RAY'S MATHEMATICAL SERIES.

SURVEYING

AND

NAVIGATION,

WITH A PRELIMINARY TREATISE ON

TRIGONOMETRY AND MENSURATION,

BY

A. SCHUYLER, M. A.,

Professor of Applied Mathematics and Logic in Baldwin University; Author of Higher Arithmetic, Principles of Logic, and Complete Algebra.

VAN ANTWERP, BRAGG & CO.,

137 WALNUT STREET, CINCINNATI.

28 BOND STREET, NEW YORK.

RAY'S SERIES,

EMBRACING

A Thorough and Progressive Course in Arithmetic, Algebra, the Higher Mathematics.

Primary Arithmetic.	**Higher Arithmetic.**
Intellectual Arithmetic.	**Test Examples in Arithmetic.**
Rudiments of Arithmetic.	**New Elementary Algebra.**
Practical Arithmetic.	**New Higher Algebra.**

Plane and Solid Geometry. By ELI T. TAPPAN, A.M., *Pres't Kenyon College.* 12*mo, cloth*, 276 *pp.*

Geometry and Trigonometry. By ELI T. TAPPAN, A.M. *Pres't Kenyon College.* 8*vo, sheep*, 420 *pp.*

Analytic Geometry. By GEO. H. HOWISON, A.M., *Prof. in Mass. Institute of Technology.* Treatise on Analytic Geometry, especially as applied to the Properties of Conics: including the Modern Methods of Abridged Notation.

Elements of Astronomy. By S. H. PEABODY, A.M., *Prof. of Physics and Civil Engineering, Amherst College.* Handsomely and profusely illustrated. 8*vo, sheep*, 336 *pp.*

KEYS.

Ray's Arithmetical Key (*To Intellectual and Practical*);
Key to Ray's Higher Arithmetic;
Key to Ray's New Elementary and Higher Algebras.

The Publishers furnish Descriptive Circulars of the above Mathematical Text-Books, with Prices and other information concerning them.

PREFACE.

Nearly twenty years ago the Publishers made the following announcement: "*Surveying and Navigation;* containing Surveying and Leveling, Navigation, Barometric Heights, etc."

To redeem this promise, the present work now appears.

It is customary to preface works on Surveying by a meager sketch of Plane Trigonometry, but it has been thought best to include in this work a thorough treatment of Plane and Spherical Trigonometry and Mensuration. These subjects have been treated in view of the wants of our best High Schools and Colleges.

Certain modern writers have defined the Trigonometric functions as ratios; for example, in a right triangle, the sine of an angle is the ratio of the opposite side to the hypotenuse, etc.

The historical method of considering the sine, co-sine, tangent, etc., as linear functions of the arc, explains the origin of these terms—avoids the ambiguity of the word *ratio;* explains how the logarithm of the sine, for example, can reach the limit 10, which would be impossible if the limit of the sine itself is 1, and is much more readily apprehended by the student.

The advantages in analytic investigations resulting from defining these functions as ratios have been secured in the principles relating to the Right Triangle, Art. 64.

Each of the circular functions has, in the first place, been considered by itself, and its value traced, for all arcs, from 0° to 360°.

Then follows the solution of triangles, right and oblique, the general relations of the circular functions, the functions of the sum or difference of two angles, and a variety of interesting practical applications.

It is hoped that Spherical Trigonometry has been made intelligible to the diligent student. More than ordinary care has been given to the development of Napier's principles, and to the discussion of the species of the parts of both right and oblique spherical triangles, Arts. 126, 129, 145, 148, 151.

Mensuration, a subject at once interesting and practically important, has been discussed at length, and formulas have been developed instead of rules for the solution of the problems.

In the Surveying, the instruments are first represented and described, and the methods of making the adjustments given in detail.

The Author takes this opportunity to express his obligations to Messrs. W. & L. E. Gurley, Manufacturers of Surveying and Engineering Instruments, Troy, N. Y., who have kindly granted him the use of their Manual for the delineation and description of the instruments. In consequence of this courtesy, much better drawings and descriptions have been made than would otherwise have been possible.

The instruments themselves should, however, be accessible to the student, who should study them in connection with the descriptions in the book, and learn to use them in practical work, guided by a competent instructor.

The Rectangular method of surveying the Public lands, now brought to great perfection under the direction of the Government, has been minutely explained, and illustrated by field notes of actual surveys. In this portion of the work, the United States Manual of Surveying Instructions has been taken as authority, and thus the authorized methods, which must form the basis for subsequent surveys, have been made accessible to the student.

The methods of finding the true meridian and the variation of the needle have been given at length; also specific directions for finding corners, taking bearings, measuring lines, recording field notes, and plotting.

In addition to the ordinary method of finding the area, a new method, developed by E. M. Pogue, of Kentucky, is given in Art. 304. This method has the merit of giving always a uniform result from the same field notes, and thus avoids disputes about the different results of the ordinary method, unavoidably attending the various distribution of errors by different calculators.

The methods of supplying omissions are explained and illustrated by examples.

Laying out and dividing land, operations admitting of an unlimited variety of applications, have been treated in view of the wants of the practical surveyor. The subject is also full of interest to the student, who can not fail to receive from it new views of the resources of mathematical science.

Leveling, the construction of railroad curves, embankments and excavations, the method of making Topographical surveys, with the authorized conventional symbols, Barometric heights, etc., have been explained and illustrated by diagrams and examples.

It has been thought best to give a clear, elementary treatment of Navigation, not only on account of those who may desire to pursue the subject further, but for the sake of gratifying the wishes of intelligent persons who may desire to know something of Navigation. The limits of the work, however, forbid the discussion of Nautical Astronomy. The examples in Navigation have been selected from the English work of J. R. Young.

The tables of Logarithms, Natural and Logarithmic sines, etc., have been carried only to five decimal places, and for the purposes intended will be found practically better than tables to six or seven places.

The Traverse table has been thrown into a new form, at once condensed and convenient.

These tables have been compiled by Mr. Henry H. Vail, and by him compared with Babbage's and Wittstein's tables, then by the Author with Vega's tables to seven decimal places. It is hoped that by this double comparison perfect accuracy has been attained.

The table of Meridional Parts, taken from "Projection Tables for the use of the United States Navy," prepared by the Bureau of Navigation, and issued from the Government Printing office, was calculated in the Hydrographic office for the terrestrial spheroid, compression $\frac{1}{299.1528}$. This table, now for the first time published in a text-book, is believed to be more correct than those in general use.

The Author takes pleasure in acknowledging his obligations to Prof. E. H. Warner for critical suggestions and acceptable aid in reading proof and testing the accuracy of the answers.

With the hope that the book will be attractive and useful to the student, teacher, and practical surveyor, it is sent forth to accomplish its work.

A. SCHUYLER.

BALDWIN UNIVERSITY,
BEREA, O., June 12, 1873.

INDEX.

INTRODUCTION.

LOGARITHMS.

1. Definition.

A logarithm of a number is the exponent denoting the power to which a fixed number, called the base, must be raised in order to produce the given number.

Thus, in the equation, $b^x = n$, b is the base of the system, n is the number whose logarithm is to be taken, and x is the logarithm of n to the base b, which may be written: $x = \log_b n$.

Any positive number, except 1, may be assumed as the base, but when assumed, it remains fixed for a system; hence, there may be an infinite number of systems, since there may be an infinite number of bases.

2. Common Logarithms.

Common logarithms are the logarithms of numbers in the system whose base is 10.

$$10^0 = 1; \quad \therefore \text{ by def., } \log 1 = 0.$$
$$10^1 = 10; \quad \therefore \text{ by def., } \log 10 = 1.$$
$$10^2 = 100; \quad \therefore \text{ by def., } \log 100 = 2.$$
$$10^3 = 1000; \quad \therefore \text{ by def., } \log 1000 = 3.$$

. . . .

Hence, *In the common system, the logarithm of an exact power of* 10 *is the whole number equal to the exponent of the power.*

3. Consequences.

1. If the number is greater than 1 and less than 10, its logarithm is greater than 0 and less than 1, or is 0 + a decimal.

2. If the number is greater than 10 and less than 100, its logarithm is greater than 1 and less than 2, or is 1 + a decimal.

3. In general, if the number is not an exact power of 10, its logarithm, in the common system, will consist of two parts—an entire part and a decimal part.

The entire part is called the *characteristic* and the decimal part is called the *mantissa.*

4. Problem.

To find the laws for the characteristic.

Let (1) $10^x = n$; then, by def., $\log n = x$.
But (2) $10^1 = 10$.

$$(1) \div (2) = (3)\ 10^{x-1} = \frac{n}{10};\ \text{then, by def.,}\ \log \frac{n}{10} = x - 1.$$

$$\therefore\ \log \frac{n}{10} = \log n - 1.$$

Hence, *The logarithm of the quotient of any number by* 10 *is less by* 1 *than the logarithm of the number.*

Let us now take the number 8979 and its logarithm 3.95323, as given in a table of logarithms, and divide the number successively by 10, and for each division subtract 1 from the logarithm of the dividend, then we have,

Log	8979 = 3.95323.	Log	.8979 = $\bar{1}$.95323.
"	897.9 = 2.95323.	"	.08979 = $\bar{2}$.95323.
"	89.79 = 1.95323.	"	.008979 = $\bar{3}$.95323.
"	8.979 = 0.95323.		. . . .

The minus sign applies only to the characteristic over which it is placed.

The mantissa is always positive, and is the same for all positions of the decimal point.

An inspection of the above will reveal the following laws:

1. *If the number is integral or mixed, the characteristic is positive and is one less than the number of integral figures.*

2. *If the number is entirely decimal, the characteristic is negative and is one greater, numerically, than the number of 0's immediately following the decimal point.*

5. Exercises on the Characteristic.

1. What is the characteristic of the logarithm of 7?

2. What is the characteristic of the logarithm of 465?

3. What is the characteristic of the logarithm of 4678?

4. What is the characteristic of the logarithm of 34.75?

5. What is the characteristic of the logarithm of .65?

6. What is the characteristic of the logarithm of .0789?

7. What is the characteristic of the logarithm of .00084?

8. If the characteristic of the logarithm of a number is 2, how many integral places has that number?

9. If the characteristic of the logarithm of a number is 5, how many integral places has that number?

10. If the characteristic of the logarithm of a number is 1, how many integral places has that number?

11. If the characteristic of the logarithm of a number is 0, how many integral places has that number?

12. If the characteristic of the logarithm of a number is negative, is the number integral, decimal, or mixed?

13. If the characteristic of the logarithm of a number is $\overline{4}$, how many 0's immediately follow the decimal point?

14. If the characteristic of the logarithm of a number is $\overline{2}$, how many 0's immediately follow the decimal point?

15. If the characteristic of the logarithm of a number is $\overline{1}$, how many 0's immediately follow the decimal point?

TABLE OF LOGARITHMS.

6. Description of the Table.

The table of logarithms annexed gives the mantissa of the logarithm of every number from 1000 to 10900. The characteristic can be found by the preceding laws.

It follows, from Art. 4, that the mantissa of the logarithm of a number is the same as the mantissa of the logarithm of the product or quotient of that number by any power of 10. Thus:

$$\begin{aligned} \text{Log } 12 &= 1.07918. \\ \text{“} \quad 120 &= 2.07918. \\ \text{“} \quad .012 &= \overline{2}.07918. \end{aligned}$$

Hence, we can determine from the table the logarithm of any number less than 1000. Thus, the mantissa of the logarithm of 8 is the same as that of the logarithm of 8000.

In the table, the first three or four figures of each number are given in the left-hand column, marked *N*. The next figure is given at the head and foot of one of the columns of mantissas.

The mantissas, in the column under 0, are given to five decimal places. The first and second decimal figures of this column are understood to be repeated in the spaces below, and to be prefixed, across the page, to the three figures of the remaining columns.

When the third decimal digit changes from 9 to 0, the second is increased by the 1 carried; and the corresponding mantissa, and all to the right, commence with a smaller figure, to indicate that the first two decimal figures, to be prefixed, are to be taken from the line below.

The last column, marked *D*, contains the difference of two successive mantissas, called the *tabular difference.*

7. Problem.

To find the logarithm of a given number.

1. Find the logarithm of 3675.

The characteristic is 3. Opposite 367, in the column headed *N*, and under the column headed 5, we find 526, to which prefix the two figures, 56, in the column headed 0, and we have for the mantissa .56526.

$$\therefore \log 3675 = 3.56526.$$

2. Find the logarithm of 76.

The characteristic is 1, and the mantissa is the same as that of 7600, which is .88081.

$$\therefore \log 76 = 1.88081.$$

3. Find the logarithm of .004268.

The characteristic is $\bar{3}$, and the mantissa is the same as that of 4268. Looking opposite 426, and under 8, we find 022, of which the 0 is a small figure. Prefixing

63, from the line below, in the column headed 0, we have for the mantissa .63022.

$$\therefore \log .004268 = \bar{3}.63022.$$

4. Find the logarithm of 109684.

The characteristic	$=5.$	
The mantissa of log 1096	$=$	.03981
Tab. diff. is 40; and $40 \times .84 =$		34
	log 109684 $=$	5.04015

The reason for multiplying the tabular difference by .84 will be apparent from the following:

$$\log 109600 = 5.03981.$$
$$\log 109700 = 5.04021.$$

The difference of the logarithms is 40 hundred-thousandths, and the difference of the numbers is 100; but the difference of 109600 and 109684 is 84, which is .84 of 100; hence, the difference of the logarithms of 109600 and 109684 is .84 of 40 hundred-thousandths, which is 40 hundred-thousandths $\times .84 = 34$ hundred-thousandths, nearly.

It is assumed that the difference of the logarithms of two numbers is proportional to the difference of the numbers, which is approximately true, especially if the numbers are large.

5. Find the logarithm of 123.613.

The characteristic	$=2.$	
The mantissa of log 1236	$=$	.09202
Tab. diff. is 35; and $35 \times .13 =$		5
	$\therefore$ log 123.613 $=$	2.09207

The tabular difference is .00035, and $.00035 \times .13 = .0000455$. But since the logarithms in this table are taken only to five decimal places, the two last figures, 55, are rejected, and 1 is carried to .00004, making .00005 for the correction.

In general, when the left-hand figure of the part rejected exceeds 4, carry 1.

When the tabular difference is large, as in the first part of the table, there may be small errors. Accordingly, for numbers between 10000 and 10900, it will be better to use the last two pages instead of the first page.

8. Rule.

1. *If the number, or the product of the number by any power of* 10, *is found in the table, take the corresponding mantissa from the table, and prefix the proper characteristic.*

2. *If the number, without reference to the decimal point or* 0's *on the right, is expressed by more than five figures, take from the table the mantissa corresponding to the first four or five figures on the left, multiply the corresponding tabular difference by the number expressed by the remaining figures, considered as a decimal, reject from the product as many figures on the right as are in the multiplier, carrying to the nearest unit, and add the result as so many hundred-thousandths to the mantissa before found, and to the sum prefix the proper characteristic.*

9. Examples.

1. What is the logarithm of 2347? *Ans.* 3.37051.
2. What is the logarithm of 108457? *Ans.* 5.03526.
3. What is the logarithm of 376542? *Ans.* 5.57581.
4. What is the logarithm of 229.7052? *Ans.* 2.36117.
5. What is the logarithm of 1128737? *Ans.* 6.05260.
6. What is the logarithm of .30365? *Ans.* $\bar{1}.48237$.
7. What is the logarithm of .0042683? *Ans.* $\bar{3}.63025$.
8. What is the logarithm of 1245400? *Ans.* 6.09531.

10. Problem.

To find the number corresponding to a given logarithm.

1. What number corresponds to logarithm $\overline{2}.03262$?

The mantissa is found in the column headed 8, and opposite 107 in the column headed N. Hence, without reference to the decimal point, the number corresponding is 1078; but since the characteristic is $\overline{2}$, the number is entirely decimal, and one 0 immediately follows the decimal point. Hence, the number corresponding is .01078.

2. What number corresponds to logarithm 2.83037?

Since this logarithm can not be found in the table, take the next less, which is 2.83033, and the corresponding number, without reference to the decimal point, which is 6766.

The difference between the given logarithm and the next less is 4, and the tabular difference is 6, which is the difference of the logarithms of the two numbers, 6766 and 6767, whose difference is 1.

If the tabular difference of the logarithms, 6, corresponds to a difference in the numbers of 1, the difference of the logarithms, 4, will correspond to a difference of $\frac{4}{6}$ of 1; which, reduced to a decimal, and annexed to 6766, will give for the number, without reference to the decimal point, 676666. But since the characteristic is 2, there will be three integral places; hence, 676.666 is the number required.

3. What number corresponds to logarithm 2.76398?

$$\begin{array}{ll} \text{The given log} = 2.76398 & \therefore \text{ number} = 580.737 \\ \text{Next less log } = \underline{2.76395} & \therefore \text{ number} = 580.7 \end{array}$$

$$\text{Tab. difference} = 8\underline{)300} = \text{difference.}$$

$$37 = \text{correction.}$$

It is necessary to write only that part of the next less logarithm which differs from the given logarithm. Conceive 0's annexed to the difference, and divide by the tabular difference; and annex the quotient to the number corresponding to the next less logarithm.

In practical work abbreviate thus: Let l denote the given logarithm; l', the next less logarithm; n and n', the corresponding numbers; t, the tabular difference; d, difference of logarithms; c, the correction.

4. What number corresponds to logarithm $\bar{1}.73048$?

$$l = \bar{1}.73048 \quad \therefore \; n = .537625$$
$$l' = \bar{1}.73046 \quad \therefore \; n' = .5376$$
$$t = 8)\,2 = d.$$
$$25 = c.$$

n' is found first, then n by annexing c.

11. Rule.

1. *If the given mantissa can be found in the table, take the number corresponding, and place the decimal point according to the law for the characteristic.*

2. *If the given mantissa can not be found in the table, take the next less and the corresponding number. Subtract this mantissa from the given mantissa, annex 0's to the remainder, divide the result by the tabular difference, annex the quotient to the number corresponding to the logarithm next less than the given logarithm, and place the decimal point according to the law for the characteristic.*

12. Examples.

1. What number corresponds to logarithm 4.55703? *Ans.* 36060.

2. What number corresponds to logarithm 3.95147? *Ans.* 8942.8.

3. What number corresponds to logarithm $\bar{2}.41130$? *Ans.* .025781.

4. What number corresponds to logarithm $\bar{1}.48237$?

Ans. .30365.

5. What number corresponds to logarithm $\bar{3}.63025$?

Ans. .0042683.

MULTIPLICATION BY LOGARITHMS.

13. Proposition.

The logarithm of the product of two numbers is equal to the sum of their logarithms.

Let $\begin{cases} (1)\ b^x = m; \text{ then, by def., } \log m = x. \\ (2)\ b^y = n; \text{ then, by def., } \log n = y. \end{cases}$

$(1)\times(2) = (3)\ b^{x+y} = mn;$ then, by def., $\log mn = x + y.$

$$\therefore \log mn = \log m + \log n.$$

14. Rule.

1. *Find the logarithms of the factors and take their sum, which will be the logarithm of the product.*

2. *Find the number corresponding which will be their product.*

15. Examples.

1. Find the product of 57846 and .003927.

$\log 57846 = 4.76228$
$\log .003927 = \bar{3}.59406$
$\log \text{product} = 2.35634, \quad \therefore \text{product} = 227.16.$

2. Find the product of 37.58 and 75864.

Ans. 2851000.

3. Find the product of .3754 and .00756.

Ans. .002838.

4. Find the product of 999.75 and 75.85.

Ans. 75831.667.

5. Find the product of 85, .097, and .125. *Ans.* 1.03062.

DIVISION BY LOGARITHMS.

16. Proposition.

The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.

$$\text{Let} \begin{cases} (1) \quad b^x = m; \text{ then, by def., } \log m = x. \\ (2) \quad b^y = n; \text{ then, by def., } \log n = y. \end{cases}$$

$$(1) \div (2) = (3) \quad b^{x-y} = \frac{m}{n}; \text{ then, by def., } \log \frac{m}{n} = x - y.$$

$$\therefore \quad \log \frac{m}{n} = \log m - \log n.$$

17. Rule.

1. *Find the logarithms of the numbers, subtract the logarithm of the divisor from the logarithm of the dividend, and the remainder will be the logarithm of the quotient.*

2. *Find the number corresponding which will be the quotient.*

18. Examples.

1. Divide 73.125 by .125.

$$\begin{aligned} \log \quad 73.125 &= 1.86407 \\ \log \quad .125 &= \bar{1}.09691 \\ \hline \log \text{ quotient} &= 2.76716, \quad \therefore \quad \text{quotient} = 585. \end{aligned}$$

2. Divide 7.5 by .000025. *Ans.* 300000.

3. Divide 87.9 by .0345. *Ans.* 2547.824.

4. Divide .34852 by .00789. *Ans.* 44.171.

5. Divide 85734 by 12.7523. *Ans.* 6723.

ARITHMETICAL COMPLEMENT.

19. Definition.

The arithmetical complement of a logarithm is the result obtained by subtracting that logarithm from 10. Thus, denoting the logarithm by *l.*, and its arithmetical complement by *a. c. l.*, we shall have the formula,

$$a.\ c.\ l. = 10 - l.$$

The arithmetical complement of a logarithm is most readily found by commencing at the left of the logarithm, and subtracting each digit from 9 till we come to the last numeral digit, which must be subtracted from 10.

Thus, to find the *a. c.* of 3.47540, we say: 3 from 9, 6; 4 from 9, 5; 7 from 9, 2; 5 from 9, 4; 4 from 10, 6; 0 from 0, 0.

$$\therefore\ a.\ c. \text{ of } 3.47540 = 6.52460.$$

20. Proposition.

The difference of two logarithms is equal to the minuend, plus the arithmetical complement of the subtrahend, minus 10.

$$\text{For, } l - l' = l + (10 - l') - 10.$$

It is convenient to use the *a. c.* in division when either the dividend or the divisor is the indicated product of two or more factors. Thus, let it be required to find x in the proportion:

$$37.5 : 678.5 :: 27.56 : x; \quad \therefore\ x = \frac{678.5 \times 27.56}{37.5}.$$

$$\therefore\ \log x = \log 678.5 + \log 27.56 + a.\ c.\ \log 37.5 - 10.$$

$$\begin{aligned} \log 678.5 &= 2.83155 \\ \log 27.56 &= 1.44028 \\ a.\ c.\ \log 37.5 &= 8.42597 \\ \hline \log x &= 2.69780 \quad \therefore\ x = 498.656. \end{aligned}$$

21. Examples.

1. Given 125.5 : .0756 : : x : .0034532, to find x.
Ans. 5.7325.

2. Given 843 : x : : 732.534 : .759, to find x.
Ans. .87346.

3. Given x : .034 : : .784 : .00489, to find x.
Ans. 5.451125.

4. Given $x = \frac{32.015 \times .874}{.000216 \times 90257}$, to find x. *Ans.* 1.4353.

5. Given $.753 \times 12.234 : 87.5 \times 3.7547$: : 56.5 : x, to find x. *Ans.* 2014.96.

INVOLUTION BY LOGARITHMS.

22. Proposition.

The logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.

Let (1) $b^x = n$; then, by def., $\log n = x$.
$(1)^p = (2)$ $b^{px} = n^p$; then, by def., $\log n^p = px$.

$$\therefore \log n^p = p \log n.$$

23. Rule.

1. *Find the logarithm of the number and multiply it by the exponent of the power, and the product will be the logarithm of the power.*

2. *Find the number corresponding which will be the power.*

24. Examples.

1. Find the cube of .034.

(1) $\log .034 = \overline{2}.53148$
$(1) \times 3 = (2)$ $\log .034^3 = \overline{5}.59444$ $\therefore .034^3 = .000039305$.

2. Find the square of 25.7. *Ans.* 660.47.

3. Find the fourth power of .75. *Ans.* .3164.

4. Find the cube of 8.07. *Ans.* 525.55.

5. Find the fifth power of .9. *Ans.* .59047.

EVOLUTION BY LOGARITHMS.

25. Proposition.

The logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root.

Let (1) $b^x = n$; then, by def., $\log n = x$.

$\sqrt[r]{(1)} = (2)$ $b^{\frac{x}{r}} = \sqrt[r]{n}$; then, by def., $\log \sqrt[r]{n} = \frac{x}{r}$.

$$\therefore \log \sqrt[r]{n} = \frac{\log n}{r}.$$

26. Rule.

1. *Find the logarithm of the number, divide it by the index of the root, and the quotient will be the logarithm of the root.*

2. *Find the number corresponding which will be the root.*

27. Examples.

1. Extract the square root of .75.

(1) $\log .75 = \bar{1}.87506$

$(1) \div 2 = (2)$ $\log \sqrt{.75} = \bar{1}.93753$ $\therefore \sqrt{.75} = .86602.$

Scholium. $\bar{1}.87506 \div 2 = (\bar{2} + 1.87506) \div 2 = \bar{1}.93753.$

2. Extract the cube root of 91125. *Ans.* 45.

3. Find the value of $\frac{2}{5}\sqrt{5}$. *Ans.* .89443.

4. Extract the fifth root of .075. *Ans.* .59569.

5. Find the value of $\sqrt[3]{\frac{37.5 \times (.78)^2}{12.5 \times 5.9}}$. *Ans.* .676317.

TRIGONOMETRY.

28. Definition and Classification.

Trigonometry is that branch of Mathematics which treats of the solution of triangles.

Trigonometry is divided into two branches — *Plane* and *Spherical*.

PLANE TRIGONOMETRY.

29. Definition.

Plane Trigonometry is that branch of Trigonometry which treats of the solution of plane triangles.

30. Parts of a Triangle.

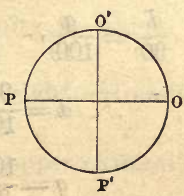
Every triangle has six parts—three sides and three angles.

If three parts are given, one being a side, the remaining parts can be computed.

If the three angles only are given, the triangle is indeterminate, since an infinite number of similar triangles will satisfy the conditions.

31. Sexagesimal Division of Angles and Arcs.

The horizontal diameter, $O\ P$, called the *primary diameter*, and the vertical diameter, $O'\ P'$, called the *secondary diameter*, divide the circumference into four equal parts, called quadrants.



$O\ O'$ is the *first quadrant*, $O'\ P$ the *second*, $P\ P'$ the *third*, and $P'\ O$ the *fourth*.

A degree is **one-ninetieth** of a right angle, or of a quadrant.

A minute is **one-sixtieth** of a degree.

A second is **one-sixtieth** of a minute.

Thus, $25° \; 34' \; 46''$ denote 25 degrees, 34 minutes, and 46 seconds.

An angle, whose vertex is at the center, has the same *numerical measure*, or contains the same number of degrees, minutes, and seconds, as the arc of the circumference intercepted by its sides.

32. Centesimal Division of Angles and Arcs.

A grade is **one-hundreth** of a right-angle, or of a quadrant.

A minute is **one-hundreth** of a grade.

A second is **one-hundreth** of a minute.

Thus, $7^g \; 24^{\backprime} \; 40^{\backprime\backprime}$ denotes 7 grades, 24 minutes, and 40 seconds.

$$1° = \frac{10^g}{9}, \quad 1' = \frac{50^{\backprime}}{27}, \quad 1'' = \frac{250^{\backprime\backprime}}{81}.$$

$$1^g = \frac{9°}{10}, \quad 1^{\backprime} = \frac{27'}{50}, \quad 1^{\backprime\backprime} = \frac{81''}{250},$$

Let d, m, s, respectively, denote an angle expressed in degrees, sexagesimal minutes and seconds, and let g, μ, σ, respectively, denote the same angle expressed in grades, centesimal minutes and seconds, then expressing the ratio of the angle to a right angle in each kind of units, we shall have:

$$\frac{d}{90} = \frac{g}{100}, \qquad \frac{m}{5400} = \frac{\mu}{10000}, \qquad \frac{s}{324000} = \frac{\sigma}{1000000}.$$

$$\therefore \; d = \frac{9}{10}g, \qquad m = \frac{27}{50}\mu, \qquad s = \frac{81}{250}\sigma,$$

$$\therefore \; g = \frac{10}{9}d, \qquad \mu = \frac{50}{27}m, \qquad \sigma = \frac{250}{81}s.$$

Let r denote the radius, and $\pi = 3.14159265358979 \ldots$

$\pi r =$ a semi-circumference $= 180° = 200^g =$ two right angles.

$\frac{\pi}{2} r =$ a quadrant $= 90° = 100^g =$ one right angle.

$2 \pi r =$ a circumference $= 360° = 400^g =$ four right angles.

If $r = 1$, the above expressions become, respectively, $\pi, \frac{\pi}{2}, 2\pi$.

33. Unit of Circular Measure.

The unit of circular measure is that angle at the center whose intercepted arc is equal in length to the radius.

Let u denote the unit of circular measure, and r the radius.

Then, since $\pi r =$ the semi-circumference, $\pi u = 180° = 200^g$.

$$u = \frac{180°}{\pi} = 57°.\ 29577951 \ldots = \frac{200^g}{\pi} = 63^g.\ 6619772 \ldots$$

Let d, g, c, respectively, denote the number of degrees, grades, and units of circular measure in an angle; then,

$$d = \frac{180}{\pi} c, \quad g = \frac{200}{\pi} c, \quad c = \frac{\pi}{180} d, \quad c = \frac{\pi}{200} g.$$

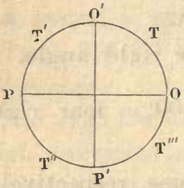
34. Origin, Termini and Situation of Arcs.

The origin of an arc is the extremity at which it begins.

The primary origin of arcs is at the right extremity of the primary diameter.

The secondary origin of arcs is at the upper extremity of the vertical diameter.

The terminus of an arc is the extremity at which it ends.



An arc is said to be situated in that quadrant in which its terminus is situated, thus:

The arc OT is in the first quadrant.

The arc $OO'T'$ is in the second quadrant.

The arc OPT'' is in the third quadrant.

The arc OPT''' is in the fourth quadrant.

35. Positive and Negative Arcs.

Positive arcs are those which are estimated in the direction contrary to that of the motion of the hands of a watch.

Negative arcs are those which are estimated in the same direction as that of the motion of the hands of a watch.

Thus, OT, OT', OT'', OT''', estimated to the left, are positive, and OT''', OT'', OT', OT, estimated to the right, are negative.

36. The Complement of an Arc.

The complement of an arc or angle is 90° minus that arc or angle.

If the arc or angle is less than 90°, its complement is *positive.*

If the arc or angle is greater than 90°, its complement is *negative.*

The complement of an arc, geometrically considered, is the arc estimated from the terminus of the given arc to the secondary origin. Therefore, by the preceding article, the complement of an arc will be positive or negative, according as the arc is less or greater than 90°.

TO′ is the complement of OT, and is positive.

T′O′ is the complement of OT′, and is negative.

T″O′ is the complement of OT″, and is negative.

T‴O′ is the complement of OT‴, and is negative.

37. The Supplement of an Arc.

The supplement of an arc or angle is 180° minus that arc or angle.

If the arc or angle is less than 180°, its supplement is *positive.*

If the arc or angle is greater than 180°, its supplement is *negative.*

The supplement of an arc, geometrically considered, is the arc estimated from the terminus of the given arc to the left-hand extremity of the primary diameter. Therefore, by article 35, the supplement of an arc will be positive or negative, according as the arc is less or greater than 180°.

TP is the supplement of OT, and is positive.

$T'P$ is the supplement of OT', and is positive.

$T''P$ is the supplement of OT'', and is negative.

$T'''P$ is the supplement of OT''', and is negative.

TRIGONOMETRICAL FUNCTIONS.

38. Preliminary Definitions and Remarks.

1. **A function** of a quantity is a quantity whose value depends on the given quantity.

2. **The trigonometrical functions,** called also *circular functions,* are auxiliary lines, which are functions of an arc or of the angle which has the same measure as that arc.

3. These functions are eight in number, and are called the *sine*, *co-sine*, *versed-sine*, *co-versed-sine*, *tangent*, *co-tangent*, *secant* and *co-secant*, which are abbreviated thus, *sin*, *cos*, *vers*, *covers*, *tan*, *cot*, *sec*, *cosec*.

4. The solution of triangles is accomplished by the aid of these functions, since they enable us to ascertain the relations which exist between the sides and angles of triangles.

5. The primary origin will be taken as the common origin of the arcs, unless the contrary is stated.

6. The origin of any arc, wherever situated, may be considered the primary origin of that arc; and its secondary origin is a quadrant's distance from the primary origin, in the direction of the positive or negative arcs, according as the given arc is positive or negative.

7. An arc will be considered positive unless the contrary is stated.

8. The primary diameter passes through the primary origin; and the secondary diameter, through the secondary origin.

9. Lines estimated *upward, toward the right*, or *from the center toward the terminus of the arc, are considered positive.*

10. Lines estimated *downward*, *toward the left*, *or from the center and the terminus of the arc*, *are considered negative.*

11. The limiting values of the circular functions are their values for the arcs 0°, 90°, 180°, 270°, 360°.

12. The sign of a varying quantity, up to a limit, is its sign at the limit.

13. Point out positive arcs in the following diagram, and the origin and terminus of each.

14. Point out negative arcs, the origin, terminus and primary diameter of each.

15. Point out the positive lines, also the negative.

39. The Sine of an Arc.

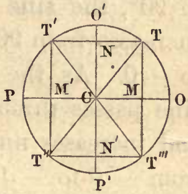
The sine of an arc is the perpendicular distance of its terminus from the primary diameter.

MT is the sine of the arc OT.

$M'T'$ is the sine of the arc OT'.

$M'T''$ is the sine of the arc OT''.

MT''' is the sine of the arc OT'''.



By the arcs OT'' and OT''', we are to understand the *positive* arcs, and not the negative arcs designated by the same letters.

The sine of an arc is the sine of the angle measured by that arc.

Thus, MT, the sine of the arc OT, is the sine of the angle OCT, which is measured by the arc OT; and similarly for the other arcs and angles.

The arcs OT and OT' are in the first and second quadrants, respectively, and their sines MT and $M'T'$ are estimated *upward*, and are therefore *positive;* hence,

The sine of an arc in the first or second quadrant is positive.

The arcs OT'' and OT''' are in the third and fourth quadrants, respectively, and their sines, $M'T''$ and MT''', are estimated *downward*, and are therefore *negative;* hence,

The sine of an arc in the third or fourth quadrant is negative.

Let the chord TT' be parallel to the primary diameter OP, then will $M'T'$ be equal to MT, and the arc OT will be equal to the arc $T'P$; but the arc $T'P$ is the supplement of the arc OT'; therefore, the arc OT is the supplement of the arc OT'; but $M'T'$, the sine of the arc OT', is equal to MT, the sine of the arc OT, the supplement of OT'; hence,

The sine of an arc is equal to the sine of its supplement.

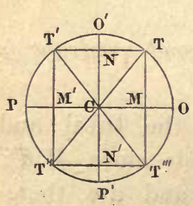
The sine of 0° is 0. As the arc increases from 0° to 90°, the sine increases from 0 to +1. As the arc increases from 90° to 180°, the sine decreases from +1 to +0. As the arc increases from 180° to 270°, the sine passes through 0, changes its sign from + to —, and increases numerically, but decreases algebraically from − 0 to − 1. As the arc increases from 270° to 360°, the sine decreases numerically, but increases algebraically from −1 to −0.

Hence, for the limiting values of the sine, we have

$$\sin 0° = 0, \qquad \sin 90° = +1, \qquad \sin 180° = +0,$$
$$\sin 270° = -1, \quad \sin 360° = -0.$$

40. The Co-sine of an Arc.

The co-sine of an arc is the perpendicular distance of its terminus from the secondary diameter.



NT is the co-sine of the arc OT.

NT' is the co-sine of the arc OT'.

$N'T''$ is the co-sine of the arc OT''.

$N'T'''$ is the co-sine of the arc OT'''.

The arcs OT and OT''' are in the first and fourth quadrants, respectively, and their co-sines NT and $N'T'''$ are estimated toward the *right*, and are therefore *positive;* hence,

The co-sine of an arc in the first or fourth quadrant is positive.

The arcs OT' and OT'' are in the second and third quadrants, respectively, and their co-sines, NT' and $N'T''$, are estimated toward the *left*, and are therefore *negative;* hence, *The co-sine of an arc in the second or third quadrant is negative.*

The word *co-sine* is an abbreviation of *complementi sinus*, the sine of the complement. In fact, NT, the co-sine of OT, is the sine of $O'T$, the complement of OT; hence,

The co-sine of an arc is the sine of its complement.

MT, the sine of OT, is the co-sine of $O'T$, the complement of OT; hence,

The sine of an arc is the co-sine of its complement.

Since the radius CO' is perpendicular to the chord TT', NT and NT' are numerically equal; but since NT is estimated toward the right, and NT' toward the left, they have contrary signs; hence, $NT = -NT'$; but NT is the co-sine of OT, and NT' is the co-sine of OT', the supplement of OT; hence,

The co-sine of an arc is equal to minus the co-sine of its supplement.

It is evident that CN is equal to the sine of OT, or of OT', and that CN' is equal to the sine of OT'', or of OT'''; hence,

The sine of an arc is equal to that part of the secondary diameter from the center to the foot of the co-sine.

It is evident that CM is equal to the co-sine of OT, or of OT''', and that CM' is equal to the co-sine of OT' or of OT''; hence,

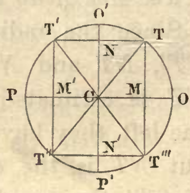
The co-sine of an arc is equal to that part of the primary diameter from the center to the foot of the sine.

The co-sine of 0° is + 1. As the arc increases from 0° to 90°, the co-sine decreases from + 1 to + 0. As the arc increases from 90° to 180°, the co-sine passes through 0, changes its sign from + to —, and increases numerically, but decreases algebraically from — 0 to — 1. As the arc increases from 180° to 270°, the co-sine decreases numerically, but increases algebraically from — 1 to — 0. As the arc increases from 270° to 360°, the co-sine passes through 0, changes its sign from — to +, and increases from + 0 to + 1.

Hence, for the limiting values of the co-sine, we have
$\cos 0° = +1$, $\cos 90° = +0$, $\cos 180° = -1$,
$\cos 270° = -0$, $\cos 360° = +1$.

41. The Versed-Sine of an Arc.

The versed-sine of an arc is the perpendicular distance of the primary origin from the sine.



MO is the versed-sine of the arc *OT*, and of the arc *OT'''*.

M'O is the versed-sine of the arc *OT'*, and of the arc *OT''*.

The versed-sine of an arc, in any quadrant, is estimated to the *right*, and is therefore *positive;* hence,

The versed-sine is always positive.

The versed-sine of 0° is 0. As the arc increases from 0° to 90°, the versed-sine increases from 0 to + 1. As the arc increases from 90° to 180°, the versed-sine increases from + 1 to + 2. As the arc increases from 180° to 270°, the versed-sine decreases from + 2 to + 1. As the arc increases from 270° to 360°, the versed-sine decreases from + 1 to + 0.

Hence, the limiting values of the versed-sine are
$\text{vers } 0° = 0$, $\text{vers } 90° = +1$, $\text{vers } 180° = +2$,
$\text{vers } 270° = +1$, $\text{vers } 360° = +0$.

What are the least and greatest values of the sine, and what are the corresponding arcs?

What are the least and greatest values of the co-sine, and what are the corresponding arcs?

What are the least and greatest values of the versed-sine, and what are the corresponding arcs?

42. The Co-versed-sine of an Arc.

The co-versed-sine of an arc is the perpendicular distance of the secondary origin from the co-sine.

Thus, see diagram of the last article, NO' is the co-versed-sine of the arc OT, and of the arc OT'; $N'O'$ is the co-versed-sine of the arc OT'', and of the arc OT'''.

The co-versed-sine of an arc in any quadrant is estimated *upward*, and is therefore *positive;* hence,

The co-versed-sine is always positive.

The word *co-versed-sine* is an abbreviation of *complementi versatus sinus*, the *versed* or *turned* sine of the complement. In fact, NO', the co-versed-sine of OT, is the versed-sine of $O'T$, the complement of OT; hence,

The co-versed-sine of an arc is the versed-sine of its complement.

MO, the versed-sine of OT, is the co-versed-sine of $O'T$, the complement of OT; hence,

The versed-sine of an arc is the co-versed-sine of its complement.

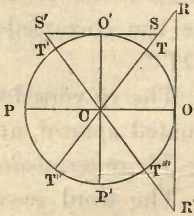
The co-versed-sine of 0° is 1. As the arc increases from 0° to 90°, the co-versed-sine decreases from + 1 to + 0. As the arc increases from 90° to 180°, the co-versed-sine increases from + 0 to + 1. As the arc increases from 180° to 270°, the co-versed-sine increases from + 1 to + 2. As the arc increases from 270° to 360°, the co-versed-sine decreases from + 2 to + 1. Hence, the limiting values of the co-versed-sine are, covers $0° = +1$, covers $90° = +0$, covers $180° = +1$, covers $270° = +2$, covers $360° = +1$.

What are the least and greatest values of the co-versed-sine, and what are the corresponding arcs?

Trace the arcs from 0° to 360°, and the changing functions.

43. The Tangent of an Arc.

The tangent of an arc is the perpendicular to the primary diameter, produced from the primary origin, till it meets the prolongation of the diameter through the terminus of the arc.



OR is the tangent of the arcs OT and OT''.

OR' is the tangent of the arcs OT' and OT'''.

The arcs OT and OT'' are in the first and third quadrants, respectively, and their tangent, OR, is estimated *upward*, and is therefore *positive;* hence,

The tangent of an arc in the first or third quadrant is positive.

The arcs OT' and OT''' are in the second and fourth quadrants, respectively, and their tangent, OR', is estimated *downward*, and is therefore *negative;* hence,

The tangent of an arc in the second or fourth quadrant is negative.

Let the arc OT be equal to the arc $T'P$. Then, since $T'P$ is the supplement of OT', OT will be the supplement of OT'; but the arc $T'''O$ is the supplement of OT'; hence, $OT = T'''O$, and the angle OCT is equal to the angle OCT'''. The angle COR is equal to the angle COR', since each is a right angle. Hence, the two triangles COR and COR' have two angles, and the included side of the one equal to two angles and the included side of the other, each to each, and are therefore equal in all their parts. Hence, OR, opposite the angle OCR, is equal to OR', opposite the equal angle OCR'. Since OR is estimated *upward*, and OR' *downward*, they have contrary signs; hence, $OR = -OR'$. But OR is the tangent of the arc OT, and OR' is the tangent of the arc OT', the supplement of OT; hence,

The tangent of an arc is equal to minus the tangent of its supplement.

The tangent of 0° is 0. As the arc increases from 0° to 90°, the tangent increases from 0 to $+\infty$. As the arc increases from 90° to 180°, the tangent passes through ∞, changes its sign from + to —, and decreases numerically, but increases algebraically from $-\infty$ to -0. As the arc increases from 180° to 270°, the tangent passes through 0, changes its sign from — to +, and increases from $+0$ to $+\infty$. As the arc increases from 270° to 360°, the tangent passes through ∞, changes its sign from + to —, and decreases numerically, but increases algebraically from $-\infty$ to -0. Hence, for the limiting values of the tangent we have $\tan 0° = 0$, $\tan 90° = +\infty$, $\tan 180° = -0$, $\tan 270° = +\infty$, $\tan 360° = -0$.

44. The Co-tangent of an Arc.

The co-tangent of an arc is the perpendicular to the secondary diameter, produced from the secondary origin, till it meets the prolongation of the diameter through the terminus of the arc.

$O'S$ is the co-tangent of OT and OT''.

$O'S'$ is the co-tangent of OT' and OT'''.

The arcs OT and OT'' are in the first and third quadrants, respectively, and their co-tangent, $O'S$, is estimated to the *right*, and is therefore *positive;* hence,

The co-tangent of an arc in the first or third quadrant is positive.

The arcs OT' and OT''' are in the second and fourth quadrants, respectively, and their co-tangent, $O'S'$, is estimated to the *left*, and is therefore *negative;* hence, *The co-tangent of an arc in the second or fourth quadrant is negative.*

The word *co-tangent* is an abbreviation of *complementi tangens*, the tangent of the complement. In fact, $O'S$, the co-tangent of OT, is the tangent of $O'T$, the complement of OT; hence,

The co-tangent of an arc is the tangent of its complement.

OR, the tangent of OT, is the co-tangent of $O'T$, the complement of OT; hence,

The tangent of an arc is the co-tangent of its complement.

Let the arcs OT and $T'P$ be equal. Then, since $T'P$ is the supplement of OT', OT will be the supplement of OT'.

The arcs $O'T$ and $O'T'$ are equal, since they are complements of the equal arcs OT and $T'P$; hence, the angles $O'CT$ and $O'CT'$, measured by these equal arcs, are equal. The angles $CO'S$ and $CO'S'$ are equal, since each is a right angle. Hence, the two triangles $CO'S$ and $CO'S'$ have the common side CO', and the two adjacent angles equal, and are therefore equal in all their parts; and $O'S$, opposite the angle $O'CS$, is equal to $O'S'$, opposite the equal angle $O'CS'$.

Since $O'S$ is estimated to the *right*, and $O'S'$ to the *left*, they have contrary signs; hence, $O'S = -O'S'$. But $O'S$ is the co-tangent of OT, and $O'S'$ is the co-tangent of OT', the supplement of OT; hence,

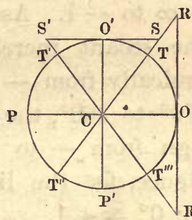
The co-tangent of an arc is equal to minus the co-tangent of its supplement.

The co-tangent of 0° is $+\infty$. As the arc increases from 0° to 90°, the co-tangent decreases from $+\infty$ to $+0$. As the arc increases from 90° to 180°, the co-tangent passes through 0, changes its sign from + to −, and increases numerically, but decreases algebraically from -0 to $-\infty$. As the arc increases from 180° to 270°, the co-tangent passes through ∞, changes its sign from $-$ to $+$, and decreases from $+\infty$ to $+0$. As the arc increases from 270° to 360°, the co-tangent passes through 0, changes its sign from $+$ to $-$, and increases numerically, but decreases algebraically from -0 to $-\infty$.

Hence, the limiting values of the co-tangent are $\cot 0° = +\infty$, $\cot 90° = +0$, $\cot 180° = -\infty$, $\cot 270° = +0$, $\cot 360° = -\infty$.

45. The Secant of an Arc.

The secant of an arc is the line drawn from the center of the circle to the terminus of the tangent.



CR is the secant of *OT* and *OT''*.

CR' is the secant of *OT'* and *OT'''*.

The arcs *OT* and *OT'''* are in the first and fourth quadrants, respectively, and their secants, *CR* and *CR'* are estimated from the center toward the termini of the arcs, and are therefore positive; hence,

The secant of an arc in the first or fourth quadrant is positive.

The arcs *OT'* and *OT''* are in the second and third quadrants, respectively, and their secants, *CR'* and *CR*, are estimated from the center, from the termini of the arcs, and are therefore *negative;* hence,

The secant of an arc in the second or third quadrant is negative.

Let the arcs *OT* and *T'P* be equal. Then, since *T'P* is the supplement of *OT'*, *OT* is the supplement of *OT';* but *T'''O* is the supplement of *OT';* therefore, *T'''O* is equal to *OT*, and the angle *T'''CO*, measured by *T'''O*, is equal to the angle *OCT*, measured by the equal arc *OT*. The right angles *COR* and *COR'* are equal. Hence, in the triangles having the common side CO, and the two adjacent angles equal, CR is equal to CR'; but CR, the secant of OT, is positive; and CR', the secant of OT', the supplement of OT, is negative; hence, $CR = -CR'$; hence,

The secant of an arc is equal to minus the secant of its supplement.

The secant of 0° is +1. As the arc increases from 0° to 90°, the secant increases from +1 to +∞. As the arc increases from 90° to 180°, the secant passes through ∞, changes its sign from + to —, and decreases numerically, but increases algebraically from —∞ to —1. As the arc increases from 180° to 270°, the secant increases numerically, but decreases algebraically from —1 to —∞. As the arc increases from 270° to 360°, the secant passes through ∞, changes its sign from — to +, and decreases from +∞ to +1. Hence, for the limiting values of the secant we have

$\sec 0° = +1$, $\sec 90° = +\infty$, $\sec 180° = -1$, $\sec 270° = -\infty$, $\sec 360° = +1$.

46. The Co-secant of an Arc.

The co-secant of an arc is the line drawn from the center of the circle to the terminus of the co-tangent.

CS is the co-secant of OT and OT''.

CS' is the co-secant of OT' and OT'''.

The arcs OT and OT' are in the first and second quadrants, respectively, and their co-secants CS and CS' are estimated from the center toward the termini of the arcs, and are therefore *positive;* hence,

The co-secant of an arc in the first or second quadrant is positive.

The arcs OT'' and OT''' are in the third and fourth quadrants, respectively, and their co-secants, CS and CS', are estimated from the center and the termini of the arcs, and are therefore *negative;* hence,

The co-secant of an arc in the third or fourth quadrant is negative.

The word *co-secant* is an abbreviation of *complementi secans*, the secant of the complement. In fact, CS, the co-secant of OT, is the secant of $O'T$, the complement of OT; hence,

The co-secant of an arc is the secant of its complement.

CR, the secant of OT, is the co-secant of $O'T$, the complement of OT; hence,

The secant of an arc is the co-secant of its complement.

Let the arcs OT and $T'P$ be equal. Then, since $T'P$ is the supplement of OT', OT will be the supplement of OT'. $O'T = O'T'$, since they are complements of equal arcs. Hence, the angle $O'CT$, measured by the arc $O'T$, is equal to the angle $O'CT'$, measured by the equal arc $O'T'$. The right angles, $CO'S$ and $CO'S'$, are equal.

Hence, in the triangles having the common side CO', and the two adjacent angles equal, CS is equal to CS'; but CS is the co-secant of OT, and positive, and CS' is the co-secant of OT', and positive; hence,

The co-secant of an arc is equal to the co-secant of its supplement.

The co-secant of $0°$ is $+\infty$. As the arc increases from $0°$ to $90°$, the co-secant decreases from $+\infty$ to $+1$. As the arc increases from $90°$ to $180°$, the co-secant increases from $+1$ to $+\infty$. As the arc increases from $180°$ to $270°$, the co-secant passes through ∞, changes its sign from $+$ to $-$, and decreases numerically, but increases algebraically from $-\infty$ to -1. As the arc increases from $270°$ to $360°$, the co-secant increases numerically, but decreases algebraically from -1 to $-\infty$. Hence, the limiting values of the co-secant are cosec $0° = +\infty$, cosec $90° = +1$, cosec $180° = +\infty$, cosec $270° = -1$, cosec $360° = -\infty$.

To aid the memory, and for convenience of reference, we give the following tabular summaries:

47. Signs of the Circular Functions.

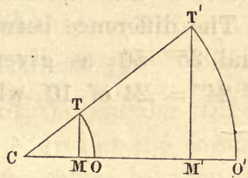
Functions.	1*st* *q*.	2*d* *q*.	3*d* *q*.	4*th* *q*.
sine.	+	+	−	−
co-sine.	+	−	−	+
versed-sine.	+	+	+	+
co-versed-sine.	+	+	+	+
tangent.	+	−	+	−
co-tangent.	+	−	+	−
secant.	+	−	−	+
co-secant.	+	+	−	−

48. Limiting Values of the Circular Functions.

0°	90°	180°	270°	360°
sin $= +0$	sin $= +1$	sin $= +0$	sin $= -1$	sin $= -0$
cos $= +1$	cos $= +0$	cos $= -1$	cos $= -0$	cos $= +1$
vsin $= +0$	vsin $= +1$	vsin $= +2$	vsin $= +1$	vsin $= +0$
cvs $= +1$	cvs $= +0$	cvs $= +1$	cvs $= +2$	cvs $= +1$
tan $= +0$	tan $= +\infty$	tan $= -0$	tan $= +\infty$	tan $= -0$
cot $= +\infty$	cot $= +0$	cot $= -\infty$	cot $= +0$	cot $= -\infty$
sec $= +1$	sec $= +\infty$	sec $= -1$	sec $= -\infty$	sec $= +1$
cose $= +\infty$	cose $= +1$	cose $= +\infty$	cose $= -1$	cose $= -\infty$

49. Problem.

To find any function of an angle to the radius R, in terms of the corresponding function of the same angle to the radius 1, *and the reverse.*



Let sin C_1 denote sin C to the radius $CT = 1$, and sin C_R denote sin C to the radius $CT' = R$.

From similar triangles,

$$CT : CT' :: MT : M'T',$$
$$\text{or } 1 : R :: \sin C_1 : \sin C_R.$$

$$\therefore (1) \quad \sin C_R = \sin C_1 \times R. \qquad \therefore (2) \quad \sin C_1 = \frac{\sin C_R}{R}.$$

Let formulas for other functions be deduced; hence,

1. *Any function of an angle to the radius R is equal to the corresponding function of the same angle to the radius* 1, *multiplied by R.*

2. *Any function of an angle to the radius* 1 *is equal to the corresponding function of the same angle to the radius R, divided by R.*

TABLE OF NATURAL FUNCTIONS.

50. Description of the Table.

This table gives, to the radius 1, the values of the sine, co-sine, tangent, and co-tangent, to five decimal places, for every 10′ from 0° to 90°.

For sines and tangents, the degrees are given in the left column, and the minutes at the top.

For co-sines and co-tangents, the degrees are given in the right-hand column, and the minutes at the bottom.

51. Problem.

To find the natural sine, co-sine, tangent, or co-tangent of a given arc or angle.

Let us find the natural sine of 35° 42′ 24″.

The difference between the natural sines of 35° 40′ and 35° 50′, as given in the table, is .00236. Now 2′ 24″ = .24 of 10′, which is found thus:

60	24
10	2.4
	.24

Then take Nat sin 35° 40′ = .58307

Correction for 2′ 24″ = .00236 × .24 = .00057

∴ Nat sin 35° 42′ 24″ = .58364

In case of co-sine or co-tangent, the correction must be subtracted, since, between 0° and 90°, the greater the angle, the less the co-sine and co-tangent.

52. Examples.

1. Find the natural sine of 75° 45′ 30″.

Ans. .96927.

2. Find the natural co-sine of 15° 36′ 12″.

Ans. .96315.

3. Find the natural tangent of 43° 33′ 18″.

Ans. .95079.

4. Find the natural co-tangent of 84° 28′ 30″.

Ans. .09673.

53. Problem.

To find the angle corresponding to a given natural sine, co-sine, tangent, or co-tangent.

1. Find the angle corresponding to the natural sine .50754.

Looking in the table we find the angle 30° 30′.

2. Find the angle whose natural sine $= .82468$.

The next less sine, $\sin 55° \ 30' = .82413$.

Difference $= \quad 55$

Difference corresponding to $10' = \quad 164$

$\therefore \quad \text{Correction} = 10' \times \frac{55}{164} = 3' \ 21''$.

$\therefore \quad \text{Angle} = 55° \ 30' + 3' \ 21'' = 55° \ 33' \ 21''$.

In case of co-sine and co-tangent, the angular difference must be subtracted, since the greater the co-sine or co-tangent, the less the angle, for values between 0° and 90°.

54. Examples.

1. Find the angle whose sine is .75684.

Ans. 49° 11′ 13″.

2. Find the angle whose co-sine is .67898.

Ans. 47° 14′ 10″.

3. Find the angle whose tangent is 1.34567.

Ans. 53° 22′ 59″.

4. Find the angle whose co-tangent is .98765.

Ans. 45° 21′ 22″.

TABLE OF LOGARITHMIC FUNCTIONS.

55. Description of the Table.

The table of logarithmic functions gives to the radius 10,000,000,000 the logarithm of the sine, co-sine, tangent, and co-tangent, for every minute, from 0° to 90°.

The expression, *logarithmic sine*, *tangent*, etc., is equivalent to *the logarithm of the sine*, *of the tangent*, *etc.*

For sines and tangents, the degrees are given at the top of the page, and the minutes in the left-hand column.

For co-sines and co-tangents, the degrees are given at the bottom of the page, and the minutes in the right-hand column.

The columns marked D 1″ contain the difference for 1″.

56. Problem.

Find the logarithmic sine of 48° 25′ 30″.

$$\log \sin 48^\circ\ 25' = 9.87390.$$
$$D\ 1'' = .19. \quad \therefore \text{Correc. for } 30'' = .19 \times 30 = \qquad 6$$
$$\therefore \log \sin 48^\circ\ 25'\ 30'' = 9.87396$$

In case of co-sine or co-tangent, the correction must be subtracted, since between 0° and 90°, the greater the angle, the less the co-sine and co-tangent.

57. Examples.

1. Find the logarithmic sine of 75° 35′.

Ans. 9.98610.

2. Find the logarithmic sine of 25° 40′ 24″.

Ans. 9.63673.

3. Find the logarithmic co-sine of 29° 55′ 55″.

Ans. 9.93782.

4. Find the logarithmic tangent of 50° 50′ 50″.

Ans. 10.08927.

5. Find the logarithmic co-tangent of 65° 45′ 30″.

Ans. 9.65349.

58. Problem.

To find the angle corresponding to a given logarithmic sine, co-sine, tangent, or co-tangent.

Find the angle whose logarithmic sine $= 9.84567$

For next less we have $\sin 44° 30' = 9.84566$

$$D\ 1'' = .21 \quad \therefore \text{ Correc.} = 1'' \times \frac{1}{.21} = 5'', \quad .21)1.00(5.$$

$$\therefore \text{ Angle} = 44° 30' 05''.$$

In case of co-sine and co-tangent, the correction for seconds must be subtracted, since the greater the co-sine or co-tangent, and consequently the greater the logarithm, the less the angle for values between 0° and 90°.

59. Examples.

1. Find the angle whose logarithmic sine is 9.98437. *Ans.* 74° 43′ 17″.
2. Find the angle whose logarithmic co-sine is 9.78456. *Ans.* 52° 29′ 19″.
3. Find the angle whose logarith. tangent is 10.12346. *Ans.* 53° 02′ 11″.
4. Find the angle whose logarith. co-tangent is 9.99999. *Ans.* 45° 00′ 03″.

60. Problem.

Given any natural function, to find the corresponding logarithmic function.

1st Solution.

Find from the natural function the corresponding angle; then, from the angle, the corresponding logarithmic function.

2d Solution.

Let a denote any arc or angle, $f(a)_1$ any function of a to the radius 1, and $f(a)_R$ the corresponding function of a to the radius R. Then, by article 49 we have,

$$f(a)_R = f(a)_1 \times R.$$

Substituting the value of R in the second member,

$$f(a)_R = f(a)_1 \times 10{,}000{,}000{,}000.$$

$$\therefore \quad \log f(a)_R = \log f(a)_1 + 10.$$

Hence, *Add* 10 *to the logarithm of the natural function.*

61. Examples.

1. Given nat. sin $a = .98457$, required a and log sin a. *Ans.* $a = 79° \; 55' \; 25''$, log sin $a = 9.99325$.
2. Given nat. cos $a = .63878$, required a and log cos a. *Ans.* $a = 50° \; 17' \; 52''$, log cos $a = 9.80536$.
3. Given nat. tan $a = 1.68685$, required a and log tan a. *Ans.* $a = 59° \; 20' \; 23''$, log tan $a = 10.22708$.
4. Given nat. cot $a = 1.41987$, required a and log cot a. *Ans.* $a = 35° \; 09' \; 24''$, log cot a $= 10.15225$.

62. Problem.

Given any logarithmic function, to find the corresponding natural function.

1st Solution.

Find from the logarithmic function the corresponding angle; then, from the angle, the corresponding natural function.

2d Solution.

From article 49 we have,

$$f(a)_1 = \frac{f(a)_R}{R}.$$

$$\therefore \quad \log f(a)_1 = \log f(a)_R - 10.$$

Hence, *Subtract* 10 *from the logarithmic function, and find the number corresponding to the resulting logarithm.*

63. Examples.

1. Given log sin $a = 9.87654$, required a and nat. sin a. *Ans.* $a = 48° 48' 44''$, nat. sin $a = .75255$.

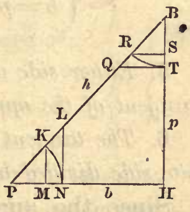
2. Given log cos $a = 9.84877$, required a and nat. cos a. *Ans.* $a = 45° 05' 41''$, nat. cos $a = .70595$.

3. Given log tan $a = 10.22708$, required a and nat. tan a. *Ans.* $a = 59° 20' 23''$, nat. tan $a = 1.68685$.

4. Given log cot $a = 10.15225$, required a and nat. cot a. *Ans.* $a = 35° 09' 24''$, nat. cot $a = 1.41987$.

RIGHT TRIANGLES.

64. Principles.



$PB : PK :: HB : MK,$

or $h : 1 :: p : \sin P.$

$BP : BR :: HP : SR,$

or $h : 1 :: b : \sin B.$

$$\therefore \quad (1) \begin{cases} p = h \sin P. \\ b = h \sin B. \end{cases} \qquad \therefore \quad (2) \begin{cases} \sin P = \dfrac{p}{h}. \\ \sin B = \dfrac{b}{h}. \end{cases}$$

1. *Either side adjacent to the right angle is equal to the sine of the opposite angle multiplied by the hypotenuse.*

2. *The sine of either acute angle is equal to the opposite side divided by the hypotenuse.*

Since the angles P and B are complements of each other, sin $P = \cos B$, and sin $B = \cos P$; $\therefore$ (1) and (2) become,

$$(3) \left\{ \begin{array}{l} p = h \cos B. \\ b = h \cos P. \end{array} \right\} \text{ and } (4) \left\{ \begin{array}{l} \cos B = \dfrac{p}{h}. \\ \cos P = \dfrac{b}{h}. \end{array} \right\}.$$

3. *Either side adjacent to the right angle is equal to the co-sine of the adjacent acute angle multiplied by the hypotenuse.*

4. *The co-sine of either acute angle is equal to the adjacent side divided by the hypotenuse.*

$PH : PN :: HB : NL$, or $b : 1 :: p : \tan P$.

$BH : BT :: HP : TQ$, or $p : 1 :: b : \tan B$.

$$\therefore \ (5) \left\{ \begin{array}{l} p = b \tan P. \\ b = p \tan B. \end{array} \right\} \ \therefore \ (6) \left\{ \begin{array}{l} \tan P = \dfrac{p}{b}. \\ \tan B = \dfrac{b}{p}. \end{array} \right\}$$

5. *Either side adjacent to the right angle is equal to the tangent of the opposite angle multiplied by the other side.*

6. *The tangent of either acute angle is equal to the opposite side divided by the adjacent side.*

Since the angles P and B are complements of each other, tan $P = \cot B$, and tan $B = \cot P$; $\therefore$ (5) and (6) become,

$$(7) \left\{ \begin{array}{l} p = b \cot B. \\ b = p \cot P. \end{array} \right\} \text{ and } (8) \left\{ \begin{array}{l} \cot B = \dfrac{p}{b}. \\ \cot P = \dfrac{b}{p}. \end{array} \right\}.$$

7. *Either side adjacent to the right angle is equal to the co-tangent of the adjacent acute angle multiplied by the other side.*

8. *The co-tangent of either acute angle is equal to the adjacent side divided by the opposite side.*

$BH : BT :: BP : BQ$, or $p : 1 :: h : \sec B$.

$PH : PN :: PB : PL$, or $b : 1 :: h : \sec P$.

$$(9) \left\{ \begin{aligned} p &= \frac{h}{\sec B}. \\ b &= \frac{h}{\sec P}. \end{aligned} \right\} \therefore \quad (10) \left\{ \begin{aligned} \sec B &= \frac{h}{p}. \\ \sec P &= \frac{h}{b}. \end{aligned} \right\}$$

9. *Either side adjacent to the right angle is equal to the hypotenuse divided by the secant of the adjacent acute angle.*

10 *The secant of either acute angle is equal to the hypotenuse divided by the adjacent side.*

Since the angles B and P are complements of each other $\sec B = \operatorname{cosec} P$, $\sec P = \operatorname{cosec} B$; $\therefore$ (9) and (10) become,

$$(11) \left\{ \begin{aligned} p &= \frac{h}{\operatorname{cosec} P}. \\ b &= \frac{h}{\operatorname{cosec} B}. \end{aligned} \right\} \text{ and } (12) \left\{ \begin{aligned} \operatorname{cosec} P &= \frac{h}{p}. \\ \operatorname{cosec} B &= \frac{h}{b}. \end{aligned} \right\}$$

11. *Either side adjacent to the right angle is equal to the hypotenuse divided by the co-secant of the angle opposite that side.*

12. *The co-secant of either acute angle is equal to the hypotenuse divided by the side opposite that angle.*

Scholium. By some authors, principles 2, 4, 6, 8, 10, and 12, have been given in the form of definitions.

Introducing radius into these formulas, by substituting for any function to the radius 1, the corresponding function to the radius R divided by R, and reducing, we have:

$$(1)\ \left\{\begin{array}{l} p=\dfrac{h \sin P}{R}. \\ b=\dfrac{h \sin B}{R}. \end{array}\right\} \qquad (2)\ \left\{\begin{array}{l} \sin P=\dfrac{Rp}{h}. \\ \sin B=\dfrac{Rb}{h}. \end{array}\right\}$$

$$(3)\ \left\{\begin{array}{l} p=\dfrac{h \cos B}{R}. \\ b=\dfrac{h \cos P}{R}. \end{array}\right\} \qquad (4)\ \left\{\begin{array}{l} \cos B=\dfrac{Rp}{h}. \\ \cos P=\dfrac{Rb}{h}. \end{array}\right\}$$

$$(5)\ \left\{\begin{array}{l} p=\dfrac{b \tan P}{R}. \\ b=\dfrac{p \tan B}{R}. \end{array}\right\} \qquad (6)\ \left\{\begin{array}{l} \tan P=\dfrac{Rp}{b}. \\ \tan B=\dfrac{Rb}{p}. \end{array}\right\}$$

$$(7)\ \left\{\begin{array}{l} p=\dfrac{b \cot B}{R}. \\ b=\dfrac{p \cot P}{R}. \end{array}\right\} \qquad (8)\ \left\{\begin{array}{l} \cot B=\dfrac{Rp}{b}. \\ \cot P=\dfrac{Rb}{p}. \end{array}\right\}$$

$$(9)\ \left\{\begin{array}{l} p=\dfrac{Rh}{\sec B}. \\ b=\dfrac{Rh}{\sec P}. \end{array}\right\} \qquad (10)\ \left\{\begin{array}{l} \sec B=\dfrac{Rh}{p}. \\ \sec P=\dfrac{Rh}{b}. \end{array}\right\}$$

$$(11)\ \left\{\begin{array}{l} p=\dfrac{Rh}{\operatorname{cosec} P}. \\ b=\dfrac{Rh}{\operatorname{cosec} B}. \end{array}\right\} \qquad (12)\ \left\{\begin{array}{l} \operatorname{cosec} P=\dfrac{Rh}{p}. \\ \operatorname{cosec} B=\dfrac{Rh}{b}. \end{array}\right\}$$

Applying logarithms to these formulas, we have:

$$(1)\ \left\{\begin{array}{l} \log p=\log h+\log \sin P-10. \\ \log b=\log h+\log \sin B-10. \end{array}\right\}$$

$$(2)\ \left\{\begin{array}{l} \log \sin P=10+\log p-\log h. \\ \log \sin B=10+\log b-\log h. \end{array}\right\}$$

$$(3)\ \left\{\begin{array}{l} \log p=\log h+\log \cos B-10. \\ \log b=\log h+\log \cos P-10. \end{array}\right\}$$

(4) $\begin{cases} \log \cos B = 10 + \log p - \log h. \\ \log \cos P = 10 + \log b - \log h. \end{cases}$

(5) $\begin{cases} \log p = \log b + \log \tan P - 10. \\ \log b = \log p + \log \tan B - 10. \end{cases}$

(6) $\begin{cases} \log \tan P = 10 + \log p - \log b. \\ \log \tan B = 10 + \log b - \log p. \end{cases}$

(7) $\begin{cases} \log p = \log b + \log \cot B - 10. \\ \log b = \log p + \log \cot P - 10. \end{cases}$

(8) $\begin{cases} \log \cot B = 10 + \log p - \log b. \\ \log \cot P = 10 + \log b - \log p. \end{cases}$

(9) $\begin{cases} \log p = 10 + \log h - \log \sec B. \\ \log b = 10 + \log h - \log \sec P. \end{cases}$

(10) $\begin{cases} \log \sec B = 10 + \log h - \log p. \\ \log \sec P = 10 + \log h - \log b. \end{cases}$

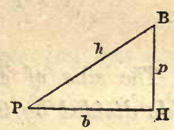
(11) $\begin{cases} \log p = 10 + \log h - \log \operatorname{cosec} P. \\ \log b = 10 + \log h - \log \operatorname{cosec} B. \end{cases}$

(12) $\begin{cases} \log \operatorname{cosec} P = 10 + \log h - \log p. \\ \log \operatorname{cosec} B = 10 + \log h - \log b. \end{cases}$

65. Case I.

Given the hypotenuse and one acute angle, required the remaining parts.

1. Given $\begin{cases} h = 365. \\ P = 33° \ 12'. \end{cases}$ Requir. $\begin{cases} B. \\ p. \\ b. \end{cases}$



$$B = 90° - P = 90° - 33° \ 12' = 56° \ 48'.$$

Either side adjacent to the right angle is equal to the sine of the opposite angle, multiplied by the hypotenuse.

$$\therefore \quad p = h \sin P.$$

Introducing radius, we have, $p = \frac{h \sin P}{R}$.

Applying logarithms, we have,

$$\log p = \log h + \log \sin P - 10.$$

$$\begin{aligned} \log h\ (365) &= 2.56229 \\ \log \sin P\ (33^\circ\ 12') &= \underline{9.73843} \\ \log p &= 2.30072 \quad \therefore\ p = 199.85. \end{aligned}$$

In like manner, from either formula, $b = h \sin B$, or $b = h \cos P$, we find $b = 305.41$.

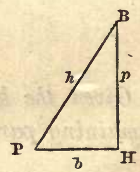
2. Given $\left\{ \begin{array}{l} h = 73.26. \\ B = 49^\circ\ 12'\ 20''. \end{array} \right\}$ Requir. $\left\{ \begin{array}{l} P = 40^\circ\ 47'\ 40''. \\ b = 55.4625. \\ p = 47.8644. \end{array} \right.$

3. Given $\left\{ \begin{array}{l} h = 2195. \\ P = 27^\circ\ 38'\ 50''. \end{array} \right\}$ Requir. $\left\{ \begin{array}{l} B = 62^\circ\ 21'\ 10''. \\ p = 1018.512. \\ b = 1944.364. \end{array} \right.$

66. Case II.

Given the hypotenuse and one side adjacent to the right angle, required the remaining parts.

1. Given $\left\{ \begin{array}{l} h = 112. \\ p = 97. \end{array} \right\}$ Required $\left\{ \begin{array}{l} P. \\ B. \\ b. \end{array} \right.$



The sine of either acute angle is equal to the opposite side divided by the hypotenuse.

$$\therefore\ \sin P = \frac{p}{h}.$$

Introducing radius, and multiplying by R, we have,

$$\sin P = \frac{Rp}{h}.$$

Applying logarithms, we have,

$$\log \sin P = 10 + \log p - \log h.$$

$$\begin{aligned} \log p \ (97) &= 1.98677 \\ \log h \ (112) &= 2.04922 \\ \log \sin P &= 9.93755 \quad \therefore \quad P = 60^\circ\ 00'\ 17''. \end{aligned}$$

$$B = 90^\circ - P = 90^\circ - 60^\circ\ 00'\ 17'' = 29^\circ\ 59'\ 43''.$$

$$b = h \sin B, \text{ or } b = h \cos P, \ \therefore \ b = 55.991.$$

We can also find b as follows:

$$b = \sqrt{h^2 - p^2} = \sqrt{(h+p)(h-p)}.$$

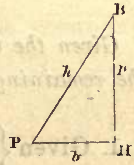
$$\log b = \tfrac{1}{2}[\log (h+p) + \log (h-p)].$$

2. Given $\left\{ \begin{array}{l} h = 7269. \\ b = 3162. \end{array} \right\}$ Required $\left\{ \begin{array}{l} B = 25^\circ\ 47'\ 07''. \\ P = 64^\circ\ 12'\ 53''. \\ p = 6545. \end{array} \right.$

3. Given $\left\{ \begin{array}{l} h = 444.4 \\ p = 150. \end{array} \right\}$ Required $\left\{ \begin{array}{l} P = 19^\circ\ 43'\ 36''. \\ B = 70^\circ\ 16'\ 24''. \\ b = 418.33. \end{array} \right.$

67. Case III.

Given one side adjacent to the right angle and one acute angle, required the remaining parts.



1. Given $\left\{ \begin{array}{l} b = 152.67. \\ P = 50^\circ\ 18'\ 32''. \end{array} \right\}$ Requir. $\left\{ \begin{array}{l} B. \\ p. \\ h. \end{array} \right.$

$$B = 90^\circ - P = 90^\circ - 50^\circ\ 18'\ 32'' = 39^\circ\ 41'\ 28''.$$

Either side adjacent to the right angle is equal to the tangent of the opposite angle multiplied by the other side.

$$\therefore\ p = b \tan P.$$

Introducing radius and applying logarithms, as in the preceding cases, we find $p = 183.95$.

Either side adjacent to the right angle is equal to the co-sine of the adjacent acute angle multiplied by the hypotenuse.

$$\therefore\ b = h \cos P;\quad \therefore\ h = \frac{b}{\cos P}.$$

Introducing radius and applying logarithms, as above, we shall find $h = 239.05$.

2. Given $\begin{cases} p = 3963.35 \text{ miles} = \text{the earth's radius.} \\ P = 57'\ 2.3'' = \text{the moon's horizontal parallax.} \end{cases}$

Required h, the distance of the moon from the earth.
Ans. $h = 238889$ miles.

3. Given $\begin{cases} p = 3963.35 \text{ miles} = \text{the earth's radius.} \\ P = 8.9'' = \text{the sun's horizontal parallax.} \end{cases}$

Required h, the distance of the sun from the earth.
Ans. $h = 91852000$ miles.

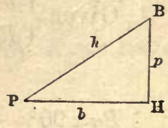
Scholium. $\text{Sin } 8.9'' = \sin\ 1' \times \frac{8.9}{60}$.

$\therefore\ \log \sin 8.9'' = \log \sin 1' + \log 8.9 + a.c. \log 60 - 10.$

68. Case IV.

Given the two sides adjacent to the right angle, required the remaining parts.

1. Given $\begin{Bmatrix} p = 29.37. \\ b = 37.29. \end{Bmatrix}$ Requir. $\begin{cases} P. \\ B. \\ h. \end{cases}$



The tangent of either acute angle is equal to the opposite side divided by the adjacent side.

$$\therefore \quad \tan P = \frac{p}{b}.$$

Introducing radius and applying logarithms, we shall find that $P = 38° \ 13' \ 28''$.

$B = 90° - P = 90° - 38° \ 13' \ 28'' = 51° \ 46' \ 32''$.

Either side adjacent to the right angle is equal to the sine of the opposite angle multiplied by the hypotenuse.

$$\therefore \quad p = h \sin P. \qquad \therefore \quad h = \frac{p}{\sin P}.$$

Introducing radius and applying logarithms, we find $h = 47.466$.

2. Given $\begin{Bmatrix} p = 694.73. \\ b = 8372.1. \end{Bmatrix}$ Required $\begin{cases} P = 4° \ 44' \ 37''. \\ B = 85° \ 15' \ 23''. \\ h = 8401. \end{cases}$

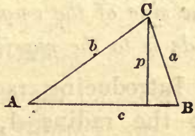
3. Given $\begin{Bmatrix} p = 101. \\ b = 103. \end{Bmatrix}$ Required $\begin{cases} P = 44° \ 26' \ 17''. \\ B = 45° \ 33' \ 43''. \\ h = 144.253. \end{cases}$

4. Given $\begin{Bmatrix} p = 1728. \\ b = 1575. \end{Bmatrix}$ Required $\begin{cases} P = 47° \ 39' \ 07''. \\ B = 42° \ 20' \ 53''. \\ h = 2338.1. \end{cases}$

OBLIQUE TRIANGLES.

69. Case I.

Given one side and two angles, required the remaining parts.



Let ABC be an oblique triangle, and let the sides opposite the angles A, B, and C be denoted respectively by a, b and c.

Let the angles A and B and the side a be given, and the angle C and the sides b and c be required.

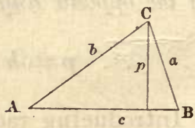
We find C from the formula,

$$C = 180° - (A + B).$$

Draw the perpendicular p from the vertex C to the side c, thus forming two right triangles. There are two cases:

1*st.* When the perpendicular falls on the side c.

From the principles of the right triangle we have,



$$p = b \sin A \text{ and } p = a \sin B.$$

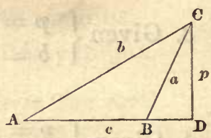
$$\therefore \quad b \sin A = a \sin B.$$

$$\therefore \quad (1) \quad \sin A : \sin B :: a : b.$$

2*d.* When the perpendicular falls on c produced.

$$p = b \sin A \text{ and } p = a \sin CBD.$$

But CBD is the supplement of CBA, or B of the triangle. Since the sine of an angle is equal to the sine of its supplement,



$$\sin CBD = \sin B; \quad \therefore \quad p = a \sin B.$$

$$\therefore \quad b \sin A = a \sin B.$$

$$\therefore \quad (1) \quad \sin A : \sin B :: a : b.$$

In like manner we may find,

$$(2) \quad \sin A : \sin C :: a : c.$$

Hence, *The sine of the angle opposite the given side is to the sine of the angle opposite the required side as the given side is to the required side.*

Introducing radius by substituting for the function to the radius 1, the corresponding function to the radius R divided by R, and reducing, the proportions (1) and (2) will be of the same form as before substitution, and hence are true for any radius.

From proportions (1) and (2), we find,

$$(3)\quad b=\frac{a \sin B}{\sin A}, \qquad (4)\quad c=\frac{a \sin C}{\sin A}.$$

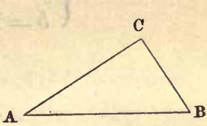
Applying logarithms to (3) and (4), we have,

$$(5)\quad \log b=\log a+\log \sin B+a.\,c.\,\log \sin A-10.$$

$$(6)\quad \log c=\log a+\log \sin C+a.\,c.\,\log \sin A-10.$$

70. Examples.

1. Given $\begin{cases} A=35^\circ\ 45'. \\ B=45^\circ\ 28'. \\ a=7985. \end{cases}$ Req. $\begin{cases} C. \\ b. \\ c. \end{cases}$



$$C=180^\circ-(A+B)=180^\circ-81^\circ\ 13'=98^\circ\ 47'.$$

Since the sine of the angle opposite the given side is to the sine of the angle opposite the required side as the given side is to the required side, we have the proportion,

$$\sin A : \sin B :: a : b, \quad \therefore\ b=\frac{a \sin B}{\sin A}.$$

$$\therefore\quad \log b=\log a+\log \sin B+a.\,c.\,\log \sin A-10.$$

$\log a$ (7985)	$=3.90227$
$\log \sin B$ (45° 28′)	$=9.85299$
$a.\,c.\,\log \sin A$ (35° 45′)	$=0.23340$
$\log b$	$=3.98866 \quad \therefore\ b=9742.25.$

In like manner we have the proportion,

$$\sin A : \sin C :: a : c, \quad \therefore\ c=\frac{a \sin C}{\sin A}.$$

$\therefore \log c = \log a + \log \sin C + a.c. \log \sin A - 10.$

$$\begin{aligned} &\log a \ (7985) &&= 3.90227 \\ &\log \sin C \ (98° \ 47') &&= 9.99488 \\ a.c. \ &\log \sin A \ (35° \ 45') &&= \underline{0.23340} \\ &\log c &&= 4.13055 \quad \therefore c = 13506.88. \end{aligned}$$

In finding log sin 98° 47′, take the supplement of 98° 47, which is 81° 13′, and find log sin 81° 13′.

