explanation of irreversibility of the mean motion and the irreversible dissipation of energy could not have been obtained until the analytical separation of the abstract mean motion from the relative motion had been accomplished. And this fact fully explains the obscurity which has hitherto surrounded dissipation of energy.

The general reasoning in this article, although sufficient to afford a general explanation, is, of necessity, supplemented by the definite analysis by which the inequalities in the *vis viva* of relative motion are determined in the next article.

190. *The determination, in terms of the quantities which define the condition of the medium, of the inequalities maintained in the vis viva of relative motion, and in the rates of conduction, by the combined actions of institution by transformation, and redistribution by relative relative-motion.*

In entering upon this undertaking it is in the first place necessary, in order to render the course of procedure intelligible, to point out that as far as mechanical analysis has as yet been developed, including the present research, it has not included such analysis as is necessary to express the means of the instantaneous transmission of accelerations, and thus we are unable to deal definitely with continuous initiation from rest of continuous inequalities. This inability, which is generally recognised, was discussed in a paper read before Section A of the British Association at Southport, though not further published. In this paper it was suggested that such inability was evidence of some property in the constitution of the medium necessary for the instantaneous transmission of acceleration, and showed that if the medium consisted of rigid particles as in Maxwell's Kinetic Theory (1860), then since any acceleration at a point would, necessarily, extend through the thickness of the grain, it would therefore afford instantaneous linear transmission of acceleration, and so render the necessary analysis for dealing with initiation possible. As we are here dealing with a granular medium, this analysis, if fully developed, would remove the disability. But, having assurance of this, we may avoid the development of the analysis by following the method of Stokes—considering only such inequalities as are steady or periodic when referred to moving axes. Under such conditions the determination of the inequalities maintained is practicable, and indicates the general form of the equations for the general inequalities.

The incompleteness of the analysis for the expression of the linear instantaneous transmission of accelerations is not the only reason for confining the application of the analysis to steady or periodic inequalities.

Putting aside uniform continuous strains and rotations in the case of a granular medium, of which the mean condition is uniform and indefinitely continuous, it is the properties of such a medium, of transmitting undulations, that first claim our attention. And as such undulations are the only motions, in such a medium, that can extend to infinity throughout an infinite space, they must be considered as the principal form of mean motion.

However, before proceeding to consider the undulations, it may be well to point out the several classes of mean motion which may be recognised at this stage of the analysis.

Other than undulations, the only possible mean motions, including mean strains, are such as involve some local disarrangement of the medium, together with displacement of portions of the medium from their previous neighbourhood—as in the vortex ring—which may have a temporary existence when σ/λ is small; or, of far greater interest, local disarrangement of the grains when so close together that diffusion is impossible, except at inclosed spaces or surfaces of disarrangement, depending, as already explained, on the value of G being greater than $6/\sqrt{2}\,.\,\pi$. Under which condition it is possible that, about the local centres, there may be singular surfaces of freedom, which admit of their motion in any direction through the medium by propagation, combined with convection, together with strains throughout the medium which result from the local disarrangement, without any change in the mean arrangement of the grains about the local centres; the grains moving so as to preserve the mean arrangement.

191. *Steady continuous uniform strains or undulations extending throughout the medium otherwise in normal condition.*

We have:

(1) Equations for the angular inequalities maintained in the *vis viva* of relative motion.

(2) Equations for the angular inequalities maintained in the rates of conduction.

(3) Equations for the component linear inequalities maintained in the mean *vis viva*.

(4) Equations for the linear inequalities maintained in the rates of conduction.

(5) Equations for the rates of increase of mean *vis viva*—$\alpha^2/2$—resulting from angular dispersion by convection.

(6) Equations for the rates of increase of mean *vis viva* resulting from angular dispersion by conduction.

(7) Equations for the rates of increase of mean *vis viva* by linear displacement resulting from inequalities in the mean *vis viva*.

(8) Equations for the rates of increase of mean *vis viva* by linear displacements resulting from inequalities in the mean pressures.

192. THEOREM. *To a first approximation the first four of these eight equations all have the same general form as long as the space and time variations of the mean motion are constant, simple harmonic, or logarithmic functions of time and space, in which case the constants of frequency and the hyperbolic variations are such as may be neglected as compared with* σ/λ *and* $1/\lambda$. *And the same for the last four equations.*

It is to be noticed that the condition in the theorem as to smallness of the constants is necessary when treating the variations of the mean motion as arbitrary, since the condition is, as shown in Section V., a necessity for the maintenance of the mean and relative systems.

To prove the first part of the theorem :

The equations for any one of the six partial angular inequalities in vis viva of relative motion.

Putting

$\check{I}$ for the inequality in *vis viva* of relative motion.

I'' ,, ,, ,, ,, ,, in mean motion.

A_1^2 for the coefficient by which I'' is multiplied to represent the rate of institution.

A_2^2 for the coefficient by which $\check{I}$ must be multiplied to express redistribution.

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, to represent distances in directions x, y, z, which are the parameters of the component harmonic inequalities in the mean motion; the equation for the maintenance of $\check{I}$ becomes:

$$\left.\begin{aligned} \frac{\partial_1 \check{I}}{\partial_1 t} + \frac{\partial_2 \check{I}}{\partial_2 t} &= - A_1^2 I'' - A_2^2 \check{I} \\ \frac{\partial_1 \check{I}}{\partial_1 t} + A_2^2 \check{I} &= 2A_1^2 I'' \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots(276).$$

In this case where I'' and $\check{I}$ are component inequalities in the mean-motion, and in the *vis viva* of relative-motion, the coefficients A_1^2, A_2^2, are respectively, as in equation (263) Section XIII. and (195) Section XI.:

$$A_1^2 = - 2\rho \frac{\alpha^2}{2}, \qquad A_2^2 = - \frac{3}{4} \frac{\sqrt{\pi}}{\lambda} \alpha \ldots\ldots\ldots\ldots\ldots(277).$$

Then if I'' is as before, and $\check{I}$ is taken for the inequality in conduction corresponding to the inequality in convection in the same direction, the equation will become the equation for the inequality in conduction. If B_1^2, B_2^2 are put for the coefficients of conduction corresponding to A_1^2 and A_2^2,

$$B_1^2 = \frac{0{\cdot}32p''^2}{\rho\frac{\alpha^2}{2}}, \qquad B_2^2 = \frac{3}{4} \frac{\sqrt{\pi}}{\lambda} \alpha \frac{1 - f(G)\rho}{1 + e^{-x\{1-f(G)\}}} \ldots\ldots\ldots(278),$$

as in equation (205) Section XI.

Also, if I'' is taken to express the linear inequality in mean-motion in any direction, say that of x, in the rate of volumetric strain in the mean-motion, and $\check{I}$ is taken to express the linear inequality in the mean *vis viva* of relative-motion, since $d^2\check{I}/dx^2$, &c. take the forms $-a^2\check{I}_{xx}$, $-b^2\check{I}_{yy}$, $-c^2\check{I}_{zz}$, where $1/a$, $1/b$, $1/c$ are components of some constant parameter, the equation will become the equation for the linear inequality maintained in direction x in the mean *vis viva* when λ/σ is large.

Putting C_1^2 and $a^2C_2^2$ to correspond to A_1^2 and A_2^2 in (277),

$$C_1^2 = \frac{5}{3} \rho \frac{\alpha^2}{2}, \qquad a^2C_2^2 = \frac{3a^2\lambda\alpha}{\sqrt{\pi}},$$

$$- C_2^2 a^2 \check{I} = - C_2^2 \frac{d^2}{dx^2}(\check{I}) \ldots\ldots\ldots\ldots\ldots\ldots(279).$$

And I'' being the linear inequality in the same direction in the rate of volumetric strain of mean-motion; if $\check{I}$ is taken to express the linear inequality in the rate of mean-conductivity (p''), equal in all directions, the equation becomes the equation for the inequality in the mean-conduction if D_1^2, $a^2D_2^2$ correspond to A_1^2 and A_2^2 in equation (277),

$$D_1^2 = \frac{5}{3}\frac{p''^2}{\rho\frac{\alpha^2}{2}}, \qquad a^2D_2^2 = a^2\frac{4}{3}\frac{\sigma}{\lambda}\frac{\alpha}{\sqrt{\pi}}\frac{G}{4} \quad \text{..............(280),}$$

since, as in equation (279), $a^2\check{I} = \frac{d^2}{dx^2}(\check{I})$.

Thus as long as the inequalities in the mean-motion can be expressed as simple finite harmonic or logarithmic functions of time and displacement, the equations for the dispersive inequalities have the common form as in equation (276).

The second part of the theorem follows as a consequence of the first for, since the equations for the dispersive inequalities have the same form, the general solution of this form of equation will apply to all the inequalities.

Then if such solution can be found for the dispersive inequalities, since the rate of increase of the mean *vis viva* at a point, at any instant, is the result of the action of the inequality on the space rate of variation of the mean strain which institutes the inequality, the rates of increase of the mean *vis viva* ($\alpha^2/2$) are the products of the inequalities ($\check{I}$) by the corresponding inequalities (I'') in the mean-motion. And these are expressed in a general form.

193. *The approximate solution of the general differential equation for the inequalities in mean vis viva of relative-motion and rate of conduction resulting from steady or periodic inequalities in the mean-motion.*

In all probability the equation (276) does admit of complete solution. But the analysis is greatly simplified by recognising that any secondary effects, resulting from the existence of inequalities, to vary the mean *vis viva* of relative-motion ($\alpha^2/2$) by transformation from mean-motion, and thus to vary the coefficients A_1^2 and A_2^2, are proportional to α^2I''. And consequently, since by definition α^2 is finite, by taking I'' sufficiently small the secondary effects of I'' and α'' may be rendered as small as we please, and the integral effects indefinitely small as compared with the finite value of α^2.

In this way the coefficients A_1^2 and A_2^2 may be taken as constant, and there is no loss of generality in the solution; while the expression for the rate of increase of α^2, as determined by the approximate solution of the equation of transformation, may be subsequently introduced as a small quantity.

Solution to a first approximation, I'' small.

Since according to the theorem the space and time variations of I'' are constant or periodic, we may transform the equation (276) by putting

q_{xx}, &c. for the maximum values of I''_{xx}, &c., which are constant. And I''_{xx}/a is the maximum value of u''. Hence

$$I''_{xx} = q_{xx} \sin(mt - ax),$$

where q_{xx} is constant in time and space.

We then have for the angular inequalities and linear inequalities respectively:

$$\left.\begin{aligned} \frac{\partial}{\partial t}(\check{I}) + A_2^2 \check{I} &= A_1^2 q_{xx} \sin(mt - ax), \text{ \&c.} \\ \frac{\partial}{\partial t}(\check{I}) + a^2 C_2^2 \check{I} &= C_1^2 q_{xx} \sin(mt - ax), \text{ \&c.} \end{aligned}\right\} \text{..........(281).}$$

The introduction of the two forms is only a matter of convenience in keeping the partial constants distinct.

Then if we put $\check{I} = Ce^{st}$ and eliminate by differentiation with respect to time, A_1^2, A_2^2 being constant, it can be shown that for steady or periodic motion

$$\left.\begin{aligned} \check{I} &= \frac{1}{(A_2)^4 + m^2} A_1^2 \left[A_2^2 I'' - \frac{\partial}{\partial t}(I'')\right] \\ \text{or}\qquad \check{I} &= \frac{A_1^2}{CA_2^4 + m^2}\left[a^2 C_2^2 - \frac{\partial}{\partial t}(I'')\right] \end{aligned}\right\} \text{..............(282),}$$

and that this is the only solution if A_1^2, A_2^2, &c. are constant. The analysis is somewhat long. But if we recognise that all the terms in the equation (281) must have the same frequency m, the same result is obtained by differentiating both members of (281) and substituting the result from

$$A_2^2 \check{I} - \frac{\partial_2}{\partial t^2}(\check{I}) = A_1^2 q_{xx} \{A_2^2 \sin(mt - ax) - m \cos(mt - ax)\} \text{ ...(283),}$$

whence, since $\partial^2 \check{I}/\partial t^2 = -m^2 \check{I}$ is of the same form as equation (282),

$$\check{I} = \frac{1}{A_2^4 + m^2} A_1^2 q_{xx} \{A_2^2 \sin(mt - ax) - m \cos(mt - ax)\} \text{ ...(284),}$$

which will be the general form on substituting B_1^2, B_2^2 for C_1^2, $a^2 C_2^2$, and D_1^2, $a^2 D_2^2$ for A_1^2, A_2^2. Q. E. D.

The equation for the rate of increase of the mean vis viva ($\alpha^2/2$).

Multiplying the expression for $\check{I}$, equation (284), by the corresponding expression for I'', it at once appears that $\check{I}$ consists of two parts, the one being continuously positive and the other periodic.

Thus:

$$\begin{aligned} \check{I} I'' = {} & \frac{1}{m^2 + A_2^4} A_1^2 q A_2^2 \sin(mt - ax) \\ & - \frac{1}{m^2 + A_2^4} A_1^2 q m \cos(mt - ax) \text{ ....(285),} \end{aligned}$$

from which it appears that the dispersive inequality in equation (284) is expressed by

$$\frac{1}{m^2 + A_2^4} A_1^2 q A_2^2 \sin(mt - ax):$$

the remaining part of $\check{I}$,

$$-\frac{1}{m^2 + A_2^4} A_1^2 qm \cos(mt - ax),$$

representing that part of the inequality the effect of which is purely periodic, or non-dispersive. Therefore the equation for the rate of increase of the mean *vis viva* is

$$\check{I}I'' = \frac{1}{m^2 + A_2^4} A_1^2 q A_2^2 \sin(mt - ax) \ldots\ldots\ldots\ldots\ldots(286),$$

which is a general form for all rates of dispersion of mean *vis viva*.

Q. E. D.

194. Having, in Art. 193, obtained the general expression for total inequalities maintained by relative-motion as the result of institution by transformation and redistribution, as well as the general expressions for the dispersive and periodic components of the inequalities, it appears that the analytical distinction between the corresponding inequalities in *vis viva*, and rates of conduction, may be expressed by substitution for A_1^2 and A_2^2, &c., the values of these constants as expressed:—

$$\text{for angular inequalities in} \begin{cases} \text{convection, in equation (277),} \\ \text{conduction, ,, ,, (278),} \end{cases}$$

$$\text{for linear inequalities in} \begin{cases} \text{convection, ,, ,, (279),} \\ \text{conduction, ,, ,, (280).} \end{cases}$$

They are, for angular inequalities in convection:

$$\check{I}_{xx} = \frac{\rho\alpha^2}{m^2 + \left(\frac{3}{4}\frac{\sqrt{\pi}}{\lambda}\alpha\right)^2} q \left\{\frac{3}{4}\frac{\sqrt{\pi}}{\lambda}\alpha \sin(mt - ax) - m\cos(mt - ax)\right\} \ldots(287);$$

for angular inequalities in conduction:

$$\check{I}_{xx} = \frac{0{\cdot}32\left(\dfrac{p''}{\rho\,\dfrac{\alpha^2}{2}}\right)\rho\,\dfrac{\alpha^2}{2}}{m^2 + \left\{\dfrac{3}{4}\dfrac{\sqrt{\pi}}{\lambda}\alpha\,\dfrac{1 - f(G)\rho}{1 + e^{-x\{1 - f(G)\rho\}}}\right\}^2} q \left[\frac{3}{4}\frac{\sqrt{\pi}}{\lambda}\alpha\,\frac{1 - f(G)\rho}{1 + e^{-x\{1 - f(G)\rho\}}}\sin(mt - ax) - m\cos(mt - ax)\right] \ldots(288);$$

for linear inequalities in $\alpha^2/2$ in convection:

$$\check{I}_{vx} = \frac{\frac{5}{3}\rho\frac{\alpha^2}{2}}{m^2 + \left(\frac{\alpha^2 3\lambda\alpha}{\sqrt{\pi}}\right)^2} q \left\{\frac{\alpha^2 3\lambda\alpha}{\sqrt{\pi}} \sin(mt - ax) - m\cos(mt - ax)\right\} \ldots(289);$$

for linear inequalities in $\alpha^2/2$ in conduction:

$$\check{I}_{p''x} = \frac{\frac{5}{3}\left(\frac{\frac{p''^2}{\alpha^2}}{\rho\frac{\alpha^2}{2}}\right)\rho\alpha^2}{m^2 + \left(\frac{4}{3\sqrt{\pi}}\frac{\sigma^2}{\lambda}\alpha\frac{G}{4}a^2\right)^2} q \left\{\frac{4\sigma^2\alpha}{3\sqrt{\pi}\lambda}\frac{G}{4}a^2 \sin(mt - ax) - m\cos(mt - ax)\right\}$$

$$\ldots\ldots(290).$$

The equations for angular inequalities are general for all states of the medium. But the expressions for the linear inequalities are those to which linear inequalities approximate according as λ/σ is less than the limit at which diffusion ceases, or is greater than that at which diffusion is general. [See Art. 145 and Art. 155, Section XI.]

In considering periodic inequalities in a medium of unlimited extent, which is, except for the inequalities, uniform and isotropic, it will simplify the analysis to recognise, that such inequalities as can be propagated through the medium, must have directions of propagation which are normal to continuous surfaces which are either spherical closed surfaces, or of such extent that their boundaries are at distances large compared with the periodic parameters.

This in the first instance confines our attention to directions of propagation everywhere normal to an infinite plane. We notice that the classes of inequalities in the mean motion are reduced to two: those in which the mean motion is in the direction of propagation, and those in which the mean motion is normal to this direction.

We also notice that these two resultant inequalities are to a first approximation independent, although they may have the same direction of propagation, and therefore may be dealt with separately.

195. *Expressions for the resultant institutions of inequalities of mean motion when the motion is in the direction of propagation.*

Putting x_1 and u_1'' as the direction of propagation and motion for institution of angular inequalities we have, since

$$\left(\frac{du''}{dx} + \frac{dv''}{dy} + \frac{dw''}{dz}\right)$$

is an invariant, for the inequalities of mean motion for the inequalities $(u'u', v'v', w'w')$, &c.

$$-\left[\frac{du_1''}{dx_1} - \frac{1}{3}\left(\frac{du_1''}{dx_1} + \frac{dv_1''}{dy_1} + \frac{dw_1''}{dz_1}\right)\right], \quad +\frac{1}{3}\frac{du_1''}{dy_1}, \quad +\frac{1}{3}\frac{du_1''}{dz_1}.$$

Then, taking x_1, y_1, z_1 as principal axes, l_1, m_1, n_1 as the direction cosines of x_1, y_1, z_1 referred to any rectangular system x, y, z, the components are, since

$$\frac{1}{3}\frac{du_1''}{dx_1} = \frac{du_1''}{dx_1} + \frac{dv_1''}{dy_1} + \frac{dw_1''}{dz_1},$$

$$-\left[l_1^2\frac{du_1''}{dx_1} - \frac{1}{3}\left(\frac{du_1''}{dx_1} + \frac{dv_1''}{dy_1} + \frac{dw_1''}{dz_1}\right)\right] - l_1m_1\frac{du_1''}{dx_1}$$

$$-n_1l_1\frac{du_1''}{dx_1}, \text{ \&c., \&c. } \ldots\ldots (290 \text{ A}).$$

&c. &c.

For the linear inequality of mean motion, taking the principal axes the same as for the angular inequality, we have

$$-\left(\frac{du_1''}{dx_1} + \frac{dv_1''}{dy_1} + \frac{dw_1''}{dz_1}\right),$$

where

$$\frac{dv_1}{dy_1} = \frac{dw_1}{dz_1} = 0.$$

And transforming to the axes x, y, z, we have for the components in directions x, y, z,

$$-\left(\frac{du''}{dx} + \frac{dv''}{dy} + \frac{dw''}{dz}\right), \text{ \&c., \&c.}$$

Expressions for the resultant institutions of inequalities of mean motion when the direction of propagation is perpendicular to the direction of motion.

If x_1, y_1 are measured in the directions of propagation and mean motion respectively, the resultant rate of shear strain is expressed by

$$-\frac{dv_1''}{dx_1}.$$

Then taking x_1, y_1, z_1 for the principal axes, l_1, m_1, n_1 for the direction-cosines of the principal axes referred to x_1, y_1, z_1, we have, resolving for the principal strains,

$$\frac{du_1''}{dx_1} = l_1m_1\frac{dv_1''}{dx_1}, \quad \frac{dv_1''}{dx_1} = l_2m_2\frac{dv_1}{dx_1} = \frac{dw_1}{dx_1} = 0.$$

And since

$$n_1 = n_2 = n_3 = 0, \qquad l_1^2 = l_2^2 = m_1^2 = m_2^2 = \tfrac{1}{2} l_1 m_1 = -l_2 m_2 = \pm \tfrac{1}{2},$$

$$du_1/dx_1 = -dv_1/dx_1 = \pm \tfrac{1}{2} dv_0/dx_0,$$

and referring to any rectangular axes x, y, z, the partial inequalities are

$$-\left\{\frac{du''}{dx} - \left(\frac{du''}{dx} + \frac{dv''}{dy} + \frac{dw''}{dz}\right)\right\} - \frac{1}{2}\left(\frac{dv''}{dx} + \frac{du''}{dy}\right)$$

$$- \frac{1}{2}\left(\frac{dw''}{dx} + \frac{du''}{dz}\right), \text{ \&c., \&c.} \quad\ldots\ldots(290\text{ B}).$$

196. *The equations of motion of the mean system in terms of the quantities defining the state of the medium.*

Having obtained the four general expressions for:

The total	angular	inequality	in	convection:—	equation	(287)
,,	,, linear	,,	,,	,,	,,	(289)
,,	,, angular	,,	,,	conduction	,,	(288)
,,	,, linear	,,	,,	,,	,,	(290)

Adding the two first together we have the total inequality in *vis viva*.

And in the same way adding the last two together we have the total inequality in conduction.

Then again adding we have the total inequality.

Thus reverting to the forms A_1^2, B_1^2, &c., for the respective constants, and introducing the actual expressions for the general expressions I'', or the harmonic expressions $\rho(u'u')$, &c., for the inequalities, we have, for angular and linear inequalities in *vis viva*,

$$\rho(u'u') = -\frac{A_1^2}{m^2 + A_2^4}\left[A_2^2 - \frac{\partial}{\partial t}\right]\left\{\frac{du''}{dx} - \frac{1}{3}\left(\frac{du''}{dx} + \frac{dv''}{dy} + \frac{dw''}{dz}\right)\right\}$$

$$-\frac{C_1^2}{m^2 + (aC_2)^4}\left[a^2 C_2^2 - \frac{\partial}{\partial t}\right]\left\{\frac{du''}{dx} + \frac{dv''}{dy} + \frac{dw''}{dz}\right\}, \text{ \&c., \&c.} \ldots(291),$$

$$\rho(v'u') = -\frac{A_1^2}{m^2 + A_2^4}\left[A_2^2 - \frac{\partial}{\partial t}\right]\frac{1}{2}\left(\frac{dv''}{dx} + \frac{du''}{dy}\right), \text{ \&c., \&c.} \ldots\ldots\ldots\ldots\ldots(292),$$

$$\rho(w'u') = -\frac{A_1^2}{m^2 + A_2^4}\left[A_2^2 - \frac{\partial}{\partial t}\right]\frac{1}{2}\left(\frac{dw''}{dx} + \frac{du''}{dz}\right), \text{ \&c., \&c.} \ldots\ldots\ldots\ldots\ldots(293).$$

And for angular and linear inequalities in conductions

$$p''_{xx} - \frac{m}{2\pi}\int_t^{t+\frac{2\pi}{m}} p''dt = -\frac{B_1^2}{m^2+B_2^4}\left\{B_2^2 - \frac{\partial}{\partial t}\right\}\left\{\frac{du''}{dx} - \frac{1}{3}\left(\frac{du''}{dx} + \frac{dv''}{dy} + \frac{dw''}{dz}\right)\right\}$$

$$-\frac{D_1^2}{m^2+(aD_2)^4}\left\{a^2D_2^2 - \frac{\partial}{\partial t}\right\}\left\{\frac{du''}{dx} + \frac{dv''}{dy} + \frac{dw''}{dz}\right\}, \text{ \&c., \&c.} \quad \ldots\ldots(294),$$

$$p''_{yx} = -\frac{B_1^2}{m^2+B_2^4}\left\{B_2^2 - \frac{\partial}{\partial t}\right\}\frac{1}{2}\left\{\frac{dv''}{dx} + \frac{du''}{dy}\right\}, \text{ \&c., \&c.} \ldots (295),$$

$$p''_{zx} = -\frac{B_1^2}{m^2+B_2^4}\left\{B_2^2 - \frac{\partial}{\partial t}\right\}\frac{1}{2}\left\{\frac{dw''}{dx} + \frac{du''}{dz}\right\}, \text{ \&c., \&c.} \ldots(296),$$

and two corresponding equations in directions y and z for convections and conductions.

