# SUGGESTIONS
## ON THE
# TEACHING of GEOMETRY

With Especial Reference to the Use of
DURELL and ARNOLD'S GEOMETRY



BY

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## DURELL AND ARNOLD'S GEOMETRY

**1. Introductory Remarks.**—In recent years pupils in the high school on the average are noticeably more immature than they were ten years ago. The subject matter also of geometry has been changed, and to a certain extent vocationalized and humanized. These changes call for modifications in class-room methods of teaching geometry, in order to obtain maximum results. It is hoped that the suggestions which follow may help to meet the new situation.

It is to be understood that this pamphlet is written merely by way of suggestion and for teachers who desire any help they can get from any source. It is not intended for teachers who have already worked out methods of their own adequate to their needs. Suggestions of any kind looking to the betterment of the following remarks will be gladly received.

While the mind of the average high school pupil is immature and occupied with many interesting concrete things such as the movie, kodak, and automobile, it is also alert, eager, and quick to grow when interested. Hence the best general course to follow in teaching such pupils is to give them at the outset, a large amount of simple and easily appreciated work. This arouses natural growth processes in their minds, so that in time, and often without serious effort, they develop the power to do more difficult work and form an active appetite for it.

**2. The Teaching of Definitions and Elementary Principles.**—Under present conditions, in teaching the first groups of definitions in geometry, it is usually best at first to concentrate attention on certain leading or specially important ones, as on the definitions of geometry, curved line, parallel lines, circle, right angle, perpendicular, and complementary and supplementary angles, on pp. 7–20 of Durell and Arnold's Plane Geometry. The other definitions and discussions on these pages are to be read, con-considered, queried, and commented upon, till they are fully understood, but attention is to be concentrated on the list given above, the ideal result aimed at being that each of these pivotal definitions shall be so thoroughly memorized that each member of the class shall be able to give it instantly at any time and shall not be able to forget it. When these definitions have been thoroughly assimilated, they form a foundation or starting point to which the neighboring definitions can be readily attached. Some of these will come to the pupils as natural consequences of those first learned and can readily be picked up in review.

The learning of definitions like that of vertical angles may be postponed till Prop. I is studied. In like manner the geometric axioms and postulates (p. 23) in the first study are to be read and discussed but not committed to memory till later, as they are used in the course of demonstrations.

Similarly the principles enunciated on pp. 26–28, as matters of direct inference, are not to be committed to memory in a mass, but each is to be learned in connection with its use.

Thus, in working the examples in Group 8, each principle as it is taken from §§ 55–68 and used, is to be quoted in exact language.

It is usually best to have pupils learn corollaries not as they occur as consequences of propositions, but as they are used in connection with later propositions.

This is not only a great economy of time, but is also an illustration of the guiding principle of the book; viz., that geometry is not merely a set of correct deductions, but is a series of improving tools, the relation of each of which to its uses should be made as close as possible.

By following the above method, pp. 7–28 of the textbook can be covered in five lessons. In the last ten or fifteen minutes of each recitation, have the class do written work, such as defining certain terms, or making simple drawings, like those on page 14, or solving examples like those in Groups 1–8.

At this point before taking up the further detailed treatment of the subject matter of geometry, it may be well to consider the form of recitation on theorems and originals, best suited to the new conditions under which geometry must now be taught.

**3. Form of a Typical Recitation on Theorems.**—When the study of the propositions demonstrated (or solved) in the text is begun, it is of course important to make clear to a pupil the meaning and parts of a proof and also to teach him how to study a proof. He is to be brought to see that mere repeated reading of a theorem and its proof usually will not give real mastery of a proposition, but that he is to study the proof till he understands it, then try to write it out on paper; and afterward compare his written work with the demonstration as given in the textbook. If he finds his proof is faulty, he is to repeat the process till he can prove or solve the proposition correctly.

It is well sometimes to have pupils hand in at the beginning of the recitation the written proof which is the result of their study, with the following or a similar statement attached: "I have written the above proof without consulting the book or receiving any aid from any person," or more briefly, "no help," or "a little help." Some pupils seem to need a definite end like this to work for.

We shall now give a statement of what we consider as, on the whole, the best general method of conducting a recitation on text propositions, and follow this statement by a mention of some variations from the general method which will tend to stimulate interest and add further effectiveness to the work.