2. Given $\begin{cases} A = 50° \ 30' \ 40''. \\ B = 70° \ 45' \ 30''. \\ a = 478.35 \ yd. \end{cases}$ Req. $\begin{cases} C = 58° \ 43' \ 50''. \\ b = 585.2 \ yd. \\ c = 529.8 \ yd. \end{cases}$

3. Given $\begin{cases} B = 65° \ 25' \ 35''. \\ C = 60° \ 28' \ 34''. \\ b = 12.25 \text{ miles}. \end{cases}$ Req. $\begin{cases} A = 54° \ 05' \ 51''. \\ c = 11.72 \text{ miles}. \\ a = 10.91 \text{ miles}. \end{cases}$

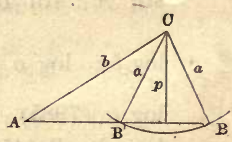
71. Case II.

Given two sides and an angle opposite one of them, required the remaining parts.

1. When the Given Angle is Acute.

Let the sides a and b and the angle A be given, and the remaining parts be required.

Let the perpendicular p be drawn from C to the opposite side. Then we shall have,

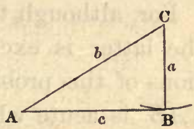


$$p = b \sin A.$$

1*st*. If $a > p$ and $a < b$, there will be two solutions.

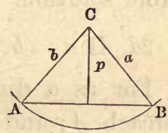
For, if with C as a center and a as radius a circumference be described, it will intersect the side opposite C in two points, B and B', and either triangle, ABC or $AB'C$ will fulfill the conditions of the problem, since it will have two sides and an angle opposite one of them the same as those given. Hence, there will be two solutions if a has any value between the limits p and b.

2d. If $a = p$, there will be but one solution.



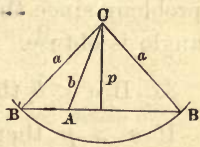
For, as a diminishes and approaches p, the two points B and B' approach; and if $a = p$, B and B' will unite, the arc will be tangent to c, and the two triangles will become one, and there will be one solution.

3d. If $a = b$, there will be but one solution.



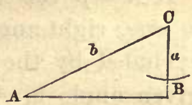
For, as a increases and approaches b, the points B and B' separate, the triangle ABC increases, and the triangle $AB'C$ decreases; and when a becomes equal to b, the triangle $AB'C$ vanishes, and there remains but one triangle, or there is but one solution.

4th. If $a > b$, there will be but one solution.



For, although there are two triangles ABC and $AB'C$, the latter is excluded by the condition that the given angle A is acute, since CAB' is obtuse, and there remains but one triangle ABC which satisfies the conditions, or there is but one solution.

5th. If $a < p$, there will be no solution.

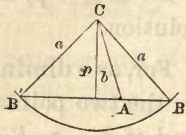


For the arc described with C as center and a as radius will neither intersect the opposite side nor be tangent to it. The triangle can not be constructed, or there will be no solution.

2. When the Given Angle is Obtuse.

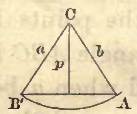
1*st*. If $a > b$ there will be but one solution.

For, although there are two triangles ABC and $AB'C$, the latter is excluded by the conditions of the problem, since the angle CAB' is acute while the given angle is obtuse. There remains but one triangle, ABC, which satisfies all the conditions of the problem, or there is but one possible solution.



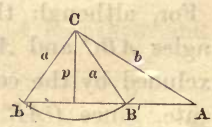
2*d*. If $a = b$ there will be no solution.

For as a diminishes and approaches b, B will approach A; and when a becomes equal to b, B will unite with A, and the triangle ABC will vanish. The triangle $AB'C$ will remain, but will be excluded by the conditions of the problem, since the angle CAB' is acute while the given angle is obtuse.

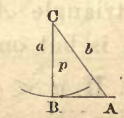


3*d*. If $a < b$ there will be no solution; for then,

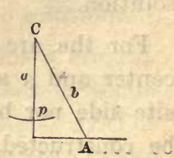
If $a > p$ there will be two triangles, $AB'C$ and $AB''C$, but both are excluded by the condition that the given angle is obtuse.



If $a = p$ the two triangles reduce to one, right-angled at B, which is excluded by the condition that the given angle is obtuse.



If $a < p$ no triangle can be constructed with the given parts, and there will be no solution.



72. Summary of Results.

1. *When* $A < 90°$.

Two Solutions, If $a > p$ and $a < b$.

One Solution, $\begin{cases} 1st. \text{ If } a = p. \\ 2d. \text{ If } a = b. \\ 3d. \text{ If } a > b. \end{cases}$

No Solution, If $a < p$.

2. *When* $A > 90°$.

One Solution, If $a > b$.

No Solution, $\begin{cases} 1st. \text{ If } a = b. \\ 2d. \text{ If } a < b. \end{cases}$

73. Method of Computation.

Reversing the order of the couplets of the proportion in Case I, we have

$$(1) \quad a : b :: \sin A : \sin B.$$

Hence, *The side opposite the given angle is to the side opposite the required angle, as the sine of the given angle is to the sine of the required angle.*

$$(1) \text{ gives} \quad (2) \quad \sin B = \frac{b \sin A}{a}.$$

$$\therefore (3) \quad \log \sin B = \log b + \log \sin A + a.\ c.\ \log a - 10.$$

If there is but one solution, take from the table the angle B corresponding to $\log \sin B$; if there are two solutions, take B and its supplement B', for both correspond to $\log \sin B$.

We find C from the formula,

$$C = 180° - (A + B) \text{ or } C = 180° - (A + B').$$

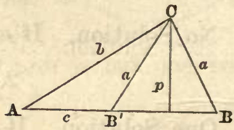
We find c from the proportion,

$$\sin A : \sin C :: a : c, \quad \therefore c = \frac{a \sin C}{\sin A}.$$

$$\therefore \log c = \log a + \log \sin C + a.\ c.\ \log \sin A - 10.$$

74. Examples.

1. Giv. $\begin{cases} a = 9.25. \\ b = 12.56. \\ A = 30° 25'. \end{cases}$ Req. $\begin{cases} B. \\ C. \\ c. \end{cases}$



$$p = b \sin A.$$

Introducing R and applying logarithms, we have

$$\log p = \log b + \log \sin A - 10.$$

log b (12.56)	= 1.09899
log sin A (30° 25′)	= 9.70439
log p	= 0.80338 $\therefore p = 6.3589.$

Since $a > p$ and $a < b$, there are two solutions.

Since the side opposite the given angle is to the side opposite the required angle as the sine of the given angle is to the sine of the required angle, we have the proportion,

$$a : b :: \sin A : \sin B, \quad \therefore \sin B = \frac{b \sin A}{a}.$$

$$\log \sin B = \log b + \log \sin A + a.\ c.\ \log a - 10.$$

log b (12.56)	= 1.09899	
log sin A (30° 25′)	= 9.70439	
$a.\ c.$ log a (9.25)	= 9.03386	
log sin B	= 9.83724	$\therefore \begin{cases} B = 43° 25' 41''. \\ B' = 136° 34' 19''. \end{cases}$

$$C = 180° - (A + B) = 106° \ 9' \ 19'',$$

$$C' = 180° - (A + B') = 13° \ 0' \ 41''.$$

$$\sin A : \sin C :: a : c, \quad \therefore c = \frac{a \sin C}{\sin A}.$$

$$\log c = \log a + \log \sin C + a.c. \log \sin A - 10.$$

Taking the value of C, we have,

$$\begin{aligned} &\log a \ (9.25) &&= 0.96614 \\ &\log \sin C \ (106° \ 9' \ 19'') &&= 9.98250 \\ a.c. &\log \sin A \ (30° \ 25') &&= \underline{0.29561} \\ &\log c &&= 1.24425 \quad \therefore c = 17.549. \end{aligned}$$

Taking the value of C', we have,

$$\begin{aligned} &\log a \ (9.25) &&= 0.96614 \\ &\log \sin C' \ (13° \ 0' \ 41'') &&= 9.35246 \\ a.c. &\log \sin A \ (30° \ 25') &&= \underline{0.29561} \\ &\log c &&= 0.61421 \quad \therefore c = 4.1135. \end{aligned}$$

2. Given $\left\{ \begin{array}{l} a = 20.35. \\ b = 20.35. \\ A = 52° \ 35' \ 27''. \end{array} \right\}$ Req. $\left\{ \begin{array}{l} B = 52° \ 35' \ 27''. \\ C = 74° \ 49' \ 06''. \\ c = 24.725. \end{array} \right.$

3. Given $\left\{ \begin{array}{l} a = 645.8. \\ b = 234.5. \\ A = 48° \ 35'. \end{array} \right\}$ Req. $\left\{ \begin{array}{l} B = 15° \ 48' \ 04''. \\ C = 115° \ 36' \ 56''. \\ c = 776.53. \end{array} \right.$

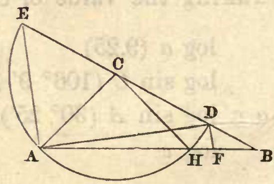
4. Given $\left\{ \begin{array}{l} a = 17. \\ b = 40.25. \\ A = 27° \ 43' \ 15''. \end{array} \right\}$ Req. $\left\{ \begin{array}{l} B. \\ C. \\ c. \end{array} \right\}$ No Solution.

5. Given $\left\{ \begin{array}{l} a = 94.26. \\ b = 126.72. \\ A = 27° \ 50'. \end{array} \right\}$ Req. $\left\{ \begin{array}{l} B = \left\{ \begin{array}{l} 38° \ 52' \ 46''. \\ 141° \ 7' \ 14''. \end{array} \right. \\ C = \left\{ \begin{array}{l} 113° \ 17' \ 14''. \\ 11° \ 2' \ 46''. \end{array} \right. \\ c = \left\{ \begin{array}{l} 185.439. \\ 38.682. \end{array} \right. \end{array} \right.$

6. Given $\begin{cases} a = 1800. \\ b = 2000. \\ B = 111° \ 15'. \end{cases}$ Req. $\begin{cases} A = 57° \ 0' \ 50''. \\ C = 11° \ 44' \ 10''. \\ c = 436.49. \end{cases}$

75. Case III.

Given two sides and their included angle, required the remaining parts.



Let ABC be a triangle, and let the sides opposite the angles A, B, C, be denoted, respectively, by a, b, c. Let a and b, and their included angle C, be given, and the remaining parts, A, B, and c, required.

The sum of the angles A and B is found from the formula,

$$A + B = 180° - C.$$

With C as a center, and b, the shorter of the two given sides, as a radius, describe a circumference cutting a in D, a produced in E, and c in H. Draw AE, AD, CH, and DF parallel to AE. The angle DAE is a right angle, since it is inscribed in a semi-circle; hence, its alternate angle, ADF, is also a right angle.

The angle ACE being exterior to the triangle ABC, is equal to $A + B$. But ACE having its vertex at the center, is measured by the intercepted arc AE. The inscribed angle ADE is measured by one-half the arc AE; hence, $ADE = \frac{1}{2} ACE = \frac{1}{2}(A + B)$.

$CH = CA$, since they are radii of the same circle; hence, the angle $CHA = A$. The angle CHA being exterior to the triangle CHB is equal to $HCB + B$; hence,

$$HCB + B = A. \quad \therefore \quad HCB = A - B.$$

But HCB, having its vertex at the center, is measured by the intercepted arc DH; and DAF, being an inscribed angle, is measured by one-half the arc DH; hence, $DAF = \frac{1}{2} HCB = \frac{1}{2}(A - B)$.

In the right triangles ADE and ADF we have

$$AE = AD \tan ADE = AD \tan \tfrac{1}{2}(A + B).$$

$$DF = AD \tan DAF = AD \tan \tfrac{1}{2}(A - B).$$

From the similar triangles, ABE and FBD, we have

$$BE : BD :: AE : DF.$$

Since $CE = CA$, $BE = BC + CA = a + b$.

Since $CD = CA$, $BD = BC - CA = a - b$.

Substituting the values of BE, BD, AE, and DF in the above proportion, and omitting the common factor AD in the second couplet, we have

$$a + b : a - b :: \tan \tfrac{1}{2}(A + B) : \tan \tfrac{1}{2}(A - B).$$

Hence, *In any plane triangle, the sum of the sides including an angle is to their difference as the tangent of half the sum of the other two angles is to the tangent of half their difference.*

We find from the proportion, the equation

$$\tan \tfrac{1}{2}(A - B) = \frac{(a - b) \tan \frac{1}{2}(A + B)}{a + b}.$$

$$\therefore \log \tan \tfrac{1}{2}(A - B) = \log (a - b) + \log \tan \tfrac{1}{2}(A + B) + a.\,c. \log (a + b) - 10.$$

We have now found $\frac{1}{2}(A + B)$ and $\frac{1}{2}(A - B)$.

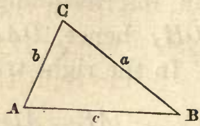
$$A = \tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B), \quad B = \tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B).$$

$$\sin A : \sin C :: a : c, \quad \therefore c = \frac{a \sin C}{\sin A}.$$

$$\therefore \log c = \log a + \log \sin C + a.\,c. \log \sin A - 10.$$

76. Examples.

1. Given $\begin{cases} a = 37.56. \\ b = 23.75. \\ C = 68° \ 25'. \end{cases}$ Req. $\begin{cases} A. \\ B. \\ c. \end{cases}$



$$A + B = 180° - C = 111° \ 35'.$$

$$a + b : a - b :: \tan \tfrac{1}{2}(A + B) : \tan \tfrac{1}{2}(A - B).$$

$$\therefore \quad \tan \tfrac{1}{2}(A - B) = \frac{(a - b) \tan \frac{1}{2}(A + B)}{a + b}.$$

$$\therefore \quad \log \tan \tfrac{1}{2}(A - B) = \log (a - b) + \log \tan \tfrac{1}{2}(A + B) + a.\,c.\, \log (a + b) - 10.$$

$\log (a - b)$ (13.81)	$= 1.14019$
$\log \tan \frac{1}{2}(A + B)$ (55° 47′ 30″)	$= 10.16761$
$a.\,c.\ \log (a + b)$ (61.31)	$= 8.21247$
$\log \tan \frac{1}{2}(A - B)$	$= 9.52027$

$$\therefore \quad \tfrac{1}{2}(A - B) = 18° \ 19' \ 55''.$$

$$A = \tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B) = 74° \ 7' \ 25''.$$

$$B = \tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B) = 37° \ 27' \ 35''.$$

$$\sin A : \sin C :: a : c, \quad \therefore \ c = \frac{a \sin C}{\sin A}.$$

$$\log c = \log a + \log \sin C + a.\,c.\, \log \sin A - 10.$$

$\log a$ (37.56)	$= 1.57473$
$\log \sin C$ (68° 25′)	$= 9.96843$
$a.\,c.\ \log \sin A$ (74° 7′ 25″)	$= 0.01689$
$\log c$	1.56005, $\therefore \ c = 36.312.$

2. Given $\begin{cases} a = 996.63. \\ b = 712.83. \\ C = 72° \ 29' \ 48''. \end{cases}$ Req. $\begin{cases} A = 66° \ 30' \ 37''. \\ B = 40° \ 59' \ 35''. \\ c = 1036.35. \end{cases}$

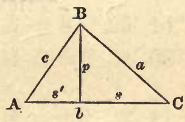
3. Given $\begin{cases} b = 776.525. \\ c = 234.5. \\ A = 48° \ 35'. \end{cases}$ Req. $\begin{cases} B = 115° \ 36' \ 56''. \\ C = 15° \ 48' \ 04''. \\ a = 645.8. \end{cases}$

4. Given $\begin{cases} a = 11.7209. \\ c = 10.9232. \\ B = 65° \ 25' \ 35''. \end{cases}$ Req. $\begin{cases} A = 60° \ 25' \ 34''. \\ C = 54° \ 08' \ 51''. \\ b = 12.256. \end{cases}$

77. Case IV.

Given the three sides of a triangle, required the angles.

Let ABC be a triangle, take the longest side for the base, and draw the perpendicular p from the vertex B to the base.



Denote the segments of the base by s and s' respectively.

Then, (1) $c^2 - s'^2 = p^2$, and (2) $a^2 - s^2 = p^2$.

$\therefore$ (3) $c^2 - s'^2 = a^2 - s^2$, $\therefore$ (4) $s^2 - s'^2 = a^2 - c^2$.

$\therefore$ (5) $(s + s')(s - s') = (a + c)(a - c)$.

$\therefore$ (6) $s + s' : a + c :: a - c : s - s'$.

Hence, *The sum of the segments of the base is to the sum of the other sides as the difference of those sides is to the difference of the segments.*

$$(6) \text{ gives } (7) \ s - s' = \frac{(a + c)(a - c)}{s + s'}.$$

$$\therefore (8) \ \log (s - s') = \log (a + c) + \log (a - c) + a.\,c.\,\log (s + s') - 10.$$

In case the sides of the triangle are small, find $s - s'$ from (7); otherwise, it will be more convenient to employ (8).

Having $s+s'$ and $s-s'$, we find s and s' thus,

(9) $s=\frac{1}{2}(s+s')+\frac{1}{2}(s-s')$, (10) $s'=\frac{1}{2}(s+s')-\frac{1}{2}(s-s')$.

(11) $\cos A=\frac{s'}{c}$, (12) $\cos C=\frac{s}{a}\cdot$

Introducing R, reducing, and applying logarithms,

(13) $\log\cos A=10+\log s'-\log c$.

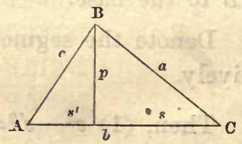
(14) $\log\cos C=10+\log s-\log a$.

From which we find A and C.

Then, (15) $B=180°-(A+C)$.

78. Examples.

1. Given $\begin{cases} a=125. \\ b=150. \\ c=100. \end{cases}$ Req. $\begin{cases} A. \\ B. \\ C. \end{cases}$



$$s+s' : a+c :: a-c : s-s'.$$

$$\therefore \quad s-s'=\frac{(a+c)\,(a-c)}{s+s'}=\frac{225\times 25}{150}=37.5.$$

$$s=\tfrac{1}{2}(s+s')+\tfrac{1}{2}(s-s')=75+18.75=93.75.$$

$$s'=\tfrac{1}{2}(s+s')-\tfrac{1}{2}(s-s')=75-18.75=56.25.$$

$\cos A=\frac{s'}{c}$, or introducing R, $\cos A=\frac{Rs'}{c}\cdot$

$$\therefore \quad \log\cos A=10+\log s'-\log c.$$

$\log s'$ (56.25)	$=1.75012$	
$\log c$ (100)	$=2.00000$	
$\log\cos A$	$=9.75012$	$\therefore A=55°\ 46'\ 18''$.

$\cos C=\frac{s}{a}$, or introducing R, $\cos C=\frac{Rs}{a}\cdot$

$\therefore \log \cos C = 10 + \log s - \log a.$

$$\begin{aligned}
\log s\ (93.75) &= 1.97197 \\
\log a\ (125) &= 2.09691 \\
\log \cos C &= 9.87506 \quad \therefore\ C = 41^\circ\ 24'\ 34''.
\end{aligned}$$

$$B = 180 - (A + C) = 82^\circ\ 49'\ 08''.$$

2. Given $\begin{cases} a = 332.21. \\ b = 345.46. \\ c = 237.61. \end{cases}$ Required $\begin{cases} A = 66^\circ\ 30'\ 35''. \\ B = 72^\circ\ 29'\ 53''. \\ C = 40^\circ\ 59'\ 32''. \end{cases}$

3. Given $\begin{cases} a = 864. \\ b = 1308. \\ c = 1086. \end{cases}$ Required $\begin{cases} A = 41^\circ\ 00'\ 38''. \\ B = 83^\circ\ 25'\ 14''. \\ C = 55^\circ\ 34'\ 08''. \end{cases}$

4. Given $\begin{cases} a = 251.25. \\ b = 302.5. \\ c = 342. \end{cases}$ Required $\begin{cases} A = 45^\circ\ 22'\ 41''. \\ B = 58^\circ\ 58'\ 20''. \\ C = 75^\circ\ 38'\ 59''. \end{cases}$

APPLICATION TO HEIGHTS AND DISTANCES.

79. Definitions.

1. **A horizontal plane** is a plane parallel to the horizon.

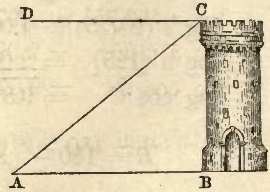
2. **A vertical plane** is a plane perpendicular to a horizontal plane.

3. **A horizontal line** is a line parallel to a horizontal plane.

4. **A vertical line** is a line perpendicular to a horizontal plane.

5. **A horizontal angle** is an angle whose plane is horizontal.

6. **A vertical angle** is an angle whose plane is vertical.



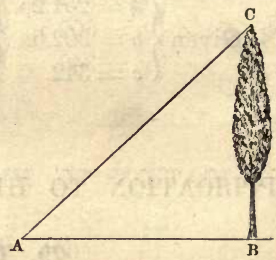
7. **An angle of elevation** is a verticle angle, one of whose sides is horizontal, and the inclined side above the horizontal side. Thus, *BAC*.

8. **An angle of depression** is a vertical angle, one of whose sides is horizontal, and the inclined side below the horizontal side. Thus, *DCA*.

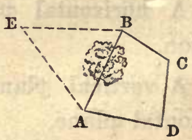
80. Problems.

1. Wishing to know the height of a tree standing on a horizontal plane, I measured from the tree the horizontal line *BA*, 150 ft., and found the angle of elevation, *BAC*, to the top of the tree to be 35° 20′. Required the height of the tree.

Ans. 106.335 ft.



2. In surveying a tract of land, I found it impracticable to measure the side *AB* on account of thick brushwood lying between *A* and *B*. I therefore measured *AE*, 7.50 ch., and *EB*, 8.70 ch., and found the angle $AEB = 38° \ 46'$. Required *AB*.

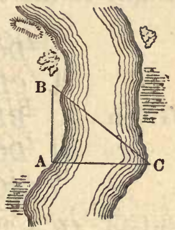


Ans. 5.494 ch.

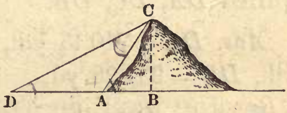
3. One side of a triangular field is double another, their included angle is 60°, and the third side is 15 ch. Required the longest side.

Ans. 17.32 ch.

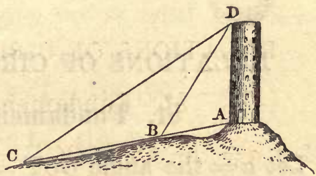
4. Wishing to know the width of a river, from the point A on one bank to the point C on the other bank, I measure the distance AB, 75 yd., and find the angle $BAC = 87° \; 28' \; 30''$, and the angle $ABC = 47° \; 38' \; 25''$. Required AC, the width of the river. *Ans.* 78.53 yd.



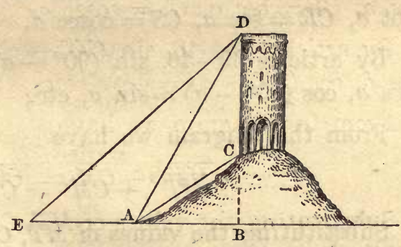
5. I find the angle of elevation, BAC, from the foot of a hill to the top to be $46° \; 25' \; 30''$. Measuring back from the hill, $AD = 500$ ft., I find the angle of elevation $ADC = 25° \; 38' \; 40''$. Required BC, the vertical height of the hill. *Ans.* 441.87 ft.



6. From the foot of a tower standing at the top of a declivity, I measured $AB = 45$ ft., and the angle $ABD = 50° \; 15'$. I also measured, in a straight line with AB, $BC = 68$ ft., and the angle $BCD = 30° \; 45'$. Required AD, the height of the tower. *Ans.* 82.94 ft.



7. Wishing to know the height of a tower standing on a hill, I find the angle of elevation, BAC, to the top of the hill to be $44° \; 35'$, and the angle of elevation to the top of the tower to be $59° \; 48'$. Measuring the horizontal line AE, 275 ft., I find the angle of eleva-

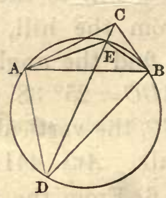


tion to the top of the tower to be 46° 25′. Required the height of the tower. *Ans.* 317.143 ft.

8. Given $\begin{cases} DC = 24 \text{ ch.} \\ CDB = 45°. \\ BDA = 50°. \\ DCA = 48°. \\ ACB = 60°. \end{cases}$

Required $AB = 38.61$ ch.

9. Given $AB = 800$ yd., $AC = 600$ yd., $BC = 400$ yd., $ADC = 33°\ 45'$, $BDC = 22°\ 30'$. Required DA, DC, DB.



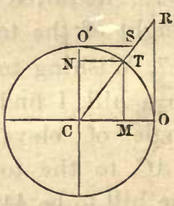
Ans. $DA = 710.15$ yd., $DC = 1042.5$ yd., $DB = 934.28$ yd.

Remark.—Describing the circumference through A, B, D, and drawing AE and BE, $EAB = BDC$, $EBA = ADC$.

RELATIONS OF CIRCULAR FUNCTIONS.

81. Fundamental Formulas.

Let $a =$ the angle $OCT =$ the arc OT, and $CO = CT = 1$. Then, we have $MT = CN = \sin a$, $NT = CM = \cos a$, $MO = \text{vers } a$, $NO' = \text{covers } a$, $OR = \tan a$, $O'S = \cot a$, $CR = \sec a$, $CS = \text{cosec } a$.



By articles 39—46, $\sin(90° - a) = \cos a$, $\cos(90° - a) = \sin a$, etc.

From the diagram we have

$$\overline{MT}^2 + \overline{CM}^2 = \overline{CT}^2.$$

Substituting the values of MT, CM, and CT, we have

$$(1) \quad \sin^2 a + \cos^2 a = 1.$$

Hence, *The square of the sine of any arc plus the square of its co-sine is equal to* 1.

From (1) we have, by transposition,

$$(2) \quad \sin^2 a = 1 - \cos^2 a,$$

$$(3) \quad \cos^2 a = 1 - \sin^2 a. \quad \text{Hence,}$$

1. *The square of the sine of any arc is equal to* 1 *minus the square of its co-sine.*

2. *The square of the co-sine of any arc is equal to* 1 *minus the square of its sine.*

From the diagram we have

$$MO = CO - CM.$$

Substituting the values of MO, CO, and CM, we have

$$(4) \quad \text{vers } a = 1 - \cos a.$$

Hence, *The versed-sine of any arc is equal to* 1 *minus its co-sine.*

$$\therefore \quad \text{vers } (90° - a) = 1 - \cos (90° - a).$$

$$\therefore \quad (5) \quad \text{covers } a = 1 - \sin a.$$

Hence, *The co-versed-sine of any arc is equal to* 1 *minus its sine.*

From the diagram we have

$$CM : CO :: MT : OR,$$

$$\text{or } \cos a : 1 :: \sin a : \tan a.$$

$$\therefore \quad (6) \quad \tan a = \frac{\sin a}{\cos a}.$$

Hence, *The tangent of any arc is equal to its sine divided by its co-sine.*

$$\therefore \quad \tan\,(90^\circ - a) = \frac{\sin\,(90^\circ - a)}{\cos\,(90^\circ - a)}.$$

$$\therefore \quad (7) \quad \cot a = \frac{\cos a}{\sin a}.$$

Hence, *The co-tangent of any arc is equal to its co-sine divided by its sine.*

$$(6) \times (7) = (8) \quad \tan a \cot a = 1.$$

Hence, *The tangent of any arc into its co-tangent is equal to* 1.

$$(8) \div \cot a = (9) \quad \tan a = \frac{1}{\cot a}.$$

Hence, *The tangent of any arc is equal to the reciprocal of its co-tangent.*

$$(8) \div \tan a = (10) \quad \cot a = \frac{1}{\tan a}.$$

Hence, *The co-tangent of any arc is equal to the reciprocal of its tangent.*

$$CM : CO :: CT : CR, \text{ or } \cos a : 1 :: 1 : \sec a.$$

$$\therefore \quad (11) \quad \sec a = \frac{1}{\cos a}.$$

Hence, *The secant of any arc is equal to the reciprocal of its co-sine.*

$$\therefore \quad \sec\,(90^\circ - a) = \frac{1}{\cos\,(90^\circ - a)}.$$

$$\therefore \quad (12) \quad \operatorname{cosec} a = \frac{1}{\sin a}.$$

Hence, *The co-secant of any arc is equal to the reciprocal of its sine.*

$$\because \quad \overline{CR}^2 = \overline{CO}^2 + \overline{OR}^2,$$

$$\therefore \quad (13) \quad \sec^2 a = 1 + \tan^2 a.$$

Hence, *The square of the secant of any arc is equal to* 1, *plus the square of its tangent.*

$$\therefore \quad \sec^2 (90° - a) = 1 + \tan^2 (90° - a).$$

$$\therefore \quad (14) \quad \operatorname{cosec}^2 a = 1 + \cot^2 a.$$

Hence, *The square of the co-secant is equal to* 1, *plus the square of the co-tangent.*

82. Summary of Fundamental Formulas.

1. $\sin^2 a + \cos^2 a = 1.$	9. $\tan a = \dfrac{1}{\cot a}.$
2. $\sin^2 a = 1 - \cos^2 a.$	10. $\cot a = \dfrac{1}{\tan a}.$
3. $\cos^2 a = 1 - \sin^2 a.$	11. $\sec a = \dfrac{1}{\cos a}.$
4. $\text{vers } a = 1 - \cos a.$	12. $\operatorname{cosec} a = \dfrac{1}{\sin a}.$
5. $\text{covers } a = 1 - \sin a.$	
6. $\tan a = \dfrac{\sin a}{\cos a}.$	13. $\sec^2 a = 1 + \tan^2 a.$
7. $\cot a = \dfrac{\cos a}{\sin a}.$	
8. $\tan a \cot a = 1.$	14. $\operatorname{cosec}^2 a = 1 + \cot^2 a.$

83. Problems.

1. Prove that the above formulas become homogeneous by the introduction of *R*.

2. Deduce formulas (5), (7), (12) and (14) from the diagram.

3. Prove that the above formulas are true if *a* is in the second, third, or fourth quadrant.

84. Each Function in Terms of the Others.

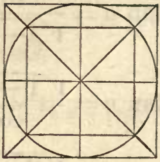
$\sin a = \sqrt{1 - \cos^2 a}.$	$\text{vers } a = 1 - \sqrt{1 - \sin^2 a}.$
$\sin a = \sqrt{2 \text{ vers } a - \text{vers}^2 a}.$	$\text{vers } a = 1 - \cos a.$
$\sin a = 1 - \text{covers } a.$	$\text{vers } a = 1 - \sqrt{2 \text{ cvs } a - \text{cvs}^2 a}.$
$\sin a = \frac{\tan a}{\sqrt{1 + \tan^2 a}}.$	$\text{vers } a = 1 - \frac{1}{\sqrt{1 + \tan^2 a}}.$
$\sin a = \frac{1}{\sqrt{1 + \cot^2 a}}.$	$\text{vers } a = 1 - \frac{\cot a}{\sqrt{1 + \cot^2 a}}.$
$\sin a = \frac{\sqrt{\sec^2 a - 1}}{\sec a}.$	$\text{vers } a = \frac{\sec a - 1}{\sec a}.$
$\sin a = \frac{1}{\text{cosec } a}.$	$\text{vers } a = 1 - \frac{\sqrt{\text{cosec}^2 a - 1}}{\text{cosec } a}.$
$\cos a = \sqrt{1 - \sin^2 a}.$	$\text{covers } a = 1 - \sin a.$
$\cos a = 1 - \text{vers } a.$	$\text{covers } a = 1 - \sqrt{1 - \cos^2 a}.$
$\cos a = \sqrt{2 \text{ cvs } a - \text{cvs}^2 a}.$	$\text{cvs } a = 1 - \sqrt{2 \text{ vs } a - \text{vs}^2 a}.$
$\cos a = \frac{1}{\sqrt{1 + \tan^2 a}}.$	$\text{covers } a = 1 - \frac{\tan a}{\sqrt{1 + \tan^2 a}}.$
$\cos a = \frac{\cot a}{\sqrt{1 + \cot^2 a}}.$	$\text{covers } a = 1 - \frac{1}{\sqrt{1 + \cot^2 a}}$
$\cos a = \frac{1}{\sec a}.$	$\text{covers } a = 1 - \frac{\sqrt{\sec^2 a - 1}}{\sec a}.$
$\cos a = \frac{\sqrt{\text{cosec}^2 a - 1}}{\text{cosec } a}.$	$\text{covers } a = \frac{\text{cosec } a - 1}{\text{cosec } a}.$

84. Each Function in Terms of the Others.

$\tan a = \dfrac{\sin a}{\sqrt{1 - \sin^2 a}}.$	$\sec a = \dfrac{1}{\sqrt{1 - \sin^2 a}}.$
$\tan a = \dfrac{\sqrt{1 - \cos^2 a}}{\cos a}.$	$\sec a = \dfrac{1}{\cos a}.$
$\tan a = \dfrac{\sqrt{2 \text{ vs } a - \text{vs}^2 a}}{1 - \text{vs } a}$	$\sec a = \dfrac{1}{1 - \text{vers } a}.$
$\tan a = \dfrac{1 - \text{cvs } a}{\sqrt{2 \text{ cvs } a - \text{cvs}^2 a}}.$	$\sec a = \dfrac{1}{\sqrt{2 \text{ cvs } a - \text{cvs}^2 a}}$
$\tan a = \dfrac{1}{\cot a}.$	$\sec a = \sqrt{1 + \tan^2 a}.$
$\tan a = \sqrt{\sec^2 a - 1}.$	$\sec a = \dfrac{\sqrt{1 + \cot^2 a}}{\cot a}.$
$\tan a = \dfrac{1}{\sqrt{\text{cosec}^2 a - 1}}.$	$\sec a = \dfrac{\text{cosec } a}{\sqrt{\text{cosec}^2 a - 1}}.$
$\cot a = \dfrac{\sqrt{1 - \sin^2 a}}{\sin a}.$	$\text{cosec } a = \dfrac{1}{\sin a}.$
$\cot a = \dfrac{\cos a}{\sqrt{1 - \cos^2 a}}.$	$\text{cosec } a = \dfrac{1}{\sqrt{1 - \cos^2 a}}$
$\cot a = \dfrac{1 - \text{vs } a}{\sqrt{2 \text{ vs } a - \text{vs}^2 a}}.$	$\text{cosec } a = \dfrac{1}{\sqrt{2 \text{ vs } a - \text{vs}^2 a}}.$
$\cot a = \dfrac{\sqrt{2 \text{ cvs } a - \text{cvs}^2 a}}{1 - \text{cvs } a}.$	$\text{cosec } a = \dfrac{1}{1 - \text{covers } a}.$
$\cot a = \dfrac{1}{\tan a}.$	$\text{cosec } a = \dfrac{\sqrt{1 + \tan^2 a}}{\tan a}.$
$\cot a = \dfrac{1}{\sqrt{\sec^2 a - 1}}$	$\text{cosec } a = \sqrt{1 + \cot^2 a}.$
$\cot a = \sqrt{\text{cosec}^2 a - 1}.$	$\text{cosec } a = \dfrac{\sec a}{\sqrt{\sec^2 a - 1}}.$

85. Functions of Negative Arcs.

We first find the sine and co-sine of $-a$, in terms of the functions of a from the diagram. Then, dividing the sine by the co-sine, the cosine by the sine, taking the reciprocal of the co-sine and the reciprocal of the sine, we have



$$\begin{aligned} \sin(-a) &= -\sin a, & \cos(-a) &= \cos a, \\ \tan(-a) &= -\tan a, & \cot(-a) &= -\cot a, \\ \sec(-a) &= \sec a, & \operatorname{cosec}(-a) &= -\operatorname{cosec} a. \end{aligned}$$

86. Functions of ($n\ 90° \mp a$).

1. Let n be 1 and a be negative.

From the figure of the last article, and by similar processes,

$$\begin{aligned} \sin(90° - a) &= \cos a, & \cos(90° - a) &= \sin a, \\ \tan(90° - a) &= \cot a, & \cot(90° - a) &= \tan a, \\ \sec(90° - a) &= \operatorname{cosec} a, & \operatorname{cosec}(90° - a) &= \sec a. \end{aligned}$$

These relations have already been found, articles 39—46.

2. Let n be 1 and a be positive.

$$\begin{aligned} \sin(90° + a) &= \cos a, & \cos(90° + a) &= -\sin a, \\ \tan(90° + a) &= -\cot a, & \cot(90° + a) &= -\tan a, \\ \sec(90° + a) &= -\operatorname{cosec} a, & \operatorname{cosec}(90° + a) &= \sec a. \end{aligned}$$

3. Let n be 2, and a be negative.

$$\begin{aligned} \sin(180° - a) &= \sin a, & \cos(180° - a) &= -\cos a, \\ \tan(180° - a) &= -\tan a, & \cot(180° - a) &= -\cot a, \\ \sec(180° - a) &= -\sec a, & \operatorname{cosec}(180° - a) &= \operatorname{cosec} a. \end{aligned}$$

4. Let n be 2, and a be positive.

$\sin(180°+a) = -\sin a$, $\quad \cos(180°+a) = -\cos a$,
$\tan(180°+a) = \tan a$, $\quad \cot(180°+a) = \cot a$,
$\sec(180°+a) = -\sec a$, $\quad \operatorname{cosec}(180°+a) = -\operatorname{cosec} a$.

5. Let n be 3, and a be negative.

$\sin(270°-a) = -\cos a$, $\quad \cos(270°-a) = -\sin a$,
$\tan(270°-a) = \cot a$, $\quad \cot(270°-a) = \tan a$,
$\sec(270°-a) = -\operatorname{cosec} a$, $\quad \operatorname{cosec}(270°-a) = -\sec a$.

6. Let n be 3, and a be positive.

$\sin(270°+a) = -\cos a$, $\quad \cos(270°+a) = \sin a$,
$\tan(270°+a) = -\cot a$, $\quad \cot(270°+a) = -\tan a$,
$\sec(270°+a) = \operatorname{cosec} a$, $\quad \operatorname{cosec}(270°+a) = -\sec a$.

7. Let n be 4, and a be negative.

$\sin(360°-a) = -\sin a$, $\quad \cos(360°-a) = \cos a$,
$\tan(360°-a) = -\tan a$, $\quad \cot(360°-a) = -\cot a$,
$\sec(360°-a) = \sec a$, $\quad \operatorname{cosec}(360°-a) = -\operatorname{cosec} a$.

8. Let n be 4, and a be positive.

$\sin(360°+a) = \sin a$, $\quad \cos(360°+a) = \cos a$,
$\tan(360°+a) = \tan a$, $\quad \cot(360°+a) = \cot a$,
$\sec(360°+a) = \sec a$, $\quad \operatorname{cosec}(360°+a) = \operatorname{cosec} a$.

It will be observed that when n is *even*, the functions in the two members of the equations have the same name; and that when n is odd, they have contrary names. The algebraic sign attributed to the second member is determined by the quadrant in which the arc is situated.

Let this article be reviewed, and these principles applied in determining the names and algebraic signs of the second members.

Hence, functions of arcs greater than 90° can be found in terms of functions of arcs less than 90°. Thus,

1. $\sin 120^\circ = \sin(90^\circ + 30^\circ) = \cos 30^\circ.$
2. $\cos 290^\circ = \cos(270^\circ + 20^\circ) = \sin 20^\circ.$
3. $\tan 165^\circ = \tan(180^\circ - 15^\circ) = -\tan 15^\circ.$

If n is integral and positive, prove the following:

4. $\sin[n\ 180^\circ + (-1)^n a] = \sin a.$
5. $\cos(n\ 360^\circ \pm a) = \cos a.$
6. $\tan(n\ 180^\circ + a) = \tan a.$
7. Any function of $(n\ 360^\circ + a) =$ the same function of a, whatever be the value of a.

87. Values of Functions of Particular Arcs.

1. *To find the functions of* 30°.

Since 60° is one-sixth of the circumference, the chord of 60° is equal to one side of a regular inscribed hexagon, which is equal to the radius or 1. But the sine of 30° is equal to one-half the chord of 60°.

$$\therefore (1)\ \sin 30^\circ = \tfrac{1}{2}, \quad \therefore (2)\ \cos 30^\circ = \sqrt{1 - \tfrac{1}{4}} = \tfrac{1}{2}\sqrt{3}.$$

Dividing (1) by (2), then (2) by (1), taking the reciprocals of (2) and (1), we have

$$(3)\ \tan 30^\circ = \frac{1}{\sqrt{3}}, \qquad (4)\ \cot 30^\circ = \sqrt{3}.$$

$$(5)\ \sec 30^\circ = \frac{2}{\sqrt{3}}, \qquad (6)\ \operatorname{cosec} 30^\circ = 2.$$

2. *To find the functions of* 60°.

From article 40, $\sin 60^\circ = \sin(90^\circ - 30^\circ) = \cos 30^\circ$, $\cos 60^\circ = \cos(90^\circ - 30^\circ) = \sin 30^\circ$. Hence,

(1) $\sin 60° = \frac{1}{2}\sqrt{3}$, (2) $\cos 60° = \frac{1}{2}$,

(3) $\tan 60° = \sqrt{3}$, (4) $\cot 60° = \frac{1}{\sqrt{3}}$,

(5) $\sec 60° = 2$, (6) $\operatorname{cosec} 60° = \frac{2}{\sqrt{3}}$.

3. *To find the functions of* 45°.

From Art. 40, $\sin 45° = \sin(90° - 45°) = \cos 45°$; but $\sin^2 45° + \cos^2 45° = 1$,

$\therefore\ 2\sin^2 45° = 1$, $\therefore\ \sin^2 45° = \frac{1}{2}$. Hence,

(1) $\sin 45° = \frac{1}{2}\sqrt{2}$, (2) $\cos 45° = \frac{1}{2}\sqrt{2}$,

(3) $\tan 45° = 1$, (4) $\cot 45° = 1$,

(5) $\sec 45° = \sqrt{2}$, (6) $\operatorname{cosec} 45° = \sqrt{2}$.

Prove the following:

1. $\sec 120° = -2$.
2. $\cos 135° = -\frac{1}{2}\sqrt{2}$.
3. $\sin 300° = -\frac{1}{2}\sqrt{3}$.
4. $\tan 225° = 1$.
5. $\operatorname{cosec} 210° = -2$.
6. $\cot 240° = \frac{1}{\sqrt{3}}$.
7. $\sin 390° = \frac{1}{2}$.
8. $\cos(-120°) = -\frac{1}{2}$.
9. Construct an angle whose tangent is -1.
10. Construct an angle whose sine is $-\frac{1}{2}$.
11. Find all the functions of 150°.

88. Inverse Trigonometric Functions.

If $x = \sin a$, then a is the angle or arc whose sine is x, which is written $a = \sin^{-1} x$, and read a equals the arc whose sine is x.

It must not be supposed that $^{-1}$ is an exponent, and that $\sin^{-1} x = \frac{1}{\sin x}$; this would be a grievous error.

Let the following be read:

$\cos^{-1}x$, $\tan^{-1}x$, $\sec^{-1}x$, $\operatorname{cosec}^{-1}x$, $\sin^{-1}(\cos x)$, $\sin(\sin^{-1}x)$,

$\sin^{-1}x = \operatorname{cosec}^{-1}\frac{1}{x}$, $\cos^{-1}x = \sec^{-1}\frac{1}{x}$, $\tan^{-1}x = \cot^{-1}\frac{1}{x}$.

The above notation is not altogether arbitrary; for let $f(x)$ be any function of x, and let $f[f(x)]$, or, more briefly, let $f^2(x)$ be the same function of $f(x)$, which notation denotes, not the square of $f(x)$, that is, not $[f(x)]^2$, but that the same function is taken of $f(x)$ as of x. Thus, if $f(x) = \sin x$, $f[f(x)] = \sin(\sin x)$, then, in general,

$$(1) \quad f^m f^n(x) = f^{m+n}(x).$$

If $n = 0$, (1) becomes,

$$(2) \quad f^m f^0(x) = f^m(x).$$

$$\therefore \quad (3) \quad f^0(x) = x.$$

If $m = 1$, and $n = -1$, (1) becomes,

$$(4) \quad f f^{-1}(x) = f^0(x) = x.$$

Hence, $f^{-1}(x)$ denotes a quantity whose like function is x.

Hence, if $y = \sin^{-1}x$, $\sin y = \sin(\sin^{-1}x) = x$; that is, y or $\sin^{-1}x$ is an arc whose sine is x.

It would follow from the above that $\sin^2 a$ ought to signify $\sin(\sin a)$, and not $(\sin a)^2$; but since we rarely have $\sin(\sin a)$, it is customary to write $\sin^2 a$ for $(\sin a)^2$, as we are thus saved the trouble of writing the parenthesis.

It would not, of course, do to write $\sin a^2$ for $(\sin a)^2$, for then we should have the sine of the square of an arc for the square of the sine of an arc.

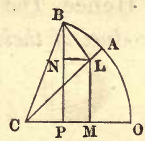
Let the following equations be proved:

1. $\sin^{-1}\frac{\sqrt{3}}{2} = \cos^{-1}\frac{1}{2}$.
2. $\sin^{-1}\frac{1}{2} = \frac{1}{2}\tan^{-1}\sqrt{3}$.
3. $\tan^{-1}\sqrt{3} = \sec^{-1} 2$.
4. $\cos^{-1}\frac{1}{2} = 2\cot^{-1}\sqrt{3}$.
5. $\sin^{-1} 1 = 2\tan^{-1} 1$.
6. $\sec^{-1} 2 = \frac{1}{2}\sec^{-1}(-2)$.

89. Problem.

To find the sine and co-sine of the sum of two angles.

Let $a =$ the angle OCA, and $b =$ the angle ACB. Draw BL perpendicular to CA, BP and LM perpendicular to CO, and LN parallel to CO.



The triangles NBL and MCL are similar, since their sides are respectively perpendicular; hence, the angle NBL opposite the side NL equals the angle MCL opposite the homologous side ML. But $MCL = a$; hence $NBL = a$.

From the diagram we find the following relations:

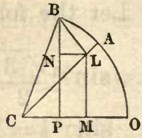
(1) $LB = \sin b$.
(2) $CL = \cos b$.
(3) $PB = ML + NB$.
(4) $PB = \sin OCB = \sin (a + b)$.
(5) $ML = \sin MCL \times CL = \sin a \cos b$.
(6) $NB = \cos NBL \times LB = \cos a \sin b$.

Substituting the values of PB, ML, and NB, found in (4), (5), and (6), in (3), and denoting the formula by (a), we have

$$(a) \quad \sin (a + b) = \sin a \cos b + \cos a \sin b.$$

Hence, *The sine of the sum of two angles is equal to the sine of the first into the co-sine of the second, plus the co-sine of the first into the sine of the second.*

From the diagram we find the following relations:



(1) $CP = CM - NL$.

(2) $CP = \cos OCB = \cos(a + b)$.

(3) $CM = \cos MCL \times CL = \cos a \cos b$.

(4) $NL = \sin NBL \times LB = \sin a \sin b$.

Substituting the values of CP, CM, and NL, found in (2), (3), and (4), in (1), we have

$$(b) \quad \cos(a + b) = \cos a \cos b - \sin a \sin b.$$

Hence, *The co-sine of the sum of two angles is equal to the product of their co-sines minus the product of their sines.*

90. Problems.

1. Prove that formulas (a) and (b) become homogeneous by introducing R.

2. Prove that formulas (a) and (b) are true when $(a + b)$ is in the second quadrant.

3. Prove that formulas (a) and (b) are true when $(a + b)$ is in the third quadrant.

4. Prove that formulas (a) and (b) are true when $(a + b)$ is in the fourth quadrant.

5. Deduce formula (b) from formula (a) by substituting $90° - a$ for a, and $-b$ for b, and reducing by articles 85—86.

6. Develop $\sin(45° + 30°)$ by formula (a).

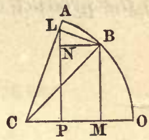
7. Develop $\cos 105°$ by formula (b).

91. Problem.

To find the sine and co-sine of the difference of two angles.

Let $a =$ the angle OCA, and $b =$ the angle BCA.

Draw BL perpendicular to CA, LP and BM perpendicular to CO, and BN parallel to CO.



The triangles NLB and PCL are similar, since their sides are respectively perpendicular; hence, the angle NLB, opposite the side NB, equals the angle PCL opposite the homologous side PL. But the angle $PCL = a$; hence, the angle $NLB = a$. Then we shall have

(1) $LB = \sin b$.
(2) $CL = \cos b$.
(3) $MB = PL - NL$.
(4) $MB = \sin OCB = \sin (a - b)$.
(5) $PL = \sin PCL \times CL = \sin a \cos b$.
(6) $NL = \cos NLB \times LB = \cos a \sin b$.

Substituting the values of MB, PL, and NL, found in (4), (5), and (6), in (3), we have

$$(c) \quad \sin (a - b) = \sin a \cos b - \cos a \sin b.$$

Hence, *The sine of the difference of two angles is equal to the sine of the first into the co-sine of the second, minus the co-sine of the first into the sine of the second.*

From the diagram we find the following relations:

(1) $CM = CP + NB$.
(2) $CM = \cos OCB = \cos (a - b)$.
(3) $CP = \cos PCL \times CL = \cos a \cos b$.
(4) $NB = \sin NLB \times LB = \sin a \sin b$.

Substituting in (1) the values of CM, CP, and NB found in (2), (3), and (4), we have

$$(d) \quad \cos(a-b) = \cos a \cos b + \sin a \sin b.$$

Hence, *The co-sine of the difference of two angles is equal to the product of their co-sines, plus the product of their sines.*

92. Problems.

1. Prove that formulas (c) and (d) become homogeneous by introducing R.

2. Deduce formulas (c) and (d) from (a) and (b), respectively, by substituting $-b$ for b, and reducing by article 85.

3. Prove that formulas (c) and (d) are true when $(a-b)$ is in the second quadrant.

4. Prove that formulas (c) and (d) are true when $(a-b)$ is in the third quadrant.

5. Prove that formulas (c) and (d) are true when $(a-b)$ is in the fourth quadrant.

93. Problem.

To find the tangent and co-tangent of the sum or difference of two angles.

Dividing (a) by (b), we have

$$\frac{\sin(a+b)}{\cos(a+b)} = \frac{\sin a \cos b + \cos a \sin b}{\cos a \cos b - \sin a \sin b}.$$

Dividing both terms of the fraction in the second member by $\cos a \cos b$, reducing, and recollecting that the sine of an arc divided by its co-sine is equal to its tangent, we have

$$(e)\quad \tan(a+b)=\frac{\tan a+\tan b}{1-\tan a\tan b}.$$

Hence, *The tangent of the sum of two angles is equal to the sum of their tangents, divided by* 1 *minus the product of their tangents.*

Dividing (*b*) by (*a*), and reducing, we have

$$(f)\quad \cot(a+b)=\frac{\cot a\cot b-1}{\cot a+\cot b}.$$

Hence, *The co-tangent of the sum of two angles is equal to the product of their co-tangents, minus* 1, *divided by the sum of their co-tangents.*

Dividing (*c*) by (*d*), and reducing, we have

$$(g)\quad \tan(a-b)=\frac{\tan a-\tan b}{1+\tan a\tan b}.$$

Hence, *The tangent of the difference of two angles is equal to the tangent of the first minus the tangent of the second, divided by* 1 *plus the product of their tangents.*

Dividing (*d*) by (*c*), and reducing, we have

$$(h)\quad \cot(a-b)=\frac{\cot a\cot b+1}{\cot b-\cot a}.$$

Hence, *The co-tangent of the difference of two angles is equal to the product of their co-tangents, plus* 1, *divided by the co-tangent of the second, minus the co-tangent of the first.*

94. Problems.

1. Prove that (*e*), (*f*), (*g*), (*h*) become homogeneous by introducing *R*.

2. Deduce (g) from (e) by substituting $-b$ for b.

3. Deduce (h) from (f) by substituting $-b$ for b.

4. Deduce (f) from (e) by taking the reciprocal of each member, substituting $\frac{1}{\cot a}$ for $\tan a$, $\frac{1}{\cot b}$ for $\tan b$, and reducing.

5. Deduce, in like manner, (h) from (g).

6. Find the value of $\sin(a+b+c)$ by substituting $b+c$ for b in (a).

7. Find the value of $\cos(a+b+c)$ by substituting $b+c$ for b in (b).

8. Find the value of $\tan(a+b+c)$ by substituting $b+c$ for b in (e).

9. Find the value of $\cot(a+b+c)$ by substituting $b+c$ for b in (f).

95. Functions of Double and Half Angles.

Making $b=a$ in (a), (b), (e), and (f), we have

(1) $\sin 2a = 2\sin a \cos a.$

(2) $\cos 2a = \cos^2 a - \sin^2 a.$

(3) $\tan 2a = \dfrac{2\tan a}{1-\tan^2 a}.$

(4) $\cot 2a = \dfrac{\cot^2 a - 1}{2\cot a}.$

Substituting $\frac{1}{2}a$ for a in (1), (2), (3), (4), we have

(5) $\sin a = 2\sin\frac{1}{2}a \cos\frac{1}{2}a.$

(6) $\cos a = \cos^2\frac{1}{2}a - \sin^2\frac{1}{2}a.$

$$(7)\quad \tan a = \frac{2\tan\frac{1}{2}a}{1-\tan^2\frac{1}{2}a}.$$

$$(8)\quad \cot a = \frac{\cot^2\frac{1}{2}a - 1}{2\cot\frac{1}{2}a}.$$

Substituting $1-\sin^2\frac{1}{2}a$ for $\cos^2\frac{1}{2}a$, then $1-\cos^2\frac{1}{2}a$ for $\sin^2\frac{1}{2}a$, in (6), and reducing, we have

$$(9)\quad 1-\cos a = 2\sin^2\frac{1}{2}a.$$

$$(10)\quad 1+\cos a = 2\cos^2\frac{1}{2}a.$$

$$\therefore\quad (11)\quad \sin\frac{1}{2}a = \sqrt{\frac{1-\cos a}{2}}.$$

$$\therefore\quad (12)\quad \cos\frac{1}{2}a = \sqrt{\frac{1+\cos a}{2}}.$$

Dividing (11) by (12), then (12) by (11), we have

$$(13)\quad \tan\frac{1}{2}a = \sqrt{\frac{1-\cos a}{1+\cos a}}.$$

$$(14)\quad \cot\frac{1}{2}a = \sqrt{\frac{1+\cos a}{1-\cos a}}.$$

Dividing (5) first by (10), then by (9), and transposing, we have

$$(15)\quad \tan\frac{1}{2}a = \frac{\sin a}{1+\cos a}.$$

$$(16)\quad \cot\frac{1}{2}a = \frac{\sin a}{1-\cos a}.$$

Taking the reciprocal of (16), then of (15), we have

$$(17)\quad \tan\frac{1}{2}a = \frac{1-\cos a}{\sin a}.$$

$$(18)\quad \cot\frac{1}{2}a = \frac{1+\cos a}{\sin a}.$$

Let the formulas of this article be expressed in words.

96. Consequences of (a), (b), (c), (d).

Taking the sum and difference of (*a*) and (*c*), (*d*) and (*b*), we have

(1) $\sin(a+b)+\sin(a-b)=2\sin a\cos b.$
(2) $\sin(a+b)-\sin(a-b)=2\cos a\sin b.$
(3) $\cos(a+b)+\cos(a-b)=2\cos a\cos b.$
(4) $\cos(a-b)-\cos(a+b)=2\sin a\sin b.$

$$\text{Let } \left\{\begin{array}{l} a+b=s, \\ a-b=d, \end{array}\right\} \quad \text{then } \left\{\begin{array}{l} a=\tfrac{1}{2}(s+d). \\ b=\tfrac{1}{2}(s-d). \end{array}\right\}$$

Substituting the values of $a+b$, $a-b$, a, and b, in (1), (2), (3), and (4), we have

(5) $\sin s+\sin d=2\sin\tfrac{1}{2}(s+d)\cos\tfrac{1}{2}(s-d).$
(6) $\sin s-\sin d=2\cos\tfrac{1}{2}(s+d)\sin\tfrac{1}{2}(s-d).$
(7) $\cos s+\cos d=2\cos\tfrac{1}{2}(s+d)\cos\tfrac{1}{2}(s-d).$
(8) $\cos d-\cos s=2\sin\tfrac{1}{2}(s+d)\sin\tfrac{1}{2}(s-d).$

By formula (5) of the preceding article we have

(9) $\sin(s+d)=2\sin\tfrac{1}{2}(s+d)\cos\tfrac{1}{2}(s+d).$
(10) $\sin(s-d)=2\sin\tfrac{1}{2}(s-d)\cos\tfrac{1}{2}(s-d).$

Dividing each of these formulas by each of the following, we have

(11) $$\frac{\sin s+\sin d}{\sin s-\sin d}=\frac{\sin\frac{1}{2}(s+d)\cos\frac{1}{2}(s-d)}{\cos\frac{1}{2}(s+d)\sin\frac{1}{2}(s-d)}=\frac{\tan\frac{1}{2}(s+d)}{\tan\frac{1}{2}(s-d)}.$$

(12) $$\frac{\sin s+\sin d}{\cos s+\cos d}=\frac{\sin\frac{1}{2}(s+d)}{\cos\frac{1}{2}(s+d)}=\tan\tfrac{1}{2}(s+d).$$

(13) $$\frac{\sin s+\sin d}{\cos d-\cos s}=\frac{\cos\frac{1}{2}(s-d)}{\sin\frac{1}{2}(s-d)}=\cot\tfrac{1}{2}(s-d).$$

(14) $$\frac{\sin s+\sin d}{\sin(s+d)}=\frac{\cos\frac{1}{2}(s-d)}{\cos\frac{1}{2}(s+d)}.$$

(15) $$\frac{\sin s + \sin d}{\sin (s - d)} = \frac{\sin \frac{1}{2}(s+d)}{\sin \frac{1}{2}(s-d)}.$$

(16) $$\frac{\sin s - \sin d}{\cos s + \cos d} = \frac{\sin \frac{1}{2}(s-d)}{\cos \frac{1}{2}(s-d)} = \tan \tfrac{1}{2}(s-d).$$

(17) $$\frac{\sin s - \sin d}{\cos d - \cos s} = \frac{\cos \frac{1}{2}(s+d)}{\sin \frac{1}{2}(s+d)} = \cot \tfrac{1}{2}(s+d).$$

(18) $$\frac{\sin s - \sin d}{\sin (s + d)} = \frac{\sin \frac{1}{2}(s-d)}{\sin \frac{1}{2}(s+d)}.$$

(19) $$\frac{\sin s - \sin d}{\sin (s - d)} = \frac{\cos \frac{1}{2}(s+d)}{\cos \frac{1}{2}(s-d)}.$$

(20) $$\frac{\cos s + \cos d}{\cos d - \cos s} = \frac{\cot \frac{1}{2}(s+d)}{\tan \frac{1}{2}(s-d)}.$$

(21) $$\frac{\cos s + \cos d}{\sin (s + d)} = \frac{\cos \frac{1}{2}(s-d)}{\sin \frac{1}{2}(s+d)}.$$

(22) $$\frac{\cos s + \cos d}{\sin (s - d)} = \frac{\cos \frac{1}{2}(s+d)}{\sin \frac{1}{2}(s-d)}.$$

(23) $$\frac{\cos d - \cos s}{\sin (s + d)} = \frac{\sin \frac{1}{2}(s-d)}{\cos \frac{1}{2}(s+d)}.$$

(24) $$\frac{\cos d - \cos s}{\sin (s - d)} = \frac{\sin \frac{1}{2}(s+d)}{\cos \frac{1}{2}(s-d)}.$$

(25) $$\frac{\sin (s + d)}{\sin (s - d)} = \frac{\sin \frac{1}{2}(s+d) \cos \frac{1}{2}(s+d)}{\sin \frac{1}{2}(s-d) \cos \frac{1}{2}(s-d)}.$$

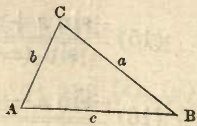
Formula (11) gives the proportion,

$$\sin s + \sin d : \sin s - \sin d :: \tan \tfrac{1}{2}(s+d) : \tan \tfrac{1}{2}(s-d).$$

Hence, *The sum of the sines of two angles is to their difference as the tangent of one-half the sum of the angles is to the tangent of one-half their difference.*

Let us apply this principle in solving triangles when two sides and their included angle are given. Article 75.

$a : b :: \sin A : \sin B.$



$\therefore \quad a+b : a-b :: \sin A + \sin B : \sin A - \sin B.$

$\sin A + \sin B : \sin A - \sin B :: \tan\frac{1}{2}(A+B) : \tan\frac{1}{2}(A-B).$

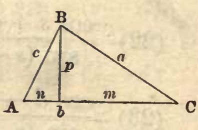
$\therefore \quad a+b : a-b :: \tan \frac{1}{2}(A+B) : \tan \frac{1}{2}(A-B).$

97. Theorem.

The square of any side of a triangle is equal to the sum of the squares of the other sides, minus twice their product into the co-sine of their included angle.

1*st*. When the angle is acute.

(1) $m = b - n.$



$(1)^2 =$ (2) $m^2 = b^2 + n^2 - 2\,bn.$

(3) $p^2 = p^2.$

(2)+(3)=(4) $m^2 + p^2 = b^2 + n^2 + p^2 - 2\,bn.$

But $m^2 + p^2 = a^2$ and $n^2 + p^2 = c^2$, $\therefore$ (4) becomes

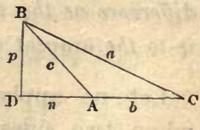
(5) $a^2 = b^2 + c^2 - 2\,bn.$

But $n = c \cos A$, which substituted in (5) gives

(6) $a^2 = b^2 + c^2 - 2\,bc \cos A.$

2*d*. When the angle is obtuse.

(1) $m = b + n.$



$(1)^2 =$ (2) $m^2 = b^2 + n^2 + 2\,bn.$

(3) $p^2 = p^2.$

(2) + (3) = (4) $m^2 + p^2 = b^2 + n^2 + p^2 + 2\,bn.$

But $m^2+p^2=a^2$ and $n^2+p^2=c^2$, $\therefore$ (4) becomes

$$(5) \quad a^2=b^2+c^2+2\,bn.$$

But $n=c\cos BAD=-c\cos BAC=-c\cos A$.

$$\therefore \quad (6) \quad a^2=b^2+c^2-2\,bc\cos A.$$

98. Problem.

To find the angles of a triangle when the sides are given.

From either formula (6) of the last article we have

$$(1) \quad \cos A=\frac{b^2+c^2-a^2}{2\,bc}.$$

Hence, *The co-sine of any angle of a triangle is equal to the sum of the squares of the adjacent sides, minus the square of the opposite side, divided by twice the rectangle of the adjacent sides.*

Formula (1) gives the natural co-sine of A; hence, A can be found. But it is best to place the formula under such a form as to adapt it to logarithmic computation.

Adding 1 to both members of (1) we have

$$(2) \quad 1+\cos A=\frac{(b+c)^2-a^2}{2\,bc}=\frac{(a+b+c)\,(b+c-a)}{2\,bc}.$$

But $1+\cos A=2\cos^2\tfrac{1}{2}A$. Article 95, (10).

Let $a+b+c=p$, then $\dfrac{(a+b+c)\,(b+c-a)}{2\,bc}=\dfrac{p\,(p-2a)}{2\,bc}$.

Substituting these values in (2), and dividing by 2, we have

$$(3) \quad \cos^2\tfrac{1}{2}A=\frac{\tfrac{1}{2}p\,(\tfrac{1}{2}p-a)}{bc}.$$

$$\sqrt{(3)}=(4) \quad \cos\tfrac{1}{2}A=\sqrt{\frac{\tfrac{1}{2}p\,(\tfrac{1}{2}p-a)}{bc}}$$

In like manner, (5) $\cos\frac{1}{2}B = \sqrt{\dfrac{\frac{1}{2}p(\frac{1}{2}p-b)}{ac}}.$

Also, (6) $\cos\frac{1}{2}C = \sqrt{\dfrac{\frac{1}{2}p(\frac{1}{2}p-c)}{ab}}.$

Introducing R, applying logarithms, and reducing, (4) becomes

$$\log\cos\tfrac{1}{2}A = \tfrac{1}{2}[\log\tfrac{1}{2}p + \log(\tfrac{1}{2}p - a) + a.c.\log b + a.c.\log c].$$

In like manner introduce R and apply logarithms to (5) and (6).

By subtracting both members of (1) from 1 and reducing we find

$$(7)\quad \sin\tfrac{1}{2}A = \sqrt{\frac{(\frac{1}{2}p-b)(\frac{1}{2}p-c)}{bc}}.$$

$$(8)\quad \sin\tfrac{1}{2}B = \sqrt{\frac{(\frac{1}{2}p-a)(\frac{1}{2}p-c)}{ac}}.$$

$$(9)\quad \sin\tfrac{1}{2}C = \sqrt{\frac{(\frac{1}{2}p-a)(\frac{1}{2}p-b)}{ab}}.$$

$$(7)\div(4)=(10)\quad \tan\tfrac{1}{2}A = \sqrt{\frac{(\frac{1}{2}p-b)(\frac{1}{2}p-c)}{\frac{1}{2}p(\frac{1}{2}p-a)}}.$$

$$(8)\div(5)=(11)\quad \tan\tfrac{1}{2}B = \sqrt{\frac{(\frac{1}{2}p-a)(\frac{1}{2}p-c)}{\frac{1}{2}p(\frac{1}{2}p-b)}}.$$

$$(9)\div(6)=(12)\quad \tan\tfrac{1}{2}C = \sqrt{\frac{(\frac{1}{2}p-a)(\frac{1}{2}p-b)}{\frac{1}{2}p(\frac{1}{2}p-c)}}.$$

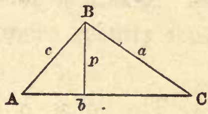
99. Examples.

1. Given $\begin{cases} a = 125. \\ b = 150. \\ c = 100. \end{cases}$ Required $\begin{cases} A = 55°\ 46'\ 18''. \\ B = 82°\ 49'\ 08''. \\ C = 41°\ 24'\ 34''. \end{cases}$

2. Given $\begin{cases} a = 864. \\ b = 1308. \\ c = 1086. \end{cases}$ Required $\begin{cases} A = 41°\ 00'\ 38''. \\ B = 83°\ 25'\ 14''. \\ C = 55°\ 34'\ 08''. \end{cases}$

100. Problem.

To find the area of a triangle when two sides and their included angle are given.



Let k denote the area of the triangle ABC, of which the two sides b and c and their included angle A are given.

$$(1)\quad 2k = bp.$$

$$(2)\quad p = c \sin A.$$

$$\therefore (3)\quad 2k = bc \sin A.$$

Introducing R, and applying logarithms, we have

$$\log (2k) = \log b + \log c + \log \sin A - 10.$$

101. Examples.

1. Two sides of a triangle are 345.6 and 485, respectively, and their included angle is 38° 45′ 40″; what is the area? *Ans.* 52468.

2. Two sides of a triangle are 784.25 and 1095.8, respectively, and their included angle is 85° 40′ 20″; what is the area. *Ans.* 428470.

102. Problem.

To find the area of a triangle when the three sides are given.

By the last problem we find

$$(1)\quad k = \tfrac{1}{2} bc \sin A,$$

$$(2)\quad \sin A = 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A. \quad \text{Article 95, (5).}$$

$$(3)\quad \sin \tfrac{1}{2} A = \sqrt{\frac{(\tfrac{1}{2}p - b)(\tfrac{1}{2}p - c)}{bc}}. \quad \text{Article 98, (7).}$$

(4) $\cos \frac{1}{2} A = \sqrt{\dfrac{\frac{1}{2}p(\frac{1}{2}p-a)}{bc}}$. Article 98, (4).

$\therefore$ (5) $\sin A = \dfrac{2\sqrt{\frac{1}{2}p(\frac{1}{2}p-a)(\frac{1}{2}p-b)(\frac{1}{2}p-c)}}{bc}$.

$\therefore$ (6) $k = \sqrt{\frac{1}{2}p(\frac{1}{2}p-a)(\frac{1}{2}p-b)(\frac{1}{2}p-c)}$.

103. Examples.

1. The sides of a triangle are 40, 45, 55, required the area. *Ans.* 887.412.

2. The sides of a triangle are 467, 845, 756, required the area. *Ans.* 175508.

104. Problem.

Given the perimeter and angles of a triangle, required the sides.

(1) $\dfrac{b}{a} = \dfrac{\sin B}{\sin A}$. (2) $\dfrac{c}{a} = \dfrac{\sin C}{\sin A}$.

Adding and reducing by Articles 96, (5) and 95, (5), we have

(3) $\dfrac{b+c}{a} = \dfrac{\sin \frac{1}{2}(B+C) \cos \frac{1}{2}(B-C)}{\sin \frac{1}{2} A \cos \frac{1}{2} A}$.

$\sin \frac{1}{2}(B+C) = \cos \frac{1}{2} A$, and $\sin \frac{1}{2} A = \cos \frac{1}{2}(B+C)$.

$\therefore$ (3) $\dfrac{b+c}{a} = \dfrac{\cos \frac{1}{2}(B-C)}{\cos \frac{1}{2}(B+C)}$.

Adding 1 to both members, we have

(4) $\dfrac{a+b+c}{a} = \dfrac{\cos \frac{1}{2}(B+C) + \cos \frac{1}{2}(B-C)}{\cos \frac{1}{2}(B+C)}$.

Let $p = a+b+c$, and reduce by 96, (7), we have

(5) $\dfrac{p}{a} = \dfrac{2 \cos \frac{1}{2} B \cos \frac{1}{2} C}{\sin \frac{1}{2} A}$.

$\therefore$ (6) $a = \dfrac{\frac{1}{2} p \sin \frac{1}{2} A}{\cos \frac{1}{2} B \cos \frac{1}{2} C}$.

Introducing R and applying logarithms, we have

$$\log a = \log \tfrac{1}{2} p + \log \sin \tfrac{1}{2} A + a.c. \log \cos \tfrac{1}{2} B + a.c. \log \cos \tfrac{1}{2} C - 10.$$

Similar formulas can be found for b and c. But, after a is found, b and c can be more readily found by article 69.

105. Examples.

1. Given $p = 150$, $A = 70°$, $B = 60°$, $C = 50°$, required a, b, c.

Ans. $a = 54.81$, $b = 50.51$, $c = 44.68$.

2. Given $p = 31234.36$, $A = 35° \ 45'$, $B = 45° \ 28'$, $C = 98° \ 47'$, required a, b, c.

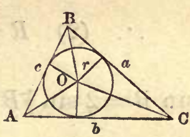
Ans. $a = 7985$, $b = 9742.5$, $c = 13506.86$.

3. Given $p = 375$, $A = 55° \ 46' \ 18''$, $B = 82° \ 49' \ 08''$, $C = 41° \ 24' \ 34''$, required a, b, c.

Ans. $a = 125$, $b = 150$, $c = 100$.

106. Problem.

Given the three sides of a triangle, to find the radius of the inscribed circle.



(1) $BOC + AOC + AOB = ABC$.

(2) $BOC = \tfrac{1}{2} ar$.

(3) $AOC = \tfrac{1}{2} br$.

(4) $AOB = \tfrac{1}{2} cr$.

$\therefore$ (5) $BOC + AOC + AOB = \tfrac{1}{2}(a + b + c) r = \tfrac{1}{2} pr$.

But (6) $ABC = \sqrt{\tfrac{1}{2} p (\tfrac{1}{2} p - a)(\tfrac{1}{2} p - b)(\tfrac{1}{2} p - c)}$.

$$\therefore \quad (7) \quad \tfrac{1}{2}pr = \sqrt{\tfrac{1}{2}p(\tfrac{1}{2}p-a)(\tfrac{1}{2}p-b)(\tfrac{1}{2}p-c)}.$$

$$\therefore \quad (8) \qquad r = \sqrt{\frac{(\frac{1}{2}p-a)(\frac{1}{2}p-b)(\frac{1}{2}p-c)}{\frac{1}{2}p}} = \frac{k}{\frac{1}{2}p}.$$

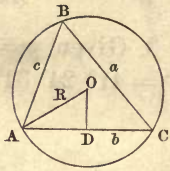
107. Examples.

1. The three sides of a triangle are 20, 30, 40, respectively, required the radius of the inscribed circle.
Ans. 6.455.

2. The three sides of a triangle are 100, 150, 200, respectively, required the radius of the inscribed circle.
Ans. 32.275.

108. Problem.

Given the three sides of a triangle to find the radius of the circumscribed circle.



Let O be the center of the circle, and R the radius.

Let OD be perpendicular to b, then $AD = \frac{b}{2}$.

The angle $O =$ the angle B, since each is measured by one-half the arc AC.

$$(1) \quad AD = \frac{b}{2} = AO \sin O = R \sin B.$$

$$\therefore \quad (2) \quad R = \frac{b}{2 \sin B}.$$

$$\sin B = 2 \sin \tfrac{1}{2}B \cos \tfrac{1}{2}B = \frac{2\sqrt{\frac{1}{2}p(\frac{1}{2}p-a)(\frac{1}{2}p-b)(\frac{1}{2}p-c)}}{ac}.$$

$$\therefore \quad (3) \quad R = \frac{abc}{4\sqrt{\frac{1}{2}p(\frac{1}{2}p-a)(\frac{1}{2}p-b)(\frac{1}{2}p-c)}} = \frac{abc}{4k}.$$

Prove that the formula will be the same if the center is without the triangle.

109. Examples.

1. The sides of a triangle are 7, 9, 10, respectively, required the radius of the circumscribed circle.

Ans. 5.148.

2. The sides of a triangle are 50, 60, 70, respectively, required the radius of the circumscribed circle.

Ans. 35.72.

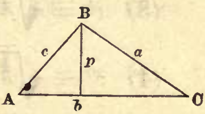
110. Theorem.

The perpendicular let fall on either side of a triangle from the vertex of the opposite angle is equal to that side into the product of the sines of the adjacent angles divided by the sine of the sum of those angles.

$$(1)\quad p = c \sin A.$$

$$(2)\quad \sin B : \sin C :: b : c, \quad \therefore \quad c = \frac{b \sin C}{\sin B}.$$

$$\therefore \quad (3)\quad p = \frac{b \sin A \sin C}{\sin B}.$$



$$(4)\quad \sin B = \sin\,[180° - (A + C)] = \sin\,(A + C).$$

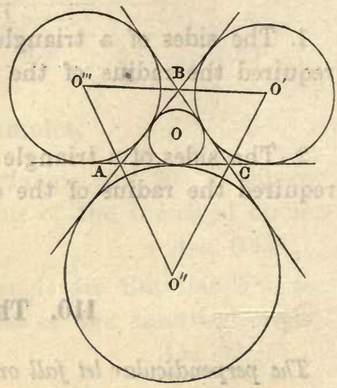
$$\therefore \quad (5)\quad p = \frac{b \sin A \sin C}{\sin\,(A + C)}.$$

111. Problem.

Given the three sides of a triangle to find the radii of the escribed circles.

The escribed circles are the three circles external to the triangle, each tangent to one side and to the prolongation of the other sides.

The centers of the escribed circles are the points of intersection of the lines bisecting the external angles.



The radii r', r'', r''', of the escribed circles, will be the perpendiculars let fall from their centers O', O'', O''', respectively, on the three sides a, b, c.

Hence, by the last article,

$$(1)\quad r' = \frac{a \sin(90° - \frac{1}{2}B)\sin(90° - \frac{1}{2}C)}{\sin[180° - \frac{1}{2}(B+C)]}.$$

$$\therefore\ (2)\quad r' = \frac{a \cos\frac{1}{2}B \cos\frac{1}{2}C}{\cos\frac{1}{2}A} = \tfrac{1}{2}p \tan\tfrac{1}{2}A. \quad \text{Art. 104.}$$

Substituting the value of $\tan\frac{1}{2}A$, article 98, we have

$$(3)\quad r' = \sqrt{\frac{\frac{1}{2}p(\frac{1}{2}p-b)(\frac{1}{2}p-c)}{\frac{1}{2}p-a}} = \frac{k}{\frac{1}{2}p-a}.$$

$$\therefore\ (4)\quad r'' = \sqrt{\frac{\frac{1}{2}p(\frac{1}{2}p-a)(\frac{1}{2}p-c)}{\frac{1}{2}p-b}} = \frac{k}{\frac{1}{2}p-b}.$$

$$\therefore\ (5)\quad r''' = \sqrt{\frac{\frac{1}{2}p(\frac{1}{2}p-a)(\frac{1}{2}p-b)}{\frac{1}{2}p-c}} = \frac{k}{\frac{1}{2}p-c}.$$

112. Examples.

1. Given the sides of a triangle, 6, 9, 11, required the radii of the three escribed circles.

Ans. 3.854, 6.745, 13.49.

2. Given $p = 100$, $A = 55°$, $B = 60°$, $C = 65°$, required the radii of the three escribed circles.

[See (2), Art. 111.] *Ans.* 26.028, 28.867, 31.854.

113. Theorem.

The product of the radius of the inscribed circle and the radii of the three escribed circles is equal to the square of the area of the triangle.

The product of (8), article 106, and (3), (4), (5), article 111, gives

$$r\,r'r''r'''=\frac{k^4}{\frac{1}{2}p\,(\frac{1}{2}p-a)\,(\frac{1}{2}p-b)\,(\frac{1}{2}p-c)}=\frac{k^4}{k^2}=k^2.$$

114. Theorem.

The reciprocal of the radius of the inscribed circle, the sum of the reciprocals of the radii of the escribed circles, and the sum of the reciprocals of the perpendiculars let fall from the vertices of the three angles on the opposite sides of a triangle are equal to each other.

Taking the reciprocal of (8), article 106, we have

$$(1)\quad \frac{1}{r}=\frac{p}{2\,k}.$$

Taking the sum of the reciprocals of (3), (4), (5), article 111,

$$(2)\quad \frac{1}{r'}+\frac{1}{r''}+\frac{1}{r'''}=\frac{p-2\,a}{2\,k}+\frac{p-2\,b}{2\,k}+\frac{p-2\,c}{2\,k}=\frac{p}{2\,k}.$$

Let p', p'', p''', respectively, be the perpendiculars let fall from the vertices of the three angles on the sides a, b, and c. Then we have

$$a\,p'=2\,k.\quad \therefore\quad \frac{1}{p'}=\frac{a}{2\,k}.$$

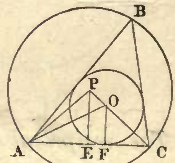
In like manner, $\frac{1}{p''}=\frac{b}{2\,k}$. Also, $\frac{1}{p'''}=\frac{c}{2\,k}$.

$$\therefore \quad (3) \quad \frac{1}{p'} + \frac{1}{p''} + \frac{1}{p'''} = \frac{a+b+c}{2k} = \frac{p}{2k}.$$

$$\therefore \quad (4) \quad \frac{1}{r} = \frac{1}{r'} + \frac{1}{r''} + \frac{1}{r'''} = \frac{1}{p'} + \frac{1}{p''} + \frac{1}{p'''}.$$

115. Problem.

To find the distance between the centers of the circumscribed and inscribed circles of a triangle.