N.B. The linear inequalities which form the second member of equations (291) and (294), and the corresponding terms of the equations for directions y and z, do not include such linear inequalities in the *vis viva* and conductions as are instituted by dispersion of angular inequalities, since these, being secondary effects of the mean inequalities which are themselves small, are altogether negligible. And thus equations (291) to (296) are the equations for the inequalities in *vis viva* of relative motion to a first approximation. Q. E. F.

As to these inequalities it may be well at this stage to point out:

(1) That if m^2 and a^2, b^2, c^2, which express the frequencies in time and space are zero, the angular inequalities in the mean motion are severally constant, while the linear inequalities are zero.

(2) If the direction of propagation is in the direction of motion, or is normal to a shearing motion, all the inequalities in mean motion are zero except that one, whether it be

$$\frac{du}{dx}, \quad \frac{du}{dy}, \quad \frac{du}{dz}, \text{ \&c., \&c., \&c.}$$

But otherwise the inequalities of mean motion as expressed in equation (291) are partial.

(3) The coefficients of these partial equations must be such as will, within the limits of approximation, resolve into the resultant equations for the resultant inequalities.

(4) The coefficients in the partial equations which express component angular inequalities satisfy the condition of resolution stated in (3) as a matter of form.

(5) The coefficients in the partial equations which express component linear inequalities do not obviously, as a matter of form, satisfy the condition of resolution to a first approximation unless $a^2C_2^2/m^2$ is small. But treating

this quantity as small, it can be shown that they do satisfy the condition even to a second approximation. Thus omitting the square of $a^2C_2^2/m^2$ as a first approximation, and putting $(a^2+b^2+c^2)^2 C_2^4/4m^2$, the mean value of $a^4C_2^2$, in the second approximation, the terms expressing component linear inequalities take the form

$$\frac{C_1^2}{m^2}\left(a^2C_2^2 - \frac{\partial}{\partial t}\right) I'' \left(1 - \frac{(a^2+b^2+c^2)^2 C_2^4}{4m^2}\right), \text{ \&c., \&c.} \ldots (297),$$

and these obviously satisfy the conditions of resolution for inequalities in both *vis viva* and conduction:

$$\frac{C_1^2}{m^2}\left(a^2C_2^2 - \frac{\partial}{\partial t}\right)\left(\frac{du''}{dx} + \frac{dv''}{dy} + \frac{dw''}{dz}\right)\left\{1 - \frac{(a^2+b^2+c^2)^2 C_2^4}{a^2}\right\}, \text{ \&c., \&c.} \ldots (298),$$

$$\frac{D_1^2}{m^2}\left(a^2D_2^2 - \frac{\partial}{\partial t}\right)\left(\frac{du''}{dx} + \frac{dv''}{dy} + \frac{dw''}{dz}\right)\left\{1 - \frac{(a^2+b^2+c^2)^2 D_2^4}{m^2}\right\}, \text{ \&c., \&c.} \ldots (299),$$

which satisfy the conditions of resolution, and the second approximation may be neglected.

(6) The proof that these—$a^2C_2^2/m^2$—are small, is not possible as long as m^2 and a^2 are considered as arbitrary, and subject only to the conditions of being small as compared with σ/λ and $1/\lambda$, since the proof depends on dynamical analysis which is effected in a subsequent article, in which it is shown that for any disturbance propagated through the medium these constants are extremely small.

(7) Although small the second approximation is finite as long as the first approximation to the inequalities is finite. Beyond reminding us of this fact there is no object in retaining this second approximation.

197. *The equations of motion to a first approximation.*

Substituting in the equation of mean-motion (119) from equations (291) to (296) for the inequalities in the relative *vis viva* and rate of conduction, these take the form:

$$\begin{aligned}
\rho\frac{du''}{dt} = &\left\{\frac{A_1^2}{m^2+A_2^4}\left[A_2^2 - \frac{\partial}{\partial t}\right] + \frac{B_1^2}{m^2+B_2^4}\left[B_2^2 - \frac{\partial}{\partial t}\right]\right\} \\
&\qquad\qquad \left\{\tfrac{1}{2}\nabla^2(u'') + \frac{1}{6}\frac{d}{dx}\left(\frac{du''}{dx} + \frac{dv''}{dy} + \frac{dw''}{dz}\right)\right\} \\
&+ \text{\&c. \&c.} \\
&+ \left\{\frac{C_1^2}{m^2+(aC_2)^4}\left[a^2C_2^2 - \frac{\partial}{\partial t}\right] + \frac{D_1^2}{m^2+(aD_2)^4}\left[a^2D_2^2 - \frac{\partial}{\partial t}\right]\left[\frac{du''}{dx} + \frac{dv''}{dy} + \frac{dw''}{dz}\right]\right\} \\
&\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (300),
\end{aligned}$$

with two similar partial equations for the rates of increase of dv''/dt and dw''/dt, and the conditions

$$\frac{dw}{dy} + \frac{dv}{dz} + \text{\&c.} = 0.$$

As explained in (7) in the last article the last factor in the second term on the right, which adds the second approximation, may be omitted within limits of a first approximation.

Substituting for the coefficients A_1^2, A_2^2, &c. their values in terms of the quantities which define the state of the medium, as given in equations (277) to (280) and (287) to (290), we have, to a first approximation, the equations of motion in the mean system in terms of the quantities, referred to axes moving with the mean-motion of the medium, the general expressions for which are stated in equations (119). Q. E. F.

From these partial equations (300), we get the partial equations for the component *vis viva* of mean-motion, in terms of the quantities which define the state of the medium, by multiplying the partial equations of motion by u'', v'', w'' respectively, as in equation (122), and these added together resolve into the several equations of *vis viva* in terms of the quantities the general expression for which is given in equations (125).

198. *The equations of the components of energy of the relative system in steady or periodic motion.*

It has already been shown, equation (285), that the rate at which the component of energy of relative motion is increasing, at a point moving with the mean-motion of the medium, is the product of the total partial component of the inequality in relative motion multiplied by the inequality of mean-motion in the general form:

$$\tfrac{1}{2}\rho\frac{\partial}{\partial t}\left(\frac{\alpha^2}{2}\right)+\frac{A_1^2}{m^2+(aA_2)^4}\left(a^2A_2^2-\frac{\partial}{\partial t}\right)I''^2.$$

Therefore, proceeding as in the last article to take account of all the inequalities angular and linear, since the constants are the same, and the linear inequalities a, b, c are the parameters of the variations, the equations for the partial rates of increase of the energy of relative motion by transformation from the mean-motion become

$$\begin{aligned}\frac{1}{2}\rho\frac{\partial}{\partial t}\left(\frac{\alpha^2}{2}\right)=&\left\{\frac{A_1^2}{m^2+A_2^4}\left[A_2^2-\frac{1}{2}\frac{\partial}{\partial t}\right]+\frac{B_1^2}{m^2+B_2^4}\left[B_2^2-\frac{1}{2}\frac{\partial}{\partial t}\right]\right\}\\&\cdot\left\{\left[\frac{du}{dx}-\frac{1}{3}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\right]^2+\frac{1}{4}\left(\frac{dv}{dx}+\frac{du}{dy}\right)^2+\frac{1}{4}\left(\frac{dw}{dx}+\frac{du}{dz}\right)^2\right\}\\&-\frac{C_1^2}{m^2+(aC_2)^4}\left[a^2C_2^2-\frac{1}{2}\frac{\partial}{\partial t}\right]\\&+\frac{D_1^2}{m^2+(aD_2)^4}\left[a^2D_2^2-\frac{1}{2}\frac{\partial}{\partial t}\right]\left[\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right]^2\ \ldots\ldots\ldots\ldots\ (301),\end{aligned}$$

with two corresponding equations for the directions y and z.

Then substituting for the coefficients from equations (287) to (290) we have, to a first approximation, the partial equations for the *vis viva* of relative motion in terms of the quantities which define the state of the medium, terms for which the general expressions are given in equations (123).

Then considering the partial equations (298) we have for the resultant equation of relative *vis viva*, the general expression for which is given by equation (126),

$$\begin{aligned}\frac{1}{2}\rho\frac{\partial}{\partial t}\left(\frac{3\alpha^2}{2}\right) = &\left\{\frac{A_1^2}{m^2+A_2^4}\left[A_2^2 - \frac{1}{2}\frac{\partial}{\partial t}\right]\right.\\ &\left.+\frac{B_1^2}{m^2+B_2^4}\left[B_2^2-\frac{1}{2}\frac{\partial}{\partial t}\right]\right\}\left\{\left(\frac{du''}{dx}\right)^2+\left(\frac{dv''}{dy}\right)^2+\left(\frac{dw''}{dz}\right)^2\right\}\\ &-\frac{1}{3}\left(\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right)^2\\ &+\frac{1}{2}\left\{\left(\frac{du''}{dy}+\frac{dv''}{dx}\right)^2+\left(\frac{dv''}{dz}+\frac{dw''}{dy}\right)^2+\left(\frac{dw''}{dx}+\frac{du''}{dz}\right)^2\right\}\\ &+\left\{\frac{C_1^2}{m^2}\left[C_2^2(a^2+b^2+c^2)-\frac{3}{2}\frac{\partial}{\partial t}\right]\right.\\ &\left.+\frac{D_1^2}{m^2}\left[D_2^2(a^2+b^2+c^2)-\frac{3}{2}\frac{\partial}{\partial t}\right]\right\}\left\{\frac{du''}{dx}+\frac{dv''}{dy}+\frac{dw''}{dz}\right\}^2 \quad\text{.........(302).}\end{aligned}$$

And putting for the right-hand member its equivalent

$$\frac{1}{2}\frac{d}{dt}\left[\rho''\left(u'^2+v'^2+w'^2\right)\right]-\frac{1}{2}\frac{d_c}{dt}\left[\rho''\left(u'^2+v'^2+w'^2\right)\right],$$

we have the expression which would constitute the first member of equation (126).

Therefore we have, in the second member of equation (302), the expression, to a first approximation, for the rate of variation of the energy of the relative system in terms of the quantities which define the state of the medium.

Thus equations (300), (301) and (302) are, to a first approximation, respectively the partial equation of momentum of mean-motion, the partial equation of energy of relative motion, and the resultant equation of energy of the relative system.

And it may be noticed that the equation of energy of mean-motion corresponding to equation (125) Section VI. is at once obtained by multiplying equations (300) by u'', v'', w'' respectively.

And thus the dynamical theory of a purely mechanical medium is established and defined for periodic inequalities to a first approximation.

Q. E. D.

It is to be noticed here that the three equations (300) of momentum in the mean system, to a first approximation, when multiplied by the respective components of mean motion, become the component equations of energy of mean motion, and on being reduced and added together form the resultant equation of mean energy.

And since, in a conservative system, such as that under consideration, the only exchanges between the two systems are between the energy of mean motion and the energy of relative motion, we should have as the sum

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{3}{2}\alpha^2\right)+\frac{1}{2}\frac{\partial}{\partial t}(u''^2+v''^2+w''^2)=0,$$

if the approximation is complete; and this is the case.

That is to say, the approximate expressions for energy of mean motion obtained from equation (128) become, on changing the sign, the equations for energy of relative motion.

It thus appears that there is only one equation of energy although there may be two systems of partial equations for the energy of the components of mean and relative motion.

There are, however, two systems of equations for momentum, one for momentum of mean motion, and the other for the mean momentum of relative motion, the second of which is expressed by

$$(u')''=0, \quad (v')''=0, \quad (w')''=0,$$

while the first is the system expressed by equations (300).

This affords a check on the method of approximation which only becomes apparent at this stage.

199. *The equations of motion to a second approximation.*

In proceeding to a second approximation, it is to be noticed that the rates of increase of α or α^2, A_1^2, B_1^2, C_1^2, and D_1^2, the coefficients in the first approximation, are the result of the irreversible dissipation from *vis viva* of mean motion in consequence of the inequalities in mean motion, as considered in the first approximation, tending to increase the value of α, and to institute linear inequalities in the value of α or α^2; such secondary inequalities are instituted both by angular and linear inequalities in the first approximation.

But it is not in taking account of these secondary inequalities that the second approximation consists, for, as will appear as we proceed, such secondary inequalities are of no account as compared with the first.

The second approximation consists in taking account of the rate of irreversible dissipation of energy resulting from each of the several actions, as expressed in the first approximation, as cause logarithmic rates of diminution in the linear inequalities of mean motion.

In this portion of the analysis, since the general expression for the equations to a first approximation has been effected, attention may be confined to the two primary undulations, approximately simple harmonic, referred to axes in the direction of mean strain; taking the axis of x for that of propagation and the axis of y for that of shear, so that the inequalities (I'') in mean motion are expressed by

$$\frac{du''}{dx} \text{ and } \frac{dv''}{dx}.$$

The equations for the undulations are obtained to a first approximation by taking all the rates of variation of the mean motion zero, except those which enter into the two expressions respectively in the equations (300), (301) and (302).

200. *The determination of the mean approximate rates of logarithmic decrement.*

To do this it is necessary to know two quantities:—

(1) The ratio which the mean of the total undulatory energy bears to the mean of the energy of mean motion, including resilience, per unit volume.

(2) The rate of irreversible dissipation per unit volume in terms of the energy of mean motion to which it is proportional.

Let R be the ratio of the total energy of undulation to the total, including resilience, per unit volume;

T the coefficient by which mean energy of mean motion must be multiplied to express the rate of dissipation.

Then, the bar indicating the mean,

$$\left.\begin{array}{l} \rho R \dfrac{\partial}{\partial t}\left(\dfrac{\overline{u''^2} + \overline{v''^2} + \overline{w''^2}}{2}\right) = T^* \left(\dfrac{\overline{u''^2} + \overline{v''^2} + \overline{w''^2}}{2}\right). \\ \text{The logarithmic rate of decrement is} \\ \sqrt{u''^2 + v''^2 + w''^2} = e^{\frac{-T}{2R}} \end{array}\right\} \quad \ldots\ldots(303).$$

The values of T are all to be obtained from equation (302) omitting the $\partial/\partial t$.

The values of R are a little more complex. But as in the first

* No connection with τ (tau)—the rate of propagation of light.

approximation the motions are a simple harmonic function of t and x or x and y,

$R = 2$ for normal waves,

$R = 2$ for transverse waves when there is no diffusion,

$R = 1$ for transverse waves when diffusion becomes easy.

This last case, whatever other interest it may have, is of great interest in affording a check on the correctness of the approximation, since Stokes has obtained a complete solution of this case for a gas as well as any viscous fluid, and as σ/λ is small in this case it enables us to compare this approximation, and, as will appear, to show that the results are identical. In this case total mean energy is the same as the energy of mean motion.

The only values of R which are not included in the list above are the values of R for transverse waves for the region between the state of no diffusion and that at which diffusion becomes easy, and in this case the value of R varies, very rapidly at first, but at a diminishing rate, from 2 to 1.

201. *The rates of decrement in a normal wave.*

Taking x for the direction of propagation and motion, the motion harmonic and $u_1''^2$ for the maximum value of u''^2; the mean value is $u_1''^2/2$, and the mean energy $u_1''^2/4$.

The two rates of irreversible dissipation of energy by angular inequalities and linear inequalities are obtained by omitting the $\partial/\partial t$ in the coefficients of both the terms of equation (302) and dividing by ρ.

For convenience putting A for the sum of the coefficients for the angular inequalities, and L for the sum of the coefficients for the linear inequalities, resolving in direction x, we have for the respective rates of dissipation

$$R\frac{\partial}{\partial t}\left(\frac{u''^2}{2}\right) = -\left(\tfrac{2}{3}A + L\right)\left(\frac{du''}{dx}\right)^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(304).$$

And we have for the mean square of the inequality, mean energy of motion, and total energy,

$$q^2/2,\ q^2/2a^2,\ \text{and}\ q^2/a^2\ \text{respectively.}$$

Thus $R = 2$ and $\dfrac{Tq^2}{2a^2} = \dfrac{-q^2}{2a^2}\left(\tfrac{2}{3}A + L\right)a^2,$

$$\frac{T}{R} = -\left(\frac{1}{3}A + \frac{L}{2}\right)a^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(305).$$

And the equation for the normal wave is

$$u'' = \frac{q_1}{a}e^{-\left(\frac{1}{3}A+\frac{L}{2}\right)a^2 t}\sin(mt - ax) \quad\ldots\ldots\ldots\ldots\ldots\ldots(306).$$

In a similar manner for the transverse wave

$$R\frac{\partial}{\partial t}\left(\frac{v''^2}{2}\right) = -A\left(\frac{dv''}{dx}\right)^2 \quad\text{.........................(307).}$$

The mean values of $-(dv''/dx)^2$, $(v'')^2$, and total energy are, when σ/λ is large, and since there is no linear inequality,

$$q^2_{yx}/2, \quad q^2_{yx}/2a^2, \quad\text{and}\quad q^2/a^2,$$
$$T = -Aa^2, \quad R = 2,$$

and the equation for the transverse wave becomes

$$v'' = \frac{q_{yx}}{a} e^{-\frac{A}{2}t} \sin(mt - ax) \quad\text{.........................(308).}$$

If σ/λ is small, R is 1 and

$$v'' = q_{yx}\, e^{-Aa^2 t} \sin(mt - ax) \quad\text{.....................(309).}$$

When σ/λ is large the equation for undulations in the direction of the propagation is

$$u'' = \frac{q_1}{a} e^{-\left\{\frac{4}{9}\lambda\alpha a^2 + \frac{1}{2}\frac{5}{3}\frac{p^2}{\rho^2\frac{\alpha^2}{2}}\frac{4}{3}\frac{\sigma^2}{\lambda}\frac{\alpha}{\sqrt{\pi}}\frac{G}{4}\frac{a^4}{m^2}\right\}t} \cos(mt - ax)\quad\text{.........(310),}$$

and the equation for transverse undulations

$$v'' = \frac{q_{yx}}{a} e^{-\left(\frac{2}{3}\frac{\lambda\alpha}{\sqrt{\pi}}a^2\right)t} \cos(mt - ax)\quad\text{..................(311).}$$

In the same way if σ/λ is small the equation for the normal undulations is

$$u'' = \frac{q_1}{a} e^{-\frac{1}{\rho}\left(\frac{4}{9}\frac{\rho\lambda}{\sqrt{\pi}}a + \frac{5}{4}\rho\frac{\lambda\alpha^3 a^2}{\sqrt{\pi}\,m^2}\right)a^2 t} \cos(mt - ax) \quad\text{............(312),}$$

and for transverse undulations

$$v'' = \frac{q_{yx}}{a} e^{-\left(\frac{4}{3}\frac{\lambda\alpha}{\sqrt{\pi}}a^2\right)t} \cos(mt - ax)\quad\text{..................(313).}$$

From equation (310) the coefficients A, B, L, are

$$\left.\begin{array}{ll} & A = \dfrac{4}{3}\dfrac{\lambda\alpha}{\sqrt{\pi}}, \quad B = 0; \\ \text{and for } \dfrac{\sigma}{\lambda} \text{ small} & L = \dfrac{1}{m^2}\dfrac{5}{2}\dfrac{\lambda\alpha^3 a^2}{\sqrt{\pi}}; \\ \text{and for } \dfrac{\sigma}{\lambda} \text{ large} & L = \dfrac{5}{3m^2}\dfrac{p^2}{\rho^2\dfrac{\alpha^2}{2}}\dfrac{4}{3}\dfrac{\sigma^2\alpha}{\lambda\sqrt{\pi}}\dfrac{G}{4}a^2 \end{array}\right\} \quad\text{......................(314).}$$

We have thus obtained the complete equations for indefinitely small steady continuous undulations, including rates of decrement for normal and transverse waves, in terms of the quantities α, λ, σ which define the condition of the medium.

These equations are thus available for obtaining the rates of propagation and the rates of decrement for normal as well as transverse undulations for any specified values of α, λ, σ.

Also if the rates of propagation together with the rates of decrement for both the normal and transverse waves are known, the values of α, λ, σ may be found from the equations.

At this stage of the analysis, however, we have not before us all the data necessary to make a complete determination of the values of α, λ, σ, so that the equations would be the equations of light, as this would require a knowledge of actual rates of decrement as to which we have no certain knowledge, and further, these equations have been obtained by neglecting all secondary actions (see note, Art. 196). And thus these equations afford no evidence as to the limits of the possible magnitudes of the undulations.

The conditions which limit the possible magnitudes of the undulatory strains have been generally discussed in Art. 91, Section VII. From which discussion it appears that, when the medium in normal piling has relative motion, however small λ/σ may be, the medium yields in proportion to the stress when subject to indefinitely small variations of stress; so that such stress is equal to the strain multiplied by a coefficient which is constant if the terms involving the square and higher powers of the strain are neglected as small compared with the first term; and in this case the medium has the properties of an elastic solid within the limits of such strain. It has no finite stability and only such dilatation as would correspond to the elastic solid as long as the terms involving the square and higher powers of the strain are small.

On account of both these the further consideration of the undulations is continued in the section next but one to this—after the consideration of the possible strains, other than the undulatory strains, which afford further evidence.

SECTION XIV.

THE CONSERVATION OF MEAN INEQUALITIES, AND THEIR MOTIONS ABOUT LOCAL CENTRES, IN THE MEAN MASS.

202. In the last section we obtained the equations for continuous steady undulations, including the rates of decrement, for normal and transverse waves in terms of a'', λ'' and σ, the only quantity undetermined being the superior limit to the amplitude; while from the same section it is evident that undulatory strains have characteristics which differentiate them from strains other than undulatory, and that they are essentially elastic strains maintained only by the inequalities of the mean motion, and independent of motion by propagation. It remains to effect such analysis of the strains other than undulatory, the possibility of which has been pointed out in Art. 190, Section XIII. These are:

(i) Some local disarrangement of the medium together with some displacement of portions of the medium from their previous neighbourhood, such as vortex rings, which may have a temporary existence if λ''/σ is large.