If a class has a lesson of, say, three text propositions in advance and three in review, good results are obtained by having the proof of each advance proposition written on the blackboard at the beginning of the recitation by at least two members of the class. While these six pupils are engaged in thus writing the proofs on the board, have the figures for the three review propositions put on the blackboard where all the rest of the class can see them readily (a teacher can often save time by himself drawing the diagrams on the board before the recitation begins; he thus also has an opportunity to stimulate interest by varying the diagrams in form, position, and lettering). Then let the pupils who are at their seats demonstrate the review propositions orally from the figures thus drawn. It is often, or even usually, best to divide each oral recitation of a proof into two parts, each part assigned to a different pupil: viz., 1st. A careful and accurate statement of the general and particular enunciation of the proposition; 2d. The proof proper (or in case of a problem, the construction and proof).

By this means the learning of the exact language of a proposition, and the acquiring of a clear grasp of hypothesis and conclusion, which are matters likely to be neglected and slurred over by the pupil, are brought out into due prominence. (Hence, for instance, it is not necessary to have special pads or forms on which pupils do their written work, in order to attain this end.) Also the above method is an aid in teaching a large class, since more pupils are called on to recite in a given time; also, a weak pupil who has difficulty in making demonstrations, soon learns to enunciate propositions clearly and with under-standing, and thus lays a good foundation on which to base the learning of demonstrations proper.

By the time that the oral recitation of the review propositions has been finished, the written work on the board will be ready for inspection.

Time may often be saved by using the following plan in inspecting such written work on the blackboard. Read aloud one written statement of each proposition, carefully correcting or having the class correct each defect in the proof read, and making sure that the whole class understands each step of the proof. The duplicate proof or proofs may then be read silently, or quickly glanced over, by the teacher, errors being pointed out and corrected, by the teacher if the time is short, or by the class if time is available.

If this plan of conducting a recitation on text theorems be followed, an interval of from ten to fifteen minutes will usually be left at the end of the recitation which can be utilized for various purposes, after the next lesson has been announced, as for review, for sight work in originals, or for written work on paper by the whole class. These methods of utilizing the last ten or fifteen minutes of the recitation hour will be described in more detail (see § 5 of this pamphlet) after we have given some methods of varying the first part of our typical recitation on text propositions.

**4. Variations in the Methods of Conducting a Recitation on Text Propositions.**—The variations next to be mentioned are not merely useful as a means of arousing interest but are often necessary to a degree owing to radical differences in aptitudes of teachers and classes, and to various other circumstances.

(1) In the oral recitation of review propositions it is well occasionally to dispense with diagrams drawn on the blackboard. Have the pupil, when reciting, picture and letter his diagram mentally and have the class follow him in his proof and correct every misstatement; or after a

figure has been drawn on the blackboard and used once, rub it out and have the pupil give the proof, using mentally the figure as it had been drawn and lettered on the blackboard.

(2) Send the whole class (if it is not too large) to the blackboard and have each member of the class write on the board the proof of one theorem after another during the entire recitation period.

(3) Use the recitation period in a written test (on paper) not only on the advance work, but on the last ten (or more) propositions that have been studied.

(4) If the class is large have part of the class do written work on paper at their seats, while the other part of the class does written work (on originals often) at the blackboard.

**5. Methods of Using the Final Ten or Fifteen Minutes of the Typical Recitation on Text Theorems.**—In the first few lessons in the study of theorems and originals, a considerable part of the final fifteen minutes must be used in explaining the nature of a proof, methods of study, etc. (See §§ 3, 6, 7.) After that the work may be varied in one of the following ways:

(1) Have the entire class write out the statement and proof of one or more propositions or originals on paper at their seats, such written proof to be carefully corrected later by the teacher in red pencil or ink and returned to the pupil at the next recitation.

It will be a considerable economy of the teacher's time and strength to indicate the proposition to be proved by the entire class, not by writing on the blackboard the proposition to be proved, but by drawing on the board the figure (or using one of the figures already drawn there) belonging to this proposition and having the class write out the theorem, enunciation, and proof, all members of the class using the figure as lettered on the blackboard. The economy to the teacher comes from the fact that in correcting the proofs the teacher quickly commits this lettering to memory and is able to correct a paper in half the time otherwise required.