Let R and r be the radii, and P and O the centers of the circles, and let $D = OP$.

Draw PE perpendicular to AC. The angle $APE = B$, since each is measured by one-half the arc AC; but $PAE = 90° - APE$, $\therefore PAE = 90° - B$. $OAC = \frac{1}{2}A$. $PAO = PAE - OAC$.

$$\therefore \quad PAO = 90° - B - \tfrac{1}{2}A = \tfrac{1}{2}(C - B). \quad AO = \frac{r}{\sin \frac{1}{2}A}.$$

$$(1) \quad \overline{OP}^2 = \overline{AP}^2 + \overline{AO}^2 - 2\, AP \times AO \cos PAO. \quad \text{Art. 97.}$$

Substituting the values of OP, AP, AO, and PAO, we have

$$(2) \quad D^2 = R^2 + \frac{r^2}{\sin^2 \frac{1}{2}A} - \frac{2\,Rr \cos \frac{1}{2}(C - B)}{\sin \frac{1}{2}A}.$$

$$(3) \quad R = \frac{b}{2 \sin B} = \frac{b}{4 \sin \frac{1}{2}B \cos \frac{1}{2}B}. \quad \text{Arts.} \begin{cases} 108, (2). \\ 95, (5). \end{cases}$$

$$(4) \quad r = \frac{b \sin \frac{1}{2}A \sin \frac{1}{2}C}{\sin \frac{1}{2}(A + C)} = \frac{b \sin \frac{1}{2}A \sin \frac{1}{2}C}{\cos \frac{1}{2}B}. \quad \text{Art. 110.}$$

$$\therefore \quad (5) \quad \frac{r^2}{\sin^2 \frac{1}{2}A} = \frac{4\,Rr \sin \frac{1}{2}B \sin \frac{1}{2}C}{\sin \frac{1}{2}A}.$$

Substituting in (2), and reducing by article 91, (d), and 89, (b), we have

$$(6) \quad D^2 = R^2 - \frac{2\ Rr \cos \frac{1}{2}(B+C)}{\sin \frac{1}{2}A} = R^2 - 2\ Rr.$$

$$\therefore \quad (7) \quad D = \sqrt{R^2 - 2\ Rr}.$$

116. Examples.

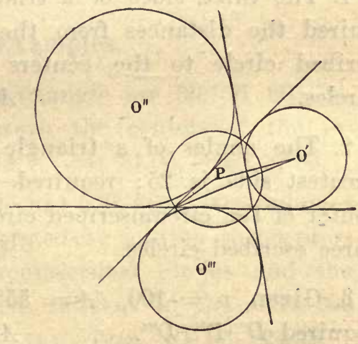
1. The sides of a triangle are 12, 13, 15; required the distance between the centers of the circumscribed and inscribed circles. *Ans.* 1.616.

2. Two sides of a triangle are 35 and 37, and their included angle is 50°; required the distance between the centers of the circumscribed and inscribed circles. *Ans.* 3.266.

3. The perimeter of a triangle is 120, the angles are 40°, 60°, and 80°, respectively; required the distance between the centers of the circumscribed and inscribed circles. *Ans.* 8.353.

117. Problem.

To find the distance between the centers of the circumscribed and escribed circles.



Let r', r'', r''' be the radii of the escribed circles, and D', D'', D''', be the distances of their centers, O', O'', O''', respectively, from P, the center of the circumscribed circle, whose radius is R.

As in the last Problem, we find

$$(1)\quad D'^2 = R^2 + \frac{r'^2}{\sin^2 \frac{1}{2}A} - \frac{2\,Rr' \cos \frac{1}{2}(C-B)}{\sin \frac{1}{2}A}.$$

$$(2)\quad R = \frac{a}{2\sin A} = \frac{a}{4\sin\frac{1}{2}A\cos\frac{1}{2}A}.\quad \text{Arts.} \begin{cases} 108, (2). \\ 95, (5). \end{cases}$$

$$(3)\quad r' = \frac{a\cos\frac{1}{2}B\cos\frac{1}{2}C}{\cos\frac{1}{2}A}.\quad \text{Art. 111, (2).}$$

$$\therefore\ (4)\quad \frac{r'^2}{\sin^2\frac{1}{2}A} = \frac{4\,Rr'\cos\frac{1}{2}B\cos\frac{1}{2}C}{\sin\frac{1}{2}A}, \text{ by (2) and (3).}$$

Substituting (4) in (1), and reducing by (*d*) and (*b*), we have

$$(5)\quad D'^2 = R^2 + \frac{2\,Rr'\cos\frac{1}{2}(B+C)}{\sin\frac{1}{2}A} = R^2 + 2\,Rr'.$$

$$\therefore\ (6)\quad D' = \sqrt{R^2 + 2\,Rr'}.$$

$$\therefore\ (7)\quad D'' = \sqrt{R^2 + 2\,Rr''}.$$

$$\therefore\ (8)\quad D''' = \sqrt{R^2 + 2\,Rr'''}.$$

118. Examples.

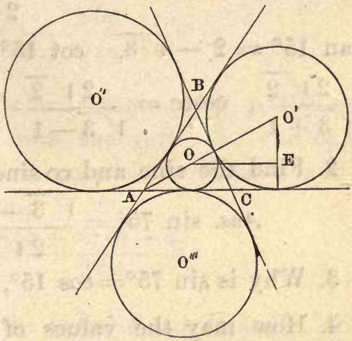
1. The three sides of a triangle are 21, 23, 26; required the distances from the center of the circumscribed circle to the centers of the three escribed circles. *Ans.* 25.19, 26.64, 29.73.

2. The angles of a triangle are 56°, 60°, 64°, the greatest side is 25; required the distances from the center of the circumscribed circle to the centers of the three escribed circles. *Ans.* 26.96, 27.80, 28.65.

3. Given $p = 100$, $A = 55°$, $B = 60°$, $C = 65°$, required D', D'', D'''. *Ans.* 37.10, 38.55, 40.01.

119. Problem.

To find the distance between the centers of the inscribed and escribed circles.



Let D_1, D_2, D_3, be the distances.

In the triangle $OO'E$, we have

(1) $D_1 = \frac{r' - r}{\sin \frac{1}{2}A}$.

Substituting the values of r', r, and $\sin \frac{1}{2}A$, we have

$$(2) \quad D_1 = \frac{a}{\sqrt{\frac{\frac{1}{2}p\,(\frac{1}{2}p - a)}{bc}}}.$$

$$(3) \quad D_2 = \frac{b}{\sqrt{\frac{\frac{1}{2}p\,(\frac{1}{2}p - b)}{ac}}}.$$

$$(4) \quad D_3 = \frac{c}{\sqrt{\frac{\frac{1}{2}p\,(\frac{1}{2}p - c)}{ab}}}.$$

120. Examples.

1. The three sides of a triangle are 30, 50, 60; required the distances between the centers of the inscribed and escribed circles. *Ans.* 31.05, 56.69, 87.83.

2. The sides of a triangle are 500, 600, 700; required the sides of the triangle formed by joining the centers of the inscribed and circumscribed circles and the center of the escribed circle, tangent to the sides 600 and 700 produced. *Ans.* 540.06, 104.58, 624.58.

121. Miscellaneous Exercises.

1. Prove that $\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$, $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$, $\tan 15° = 2 - \sqrt{3}$, $\cot 15° = 2 + \sqrt{3}$, $\sec 15° = \frac{2\sqrt{2}}{\sqrt{3}+1}$, $\text{cosec} = \frac{2\sqrt{2}}{\sqrt{3}-1}$.

2. Find the sine and co-sine of 75°.

Ans. $\sin 75° = \frac{\sqrt{3}+1}{2\sqrt{2}}$, $\cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

3. Why is $\sin 75° = \cos 15°$, and $\cos 75° = \sin 15°$?

4. How may the values of tangent, co-tangent, secant, and co-secant of 75° be found from the values of the sine and co-sine?

5. Find the functions of 150°.

Ans. $\sin 150° = \frac{1}{2}$, $\cos 150° = -\frac{\sqrt{3}}{2}, \cdots$

6. Given $\sin a + \cos a = \sqrt{2}$, to find a.

Ans. 45°, or 45° + 360°; or, in general, $\frac{\pi}{4} + 2\pi n$.

7. Given $\sin 2a = \cos a$, to find a.

Ans. $\frac{\pi}{6} + 2\pi n$, or $\frac{5}{6}\pi + 2\pi n$.

8. Prove that the sum of the tangents of the three angles of a plane triangle is equal to their product.

9. Prove that the sum of the co-tangents of one-half the angles of a plane triangle is equal to their product.

10. Prove that ABC is isosceles if $\cos A = \frac{\sin B}{2 \sin C}$.

11. Prove that the sum of the diameters of the inscribed and circumscribed circles of any plane triangle ABC is

$$a \cot A + b \cot B + c \cot C.$$

12. If b is the base of the triangle ABC, p, the perpendicular to the base from the vertex of the opposite angle, and s, the sum of the sides a and c, prove that

$$\tan \tfrac{1}{2}B = \frac{2\,bp}{(s+b)\,(s-b)}.$$

13. If b is the base of the triangle ABC, p, the perpendicular to the base from the vertex of the opposite angle, and d, the difference of the sides a and c, prove that

$$\tan \tfrac{1}{2}B = \frac{(b+d)\,(b-d)}{2\,bp}.$$

14. If a, b, and c be the sides of the triangle ABC, s, the sum of the sides a and c, and r, the radius of the inscribed circle, prove that

$$\tan \tfrac{1}{2}B = \frac{2\,r}{s-b}.$$

122. Computation of Natural Functions.

Dividing the length of the semi-circumference to the radius 1, which is $\pi = 3.14159265358979З\ldots$ by 1080, the number of minutes in 180°, the quotient, which is .0002908882 . . . , will be the length of the arc 1′, and will differ insensibly from its sine.

$\therefore$ (1) $\sin 1' = .0002908882.$

$\therefore$ (2) $\cos 1' = \sqrt{1 - \sin^2 1'} = .9999999577.$

Adding (a) and (c), then (b) and (d), articles 89, 91, and transposing,

(3) $\sin(a+b) = 2 \sin a \cos b - \sin(a-b).$

(4) $\cos(a+b) = 2 \cos a \cos b - \cos(a-b).$

If in (3) and (4) $b = 1$, $a = 1, 2, 3 \ldots$, in succession, we have

$$\sin 2' = 2 \cos 1' \sin 1' - \sin 0' = .0005817764.$$
$$\sin 3' = 2 \cos 1' \sin 2' - \sin 1' = .0008726646.$$
$$\sin 4' = 2 \cos 1' \sin 3' - \sin 2' = .0011635526.$$
$$\cdots\cdots\cdots\cdots$$
$$\cos 2' = 2 \cos 1' \cos 1' - \cos 0' = .9999998308.$$
$$\cos 3' = 2 \cos 1' \cos 2' - \cos 1' = .9999996193.$$
$$\cdots\cdots\cdots\cdots$$

To facilitate computation, for $2 \cos 1' = 1.9999999154$, use its equal, $2 - .0000000846$. Then we have

$$\sin 2' = 2 \sin 1' - .0000000846 \sin 1' - \sin 0'.$$
$$\sin 3' = 2 \sin 2' - .0000000846 \sin 2' - \sin 1'.$$
$$\cdots\cdots\cdots\cdots$$

After finding the sines and co-sines, the tangents and co-tangents can be calculated from the formulas:

$$(5)\quad \tan a = \frac{\sin a}{\cos a}. \qquad (6)\quad \cot a = \frac{\cos a}{\sin a}.$$

It is not necessary to carry the computation beyond 45°, since $\sin a = \cos (90° - a)$, etc.

The logarithmic functions can be found from the corresponding natural functions by the method of article 60.

SPHERICAL TRIGONOMETRY.

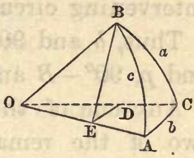
123. Definition and Remarks.

Spherical Trigonometry is that branch of Trigonometry which treats of the solution of spherical triangles.

If any three of the six parts of a spherical triangle are given, the remaining parts can be computed.

The radius of the sphere is taken equal to 1, and each side has the same numerical measure as the subtended angle whose vertex is at the center of the sphere. Thus,

$$a = BOC, b = AOC, c = AOB.$$



An angle of a spherical triangle is the angle included by the planes of its sides which is measured by the angle included by two lines, one line in one plane, the other in the other, both perpendicular to the common intersection of the planes at the same point.

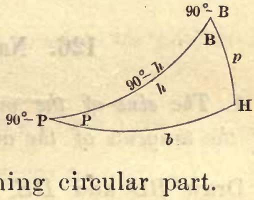
Thus, if BE, in the plane AOB, is perpendicular to OA, and if ED, in the plane AOC, is perpendicular to OA, then the angle BED will measure the inclination of the planes AOB and AOC, and will be equal to the angle A of the spherical triangle.

RIGHT TRIANGLES.

124. Napier's Circular Parts.

Napier's circular parts are the two sides adjacent to the right angle, the complements of their opposite angles, and the complement of the hypotenuse.

Thus, if HBP is a spherical triangle, right-angled at H, the circular parts are b, p, $90° - B$, $90° - P$, and $90° - h$.



Adjacent parts are those which are not separated by an intervening circular part.

Thus, b and $90° - P$, $90° - P$ and $90° - h$, $90° - h$ and $90° - B$, $90° - B$ and p, p and b are adjacent parts.

The right angle H is not regarded as a circular part, nor as separating the parts b and p.

Opposite parts are those which are separated by an intervening circular part.

Thus, b and $90° - h$, $90° - P$ and $90° - B$, $90° - h$ and p, $90° - B$ and b, p and $90° - P$ are opposite parts.

Any one of these five circular parts is adjacent to two of the remaining parts, and opposite the other two parts.

Of any three circular parts, one part is either adjacent to both the others or opposite both.

A middle part is that which is adjacent to two other parts, or opposite two other parts.

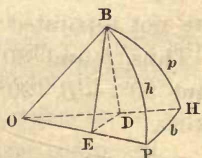
125. Exercises.

Tell which is the middle part, and whether the other parts are adjacent to, or opposite, the middle in the following:

1. $90° - B$, $90° - P$, $90° - h$.
2. b, $90° - h$, p.
3. $90° - h$, $90° - B$, p.
4. $90° - P$, $90° - B$, b.
5. b, $90° - B$, p.
6. $90° - P$, $90° - h$, p.
7. b, $90° - P$, p.
8. $90° - B$, $90° - h$, b.
9. $90° - h$, $90° - P$, b.
10. $90° - P$, $90° - B$, p.

126. Napier's Principles.

1. *The sine of the middle part is equal to the product of the tangents of the adjacent parts.*



Draw BD and DE, respectively perpendicular to OH and OP, and draw BE. BDE is a right angle, since the plane BOH is perpendicular to the plane POH, and BD is perpendicular to OH. The angle BED is equal to P.

$EB = \sin h$, $OE = \cos h$, $DB = \sin p$, and $OD = \cos p$.

$$\frac{ED}{EB} = \frac{OE}{EB} \times \frac{ED}{OE}, \text{ or } \cos P = \cot h \tan b.$$

$$\therefore \quad (1) \quad \sin (90° - P) = \tan (90° - h) \tan b.$$

$$\frac{ED}{OD} = \frac{DB}{OD} \times \frac{ED}{DB}, \text{ or } \sin b = \tan p \cot P.$$

$$\therefore \quad (2) \quad \sin b = \tan p \tan (90° - P).$$

By changing P, b, p into B, p, b, (1) and (2) become

$$(3) \quad \sin (90° - B) = \tan (90° - h) \tan p.$$

$$(4) \quad \sin p = \tan b \tan (90° - B).$$

Multiplying (2) by (4), member by member, we have

$$\sin b \sin p = \tan b \tan p \tan (90° - B) \tan (90° - P).$$

Dividing by $\tan b \tan p$, and reducing, we have

$$\cos b \cos p = \tan (90° - B) \tan (90° - P).$$

$$\cos b \cos p = \cos EOD \times OD = OE = \cos h = \sin (90° - h).$$

$$\therefore \quad (5) \quad \sin (90° - h) = \tan (90° - B) \tan (90° - P).$$

2. *The sine of the middle part is equal to the product of the co-sines of the opposite parts.*

$$OE = \cos EOD \times OD, \text{ or } \cos h = \cos b \cos p.$$

$$\therefore \quad (6) \quad \sin (90° - h) = \cos b \cos p.$$

$$DB = EB \sin DEB, \text{ or } \sin p = \sin h \sin P.$$

$$\therefore \quad (7) \quad \sin p = \cos (90° - h) \cos (90° - P).$$

$$(3) \text{ gives } \sin (90° - B) = \frac{\sin (90° - h) \sin p}{\cos (90° - h) \cos p}.$$

This, by substituting $\cos b \cos p$ for $\sin (90° - h)$, $\cos (90° - h) \cos (90° - P)$ for $\sin p$, and reducing, gives

$$(8) \quad \sin (90° - B) = \cos b \cos (90° - P).$$

By changing p, P, B, b into b, B, P, p, (7) and (8) become

$$(9) \quad \sin b = \cos (90° - h) \cos (90° - B).$$

$$(10) \quad \sin (90° - P) = \cos p \cos (90° - B).$$

These ten formulas are thus reduced to two principles, from which the formulas can be written.

The memory will be further aided by observing the common vowel *a* in the first syllables of the words t*a*ngent and *a*djacent of the first principle, and the common vowel *o* in the first syllables of the words c*o*-sine and *o*pposite of the second principle; that is, we take the product of the *tangents* of the parts *adjacent* to the middle, and the product of the *co-sines* of the parts *opposite* the middle.

127. Mauduit's Principles.

If we take, as circular parts, the complements of the two sides adjacent to the right angles, their opposite angles, and the hypotenuse, we can readily deduce from the diagram, or from Napier's principles, the following principles:

1. *The co-sine of the middle part is equal to the product of the co-tangents of the adjacent parts.*

2. *The co-sine of the middle part is equal to the product of the sines of the opposite parts.*

Let the ten formulas be written and compared with those of the last article.

128. Analogies of Plane and Spherical Triangles.

The formulas which demonstrate Napier's principles may be placed under forms which will exhibit the analogies existing between Plane and Spherical Triangles, as in the subjoined table.

Plane Right Triangles.	*Spherical Right Triangles.*
1. $\sin P = \frac{p}{h}\cdot$	1. $\sin P = \frac{\sin p}{\sin h}\cdot$
2. $\sin B = \frac{b}{h}\cdot$	2. $\sin B = \frac{\sin b}{\sin h}\cdot$
3. $\cos P = \frac{b}{h}\cdot$	3. $\cos P = \frac{\tan b}{\tan h}\cdot$
4. $\cos B = \frac{p}{h}\cdot$	4. $\cos B = \frac{\tan p}{\tan h}\cdot$
5. $\tan P = \frac{p}{b}\cdot$	5. $\tan P = \frac{\tan p}{\sin b}\cdot$
6. $\tan B = \frac{b}{p}\cdot$	6. $\tan B = \frac{\tan b}{\sin p}\cdot$
7. $\sin P = \cos B.$	7. $\sin P = \frac{\cos B}{\cos b}\cdot$
8. $\sin B = \cos P.$	8. $\sin B = \frac{\cos P}{\cos p}\cdot$
9. $h^2 = b^2 + p^2.$	9. $\cos h = \cos b \cos p.$
10. $1 = \cot B \cot P.$	10. $\cos h = \cot B \cot P.$

These formulas can be committed and applied instead of Napier's principles by those who prefer to do so. The analogies will assist the memory.

129. Species of the Parts.

Two parts of a spherical triangle are of the *same species* when both are less than 90° or both greater than 90°.

Two parts of a spherical triangle are of *different species* when one part is less than 90° and the other part greater than 90°.

We shall, at present, consider those triangles only whose parts do not exceed 180°.

Let it be remembered that the sine is positive from 0° to 180°, and that the co-sine, the tangent, and the co-tangent are positive from 0° to 90°, and negative from 90° to 180°. Hence, if the co-sines, tangents, or co-tangents of two parts have like signs, these parts will be of the same species; if they have unlike signs, these parts will be of different species.

$$\sin P = \frac{\cos B}{\cos b} \text{ and } \sin B = \frac{\cos P}{\cos p}\cdot \quad \text{Art. 128, 7, 8.}$$

Since neither P nor B exceeds 180°, sin P and sin B are both positive; hence, cos B and cos b have like signs, so also have cos P and cos p. Therefore, B and b are of the same species; so also are P and p.

Hence, *The sides adjacent to the right angle are of the same species as their opposite angles.*

$$\cos h = \cos b \cos p. \quad \text{Art. 128, 9.}$$

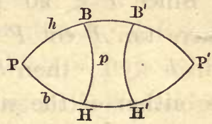
If $h < 90°$, cos h is positive; hence, cos b cos p is positive; ∴ cos b and cos p have like signs; ∴ b and p are of the same species; ∴ B and P are of the same species.

Hence, *If the hypotenuse is less than* 90°, *the two sides adjacent to the right angle are of the same species; so also are their opposite angles.*

If $h > 90°$, cos h is negative; hence, cos b cos p is negative; $\therefore$ cos b and cos p have unlike signs; $\therefore$ b and p are of different species; $\therefore$ B and P are of different species.

Hence, *If the hypotenuse is greater than* 90°, *the two sides adjacent to the right angle are of different species; so also are their opposite angles.*

Let us now investigate the case in which a side adjacent to the right angle and its opposite angle are given.



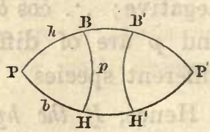
Let p and P be given. Produce the sides PH and PB till they meet in P'. The angles P and P' are equal, since each is the angle included by the plane of the arcs PHP' and PBP'. Take $P'H' = PH = b$ and $P'B' = PB = h$. The two triangles, PHB and $P'H'B'$, have the two sides PH and PB and the included angle P of the one, equal to $P'H'$ and $P'B'$ and the included angle P' of the other; hence, they are equal in all their corresponding parts; $\therefore$ $H' = H$, $B' = B$, and $H'B' = HB$. But H is a right angle; $\therefore$ H' is a right angle. Hence, either triangle, PHB or $PH'B'$, will answer to the given conditions.

Since $P'H'$ and PH are equal, and $P'H'$ and PH' are supplements of each other, PH and PH' are supplements of each other. In like manner it can be shown that PB and PB' are supplements of each other.

When, therefore, a side adjacent to the right angle and an opposite angle are given, there are apparently two solutions. The conditions of the problem, however, may be such as to render the two solutions possible, reduce them to one, or render any solution impossible.

Let us now proceed to investigate these conditions.

1. When $P < 90°$ and $p < P$.



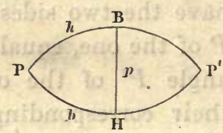
We have from Napier's principles,

$\sin b = \tan p \tan (90° - P)$, or $\sin b = \tan p \cot P$.

Since $P < 90°$ and $p < P$, $\tan p < \tan P$; but we have $\tan P \cot P = 1$; $\therefore$ $\tan p \cot P < 1$; hence, $\sin b < 1$; then $b < 90°$ or $b > 90°$; hence, b may be either of the supplementary arcs PH or PH' which have the same sine equal to $\tan p \cot P$.

If $b < 90°$, since $p < 90°$, $h < 90°$; if $b > 90°$, since $p < 90°$, $h > 90°$. Hence, if $P < 90°$ and $p < P$, either triangle, PHB or $PH'B'$, will satisfy the conditions, and there will be two solutions.

2. When $P < 90°$ and $p = P$.



We have $\sin b = \tan p \cot P$, as before.

Since $p = P$, $\tan p \cot P = \tan P \cot P = 1$; therefore, $\sin b = 1$; $\therefore$ $b = 90°$, or $PH = 90°$.

From Napier's principles, we have

$\sin (90° - h) = \cos b \cos p$, or $\cos h = \cos b \cos p$.

Since $b = 90°$, $\cos b = 0$; $\therefore$ $\cos b \cos p = 0$; hence, $\cos h = 0$; $\therefore$ $h = 90°$, or $PB = 90°$.

$\sin (90° - B) = \tan p \tan (90° - h)$, which reduces to $\cos B = \tan p \cot h$.

Since $h = 90°$, $\cot h = 0$; $\therefore$ $\tan p \cot h = 0$; $\therefore$ $\cos B = 0$; $\therefore$ $B = 90°$.

$$PH' = 180° - PH = 90°; \quad \therefore PH' = PH.$$
$$PB' = 180° - PB = 90°; \quad \therefore PB' = PB.$$

Hence, if $P < 90°$ and $p = P$, $b = 90°$, $h = 90°$, $B = 90°$, the two triangles reduce to the bi-rectangular triangle PHB, and there is but one solution.

3. When $P < 90°$ and $p > P$.

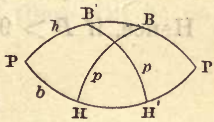
As before, we have $\sin b = \tan p \cot P$.

Since p and P are of the same species, $p < 90°$.

Then, if $p > P$, $\tan p > \tan P$; but $\tan P \cot P = 1$; $\therefore \tan p \cot P > 1$; $\therefore \sin b > 1$, which is impossible.

Hence, if $P < 90°$ and $p > P$, no solution is possible.

4. When $P > 90°$ and $p > P$.

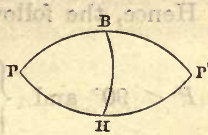


We have $\sin b = \tan p \cot P$, as before. $\tan p$ and $\cot P$ are both negative, and $\tan p < \tan P$, numerically; but $\tan P \cot P = 1$; $\therefore \tan p \cot P < 1$; hence, $\sin b < 1$; $\therefore b < 90°$, or $b > 90°$; hence, b may be either of the supplementary arcs PH or PH' which have the common sine equal to $\tan p \cot P$.

If $b < 90°$, since $p > 90°$, $h > 90°$; if $b > 90°$, since $p > 90°$, $h < 90°$.

Hence, if $P > 90°$ and $p > P$, either triangle, PHB or $PH'B'$ will satisfy the conditions, and there will be two solutions.

5. When $P > 90°$ and $p = P$.



We have $\sin b = \tan p \cot P$, as before.

$\therefore \sin b = \tan P \cot P = 1$; $\therefore b = 90°$.

$\therefore \cos b = 0$; $\therefore \cos h = \cos b \cos p = 0$; $\therefore h = 90°$.

$\therefore \cot h = 0$; $\therefore \cos B = \tan p \cot h = 0$; $\therefore B = 90°$.

Hence, if $P > 90°$ and $p = P$, $b = 90°$, $h = 90°$, $B = 90°$, the two triangles reduce to the bi-rectangular PHB, and there is but one solution.

6. When $P > 90°$ and $p < P$.

As before, we have $\sin b = \tan p \cot P$.

Since p and P are of the same species, and since $P > 90°$, $p > 90°$; hence, $\tan p$, $\cot P$ are both negative, and $\tan p > \tan P$, numerically; but since $\tan P \cot P = 1$, $\tan p \cot P > 1$; $\therefore \sin b > 1$, which is impossible.

Hence, if $P > 90°$ and $p < P$, there is no solution.

7. When $P = 90°$.

$$\tan p = \frac{\sin b}{\cot P} = \frac{\sin b}{0} = \infty; \quad \therefore p = 90°.$$

$$\therefore \cos p = 0; \quad \therefore \cos h = \cos b \cos p = 0; \quad \therefore h = 90°.$$

$$\sin b = \tan p \cot P = \infty \times 0; \quad \therefore \sin b \text{ is indeterminate.}$$

$$\sin B = \frac{\cos P}{\cos p} = \frac{0}{0}; \quad \therefore \sin B \text{ is indeterminate.}$$

Hence, if $P = 90$, then $p = 90°$, $h = 90°$, b and B are indeterminate; the triangle is bi-rectangular, and there is an infinite number of solutions.

Hence, the following results:

$$P < 90° \text{ and } \begin{cases} p < P, & \text{Two solutions.} \\ p = P, & \text{One solution.} \\ p > P, & \text{No solution.} \end{cases}$$

$$P > 90° \text{ and } \begin{cases} p > P, & \text{Two solutions.} \\ p = P, & \text{One solution.} \\ p < P, & \text{No solution.} \end{cases}$$

$$P = 90^\circ \text{ then } \left\{ \begin{array}{l} p = 90^\circ, \\ h = 90^\circ, \\ b \text{ indeterminate}, \\ B \text{ indeterminate}, \end{array} \right\} \begin{array}{c} \text{Infinite number} \\ \text{of solutions.} \end{array}$$

By a comparison of these results, we find,

1. If p differs more from 90° than P, there will be two solutions.

2. If $p = P$, and $P < 90^\circ$ or $P > 90^\circ$, there will be one solution.

3. If $p = P = 90^\circ$, there will be an infinite number of solutions.

4. If p differs less from 90° than P, there will be no solution.

130. Remarks.

1. Napier's principles render it unnecessary to divide the subject of right-angled spherical triangles into cases.

2. Two parts will be given, and three required.

3. These parts or their complements will be circular parts.

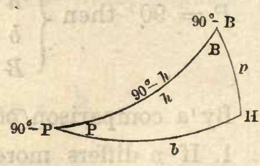
4. Take the two given parts, if they are circular parts, otherwise their complements, and any one part required, if it is a circular part, otherwise its complement, and observe which is the middle part, and whether the other parts are adjacent to, or opposite, the middle part: if adjacent, the first of Napier's principles will give the formula; if opposite, the second.

5. Introduce R and apply logarithms.

6. Apply the principles which determine the species of the required part.

131. Examples.

1. Giv. $\begin{cases} h = 110° \ 30'. \\ p = \ \ 50° \ 45'. \end{cases}$ Req. $\begin{cases} b. \\ B. \\ P. \end{cases}$



1. To find b.

From the second of Napier's principles, we have

$\sin(90° - h) = \cos b \cos p$, or $\cos h = \cos b \cos p$.

Finding $\cos b$ and introducing R, we have

$$\cos b = \frac{R \cos h}{\cos p}.$$

$\therefore \ \log \cos b = 10 + \log \cos h - \log \cos p.$

$\log \cos h \ (110° \ 30') = 9.54433 -$
$\log \cos p \ (\ 50° \ 45') = 9.80120 +$
$\log \cos b \qquad\qquad\quad = 9.74313 - \ \therefore b = 123° \ 36' \ 31''.$

Since the hypotenuse is greater than 90°, the sides b and p are of different species; but $p < 90°$; $\therefore b > 90°$. But $\log \cos b$ corresponds to 56° 23′ 29″, and to its supplement 123° 36′ 31″ which must be taken, since $b > 90°$.

The species of b can also be determined by the formula,

$$\cos b = \frac{\cos h}{\cos p}.$$

Since $h > 90°$, $\cos h$ is negative, and since $p < 90°$, $\cos p$ is positive; $\therefore \cos b$ is negative; $\therefore b > 90°$. The signs of the functions may be conveniently indicated by placing the signs after their logarithms.

2. To find B.

$$\sin\ (90° - B) = \tan\ p\ \tan\ (90° - h),$$

$$\therefore\ \cos\ B = \frac{\tan\ p\ \cot\ h}{R}.$$

$$\therefore\ \log\ \cos\ B = \log\ \tan\ p + \log\ \cot\ h - 10.$$

$$\begin{array}{lll}\log\ \tan\ p\ (\ 50°\ 45') & = 10.08776 + & \\ \log\ \cot\ h\ (110°\ 30') & = \ \ 9.57274 - & \\ \hline \log\ \cos\ B & = \ \ 9.66050 - & \therefore\ B = 117°\ 14'.\end{array}$$

Since b and B are of the same species, and since $b > 90°$, $B > 90°$. The species of B can also be determined from the sign of $\cos B$.

3. To find P.

$$\sin\ p = \cos\ (90° - h)\ \cos\ (90° - P),\ \text{or}\ \sin\ p = \sin\ h\ \sin\ P.$$

$$\therefore\ \sin\ P = \frac{R\ \sin\ p}{\sin\ h};$$

$$\therefore\ \log\ \sin\ P = 10 + \log\ \sin\ p - \log\ \sin\ h.$$

$$\begin{array}{lll}\log\ \sin\ p\ (\ 50°\ 45') & = 9.88896 + & \\ \log\ \sin\ h\ (110°\ 30') & = 9.97159 + & \\ \hline \log\ \sin\ P & = 9.91737 + & \therefore\ P = 55°\ 45'\ 57''.\end{array}$$

P is of the same species as p, and since $p < 90°$, $P < 90°$. The species of P can not be determined by the sign of $\sin P$, since the sign of $\sin P$ is plus from 0° to 180°.

2. Given $\left\{\begin{array}{l} h = \ \ 94°\ 05'. \\ p = 100°\ 45'. \end{array}\right\}$ Req. $\left\{\begin{array}{l} b = \ 67°\ 33'\ 27''. \\ B = \ 67°\ 54'\ 47''. \\ P = \ 99°\ 57'\ 35''. \end{array}\right.$

3. Given $\left\{\begin{array}{l} h = 110°\ 46'\ 26''. \\ B = \ \ 80°\ 10'\ 36''. \end{array}\right\}$ Req. $\left\{\begin{array}{l} b = \ \ 67°\ 06'\ 44''. \\ p = 155°\ 47'\ 05''. \\ P = 153°\ 58'\ 45''. \end{array}\right.$

4. Given $\begin{cases} b = 29° \; 46' \; 08''. \\ P = 137° \; 24' \; 21''. \end{cases}$ Req. $\begin{cases} B = 54° \; 01' \; 15''. \\ h = 142° \; 09' \; 12''. \\ p = 155° \; 27' \; 55''. \end{cases}$

5. Given $\begin{cases} b = 63° \; 15'. \\ p = 55° \; 28'. \end{cases}$ Req. $\begin{cases} h = 75° \; 13' \; 01''. \\ B = 67° \; 27' \; 01''. \\ P = 58° \; 25' \; 45''. \end{cases}$

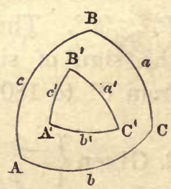
6. Given $\begin{cases} B = 52° \; 32' \; 55''. \\ P = 66° \; 20' \; 40''. \end{cases}$ Req. $\begin{cases} h = 70° \; 23' \; 41''. \\ b = 48° \; 24' \; 18''. \\ p = 59° \; 38' \; 27''. \end{cases}$

7. Giv. $\begin{cases} P = 75° \; 30'. \\ p = 50° \; 15'. \end{cases}$ Req. $\begin{cases} b = 18° \; 07' \; 02'' \text{ or } 161° \; 52' \; 58''. \\ h = 52° \; 34' \; 31'' \text{ or } 127° \; 25' \; 29''. \\ B = 23° \; 03' \; 06'' \text{ or } 156° \; 56' \; 54''. \end{cases}$

8. If a line make an angle of 40° with a fixed plane, and a plane embracing this line be perpendicular to the fixed plane, how many degrees from its first position must the plane embracing the line revolve about it in order that it may make an angle of 45° with the fixed plane? *Ans.* 67° 22′ 44″ or 112° 37′ 16″.

132. Polar Triangles.

The polar triangle of a given triangle is the triangle formed by the intersection of three arcs of great circles described about the vertices of the given triangle as poles.



If one triangle is the polar of another, the second is the polar of the first.

Thus, if $A'B'C'$ is the polar of the triangle ABC, then ABC is the polar of $A'B'C'$.

Each angle in one of two polar triangles is the supplement of the side lying opposite to it in the other;

and each side is the supplement of the angle lying opposite to it in the other. Thus,

$$A = 180° - a', \quad B = 180° - b', \quad C = 180° - c'.$$
$$a = 180° - A', \quad b = 180° - B', \quad c = 180° - C'.$$
$$A' = 180° - a, \quad B' = 180° - b, \quad C' = 180° - c.$$
$$a' = 180° - A, \quad b' = 180° - B, \quad c' = 180° - C.$$

Cor.—If $a' = 90°$, $A = 90°$; hence, if one side of a triangle is 90°, one angle of its polar triangle is 90°.

133. Quadrantal Triangles.

A quadrantal triangle is a triangle one side of which is 90°.

By the corollary of the last article, it follows that the polar of a quadrantal triangle is a right-angled triangle.

A quadrantal triangle is solved by passing to its polar triangle, which is solved as a right-angled triangle, then by passing back to the quadrantal triangle, which is the polar of the right-angled triangle.

134. Examples.

1. Given $\left\{\begin{array}{l} h' = 90°. \\ P' = 129° \ 15'. \\ b' = 62° \ 46' \ 01''. \end{array}\right\}$ Req. $\left\{\begin{array}{l} H' = 69° \ 30'. \\ B' = 56° \ 23' \ 30''. \\ p' = 124° \ 14' \ 03''. \end{array}\right.$

Passing to the polar triangle, which is right-angled, we have

Given $\left\{\begin{array}{l} H = 90°. \\ p = 50° \ 45'. \\ B = 117° \ 13' \ 59''. \end{array}\right\} \therefore \left\{\begin{array}{l} h = 110° \ 30'. \\ b = 123° \ 36' \ 30''. \\ P = 55° \ 45' \ 57''. \end{array}\right.$

Passing back to the quadrantal triangle, we find H', B', p'.

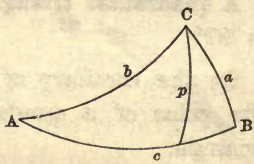
2. Given $\begin{cases} a' = 90°. \\ c' = 99° 20'. \\ B' = 30° 12' 23''. \end{cases}$ Req. $\begin{cases} A' = 74° 26'. \\ C' = 108° 05' 26''. \\ b' = 31° 29' 14''. \end{cases}$

OBLIQUE TRIANGLES.

135. Proposition I.

The sines of the sides of a spherical triangle are proportional to the sines of their opposite angles.

Let ABC be a spherical triangle. From C draw p, the arc of a great circle perpendicular to the opposite side or to the opposite side produced.



In the first case we have, by Napier's principles,

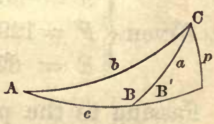
$$\sin p = \cos (90° - a) \cos (90° - B) = \sin a \sin B.$$

$$\sin p = \cos (90° - b) \cos (90° - A) = \sin b \sin A.$$

$$\therefore \quad \sin a \sin B = \sin b \sin A.$$

$$\therefore \quad \sin a : \sin b :: \sin A : \sin B.$$

In the second case we have, by Napier's principles,



$\sin p = \cos (90° - a) \cos (90° - B') =$
$\sin a \sin B' = \sin a \sin B.$

$$\sin p = \cos (90° - b) \cos (90° - A) = \sin b \sin A.$$

$$\therefore \quad \sin a \sin B = \sin b \sin A.$$

$$\therefore \quad \sin a : \sin b :: \sin A : \sin B.$$

In like manner other proportions may be deduced, giving the group,

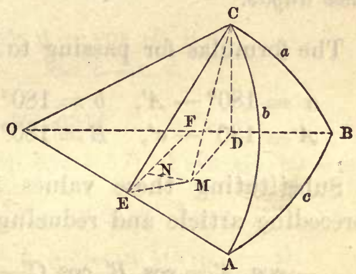
(1) $\sin a : \sin b :: \sin A : \sin B.$

(2) $\sin a : \sin c :: \sin A : \sin C.$

(3) $\sin b : \sin c :: \sin B : \sin C.$

136. Proposition II.

The co-sine of any side of a spherical triangle is equal to the product of the co-sines of the other sides, plus the product of their sines into the co-sine of their included angle.



Let ABC be a spherical triangle, and O the center of the sphere.

Let CM be perpendicular to the plane AOB. Draw MD and ME, respectively perpendicular to OB and OA, and draw CD and CE, which will be respectively perpendicular to OB and OA; hence, the angle $CEM = A$. and $CDM = B$. Draw EF perpendicular to OB, and MN perpendicular to EF. Each of the angles MEN and EOF is the complement of OEF; $\therefore$ $MEN = EOF = c$.

$$OD = OF + NM.$$

$$OD = \cos a.$$

$$OF = OE \cos EOF = \cos b \cos c.$$

$$NM = EM \sin MEN = \sin b \cos A \sin c.$$

Substituting the values of OD, OF, and NM, we have

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

In like manner other formulas may be deduced, giving the group,

$$(1) \quad \cos a = \cos b \cos c + \sin b \sin c \cos A.$$
$$(2) \quad \cos b = \cos a \cos c + \sin a \sin c \cos B.$$
$$(3) \quad \cos c = \cos a \cos b + \sin a \sin b \cos C.$$

137. Proposition III.

The co-sine of any angle of a spherical triangle is equal to the product of the sines of the other angles into the co-sine of their included side, minus the product of the co-sines of these angles.

The formulas for passing to the polar triangle are,

$$a = 180° - A', \quad b = 180° - B', \quad c = 180° - C'.$$
$$A = 180° - a', \quad B = 180° - b', \quad C = 180° - c'.$$

Substituting these values in the formulas of the preceding article and reducing, we have

$$-\cos A' = \cos B' \cos C' - \sin B' \sin C' \cos a'.$$
$$-\cos B' = \cos A' \cos C' - \sin A' \sin C' \cos b'.$$
$$-\cos C' = \cos A' \cos B' - \sin A' \sin B' \cos c'.$$

Changing the signs and omitting the accents, since the formulas are true for any triangle, we have

$$(1) \quad \cos A = \sin B \sin C \cos a - \cos B \cos C.$$
$$(2) \quad \cos B = \sin A \sin C \cos b - \cos A \cos C.$$
$$(3) \quad \cos C = \sin A \sin B \cos c - \cos A \cos B.$$

138. Proposition IV.

The co-sine of one-half of any angle of a spherical triangle is equal to the square root of the quotient obtained by *dividing the sine of one-half the sum of the sides into the sine of one-half the sum minus the side opposite the angle, by the product of the sines of the adjacent sides.*

The first formula of article 136 gives

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Adding 1 to both members, we have

$$1 + \cos A = \frac{\cos a + \sin b \sin c - \cos b \cos c}{\sin b \sin c}.$$

$$1 + \cos A = 2 \cos^2 \tfrac{1}{2} A. \quad \text{Article 95, (10).}$$

$$\sin b \sin c - \cos b \cos c = - \cos (b + c). \quad \text{Art. 89, } (b).$$

$$\therefore \quad 2 \cos^2 \tfrac{1}{2} A = \frac{\cos a - \cos (b + c)}{\sin b \sin c}.$$

But by article 96, (8), we have

$$\cos a - \cos (b + c) = 2 \sin \tfrac{1}{2}(a + b + c) \sin \tfrac{1}{2}(b + c - a).$$

Substituting and dividing by 2, we have

$$\cos^2 \tfrac{1}{2} A = \frac{\sin \tfrac{1}{2}(a + b + c) \cdot \sin \tfrac{1}{2}(b + c - a)}{\sin b \sin c}.$$

Let $s = a + b + c$, then will $\frac{1}{2} s = \frac{1}{2}(a + b + c)$, $\frac{1}{2} s - a = \frac{1}{2}(b + c - a)$.

Substituting in the value of $\cos^2 \frac{1}{2} A$, and in the similar values for $\cos^2 \frac{1}{2} B$ and $\cos^2 \frac{1}{2} C$, and extracting the square root, we have

$$(1) \quad \cos \tfrac{1}{2} A = \sqrt{\frac{\sin \tfrac{1}{2} s \sin (\tfrac{1}{2} s - a)}{\sin b \sin c}}.$$

$$(2) \quad \cos \tfrac{1}{2} B = \sqrt{\frac{\sin \tfrac{1}{2} s \sin (\tfrac{1}{2} s - b)}{\sin a \sin c}}.$$

$$(3) \quad \cos \tfrac{1}{2} C = \sqrt{\frac{\sin \tfrac{1}{2} s \sin (\tfrac{1}{2} s - c)}{\sin a \sin b}}.$$

139. Proposition V.

The sine of one-half of any side of a spherical triangle is equal to the square root of the quotient obtained by dividing minus the co-sine of one-half the sum of the angles into the co-sine of one-half the sum minus the angle opposite the side, by the product of the sines of the adjacent angles.

Taking the formulas of the last article, passing to the polar triangle, making $S = A' + B' + C'$, substituting in these formulas, reducing, and omitting the accents, we have

$$(1)\quad \sin \tfrac{1}{2} a = \sqrt{\frac{-\cos \frac{1}{2} S \cos (\frac{1}{2} S - A)}{\sin B \sin C}}.$$

$$(2)\quad \sin \tfrac{1}{2} b = \sqrt{\frac{-\cos \frac{1}{2} S \cos (\frac{1}{2} S - B)}{\sin A \sin C}}.$$

$$(3)\quad \sin \tfrac{1}{2} c = \sqrt{\frac{-\cos \frac{1}{2} S \cos (\frac{1}{2} S - C)}{\sin A \sin B}}.$$

140. Proposition VI.

The sine of one-half of any angle of a spherical triangle is equal to the square root of the quotient obtained by dividing the sine of one-half the sum of the sides minus one adjacent side into the sine of one-half the sum minus the other adjacent side, by the product of the sines of the adjacent sides.

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \quad \text{Article 136, (1).}$$

Subtracting both members from 1, we have

$$1 - \cos A = \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c}.$$

$$1 - \cos A = 2 \sin^2 \tfrac{1}{2} A. \quad \text{Article 95, (9).}$$

$\cos b \cos c + \sin b \sin c = \cos (b - c)$. Article 91, (*d*).

$$\therefore\ 2 \sin^2 \tfrac{1}{2}A = \frac{\cos (b - c) - \cos a}{\sin b \sin c}.$$

But by article 96, (8), we have

$$\cos (b - c) - \cos a = 2 \sin \tfrac{1}{2}(a + c - b) \sin \tfrac{1}{2}(a + b - c).$$

Substituting and dividing by 2, we have

$$\sin^2 \tfrac{1}{2}A = \frac{\sin \tfrac{1}{2}(a + c - b) \sin \tfrac{1}{2}(a + b - c)}{\sin b \sin c}.$$

But $\tfrac{1}{2}(a + c - b) = \tfrac{1}{2}s - b$ and $\tfrac{1}{2}(a + b - c) = \tfrac{1}{2}s - c$.

Substituting in the value of $\sin^2 \tfrac{1}{2}A$, and in the similar values for $\sin^2 \tfrac{1}{2}B$ and $\sin^2 \tfrac{1}{2}C$, and extracting the square root, we have

$$(1)\quad \sin \tfrac{1}{2}A = \sqrt{\frac{\sin (\tfrac{1}{2}s - b) \sin (\tfrac{1}{2}s - c)}{\sin b \sin c}}.$$

$$(2)\quad \sin \tfrac{1}{2}B = \sqrt{\frac{\sin (\tfrac{1}{2}s - a) \sin (\tfrac{1}{2}s - c)}{\sin a \sin c}}.$$

$$(3)\quad \sin \tfrac{1}{2}C = \sqrt{\frac{\sin (\tfrac{1}{2}s - a) \sin (\tfrac{1}{2}s - b)}{\sin a \sin b}}.$$

141. Proposition VII.

The co-sine of one-half of any side of a spherical triangle is equal to the square root of the quotient obtained by dividing the co-sine of one-half the sum of the angles minus one adjacent angle into the co-sine of half the sum minus the other adjacent angle, by the product of the sines of the adjacent angles.

Taking the formulas of the last article, passing to the polar triangle, making $S = A' + B' + C'$, substituting, reducing, and omitting the accents, we have

$$(1) \quad \cos \tfrac{1}{2} a = \sqrt{\frac{\cos(\frac{1}{2}S - B) \cos(\frac{1}{2}S - C)}{\sin B \sin C}}.$$

$$(2) \quad \cos \tfrac{1}{2} b = \sqrt{\frac{\cos(\frac{1}{2}S - A) \cos(\frac{1}{2}S - C)}{\sin A \sin C}}.$$

$$(3) \quad \cos \tfrac{1}{2} c = \sqrt{\frac{\cos(\frac{1}{2}S - A) \cos(\frac{1}{2}S - B)}{\sin A \sin B}}.$$

142. Proposition VIII.

The tangent of one-half of any angle of a spherical triangle is equal to the square root of the quotient obtained by dividing the sine of one-half the sum of the sides minus one adjacent side into the sine of one-half the sum minus the other adjacent side, by the sine of one-half the sum of the sides into the sine of one-half the sum minus the opposite side.

Dividing (1), (2), (3), article 140, respectively, by (1), (2), (3), article 138, we have

$$(1) \quad \tan \tfrac{1}{2} A = \sqrt{\frac{\sin(\frac{1}{2}s - b) \sin(\frac{1}{2}s - c)}{\sin \frac{1}{2}s \sin(\frac{1}{2}s - a)}}.$$

$$(2) \quad \tan \tfrac{1}{2} B = \sqrt{\frac{\sin(\frac{1}{2}s - a) \sin(\frac{1}{2}s - c)}{\sin \frac{1}{2}s \sin(\frac{1}{2}s - b)}}.$$

$$(3) \quad \tan \tfrac{1}{2} C = \sqrt{\frac{\sin(\frac{1}{2}s - a) \sin(\frac{1}{2}s - b)}{\sin \frac{1}{2}s \sin(\frac{1}{2}s - c)}}.$$

143. Proposition IX.

The tangent of one-half of any side of a spherical triangle is equal to the square root of the quotient obtained by dividing *minus the co-sine of one-half the sum of the angles into the co-sine of one-half the sum minus the angle opposite the side, by the co-sine of one-half the sum of the angles minus one adjacent angle into the co-sine of one-half the sum minus the other adjacent angle.*

Dividing (1), (2), (3), article 139, respectively, by (1), (2), (3), article 141, we have

$$(1)\quad \tan \tfrac{1}{2} a = \sqrt{\frac{-\cos \frac{1}{2}S \cos (\frac{1}{2}S - A)}{\cos (\frac{1}{2}S - B) \cos (\frac{1}{2}S - C)}}.$$

$$(2)\quad \tan \tfrac{1}{2} b = \sqrt{\frac{-\cos \frac{1}{2}S \cos (\frac{1}{2}S - B)}{\cos (\frac{1}{2}S - A) \cos (\frac{1}{2}S - C)}}.$$

$$(3)\quad \tan \tfrac{1}{2} c = \sqrt{\frac{-\cos \frac{1}{2}S \cos (\frac{1}{2}S - C)}{\cos (\frac{1}{2}S - A) \cos (\frac{1}{2}S - B)}}.$$

The reciprocals of (1), (2), (3), articles 142, 143, will give formulas for co-tangents, which may be written and expressed in words.

144. Napier's Analogies.

Dividing (1), article 142, by (2), we have

$$\frac{\tan \frac{1}{2} A}{\tan \frac{1}{2} B} = \frac{\sin (\frac{1}{2} s - b)}{\sin (\frac{1}{2} s - a)}.$$

This, as a proportion taken by composition and division, gives

$$\frac{\tan \frac{1}{2} A + \tan \frac{1}{2} B}{\tan \frac{1}{2} A - \tan \frac{1}{2} B} = \frac{\sin (\frac{1}{2} s - b) + \sin (\frac{1}{2} s - a)}{\sin (\frac{1}{2} s - b) - \sin (\frac{1}{2} s - a)}.$$

$$\frac{\tan \frac{1}{2} A + \tan \frac{1}{2} B}{\tan \frac{1}{2} A - \tan \frac{1}{2} B} = \frac{\dfrac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} + \dfrac{\sin \frac{1}{2} B}{\cos \frac{1}{2} B}}{\dfrac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} - \dfrac{\sin \frac{1}{2} B}{\cos \frac{1}{2} B}}.$$

Multiplying both terms of the second member by $\cos \frac{1}{2}A \cos \frac{1}{2}B$,

$$\frac{\tan \frac{1}{2}A + \tan \frac{1}{2}B}{\tan \frac{1}{2}A - \tan \frac{1}{2}B} = \frac{\sin \frac{1}{2}A \cos \frac{1}{2}B + \cos \frac{1}{2}A \sin \frac{1}{2}B}{\sin \frac{1}{2}A \cos \frac{1}{2}B - \cos \frac{1}{2}A \sin \frac{1}{2}B}.$$

Reducing the second member by articles 89, (a), and 91, (c),

$$\frac{\tan \frac{1}{2}A + \tan \frac{1}{2}B}{\tan \frac{1}{2}A - \tan \frac{1}{2}B} = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)}.$$

$$\frac{\sin(\frac{1}{2}s - b) + \sin(\frac{1}{2}s - a)}{\sin(\frac{1}{2}s - b) - \sin(\frac{1}{2}s - a)} = \frac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(a-b)}. \text{ Art. 96, (11).}$$

$$\therefore \quad \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(a-b)}.$$

$\therefore$ (1) $\sin \frac{1}{2}(A+B) : \sin \frac{1}{2}(A-B) :: \tan \frac{1}{2}c : \tan \frac{1}{2}(a-b)$.

The reciprocal of (1) $\times$ (2), article 142, gives

$$\frac{1}{\tan \frac{1}{2}A \tan \frac{1}{2}B} = \frac{\sin \frac{1}{2}s}{\sin(\frac{1}{2}s - c)}.$$

By division and composition, we have

$$\frac{1 - \tan \frac{1}{2}A \tan \frac{1}{2}B}{1 + \tan \frac{1}{2}A \tan \frac{1}{2}B} = \frac{\sin \frac{1}{2}s - \sin(\frac{1}{2}s - c)}{\sin \frac{1}{2}s + \sin(\frac{1}{2}s - c)}.$$

Reducing both members as before, we have

$$\frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(a+b)}.$$

$\therefore$ (2) $\cos \frac{1}{2}(A+B) : \cos \frac{1}{2}(A-B) :: \tan \frac{1}{2}c : \tan \frac{1}{2}(a+b)$.

Passing from (1) and (2) to the polar triangle, we have

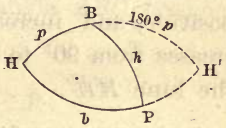
(3) $\sin \frac{1}{2}(a+b) : \sin \frac{1}{2}(a-b) :: \cot \frac{1}{2}C : \tan \frac{1}{2}(A-B)$.

(4) $\cos \frac{1}{2}(a+b) : \cos \frac{1}{2}(a-b) :: \cot \frac{1}{2}C : \tan \frac{1}{2}(A+B)$.

145. Proposition.

In a right-angled spherical triangle, as b increases from $0°$ *to* $90°$, *from* $90°$ *to* $180°$, *from* $180°$ *to* $270°$, *and from* $270°$ *to* $360°$, *if* $p < 90°$, *h increases from p to* $90°$, *from* $90°$ *to* $180° - p$, *decreases from* $180° - p$ *to* $90°$, *and from* $90°$ *to p; if* $p > 90°$, *h decreases from p to* $90°$, *from* $90°$ *to* $180° - p$, *increases from* $180° - p$ *to* $90°$, *and from* $90°$ *to p; if* $p = 90°$, $h = 90°$ *for all values of b.*

1. $p < 90°$; $\therefore \cos p$ is positive.



$\cos h = \cos b \cos p.$

If $b = 0$, $\cos b = 1$; therefore, $\cos h = \cos p$; $\therefore h = p$.

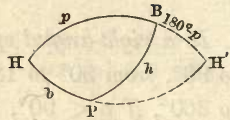
As b increases from $0°$ to $90°$, $\cos b$ is positive, and diminishes from 1 to 0; $\therefore \cos h$ is positive, and diminishes from $\cos p$ to 0; $\therefore h$ increases from p to $90°$.

As b increases from $90°$ to $180°$, $\cos b$ is negative, and increases numerically from 0 to -1; $\therefore \cos h$ is negative, and increases numerically from 0 to $-\cos p$; $\therefore h$ increases from $90°$ to $180° - p$, and the triangle becomes the lune HH'.

As b increases from $180°$ to $270°$, $\cos b$ is negative, and decreases numerically from -1 to 0; $\therefore \cos h$ is negative, and decreases numerically from $-\cos p$ to 0; $\therefore h$ decreases from $180° - p$ to $90°$.

As b increases from $270°$ to $360°$, $\cos b$ is positive, and increases from 0 to 1; $\therefore \cos h$ is positive, and increases from 0 to $\cos p$; $\therefore h$ decreases from $90°$ to p, and the triangle becomes the hemisphere.

2. $p > 90°$; $\therefore$ $\cos p$ is negative.



$\cos h = \cos b \cos p$.

If $b = 0$, $\cos b = 1$; therefore,
$\cos h = \cos p$; $\therefore h = p$.

As b increases from 0° to 90°, $\cos b$ is positive, and decreases from 1 to 0; $\therefore$ $\cos h$ is negative, and decreases numerically from $\cos p$ to 0; $\therefore$ h decreases from p to 90°.

As b increases from 90° to 180°, $\cos b$ is negative, and increases numerically from 0 to -1; $\therefore$ $\cos h$ is positive, and increases from 0 to $-\cos p$; $\therefore$ h decreases from 90° to $180° - p$, and the triangle becomes the lune HH'.

As b increases from 180° to 270°, $\cos b$ is negative, and decreases numerically from -1 to 0; $\therefore$ $\cos h$ is positive, and decreases from $-\cos p$ to 0; $\therefore$ h increases from $180° - p$ to 90°.

As b increases from 270° to 360°, $\cos b$ is positive, and increases from 0 to 1; $\therefore$ $\cos h$ is negative, and increases numerically from 0 to $\cos p$; $\therefore$ h increases from 90° to p, and the triangle becomes the hemisphere.

3. $p = 90°$; $\therefore$ $\cos p = 0$.

$\therefore$ $\cos h = \cos b \cos p = 0$; $\therefore$ $h = 90°$.

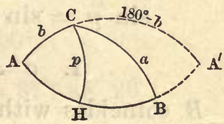
Cor.—Since B and b are of the same species, B may be substituted for b in the preceding proposition.

In the application of these principles to the discussion of Case I, in which two sides and an angle opposite one of them are given, a corresponds to h, and HB to b.

146. Case I.

Given two sides of a spherical triangle, and the angle opposite one of them; required the remaining parts.

Let a and b be the given sides and A the given angle.



I. $A < 90°$; $\therefore p < 90°$.

$\sin p = \sin b \sin A$.

1. $a = p$.

B coincides with H, and the triangle ABC becomes the right triangle AHC.

2. $a < 90°$ and $a > p$.

By the last proposition the point B lies in the first or fourth quadrant, estimated from H.

3. $a = 90°$.

$HB = 90°$ or $270°$, and $HCB = 90°$ or $270°$.

4. $a > 90°$ and $a < 180° - p$.

B lies in the second or third quadrant from H.

5. $a = 180° - p$.

$HB = 180°$, and $ABC = AHC + \frac{1}{2}$ the hemisphere.

6. $a = 180° - b$.

$HB = HA'$ or $360° - HA'$, and then the first triangle becomes the lune AA'.

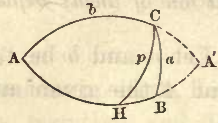
7. $a = b$.

$HB = AH$ or $360° - AH$, and the second triangle becomes the hemisphere.

8. $a < p$ or $a > 180° - p$.

The triangle is impossible, since p is the least, and $180° - p$ is the greatest value of a.

II. $A > 90°$; $\therefore p > 90°$.



$\sin p = \sin b \sin A$.

1. $a = p$.

B coincides with H, and ABC becomes AHC.

2. $a > 90°$ and $a < p$.

B lies in the first or fourth quadrant from H.

3. $a = 90°$.

$HB = 90°$ or $270°$, and $HCB = 90°$ or $270°$.

4. $a < 90°$ and $a > 180° - p$.

B lies in the second or third quadrant from H.

5. $a = 180° - p$.

$HB = 180°$, and $ABC = AHC + \frac{1}{2}$ the hemisphere.

6. $a = 180° - b$.

$HB = HA'$ or $360° - HA'$, and the first triangle becomes the lune AA'.

7. $a = b$.

$HB = AH$ or $360° - AH$, and the second triangle becomes the hemisphere.

8. $a > p$ or $a < 180° - p$.

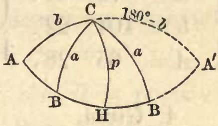
The triangle is impossible, since p is the greatest, and $180° - p$ is the least value of a.

III. $A = 90°$.

The triangle is right-angled, and is solved as in article 131.

147. Examples.

1. Given $\left\{\begin{array}{l} a = 60° \ 20'. \\ b = 80° \ 35'. \\ A = 38° \ 25'. \end{array}\right\}$ Req. $\left\{\begin{array}{l} B. \\ C. \\ c. \end{array}\right.$



$$A < 90°; \ \therefore \ p < 90°.$$

$$\sin p = \sin b \sin A, \ \therefore \ p = 37° \ 48' \ 26''.$$

Since $a > p$ and $a < 180° - p$, the triangle is possible.

Since $a < b$ and $a < 180° - b$, B lies between H and A or H and A'.

$$\sin p = \sin a \sin B, \quad \therefore \ B = 44° \ 52' \ 05''.$$

$$\cos HCB = \tan p \cot a, \quad \therefore \ HCB = 63° \ 46' \ 18''.$$

$$\cos a = \cos p \cos HB, \quad \therefore \ HB = 51° \ 12' \ 41''.$$

$$\cos ACH = \tan p \cot b, \quad \therefore \ ACH = 82° \ 36' \ 25''.$$

$$\cos b = \cos p \cos AH, \quad \therefore \ AH = 78° \ 02' \ 54''.$$

$$C = ACH \pm HCB = 146° \ 22' \ 43'' \text{ or } 18° \ 50' \ 07''.$$

$$c = AH \pm HB = 129° \ 15' \ 35'' \text{ or } 26° \ 50' \ 13''.$$

$$\text{In } ACB, \ ABC = 180° - HBC = 135° \ 07' \ 55''.$$

We can also find B from the proportion,

$$\sin a : \sin b :: \sin A : \sin B.$$

C and c can be found from the proportions,

$$\sin \tfrac{1}{2}(b + a) : \sin \tfrac{1}{2}(b - a) :: \cot \tfrac{1}{2}C : \tan \tfrac{1}{2}(B - A).$$

$$\sin A : \sin C :: \sin a : \sin c.$$

2. *Given.* *Required.*

$$\left\{\begin{array}{l} a = 63° \ 50'. \\ b = 80° \ 19'. \\ A = 51° \ 30'. \end{array}\right\} \left\{\begin{array}{l} B = 59° \ 16' \ 00'' \text{ or } 120° \ 44' \ 00''. \\ C = 131° \ 29' \ 42'' \text{ or } 24° \ 37' \ 30''. \\ c = 120° \ 47' \ 50'' \text{ or } 28° \ 32' \ 44''. \end{array}\right.$$

3. *Given.* *Required.*

$$\left\{\begin{aligned} a &= 75^\circ\ 38'.\\ b &= 104^\circ\ 22'.\\ A &= 65^\circ\ 28'.\end{aligned}\right\}\left\{\begin{aligned} B &= 65^\circ\ 28' &&\text{or } 114^\circ\ 32'.\\ C &= 180^\circ &&\text{or } 57^\circ\ 03'\ 32''.\\ c &= 180^\circ &&\text{or } 63^\circ\ 20'\ 18''.\end{aligned}\right.$$

4. *Given.* *Required.*

$$\left\{\begin{aligned} a &= 99^\circ\ 40'\ 48''.\\ b &= 64^\circ\ 23'\ 15''.\\ A &= 95^\circ\ 38'\ 04''.\end{aligned}\right\}\left\{\begin{aligned} B &= 114^\circ\ 26'\ 50'' &&\text{or } 65^\circ\ 33'\ 10''.\\ C &= 236^\circ\ 51'\ 27'' &&\text{or } 97^\circ\ 27'\ 13''.\\ c &= 236^\circ\ 01'\ 51'' &&\text{or } 100^\circ\ 49'\ 49''.\end{aligned}\right.$$

5. *Given.* *Required.*

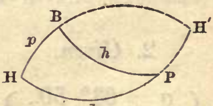
$$\left\{\begin{aligned} a &= 100^\circ.\\ b &= 85^\circ.\\ A &= 50^\circ.\end{aligned}\right\}\left\{\begin{aligned} B &= 50^\circ\ 47'\ 41'' &&\text{or } 129^\circ\ 12'\ 19''.\\ C &= 186^\circ\ 05'\ 16'' &&\text{or } 342^\circ\ 03'\ 12''.\\ c &= 187^\circ\ 50'\ 09'' &&\text{or } 336^\circ\ 39'\ 45''.\end{aligned}\right.$$

6. If $A < 90^\circ$, what is the relation of a to p, or to $180^\circ - p$, when there is no solution?

7. If $A > 90^\circ$, what is the relation of a to p, or to $180^\circ - p$, when there is no solution?

148. Proposition.

In a right-angled spherical triangle, as B increases from 0° to 90°, from 90° to 180°, from 180° to 270°, and from 270° to 360°; if $p < 90^\circ$, P decreases from 90° to p, increases from p to 90°, increases from 90° to $180^\circ - p$, and decreases from $180^\circ - p$ to 90°; if $p > 90^\circ$, P increases from 90° to p, decreases from p to 90°, decreases from 90° to $180^\circ - p$, and increases from $180^\circ - p$ to 90°; if $p = 90^\circ$, $P = 90^\circ$, for all values of B.



1. $p < 90^\circ$; $\therefore$ $\cos p$ is positive.

$$\cos P = \cos p \sin B.$$

If $B = 0^\circ$, $\sin B = 0$; $\therefore \cos P = 0$; $\therefore P = 90^\circ$.

As B increases from 0° to 90°, sin B is positive, and increases from 0 to 1; $\therefore$ cos P is positive, and increases from 0 to cos p; $\therefore$ P decreases from 90° to p.

As B increases from 90° to 180°, sin B is positive, and decreases from 1 to 0; $\therefore$ cos P is positive, and decreases from cos p to 0; $\therefore$ P increases from p to 90°, and the triangle becomes the lune HH'.

As B increases from 180° to 270°, sin B is negative, and increases numerically from 0 to -1; $\therefore$ cos P is negative, and increases numerically from 0 to $-$ cos p; $\therefore$ P increases from 90° to $180° - p$.

As B increases from 270° to 360° sin B is negative, and decreases numerically from -1 to 0; $\therefore$ cos P is negative, and decreases numerically from $-$ cos p to 0; $\therefore$ P decreases from $180° - p$ to 90°, and the triangle becomes the hemisphere.

2. $p > 90°$; $\therefore$ cos p is negative.

$$\cos P = \cos p \sin B.$$

If $B = 0°$, sin $B = 0$; $\therefore$ cos $P = 0$; $\therefore$ $P = 90°$.

As B increases from 0° to 90°, sin B is positive, and increases from 0 to 1; $\therefore$ cos P is negative, and increases numerically from 0 to cos p; $\therefore$ P increases from 90° to p.

As B increases from 90° to 180°, sin B is positive, and decreases from 1 to 0; $\therefore$ cos P is negative, and decreases numerically from cos p to 0; $\therefore$ P decreases from p to 90°, and the triangle becomes the lune.

As B increases from 180° to 270°, sin B is negative, and increases numerically from 0 to -1; $\therefore$ cos P is positive, and increases from 0 to $-$ cos p; $\therefore$ P decreases from 90° to $180° - p$.

As B increases from 270° to 360°, sin B is negative, and decreases numerically from -1 to 0; $\therefore$ cos P is positive, and decreases numerically from $-\cos p$ to 0; $\therefore$ P increases from $180° - p$ to 90°, and the triangle becomes the hemisphere.

$$3.\quad p = 90°;\quad \therefore \cos p = 0.$$

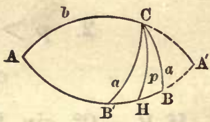
$$\therefore \cos P = \cos p \sin B = 0;\quad \therefore P = 90°.$$

Cor.— Since b and B are of the same species, b may be substituted for B in the preceding proposition.

149. Case II.

Given two angles of a spherical triangle and the side opposite one of them; required the remaining parts.

Let A and B be the given angles, and b the given side.



I. $A < 90°$; $\therefore p < 90°$.

$$\sin p = \sin b \sin A.$$

$$1.\quad B > p \text{ and } B < 90°.$$

By the last proposition, the point B lies in the first or second quadrant estimated from H as origin.

$$2.\quad B = p.$$

The angle $HCB = 90°$, and the arc $HB = 90°$.

$$3.\quad B < 180° - p, \text{ and } B > 90°.$$

B lies in the third or fourth quadrant from H.

$$4.\quad B = 180° - p.$$

The angle $HCB = 270°$, and the arc $HB = 270°$.

5. $B = 90°$.

$HB = 0°$, $180°$, or $360°$, and the triangle becomes ACH, $ACH + \frac{1}{2}$ of a hemisphere, or a hemisphere + ACH.

6. $B = A$.

B lies in the first or second quadrant from H, and one of the triangles becomes the lune AA'.

7. $B = 180° - A$.

B lies in the third or fourth quadrant from H, and one of the triangles becomes the hemisphere.

8. $B < p$ or $B > 180° - p$.

The triangle is impossible, since p is the least, and $180° - p$ is the greatest value of B.

II. $A > 90°$; $\therefore p > 90°$.

$$\sin p = \sin b \sin A.$$

1. $B < p$ and $B > 90°$.

B lies in the first or second quadrant from H.

2. $B = p$.

The angle $HCB = 90°$, and the arc $HB = 90°$.

3. $B > 180° - p$ and $B < 90°$.

B lies in the third or fourth quadrant from H.

4. $B = 180° - p$.

The angle $HCB = 270°$, and the arc $HB = 270°$.

5. $B = 90°$.

$HB = 0°$, $180°$, or $360°$, and the triangle becomes ACH, $ACH + \frac{1}{2}$ of a hemisphere, or a hemisphere + ACH.

6. $B = A$.

B lies in the first or second quadrant from H, and one of the triangles becomes the lune AA'.

7. $B = 180° - A$.

B lies in the third or fourth quadrant from H, and one of the triangles becomes the hemisphere.

8. $B > p$ or $B < 180° - p$.

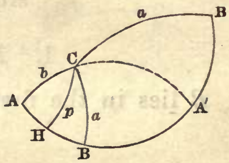
The triangle is impossible, since p is the greatest, and $180° - p$ is the least value of B.

III. $A = 90°$.

The triangle is right-angled, and is solved as in article 131.

150. Examples.

1. Giv. $\left\{\begin{array}{l} A = 75° \ 30'. \\ B = 80° \ 40'. \\ b = 70° \ 50'. \end{array}\right\}$ Req. $\left\{\begin{array}{l} a. \\ C. \\ c. \end{array}\right.$



$$A < 90°; \quad \therefore p < 90°.$$

$$\sin p = \sin b \sin A; \quad \therefore p = 66° \ 07' \ 56''.$$

Since $B > p$ and $< 180° - p$, the triangle is possible.

Since $B < 90°$ and $> p$, B lies in the first or second quadrant from H.

$$\sin p = \sin a \sin B, \quad \therefore a = \left\{\begin{array}{l} 67° \ 56'. \\ 112° \ 04'. \end{array}\right.$$

The second value of a, the supplement of the first, is taken when B lies in the second quadrant from H.

$$\cos B = \cos p \sin HCB, \therefore HCB = \begin{cases} 23^\circ\ 37'\ 44''. \\ 156^\circ\ 22'\ 16''. \end{cases}$$

$$\sin HB = \tan p \cot B, \quad \therefore HB = \begin{cases} 21^\circ\ 48'\ 19''. \\ 158^\circ\ 11'\ 41''. \end{cases}$$

$$\cos ACH = \tan p \cot b, \quad \therefore ACH = 38^\circ\ 13'\ 36''.$$

$$\cos b = \cos p \cos AH, \quad \therefore AH = 35^\circ\ 46'.$$

$$C = ACH + HCB = 61^\circ\ 51'\ 20'' \text{ or } 194^\circ\ 35'\ 52''.$$

$$c = AH + HB = 57^\circ\ 34'\ 19'' \text{ or } 193^\circ\ 57'\ 41''.$$

We can find a, c, and C from the proportions,

$$\sin B : \sin A :: \sin b : \sin a.$$

$$\sin \tfrac{1}{2}(B + A) : \sin \tfrac{1}{2}(B - A) :: \tan \tfrac{1}{2}c : \tan \tfrac{1}{2}(b - a).$$

$$\sin b : \sin c :: \sin B : \sin C.$$

2. *Given.* *Required.*

$$\begin{cases} A = 33^\circ\ 15'. \\ B = 31^\circ\ 34'\ 38''. \\ b = 70^\circ\ 10'\ 30''. \end{cases} \quad \begin{cases} a = 80^\circ\ 03'\ 25'' \text{ or } 99^\circ\ 56'\ 35''. \\ C = 161^\circ\ 24'\ 52'' \text{ or } 173^\circ\ 30'\ 52''. \\ c = 145^\circ\ 03'\ 13'' \text{ or } 168^\circ\ 18'\ 23''. \end{cases}$$

3. *Given.* *Required.*

$$\begin{cases} A = 132^\circ\ 16'. \\ B = 139^\circ\ 44'. \\ b = 127^\circ\ 30'. \end{cases} \quad \begin{cases} a = 65^\circ\ 16'\ 30'' \text{ or } 114^\circ\ 43'\ 30''. \\ C = 165^\circ\ 41'\ 46'' \text{ or } 126^\circ\ 40'\ 44''. \\ c = 162^\circ\ 20'\ 55'' \text{ or } 100^\circ\ 07'\ 25''. \end{cases}$$

4. *Given.* *Required.*

$$\begin{cases} A = 48^\circ\ 50'. \\ B = 131^\circ\ 10'. \\ b = 75^\circ\ 48'. \end{cases} \quad \begin{cases} a = 75^\circ\ 48' \text{ or } 104^\circ\ 12'. \\ C = 360^\circ \text{ or } 328^\circ\ 39'\ 28''. \\ c = 360^\circ \text{ or } 317^\circ\ 56'\ 42''. \end{cases}$$

Scholium.—In the two preceding cases some of the parts are found to be greater than 180°; but the corresponding triangles conform to the conditions of the problem, and are therefore true solutions.

Parts greater than 180° are usually excluded, in which case the principles of the following article will aid in determining the species of the parts.

The principles established in Geometry are given without demonstration.

151. Principles.

1. *Each part of a spherical triangle is less than* 180°.

2. *The greater side is opposite the greater angle, and conversely.*

3. *Each side is less than the sum of the other sides.*

4. *The sum of the sides is less than* 360°.

5. *The sum of the angles is greater than* 180°, *and less than* 540°.

6. *Each angle is greater than the difference between* 180° *and the sum of the other angles.*

$$\text{For,}\quad A + B + C > 180°. \quad \text{Principle 5.}$$

$$\therefore\quad A > 180° - (B + C).$$

The last formula is always *algebraically* true; but in case $B + C > 180°$, it might be doubted whether it is numerically true.

Passing to the polar triangle, we have, by principle 3,

$$a' < b' + c'.$$

$$\text{or}\quad 180° - A < 180° - B + 180° - C.$$

$$\text{or}\quad -A < 180° - (B + C).$$

$$\therefore\quad A > B + C - 180°.$$

7. *A side differing more from* 90° *than another side is of the same species as its opposite angle.*

By article 136, we have

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

$$\therefore \quad \cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

But $\sin b \sin c$ is positive, since b and c are each less than 180°.

If a differs more from 90° than b or c, then we shall have

$$\cos a > \cos b, \text{ or } \cos a > \cos c, \text{ numerically;}$$

and since neither $\cos b$ nor $\cos c$ exceeds 1, we have

$$\cos a > \cos b \cos c.$$

$\therefore$ $\cos A$ and $\cos a$ have the same sign, $\therefore$ A and a are of the same species.

8. *An angle differing more from* 90° *than another angle is of the same species as its opposite side.*

By article 137, we have

$$\cos A = \sin B \sin C \cos a - \cos B \cos C.$$

$$\therefore \quad \cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

If A differs more from 90° than B or C, then, as before, $\cos A$ and $\cos a$ have the same sign, or A and a are of the same species.

9. *Two sides, at least, are of the same species as their opposite angles, and conversely.*

If each of two sides differs more from 90° than the remaining side, they will be of the same species as their opposite angles, as is evident from principle 7.

If the triangle is isosceles, and the equal sides less than 90°, the perpendicular from the vertex to the third side will be less than 90°, since one-half the third side is less than 90°, and the angles opposite this perpendicular will be less than 90°, article 129, or of the same species as their opposite sides.

If the equal sides are greater than 90°, the perpendicular will be greater than 90°, since one-half the third side is less than 90°, and the angles opposite the perpendicular will be greater than 90°, article 129, or of the same species as their opposite sides.

If one side exceeds 90° by as much as 90° exceeds another side, and the third side is greater or less than each of the other sides, this third side is of the same species as its opposite angle by principle 7.

If the greater of the two sides is of the same species as its opposite angle, then we shall have two sides of the same species as their opposite angles.

If the greater of the two sides is not of the same species as its opposite angle, this angle will be of the same species as the other side, or less than 90°; but the angle opposite this other side is less than the angle opposite the greater side, and hence less than 90°, or of the same species as its opposite side, and again we have two sides of the same species as their opposite angles.

10. *The sum of two sides is greater than, equal to, or less than,* 180°, *according as the sum of their opposite angles is greater than, equal to, or less than,* 180°.

$\tan \frac{1}{2}(a+b) \cos \frac{1}{2}(A+B) = \tan \frac{1}{2}c \cos \frac{1}{2}(A-B)$. Art. 144.

But $c < 180°$, $\therefore \frac{1}{2}c < 90°$, $\tan \frac{1}{2}c > 0$,

and $A-B < 180°$, $\therefore \frac{1}{2}(A-B) < 90°$, $\cos \frac{1}{2}(A-B) > 0$.

$\therefore \tan \frac{1}{2}c \cos \frac{1}{2}(A-B) > 0$, $\tan \frac{1}{2}(a+b) \cos \frac{1}{2}(A+B) > 0$.

$\therefore \tan \frac{1}{2}(a+b)$ and $\cos \frac{1}{2}(A+B)$ have like signs.

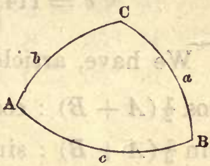
$\therefore$ If $\frac{1}{2}(A+B) >, =$ or $< 90°$, $\frac{1}{2}(a+b) >, =$ or $< 90°$.

$\therefore$ If $A+B >, =$ or $< 180°$, $a+b >, =$ or $< 180°$.

152. Case III.

Given two sides and the included angle of a spherical triangle; required the remaining parts.