(ii) Local abnormal arrangements of the grains when so close that diffusion is impossible except in spaces or at closed surfaces of disarrangement, depending, as already explained, on the value of G being greater than $6/\sqrt{2}\,\pi$, under which conditions it is possible that, about the local centres, there may be singular surfaces of freedom, which admit of their motion in any direction through the medium by propagation, combined with strains throughout the medium, which strains result from the local disarrangement without change in the mean arrangement of the grains about the local centres—the grains moving so as to preserve the similarity of the arrangement.

203. The character of these two general classes of strain must depend primarily on the state of the medium, where uniform, as indicated by the value of σ/λ''.

When σ/λ'' is small there is no dilatation, and there is diffusion, hence there are no singular surfaces except such temporary surfaces as result from vortex motion. Therefore this class of strain may be considered as belonging to the undulatory class which does not concern us in this section.

The second of these classes of local disturbance, in which σ/λ is large, so that there is no diffusion except about centres of disturbance, includes all local disarrangement of the normal piling that can under any circumstances be permanent.

(i) Such permanence belongs to all local disarrangements of the grains from the normal piling, which result from the absence of any particular number of grains at some one or more places in the medium which would otherwise be in normal piling. The centres of such local disturbance may be called centres of negative disturbance, or centres of negative inequalities in the mean density.

(ii) We can also conceive disarrangement resulting from excess of grains in the otherwise uniform medium—a definite number of grains over and above the number which constitute the uniform piling, and such, whether or not capable of independent existence, will be called a positive disturbance.

These positive and negative centres are the principal centres of disturbance, as well as the simple centres of disturbance.

There are other classes of disturbance which, although more or less complex, are to some extent permanent.

(iii) If by any action on the medium in normal piling a number (n) grains were displaced from their previous neighbourhood when in normal piling, to some other neighbourhood previously in normal piling, the disturbance would be reciprocal, and, if there were no further displacement, would be permanent if there were no further action.

It should be noticed that such displacement might correspond exactly with that of a negative disturbance resulting from the absence of (n) grains, and a positive disturbance from introduction of (n) grains in positions corresponding to those from and to which the (n) grains were displaced.

It should be noticed however that, assuming the possibility of the displacement and that of the simultaneous existence of equal negative disturbances, this in no way proves the possibility of the existence of a solitary positive disturbance.

(iv) Another class of possible local disarrangement of the normal piling in an otherwise uniform medium is that class which does not depend on the absence, presence, or linear displacement of grains, but does depend on the rotational displacement of the grains about some axis.

If we conceive a finite spherical surface in the medium, and further conceive that for 30° on either side of a diametral plane the medium immediately external to this surface is, owing to rotational disarrangement, resisting positive rotation of the surface, while the medium immediately internal to the surface, that which extends from each of the poles to within

30° of the diametral plane, is resisting negative rotation, then it will appear, since owing to the relative motion the medium is to some degree elastic, there will be positive rotational strains extending outwards in the external medium within 30° of the equator, and negative rotational strains extending outwards over both the surfaces from the poles to within 30° of the diametral plane.

These represent a state of polarisation in the strains of the medium, inside and outside, and if we had two such polarising surfaces with similar poles in contact the strains would superimpose, while if the opposite poles were in contact the strains would cancel.

204. With regard to the conservation of similarity in the arrangement of the grains within and without singular surfaces, we may prove the following theorem.

THEOREM 1. *When the condition of the medium is such that there is no diffusion except at a singular surface, where G is greater than $6/\sqrt{2}\,\pi$ as a result of the absence of n grains, the replacement of which would restore the uniformity of the medium to that of unstrained normal piling, there will result inward strains extending from an infinite distance to some spherical surface within the singular surface; then whatsoever may be the inward strains in the normal piling and the disarrangement of the grains, with the surface at which the strained normal piling ceased and abnormal piling commenced, the number of grains absent would be the same (n) and the strains in normal piling would be the same.*

To prove this we have only to consider that, owing to the pressure from the outside and the mobility of the grains due to the relative motion, α'', however small, would secure that in the first instance the arrangement of the grains was such as to cause the minimum dilatation, and hence would secure the maximum normal inward strain and then would be in equilibrium. Then since there would be no outside disturbance, if there are to be any exchanges of neighbourhood owing to relative motion, these exchanges must be such as do not entail any increase in the mean dilatation. Whence it follows either that all the grains within the singular surface must maintain their neighbourhood, in which case the centre of disturbance would remain unchanged, following whatever uniform motion the medium might have, or the arrangement of the grains immediately inside and outside the singular surface must be such that the dilatation caused by any influx of grains into the singular surface from one side would be simultaneously compensated by the contraction caused by the efflux of the same number of grains from the opposite side, in which case the centre of disturbance, together with its attendant strains extending from infinity to the abnormal piling, would be free to move in any direction and maintain the same minimum dilatation. Q. E. D.

It is to be noticed that the second alternative requires conditions as to the possibility of which nothing has been affirmed in the proof of the theorem, while the first is general.

Then again we have as a corollary to the last theorem: *If two negative centres of disturbance exist within any finite distance of each other, the numbers of the grains absent in each of the centres would remain the same.* But it does not follow, as a necessity, that the strains in the normal piling in the respective centres should be the same as if the other centre of disturbance was absent.

Then again we have a theorem with respect to a more complex disturbance:

THEOREM 2. *When the disturbance is such as would result from the removal of n grains from one place in a uniform medium and their introduction to another place at any finite distance, which is the same thing as two equal centres of disturbance at a finite distance, one negative as the result of n grains being absent, and one positive as the result of n grains in excess. Then whatever may be the resulting strain or motion in and about the two centres, the number of grains absent in the negative disturbance must always be the same as the number of grains in excess in the positive disturbance however this number may be changed by exchanges between the centres.*

This theorem being self-evident needs no demonstration.

205. The dilatations which result from strains in the normal piling in the otherwise uniform continuous granular medium have been subjected to somewhat full discussion in Arts. 86 to 92, Section VII. This discussion includes the ideal case ($\alpha'' = 0$), in which there is no relative-motion, as well as that (α'' finite) in which there is relative relative-motion.

It is with the second of these cases that we are directly concerned, but it appears that the only process of effecting the analysis necessary for determining the coefficients for the dilatations in the medium with relative motion is, in the first instance, to determine the coefficients of dilatation, when $\alpha'' = 0$, for small strains in the directions of the axes of distortion. Then by examining the effects of relative motion on these to arrive at the general coefficients of dilatation for small strains in all directions in the medium with relative motion.

206. In Art. 90, Section VII. it appears that in the uniform kinematical medium ($\lambda = 0$) there are six axes symmetrically placed, which are axes of no contraction, and bisect the middle points of the edges of the cube of reference, and all pass through the centre. Between these axes and at angles of 45° to them, that is in directions parallel to the axes of reference, or the edges of the cube, there are three axes of possible symmetrical distortion; hence this medium under any mean stress p'', equal in all directions, has stability and crystalline properties. If however the stability resulting from uniform stress is overcome, say by uniform superimposed stress in the direction of one of the axes of reference, the dilatation resulting from the initial small strain is positive, and can be shown to be equal to the normal contraction, *i.e.* the result of the normal contraction and lateral extensions is to increase the volume by a quantity equal to the small normal strain multiplied by the initial volume. Hence the coefficient is unity.

As the strain increases the coefficient diminishes according to a definite law (which will be expressed) slowly at first, then more rapidly until maximum dilatation is reached, when the coefficient is zero, and $G = 6/\pi$. The medium is then unstable, and under the mean pressure equal in all directions would revert to some second state of normal piling.

207. To prove the statements in the last article as to the coefficients of the dilatations resulting from small strain in the direction of one of the axes of dilatation in a kinematical medium:

Let OA, OB, $OC = a_1, b_1, c_1$, respectively be the principal axes of strain.

Let AB, AC, &c. the generating lines of the conical surface be the lines of no contraction.

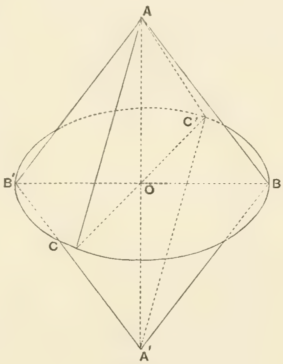


Fig. 2.

Put

$\theta = OAB$, $\phi = OAC$, $L_B = AB$, $L_C = AC$.

Then

$$\left.\begin{aligned} a &= L_B \cos\theta = L_C \cos\phi \\ b &= L_B \sin\theta = C = L_C \sin\phi \end{aligned}\right\} \quad \text{.........................(315)},$$

$$\frac{da}{d\theta} = -L_B \sin\theta, \quad \frac{da}{d\phi} = -L_C \sin\phi \quad \text{...............(316)},$$

$$V = \frac{\pi}{3} \cdot a \cdot b \cdot c = \frac{\pi}{3} \cdot a \cdot L_B L_C \sin\theta \sin\phi \quad \text{...........(317)},$$

$$\frac{a}{-da} \cdot \frac{dV}{V} = -1 + \cot^2\theta + \cot^2\phi \quad \text{..................(318)}.$$

Then, since dV/V is the dilatation and $-da/a$ the strain, the coefficient of dilatation is by equation (318)

$$\frac{a}{-da} \cdot \frac{dV}{V} = -1 + \cot^2\theta + \cot^2\phi \quad \text{...................(319)}.$$

Whence it appears, since $\theta = \phi$ and $\cot\theta$ diminishes as θ increases, we have for the maximum coefficient

$$\cot^2\theta + \cot^2\phi - 1 = 1,$$

and this is when the axes of no contraction are inclined to the axes of distortion at 45°.

Further, it appears that as θ increases from 45°, $\cot^2\theta$ diminishes until dilatation is zero, when the condition of the medium is unstable.

This may be demonstrated graphically. In Figs. 3 and 4 AA, BB and CC are the three axes of symmetrical distortion, and the full-line circles represent the spherical grains in contact. (See also Fig. 1, page 83.)

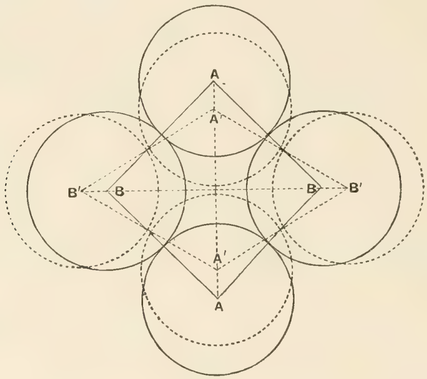


Fig. 3.

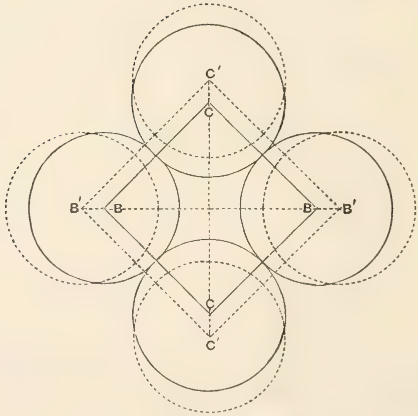


Fig. 4.

Fig. 3 shows a loss $2AA'$ in height. Fig. 4 shows a gain $4AA'$ in plan.

These losses and gains are taken on the three axes at right angles of which the dimensions are AA, BB, CC.

The normal strain is $2AA'/AA$.

The volume is $AA \,.\, BB \,.\, CC$ or $(AA)^3$.

The increase of volume $(AA)^2 \,.\, 4AA' - (AA)^2 \,.\, 2AA' = (AA)^2 \,.\, 2AA'$.

Whence we have the dilatation

$$\frac{dV}{V} = \frac{(AA)^2 \,.\, 2AA'}{(AA)^3}.$$

And dividing by the strain $-2AA'/AA$ and changing the sign, we have for the coefficient of dilatation

$$\frac{AA}{2AA'} \cdot \frac{(AA)^2 \,.\, 2AA'}{(AA)^3} = 1.$$

207 A. Then as regards the inequalities of pressure $p_r = 2p_t = \frac{3}{2}p''$, resulting from such symmetrical distortional strains in the principal axes of strain, since there is no work done on the grains it follows directly, putting p'' for the mean pressure, p_r for the normal in the direction of the strain, and p_t for either one of the tangential since these are principal stresses

$$p_r + 2p_t = 3p'' \quad\text{.............................}(320),$$

and since there is no work done on the grains,

$$p_r = 2p_t \quad\text{.................................}(321),$$

whence by (320)

$$p_r = \tfrac{3}{2}p'', \quad p_t = \tfrac{3}{4}p'' \quad\text{..........................}(322).$$

208. It is to be noticed that contraction strains, such as that discussed in the last article, the strain being in the direction of one of the axes of distortion, are the only symmetrical strains when $\alpha = 0$, and it does not follow that the coefficient of dilatation for small unsymmetrical strains is unity. But it does follow from virtual velocities that if p'' is the mean pressure in a kinematical medium without limit, that the normal pressure resulting from a local disturbance cannot be greater than $2p''$ and must be greater than zero if p'' is finite.

From this we have the proof of the important theorem:

That whatever the coefficient of dilatation may be, a disturbance such as might be caused by the removal of any number of grains from a space in an otherwise uniform medium, without relative motion, would be attended with inward radial displacement of the grains from infinity throughout the entire medium.

For, as has just been shown, p_r must be greater than zero; so that there can be no cavity greater than the space from which the grains can exclude other grains, and there can be no dilatation without the displacement of grains, so that as the ideal excavations proceeded the grains would follow inwards, and as there is no elasticity and the grains are all under pressure, each grain as it disappears must cause inward movement from infinity; for as the coefficient of dilatation cannot be infinite, the grains being smooth spheres without friction (so that any binding or jamming would be impossible) every grain would be under pressure. Q. E. D.

Thus the relation between the tangential and normal pressures would depend upon nothing but the coefficients of dilatation, and if these were constant the normal and tangential pressures would be constant. But such constancy would depend on there being angular similarity in the arrangement of the grains about every axis through the centre of disturbance, which similarity does not exist in the normal piling. It is therefore certain that the inward strains, although having six axes of similar arrangement symmetrically placed, would be influenced by the crystalline formation of the uniform piling; particularly at great distances from the centre of disturbance. For when the distances from the centre are large the strains would be so small that the crystalline characteristics of the uniform medium would have undergone very slight modification, whereas near the centre where the displacements are greatly larger the unsymmetrical characteristics would be greatly modified.

On these grounds it appears certain that the coefficients of dilatation would be greatest at an infinite distance from the centre and would gradually diminish; in which case the tangential pressure would fall and the normal pressure rise gradually as they neared the centre, satisfying the conditions of virtual velocities and the condition for equilibrium, which latter requires that at any distance r from the centre $p_r + 2p_t = p''$. What the mean of such coefficients might be is doubtful, but it seems probable that they would not differ greatly from the coefficient unity, which is the smallest coefficient for symmetrical distortion.

Whatever these coefficients may be it follows from the paragraph last but one, that the dilatation resulting from the inward strain must occupy the space from which the grains were absent, so that the sum of the normal and tangential stresses would be equal to the mean pressure of the medium, or $p_r + 2p_t = 3p''$.

209. From the conditions of geometrical similarity in the case of uniform continuous media it appears:

(i) *The size of the uniform grains has no effect on the dilatation or mean pressures resulting from continuous uniform distortions. Therefore similar and equal continuous finite distortional strains will produce similar* *and equal dilatations whether the grains are indefinitely small or of any finite size.*

(ii) *The size of the uniform grains in a continuous medium does affect the dilatations resulting from strains other than continuous uniform distortional strains.*

To prove these theorems.

If we consider two finite media of which the parts are exactly similar in shape, number, and relative position, but in one of which the scale is A and the other B, these media will be geometrically similar except as to scale.

Thus whatever strains in proportion to the constant parameters, A and B respectively, these media may undergo, the proportional similarity will hold, and this extends to the dilatations, the coefficients of which will be equal. Q. E. D.

If however instead of considering these similar actions within spaces proportional to the scales A and B, we consider these proportional actions within equal spaces, the principle of similarity disappears unless the positions and strains are such that there is perfect uniformity throughout the medium. This proves the first theorem. Perfect uniformity exists in the case of grains in uniform piling subject to equal distortional strains whatever the values of A and B, provided the spaces are such that there is no sensible effect from the boundaries. Q. E. D.

It is thus proved that for other than equal uniform strain there cannot be similarity in the effects *in equal spaces* in media of which the scales of similarity A and B differ.

Thus if the strains in the medium in which the scale is A are subject to variation on that scale, while those on the scale B are subject to similar strains on that of B, then the effects of these variations taken over equal spaces will of necessity differ. Q. E. D.

Then since the dilatations resulting from parallel continuous strains are in no way dependent on the size of the grains, even if these are infinitely small or have any finite size, the question arises as to what would be the difference in the dilatations resulting from finite similar local disturbances about negative centres in two media in one of which the grains are infinitely small and in the other finite.

In the first place it appears that as far as regards the dilatations resulting from uniform parallel distortional strain these would be independent of the size σ.

And it can be shown that these are the only dilatations if σ is indefinitely small as compared with the reciprocal of the curvature.

For since σ is indefinitely small when the scale of disturbance is finite, if we conceive all dimensions including σ to be exaggerated so that σ becomes finite, and the distances between the grains exaggerated on the same scale, then, since the mean strains before exaggeration vary continuously without crossing, so that in the strains the finite paths of two grains which were neighbours before the strain would still be neighbours after the finite strain although separated by any distance which is less than the finite distance σ, their two paths would still be parallel lines of infinite length and at any finite distance apart.

It is thus shown that if the grains are indefinitely small as compared with the dimensions of the disturbance, the only dilatations would be those resulting from uniform parallel distortional strains. Q. E. D.

Again in the case of the medium in which the grains are finite it has been shown, Art. 207, that when the grains are finite, however small as compared with the dimensions of the finite volume from which grains are absent, that the effects must differ from those resulting from uniform parallel distortion.

And by the last theorem, putting $4\pi r_0^3/3$ for the volume the absent grains would occupy in normal piling, it appears, since σ/r_0 is indefinitely small, that the dilatations result solely from uniform parallel distortional strains. And hence whatever finite curvature may result from finite strains, this curvature does not, as curvature, produce any effect on the dilatation; so that there are no curvature effects.

Then since it is shown that when σ is finite, however small compared with the reciprocal of the curvature in the strained normal piling, the dilatation resulting from curvature depends solely on the existence of a finite value of the product of σ multiplied by the curvature, the dilatation will equal σ multiplied by the curvature.

Further, it follows that for any given strain, this dilatation resulting from curvature will be in excess of the dilatations resulting from uniform parallel strains.

210. The analytical separation of the dilatation resulting from uniform strain and that resulting from the curvature would be perfectly general if σ might have any value as compared with the curvature. But, in that case, any analytical separation of the dilatation resulting from distortions from that resulting from the size of the grains would be different on account of the reaction of the dilatation resulting from the size of the grains on that resulting from distortion. But we are only concerned with cases in which σ is such that σ multiplied by the curvature is so small that to a first approximation any reaction from the dilatation resulting from the curvature may be neglected.

Whence it appears that, to a first approximation the only curvature is that instituted by a uniform distortional strain—as if σ multiplied by the curvature were indefinitely small—the dilatation resulting from small inward radial displacements about a centre being of necessity equal to the curvature at each point. It follows as a necessity that, taking A as the dilatation resulting from the uniform distortional strains, the dilatation resulting from curvature owing to the finite size of the grains at the same point is expressed by $A\sigma/2r_1$, where r_1 is the radius of the singular surface, whence we have for the total dilatation

$$A\left(1+\frac{\sigma}{2r_1}\right).$$

211. Granular media with relative motion have this fundamental difference from media without relative motion, that when in normal piling the medium with relative motion is within certain limits perfectly elastic without crystalline properties, that without relative motion is perfectly rigid and crystalline.

When the media are both under strain this difference is not so apparent, as the medium without relative motion is then also without rigidity. But the difference is still fundamental, and the fundamentality of the difference in no way depends upon the degree of relative motion. For in the one the medium satisfies the condition of virtual velocities, while in the other state, owing to its elasticity, this condition cannot be absolutely satisfied however near the approximation may be.

The crucial difference between the two states is virtually reduced to the existence of a state of absolute rigidity in the one, however limited, when the piling is normal, and the absence of such rigidity in the other however small may be the relative motion.

For as has been shown in Art. 207 the medium without relative motion while satisfying the condition of virtual velocities when strained from the normal piling, will also satisfy the condition of equilibrium—that the sum of the normal and tangential pressures equals three times the mean pressure, or that

$$p_r + 2p_t = 3p'' \quad\text{..............................(323).}$$

Another medium will also satisfy the conditions that the pressure between the grains cannot be negative, and that every grain is in contact with at least four grains, whence it follows (since the last three of the four preceding conditions are satisfied in the strained medium without relative motion they are of necessity satisfied by the strained or unstrained medium with relative motion) that if, as has been shown, the condition of virtual velocities can be satisfied to any degree of approximation in the medium with relative motion, such medium has to any degree of approximation all the properties of the medium without relative motion, except those depending on the limited stability on which the crystalline properties depend.

It is thus shown that the necessary distinction between the two states is that of finite rigidity when there is no relative motion.

In regard to this statement it is perhaps necessary to call attention to the fact already demonstrated, that in the case of a medium with relative motion, the relative motion as expressed by α in a steady state of strain must be constant, since any inequalities in α are subject to redistribution, so that the mean energy of every grain remains constant. Therefore the energy of the medium after the grain has been removed and the inward strain established would be constant, and there would be no change in the mean relative kinetic energy of the grains $\frac{\sigma}{\sqrt{2}} \cdot \frac{\alpha^2}{2}$, and it is the state after the grains have been removed with which we are alone concerned.

This although, for the purpose of analysis, an ideal action—that of removing grains from a medium in otherwise uniform normal piling—such action has no existence. This appears from Theorem 1 in this section, from which it follows that whatever may be the volume occupied by the absent grains when in normal piling such accident is permanent.

It has thus been shown that the inward strains resulting from the absence of grains which would occupy the volume $4\pi r_0^3/3$ in normal piling about any centre in the infinite, elastic medium, must cause dilatations extending from an infinite distance to the singular surface about the centre of disturbance, which dilatations occupy a volume equal to $4\pi r_0^3/3$, the volume from which the grains are absent; and they are such as satisfy the conditions of equilibrium under the same mean pressures normal and tangential expressed by

$$p_r + 2p_t = 3p'' \qquad \text{.............................(324)},$$

p'' being the mean pressure equal in all directions.

212. It also follows from Art. 210 that these dilatations, notwithstanding the relative motion of the medium, admit of analytical separation into the two classes:

(i) Dilatation resulting from uniform distortional strains such as would result if σ were indefinitely small.

(ii) Dilatation which results from the finite value of σ and the curvature induced by the uniform distortional strains.

The relations of these dilatations are those expressed in Art. 210 by

$$A\left(1+\frac{\sigma}{2r_1}\right) = \left\{\begin{array}{c}\text{the total dilatation per unit}\\ \text{of volume at the point}\end{array}\right\} \qquad \text{..........(325)},$$

for the only difference resulting from the relative motion is the absence of any limited stability.

213. From the conclusions arrived at in Art. 211 it follows, if p'' is constant, that the total dilatation resulting from the inward strains does not depend in any degree upon the coefficients of dilatation, nor upon the relative motion α, as long as σ/λ is within the limits of no diffusion, whatever may be the value of α.