While the members of the class are working at their seats, an opportunity is afforded the teacher for a rapid grouping and inspection of the papers which contain the work done outside the class and which have been handed in at the beginning of the recitation. If any member has failed properly to do this work, he can at once be called to the desk and assigned to deficiency study. (See § 13.)

(2) The whole class may be sent to the blackboard to write out demonstrations. This procedure often has a peculiarly stimulating effect on the class. It seems to arouse in a striking way a sense of unity and co-operation.

(3) While the class remain at their seats the teacher may stand at the blackboard and rapidly sketch one after another the diagrams used in the last fifteen or twenty propositions and ask individual pupils to state the proposition relating to each diagram.

The teacher may also in case of a certain diagram ask a pupil to draw the required auxiliary lines, and to start the proof, or even to give an outline of the proof.

(4) The time in whole or part may be spent in simple construction problem exercises. If the class is small, all pupils may be sent to the blackboard. If the class is too large for this, some of the pupils may work at the blackboard and others at their seats doing the work on paper.

The class may be asked to draw accurately the figures used in recent propositions or to review previous construction problems by drawing accurately the diagrams used in them. Or examples like the following may be devised and used:

Ex. **1.** Construct a quadrilateral $ABCD$ in which $AB=1$ in., $BC=1\frac{1}{2}$ in., $CD=2$ in., $AD=2\frac{1}{2}$ in., and the diagonal $AC=1\frac{1}{2}$ in.

Ex. **2.** Construct a triangle whose sides are $2\frac{1}{2}$ in., $1\frac{3}{4}$ in., and 1 in. Then construct the three altitudes of the triangle, using the concurrence of the three altitudes as a test of the accuracy of the work.

In construction work at the blackboard, have the pupils construct arcs by use of a string with a piece of crayon attached to one end, the crayon being sharpened to a point and held perpendicular to the board. The room should also be provided with a number of rulers two or three feet long divided into inches. In constructing on the blackboard diagrams like those called for in Exs. 17 and 18, p. 54, inches should be changed into feet.

(5) The time may be employed in doing the exercises given as sight work at the foot of different pages in the textbook.

The pupils may be asked to solve these exercises at the blackboard, or the teacher may stand at the blackboard, draw the figure for each exercise, and have the pupil state the solution, the teacher doing on the blackboard such mechanical work as may be necessary.

(6) Frequently the time may be profitably spent on sight work obtained from outside sources, on originals, or on numerical exercises, the teacher standing at the blackboard, drawing the diagrams, and bringing out and recording what is given concerning the diagram and what is to be proved, the actual demonstration being oral. See remarks on oral drill in originals, § 8.

(7) The teacher may spend part or all of the time discussing simple practical applications of geometry. (See § 9.)

(8) The final interval which we are considering may be spent in rapid oral review of definitions.

(9) Or it may be spent in a combination of two or more of the preceding methods.

Utilization of the preceding methods of varying the form and content of a recitation are especially valuable in dealing with the type of pupil described in § 1. A certain amount of change and unexpectedness gives spice to the work and arouses to the maximum the natural growth processes in the child.

**6. First Theorems.**—On taking up the study of theorems (text-book, p. 29) let us remember that the child has no idea of what a demonstration is, or of its utility. The following has been found a satisfactory way of approaching, Prop. I, p. 29. Draw two intersecting lines on the blackboard. Say that an engineer has two intersecting lines like these, and wants to know what is the least number of the four angles which he must measure in order to know the size of all the angles. The child has already had some experience in measuring angles and is quick to answer "one angle." (Let the teacher mark on the diagram one angle as, say, 47° and ask the pupil to give the size of the other angles.) This opens the way for the statement, "Let us prove that in all cases, no matter what the size of the angles of intersection, the property which you have used, viz.: that vertical angles are equal, is true." Let the teacher draw the figure on the blackboard and write out the proof exactly as it is in the textbook, explaining it step by step. Then erase this and ask some pupil to write out the proof on the blackboard, using a diagram lettered differently.

After this proof has been worked out, draw three lines intersecting at one point, and ask how many of the six angles thus formed must be measured in order to determine the rest without measuring them, and so on.

The first lesson on theorems comprises Prop. I and Exs. 1–7 which follow.