1. Given $\begin{cases} a = 85° \; 30'. \\ b = 65° \; 40'. \\ C = 95° \; 50'. \end{cases}$ Req. $\begin{cases} A. \\ B. \\ C. \end{cases}$



We have, article 144,

$$\cos \tfrac{1}{2}(a+b) : \cos \tfrac{1}{2}(a-b) :: \cot \tfrac{1}{2} C : \tan \tfrac{1}{2}(A+B).$$

$$\sin \tfrac{1}{2}(a+b) : \sin \tfrac{1}{2}(a-b) :: \cot \tfrac{1}{2} C : \tan \tfrac{1}{2}(A-B).$$

$$\therefore \begin{cases} \frac{1}{2}(A+B) = 74° \; 21' \; 49''. \\ \frac{1}{2}(A-B) = \;\; 9° \; 07' \; 21''. \end{cases} \quad \therefore \begin{cases} A = 83° \; 29' \; 10''. \\ B = 65° \; 14' \; 28''. \end{cases}$$

We also have, article 144,

$$\sin \tfrac{1}{2}(A+B) : \sin \tfrac{1}{2}(A-B) :: \tan \tfrac{1}{2} c : \tan \tfrac{1}{2}(a-b).$$

$$\therefore \tfrac{1}{2} c = 46° \; 43' \; 09'', \quad \therefore c = 93° \; 26' \; 14''.$$

We can also find c from the proportion,

$$\sin A : \sin C :: \sin a : \sin c.$$

But the species of c is more readily determined from the proportion employed; for if we take the supplement of $46° \; 43' \; 09''$, then c would be greater than $180°$.

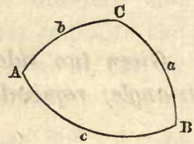
Again, all the known terms of the proportion are positive; hence, $\tan \frac{1}{2} c$ is positive, $\therefore \frac{1}{2} c < 90°$.

2. Given $\begin{cases} a = 120° \; 30' \; 30''. \\ b = 70° \; 20' \; 20''. \\ C = 50° \; 10' \; 10''. \end{cases}$ Req. $\begin{cases} A = 135° \; 05' \; 29''. \\ B = \;\; 50° \; 30' \; 09''. \\ c = \;\; 69° \; 34' \; 58''. \end{cases}$

153. Case IV.

Given two angles and the included side of a spherical triangle; required the remaining parts.

1. Giv. $\left\{\begin{array}{l} A = 62° 54'. \\ B = 48° 30'. \\ c = 114° 29' 58''. \end{array}\right\}$ Req. $\left\{\begin{array}{l} a. \\ b. \\ C. \end{array}\right.$



We have, article 144,

$$\cos \tfrac{1}{2}(A+B) : \cos \tfrac{1}{2}(A-B) :: \tan \tfrac{1}{2}c : \tan \tfrac{1}{2}(a+b).$$

$$\sin \tfrac{1}{2}(A+B) : \sin \tfrac{1}{2}(A-B) :: \tan \tfrac{1}{2}c : \tan \tfrac{1}{2}(a-b).$$

$$\therefore \left\{\begin{array}{l} \tfrac{1}{2}(a+b) = 69° 55' 48''. \\ \tfrac{1}{2}(a-b) = 13° 16' 18''. \end{array}\right\} \therefore \left\{\begin{array}{l} a = 83° 12' 06''. \\ b = 56° 39' 30''. \end{array}\right.$$

We also have, article 144,

$$\sin \tfrac{1}{2}(a+b) : \sin \tfrac{1}{2}(a-b) :: \cot \tfrac{1}{2}C : \tan \tfrac{1}{2}(A-B).$$

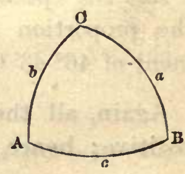
$$\therefore \tfrac{1}{2}C = 62° 40', \quad \therefore C = 125° 20'.$$

2. Given $\left\{\begin{array}{l} A = 126° 35' 02''. \\ B = 61° 43' 58''. \\ c = 57° 30'. \end{array}\right\}$ Req. $\left\{\begin{array}{l} a = 115° 19' 57''. \\ b = 82° 27' 59''. \\ C = 48° 31' 38''. \end{array}\right.$

154. Case V.

Given the three sides of a spherical triangle; required the angles.

1. Giv. $\left\{\begin{array}{l} a = 100° 49' 30''. \\ b = 99° 40' 48''. \\ c = 64° 23' 15''. \end{array}\right\}$ Req. $\left\{\begin{array}{l} A. \\ B. \\ C. \end{array}\right.$



By article 138, we have

$$\cos \tfrac{1}{2}A = \sqrt{\frac{\sin \tfrac{1}{2}s \sin (\tfrac{1}{2}s - a)}{\sin b \sin c}}.$$

Introducing R and applying logarithms, we have

$$\log\cos\tfrac{1}{2}A = \tfrac{1}{2}[\log\sin\tfrac{1}{2}s + \log\sin(\tfrac{1}{2}s - a) + a.c.\log\sin b + a.c.\log\sin c].$$

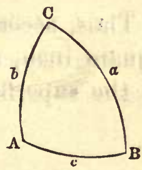
$\therefore \tfrac{1}{2}A = 48° 43' 14''$, $\quad \therefore A = 97° 26' 28''$.

In like manner we find $\begin{cases} B = 95° 38' 00''. \\ C = 65° 33' 04''. \end{cases}$

2. Given $\begin{cases} a = 85° 30'. \\ b = 65° 40'. \\ c = 93° 26' 18''. \end{cases}$ Req. $\begin{cases} A = 83° 29' 08''. \\ B = 65° 14' 20''. \\ C = 95° 50'. \end{cases}$

155. Case VI.

Given the three angles of a spherical triangle; required the sides.



1. Given $\begin{cases} A = 119° 15'. \\ B = 70° 39'. \\ C = 48° 36'. \end{cases}$ Req. $\begin{cases} a. \\ b. \\ c. \end{cases}$

By article 139, we have

$$\cos\tfrac{1}{2}a = \sqrt{\frac{\cos(\tfrac{1}{2}S - B)\cos(\tfrac{1}{2}S - C)}{\sin B \sin C}}.$$

Introducing R and applying logarithms, we have

$$\log\cos\tfrac{1}{2}a = \tfrac{1}{2}[\log\cos(\tfrac{1}{2}S - B) + \log\cos(\tfrac{1}{2}S - C) + a.c.\log\sin B + a.c.\log\sin C].$$

$\therefore \tfrac{1}{2}a = 56° 11' 31''$, $\quad \therefore a = 112° 23' 02''$.

In like manner we find $\begin{cases} b = 89° 16' 54''. \\ c = 52° 39' 00''. \end{cases}$

2. Given $\begin{cases} A = 121° 36' 24''. \\ B = 42° 15' 13''. \\ C = 34° 15' 03''. \end{cases}$ Req. $\begin{cases} a = 76° 36' 00''. \\ b = 50° 10' 40''. \\ c = 40° 00' 20''. \end{cases}$

MENSURATION.

156. Definition and Classification.

Mensuration is the art of calculating the values of geometrical magnitudes.

Mensuration is divided into two branches — *Mensuration of surfaces* and *Mensuration of volumes.*

MENSURATION OF SURFACES.

157. Unit of Superficial Measure.

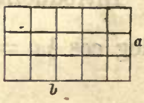
A unit of superficial measure is a square each side of which is a linear unit.

Thus, according to the object to be accomplished, a square inch, a square foot, a square yard, an acre, etc., is the superficial unit taken.

158. Problem.

To find the area of a rectangle.

Let k denote the area, b the base, and a the altitude of a rectangle.



There are a rows of b superficial units each.

Since there are b superficial units in one row, in a such rows there will be a times b or ab superficial units.

$$\therefore \quad (1) \quad k = ab.$$

The above demonstration applies only in case the base and altitude are commensurable, or have a common unit.

If the base and altitude are incommensurable, denote the area by k', the base by b', and the altitude by a'. Then, since by Geometry any two rectangles are to each other as the products of their bases and altitudes, we have

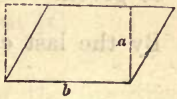
$$k : k' :: ab : a'b'.$$

$$\text{But } k = ab, \quad \therefore k' = a'b'.$$

159. Problem.

To find the area of a parallelogram.

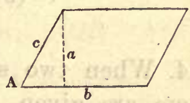
1. When the base and altitude are given.



Let k denote the area, b the base, and a the altitude of a parallelogram.

Since a parallelogram is equal to a rectangle, having the same base and altitude, and since the area of the rectangle is equal to the product of its base and altitude, the area of the parallelogram is equal to the product of its base and altitude.

$$\therefore \quad (1) \quad k = ab.$$



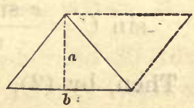
2. When two sides and their included angle are given.

$$a = c \sin A. \qquad \therefore \quad (2) \quad k = bc \sin A.$$

160. Problem.

To find the area of a triangle.

1. When the base and altitude are given.

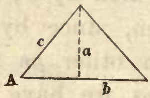


Since a triangle is one-half the parallelogram having the same base and altitude, we have for the triangle,

$$(1) \quad k = \tfrac{1}{2} ab.$$

2. When two sides and their included angle are given.

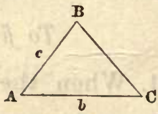
Since a triangle is one-half the parallelogram, having an equal angle and equal adjacent sides, we have for the triangle,



$$(2) \quad k = \tfrac{1}{2} bc \sin A.$$

3. When two angles and a side are given.

The third angle is equal to 180° minus the sum of the given angles.



Let, then, the angles and the side b be given.

By the last case, we have

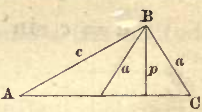
$$k = \tfrac{1}{2} bc \sin A.$$

But $\sin B : \sin C :: b : c, \quad \therefore c = \dfrac{b \sin C}{\sin B}.$

Substituting this value of c, we have

$$(3) \quad k = \frac{b^2 \sin A \sin C}{2 \sin B}.$$

4. When two sides and an angle opposite one of them are given.



Let a and c be the given sides, and A the given angle.

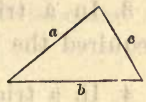
In case of one or two solutions determined by article 72, find the value or values of C and B from the formulas,

$$\sin C = \frac{c \sin A}{a}, \quad \text{and } B = 180° - (A + C).$$

Then, by (2), we have

$$(4) \quad k = \tfrac{1}{2} ac \sin B.$$

5. When the three sides are given.

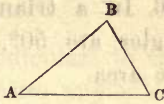


Let p = the perimeter $= a + b + c$.

Then, by article 102, we have

$$(5)\quad k = \sqrt{\tfrac{1}{2}p\,(\tfrac{1}{2}p - a)\,(\tfrac{1}{2}p - b)\,(\tfrac{1}{2}p - c)}.$$

6. When the perimeter and angles are given.



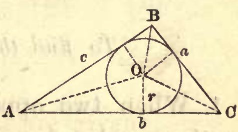
Let p be the perimeter, and A, B, and C the angles.

By article 98, (10), (11), (12),

$$\tfrac{1}{4}p^2 \tan\tfrac{1}{2}A \tan\tfrac{1}{2}B \tan\tfrac{1}{2}C = \sqrt{\tfrac{1}{2}p(\tfrac{1}{2}p - a)(\tfrac{1}{2}p - b)(\tfrac{1}{2}p - c)}.$$

$$\therefore\quad (6)\quad k = \tfrac{1}{4}p^2 \tan\tfrac{1}{2}A \tan\tfrac{1}{2}B \tan\tfrac{1}{2}C.$$

7. When the perimeter and radius of the inscribed circle are given.



Let $p = a + b + c$, and r be the radius of the inscribed circle.

$$ABC = BOC + AOC + AOB.$$

$$ABC = k,\quad BOC = \tfrac{1}{2}ar,\quad AOC = \tfrac{1}{2}br,\quad AOB = \tfrac{1}{2}cr.$$

$$\therefore\quad k = \tfrac{1}{2}(a + b + c)\,r;\ \text{ but } a + b + c = p.$$

$$\therefore\quad (7)\quad k = \tfrac{1}{2}pr.$$

161. Examples.

1. Find the area of a triangle whose base is 75 ft., and altitude is 24 ft. *Ans.* 900 sq. ft.

2. Two sides of a triangle are 25 yds. and 30 yds., respectively, and their included angle is 50°; required the area. *Ans.* 287.2665 sq. yds.

3. In a triangle, $b = 100$ ft., $A = 50°$, $C = 60°$; required the area. *Ans.* 3529.9 sq. ft.

4. In a triangle, $a = 40$ yds., $c = 50$ yds., $A = 40°$; required the area. *Ans.* 998.18, or 232.83 sq. yds.

5. In a triangle, $a = 12$ ft., $b = 15$ ft., $c = 17$ ft.; required k. *Ans.* 87.75 sq. ft.

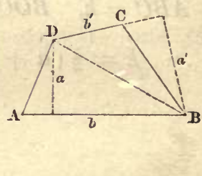
6. In a triangle the perimeter is 20 ft., and the angles are 50°, 60°, and 70°, respectively; required the area. *Ans.* 18.85 sq. ft.

7. In a triangle the perimeter is 60 ft., and the radius of the inscribed circle is 5 ft.; required the area. *Ans.* 150 sq. ft.

162. Problem.

To find the area of a quadrilateral.

1. When two opposite sides and the perpendiculars to these sides from the vertices of the angles at the extremities of a diagonal are given.



Let b and b' be two opposite sides, and a and a' the perpendiculars to these sides from the vertices of the angles D and B.

$$ABCD = ABD + DCB.$$

$$ABCD = k, \quad ABD = \tfrac{1}{2}ab, \quad DCB = \tfrac{1}{2}a'b'.$$

$$\therefore \quad (1) \quad k = \tfrac{1}{2}ab + \tfrac{1}{2}a'b'.$$

Corollary 1.—If b' is parallel to b, the quadrilateral becomes a trapezoid, $a' = a$, and (1) becomes

$$(2) \quad k = \tfrac{1}{2}a\,(b + b').$$

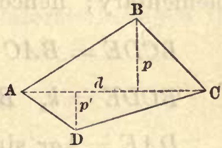
Corollary 2.—If $b' = b$, the trapezoid becomes a parallelogram, and (2) becomes

$$(3) \quad k = ab.$$

Corollary 3.—If $b' = 0$, the trapezoid becomes a triangle, and (2) becomes

$$(4) \quad k = \tfrac{1}{2} ab.$$

2. When a diagonal and the perpendiculars to the diagonal from the vertices of the opposite angles are given.



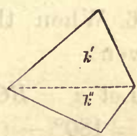
Let d denote the diagonal, and p and p' the perpendiculars.

$$ABCD = ABC + ADC.$$

$$ABCD = k, \qquad ABC = \tfrac{1}{2} dp, \qquad ADC = \tfrac{1}{2} dp'.$$

$$\therefore \quad (5) \quad k = \tfrac{1}{2} d\,(p + p').$$

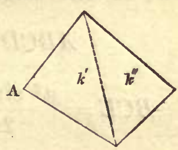
3. When the sides and a diagonal are given.



Let the areas of the triangles be denoted by k' and k'', which are found by article 160, (5).

$$\therefore \quad (6) \quad k = k' + k''.$$

4. When the sides and one angle are given.



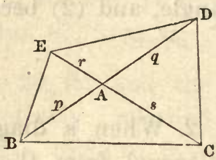
Draw the diagonal opposite the given angle, and call the areas of the triangles k' and k''.

In one triangle we have two sides and their included angle, from which we find the area and the diagonal.

Then, in the other triangle, we have the three sides, from which we find the area.

$$\therefore \quad (7) \quad k = k' + k''.$$

5. When the diagonals and their included angle are given.



Let d and d' denote the diagonals p and q, r and s their segments, and A their included angle.

The angles at A are equal or supplementary; hence their sines are equal.

$$BCDE = BAC + CAD + DAE + EAB.$$

$$BCDE = k, \quad BAC = \tfrac{1}{2}ps \sin A, \quad CAD = \tfrac{1}{2}qs \sin A.$$

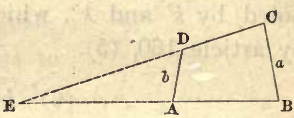
$$DAE = \tfrac{1}{2}qr \sin A, \quad EAB = \tfrac{1}{2}pr \sin A.$$

$$\therefore \quad k = \tfrac{1}{2}(ps + qs + qr + pr) \sin A.$$

But $ps + qs + qr + pr = (p+q)\cdot(r+s) = dd'$.

$$\therefore \quad (8) \quad k = \tfrac{1}{2}dd' \sin A.$$

6. When the angles and two opposite sides are given.



Let $a = BC$, and $b = AD$.
$E = 180° - (B + C)$.

The angles at A being supplementary, their sines are equal. The same is true of the angles at D.

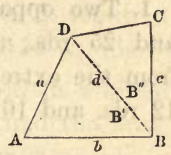
$$ABCD = BCE - ADE, \quad ABCD = k.$$

$$BCE = \frac{a^2 \sin B \sin C}{2 \sin E}, \quad ADE = \frac{b^2 \sin A \sin D}{2 \sin E}.$$

$$\therefore \quad (9) \quad k = \frac{a^2 \sin B \sin C}{2 \sin E} - \frac{b^2 \sin A \sin D}{2 \sin E}.$$

7. When three sides and their included angles are given.

Let a, b, and c be the given sides, and A and B their included angles.



$$ABCD = ABD + DBC.$$

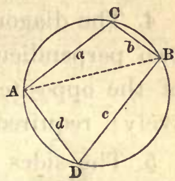
$$ABCD = k, \quad ABD = \tfrac{1}{2}ab \sin A.$$

Find B' and d, $B'' = B - B'$, $DBC = \frac{1}{2}cd \sin B''$.

$$\therefore \quad (10) \quad k = \tfrac{1}{2}ab \sin A + \tfrac{1}{2}cd \sin B''.$$

8. When the sides of a quadrilateral inscribed in a circle are given.

Let a, b, c, d be the given sides.



$$ACBD = ACB + ADB.$$

$$ACBD = k, \quad ACB = \tfrac{1}{2}ab \sin C.$$

$$ADB \quad = \tfrac{1}{2}cd \sin D = \tfrac{1}{2}cd \sin C,$$

since $D = 180° - C$.

$$\therefore \quad k = \tfrac{1}{2}(ab + cd) \sin C.$$

$\overline{AB}^2 = a^2 + b^2 - 2\,ab \cos C$, article 97.

$$\overline{AB}^2 = c^2 + d^2 - 2\,cd \cos D = c^2 + d^2 + 2\,cd \cos C.$$

$$\therefore \quad c^2 + d^2 + 2\,cd \cos C = a^2 + b^2 - 2\,ab \cos C.$$

$$\cos C = \frac{a^2 + b^2 - c^2 - d^2}{2\,(ab + cd)}.$$

$$\sin C = \sqrt{1 - \cos^2 C}, \quad \text{Let } s = a + b + c + d.$$

$$\therefore \quad \sin C = \frac{2\sqrt{(\frac{1}{2}s - a)(\frac{1}{2}s - b)(\frac{1}{2}s - c)(\frac{1}{2}s - d)}}{ab + cd}.$$

$$\therefore \quad (11) \quad k = \sqrt{(\tfrac{1}{2}s - a)(\tfrac{1}{2}s - b)(\tfrac{1}{2}s - c)(\tfrac{1}{2}s - d)}.$$

163. Examples.

1. Two opposite sides of a quadrilateral are 35 rds. and 25 rds., and the perpendiculars to these sides from the extremities of the diagonal are, respectively, 12 rds. and 16 rds.; required the area.

Ans. 410 sq. rds.

2. Find the area of a trapezoid whose bases are 15 rds. and 20 rds., and whose altitude is 18 rds.

Ans. 315 sq. rds.

3. Two adjacent sides of a parallelogram are 30 rds. and 40 rds., and their included angle is 30°; required the area. *Ans.* 600 sq. rds.

4. The diagonal of a quadrilateral is 40 rds., and the two perpendiculars to the diagonal from the vertices of the opposite angles are 10 rds. and 15 rds., respectively; required the area. *Ans.* 500 sq. rds.

5. The sides of a quadrilateral are 30 rds., 40 rds., 50 rds., and 60 rds., and the diagonal drawn from the intersection of the sides, whose lengths are 30 rds. and 40 rds., is 70 rds.; required the area.

Ans. 1874.22 sq. rds.

6. The sides of a quadrilateral are 25 rds., 35 rds., 45 rds., 55 rds., and the angle included by the sides, whose lengths are 35 rds. and 45 rds., is 50°; required the area. *Ans.* 927.47 sq. rds.

7. The diagonals of a quadrilateral are 30 rds. and 40 rds., and their included angle is 30°; required the area. *Ans.* 300 sq. rds.

8. The angles of a quadrilateral are 80°, 110°, 88°, 82°, the side included by the first and second of these angles is 25 rds., and the side included by the third and fourth angles is 45 rds.; required the area.

Ans. 4105.08 sq. rds.

9. Three sides of a quadrilateral are 20 rds., 30 rds., 40 rds., the angle included by the first and second is 60°, and between the second and third, 80°; required the area. *Ans.* 593.58 sq. rds.

10. The sides of a quadrilateral inscribed in a circle are 40 rds., 50 rds., 60 rds., 70 rds.; required the area. *Ans.* 2898.28 sq. rds.

11. The area of a parallelogram is 47.055 sq. ft., the sides are 6 ft. and 8 ft.; required the diagonal. *Ans.* 9 ft., or 10.906 ft.

12. If the adjacent sides of a parallelogram are b and c, and their included angle A, find A and k when k is a maximum. *Ans.* $A = 90°$, $k = bc$.

13. The sides and angles being expressed as in the last example, find A and k when k is a minimum. *Ans.* $A = 0°$ or $180°$, $k = 0$.

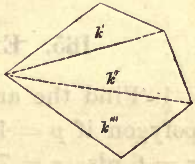
14. If only two adjacent sides, b and c, of a parallelogram be given, prove that k is indeterminate between the limits 0 and bc.

15. Prove that the diagonals of a parallelogram divide it into four equal triangles.

164. Problem.

To find the area of an irregular polygon.

1. When the sides and diagonals from the same vertex are given.

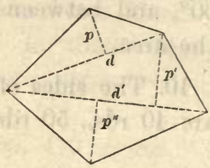


The diagonals divide the polygon into triangles whose sides are given.

The areas of these triangles, k', k'', k''', ... are found by article 160, (5).

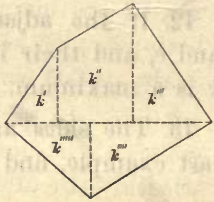
$$\therefore \quad (1) \quad k = k' + k'' + k''' + \ldots$$

2. When the diagonals from the same vertex, and the perpendiculars to these diagonals from the opposite vertices are given.



(2) $k = \frac{1}{2}dp + \frac{1}{2}d'p' + \frac{1}{2}d'p'' + \ldots$

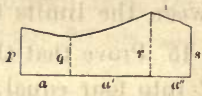
3. When the perpendiculars to a diagonal from the vertices of the opposite angles and the segments of the diagonal made by these perpendiculars are given.



The polygon is divided into right triangles and trapezoids, whose areas k', k'', k''', ... are found by article 162, (2), (4).

(3) $k = k' + k'' + k''' + \ldots$

4. When one side of a figure is a straight line, and the opposite side is an irregular curve or broken line.



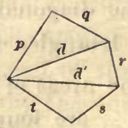
Let the straight line be divided into the parts a, a', a'', ..., and let the perpendiculars be $p, q, r, \ldots$ dividing the figure into parts which may be considered trapezoids.

$\therefore$ (4) $k = \frac{1}{2}a(p + q) + \frac{1}{2}a'(q + r) + \frac{1}{2}a''(r + s)$.

If $a' = a$ and $a'' = a$, (4) becomes,

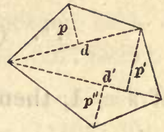
(5) $k = \frac{1}{2}a(p + 2q + 2r + s)$.

165. Examples.

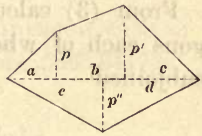


1. Find the area of the annexed polygon if $p = 10$ rds., $q = 6$ rds., $r = 6$ rds., $s = 7$ rds., $t = 15$ rds., $d = 14$ rds., $d' = 16$ rds. *Ans.* 119.86 sq. rds.

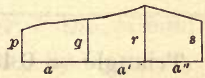
2. Find the area of the annexed polygon if $p = 3$ rds., $d = 9$ rds., $p' = 4$ rds., $d' = 12$ rds., and $p'' = 5$ rds. *Ans.* 67.5 sq. rds.



3. Find the area of the annexed polygon if $p = 3$ ft., $p' = 5$ ft., $p'' = 4$ ft., $a = 5$ ft., $b = 6$ ft., $c = 6$ ft., $d = 9$ ft., $e = 8$ ft. *Ans.* 80.5 sq. ft.



4. Find the area of the annexed figure. $p = 2$ rds., $q = 3$ rds., $r = 4$ rds., $s = 3$ rds., $a = a' = a'' = 5$ rds.



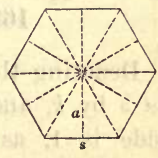
Ans. 47.5 sq. rds.

166. Problem.

To find the area of a regular polygon.

1. When the perimeter and apothegm are given.

Let p be the perimeter, a the apothem, and s one side of the polygon.



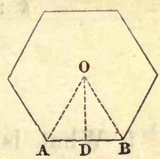
$$k = \tfrac{1}{2}as + \tfrac{1}{2}as + \tfrac{1}{2}as + \tfrac{1}{2}as + \dots$$

$$k = \tfrac{1}{2}a(s + s + s + s + \dots)$$

$$\therefore \quad (1) \quad k = \tfrac{1}{2}ap.$$

2. When the value of each side and the number of sides are given.

Let s be one side, n the number of sides, a the apothem, and p the perimeter.



$$p = ns. \quad DOB = \frac{360°}{2n} = \frac{180°}{n}.$$

$$OD = DB \cot DOB, \text{ or } a = \tfrac{1}{2}s \cot \frac{180°}{n}.$$

$$\therefore \quad (2) \quad k = \tfrac{1}{4} ns^2 \cot \frac{180°}{n}.$$

If $s = 1$, then (3) $k = \frac{1}{4} n \cot \frac{180°}{n}$.

From (3) calculate the areas of the regular polygons each of whose sides is 1, as given in the table subjoined.

167. Table.

Triangle = 0.4330127.	Octagon = 4.8284271.
Square = 1.0000000.	Enneagon = 6.1818242.
Pentagon = 1.7204774.	Decagon = 7.6942088.
Hexagon = 2.5980762.	Hendecagon = 9.3656399.
Heptagon = 3.6339124.	Dodecagon = 11.1961524.

168. Application of the Table.

Denoting the area of a regular polygon whose side is s by k, and the area of a similar polygon whose side is 1, as given in the table by k', and applying the principle that the areas of similar polygons are to each other as the squares of the homologous sides, we have the proportion,

$$k : k' :: s^2 : 1^2. \quad \therefore \quad k = k's^2.$$

169. Examples.

1. What is the area of a regular hexagon each of whose sides is 6? *Ans.* 93.5307432.

2. What is the area of a regular pentagon each of whose sides is 10? *Ans.* 172.04774.

3. What is the area of a regular decagon each of whose sides is 20? *Ans.* 3077.68352.

4. What is the area of a regular dodecagon each of whose sides is 100? *Ans.* 111961.524.

5. What is the area of a regular enneagon each of whose sides is 30? *Ans.* 5563.64178.

170. Formulas for the Circle.

Let r be the radius, d the diameter, c the circumference, and k the area of a circle, then, by Geometry, we have

$$d = 2\,r, \quad c = \pi d, \quad k = \tfrac{1}{2} rc.$$

From which verify the following table of formulas:

1. $r = \frac{1}{2} d.$	7. $c = 2\,\pi r.$
2. $r = \frac{c}{2\,\pi}.$	8. $c = \pi d.$
3. $r = \sqrt{\frac{k}{\pi}}.$	9. $c = 2\sqrt{k\pi}.$
4. $d = 2\,r.$	10. $k = \pi r^2.$
5. $d = \frac{c}{\pi}.$	11. $k = \frac{1}{4}\pi d^2.$
6. $d = 2\sqrt{\frac{k}{\pi}}.$	12. $k = \frac{c^2}{4\,\pi}.$

171. Examples.

1. Given the radius of a circle = 10 rds.; required d, c, and k.

2. Given the diameter of a circle = 20 rds.; required r, c, and k.

3. Given the circumference of a circle = 150 rds.; required r, d, and k.

4. Given the area of a circle = 1000 sq. rds.; required r, d, and c.

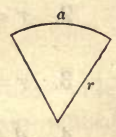
5. Find the diameter of a circle whose area is equal to that of a regular decagon, each side of which is 10 ft. *Ans.* 31.3.

6. The radius of a circle is 10 ft., the diagonals of an equal parallelogram are 24 ft. and 30 ft.; required their included angle. *Ans.* 60° 46′ 17″.

7. The radii of two concentric circles are r and r'; find the area of the ring included by their circumferences. *Ans.* $\pi(r+r')(r-r')$.

172. Problem.

To find the area of a sector of a circle.



Let a be the arc of a sector, d the degrees in the arc, r the radius, and k the area.

By Geometry, (1) $k = \frac{1}{2} ra$.

$\pi r =$ the semi-circumference,

$$\frac{\pi r}{180} = \text{the arc of } 1°. \quad \therefore \quad \frac{d\pi r}{180} = \text{the arc of } d°.$$

$$\therefore \quad (2) \quad a = \frac{d\pi r}{180}. \quad \therefore \quad (3) \quad k = \frac{d\pi r^2}{360}.$$

173. Examples.

1. Find the area of a sector whose arc is 40° and radius is 10 ft. *Ans.* 34.907 sq. ft.

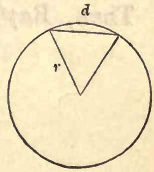
2. Find the area of a sector whose arc is 60° 24′ 30″ and radius is 100 rds. *Ans.* 5271 64 sq. rds.

3. The area of a sector is 345 sq. ft., the radius is 20 ft.; required the arc. *Ans.* 98° 50′ 06″.

4. The area of a sector is 1000 sq. rds., the arc is 30° 45′; required the radius. *Ans.* 61.04 rds.

174. Problem.

To find the area of a segment of a circle.



Let d be the degrees in the arc of the segment, r the radius, and k the area.

By the last problem,

$$\frac{d\pi r^2}{360} = \text{the area of the sector.}$$

$$\tfrac{1}{2} r^2 \sin d = \text{the area of the triangle.}$$

$$\therefore \quad k = \frac{d\pi r^2}{360} - \tfrac{1}{2} r^2 \sin d.$$

If d is greater than 180, $\sin d$ is negative, and the second term in the value of k becomes positive, as it should, since, in this case, the segment is equal to the corresponding sector plus the triangle.

175. Examples.

1. Find the area of the segment of a circle whose arc is 36° and radius 10 ft. *Ans.* 2.027 sq. ft.

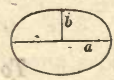
2. Find the area of a segment whose chord is 36 ft. and radius 30 ft. *Ans.* 147.30 sq. ft.

3. Find the area of a segment whose altitude is 36 rds. and radius 50 rds. *Ans.* 2545.85 sq. rds.

4. The area of a segment is 2545.85 sq. rds., the radius is 50 rds.; required the number of degrees in the arc.

176. Problem.

To find the area of an ellipse.



Let a be the semi-major axis, and b the semi-minor axis.

Then, Ray's Analytic Geometry, article 446,

$$k = \pi ab.$$

177. Examples.

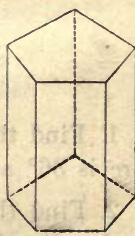
1. The semi-axes of an ellipse are 10 in. and 7 in.; required the area. *Ans.* 219.912 sq. in.

2. The area of an ellipse is 125 sq. rds.; find the axes if they are to each other as 3 is to 2.

Ans. 15.45; 10.30.

178. Problem.

To find the area of the entire surface of a right prism.



Let p be the perimeter of the base, a the altitude, s one side of the base, k' the area of a polygon similar to the base, each side of which is unity, article 167, and k the area of the entire surface.

$ap =$ the convex surface.

$2\,k's^2 =$ the areas of the bases. Article 168.

$$\therefore\ k = ap + 2\,k's^2.$$

179. Examples.

1. What is the entire surface of a right prism whose altitude is 20 ft., and base a regular octagon each side of which is 10 ft.? *Ans.* 2565.68542 sq. ft.

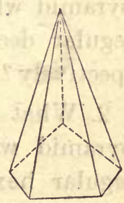
2. What is the entire surface of a right hexagonal prism whose altitude is 12 ft., and each side of the base is 6 ft.? *Ans.* 619.0614864 sq. ft.

3. What is the entire surface of a right prism whose altitude is 15 in., and base a regular triangle each side of which is 3 in.? *Ans.* 142.7942286 sq. in.

180. Problem.

To find the area of the surface of a regular pyramid.

Let p be the perimeter of the base, a the slant height, s one side of the base, k' and k as in the last problem.



$\frac{1}{2}ap =$ the convex surface.

$k's^2 =$ the area of the base.

$\therefore \quad k = \frac{1}{2}ap + k's^2.$

181. Examples.

1. What is the entire surface of a regular pyramid whose slant height is 12 ft., and base a regular triangle each side of which is 5 ft.?

Ans. 100.82532 sq. ft.

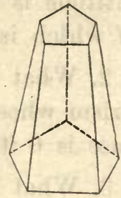
2. What is the entire surface of a right pyramid whose slant height is 100 ft., and base a regular decagon each side of which is 20 ft.?

Ans. 13077.68352 sq. ft.

182. Problem.

To find the entire surface of a frustum of a right pyramid.

Let p be the perimeter of the lower base, p' the perimeter of the upper base, a the slant height, s one side of the lower base, s' one side of the upper base, k' and k as in Art. 178.



$\frac{1}{2}a(p + p') =$ the convex surface.

$k's^2 =$ the area of lower base.

$k's'^2 =$ the area of upper base.

$$\therefore \quad k = \tfrac{1}{2}a(p + p') + k'(s^2 + s'^2).$$

183. Examples.

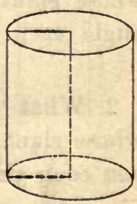
1. What is the entire surface of a frustum of a pyramid whose slant height is 12 ft., and the bases regular decagons whose sides are 8 ft. and 5 ft., respectively? *Ans.* 1464.78458 sq. ft.

2. What is the entire surface of a frustum of a pyramid whose slant height is 15 ft., and the bases regular hexagons whose sides are 10 ft. and 6 ft., respectively? *Ans.* 1073.338 sq. ft.

184. Problem.

To find the area of the entire surface of a cylinder.

Let r be the radius of the cylinder, a its altitude, and k the area of the entire surface.



$2\,\pi ra =$ the convex surface.

$2\,\pi r^2 =$ the area of the bases.

$$\therefore \quad k = 2\,\pi r(a + r).$$

185. Examples.

1. What is the entire surface of a cylinder whose altitude is 6 ft. and radius 2 ft.?

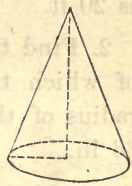
Ans. 100.5312 sq. ft.

2. What is the entire surface of a cylinder whose altitude is 100 ft. and radius 20 ft.?

Ans. 15079.68 sq. ft.

186. Problem.

To find the area of the entire surface of a cone.



Let r be the radius of the base of the cone, a the slant height, and k the area of the entire surface.

$\pi ra =$ the convex surface.

$\pi r^2 =$ the area of the base.

$$\therefore \quad k = \pi r (a + r).$$

187. Examples.

1. What is the entire surface of a cone whose slant height is 10 ft. and radius 5 ft.? *Ans.* 235.62 sq. ft.

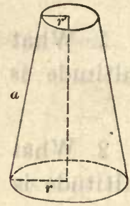
2. What is the entire surface of a cone whose altitude is 100 ft. and radius 25 ft.?

Ans. 10059.1675 sq. ft.

188. Problem.

To find the area of the entire surface of the frustum of a cone.

Let r be the radius of the lower base, r' be the radius of the upper base, a the slant height, and k the area of the entire surface.



$\pi a(r + r') =$ the convex surface.

$\pi r^2 =$ the area of the lower base.

$\pi r'^2 =$ the area of the upper base.

$$\therefore \quad k = \pi[a(r + r') + r^2 + r'^2].$$

189. Examples.

1. Find the entire surface of the frustum of a cone of which the radius of the lower base is 10 ft., the radius of the upper base is 6 ft., and slant height is 20 ft. *Ans.* 1432.5696 sq. ft.

2. Find the entire surface of the frustum of a cone of which the radius of the lower base is 25 in., the radius of the upper base 12 in., and the slant height 36 in. *Ans.* 45.8368 sq. ft.

190. Problem.

To find the area of the surface of a sphere.

Let r be the radius, d the diameter, c the circumference, and k the area. Then, by Geometry,

$$(1)\quad k = 4\pi r^2. \qquad (2)\quad k = \pi d^2.$$

$$(3)\quad k = \frac{c^2}{\pi}. \qquad (4)\quad k = cd.$$

191. Examples.

1. The radius of a sphere is 10 ft.; required the area. *Ans.* 1256.64 sq. ft.

2. The diameter of a sphere is 25 ft.; required the area. *Ans.* 1963.5 sq. ft.

3. The circumference of a sphere is 100 in.; required the area. *Ans.* 3183.0914 sq. in.

4. The circumference of a sphere is 62.832, and diameter 20; required the area. *Ans.* 1256.64.

192. Problem.

To find the area of a zone.

By Geometry, the area of a zone is equal to the circumference of a great circle multiplied by the altitude of the zone.

Let a denote the altitude of the zone, r the radius of the sphere, and k the area of the zone.

$$\therefore \quad k = 2\,\pi r a.$$

193. Examples.

1. What is the area of the torrid zone, calling its width 46° 56′, and the earth a perfect sphere whose radius is 3956.5 mi.? *Ans.* 78333333. sq. mi.

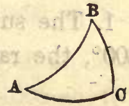
2. What is the area of the two frigid zones if the polar circles are 23° 28′ from the poles? *Ans.* 16270370. sq. mi.

3. What is the area of the two temperate zones? *Ans.* 102109933. sq. mi.

194. Problem.

To find the area of a spherical triangle.

Let $s = A + B + C$, and $\frac{1}{2}\pi r^2 =$ the tri-rectangular triangle.



Then, by Geometry,

$$k = \tfrac{1}{2}\pi r^2\left(\frac{s}{90°} - 2\right).$$

In this formula, $\frac{s}{90^\circ} - 2$ is to be regarded as an abstract number. Minutes and seconds are to be reduced to the decimal of a degree.

195. Examples.

1. Find the area of the spherical triangle whose angles are 60°, 80°, 100°, and the radius 3956.5 mi.
Ans. 16392592 sq. mi.

2. Find the area of a spherical triangle whose sides are 70°, 90°, 100°, respectively, and radius 100 in.
Ans. 10942.1928 sq. in.

196. Problem.

To find the area of a spherical polygon.

Let s be the sum of the angles, n the number of sides, k the area of the polygon, and r the radius of the sphere.

Then, by Geometry,

$$k = \tfrac{1}{2}\pi r^2 \left[\frac{s}{90^\circ} - 2(n-2)\right].$$

197. Examples.

1. The sum of the angles of a spherical hexagon is 800°, the radius is 100 ft.; required the area.
Ans. 13963. sq. ft.

2. Each angle of a spherical pentagon is 120°, the radius is 50 ft.; required the area. *Ans.* 2618. sq. ft.

3. The angles of a spherical polygon are 90°, 100°, 110°, 150°, respectively, the radius is 10 ft.; required the area. *Ans.* 157.08 sq. ft.

4. Each angle of a spherical decagon is 150°, the radius is 1 ft.; required the area. *Ans.* 1.0472 ft.

198. Problem.

To find the area of the surface of a regular polyhedron.

Let e be one edge, n the number of faces, k' the area of a polygon whose side is 1, and similar to one face, and k the area of the entire surface.

$k'e^2 =$ the area of one face. Article 168.

$$\therefore\ k = nk'e^2.$$

199. Examples.

1. What is the area of the entire surface of a tetrahedron whose edge is 10 ft.? *Ans.* 173.20508 sq. ft.

2. What is the area of the entire surface of a hexahedron whose edge is 5 ft.? *Ans.* 150 sq. ft.

3. What is the area of the entire surface of an octahedron whose edge is 20 ft.? *Ans.* 1385.64064 sq. ft.

4. What is the area of the entire surface of a dodecahedron whose edge is 15 in.? *Ans.* 32.25895 sq. ft.

5. What is the area of the entire surface of an icosahedron whose edge is 100 in.? *Ans.* 601.4065 sq. ft.

MENSURATION OF VOLUMES.

200. Problem.

To find the volume of a prism.

Let k be the area of the base, a the altitude, and v the volume. Then, by Geometry,

$$v = ak.$$

201. Examples.

1. What is the volume of a regular hexagonal prism whose altitude is 20 ft., and each side of the base 10 ft.? *Ans.* 5196.1524 cu. ft.

2. What is the volume of a triangular prism whose altitude is 6 ft., and the sides of its base 3 ft., 4 ft., and 5 ft., respectively? *Ans.* 36 cu. ft.

3. What is the volume of a regular octagonal prism whose altitude is 120 ft., and each side of the base 20 ft.? *Ans.* 231764.5008 cu. ft.

202. Problem.

To find the volume of a pyramid.

Let k be the area of the base, a the altitude, and v the volume.

$$v = \tfrac{1}{3} ak.$$

203. Examples.

1. What is the volume of a pyramid whose altitude is 15 ft., and whose base is a regular heptagon each side of which is 5 ft.? *Ans.* 454.23905 cu. ft.

2. What is the volume of a pyramid whose altitude is 21 in., and whose base is a triangle each side of which is 30 in.? *Ans.* 2727.98 cu. in.

204. Problem.

To find the volume of the frustum of a pyramid.

Let k and k_1 be the areas of the bases, a the altitude, and v the volume. Then, by Geometry,

$$(1) \quad v = \tfrac{1}{3} a (k + k_1 + \sqrt{kk_1}).$$

If the bases are regular polygons whose sides are s and s', we shall have, by article 168, $k = k's^2$, and $k_1 = k's'^2$, in which k' is given in the table of article 167, and (1) becomes

$$(2) \quad v = \tfrac{1}{3} a (s^2 + s'^2 + ss') k'.$$

205. Examples.

1. What is the volume of the frustum of a pyramid whose altitude is 9 ft., and whose bases are regular triangles, one side of the lower being 8 ft., and one side of upper, 5 ft.? *Ans.* 167.576 cu. ft.

2. What is the volume of the frustum of a pyramid whose altitude is 27 in., and the bases regular hexagons, the sides of which are 10 in. and 6 in., respectively? *Ans.* 4583.0064 cu. in.

206. Problem.

To find the volume of a cylinder.

Let r represent the radius, a the altitude, and v the volume.

$$v = a\pi r^2.$$

207. Examples.

1. What is the volume of a cylinder whose altitude is 50 in., and radius 15 in.? *Ans.* 20.453 cu. ft.

2. What is the volume of a cylinder whose altitude is 25 ft., and radius 4 ft.? *Ans.* 1256.64 cu. ft.

208. Problem.

To find the volume of a cone.

Let r be the radius of the base, a the altitude, and v the volume.

$$v = \tfrac{1}{3} a\pi r^2.$$

209. Examples.

1. What is the volume of a cone whose altitude is 21 in., and radius 10 in.? *Ans.* 2199.12 cu. in.

2. What is the volume of a cone whose altitude is 30 ft., and radius is 10 ft.? *Ans.* 31416. cu. ft.

210. Problem.

To find the volume of the frustum of a cone.

Let r and r' be the radii of the bases, a the altitude, and v the volume.

$$v = \tfrac{1}{3} a\pi (r^2 + r'^2 + rr').$$

211. Examples.

1. What is the volume of the frustum of a cone whose altitude is 15 ft., and the radii of whose bases are 9 ft. and 4 ft, respectively? *Ans.* 2089.164 cu. ft.

2. How many barrels will that cistern contain whose altitude is 8 ft., the diameter at the bottom 4 ft., and at the top 6 ft.? *Ans.* 37.8 bbl.

212. Formulas for the Sphere.

Let r be the radius, d the diameter, c the circumference, k the area of the surface, and v the volume of a sphere, then, by Geometry, we have

$$d = 2r, \quad c = \pi d, \quad k = 4\pi r^2, \quad v = \tfrac{1}{3} rk.$$

From which verify the following table of formulas:

1. $r = \frac{1}{2} d.$	11. $c = \sqrt{\pi k}.$
2. $r = \frac{c}{2\pi}.$	12. $c = \sqrt[3]{6 v \pi^2}.$
3. $r = \frac{1}{2}\sqrt{\frac{k}{\pi}}.$	13. $k = 4\pi r^2.$
4. $r = \frac{1}{2}\sqrt[3]{\frac{6v}{\pi}}.$	14. $k = \pi d^2.$
5. $d = 2r.$	15. $k = \frac{c^2}{\pi}.$
6. $d = \frac{c}{\pi}.$	16. $k = \sqrt[3]{36 \pi v^2}.$
7. $d = \sqrt{\frac{k}{\pi}}.$	17. $v = \frac{4}{3}\pi r^3.$
8. $d = \sqrt[3]{\frac{6v}{\pi}}.$	18. $v = \frac{1}{6}\pi d^3.$
9. $c = 2\pi r.$	19. $v = \frac{c^3}{6\pi^2}.$
10. $c = \pi d.$	20. $v = \frac{k}{6}\sqrt{\frac{k}{\pi}}.$

213. Examples.

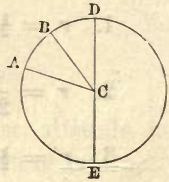
1. Calling the diameter of the earth 7913 mi., and the diameter of the sun 856,000, find the ratio of their surfaces, also the ratio of their volumes.

2. What is the volume of the shell of a hollow sphere whose radius is 8 ft. 4 in., and the thickness of the shell 3 ft. 6 in.? *Ans.* 1951.1081 cu. ft.

214. Problem.

To find the volume of a spherical sector.

A spherical sector is the volume generated by the revolution of any circular sector, ABC, about any diameter, DE. By Geometry, the volume of a spherical sector is equal to the zone which forms its base, multiplied by one-third of the radius.



Let a be the altitude of the zone, and r the radius.

$$\therefore \quad v = \tfrac{2}{3}\pi r^2 a.$$

215. Examples.

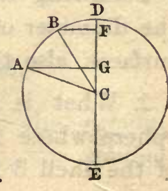
1. The altitude of the zone which forms the base of a sector is 6 ft., the radius is 12 ft.; required the volume. *Ans.* 1809.5616 cu. ft.

2. The angle BCD, in the diagram of last article, is 20°, ACB is 35°, $r = 20$ ft.; required the volume. *Ans.* 6134.25 cu. ft.

216. Problem.

To find the volume of a spherical segment.

A spherical segment is the portion of a sphere included between two parallel planes.



Let $r' = BF$ perpendicular to DE, and $r'' = AG$ perpendicular to DE.

$r =$ the radius, $d' = CF$, and $d'' = CG$.

$v =$ the vol. generated by $ABFG$.

$v' =$ the vol. generated by $ABC = \frac{2}{3}\pi r^2 a$.

$v'' =$ the vol. generated by $BFC = \frac{1}{3}d'\pi r'^2$.

$v''' =$ the vol. generated by $AGC = \frac{1}{3}d''\pi r''^2$.

$$v = v' + v'' \mp v'''.$$

The sign of v''' is $-$ or $+$ according as AG is on the same or opposite side of the center as BF.

$$\therefore \quad v = \tfrac{1}{3}\pi(2\,ar^2 + d'r'^2 \mp d''r''^2).$$

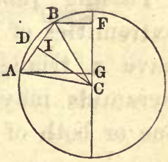
217. Examples.

1. $r = 12$ in., $r' = 3$ in., $r'' = 10$ in.; required v.

2. Two parallel planes divide a sphere whose diameter is 36 in. into three equal segments; required the altitude of each. *Ans.* 13.93 in.; 8.14 in.; 13.93 in.

218. Problem.

To find the volume generated by the revolution of a circular segment about a diameter exterior to it.



Let $v =$ vol. generated by ADB.

$v' =$ vol. generated by $ADBC$.

$v'' =$ vol. generated by ABC.

$$v = v' - v''.$$

Let $a = FG$, $c = AB$, $p = CI$, perpendicular to AB.

$$v' = \tfrac{2}{3}\pi ar^2, \qquad v'' = \tfrac{2}{3}\pi ap^2.$$

$$\therefore \quad v' - v'' = \tfrac{2}{3}\pi a(r^2 - p^2) = \tfrac{1}{6}\pi ac^2.$$

$$\therefore \quad v = \tfrac{1}{6}\pi ac^2.$$

219. Examples.

1. $a = 5$ in., $c = 8$ in.; find v. *Ans.* 167.552 cu. in.

2. A sphere 6 in. in diameter is bored through the center with a 3-inch auger; required the volume remaining. *Ans.* 73.457 cu. in.

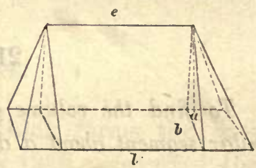
3. Prove that the volume generated by the segment whose altitude is a and chord c is to the sphere whose diameter is c as $a : c$.

4. Prove that if c is parallel to the diameter about which it is revolved, the volume generated by the segment is equal to the volume of a sphere whose diameter is c.

220. Problem.

To find the volume of a wedge.

The base is a rectangle, the sides are trapezoids, the ends, triangles.



Let e be the edge, l the length of base, b the breadth of base, and a the altitude.

Passing planes through the extremities of the edge perpendicular to the base, we have a triangular prism and two pyramids. These pyramids may fall within or without the wedge, or one or both of the pyramids may vanish.

But in all cases the formula is the same.

$$\tfrac{1}{2}\,abe = \text{the volume of the prism.}$$

$$\tfrac{1}{3}\,a\,(l - e)\,b = \text{the volume of the pyramids.}$$

$$\therefore \quad v = \tfrac{1}{6}\,ab\,(2\,l + e).$$

221. Examples.

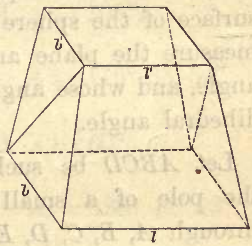
1. The edge of a wedge is 6 in., the altitude 12 in., the length of base 9 in., and the breadth of base 5 in.; what is the volume? *Ans.* 240 cu. in.

2. The edge of a wedge is 20 ft., the altitude 24 ft., the length of base 15 ft., the breadth of base 10 ft.; what is the volume? *Ans.* 2000 cu. ft.

222. Problem.

To find the volume of a rectangular prismoid.

The bases are parallel rectangles, the other faces are trapezoids.



Let l and b be the length and breadth of the lower base, l' and b' the length and breadth of the upper base, and a the altitude.

Passing the plane as indicated, the prismoid is divided into two wedges.

$\frac{1}{6}ab(2l + l')$ = the vol. of wedge whose base is bl.

$\frac{1}{6}ab'(2l' + l)$ = the vol. of wedge whose base is $b'l'$.

$$\therefore \quad v = \tfrac{1}{6}a[b(2l + l') + b'(2l' + l)].$$

223. Examples.

1. The length and breadth of the lower base of a rectangular prismoid are 25 ft. and 20 ft., the length and breadth of the upper base are 15 ft. and 10 ft., and the altitude is 18 ft.; what is the volume? *Ans.* 5550 cu. ft.

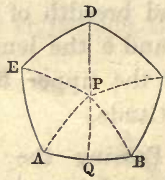
2. The length and breadth of the lower base of a rectangular prismoid are 15 yds. and 10 yds., the length and breadth of the upper base are 9 yds. and 6 yds., and the altitude is 18 yds.; what is the volume? *Ans.* 1764 cu. yds.

224. Problem.

To find the dihedral angle included by the faces of a regular polyhedron.

Conceive a sphere whose radius is 1 so placed that its center shall be at any vertex of the polyhedron.

The faces of the polyhedral angle will intersect the surface of the sphere in a regular polygon, whose sides measure the plane angles that include the polyhedral angle, and whose angles are each equal to the required dihedral angle.



Let *ABCD* be such a polygon, *P* the pole of a small circle passing through *A*, *B*, *C*, *D*, *E*. Join *P* with the vertices and with the middle of *AB* by arcs of great circles.

Let n denote the number of sides of the polygon, $s =$ one side, and $A =$ a dihedral angle.

$$\therefore \quad APQ = \frac{360°}{2\,n} = \frac{180°}{n}, \text{ and } AQ = \tfrac{1}{2}\,s.$$

By Napier's circular parts, we have

$$\sin\,(90° - APQ) = \cos AQ \cos\,(90° - PAQ).$$

$$\text{or } \sin\,(90° - \frac{180°}{n}) = \cos \tfrac{1}{2}\,s \;\cos\,(90° - \tfrac{1}{2}\,A).$$

$$\text{or } \cos \frac{180°}{n} = \cos \tfrac{1}{2}\,s \sin \tfrac{1}{2}\,A.$$

$$\therefore \quad \sin \tfrac{1}{2}\,A = \frac{\cos \frac{1}{n}\,180°}{\cos \frac{1}{2}\,s}.$$

In the Tetrahedron, $n = 3$, and $s = 60°$,

$$\therefore \quad \sin \tfrac{1}{2}\,A = \frac{\cos\,60°}{\cos\,30°}. \quad \therefore \quad A = 70°\;31'\;42''.$$

In the Hexahedron, $n = 3$, and $s = 90°$,

$$\therefore \quad \sin \tfrac{1}{2} A = \frac{\cos 60°}{\cos 45°}. \qquad \therefore \quad A = 90°,$$

In the Octahedron, $n = 4$, and $s = 60°$,

$$\therefore \quad \sin \tfrac{1}{2} A = \frac{\cos 45°}{\cos 30°}. \qquad \therefore \quad A = 109° \; 28' \; 18''.$$

In the Dodecahedron, $n = 3$, and $s = 108°$,

$$\therefore \quad \sin \tfrac{1}{2} A = \frac{\cos 60°}{\cos 54°}. \qquad \therefore \quad A = 116° \; 33' \; 54''.$$

In the Icosahedron, $n = 5$, and $s = 60°$,

$$\therefore \quad \sin \tfrac{1}{2} A = \frac{\cos 36°}{\cos 30°}. \qquad \therefore \quad A = 138° \; 11' \; 23''.$$

225. Problem.

To find the volume of a regular polyhedron.

If planes be passed through the edges of the polyhedron and the center, they will bisect the dihedral angles and divide the polyhedron into as many pyramids as it has faces. The faces will be the bases of the pyramids, the center will be their common vertex, the line drawn from the center of the polyhedron to the center of any base will be perpendicular to the base, and will be the altitude of the pyramid.

From the foot of the perpendicular draw a perpendicular to one side of the base, and join the foot of this perpendicular with the center. We thus have a right triangle whose perpendicular is the altitude of the pyramid, the base the apothem of one face of the polyhedron, the angle opposite the perpendicular one-half the dihedral angle of the polyhedron.

Let p be the perpendicular, a the apothem of one face, $\frac{1}{2}A$ one-half of a dihedral angle, n' the number of sides of one face, and e one edge.

$$p = a \tan \tfrac{1}{2} A, \quad a = \tfrac{1}{2} e \cot \tfrac{1}{n'} 180°. \quad \text{Article 166.}$$

$$\therefore \quad p = \tfrac{1}{2} e \cot \tfrac{1}{n'} 180° \tan \tfrac{1}{2} A.$$

Let k', n, and k be the same as in article 198.

Then, $\frac{1}{3} pk =$ the volume of the polyhedron.

$$\therefore \quad v = \tfrac{1}{6} nk'e^3 \cot \tfrac{1}{n'} 180° \tan \tfrac{1}{2} A.$$

Let $e = 1$, and verify the table subjoined:

226. Table.

Names.	*Surfaces.*	*Volume.*
Tetrahedron	1.7320508	0.1178513
Hexahedron	6.0000000	1.0000000
Octahedron	3.4641016	0.4714045
Dodecahedron	20.6457288	7.6631189
Icosahedron	8.6602540	2.1816950

227. Application of the Table.

Let v' and v denote similar regular polyhedrons whose edges are 1 and e, respectively. Then we have

$$v' : v :: 1^3 : e^3. \quad \therefore \quad v = v'e^3.$$

228. Examples.

1. What is the volume of a tetrahedron whose edge is 10 ft.? *Ans.* 117.8513 cu. ft.

2. The volume of a hexahedron is 134217728 cu. in.; what is its surface? *Ans.* 1572864 sq. in.

SURVEYING.

229. Definition and Classification.

Surveying is the art of laying out, measuring, and dividing land, and of representing on paper its boundaries and peculiarities of surface.

There are three branches—*Plane, Geodesic, and Topographical.*

Plane surveying is that branch in which the portion surveyed is regarded as a plane, as is the case in small surveys.

Geodesic surveying is that branch in which the curvature of the surface of the earth is taken into consideration, as is the case in all extensive surveying.

Topographical surveying is that branch in which the slope and irregularities of the surface, the course of streams, the position and form of lakes and ponds, the situation of trees, marshes, rocks, buildings, etc., are considered and delineated.

INSTRUMENTS.

230. Classification.

The instruments employed in surveying may be classed as *Field instruments* and *Plotting instruments.*

The principal field instruments are the *chain* and *tally pins*, *marking tools*, *field-book* and *pencil*, the *magnetic* *compass*, the *solar compass*, the *transit compass*, the *level*, and the *theodolite.*

The principal plotting instruments are the *dividers*, the *ruler* and *triangle*, *parallel rulers*, the *diagonal scale*, the *semicircular protractor.*

231. The Chain and Tally Pins.

The chain is 4 rods or 66 feet in length, and is divided into 100 links, each equal to 7.92 inches.

After every tenth link from each end is a piece of brass, notched so as to indicate the number of links from the end of the chain, thus facilitating the counting of the links.

A half-chain of 50 links is sometimes used, especially in rough or hilly districts.

The tally pins are made of iron or steel, about 12 inches in length and one-eighth of an inch in thickness, heavier toward the point, with a ring at the top in which is fastened a piece of cloth of some conspicuous color.

These pins are conveniently carried by stringing them on an iron ring attached to a belt which is passed over the right shoulder, leaving the pins suspended at the left side.

In Government surveys eleven tally pins are used.

232. Marking Tools.

A surveying party will need an *ax* for cutting notches, cutting and driving stakes and posts; *a spade* or *mattock* for planting or finding corners; *knives*, or other tools, for cutting letters or figures; and a *file* and *whetstone* for keeping the tools in order.

233. Field-Book and Pencil.

In ordinary practice one field-book will be sufficient; but in surveying the public lands, four different books are required—one for meridian and base lines, another for standard parallels or correction lines, another for exterior or township lines, and another for subdivision or section lines, as designated on the title-page.

A good pencil, number 2 or 3, well sharpened, should be used, so that the notes may be legible.

A temporary book may be used on the ground, and the notes taken with a pencil. These notes can then be carefully transcribed with pen and ink into the permanent field-book.

234. The Magnetic Compass.

The vernier magnetic compass is exhibited in the drawing on page 189.

The needle turns freely on a pivot at the center, and settles in the magnetic meridian.

The compass circle is divided, on its upper surface, to half-degrees, numbered from 0° to 90° each side of the line of zeros.

The sight standards are firmly fastened at right angles to the plate by screws, and have slits cut through nearly their whole length, terminated at intervals by apertures through which the object toward which the sights are directed can be readily found.

Two spirit levels at right angles to each other are attached to the plate.

Tangent scales are scales on the right and left edges of the north sight standard, the one on the right being used in taking angles of elevation, and the one on the left in taking angles of depression.

Eye-pieces are placed on the right and left sides of the south sight standard — the one on the right near the bottom, the one on the left near the top — each on a level, when the compass is level, with the zero of its tangent scale. These eye-pieces are centers of arcs tangent to the tangent scales at the zero point.

The vernier is a scale movable by the side of another scale, and divided into parts each a little greater or a little less than a part of the other, and having a known ratio to it. In the drawing the vernier is represented on the plate near the south sight.

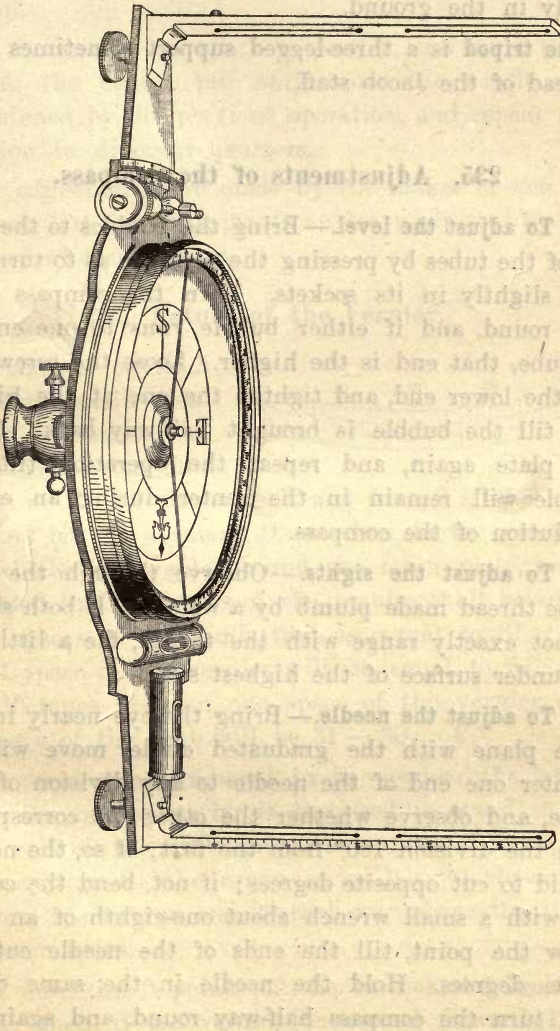
The needle lifter is a concealed spring, moved from beneath the main plate, by which the needle may be lifted to avoid blunting the point of the pivot in transporting the instrument.

The out-keeper is a small dial plate, having an index turned by a milled head, and is used in keeping tally in chaining.

The ball spindle is a small shaft, slightly conical, to which the compass is fitted, having on its lower end a ball confined in a socket by a light pressure, so that the ball can be moved in any direction in leveling the instrument.

The clamp screw is a screw in the side of the hollow cylinder or socket, which fits to the ball spindle, by which the compass may be clamped to the spindle in any position.

A spring catch, fitted to the socket, slips into a groove when the instrument is set on the spindle, and secures it from slipping from the spindle when carried.



THE MAGNETIC COMPASS.

The Jacob staff is a single staff to support the compass, about $5\frac{1}{2}$ feet long, having at the upper end the ball and socket joint, and terminating at the lower end in a sharp steel point, so as to be set firmly in the ground.

The tripod is a three-legged support sometimes used instead of the Jacob staff.

235. Adjustments of the Compass.

1. **To adjust the level.**—Bring the bubbles to the center of the tubes by pressing the plates so as to turn the ball slightly in its sockets. Turn the compass half-way round, and if either bubble runs to one end of its tube, that end is the higher. Loose the screw under the lower end, and tighten the one at the higher end till the bubble is brought half-way back. Level the plate again, and repeat the operation till the bubble will remain in the center during an entire revolution of the compass.

2. **To adjust the sights.**—Observe through the slits a fine thread made plumb by a weight. If both sights do not exactly range with the thread, file a little off the under surface of the highest side.

3. **To adjust the needle.**—Bring the eye nearly in the same plane with the graduated circle, move with a splinter one end of the needle to any division of the circle, and observe whether the other end corresponds with the division 180° from the first; if so, the needle is said to cut opposite degrees; if not, bend the center pin with a small wrench about one-eighth of an inch below the point, till the ends of the needle cut opposite degrees. Hold the needle in the same direction, turn the compass half-way round, and again see

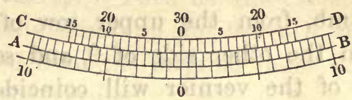
whether the needle cuts opposite degrees; if not, correct half the error by bending the needle, and the remainder by bending the center pin, and repeat the operation till perfect reversion is secured in the first position.

Try the needle in another quarter, and correct by bending the center pin only, since the needle was straightened by the previous operation, and repeat the operation in different quarters.

The adjustments are made by the maker of the instrument, but the instrument can be re-adjusted by the surveyor when necessary.

236. Nature of the Vernier.

Let the arc or limb *AB*, on the main plate of the instrument, be graduated to one-half degrees or 30′, numbered each way from 0 at the middle; and let the vernier *CD*, attached to the compass box, which is movable around the main plate, be so graduated that 30 spaces of the vernier shall be equal to 31 spaces of the limb, that is, equal to $31 \times 30'$; then 1 space of the vernier will be equal to 31′, and the difference between one space of the vernier and one space of the limb will be $31' - 30' = 1'$.



The vernier is numbered in two series: the lower, nearer the spectator, who is supposed to stand at the south end of the instrument, is numbered 5, 10, 15, each way from 0; the upper series has 30 above the 0, from the observer, and 20 each way above the 10 of the lower series.

Let, now, the 0 points of the vernier and limb coincide; then, if the vernier be moved forward 1′ to the right, which is done by means of a tangent screw, the first division line of the vernier at the left of its 0 will coincide with the first division line of the limb at the left of its 0; if the vernier be moved forward 2′ to the right, then the second division line of the vernier at the left of its 0 will coincide with the second division line of the limb at the left of its 0.

If the vernier be moved to the right so that its fifteenth division line at the left of its 0 shall coincide with the fifteenth division line of the limb at the left of its 0, the vernier will have moved forward 15′.

If the vernier be moved more than 15′, the excess over 15′ is found by reading the division line, in the vernier, which coincides with a division line of the limb, from the upper row of figures on the vernier, on the other side of 0, and so on, up to 30′, when the 0 of the vernier will coincide with the first division line from the 0 of the limb.

If the vernier is moved more than 30′, the excess over 30′, up to 15′ and then to 30′ is found as before.

If the 0 of the vernier coincides with a division line of the limb, the reading of the division line of the limb will be the true reading.

If the 0 of the vernier has passed one or more division lines of the limb, and does not coincide with any, read the limb from its 0 point up to its division next preceding the 0 of the vernier; to this add the reading of the vernier, and the sum will be the true reading.

If the vernier be moved to the left, the minutes must be read off on the vernier scale to the right.

Sometimes the spaces of the vernier are less than the spaces of the limb; then if the vernier be moved either way, the minutes must be read off the same way from the 0 of the vernier. Verniers may be so graduated as to read to any appreciable angle; but the graduation which reads to minutes is the most common.

237. Uses of the Vernier.

1. **To turn off the variation.** — Let the instrument be placed on some definite line of an old survey, and the tangent screw be turned till the needle indicates the same bearing for the line as that given in the field notes of the original survey.

Then will the reading of the limb and vernier indicate the variation.

2. **To retrace an old survey.** — Turn off the variation as above, and screw up the clamping nut beneath, then old lines can be retraced from the original notes without further change of the vernier.

3. **To run a true meridian.** — The absolute variation of the needle being known, not simply its change since a given date, move the vernier to the right or left, according as the variation is west or east, till the given variation is turned off, screw up the clamping nut beneath, and turn the compass till the needle is made to cut zeros, then will the line of sights indicate a true meridian.

Such a change in the position of the vernier is necessary in subdividing the public lands, after the principal lines have been truly run with the solar compass.

4. **To read the needle to minutes.** — Note the degrees given by the needle, then turn back the compass circle, with the tangent screw, till the nearest whole degree mark coincides with the point of the needle; the space

passed over by the vernier will be the minutes which, added to the degrees, will give the reading of the needle to minutes.

This operation is simplified when the 0 of the vernier is first made to coincide with the 0 of the limb; otherwise the difference of the two readings of the vernier must be taken.

238. Uses of the Compass.

1. **To take the bearing of a line.** — Place the compass on the line, turn the north end in the direction of the course, and, standing at the south end, direct the sights to some well-defined object, as a flag-staff, in the course. Read the bearing from the north end of the needle, which can be done accurately to quarter-degrees by observing the position of the point of the needle, since the compass circle is divided into half-degrees.

It will be observed that the letters *E* and *W*, on the face of the compass, are reversed from their true position. This is as it should be; for if the sights are turned toward the west, the north end of the needle is turned toward the letter *W*. If the north end of the needle is turned toward *E*, the sights will be turned toward the east. If the north end of the needle point exactly to either letter *E* or *W*, the sights will range east or west.

In general, to guard against error, let the surveyor turn the letter *S* toward himself, and read the arc cut off by the north end of the needle from the nearest zero of the compass circle. If, for example, the nearest 0 is at *S*, and the north end of the needle is turned toward *E*, cutting off 25° from this 0, then the course is *S* 25° *E*.

If it is desired to find the bearing to minutes, the vernier must be used.

2. **To run from a given point a line having a given bearing.** — Place the compass over the point, and turn it so that the reading of the needle shall be the given bearing; the line of sights observed from the south end of the compass will be the required line.

3. **To take angles of elevation.** — Level the compass, bring the south end toward you, place the eye at the eye-piece on the right side of the south sight, and, with the hand, fix a card on the front surface of the north sight, so that its top edge shall be at right angles to the divided edge and coincide with the zero mark; then, sighting over the top of the card, note upon a flag-staff the height cut by the line of sight, move the staff up the elevation, and carry the card along the sight until the line of sight again cuts the same height on the staff, read off the degrees and half-degrees passed over by the card, and the result will be the angle required.

4. **To take angles of depression.** — Proceed in the same manner, using the eye-piece and scale on the opposite sides of the sights, and reading from the top of the standard.

239. Surveyor's Transit.

The Surveyor's transit exhibited in the drawing on page 197 is, in fact, a *transit theodolite*, combining the advantages of the ordinary transit and the theodolite.

The vernier plate, carrying two horizontal verniers, two spirit levels at right angles, the telescope and attachments, moves around a circle graduated to half-degrees, so that, by the vernier, horizontal angles can be taken to minutes, and any variation turned off.

The telescope and its attachments, the clamp and tangent, the vertical circle, the level, and the sights, give to this instrument a great advantage over the ordinary compass.

The cross wires, two fine fibers of spider's web, extending across the tube at right angles, intersect in a point which, when the wires are adjusted, determines the optical axis or line of collimation of the telescope, and enables the surveyor to fix it upon an object with great precision.

The clamp and tangent screw consist of a ring encircling the axis of the telescope, having two projecting arms—the one above, slit through the middle, holding the clamp screw; the other, longer, connected below with the tangent screw.

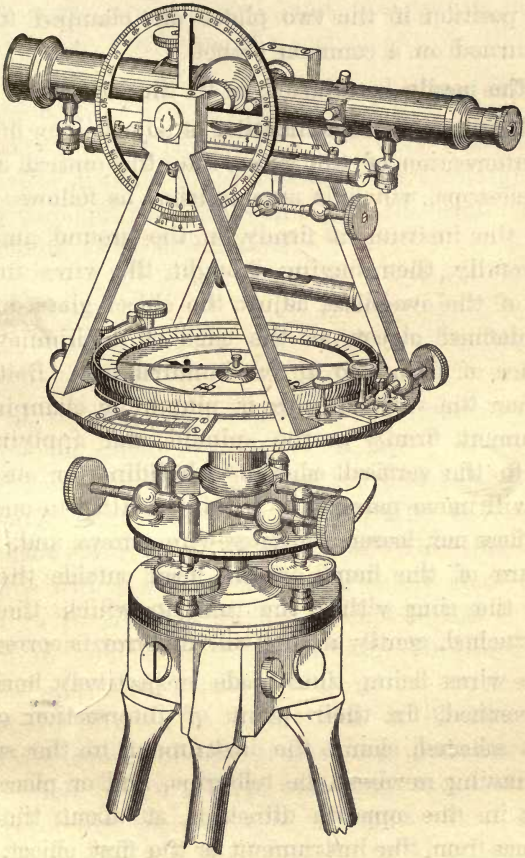
The ring is brought firmly around the axis by means of the clamp screw, and the telescope can be moved up or down by turning the tangent screw.

The vertical circle, graduated to half-degrees, is attached to the axis of the telescope, and, in connection with the vernier, gives the means of measuring vertical angles to minutes with great facility.

The level attached to the telescope enables the surveyor to run horizontal lines, or to find the difference of level between two points.

Sights on the telescope are useful in taking backsights without turning the telescope, and in sighting through bushes or woods.

Sights for right angles attached to the plate of the instrument, or to the standards supporting the telescope, afford the means of laying off right angles, or running out offsets without changing the position of the instrument.



SURVEYOR'S TRANSIT.

240. Adjustments.

1. **The levels** are adjusted in the same manner as those of the compass, and when adjusted should keep their position if the two plates are clamped together and turned on a common socket.

2. **The needle** is adjusted as in the compass.

3. **The line of the collimation** is adjusted by bringing the intersection of the wires into the optical axis of the telescope, which is accomplished as follows:

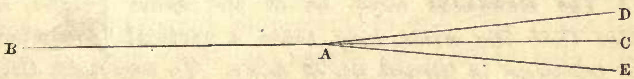
Set the instrument firmly on the ground and level it carefully, then, having brought the wires into the focus of the eye-piece, adjust the object glass on some well defined object, as the edge of a chimney, at a distance of from two to five hundred feet. Determine whether the vertical wire is plumb by clamping the instrument firmly to the spindle, and applying the wire to the vertical edge of a building, or observing if it will move parallel to a point a little to one side; if it does not, loosen the cross-wire screws, and, by the pressure of the hand on the head outside the tube, move the ring within the tube, to which the wires are attached, gently around till the error is corrected.

The wires being thus made respectively horizontal and vertical, fix their point of intersection on the object selected, clamp the instrument to the spindle, and, having revolved the telescope, find or place some object in the opposite direction, at about the same distance from the instrument as the first object.

Great care should be taken in turning the telescope not to disturb the position of the instrument upon the spindle.

Having found an object which the vertical wire bisects, unclamp the instrument, turn it half-way round, and direct the telescope to the first object selected, and having bisected this with the wires, again clamp the instrument, revolve the telescope and note if the vertical wire bisects the second object observed; if so, the wires are adjusted, and the points bisected are, with the center of the instrument, in the same straight line.

If the vertical wire does not bisect the second point, the space which separates this wire from that point is double the distance of that point from a straight line drawn through the first point and the center of the instrument, as is shown thus:



Let *A* represent the center of the instrument, *BC* the line on whose extremities, *B* and *C*, the line of collimation is to be adjusted, *B* the first object, and *D* the point which the wires bisected after the telescope was made to revolve on its axis. The side of the telescope which was up when the object glass was directed to *B*, is down when the object glass is turned toward *D*. When the telescope is unclamped from its spindle and turned half-way round its vertical axis, and again directed to *B*, the side of its tube which was down when the object glass was first directed to *B* will now be up. Then clamping the instrument, and revolving the telescope about its axis, and directing it toward *D*, the side of its tube which was down when the object glass was first turned toward *D* will now be up, or the telescope will virtually have revolved about its optical axis, and the vertical wire will appear at *E* as far on one side of *C* as *D* is on the other side.

To move the vertical wire to its true position, turn the capstan head screws on the sides of the telescope, remembering that the eye-piece inverts the position of the wire, and, therefore, that in loosening one of the screws and in tightening the other the operator must proceed as if to increase the error. Having moved back the vertical wire, as nearly as can be judged, so as to bisect the space *ED*, unclamp the instrument, direct the telescope as at first, so that the cross wires bisect *B*, proceed as before, and continue the operation till the two points *D* and *E* coincide at *C*.

4. **The standards** must be of the same height, in order that the wires may trace a vertical line when the telescope is turned up or down. To ascertain this, and to make the correction, proceed as follows:

Having the line of collimation previously adjusted, set the instrument in a position where points of observation, such as the point and base of a lofty spire, can be selected, giving a long range in a vertical direction.

Level the instrument, fix the wires on the top of the object, and clamp to the spindle; then bring the telescope down till the wires bisect some good point, either found or marked at the base; turn the instrument half around, fix the wires on the lower point, clamp to the spindle, and raise the telescope to the highest object, and if the wires bisect it, the vertical adjustment is effected.

If the wires are thrown to one side, the standard opposite that side is higher than the other.

The correction is made by turning a screw underneath the sliding piece of the bearing of the movable axis.

5. **The vertical circle** is adjusted thus: First carefully level the instrument, bring the zeros of the wheel and vernier into line, and find or place some well defined point which is cut by the horizontal wire; then turn the instrument half-way around, revolve the telescope, fix the wire on the same point as before, note if the zeros are again in line.

If not, loosen the screws, move the zero over half the error, and again bring the zeros into coincidence, and proceed as before till the error is corrected.

6. **The level** on the telescope can be adjusted thus: First level the instrument carefully, and with the clamp and tangent movement to the axis make the telescope horizontal as nearly as possible with the eye. Then, having the line of collimation previously adjusted, drive a stake at a distance of from one to two hundred feet, and note the height cut by the horizontal wire upon a staff set on the top of the stake.

Fix another stake in the opposite direction, at the *same* distance from the instrument, and, without disturbing the telescope, turn the instrument upon its spindle, set the staff upon the stake and drive in the ground till the same height is indicated as in the first observation.

The top of the two stakes will then be in the same horizontal line, whether the telescope is level or not.

Now remove the instrument to a point on the same side of both stakes, in a line with them, and from fifty to one hundred feet from the nearest one; again level the instrument, clamp the telescope as nearly horizontal as possible, and note the heights indicated on the staff placed first on the nearest, then on the more distant stake.

If both agree, the telescope is level; if they do not agree, then with the tangent screw move the wire over nearly the whole error, as shown at the distant stake, and repeat the operation just described till the horizontal wire will indicate the same height at both stakes, when the telescope will be level. Bring the bubble into the center by the leveling nuts at the end, taking care not to disturb the position of the telescope, and the adjustment will be completed.

The adjustments above described are always made by the maker of the instrument, but the instrument may need re-adjusting.

241. Uses of the Transit.

1. **The transit** may be used for all the purposes for which the compass is employed, and, in general, with much greater precision.

2. **Horizontal angles** can be taken by the needle, or without reference to the needle, as follows: Level the plate, set the limb at zero, direct the telescope so that the intersection of the wires shall fall upon one of the objects selected, clamp the instrument firmly to the spindle, unclamp the vernier plate, turn it with the hand till the intersection of the wires is nearly upon the second object; then clamp to the limb, and with the tangent screw fix the intersection of the wires precisely upon the second object. The reading of the vernier will give the angle whose vertex is at the center of the instrument, and whose sides pass through the objects respectively.

3. **Vertical angles** can be measured thus: Level the instrument, fix the zeros of the vertical circle and vernier in a line, note the height cut upon the staff

by the horizontal wire, carry the staff up the elevation or down the depression, fix the wire again upon the same point, and the angle will be read off by the vernier. Sometimes, of course, it will be impossible to carry the staff up the elevation, as in taking the angle of elevation of the top of a steeple from a given point in a horizontal plane.

4. **Horizontal lines** can be run, or the difference of level easily found, by means of the level attached to the telescope.

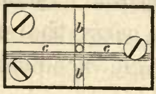
242. The Solar Compass.

Burt's solar compass, represented in the drawing on page 205, includes the essential parts of the magnetic compass, together with the solar apparatus, which consists mainly of three arcs of circles by which the latitude of the place, the declination of the sun, and the hour of the day can be set off.

The latitude arc, *a*, graduated to quarter-degrees and read to minutes by a vernier, has its center of motion in two pivots, one of which is seen at *d*, and is moved by the tangent screw, *f*, up or down a fixed arc of similar curvature through a range of about 35°.