It does not however follow from this that the distribution of the strains is independent of the variations in the coefficients of dilatation, since it has been shown (Art. 207) that if there is no relative motion the coefficients of dilatation must increase with the distance from the centre of disturbance.

But in the absence of any limited stability as in the case of α being finite, since we need consider those cases only in which the coefficients of dilatation from small strains are unity, the circumstances may be so chosen that the strains follow some regular law.

However, before discussing these circumstances, we may with advantage consider what further conclusions as to the relation between the strains and dilatations, as well as the relation between the normal and tangential pressures, are afforded by the adoption of unity as the general coefficient of dilatation in the medium with relative motion.

Since the coefficients are constant and equal to unity, the mean strains resulting from the absence of a volume of grains expressed both in magnitude and shape by the sphere $4\pi r_0^3/3$, will be radial and symmetrical. Then by the theorem of Art. 212, if σ is small compared with r_0, since the strains must be everywhere very small, the relations between the inward strain and the dilatation will be such (if at any point we take α* for the principal strain in the direction of any radius and β and γ for the principal strains tangential to the surface of the sphere, since the strains are inwards β and γ are negative and equal) as are expressed by

$$\beta + \gamma = -\tfrac{1}{2}\alpha, \text{ or } \frac{\alpha + \beta + \gamma}{\beta + \gamma} = -1 \quad \text{.................. (326).}$$

Then adding $(\beta + \gamma)$ the negative or contraction strains to α the positive or expansion strain, we have the dilatation

$$\left.\begin{aligned} -(\beta + \gamma) &= \frac{\alpha}{2} \\ \alpha &= -2(\beta + \gamma) \end{aligned}\right\} \quad \text{..................... (327).}$$

Then we have from these equations the general relation that the dilatation resulting from tangential contraction $-(\beta + \gamma)$ is equal to half, and can only be half, the normal elongation resulting from the tangential contraction, together with the dilatation caused by the contraction strain.

* α, β, γ are here used to express principal strains.

The dilatation expressed by either member of equation (327) is the total dilatation resulting from the uniform distortional strains, as well as that resulting from the curvature on account of the finite size of the grains. And to complete the analysis of the relations between the dilatations and the strains it is necessary to effect the analytical separation of these two dilatations.

The separation of the dilatations follows at once from equation (324).

By equation (327) we have for the total dilatation per unit of volume at a point

$$-(\beta+\gamma).$$

And from equation (325) the total dilatation is

$$A\left(1+\frac{\sigma}{2r_1}\right).$$

Therefore

$$\left.\begin{aligned} A &= \frac{-(\beta+\gamma)}{1+\dfrac{\sigma}{2r_1}} \\ A\frac{\sigma}{2r_1} &= \frac{-(\beta+\gamma)}{1+\dfrac{\sigma}{2r_1}}\cdot\frac{\sigma}{2r_1} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(328).$$

The first and second of equations (328) are respectively for the dilatations resulting from uniform strains and from the size of the grains.

These involve the squares of $\sigma/2r_1$; neglecting this term we have as approximations:

For the dilatations resulting from uniform strains

$$-(\beta+\gamma)\left(1-\frac{\sigma}{2r_1}\right).$$

And for the dilatations resulting from the size of the grains

$$-(\beta+\gamma)\frac{\sigma}{2r_1}.$$

Adding these two last expressions we have

$$-(\beta+\gamma)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(329),$$

which expresses the total dilatation per unit of volume at a point in the medium.

Then integrating the partial dilatations from ∞ to r_1 over the medium, since the total integral dilatation is $4\pi r_0^3/3$ we have for the integral dilatation resulting from uniform distortion

$$\int_{\infty}^{r_0} -(\beta+\gamma)\left(1-\frac{\sigma}{2r_1}\right)r^2dr = \left(1-\frac{\sigma}{2r_1}\right)\frac{4\pi}{3}r_0^3 \ldots\ldots\ldots\ldots(330).$$

And for the dilatation resulting from the size of the grains

$$\int_{\infty}^{r_0} -(\beta+\gamma)\frac{\sigma}{2r_1} r^3 dr = \left(1 - \frac{\sigma}{r_1}\right) \sigma \frac{4\pi}{3} r_0^3 \text{ ....(331).}$$

The relations between the strains and the resulting dilatations, as expressed in equations (326 to 331), are the complete relations to a first approximation as long as there is no other disturbance in the normal piling than the spherical disturbance which gives rise to the radial inward strains. And they have been obtained by taking the coefficients of dilatation as unity.

The relations between the principal stresses are such as satisfy the equation of equilibrium

$$p_r'' + 2p_t'' = 3p'' \text{..............................(332),}$$

and are also such as satisfy the condition of virtual velocities approximately, which on the assumption that the coefficients of dilatation are unity, since the contraction strains are tangential, requires that

$$p_t'' = 2p_r'' \text{................................(333).}$$

Therefore from (332) and (333) we have

$$p_t'' = \tfrac{6}{5}p'' \text{ and } p_r'' = \tfrac{3}{5}p'' \text{........................(334).}$$

Equations (332) and (333) express completely, to a first approximation, the relations between the constant mean pressure, equal in all directions, and the constant mean tangential and normal principal stresses resulting from a negative spherical disturbance about an only centre on the supposition that the coefficients of dilatation are unity.

214. Having in the last article effected the analysis of the relations between the dilatations and strains, as well as between the mean tangential and normal principal stresses and the mean pressures, equal in all directions, about an only negative centre, on the supposition that the coefficients of dilatation are unity, it remains to consider that choice pointed out (Art. 213) of the circumstances under which this condition can be realised.

The definition of a negative local disturbance (Theorem (i), Art. 203) involves the absence of a certain number of grains, which if present in normal piling would render the piling in the medium normal, reverse the strains, and so obliterate all trace of disturbance about the centre.

There is nothing in the definition of such local centres that defines the mean distance from the local centre at which the grains may be absent, nor is there any obligation that the space from which the grains are absent shall be continuous, as long as there is some symmetry about the centre.

It is therefore open for us to consider such arrangement of the position about the centre from which the grains are absent as will result in the least analytical complexity.

It would seem at first sight that the greatest simplicity would be secured by assuming that the grains were removed from a spherical space. But in that case it at once appears that the inward radial displacement would extend to the centre of the sphere. And it also appears (Art. 207) that the contraction strains as the centre was approached would be such that instability would come in, and the arrangement near the centre would revert to some more nearly normal piling, forming a nucleus of grains in normal piling without dilatation. In this case the dilatation would commence in the grains outside the spherical nucleus, there being a spherical shell of grains in abnormal piling constituting a broken joint between the nucleus and the medium outside, which, although strained inwards, would still be such that the grains had not changed their neighbourhood. Thus it appears that the abstraction of grains from a spherical space would not entail that this strained normal piling would reach the centre.

The arrangement instituted as a result of this abstraction from a spherical space seems most natural and, with a little modification, such arrangement presents the least analytical difficulty.

If we adopt the nucleus in an exaggerated form and the spherical shell of grains in abnormal piling, no matter how thin, also take r_1 for the radius of the singular surface which is somewhere within the spherical shell of grains in abnormal piling, since the volume of grains absent is $4\pi r_0^3/3$ which volume as a spherical shell of radius r_1 would have a thickness approximating to $r_0^3/3r_1^2$, we have as an expression for the inward radial displacement of the grains in strained normal piling which are adjacent to the singular surface

$$\frac{r_0^3}{3r_1^2} = \frac{r_1 r_0^3}{3r_1^3} \qquad \text{(335)}.$$

Then since this is the greatest possible radial displacement, and being adjacent to the singular surface is independent of dilatation, the contraction strain, owing to the displacement, would be the largest contraction strain possible. Whence, if this is small, all the contraction strains will be very small, and as the dilatations are equal to the contraction strains, though of opposite sign, the dilatation would be very small, and by Art. 207 the coefficients of dilatation would approximate to unity.

In order to show that the contraction strains at the singular surface resulting from radial displacement

$$\frac{r_0^3}{3r_1^2}$$

would be very small; let the outer circle (Fig. 4 A) represent a section through the centre of disturbance before the volume $4\pi r_0^3/3$ is removed, and the inner circle represent the section through the centre after the

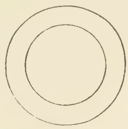


Fig. 4 A.

volume is removed. Then if the inner circle is taken to represent the section of the singular surface through the centre of disturbance, since the radial displacement $[\alpha = -2(\beta + \gamma)]$ of the grains at that surface has been shown to be (equation 335) $r_0^3/3r_1^2$, the contraction at the singular surface is

$$\frac{\left\{\frac{r_0^3}{6r_1^2} + r_1\right\}^2 - r_1^2}{\left\{\frac{r_0^3}{3r_1^2} + r_1\right\}^2} \text{..........................(336)}.$$

Then since r_0/r_1 is small, according to powers of r_0/r_1, we get a rapidly converging series, the first term of which is

$$-\frac{r_0^3}{3r_1^3} = \beta + \gamma \text{..............................(337)}.$$

Then by equation (327) we have as a first approximation to the dilatation resulting from the contraction at the singular surface $r_0^3/3r_1^3$. And as this is, approximately, the greatest possible dilatation, it follows that under the conditions as stated above the radial displacement and inward strains are such that the coefficients of dilatation would to a first approximation be unity.

It is thus shown that the conditions assumed in the present article are not only possible but are also the most probable.

215. In order to complete the analysis for an only negative centre it remains to obtain the expressions for the contraction strains and dilatations at any distance from the singular surface corresponding to those found in the last article for the contraction strains and dilatations at the singular surface.

This problem differs essentially from that of determining the strains at the singular surface; this difference appears at once when we realise, as already pointed out, that the radial displacement which the grains at the singular surface have undergone is definitely expressed by $r_0^3/3r_1^2$, since it is subject to no displacement from dilatation, whereas the radial displacement which the grains at an arbitrary distance r from the centre have undergone depends on the dilatation between r and r_1.

There are however two definite conditions that the radial displacements must satisfy to a first approximation.

(1) The condition (Art. 207) that whatever the radial displacement may be it must be such that the integral of the dilatations taken from r_1 to ∞ shall be equal to the volume from which the grains are absent.

(2) That the radial displacement must be such that at any distance greater than r_1 the resulting tangential contractions will cause dilatation which, integrated over the volume of the spherical shell $4\pi(r_1^3 - r_0^3)/3$, will express when divided by r_1^2 radial displacements corresponding to those assumed.

If instead of taking $-r_0^3/3r^2$ or $-r_1r_0^3/3r_1r^2$ we take

$$-\frac{r_1r_0^3}{3r^3}$$

for the radial displacement, we have for the contraction strains, since they are negative and only half the total elongation,

$$\frac{\left(\dfrac{r_1r_0^3}{6r^3}+r\right)^2 - r^2}{\left(\dfrac{r_0^3}{3r^3}+r\right)^2}.$$

From which to a first approximation we have for the contraction strain

$$-\frac{1}{3}\frac{r_1r_0^3}{r^4}.$$

Then changing the sign, multiplying by π and integrating from r_1 to r

$$4\pi r_1r_0^3\int_{r_1}^{r}\frac{1}{3r^4}r^2dr = \frac{4\pi r_0^3}{3} - \frac{4\pi}{3}\frac{r_1r_0^3}{r} \qquad (338).$$

The result arrived at in equation (338) admits of more general proof, from which it appears that this result is the only result possible.

Putting X for the radial displacement; since the dilatation is expressed by X/r we have to obtain the expression for X satisfying the condition

$$4\pi\int_{\infty}^{r_1}\frac{X}{r}r^2dr = \frac{4\pi}{3}r_0^3 \qquad (339),$$

whence it appears that

$$X = -\frac{r_1r_0^3}{3r^3} \qquad (340).$$

Also dividing the last term in equation (338) by r^2 we have for the radial displacement at a distance r

$$-\frac{r_1r_0^3}{3r^3},$$

which is the same expression for the radial displacement as that assumed. So that both conditions are completely satisfied.

216. In this section it is assumed that there is no diffusion. Having in the previous articles in this section effected the analysis of the inward strains and the consequent dilatations for only negative spherical disturbances resulting from the absence of grains, before proceeding to consider the corresponding analysis for the other inequalities in the density of mean matter, it seems convenient to proceed with the analysis necessary to determine the effects such negative disturbances may have on each other when existing within finite distances of each other.

Any such action must depend on the interference of the strains outside the respective singular surfaces, and any attraction of the centres resulting from such interference must be a function of the distance between the centres.

From Arts. 209 and 212 we have perfect similarity in the strain resulting from uniform distortions, from which it follows that such strains from different negative centres superimpose without affecting their respective dilatations, and hence can in no way interfere or attract one another.

In the case of the strains resulting from finite values of σ owing to the curvature resulting from distortions, the strains from different negative centres at any finite distance must interfere.

This appears in Arts. 209 and 212, in which it is shown that for other than equal uniform strains there cannot be geometrical similarity in the effects in equal spaces, in media of which the scales are different.

For, applying this to the case in hand, since the diameter of the grains, σ_1 say, is common to all the grains, while the number of grains absent as well as the radii of the singular surfaces may differ in almost any degree, the dissimilarity at once appears.

For the sake of clearness we may consider in the first place two cases in both of which the σ has the value σ_1, and the singular surfaces both of radii r_1, but in one of which the volume of the grains absent is $\frac{4\pi}{3} r_a^3$, and in the other $\frac{4\pi}{3} r_b^3$.

Then by equation (331) we have for the dilatation at a distance r for the centre a

$$\frac{4\pi}{3} r_a^3 \frac{\sigma_1}{r^4}\left(1 - \frac{\sigma_1}{2r_1}\right),$$

and for the centre b

$$\frac{4\pi}{3} r_b^3 \frac{\sigma_1}{r^4}\left(1 - \frac{\sigma_1}{2r_1}\right),$$

and neglecting $\sigma_1/2r_1$ for the present, as small, multiplying by r^2dr and integrating from r_1 to $r=\infty$ we have for the dilatation, taking ω to express it,

$$\left.\begin{aligned}\omega_a &= \frac{4\pi r_a{}^3}{3}\frac{\sigma_1}{r_1}\\ \omega_b &= \frac{4\pi r_b{}^3}{3}\frac{\sigma_1}{r_1}\end{aligned}\right\} \quad\text{.............................(341).}$$

From the expressions in the preceding paragraph for the total dilatation resulting respectively from the two centres considered as if each were the only centre within an infinite distance, it appears in the first place that the dilatation resulting from the product σ into the curvature is directly proportional to the volume occupied in normal piling by the grains absent. And in the second place from the form of the expressions obtained, that the total dilatation is inversely as the radius of the singular surface.

It is this fact, that whatever may be the volume occupied by the absent grains in normal piling, the dilatation will be inversely as the radius of the singular surface, which proves the effect of dissimilarity between the constant value of σ and the different values of r_1, namely that for any particular volume of grains absent the dilatation resulting from the small centre will be greater than that resulting from the large centre in the inverse ratio of the radii of the centres.

So far we have only considered the effect of dissimilarity in σ/r_1 on the supposition that each centre is the only centre within finite distance.

We may now proceed to prove that negative centres at finite distances attract each other.

Taking ω to express the total dilatation from r_1 to $r=\infty$ resulting from a single negative centre, then as has just been shown

$$\omega_1 = \frac{4\pi r_0{}^3}{3}\frac{\sigma}{r_1} \quad\text{.................................(342).}$$

Then the number of such singular surfaces which would occupy an empty spherical shell of radius r_B when arranged in closest order would be approximately

$$N' = \frac{0{\cdot}75 r_B{}^3}{r_1{}^3} \quad\text{.................................(343).}$$

And by equation (341) the total dilatation of each of the N' surfaces outside the surface $4\pi r_0{}^2$ is

$$\omega_B = \frac{4\pi r_0{}^3}{3}\frac{\sigma}{r_B} \quad\text{.............................(344).}$$

Multiplying ω_1 and ω_B by N' we have for the respective total dilatations

$$N'\omega_1 = N' \frac{4\pi r_0^3}{3} \frac{\sigma}{r_1} \text{ ....(345)}$$

and

$$N'\omega_B = N' \frac{4\pi r_0^3}{3} \frac{\sigma}{r_B} \text{ ....(346).}$$

Subtracting these equations as they stand we have

$$N'(\omega_1 - \omega_B) = N' \frac{4\pi r_0^3}{3} \left(\frac{\sigma}{r_1} - \frac{\sigma}{r_B}\right) \text{ ....(347).}$$

Then from equation (347) it follows that the dilatation resulting from any number of negative similar disturbances (if the singular surfaces are at an infinite distance from each other) will be

$$N' \frac{4\pi r_0^3}{3} \frac{\sigma}{r_1},$$

while if these surfaces are arranged in closest order the dilatation will be

$$N' \frac{4\pi r_0^3}{3} \frac{\sigma}{r_B}.$$

Whence since r_B is greater than r_1 it is shown that, no matter how accomplished, the dilatation resulting from negative centres diminishes in the ratio

$$\frac{r_1}{r_B},$$

as the centres of the singular surfaces approach until they are arranged in closest order.

This proves the diminution of the dilatation owing to the diminution of the variations of strain as the centres approach—or the diminution of the dilatation owing to the diminution of the curvature of the normal piling in the medium due to dissimilarity. Q. E. D.

From the proof of the foregoing theorem it also appears how it is that the dilatations resulting from distortion do not interfere however much they superimpose, for since the dilatations resulting from distortion in no way depend on the curvature in the medium, as curvature, they depend only on the strain, whereas the diminution is in the variations of the strain.

In order to prove the attraction of the negative centres it is necessary to consider the effects of the pressures in the medium. These have already been discussed in Art. 213, equations (332) to (334), in which it is shown that the dilatations resulting from curvature are subject to the mean pressure p'' and satisfy the condition of virtual velocities. In dealing with attraction it might seem necessary first to prove or assume that the singular surfaces are also surfaces of freedom which can be propagated in any direction through the medium, for as the medium is elastic in consequence of the finite relative motion, if we can find the variation of the work done by the external media on the singular surfaces owing to variation of their distances, it becomes possible to separate the active effort from the passive resistance.

Multiplying the member on the right of equation (347) by p'' we have

$$p'' N' \frac{4\pi r_0^3}{3} \left(\frac{\sigma}{r_1} - \frac{\sigma}{r_B}\right)$$

as the expression for the difference of the energies in the media when the N' singular surfaces of radius r_1 are at an infinite distance from each other, and when the N' singular surfaces of radius r_1 are arranged in closest order within the surface r_B.

This difference in the energy proves the existence of attractions whatever may be the passive resistance owing to want of mobility of the singular surfaces.

These attractions as obtained by neglecting σ^2 are the only attractions between negative centres of disturbance which are small compared with their distances apart, as follows from the fact already proved that the aggregate dilatation resulting from distortional strains depends only on the volume of the absent grains.

217. The law of the attraction of negative centres appears at once from the analysis.

If instead of taking the total dilatation from r_B to $r = \infty$, as in equation (346), we take the dilatation from r_B to r, where r is greater than r_B, the dilatation from the N' singular surfaces in closest order is

$$N' \frac{4\pi r_0^3}{3} \left(\frac{\sigma}{r_B} - \frac{\sigma}{r}\right).$$

Then if there is another singular surface of radius r_1 in which the volume of grains absent is $4\pi r_0^3/3$ at the distance r the variations of the strains of the outside singular surfaces interfere with those from the centre r_B; and multiplying the dilatation outside r_B less the dilatation outside r by minus the volume of the grains absent in the outside centre, we have the expression

$$-N' \left(\frac{4\pi r_0^3}{3}\right)^2 \left(\frac{\sigma}{r_B} - \frac{\sigma}{r}\right),$$

and differentiating this expression with respect to r we have

$$-N' \left(\frac{4\pi r_0^3}{3}\right) \frac{\sigma}{r^2},$$

whence multiplying by p'', since σ/λ is large so that the density within the singular surfaces is unity, we have for the acceleration

$$p''N'\left(\frac{4\pi r_0^3}{3}\right)^2\frac{d}{dr}\left(\frac{\sigma}{r_B}-\frac{\sigma}{r}\right)=-p''N'\left(\frac{4\pi r_0^3}{3}\right)^2\frac{\sigma}{r^2}\ \ldots\ldots\ldots(348).$$

This expresses the *space* rate of variation in the work, or energy in the system, with the distance, that is the effort to bring the centres together whatever may be the passive resistance.

It is thus shown that the law of attraction, that is the effort to bring the surfaces together, whatever may be the passive resistance, is the product of the masses of the grains absent multiplied by σ and again by minus the reciprocal of the square of the distance.

This law of attraction, which satisfies all the conditions of gravitation, is now shown by definite analysis to result from negative local inequalities in an otherwise uniform granular medium under a mean pressure equal in all directions, as a consequence of the property of dilatancy in such media when the grains are so close that there is no diffusion and infinite relative motion; and further it is shown to be the only attraction which satisfies the conditions of gravitation in a purely mechanical system.

The mechanical actions on which this attraction depends are completely exposed in the foregoing analysis, and offer a complete explanation of the cause of gravitation.

In this explanation of the cause of gravitation there are some things which are at variance with previous conceptions, besides the fundamental facts, (i) that the attraction of the singular surface which corresponds to that of gravitation is not the effect of masses present but of masses absent, which has already been revealed in the previous analysis, and (ii) that the volume enclosed within the singular surfaces, which is the volume from which the singular surfaces shut each other out, has no proportional relation to the number of grains absent, but, as will at a later stage appear, depends on the possibility of some one definite arrangement of the grains absent, out of a finite number of possible different arrangements.

218. In the analyses of Newton, Laplace, Poisson, and Green, for defining the consequence which would result if distant masses attracted each other according to the product of the masses divided by the squares of the distances, the attraction is taken as inherent in the masses. This assumption assumed that there was something that was not force, but which varied with the distance from a solitary mass, and this something after various names is now generally called the potential. That any of the philosophers named believed in force at a distance is more than doubtful, as Hooke and Newton and Faraday repudiated any such idea. Maxwell went a stage further and showed that such attractions might be a result of a certain law of varying stresses in a medium—as to this he writes, "It must be carefully borne in mind we have made only one step in the theory of the medium. We have supposed it to be in a state of stress, but we have not in any way accounted for the stress or explained how it could be maintained."......"I have not been able to make the next step, namely to account by mechanical considerations for the stresses in the dielectric*."

Maxwell is here writing of electricity, which is not the same thing as gravitation, as will presently appear.

This second step, namely that of accounting by mechanical considerations for the stresses in the medium, has now been overcome; as we have the mechanical interpretation of the potential as the product of the uniform pressure p'' multiplied by the integral of the dilatation over the medium r_B to r_1, or

$$V = p''N'\frac{4\pi r_0^3}{3}\sigma\left(\frac{1}{r_B} - \frac{1}{r}\right) \text{......................(349)},$$

or, omitting the constants,

$$V = -p''\sigma\frac{N'\frac{4}{3}\pi r_0^3}{r} \text{.........................(350)}.$$

This is entirely rational and when multiplied by $-4\pi r_0^3/3$ and differentiated gives us the attraction hitherto expressed by R†.

And it thus appears that the thing to which the name potential has been applied is the product of p'' multiplied by the total dilatation between the surface of radius r_B and the surface of radius r (greater than r_B).

It is to be noticed that in so far as we are concerned with the effort of attraction and not with acceleration, it is only the volume of the space from which the grains are absent, and not the mass within the space, that we have to take into account.