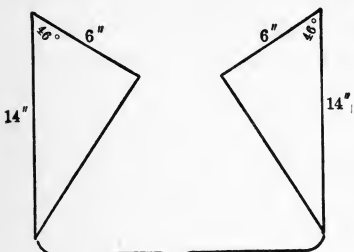
Proposition II is approached in a similar way. An engineer wishes to determine whether two given triangles are equal. He measures two corresponding sides in the triangles and finds each of them to be 123 ft. (Insert each number when named in the proper place in a diagram drawn on the board.) Then two other sides and finds each of them to be 117 ft.; then the angles between these pairs of sides and finds each of them to be 63°. Does he need to measure and compare any more parts of the triangle? The instant reply from the class is, "No." This again opens the way for the teacher to say "Let us prove that this is so."

In the next lesson Proposition III is treated in like manner.

The second lesson in Book I may comprise the definitions on pp. 30–32, and Prop. II, with a review of Prop. I. Exs. 1–10, p. 32, may be treated as sight work. The third lesson will be Prop. III, and Exs. 1 and 2, top of p. 35, and a review of Props. I and II.

In introducing Proposition IV (p. 37), we may say that an engineer knows that $AB$ and $BC$ are each 110 ft.; does he need to measure angles $A$ and $C$ in order to determine whether they are equal? A similar method may be used with advantage in many places throughout Plane Geometry. Thus, in introducing the subject of parallel lines draw two parallel lines intersected by a transversal and ask how many of the eight angles formed the engineer must measure in order to know the rest. So when first treating similar triangles, draw pairs of triangles, assign numerical values to certain sides and angles and ask whether each pair of triangles is similar.

**7. First Originals.**—After studying Propositions I, II, and III, it is well next to stimulate the growth processes in pupils and to arouse the pleasure which comes from a sense of mastery by having the class prove a number of simple originals, by the use of Propositions II and III. If left to himself, the pupil will usually try to prove these exercises by placing one of two given triangles upon the other. This is natural since it is the only method of proving triangles equal that he has seen. Hence, before asking him to prove such exercises, it is well to have a preliminary blackboard drill, in which the teacher draws in succession many pairs of triangles in different positions, with the equal parts written on the triangles, and asks whether each pair is equal and why. The following are illustrations:

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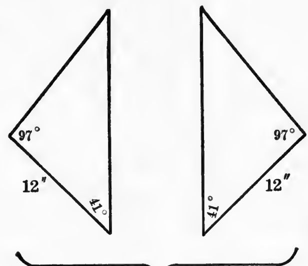


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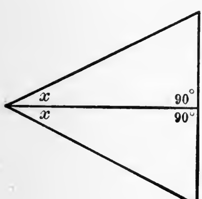


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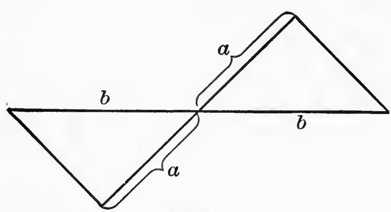



(4)

In some cases the corresponding parts of triangles as given should not be equal, and the pupil should be asked to change the given numbers so as to make the triangles equal.

The first originals assigned should be as simple and clear as possible. The work can hardly be made too easy at the start. Do not hesitate to do all the preliminary drudgery for the pupil. Draw and letter the diagram; state the parts *given,* and to be *proved,* leave to the child the pleasure of merely discovering the proof. Arouse in the child the pleasure of attaining large results from small effort. As his powers grow, it will be an added pleasure to him later to do more difficult work, as by converting abstract language into a diagram, or by supplying the definite hypothesis and conclusion. Step by step he will naturally grow into the power of dealing with triangles that overlap, of proving their unknown parts equal, and of devising auxiliary lines by which to obtain a demonstration. At the end of a month's study of geometry in this way, ninety per cent of a class proved the following in a written test with the greatest zest and pride: Given $AB$ and $CD$, two lines intersecting at $O$, $AC$ parallel to $DB$ and $AO=CO$; prove $OD=OB$. This is a result far beyond that ever obtained by the same teacher by use of the old forcing process, when the beginning in the study was made in a much more difficult and formal way.

The first lesson on originals may comprise Exs. 1–7 in Group 9. In the next lesson Exs. 8–11 may be given along with Prop. IV.