The declination arc, *b*, having a range of about 24°, is graduated to quarter-degrees and read to minutes by the vernier, *v*, fixed to the movable arm, *h*, which has its center of motion in the center of the declination arc at *g*. The vernier may be set to any reading by the tangent screw, *k*, and the arm clamped in any position by a screw concealed in the engraving.

A solar lens, set in a rectangular block of brass at each end of the arm, *h*, has its focus at the inside of the opposite block on the surface of a silver plate on which are drawn certain lines, as shown in the annexed figure. The lines *bb*, called hour lines, and the lines *cc*, called equatorial lines, intersect each other at right angles. The rectangular space between the lines is just sufficient to include the circular image of the sun formed by the solar lens on the opposite end of the arm.



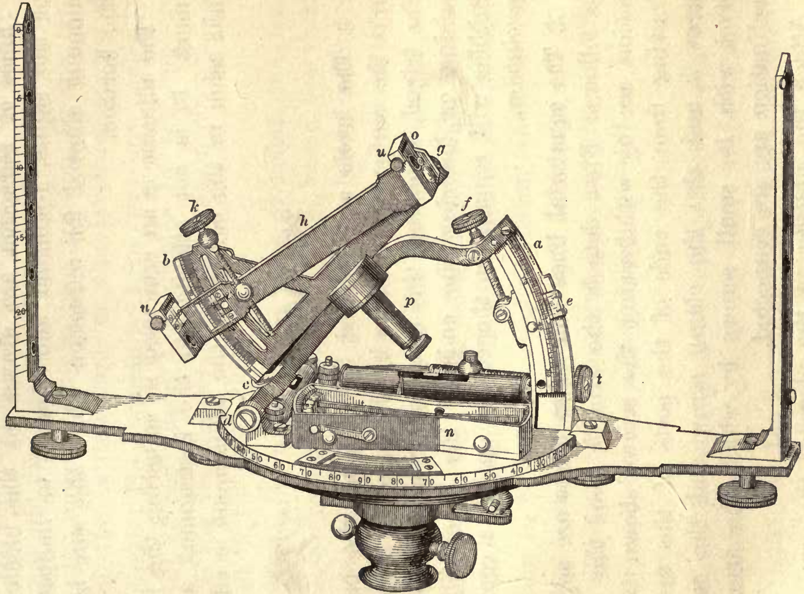
The three other lines below the equatorial lines are five minutes apart, and are used in making allowance for refraction.

An equatorial sight, used in adjusting the solar apparatus, is placed on the top of each rectangular block by a small milled head screw, so as to be detached at pleasure.

The hour arc, *c*, supported by the pivots of the latitude arc, and connected with that arc by a curved arm, has a range of 120°, graduated to half-degrees and figured in two series, designating both the hours and the degrees; the middle division being marked 12 and 90 on either side of the graduated lines.

The polar axis, *p*, consists of a hollow socket containing the spindle of the declination arc, around which this arc can be moved over the hour arc, which is read by the lower edge of the graduated side of the declination arc. The declination arc may be turned half round, if required, and the hour arc read by a point below *g*.

The needle box, *n*, with an arc of 36°, graduated to half-degrees, and numbered from the center as zero, is attached by a projecting arm to a tangent screw, *t*, by which it is moved about its center, and its needle



THE SOLAR COMPASS.

set to any variation which may be read to minutes by the vernier at the end of the arm.

The levels are similar to those of the ordinary compass.

Lines of refraction are drawn on the inside faces of the sights, graduated and figured to indicate the amount allowed for refraction when the sun is near the horizon.

The adjuster is an arm used in adjusting the instrument. It is not attached to the instrument, and is laid aside in the box when the adjustment is effected.

243. Adjustments.

1. **The levels** are adjusted by bringing the bubbles into the center of the tubes by the leveling screws of the tripod, reversing the instrument on the spindle, raising or lowering the ends of the tubes till the bubbles will remain in the center during a complete revolution.

2. **The equatorial lines and solar lenses** are adjusted as follows: First detach the arm, *h*, from the declination arc by withdrawing the screws shown in the drawing from the ends of the posts of the tangent screw, *k*, and also the clamp screw, and the conical pivot with its small screws by which the arm and declination arc are connected.

Attach the adjuster in the place of the arm, *h*, by replacing the conical pivot and screws, and insert the clamp screw so as to clamp the adjuster at any point on the declination arc.

Now level the instrument, place the arm, *h*, on the adjuster, with the same side resting against the surface of the declination arc as before it was detached, turn the instrument on its spindle, so as to bring the solar lens to be adjusted in the direction of the sun, raise or lower the adjuster on the declination arc till it can be clamped in such a position as to bring the sun's image, as near as may be, between the equatorial lines on the opposite silver plate, and bring the image precisely into position by the tangent of the latitude arc, or the leveling screws of the tripod. Then carefully turn the arm half-way over, till it rests upon the adjuster by the opposite faces of the rectangular blocks, and again observe the position of the sun's image.

If it remains between the lines as before, the lens and plate are in adjustment; if not, loosen the three screws which confine the plate to the block, and move the plate under their heads till one-half the error in the position of the sun's image is removed.

Again bring the image between the lines, and repeat the operation till it will remain in the same situation in both portions of the arm, when the adjustment will be complete.

To adjust the other lens and plate, reverse the arm, end for end, on the adjuster, and proceed as in the former case.

Remove the adjuster, and replace the arm, *h*, with its attachments.

In tightening the screws over the silver plate, care must be taken not to move the plate.

3. **The vernier of the declination arc** is adjusted as follows: Having leveled the instrument, and turned its lens in the direction of the sun, clamp to the spindle, and set the vernier, *v*, of the declination arc at zero, by means of the tangent screw, *k*, and clamp to the arc.

See that the spindle moves easily and truly in the socket, or polar axis, and raise or lower the latitude arc by turning the tangent screw, *f*, till the sun's image is brought between the equatorial lines on one of the plates; clamp the latitude arc by the screw, and bring the image precisely into position by the leveling screws of the tripod or socket, and without disturbing the instrument carefully revolve the arm, *h*, till the opposite lens and plate are brought in the direction of the sun, and note if the sun's image comes between the lines as before.

If the sun's image comes between the lines, there is no index error of the declination arc; if not, then with the tangent screw, *k*, move the arm till the sun's image passes over half the error, and again bring the image between the lines, and repeat the operation as before till the image will occupy the same position on both plates.

We shall now find that the zero marks on the arc and the vernier do not correspond; and to remedy this error, the little flat-head screws above the vernier must be loosened till it can be moved so as to make the zeros coincide, when the operation will be complete.

4. **The solar apparatus** is adjusted to the sights as follows: First level the instrument, then with the clamp and tangent screws set the main plate at 90° by the verniers and horizontal limb. Then remove the clamp screw, and raise the latitude arc till the polar axis is by estimation very nearly horizontal, and, if necessary, tighten the screws on the pivots of the arc so as to retain it in this position.

Fix the vernier of the declination arc at zero, and direct the equatorial sights to some distant and well-marked object, and observe the same through the compass sights. If the same object is seen through both, and the verniers read to 90° on the limb, the adjustment is complete; if not, the correction must be made by moving the sights or changing the position of the verniers.

The adjustments are all made by the maker of the instrument, and, ordinarily, need not concern the surveyor, as the instrument is very little liable to derangement.

244. Use of the Solar Compass.

The declination of the sun, or its angular distance from the celestial equator, must be set off on the declination arc.

The declination of the sun for apparent noon at Greenwich, England, is given from year to year in the Nautical Almanac.

To determine the declination for another place and hour, allowance must be made for the difference of time arising from longitude, and for the change of declination from hour to hour.

The longitude of the place can be determined with sufficient accuracy by reference to that of given prominent places which are situated nearly on the same meridian.

The difference of longitude, divided by 15, will, by changing degrees, minutes, and seconds into hours, minutes, and seconds, give the difference of time, which is usually taken to the nearest hour, as it will be sufficiently accurate.

In practice, surveyors in states just east of the Mississippi allow a difference of 6 hours for longitude;

7 hours for about the longitude of Santa Fé; 8 hours for California and Oregon; 5 hours for the eastern portions of the United States.

Having found the hour at any place from its longitude when it is noon at Greenwich, the declination for noon at Greenwich will be the declination for the determined hour at the given place.

To find the declination for the following hours of the day, add or subtract, for each succeeding hour, the difference of declination for 1 hour, as given in the almanac.

Thus, let it be required to find the declination of the sun for the different hours of May 20th, 1873. W. lon. 95°. 95° = 6 h. 20 m., practically 6 h.

	Sun's dec., Greenwich, noon	= 20° 3′ 14″.6
∴	Sun's dec., lon. 95°, 6 A. M.	= 20° 3′ 14″.6
	Add difference for 1 h.	= 31″.03
	Sun's dec. 7 A. M.	= 20° 3′ 45″.63
	Add difference for 1 h.	= 31″.03
	Sun's dec. 8 A. M.	= 20° 4′ 16″.66

In like manner proceed for the remaining hours.

Such a calculation should be made before beginning the work of the day.

Refraction, or the bending of the sun's rays as they pass obliquely through the atmosphere, affects its declination by increasing its apparent altitude.

The amount of refraction depends upon the altitude, being less as the altitude is greater. At the horizon the refraction is 35′; at the altitude of 45°, 1′; at the zenith, 0.

Meridional refraction, by increasing the apparent altitude of the sun, when on the meridian, increases or diminishes its apparent declination according as it is north or south of the equator.

To find the amount of meridional refraction, we must first find the meridional altitude of the sun for the given latitude, which is equal to the complement of the latitude, plus or minus the declination, according as the sun is north or south of the equator.

The meridional altitude of the sun being given, the tables will give the refraction.

The meridional refraction, being quite small, may be disregarded in practice except when great accuracy is required, as in running great standard meridians or base lines.

Incidental refraction, as affected by the hour of the day and the state of the atmosphere, can not, in practice, be determined by a precise calculation.

It will about compensate for incidental refraction to keep the image of the sun square between the equinoctial lines for the middle of the day, but toward morning or evening, to run the image, which is then hazy round the edge, so that the hazy edge shall overlap one or two lines of the spaces below.

To set off the latitude, find the declination of the sun for the given day at noon, and set it off on the declination arc, and clamp the arm firmly to the arc.

Find in the almanac the equation of time for the given day, in order to ascertain the time when the sun will reach the meridian.

About twenty minutes before noon, set up the instrument, level it carefully, fix the divided surface of the declination arc at 12 on the hour circle, and turn the instrument on its spindle till the solar lens is brought into the direction of the sun.

Loosen the clamp screw of the latitude arc, raise or lower this arc with the tangent screw till the image of the sun is brought precisely between the equatorial lines, and turn the instrument so as to keep the image between the hour lines on the plate.

As the sun ascends, in approaching the meridian, its image will move below the lines, and the arc must be moved to follow it. Keep the image between the two sets of lines till it begins to pass above the equatorial, which is the moment after it passes the meridian.

Read off the vernier of the arc, and we have the latitude of the place which is to be set off on the latitude arc.

To run lines with the solar compass.—Having adjusted the instrument and set off the declination and latitude, the surveyor places the instrument over the station, levels it carefully, clamps the plates at zero on the horizontal limb, and directs the sights north and south, approximately, by the needle.

The solar lens is then turned toward the sun, and with one hand on the instrument, and the other on the revolving arm, both are moved from side to side till the image of the sun is made to appear on the silver plate, and is brought precisely within the equatorial lines, *when the line of sights will indicate the true meridian.*

In running an east and west line, the verniers of the horizontal limb are set at 90°, and the sun's image kept between the equatorial lines.

The needle is made to indicate zero on the arc of the compass box by turning the tangent screw. Lines can then be run by the needle in the temporary disappearance of the sun.

The variation of the needle, which should be noted at every station, is read off to minutes on the arc along the edge of which the vernier of the needle box moves.

Since the limb must be clamped at 0 when the sun's image is in position, in order that the sights may indicate the meridian, it is evident that the bearing of any line may be found by the solar compass, as well as by the compass or transit.

In running long lines, allowance must be made for the curvature of the earth. Thus, in running north or south the latitude changes 1′ for 92.30 ch.; and six miles, or one side of a township, requires a change of 5′ 12″ on the latitude arc.

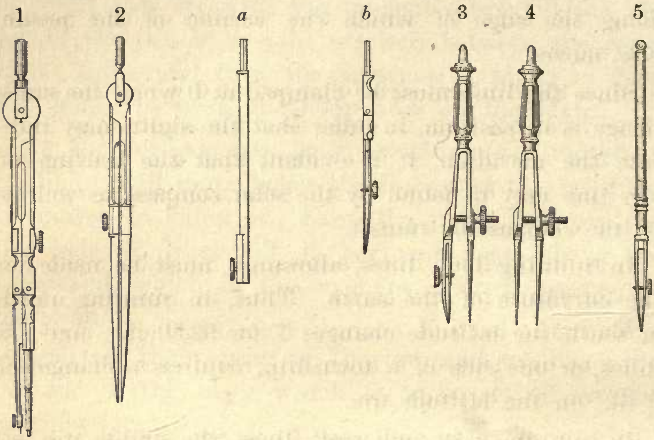
In running east and west lines, the sights are set at 90° on the limb, and the line run at right angles to the meridian; but this line, if sufficiently produced, would cross the equator. Hence, at the next station, a backsight is taken, and one-half the error is set off for the next foresight on the side toward the pole.

The most favorable season for using the solar compass is the summer; and the most favorable time of day, between 8 and 11 A. M., and 1 and 5 P. M.

A solar telescope compass is sometimes used; and, in this case, the telescope is placed at one side of the center. All error from this position of the telescope is avoided by an offset from the flag-staff.

The solar compass, while indispensable in the survey of public lands, can be used, in common practice, with considerable advantage over ordinary needle instruments, since lines can be run by it without regard to the variation of the needle or local attraction, and the bearings being taken from the true meridian will remain constant for all time.

245. Dividers and Pens.

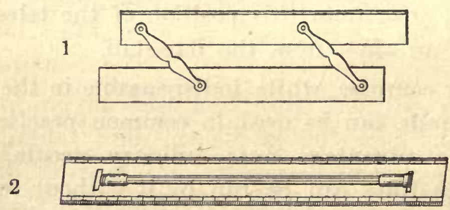


1. Dividers with lead-pencil.
2. Hair dividers with one leg movable by screw.

a, *b*. Lengthening bar and pen which may be inserted together or the pen alone instead of pencil leg.

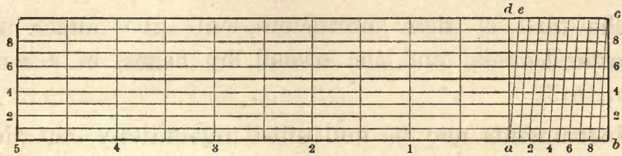
3. Bow pen with spring and adjusting screw.
4. Spacing dividers.
5. Drawing pen.

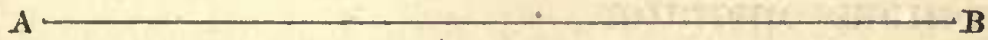
246. Parallel Rulers.



1. Parallel ruler for drawing parallel lines.
2. Sliding parallel ruler with scales.

247. Diagonal Scale.





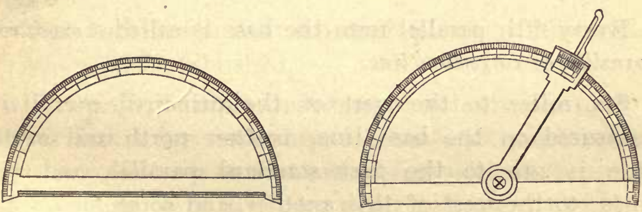
Let *de* be .1, then the distance from *ad* to *ae* on the first line above *ab* is .01, on the second line .02, etc.

Let it be required to lay off on *AB* 4.63.

Place one foot of the dividers at the intersection of the diagonal line, 6, and the horizontal line, 3. Extend the other foot till the horizontal line, 3, intersects the vertical line, 4, then will the distance from one point of the dividers to the other be 4.63.

Now place one foot of the dividers at *A*, and the other at *B*, then *AB* will be 4.63.

248. Protractors.



These protractors are used in laying off or measuring angles. The vertex of the angle is at the center, and one side is made to coincide with the horizontal line passing through the center; then, counting the degrees from the horizontal line round the circumference till the required degree is reached, and drawing a line from this degree to the center, we shall have the angle required.

The first of these protractors will give angles to quarter-degrees; and the second, by means of a vernier, to 3′.

Instruments may be multiplied indefinitely, but the manner of using them will be readily discovered by the ingenious operator.

SURVEY OF PUBLIC LANDS.

249. Division into Townships.

In the rectangular system of surveying the public lands, adopted by the government, two principal lines —an east and west line, called a *base line*, and a north and south line, called a *principal meridian*—are established before the survey of the townships.

Six miles to the north of the base line another east and west line is run, and six miles to the north of this another, and so on.

Every fifth parallel from the base is called a *standard parallel*, or *correction line.*

Six miles to the west of the principal meridian, measured on the base line, another north and south line is run to the first standard parallel, and six miles to the west of this another, and so on.

The intersection of the east and west with the north and south lines divides the tract into *townships*, which would be exactly six miles square were it not for the convergence of the meridians.

To preserve as nearly as possible the form and size of the townships, the standard parallels before mentioned are established, which serve as base lines for the townships north up to the next standard parallel.

Tiers of townships north and south are called *ranges*, and are numbered east or west, as the case may be, from the principal meridian.

Lines running north and south, bounding the townships on the east and west, are called *range lines.*

Lines running east and west, bounding the townships on the north and south, are called *township lines.*

A township marked thus, *T.* 5 *N.*, *R.* 4 *W.*, read township five north, range four west, would be in the fifth tier north of the base line, and in the fourth tier west of the principal meridian.

Townships are divided into *sections*, or *square miles*, containing 640 acres; each section into four *quarter sections*, each quarter section into two *half-quarter sections*, and each half-quarter section into two *quarter-quarter sections.* These are called legal subdivisions, and are the only divisions recognized by the government, except pieces made fractional by water-courses or other natural agencies.

On base lines and standard parallels two sets of corners are established.

1. **Standard corners,** established when these lines are run, embracing township, section, and quarter-section corners, common to two townships, sections, or quarter sections north of the base line or standard parallels.

2. **Closing corners,** established when exterior and subdivision lines close on them from the south, embracing township and section corners, common to two townships or sections south of the standard parallels.

In consequence of the convergence of the meridians, the north and south lines, produced to the standard parallels, will not close on the standard corners previously established, but will strike the standard parallels to the east or west of the standard corners, making the closing corners east or west of the standard corners, according as the field of operation is west or east of the principal meridian.

The following diagram will illustrate the subject:

AB is the base line.

AC, the principal meridian.

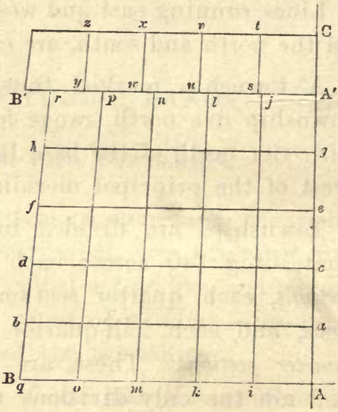
A′B′, a standard parallel.

ab, *cd*, etc., township lines.

ij, *kl*, etc., range lines.

s, *u*, *w*, etc., standard corners.

j, *l*, *n*, etc., closing corners.



The distances *js*, *lu*, etc., are measured and recorded in the field book.

The details of running lines will be given after describing the methods of perpetuating corners, the process of chaining, and the method of marking lines.

Burt's improved solar compass is used in surveying standard and township lines, but the ordinary compass may be used in subdividing.

250. Methods of Perpetuating Corners.

1. **Corner trees.**—A sound tree, five inches or more in diameter, standing exactly at a corner, is the best monument.

2. **Corner stones.**—A stone, at least 14 inches long and 6 inches square, set from two-thirds to three-fourths in the ground, is preferred to other monuments, except a tree.

3. **Posts and witness trees.**—In the absence of corner trees and stones, when trees are near, a post may be planted and witnessed by taking the bearing and distance of two or more trees in different directions from the corner. These trees are marked by a blaze in which is marked the number of the township, range, and section. A notch is cut in the lower end of the blaze, under which another blaze is made in which are cut the letters *B. T.*, signifying bearing tree.

4. **Posts, mounds, and witness pits.**—When neither corner trees, stones, nor witness trees are available, corners may be marked by posts, mounds, and witness pits. The posts are planted 12 inches in the ground, and at the lower end, on the north or west side, according as the course is north or west, a marked stone, a small quantity of charcoal, or a charred stake must be deposited. Four pits are dug, 6 feet from the post, on opposite sides, 2 feet square and 1 foot deep, and the excavated earth packed round the post within 1 foot of the top. If sod is to be had, it is to be used in covering the mounds.

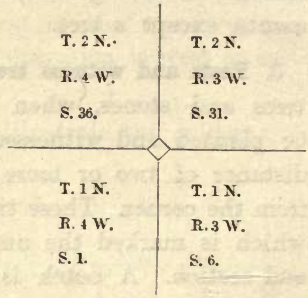
The method of marking the corner is to be carefully noted in the field book.

251. Township Corners.

1. Posts used in marking township corners must be 4 feet in length, and 5 inches, at least, in diameter. These posts are to be set 2 feet in the ground, and

the upper part squared to receive the marks to be cut on them.

If the corner is common to four townships, the post is set so as to present the angles in the direction of the line; and the number of the township, range, and section must be marked on the side facing, and six notches cut on each of the four edges.



If the township corner is on a base line or standard parallel, unless it is also on the principal meridian, it will be common to two townships only; and if these are on the north, the corner will be a standard corner. In this case, six notches are cut on the east, north, and west edges, but not on the south edge, and the letters *S. C.*, signifying standard corner, cut on the flat surface.

If the corner is common to two townships on the south, but not on the north, it will be a closing corner, and six notches are cut on the east, south, and west edges, but not on the north edge, and the letters *C. C.*, signifying closing corner, cut on the flat surface.

2. Township corner stones should be inserted at least 10 inches in the ground, with their sides facing the cardinal points of the compass, and small mounds of stones heaped against them.

These corner stones are notched in the same manner as posts in similar circumstances, but are not otherwise marked.

3. A tree of proper size on the corner is marked in the same manner as a post.

The mounds, when made round the posts, must be 5 feet in diameter at the base, and 2½ feet high. The posts, therefore, must be 4½ feet long, so as to be 1 foot in the ground and 1 foot above the top of the mound.

Witness pits for township corners must be 2 feet long, 1½ feet wide, and 1 foot deep. If the corner is common to four townships, there will be four pits placed lengthwise on the lines; but if the corner is common to only two townships, only three pits are dug, and are placed lengthwise on the lines. Thus the kind of township corners are readily distinguished.

These pits are made only in the absence of witness trees, which are to be selected, if possible, one from each township.

252. Section Corners.

Section corners are established at intervals of 80 chains or 1 mile, and are perpetuated by the following methods:

1. **Section corner posts** are 4 feet in length, and at least 4 inches in diameter. They are planted 2 feet in the ground, and the part above the ground squared to receive the marks.

If the corner is common to four sections, the post is set cornerwise to the lines, the number of the section is marked on the side facing it, and the number of the township and range on the north-east face.

Mile-posts on township lines have as many notches on the corresponding edges as they are miles from the respective township corners.

Section posts within a township have as many notches on the south and east edges as they are miles from the south and east boundaries of the township; but no notches are cut on the north and west edges.

Section posts must be witnessed by trees, one in each section, or, in the absence of trees, by pits 18 inches square and 12 inches deep.

2. **Section corner mounds** are $4\frac{1}{2}$ feet in diameter at the base, and 2 feet high. The post must be 4 feet long, 1 foot in the ground, and 1 foot high above the mound, and at least 3 inches square.

At corners common to four sections, the edges are in the direction of the cardinal points; but at corners common only to two sections, the flattened sides face the cardinal points.

Section posts in mounds are to be marked and witnessed in the same manner as the post without the mound.

3. **Stones** used to mark section corners on township lines are set with their edges in the direction of the line; but for interior sections they face the north. They are witnessed in the same manner as posts, but are not marked except by notches.

4. **Section corner trees** are marked and witnessed the same as posts.

253. Quarter Section Corners.

Quarter section corners are established at intervals of 40 chains or half a mile, except in the north or west tiers of sections of a township.

In subdividing these sections, the quarter post is placed 40 chains from the interior section corner, so that the excess or deficiency shall fall in the last half mile.

Quarter section corners are not required to be established on base or standard parallel lines on the north.

The methods of perpetuating these corners are the following:

1. **Quarter section posts,** 4 feet in length and 4 inches in diameter, are planted or driven 2 feet into the ground, and the part above the ground squared and marked ¼ *S.*, signifying quarter section. These corners are witnessed by two bearing trees.

2. **Quarter section mounds** are, like section mounds, packed round the posts, and pits may be used in the absence of witness trees.

3. **Quarter section stones** have ¼ cut on the west side of north and south lines, and on the north side of east and west lines, and are witnessed by two bearing trees or pits.

4. **A quarter section tree** is marked and witnessed in the same manner as a post.

254. Meander Corners.

Meander corners are the intersections of township or section lines with the banks of lakes, bayous, or navigable rivers.

These corners are marked by the following methods:

1. **Meander posts** of the same size as section posts, are planted firmly in the ground, and witnessed by two bearing trees or pits, but are not marked.

2. **Mounds** of the same size as those for section corners are, in the absence of witness trees, formed round

the posts, and a pit dug exactly on the line, 8 links further from the water than the mound.

3. **Stones or trees,** witnessed in the same manner as posts, may be employed.

255. Chaining.

Eleven tally pins are employed, ten of which are taken by the fore chainman, or leader, and the remaining one by the hind chainman, or follower, who sticks it at the beginning of the course, and against it brings the handle at one end of the chain.

The leader, holding the other handle of the chain and one pin in his right hand, draws out the chain to its full length in the direction of the course; both taking care that the chain is free from kinks.

The leader standing to the left of the line, so as not to obstruct the range, with his right arm extended, draws the chain tight, brings the pin into line according to the order "right" or "left," from the follower, sticks it at the order "down" by pressing his left hand on the top of the pin, and replies "down."

The follower then withdraws his pin, and both advance, the leader drawing the chain in the direction of the course, but a little to one side to avoid dragging out the pin, till the follower comes up to the pin, against which he brings the handle at his end of the chain, and directs the sticking of another pin, as before, and so on.

When the leader has stuck his last pin, he cries "tally," which is repeated by the other, and each registers the tally by slipping a ring on a belt.

The follower then comes forward, and counting in presence of his fellow, to avoid mistake, the pins taken up, takes the foreward end of the chain and proceeds, as the leader, for another tally.

If a whole chain is employed, a tally is ten chains; and accordingly four tallies make half a mile, and eight tallies a mile.

If a half-chain is employed, a tally is five chains, eight tallies are half a mile, and sixteeen tallies a mile.

In measuring up or down a hill, the chain must be kept horizontal, so that it is often necessary to use but a portion of the chain.

The chain employed in the field must be compared, from day to day, with a *standard chain* furnished by the Surveyor-General, and any variation promptly corrected.

256. Marking Lines.

Line trees, called also "station trees," or "sight trees," are marked by two notches on each side of the tree, in the direction of the line.

The line is marked, so as to be easily followed, by blazing a sufficient number of trees near the line on two sides quartering toward the line.

Saplings near the line are cut partly off by a blow from the ax, at the usual height of blazes, and bent at right angles to the line.

Random lines are not marked by blazing trees; but to enable the surveyor to retrace the line on his return, bushes are lopped and bent in the direction of the line, and stakes are driven every ten chains, which are pulled up when the true line is established.

Insuperable objects, such as ponds, marshes, etc., are passed by making right-angled offsets, or by trigonometrical operations, a complete record of which must be made in the field book.

257. Initial Point and Principal Lines.

1. **The initial point,** which is usually some permanent natural object, as the confluence of two rivers, or an isolated mountain, is first selected.

2. **Principal meridians** are run from the initial points due north or due south, and the quarter section, section, and township corners on these lines are accurately located and perpetuated.

The following are the principal meridians already established:

1st. The first runs north from the mouth of the Great Miami river, between Ohio and Indiana, to the south line of Michigan.

2d. The second runs north from the mouth of the Little Blue river through the center of Indiana to its north line.

3d. The third runs north from the mouth of the Ohio river through Illinois to its north line.

4th. The fourth runs north from the Illinois river through the western part of Illinois and the center of Wisconsin to Lake Superior.

5th. The fifth runs north from the mouth of the Arkansas river through the eastern portion of Arkansas, Missouri, and Iowa, and regulates the surveys in Minnesota west of the Mississippi river, and the surveys in Dakota east of the Missouri river.

6th. The sixth commences on the Arkansas river, in Kansas, and runs north through the eastern part of Kansas and Nebraska to the Missouri river.

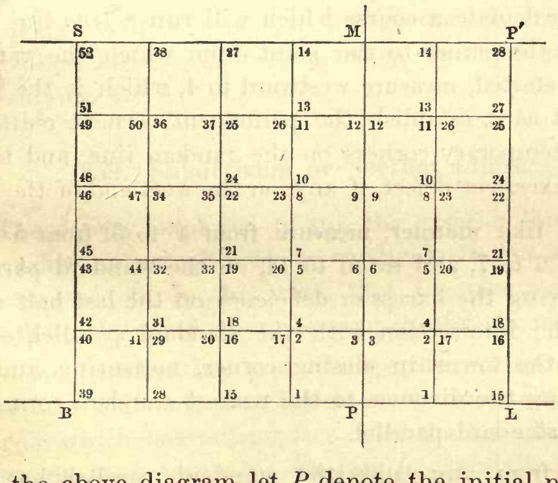
7th. *Independent meridians.*—These are the *Independent* meridian of *New Mexico*, the *Salt Lake* meridian in *Utah*, the *Willamette* meridian of *Oregon* and *Washington*, and the *Humboldt* meridian, the *Mt. Diablo* meridian, and the *St. Bernardino* meridian of *California.*

3. **Base lines** are run from the initial points due east or due west, and the quarter section, section, and township corners, for the land north of the line, are accurately located, at full measure, and perpetuated.

4. **Standard parallels** are also run due east or due west thirty miles north of the base line or other standard parallel, and the corners located and perpetuated as on the base line.

5. **Range lines** are run between the ranges of townships due north from a base line or standard parallel to the next standard parallel.

258. Exterior or Township Lines.



In the above diagram let *P* denote the initial point, *PM* the principal meridian, *BL* the base line, *SP′* the first standard parallel north, and let the squares denote townships.

1. For townships west of the meridian, begin at the first pre-established township corner on the base line west of the meridian. This is the *S. W.* corner of *T.* 1 *N.*, *R.* 1 *W.*, and is marked 1 in the diagram.

Measure thence due north 480 chains, establishing the quarter section and section corners, to 2, at which point establish the corner common to *T.'s* 1 and 2 *N.* and *R.'s* 1 and 2 *W.;* thence east on a random line, setting temporary quarter section and section stakes to 3.

If the random line should overrun, or fall short, or intersect the meridian north or south of the true corner, more than 3.50 chains, a material error has been committed, and the lines must be retraced.

If the random line should terminate within 3.50 chains of the corner, measure the distance at which the meridian is intersected north or south of the corner, calculate a course which will run a true line back from the corner to the point from which the random line started, measure westward to 4, which is the same point as 2, establish the permanent corners, obliterate the temporary corners on the random line, and throw the excess or defect, if any, on the west end of the line.

In like manner, measure from 4 to 5, from 5 to 6, from 6 to 7, and so on to 14, on the standard parallel, throwing the excess or deficiency on the last half mile. At the intersection with the standard parallel, establish the township closing corner, measuring and recording the distance to the nearest standard corner on said standard parallel.

If from any cause the standard parallel has not been run, the surveyor will plant the corner of the township in place, subject to removal north or south when the standard parallel shall have been run.

The surveyor then proceeds to the *S. W.* corner of *T.* 1 *N.*, *R.* 2 *W.*, on the base line at 15, and proceeds in a similar manner with another range of townships, and so on.

2. For townships east of the meridian, begin at the *S. E.* corner of *T.* 1 *N.*, *R.* 1 *E.*, at 1 on the base line, and proceed as on the west of the meridian, except that the random lines are run west and the true lines east, throwing the excess over 480 chains, or the deficiency, on the west end of the line in measuring the first quarter section boundary on the north, the remaining distances will be exact half-miles and miles.

With the field notes of the exterior or township lines, a plot of the lines, run on a scale of 2 inches to the mile, must be submitted, on which are noted all objects of topography, which will illustrate the notes, as the direction of streams, by arrow-heads pointing down stream, the intersection of the lines by lakes, streams, ponds, marshes, swamps, ravines, mountains, etc.

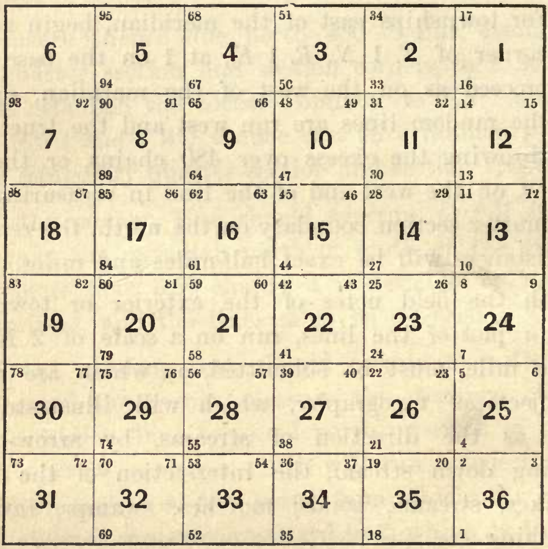
259. Subdivision or Section Lines.

The deputy employed to run the exterior lines of a township is not allowed to subdivide it, but another is employed to do this work, that the one may be a check to the other, thus securing greater accuracy.

Before subdividing a township, the surveyor must ascertain and note the change in the variation of the needle which has taken place since the township lines were run, and adjust his compass to a variation which will retrace the eastern boundary of the township.

He must also compare his own chaining with the original by remeasuring the first mile both of the south and east lines of the township, and note the discrepancies, if any.

The following is a diagram of a township:



The sections are designated by beginning at the *N. E.* corner and numbering west, 1, 2, 3, 4, 5, 6, then east on the next tier, 7, 8, . . . , then west, and so on.

In running the subdivision lines, begin on the south line of the township, at the first section corner west of the east line, numbered 1 in the diagram, and common to sections 35 and 36.

Measure thence due north 40 chains, at which point establish a quarter section corner; thence due north another 40 chains to 2, where establish a section corner common to sections 25, 26, 35, and 36.

Run a random line from 2 due east to the township line, setting up a temporary quarter section stake 40 chains from 2.

If the random line intersect the township line precisely at the pre-established section corner at 3, it may be established as the true line by blazing back and making the quarter section corner permanent.

If the random line intersect the township line either north or south of the section corner, measure and note the distance of the intersection from said corner, and calculate a course which will run a true line from the corner back to 4, where the random line started.

Let A correspond to section corner 2, B to 3, and C to the intersection of the township and random lines, and north, for example, of B the section corner.

$$\text{Then, } \tan A = \frac{BC}{AB}.$$

Let $l =$ the number of links in BC, and m the number of minutes in A. Then, practically, we shall have,

If $AB = \frac{1}{2}$ mile, $m = l - \frac{1}{7}l$.

If $AB = 1$ mile, $m = \frac{1}{2}l - \frac{1}{14}l$.

If $AB = 3$ miles, $m = \frac{1}{7}l$.

If $AB = 6$ miles, $m = \frac{1}{7}$ of $\frac{1}{2}l$.

Let us suppose that we have found $A = 10\frac{1}{2}'$.

Now, as CA is west by the compass, BA is *N.* 89° 49½′ *W.* Run this line and establish the quarter section at a point equidistant from the two section corners, which will be, with sufficient accuracy, one-half the length of the random line from 2. Pull up the temporary quarter section stake on the random line.

Proceed from 4 to 5, then on a random line to 6, and back on a true line to 7, and so on to 16.

From 16 run due north on a random line to the north line of the township, setting up a temporary quarter section stake at 40 chains.

If the random line intersect the north line of the township at the pre-established section corner, the random line will be the true line, and is made permanent by blazing back, and making the quarter section corner permanent.

If the random line does not close exactly on the pre-established section corner, measure and note the distance of the intersection from said corner, calculate a course that will run a true line southward from the corner to 16, run this line, and establish the quarter section corner on it just 40 chains from 16, throwing the excess or deficiency, if any, on the last half mile.

If the north township line is a base line or standard parallel, no random line is run, but a true line due north, on which a quarter section post is established 40 chains from 16; and at the intersection with said base line or standard parallel, establish a closing corner, measuring and noting its distance from the corresponding standard corner.

Pass from 17 to 18, and survey the second tier of sections in the same manner as the first, closing on the interior section corners before established as upon those on the east line of the township.

In running the line between the fifth and sixth tiers of sections, not only is a random line run east as before, but one is run west to the range line, and a true line run back, and the permanent quarter section corner established on it just 40 chains from the interior corner, throwing the excess or deficiency on the west half mile.

The Surveyor-General furnishes the outline of the diagram, and the deputy fills it out, and makes the appropriate topographical sketches.

260. Meandering.

Navigable rivers, lakes, and bayous, being public highways, are meandered and separated from the adjoining land.

Standing with the face down stream, the bank on the right hand is called the *right bank;* the bank on the left, the *left bank.*

If a river is navigable, both banks are meandered, care being taken not to mistake, in high water, the border of bottom-land for the true bank.

Commence at a meander corner of the township line, take the bearing along the bank of the river, and measure the distance of the longest possible straight course to the nearest chain, if the distance exceeds 10 chains; otherwise, to the nearest ten links; and so on to the next meander corner on another boundary line of the township.

Enter in the field book, after the township notes, keeping the notes separate through each fractional section, the date, the point of beginning, the bearings and distances in order, the intersections with all intermediate meander corners, the height of falls, the length of rapids, the location and width at the mouth of streams running into the water you are meandering, the location of springs on the banks, the nature of their waters, the location of islands, the elevation of banks, etc.

If the river is not navigable, meander the right bank, unless it presents formidable obstacles not found on the left bank; but the crossing of the stream, in meandering, must be made from a pre-established meander corner on one bank to the corner on the other bank, and the width of the river between the corners computed trigonometrically.

Wide flats, whose area is more than 40 acres, permanently covered with water, along rivers not navigable, are meandered on both banks.

The position of islands in rivers is determined by measuring, on or near the bank, a base line, connected with the surveyed lines, and taking the proper bearings to a flag or other object on the island, and computing the distance from the meander corners of the river to points on the bank of the island. The island can be meandered from such points.

In meandering lakes, ponds, or bayous, commence at a meander corner of the township line, and proceed as in case of a river. If, however, the body of water is entirely within a township, begin at a meander corner established in subdividing.

In meandering a pond lying entirely within the boundaries of a section, run to the pond two lines from the nearest section or quarter section corners, on opposite sides of the pond, giving their bearings and distances, and at the intersection of these lines with the bank of the pond establish witness points by planting posts, witnessed by bearing trees or mounds and pits, then commence to meander at one of these points, and proceed around to the other, and thence to the point of beginning.

No blazes or marks are made on meander lines between established corners.

261. Swamp Lands.

By the act of Congress approved Sept. 28th, 1850, swamp and overflowed lands, unfit for cultivation, are granted to the state in which they are situated.

If the larger part of the smallest legal subdivision is swamp, it goes to the state; if not, it is retained by the Government.

In order to determine what lands fall to the state under the swamp act, it is required that the field notes, beside other things required to be noted, should indicate the points where the public lines enter and leave all such land.

The aforesaid grant does not embrace lands subject to casual inundation, but those only where the overflow would prevent the raising of crops without artificial aid, such as levees, etc. The surveyor should therefore state whether such lands are continually and permanently wet, or subject to overflow so frequently as to render them totally unfit for cultivation.

The depth of inundation is to be stated, as determined from indications on the trees, and the frequency of inundation should be given as accurately as possible, from the nature of the case or reliable testimony. The character of the timber, shrubs, plants, etc., growing on such lands, and on the land near rivers, lakes, or other bodies of water, should be stated.

The words "unfit for cultivation" should be employed, in connection with the usual phraseology, in the notes, on entering or leaving such lands.

If the margin of bottoms, swamps, or marshes, in which such uncultivable land exists, is not identical with the body of land unfit for cultivation, a separate entry must be made opposite the marginal distance.

In case the land is overflowed by artificial means, such as dams for milling, logging, etc., such overflow will not be officially regarded, but the lines of the public surveys will be continued across the same without setting meander posts, stating particularly in the notes the depth of the water, and how the overflow was caused.

262. Field Books.

The field books are the original and official records of the location and boundaries of the public lands, and afford the elements from which the plots are constructed.

They should, therefore, contain an accurate record of every thing officially done by the surveyor, pursuant to instructions in running, measuring, and marking lines, and establishing corners, and should present a full topographical description of the tract surveyed.

There are four distinct field books.

1. A field book for the *meridian* and *base lines*, exhibiting the establishment of the township, section, and quarter section corners on these lines, the crossing of streams, ravines, hills, and mountains, the character of the soil, timber, minerals, etc.

2. A field book for *standard parallels* or *correction lines*, showing the township, section, and quarter section corners on the lines, and the topography of the country through which the lines pass.

3. A field book for *exterior* or *township lines*, showing the establishment of corners on the lines, and the topography.

4. A field book for *subdivision* or *section lines*, giving the corners and topography as aforesaid.

The variations of the needle must be stated in a separate line, preceding the notes of measurement, which must be recorded in the order in which the work is done, and the date must immediately follow the notes of each day's work.

The exhibition of every mile surveyed must be complete in itself, and be separated from the preceding and following notes by a line drawn across the paper.

The topographical description must follow the notes for each mile, and not be mixed up with them.

No abbreviations are allowed, except for words constantly occurring, as *sec.* for section, *ch.* for chains, *ft.* for feet, ¼ *sec. cor.* for quarter section corner.

Proper names are never to be abbreviated.

The field books must be so kept as to show the amount of work done in each fiscal year.

The notes should be expressed in clear and precise language, and the writing legible.

No record is to be obliterated, or leaf mutilated or taken out.

The title-page of each book should designate the kind of lines run, giving prominently the name of the state or territory and surveyor, the dates of contract, and of commencing and completing the work.

The second page should contain the names and duties of assistants; and whenever a new assistant is employed, or the duties of any of them changed, such facts, with the reason, should be stated in an appropriate entry, immediately preceding the notes taken under such changed arrangements.

An index, in the form of a diagram or plot of the survey, with number on each line, referring to the page of the field notes on which is found the description of the line, must accompany the notes.

263. Records in the Field Book.

1. **General heading of the pages.**—The number of the township and range, and the name of the principal meridian of reference, stand at the head of each page.

2. **Heading for each mile.**—The bearing, location, and kind of line run, whether random or true, must be stated in a line; and the variation of the needle, in a separate line on the page at the head of the notes, for each mile run.

3. **Courses and distances.**—The course and length of each line run, noting all necessary offsets therefrom, with the reason and mode thereof.

4. **The method of perpetuating corners.**—If a tree, note the kind and diameter; if a stone, its dimensions, as factors in the order of length, breadth, and thickness; if a post, its dimensions, the kind of timber, the kind of memorial, if any, buried by its side, and if surrounded by a mound, the material of which the mound is constructed, whether of stones or earth; the course and distance of the pits from the center of the mound where a necessity exists for deviating from the general rule of witness trees.

5. **Bearing trees.**—The kind and diameter of all bearing trees, with the course and distance of the same from their respective corners, and the precise relative position of the witness corners with respect to the true corners.

6. **Line trees.**—The kind, diameter, and distance on the line, from the corner, of all trees which the line intersects.

7. **Intersection of land objects.**—The distance at which the line first intersects and then leaves every settler's claim and improvement, prairie, bottom-land, swamp,

marsh, grove, or windfall, with the course of the same at both points of intersection; the distance at which a line begins to ascend, arrives at the top, or reaches the foot of all remarkable hills and ridges, with their courses and estimated height above the surrounding country.

8. **Intersection of water objects.**—The distance at which the line intersects rivers, creeks, or other bodies of water, the width of navigable streams, and small lakes or ponds between the meander corners, the height of banks, the depth and nature of the water.

9. **Surface.**—Level, rolling, broken, or hilly.

10. **Soil.**—First, second, or third-rate; clay, sand, loam, or gravel.

11. **Timber.**—Kind, in order of abundance, and undergrowth.

12. **Bottom-lands.**—Wet or dry; whether subject to inundation, and to what depth.

13. **Springs.**—Fresh, saline, or mineral; and course of their streams.

14. **Improvements.**—Towns and villages, Indian villages and wigwams, houses and cabins, fields, fences, sugar-tree groves, mill-seats, forges or factories.

15. **Coal beds.**—Note the quality of coal beds, and their extent to the nearest legal subdivision.

16. **Roads and trails.**—Whence, whither, and direction.

17. **Rapids, cascades.**—Length of rapids, height of falls in feet.

18. **Precipices.**—Describe precipices, caves, ravines, sink-holes.

19. **Quarries.**—Whether marble, granite, lime-stone or sand-stone.

20. **Natural curiosities.** — Interesting fossils, ancient works, as mounds, fortifications, embankments, etc.

21. **Change of variation.**—Any material change in the variation of the needle must be noted, and the exact points where such variation occurs.

22. **Dates.** — State the date of each day's work in a separate line, immediately after the notes for that day.

23. **General description.** — At the conclusion of the notes for the subdivisional work, taken on the line, the deputy must subjoin a general description of the township in the aggregate, in reference to the face of the country, its soil, timber, geological features, etc.

24. **Verification of Deputy Surveyor.**—The deputy must append to each separate book of field notes his affidavit that all the lines therein described have been run, and all the corners established and perpetuated according to the instructions and laws, and that the foregoing notes are the true and original field notes of such survey.

25. **Verification of Assistants.**—The compassman, flagman, chainmen, and axman must also attest, under oath, that they assisted said deputy in executing said surveys, and that, to the best of their knowledge and belief, the work has been strictly performed according to the instructions furnished by the Surveyor-General.

26. **Approval and certificate of the Surveyor-General.**— The Surveyor-General will attach his official approval to each of the original field books, and affix his official certificate to the copies of the field notes transmitted to the general land office, that they are true copies of the originals on file in his office.

The following specimen pages of field notes, taken from the *United States Manual of Surveying Instructions*, will illustrate the subject:

FIELD NOTES

OF THE

Exterior and Subdivision Lines

OF TOWNSHIP 25 NORTH, RANGE 2 WEST,

WILLAMETTE MERIDIAN,

OREGON.

Surveyed by Robert Acres, Deputy Surveyor,

Under his contract, dated ——, 18—.

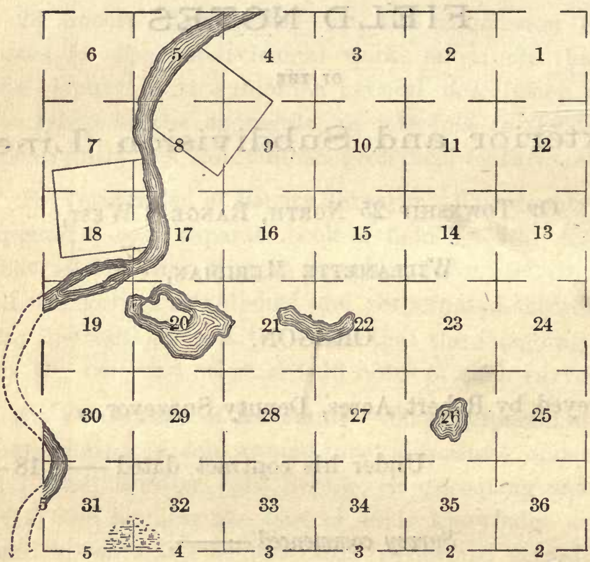
Survey commenced ———.

Survey completed ———.

264. Index.

Referring the lines to the pages of the field notes.

T. 25 N., R. 2 W., Willamette Meridian.



The lines numbered are described in the notes on the pages indicated by the numbers.

NAMES OF SURVEYOR AND ASSISTANTS.

Robert Acres, Surveyor.
Peter Long, Chainman.
John Short, Chainman

George Sharp, Axman.
Adam Dull, Axman.
Henry Flagg, Compassman.

265. Field Notes.

South Boundary, T. 25 N., R. 2 W., Willamette Meridian.

Chains.	
	Begin at the post, the established corner to Townships 24 and 25 North, in Ranges 2 and 3 West. The witness trees all standing, and agree with the description furnished me by the office, viz:
	A Black Oak, 20 in. dia., N. 37° E. 27 links,
	A Burr-oak, 24 in. dia., N. 43° W. 35 links,
	A Maple, 18 in. dia., S. 27° W. 39 links,
	A White Oak, 15 in. dia., S. 47° E. 41 links.
	East on a random line on the South Boundaries of sections 31, 32, 33, 34, 35, and 36.
	Variation by Burt's improved solar compass, 18° 41′ E.
	I set temporary half-mile and mile posts at every 40 and 80 chains, and at 5 miles, 74 chains 53 links, to a point 2 chains and 20 links north of the corner to Townships 24 and 25 North, Ranges 1 and 2 W.
	(Therefore, the correction will be 5 chains, 47 links *West*, and 37 links *South* per mile.)
	I find the corner post standing and the witness trees to agree with the description furnished me by the Surveyor-General's office, viz:
	A Burr-oak, 17 in. dia., bears N. 44° E. 31 links,
	A White Oak, 16 in. dia., bears N. 26° W. 21 links,
	A Linden, 20 in. dia., bears S. 42° W. 15 lks.,
	A Black Oak, 24 in. dia., bears S. 27° E. 14 links.

(2)

South Boundary, T. 25 N., R. 2 W., Willamette Meridian.

Chains.	
	From the corner to Townships 24 and 25 N., Ranges 1 and 2 W., I run (at a variation of 18° 41′ East) [See Arts. 258, 289.]
	N. 89° 44′ W., on a true line along the
40.00	South Boundary of section 36, set a post for quarter section corner, from which
	A Beech, 24 in. dia., bears N. 11° E. 38 links dist.
	A Beech, 9 in. dia., bears S. 9° E. 17 links dist.
62.50	A Brook, 6 links wide, runs North.
80.00	Set a post for corner to sections 35 and 36, 1 and 2, from which
	A Beech, 9 in. dia., bears N. 22° E. 16 links dist.
	A Beech, 8 in. dia., bears N. 19° W. 14 links dist.
	A White Oak, 10 in. dia., bears S. 52° W. 7 links dist.
	A Black Oak, 14 in. dia., bears S. 46° E. 8 links dist.
	Land—level, good soil, fit for cultivation.
	Timber—Beech, various kinds of Oak, Ash, Hickory.
	N. 89° 44′ W., on a *true* line along the South Boundary of section 35, Variation 18° 41′ E.
40.00	Set a post for quarter section corner, from which
	A Beech, 8 in. dia., bears N. 20° E. 8 links dist.
	No other tree convenient; made a trench around post.

(3)

South Boundary, T. 25 N., R. 2 W., Willamette Meridian.

Chains.	
65.00	Begin to ascend a moderate hill; bears N. and S.
80.00	Set a post with trench, for corner of sections 34 and 35, 2 and 3, from which A Beech, 10 in. dia., bears N. 56° W. 9 links dist. A Beech, 10 in. dia., bears S. 51° E. 13 links dist. No other tree convenient to mark. Land—level, or gently rolling, and good for farming. Timber—Beech, Oak, Ash, and Hickory; some Walnut and Poplar.
	N. 89° 44′ W. on a *true* line along the South Boundary of section 34, Variation 18° 41′ E.
40.00	Set a quarter section post with trench, from which A Black Oak, 10 in. dia., bears N. 2° E. 635 links dist. No other tree convenient to mark.
80.00	To point for corner sections 33, 34, 3 and 4. Drove charred stakes, raised mounds with trenches, as *per* instructions, from which A Burr-oak, 16 in. dia., bears N. 31° E. 344 links. A Hickory, 12 in. dia., bears S. 43° W. 231 links. No other tree convenient to mark. Land—level, rich, and good for farming. Timber—some scattering Oak and Walnut.

(4)

South Boundary, T. 25 N., R. 2 W., Willamette Township.

Chains.	
	N. 89° 44′ W. on a *true* line along the South Boundary of section 33, Variation 18° 41′ E.
37.51	A Black Oak, 24 in. dia.
40.00	Set a post for quarter section corner, from which
	A Black Oak, 18 in. dia., bears N. 25° E. 32 links dist.
	A White Oak, 15 in. dia., bears N. 43° W. 22 links dist.
62.00	To foot of steep hill, bears N. E. and S.W.
80.00	Set a post for corner to sections 32, 33, 4 and 5, from which
	A White Oak, 15 in. dia., bears N. 23° E. 27 links dist.
	A Black Oak, 20 in. dia., bears N. 82° W. 75 links dist.
	A Burr-oak, 20 in. dia., bears S. 37° W. 92 links dist.
	A White Oak, 24 in. dia., bears S. 26° E. 42 links dist.
	Land — gently rolling; rich farming land.
	Timber — Oak, Hickory, and Ash.
	N. 89° 44′ W. on a *true* line along the South Boundary of section 32, Variation 18° 41′ E.
37.50	A Creek, 20 links wide, runs North.
40.00	Set a granite stone, 14 in. long, 10 in. wide, and 4 in. thick, for quarter section corner, from which
	A Maple, 20 in. dia., bears N. 41° E. 25 links dist.
	A Birch, 24 in. dia., bears N. 35° W. 22 links dist.

(5)

South Boundary, T. 25 N., R. 2 W., Willamette Meridian.

Chains.	
76.00	To S. E. edge of swamp.
	As it is impossible to establish *permanently* the corner to sections 31, 32, 5 and 6, in the swamp, I therefore, at this point, 4.00 chains east of the true point for said section corner, raise a witness mound with trench, as *per* instructions, from which
	A Black Oak, 20 in. dia., bears N. 51° E. 115 links.
80.00	A point in deep swamp for corner to sections 31, 32, 5 and 6.
	Land—rich bottom; *west* of creek, part wet; *east* of creek, good for farming.
	Timber—good; Oak, Hickory, and Walnut.
	N. 89° 44′ W. on a *true* line along the South Boundary of section 31, Variation 18° 41′ E.
11.00	Leave swamp and rise bluff 30 feet high, bears N. and S.
40.00	Set post for quarter section corner, from which
	A Sugar tree, 27 in. dia., bears S. 81° W. 42 links dist.
	A Beech, 24 in. dia., bears S. 71° E. 24 links dist.
54.00	Foot of rocky bluff 30 feet high, bears N. E. and S. W.
57.50	A spring branch comes out at the foot of the bluff, 5 links wide; runs N. W. into swamp.
61.00	Enter swamp; bears N. and S.
70.00	Leave swamp; bears N. and S.

(6)

South Boundary, T. 25 N., R. 2 W., Willamette Meridian.

Chains.	
	The swamp contains about 15 acres, the greater part in section 31.
74.73	The corner to Townships 24 and 25 N., Ranges 2 and 3 W.
	Land — except the swamp, rolling, good, rich soil.
	Timber — Sugar-tree, Beech, Swamp Maple. Jan. 25th, 1854.
	Between Ranges 2 and 3 West, from corner to Townships 24 and 25 N., I run
	North, on the range line between sections 31 and 36, Variation 18° 56′ East.
8.56	Set a post on the left bank of Chickeeles river, for corner to fractional sections 31 and 36, from which
	A Hackberry, 11 in. dia., bears N. 50° E. 11 links dist.
	A Sycamore, 60 in. dia., bears S. 15° W. 24 links dist.
	I now cause a flag to be set on the right bank of the river, and in the line between sections 31 and 36. I now cross the river, and from a point on the right bank thereof, west of the corner just established on the left bank, I run *North* on an offset line, 25 chains and 94 links, to a point 8 chains and 56 links west of the flag. I now set a post in the place of the flag, for corner to fractional sections 31 and 36, from which
	A Beech, 10 in. dia., bears N. 2° E. 12 links dist.

(7)

Between Ranges 2 and 3 W., T. 25 N., Willamette Meridian.

Chains.	
	A Black Oak, 12 in. dia., bears N. 80° W. 16 links dist.
34.50	The corner above described.
40.00	Set a post for ¼ section corner, from which
	A Burr-oak, 20 in. dia., bears N. 37° E. 26 links dist.
	A Black Oak, 24 in. dia., bears N. 80° W. 16 links dist.
43.41	A Black Walnut, 30 in. dia.
80.00	Set a post for corner to sections 30, 31, 25, and 36, from which
	A Beech, 14 in. dia., bears N. 20° E. 14 links dist.
	A Hickory, 9 in. dia., bears N. 25° W. 12 links dist.
	A Beech, 16 in. dia., bears S. 40° W. 16 links dist.
	A White Oak, 10 in. dia., bears S. 44° E. 20 links dist.
	Land—level; rich bottom; not inundated.
	Timber—Oak, Hickory, Beech, and Ash.

In like manner all the other Township lines are run.

General Description.

This township contains a large amount of first-rate land for farming. It is well timbered with Oak, Hickory, Sugar-tree, Walnut, Beech, and Ash.

Chickeeles river is navigable for small boats in low water, and does not often overflow its banks, which are from ten to fifteen feet high.

The township will admit of a large settlement, and should therefore be subdivided.

(8)

Field Notes of the Subdivision Lines and Meanders of Chickeeles River, in Township 25 N., R. 2 W., Willamette Meridian.

Chains.	
	To determine the proper adjustment of my compass for subdividing this township, I commence at the corner to Townships 24 and 25 N., R. 1 and 2 W., and run
	North, on a blank line along the East Boundary of section 36, Variation 17° 51′ East,
40.05	To a point 5 links west of the quarter section corner.
80.09	To a point 12 links west of the corner to sections 25 and 36.
	To retrace this line, or run parallel thereto, my compass must be adjusted to a variation of 17° 46′ East.
	Subdivision commenced Feb. 1, 1854.
	From the corner to sections 1, 2, 35, and 36, on the South Boundary of the Township, I run
	North, between sections 35 and 36, Variation 17° 46′ East,
9.19	A Beech, 30 in. dia.
29.97	A Beech, 30 in. dia.
40.00	Set a post for quarter section corner, from which
	A Beech, 8 in. dia., bears N. 23° W. 45 links dist.
	A Beech, 15 in. dia., bears S. 48° E. 12 links dist.
51.00	A Beech, 18 in. dia.
76.00	A Sugar-tree, 30 in. dia.

(9)

Township 25 N., Range 2 W., Willamette Meridian.

Chains.	
80.00	Set a post for corner to sections 25, 26, 35, and 36, from which A Beech, 28 in. dia., bears N. 60° E. 45 links dist. A Beech, 24 in. dia., bears N. 62° W. 17 links dist. A Poplar, 20 in. dia., bears S. 70° W. 50 links dist. A Poplar, 36 in. dia., bears S. 66° E. 34 links dist. Land — level, second-rate. Timber — Poplar, Beech, Sugar-tree, and some Oak; undergrowth — same, and Hazel.
	East, on a *random* line between sections 25 and 36, Variation 17° 46′ East.
9.00	A Brook, 20 links wide, runs north.
15.00	To foot of hills, bears N. and S.
40.00	Set a post for temporary quarter section corner.
55.00	To opposite foot of hill, bears N. and S.
72.00	A brook, 15 links wide, runs N.
80.00	Intersected East Boundary at post corner to sections 25 and 36, from which corner I run West, on a *true* line between sections 25 and 36, Variation 17° 46′ East.
40.00	Set a post on top of hill, bears N. and S., from which A Hickory, 14 in. dia., bears N. 60° E. 27 links dist. A Beech, 15 in. dia., bears S. 74° W. 9 links dist.

(10)

Township 25 N., Range 2 W., Willamette Meridian.

Chains.	
80.00	The corner to sections 25, 26, 35, and 36.
	Land — east and west parts, level, first-rate; middle part, broken, third-rate.
	Timber — Beech, Oak, Ash, etc.; undergrowth — same, and Spice in the bottoms.
	North, between sections 25 and 26, Variation 17° 46′ East.
7.00	A Poplar, 40 in. dia.
17.20	A Brook, 25 links wide, runs N. W.
18.05	A Walnut, 30 in. dia.
23.44	A Brook, 25 links wide, runs N. E.
40.00	Set a post for ¼ sec. corner, from which
	A Burr-oak, 36 in. dia., bears N. 42° E. 18 links dist.
	A Beech, 30 in. dia., bears S. 72° W. 9 links dist.
60.15	A Beech, 30 in. dia.
80.00	Set a post for corner to sections 23, 24, 25, 26, from which
	A White Oak, 14 in. dia., bears N. 50° E. 40 links.
	A Sugar-tree, 12 in. dia., bears N. 14° W. 31 links.
	A White Oak, 13 in. dia., bears S. 38° W. 32 links.
	A Sugar-tree, 12 in. dia., bears S. 42° E. 14 links.
	Land — level on the line; high ridge of hills through the middle of section 25, running N. and S.
	Timber — Beech, Walnut, Ash, Maple, etc.

(11)

Township 25 *N.*, *Range* 2 *W.*, *Willamette Meridian.*

Chains.

In like manner other subdivison lines are run.

Notes of the Meanders of a Small Lake in Section 26.

Begin at the ¼ sec. cor. on the line between sections 23 and 26, run thence South

24.00 To the margin of the lake, where set a post for meander corner, from which

A Beech, 14 in. dia., bears N. 45° E. 10 links dist.

A Beech, 9 in. dia., bears N. 15° W. 14 links dist.

Thence meander around the lake as follows:

S. 53° E. 17.75. At 75 links, cross outlet to lake 10 links wide, runs N. E.

S. 3^{2} E. 13.00.

S. 30′ W. 8.00.

S. 65° W. 12.00 to a point previously determined 20.30 chains *North* of the quarter section corner on the line between sections 26 and 35.

Set post meander corner, Maple, 16 in. dia., bears S. 15° W. 20 links dist.

Ash, 12 in. dia., bears S. 21° E. 15 links dist.

N. 63° W. 10.00
N. 13° W. 21.00

In this vicinity we discovered remarkable fossil remains of animals well worth the attention of naturalists.

(12)

Township 25 N., Range 2 W., Willamette Meridian.

Chains.	
	N. 52° E. 17.30 to the place of beginning. This is a beautiful lake, with well-defined banks from 6 to 10 feet high. Land — first-rate.

Meanders of the left bank of Chickeeles River.

Begin at the corner to fractional sections 4 and 33, in the North Boundary of the Township, and on the left and S. E. bank of the river, and run thence down the stream with the meanders of the left bank of said river, in fractional section 4, as follows:

Courses.	Dist.	Remarks.
S.76°W.	18.50	
S.61°W.	10.00	
S.59°W.	8.30	To the corner to fractional sections 4 and 5; thence in section 5,
S.54°W.	10.70	
S.40°W.	5.60	
S.50°W.	8.50	
S.37°W.	17.00	
S.44°W.	22.00	
S.38°W.	26.72	To the corner to fractional sections 5 and 8; thence in section 8,
S.21°W.	16.00	
S.10°W.	13.00	
South	8.50	To the head of rapids.
S.9°E.	5.00	
S.17°E.	20.00	
S.10°E.	12.00	To the foot of rapids.
S.22¼°E.	8.46	To the corner to fractional sections 8 and 17.
		Land, along fractional section 8,

(13)

Township 25 N., Range 2 W., Willamette Meridian.

Courses.	Dist.	Remarks.
		high, rich bottom; not inundated. The rapids are 37.00 chains long; rocky bottom; estimated fall, 10 feet.
		Meanders in Section 17.
S.17°E.	15.00	At 5 chains, discovered a vein of coal, which appears to be 5 feet thick, and may be readily worked.
S.8°E.	12.00	
S.4°W.	22.00	At 3 chains, the ferry across the river to Williamsburgh, on the opposite side of the river.
S.25°W.	17.00	
S.78°W.	12.00	
S.71°W.	9.55	To the corner to fractional sections 17 and 18; thence in section 18,
S.65°W.	15.00	
S73¾°W.	15.93	To the corner to fractional sections 18 and 19.
S.65°W.	14.00	In section 19.
S.60°W.	23.00	
S.42°W.	10.00	
S.20°W.	10.00	
S16½°W.	13.83	☞ At 2 chains, cross outlet to pond and lake, 50 links wide, to the corner to fractional sections 19 and 24, on the range line, 32.50 chains North of the corner to sections 19, 30, 24, and 25.

The above selections will serve as specimens of the manner of taking the field notes.

266. General Description.

The quality of the land in this township is considerably above the average. There is a fair proportion of rich bottom-land, chiefly situated on both sides of Chickeeles river, which is navigable, through the township, for steamboats of light draft, except over the rapids in Section 8.

The uplands are generally rolling, good first and second rate land, etc.

267. Certificates.

I, Robert Acres, Deputy Surveyor, do solemnly swear that, in pursuance of a contract with , Surveyor of the public lands of the United States, in the State [or Territory] of , bearing date the day of , 18 , and in strict conformity to the laws of the United States and the instructions furnished by the said Surveyor-General, I have faithfully surveyed the exterior boundaries [or subdivision and meanders, as the case may be] of Township number twenty-five North of the base line of Range number two West of the Willamette Meridian, in the aforesaid; and do further solemnly swear that the foregoing are the true and original field notes of such survey.

ROBERT ACRES,
Deputy Surveyor.

Subscribed by said Robert Acres, Deputy Surveyor, and sworn to before me, a Justice of the Peace for the County, in the State [or Territory] of this day of , 18 .

HENRY DOOLITTLE,
Justice of the Peace.

We hereby certify that we assisted Robert Acres, Deputy Surveyor, in surveying the exterior boundaries, and subdividing Township number twenty-five North of the base line of Range number two West of the Willamette Meridian, and that said Township has been, in all respects, to the best of our knowledge and belief, well and faithfully surveyed, and the boundary monuments planted according to the instructions furnished by the Surveyor-General.

PETER LONG, *Chainman.*
JOHN SHORT, *Chainman.*
GEORGE SHARP, *Axman.*
ADAM DULL, *Axman.*
HENRY FLAGG, *Compassman.*

Subscribed and sworn to by the above named persons, before me, a Justice of the Peace for the county of , in the State [or Territory] of , this day of , 18 .

HENRY DOOLITTLE,
Justice of the Peace.

SURVEYOR'S OFFICE AT , 18 .

The foregoing field notes of the Survey of [here describe the survey], executed by Robert Acres, under his contract of the day of , 18 , in the month of , 18 , having been critically examined, the necessary corrections and explanations made, the said field notes, and the surveys they describe, are hereby approved. A. B.,
Surveyor-General.

To the notes of each Township, in the copies of the field notes transmitted to the seat of government, the Surveyor-General will append the following certificate:

I certify that the foregoing transcript of the field notes of the Survey of the [here describe the character of the surveys, whether meridian, base line, standard parallel, exterior township lines, or subdivision lines and meanders of a particular township], in the State [or Territory] of , has been correctly copied from the original notes on file in this office. A. B.,

Surveyor-General.

268. Corners and Boundaries Unchangeable.

According to an act of Congress, entitled "An act concerning the mode of Surveying the Public Lands of the United States," approved February 11th, 1805, and still in force,

1st. "All the *corners* marked in the surveys returned by the Surveyor-General, shall be established as the proper corners of sections or subdivisions of sections which they were intended to designate; and the corners of half and quarter sections, not marked on said surveys, shall be placed, as nearly as possible, equidistant from those two corners which stand on the same line."

2d. "The boundary lines actually run and marked in the surveys returned by the Surveyor-General, shall be established as the proper boundary lines of the sections or subdivisions for which they were intended; and the length of such lines, as returned by the Surveyor-General aforesaid, shall be held and considered as the true length thereof."

If it is afterward found that a post is out of line, or that the line has been unequally subdivided, the general government only has the power of correction, and that only while it holds the title to the lands affected.

Such boundaries only as are established by the Surveyor-General, or the deputy, in the performance of his official duties, and in accordance with law, come under the above rules.

269. Restoring Lost Boundaries.

Lost boundaries must be restored in conformity with the laws under which they were originally established.

At an early day, three sets of section corners were established on the range lines; later, two sets on all the township boundaries; at present, the section lines close on previously established corners on township corners, making one set of corners, except on the base lines and standard parallels, where double corners — standard corners and closing corners — are established.

In order to restore lost boundaries correctly, the surveyor must know the manner in which townships were originally subdivided.

In case of three sets of corners on the range lines, one set was planted when the exteriors were run.

Corners on the east and west lines between two townships, belong to the sections of the township north.

From these corners, section lines were run due north, which would not, in general, close on the corners of the township line on the north, thus making two sets of corners on the north and south boundaries of the township.

The east and west lines were run due east and west from the last interior section corner, and new corners established at the intersections with the range lines.

In case of two sets of corners, the subdivisions were made as above, except that the east and west lines were closed on the corners previously established on the east boundary, but were run due west from the last interior section corner to the range line, and new section corners established at the intersection with the range line.

The method of making but one set of corners, except on the base line and standard parallels, is the one now in vogue, and has been sufficiently considered.

270. Restoring Lost Corners.

Lost corners must be restored, if possible, to their exact original position.

The surveyor should seek to accomplish this, first, by the aid of bearing trees, mounds, etc., described in the original field notes.

If the corner can not be located in this way, good testimony may be taken.

It often happens that in retracing lines, the measurements do not agree with the field notes. When such cases occur, from whatever cause, the surveyor must establish his corners at intervals proportional to those given in the original field notes.

1. *To restore a lost corner common to four sections.*

Find the distances between the nearest noted line trees or well-defined corners, north and south, and east and west of the lost corner. Establish the corner between them at a point intercepting distances proportional to those given in the original notes.

2. *To restore one of a double corner when the other is standing.*

First ascertain to which sections the existing corner belongs. Then re-establish the lost corner in the direction and at the distance stated in the original notes. Verify the work by chaining to noted line trees or corners, having previously compared your chaining with that of the United States deputy by rechaining between corners noted in the original survey, and making all distances proportional.

3. *To restore that one of a double corner established in running the township lines when both are missing.*

Run a straight line between the nearest noted line trees or corners on the line, and, at the distance given in the notes, establish the corner which will be common to two sections north or west of the line.

Let the accuracy of the result be verified by measuring to the next section corner west or north.

4. *To restore that one of a double corner established in subdividing the township when both are missing.*

Retrace the section line which closed on the corner, and establish the section post at the intersection with the township line. Verify the result by measuring on the township line to noted objects.

The restored corner will be common to two sections south or east of the line.

5. *To restore one of a triple corner, on a range line when one at least remains standing.*

The one of the triple corner, established when the range line was run, is not a section corner.

First identify the existing corners, then establish the lost corner, according to the field notes, north or south of the existing corner, on the line, and verify the result.

If the field notes do not give the distances between the triple corners, retrace the section line closing on said corner.

6. *To restore a triple corner when all are lost.*

Rechain the range line, and retrace the section lines closing on the range line.

7. *To restore lost quarter section corners.*

1st. Except on those section lines which close on the north or west boundaries of a township, quarter section corners are equidistant between the two section corners. Hence, rechain the section line, then chain back one-half the distance.

2d. On township lines, where there may be double section corners, only one set of quarter section corners are actually marked in the field — those established when the exteriors are run half-way between the section corners established at the same time. These are restored as above.

The same will apply when there are triple corners.

3d. If the section line closes on the north or west boundary of a township, the quarter section corner must be established 40 chains of the original measurement from the last interior section corner.

8. *To restore lost township corners.*

1st. If the corner is common to four townships, retrace the township and range lines, and establish the corner at their intersection.

2d. If the corner is common only to two townships, as may be the case on the base line or standard parallels, retrace the base line or standard parallel from the last standing corner, if the lost corner is common to two townships north; but if the lost corner is common to two townships south, retrace also the range line.

9. To restore lost meander corners.

Retrace the lines which close upon the banks in the direction they were originally run.

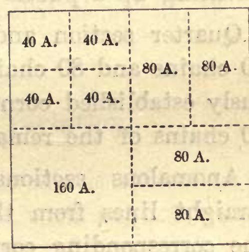
Fractional section lines closing on Indian boundaries, private grants, etc., should be retraced, and the corners established in the same manner.

Remark.— If, in restoring a lost corner, the original corner is found by some unmistakable trace, it must stand, and the resurvey be made to correspond.

271. Subdividing Sections.

The United States deputy runs only the exterior or section lines, and makes the section and quarter section corners.

Lines joining the opposite quarter section corners divide the section into quarter sections of 160 acres each.



These quarter sections are divisible into half-quarters of 80 acres, and these into quarter-quarters of 40 acres.

These are the legal subdivisions of a section, and are exhibited in the annexed diagram.

If private parties wish the subdivision lines traced on the ground, they employ the county surveyor, or a private surveyor, who must be governed by the section and quarter section corners previously established.

The following rules will enable the surveyor to subdivide a section in accordance with the laws of the United States:

1. The original section and quarter section corners must stand where they were established by the government surveyor.

2. The quarter-quarter corners must be established equidistant, and on the line between the section and quarter section corners of the exterior lines of the section, and equidistant and on the line between quarter section corners of internal lines of the section.

3. All subdivision lines must run straight from the proper corner in one exterior line of the section to the corresponding corner in the opposite exterior line.

4. In fractional sections, where no opposite corresponding corner has been established, the subdivision line must be run from the given corner due north and south, or east and west, to the exterior boundary of said fractional section.

5. Anomalous sections or sections larger than a mile, sometimes close on a previously established line, in finishing up a public survey.

Quarter section and section corners are established 40 chains and 80 chains, respectively, from the previously established corners, and posts are planted every 20 chains of the remaining distance.

Anomalous sections are subdivided by running straight lines from the corners on the south line to the corresponding corners on the north, and east, and west lines, the same as in regular sections.

VARIATION OF THE NEEDLE.

272. Definitions and Illustrations.

The variation of the needle is the angle which the magnetic meridian makes with the true meridian.

The variation is *east* or *west*, according as the north end of the needle is east or west of the true meridian.

The variation is different at different places, and it does not remain the same at the same place.

The line of no variation is that line traced through those points on the surface of the earth where the needle points due north.

At all places east of this line, the variation is west; and at all places west of this line, the variation is east.

West variation is designated by the sign *plus*, and east variation by the sign *minus*.

In the year 1840, at a point whose latitude is 40° 53′, and longitude 80° 13′, being a little S. E. of Cleveland, O., the variation was nothing. The line of no variation passed through this point N. 24° 35′ W., and S. 24° 35′ E.

273. Changes of Variation.

1. **Irregular changes.** — The needle is subject to sudden changes coincident, in time, with a thunder storm, an aurora borealis, solar changes, etc.

2. **Diurnal changes.** — In the northern hemisphere, the north end of the needle moves from 10′ to 15′ west from about 8 A. M. to 2 P. M., and then gradually returns to its former position.

3. **Annual changes.**—The diurnal changes vary with the season, being about twice as great in the summer as in the winter.

4. **Secular changes.**—In addition to the above changes, there is a change of variation, in the same direction, running with considerable regularity through a period of about 234 years, as is indicated by observations at Paris.

In the United States, the north end of the needle was moving east from the earliest recorded observations till about the year 1810, since which time the movement has been west, at the rate, on an average, of about 5′ per annum.

We give the following tables of places, their latitude and longitude, and variation as it was in 1840, and the annual change of variation, from the tables prepared by Professor Loomis for the 39th and 42d volumes of Silliman's Journal:

Places near the Line of no Variation.

Places.	*Lat.*	*Lon.*	*Var.*	*An. Mo.*
A Point.	40° 53′	80° 13′	0° 00′	+ 4′.4
Cleveland, O.	41° 31′	81° 45′	—0° 19′	4′.4
Mackinaw.	45° 51′	84° 41′	—2° 08′	3′.9
Charlottesville, Va.	39° 02′	78° 30′	+ 0° 19′	3′.7

Assuming the annual motion uniform, and correctly found for 1840, the variation for any subsequent time can be found by multiplying the annual motion by the number of years since 1840, and taking the algebraic sum of the product and the variation at that date.

Places where the Variation was West.

Places.	*Lat.*	*Lon.*	*Var.*	*An. Mo.*
Point in Maine.	48° 00′	67° 37′	+19° 30′	+8′.8
Waterville, Me.	44° 27′	69° 32′	12° 36′	5′.7
Montreal.	45° 31′	73° 35′	10° 18′	5′.7
Burlington, Vt.	44° 27′	73° 10′	9° 27′	5′.3
Hanover, N. H.	43° 42′	72° 14′	9° 20′	5′.2
Cambridge, Mass.	42° 22′	71° 08′	9° 12′	5′.
Hartford, Conn.	41° 46′	72° 41′	6° 58′	5′.
Newport, R. I.	41° 28′	71° 21′	7° 45′	5′.
Geneva, N. Y.	42° 52′	77° 03′	4° 18′	4′.1
West Point.	41° 25′	74° 00′	6° 52′	4′.
New York City.	40° 43′	71° 01′	5° 34′	3′.6
Philadelphia.	39° 57′	75° 11′	4° 08′	3′.2
Buffalo, N. Y.	42° 52′	79° 06′	1° 37′	4′.1

Places where the Variation was East.

Places.	*Lat.*	*Lon.*	*Var.*	*An. Mo.*
Jacksonville, Ill.	39° 43′	90° 20′	−8° 28′	+2′.5
St. Louis, Mo.	38° 37′	90° 17′	8° 37′	2′.3
Nashville, Tenn.	36° 10′	86° 52′	6° 42′	2′.
Louisiana.	29° 40′	94° 00′	8° 41′	1′.4
Mobile, Ala.	30° 42′	88° 16′	7° 05′	1′.4
Tuscaloosa, Ala.	33° 12′	87° 43′	7° 26′	1′.6
Columbus, Ga.	32° 28′	85° 11′	5° 28′	2′.
Milledgeville, Ga.	33° 07′	83° 24′	5° 07′	2′.4
Savannah, Ga.	32° 05′	81° 12′	4° 13′	2′.7
Tallahassee, Fa.	30° 26′	84° 27′	5° 03′	1′.8
Pensacola, Fa.	30° 24′	87° 23′	5° 53′	1′.4
Logansport, Ind.	40° 45′	86° 22′	5° 24′	2′.7
Cincinnati, O.	39° 06′	84° 27′	4° 46′	3′.1

274. Methods of Ascertaining the Variation.

First establish a true meridian, which may be done

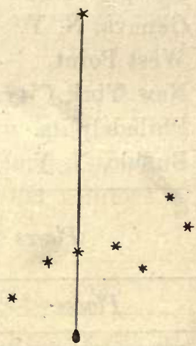
1. *By means of Burt's Solar Compass.*

2. *By observation of the North star, when on the meridian.*

The north star is about 1° 22′ from the true pole, around which it revolves in a siderial day, or 23 h., 56 m., 4 s.

Twice in this period the star will be on the meridian.

The exact moment of its passage can be determined very nearly, from the fact that it reaches the meridian almost at the same instant as Alioth in the tail of the Great Bear, or the first star in the handle of the Dipper.



Suspend a plumb line a few feet in front of the telescope, and place a faint light near the object glass of the telescope, so that the spider lines may be seen.