And it is for this reason that in the foregoing analysis, in this section, ρ has not been introduced. But since, in states of the medium under consideration, in our present notation ρ is, to a first approximation, equal to unity, it would have made no difference if we had taken it into account (when we have to consider the displacement of mass owing to the effort, the fact that ρ'' is unity is of primary importance), since whatever the effort to acceleration, the acceleration is inversely proportional to the density—and this will appear at a later stage.

In order to render the expression for attraction intelligible it should here be noticed that strains, and consequent dilatations in the medium, which have

* *Electricity and Magnetism*, Vol. I. Arts. 110 and 111.

† This R has no connection with the R used in Arts. 200 and 201.

no dimension, and which are the only actions, are outside the singular surfaces; so that we are not dealing with two or more independent masses, but with the variations in the displacements in the entire medium, all the mechanism, so to speak, being in elastic connection controlled by the pressures, as conditioned by the positions of negative inequalities in the mean mass represented by $4\pi r_0^3/3$.

There is no complete freedom of inequalities as long as there are other inequalities within a finite distance.

Thus it appears that the singular surfaces are virtually the handles of the mechanical train.

219. Having effected the analysis for the attractions and the potential, we may now return to the inequalities in mass as mentioned in the schedule, Art. 203.

The second inequality in the mean mass in that schedule is that which may be conceived to result from an excess of grains, instituting a positive centre.

The analysis for the effects of such positive centres is precisely similar to that already effected for the negative centre, except that in the case of the positive centre the curvature would be reversed, the curvature being away from instead of towards the centre.

The effect of this appears to be to cause positive centres to repel instead of attract each other. Such repulsions would as in the case of negative centres depend on the product σ multiplied by the curvature, which is of opposite sign to that for positive centres, and thus the effort of repulsion between two positive centres would be expressed by

$$\rho''\left(\frac{4\pi r_0^3}{3}\right)\frac{\sigma}{2r}.$$

The coefficient of dilatation is the same—unity. There is thus no necessity to repeat the analysis. This concludes the approximate analysis of the actions between centres having similar signs.

It may however be remarked that there are reasons why it is probable that positive centres should exist, as will appear at a later stage.

220. The first of the class of complex local inequalities ((iii), Art. 203) is that which would be instituted if by action on the medium in normal piling a number of grains (n) were displaced from their previous neighbourhood when in normal piling to some other neighbourhood previously in normal piling.

Such complex inequalities are only second in importance to groups of negative inequalities at finite distances, such as have already been discussed. In the case of complex inequalities there is no difficulty in conceiving that owing to the mean pressure there would be an effort to reverse the displacement, as nothing would seem more natural if we have an absence of grains in one place and an excess in another, under pressure, than that there should be strains from the place of excess to the place in which the grains are absent, and *vice versa*.

It also appears at once as pointed out in Art. 203 that the case is identical with that which would result from the existence at finite distance of equal positive and negative centres, having the same number of grains absent and present respectively.

This identity indicates the direction of the analysis necessary in order to obtain the expressions for the effort to reverse the displacement.

We have already obtained the expressions for the dilatations per unit of volume at any point distant r from a negative centre resulting both from the distortional strain and from the curvature owing to the finite size of the grain

$$\frac{4\pi r_0^3}{3}\frac{r_1}{r^4} \text{ and } \frac{4\pi r_0^3}{3}\frac{\sigma}{r^4}.$$

And it has also been shown that there is no diminution in the dilatations in the former as the centres approach.

It has also been shown, Art. 217, that multiplying the dilatations at a point resulting from a negative centre by $p''r^2dr$ and integrating from r_1 to r, we have the equation

$$p''\frac{4\pi r_0^3}{3}\int_{r_1}^{r}\frac{\sigma}{r^4}\cdot r^2dr = p''\frac{4\pi r_0^3}{3}\left(\frac{\sigma}{r_1}-\frac{\sigma}{r}\right)\ldots\ldots\ldots\ldots\ldots(351),$$

the second member of which expresses the potential of attraction between the two equal negative centres. This multiplied by a second negative inequality and differentiated with respect to the distance between the centres expresses the effort of attraction of the centres as

$$R = -p''\left(\frac{4\pi r_0^3}{3}\right)^2\frac{\sigma}{r^2}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(352).$$

And again, although not previously noticed, it appears at once from equation (351) that, if instead of the limits of integration being from r_1 to r, they are taken from r to $r=\infty$, we have

$$p''\frac{4\pi r_0^3}{3}\int_{r}^{\infty}\frac{\sigma}{r^4}\cdot r^2dr = p''\frac{4\pi r_0^3}{3}\cdot\frac{\sigma}{r}\ldots\ldots\ldots\ldots\ldots\ldots(353).$$

This integral must have some significance as a potential. And it appears on multiplying equation (353) by $4\pi r_0^3/3$, which is an expression for a positive inequality equal to the negative inequality, and differentiating with respect to the distance between the centres, when the equation becomes:

$$\frac{d}{dr}\left[p''\left(\frac{4\pi r_0^3}{3}\right)^2\int_r^\infty \frac{\sigma}{r^4}\cdot r^2 dr\right] = -p''\left(\frac{4\pi r_0^3}{3}\right)^2\frac{\sigma}{r^2} \quad \text{..........(354).}$$

The second member expresses an attraction between the positive and negative centres.

221. *The significance of the two integrals.*

In Art. 216 from equation (346) it is shown that negative centres attract, therefore if there were a choice of two general integrals of the dilatation from a negative centre, from one of which in the case of negative centres there would result a repulsion, while the other would result in attraction, it is certain that the integration which would result in the attraction is the only one between negative centres whatever might be the significance of the other integration. And this is what actually occurs.

If instead of the limits from r_1 to r as in equation (351) the limits are taken from r to ∞ as in equation (353), then taking account of a second negative singular surface we should have for the complete potential:

$$-p''\left(\frac{4\pi r_0^3}{3}\right)\frac{\sigma}{r},$$

which differentiated with respect to r is:

$$p''\left(\frac{4\pi r_0^3}{3}\right)\frac{\sigma}{r^2},$$

which expresses a repulsion. Hence this cannot be the integral for the attraction of one negative centre for another.

As already remarked this form of integral of the dilatation from a negative centre must have a significance, and significance appears when we substitute a positive inequality $4\pi r_0^3/3$ in place of the negative inequality $-4\pi r_0^3/3$ in the last expression for the attraction, which becomes

$$p''\left(\frac{4\pi r_0^3}{3}\right)\frac{\sigma}{r^2}.$$

Thus we have the expression for the attraction of equal positive and negative centres resulting from the finite size of the grains.

222. *The intensity of the attractions of equal positive and negative inequalities.*

In the first place it is to be noticed that the intensity of the attraction between equal positive and negative inequalities as in the last expression

(Art. 221) is as σ to r_1 of the total intensity of attraction between positive and negative surfaces. Indeed the expressions last but one and last (Art. 221) only indicate the significance of the two integral potentials. And such intensity as they express in no way depends on the curvature.

This becomes clear if we recognise that in the case of a displacement of n grains the strains from the negative centres are negative and extend to infinity, while the strains resulting from the positive centres are positive and extend to infinity. The components of the negative strains cancel with the components of the positive strains with which they are parallel; hence the diminution of the dilatation as the displacement diminishes in no way depends on the curvature but wholly on the cancelling of the distortional strains.

It thus appears that in order to express the effort to restore the normal piling in the medium, we have only to substitute the radius of the singular surface in the place of σ in the last expression (Art. 221).

Thus for the total effort, in the complex inequality resulting from the displacement of a volume of grains $4\pi r_0^3/3$ through a distance r, to restore the normal piling we have

$$R = -p''\left(\frac{4\pi r_0^3}{3}\right)^2 \frac{r_1}{r^2} \qquad (355).$$

Q. E. F.

223. It may be noticed that in obtaining equation (355) no use has been made of the potential of attraction. This is because the inequality caused by a displacement of a volume of grains under the pressure p'', which has the dimensions ML^3T^2, is essentially one displacement, not two equal and opposite displacements as in the case of two equal negative centres, in which the relative displacements of energy have no effect on the mean position of energy in the medium.

This may be shown by subjecting the expressions for the effort of attraction between negative centres, and the effort to reverse the displacement in the case of complex inequality, respectively, to further analysis.

Taking the effort of attraction of two equal negative centres, as in equation (354), to be:

$$-p''\frac{4\pi r_0^3}{3}\cdot\frac{\sigma}{r^2},$$

and the effort to reverse the displacement in the complex inequality, as in equation (355), to be:

$$-p''\left(\frac{4\pi r_0^3}{3}\right)^2 \frac{r_1}{r^2},$$

and then integrating each of these expressions from r_1 to ∞, we have as the energies resulting from the dilatation from outside the singular surfaces of radius r_1,

$$p''\sigma\left(\frac{4\pi r_0^3}{3}\right)^2\frac{1}{r_1},$$

and

$$-p''r_1\left(\frac{4\pi r_0^3}{3}\right)^2\frac{1}{r_1}.$$

Then to obtain the expressions for the potential of attraction for either of these respective energies, the factor $1/r$ must be separated into two factors proportional to two inequalities of the same or opposite sign in accordance with the sign of the product of the inequalities. Then multiplying the factor which has the positive sign by $1/r$ we have the potential, while the other factor is numerical and represents the attraction of the centres.

In the case of two negative centres, taken as equal for simplicity, as the signs of the inequalities are the same we have for the potential:

$$\sqrt{p''\sigma\left(\frac{4\pi r_0^3}{3}\right)^2}\frac{1}{r},$$

and for the attraction:

$$\sqrt{p''\sigma\left(\frac{4\pi r_0^3}{3}\right)^2}.$$

And in the case of the complex centre, since the product of the centres is negative, we have for the potential:

$$i\sqrt{p''r_1\left(\frac{4\pi r_0^3}{3}\right)^2}\frac{1}{r},$$

and for the attraction:

$$-i\sqrt{p''r_1\left(\frac{4\pi r_0^3}{3}\right)^2}.$$

Whence it appears that in the complex inequality both the potential and the attraction are irrational. Whence it is proved, since the effort is real, that the absolute displacement of energy is one displacement and not two.

224. The electrostatic unit of electricity is defined as the quantity of positive electricity which will attract an equal quantity of negative electricity at unit distance with unit effort. This unit as is shown in Art. 223 is irrational. An expression for the unit corresponding to the electrostatic unit is obtained from either of the last two expressions in Art. 223.

Thus from the first of these, putting $r_1 = r_0$ and $r = 1$, we get:

$$i\sqrt{p''\left(\frac{4\pi}{3}\right)^2 r_0^7} = 1.$$

And from this, since all the quantities under the radical are positive, we have the condition

$$p''\left(\frac{4\pi}{3}\right)^2 r_0^7 = 1 \quad \text{.........................(356)},$$

from which if p'' is known r_0 may be found.

225. From the analysis in Art. 223 it is easily realised that there is a fundamental difference in attractions between two negative centres, and the attraction of two equal centres one positive and one negative. It has been shown (Art. 217), that the attraction of two negative centres corresponds, in every particular, to the attraction of gravitation as derived from experience. And it now appears that the alteration from a positive to a negative inequality correspond to the statical attraction of the positive for the negative electricity. Not only then has the step at which Maxwell was arrested—that of accounting by mechanical considerations for the stresses in the dielectric—been achieved, and a moot point of historical interest settled, but as now appears a definite error as to the actual attractions has been revealed.

This error is in the general assumption that electrified bodies repel each other. As this may not be at once obvious it will be discussed in the next article.

226. *To show that positively electrified bodies do not repel.*

It has been shown in Art. 225, neglecting the small attractions of two positive or two negative centres, that the efforts of attraction between equal positive and negative centres, at any distance r, are equal and opposite.

If then in the same line we have two equal complex inequalities arranged so that their displacements are opposite, the negative centres being outwards as $+ - - +$, the effort of attraction of one of these complex inequalities would not in the least be affected by the other complex centre.

Hence there is no attraction between two positive centres, the only effort to separation of the two positive centres being between those of the two complex inequalities, the effort in either being the same as if the other was not there. Hence the only efforts are those of attraction. Q.E.D.

It should be noticed that these attractions are quite apart from the repulsions resulting from two positive centres owing to the curvature and finite size of the grains as in gravitation, and further that, other things being the same, the ratio of the attractions between positive and negative and the repulsions between positive centres is as r_1/σ, and hence the repulsion may be neglected as compared with the attraction.

227. In the analysis for the effort of attraction of negative inequalities and that to reverse the displacement of a complex inequality the terms in the expressions for the contraction strains which involve powers of r_0^3/r_1^3—the ratio of the volume of grains absent divided by the volume enclosed by the singular surface—have been neglected (Art. 214, equation (337)) and it is this simplification only which renders the law of attraction—as the inverse square—the law of attraction of the singular surface at a distance.

But this in no way limits the variation of the stresses over those portions of the space in and between the parts of the two singular surfaces which are within indefinitely small distance of each other. Such limits can only be determined by taking into account the higher terms which have been neglected.

This analysis I have not attempted. But it seems to me very important to notice this omission, as it appears that the attractions or repulsions expressed by the higher powers of $1/r$, when the surfaces are indefinitely near, must be of great intensity, so that owing to sudden variations the work done in separating the surfaces must be extremely small.

These characteristics are those of cohesion and surface tension and they promise to account by mechanical considerations for the hitherto obscure cohesion between the molecules as belonging to the attractions resulting from the finite value of the diameter of the molecules divided by the curvature resulting from distortion, or, we might say the complement of gravitation.

228. The fourth and last class of possible local disarrangements causing strain in the normal piling, with some degree of permanence, in the schedule (Art. 203), is that which does not depend on the absence, presence, or linear displacement of grains, but does depend on local rotational displacement of grains about some axis.

Then since there are no resultant rotational stresses or rotational strains in the medium, or rotation of the medium, the rotational inequalities must be arranged so as to balance.

Any such rotation of a portion of the medium would be attended with dilatations. But it does not follow that the dilatations would in all cases be so small that the coefficient would be unity.

Then noting that the medium in virtue of relative motion of the grains is in some degree elastic, if we conceive that by two opposite couples about parallel axes at a finite distance two equal spheres of grains in normal piling having their centres on the respective axes, could be caused to turn about their axes through opposite but equal angles θ and $-\theta$, the actions would be reciprocal, and supposing the actions to start from the medium in normal piling, when the angles were so small that at the surfaces there was no change of neighbours, the only effects would be strains attended by dilatation about the axes, which on removal of the couples would revert, restoring the unstrained medium. And in this case the coefficient of dilatation would be unity.

Then if the angles were increased the strains would be such that over the equators of the spheres the grains would change neighbours, diminishing the dilatation; so that on the couples being removed the spheres would not revert and would not restore the unstrained medium, nor would the angles θ and $-\theta$ be zero.

Those portions of the surfaces of the spheres nearer the axes, where the strains had not been sufficient to cause a change of neighbouring grains, would be subject to stress tending to diminish the angles θ and $-\theta$, while in those portions where the grains had changed their neighbours the stresses would be resisting this change, so that the result would be a balance of strains and stresses, leaving the system in equilibrium under the relative rotational strains and stresses and dilatations extending outwards from the surfaces of each till they vanish at an indefinite distance.

The strains and stresses extending from the sphere of which the residual angle was θ, since the axes are at a finite distance, could not in any way affect strains of shear having the angle $-\theta$. But if the shears were in a plane perpendicular to the axes and at a finite distance from each other, the strains and stresses being opposite would cancel, and the dilatations would diminish in such manner and proportions that there would be efforts to approach proportional to the inverse square of the distance. Or, if, other things being the same, the spheres were at finite distances on the same axes, they would still be under efforts to approach, owing to the cancelling of the strains and diminution of the dilatation. And in either case, other things being the same, if one of the poles at the axis of either one of the spheres were reversed the result would be an effort of repulsion. Q.E.F.

Thus efforts of attraction correspond exactly with those of fixed magnets, and thus we have been able to account by mechanical considerations for the magnetism which has any degree of permanence.

229. Having in the foregoing articles of this section accomplished the analysis necessary for the determination of the attraction of negative centres of disturbance, the efforts to reverse the displacement in the complex inequalities, discussed the probability of cohesion as the result of the terms neglected in the analysis for the efforts of the negative centres, and effected the analysis for the efforts of attraction resulting from opposite rotational strains about parallel axes at a distance; it remains to complete the section by effecting the analysis for determining the mobility of the singular surfaces.

230. From Theorems 1 and 2, Art. 204, and more particularly in Art. 214, we have defined the effects of local inequalities in the mean mass, when σ/λ is large, on the arrangement of the grains and the distribution of the strains in the medium about both negative and positive centres. Thus it has been shown in the case of a negative centre that the inward strains would be such that the resulting dilatation would pass the point of stability and reform, causing a nucleus of grains in normal piling which might increase until it was stopped by meeting the inward strained, and consequently dilated, normal piling.

This meeting of the two closed surfaces, the outer surface of the nucleus in normal piling with the inner surface of the inwardly strained normal piling, affords the first clue to the possibility of a surface of freedom. For, since the grains are uniform equal spheres, there can be no fit between the grains in normal piling at the one surface and the grains in strained normal piling at the other. To use a mechanical expression the grains cannot pitch, and consequently there is a spherical shell of grains in abnormal piling which constitutes the singular surface a surface of weakness if not a surface of freedom. Then by Theorem 1 it follows, whatever may be the arrangement of the grains and whatever the exchange, there can be no change in the arrangement or number of the grains. Therefore these surfaces of misfit are fundamental to all inequalities in the mean mass.

231. Since there is no regular fit in the shell of abnormal piling at the singular surface, say of a negative centre, and each of the grains is in a state of relative motion, each of the grains is in a state of mean elastic equilibrium such that half the grains are on the verge of instability one way and half in another. If, as by the existence of another negative centre at finite distance there is an effort of attraction, however small, it would, since there is no finite stability, in the first instance cause change of neighbours, and if sufficiently strong it would entirely break down the stability and cause one or both the centres to approach at rates increasing according to the inverse square of the distance, since as by Theorem 1 there would be no change in the mean arrangement of the grains and the viscosity may be neglected.

232. This brings us face to face with questions as to the mode of displacement of the singular surfaces, as well as the manner of motion of the inequalities in the mean mass which constitutes the centre, which have not as yet been discussed.

In the first place it appears at once, however strange it may seem, that in the case of a negative inequality, to secure similarity in the arrangement of the infinite medium the mass must move in the opposite direction to the inequality, otherwise there would be no displacement. And further the opposite displacements of the positive and negative masses must be equal, subject to the condition that for every indefinitely small displacement of the negative inequality there should be an equal and opposite and exactly similar and similarly placed displacement of positive mass.

233. Then, apart from vortex rings which cannot exist in a medium in which σ/λ is so large that there is no diffusion of the grains, it appears that the only way in which the conditions in the last paragraph are realised is by propagation. This admits of definite proof.

If we conceive a singular surface about a negative centre to be moving upwards through the medium, as it rises the upper surface will be continuously meeting fresh grains. Then if the motion continues one of two things must happen. The grains must be shoved out of the way, in which case all similarity of the arrangement would be destroyed, or the grains must cross into the singular surface. If this were all we should again have the similarity upset, as the singular surface must increase to accommodate grains coming in. But if at the same time as the grains enter the singular surface from above grains cross out of the singular surface in exactly the same numbers and vertically under the grains which enter from above, the motion of the singular surface would not disturb the similarity of the arrangement beyond such limits as the elasticity of the medium admits.

This manner of progress of a singular surface is that which has several times been referred to as propagation. It is strictly propagation. For if there is no general uniform mean motion the grains within the singular surface are at rest, while if the medium has such mean motion it would not affect the motion of the singular surface though it would affect the rate of propagation since that would include the propagation through the moving medium.

This then is the only mode of displacement of a singular surface—the propagation.

N.B. This law of propagation would not prevent strains in the singular surfaces such as might be caused by undulations in the medium corresponding to those of light.

234. It may seem that displacement by propagation does not of necessity entail displacement of mass; nor would it if there could be propagation without local inequalities in the mean density of the medium. But in a uniform medium, without inequalities, there can be no propagation as there is nothing to propagate.

Thus it is that the inequality in density, the integral of which is the volume of the grains, the replacement of which would restore the uniformity of the medium, obliterating the inequality, constitutes the mass propagated. And as this, for a negative centre, is negative, its propagation requires the displacement of an equivalent positive mass in the opposite direction to that of propagation of the negative inequality.

235. It thus appears that the distribution of the density of the positive moving mass is at all points the same as the distribution of the density of the negative inequality, and as this on changing the sign is the same as the dilatation at all points, the density of the positive moving mass is equal to the dilatation.

The dilatation at any point in the medium resulting from a negative centre is expressed by:

$$\frac{4}{3}\frac{\pi r_1 r_0^3}{r^4}$$

in which r is greater than r_1, while r_0/r is small.

It thus appears that, since the density of the medium is unity, the motions of the medium of unit density necessary to equal the displacements of the positive mass at density $4\pi r_1 r_0^3/3r^4$, which can under no circumstances be greater than $4\pi r_0^3/3r_1^3$ are almost indefinitely small.

236. Taking U_s as the velocity of the singular surface and u'' as the velocity of the medium at any point outside the singular surface, since there is no mean motion of the grains within the singular surface, u'' is everywhere small compared with U_s.

Of course this does not affect the integral displacement of mass integrated over the medium from r_1 to ∞. But it does affect the displacement of the apparent energy of the motion of the inequality which is taken to be $4\pi r_0^3/3$. For if we integrate u''^2 over the medium it is small compared with

$$U_s^2 \cdot \frac{4\pi r_0^3}{3}.$$

This apparent paradox, however, is explained on recognising that the grains being uniform, since σ/λ is very large, the conduction of energy is nearly perfect; so that the rate of displacement of momentum does not depend only on the convections of the order $u''^2\rho$ but depends also on the conductions

$$\frac{\sigma}{\lambda} a u'' \rho,$$

since these actions are the direct result of the propagation of the singular surface through the medium, so that there is no change in the strains, dilatations, or the mean arrangement within or about the singular surface for an infinite distance. It is easy to realise the way in which the strains at any fixed point contract and expand as the singular surface moves away from or approaches the point.

237. In the foregoing reasoning in this section no account has been taken of the possibility or impossibility of any lateral motions of the grains which might be necessary to maintain the arrangement. That such lateral motions of the individual grains would be necessary is certain; but it does not follow as a matter of course that they would be possible without creating temporary strains which would in the first instance require a certain acceleration to start them. But once started the action, since it involves a certain definite rate of displacement of mass, would proceed at a uniform rate, supposing no viscosity, and the medium unstrained by other centres.

That the necessary acceleration to effect the start must depend on the particular arrangements inside and outside the singular surfaces, is clear. And from this it may be definitely inferred that the number of definite *primary* arrangements in which the stability to be overcome by acceleration is within finite limits, is finite.

Whence it follows that the number of singular surfaces having different numbers of grains absent, in which the limits of stability are within finite limits, is finite; and these would be the only surfaces of freedom. Q.E.D.

It should be noticed that the expression "primary arrangements" is here used to distinguish those singular surfaces which do not admit of separation into two or more singular surfaces of freedom.

It is thus shown that singular surfaces about negative inequalities admit of motion in all directions, by a process of propagation, without any mean motion of the grains within the singular surfaces, while the motion of the mass outside the singular surfaces, when there is no other inequality within finite distance, is such as to maintain the similarity in the arrangement about the centre entailing the displacement of the mass $(4\pi r_0^3/3)$ in the direction opposite to that in which the singular surface is displaced by propagation.