**8. Typical Recitation on Originals.**—In order to obtain the best results the typical recitation as described in § 3 should be modified when teaching originals. At the beginning of a recitation on originals, when the written work done by pupils during the last fifteen minutes of the preceding recitation is returned with corrections to the class, there should be some discussion of this work. When this discussion is concluded, the proof of the originals assigned as the advance lesson should be written on the blackboard by the pupils who have succeeded in solving them. While this is being done, the part of the class at their seats may be employed in oral review of originals previously studied, or in the study of originals which are to form part of the next lesson. After the work written on the blackboard by pupils has been discussed, and the next lesson explained and assigned, during the last fifteen minutes on, say, alternate days, have the class work from three to five originals at the board. The largest number of pupils which can be handled usually in this way is about twenty-five. If some of the pupils finish before others, sometimes have them help backward pupils that are in difficulty.

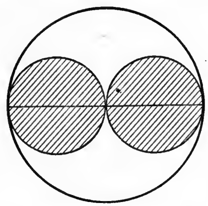
On other days, during the last fifteen minutes, have pupils do written work at their seats. When a pupil in this written work makes some glaring fallacy, or devises some specially meritorious proof, it is well to call attention to these facts in returning the corrected papers at the beginning of the next recitation. Thus the teachers may draw the diagram on the blackboard and ask the pupils to state his proof. When different proofs have been devised for the same theorem, it is peculiarly instructive and stimulating to the class to have all of these proofs presented. See for instance, the different proofs for Ex. 4, p. 82, or for Ex. 7, p. 146.

The above methods of teaching originals, with modifications which will suggest themselves to the thoughtful teacher may be used throughout the whole subject of plane geometry.

It may be well, however, to make a brief statement concerning special drill in preparation for the theorem that the sum of the angles of an $n$-gon is $(2n-4)$ right angles, the proof of which is difficult for the average pupil. In the next lesson after proving that the sum of the angles of a triangle is two right angles, draw a quadrilateral on the board and ask for the sum of its angles; a day or two later ask for the sum of the angles of a pentagon, and of a hexagon; develop the fact that an $n$-gon can be divided into $(n\text{-}2)$ triangles, and possibly by informal discussion develop the formula $(2n\text{-}4)$ right angles for the sum of the angles of an $n$-gon. Also exercises like the following may be given. Draw a triangle, denote each of its sides by $a$ and one of its angles by $x$, and ask the pupil to determine the number of degrees in $x$. Also draw an appropriate pentagon (equiangular, not equilateral), denote each of its angles by $x$, and ask the pupil to determine the number of degrees in $x$. This instance shows how a series of originals may be made to lead up to an important theorem, instead of having the reverse relation of following the theorem as applications of it.

**9. Teaching the Practical Applications of Geometry.**— The principle of giving much easy work at first and utilizing natural growth processes also applies when instructing a class in the applications of geometry. Feed pupils simple work at first, and they will naturally grow into the power of doing more difficult work. It is worse than useless to present at first abstruse or technical illustrations of the utility of geometry, involving scientific or engineering principles quite outside the scope of the pupil's past experience. Even an example like Ex. 2, p. 105, if used too early, takes up considerable time in its explanation and discourages and repels by the difficulty and unfamiliarity in its application as an aid in measuring the velocity of light. The thing to do is to give an abundance of simple applications, till the pupil becomes interested and begins to observe and suggest like applications for himself, and finally grows into an appreciation of more difficult cases. Illustrations of the simple applications to be given at first are found in Ex. 10, p. 28; Ex. 5, p. 35; Ex. 10, p. 36; Ex. 1, p. 82; Ex. 3, p. 106; Ex. 20, p. 86 and Ex. 4, p. 106; Exs. 12 and 13, p. 132; Exs. 1–4, Group 38; Ex. 3, p. 160; Exs. 1–7, p. 175; Ex. 16, p. 177; Ex. 7, p. 186; Ex. 9, p. 197; Exs. 1, 2, 3, p. 218; Ex. 13, p. 234, etc.