Just 17 minutes after the plumb line, the North star, and Alioth all fall on the vertical spider line, the North star is on the meridian.

The horizontal limb of the instrument is then firmly clamped, and the telescope is turned down horizontally.

A light, shining through a small aperture in a board, at some distance, say ten rods, is moved by an assistant, according to signals, till it ranges with the intersection of the spider lines.

A stake driven into the ground directly under the light, and another directly under the telescope, will mark, on the ground, the true meridian.

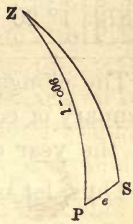
The season of the year may be such that Alioth may be above instead of below the North star, when both are on the meridian at night. With the telescope, the stars can be seen in the day-time.

3. *By the azimuth of the North star.*

When the North star is farthest from the meridian, east or west, it is said to be at its greatest eastern or western elongation.

The azimuth of a star is the angle which a vertical plane, through the star, makes with the meridian plane.

Let us now find the azimuth of the North star at its greatest elongation.



Let Z be the zenith, P the pole, S the North star at its greatest elongation, ZP, ZS, and PS arcs of great circles. Then ZPS will be a spherical triangle, right-angled at S, and the angle Z will be the azimuth, PS the greatest elongation, and ZP the complement of latitude; since the elevation of the pole above the horizon is equal to the latitude.

Now, from Napier's principles, we have

$$\sin e = \cos l \cos (90° - Z).$$

$$\therefore \quad \sin Z = \frac{\sin e}{\cos l}.$$

Introducing R and applying logarithms, we have

$$\log \sin Z = 10 + \log \sin e - \log \cos l.$$

Hence, the azimuth is readily computed if we know the greatest elongation of the star and the latitude of the place.

Greatest Elongation of Polaris.

Date.	*Elongation.*	*Date.*	*Elongation.*	*Date.*	*Elongation.*
1870	1° 23′ 01″.	1880	1° 19′ 50″.4	1890	1° 16′ 40″.7
1871	1° 22′ 41″.9	1881	1° 19′ 31″.4	1891	1° 16′ 21″.8
1872	1° 22′ 22″.9	1882	1° 19′ 12″.5	1892	1° 16′ 03″
1873	1° 22′ 03″.8	1883	1° 18′ 53″.5	1893	1° 15′ 44″.1
1874	1° 21′ 44″.8	1884	1° 18′ 34″.5	1894	1° 15′ 25″.3
1875	1° 21′ 25″.7	1885	1° 18′ 15″.5	1895	1° 15′ 06″.4
1876	1° 21′ 06″.6	1886	1° 17′ 56″.6	1896	1° 14′ 47″.6
1877	1° 20′ 47″.6	1887	1° 17′ 37″.6	1897	1° 14′ 28″.7
1878	1° 20′ 28″.5	1888	1° 17′ 18″.6	1898	1° 14′ 09″.9
1879	1° 20′ 09″.5	1889	1° 16′ 59″.7	1899	1° 13′ 51″

The elongation in the table is given for the 1st of January of each year; but the elongation for any month of the year can be readily found.

Thus, let us find the elongation for May 1st, 1873.

Jan. 1st, 1873, Elongation = 1° 22′ 03″.8
Jan. 1st, 1874, Elongation = 1° 21′ 44″.8

Change for 12 months = 19″
Change for 4 months = 6.3″

∴ Then, for May 1st, 1873, we shall have,

$$\text{Elongation} = 1° \, 22' \, 03''.8 - 6''.3 = 1° \, 21' \, 57''.5.$$

1. Find the azimuth of the North star at its greatest elongation, May 1st, 1873 — latitude 40°. *Ans.* 1° 47′.

2. Find the azimuth of the North star at its greatest elongation, July 1st, 1875 — latitude 42°. *Ans.* 1° 49¼′.

3. Find the azimuth of the North star at its greatest elongation, Sept. 21st, 1880 — latitude 45° 45′.

Ans. 1° 54¼′.

It will be necessary to know the times of the greatest elongation. These times are given in the following tables, for the 1st, 11th, and 21st of each month of the year 1880, which will answer the purpose for the rest of the century, since the change of time is very slow, being only about 16 minutes in 50 years.

Eastern Elongation.

Month.	1*st day.*	11*th day.*	21*st day.*
April.	6h. 40m. A.M.	6h. 01m. A.M.	5h. 22m. A.M.
May.	4h. 42m. A.M.	4h. 03m. A.M.	3h. 24m. A.M.
June.	2h. 41m. A.M.	2h. 01m. A.M.	1h. 22m. A.M.
July.	0h. 43m. A.M.	0h. 00m. A.M.	11h. 21m. P.M.
August.	10h. 38m. P.M.	9h. 59m. P.M.	9h. 19m. P.M.
Sept.	8h. 36m. P.M.	7h. 57m. P.M.	7h. 17m. P.M.

Western Elongation.

Month.	1*st day.*	11*th day.*	21*st day.*
Oct.	6h. 31m. A.M.	5h. 52m. A.M.	5h. 13m. A.M.
Nov.	4h. 30m. A.M.	3h. 50m. A.M.	3h. 11m. A.M.
Dec.	2h. 31m. A.M.	1h. 52m. A.M.	1h. 13m. A.M.
Jan.	0h. 28m. A.M.	11h. 44m. P.M.	11h. 04m. P.M.
Feb.	10h. 22m. P.M.	9h. 42m. P.M.	9h. 03m. P.M.
March.	8h. 31m. P.M.	7h. 52m. P.M.	7h. 13m. P.M.

About half an hour before the greatest eastern or western elongation, place the transit in a convenient position, and level it carefully.

Paste white paper on a board about one foot square, and perforate the board through the center with a two-inch auger, and, on the lower edge, fix some contrivance for holding a candle.

Let this board be fixed to a vertical staff, so as to slide freely up and down, and let it be placed about one foot in front of the telescope, so that the light reflected from the paper will render the spider lines visible.

Slide the board up or down the staff till the North star is visible through the telescope and orifice in the board, and bring the vertical spider line in range with the star.

As the star approaches its greatest elongation, move the telescope by a tangent screw, so as to keep the vertical line in range with the star. When the star reaches its greatest elongation, it will appear, for some time, to coincide with the spider line, and then leave it in the opposite direction.

Clamp the horizontal limb, and turn the telescope down till it is horizontal.

Let now a staff, with a light on its upper end, be carried ten or fifteen rods distant, toward the star, and placed so as to range, when vertical, with the vertical spider line of the telescope.

Drive a stake at the foot of the staff, and another directly under the instrument, then will the line determined by the stakes make an angle with the true meridian, equal to the azimuth of the North star. The true meridian will lie west or east of the line of stakes, north of the telescope, according as the elongation was east or west, and may readily be located by the instrument.

The location of the meridian can be verified thus:

Let AB be the line of the stakes produced to a considerable distance, say from 20 to 40 chains, A the azimuth angle, AC the true meridian, and BC perpendicular to AB.

BC can be found from the formula,

$$BC = AB \tan A.$$

Then laying off BC on the ground, and driving a stake at C, the stakes A and C will trace the true meridian.

Having found the true meridian, the variation of the needle can be readily determined by turning the telescope or the sights of the compass in the direction AC.

Without finding the true meridian, the bearing of AB being equal to the known azimuth of the North star at its greatest elongation, the variation of the needle can be found by directing the telescope or the sights of the compass in the direction AB.

The following method may be resorted to by the surveyor who does not possess an instrument with a telescope.

Fix a plank, firmly level, east and west, about three feet above the ground; then take a board about six inches square, and having detached one of the compass sights, fix it to the board, at right angles with its upper edge. Drive a nail obliquely a little way into the board, so that it can be tacked to the plank.

About fifteen feet north of the plank suspend a plumb line, from the top of an inclined stake of such height that the North star, when seen through the sight while the board rests on the plank, will appear about one foot below the upper end of the plumb line.

Suspend the plumb in a vessel of water to prevent the line from vibrating, and let an assistant hold a light near it, so that it can be seen through the sight.

About half an hour before the time of the greatest elongation of the North star, place the board on the plank, and slide so that the star and plumb line shall range when seen through the sight. As the star approaches its greatest elongation, move the board along the plank in the opposite direction, so as to keep the range.

When the star reaches its greatest elongation, it will appear to keep the range for several minutes, then it will move slowly in the opposite direction.

Tack the board to the plank, taking care not to change its position. Then let a staff with a light on its top be placed about ten rods farther to the north, so as to range, when vertical, through the sight, with the plumb line.

Drive a stake at the foot of the staff, and one directly under the plumb line, then will the line of the stakes make, with the meridian, an angle equal to the azimuth of the North star at its greatest elongation.

The true meridian, and the variation of the compass, can then be found as above.

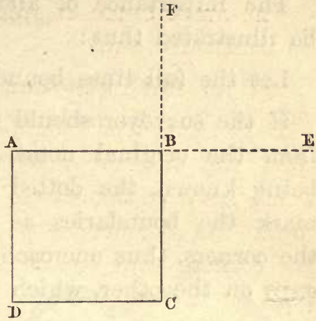
FIELD OPERATIONS.

275. Finding Corners.

In searching for a corner, first seek for the monument, whether tree, post, stake, or stone, as given and witnessed in the original field notes, which, if found, must be considered decisive in establishing the corner.

If no monument can be found, the corner can often be found by indirect methods, of which the following are the most available:

Thus, if a monument can be found at each of the corners *A*, *C*, *D*, but not at *B*, find the corners *E* and *F*, at each of which set up a flag-staff or high pole, and send the flag-man as near to *B* as possible, and let him stand facing *D*, so that he can see signals made both at *A* and *C*.



The observer at *A* can, by waving his hand, bring the flag-man in the line *AE*, and the observer at *C* can bring him in the line *CF*, and being in both lines, *AE* and *CF*, at the same time, he will be at their intersection *B*, the corner required.

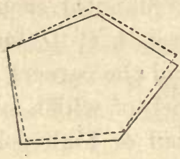
If the corner *E* can be found, but not *F*, measure *AB* the required distance in the line *AE*. If the distance *AB* is not known, but it is simply known that *AB* is equal to *DC*, first measure *DC*. If neither *E* nor *F* can be found, run *AB* parallel to *DC*, and *CB* parallel to *DA*, and the intersection of these lines will determine *B*, if the field is a parallelogram.

If the field is not a parallelogram, retrace one of the lines terminated by known corners, and compare the bearing with the bearing in the original notes, which will give the variation of the needle. Then run the lines *AB* and *CB* from the notes, allowing for the variation, and the intersection will determine *B*.

In like manner two or more lost corners may be found.

If the bearings and distances are given in the original notes, and but one corner can be found, retrace some established line in the neighborhood to find the variation, and, beginning at the known corner, run the lines from the notes, allowing for the variation.

The importance of allowing for the variation may be illustrated thus:



Let the full lines bound the lot.

If the surveyor should run this lot from the original notes, one corner being known, the dotted lines would mark the boundaries as run, and their intersections the corners, thus encroaching on one side, and leaving gaps on the other, which of course would never do.

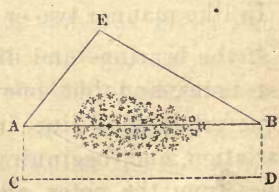
276. Finding Bearings and Distances.

After finding the corners, set a stake at each, and, beginning at any corner, place the compass or transit directly over the stake, and send the flag-man to the next corner, who must place the flag-staff on the stake.

Take the bearing, and measure the distance as heretofore directed; and, in like manner, find the bearings and distances of the remaining sides.

If obstacles should prevent the taking of the bearing of any line, measure the same distance from each corner, at right angles to the line, on the same side, so as to secure a line free from obstacles, and take the bearing of this line, which will be the bearing of the required line, since they are parallel.

Lines are measured a little to one side when fences, ponds, or other obstacles, are in the line.



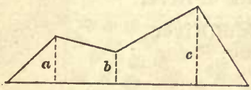
Thus, if the perpendiculars AC and BD are equal,

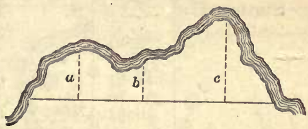
$$CD = AB.$$

AB can be found by Trigonometry, if AE and EB and two angles be measured.

277. Offsets.

Offsets are perpendiculars measured from a line to the angles of a neighboring broken line, or to the banks or centers of creeks, rivers, or other bodies of water. Thus, *a*, *b*, *c*.



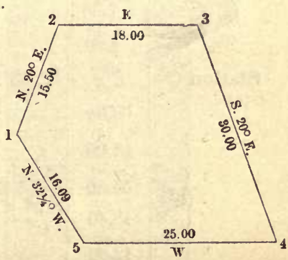


278. Taking Field Notes.

First Method.

Sta.	*Bearings.*	*Dist.*
1	N. 20° E.	15.50
2	E.	18.00
3	S. 20° E.	30.00
4	W.	25.00
5	N. $32\frac{1}{4}$° W.	16.09

Second Method.



The first method is in the proper form for calculation, and may be conveniently employed when it is not important to make a map of the lot surveyed.

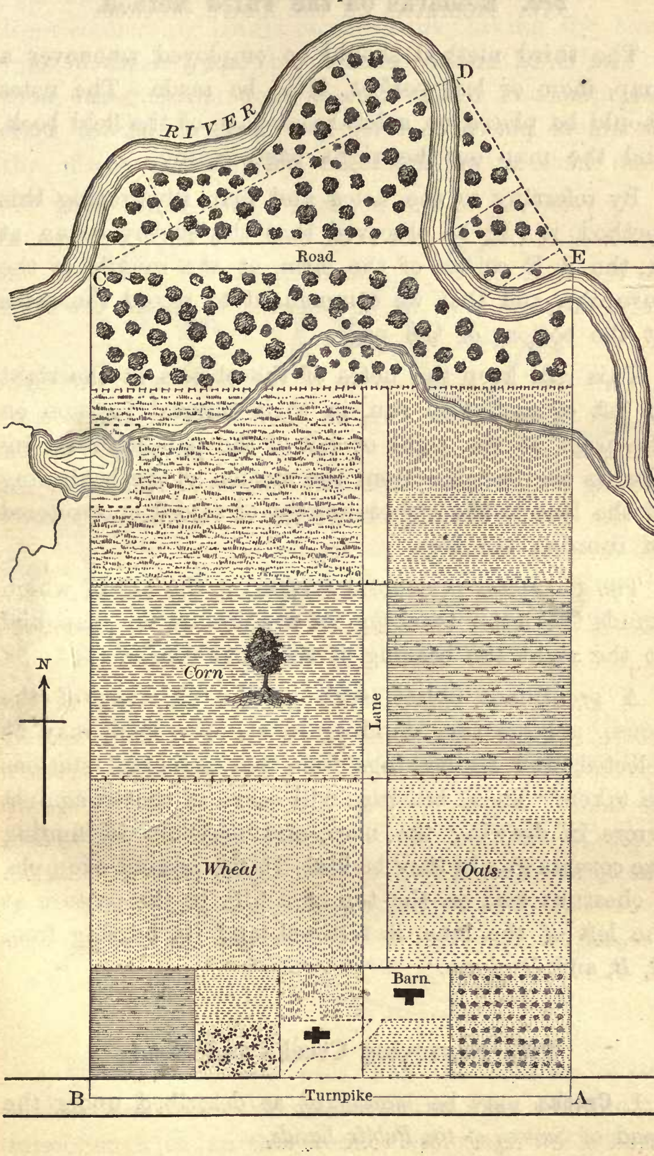
The second method, being a random outline with bearings and distances indicated, may be employed when it is desirable for the surveyor to keep before him, while at work, an outline of the lot.

		Offset	Bearing	
	68.00			Station A
	57.60			Orchard fence
	42.00			Oatfield fence
	26.00			Meadow fence
	14.00			S. Bank of Creek
	13.20			N. Bank of Creek
	10.80			Pasture fence
	4.80			S. Bank of River
Station E	△		S.	Left Bank of River
	18.40			N. Bank of River
	17.40	9.60		Offset
	16.40	9.60		Offset
	10.50			
Station D	△		S. 32°E.	Left Bank of River
	34.00			Left Bank of River
	30.00			Right Bank of River
	26.00	6.		Offset
	16.40	2.		Offset
	7.30	5.60		Offset
	4.80			N. Line of Road
Station C	△		N. 63°E.	Right Bank of River
	68.00			Road East
	58.00	4.		Woods
	55.20	6.80		Pond
	48.40	4.		
	42.00			Pasture fence
	26.00			Cornfield fence
	10.52			Wheatfield fence
Station B	△		N.	Middle of Turnpike
	40.00			Lot Line.
	31.20			Meadow fence
	24.00			Grove fence
	17.20			Dooryard fence
	10.08			Orchard fence
Station A	△		W.	Middle of Turnpike

S. 18° W.
N. 24° E.
N. 37° W.

MAP OF FARM

Scale 16 p. to 1 inch.



279. Remarks on the Third Method.

The third method should be employed whenever a map, more or less perfect, is to be made. The notes should be placed on a left-hand page of the field book, and the map on the right page, facing.

By referring to the notes and map illustrating this method, it will be observed that the survey began at A, the S. E. corner of the farm, at the middle of the turnpike, and that we commenced to record the notes at the bottom of the page.

This will keep the notes of the objects, at the right or left of each line run, in their natural position on the page, at the right or left of the parallel lines inclosing the distance from the station at the beginning of the line to the objects worthy of record encountered in running the line.

The character △ denotes station, at the left of which stands the letter marking its position on the map, and at the right the bearing of the next course.

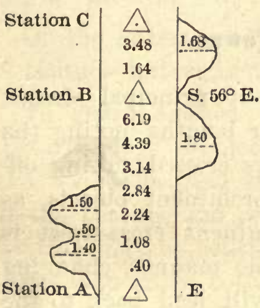
A prominent object, such as the chimney of the house, a large tree standing in an open field, may be selected, and its bearings from the principal stations be taken. These bearings will serve as checks against errors in drawing the map, and may aid in finding the corners should they be lost. In the present example, a chestnut tree on the top of a hill, in the pasture at the left of the lane, is selected, and its bearing from *A*, *B*, and *D* given.

280. Surveying Creeks and Roads.

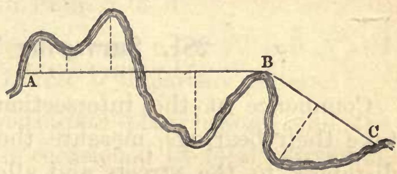
1. **Creeks** may be *meandered* as described under the head of *Survey of the Public Lands*.

2. They may also be surveyed by running straight lines connecting points on the bank, taking the bearings of these lines, the distances from the origin of these lines to the perpendicular offsets run from the lines to the bank of the river, and the length of the offsets, as exhibited in the following field notes and plot.

Field Notes.

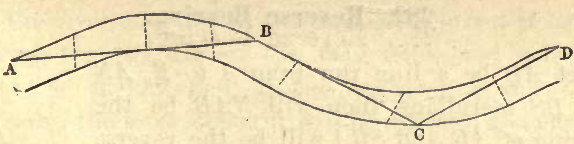


Plot.



The name of stations and the left-hand offsets are noted on the left of the parallels, the right-hand offsets and bearings on the right, the distance from the station to the offsets, and the sign for station, between the parallels.

3. In surveying an existing winding road, keep in the road, run straight lines as far as possible, without running out of the road, note the bearing of these lines, the distances to the offsets at different points to the sides of the road, the lengths of these offsets, and make an accurate plot of the road.



4. To survey a new road, find the bearing of the middle line from the origin to the next angle or intersection with another road, measuring the distance

from the origin to the lines of farms, creeks, etc., which it intersects.

Set temporary stakes at the angles, and at convenient distances along the middle line, to guide in making the road, and plant monuments at a given distance and bearing from the angular points, so that they will not be disturbed in making or working the road. Take notes, and make a correct plot of the road.

281. Surveying Towns.

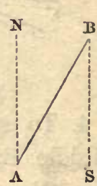
Commence at the intersection of principal streets, take their bearings, measure their lengths, noting the distances to the streets and alleys crossed, taking offsets to corners of streets and prominent objects, as public buildings, etc., till a prominent cross-street is reached, which survey in the same manner, changing the courses at such stations as will lead back to the original station.

Survey all the streets and alleys enclosed. Then survey an adjoining district, and so on, till the entire town or city has been surveyed.

Take notes, and make an accurate map of the town, on which locate not only the streets and alleys, but public buildings, parks, fountains, monuments, etc.

282. Reverse Bearing.

Let *AB* be a line run from *A* to *B*, *AN* and *BS* meridians, then will *NAB* be the bearing of *AB*, and *SBA* will be the reverse bearing.



Since the meridians *AN* and *BS* may be regarded as parallel, the bearing and reverse bearing are equal. Thus, if the bearing of *AB* is N. 30° E., the reverse bearing is S. 30° W.

The bearing and reverse bearing agree in the value of the angle, and differ in both the letters which indicate the general direction of the line. In fact, the reverse bearing of a line is the bearing of the line if run in the opposite direction. Thus, *SBA*, the reverse bearing of the line *AB*, run from *A* to *B*, is the bearing of the line *BA*, run from *B* to *A*.

Of the letters used in bearings, we shall call *N* and *S* latitude letters, and *E* and *W* departure letters.

To guard against inaccurate observations, and the disturbance of the needle occasioned by local attraction, the reverse bearing should be taken at every station. If the bearing and reverse bearing agree in value, the bearing may be considered as correctly taken; if they differ materially, both should be taken again. If they still differ, the difference may be regarded as occasioned by local attraction.

To ascertain at which station the local attraction exists, place the instrument at a third station, at a considerable distance from each of the doubtful stations, and sight to each, then from these back to the third station. The local attraction may be considered to exist at the station where the bearing of the third station disagrees with its bearing taken at the third station.

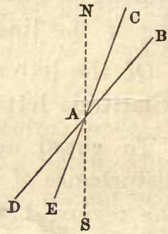
If the error occurred in the foresight, correct it before entering the bearing in the field notes, and note the amount of disturbance; if the error occurred in the backsight, the next foresight will be affected, and should be corrected before entered.

PRELIMINARY CALCULATIONS.

283. Angles between Courses.

1. *If the latitude letters are alike, also the departure letters, the included angle is equal to the difference of the bearings.*

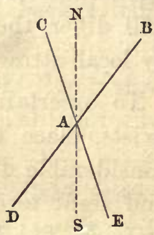
If AB bears N. 40° E., and AC N. 20° E., $BAC = BAN - CAN = 40° - 20° = 20°$.



If AD bears S. 40° W., and AE S. 20° W., $DAE = DAS - EAS = 40° - 20° = 20°$.

2. *If the latitude letters are alike, and the departure letters unlike, the included angle is equal to the sum of the bearings.*

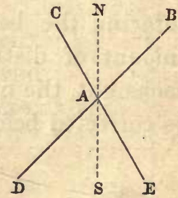
If AB bears N. 38° E., and AC N. 18° W., $BAC = BAN + NAC = 38° + 18° = 56°$.



If AD bears S. 38° W., and AE S. 18° E., $DAE = DAS + SAE = 38° + 18° = 56°$.

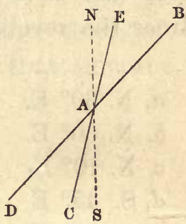
3. *If the latitude letters are unlike, and the departure letters alike, the included angle is equal to* 180° *minus the sum of the bearings.*

If AB bears N. 45° E., and AE S. 30° E., $BAE = 180° - (NAB + SAE) = 180° - 75° = 105°$.



If AD bears S. 45° W., and AC N. 30° W., $DAC = 180° - (DAS + CAN) = 180° - 75° = 105°$.

4. *If the latitude letters are unlike, also the departure letters, the included angle is equal to* 180° *minus the difference of the bearings.*



If AB bears N. 45° E., and AC S. 15° W., $BAC = 180° - (NAB - SAC) = 180° - 30° = 150°$.

If AD bears S. 45° W., and AE N. 15° E., $DAE = 180° - (SAD - NAE) = 180° - 30° = 150°$.

Remark.—These principles apply when both courses run *from* or *toward* the vertex; if one runs *from* the vertex, and the other *toward* it, reverse the bearing of one side before applying the principles.

284. Examples.

1. Find the angle A, if AB bears N. 78° E., and AC N. 24° E. *Ans.* 54°.

2. Find the angle A, if BA bears S. 34° E., and AC S. 48° W. *Ans.* 98°.

3. Find the angle A, if BA bears S. 70° W., and CA N. 25° E. *Ans.* 135°.

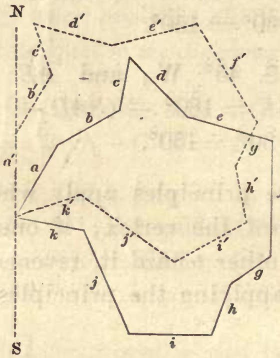
4. Find the angles of the polygon $ABCDE$, if AB bears N. 30° E.; BC, N. 60° E.; CD, S. 50° E.; DE, S. 40° W.; EA, N. 78° W.

$A = 72°$, $B = 150°$, $C = 110°$, $D = 90°$, $E = 118°$.

285. Problem.

Given the bearings of the sides of a field, to find the bearings if the field be supposed to revolve, so as to cause one of the sides to become a meridian.

In the following diagram let the full lines denote the original position of the sides of the field, *a* the side that is to become the meridian, and the dotted lines the revolved position of the sides.



a, N. 30° E.	*a′*, N.
b, N. 60° E.	*b′*, N. 30° E.
c, N. 10° E.	*c′*, N. 20° W.
d, S. 45° E.	*d′*, S. 75° E.
e, S. 75° E.	*e′*, N. 75° E.
f, S.	*f′*, S. 30° E.
g, S. 55° W.	*g′*, S. 25° W.
h, S. 20° W.	*h′*, S. 10° E.
i, W.	*i′*, S. 60° W.
j, N. 25° W.	*j′*, N. 55° W.
k, N. 80° W.	*k′*, S. 70° W.

From the above illustration we derive the following principles:

1. If the letters which indicate the general direction of the side which is to be made a meridian are both alike or both unlike those of another side, then,

1*st.* If the bearing of the former is less than that of the latter, the difference of the bearings will be the bearing of the latter, the letters remaining the same as before.

2*d.* If the bearing of the former is greater than that of the latter, the difference of the bearings will be the bearing of the latter, the departure letter being changed.

2. If one of the letters which indicate the general direction of the side which is to be made a meridian is like and the other unlike the corresponding letter of another side, then,

1st. The sum of the bearings, if less than 90°, will be the bearing of that side, the letters remaining the same as before.

2d. If the sum of the bearings is greater than 90°, its supplement will be the bearing of that side, the latitude letter being changed.

286. Examples.

1. The bearings of the sides of a field are as follows: 1st, N. 30° E.; 2d, N. 60° E.; 3d, S. 40° E.; 4th, S. 30° W.; 5th, W.; 6th, N. 18¾° W. Find the bearings of the sides if the second side becomes a meridian.

Ans. 1st, N. 30° W.; 2d, N.; 3d, N. 80° E.; 4th, S. 30° E.; 5th, S. 30° W.; 6th, N. 78¾° W. End

2. The bearings of the sides of a field are as follows: 1st, N. 45° W.; 2d, N. 18° E.; 3d, E.; 4th, N. 32° E.; 5th, S. 42½° E.; 6th, S.; 7th, S. 65¼° W. Find the bearings if the first side be made a meridian.

Ans. 1st, N.; 2d, N. 63° E.; 3d, S. 45° E.; 4th, N. 77° E.; 5th, S. 2½° W.; 6th, S. 45° W.; 7th, N. 69¾° W.

3. The bearings of the sides of a field are as follows: 1st, N. 20° E.; 2d, N. 70° E.; 3d, E.; 4th, S. 45° E.; 5th, S.; 6th, S. 45° W.; 7th, W.; 8th, N. 3¾° W. Find the bearings if the sixth side be made a meridian.

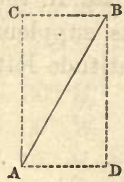
Ans. 1st, N. 25° W.; 2d, N. 25° E.; 3d, N. 45° E.; 4th, E.; 5th, S. 45° E.; 6th, S.; 7th, S. 45° W.; 8th, N. 48¾° W.

287. Latitude and Departure.

The latitude of a course is the distance between the two parallels of latitude passing through the extremities of the course.

The departure of a course is the distance between the two meridians passing through the extremities of the course.

Let AB be a course, AD and BC parallels of latitude, and AC and BD meridians. Then will AC or DB be the latitude of the course, and CB or AD its departure.



But $AC = AB \times \cos CAB$,
and $CB = AB \times \sin CAB$.

Hence, *latitude* = *course* × *cosine of bearing*,
and *departure* = *course* × *sine of bearing*.

If the line runs due east or west, its latitude is 0.

If the line runs due north or south its departure is 0.

Latitude *north* is considered *plus;* latitude *south*, *minus.*

Departure *east* is considered *plus;* departure *west*, *minus.*

For brevity let us designate the bearing by b, the course by c, the latitude by l, and departure by d, then we shall have the cases given in the following article:

288. Table of Cases.

	Given.	*Req.*	*Formulas.*	
1	$b, c,$	$l, d.$	$l = c \cos b,$	$d = c \sin b.$
2	$b, l,$	$c, d.$	$c = \frac{l}{\cos b},$	$d = l \tan b.$
3	$b, d,$	$c, l.$	$c = \frac{d}{\sin b},$	$l = \frac{d}{\tan b}.$
4	$c, l,$	$b, d.$	$\cos b = \frac{l}{c},$	$d = \sqrt{c^2 - l^2}.$
5	$c, d,$	$b, l.$	$\sin b = \frac{d}{c},$	$l = \sqrt{c^2 - d^2}.$
6	$l, d,$	$b, c.$	$\tan b = \frac{d}{l},$	$c = \sqrt{l^2 + d^2}.$

289. Examples.

1. Given $b =$ N. 53° 20′ E., and $c = 26.50$ ch.; required l and d. *Ans.* $l = 15.82$ ch. N., $d = 21.26$ ch. E.

2. Given $b =$ S. 75° 47′ W., and $l = 22.04$ ch. S.; required c and d. *Ans.* $c = 89.75$ ch., $d = 87$ ch. W.

3. Given $b =$ N. 35° W., and $d = 1.55$ ch. W.; required c and l. *Ans.* $c = 2.70$ ch., $l = 2.21$ ch. N.

4. Given $c = 35.35$ ch., and $l = 31$ ch. N.; required b and d.

Ans. $b =$ N. 28° 44′ E. or W., $d = 16.99$ ch. E. or W.

5. Given $c = 31.30$ ch., and $d = 22.89$ ch. W.; required b and l.

Ans. $b =$ N. or S. 47° W., and $l = 21.35$ ch. N. or S.

6. Given $l = 7.02$ ch. S., and $d = 7.14$ ch. W.; required b and c. *Ans.* $b =$ S. 45° 29′ W., $c = 10.01$ ch.

290. Traverse Table.

The traverse table affords a ready method of finding the latitude and departure of a course whose distance and bearing are given.

Let us find the l and d of a line whose b is N. 35° 15′ E., and $c = 47.85$ ch.

Turning to the traverse table, under 35° 15′ we find

$$c = 40 \text{ gives } l = 32.67,\ d = 23.09.$$
$$c = 7 \text{ gives } l = 5.72,\ d = 4.04.$$
$$c = .8 \text{ gives } l = .65,\ d = .46.$$
$$c = .05 \text{ gives } l = .04,\ d = .03.$$
$$\therefore\ c = 47.85 \text{ gives } l = 39.08,\ d = 27.62.$$

The l and d for 40 are found from the l and d of 4, as given in the table, by multiplying by 10, or removing the decimal point one place to the right.

The l and d for the distance 7 are given in the table, but the right hand figure is dropped, and 1 is carried if the figure dropped exceeds 5.

The l and d for the distance .8 are found from the l and d for the distance 8 by removing the decimal point one place to the left, rejecting the figures at the right of the second decimal place, carrying as above.

For the distance .05, remove the decimal point two places to the left, reject and carry as before.

If the bearing exceeds 45°, the l and d will be found in columns marked at the bottom of the page.

291. Examples.

1. Given $b =$ N. 28° 45′ E., and $c = 35.35$ ch.; required l and d. *Ans.* $l = 30.98$ ch. N., $d = 17$ ch. E.

2. Given $b =$ S. $36\frac{3}{4}$° E., and $c = 19.36$ ch.; required l and d. *Ans.* $l = 15.51$ ch. S., $d = 11.59$ ch. E.

3. Given $b =$ N. 53° 15′ E., $c = 11.60$ ch.; required l and d. *Ans.* $l = 6.94$ ch. N., $d = 9.29$ ch. E.

4. Given $b =$ S. $74\frac{1}{2}$° E., $c = 30.95$ ch.; required l and d. *Ans.* $l = 8.27$ ch. S., $d = 29.83$ ch. E.

5. Given $b =$ N. $33\frac{1}{4}$° W., $c = 37$ ch.; required l and d. *Ans.* $l = 30.94$ ch. N., $d = 20.29$ ch.W.

6. Find the l and d of the sides of a lot of which the following are the field notes: Commencing at the most westerly station, and running thence N. 52° E., 21.28 ch.; thence S. $29\frac{3}{4}$° E., 8.18 ch.; thence S. $31\frac{3}{4}$° W., 15.36 ch.; thence N. 61° W., 14.48 ch., to the point of beginning.

The work is written thus:

Sta.	*Bearings.*	*Dist.*	*N. Lat.*	*S. Lat.*	*E. Dep.*	*W. Dep.*
1	N. 52° E.	21.28	13.10		16.77	
2	S. 29¾° E.	8.18		7.11	4.06	
3	S. 31¾° W.	15.36		13.06		8.08
4	N. 61° W.	14.48	7.02			12.67

292. Balancing the Work.

It is evident that in passing around a field to the point of beginning, we have gone just as far north as south, and just as far east as west. Hence, the sum of the northings should be equal to the sum of the southings, and the sum of the eastings to the sum of the westings.

In practice, however, this is seldom the case, owing to the fact that the bearings are taken only to quarter degrees, and that the chaining is not perfectly correct.

It is not a settled point among surveyors how great an error in latitude or departure can be allowed without resurveying the lot. Some would admit an error of 1 link for every 10 chains in the sum of the courses; others, 1 link for every 3 chains. Each surveyor must settle this point for himself by ascertaining, by experience, how nearly he can make his work balance.

When an error is as likely to occur in one course as in another, the errors of latitude and departure are distributed among the courses in proportion to their length.

It will not, in general, be necessary to make all the proportions, for after making one for latitude and one for departure, the remaining corrections can be made by a comparison of distances.

Let us take example 6 of the last article.

Sta.	Bearings.	Dist.	N Lat.	S Lat.	E Dep.	W Dep.	CNL.	CSL.	CED.	CWD.
1	N.52°E.	21.28	13.10		16.77		13.12		16.74	
2	S.29¾°E.	8.18		7.11	4.06			7.10	4.05	
3	S.31¾°W.	15.36		13.06		8.08		13.05		8.10
4	N.61°W.	14.48	7.02			12.67	7.03			12.69
		59.30	20.12	20.17	20.83	20.75	20.15	20.15	20.79	20.79

Error in Lat. = 20.17 — 20.12 = .05.
Error in Dep. = 20.83 — 20.75 = .08.

Corrections for Latitude.	*Corrections for Departure.*
59.30 : 21.28 :: .05 : .02.	59.30 : 21.28 :: .08 : .03.
59.30 : 8.18 :: .05 : .01.	59.30 : 8.18 :: .08 : .01.
59.30 : 15.36 :: .05 : .01.	59.30 : 15.36 :: .08 : .02.
59.30 : 14.48 :: .05 : .01.	59.30 : 14.48 :: .08 : .02.

The corrections are made to the nearest link or hundredth.

Since the north latitude is too small, and the south latitude too great, add to each north latitude the corresponding correction, and subtract from the south latitude. In a similar manner correct the departure.

If one side is much more difficult to measure than the remaining sides, it is to be presumed that the error occurred chiefly in measuring that side, and the corrections should be made accordingly.

If, in taking one bearing, the object could not be distinctly seen, the error probably occurred in that bearing; then correct mainly in the latitude and departure of that course.

In practice it will not be necessary to make additional columns for the corrected latitude and departure, since they may be written in the same columns, over the others, with different colored ink.

293. Examples.

1. Find the l and d, and balance the work from the following notes:

1st, N. $34\frac{1}{4}$° E., 8.19 ch.; 2d, N. 85° E., 3.84 ch.; 3d, S. $56\frac{3}{4}$° E., 6.60 ch.; 4th, S. $34\frac{1}{4}$° W., 10.59 ch.; 5th, N. 56° W., 9.60 ch.

2. Find the l and d, and balance the work from the following notes:

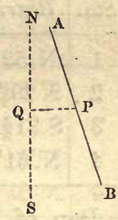
1st, N. 5° E., 22.50 ch.; 2d, S. 83° E., 12.96 ch.; 3d, N. 50° E., 19.20 ch.; 4th, S. 32° E., 32.76 ch.; 5th, S. 41° W., 12.60 ch.; 6th, W., 16.86 ch.; 7th, N. 79° W., 21.84 ch.

3. Find the balanced l and d of the following:

1st, N. 30° E., 10 ch.; 2d, N. 60° E., 18.18 ch.; 3d, S. 40° E., 20.10 ch.; 4th, S. 30° W., 24.50 ch.; 5th, W, 15 ch.; 6th, N. $18\frac{3}{4}$° W., 19.92 ch.

294. Double Meridian Distance.

The double meridian distance of a course is double the distance of its middle point from a given meridian.

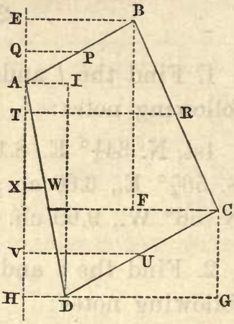


Let AB be a given course, NS the given meridian, P the middle point of AB, PQ perpendicular to NS.

Then will $2\,QP$ be the double meridian distance of AB.

In the following illustration we shall assume that the meridian of reference passes through the most westerly station, which we shall call the principal station, that departures *east* are *plus*, and *west*, *minus*, that the lines were run in the direction

ABCD, so as to keep the field on the right.



The following relations can be verified from the diagram:

1. $2QP = EB.$
2. $2TR = 2QP + EB + FC.$
3. $2VU = 2TR + FC + (-GD).$
4. $2XW = 2VU + (-GD) + (-IA) = AI.$

1. *The double meridian distance of the first course is equal to its departure.*

2. *The double meridian distance of the second course is equal to the double meridian distance of the first course, plus the departure of the first course, plus the departure of the second course.*

3. *The double meridian distance of any course is equal to the double meridian distance of the preceding course, plus the departure of that course, plus the departure of the given course.*

4. *The double meridian distance of the last course is equal to its departure with its sign changed.*

Take the example of a preceding article, as balanced.

Sta.	*Bearings.*	*Dist.*	*NLat.*	*SLat.*	*EDep.*	*WDep.*	*DMD.*
1	N. 52° E.	21.28	13.12		16.74		16.74
2	S. 29¾° E.	8.18		7.10	4.05		37.53
3	S. 31¾° W.	15.36		13.05		8.10	33.48
4	N. 61° W.	14.48	7.03			12.69	12.69

Dep. of 1st course = 16.74 = *D.M.D.* of 1st course.
+ dep. of 1st course = 16.74
+ dep. of 2d course = 4.05
37.53 = *D.M.D.* of 2d course.

+ dep. of 2d course =	4.05
	41.58
+ dep. of 3d course =	− 8.10
	33.48 = *D.M.D.* of 3d course.
+ dep. of 3d course =	− 8.10
	25.38
+ dep. of 4th course =	− 12.69
	12.69 = *D.M.D.* of 4th course.

The principal or most westerly station is not always the first station in the field notes.

It will be observed that the word *plus*, in the above principles and illustrations, is used in the algebraic sense, that *east* departure is considered *plus* and *west* departure *minus*; that plus, an east departure, is a plus quantity, and plus a west departure a minus quantity; and that the double meridian distance of the last course is equal to its departure with its sign changed, which will serve as a verification of the work.

The first station of the notes, in the preceding example, is the most westerly, and was therefore taken for the principal station.

The most westerly station can readily be determined by inspecting the bearings of the courses as given in the field notes, and should be taken as the principal station, and the corresponding course as the first course in finding the double meridian distances.

295. Examples.

1. Given the following field notes:

1st, N. 30° E., 10 ch.; 2d, N. 60° E., 18.18 ch.; 3d, S. 40° E., 20.10 ch,; 4th, S. 30° W., 24.50 ch.; 5th, W., 15 ch.; 6th, N. 18° 45′ W., 19.92 ch.: Required the latitude and departure; balance the work, and find the double meridian distances.

2. Given the following field notes:

1st, N. 45° W., 20 ch.; 2d, N. 18° E., 12.25 ch.; 3d, E., 12.80 ch.; 4th, N. 32° E., 6.50 ch.; 5th, S. $42\frac{1}{2}$° E., 13.20 ch.; 6th, S., 14.75 ch.; 7th, S. $65\frac{1}{4}$° W., 16.30 ch.: Required the corrected latitude and departure, and the double meridian distances.

AREA OF LAND.

296. Table of Linear Measure.

Mi.	*Ch.*	*Rds.*	*Yds.*	*Ft.*	*Lks.*	*In.*
1 =	80 =	320 =	1760 =	5280 =	8000 =	63360.
	1 =	4 =	22 =	66 =	100 =	792.
		1 =	$5\frac{1}{2}$ =	$16\frac{1}{2}$ =	25 =	198.
			1 =	3 =	$4\frac{6}{11}$ =	36.
				1 =	$1\frac{17}{33}$ =	12.
					1 =	$7\frac{23}{25}$.

297. Table of Superficial Measure.

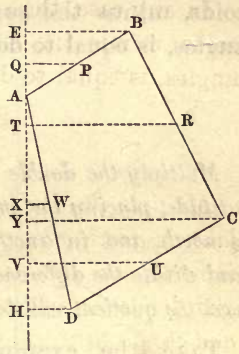
Mile.	*Acres.*	*Roods.*	*Chains.*	*Perches.*	*Links.*
1 =	640 =	2560 =	6400 =	102400 =	64000000.
	1 =	4 =	10 =	160 =	100000.
		1 =	$2\frac{1}{2}$ =	40 =	25000.
			1 =	16 =	10000.
				1 =	625.

Note 1.—It should be remembered that in finding the area of a tract of land the inequalities of its surface are not considered, but the tract is treated as a horizontal plane.

Note 2.—The area of a portion of land can, in a great variety of cases, be calculated by the rules already given for *Mensuration of Plane Surfaces.*

298. Problem.

To find the area of a tract of land when the length and direction of the bounding lines are given.



It is evident from the diagram that the area of *ABCD* is equal to the sum of the trapezoids *EBCY* and *YCDH*, minus the sum of the triangles *AEB* and *ADH;* and that twice the sum of the trapezoids, minus twice the sum of the triangles, is equal to twice *ABCD.*

The following table will exhibit the general form of operation:

Sta.	*Cour.*	*NLat.*	*SLat.*	*DMD.*	*Triangles.*	*Trapezoids.*
1	*AB*	*AE*		$2QP$	$2QP \times AE$	
2	*BC*		*EY*	$2TR$		$2TR \times EX$
3	*CD*		*YH*	$2VU$		$2VU \times XH$
4	*DA*	*HA*		$2XW$	$2XW \times HA$	

It will be observed that we have taken the most westerly station for the principal station, and have multiplied the double meridian distance of each course by its latitude, and that the product is double the area of a triangle when the latitude is north, and double the area of a trapezoid when the latitude is south.

If we had taken the most easterly station for the principal station, the reverse would be true.

In the above we have supposed that the lines were run in such direction as to keep the lot at the right.

If the lines were run in the opposite direction, so as to keep the lot at the left, the reverse would be true.

In any case, the sum of the double areas of the trapezoids, minus the sum of the double areas of the triangles, is equal to double the area required.

299. Rule.

Multiply the double meridian distance of each course by its latitude, placing the product in one column when the latitude is north, and in another column when the latitude is south, and divide the difference of the sums of the two columns by 2, and the quotient will be the area required.

Take the example of a preceding article whose *D. M. D.'s* have been found.

Sta.	*Bearings.*	*Dist.*	*NLat.*	*SLat.*	*EDep.*	*WDep.*	*DMD.*	*Triang.*	*Trap.*
1	N.52°E.	21.28	13.12		16.74		16.74	219.6288	
2	S.29¾°E.	8.18		7.10	4.05		37.53		266.4630
3	S.31¾°W.	15.36		13.05		8.10	33.48		436.9140
4	N.61°W.	14.48	7.03			12.69	12.69	89.2107	
								308.8395	703.3770

Area = 19 A. 2 R. 36 P.

703.3770
308.8395
2)394.5375
10)197.26875
19.726875
4
2.907500
40
36.300000

Triangles.	*Trapezoids.*
$16.74 \times 13.12 = 219.6288$.	$37.53 \times 7.10 = 266.4630$.
$12.69 \times 7.03 = 89.2107$.	$33.48 \times 13.05 = 436.9140$.

Divide double the area by 2, the result by 10 to reduce the chains to acres, multiply the decimal by 4 to reduce to roods, and the next decimal by 40 to reduce to perches.

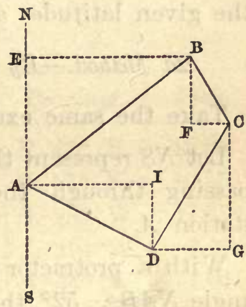
300. Plotting.

Plotting is the process of representing, to a given scale, the length, direction, and relative position of the bounding lines of a tract of land.

1*st Method.—By means of latitudes and departures.*

Take the example of the last article.

Let *NS* represent the meridian passing through the principal station *A*.



Select a scale whose unit shall represent 1 ch., and take $AE =$ 13.12 ch., the lat. of first course.

Through *E* draw a line perpendicular to *NS;* take $EB = 16.74$ ch., the dep. of first course, and draw *AB*.

Through *B* draw a meridian, and take $BF = 7.10$, the lat. of second course.

Through *F* draw a line perpendicular to *BF;* take $FC = 4.05$ ch., the dep. of second course, and draw *BC*.

Through *C* draw a meridian, and take $CG = 13.05$, the lat. of third course.

Through *G* draw a line perpendicular to *CG*, and take $GD = 8.10$ ch., the dep. of third course, and draw *CD*.

Through *D* draw a meridian, and take $DI = 7.03$ ch., the lat. of fourth course.

Through *I* draw a line perpendicular to *DI;* take *IA* $= 12.69$ ch., the dep. of fourth course, and draw *DA*.

Remark 1.—If the departure of fourth course terminates at *A*, the work will be verified.

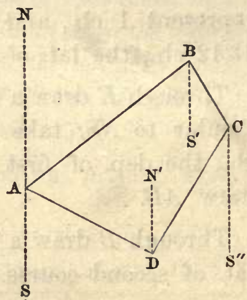
2. It will be observed that N. lat. is laid off upward, S. lat. downward, E. dep. to the right, and W. dep. to the left.

3. The auxiliary lines can be drawn with a pencil and afterward erased.

4. If every scale in possession of the surveyor should make the diagram too large or too small, all the latitudes and departures can be divided or multiplied by the same number, and the results taken instead of the given latitudes and departures.

2d Method.—By means of bearings and distances.

Take the same example.



Let NS represent the meridian passing through the principal station A.

With a protractor lay off the angle $NAB = 52°$, the bearing of first course, and take $AB = 21.28$ ch., the first course.

Through B draw a meridian, and lay off $S'BC = 29\frac{3}{4}°$, the bearing of second course, and take $BC = 8.18$ ch., the second course.

Through C draw a meridian, and lay off $S''CD = 31\frac{3}{4}°$, the bearing of third course, and take $CD = 15.36$ ch., the third course.

Through D draw a meridian, and lay off $N'DA = 61°$, the bearing of fourth course, and take $DA = 14.48$ ch., the fourth course, which will terminate at A if the work is correct.

Remark 1.—The latitude and departure letters indicate the general direction of the lines, and the degrees the exact direction.

2. Let the examples of the following article be carefully plotted, and the area be found.

3. By a careful inspection of the bearings, the most westerly station can be found, which take for the principal station.

4. The distances are all given in chains.

301. Examples.

1.

Sta.	*Bearings.*	*Dist.*
1	N. 30° E.	10.
2	N. 60° E.	18.18
3	S. 40° E.	20.10
4	S. 30° W.	24.50
5	W.	15.
6	N. $18\frac{3}{4}$° W.	19.92

Ans. 80 A. 1 R. 25 P.

2.

Sta.	*Bearings.*	*Dist.*
1	N. 47° E.	15.65
2	S. 57° E.	10.55
3	S. $28\frac{3}{4}$°W.	17.67
4	S. $29\frac{1}{4}$°W.	1.11
5	S. 54° W.	1.04
6	N. $40\frac{1}{2}$°W.	15.90

Ans. 23 A. 0 R. 38 P.

3.

Sta.	*Bearings.*	*Dist.*
1	N. 45° W.	20.
2	N. 18° E.	12.25
3	E.	12.80
4	N. 32° E.	6.50
5	S. $42\frac{1}{2}$°E.	13.20
6	S.	14.75
7	S. $65\frac{1}{4}$° W.	16.30

Ans. 58 A. 3 R. 30 P.

4.

Sta.	*Bearings.*	*Dist.*
1	N. 58° E.	12.97
2	S. $27\frac{3}{4}$°E.	3.30
3	S. $85\frac{1}{4}$°E.	11.65
4	S. 19° E.	15.56
5	S. $66\frac{1}{2}$°W.	14.03
6	N. 64° W.	14.86
7	N. $15\frac{1}{2}$°W.	11.23

Ans. 45 A. 2 R. 5 P.

5.

Sta.	Bearings.	Dist.
1	N. 20° E.	12.20
2	N. 70° E.	15.50
3	E.	18.25
4	S. 45° E.	20.00
5	S.	20.00
6	S. 45° W.	20.00
7	W.	18.25
8	N. $30\frac{3}{4}$° W.	36.66

Ans. 188 A. 3 R. 20 P.

6.

Sta.	Bearings.	Dist.
1	S. 34° E.	4.56
2	S. $66\frac{1}{4}$° W.	13.84
3	N. $12\frac{3}{4}$° E.	12.15
4	N. $48\frac{1}{4}$° W.	12.30
5	N. $58\frac{3}{4}$° E.	9.92
6	N. $39\frac{1}{2}$° E.	5.22
7	S. $45\frac{1}{4}$° E.	18.63
8	S. $52\frac{1}{2}$° W.	10.76

Ans. 32 A. 2 R. 26 P.

7.

Sta.	Bearings.	Dist.
1	N. 30° E.	15.
2	N. 60° E.	15.
3	E.	15.
4	S. 60° E.	15.
5	S. 30° E.	15.
6	S.	15.
7	S. 30° W.	15.
8	S. 60° W.	15.
9	W.	15.
10	N. 60° W.	15.
11	N. 30° W.	15.
12	N.	15.

Ans. 251.9 A.+

8.

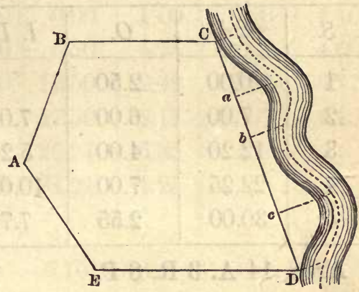
Sta.	Bearings.	Dist.
1	S. $76\frac{1}{2}$° E.	6.69
2	S. $14\frac{3}{4}$° W.	5.96
3	S. 38° E.	9.82
4	N. $30\frac{1}{2}$° E.	8.63
5	S. $73\frac{1}{4}$° E.	9.43
6	S. $10\frac{3}{4}$° W.	15.70
7	S. $42\frac{1}{2}$° W.	13.06
8	N. 64° W.	11.93
9	S. $79\frac{1}{4}$° W.	10.45
10	N. $22\frac{1}{2}$° W.	11.60
11	N. $37\frac{1}{4}$° E.	14.37
12	N. $22\frac{3}{4}$° E.	10.79

Ans. 76.14 A.—

302. Problem.

To find the area when offsets are taken.

Find the area of the tract of land bounded by the full lines and middle of the river, as shown in the annexed diagram.



Having run the stationary line *CD*, we have the following notes.

For ABCDE.

Sta.	*Bearings.*	*Dist.*
1	N. 20° E.	15.50
2	E.	18.00
3	S. 20° E.	30.00
4	W.	25.00
5	N. $32\frac{1}{4}$° W.	16.09

For Offsets.

Sta.	*Dist.*	*Offsets.*
1	0.00	2.50
2	7.00	6.00
3	12.20	4.00
4	22.25	7.00
5	30.00	2.55

Area = 70 A. 1 R. 33 P. + 14 A. 3 R. 8 P. = 85 A. 1 R. 1 P.

We find, as in the last article, *ABCDE* = 70 A. 1 R. 33 P.

To calculate the area included between the stationary line *CD* and the line passing along the middle of the river, we find $Ca = 7$, $ab = Cb - Ca = 12.20 - 7 = 5.20$, etc., which gives the altitudes of the trapezoids. The parallel sides are given under the head of offsets.

The altitude of a trapezoid multiplied by the sum of the parallel sides will give twice its area.

The calculation is made as in the subjoined table, the letters, *S.*, *S. D.*, *O.*, *I. D.*, *S. O.*, *D. T.*, heading the columns of the table, denoting stations, station distances or distances from *C*, offsets, intercepted distances, sum of offsets, and double trapezoids.

S.	*S. D.*	*O.*	*I. D.*	*S. O.*	*D. T.*
1	0.00	2.50			
2	7.00	6.00	7.00	8.50	59.5000
3	12.20	4.00	5.20	10.00	52.0000
4	22.25	7.00	10.05	11.00	110.5500
5	30.00	2.55	7.75	9.55	74.0125

Area, 14 A. 3 R. 8 P.

2) 296.0625
10) 148.03125
14.803125
4
3.212500
40
8.500000

If the offsets fall within the stationary line, the sum of the trapezoids must be subtracted.

In general, if the lines are run so as to keep the field on the right, the sum of the trapezoids must be added in case of left-hand offsets, and subtracted in case of right-hand offsets.

In case of navigable rivers, the bank is, in general, the boundary — the first and last offsets become 0, and the first and last trapezoids become triangles, but the form of the computation is the same.

303. Examples.

1. Find the area of the lot of which the following are the field notes, and make a plot of the survey.

Rectilinear Area.			*L.H. Offsets.**		*R.H. Offsets.***	
Sta.	*Bearings.*	*Dist.*	*St.Dist.*	*Offsets.*	*St.Dist.*	*Offsets.*
1	N. 45° E.	10.00	0.00	1.00	0.00	1.10
2	N.	10.00	6.50	4.25	5.62	4.00
3	N. 45° E.	10.00	12.50	2.43	12.62	5.27
4	E.	10.00	17.50	5.17	17.07	1.13
5*	S.	31.21	26.21	5.83		
6**	W.	17.07	31.21	1.25		
7	N. 45° W.	10.00				

55.774715 A. + 12.17075 A. − 6.10160 A. = 61 A. 3 R. 15 P.

The left-hand offsets were made from the fifth course, as indicated by the single star; and the right-hand offsets from the sixth course, as indicated by the double star.

2. Find the area of the lot of which the following are the field notes, and make a plot of the survey.

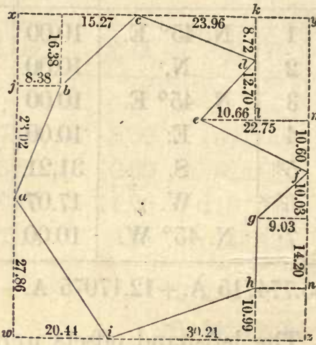
Rectilinear Area.			*L.H. Offsets.**		*R.H. Offsets.***	
Sta.	*Bearings.*	*Dist.*	*St.Dist.*	*Offsets.*	*St.Dist.*	*Offsets.*
1	N. 30° E.	20.	0.00	0.00	0.00	0.00
2	E.	20.	6.00	3.00	6.00	4.00
3*	S. 30° E.	20.	10.00	2.00	14.00	4.00
4**	S. 30° W.	20.	15.00	3.50	20.00	0.00
5	W.	20.	20.00	0.00		
6	N. 30° W.	20.				

Ans. 102 A. 1 R. 36 P.

304. Pogue's Method of Finding the Area.

This method is illustrated by the following example:

1. N. 20° E., 24.50 ch.
2. N. 43° E., 22.40 ch.
3. S. 70° E., 25.50 ch.
4. S. 40° W., 16.58 ch.
5. S. 65° E., 25.10 ch.
6. S. 42° W., 13.50 ch.
7. S., 14.20 ch.
8. S. 70° W., 32.15 ch.
9. N. 36$\frac{1}{4}$° W., 34.55 ch.



Make a plot from the field notes, draw meridians through the most easterly and westerly stations, and parallels of latitude through the most northerly and southerly, thus enclosing the whole figure in a rectangle.

Find, from the traverse table, the latitudes and departures as in diagram.

To find *xy*, pass from the most westerly station, round the north, to the most easterly, taking the sum of the eastings minus the sum of the westings; and to find *zw*, pass from the most easterly station, round the south, to the most westerly, taking the sum of the westings minus the sum of the eastings, thus:

$$xy = 8.38 + 15.27 + 23.96 - 10.66 + 22.75 = 59.70$$
$$zw = 9.03 + 30.21 + 20.44 = 59.68$$
$$2)\overline{119.38}$$
$$\tfrac{1}{2}(xy + zw) = \text{the average base} = 59.69$$

To find *wx*, pass from the most southerly station, round the west, to the most northerly, taking the sum of the northings minus the sum of the southings; and to find

yz, pass from the most northerly station, round the east, to the most southerly, taking the sum of the southings minus the sum of the northings, thus:

$$wx = 27.86 + 23.02 + 16.38 = 67.26$$
$$yz = 8.72 + 12.70 + 10.60 + 10.03 + 14.20 + 10.99 = 67.24$$
$$2)\underline{134.50}$$
$$\tfrac{1}{2}(wx + yz) = \text{the average altitude} = 67.25$$

Area of rectangle $= 59.69 \times 67.25 = 4014.1525$.

From the area of the rectangle we must deduct the area included between $wxyz$ and $abcdefghi$, thus found.

$$abj = \frac{23.02 \times 8.38}{2} = 96.4538$$
$$bjxc = \frac{8.38 + 8.38 + 15.27}{2} \times 16.38 = 262.3257$$
$$ckd = \frac{23.96 \times 8.72}{2} = 104.4656$$
$$del = \frac{12.70 \times 10.66}{2} = 67.6910$$
$$kyml = (8.72 + 12.70)(22.75 - 10.66) = 258.9678$$
$$emf = \frac{22.75 \times 10.60}{2} = 120.5750$$
$$fnhg = \frac{10.03 + 14.20 + 14.20}{2} \times 9.03 = 173.5114$$
$$hnzi = \frac{9.03 + 9.03 + 30.21}{2} \times 10.99 = 265.2436$$
$$iwa = \frac{20.44 \times 27.86}{2} = 284.7292$$
$$1633.9631$$

$$abcdefghi = 4014.1525 \text{ sq. ch.} - 1633.9631 \text{ sq. ch.}$$
$$= 2380.1894 \text{ sq. ch.} = 238.02 \text{ A.}$$

For additional exercises, work the examples of articles 301 and 303, and compare the answers obtained by the two methods.

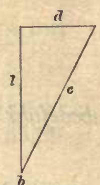
SUPPLYING OMISSIONS.

305. Case I.

When the bearing and length of one side are wanting.

The wanting side must be such that its latitude and departure will make the work balance. Hence, its latitude must be the difference between the sum of the northings and the sum of the southings of the given sides, and of the same name as the less; and its departure must be the difference between the sum of the eastings and the sum of the westings of the given sides, and of the same name as the less.

Having found the latitude and departure of the wanting side, construct a right-angle triangle by drawing on the paper, to represent the latitude, a line, up or down, according as the latitude is north or south; and at the terminus of the line, draw, to represent the departure, a horizontal line, to the right or left, according as the departure is east or west, and join the origin of the line representing the latitude with the terminus of the line representing the departure, and this last line will be the hypotenuse which will represent the course or length of the line sought, and the angle which it makes with the vertical line will be the bearing.



Denote the latitude by l, the departure by d, the course by c, and the bearing by b, then we have,

(1) $c = \sqrt{l^2 + d^2}$. (2) $\tan b = \frac{d}{l}$.

Having found the bearing and distance, enter them in the notes and find the area.

306. Examples.

Supply the omissions in the following field notes, calculate the areas, and plot the surveys.

1.

Sta.	*Bearings.*	*Dist.*
1	N. 18° E.	9.25
2	N. 71° E.	8.33
3	S. $43\frac{1}{4}$°E.	12.37
4	S. $36\frac{1}{2}$°W.	16.00
5	Wanting.	Want'g.

Ans. { N. 43° W., 14.18 ch.
23 A. 3 R. 32 P.

2.

Sta.	*Bearings.*	*Dist.*
1	N. 24° W.	15.50
2	N. 31° E.	17.07
3	E.	20.
4	Wanting.	Want'g.
5	S. 56° W.	30.30

Ans. { S. $12\frac{1}{2}$° E., 12.13 ch.
56 A. 3 R. 0 P.

307. Case II.

When the lengths of two sides are wanting.

Revolve the field so that one of the sides whose bearing only is given shall become a meridian, and find, by article 285, the bearings of all the sides in their new position.

The departure of the side made a meridian will then be 0, and the difference of the sums of the columns of the departures will be the departure, in the new position, of the other side whose distance is wanting.

Knowing the bearing and departure of this side, we can find its distance and latitude. Then the difference between the sums of the columns of latitudes will be the length of the side made a meridian.

Revolve the field to its original position, calculate its area, and make a plot of it; or, if the area only is required after supplying omissions, it may be computed more readily without revolving the field to its original position.

308. Examples.

1.

Sta.	*Bearings.*	*Dist.*
1	N. 30° E.	10.00
2	N. 60° E.	18.18
3	S. 40° E.	Want'g.
4	S. 30° W.	Want'g.
5	W.	15.00
6	N. 18¾° W.	19.92

Ans. { 3d. 20.08 ch.; 4th. 24.52 ch.; 80 A. 1 R. 25 P.

2.

Sta.	*Bearings.*	*Dist.*
1	N. 47° E.	15.65
2	S. 57° E.	10.55
3	S. 28¾° W.	Want'g.
4	S. 29¼° W.	1.11
5	S. 54° W.	1.04
6	N. 40½° W.	Want'g.

Ans. { 3d. 17.69 ch.; 6th. 16.01 ch.; 23 A. 1 R. 14 P.

309. Case III.

When the bearings of two sides are wanting.

If the sides whose bearings are wanting are separated from each other by one or more intervening sides, suppose one of these sides and a side adjacent to the other to change places, so as to bring the sides under consideration together without changing the bearings or lengths of the sides transposed.

Then, throwing these sides out of consideration, find, by Case I, the bearing and length of the line joining the extremities of the sides whose bearings are wanting.

This line with those sides form a triangle whose sides are known, from which the angles can be computed.

Knowing the angles and the bearing of one side, the bearings of the other sides can be found.

Restore to their original position the sides which have changed places, if such is the fact, calculate the area, and make a plot of the field.

310. Examples.

1.

Sta.	Bearings.	Dist.
1	N. 45° W.	20.00
2	N. 18° E.	12.25
3	E.	12.80
4	N. 32° E.	6.50
5	S. $42\frac{1}{2}$°E.	13.20
6	Wanting.	14.75
7	Wanting.	16.30

Ans. 6th. S.; 7th. S. $65\frac{1}{4}$° W.; 59 A.

2.

Sta.	Bearings.	Dist.
1	N. 58° E.	12.97
2	S. $27\frac{3}{4}$°E.	3.30
3	S. $85\frac{1}{4}$°E.	11.65
4	S. 19° E.	15.56
5	Wanting.	14.03
6	N. 64° W.	14.86
7	Wanting.	11.23

Ans. 5th. S. $66\frac{1}{2}$° W.; 7th. N. $15\frac{1}{2}$° W.; 45 A. 2 R. 5 P.

311. Case IV.

When the bearing of one side and the length of another are wanting.

Revolve the field so that the side whose bearing only is given shall become a meridian.

The departure of this side will then be 0, and the difference of the sums of the columns of departures will be the departure, in its new position, of the side whose bearing is wanting.

Knowing the length and departure of this side, its bearing and latitude can be found.

Then the difference of the sums of the columns of latitudes will be the length of the side made a meridian.

Revolve the field to its original position, compute the area and plot the work.

Remark 1.—In finding the bearing of the side whose distance only is given, though the angle can be readily found, the bearing, and consequently the latitude, may be either north or south, since either will comply with the condition. The length of the side whose bearing only is given will therefore be ambiguous, and there will be two solutions to the problem. If but one solution is admissible, the omission should be supplied by a remeasurement; and if the lost bearing or distance can not be taken directly, auxiliary lines may be run, and the omissions supplied by Trigonometry.

2. From the fact that two omissions can be supplied, the surveyor should not deem it unimportant to find all the measurements on the ground, since thus he can ascertain the correctness of his notes by balancing his work—a test not applicable when omissions are supplied.

312. Examples.

1.

Sta.	*Bearings.*	*Dist.*
1	N. 20° E.	12.20
2	N. 70° E.	15.50
3	E.	18.25
4	S. 45° E.	20.00
5	S.	20.00
6	Wanting.	20.00
7	W.	Want'g.
8	N. $30\frac{3}{4}$°W.	36.66

Ans. { 6th. S. 45° W.
7th. 18.25.
188 A. 3 R. 20 P.

2.

Sta.	*Bearings.*	*Dist.*
1	S. 34° E.	4.56
2	S. $66\frac{1}{4}$°W.	13.84
3	N. $12\frac{3}{4}$° E.	12.15
4	Wanting.	12.30
5	N. $58\frac{3}{4}$°E.	9.92
6	N. $39\frac{1}{2}$°E.	5.22
7	S. $45\frac{1}{4}$°E.	Want'g.
8	S. $52\frac{1}{2}$°W.	10.76

Ans. { 4th. N. $48\frac{1}{4}$° W.
7th. 18.63.
32 A. 2 R. 26 P.

LAYING OUT LAND.

313. Laying out Squares.

To lay out a given quantity of land in the form of a square.

Let a be the area of the square, and x one side.

$$\text{Then, } x^2 = a, \quad \therefore x = \sqrt{a}.$$

Reduce the given area to square chains, extract the square root, and the result will be the length of one side.

With the chain and transit lay out the square on the ground.

EXAMPLES.

1. Lay out 12 A. 3 R. 20 P. in the form of a square.

2. Find the side of a square containing 1 A., and lay out the square on the ground.

314. Laying out Rectangles.

1. *To lay out a given quantity of land in the form of a rectangle, one side of which is given.*

Let a be the area of the rectangle, b the given side, and x an adjacent side.

$$\text{Then, } bx = a, \quad \therefore x = \frac{a}{b}.$$

2. *To lay out a given quantity of land in the form of a rectangle whose length is to its breadth in a given ratio.*

Let a denote the area of the rectangle, x its length, y its breadth, and $m : n$ the ratio of x to y.

$$\text{Then, } \begin{cases} xy = a. \\ x : y :: m : n. \end{cases} \quad \therefore \begin{cases} x = \sqrt{\dfrac{am}{n}}. \\ y = \sqrt{\dfrac{an}{m}}. \end{cases}$$

3. *To lay out a given quantity of land in the form of a rectangle when the sum of its length and breadth is given.*

Let a be the area of the rectangle, x the length, y the breadth, and s the sum of x and y.

$$\text{Then, } \begin{cases} x + y = s. \\ xy = a. \end{cases} \quad \therefore \quad \begin{cases} x = \frac{1}{2}s + \frac{1}{2}\sqrt{s^2 - 4a}. \\ y = \frac{1}{2}s - \frac{1}{2}\sqrt{s^2 - 4a}. \end{cases}$$

4. *To lay out a given quantity of land in the form of a rectangle when the difference of the length and breadth is given.*

Let a denote the area of the rectangle, x its length, y its breadth, and d the difference of x and y.

$$\text{Then, } \begin{cases} x - y = d. \\ xy = a. \end{cases} \quad \therefore \quad \begin{cases} x = \frac{1}{2}\sqrt{d^2 + 4a} + \frac{1}{2}d. \\ y = \frac{1}{2}\sqrt{d^2 + 4a} - \frac{1}{2}d. \end{cases}$$

315. Examples.

1. The area of a rectangle is 3 A., one side is 4 ch. Find an adjacent side and lay out the rectangle.

2. The area of a rectangle is 8 A.; the length is to the breadth as 3 is to 2. Find the sides and lay out the rectangle. *Ans.* 10.95 ch. and 7.30 ch.

3. The area of a rectangle is 4.8 A.; the sum of the length and breadth is 14 ch. Find the sides and lay out the rectangle. *Ans.* 8 ch. and 6 ch.

4. The area of a rectangle is 18 A.; the difference of the length and breadth is 3 ch. Find the sides and lay out the rectangle. *Ans.* 15 ch. and 12 ch.

316. Laying out Parallelograms.

1. *To lay out a given quantity of land in the form of a parallelogram when the base is given.*

Let a be the area, b the base, and x the altitude.

$$\text{Then } bx = a, \quad \therefore \; x = \frac{a}{b}.$$

Measure the base, from any point of which erect a perpendicular equal to the calculated altitude.

Through the extremity of the perpendicular run a line parallel to the base, any point of which may be taken for one extremity of the upper base, which may then be measured off on this line.

2. *When one side and an adjacent angle are given.*

Let a be the area, b the given side, A the given angle, and x the other side adjacent to this angle.

$$\text{Then } bx \sin A = a, \quad \therefore \; x = \frac{a}{b \sin A}.$$

3. *When two adjacent sides are given.*

Let a be the area, b and c the given sides, and x their included angle.

$$\text{Then } bc \sin x = a, \quad \therefore \; \sin x = \frac{a}{bc}.$$

Remark.—If $bc = a$, then $\sin x = 1$, $x = 90°$, and the parallelogram becomes a rectangle.

If $bc < a$, the solution is impossible.

317. Examples.

1. The area of a parallelogram is 6 A., the base is 6 ch. Find the altitude and lay out the land.

2. The area of a parallelogram is 12 A., one side is 12 ch., and an adjacent angle is 60°. Find the other side adjacent to the given angle and lay out the land.

3. The area of a parallelogram is 8 A., two adjacent sides are 8 ch. and 12 ch. Find their included angle and lay out the land.

318. Laying out Triangles.

1. *To lay out a given quantity of land in the form of a triangle when the base is given.*

Let a denote the area, b the base, and x the altitude.

$$\text{Then, } \tfrac{1}{2}bx = a, \quad \therefore\ x = \frac{2a}{b}.$$

Measure the base, at any point of which erect a perpendicular equal to the calculated altitude.

Through the extremity of this perpendicular draw a line parallel to the base. This parallel will be the *locus* of the vertex, any point of which may be taken for the vertex.

2. *When the base is to the altitude in a given ratio.*

Let a denote the area, x the base, y the altitude, and $m : n$ the ratio of the base to the altitude.

$$\text{Then, } \left\{ \begin{array}{l} \tfrac{1}{2}xy = a. \\ x : y :: m : n. \end{array} \right\} \quad \therefore \quad \left\{ \begin{array}{l} x = \sqrt{\dfrac{2am}{n}}. \\ y = \sqrt{\dfrac{2an}{m}}. \end{array} \right.$$

3. *When the triangle is equilateral.*

Let a denote the area and x one side.

$$\text{Then, } .4330127\, x^2 = a, \quad \therefore\ x = \sqrt{\frac{a}{.4330127}}.$$

4. *When one side and an adjacent angle are given.*

Let a denote the area, b the given side, x the adjacent side, and A the included angle.

$$\text{Then, } \tfrac{1}{2}bx \sin A = a, \quad \therefore\ x = \frac{2a}{b \sin A}.$$

5. *When two sides are given.*

Let a denote the area, b and c the given sides, and x their included angle.

$$\text{Then, } \tfrac{1}{2} bc \sin x = a, \quad \therefore \sin x = \frac{2a}{bc}.$$

319. Examples.

1. The area of a triangle is 3 A., the base is 5 ch. Find the altitude and lay out the triangle on the ground.

2. The area of a triangle is 12 A., the base is to the altitude as 3 is to 2. Find the base and altitude and lay out the triangle on the ground.

3. The area of an equilateral triangle is 1 A. Find a side and lay out the triangle.

4. The area of a triangle is 1.2 A., one side is 2 ch., an adjacent angle is 45°. Find the other side adjacent to the given angle and lay out the land.

5. The area of a triangle is 2 A., two sides are 6 ch. and 10 ch. Find the included angle and lay out the triangle.

320. Laying out Circles or Regular Polygons.

1. Let a be the area of the circle, and x the radius.

$$\text{Then, } 3.1416\, x^2 = a, \quad \therefore x = \sqrt{\frac{a}{3.1416}}.$$

2. Let a be the area of a regular polygon, x one side, y one angle, n the number of sides, and a' the area of a similar polygon whose side is 1. Article 167.

$$\text{Then, } a'x^2 = a, \quad \therefore x = \sqrt{\frac{a}{a'}}. \quad y = \frac{180^\circ (n-2)}{n}.$$

321. Examples.

1. Find the radius of a circle whose area is 1 A. and lay out the circle.

2. Find the sides and angles of a regular hexagon containing 1 A. and lay out the hexagon.

3. Find the sides and angles of a regular octagon containing 1 A. and lay out the octagon.

DIVIDING LAND.

322. Division of Rectangles or Parallelograms.

1. *To cut off a given area by a line parallel to a given side.*

Let a be the area, b the given side, x the distance to be cut off on the sides adjacent to b, and A the acute angle of the parallelogram.

For the rectangle, $bx = a, \quad \therefore \; x = \frac{a}{b}$.

For the parallelogram, $bx \sin A = a, \quad \therefore \; x = \frac{a}{b \sin A}$.

2. *When the lot is to be divided into parts having a given ratio, by lines parallel to two of the sides, divide the other sides into parts having the same ratio.*

323. Examples.

1. The sides of a rectangle are 15 ch. and 10 ch.; cut off 8 A. by a line parallel to the shorter sides.

2. The adjacent sides of a parallelogram are 12 ch. and 20 ch., and their included angle is 65°; cut off 10 A. by a line parallel to the shorter sides.

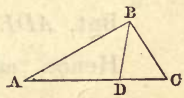
3. A man willed that his farm, which was 1 mile long and ½ mile wide, be divided among his three

sons, A, B, and C, aged 21 yrs.. 18 yrs.. 15 yrs., respectively, in proportion to their ages, by lines parallel to the shorter sides. Make the divisions.

324. Division of Triangles.

1. *To find a point on a given side of a triangle from which a line drawn to the vertex of the opposite angle will divide the triangle into parts having a given ratio.*

Let $b = AC$, the given side;
D, the required point; $x = AD$,
and $ABD : DBC :: m : n$.



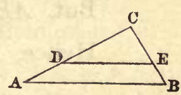
By composition we have,

$ABC : ABD :: m + n : m$; but $ABC : ABD :: b : x$.

$$\text{Hence, } m + n : m :: b : x, \quad \therefore x = \frac{bm}{m+n}.$$

2. *Two sides of a triangle being given, to divide the triangle into parts having a given ratio by a line parallel to the third side.*

Let $a = BC$, $b = AC$, the given sides;
$x = CE$, $y = CD$,
and $DEC : ABED :: m : n$.



By composition we have,

$$ABC : DEC :: m + n : m;$$

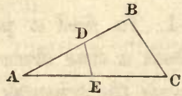
$$\text{but } ABC : DEC :: a^2 : x^2 :: b^2 : y^2.$$

$$\text{Hence, } \begin{cases} m + n : m :: a^2 : x^2. \\ m + n : m :: b^2 : y^2. \end{cases} \therefore \begin{cases} x = a\sqrt{\dfrac{m}{m+n}}. \\ y = b\sqrt{\dfrac{m}{m+n}}. \end{cases}$$

If, for example, the triangle is to be divided into three equal parts by lines parallel to the third side, then, The distances cut off on a are $a\sqrt{\frac{1}{3}}$, $a\sqrt{\frac{2}{3}}$.

The distances cut off on b are $b\sqrt{\frac{1}{3}}$, $b\sqrt{\frac{2}{3}}$.

3. *Two sides of a triangle being given, to cut off, by a line intersecting the given sides, an isosceles triangle having a given ratio to the given triangle.*



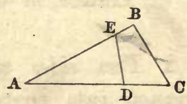
Let $b = AC$, $c = AB$, the two given sides; $x = AE = AD$, and

$$ADE : ABC :: m : n.$$

$$\text{But, } ADE : ABC :: x^2 : bc.$$

$$\text{Hence, } m : n :: x^2 : bc, \quad \therefore \; x = \sqrt{\frac{bcm}{n}}.$$

4. *Two sides of a triangle being given, to cut off a triangle having a given ratio to the given triangle by a line running from a given point in one of the given sides to the other given side.*

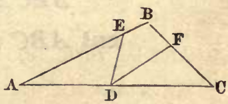


Let $b = AC$, $c = AB$, the given sides; D, the given point; $d = AD$, $x = AE$, and $AED : ABC :: m : n$.

$$\text{But, } AED : ABC :: dx : bc.$$

$$\text{Hence, } m : n :: dx : bc, \quad \therefore \; x = \frac{bcm}{dn}.$$

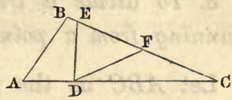
5. *The three sides being given, to divide the triangle into three equal parts by lines running from a given point in one of the sides.*



Let a, b, c be the sides of the triangle, respectively, opposite the angles A, B, C; $p = AD$, $q = CD$, $x = AE$, and $y = CF$.

$$\text{Then, } \begin{Bmatrix} 3 : 1 :: bc : px. \\ 3 : 1 :: ab : qy. \end{Bmatrix} \quad \therefore \quad \begin{cases} x = \dfrac{bc}{3p}. \\ y = \dfrac{ab}{3q}. \end{cases}$$

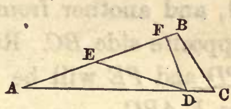
If x, thus found, is greater than c, both lines will intersect a. Then find y as above.



Let $x = CE$.

$$\text{Then, } 3 : 2 :: ab : qx, \therefore x = \frac{2ab}{3q}.$$

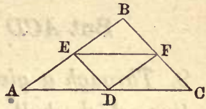
If y, found above, is greater than a, both lines will intersect c. Then find x as in first case.



Let $AF = y$.

$$\text{Then, } 3 : 2 :: bc : py, \therefore y = \frac{2bc}{3p}.$$

6. *To divide a triangle into four equal triangles, join the middle points of the sides.*



The lines ED, EF, and DF are, respectively, parallel to BC, AC, and AB.

$EBF = EDF$, since each is $\frac{1}{2}$ the parallelogram BD.

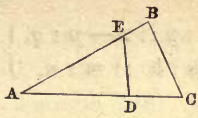
$ADE = EDF$, since each is $\frac{1}{2}$ the parallelogram AF.

$CDF = EDF$, since each is $\frac{1}{2}$ the parallelogram CE.

Hence, the triangles are all equal, and each is $\frac{1}{4}$ ABC.