238. We have thus effected the analysis for the determination of the mobility of solitary negative centres. And it may be taken that the analysis for positive centres would follow on the same lines with the exception of the sign of the inequalities.

There still remains to consider the possibility of the combination of primary singular surfaces, forming singular surfaces with limited stability in which the grains absent or present are the sum of the grains, the absence or presence of which constitutes the inequalities of the primary singular surfaces combined.

It has been shown by neglecting certain terms (equation 337) that negative inequalities attract according to the inverse square of the distance and in Art. 227 it has been pointed out that the terms neglected are such as would indicate cohesion or repulsion between the singular surfaces when closest; and in such conditions there would be a connected singular surface however many were the primary singular surfaces cohering, so that mobility of the whole group would be secured.

In the case of two primary negative inequalities in which the numbers of grains absent are different, although neither of these admit of separation into two or more separate inequalities, there does not appear any impossibility, except such as results from their limited stability, why they should not combine if their velocities are sufficient to break down the limited stability.

In such case it seems that one or other of two results must happen; either the breakdown would be temporary, the two centres immediately reforming as by the rebound, setting up a disturbance in the medium which would be propagated through the medium, or they would reform into a single negative centre, in which the volume inside the reformed singular surface would be less than that of the sum of the volumes within the two singular surfaces of the two primary inequalities, or in some other way manage to diminish the dilatation; and in this case also there would be a disturbance in the medium.

239. It is certain that when negative inequalities are arranged in their closest order, there is cohesion between the adjacent singular surfaces which resists the separation of the adjacent singular surfaces but does not cause attraction between the singular surfaces when these are at a distance which is greater than some small fraction of the radius (r_1) of the singular surface (Art. 227). It is also certain that, when under the conditions stated, the singular surfaces would still attract one another at a distance—as in equation (348):

$$-p''N'\left(\frac{4\pi r_0^3}{3}\right)^2 \cdot \frac{\sigma}{r^2}.$$

And thus if we consider N—the number of such negative centres within a distance r_3—to be indefinitely large as compared with r_1, since they are in closest order the centres would be in stable equilibrium under normal and tangential pressure, as in the case of gravitation.

240. If the number of grains absent about each of the centres which constitute the total negative inequality is the same, and by some shearing stress the inequality is subject to a shearing strain, there would result dilatation, doing work on the medium outside, which would be maintained as long as the shearing stress; but since all the centres are equal, whatever arrangements of the grains under the stress take place between the centres, there would be no absolute displacement of mass.

And the result would be the same whatever might be the number of grains absent in the primary inequality.

241. Thus we may consider what the action would be if we had two such total inequalities A and B differing in respect to the number of grains absent in their primary inequalities—say that the number of grains absent is greatest in A.

If these total inequalities are brought together by their attractions the grains in abnormal piling which separate the two total inequalities A and B

may be, for simplicity, taken parallel to a plane which is a *plane of weakness* in the medium. If, then, there are shearing strains parallel to this plane such as cause grains from the inequality A to pass to the inequality B in the abnormal piling in the plane of weakness, so that in this piling the arrangement, instead of the two primary inequalities in which the numbers of grains absent are A and B, is two equal negative inequalities in each of which the number of grains absent is:

$$\frac{A+B}{2}, \quad \frac{A+B}{2},$$

and one complex inequality in which the numbers of grains absent in the positive and negative centres are:

$$\frac{A-B}{2}, \quad \frac{B-A}{2},$$

in this case it at once appears that besides the attraction corresponding to gravitation and cohesion, the effect of the rotational strain would be to cause absolute displacements of mass, which, by Art. 225, would cause efforts of reinstitution between the strained aggregate inequalities, corresponding to electric attractions. But as the attraction would be normal to the surface of weakness, while for reinstitution the action must be tangential, the rotational strain might be stable, and the attraction might hold when the strained aggregate inequalities were forced apart. If the rotational strains were sufficient the normal attractions might overcome the normal stability of the complex inequalities, and in that case there would be a sudden tangential reversion, which, as there is absolute displacement of mass, would in the recoil reverse the complex inequality and so on, oscillating until the energy was exhausted in setting up undulations in the medium which would be propagated through the medium at the velocities of the normal or transverse waves as in light.

If we have two aggregate inequalities in one of which the primary inequalities are not combined, while in the other the different primary inequalities are combined, we should have three total inequalities A, $B/2$, $C/2$ in the arrangement:

$$\frac{A+\frac{B}{2}+\frac{C}{2}}{2}, \quad \frac{A+\frac{B}{2}+\frac{C}{2}}{2},$$

and two complex inequalities:

$$\frac{A-\left(\frac{B}{2}+\frac{C}{2}\right)}{2}, \quad \frac{\frac{B}{2}+\frac{C}{2}-A}{2}.$$

Then if the strains were sufficient the normal attraction might overcome the normal stability of the complex inequalities, causing a reversion. In this case however it does not follow that the reversion would be complete and so reinstitute A, $B/2$, $C/2$; for since the work done by the strains might be sufficient to overcome the resistance to combination of $B/2$ and $C/2$, the recoil from the breakdown would cause a total or partial combination of $B/2$, $C/2$, instituting B the aggregate inequality and so diminishing the energy available for undulations, thus affording an explanation by mechanical considerations of the part electricity plays in instituting the combination of molecules into compound molecules with limited stability.

It is to be noticed that the effects of rotational strain between the aggregate negative inequalities which differ as to the number of grains in the primary inequalities, correspond to the effects produced when resin is rubbed by silk—or frictional electricity—and thus the so-called separation of the two electricities by friction is accounted for by mechanical considerations.

Having shown that negative inequalities may not only attract, but may also cohere when in contact, we may return to the consideration of the significance of the fact mentioned in Art. 217, that the attractions corresponding to gravitation as well as cohesion depend solely on the numbers of grains absent, while the volume within the singular surfaces, which determines the volume from which one centre excludes other centres, depends on the possibility of some arrangement between the grains in abnormal piling and those in strained normal piling (Art. 214).

241 A. It is shown in Art. 217 that for any displacement of a negative inequality there must be a corresponding displacement of positive mass in the same plane and in the opposite direction. From this it follows that as two negative centres approach under their mutual attractions the mass in the medium recedes, which is an inversion of the preconceived ideas. Such action however is not outside experience, since every bubble which ascends from the bottom in a glass of soda-water involves the same action. The matter in the bubble having the density of the air requires the descent of an equal volume of water at a density 800 times greater than that of the air. It is the negative inequality in the density of matter which under the varying pressure of the water causes the negative or downward displacement of the material medium—water—and the positive or upward displacement of the negative inequality in the density within the singular surface.

In order to recognise the significance of the parallel drawn in the last paragraph it must be noticed that in this research we have adopted a definition of mass, which, although satisfying the laws of motion and the conservation of energy, is independent of any other definition of matter. Hence it is open to us to suppose that what we call matter may be such, that if expressed in the notation so far used in this research, would represent local negative inequalities in the mean density of the medium.

Then since, as has already been shown, and will be confirmed in what is to follow, the definition of matter as representing negative local inequalities in the mean density of the granular medium completes the inversion and removes all paradox, this definition of matter is adopted as the only possible definition.

We then have for the negative inequality:

$$-\frac{4\pi r_0^3}{3}\cdot\rho'',$$

where $\rho''=1$.

And for the volume from which one negative inequality excludes other similar inequalities, when in closest order, we have by equation (343):

$$\frac{4}{3}\cdot\frac{4}{3}\pi\cdot r_1^3.$$

Then dividing the negative inequality by the volume from which other centres are excluded we have as the expression for the mean density of the negative inequalities when in closest order:

$$\frac{3}{4}\cdot\frac{r_0^3}{r_1^3}\cdot\rho''=\Pi \qquad (357).$$

Then again dividing ρ'' the density of the uniform medium by Π, the mean density of the inequality, we have in the ratio of the two densities a number without dimensions as expressed by

$$\frac{\rho''}{\Pi}=\frac{4}{3}\frac{r_1^3}{r_0^3} \qquad (358).$$

In equations (357) and (358) Π is used to express the mean density of the negative centres when in closest order. Thus Π is the maximum mean density of the negative centres for any particular negative centres.

It does not however follow that Π expresses the maximum mean density of negative inequalities for all negative inequalities when in closest order. For as pointed out there is no proportional relation between the number of grains absent and the volume within the singular surfaces for inequalities which differ.

But it does follow, from the fact that the number of centres which have surfaces of freedom is finite, that there must be some negative inequality of which the mean density is a maximum. And from this it again follows that ρ''/Π must have a minimum value.

Then taking Ω to express the minimum value which, whatever it may be, is constant and without dimensions, we may express the densities of all the other negative inequalities in terms of Ω, making use of any system of units.

Then if, as before, the density of the medium is unity, the maximum density of negative inequalities is:

$$\frac{1}{\Omega},$$

and if the mean density of an inequality is n times less than the maximum inequality it is expressed by:

$$\frac{1}{n\Omega}.$$

And again, if, changing the unit of density, the density of the medium becomes $n\Omega$, the maximum density of negative inequalities is expressed by n.

The proof that the quotient Ω of the density of the uniform medium divided by the maximum mean density of the negative inequalities is a numerical constant, independent of units, giving us, as it were, the gauge by which we can compare the quantities, as obtained, in this and the previous sections, with the evidence derived from actual experience, completes the consideration of the possible strains other than the undulatory strains (considered in Section XIII.) resulting from the conservation of inequalities in the mean mass, which formed the subject of this section.

SECTION XV.

THE DETERMINATION OF THE RELATIVE QUANTITIES a'', λ'', σ, G, WHICH DEFINE THE CONDITION OF THE GRANULAR MEDIUM BY THE RESULTS OF EXPERIENCE. THE GENERAL INTEGRATION OF THE EQUATIONS.

242. In the last paragraph of Section XIII. it was noticed that, up to that stage, it was not possible, for want of evidence as to the actual rates of degradation of light, to complete the determination of the values of a'', σ, λ''. And further, that as the equations (310—313) have been obtained by neglecting all secondary inequalities, they afford no evidence as to the limits imposed by dilatation on the shearing and normal strains. These disabilities have not as yet been altogether removed. But we have, in the last section, obtained expressions, in terms of p'', a'', σ, λ'', for the attraction of negative centres, which correspond to those of gravitation. Also in the last article it is shown that what is known as "matter" corresponds with the inequality in the medium resulting from absence of grains. Also it is proved that there must be a finite maximum mean density for negative inequalities when in close order, which corresponds to the mean of the heaviest matter. And further, it is shown that the mean density of the uniform granular medium, divided by the maximum density of negative inequalities, is a number without dimensions—expressed by Ω—whence we are enabled to measure the density of any inequalities in closest order, in any system of units. We are thus in a very different position, as regards evidence, from what we were at the end of Section XIII.

243. By the last article of Section XIV., taking 22 as expressing in C.G.S. units the density of the matter platinum, which is approximately the densest form of matter, we have unity for the density of the matter water in C.G.S. units.

Then for the density of the granular medium in C.G.S. units we have

$$22\Omega,$$

where the constant number Ω has still to be determined.

The change of units of density, from that in which the density of the medium was taken as unity, to the density as measured in units of matter, has thus been effected.

244. From the last article it follows that, measured in C.G.S. units of matter, the mean pressure in the medium, equal in all directions, becomes

$$p = 22\Omega p'' \quad \text{..............................(359).}$$

Also the mean density of the medium ρ'' or unity becomes

$$\rho = 22\Omega\rho'' \quad \text{..............................(360).}$$

And, if in C.G.S. units of matter, ρ expresses the mean density of any negative inequalities in closest order, however complex, such as the mean density of the earth—5·67, the corresponding expression, when ρ'' is taken as unity, is

$$\rho = \frac{5{\cdot}67}{22\Omega} \quad \text{..............................(361).}$$

245. From equation (359) we may now proceed to find an expression for the mean pressure in terms of the rate of degradation in the transverse undulations when σ/λ'' is large.

From equation (311) the rate of degradation of *transverse* waves is expressed by

$$\frac{1}{v''} \cdot \frac{dv''}{dt} = -\frac{2}{3}\frac{\lambda''\alpha''}{\sqrt{\pi}} \cdot a^2 \quad \text{........................(362).}$$

Then if t_t is the time taken to reduce v_0'' to v_1

where $$v_1 = \frac{1}{e} \cdot v_0'',$$

$$\lambda''\alpha'' = \frac{3}{2} \cdot \frac{\sqrt{\pi}}{a^2} \cdot \frac{1}{t_t} \quad \text{............................(363),}$$

which gives one equation between the three quantities α'', λ'' and t_t.

A second equation is obtained from the dynamical condition of undulation

$$\frac{m}{a} = \tau = \sqrt{\frac{n}{\rho}} \quad \text{..............................(364),}$$

and $$n = \frac{3}{8}k,\ k \text{ being } \left(\frac{\sigma}{2\pi\lambda''}\right)^2 \cdot \rho \cdot \frac{\alpha''^2}{2} \quad \text{..................(365).}$$

Therefore, reducing,

$$\tau = \frac{\sqrt{3}}{4} \cdot \frac{\sigma}{2\pi} \cdot \frac{\alpha''}{\lambda''} \quad \text{............................(366),}$$

or $$\frac{\lambda''}{\alpha''} = \frac{\sqrt{3}}{4} \cdot \frac{\sigma}{2\pi\tau} \quad \text{............................(367).}$$

Then, L being the wave-length, if we put

$$n_2 \,.\, \sigma = L$$

since

$$L = \frac{2\pi}{a},$$

substituting $\frac{2\pi}{a n_2}$ for σ in equation (367),

$$\frac{\lambda''}{\alpha''} = \frac{\sqrt{3}}{4} \cdot \frac{1}{a n_2 \tau} \quad \text{.......................(368).}$$

Then eliminating α'' from equations (367) and (368) to find λ''

$$\lambda'' = s_1 \,.\, (n_2 t_t)^{-\frac{1}{2}} \quad \text{.......................(369),}$$

the value of the constant coefficient being

$$s_1{}^2 = \frac{\sqrt{27\pi}}{8a^3\tau}.$$

Then substituting from equation (369) in equation (367)

$$\left.\begin{aligned} \alpha'' &= \frac{1}{s_1} \cdot \frac{3}{2} \frac{\sqrt{\pi}}{a^2} (n_2 t_t)^{\frac{1}{2}} \cdot \frac{1}{t_t} \\ \text{or} \quad \alpha'' &= s_2 (n_2 t_t)^{\frac{1}{2}} \cdot \frac{1}{t_t} \end{aligned}\right\} \quad \text{.......................(370),}$$

$$\frac{\alpha''^2}{2} = \frac{s_2{}^2}{2} (n_2 t_t) \frac{1}{t_t{}^2} \quad \text{.......................(371).}$$

The equations (369) and (371) define the values of the constants λ'' and α'' which enter into the expression p'' in equation (159) in terms of a, τ, n_2 and t_t which define the wave-length and rate of propagation for any particular rate of degradation.

Thus substituting in the equation (159) which is

$$p'' = \frac{\sqrt{2}}{3} \cdot \frac{\sigma}{\lambda''} \cdot \rho \cdot \frac{\alpha''^2}{2} f\left(\frac{\sigma}{\lambda''}\right),$$

and which, under the condition σ/λ'' large, is, taking the density of the medium as unity,

$$p'' = \frac{\sqrt{2}}{3} \frac{\sigma}{\lambda''} \frac{\alpha''^2}{2} \frac{6}{4\sqrt{2}\,\pi} \quad \text{.......................(372),}$$

the equation becomes

$$p'' = \frac{\sqrt{2}}{3} \cdot \frac{L}{n_2} \cdot \frac{\sqrt{n_2 t_t}}{s_1} \cdot \frac{s_2{}^2}{2} (n_2 t_t) \frac{1}{t_t{}^2} \cdot \frac{6}{4\sqrt{2}\,\pi} \quad \text{.......................(373).}$$

Then transforming we have

$$p'' = \frac{1}{4\pi} \frac{s_2{}^2 L}{s_1} \sqrt{\frac{n_2}{t_t}} \quad \text{.......................(374).}$$

If the constants s_1 and s_2 are taken to correspond with the rate of propagation of light and with the wave-length of the ultra-violet light in the C.G.S. units

$$s_1 = 9{\cdot}7005 \times 10^{-14}$$

$$s_2 = 1{\cdot}0738 \times 10^3,$$

from which substituting in equations (369) and (371)

$$\left.\begin{aligned} \lambda'' &= 9{\cdot}7005 \times 10^{-14} \cdot \frac{1}{\sqrt{n_2 t_t}} \\ \alpha'' &= 1{\cdot}0738 \times 10^3 \sqrt{\frac{n_2}{t_t}} \\ \frac{\alpha''^2}{2} &= 5755 \times 10^5 \left(\frac{n_2 t_t}{t_t^2}\right) \end{aligned}\right\} \qquad (375).$$

And since the wave-length L is $3{\cdot}933 \times 10^{-5}$ we have, dividing by s_1 and substituting in the second expression for p'',

$$p'' = 1{\cdot}8574 \times 10^{11} \left(\frac{n_2}{t_t}\right)^{\frac{1}{2}} \qquad (376),$$

which becomes in C.G.S. units of matter (by equation 359)

$$22\Omega p'' = 22\Omega \times 1{\cdot}8574 \times 10^{11} \left(\frac{n_2}{t_t}\right)^{\frac{1}{2}} \qquad (377).$$

For convenience the expression for $\alpha''\lambda''$ may be here included:

$$\alpha''\lambda'' = 1{\cdot}0418 \times 10^{-10} \cdot \frac{1}{t_t} \qquad (378).$$

246. Having effected the translation of units and obtained an expression for the mean pressure in the uniform medium in terms of n_2/t_t, we now proceed to the evidence as to the absolute density, or, what is the same thing, the value of the number expressed by Ω.

The density of the luminiferous ether, thus far, has been an unknown quantity. Such views as have been expressed range from a density indefinitely greater than that of the heaviest material—Hooke—to a density indefinitely smaller than that of the lightest solid material—Sir Gabriel Stokes and Lord Kelvin.

But as pointed out in Art. 242 we have now the two sources of evidence—that arising from the known law of gravitation, which includes the existence of permanent negative inequalities, or molecules with surfaces of freedom, and that resulting from the limits to the intensity of waves of light; besides such evidence as may accrue from the determination made by Lord Kelvin as to the dimensions of the molecules, and such evidence as has been obtained as to the rates of degradation of the transverse and normal waves.

The equations (376) and (377) define the pressure in terms of

$$\frac{n_2}{t_t} \text{ or } 22\Omega \frac{n_2}{t_t},$$

according to whether the density of the uniform medium is taken as unity, or is expressed in C.G.S. units of matter.

247. As measured in C.G.S. units, the matter in the earth, assuming Baily's value, 5·67, for the mean density, is

$$6{\cdot}14 \times 10^{27},$$

the mean radius is $6{\cdot}3702 \times 10^8$ and the attraction of the earth on a unit of matter at the surface is

$$g = 981 \text{ ....(379).}$$

To compare with this evidence we have the expressions for the corresponding quantities as obtained from equations (348) for corresponding conditions when translated into the same units.

In the general expression for the attraction of negative centres in closest order, equation (348), where $\rho'' = 1$:

$$-p''N'\left(\frac{4\pi}{3}\,.\,r_0^3\right)^2\frac{\sigma}{r^2},$$

where $$N' = {\cdot}75\left(\frac{r_B}{r_1}\right)^3 \text{ and } r = r_B;$$

substituting, the expression for the attraction of unit mass becomes, if the ratio $\dfrac{r_0^3}{r_1^3} = \dfrac{4}{3} \times \dfrac{5{\cdot}67}{22\Omega}$ when $\rho = 1$,

$$-\pi p''\sigma\left(\frac{r_0}{r_1}\right)^3 . \, r_B.$$

Then, supposing that r_0^3/r_1^3 is a maximum, we have from equation (358)

$$\frac{r_0^3}{r_1^3} = \frac{1}{75\Omega} \text{ ....(380).}$$

And as the density of the mean negative inequality is 5·67/22 of the maximum inequality, we have for the attraction

$$\pi p''\sigma\left(\frac{r_0}{r_1}\right)^3 r_B \frac{5{\cdot}67}{22},$$

which becomes, on substituting from equation (380) and reducing,

$$\frac{4}{3}\pi p''\sigma r_B = \frac{4}{3}\pi p''\sigma\frac{5{\cdot}67}{22\Omega} r_B.$$

Then transforming so that the density of the medium is 22Ω, since r_B is $6{\cdot}37 \times 10^8$, we have for g

$$981 = 22\Omega p''\sigma\frac{4}{3}\pi\frac{5{\cdot}67}{22\Omega} . \, 6{\cdot}37 \times 10^8 \text{ ....(381).}$$

Then substituting the value of $22\Omega p''$ in equation (377) we have

$$\frac{4}{3}\pi 5{\cdot}67 \times 6{\cdot}37 \times 10^8 \times 1{\cdot}8574 \times 10^{11} \times \left(\frac{n_2}{t_t}\right)^{\frac{1}{2}} \sigma = 981 \;\ldots\ldots(382).$$

Then, cancelling and reducing the numerical factors, since

$$\sigma\,(n_2/t_t)^{\frac{1}{2}} = L/\sqrt{n_2 t_t},$$

we have

$$\left.\begin{aligned} 981 &= \frac{1{\cdot}105 \times 10^{17}}{\sqrt{n_2 t_t}} \\ \sqrt{n_2 t_t} &= 1{\cdot}126 \times 10^{14} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(383).$$

whence

And thus we have obtained the value of

$$n_2 t_t,$$

which satisfies the condition $g = 981$.

248. The evidence afforded by the limits of the intensity of light and heat does not appear to have hitherto demanded much attention. But it now appears that, if we can find a fair estimate of the maximum intensity of transverse undulations, it would afford important evidence.

For the rate of displacement of energy by the transverse waves in the uniform medium we have, taking U for the rate at which energy must be supplied to maintain the waves, and τ for the rate of propagation: since the velocity of light is independent of the wave-length, the maximum energy of mean motion over a unit surface

$$\rho \cdot \frac{v''^2}{2},$$

is, by equation (308), the mean energy of the undulation; and

$$U = \tau\,.\,\rho'' \frac{v''^2}{2} \quad \text{and} \quad v'' = \left(\frac{2U}{\rho''\tau}\right)^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots(384).$$

It must be noticed that in these expressions for U and v'' no account is taken of the secondary effects imposed by the dilatation in the granular medium. This was noticed in the last paragraph, Section XIII., as showing that there is a limit to the intensity of harmonic institutions.

Put definitely, the condition to be satisfied for harmonic undulations is that, taking x and y for the directions of propagation and mean motion respectively,

$$\frac{3}{8}\kappa\frac{dy}{dx} \text{ is small as compared with } p''.$$

Thus if the amplitude of the transverse motions is considerable, the action will not be confined to the institution of simple harmonic waves, but will include compound harmonic waves, and probably normal waves, which would proceed faster than the simple transverse harmonic waves, until, by divergence or degradation, their intensity was reduced.

Evidence from which we may form an estimate of the limit to the amplitude at which the waves cease to be sensibly harmonic may, it appears, be found. The greatest intensity of transverse waves is obtained from the carbons of the electric arc. If then we assume that U, the work expended in producing the light, is all spent in radiation of heat and light from the carbons, we have only to measure the radiation area of the carbons to obtain an outside estimate of the mean value of v''.