Of particular value are illustrations which the teacher can give from his or her own experience, or which are visible in the pupil's own life, though hitherto unobserved. The following is an instance which I have often used. Formerly water was supplied through a one-inch pipe to the house in which I live. Later the authorities decided to replace this by a two-inch pipe. When I saw the plumber putting in the larger pipe, I said to him, "That is good; now we shall have at least four times as much water." "No," he said, "you will have twice as much." I took a sheet of paper and drew on it a two-inch circle and, inside this larger circle, two smaller circles each one inch in diameter and not overlapping, thus showing that the area of the larger was



more than twice that of one of the smaller circles. On seeing this, he said, "I guess that I had better go to a night school and learn geometry."

The following are additional simple and personally observed illustrations of the practical value of geometric principles:

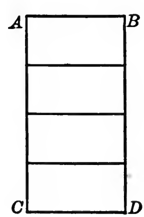
In a house heated by hot air, there was at one time something wrong with the furnace and pipes. A tinsmith was called in and his first move was to determine the areas of the cross-sections of the various hot, air pipes, in order to compare the sum of the areas with the area of the cross-section of the box supplying fresh air from the outside. The ends of the pipes being inaccessible, with a tape he measured the circumference of each pipe and from the length of the circumference deduced the area of the pipe. This shows the value of a formula for the area of a circle in terms of the circumference, or better, of a numerical table from which an area may be obtained when a circumference is known.

Or again, a teacher who was going through a certain village happened to see a boy climbing a tree and carrying a stout twine up with him. On asking the reason for this, he was told that a bet had been made as to the height of the tree and the boy was trying to measure the height by carrying a twine to its top and then measuring the length of the twine thus used. The teacher showed him that this labor might be saved by measuring the length of the shadow of the tree and also the height and shadow of a nearby post.

Or take this problem given the writer by a sergeant who had to solve it while scouting in command of a small patrol in France during the Great War. He saw a column of the enemy moving at a distance near a building which he knew to be 120 ft. high. He knew that the palm of his hand was 4 in. wide and that the length of his arm from shoulder to the bending place of his wrist was two feet. Using these facts, how could he make an approximate estimate of the distance of the enemy?

Similarly, each teacher can collect a series of personal incidents illustrating in a fundamental way, yet one that takes little time or effort to discuss or the pupil to assimilate, all of the leading principles of plane geometry.

Sometimes it is advantageous to elaborate at some length an illustration given in the text. Thus, in continuation of Ex. 20, p. 86, draw on the board a diagram of a pentagon made of rods hinged at the vertices, and ask how many diagonal rods must be inserted in order to make the figures rigid. Ask whether, if all possible diagonals were inserted and fastened together at all possible points, the figure would be made still more rigid. This naturally raises the question as to how many different diagonals a pentagon has. Treat a hexagon in like manner. Again,




ask whether the above figure $ABCD$ made of jointed rods ($AB$ and $DC$ each being one rod) is rigid; and if not, how many diagonal rods must be inserted, and where and why, to make it rigid.

Also ask the members of the class when next travelling on the railroad or in an automobile, to observe whether the trusses of all bridges and the steel framework of buildings being erected are not made up of beams and rods arranged so as to form a network of triangles. If some of the pupils later report that they have observed frameworks which were in part composed of quadrilaterals, this opens the way to ask whether the engineer has shown the highest type of skill, and whether the framework would not be stronger if cross rods had been inserted so as to divide all quadrilaterals into triangles.

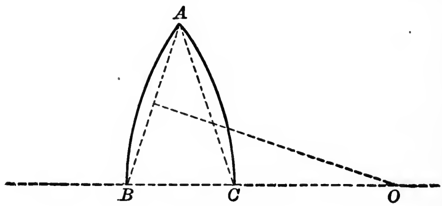
**10. Ornamental Designs.**—Similarly in drawing ornamental designs, such as the tracery of stained-glass windows and architectural outlines, most of the designs usually given at first are too complicated. It is important at the outset to have the pupil draw many simple figures till he becomes interested and develops the capacity and desire to attack the more elaborate ones.

In this connection it is a great help to have pupils draw their first ornamental or architectural figures by the aid of squared paper. In doing this, pupils are stimulated by the fact they get large results with small expenditure of time or effort. Also by noting that a simple array of parallel and perpendicular lines can be so useful, they get a new insight into the utility of geometric figures in general. This kind of drawing also has a distinct vocational value, since squared paper is being used more and more as an aid in commercial designing of artistic forms.