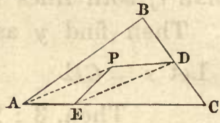
7. *The bearing of two sides being given, to cut off a triangle having a given area by a line of a given bearing intersecting the sides whose bearings are given.*

Let ADE be the triangle cut off, a the area of ADE; $x = AD$ and $y = AE$. The angles A, D, E can be determined from the bearings.



$$\text{Then, } \begin{cases} \frac{1}{2}xy \sin A = a. \\ \sin E : \sin D :: x : y. \end{cases} \therefore \begin{cases} x = \sqrt{\dfrac{2a \sin E}{\sin A \sin D}}. \\ y = \sqrt{\dfrac{2a \sin D}{\sin A \sin E}}. \end{cases}$$

8. *To divide a triangle into two equal parts by lines running from a point within.*



Let ABC be the given triangle, and P the given point.

Run a line from P to the vertex A, and another from P to D, the middle point of the opposite side BC. Run DE parallel to PA, and run PE. PD and PE will be the dividing lines, and $CDPE$ will be $\frac{1}{2}\,ABC$.

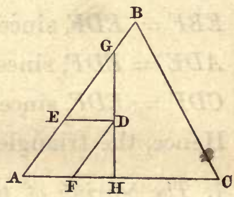
For, draw the line AD, then we have,

$$CDPE = CDE + PED, \text{ and } ACD = CDE + AED.$$

$$\text{But } PED = AED, \quad \therefore \; CDPE = ACD.$$

$$\text{But } ACD = \tfrac{1}{2}\,ACB, \quad \therefore \; CDPE = \tfrac{1}{2}\,ACB.$$

9. *Through a given point, within a given triangle, to draw a line which shall cut off a triangle having a given ratio to the given triangle.*

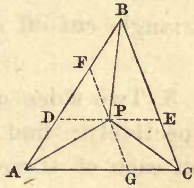


Let ABC be the given triangle; a, b, c, the sides opposite the angles A, B, C, respectively; D the point given by knowing $p = AF = ED$, parallel to AC; $q = AE = FD$, parallel to AB; $x = AH$, $y = AG$, and $AGH : ABC :: m : n$. Then,

$$\left.\begin{array}{l} x : y :: x - p : q. \\ xy : bc :: m : n. \end{array}\right\} \therefore \left\{\begin{array}{l} x = \dfrac{bcm \pm \sqrt{b^2c^2m^2 - 4\,bcmnpq}}{2\,nq}. \\ y = \dfrac{2\,bcmq}{bcm \pm \sqrt{b^2c^2m^2 - 4\,bcmnpq}}. \end{array}\right.$$

Remark.—If either $x > b$ or $y > c$, the line cuts off the triangle from another angle; and the distances cut off from the vertex of this angle can be found in a manner similar to the above.

10. *To find a point within a triangle from which the lines drawn to the vertices will divide the triangles into three equal triangles.*



Let ABC be the triangle. Take $AD = \frac{1}{3} AB$, $CE = \frac{1}{3} CB$, and draw DE. Take $BF = \frac{1}{3} BA$, $CG = \frac{1}{3} CA$, and draw FG.

P, the intersection of these lines, will be the point required.

For $AD : AB ::$ altitude of APC : altitude ABC.

But $AD = \frac{1}{3} AB$, $\therefore$ altitude $APC = \frac{1}{3}$ altitude ABC.

$$\therefore \quad APC = \tfrac{1}{3} ABC.$$

In like manner, $BPC = \frac{1}{3} ABC$.

$$\therefore \quad APB = \tfrac{1}{3} ABC.$$

Remark.—If APC, BPC, and APB are to be to each other as p, q, r, take $AD = \dfrac{p}{p+q+r}$ of AB, $CE = \dfrac{p}{p+q+r}$ of CB, $BF = \dfrac{q}{p+q+r}$ of BA, $CG = \dfrac{q}{p+q+r}$ of CA, and draw DE and FG, their intersection will be the point required.

325. Examples.

1. One side of a triangle is 15 ch.; from what point in this side must a line be drawn to the vertex of the opposite angle so as to divide the triangle into two triangles which are to each other as 2 to 3?

Ans. 6 ch. from one extremity.

2. Two sides of a triangle are 10 ch. and 15 ch., respectively; find the distance from the vertex of the

angle included by these sides, cut off on each of these sides by a line parallel to the third side, dividing the triangle into a triangle and a trapezoid, so that the triangle cut off shall be to the trapezoid as 9 to 16.

Ans. 6 ch. and 9 ch.

3. Two sides of a triangle are 4 ch. and 9 ch., respectively; find the distance from the vertex cut off on each of these sides by a line cutting off an isosceles triangle which shall be to the given triangle as 16 to 25. *Ans.* 4.80 ch.

4. Two sides of a triangle are 7 ch. and 9 ch., respectively. From a point in one side, 5 ch. from the vertex of the angle included by these sides, a line is run to the other given side, cutting off a triangle which is to the given triangle as 5 to 9. How far from the same vertex does this line intersect that side?

Ans. 7 ch.

5. The sides of a triangle, ABC, are $a = 6$ ch., $b = 12$ ch., and $c = 9$ ch. From the middle point of b two lines are run, dividing the triangle into three equal parts. To what points of what sides must the lines be run?

Ans. To c, 6 ch. from A, and to a, 4 ch. from C.

6. The sides of a triangle, ABC, are $a = 10$ ch., $b = 12$ ch., and $c = 4$ ch. From a point in b, 3 ch. from A, two lines are run, dividing the triangle into three equal parts. To what points of what side must these lines be run?

Ans. To a, 8.89 ch. from C, and to a, 4.44 ch. from C.

7. The sides of a triangle, ABC, are $a = 5$ ch., $b = 18$ ch., and $c = 15$ ch. From a point in b, 12 ch. from A, two lines are run, dividing the triangle into three equal parts. To what points must these lines be run?

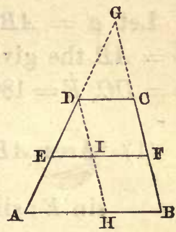
Ans. To c, 7.50 ch. from A, and to B.

8. In the triangle ABC, the side AB runs N. 50° E., AC runs E. DE, running N. 10° W., intersects these lines in D and E, and cuts off $ADE = 10$ A. Required AD and AE. *Ans.* $AD = 16.54$, $AE = 18.81$.

9. In the 9th general problem of the last article, $b = 10$ ch., $c = 12$ ch., $m = 1$, $n = 4$, $p = 2$ ch., $q =$ 3 ch. Find x and y. *Ans.* $x = 7.24$ ch., $y = 4.14$ ch.

326. Division of Trapezoids.

1. *Given the bases and a third side of a trapezoid, to divide it into parts having a given ratio by a line parallel to the bases.*



Let $ABCD$ be the trapezoid, $b = AB$, $b' = CD$, $s = AD$, $x = AE$, $y = EF$, the dividing line, parallel to the bases, and $ABFE : EFCD :: m : n$.

Produce AD and BC to G.

Then, $\begin{cases} ABG : DCG :: b^2 : b'^2. \\ EFG : DCG :: y^2 : b'^2. \end{cases}$

These proportions taken by division give,

$$ABCD : DCG :: b^2 - b'^2 : b'^2,$$

$$EFCD : DCG :: y^2 - b'^2 : b'^2.$$

Since the consequents are the same, we have,

$$ABCD : EFCD :: b^2 - b'^2 : y^2 - b'^2.$$

This proportion taken by division gives,

$$ABFE : EFCD :: b^2 - y^2 : y^2 - b'^2,$$

But $ABFE : EFCD :: m : n$.

$\therefore\ b^2 - y^2 : y^2 - b'^2 :: m : n, \quad \therefore\ y = \sqrt{\dfrac{b^2 n + b'^2 m}{m + n}}.$

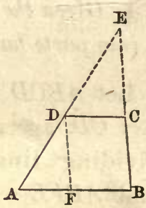
Drawing DH parallel to BC, we have,

$$AH : EI :: AD : ED,$$

or $b - b' : y - b' :: s : s - x, \quad \therefore \quad x = \frac{s}{b - b'}(b - y).$

$$\therefore \quad x = \frac{s}{b - b'}\left(b - \sqrt{\frac{b^2 n + b'^2 m}{m + n}}\right).$$

2. *Given a side and two adjacent angles of a tract of land, to cut off a trapezoid of a given area by a line parallel to the given side.*



1st. When the sum of the two angles $< 180°$.

Let $a = ABCD =$ the area cut off, $b = AB$ the given side, $x = AD$, $y = BC$, $z = DC$, $E = 180° - (A + B)$.

(1) Area $ABE = \frac{1}{2} EB \times EA \sin E$.

$$\sin E : \sin A :: b : EB, \quad \therefore \quad EB = \frac{b \sin A}{\sin E}.$$

$$\sin E : \sin B :: b : EA, \quad \therefore \quad EA = \frac{b \sin B}{\sin E}.$$

Substituting the values of EB and EA in (1), we have,

$$(2) \quad ABE = \frac{b^2 \sin A \sin B}{2 \sin E}.$$

$$\therefore (3) \quad DCE = \frac{b^2 \sin A \sin B}{2 \sin E} - a.$$

But $ABE : DCE :: b^2 : z^2$.

$$\therefore \quad \frac{b^2 \sin A \sin B}{2 \sin E} : \frac{b^2 \sin A \sin B}{2 \sin E} - a :: b^2 : z^2.$$

$$\therefore \quad z = \sqrt{b^2 - \frac{2a \sin E}{\sin A \sin B}}.$$

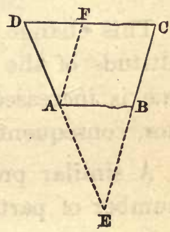
Draw DF parallel to EB, then $ADF = E$ and $DFA = B$.

$$\sin E : \sin B :: b - z : x, \therefore\ x = \frac{(b-z)\sin B}{\sin E}.$$

In like manner we shall find $y = \dfrac{(b-z)\sin A}{\sin E}$.

Since z is known, x and y are known.

2d. When the sum of the two angles $> 180°$.



E and DC lie on opposite sides of AB.

Let $a = ABCD =$ the area to be cut off, $b = AB$ the given side, $x = AD$, $y = BC$, $z = DC$, $E = A + B - 180°$.

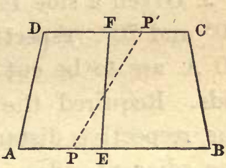
By a process similar to that employed in first case, we find,

$$z = \sqrt{b^2 + \frac{2\,a \sin E}{\sin A \sin B}}.$$

$$x = \frac{(z-b)\sin B}{\sin E}.$$

$$y = \frac{(z-b)\sin A}{\sin E}.$$

3. *To divide a trapezoid into proportional parts by a line joining the bases.*



Let $ABCD$ be the trapezoid, b and b' the bases, a the altitude, m and n the ratio of the parts.

$$\text{Take } AE = \frac{mb}{m+n}, \text{ then } EB = \frac{nb}{m+n},$$

$$\text{also } DF = \frac{mb'}{m+n}, \text{ then } FC = \frac{nb'}{m+n}.$$

Then, $AEFD = \frac{am(b+b')}{2(m+n)}$, and $EBCF = \frac{an(b+b')}{2(m+n)}$.

$$\text{But } \frac{am(b+b')}{2(m+n)} : \frac{an(b+b')}{2(m+n)} :: m : n.$$

$$\therefore \quad AEFD : EBCF :: m : n.$$

Remark. — If the line is to be drawn from a given point P, in one base, first divide as above; then, if P is on one side of E, take P' as far on the other side of F, and draw PP'.

This change in the dividing line does not affect the altitude of the parts, nor the sum of their bases, since one is increased as much as the other is diminished, nor, consequently, their area.

A similar process can be employed whatever be the number of parts.

327. Examples.

1. A trapezoid whose bases are $b = 15$ ch. and $b' = 12$ ch., and third side $s = 10$ ch., is divided by a line parallel to the bases into two parts, such that the part adjacent to b is to the part adjacent to b' as 3 to 2. Required the length of the dividing line, and the distance from b cut off on s. *Ans.* 13.28 ch., and 5.73 ch.

2. Given a side 14.30 ch., and the two adjacent angles, 60° and 70°, respectively, of a tract of land from which 10 A. are to be cut off by a line parallel to the given side. Required the length of the dividing line, and the respective distances from the given side cut off on the adjacent sides.

Ans. 4.05 ch., 12.60 ch., and 11.61 ch.

3. Given a side 10 ch., and the two adjacent angles, 120° and 115°, respectively, of a tract of land, from which 15 A. are to be cut off by a line parallel to the

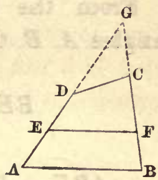
given side. Required the length of the dividing line, and the respective distances from the given side cut off on the adjacent sides.

Ans. 20.32 ch., 11.42 ch., 10.91 ch.

4. A trapezoid whose parallel sides are $AB = 14$ ch., and $DC = 7$ ch., is divided by the line PP' into two parts which are to each other as 3 to 4; $AP = 4$ ch., find DP'. *Ans.* 5 ch.

328. Division of Trapeziums.

1. *Given a side, two adjacent angles, and the area of a trapezium, to divide it, by a line parallel to the given side, into parts having a given ratio.*



Let $ABCD$ be the trapezium; $b = AB$, the given side; A and B, the given angles; $G = 180° - (A + B)$, $a =$ the area of $ABCD$, $x = AE$, $y = BF$, and $ABFE : EFCD :: m : n$.

$$\therefore \quad ABFE = \frac{ma}{m+n}, \quad EFCD = \frac{na}{m+n},$$

$$ABG = \tfrac{1}{2} BG \times AG \times \sin G.$$

$$BG = \frac{b \sin A}{\sin G} \text{ and } AG = \frac{b \sin B}{\sin G}.$$

$$\therefore \quad ABG = \frac{b^2 \sin A \sin B}{2 \sin G},$$

$$\therefore \quad EFG = \frac{b^2 \sin A \sin B}{2 \sin G} - \frac{ma}{m+n}.$$

$$ABG : EFG :: \overline{AG}^2 : \overline{EG}^2, \quad ABG : EFG :: \overline{BG}^2 : \overline{FG}^2.$$

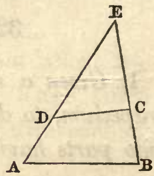
Substituting, in the proportions, the values of ABG, EFG, AG and BG, find EG and FG, and substituting the values of AG, EG, BG and FG in the equations,

$x = AG - EG$ and $y = BG - FG$, we have,

$$x = \frac{b \sin B}{\sin G} - \sqrt{\frac{b^2 \sin^2 B}{\sin^2 G} - \frac{2\,ma \sin B}{(m+n) \sin A \sin G}}.$$

$$y = \frac{b \sin A}{\sin G} - \sqrt{\frac{b^2 \sin^2 A}{\sin^2 G} - \frac{2\,ma \sin A}{(m+n) \sin B \sin G}}.$$

2. *Given the bearings of three adjacent sides of a tract of land, and the length of the middle side, to cut off, by a line running a given course, a trapezium of a given area.*



Let $a = ABCD$, the area cut off; $b = AB$, the given side; $x = AD$, $y = BC$, $z = CD$.

From the given bearings, find the angles A, B, C, D, E.

$$BE = \frac{b \sin A}{\sin E} \text{ and } AE = \frac{b \sin B}{\sin E}.$$

$$ABE = \tfrac{1}{2}\, BE \times AE \times \sin E = \frac{b^2 \sin A \sin B}{2 \sin E}.$$

$$\therefore \quad DCE = \frac{b^2 \sin A \sin B}{2 \sin E} - a.$$

$$DE = \frac{z \sin C}{\sin E} \text{ and } CE = \frac{z \sin D}{\sin E}.$$

$$DCE = \tfrac{1}{2}\, DE \times CE \times \sin E = \frac{z^2 \sin C \sin D}{2 \sin E}.$$

$$\therefore \quad \frac{z^2 \sin C \sin D}{2 \sin E} = \frac{b^2 \sin A \sin B}{2 \sin E} - a.$$

$$\therefore \quad z = \sqrt{\frac{b^2 \sin A \sin B}{\sin C \sin D} - \frac{2\,a \sin E}{\sin C \sin D}}.$$

Substituting the value of z in the values of DE and CE, then the values of AE, DE, BE and CE in the equations,

$x = AE - DE$, and $y = BE - CE$, we find,

$$x = \frac{b \sin B}{\sin E} - \sqrt{\frac{b^2 \sin A \sin B \sin C}{\sin^2 E \sin D} - \frac{2\,a \sin C}{\sin D \sin E}}.$$

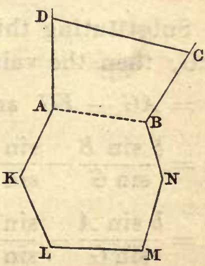
$$y = \frac{b \sin A}{\sin E} - \sqrt{\frac{b^2 \sin A \sin B \sin D}{\sin^2 E \sin C} - \frac{2\,a \sin D}{\sin C \sin E}}.$$

Remark.—If $A + B > 180°$, the values of x and y are

$$x = \sqrt{\frac{b^2 \sin A \sin B \sin C}{\sin^2 E \sin D} + \frac{2\,a \sin C}{\sin D \sin E}} - \frac{b \sin B}{\sin E}.$$

$$y = \sqrt{\frac{b^2 \sin A \sin B \sin D}{\sin^2 E \sin C} + \frac{2\,a \sin D}{\sin C \sin E}} - \frac{b \sin A}{\sin E}.$$

3. *The bearings of several adjacent sides of a tract of land being given, and the length of each, except the first and last, to cut off a given area by a line of given bearing intersecting the first and last sides.*



Let the bearings and distances of AK, KL, LM, MN, NB be given, and the bearings of AD and BC; and let a be the area cut off by CD.

Draw AB; then, in the polygon, $ABNMLK$, the bearings and distances of all the sides are known, except AB, which can be computed, and the area of $ABNMLK$ found. Subtract the area thus found from the area to be cut off by CD, and the remainder will be the area of $ABCD$.

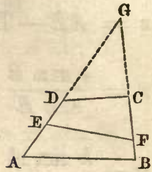
Then, by the last case, find AD and BC.

4. *The bearings of the sides of any quadrilateral tract of land and the distances of two opposite sides being given, to divide it into parts having a given ratio by a line of a given course intersecting the other sides.*

Let $b = AB$, $c = CD$,

$x = AE$, $y = BF$, $z = EF$,

and $ABFE : EFCD :: m : n$.



Find the angles A, B, C, D, E, F, G.

$$BG = \frac{b \sin A}{\sin G}, \quad AG = \frac{b \sin B}{\sin G}, \quad DG = \frac{c \sin C}{\sin G},$$

$$CG = \frac{c \sin D}{\sin G}, \quad FG = \frac{z \sin E}{\sin G}, \quad EG = \frac{z \sin F}{\sin G}.$$

$$ABFE = \frac{m(b^2 \sin A \sin B - c^2 \sin C \sin D)}{2(m+n) \sin G}.$$

$$ABFE = \frac{b^2 \sin A \sin B - z^2 \sin E \sin F}{2 \sin G}.$$

Equating these values of $ABFE$, we find,

$$z = \sqrt{\frac{nb^2 \sin A \sin B + mc^2 \sin C \sin D}{(m+n) \sin E \sin F}}.$$

Substituting this value of z in the values of FG and EG; then the values of AG, EG, BG and FG in $x = AG - EG$, and $y = BG - FG$, we have,

$$x = \frac{b \sin B}{\sin G} - \frac{\sin F}{\sin G}\sqrt{\frac{nb^2 \sin A \sin B + mc^2 \sin C \sin D}{(m+n) \sin E \sin F}}.$$

$$y = \frac{b \sin A}{\sin G} - \frac{\sin E}{\sin G}\sqrt{\frac{nb^2 \sin A \sin B + mc^2 \sin C \sin D}{(m+n) \sin E \sin F}}.$$

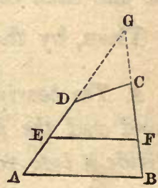
5. *The bearings and distances of the sides of any quadrilateral tract of land being given, to divide it into parts having a given ratio by a line dividing two opposite sides proportionally.*

$b = AB$, $c = CD$, $d = AD$,

$e = BC$, $x = AE$, $y = BF$,

$ABFE : EFCD :: m : n$,

$x : d - x :: y : e - y, \quad \therefore y = \dfrac{ex}{d}.$



From the bearings find the angles A, B, C, D, G.

$$BG = \frac{b \sin A}{\sin G} = p, \text{ and } AG = \frac{b \sin B}{\sin G} = q.$$

$$ABFE = \frac{m\,(b^2 \sin A \sin B - c^2 \sin C \sin D)}{2\,(m+n) \sin G}.$$

$$EFG = \frac{b^2 \sin A \sin B}{2 \sin G} - \frac{m(b^2 \sin A \sin B - c^2 \sin C \sin D)}{2\,(m+n) \sin G}.$$

$$\therefore EFG = \frac{nb^2 \sin A \sin B + mc^2 \sin C \sin D}{2\,(m+n) \sin G} = s.$$

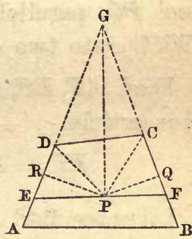
$$\text{But } EFG = \tfrac{1}{2}\,(q - x)\,(p - \frac{ex}{d}) \sin G.$$

$$\therefore \tfrac{1}{2}\,(q - x)\,(p - \frac{ex}{d}) \sin G = s.$$

$$\therefore x = \frac{dp + eq \pm \sqrt{(dp - eq)^2 + \frac{8des}{\sin G}}}{2e}.$$

$$\therefore y = \frac{dp + eq \pm \sqrt{(dp - eq)^2 + \frac{8des}{\sin G}}}{2d}.$$

6. *The bearings and distances of the sides of a quadrilateral being given, to cut off a given area by a line running through a point whose bearing and distance from the vertex of one of the angles are given.*



Let a be the area of $ABFE$, cut off by EF through P.

$b = AB$, $c = CD$, $u = EG$,
$v = FG$, $x = AE$, $y = BF$.

The bearings give the angles A, B, C, D, PCQ, PCD.

$$BG = \frac{b \sin A}{\sin G}, \quad AG = \frac{b \sin B}{\sin G}, \quad ABG = \frac{b^2 \sin A \sin B}{2 \sin G}.$$

$$EFG = \frac{b^2 \sin A \sin B}{2 \sin G} - a = a'.$$

In the triangle DCP we have given CD, CP, and DCP; hence CDP and DP can be found; then $PDR = CDR - CDP$.

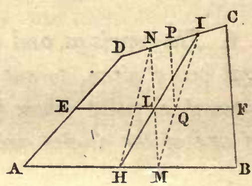
$$PR = DP \sin PDR = p, \text{ and } PQ = CP \sin PCQ = q.$$

$$EPG = \tfrac{1}{2}pu, \text{ and } FPG = \tfrac{1}{2}qv.$$

$$\left.\begin{array}{l} \therefore\ \tfrac{1}{2}pu + \tfrac{1}{2}qv = a'. \\ \text{But } \tfrac{1}{2}uv \sin G = a'. \end{array}\right\} \therefore \left\{ \begin{array}{l} u = \dfrac{a'}{p} \pm \sqrt{\left(\dfrac{a'}{p}\right)^2 - \dfrac{2a'q}{p \sin G}}. \\ v = \dfrac{a'}{q} \pm \sqrt{\left(\dfrac{a'}{q}\right)^2 - \dfrac{2a'p}{q \sin G}}. \end{array}\right.$$

$$\therefore \left\{ \begin{array}{l} x = \dfrac{b \sin B}{\sin G} - \dfrac{a'}{p} \mp \sqrt{\left(\dfrac{a'}{p}\right)^2 - \dfrac{2a'q}{p \sin G}}. \\ y = \dfrac{b \sin A}{\sin G} - \dfrac{a'}{q} \mp \sqrt{\left(\dfrac{a'}{q}\right)^2 - \dfrac{2a'p}{q \sin G}}. \end{array}\right.$$

7. *The bearings and distances of the sides of a quadrilateral being given, to divide it into four equal parts by two lines intersecting the pairs of opposite sides, respectively, one line being parallel to one side.*



Let EF, parallel to AB, and MN, parallel to BC, each divide $ABCD$ into two equal parts; and PQ, parallel to FC, divide $EFCD$ into two equal parts.

Find AE, BF, BM, CN, CP, and FQ, by problem 1 of this article.

$$EF = AB - AE \cos A - BF \cos B.$$

Likewise find MN and PQ. $NP = CN - CP$.

Produce MQ to I, draw NH parallel to IM, and draw HI; then will EF and HI be the lines required.

The line EF is evidently one of the required lines.

We are now to prove that HI is the other.

The two triangles, HNI and HNM, are equal, since they have a common base, HN, and a common altitude, their vertices being in IM, parallel to the base.

To each of these equal triangles add $AHND$, and we have $AHID = AMND = \frac{1}{2} ABCD$.

We are now to prove that HI divides $EFCD$, and also $ABFE$ into two equal parts.

$$IMH : IQL :: \overline{IM}^2 : \overline{IQ}^2.$$

$$IMN : IQP :: \overline{IM}^2 : \overline{IQ}^2.$$

$$\therefore \; IMH : IQL :: IMN : IQP.$$

$$\text{But } IMH = IMN. \quad \therefore \; IQL = IQP.$$

To each add $QFCI$, and we shall have,

$$LFCI = QFCP = \frac{1}{2} EFCD.$$

$$\text{Again, } HBCI = AHID \text{ and } LFCI = ELID.$$

Subtracting the second from the first, member from member, we have,

$$HBFL = AHLE.$$

Hence, HI is the other division line required.

Let us now find the situation of the points H and I, on the lines AB and CD, respectively.

$$NM : PQ :: NP + PI : PI.$$

$$\therefore \; NM \times PI = PQ \times NP + PQ \times PI.$$

$$(NM - PQ)\, PI = PQ \times NP.$$

$$\therefore \; PI = \frac{PQ \times NP}{NM - PQ}. \quad \text{Then, } CI = CP - PI.$$

The bearing and length of IM, and the area of $ICBM$, can be found by Art. 305. $IMH = ICBH - ICBM$.

If p be the perpendicular from I to AB,

$$p = IM \sin IMB. \quad MH = \frac{2\, IMH}{p}. \quad BH = BM + MH.$$

329. Examples.

1. A trapezium, one side of which is 20 ch., the adjacent angles 60° and 80°, respectively, and the area 10 A., is divided into two equal parts by a line parallel to the given side. Required the distance from the given side cut off on the adjacent sides.

Ans. 3.04 ch., and 2.68 ch.

2. From a tract of land, the bearings of three of whose adjacent sides are S. 20° W., E, and N. 10° W., and the distance of the middle side is 10 ch., 5 A. are cut off by a line running S. 70° W, and intersecting the first and third of the above mentioned sides. Required the distances cut off on these sides from the middle side. *Ans.* 4.91 ch., and 7.29 ch.

3. From a tract of land, the bearings of whose sides are S. 38° E., S. $29\frac{3}{4}$° E., S. $31\frac{3}{4}$° W., N. 61° W., and N. 10° W., respectively, and the distance of the second, third, and fourth sides are 8.18 ch., 15.36 ch., and 14.48 ch., respectively, 39 A. 2 R. 36 P., are cut off by a line running N. 80° E., and intersecting the first and last sides. Required the distances cut off on these sides respectively. *Ans.* 7.01 ch., 16.19 ch.

4. A tract of land, the bearing and distances of whose sides are *AB*, E. 22.21 ch.; *BC*, N.; *CD*, N. $56\frac{1}{2}$° W., 12. ch.; *DA*, S. 24° W., is cut by *EF* running S. $76\frac{1}{2}$° E., intersecting *AD* and *BC*, and dividing the field so that *ABFE* : *EFCD* :: 5 : 3. Required *AE* and *BF*.

Ans. $AE = 16.50$. ch., $BF = 11.34$ ch.

5. A trapezium whose sides are $AB = 20.45$ ch., $BC = 21.73$ ch., $CD = 13.98$ ch., $DA = 13.32$ ch., and whose angles are $A = 97\frac{1}{4}°$, $B = 64°$, $C = 89\frac{3}{4}°$, $D = 109°$, is divided into two equal parts by the line *EF*,

dividing AD and BC proportionally. Required AE and BF. *Ans.* $AE = 6.22$ ch., $BF = 10.15$ ch.

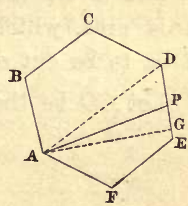
6. Within a tract of land whose sides are — 1st. E. 45.58 ch.; 2d. N. 13½° W., 40.86 ch.; 3d. S. 82° W., 30.40 ch., 4th. S. 9½° W., 36 ch. — there is a spring whose bearing and distance from the 3d corner is S. 21° W., 15.80 ch. It is required to cut off 40 A. from the north side of this tract by a line running through the spring and intersecting the 2d and 4th sides. Required the distance from the 1st corner to the point of intersection on the 4th side. *Ans.* 26.73 ch.

7. A tract of land whose boundaries are — 1st. E. 23.24 ch.; 2d. N. 11¼° W., 15.25 ch.; 3d. N. 51½° W., 11.50 ch.; 4th. S. 27° W., 24.82 ch. — is to be divided into four equal parts by two lines, one parallel to the first side, the other intersecting the first and third sides. Required the distances cut off by the parallel from the first and second corners, measured on the fourth and second sides, respectively; also the distances cut off by the other line from the first and fourth corners, measured on the first and third sides, respectively.

Ans. 8.57 ch., 7.79 ch., 10.66 ch., 3.15 ch.

330. Division of Polygons.

1. *From a given point in the boundary of a tract of land, the bearings and distances of whose sides are given, to run a line which shall cut off a given area.*



Let A be the point, and suppose it probable that the dividing line will terminate on DE. Suppose the closing line AD to be run, the bearing and distance of which can be found on the ground by observation and measurement, or, as in supplying omissions, from the bearings and distances of AB, BC, and CD. Compu e the area of $ABCD$, which, if less than the area to be cut off, subtract from that area, which gives the addition, a, to $ABCD$. The bearings of AD and DE give the angle ADE.

The perpendicular, $AG = AD \sin ADG$.

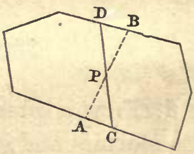
Then, if AP is the dividing line, $DP = \frac{a}{\frac{1}{2} AG}$.

If $DP > DE$, run another closing line AE, and proceed as before.

If $ABCD$ is greater than the area to be cut off, subtract the area to be cut off from $ABCD$ and divide the difference by one-half the perpendicular from A to CD, and the quotient, if less than DC, will be the distance from D to the point on DC to which the division line is to be drawn.

If the quotient is greater than DC, run another closing line, AC, and proceed as before.

2. *Through a given point within a tract of land, the bearings and distances of whose sides are given, to run a line which shall cut off a given area.*



Let P be the given point. Run a *trial line*, AB, and calculate the area which it cuts off.

Let d be the difference between this area, which we will suppose too small, and the area to be cut off.

Let CD be the division line required.

$$d = APC - BPD.$$

Let $m = AP$, and $n = PB$, which measure; find the angle PAC, also PBD. We are to find the angle P.

$$C = 180° - (A + P) \text{ and } D = 180° - (B + P).$$

$$\therefore \quad \sin C = \sin (A + P) \text{ and } \sin D = \sin (B + P).$$

$$PC = \frac{m \sin A}{\sin (A + P)}, \quad AC = \frac{m \sin P}{\sin (A + P)},$$

$$\therefore \quad APC = \frac{m^2 \sin A \sin P}{2 \sin (A + P)}.$$

$$PD = \frac{n \sin B}{\sin (B + P)}, \quad BD = \frac{n \sin P}{\sin (B + P)},$$

$$\therefore \quad BPD = \frac{n^2 \sin B \sin P}{2 \sin (B + P)}.$$

$$d = \frac{m^2 \sin A \sin P}{2 \sin (A + P)} - \frac{n^2 \sin B \sin P}{2 \sin (B + P)}.$$

$$\therefore \quad 2\,d = \frac{m^2}{\cot P + \cot A} - \frac{n^2}{\cot P + \cot B}.$$

Use natural co-tangents, find cot P, and then P.

331. Examples.

1. The boundaries of a tract of land are: AB, W. 25 ch.; BC, N. 32¼° W., 16.09 ch.; CD, N. 20° E., 15.50 ch.; DE, E. 25 ch.; EF, S. 30° E.; and FA, S. 25° W., to the point of beginning. A line is run from A, cutting-off 70 A. 1 R. 33 P. from the west side. Required the second point in which this line cuts the boundary.

Ans. The side DE, 18 ch. East of D.

2. It is required to run a line through a point, P, within a field, so as to cut off 10 A. A guess line through P, intersecting opposite sides in A and B, cuts off 9 A. Required the angle which the true division line, CD, makes with AB, if $AP = 12$ ch., $PB = 4$ ch., $PAC = 90°$, $PBD = 60°$. *Ans.* 8° 48′.

LEVELING.

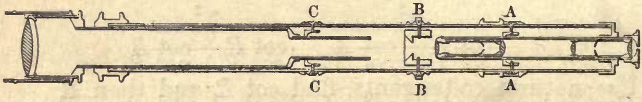
332. The Y Level.

The Y level, so called from the form of the supports in which the telescope rests, is exhibited in the annexed engraving.

The telescope is inclosed in rings, by which it can be revolved in the Y's or clamped in any position.

The Y's have each two nuts, adjustable with the steel pin, and the rings are clamped in the Y's by bringing the clips firmly on them by means of tapering Y pins.

The interior construction of the telescope is exhibited in the following figure.



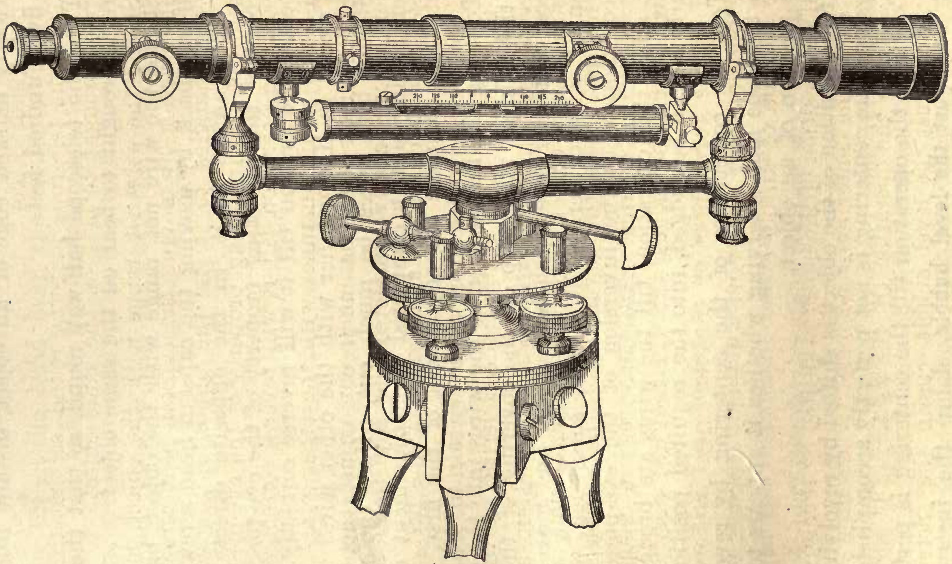
The rack and pinion, *AA* and *CC*, are contrivances, the first for centering the eye-piece, and the second for insuring the accurate projection of the object-glass in a straight line.

The level is a ground bubble tube, attached to the under side of the telescope, and furnished at each end with arrangements for the usual movements in both horizontal and vertical directions.

The tripod head is similar to that in the transit.

333. Adjustments.

1. **To adjust the line of collimation,** set the tripod firmly, remove the Y pins from the clips, so that the telescope shall turn freely, clamp the instrument to



The Y Level.

the tripod head, and by means of the leveling and tangent screws, bring either of the wires to bear on a clearly marked edge of an object, distant from two to five hundred feet.

Turn the telescope half-way round, so that the same wire is brought to bear on the same object.

Should the wire not range with the object, bring it half-way back by moving the capstan head screws, *BB*, at right angles to it, in the opposite direction, on account of the inverting property of the eye-piece, and repeat the operation till it will reverse correctly.

Proceed in like manner with the other wire.

Should both wires be much out, adjust the second after having nearly completed the adjustment of the first, then complete the adjustment of the first.

To bring the intersection of the wires into the center of the field of view, slip off the covering of the eye-piece centering screws, shown at *AA*, and move, with a small screw-driver, each pair in succession, with a direct motion, as the inversion of the eye-piece does not affect this operation, till the wires are brought, as nearly as can be judged, into the required position.

Test the correctness of the centering by revolving the telescope and observing whether it appears to shift the position of an object.

If the position of the object is shifted by revolving the telescope, the centering is not perfectly accomplished.

Continue the operation till the centering is perfect.

2. **To adjust the level bubble,** clamp the instrument over either pair of leveling screws, and bring the bubble to the middle.

Revolve the telescope in the Y's so as to bring the level tube on either side of the center of the level bar.

Should the bubble run to one end, rectify the error by bringing it, as nearly as can be estimated, half-way back with the capstan screws in the level holder.

Again bring the level over the center of the bar, and bring the bubble to the center; turn the level to one side, and, if necessary, repeat the operation till the bubble will keep its position when the tube is turned to either side of the center of the bar.

Now bring the bubble to the center with the leveling screws, and reverse the telescope in the Y's without jarring the instrument. Should the bubble run to either end, lower that end, or raise the other by turning small adjusting nuts at one end of the level till, by estimation, half the correction is made.

Again bring the bubble to the middle, and repeat the operation till the reversion can be made without causing any change in the bubble.

3. **To adjust the Y's**, or to bring the level into a position at right angles with the vertical axis, so that the bubble will remain in the center during an entire revolution of the instrument, bring the level tube directly over the center of the bar, and clamp the telescope in the Y's, placing it, as before, over two of the leveling screws, unclamp the socket, level the bubble, and turn the instrument half-way around, so that the level bar may occupy the same position with respect to the leveling screws beneath.

Should the bubble run to either end, bring it half-way back by the Y nuts on either end of the bar.

Now move the telescope over the other set of leveling screws, bring the bubble again into the center, and proceed as before, changing to each pair of screws, successively, till the adjustment is nearly completed, which may now be done over a single pair of screws.

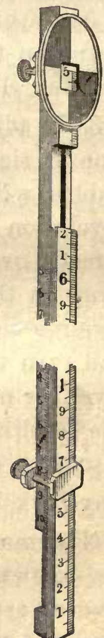
334. The Use of the Level.

Set the legs firmly in the ground, test the adjustments, making corrections if necessary.

Bring the wires precisely in the focus, and the object distinctly in view, so that the spider lines will appear fastened to the surface of the object, and will not change in position however the eye be moved.

The bubble resting in the middle, the intersection of the spider lines will indicate the line of apparent level.

335. Leveling Rod.



The New York Leveling Rod, represented in the engraving with a piece cut out of the middle, so that both ends may be exhibited, consists of two pieces, one sliding from the other.

The graduation commences at the lower end, which is to rest on the ground, and is made to tenths and hundredths of a foot.

A circular target, divided into quadrants of different colors, so as to be easily seen, moves on the front surface of the rod, which reads to six and one-half feet.

If a greater height is required, the horizontal line of the target is fixed at $6\frac{1}{2}$ feet, on the front surface, and the upper part of the rod, which carries the target, is run out of the lower, and the reading is obtained on the graduated side up to an elevation of twelve ft.

A clamp screw on the back is used to fasten the rods together in any position.

336. Definitions.

A level surface is the surface of still water, or any surface parallel to that of still water.

Such a surface is convex, and conforms to the spheroidal form of the earth.

A level line is a line in a level surface.

The difference of level of two places is the distance of one above or below the level surface passing through the other.

Leveling is the art of ascertaining the difference of level of two places.

The apparent level of any place is the horizontal plane tangent to the level surface at that place.

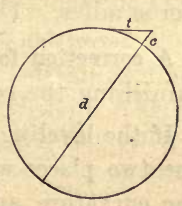
The line of apparent level of any place is a horizontal line, tangent to a level line at that place.

The Y Level indicates the line of apparent level and not the true level, which is a curved line.

The correction for curvature is the amount of deviation for a given distance of the line of apparent level from the line of true level to which it is tangent at the point from which the distance is measured.

337. Problem.

To compute the correction for curvature.



Let t denote the tangent, c the correction for curvature, d the diameter of the earth.

Then, by Geometry, we have,

$$(d+c)\,c = t^2, \quad \therefore\ c = \frac{t^2}{d+c}.$$

Since c is very small compared with d, it can be dropped from the denominator without sensibly affecting the result.

$$\therefore \quad c = \frac{t^2}{d}.$$

The arc, which is the distance measured, will not differ perceptibly from the tangent, for all distances at which observations are made, and may be substituted for it.

Calling another distance, t', and the corresponding correction, c', we have,

$$c' = \frac{t'^2}{d}. \qquad \therefore \quad c : c' :: t^2 : t'^2.$$

1. *The correction for curvature, for a given distance, is equal to the square of the distance divided by the diameter of the earth.*

2. *The corrections for different distances are to each other as the squares of the distances.*

Let us find the correction for the distance 100 chains, calling the diameter of the earth 7920 miles.

$$c = \frac{100^2 \times 66 \times 12}{7920 \times 80} = 12.5 \text{ inches.}$$

The correction for any other distance, for example, 5 ch., can be found from the proportion.

$$100^2 : 5^2 :: 12.5 : c, \quad \therefore c = .031 \text{ inches.}$$

For 1 mile, $100^2 : 80^2 :: 12.5 : c, \quad \therefore c = 8$ inches.

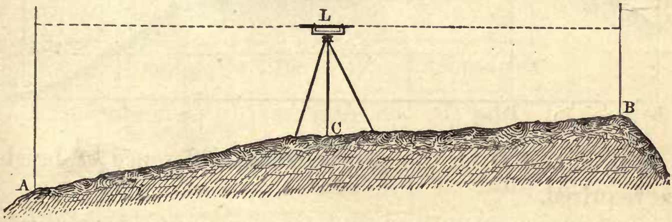
For m miles, $1^2 : m^2 :: 8 : c, \quad \therefore c = 8\ m^2$ in.

A correction for refraction is sometimes made by diminishing the correction for curvature by $\frac{1}{6}$ of itself.

If the leveling instrument is placed midway between the two places whose difference of level is to be found, the curvature and refraction on the two sides of the instrument balance, and the difference of apparent level will be the difference of true level.

338. Problem.

To find the difference of level of two places visible from a point midway between them or from each other, when the difference of level does not exceed twelve feet.



Let A and B be the two places, and C the place midway from which both are visible.

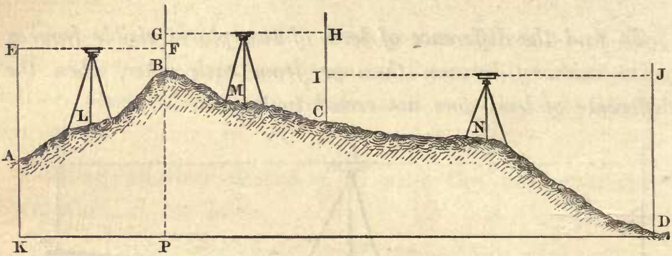
Place the level at C, and let the rod-man set up the leveling rod at A, and slide the vane till he learns, by signals from the surveyor at the level, that its horizontal line is in the line of apparent level. Let the height be accurately observed and noted, and the rod be transferred to B, and the height observed and noted as before.

The difference of these heights will be the difference of level.

If a gully intervene, so that the line of apparent level, from the intermediate station, would not cut the rod, place the instrument at one station, and take the height on the staff at the other station marked by the vane when in the line of apparent level, from which subtract the height of the instrument, and the difference corrected for curvature and refraction will be the difference of level required.

339. Problem.

To find the difference of level of two places which differ considerably in level, or which can not be seen from each other.



Let A and D be the places whose difference of level is required.

Place the level at the station L, midway between two convenient points, A and B. Take the backsight to A, and note the height of E. Send the rod to B, and note the height of the foresight at F. Remove the level to M, note the height of the backsight at G and the foresight at H. Remove the level to N, note the height of the backsight at I, and the foresight at J.

Then will the difference of the sum of the backsights and the sum of the foresights be the difference of level of A and D.

For, we find for the sum of the backsights,

$$AE + BG + CI = AE + BF + FG + CI.$$

And, we find for the sum of the foresights,

$$BF + CH + DJ = BF + CI + IH + DJ$$
$$= BF + CI + PG.$$

The sum of the backsights, minus the sum of the foresights, $= AE + FG - PG = - AK =$ difference of level, which in the field notes is denoted by $D.\ L.$

If the sum of the foresights exceeds the sum of the backsights, the point D is below A; if the reverse were true, the point D would be above A, as indicated by the sign.

It is not essential that the intermediate stations be directly between the places.

340. Field Notes.

Stations.	*Backsights.*	*Foresights.*
1	5.40	1.50
2	3.12	5.25
3	2.40	8.16
Sums ...	10.92	14.91
	14.91	
D. L. =	— 3.99	

341. Leveling for Section.

Leveling for Section is leveling for the purpose of obtaining a section or profile of the surface along a given line.

A Bench-mark is made to indicate the beginning of the line by drilling a rock or driving a nail into the upper end of a post. Such marks should be made at different points along the line, to serve as checks in case of a new survey.

It is necessary also to measure the distance between the stations. The bearings of the lines should be taken in case a map or plot is to be made, representing the horizontal surface.

In the following table of specimen field notes, *S.* denotes stations; *B.*, bearings; *D.*, distances; *B. S.*, backsights; *F. S.*, foresights; *B. S.* — *F. S.*, backsights minus foresights; *T. D. L.*, total difference of level; *R.*, remarks, and *B. M.*, bench-mark.

The numbers in the column headed *B. S.* — *F. S.* are obtained by subtracting each foresight from the corresponding backsight, observing to write the proper sign.

The numbers in the column headed *T. D. L.* are obtained by continued additions of the numbers in the column *B. S.* — *F. S.*, each being the sum of the backsights minus the sum of the foresights, up to a given point, expresses the distance of that point above or below the bench-mark at the beginning of the line.

The minus sign of a result indicates that the sum of the foresights exceeds the sum of the backsights, and hence, that the corresponding station is below the first station; the plus sign indicates the reverse.

In order to bring out prominently the difference of level, the vertical distances are usually plotted on a much larger scale than the horizontal.

Let us suppose the numbers in the column *D.* express chains, and that the numbers in the following columns express feet.

In the following profile section the horizontal distances are plotted to the scale of 20 chains to an inch, and the vertical distances to the scale of 20 feet to an inch.

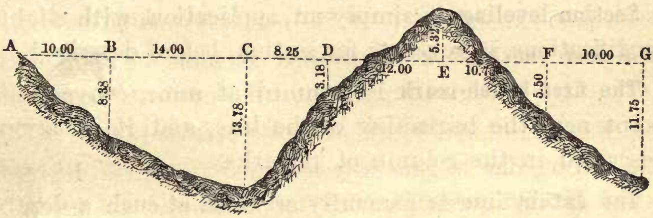
The profile of the section is therefore distorted, the vertical distances being 66 times too great to exhibit their true proportion to the horizontal distances.

The horizontal line, *AG*, through the point of beginning is called the *datum line.*

342. Field Notes.

S.	*B.*	*D.*	*B. S.*	*F. S.*	*BS.—FS.*	*T.D.L.*	*R.*
1	N.	10.00	3.25	11.63	— 8.38	— 8.38	BM. on post.
2	N.	14.00	4.80	10.20	— 5.40	—13.78	
3	N.	8.25	12.00	1.40	+ 10.60	— 3.18	BM. on rock.
4	N.10°E.	12.00	10.80	2.30	+ 8.50	+ 5.32	
5	N.10°E.	10.75	1.18	12.00	— 10.82	— 5.50	
6	N.	10.00	2.15	8.40	— 6.25	—11.75	BM. on oak.

343. Profile of Section.



SURVEYING RAILROADS.

344. General Plan.

The surveys for the construction of railroads, applicable also to canals, graded pikes, dikes, etc., are made in the following order.

1. **The reconnoissance**, to locate the route. The termini being agreed upon, sometimes several routes are examined, so that an approximate judgment can be formed in reference to the economy of construction and purchasing the right of way, the amount of stock taken at different towns along the route, and the profits from local business.

2. **The transit survey,** to determine definitely the

middle line along the surface, after the route has been decided upon by the preliminary reconnoissance.

3. **The section leveling,** to determine the profile of the middle line along the surface..

4. **The cross-section work,** to determine the position and slopes of the sides, so that the amount of earth to be removed or filled can be estimated.

345. Section Leveling.

Section leveling is simply an application, with slight modifications, of leveling for section, before described.

The first bench-mark is assumed at some convenient point near the beginning of the line, and its location described in the column of remarks.

The datum line is generally assumed at such a depth below the first bench-mark — for example, at mean high-tide water, in case one end of the route is in the vicinity of tide-water — that its whole length shall be below the section line at the surface.

The engineer's chain, 100 feet in length, is usually employed in taking the horizontal distance.

A turning-point is a hard point chosen as far in advance as possible, but not necessarily in exact line, upon which the rod rests while a careful reading is taken just before it is necessary to change the position of the instrument, whose exact height above the datum line thus becomes known in the new position.

The difference between a *turning-point* and a *bench* is this:

A turning-point is merely a temporary point, neither marked nor recorded, used to determine the height of the instrument in a new position. A bench is both marked and noted, and thus made permanent.

If, however, it is thought best to make a turning-point permanent, it is marked and recorded, and becomes a bench.

In order that a bench be not destroyed in constructing the road, it should be a little removed from the line surveyed. The location of the benches should be carefully noted, so that they may be readily found from the field notes.

The plus sights are the first readings of the rod, made after each new position of the instrument, as the rod stands on a bench or turning-point, and are taken to thousandths of a foot.

The minus sights are the other readings, and are taken to tenths, except the last minus sight, before the position of the instrument is changed, which, being taken as the rod stands on a turning-point or bench, is taken to thousandths.

The height of the instrument above the datum line is equal to a plus sight, plus the height of the corresponding bench or turning-point.

The height of the surface above the datum line, at any position of the rod, is equal to the height of the instrument, minus the corresponding backsight.

These heights are taken at intervals of 1 chain, and at intermediate points where the irregularity of the surface is deemed sufficient to render it important.

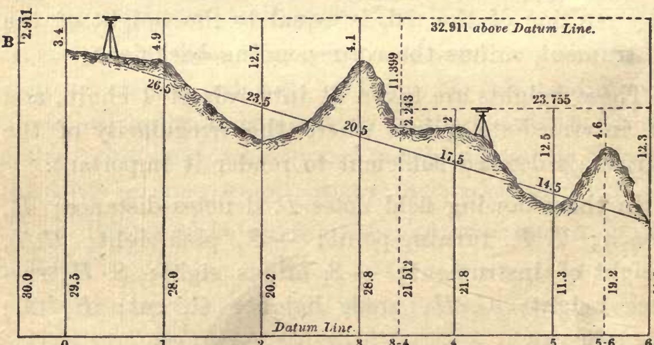
In the following field notes *D.* denotes distance; *B.*, bench; *T. P.*, turning-point; + *S.*, plus sight; *H. I.*, height of instrument; — *S*, minus sight; *S. H.*, surface height; *G. H.*, grade height; *C.*, cut; *F.*, fill; *R.*, remarks.

346. Field Notes.

D.	+ *S.*	*H. I.*	− *S.*	*S. H.*	*G. H.*	*C.*	*F.*	*R.*
B.	2.911	32.911		30.				*B.* 50 ft.
0.			3.4	29.5	29.5			E. of
1.			4.9	28.0	26.5	1.5		0 stake.
2.			12.7	20.2	23.5		3.3	
3.			4.1	28.8	20.5	8.3		
T. P.	2.243	23.755	11.399	21.512	19.2	2.3		
4.			2.0	21.8	17.5	4.3		
5.			12.5	11.3	14.5		3.2	
5.6			4.6	19.2	12.7	6.5		
6.			12.3	11.5	11.5			

The numbers in the horizontal column, *T. P.*, are found thus: The − *S.*, 11.399, is obtained from the first position of the instrument by the reading of the rod on *T. P.* 21.512 = 32.911 − 11.399. The + *S.*, 2.243, is the reading of the rod from the new position of the instrument. 23.775 = 21.512 + 2.243. The cutting or filling is the difference of *S. H.* and *G. H.*

347. Profile of Section and Grade.



348. Remarks.

1. **The grade height** at 0, minus the grade at 6, which is 29.5 — 11.5 = 18 = the descent from 0 to 6. 18 ÷ 6 = 3 = the descent for 1 chain, 29.5 — 3 = 26.5 = *G. H.* at 1; 26.5 — 3 = 23.5 = *G. H.* at 2, etc.

2. **The establishment of the grade** is influenced by the object of the work, economy, the balance of cuttings and fillings, the points desirable for termini, etc.

3. The method exhibited above may be extended to any distance.

349. Example.

Fill out the notes of the following table, and make a profile of section and grade from *S. H.* at 0 to *S. H.* at 5.

D.	+ *S.*	*H. I.*	— *S.*	*S. H.*	*G. H.*	*C.*	*F.*	*R.*
B.	6.248	36.248		30				*B.* 20 ft.
0			5.3					S. of 0.
1			9.8					
2			2.3					
T. P.	10.718		11.814					
3			7.6					
4			12.0					
5			2.1					

350. Cross-Section Work.

Excavations and embankments are constructed with sloping sides, in order to prevent the sliding of earth down the surface.

The ratio of slope is the vertical distance divided by the horizontal, and is therefore the tangent of the angle which the sloping surface makes with a horizontal plane.

The usual ratio of slope is $\frac{2}{3}$, and the angle 33° 41′.

Slope stakes are driven to mark where the sloping sides, whether of cutting or filling, will intersect the surface, and thus indicate the boundaries of the work.

The rod used in cross-section leveling is 15 feet long, graded and plainly marked to feet and tenths, and is read by the leveler at the instruments.

The assistants of the leveler are the *rodman*, *axman*, and two *tapemen*.

The Field book is ruled into four columns, headed *D.* for distance; *L.* for left; *C. C.* for center-cut; *R.* for right.

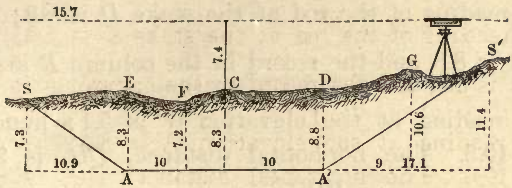
The numbers in the columns *D.* and *C. C.* are, respectively, the distance and the corresponding cut, or fill marked minus cut, taken from the field book for section leveling.

The fractions in the columns *L.* and *R.* have for their numerators the vertical distances of the cross-section, and for their denominators, the corresponding horizontal distances, from the center or from the vertex of the angle of slope, according as the vertical distance is taken within or without the limits of the horizontal portion of the road.

351. Cross-Section Excavations.

We give the following profile of cross-section, the method of performing the field operations and recording the notes.

Let us suppose the cross-section to be taken at the distance 3 of the field notes of article 343, where the center cut is 8.3; that the road bed is 20 feet wide, that the ratio of slope is $\frac{2}{3}$, and that both horizontal and vertical distances are plotted to the scale of 20 feet to 1 inch.



Take AA' for the datum line, and suppose the reading at the center stake to be 7.4. The height of the instrument above the datum line is therefore $8.3 + 7.4 = 15.7$.

The reading of the rod at the depression F, between the center and the angle A, is 8.5; hence, the cut is $15.7 - 8.5 = 7.2$. The horizontal distance, CF, is 4 feet; hence, the record in the field notes, as seen in the next article in the column L, is $\frac{7.2}{4}$.

The reading of the rod, at the temporary stake E, is 7.4; hence, the cut is $15.7 - 7.4 = 8.3$, and the entry, $\frac{8.3}{A}$.

The point S, where the slope intersects the surface, is found by trial. Since the vertical distance of the slope is $\frac{2}{3}$ of the horizontal, then ES, if horizontal, would be $\frac{3}{2}$ of EA, which is 12.4; but, on account of the inclination of the surface, ES will be less, say 10 feet. Setting the rod 10 feet out from E, the reading is 8.3, and hence the cut $= 15.7 - 8.3 = 7.4$. Now, $\frac{3}{2}$ of 7.4 is 11.1; hence, the assumed distance, 10 feet, is too small.

For a second trial, take 11 feet out from E, at which the reading of the rod is 8.4, and the cut 7.3. Now, $\frac{3}{2}$ of $7.3 = 10.9$, which lacks but .1 of 11, and is sufficiently accurate. The record for the slope stake, in the column L, is $\frac{7.3}{10.9}$.

The reading of the rod at the stake D is 6.9; hence, the cut is 8.8, and the record in the column R is $\frac{8.8}{A'}$.

The reading at the elevation G is 5.1; hence, the cut is 10.6. The horizontal distance, DG, is 9 feet; hence, the record is $\frac{10.6}{9}$.

To find S' where the slope intersects the surface, since, on account of the rising of the surface, it is more than $\frac{3}{2}$ of 8.8, which is 13.2, take, for a first trial, 18 feet out from D, at which point the reading of the rod is 4.5, and hence the cut $15.7 - 4.5 = 11.2$. Now, $\frac{3}{2}$ of $11.2 = 16.8$; hence, 18 feet is too far out.

For a second trial, take 17 feet out from D. The reading of the rod is 4.3, and the cut $15.7 - 4.3 = 11.4$. Now, $\frac{3}{2}$ of $11.4 = 17.1$, which is sufficiently accurate; hence, the record for the slope stake S', in the column R, is $\frac{11.4}{17.1}$.

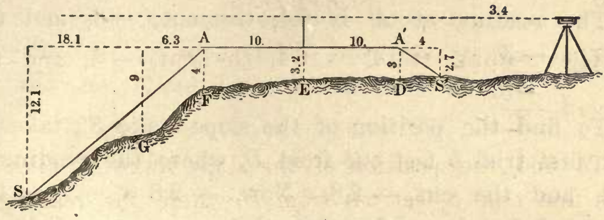
352. Field Notes.

D.	*L.*			*C. C.*	*R.*		
3	$\frac{7.3}{10.9}$	$\frac{8.3}{A}$	$\frac{7.2}{4}$	8.3	$\frac{8.8}{A'}$	$\frac{10.6}{9}$	$\frac{11.4}{17.1}$

353. Cross-Section Embankments.

The following is the profile of the cross section drawn to a scale of 20 feet to 1 inch, taken at the distance 5 of the field notes of article 346, where the filling is 3.2, now called a minus cut, and written -3.2.

Take AA', which is the horizontal top of the embankment 20 feet wide, for the datum line.



The ratio of slope, in case of embankments, is $-\frac{3}{2}$.

The reading of the rod at the center stake is 6.6, and the height of the instrument, with reference to the datum line, is the algebraic sum of the reading of the rod and the minus cut, which is $6.6 - 3.2 = 3.4$.

If the instrument should be below the datum line, the reading of the rod would be numerically less than the minus cut, and the height of the instrument would be negative.

The readings of the other points along the surface SS', subtracted from the height of the instrument, will give the corresponding minus cuts.

The reading at A is 7.4, the cut, -4, and the record, $\frac{-4}{A}$.

The reading at G is 12.4, the cut, -9, the horizontal distance FG, 6.3, and the record, $\frac{-9}{6.3}$.

To find the position of the slope stake S, take for the first trial 20 feet out from F, where the reading is 16, and the cut, -12.6. Now, $-12.6 \times -\frac{3}{2} = 18.9$; hence, 20 feet is too far out.

Next try 18 feet out, where the reading is 15.5, and the cut, -12.1. Now, $-12.1 \times -\frac{3}{2} = 18.1$, which is sufficiently accurate; hence, the record for the slope stake S is $\frac{-12.1}{18.1}$.

The reading at A' is 6.4, the cut, -3, and the record, $\frac{-3}{A'}$.

To find the position of the slope stake S', take for the first trial 5 feet out from D, where the reading is 6.2, and the cut, -2.8. Now, $-2.8 \times -\frac{3}{2} = 4.2$; hence, 5 feet is too far out.

Next take 4 feet out, where the reading is 6.1, and the cut, -2.7. Now, $-2.7 \times -\frac{3}{2} = 4$; hence, the record for the slope stake S' is $\frac{-2.7}{4}$.

354. Field Notes.

D.	*L.*			*C. C.*	*R.*	
5	$\frac{-12.1}{18.1}$	$\frac{-9}{6.3}$	$\frac{-4}{A}$	-3.2	$\frac{-3}{A'}$	$\frac{-2.7}{4}$

355. Remark.

It sometimes occurs that an excavation will be required on one side, and an embankment on the other. Guided by the stakes and field notes, the excavations and embankments can be correctly made.

356. Computation of Earth-work.

The computation of earth-work is the determination of the volume of excavation or embankment.

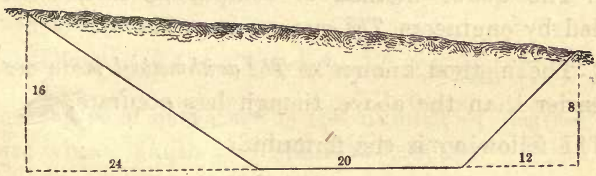
The cross-sections, being taken, wherever necessary, at every 100 feet or less, divide the excavations or embankments into blocks, which may be regarded as frustums of pyramids.

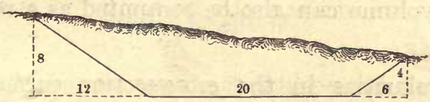
Denoting the areas of the sections regarded as bases of the frustum by b and b', respectively, the length by l, and the volume by v, we have the formula,

$$v = \tfrac{1}{3} l (b + b' + \sqrt{bb'}).$$

357. Examples.

1. The length of an excavation is 100 feet; find the volume, the two ends being thus represented:





The area required, in each case, is the area of the whole figure, regarded as a trapezoid, which is one-half the altitude multiplied by the sum of the parallel bases, minus the sum of the two triangles; hence,

$$b = 28 \times 24 - (24 \times 8 + 12 \times 4) = 432.$$

$$b' = 19 \times 12 - (12 \times 4 + \ 6 \times 2) = 168.$$

$$v = \tfrac{1}{3} \times 100 \,(432 + 168 + \sqrt{432 \times 168}).$$

$$v = 28980 \text{ cubic feet} = 1073 \text{ cubic yards}.$$

2. Compute the volume of the embankment whose horizontal breadth at the top is 16 feet, from the following field notes:

S. N. 31.

D.	L.		C. C.	R.	
5	$\frac{-11.6}{17.4}$	$\frac{-10.5}{A}$	-10	$\frac{-9.5}{A'}$	$\frac{-8.6}{13}$
6	$\frac{-17.4}{26.1}$	$\frac{-15.5}{A}$	-15	$\frac{-14.2}{A'}$	$\frac{-13}{19.5}$

Ans. 1607 cu. yds.

358. Remarks.

1. The above method of computing earth-work is called by engineers *The mean average method.*

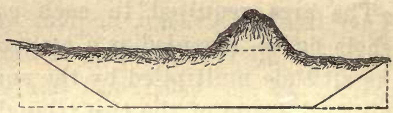
2. The method known as *The arithmetical mean method* is easier than the above, though less accurate.

The following is the formula:

$$v = \tfrac{1}{2} l (b + b').$$

3. The volume can also be computed as a rectangular prismoid.

4. Irregularities in the cross-section surface line, as elevations, depressions, or a curvature of this line, must be considered.

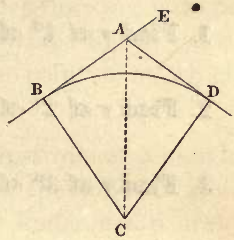


Thus, the elevation may be regarded as a triangle, its area computed and added to the trapezoid before the area of the two triangles at the right and left be deducted.

359. Railroad Curves.

In the preliminary survey of a railroad, any change in direction is made by an angle which must, in the final survey, be replaced by a curve, to which the sides of the angle are tangents.

Let the annexed diagram represent such an angle and curve.



Run out one of the tangents, as BA, to E, and let A denote the external angle EAD.

Then we shall have $C = A$, since each is the supplement of BAD, the angles B and D being right angles.

Let $r = BC$, the radius of curvature, and $t = AB$, the tangent.

Then, (1) $t = r \tan \frac{1}{2} A$, (2) $r = \dfrac{t}{\tan \frac{1}{2} A}$.

The degree of curvature is the number of degrees in an arc whose length is 1 chain or 100 feet.

360. Problem.

Given the degree of curvature, to find the radius; and, conversely, given the radius of curvature, to find the degree.

$$2\pi r = \text{the circumference},$$

$$\frac{2\pi r}{360} = \frac{\pi r}{180} = 1^\circ \text{ of circumference},$$

$$\frac{d\pi r}{180} = d^\circ \text{ of circumference}.$$

$$\text{Hence, } \frac{d\pi r}{180} = 100. \quad \therefore \begin{cases} (1) \quad r = \dfrac{18000}{d\pi}. \\ (2) \quad d = \dfrac{18000}{\pi r}. \end{cases}$$

Having found the radius of curvature, we can find t, the tangent, or the distance from the vertex of the angle to the point where the curve begins by formula (1) of the preceding article.

361. Examples.

1. Find r of 1° of curvature and t, if $A = 40°$.
   Ans. $r = 5729.58$ ft., $t = 2087.4$ ft.

2. Find r of 2° of curvature and t, if $A = 40°$.
   Ans. $r = 2864.79$ ft., $t = 1043.7$ ft.

3. Find r of 3° of curvature and t, if $A = 50°$.
   Ans. $r = 1909.86$ ft., $t = 890.6$ ft.

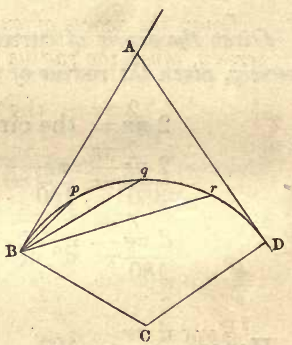
4. Find r and d, if $A = 35°$ and $t = 1000$ ft.
   Ans. $r = 3171.6$ ft., $d = 1° 48' 23''$.

5. Find r and d, if $A = 100°$ and $t = 1$ mile.
   Ans. $r = 4430.4$ ft., $d = 1° 17' 35''$.

362. Location of the Curve.

First Method.

Let each of the arcs, Bp, pq, qr, ... be 1 chain, then will the number of degrees in each, or in the corresponding angle at the center, be equal to d, the degree of curvature.



The angle ABp, formed by a tangent and a chord, is measured by one-half the arc Bp, and is therefore equal to $\frac{1}{2}d$.

Each of the inscribed angles, pBq, qBr, is measured by one-half the intercepted arc, and is therefore equal to $\frac{1}{2}d$.

Having determined the point B, where the curve begins, the transitman sets his instrument at this point, and directs it to A. He then turns it an angle equal to $\frac{1}{2}d$, on the side toward the curve.

The chainmen then take the chain, the follower placing his end at B, and the leader drawing out the chain at full length toward A, is directed by the transitman into line so as to locate the point p, at which the axman drives a stake.

The transitman again turns his instrument an angle equal to $\frac{1}{2}d$, the chainmen advance, the follower placing his end of the chain at p, the leader again drawing out the chain at full length, is directed by the transitman so as to locate the point q, at which the axman drives a stake, and so on.

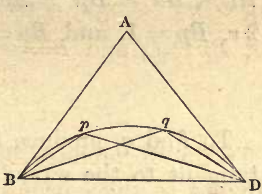
The last distance will usually not be 1 chain; but if n be the number of preceding deflections, the last angle of deflection, since the sum of all the deflections is equal to $\frac{1}{2}C = \frac{1}{2}A$, will be equal to

$$\tfrac{1}{2}A - \tfrac{1}{2}dn.$$

It is to be observed that the chord is made equal to 1 chain instead of the arc; but as the radius is much greater than the chord, the arc and chord will not differ materially, and no appreciable error arises in practice.

Second Method.

Points on the curve may be located by the use of two transits, without the use of the chain, as may be desirable, in case the curve is to be located in marshy ground or shallow water.



Let one transit be placed at B and another at D, the extremities of the curve.

Direct the transit at B to A, the one at D to B, then turn each to the right an angle equal to $\frac{1}{2}d°$.

The intersection of the lines will determine p, where the axman, directed by both transitmen, drives a stake.

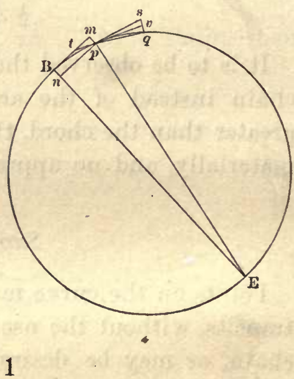
In like manner other points can be located.

If A is visible from D, but not B, direct the transit at D to A; then, to locate p, turn it to the left an angle equal to $\frac{1}{2}A° - \frac{1}{2}d°$.

To locate q, turn the transit at D from p to the right an angle equal to $\frac{1}{2}d°$, or from A to the left an angle equal to $\frac{1}{2}A° - d°$, and the transit at B to the right from p an angle equal to $\frac{1}{2}d°$, or to the right from A an angle equal to $d°$, and so on.

Third Method.

Let B be the point where the curve begins. Take Bm equal to 1 chain. Then, to find the length of the offset mp, complete the circle, draw the diameter BE, let fall the perpendicular pn to BE, and draw pE.



In the right triangle BpE, Bp is a mean proportional between BE and Bn; hence, $BE \times Bn = \overline{Bp}^2$; but $BE = 2r$, $Bp = 1$, and $Bn = mp$,

$$\therefore \quad mp = \frac{1}{2r}.$$

To find q, produce Bp till $ps = 1$ chain, and draw tv, tangent to the curve at p.

$$\text{Then, } spv = tpB = mBp = vpq,$$

For the first and second are vertical, and all the rest are included between tangents and equal chords.

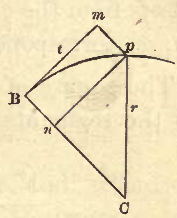
$$\therefore\ spq = 2\,mBp, \quad \therefore \text{ the arc } sq = 2 \text{ arc } mp,$$

Or, the arcs being small, do not differ materially from their chords,

$$\therefore\ sq = 2\,mp = \frac{1}{r}.$$

Hence, to locate a curve by this method without the transit, commence at B, where the curve is to begin, take $Bm = 1$ chain in the direction of the straight track, make the offset $mp = \frac{1}{2\,r}$, produce Bp till $ps = 1$ chain, make the offset sq equal to twice the first offset, produce pq till the produced part $= 1$ chain, make an offset equal to the last, and so on.

Fourth Method.



It is evident from the diagram that

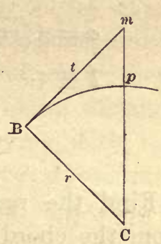
$$mp = BC - nC.$$

But $BC = r$, and $nC = \sqrt{r^2 - t^2}$.

$$\therefore\ mp = r - \sqrt{r^2 - t^2}.$$

By giving to t different values, other points of the curve can be determined.

Fifth Method.



It is evident from the diagram that

$$mp = mC - Cp.$$

But $mC = \sqrt{r^2 + t^2}$, and $Cp = r$.

$$\therefore\ mp = \sqrt{r^2 + t^2} - r.$$

In this method the offset is not made at right angles to the tangent, but in a direction toward the center, which is supposed to be visible from m.

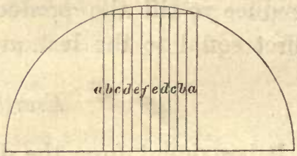
The preceding methods apply to points of the curve 1 chain or 100 feet from each other, which will be sufficient for the excavations or embankments.

Before laying the track, stakes are driven at points on the curve, distant from each other about 10 feet.

363. Problem.

To locate intermediate points on the curve.

Let the diameter in the diagram be parallel to the chord, which is equal to 1 chain = 100 feet, the ordinates $a, b, c, d, e, f, e, d, c, b, a$ be 10 feet from each other, and $v, w, x, y, z, y, x, w, v$ be offsets from the chord to the curve, corresponding to the ordinates $b, c, d, e, f, e, d, c, b$.



The square of an ordinate is equal to the rectangle of the segments into which it divides the diameter.

$$a^2 = (r-50)(r+50), \quad a = \sqrt{(r-50)(r+50)}.$$
$$b = \sqrt{(r-40)(r+40)}, \quad v = b - a.$$
$$c = \sqrt{(r-30)(r+30)}, \quad w = c - a.$$
$$d = \sqrt{(r-20)(r+20)}, \quad x = d - a.$$
$$e = \sqrt{(r-10)(r+10)}, \quad y = e - a.$$
$$f = r, \quad z = f - a.$$

364. Example.

Find the radius of a 1° curvature, and the offsets from the chord of 100 feet to the curve.

Ans. $\begin{cases} r = 5729.58 \text{ ft.}, v = .08 \text{ ft.}, w = .14 \text{ ft.} \\ x = .19 \text{ ft.}, y = .21 \text{ ft.}, z = .22 \text{ ft.} \end{cases}$

TOPOGRAPHICAL SURVEYING.

365. Definition and Method.

Topographical surveying is that branch in which the form of the surface, the situation of ponds, streams, marshes, rocks, trees, buildings, etc., are considered and delineated.

The surface is supposed to be intersected by horizontal planes equally distant from each other, and the curves formed by the intersection of the planes and the surface projected on a horizontal plane.

These projections will be nearer together or farther apart, according as the slope of the surface approaches a vertical or a horizontal position.

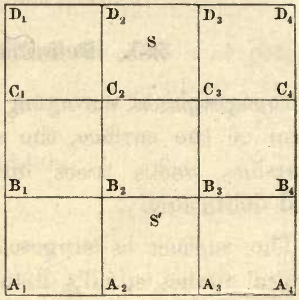
The operations are of two kinds—*field operations* and *plotting*.

366. Field Operations.

Field operations consist in finding and recording points of the curves of intersection of the surface and the horizontal planes, the course of streams, and the situation of noteworthy objects on the surface.

Range with the level, or transit theodolite, which is more convenient in topographical operations, stakes marked as in the annexed diagram, and cause them to be driven into the ground, at a horizontal distance from each other of 100 feet or less, varying with the inequality of the surface and the degree of accuracy with which it is desirable that the work be executed.

Find by the eye, or by the instrument if necessary, the lowest point in the field, at which make a *permanent bench-mark*, and assume for the plane of reference the horizontal plane passing through this point, which we will suppose to be C_1.



Place the instrument at some convenient station, S, from which take the reading of the rod at C_1, which suppose to be 10.378, and enter this as a backsight in the field notes.

Take the readings of the rod at as many stakes as possible from the station S. Suppose these readings to be C_2, 6.481; C_3, 1.214; D_1, 8.235; D_2, 6.378; D_3, 4.102; D_4, 2.304, and enter these readings in the field notes as foresights, placing the smallest reading, C_3, last.

At C_3 drive a small stake for a check.

Subtract the foresight C_2 6.481 from the backsight 10.378, and enter the difference in the column of difference, headed *D.;* also in the column of total difference of level above C_1, headed *T. D. L.*

Subtract each of the remaining foresights from the next preceding one, and enter the results, with their proper signs, in the column *D*.

Add each result to the previous total difference of level, and enter the results in the column *T. D. L.*

The total difference of level for C_3 is also found by subtracting the foresight of C_3 from the backsight of C_1, which, compared with the result before found, will serve as a *check*.

Move the instrument to S', and take a backsight to the check stake C_3, and the foresights to as many of the remaining stakes as possible, suppose all of them and enter the readings in the field notes as before.

Subtract the first of these foresights from the backsight C_3, and add the result to the total difference of level for C_3, and enter the sum in the column *T. D. L.*

Subtract each of the following foresights from the next preceding foresight, and enter the result, with its proper sign, in the column *D.*, and add it to the next preceding difference of level, and enter the sum in the column *T. D. L.*

As a check, subtract the foresight of B_3 from the backsight C_3; the difference will be the height of B_3 above C_3, which add to the former check number, which is the difference of level of C_3 and C_1, and the sum will be the total difference of level of B_3 and C_1.

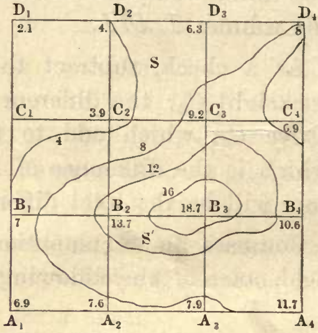
Compare the explanations of this article with the field notes of the following article.

367. Field Notes.

B. S.		*F. S.*		*D.*	*T. D. L.*		*R.*
					C_1	0.000	
C_1	10.378	C_2	6.481	+ 3.897	C_2	3.897	
		D_1	8.235	− 1.754	D_1	2.143	
		D_2	6.378	+ 1.857	D_2	4.000	
		D_3	4.102	+ 2.276	D_3	6.276	
		D_4	2.304	+ 1.798	D_4	8.074	
		C_3	1.214	+ 1.090	C_3	9.164	Check 9.164
C_3	9.687	C_4	12.000	− 2.313	C_4	6.851	
		B_1	11.845	+ 0.155	B_1	7.006	
		B_2	5.184	+ 6.661	B_2	13.667	
		B_4	8.314	− 3.130	B_4	10.537	
		A_1	12.000	− 3.686	A_1	6.851	
		A_2	11.321	+ 0.679	A_2	7.530	
		A_3	10.987	+ 0.334	A_3	7.864	
		A_4	7.125	+ 3.862	A_4	11.726	
		B_3	0.132	+ 6.993	B_3	18.719	Check 9.555
							18.719

368. Plotting.

Let the annexed diagram be a plot of the ground on which is written, with red ink, the height to tenths, taken from the field notes, of the surface, at each stake, above the plane of reference passing through C_1.



Let us suppose that the horizontal planes intersecting the surface are 4 feet apart.

The intersection of the surface and the plane 4 feet above the plane of reference crosses the line A_1 D_1 between the points B_1 C_1, at a point 4 feet above C_1.

To determine this point, observe that the rise from C_1 to B_1 is 7 feet. Then the distance on this line from C_1 to the point where the height above C_1 is 4 feet is found by the proportion,

$$7 : 4 :: 100 : x, \quad \therefore x = 57.1.$$

This method assumes the ascent to be uniform between B_1 and C_1; but this point can be tested and other points of the curve found as follows: Set up the instrument at S, and make the backsight to C_1 10.378, the same as before; then depress the vane on the rod 4 feet — that is, to the reading 6.378.

Now let the rodman set up the rod at the point between C_1 and B_1 determined from the proportion, and let the surveyor observe whether the horizontal wire of the telescope ranges with the horizontal line of the vane; if not, let the rod be moved a little toward B_1 or

C_1 till they do range, and at the point thus determined let a stake marked 4 be driven by the axman.

An inspection of the plot will show that the curve passes between B_2 and C_2 at a distance from C_2 found from the proportion,

$$9.8 : .1 :: 100 : x, \quad \therefore x = 1.$$

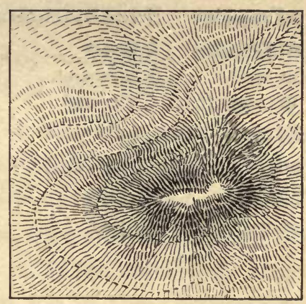
Let the rodman advance toward this point, pausing at one or two intermediate points, and at this point, whose positions are definitely determined and marked.