Thus if U per sq. cm. is $2{\cdot}29 \times 10^9$ ergs

$$2{\cdot}29 \times 10^9 = \tfrac{1}{2}\rho v_1''^2 \,.\, \tau \qquad (\tau = 3 \times 10^{10}) \;\ldots\ldots\ldots\ldots (385),$$

whence we have

$$v_1''^2 = \frac{1{\cdot}52}{\rho} \times 10^{-1} \text{ in C.G.S. units} \ldots\ldots\ldots\ldots\ldots\ldots (386),$$

where ρ is $22\Omega\rho''$ and where ρ'' is unity.

249. From this value of v_1'' we may obtain the expressions for y the amplitude of the undulations, and for x.

Taking r as an arbitrary amplitude

$$y = r\cos\theta \text{ and } dy/dt = -r\sin\theta \,.\, d\theta/dt.$$

Then since the periodic time is $2\pi/m$, differentiating θ with respect to time $d\theta/dt = m$, and

$$v'' = -mr\sin\theta \text{ and } v'' \text{ is a maximum when}$$

$$\theta = -\pi/2.$$

$$\therefore\; r = \frac{v_1''}{m}, \qquad y = \frac{v_1''}{m}\cos\theta,$$

and

$$x = \frac{\theta}{a},$$

$$\therefore\; \frac{dy}{dx} = a \,.\, \frac{dy}{d\theta} = -a\,\frac{v_1''}{m}\sin\theta = \frac{v''}{\tau} \;\ldots\ldots\ldots\ldots\ldots\ldots (387).$$

Then multiplying this by n or $\rho\tau^2$ we have for the shearing stress

$$22\Omega\,\frac{3}{8} \,.\, \kappa \,.\, \frac{dy}{dx} = \rho v''\tau \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (388),$$

and these are in gravitation units.

Then from equation (386) we have, for the maximum value of the transverse velocity v'',

$$v_1'' = \frac{390}{\sqrt{22\Omega}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (389),$$

and multiplying by 22Ω we have for the maximum shearing stress

$$\rho \,.\, v_1'' \,.\, \tau = 1{\cdot}172 \times 10^{10} \times \sqrt{22\Omega} \;\ldots\ldots\ldots\ldots\ldots\ldots (390).$$

Taking $s\,(=10^{-2})$ as the coefficient of the limit within which $22\Omega\,.\,3\kappa/8$ may approach $22\Omega p''$, we have, substituting the expression on the right of equation (377) for $22\Omega p''$,

$$22\Omega \times 1{\cdot}8574 \times 10^{11} \left(\frac{n_2}{t_t}\right)^{\frac{1}{2}} = \sqrt{22\Omega} \times 1{\cdot}172 \times 10^{12},$$

whence follows:

		logs
$\left(\frac{n_2}{t_t}\right)^{\frac{1}{2}} = \frac{6{\cdot}31}{\sqrt{22\Omega}}$,	equation (390),	$\cdot 8000 - \log\sqrt{22\Omega}$ ....(391),
$\sqrt{n_2 t_t} = 1{\cdot}126 \times 10^{14}$	......................(384),	$\cdot 0517 + 14$ ....(392),
$n_2 = \frac{7{\cdot}108 \times 10^{14}}{\sqrt{22\Omega}}$,		$\cdot 8517 + 14 - \log\sqrt{22\Omega}$ ...(393),
$t_t = 1{\cdot}785 \times 10^{13} \times \sqrt{22\Omega}$,		$\cdot 2517 + 13 + \log\sqrt{22\Omega}$ ...(394),
$\sigma = 5{\cdot}534 \times 10^{-20} \times \sqrt{22\Omega}$	.........(372),	$\cdot 7430 - 20 + \log\sqrt{22\Omega}$ ...(395),
$\alpha'' = \frac{6{\cdot}777 \times 10^{3}}{\sqrt{22\Omega}}$	......................(370),	$\cdot 8310 + 3 - \log\sqrt{22\Omega}$ ...(396).
$\lambda'' = 8{\cdot}612 \times 10^{-28}$	..................(375),	$\cdot 9351 - 28$ ...(396).

250. So far we have obtained the expressions for the limiting values of α'', λ'', σ and the logarithmic decrements for transverse and normal waves in terms of the constant coefficient Ω which enters as a factor into the expressions for the density of the medium and the potential of attraction.

Substituting from the equations (391—393) in equation (375) we have

$$\alpha'' = \frac{6{\cdot}777 \times 10^{3}}{\sqrt{22\Omega}} \quad \text{.................................(397)},$$

$$\lambda'' = 8{\cdot}612 \times 10^{-28} \quad \text{.................................(398)},$$

$$\sigma = 5{\cdot}534 \times 10^{-20} \times \sqrt{22\Omega} \quad \text{......................(399)}.$$

Then for logarithmic decrement of the transverse undulations, σ/λ'' large, substituting in equation (311) the values as given above for α'' and λ'' we have as in equation (362), t_t being the time required to reduce v'' from v_0 to v_0/e,

$$t_t = \frac{3}{2}\frac{\sqrt{\pi}}{\lambda''\alpha'' a^2} = 1{\cdot}784 \times 10^{13}\sqrt{22\Omega} \quad \text{..................(400)}.$$

N.B. This result checks the calculation, since this value corresponds with equation (394) in the first three significant figures, which is the limit of the arithmetical approximation attempted.

The value of t_t thus found in terms of the coefficient $\sqrt{22\Omega}$ expresses the time the transverse waves would travel before their amplitude was reduced in the ratio from 1 to $1/e$, or their energy in the ratio $1/e^2$.

The values of α'', λ'', σ cannot be defined except by further evidence. Such might be obtained if we could completely solve the dilatation problem and so obtain the value of Ω. Failing this, however, there remains one source of evidence from which we may obtain a close approximation to the value of the ratio $\sqrt{22\Omega}$.

251. *The conclusions to be drawn from the absence of evidence of any normal waves in the medium of space until very recent times.*

From equations (310) and (311) it appears that in a granular medium normal as well as tangential waves may exist, the only difference being in their rates of propagation and in their rates of degradation.

From this it would seem that, if the medium of space is purely mechanical, either such waves did not exist for lack of incitement or the normal waves had no effect upon our senses or on the physical properties of matter. The recent remarkable discovery of Röntgen that under certain intense electrical actions a system of waves which have the properties of normal waves in a uniform medium subject neither to refraction nor reflection, can be produced, has opened the door to different conclusions. The first suggestion by Röntgen was that these were normal waves. And although various special explanations have been attempted to avoid the admission of their being normal waves, every one of these explanations involves normal action.

It appears, from the definite analysis of the granular medium, that when the uniform medium is in the state to propagate transverse waves the degradation of which is such that the diminution from loss of energy by degradation in some millions of years is in the ratio $1/e^2$, the rate of degradation of the normal wave is such as would occupy something less than the millionth (10^{-6}) part of a second to reduce it in the same ratio; so that the normal wave would lose nine-tenths of its energy before it had traversed some thousands of metres, say x metres, and this affords crucial evidence of the purely mechanical granular structure of the medium of space. The coincidence is such, that in the absence of any definite proof to the contrary, it should carry conviction notwithstanding those things which cannot be defined for want of evidence.

252. Without attempting any general discussion of X-rays there are several very significant characteristics which afford evidence besides that of not being subject to refraction or reflection. In the first place the rays in their production are attended with very intense light, that is they are attended with transverse waves. In the second place, after the light waves have been filtered out, they can again be transformed into visible transverse waves by their passage through certain earthy substances. And in the third place, in passing through any matter they are subjected to rapid degradation which is proportional to the density and thickness of the matter through which they pass.

Thus it has been so far impossible to study these rays except by their passage through matter, while it is shown that in two ways their passage through matter is attended by degradation other than the degradation of the normal waves *in vacuo.*

Any estimate as to what might be the rate of degradation of these waves *in vacuo* is at best very difficult. But the fact that these waves, which are subject to divergence as well as the three sources of degradation, have sufficient range to permit of experiment through a distance of some metres, shows that if they are normal waves their rate of degradation *in vacuo* would be much less than it appears to be in the experiments. It thus appears that x, the distance the waves must travel *in vacuo* to reduce the energy in the ratio $1/e^2$, cannot be less than some thousand odd metres.

253. *To find the rate of decrement of the normal wave under the limits defined by equations* (221) *to* (224) *in terms of the ratio* $1/\sqrt{22\Omega}$.

From equation (310) we have, neglecting as small the first term in the index, and substituting $6/\sqrt{2\pi}$ for G,

$$\frac{1}{u''}\cdot\frac{du''}{dt} = e^{-\frac{1}{2}\left(\frac{5}{3}\frac{p^2}{\rho^2\frac{a^2}{2}} + \frac{4}{3}\frac{\sigma^2}{\lambda}\cdot\frac{a}{\sqrt{\pi}}\frac{6}{\sqrt{2}}\frac{a^4}{4\pi m^2}\right)} \qquad \text{............(401).}$$

The index in the right member of this equation represents the logarithmic rate of decrement of the normal wave.

Transforming this index and substituting the values of α, λ and σ as defined in equations (221) to (225) for the transverse wave, and of m and a for the normal wave, taking the time frequency m to have the same value as for the transverse wave and the linear frequency a to be $a'/2{\cdot}387$ where a' is the same as for the transverse wave [$2{\cdot}387$ being $\sqrt{3\kappa + 4m/3n}$]. Then taking A as expressing the numerical constant in the expression for the decrement, we find as the values of the several factors and their logarithms,

$$\left.\begin{array}{lll}
A = 1{\cdot}567 \times 10^{-2} & \log & {\cdot}1952 - 2 \\
\tau^{-2} = 1{\cdot}111 \times 10^{-21} & \text{,,} & {\cdot}0457 - 21 \\
a^2 = 2{\cdot}553 \times 10^{10} & \text{,,} & {\cdot}4068 + 10 \\
\check{I} = 3{\cdot}102 \times 10^{-2} & \text{,,} & {\cdot}4916 - 2 \\
\sigma^4 = 9{\cdot}376 \times 10^{-82} \times (22\Omega)^2 & \text{,,} & {\cdot}9720 - 82 + \log(22\Omega)^2 \\
\alpha''^3 = 3{\cdot}113 \times 10^{11} \times (22\Omega)^{-\frac{3}{2}} & \text{,,} & {\cdot}4930 + 11 + \log(22\Omega)^{-\frac{3}{2}} \\
\lambda''^{-3} = 6{\cdot}387 \times 10^{86} & \text{,,} & {\cdot}8053 + 86
\end{array}\right\} \text{...(402).}$$

The logarithm of this product being

$$\cdot 4076 + 3 + \tfrac{1}{2}\log(22\Omega) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (403),$$

log decrement log (log decrement)

$$-2{\cdot}556 \times 10^3 \times \sqrt{22\Omega}, \quad -[\cdot 4076 + 3 + \tfrac{1}{2}\log(22\Omega)] \ldots\ldots (404).$$

Then if t_n is the time to reduce u''^2 in the ratio $1/e^2$ we have

$$t_n = 3{\cdot}923 \times 10^{-4}/\sqrt{22\Omega}, \quad \log t_n = \cdot 5924 - 4 \ldots\ldots\ldots\ldots (405).$$

The product of the time t_n multiplied by the rate of propagation of the normal wave is the linear distance which the normal wave must travel so that the energy is reduced in the ratio $1/e^2$.

The rate of normal propagation is $2{\cdot}387 \times 3 \times 10^{10}$ as above.

Therefore taking x as the distance the normal wave must travel to diminish the energy in the ratio $1/e^2$ we have

$$x = 2{\cdot}801 \times 10^7 \times \frac{1}{\sqrt{22\Omega}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (406).$$

Q. E. F.

254. *Then to find the inferior limit to the value of the ratio expressed by*

$$\Omega.$$

From the evidence furnished by Röntgen rays we have in Art. 253 defined this ratio to be such that the value of x (in C.G.S. units) shall not be less than some thousand odd metres. And from the absence of any evidence of normal waves other than Röntgen it follows that there must be a superior limit; but this depends on the value of Ω and cannot be defined without further evidence.

To find the superior limit of Ω, putting for simplicity

$$x = 2{\cdot}801 \times 10^{7-q} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (407),$$

we have by equation (406) from the evidence of Röntgen rays

$$\sqrt{22\Omega} = 10^q \text{ where } q \text{ is not less than } 2,$$

whence we have for the value of Ω,

$$\Omega = \frac{10^{2q}}{22} = 4{\cdot}546 \times 10^{2q-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (408),$$

and for the density of the uniform medium

$$22\Omega = 10^{2q} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (409).$$

255. It is pointed out (Art. 254) that the superior limit to the value of Ω cannot be obtained except on further evidence; evidence which has as yet not been taken into account, and is exactly to the point, is available.

This is the evidence as determined by Lord Kelvin (and confirmed by the observation as to the area over which a definite volume of oil would destroy the ripple caused by a moderate wind on the surface of water), that the diameters of the molecules or singular surfaces are of the order of the ratio of the wave-lengths of the ultra-violet light multiplied by some ten thousandths, say 4×10^{-10}, and this evidence comes in as directly bearing on the value of q.

Although there is a degree of uncertainty about the relative value of the "atomic-volumes" of the elementary molecules, it appears certain that there is no great difference, that is to say, no difference greater than from 1 to 10 in the relative volume of the molecules, and for our purpose it is sufficient to consider that, assuming the relative volumes equal, the greatest difference of the grains absent is from 1 to 1/200.

It has been shown (Art. 230) that the probable arrangement of the grains in a negative local inequality, which has a surface of freedom, is that of a nucleus in normal piling, that is to say, a permanent nucleus on which the inward strained normal piling reaches, forming a broken joint in abnormal piling, whence it appears, in order that the singular surface may be a surface of freedom, the maximum inward strain, that is, the inward strain at the singular surface, must be greater than σ the diameter of a grain, and probably some five times σ.

In this way we have a limit to the diameter of the singular surface,

$$4 \times 10^{-10},$$

and by the last paragraph, taking 10 to be the inferior limit to the maximum inward strain, we can find a value for q which is quite independent of any evidence already adduced.

Taking $22\Omega = 10^{2q}$ for the density of the medium,

$r_1 = 2 \times 10^{-10}$ for the radius of the singular surface,

$4\pi r_0^3/3 =$ volume of grains absent.

By equation (380)

$$r_0^3 = \frac{1{\cdot}33}{10^{2q}} \cdot r_1^3 \qquad \text{.............................(410)}.$$

Then since by equation (396)

$$\sigma^3 = (5{\cdot}534)^3 \times 10^{-54+3q},$$

and

$$r_0/\sigma = n_0/2, \quad \text{also} \quad r_1/\sigma = n_1/2 \qquad \text{....................(411)},$$

$$\left(\frac{2r_0}{\sigma}\right)^3 = n_0^3, \quad \text{and} \quad 6\left(\frac{2r_1}{\sigma}\right)^2 = 6\,(n_1)^2 \qquad \text{..............(412)},$$

$$\frac{n_0^3}{6n_1^2} = \frac{\left(\dfrac{2r_0}{\sigma}\right)^3}{6\left(\dfrac{2r_1}{\sigma}\right)^2} \qquad \text{..................................(413)}.$$

Equation (413) expresses the number of diameters of a grain which would measure the inward strain at the singular surface of the maximum inequality as of platinum or 22.

Then reducing

$$\frac{n_0^3}{6n_1^2} = 1{\cdot}602 \times 10^{(9-3q)} \quad \text{.........................(414).}$$

For the minimum inequality, n_1 remains the same, and n_0^3 is divided by 200, and we have from equation (414),

$$\frac{\frac{n_0^3}{2} \times 10^{-2}}{6n_1^2} = 8{\cdot}013 \times 10^{6-3q} \quad \text{..................(415).}$$

Then if we take the number of the diameters of a grain which measure the inward strain at the singular surface of the minimum inequality to be

$$8{\cdot}013,$$

$$q = 2 \quad \text{....................................(416).}$$

We have thus found the superior limit of the square root of the density of the uniform medium to be

$$\sqrt{22\Omega} = 100.$$

256. Comparing the inferior limit of $\sqrt{22\Omega}$ in Art. 254, obtained from the evidence of Röntgen rays, with the superior limit in Art. 255 obtained from the evidence as to the size of the molecules, we see they are identical.

Too much weight must not be attached to this identity since the estimates on which they are based are somewhat wide approximations, so that they must be considered as relating rather to the order of the quantities than the actual numbers. Yet considering that the evidence of the size of the molecule, and that of the Röntgen rays, are perfectly independent, the result, which, taken as a wide approximation, would be almost infinitely improbable as a mere coincidence, when substituted in the equations (390) and (396), and (402) and (409) enables us to obtain, in C.G.S. units, the values of all the arbitrary constants which define the condition of the purely mechanical medium, and they are such as correspond with the experience—as to the rates of propagation and as to their rates of decrement—of both transverse and normal waves; they also correspond with experience as to the existence of molecules and gravitation, the limit of the intensity of the energy of light and radiant heat, besides the absence of normal waves, and the evidence of Röntgen rays.

The numerical values of these constants are for convenience given in the following table.

Description	Symbol	Quantity	Equation	Log.
Mean path of a grain in C. G. S. units	λ''	$8{\cdot}612 \times 10^{-28}$	375	$\cdot 9351 - 28$
Mean relative velocity of a grain in C. G. S. units	α''	$6{\cdot}777 \times 10$	370	$\cdot 8310 + 1$
Diameter of a grain in centimetres	σ	$5{\cdot}534 \times 10^{-18}$	372	$\cdot 7430 - 18$
Rate of decrement of transverse wave	$\frac{1}{t_t}$	$5{\cdot}603 \times 10^{-16}$	—	$\cdot 7483 - 16$
Time to reduce the energy of transverse waves in ratio $\frac{1}{e^2}$	t_t	$1{\cdot}785 \times 10^{15}$	394	$\cdot 2517 + 15$
Measure of wave-length in diameters of a grain	n_2	$7{\cdot}108 \times 10^{12}$	393	$\cdot 8517 + 12$
Parameter for	$\sqrt{n_2 t_t}$	$1{\cdot}126 \times 10^{14}$	384	$\cdot 0517 + 14$
Parameter for	$\left(\frac{n_2}{t_t}\right)^{\frac{1}{2}}$	$6{\cdot}31 \times 10^{-2}$	390	$\cdot 8000 - 2$
Mean pressure of medium in C. G. S. units	$22\Omega p''$	$1{\cdot}172 \times 10^{14}$	377	$\cdot 0690 + 14$
Ratio of density of medium to maximum density of matter	Ω	$4{\cdot}546 \times 10^{2}$	408	$\cdot 6573 + 2$
Density of medium in C. G. S. units	$22\Omega\rho''$	$1{\cdot}000 \times 10^{4}$	409	$\cdot 0000 + 4$
Rate of logarithmic decrement to normal wave	$\frac{1}{t_n}$	$2{\cdot}556 \times 10^{5}$	405	$\cdot 4076 + 5$
Time to reduce the energy of the normal wave from 1 to $\frac{1}{e^2}$	t_n	$3{\cdot}923 \times 10^{-6}$	405	$\cdot 5936 - 6$
Distance required to reduce the energy of the normal wave from 1 to $\frac{1}{e^2}$	x	$2{\cdot}801 \times 10^{5}$	406	$\cdot 4473 + 5$
The coefficient of the limit to which the shearing stress may approach the pressure of the medium	s	$1{\cdot}000 \times 10^{-2}$	390	$\cdot 0000 - 2$

It is thus shown by definite analysis that an infinite, purely mechanical, medium consisting of uniform spherical grains, in relative motion, the grains being in normal piling, except for local inequalities in the mean density, and so close that there is no diffusion, affords a complete account by purely mechanical considerations of potential energy, the propagation of transverse waves of light and the apparent absence of any rate of degradation, the lack of evidence of normal waves, the gravitation of matter and electricity, as the result of the dilatation which follows from the strains caused by local inequalities in the density of the medium.

It is also shown, by definite analysis, that this is the only explanation possible by purely mechanical considerations.

257. Having arrived at the conclusion stated in Art. 256 we might make this the end of this research, having every confidence that the evidence which has not already been adduced would confirm that which has been adduced. It is not, however, the sole purpose in undertaking this research merely to show that there is a mechanical explanation of such parts of the universe as shall render the mechanical structure of the remainder indefinitely probable, but also to obtain as much light as may accrue from the purely mechanical analysis. The analysis is therefore continued so far as it relates to effects in the medium, that is to say, it does not include electro-dynamics or electro-magnetics, since the institution of complex centres, that is, the magnetic conditions, is not a primary effect, for it results in separating the molecules, after combination, the reunion of which results in electric currents.

258. The blackness of the sky on a clear dark night would be explained if the light waves were subject to viscosity however small, or nearly so. It has been so far a moot question whether there is such viscosity. But it now appears from the rate of decrement of the transverse waves, Art. 256 ($5{\cdot}603 \times 10^{-16}$), that the time taken to reduce the energy of the wave in the ratio $1/e^2$, or $1/8$, would be more than fifty-six million years. This rate of decrement, although affording an ample account by mechanical considerations of the absence of uniform brilliance in the sky, such as would result in an infinite space from an infinite number of stars, however sparsely scattered, if there were no rate of decrement as the result of viscosity, is such as has baffled all attempts to obtain any evidence of decrement by observation.

259. The dissipation of the inequalities in the mean energy of the medium resulting from the rates of decrement of transverse and normal waves which, as shown in Art. 256, affords a complete mechanical explanation of the blackness of the sky, differs fundamentally from that dissipation which results in the increase of energy of the molecules, or singular surfaces. This is at once apparent since the degradation of the energy of the normal and transverse waves can only be a dissipation from the energy of the molecules, or mean motion, to increase the irreversible energy of the mean relative motion of the medium.

It thus appears that the dissipation of the mean motions of matter, such as the motions of the sun and planets, or vortices in fluids, until all motion ceases, does not complete the dissipation of energy, for this would go on until the only energy was irreversible relative motion of the grains, which is expressed by α''^2.

260. The electrostatic unit, or more correctly the unit corresponding to the electrostatic unit, is defined (Art. 224) by the condition

$$p''\left(\frac{4\pi}{3}\right)^2 r_0^7 = 1 \quad\text{..............................(417).}$$

This definition is on the supposition that the density of the medium is taken as unity.

Thus if the density is taken as 22Ω, we have as the condition

$$22\Omega p''\left(\frac{4\pi}{3}\right)^2 r_0^7 = 1 \quad\text{...........................(418).}$$

Then reducing the member on the left by the table (Art. 256) it is found that the complex inequality in which the number of grains is displaced is

$$1\cdot615 \times 10^{45},$$

and in which the displacement is unity; the effort to institute the normal piling is *unity* and thus corresponds to the electrostatic unit.

Comparing the effort to revert to the effort of attraction between two negative centres, each having the number of grains as above, since the radius of the shell which would contain the grains is

$$r_0 = 6\cdot493 \times 10^{-3} \quad\text{...........................(419),}$$

the ratio of the effort to reinstitute the normal piling, to the effort of attraction between gravitating mass, is approximately

$$1\cdot2 \times 10^{15}.$$

Thus the effort of attraction between the two gravitating masses, the grains absent in each of which are the same as the grains which constitute the electrostatic unit, is eighty-one thousand billion times less than unity.