The following are examples of the simple ornamental designs which may be used at first: Ex. 11, p. 36; Ex. 6, p. 39; Ex. 7, p. 57; Ex. 6, p. 74; Ex. 7, p. 82; Ex. 5, p. 117; Exs. 6 and 9, p. 120, etc.

When desirable, as when members of a given class are especially interested in ornamental designing, some of the above examples may be extended so as to form other similar designs. Thus after a class has constructed a trefoil in the manner described in Ex. 6, p. 120, it may be asked to draw a square and on it construct a quatrefoil. And later to make regular hexagons and pentagons (see Ex. 12, p. 273) and by use of these to construct sexfoils and cinquefoils.

Also, after the equilateral gothic arch has been made in the manner shown in Ex. 5, p. 117, the pupil may be taught to draw lancet Gothic arches by the aid of isosceles triangles in the way shown in the diagram




where $O$ is the center of the arc $AB$. If the pupil is required to prove that the tangent to the arc $AB$ at $B$ is perpendicular to the base $BC$, it will make the exact form of the figure clearer.

**11. Study of Numerical Exercises.**—Much that has just been said concerning the teaching of original exercises applies also to the teaching of numerical exercises. One or two additional remarks may, however, be made. The lack of power on the part of the average pupil to see a numerical computation as a whole and to make short cuts, to abbreviate work by cancellation, for instance, is something extraordinary. Some of the principal methods of shortening work are mentioned in the textbook, pp. 282–283. It requires, however, constant watchfulness and insistence on the part of the teacher to have the pupil use these short cuts till they become natural and instinctive.

Require, then that the pupil draw a figure for every numerical example, that he denote the unknown number whose value is sought by $x$ or an appropriate symbol. (Do this even in such examples as Exs. 1, 2, 3, etc., p. 284, so as to fix the habit.) Teach him to group or indicate together all the processes involved in an example; to use cancellation whenever possible (see Exs. 19, 20, etc., p. 285, etc.); and to save labor by suspending operations.

Thus in Ex. 5, p. 287, find a side of the triangle to be $\sqrt{\frac{256}{3}}$ but do not reduce this result, since our object is to find the area, and the labor of root extraction at this stage of the work would be worse than lost. Also in Ex. 24, p. 287, let $x =$ a side of square and find $x^2 = 200$, and use this, but do not find $x$ and square its value again in order to find $x^2$.

To enable some classes to obtain mastery of these short-cut processes it may be advisable for the teacher to take these labor-saving principles one by one and to dictate a group of supplementary exercises for each principle, every example in the group exemplifying the principle in hand.

It is also a matter of some importance to teach pupils to make a record of results, if such results are likely to be of use later on.

Thus teach them to record on a fly leaf at the end of the book, and have ready for future use such results as the numerical values of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc.; or of $\sqrt[3]{2}$, $\sqrt[3]{3}$, etc., in Solid Geometry. This leads to an important saving of time, and at the same time instills a valuable educational principle.

In all the earlier work with numerical examples, the pupil should not be permitted to leave his final results in the radical form. (Thus for Ex. 10, p. 284, the answer $3\sqrt{3}$ should not be accepted; the pupil should be required to carry it out to the form 5.196+.) After the pupil has acquired some power to make these reductions in the most advantageous way, time may be saved by allowing some results to remain in the radical form.

**12. Algebra an Aid in the Study of Geometry.**—When branches of mathematics are studied as separate subjects it is highly valuable to have a cumulative unification between them. Hence, when each new branch of mathematics is taken up, it is important to review and unify all preceding branches in connection with it. Thus, when studying algebra it is desirable to review arithmetic as thoroughly as possible. To do this not only makes the meaning and uses of algebra clearer, but also fixes arithmetical processes in mind and gives them new extensions and useful aspects.

Similarly on taking up the study of geometry, the two branches of mathematics previously studied: viz., arithmetic and algebra, should be reviewed and applied in every practical way.

Thus, one of the parts of arithmetic most apt to be forgotten is the topic of decimal fractions. Hence, this subject should be made prominent in the numerical examples given in the applications of geometry.

While the number of different algebraic processes that can be advantageously employed as aids in the study of geometry is not large, the particular forms usable are notably valuable in that they give room and call for initiative and inventiveness on the part of the pupil and cultivate the spirit of algebra which is after all more valuable to the average student than an extended knowledge of the technique of the subject.

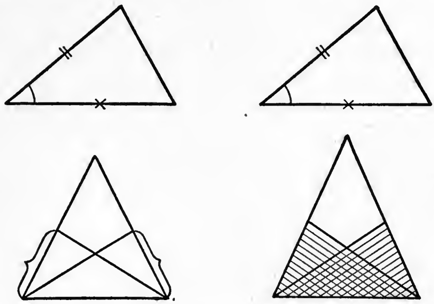
Among the algebraic processes which should be employed whereever possible, both as an aid in the rapid and thorough mastery of geometry, and as a training in algebra itself, are the following:

(1) The use of a single letter to represent an angle, line segment, or even a triangle. As examples of what is meant, see the use of *e*, *f*, *g*, on the diagram on p. 29; of the letter *a* on p. 262; of *I*, *II*, *III*, on p. 236. It is curious how reluctant, or at least inactive, pupils are in this matter, in spite of its manifest advantages.

(2) The use of marks or symbols other than letters in indicating corresponding parts of figures. See the figures on the next page. In the same connection may be introduced the shading of over-lapping triangles in order to discriminate them clearly from the rest of the figure.

If the matter be followed up, it will be found that the number of ways in which symbolisms, formal and informal, can be improvised and used to advantage is remarkably large.

(3) The use of $x$, or of some other letter, to represent an unknown magnitude, as in Ex. 24, p. 20; Ex. 8, p. 99, etc. (See also the use of $x$ in solving numerical examples, §11.)



(4) The use of the equation, or of two simultaneous equations as an aid in solving a problem. See Ex. 24, p. 20; Ex. 8, p. 99, etc.

Note that Ex. 24, p. 20, and other problems may be solved by the use of simultaneous equations.

(5) Transformations of the formulas of geometry to obtain new results.

**13. Deficiency Study.**—Before we conclude, it may be well to say a word about methods of bringing up the work of slow or negligent students. It is a growing custom among schools to make the ordinary recitation period during the day slightly shorter (to reduce it say from 45 minutes to 40 minutes) and thus to provide time for an extra period at the end of the day in which special attention may be given to laggards of any kind. This we may call the period for "**Deficiency Study.**"

There are many different ways in which this period can be utilized in teaching geometry. One good way both for those pupils who do not know how to study and for those who have neglected their work is to take the propositions in which the pupils need to be drilled (say Book I, Props. 16–25), divide them up in pairs (Props. 16–17, 18–19, 20–21, etc.), require each pupil to study the first pair till he thinks he understands them (giving such help as may be wise), and then require the pupil to write out one of the pair. If the mistakes made are serious, require the two propositions to be studied and one of them written out again. So continue till all the pairs have been treated in like manner. Drill in original exercises may be given similarly.

By means of this method of work in "deficiency study" quite a large number of pupils may receive individual attention without overtaxing the teacher. Competition among the pupils as to who can make most rapid progress and complete the work soonest will also be a material aid in the work. The method is also useful to the teacher in making clear what pupils have been dawdling over or neglecting the work of preparation. If a pupil in deficiency study can learn the demonstrations of six theorems in 40 minutes, and comes next day to class with only one proposition prepared, it is clear that the pupil's deficiency is not due to mental dullness but to lack of application, and the teacher may thus know how to proceed.

**14. The Study of Solid Geometry.**—The remarks which have been made concerning the study of Plane Geometry, apply with slight modification to Solid Geometry. After the teacher has tested the suggestions made with respect to the former subject, most of the modifications necessary in the study of Solid Geometry will occur to him without special mention here.

Much time, however, may be saved for classroom work in Solid Geometry by requiring the pupil to draw on paper all the diagrams occurring in the advance lesson and to hand them in as a part of the preparation of the lesson. The familiarity with the diagrams thus attained not only facilitates class room work, but is also an important discipline in itself. If the time at the teacher's disposal is small, most of the original exercises on construction problems may be omitted, and formal proofs may be omitted in the work on loci.

The least amount of time in which the subject of Solid Geometry can be mastered with any degree of thoroughness is a half year with five recitations a week.

In the pamphlet entitled "Suggestions Concerning the Teaching of Algebra with Especial Reference to the Use of Durell and Arnold's Algebras," certain other details will be found which may also be utilized in the teaching of geometry.