261. The conclusion arrived at in Art. 256, as to the density of the medium, does not exhaust the conclusions to be drawn from the size of the molecules. Coupled with the evidence afforded by the effects in dissociating certain compound molecules, possessed by the transverse waves of shorter length and greater frequency, it appears that there must exist certain coincidences of periods between the possible internal vibration periods of compound molecules and the periods of the shorter waves.

262. From the evidence, Art. 261, it follows that the compound molecules which are dissociated by the waves of light must have been in a state of limited stability: so that

(1) by the breakdown the total potential energy is reduced,

(2) a sudden disturbance in the medium is produced causing waves, which are of undefined length, in the medium.

263. Comparing the evidences as to the effects of waves of greater frequency in dissociating certain compound molecules, adduced in Arts. 253, 254, with the conclusions arrived at in Arts. 238—241 as to the effects of collisions between compound singular surfaces, rotational strains, and the institution of complex inequalities corresponding to electrostatic induction, it appears that the latter account for the former by mechanical considerations as will appear in the following articles.

264. Accepting the statement in Art. 263, we find ourselves face to face with the question, What is the source of light?

From the mechanical analysis it follows, Art. 238, that undulations in the medium can arise from nothing else than the relative motion of the singular surfaces. The collisions of these surfaces would set up disturbances which would be propagated through the medium with the velocity of light, and which would correspond to the waves of heat. But from Arts. 238—241 it appears that there is another effect than that of simple collision, by which undulations may be instituted.

In Art. 241 it appears that when two aggregate inequalities, separated by a surface of weakness, in which the numbers of grains absent in the primary inequalities differ, are subjected to rotational strain, parallel to the surface of weakness, the strain will cause the total aggregate inequalities to reform, instituting two fresh aggregate inequalities with limited stability, which, as the strain is gradually reduced, do not gradually revert but, owing to the limited stability, are maintained until the strain has been relaxed sufficiently to overcome the limited stability and then break down under the nearly full effort of the complex inequality; which, by Art. 260, is more than two hundred billion times greater than what would be the effort of attraction of the two equal negative inequalities at the same distance.

Such a transverse reversion as that considered would not result merely in reinstituting the normal piling. But, as it involves the absolute displacement of mass, the recoil by reversing the strain would institute a complex inequality of the opposite sign; and this would be repeated, in a gradually diminishing degree, until all the energy was spent in setting up undulations which would be transverse.

We have thus two, more or less distinct, sources of undulations; and from the evidence it appears that, whatever undulations result from the collisions of singular surfaces, the undulations corresponding to those of polarised light are those caused by the reversion of the complex inequalities.

265. Since, from Art. 264, it appears that the institution of light depends on the existence, in the medium, of compound molecules with limited stability, and it also appears that these compound molecules dissociate in the production of light, it follows that either the source of light must be continually diminishing or that there must exist some action which results in thus reassociating the primary inequalities, and as the first alternative is contrary to experience we must accept the second as a fact.

The reassociation of the primary molecules which, when associated, form compound molecules with limited stability, receives its explanation from the mechanical analysis on the same lines as that of their dissociation.

Thus if we have two aggregate inequalities in one of which the primary inequalities are not combined the differing primary inequalities are combined. These may be analysed by putting

$a+a'$ for the combined total aggregate inequality, and

$b+b'$ for the total aggregate inequality uncombined, then

$$\frac{a+a'+b+b'}{2}, \quad \frac{a+a'+b+b'}{2},$$

$$\frac{a+a'-(b+b')}{2}, \quad \frac{b+b'-(a+a')}{2}.$$

These if added together constitute the total aggregate inequalities; they express two equal total negative aggregates together with one complex aggregate inequality.

Thus putting $a+a'=A$ the total aggregate inequality in which the primary inequalities are combined, we have

$$\frac{A+b+b'}{2}, \quad \frac{A+b+b'}{2},$$

$$\frac{A-(b+b')}{2}, \quad \frac{b+b'-A}{2}.$$

Then if the strains were sufficient the normal attraction might overcome the normal stability, *i.e.* the stability in the direction of the normal, of the complex inequality, causing a reversion. In this case, however, it does not follow that the reversion would be complete and so reinstitute A, b and b', for since the work done by the strains might be sufficient to overcome the resistance to combination of b and b', the recoil from the breakdown would cause a total or partial combination of b and b', thus instituting B, the total aggregate inequality, and so diminish the energy available for the institution of undulations.

We have thus an explanation by mechanical considerations of the part played by electricity in instituting the combinations of molecules which differ into compound molecules with limited stability.

266. The absorption of the waves of light, let us say by lamp-black, presents a problem, the explanation of which, by the assumption that the molecules are capable of internal vibrations in various periods, is altogether sufficient. Thus, supposing the molecules in the lamp-black are so various that there are molecules the internal vibrations of which coincide with all periods of the incident wave, they would be set in periodic motion and absorb the energy of the waves; but this is not all. For supposing the absorption of the light continuous, the energy in the molecules would continually increase, and this is not in accordance with experience. There must therefore be some means by which the energy absorbed by the molecules may escape. This cannot be by radiation, since in that case it would only escape as light, which it does not. It is mechanically impossible that it should escape by radiation in the form of the long dark waves. And the only other mode of escape for the energy is by transmission—by convection and conduction through the molecules to the surface of the lamp-black. Nor does this altogether solve the problem—for in such an experiment as we are considering, it may be possible that the lamp-black is *in vacuo*; in which, having reached the surface, it would be arrested. And the absorption continuing the energy of the molecules would continually increase indefinitely. Since any such indefinite increase of the absorbed energy is outside experience it follows that within the limits of experience such perfect vacuum as contains no free molecules is impossible.

The evidence which follows from the theoretical explanation of Sir William Crookes' radiometer* at once illustrates the fact mentioned above, for when the light is turned on the receiver which contains the vanes, the latter almost instantly acquire a steady speed which shows that the lamp-blacked surfaces as well as the opposite surfaces, which are white, have acquired a steady difference of temperature, so that there is no further increase of temperature from the absorption of the light; the energy received from the light wave by the black surfaces of the vanes, taking the form of energy

* "Certain dimensional properties of matter in the gaseous state." *Phil. Trans. R. S.*, 1879, p. 823.

of vibration of the molecules, is transmitted to the surface beyond which the vibrating molecules do not pass, but, as the molecules at the surface are vibrating, the energy of this vibration is communicated by contact to any free molecules whose paths bring them in contact with the molecules at the surfaces of the vanes, causing reaction and conveying the energy to the inner surface of the receiver.

Thus if there were no free molecules there would be no motion imparted to the vanes, and as the stage of exhaustion at which the vanes do not revolve in unlimited light has not yet been attained, it follows that on the assumption that the waves of light are capable of communicating energy to the molecules in the mode of internal vibration, the production of an unlimited intensity of energy by the absorption of light is outside experience.

267. The assumption on which the absorption of light is based, Art. 266, has not as yet been subjected to the further analysis necessary for a mechanical explanation of the actions involved.

It therefore remains to show that, in spaces where negative inequalities exist, the state of the granular medium is so far affected by these inequalities that it no longer transmits waves which pass through the medium at the same velocity as when there are no inequalities, undisturbed, otherwise than by divergence.

To show this:

We have (Art. 230) the fundamental misfit between the nucleus in the singular surface with the grains in strained normal piling, instituting in the medium a shell of grains in abnormal piling which constitutes a shell about each singular surface which offers little or no resistance to strains tangential to the singular surface.

We have also (Art. 255) the diameter of the singular surface some ten thousand times less than the wave-length. Thus we have a free singular surface through which the medium is free to move by propagation, the diameter of which is 10000 times less than the transverse wave, but which is still subject to the undulatory motion of the medium corresponding to the light waves.

Consider next what must happen from the existence of a single negative inequality in a space through which transverse waves are passing:—

In the first place, since the surface of the inequality is a surface of freedom there would be a certain small area of the surface about an axis through the centre of the inequality which presents a nearly plane surface perpendicular to the direction of propagation, and this small surface, owing to the freedom of the inequality, offers no resistance to the transverse wave.

This area of freedom would relieve the stress in the medium in the plane normal to the direction of propagation, and so cause an increase of the undulatory motion at the small surface, the recoil from which would reverse the direction of propagation over the small area, thus instituting a partial reflection. (N.B. The amount of this reflection would admit of quantitative determination, but the analysis is long and it does not appear to be necessary.)

The reflection considered does not constitute the entire reflection which would result, for there would be similar reflections at the opposite surface of the inequality, and besides the reflections on the small surfaces nearly plane, there would be reflections resulting from the relaxation of the components of the transverse stress all over the surface of the inequality, causing reflections in all directions except in planes normal to the direction of propagation. So that there would be a general but varying scattering of the transverse wave in all directions greater than $\pi/2$ from the direction of propagation, varying from a maximum at π to nothing at $\pi/2$.

The proportion of undulations within a distance r_1 of the axis in the direction of propagation scattered by the passage of a wave by a single inequality is extremely small, for, although the small surfaces of freedom do relax, to some extent, the stresses consequent on the undulations in the medium, a singular surface is so small as compared with the wave-length, that they follow the motions of undulation, and are subject to nearly the same stresses as if there were no inequalities.

Then if we consider a space, through which the waves are passing, to be occupied with negative inequalities in somewhat close order it does not appear that the rate of propagation would be greatly altered owing to relaxation of the elasticity of the medium.

But the rates of propagation do not, as it seems, depend solely on the elasticity; for the singular surfaces, owing to their cohesion, introduce another system of possible vibrations—the internal vibrations of the negative inequalities.

That the vibrations possible in the inequalities may be instituted as the result of undulatory stresses requires only a coincidence in the periods of the waves and the vibrations of the inequalities. Then since the evidence of the existence of a considerable number of periods of vibration in all inequalities is according to evidence, and it has been shown that however small the effects of the undulations solitary grains do cause a certain disturbance in the negative inequalities, it follows that the passage of a wave through a space in which the inequalities are somewhat close will result, if continued for a sufficient time, in imparting periodic motions to the inequalities having periods coinciding with the wave periods.

Then supposing the regular undulation to cease, the vibrations of the inequalities would institute waves of the same period until their energy was exhausted. Whence it follows that in the case in which waves are passing steadily into and through a space occupied by inequalities in somewhat close order, they will maintain the vibration of the molecules and at the same time pass through the medium, and then the energy of the waves and the vibration of the inequalities together would be greater than that of the inequalities alone in the ratio

$$\frac{\text{energy of wave motion} + \text{energy of inequalities}}{\text{energy of wave}}.$$

Then supposing a steady state to have been reached, if either of these actions were diminished it would receive assistance from the other; and from this it follows directly that, while the energy in a wave-length before entering the space containing the inequalities is the only energy of the undulation, the energy in a wave-length in the space would be the energy of the undulation before passing plus the energy of the inequalities.

Then again if the mean rate of the motion of the energy of both undulation and inequality were that of the undulation, there would be more energy passing out of the space than that entering, and the state could not be maintained steady. But if, on the other hand, after entering the space with inequalities, the rate of passage of the total energy was that given by

$$\frac{\text{energy of wave}}{\text{energy of wave} + \text{energy of inequalities}},$$

the state would be steady, and the rate of propagation diminished in the same ratio.

It has thus been shown that in the granular medium waves corresponding to light waves are capable of communicating energy to the negative inequalities corresponding to molecules, which was the object in this somewhat long article.

268. Refraction of waves in the granular medium, when passing from one space to another which differs as to the closeness of the arrangement, follows directly from the paragraph last but one, Art. 267, in which it is shown that the waves pass from a space in which there are no inequalities into a space in which the inequalities are in some close order; the ratio of the rate in the space without inequalities to the rate in the space with inequalities is as

$$\frac{\text{energy of propagation} + \text{energy of inequalities}}{\text{energy of propagation}},$$

and this is the expression which corresponds with the index of refraction.

It also appears that in the main the cause of refraction is coincidence in the wave period with the period of vibration in the inequalities. The relaxation of the elasticity of the medium by the freedom of the centres must cause diminution in the velocity of propagation, but it would seem to be infinitely small.

268. The effect which, in a granular medium, corresponds to the dispersion of the rays in the spectroscope, at once receives its mechanical explanation from the explanation (Art. 266) of the effects which correspond to refraction together with the evidence (Art. 251) that as the waves become shorter their effects in dissociating the compound molecules increase. The mechanical explanation is that there are more coincidences in period in the case of short waves than in the longer. And this refraction increases as the length of the wave decreases, or, the shorter the waves the higher the index of refraction.

269. The reflection of the wave of light which results when the light passing from a space in which there are no inequalities into a space in which the inequalities are in somewhat close order, depends, when the direction of propagation is not perpendicular to the reflecting surface, on the direction of the transverse motion in the wave front. Thus supposing the direction of propagation is parallel to the paper and across the page, if the motion in the wave front is parallel to the paper and the reflecting surface is a plane perpendicular to the paper and inclined to the direction of propagation, there may be, at the same time, motion in the wave front perpendicular to the paper which motion is independent of the normal to the paper as illustrated in the figure below:

FIG. 3.

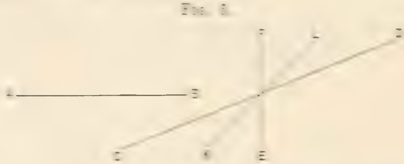
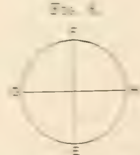


FIG. 4.



AB direction of propagation.
CD trace of plane of reflection perpendicular to the paper.
GH direction of motion parallel to plane of reflection turned at 90°.
EF direction of motion perpendicular to propagation.

The general motion of the medium thus indicated may be realised by imagining a sheet of paper to be moving with a point revolving steadily with a radius equal to half the amplitude of the wave without turning the paper in its plane, which must be perpendicular to the paper. The two motions *EF* and *GH* may be considered as analytically distinct.

The vertical harmonic motions in the medium EF in planes parallel to the paper, in varying phases, without any motion in the direction of propagation, which constitute the undulations, meet as the undulations arrive at the inclined reflecting surface of the space enclosing the inequalities undergo partial reflection in the direction KL on account of the refraction at the surface caused by the inequalities. This refraction is proportional, other things being the same, to the number of singular surfaces in unit volume.

The angle of incidence on the surface of the space enclosing the inequalities at which the reflection is a maximum is of necessity such that the reflected and refracted rays are at right angles in the plane of incidence, and in this case the motions in the medium which are reflected are parallel to the plane of incidence and thus correspond to light polarised in the plane of incidence.

Then again, the portion of the waves in which the motion of the medium is perpendicular to the plane of incidence, although subject to the same refractions as that in which the motion of the medium is parallel to the plane of incidence, undergoes no reflection at the surface CD.

The waves in the granular medium are a consequence of the motions in the medium which are transverse to the direction of propagation; these may be anything perpendicular to the direction of propagation. But they admit of being resolved in any two directions at right angles to each other and perpendicular to the direction of propagation, and when so resolved are analytically independent, that is to say, if the components of the transverse motions of the medium in one direction ceased to exist the motions of the other component would not be affected.

Thus if transverse motions in the medium were confined to one direction, and that in a plane parallel to the plane of the paper, the shearing stresses would be all parallel to the paper in planes at right angles to the direction of propagation. Hence these stresses would propagate transverse motions parallel to the paper but would not propagate motions normal to the paper, and on the other hand transverse motions in the medium normal to the paper would cause transverse shearing stresses normal to the paper which would propagate motions normal to the paper but would not propagate motions parallel to the paper.

It therefore follows that the transverse waves in which the motion is parallel to the paper can, in a granular medium, be maintained only by rotational stress in which the rotation is parallel to the paper. And such transverse waves will propagate parallel to the paper in the direction to which these planes are normal. And further, on arriving at a surface the normal to which is parallel to the paper, beyond which surface there are inequalities, the wave will be reflected according to the laws of reflection, such reflection being strictly parallel to the paper.

On the other hand, it follows that the transverse waves in which the motion is normal to the paper can, in a granular medium, be instituted only by rotational stress in which the rotation is normal to the paper; such waves propagate parallel to the paper in the direction to which their planes are normal, and are not subject to reflection at an inclined surface perpendicular to the paper, as shown in Fig. 5, since the motion in these transverse waves is entirely normal to the paper, as is shown by the line GH, turned through 90° in Fig. 6. Thus it is seen that the only reflection resulting from both components of the motion in the medium when the waves pass from a space without inequalities into a space with inequalities is the reflection resulting from the inclination of the surface parallel to the plane of incidence, as shown in Fig. 5.

It may appear from what precedes that there is a difference besides that of the motion of one of the rays being parallel and the other normal to the paper, since so far no mention has been made of any reflection of the ray in which the motion is perpendicular to the paper. This apparent difference disappears, however, since if the reflecting surface in the plane of incidence were removed and replaced by a surface normal to the paper inclined at a corresponding angle to the direction of propagation, then the reflection would be from the waves perpendicular to the plane of incidence, and there would be no reflection from the plane of incidence.

It is thus shown that in the granular medium when the transverse stresses in the medium are equal in all directions normal to the direction of propagation, when waves proceed from a space in which there are no inequalities into a space in which there are inequalities, if the separating surface is inclined to the direction of propagation there will be reflection in the plane of incidence of that component of the wave which is in the plane of incidence, in a degree depending on the closeness of the inequalities and the angle of incidence, while the other component of the wave-motion will not be subject to any reflection resulting from the inclination. And as this applies whatever the direction of propagation may be, it affords a definite proof that the motion in the medium which is reflected is in the plane of incidence.

This result in the granular medium corresponds in every particular with the experiences of polarisation except that heretofore it seems to have been a moot question whether or not the motion in the ether which is polarised by reflection was parallel or perpendicular to the plane of the medium*.

* "In the theories of Fresnel and Cauchy the vibrations are assumed to be perpendicular to the plane of polarisation—in those of MacCullagh and Neumann to be parallel to it. Stokes arrived at the conclusion that they are parallel, while by a similar experiment Holtzman arrived at the opposite conclusion." Lloyd, *Wave Theory of Light*, 1857.

Thus not only does the analysis of the granular medium account by purely mechanical considerations for the phenomena of polarisation, but also removes all doubt, if the explanation is mechanical, as to the fundamental necessity that the motion in the medium that can be reflected must be in a plane parallel to the plane of incidence.

The foregoing proof that that component of the motion of the medium which is reflected is that parallel to the plane of incidence has been based on the relaxation of the mean coefficient of rotational elasticity owing to the presence of negative inequalities, as discussed in Art. 267. This was all that was required, as the relaxation in translucent matter is comparatively very small. When, however, we come to metallic reflection, which in the case of mercury at perpendicular incidence is 0·666 as against 0·0018 for water, it appears that the relaxation is altogether of another order than in translucent substances.

In the mechanical medium such difference is accounted for by the extremely small size of the singular surfaces, the radii of which are about 2×10^{-10} or 2×10^{-5} of the length of the shorter waves. These singular surfaces as long as their arrangement is in open order will cause relaxation which is small but which increases somewhat proportionally to the number of such surfaces in unit space, each surface being, as it were, independent, so that the abnormal pilings which embrace every grain will only meet at a few points. But as the inequalities approach the closest order the rate of decrease of the relaxation increases very rapidly until the normal piling of the singular surface becomes nearly continuous. The surface of the space enclosing the inequalities then becomes a singular surface of the aggregation of inequalities outside of which the piling is abnormal.

To realise the evenness of such a boundary surface embracing the whole or any part of the aggregate inequalities it is only necessary to remember that the radii of the singular surfaces are less than one ten-thousandth of the wave-length, whence the roughness which would be less than 1×10^{-9} cm. and thus would be smoother than any artificial polish which can be imparted to metal, and hence could only compare with the surface of mercury.

It is thus shown that the granular medium not only affords an explanation of the polarisation of light but also affords an explanation of metallic reflection. And these explanations being accomplished it appears that the mechanical explanation of the rest of the phenomena of light must of necessity follow.

271. The aberration of light admits of an explanation so simple and the coincidence of the value of the velocity of light thence deduced with that derived from the observations of the eclipses of Jupiter's satellites is so remarkable as to leave no doubt in the mind as to the truth of the explanation.

But when the aberration is subjected to closer examination the explanation is found to rest on the heretofore unexplained absence of any resistance to the motion of the ether through matter; for notwithstanding the efforts made to rest the explanation on another basis this has not been completely accomplished.

The difficulties in conceiving the free motion of the ether through matter do not present themselves in the analysis of the properties of the granular medium as now accomplished. This follows from the analysis which has been effected in this and the previous section.

It is shown:—

(1) That the motions of the singular surfaces are independent of the mean-motion of the grains in the medium (Art. 233).

(2) That the institution of undulations depends on the varying strains resulting from relative motion of the singular surfaces (Art. 264).

(3) That the energy of the wave is absorbed by the singular surfaces, and that the energy thus absorbed is conducted and conveyed through the aggregate singular surfaces (Art. 266).

Whence it follows that the singular surfaces which correspond to matter are free to move in any direction through the medium without resistance, and *vice versâ* the medium is free to move in any direction through the singular surfaces without resistance. And that the waves corresponding to those of light are instituted and absorbed by the singular surfaces only. So that after institution at the place where the singular surfaces are, the motion of the waves depends solely on the mean motion of the medium, and the rate of propagation is equal in all directions until they again come to singular surfaces. Thus all paradox is removed and the explanation of aberration is established on the basis of the absence of any appreciable resistance to the medium in passing through matter.

Thus besides the explanations by definite analysis of:

the potential energy,

the propagation of transverse waves of light,

the apparent absence of any rate of degradation of light,

the lack of evidence of normal waves,

the gravitation of matter,

electricity,

which explanations render the purely mechanical substructure of the universe indefinitely probable, we have by further analysis obtained:—

The explanation of the blackness of the sky on a clear night. (Art. 258.)

The definite proof of the fundamental dissipation of the energy of the waves of light and the relative energy of the molecules to increase the mean irreversible relative motion of the grains; which dissipation is independent of that which tends to the equalisation of the mean energy of the molecules. (Art. 259.)

The number of grains, the displacement of which through a unit distance represents the electrostatic unit. (Art. 260.)

The proof of the coincidences between the periods of vibration of the molecules and the periods of the waves. (Art. 261.)

Proof that dissociation of compound molecules proves the previous state to have been one of limited stability. (Art. 262.)

Proof that light is produced by the reversion of complex inequalities. (Arts. 263—264.)

Proof that the reassociation of compound molecules results from the reversion of complex inequalities. (Art. 265.)

Proof of the absorption of the energy of light by inequalities. (Art. 266.)

Proof that negative inequalities affect the waves passing through. (Art. 267.)

Proof that refraction is caused by the vibrations of the inequalities having the same periods as the waves. (Art. 268.)

Proof that dispersion results from the greater number of coincidences as the waves get shorter. (Art. 269.)

Proof that the polarisation of light by reflection is caused only by that component of the transverse motion in the medium which is in the plane of incidence, and results from the passage of the light from a space without inequalities through a surface into a space in which there are inequalities. (Art. 270.)

Proof that metallic reflection results from the relative smallness of the dimensions of the molecules compared with the wave-length, and the closeness of their piling, when the waves pass from a space without inequalities across the surface beyond which the inequalities are in closest order. (Art. 270.)

Proof that the aberration of light results from the absence of any appreciable resistance to the motion of the medium when passing through matter. (Art. 271.)

INDEX.

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