In a similar manner determine where the curve crosses C_2 C_3 and trace it to D_2.

In like manner, trace the curves of intersection of the surface and planes, 8 feet, 12 feet, and 16 feet above the plane of reference, and let these curves be marked on the ground by stakes numbered 8, 12, and 16, respectively.

The horizontal distance of each stake from two sides of a square can be measured and recorded. From this record the surveyor can draw the curves on the plot as exhibited above.

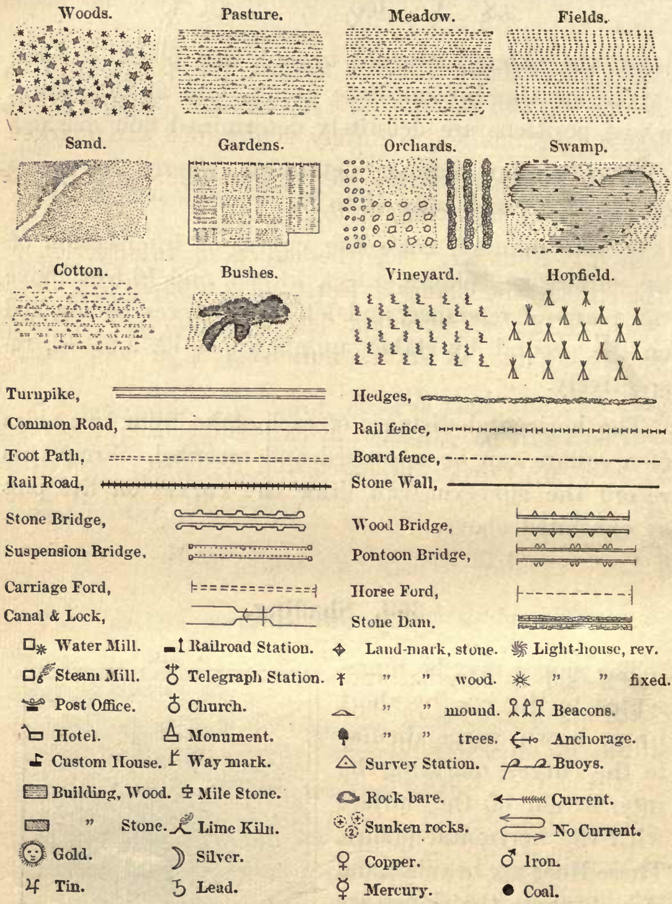
369. Shading.



The slopes may be represented to the eye by short lines drawn perpendicular to the curves, marking the intersection of the surface with the horizontal planes. These lines are heaviest and closest where the slopes are steepest, and lighter where the slopes are less abrupt.

370. Conventional Signs.

The following conventional, though not altogether arbitrary signs, are used to indicate objects worthy of note:



371. Finishing a Map.

The points of compass are indicated as is usual, the top of the map denoting the north, etc., etc.

The meridian, both true and magnetic, should be drawn, and the variation of the needle indicated.

The lettering should be executed with care, after printed models of various styles.

The border may be made by a heavy line, relieved by a light parallel.

The title, in ornamental letters, should occupy one corner of the map, with the name of the locality, the dates of the survey and drawing, and the names of the surveyor and draughtsman.

The scale of horizontal distances, for finding and comparing distances on the map, and *the scale of construction*, used in the smallest measurements required in projecting dimensions in the drawing, should be accurately drawn in some convenient position within the border.

Parallels of latitudes and meridians, in extended surveys, should be drawn in their true position.

BAROMETRIC HEIGHTS.

372. Preliminary Remarks.

The barometer affords an approximative method for finding the difference of level of two stations.

To attain to as great a degree of accuracy as possible, it is important to employ two good barometers, one at the lower and the other at the upper station.

Before using the barometers, they should be carefully compared by frequent trials, and the variation ascertained, which is to be allowed for in the observations.

Increased accuracy is attained by making repeated observations, and taking the mean of the results.

To guard against varying local conditions of the atmosphere affecting pressure, beside difference of elevation, the stations should not be distant from each other more than four or five miles; and the observations should be made when there is no wind.

373. Bailey's Formula.

The subjoined formula requires a knowledge, at both stations, of the height of the column of mercury, its temperature as indicated by an attached thermometer, the temperature of the air as indicated by a detached thermometer, and the latitude of the locality.

Let d denote the difference of level in feet;

l, the latitude of the place in degrees;

h, T, t, respectively, the height of the barometer, the temperature of the mercury, and the temperature of the air at the lower station;

h', T', t', respectively, the same at the upper station.

$$\text{Then, } d = 60345.51\,[1 + .001111\,(t + t' - 64)]$$

$$\times\,(1 + .002695 \cos 2\,l) \times \log \frac{h}{h'\,[1 + .0001\,(T - T')]}$$

$$\text{Let } A = \log \{60345.51\,[1 + .001111\,(t + t' - 64)]\},$$

$$B = \log\,(1 + .002695 \cos 2\,l),$$

$$C = \log\,[1 + .0001\,(T - T')],$$

$$D = \log h - (\log h' + C).$$

$$\therefore\ \log d = A + B + \log D.$$

This formula is applied by the aid of the tables:

374. Howlet's Tables.

Table A, for Detached Thermometer.

$t+t'$	A.	$t+t'$	A.	$t+t'$	A.	$t+t'$	A.
1°	4.74914	46°	4.77187	91°	4.79348	136°	4.81407
2°	.74966	47°	.77236	92°	.79395	137°	.81452
3°	.75017	48°	.77285	93°	.79442	138°	.81496
4°	.75069	49°	.77335	94°	.79489	139°	.81541
5°	.75120	50°	.77384	95°	.79535	140°	.81585
6°	.75172	51°	.77433	96°	.79582	141°	.81630
7°	.75223	52°	.77482	97°	.79628	142°	.81674
8°	.75274	53°	.77530	98°	.79675	143°	.81719
9°	.75326	54°	.77579	99°	.79721	144°	.81763
10°	.75377	55°	.77628	100°	.79768	145°	.81807
11°	.75428	56°	.77677	101°	.79814	146°	.81851
12°	.75479	57°	.77725	102°	.79861	147°	.81896
13°	.75531	58°	.77774	103°	.79907	148°	.81940
14°	.75582	59°	.77823	104°	.79953	149°	.81984
15°	.75633	60°	.77871	105°	.79999	150°	.82028
16°	.75684	61°	.77919	106°	.80045	151°	.82072
17°	.75735	62°	.77968	107°	.80091	152°	.82116
18°	.75786	63°	.78016	108°	.80137	153°	.82160
19°	.75837	64°	.78065	109°	.80183	154°	.82204
20°	.75888	65°	.78113	110°	.80229	155°	.82248
21°	.75938	66°	.78161	111°	.80275	156°	.82291
22°	.75989	67°	.78209	112°	.80321	157°	.82335
23°	.76039	68°	.78257	113°	.80367	158°	.82379
24°	.76090	69°	.78305	114°	.80413	159°	.82423
25°	.76140	70°	.78353	115°	.80458	160°	.82466
26°	.76190	71°	.78401	116°	.80504	161°	.82510
27°	.76241	72°	.78449	117°	.80550	162°	.82553
28°	.76291	73°	.78497	118°	.80595	163°	.82597
29°	.76342	74°	.78544	119°	.80641	164°	.82640
30°	.76392	75°	.78592	120°	.80686	165°	.82684
31°	.76442	76°	.78640	121°	.80731	166°	.82727
32°	.76492	77°	.78687	122°	.80777	167°	.82770
33°	.76542	78°	.78735	123°	.80822	168°	.82814
34°	.76592	79°	.78782	124°	.80867	169°	.82857
35°	.76642	80°	.78830	125°	.80913	170°	.82900
36°	.76692	81°	.78877	126°	.80958	171°	.82943
37°	.76742	82°	.78925	127°	.81003	172°	.82986
38°	.76792	83°	.78972	128°	.81048	173°	.83029
39°	.76842	84°	.79019	129°	.81093	174°	.83072
40°	.76891	85°	.79066	130°	.81138	175°	.83115
41°	.76940	86°	.79113	131°	.81183	176°	.83158
42°	.76990	87°	.79160	132°	.81228	177°	.83201
43°	.77039	88°	.79207	133°	.81273	178°	.83244
44°	.77089	89°	.79254	134°	.81317	179°	.83287
45°	.77138	90°	.79301	135°	.81362	180°	.83329

Table B, for Latitude.

l.	*B.*	*l.*	*B.*	*l.*	*B.*	*l.*	*B.*
0°	0.00117	27°	0.00069	50°	$\bar{1}$.99980	59°	$\bar{1}$.99945
3°	.00116	30°	.00058	51°	.99976	60°	.99941
6°	.00114	33°	.00048	52°	.99972	63°	.99931
9°	.00111	36°	.00036	53°	.99968	66°	.99922
12°	.00107	39°	.00024	54°	.99964	69°	.99913
15°	.00101	42°	.00012	55°	.99960	75°	.99899
18°	.00095	45°	.00000	56°	.99956	80°	.99890
21°	.00087	48°	$\bar{1}$.99988	57°	.99952	85°	.99885
24°	.00078	49°	.99984	58°	.99949	90°	.99883

Table C, for an Attached Thermometer.

T—T′	*C.*	*T—T′*	*C.*	*T—T′*	*C.*	*T—T′*	*C.*
0°	0.00000	12°	0.00052	24°	0.00104	36°	0.00156
1°	.00004	13°	.00056	25°	.00108	37°	.00161
2°	.00009	14°	.00061	26°	.00113	38°	.00165
3°	.00013	15°	.00065	27°	.00117	39°	.00169
4°	.00017	16°	.00069	28°	.00121	40°	.00174
5°	.00022	17°	.00074	29°	.00126	41°	.00178
6°	.00026	18°	.00078	30°	.00130	42°	.00182
7°	.00030	19°	.00082	31°	.00134	43°	.00187
8°	.00035	20°	.00087	32°	.00139	44°	.00191
9°	.00039	21°	.00091	33°	.00143	45°	.00195
10°	.00043	22°	.00095	34°	.00148	46°	.00200
11°	.00048	23°	.00100	35°	.00152	47°	.00204

375. Examples.

1. At the mountain Guanaxuato, in Mexico, lat. 21°, Humboldt made the following observations:

	Lower Station.	*Upper Station.*
Barometric column,	$h = 30.05$,	$h' = 23.66$.
Attached thermometer,	$T = 77°.6$,	$T' = 70°.4$.
Detached thermometer,	$t = 77°.6$,	$t' = 70°.4$.

$$\log d = A + B + \log D.$$

$$\begin{array}{rlrl} \log h\ (30.05) &= 1.47784 & A &= 4.81940 \\ \log h'\ (23.66) &= 1.37402 & B &= 0.00087 \\ \text{Table } C \text{ gives } C &= 0.00031 & \log.\ D &= \bar{1}.01498 \\ \log h' + C &= 1.37433 & \log.\ d &= 3.83525 \\ D = \log h - (\log h' + C) &= 0.10351 & \therefore\ d &= 6843 \text{ ft.} \end{array}$$

2. Find the difference of level of two stations, lat. 42°, from the following data:

$$\left.\begin{array}{l} h = 30,\ \ T = 75°.5,\ \ t = 75°. \\ h' = 25,\ \ T' = 70°.3,\ \ t' = 70°. \end{array}\right\}\ \textit{Ans. } 5195.\text{ ft.}$$

3. Find the difference of level of two stations, lat. 45°, from the following data:

$$\left.\begin{array}{l} h = 29.2,\ \ T = 80°.3,\ \ t = 80°. \\ h' = 27.1,\ \ T' = 77°.4,\ \ t' = 77°. \end{array}\right\}\ \textit{Ans. } 2149.9\text{ ft.}$$

4. Find d, lat. 50°, from the following data:

$$\left.\begin{array}{l} h = 29,\ \ T = 60°.1,\ \ t = 60°. \\ h' = 28,\ \ T' = 59°.1,\ \ t' = 59°.1. \end{array}\right\}\ \textit{Ans. } 973.8\text{ ft.}$$

376. Leveling with one Barometer.

Take the observations at the lower station, then proceed to the upper station and take the observations there, and note the interval of time which has intervened, then go back to the lower station and at the expiration of an equal interval repeat the observations.

Reduce the mercurial column of the second observation at the lower station to what it would have been at the temperature of the first observation, on the principle that mercury expands or contracts .0001 of its volume for each degree of increase or diminution of temperature.

Then take the arithmetical mean of this reduced height and the first observed height for the height at the lower station, the mean of the temperature denoted by the detached thermometer at the lower station for the temperature of the air at that station, and the temperature denoted by the attached thermometer at the first observation for the temperature of the mercury, then proceed as if the observations had been taken with two barometers.

377. Examples.

1. $\begin{cases} \text{Lower sta.} \begin{cases} \text{1st obv., } h = 29.62, & T = 56^\circ.5, & t = 56^\circ. \\ \text{2d obv., } h = 29.63, & T = 63^\circ, & t = 61^\circ. \end{cases} \\ \text{Lat. } 41^\circ.4\text{; upper sta., } h' = 28.94, \quad T' = 57^\circ.5, \quad t' = 57^\circ. \end{cases}$

Reducing h of 2d obv. from $T = 63^\circ$ to $T = 56^\circ.5$, we have,

$$\text{Reduced } h = 29.63\,(1 - 6.5 \times .0001) = 29.611.$$

$$\therefore \quad \text{Mean} \quad h = \frac{29.62 + 29.611}{2} = 29.6155.$$

$$\text{Mean} \quad t = \frac{56^\circ + 61^\circ}{2} = 58^\circ.5.$$

$$\therefore \quad t + t' = 58^\circ.5 + 57^\circ = 115^\circ.5.$$

$$\text{and } T - T' = 56^\circ.5 - 57^\circ.5 = -1^\circ.$$

$\log h$ (29.6155) =	1.47152	A	= 4.80481
$\log h'$ (28.94) =	1.46150	B	= 0.00014
C =	$-$0.00004	$\log D$	= $\bar{2}.00260$
$\log h' + C$ =	1.46146	$\log d$	= 2.80755
$D = \log h - (\log h' + C)$ =	0.01006	$\therefore d$	= 642 feet.

2. $\begin{cases} \text{Lower sta.} \begin{cases} \text{1st obv., } h = 29.7, & T = 60^\circ, & t = 60^\circ. \\ \text{2d obv., } h = 29.75, & T = 66^\circ, & t = 66^\circ. \end{cases} \\ \text{Lat. } 40^\circ\text{; upper sta. } h' = 28.6, \quad T' = 62^\circ, \quad t' = 62^\circ. \end{cases}$

Ans. $d = 1077$ ft.

3. $\begin{cases} \text{Lower sta.} \begin{cases} \text{1st obv., } h = 29.6, & T = 50^\circ, & t = 50^\circ. \\ \text{2d obv., } h = 29.65, & T = 46^\circ, & t = 46^\circ. \end{cases} \\ \text{Lat. } 50^\circ\text{; upper sta. } h' = 27.6, \quad T' = 45^\circ, \quad t' = 45^\circ. \end{cases}$

Ans. $d = 1909$ ft.

NAVIGATION.

PRELIMINARIES.

378. Definition and Classification.

Navigation is the art of ascertaining the place of a ship at sea, and of conducting it from port to port.

There are two methods of finding the place of a ship:

1. **By dead reckoning**; that is, by tracing from the record the courses and distances sailed.

2. **By Nautical Astronomy**; that is, by deducing the latitude and longitude of the place of the ship from celestial observations.

The first method is subdivided into the following:

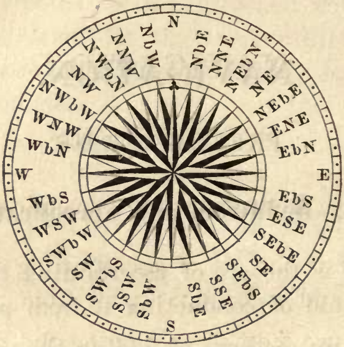
Plane sailing, parallel sailing, middle latitude sailing, Mercator's sailing, and current sailing.

379. The Mariner's Compass.

The magnetic needle rests on a pivot, so as to turn freely.

The compass box is suspended by gimbals or rings, turning on axes at right angles to each other, thus securing a horizontal position notwithstanding the rolling motion of the ship.

A circular card, whose circumference is divided into thirty-two equal parts, called *points*, each of which is subdivided into four equal parts, called *quarter points*, rests upon the needle, with which it turns freely.



N. b. E. is read north by east; *N. N. E.*, north northeast, etc.

380. Table of Points and Angles.

	North.		*South.*		*Angles.*
1	N.b.E.	N.b.W.	S.b.E.	S.b.W.	11° 15′
2	N.N.E.	N.N.W.	S.S.E.	S.S.W.	22° 30′
3	N.E.b.N.	N.W.b.N.	S.E.b.S.	S.W.b.S.	33° 45′
4	N.E.	N.W.	S.E.	S.W.	45° 0′
5	N.E.b.E.	N.W.b.W.	S.E.b.E.	S.W.b.W.	56° 15′
6	E.N.E.	W.N.W.	E.S.E.	W.S.W.	67° 30′
7	E.b.N.	W.b.N.	E.b.S.	W.b.S.	78° 45′
8	E.	W.	E.	W.	90° 0′

Note 1.— $\frac{1}{4}$ point $= 2° 48'\frac{3}{4}$, $\frac{1}{2}$ point $= 5° 37'\frac{1}{2}$, $\frac{3}{4}$ point $= 8° 26'\frac{1}{4}$.

Note 2.— The compass is placed near the helm, at the stern, and the line from the center of the compass to the ship's head indicates the track of the ship.

381. Variation and Deviation of the Compass.

The variation of the compass is the angle included between the magnetic meridian and the true meridian.

The amount of variation is ascertained by Nautical Astronomy.

The deviation of the compass is the deflection of the needle from the magnetic meridian, caused by the iron in the ship.

The amount of deviation is ascertained by special experiments.

382. Course, Leeway, Rhumb Line.

The compass course of a ship, at any point, is the angle which her track makes with the magnetic meridian at that point.

The true course of a ship, at any point, is the angle which her track makes with the true meridian at that point.

In the compass course, the deviation is supposed to be ascertained and allowed for, but not the variation; but in the true course, both the deviation and variation.

The leeway is the oblique motion of the ship, caused by a side wind driving the ship along a track oblique to the fore-and-aft line, and therefore not indicated by the compass.

The amount of leeway, under a wind of a given obliquity and velocity, for each ship with a given freight, is best found by trial.

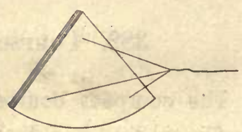
A rhumb line is the track of a ship which continues to make the same angle with the meridians. It is also called a *loxodromic curve*.

Since the meridians converge, the rhumb line is a spiral curve.

In what follows we shall suppose that proper allowances have been made for the variation and deviation of the compass, and, therefore, that the courses given are the true courses.

383. The Log and Log Line.

The log, a drawing of which is annexed, is a board in the form of a quadrant whose radius is about six inches, the circular part of which is loaded with lead, sufficient to give it a vertical position and to cause it to sink so that the vertex shall be just above the surface.



The log line is a line about 120 fathoms in length, and so attached to the log as to keep its face toward the ship, that it may, by the resistance it encounters from the water, unwind the line from a reel as the vessel advances.

The log line is divided into equal parts called *knots*, each knot being $\frac{1}{120}$ of a nautical mile, or $50\frac{2}{3}$ feet.

The time is measured by a sand glass, through which the sand passes in $\frac{1}{120}$ of an hour, or in $\frac{1}{2}$ of a minute.

Since the number of knots in a nautical mile is equal to the number of half-minutes in an hour, it follows that the number of knots run off in half a minute is equal to the number of miles the ship is sailing an hour.

The divisions of the line are marked by strings passing through the line and knotted, the number of knots in the string indicating the number of parts between it and that point of the line where the divisions commence at that end of the line next to the log.

The stray line is about 10 fathoms of the end of the line from the log to the point where the divisions begin. This portion allows the log to settle in the water, clear of the ship, before the measurement of the rate begins.

The termination of the stray line is marked by a piece of red cloth.

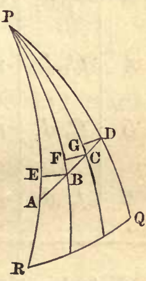
The sand glass is turned the instant this cloth passes the reel, which is stopped the moment the sand has run out.

The number of knots on the string which marks the last division run from the reel, indicates the rate of sailing.

PLANE SAILING.

384. Single Courses.

Let P be the pole of the earth; RQ, the equator; AD, a rhumb line divided into AB, BC, CD, etc., parts so small that we may regard them as straight lines; and the triangles ABE, BCF, CDG, plane triangles and similar, which give the continued proportions:



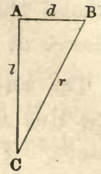
$$AB : AE :: BC : BF :: CD : CG.$$

$$AB : EB :: BC : FC :: CD : GD.$$

Since the sum of the antecedents is to the sum of the consequents as one antecedent is to its consequent, we have,

$$AD : AE + BF + CG :: AB : AE.$$

$$AD : EB + FC + GD :: AB : EB.$$



Now let a right triangle, ABC, be constructed, in which C is the course or the angle which the rhumb line makes with the meridian. $r = CB = AD$, the rhumb line of the first figure; $l = CA = AE + BF + CG =$ difference of latitude; $d = AB = EB + FC + GD =$ the sum of the elementary departures.

We may now, without supposing the ship to sail on a plane, replace the surface on which it actually sails by a plane surface, and hence the name *plane sailing.*

385. Table of Cases.

	Given.	*Req.*	*Formulas.*	
1	r, C,	l, d.	$l = r \cos C$,	$d = r \sin C$.
2	r, l,	C, d.	$\cos C = \frac{l}{r}$,	$d = \sqrt{r^2 - l^2}$.
3	r, d,	C, l.	$\sin C = \frac{d}{r}$,	$l = \sqrt{r^2 - d^2}$.
4	C, l,	r, d.	$r = \frac{l}{\cos C}$,	$d = l \tan C$.
5	C, d,	r, l.	$r = \frac{d}{\sin C}$,	$l = \frac{d}{\tan C}$.
6	l, d,	r, C.	$r = \sqrt{l^2 + d^2}$,	$\tan C = \frac{d}{l}$.

Note 1. — l in miles may be reduced to degrees by dividing by 60.

Note 2. — Examples in case I. may be solved by the Traverse table.

386. Examples.

1. A ship sails 105 miles N. E. by N., from latitude 50°; required the latitude in which the ship then is, and the departure made.

Ans. 51° 27′.3 N., $d = 58.34$ mi.

2. A ship sailed between S. and W. 148 miles, making the difference of latitude 114.4; required the course and the departure made.

Ans. $3\frac{1}{2}$ pts. W. of S., $d = 93.9$ mi.

3. A ship in latitude 3° 52′ S. sails between N. and W. 1065 miles, making a departure of 939 miles; required the course and the latitude in which she then is.

Ans. N. W. b. W. $\frac{1}{2}$W., lat. 4° 30′ N.

4. A ship ran from latitude 38° 32′ N. to latitude 36° 56′ N. on a course S. E. by S. $\frac{3}{4}$E.; required the distance sailed and the departure made.

Ans. $r = 129.56$ mi., $d = 87.009$ mi.

5. A ship sailed S. 56° 47′ E. from latitude 50° 13′ N. till her departure was 82 miles; required r and latitude in. *Ans.* $r = 98$ mi., lat. 49° 19′ N.

6. A ship from latitude 36° 12′ N. sails between S. and W. till she is in latitude 35° 1′ N., having made 76 miles of departure; required r and C.

Ans. $r = 104$ mi., $C =$ S. 46° 57′ W.

387. Compound Courses.

A compound course or traverse is the zigzag course which a ship usually takes in a voyage of considerable length.

Working the traverse is the computation of a single course and distance from the place of departure to the place of destination.

To do this, find by the Traverse table the latitude and departure of each course. The difference of the sum of the northings and the sum of the southings will be the latitude of the single course required, and the difference of the sum of the eastings and the sum of the westings will be the departure, both of the name of the greater. Then proceed as in last article.

388. Examples.

1. A ship sailed from latitude 51° 24′ N. as follows: S. E. 40 miles, N. E. 28 miles, S. W. by W. 52 miles, N. W. by W. 30 miles, S. S. E. 36 miles, S. E. by E. 58 miles; required the latitude in, and the single equivalent course and distance.

Solution.

Courses.	*Dist.*	*N. L.*	*S. L.*	*E. D.*	*W. D.*
S. E.	40		28.3	28.3	
N. E.	28	19.8		19.8	43.2
S. W. b. W.	52		28.9		24.9
N. W. b. W.	30	16.7			
S. S. E.	36		33.3	13.8	
S. E. b. E.	58		32.2	48.2	
		36.5	122.7	110.1	68.1
			36.5	68.1	
			86.2	42	

$$\tan C = \frac{d}{l} = \frac{42}{86.2}. \quad \therefore \; C = 25° \, 59'.$$

$$r = \sqrt{l^2 + d^2} = 95.87 \text{ mi.}$$

$l = 86.2$ mi. $= 1° \, 26'$. $\therefore$ $51° \, 24' - 1° \, 26' = 49° \, 58'$ N.

2. Given the following courses and distances: S. W. ¾ W. 62 miles, S. by W. 16 miles, W. ¼ S. 40 miles, S.W.

$\frac{3}{4}$ W. 29 miles, S. by E. 30 miles, S. $\frac{3}{4}$ E. 14 miles; required l, C, and r.

Ans. $l = 1° 55'$ S., $C =$ S. 43° 14′ W., $r = 158$ mi.

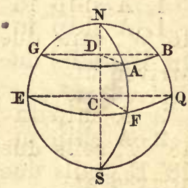
3. A ship, from latitude 1° 12′ S., has sailed as follows: E. by N. $\frac{1}{2}$ N. 56 miles, N. $\frac{1}{4}$ E. 80 miles, S. by E. $\frac{1}{2}$ E. 96 miles, N. $\frac{1}{4}$ E. 68 miles, E. S. E. 40 miles, N. N. W. $\frac{1}{2}$ W. 86 miles, E. by S. 65 miles; required the latitude in, C, and r.

Ans. Lat. in, 0° 48′ N., $C = 51° 47'$ E., $r = 193.8$ mi.

PARALLEL SAILING.

389. Definition and Principles.

Parallel sailing is that case of sailing in which the track is on a parallel of latitude.



Let EFQ be the equator;
GAB, the parallel of the track;
$r = AB =$ the distance sailed;
$L = FQ =$ the difference of longitude;
$l = QB =$ the latitude of the track.

Since similar arcs are to each other as their radii,

$$(1) \quad DB : CQ :: AB : FQ.$$

Consider the radius CQ as the unit of the first couplet, then DB will be the natural co-sine of latitude; and take 1 mile as the unit in the second couplet, put r for AB, L for FQ, then (1) becomes,

$$(2) \quad \cos l : 1 :: r : L, \quad \therefore \quad (3) \quad L = \frac{r}{\cos l}.$$

We can compute L in (3) by taking nat. cos l, or by introducing R and taking log. cos l. In either case L will be found in miles, since r is given in miles; but L can be reduced to degrees by dividing by 60.

Let r and r', measured on the parallels whose latitudes are l and l', respectively, be the distances between two meridians whose difference of longitude is L.

$$\left.\begin{matrix}\cos l : 1 :: r : L, \\ \cos l' : 1 :: r' : L,\end{matrix}\right\} \therefore \cos l : \cos l' :: r : r'.$$

Hence, *The distances between two meridians, measured on different parallels, are as the co-sines of the latitudes of those parallels.*

To find the length of a degree of longitude on any parallel, observe that at the equator 1° of lon. = 60 nautical miles, and that $\cos l = 1$, then we shall have,

$$1 : \cos l' :: 60 : r', \quad \therefore r' = 60 \cos l'.$$

390. Examples.

1. A ship in latitude 49° 32′ N., and longitude 10° 16′ W., sails due W. 118 miles; required the longitude arrived at. *Ans.* 13° 18′ W.

2. A ship in latitude 53° 36′ N., and longitude 10° 18′ E., sails due W. 236 miles; required the longitude arrived at. *Ans.* 3° 40′ E.

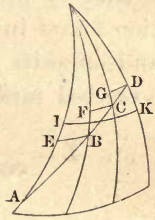
3. A ship in latitude 32° N. sails 6° 24′ due W.; required d. *Ans.* $d = 325.6$ mi.

4. A ship sails 310 miles from longitude 81° 36′ W. to longitude 91° 50′ W.; required the latitude of the track. *Ans.* 59° 41′.

MIDDLE LATITUDE SAILING.

391. Definition and Principles.

Middle latitude sailing is a combination of plane sailing and parallel sailing, on the supposition that the departure in plane sailing is equal to the distance between the meridians passing through the extreme points of the rhumb line, measured on the middle parallel between these points.



Let AD be a rhumb line; IK, the middle parallel; m, the latitude of IK; then $d = EB + FC + GD = IK$.

For r, formula (3), parallel sailing, substitude d or its value as found in plane sailing; and for $\cos l$ substitute $\cos m$, then we shall have,

$$L = \frac{d}{\cos m} = \frac{r \sin C}{\cos m} = \frac{\sqrt{r^2 - l^2}}{\cos m} = \frac{l \tan C}{\cos m}.$$

Note 1.—Remember that in these formulas l denotes the difference of latitude; L, the difference of longitude in miles; d, the departure; r, the distance run or the rhumb line; C, the course, and m, the middle latitude.

Note 2.—The middle latitude is the half sum of the extreme latitudes; or the less latitude, plus the half difference of latitude; or the greater latitude, minus the half difference of latitude.

Note 3.—That the departure is not strictly equal to the middle-latitude distance between the meridians, through the extremities of the rhumb line, is thus shown:

Suppose a ship to sail on this middle latitude from one of the meridians to the other, then the distance sailed will be the departure; but if a second ship were to sail from a lower latitude on the first meridian, and a third ship, from a higher, to the same place, the departure of the second would be greater, and the departure of the third would be less than that of the first.

It is necessary, therefore, to make the correction for middle latitude as found in the table for such corrections.

The following is the rule for correcting the middle latitude:

Add to the uncorrected middle latitude the correction found in the table under the difference of latitude, and opposite the middle latitude — the sum m' is the corrected middle latitude.

$$\therefore \quad L = \frac{d}{\cos m'} = \frac{r \sin C}{\cos m'} = \frac{\sqrt{r^2 - l^2}}{\cos m'} = \frac{l \tan C}{\cos m'}.$$

392. Examples.

1. A ship from latitude 51° 18′ N., longitude 9° 50′ W., sails S. 33° 8′ W. 1024 miles; required the latitude and longitude in.

$$l = r \cos C, \quad \therefore l = 857.4 \text{ mi.} = 14° \; 17'.$$

$$\therefore \quad 51° \; 18' - 14° \; 17' = 37° \; 1', \text{ the lat. in.}$$

$$\tfrac{1}{2}(51° \; 18' + 37° \; 1') = 44° \; 9\tfrac{1}{2}' = \text{mid. lat., correction} = 27'.$$

$$44° \; 9\tfrac{1}{2}' + 27' = 44° \; 36\tfrac{1}{2}' = m' = \text{corrected mid. lat.}$$

$$L = \frac{r \sin C}{\cos m'}, \quad \therefore L = 786.3 \text{ mi.} = 13° \; 6'.$$

$$9° \; 50' + 13° \; 6' = 22° \; 56' \text{ W., the lon. in.}$$

2. A ship, from latitude 52° 6′ N., and longitude 35° 6′ W., sails N. W. by W. 229 miles; required the latitude and longitude arrived at.

Ans. Lat. 54° 13′ N. and lon. 40° 23′ W.

3. A ship from latitude 49° 57′ N., and longitude 5° 11′ W., sails between S. and W. till she is in latitude 38° 27′ N., when she has made 440 miles departure; required C, r, and the longitude in.

Ans. C = S. 32° 32′ W.; r = 818 mi.; lon. in, 15° 28′ W.

4. A ship from latitude 37° N., longitude 22° 56′ W., sails N. 33° 19′ E. till she is in latitude 51° 18′ N. What longitude is she in? *Ans.* 9° 45′ W.

5. A ship from latitude 40° 41′ N., longitude 16° 37′ W., sails between N. and E. till she is in latitude 43° 57′ N., and finds that she has made 248 miles departure; required C, r, and longitude in.

Ans. $C = 51° 41'$ E.; $r = 316$ mi.; lon. in, 11° W.

MERCATOR'S SAILING.

393. Definitions and Principles.

Mercator's chart, so called from its originator, Gerrard Mercator, a Fleming, who first published it in 1556, is a representation of the surface of the earth on the supposition that the earth is a cylinder.

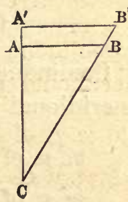
The meridians are thus represented parallel and every-where too far apart except at the equator.

To guard as much as possible against distortion, the distances between the parallels are proportionally increased.

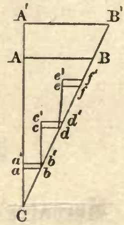
The surface is thus relatively magnified more and more toward the poles.

Mercator's sailing is the method of computing the difference of longitude from the principle on which Mercator's chart is projected.

The mathematical theory of this method was developed, and the *Table of Meridional Parts*, necessary to its application, computed by Edward Wright, an Englishman, in 1599.



Let CA and AB, respectively, be the difference of latitude and departure corresponding to the rhumb line CB, and let CA be produced to A' till $A'B'$, the corresponding departure, is equal to the difference of longitude of C and B. CA' is called the *meridional difference of latitude*, which is simply the proper difference of latitude increased till the corresponding departure is equal to the difference of longitude corresponding to the proper departure.



To find the meridional difference of latitude, let Cb, bd, df, ... be indefinitely small portions of the rhumb line CB. Ca, bc, de, ... corresponding differences of latitude; ab, cd, ef, ... corresponding differences of departure; Ca', bc', de', ... corresponding meridional differences of latitude; $a'b'$, $c'd'$, $e'f'$, ... differences of longitude corresponding to the departures ab, cd, ef, ... whose latitudes are l, l', l'', ... Then, as found in Parallel sailing,

$$ab : a'b' :: \cos l : 1.$$

$$\text{but} \quad ab : a'b' :: Ca : Ca'.$$

$$\therefore \cos l : 1 :: Ca : Ca', \quad \therefore Ca' = \frac{Ca}{\cos l}.$$

$$\text{but} \quad \frac{1}{\cos l} = \sec l, \quad \therefore Ca' = Ca \sec l.$$

In like manner, $\qquad bc' = bc \sec l'$,

$$de' = de \sec l''.$$

. . . .

$$\text{But} \quad CA' = Ca' + bc' + de' + \ldots$$

Substituting the values of Ca', bc', de', ... and making $Ca = bc = de = \ldots = 1'$, we have,

$$CA' = \sec l + \sec l' + \sec l'' + \ldots$$

Commencing at the equator, and putting *m. p.* for meridional parts, and taking natural secants, we have,

$m. p.$ of $1' = \sec 1'$.

$m. p.$ of $2' = \sec 1' + \sec 2'$.

$m.p.$ of $3' = \sec 1' + \sec 2' + \sec 3'$.

$m.p.$ of $4' = \sec 1' + \sec 2' + \sec 3' + \sec 4'$.

. . . .

By substituting and condensing, we have,

$m.p.$ of $1' = 1.0000000 \qquad\qquad\qquad\qquad = 1.0000000$

$m.p.$ of $2' = 1.0000000 + 1.0000002 = 2.0000002$

$m.p.$ of $3' = 2.0000002 + 1.0000004 = 3.0000006$

$m.p.$ of $4' = 3.0000006 + 1.0000007 = 4.0000013$

. . . .

The accuracy of the result is increased by taking the parts still smaller, as $\frac{1}{2}'$.

Having found the meridional latitude corresponding to C, and also to A, their difference will be the meridional difference of latitude found from the table; and the corresponding departure, $A'B'$, will be the difference of longitude.

Denoting the proper difference of latitude CA by l, the meridional difference of latitude by l', the departure AB by d, and the difference of longitude $A'B'$ by L, the triangles CAB and $CA'B'$ give,

$$1 : \tan C :: l' : L, \quad \therefore\ L = l' \tan C.$$

$$l : d :: l' : L, \quad \therefore\ L = \frac{l'd}{l}.$$

394. Examples in Single Courses.

1. A ship from latitude 52° 6′ N., and longitude 35° 6′ W., sails N. W. by W. 229 miles; required the latitude and longitude in.

$l = r \cos C = 229 \cos 56° 15', \quad \therefore\ l = 127.3$ mi. $= 2° 7'$.

lat. in $= 52° 6'$ N. $+ 2° 7'$ N. $= 54° 13'$ N.

$m. p.$ of $54° \ 13' = 3868$	But $L = l' \tan C$,
$m. p.$ of $52° \ \ 6' = 3657$	$\therefore \ L = 211 \tan 56° \ 15'$.
$\therefore \ l' = \ 211$	or $L = 315.8$ mi. $= 5° \ 16'$.

$\therefore$ lon. in $= 35° \ 6'$ W. $+ \ 5° \ 16'$ W. $= 40° \ 22'$ W.

2. A ship from latitude 51° 18′ N., and longitude 9° 50′ W., sails S. 33° 8′ W. 1024 miles; required the latitude and longitude in.

Ans. Lat. in 37° 1′ N.; lon. in 22° 50′ W.

3. Required the course and distance from Ushant, latitude 48° 28′ N., longitude 5° 3′ W., to St. Michael's, latitude 37° 44′ N., longitude 25° 40′ W.

Ans. S. 54° 30′ W., $r = 1106$ mi.

4. A ship from latitude 51° 9′ N. sails S. W. b. W. 216 miles; required the latitude in, and the difference of longitude made. *Ans.* Lat. 49° 9′ N., $L = 4° \ 39'$.

5. A ship sails from latitude 37° N., longitude 22° 56′ W., on the course N. 33° 19′ E., till she arrives at latitude 51° 18′ N.; required the distance sailed and the longitude arrived at. *Ans.* 1027 mi., lon. 9° 47′ W.

6. A ship sails N. E. b. E. from latitude 42° 25′ N., and longitude 15° 6′ W., till she finds herself in latitude 46° 20′ N.; required the distance sailed and the longitude in. *Ans.* Dist., 423 mi.; lon. 6° 55′ W.

395. Examples in Compound Courses.

1. A ship from latitude 60° 9′ N., and longitude 1° 7′ W., sailed as follows: N. E. b. N., 69 miles; N. N. E., 48 miles; N. b. W. ½W., 78 miles; N. E., 108 miles; S. E. b. E., 50 miles; required the latitude and longitude in, and the direct course and distance.

Courses.	Dist.	N. L.	S. L.	Lat.	m. p.	m. d. l.	E. L.	W. L.
N. E. b. N.	69	57.4		60°9′	4525			
N. N. E.	48	44.4		61°6′	4641	116	77.5	
N.b.W.½W.	78	74.6		61°50′	4733	92	38.1	
N. E.	108	76.4		63°5′	4895	162		49.
S. E. b. E.	50		27.8	64°21′	5067	172	172.0	
		252.8		63°53′	5003	64	95.8	
		27.8						

Dif. lat. $= l = 225$ mi. $= 3° 45'$ N. 383.4

49.

Dif. lon. $= L = 334.4$ mi. $=$

Lat. Left $= 60°\ \ 9'$ N. Dif. lon. $= 5° 34'$ E.

Dif. Lat. $= \ \ 3° 45'$ N. Lon. left $= 1°\ \ 7'$ W.

Lat. in $= 63° 54'$ N. Lon. in $= 4° 27'$ E.

m. p. of lat. in (63° 54′) $= 5005.$

m. p. of lat. left (60° 9′) $= 4525.$

Meridional dif. lat. $= l' = 480.$

$$\tan C = \frac{L}{l'} = \frac{334.4}{480}, \therefore C = \text{N. } 34° 53' \text{ E.}$$

$$r = \frac{l}{\cos C} = \frac{225}{\cos 34° 53'}, \therefore r = 273 \text{ mi.}$$

2. A ship from latitude 38° 14′ N., and longitude 25° 56′ W., has sailed the following courses: N. E. b. N. ¼E., 56 miles; N. N. W., 38 miles; N. W. b. W., 46 miles; S. S. E., 30 miles; S. b. W., 20 miles; N. E. b. N., 60 miles; required the latitude and longitude in, and the direct single course and distance.

Ans Lat. in, 40° 2′. 3 N.; lon. in, 25° 30′ W.; $C =$ N. 10° 33′ E., $r = 110.2$ mi.

396. Correction for Middle Latitude.

We are now prepared to understand how the correction for middle latitude, before used, is found.

l denotes the proper difference of latitude;
l', the meridional difference of latitude;
L, the difference of longitude;
m, the middle latitude uncorrected;
c, the correction;
m', the middle latitude corrected.

Then, by Plane, Middle latitude, and Mercator's sailing,

$$\tan C = \frac{d}{l} = \frac{L \cos m'}{l} = \frac{L}{l'}, \quad \therefore \cos m' = \frac{l}{l'}.$$

From which m' is readily found.

Then, $c = m' - m. \quad \therefore m' = m + c.$

CURRENT SAILING.

397. Definition and Principles.

Current sailing is the sailing of a ship as affected by a current.

Irrespective of the current the ship would move, in a certain time, a certain course and distance.

The current alone would carry the ship, in the same time, a certain other course and distance.

The actual track of the ship, which is the resultant of the two, will bring her to the same position as if she had sailed separately the two tracks.

Current sailing may therefore be treated as Plane sailing, compound courses.

The set of the current is its direction.

The drift of the current is its velocity.

The set and drift of a current may be ascertained by taking, a short distance from the ship, a boat, which is kept from being carried by the current by letting down, to a considerable depth, a heavy weight, which is attached by a rope to the stern of the boat.

The log being thrown from the boat into the current, the direction in which it is carried, or set of the current, is determined by the boat compass, and the rate at which it is carried, or drift of the current, by the number of knots of the log line run out in half a minute.

398. Examples.

1. A ship sails N. W. a distance, by the log, of 60 miles, in a current that sets S. S.W., drifting 25 miles in the same time; required the course and distance.

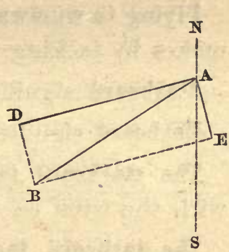
Courses.	*Dist.*	*N. L.*	*S. L.*	*E. D.*	*W. D.*
N. W.	60	42.4			42.4
S. S. W.	25		23.1		9.6

$l = 19.3.$ $d = 52.$

$$\tan C = \frac{d}{l} = \frac{52}{19.3}, \quad \therefore C = \text{N. } 69° \; 38' \text{ W.}$$

$$r = \sqrt{l^2 + d^2} = \sqrt{(19.3)^2 + (52)^2} = 55.5.$$

2. A ship, sailing 7 knots an hour, is bound to a port bearing S. 52° W., through a current S. S. E., 2 miles an hour; required the course.



Let AB be the direction of the port.

AE, the direction of the current, $= 2$.

AD, the required direction, $= 7$.

Complete the parallelogram, $DBA = BAE = 52° + 22° \; 30' = 74° \; 30'$. Then we have,

$$AD : DB :: \sin DBA : \sin DAB.$$

$$\therefore \quad \sin DAB = \frac{2 \sin DBA}{7}.$$

$$\therefore \quad DAB = 15° \ 59'. \qquad \therefore \quad C = 15° \ 59' + 52° = 67° \ 59'.$$

3. A ship runs N. E. by N. 18 miles in 3 hours, in a current W. by S. 2 miles an hour; required the course and distance. *Ans.* $C =$ N. b. E. $\frac{1}{2}$E., $r = 14$ mi.

4. In a current S. E. by S. $1\frac{1}{2}$ miles an hour, a ship sails 24 hours as follows: S.W., 40 miles; W. S.W., 27 miles; S. by E., 47 miles; required the direct course and the distance. *Ans.* $C =$ S. 11° 50′ W., $r = 117$ mi.

5. The port bears due E., the current sets S. W. by S. 3 knots an hour, the rate of sailing is 4 knots an hour; required the course steered. *Ans.* N. 51° E.

6. A ship sailing in a current has, by her reckoning, run S. by E. 42 miles, and, by observations, is found to have made 55 miles of difference of latitude, and 18 miles of departure; required the set and drift of the current. *Ans.* Set, S. 62° 12′ W.; whole drift, 30 mi.

PLYING TO WINDWARD.

399. Definitions.

Plying to windward is the zigzag course which a ship makes by tacking when she encounters a foul wind.

Starboard signifies the right side.

Larboard signifies the left side.

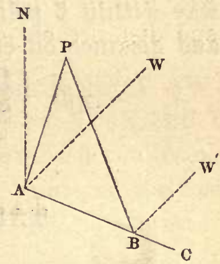
The starboard tacks are aboard when a ship plies with the wind on the right.

The larboard tacks are aboard when a ship plies with the wind on the left.

A ship is said to be *close-hauled* when she sails as nearly as possible toward the point from which the wind is blowing.

400. Examples.

1. Being within sight of my port bearing N. by E. ½E., distant 18 miles, a fresh gale sprung up from the N. E. With my larboard tacks aboard, and close-hauled within six points of the wind, how far must I run before tacking about, and what will be my distance from the port on the second board?



Let A be the place of the ship; P, the port; AB, the distance of the first board; BP, that of the second; WA or $W'B$, the direction of the wind.

Then, $WAB = W'BC = W'BP =$ 6 points.

$\therefore$ $ABP = 16$ points $-$ 12 points $= 4$ points.

$PAW = NAW - NAP = 4$ points $-$ 1½ points $=$ 2½ points.

$PAB = PAW + WAB =$ 2½ points $+$ 6 points $=$ 8½ points.

$APB = 16$ points $- (PAB + ABP) =$ 3½ points.

$\sin ABP : \sin APB :: AP : AB$, $\quad \therefore AB = 16.15$ mi.

$\sin ABP : \sin BAP :: AP : BP$, $\quad \therefore BP = 25.23$ mi.

2. If a ship can lie within 6 points of the wind on the larboard tack, and within 5½ points on the starboard tack; required her course and distance on each tack to reach a port lying S. by E. 22 miles, the wind being at S. W.

Ans. { Starboard tack, S. b. E. ½E. 23.66 mi.
Larboard tack, W. N. W. 2.79 mi.

3. A ship is bound to a port 80 miles distant, and directly to windward, which is N. E. by N. $\frac{1}{2}$E., and proposes to reach her port at two boards, each within 6 points of the wind, and to lead with the starboard tack; required her course and distance on each tack.

Ans. { Starboard tack, N. N.W. $\frac{1}{2}$W., 104.5 mi.
Larboard tack, E. S. E. $\frac{1}{2}$E., 104.5 mi.

4. Wishing to reach a point bearing N. N. W. 15 miles, but the wind being at W. by N., I was obliged to ply to windward—the ship, close-hauled, could make way within 6 points of the wind; required the course and distance on each tack.

Ans. { Larboard tack, N. b. W. 17.65 mi.
Starboard tack, S. W. b. S. 4.138 mi.

TAKING DEPARTURES.

401. Explanation.

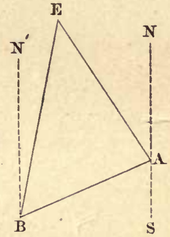
Before losing sight of land, at the beginning of a voyage, the bearing and distance of some well-known object, as a light-house or headland, is taken, the reverse bearing and distance of which are entered as the first course and distance on the log board.

The bearing is taken by the compass; but the distance is sometimes estimated by the eye, as can be done with considerable accuracy by navigators of experience.

A more correct method of taking a departure is by means of data, obtained by taking the bearing at two different positions of the ship, the distance between these positions being measured by the log.

402. Examples.

1. Sailing down the channel, the Eddystone bore N.W. by N., and after running W. S. W. 18 miles, it bore N. by E.; required the course and distance from the Eddystone to the place of the last observation.



$$E = NAE + N'BE = 4 \text{ points.}$$

$$A = 16 \text{ points} - (NAE + BAS) = 7 \text{ pts.}$$

$$\sin E : \sin A :: AB : BE,$$

$$\therefore \quad BE = 24.97.$$

2. At 3 o'clock P. M. the Lizard bore N. by W. $\frac{1}{2}$W., and after sailing 7 knots an hour, W. by N. $\frac{1}{4}$N., till 6 o'clock, the Lizard bore N. E. $\frac{3}{4}$E.; required the course and distance from the Lizard to the place of the last observation. *Ans.* S.W. $\frac{3}{4}$W., 19.35 mi.

3. In order to get a departure, I observe a headland of known latitude and longitude to bear N. E. by N., and after sailing E. by N. 15 miles, the same headland bore W. N. W.; required my distance from the headland at each place of observation.

Ans. 8.5 mi. and 10.8 mi.

Remark.—To find the latitude and longitude of a ship by means of celestial observations, requires a knowledge of Nautical Astronomy; but a thorough discussion of this subject would require an amount of space far exceeding our limits.

TABLES.

Logarithms of Numbers to 100.

1	0 00000	21	1.32222	41	1 61278	61	1.78533	81	1.90849
2	0.30103	22	1.34242	42	1.62325	62	1 79239	82	1.91381
3	0 47712	23	1.36173	43	1.63347	63	1.79934	83	1.91908
4	0.60206	24	1.38021	44	1.64345	64	1.80618	84	1.92428
5	0.69897	25	1.39794	45	1.65321	65	1.81291	85	1.92942
6	0.77815	26	1.41497	46	1.66276	66	1.81954	86	1.93450
7	0.84510	27	1.43136	47	1.67210	67	1.82607	87	1.93952
8	0.90309	28	1.44716	48	1.68124	68	1.83251	88	1.94448
9	0.95424	29	1.46240	49	1.69020	69	1 83885	89	1.94939
10	1.00000	30	1.47712	50	1.69897	70	1.84510	90	1.95424
11	1.04139	31	1.49136	51	1.70757	71	1.85126	91	1.95904
12	1.07918	32	1.50515	52	1 71600	72	1.85733	92	1.96379
13	1.11394	33	1.51851	53	1.72428	73	1.86332	93	1.96848
14	1.14613	34	1.53148	54	1.73239	74	1.86923	94	1.97313
15	1.17609	35	1.54407	55	1.74036	75	1.87506	95	1.97772
16	1.20412	36	1.55630	56	1.74819	76	1.88081	96	1.98227
17	1.23045	37	1.56820	57	1.75587	77	1.88649	97	1.98677
18	1.25527	38	1.57978	58	1.76343	78	1.89209	98	1.99123
19	1.27875	39	1.59106	59	1.77085	79	1.89763	99	1.99564
20	1.30103	40	1.60206	60	1.77815	80	1.90309	100	2.00000

N.	0	1	2	3	4	5	6	7	8	9	D.
100	00000	043	087	130	173	217	260	303	346	389	43
101	432	475	518	561	604	647	689	732	775	817	43
102	860	903	945	988	030	072	115	157	199	242	42
103	01284	326	368	410	452	494	536	578	620	662	42
104	703	745	787	828	870	912	953	995	036	078	42
105	02119	160	202	243	284	325	366	407	449	490	41
106	531	572	612	653	694	735	776	816	857	898	41
107	938	979	019	060	100	141	181	222	262	302	40
108	03342	383	423	463	503	543	583	623	663	703	40
109	743	782	822	862	902	941	981	021	060	100	40
110	04139	179	218	258	297	336	376	415	454	493	39
111	532	571	610	650	689	727	766	805	844	883	39
112	922	961	999	038	077	115	154	192	231	269	39
113	05308	346	385	423	461	500	538	576	614	652	38
114	690	729	767	805	843	881	918	956	994	032	38
115	06070	108	145	183	221	258	296	333	371	408	38
116	446	483	521	558	595	633	670	707	744	781	37
117	819	856	893	930	967	004	041	078	115	151	37
118	07188	225	262	298	335	372	408	445	482	518	37
119	555	591	628	664	700	737	773	809	846	882	36
120	918	954	990	027	063	099	135	171	207	243	36
121	08279	314	350	386	422	458	493	529	565	600	36
122	636	672	707	743	778	814	849	884	920	955	35
123	991	026	061	096	132	167	202	237	272	307	35
124	09342	377	412	447	482	517	552	587	621	656	35
125	691	726	760	795	830	864	899	934	968	003	35
126	10037	072	106	140	175	209	243	278	312	346	34
127	380	415	449	483	517	551	585	619	653	687	34
128	721	755	789	823	857	890	924	958	992	025	34
129	11059	093	126	160	193	227	261	294	327	361	34
130	394	428	461	494	528	561	594	628	661	694	33
131	727	760	793	826	860	893	926	959	992	024	33
132	12057	090	123	156	189	222	254	287	320	352	33
133	385	418	450	483	516	548	581	613	646	678	33
134	710	743	775	808	840	872	905	937	969	001	32
135	13033	066	098	130	162	194	226	258	290	322	32
136	354	386	418	450	481	513	545	577	609	640	32
137	672	704	735	767	799	830	862	893	925	956	32
138	988	019	051	082	114	145	176	208	239	270	31
139	14301	333	364	395	426	457	489	520	551	582	31
140	613	644	675	706	737	768	799	829	860	891	31
141	922	953	983	014	045	076	106	137	168	198	31
142	15229	259	290	320	351	381	412	442	473	503	31
143	534	564	594	625	655	685	715	746	776	806	30
144	836	866	897	927	957	987	017	047	077	107	30
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
145	16137	167	197	227	256	286	316	346	376	406	30
146	435	465	495	524	554	584	613	643	673	702	30
147	732	761	791	820	850	879	909	938	967	997	29
148	17026	056	085	114	143	173	202	231	260	289	29
149	319	348	377	406	435	464	493	522	551	580	29
150	609	638	667	696	725	754	782	811	840	869	29
151	898	926	955	984	013	041	070	099	127	156	29
152	18184	213	241	270	298	327	355	384	412	441	29
153	469	498	526	554	583	611	639	667	696	724	28
154	752	780	808	837	865	893	921	949	977	005	28
155	19033	061	089	117	145	173	201	229	257	285	28
156	312	340	368	396	424	451	479	507	535	562	28
157	590	618	645	673	700	728	756	783	811	838	28
158	866	893	921	948	976	003	030	058	085	112	27
159	20140	167	194	222	249	276	303	330	358	385	27
160	412	439	466	493	520	548	575	602	629	656	27
161	683	710	737	763	790	817	844	871	898	925	27
162	952	978	005	032	059	085	112	139	165	192	27
163	21219	245	272	299	325	352	378	405	431	458	27
164	484	511	537	564	590	617	643	669	696	722	26
165	748	775	801	827	854	880	906	932	958	985	26
166	22011	037	063	089	115	141	167	194	220	246	26
167	272	298	324	350	376	401	427	453	479	505	26
168	531	557	583	608	634	660	686	712	737	763	26
169	789	814	840	866	891	917	943	968	994	019	26
170	23045	070	096	121	147	172	198	223	249	274	25
171	300	325	350	376	401	426	452	477	502	528	25
172	553	578	603	629	654	679	704	729	754	779	25
173	805	830	855	880	905	930	955	980	005	030	25
174	24055	080	105	130	155	180	204	229	254	279	25
175	304	329	353	378	403	428	452	477	502	527	25
176	551	576	601	625	650	674	699	724	748	773	25
177	797	822	846	871	895	920	944	969	993	018	25
178	25042	066	091	115	139	164	188	212	237	261	24
179	285	310	334	358	382	406	431	455	479	503	24
180	527	551	575	600	624	648	672	696	720	744	24
181	768	792	816	840	864	888	912	935	959	983	24
182	26007	031	055	079	102	126	150	174	198	221	24
183	245	269	293	316	340	364	387	411	435	458	24
184	482	505	529	553	576	600	623	647	670	694	24
185	717	741	764	788	811	834	858	881	905	928	23
186	951	975	998	021	045	068	091	114	138	161	23
187	27184	207	231	254	277	300	323	346	370	393	23
188	416	439	462	485	508	531	554	577	600	623	23
189	646	669	692	715	738	761	784	807	830	852	23
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
190	27875	898	921	944	967	989	012	035	058	081	23
191	28103	126	149	171	194	217	240	262	285	307	23
192	330	353	375	398	421	443	466	488	511	533	23
193	556	578	601	623	646	668	691	713	735	758	22
194	780	803	825	847	870	892	914	937	959	981	22
195	29003	026	048	070	092	115	137	159	181	203	22
196	226	248	270	292	314	336	358	380	403	425	22
197	447	469	491	513	535	557	579	601	623	645	22
198	667	688	710	732	754	776	798	820	842	863	22
199	885	907	929	951	973	994	016	038	060	081	22
200	30103	125	146	168	190	211	233	255	276	298	22
201	320	341	363	384	406	428	449	471	492	514	22
202	535	557	578	600	621	643	664	685	707	728	21
203	750	771	792	814	835	856	878	899	920	942	21
204	963	984	006	027	048	069	091	112	133	154	21
205	31175	197	218	239	260	281	302	323	345	366	21
206	387	408	429	450	471	492	513	534	555	576	21
207	597	618	639	660	681	702	723	744	765	785	21
208	806	827	848	869	890	911	931	952	973	994	21
209	32015	035	056	077	098	118	139	160	181	201	21
210	222	243	263	284	305	325	346	366	387	408	21
211	428	449	469	490	510	531	552	572	593	613	20
212	634	654	675	695	715	736	756	777	797	818	20
213	838	858	879	899	919	940	960	980	001	021	20
214	33041	062	082	102	122	143	163	183	203	224	20
215	244	264	284	304	325	345	365	385	405	425	20
216	445	465	486	506	526	546	566	586	606	626	20
217	646	666	686	706	726	746	766	786	806	826	20
218	846	866	885	905	925	945	965	985	005	025	20
219	34044	064	084	104	124	143	163	183	203	223	20
220	242	262	282	301	321	341	361	380	400	420	20
221	439	459	479	498	518	537	557	577	596	616	20
222	635	655	674	694	713	733	753	772	792	811	19
223	830	850	869	889	908	928	947	967	986	005	19
224	35025	044	064	083	102	122	141	160	180	199	19
225	218	238	257	276	295	315	334	353	372	392	19
226	411	430	449	468	488	507	526	545	564	583	19
227	603	622	641	660	679	698	717	736	755	774	19
228	793	813	832	851	870	889	908	927	946	965	19
229	984	003	021	040	059	078	097	116	135	154	19
230	36173	192	211	229	248	267	286	305	324	342	19
231	361	380	399	418	436	455	474	493	511	530	19
232	549	568	586	605	624	642	661	680	698	717	19
233	736	754	773	791	810	829	847	866	884	903	19
234	922	940	959	977	996	014	033	051	070	088	18
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
235	37107	125	144	162	181	199	218	236	254	273	18
236	291	310	328	346	365	383	401	420	438	457	18
237	475	493	511	530	548	566	585	603	621	639	18
238	658	676	694	712	731	749	767	785	803	822	18
239	840	858	876	894	912	931	949	967	985	003	18
240	38021	039	057	075	093	112	130	148	166	184	18
241	202	220	238	256	274	292	310	328	346	364	18
242	382	399	417	435	453	471	489	507	525	543	18
243	561	578	596	614	632	650	668	686	703	721	18
244	739	757	775	792	810	828	846	863	881	899	18
245	917	934	952	970	987	005	023	041	058	076	18
246	39094	111	129	146	164	182	199	217	235	252	18
247	270	287	305	322	340	358	375	393	410	428	18
248	445	463	480	498	515	533	550	568	585	602	18
249	620	637	655	672	690	707	724	742	759	777	17
250	794	811	829	846	863	881	898	915	933	950	17
251	967	985	002	019	037	054	071	088	106	123	17
252	40140	157	175	192	209	226	243	261	278	295	17
253	312	329	346	364	381	398	415	432	449	466	17
254	483	500	518	535	552	569	586	603	620	637	17
255	654	671	688	705	722	739	756	773	790	807	17
256	824	841	858	875	892	909	926	943	960	976	17
257	993	010	027	044	061	078	095	111	128	145	17
258	41162	179	196	212	229	246	263	280	296	313	17
259	330	347	363	380	397	414	430	447	464	481	17
260	497	514	531	547	564	581	597	614	631	647	17
261	664	681	697	714	731	747	764	780	797	814	17
262	830	847	863	880	896	913	929	946	963	979	16
263	996	012	029	045	062	078	095	111	127	144	16
264	42160	177	193	210	226	243	259	275	292	308	16
265	325	341	357	374	390	406	423	439	455	472	16
266	488	504	521	537	553	570	586	602	619	635	16
267	651	667	684	700	716	732	749	765	781	797	16
268	813	830	846	862	878	894	911	927	943	959	16
269	975	991	008	024	040	056	072	088	104	120	16
270	43136	152	169	185	201	217	233	249	265	281	16
271	297	313	329	345	361	377	393	409	425	441	16
272	457	473	489	505	521	537	553	569	584	600	16
273	616	632	648	664	680	696	712	727	743	759	16
274	775	791	807	823	838	854	870	886	902	917	16
275	933	949	965	981	996	012	028	044	059	075	16
276	44091	107	122	138	154	170	185	201	217	232	16
277	248	264	279	295	311	326	342	358	373	389	16
278	404	420	436	451	467	483	498	514	529	545	16
279	560	576	592	607	623	638	654	669	685	700	16
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
280	44716	731	747	762	778	793	809	824	840	855	15
281	871	886	902	917	932	948	963	979	994	010	15
282	45025	040	056	071	086	102	117	133	148	163	15
283	179	194	209	225	240	255	271	286	301	317	15
284	332	347	362	378	393	408	423	439	454	469	15
285	484	500	515	530	545	561	576	591	606	621	15
286	637	652	667	682	697	712	728	743	758	773	15
287	788	803	818	834	849	864	879	894	909	924	15
288	939	954	969	984	000	015	030	045	060	075	15
289	46090	105	120	135	150	165	180	195	210	225	15
290	240	255	270	285	300	315	330	345	359	374	15
291	389	404	419	434	449	464	479	494	509	523	15
292	538	553	568	583	598	613	627	642	657	672	15
293	687	702	716	731	746	761	776	790	805	820	15
294	835	850	864	879	894	909	923	938	953	967	15
295	982	997	012	026	041	056	070	085	100	114	15
296	47129	144	159	173	188	202	217	232	246	261	15
297	276	290	305	319	334	349	363	378	392	407	15
298	422	436	451	465	480	494	509	524	538	553	15
299	567	582	596	611	625	640	654	669	683	698	15
300	712	727	741	756	770	784	799	813	828	842	14
301	857	871	885	900	914	929	943	958	972	986	14
302	48001	015	029	044	058	073	087	101	116	130	14
303	144	159	173	187	202	216	230	244	259	273	14
304	287	302	316	330	344	359	373	387	401	416	14
305	430	444	458	473	487	501	515	530	544	558	14
306	572	586	601	615	629	643	657	671	686	700	14
307	714	728	742	756	770	785	799	813	827	841	14
308	855	869	883	897	911	926	940	954	968	982	14
309	996	010	024	038	052	066	080	094	108	122	14
310	49136	150	164	178	192	206	220	234	248	262	14
311	276	290	304	318	332	346	360	374	388	402	14
312	415	429	443	457	471	485	499	513	527	541	14
313	554	568	582	596	610	624	638	651	665	679	14
314	693	707	721	734	748	762	776	790	803	817	14
315	831	845	859	872	886	900	914	927	941	955	14
316	969	982	996	010	024	037	051	065	079	092	14
317	50106	120	133	147	161	174	188	202	215	229	14
318	243	256	270	284	297	311	325	338	352	365	14
319	379	393	406	420	433	447	461	474	488	501	14
320	515	529	542	556	569	583	596	610	623	637	14
321	651	664	678	691	705	718	732	745	759	772	14
322	786	799	813	826	840	853	866	880	893	907	13
323	920	934	947	961	974	987	001	014	028	041	13
324	51055	068	081	095	108	121	135	148	162	175	13
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
325	51188	202	215	228	242	255	268	282	295	308	13
326	322	335	348	362	375	388	402	415	428	441	13
327	455	468	481	495	508	521	534	548	561	574	13
328	587	601	614	627	640	654	667	680	693	706	13
329	720	733	746	759	772	786	799	812	825	838	13
330	851	865	878	891	904	917	930	943	957	970	13
331	983	996	009	022	035	048	061	075	088	101	13
332	52114	127	140	153	166	179	192	205	218	231	13
333	244	257	270	284	297	310	323	336	349	362	13
334	375	388	401	414	427	440	453	466	479	492	13
335	504	517	530	543	556	569	582	595	608	621	13
336	634	647	660	673	686	699	711	724	737	750	13
337	763	776	789	802	815	827	840	853	866	879	13
338	892	905	917	930	943	956	969	982	994	007	13
339	53020	033	046	058	071	084	097	110	122	135	13
340	148	161	173	186	199	212	224	237	250	263	13
341	275	288	301	314	326	339	352	364	377	390	13
342	403	415	428	441	453	466	479	491	504	517	13
343	529	542	555	567	580	593	605	618	631	643	13
344	656	668	681	694	706	719	732	744	757	769	13
345	782	794	807	820	832	845	857	870	882	895	13
346	908	920	933	945	958	970	983	995	008	020	13
347	54033	045	058	070	083	095	108	120	133	145	13
348	158	170	183	195	208	220	233	245	258	270	12
349	283	295	307	320	332	345	357	370	382	394	12
350	407	419	432	444	456	469	481	494	506	518	12
351	531	543	555	568	580	593	605	617	630	642	12
352	654	667	679	691	704	716	728	741	753	765	12
353	777	790	802	814	827	839	851	864	876	888	12
354	900	913	925	937	949	962	974	986	998	011	12
355	55023	035	047	060	072	084	096	108	121	133	12
356	145	157	169	182	194	206	218	230	242	255	12
357	267	279	291	303	315	328	340	352	364	376	12
358	388	400	413	425	437	449	461	473	485	497	12
359	509	522	534	546	558	570	582	594	606	618	12
360	630	642	654	666	678	691	703	715	727	739	12
361	751	763	775	787	799	811	823	835	847	859	12
362	871	883	895	907	919	931	943	955	967	979	12
363	991	003	015	027	038	050	062	074	086	098	12
364	56110	122	134	146	158	170	182	194	205	217	12
365	229	241	253	265	277	289	301	312	324	336	12
366	348	360	372	384	396	407	419	431	443	455	12
367	467	478	490	502	514	526	538	549	561	573	12
368	585	597	608	620	632	644	656	667	679	691	12
369	703	714	726	738	750	761	773	785	797	808	12
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
370	56820	832	844	855	867	879	891	902	914	926	12
371	937	949	961	972	984	996	008	019	031	043	12
372	57054	066	078	089	101	113	124	136	148	159	12
373	171	183	194	206	217	229	241	252	264	276	12
374	287	299	310	322	334	345	357	368	380	392	12
375	403	415	426	438	449	461	473	484	496	507	12
376	519	530	542	553	565	576	588	600	611	623	12
377	634	646	657	669	680	692	703	715	726	738	11
378	749	761	772	784	795	807	818	830	841	852	11
379	864	875	887	898	910	921	933	944	955	967	11
380	978	990	001	013	024	035	047	058	070	081	11
381	58092	104	115	127	138	149	161	172	184	195	11
382	206	218	229	240	252	263	274	286	297	309	11
383	320	331	343	354	365	377	388	399	410	422	11
384	433	444	456	467	478	490	501	512	524	535	11
385	546	557	569	580	591	602	614	625	636	647	11
386	659	670	681	692	704	715	726	737	749	760	11
387	771	782	794	805	816	827	838	850	861	872	11
388	883	894	906	917	928	939	950	961	973	984	11
389	995	006	017	028	040	051	062	073	084	095	11
390	59106	118	129	140	151	162	173	184	195	207	11
391	218	229	240	251	262	273	284	295	306	318	11
392	329	340	351	362	373	384	395	406	417	428	11
393	439	450	461	472	483	494	506	517	528	539	11
394	550	561	572	583	594	605	616	627	638	649	11
395	660	671	682	693	704	715	726	737	748	759	11
396	770	780	791	802	813	824	835	846	857	868	11
397	879	890	901	912	923	934	945	956	966	977	11
398	988	999	010	021	032	043	054	065	076	086	11
399	60097	108	119	130	141	152	163	173	184	195	11
400	206	217	228	239	249	260	271	282	293	304	11
401	314	325	336	347	358	369	379	390	401	412	11
402	423	433	444	455	466	477	487	498	509	520	11
403	531	541	552	563	574	584	595	606	617	627	11
404	638	649	660	670	681	692	703	713	724	735	11
405	746	756	767	778	788	799	810	821	831	842	11
406	853	863	874	885	895	906	917	927	938	949	11
407	959	970	981	991	002	013	023	034	045	055	11
408	61066	077	087	098	109	119	130	140	151	162	11
409	172	183	194	204	215	225	236	247	257	268	11
410	278	289	300	310	321	331	342	352	363	374	11
411	384	395	405	416	426	437	448	458	469	479	11
412	490	500	511	521	532	542	553	563	574	584	11
413	595	606	616	627	637	648	658	669	679	690	11
414	700	711	721	731	742	752	763	773	784	794	10
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
415	61805	815	826	836	847	857	868	878	888	899	10
416	909	920	930	941	951	962	972	982	993	o03	10
417	62014	024	034	045	055	066	076	086	097	107	10
418	118	128	138	149	159	170	180	190	201	211	10
419	221	232	242	252	263	273	284	294	304	315	10
420	325	335	346	356	366	377	387	397	408	418	10
421	428	439	449	459	469	480	490	500	511	521	10
422	531	542	552	562	572	583	593	603	613	624	10
423	634	644	655	665	675	685	696	706	716	726	10
424	737	747	757	767	778	788	798	808	818	829	10
425	839	849	859	870	880	890	900	910	921	931	10
426	941	951	961	972	982	992	o02	o12	o22	o33	10
427	63043	053	063	073	083	094	104	114	124	134	10
428	144	155	165	175	185	195	205	215	225	236	10
429	246	256	266	276	286	296	306	317	327	337	10
430	347	357	367	377	387	397	407	417	428	438	10
431	448	458	468	478	488	498	508	518	528	538	10
432	548	558	568	579	589	599	609	619	629	639	10
433	649	659	669	679	689	699	709	719	729	739	10
434	749	759	769	779	789	799	809	819	829	839	10
435	849	859	869	879	889	899	909	919	929	939	10
436	949	959	969	979	988	998	o08	o18	o28	o38	10
437	64048	058	068	078	088	098	108	118	128	137	10
438	147	157	167	177	187	197	207	217	227	237	10
439	246	256	266	276	286	296	306	316	326	335	10
440	345	355	365	375	385	395	404	414	424	434	10
441	444	454	464	473	483	493	503	513	523	532	10
442	542	552	562	572	582	591	601	611	621	631	10
443	640	650	660	670	680	689	699	709	719	729	10
444	738	748	758	768	777	787	797	807	816	826	10
445	836	846	856	865	875	885	895	904	914	924	10
446	933	943	953	963	972	982	992	o02	o11	o21	10
447	65031	040	050	060	070	079	089	099	108	118	10
448	128	137	147	157	167	176	186	196	205	215	10
449	225	234	244	254	263	273	283	292	302	312	10
450	321	331	341	350	360	369	379	389	398	408	10
451	418	427	437	447	456	466	475	485	495	504	10
452	514	523	533	543	552	562	571	581	591	600	10
453	610	619	629	639	648	658	667	677	686	696	10
454	706	715	725	734	744	753	763	772	782	792	9
455	801	811	820	830	839	849	858	868	877	887	9
456	896	906	916	925	935	944	954	963	973	982	9
457	992	o01	o11	o20	o30	o39	o49	o58	o68	o77	9
458	66087	096	106	115	124	134	143	153	162	172	9
459	181	191	200	210	219	229	238	247	257	266	9
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
460	66276	285	295	304	314	323	332	342	351	361	9
461	370	380	389	398	408	417	427	436	445	455	9
462	464	474	483	492	502	511	521	530	539	549	9
463	558	567	577	586	596	605	614	624	633	642	9
464	652	661	671	680	689	699	708	717	727	736	9
465	745	755	764	773	783	792	801	811	820	829	9
466	839	848	857	867	876	885	894	904	913	922	9
467	932	941	950	960	969	978	987	997	o06	o15	9
468	67025	034	043	052	062	071	080	089	099	108	9
469	117	127	136	145	154	164	173	182	191	201	9
470	210	219	228	237	247	256	265	274	284	293	9
471	302	311	321	330	339	348	357	367	376	385	9
472	394	403	413	422	431	440	449	459	468	477	9
473	486	495	504	514	523	532	541	550	560	569	9
474	578	587	596	605	614	624	633	642	651	660	9
475	669	679	688	697	706	715	724	733	742	752	9
476	761	770	779	788	797	806	815	825	834	843	9
477	852	861	870	879	888	897	906	916	925	934	9
478	943	952	961	970	979	988	997	o06	o15	o24	9
479	68034	043	052	061	070	079	088	097	106	115	9
480	124	133	142	151	160	169	178	187	196	205	9
481	215	224	233	242	251	260	269	278	287	296	9
482	305	314	323	332	341	350	359	368	377	386	9
483	395	404	413	422	431	440	449	458	467	476	9
484	485	494	502	511	520	529	538	547	556	565	9
485	574	583	592	601	610	619	628	637	646	655	9
486	664	673	681	690	699	708	717	726	735	744	9
487	753	762	771	780	789	797	806	815	824	833	9
488	842	851	860	869	878	886	895	904	913	922	9
489	931	940	949	958	966	975	984	993	o02	o11	9
490	69020	028	037	046	055	064	073	082	090	099	9
491	108	117	126	135	144	152	161	170	179	188	9
492	197	205	214	223	232	241	249	258	267	276	9
493	285	294	302	311	320	329	338	346	355	364	9
494	373	381	390	399	408	417	425	434	443	452	9
495	461	469	478	487	496	504	513	522	531	539	9
496	548	557	566	574	583	592	601	609	618	627	9
497	636	644	653	662	671	679	688	697	705	714	9
498	723	732	740	749	758	767	775	784	793	801	9
499	810	819	827	836	845	854	862	871	880	888	9
500	897	906	914	923	932	940	949	958	966	975	9
501	984	992	o01	o10	o18	o27	o36	o44	o53	o62	9
502	70070	079	088	096	105	114	122	131	140	148	9
503	157	165	174	183	191	200	209	217	226	234	9
504	243	252	260	269	278	286	295	303	312	321	9
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
505	70329	338	346	355	364	372	381	389	398	406	9
506	415	424	432	441	449	458	467	475	484	492	9
507	501	509	518	526	535	544	552	561	569	578	9
508	586	595	603	612	621	629	638	646	655	663	9
509	672	680	689	697	706	714	723	731	740	749	9
510	757	766	774	783	791	800	808	817	825	834	9
511	842	851	859	868	876	885	893	902	910	919	9
512	927	935	944	952	961	969	978	986	995	003	9
513	71012	020	029	037	046	054	063	071	079	088	8
514	096	105	113	122	130	139	147	155	164	172	8
515	181	189	198	206	214	223	231	240	248	257	8
516	265	273	282	290	299	307	315	324	332	341	8
517	349	357	366	374	383	391	399	408	416	425	8
518	433	441	450	458	466	475	483	492	500	508	8
519	517	525	533	542	550	559	567	575	584	592	8
520	600	609	617	625	634	642	650	659	667	675	8
521	684	692	700	709	717	725	734	742	750	759	8
522	767	775	784	792	800	809	817	825	834	842	8
523	850	858	867	875	883	892	900	908	917	925	8
524	933	941	950	958	966	975	983	991	999	008	8
525	72016	024	032	041	049	057	066	074	082	090	8
526	099	107	115	123	132	140	148	156	165	173	8
527	181	189	198	206	214	222	230	239	247	255	8
528	263	272	280	288	296	304	313	321	329	337	8
529	346	354	362	370	378	387	395	403	411	419	8
530	428	436	444	452	460	469	477	485	493	501	8
531	509	518	526	534	542	550	558	567	575	583	8
532	591	599	607	616	624	632	640	648	656	665	8
533	673	681	689	697	705	713	722	730	738	746	8
534	754	762	770	779	787	795	803	811	819	827	8
535	835	843	852	860	868	876	884	892	900	908	8
536	916	925	933	941	949	957	965	973	981	989	8
537	997	006	014	022	030	038	046	054	062	070	8
538	73078	086	094	102	111	119	127	135	143	151	8
539	159	167	175	183	191	199	207	215	223	231	8
540	239	247	255	263	272	280	288	296	304	312	8
541	320	328	336	344	352	360	368	376	384	392	8
542	400	408	416	424	432	440	448	456	464	472	8
543	480	488	496	504	512	520	528	536	544	552	8
544	560	568	576	584	592	600	608	616	624	632	8
545	640	648	656	664	672	679	687	695	703	711	8
546	719	727	735	743	751	759	767	775	783	791	8
547	799	807	815	823	830	838	846	854	862	870	8
548	878	886	894	902	910	918	926	933	941	949	8
549	957	965	973	981	989	997	005	013	020	028	8
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
550	74036	044	052	060	068	076	084	092	099	107	8
551	115	123	131	139	147	155	162	170	178	186	8
552	194	202	210	218	225	233	241	249	257	265	8
553	273	280	288	296	304	312	320	327	335	343	8
554	351	359	367	374	382	390	398	406	414	421	8
555	429	437	445	453	461	468	476	484	492	500	8
556	507	515	523	531	539	547	554	562	570	578	8
557	586	593	601	609	617	624	632	640	648	656	8
558	663	671	679	687	695	702	710	718	726	733	8
559	741	749	757	764	772	780	788	796	803	811	8
560	819	827	834	842	850	858	865	873	881	889	8
561	896	904	912	920	927	935	943	950	958	966	8
562	974	981	989	997	o05	o12	o20	o28	o35	o43	8
563	75051	059	066	074	082	089	097	105	113	120	8
564	128	136	143	151	159	166	174	182	189	197	8
565	205	213	220	228	236	243	251	259	266	274	8
566	282	289	297	305	312	320	328	335	343	351	8
567	358	366	374	381	389	397	404	412	420	427	8
568	435	442	450	458	465	473	481	488	496	504	8
569	511	519	526	534	542	549	557	565	572	580	8
570	587	595	603	610	618	626	633	641	648	656	8
571	664	671	679	686	694	702	709	717	724	732	8
572	740	747	755	762	770	778	785	793	800	808	8
573	815	823	831	838	846	853	861	868	876	884	8
574	891	899	906	914	921	929	937	944	952	959	8
575	967	974	982	989	997	o05	o12	o20	o27	o35	8
576	76042	050	057	065	072	080	087	095	103	110	8
577	118	125	133	140	148	155	163	170	178	185	8
578	193	200	208	215	223	230	238	245	253	260	8
579	268	275	283	290	298	305	313	320	328	335	8
580	343	350	358	365	373	380	388	395	403	410	8
581	418	425	433	440	448	455	462	470	477	485	7
582	492	500	507	515	522	530	537	545	552	559	7
583	567	574	582	589	597	604	612	619	626	634	7
584	641	649	656	664	671	678	686	693	701	708	7
585	716	723	730	738	745	753	760	768	775	782	7
586	790	797	805	812	819	827	834	842	849	856	7
587	864	871	879	886	893	901	908	916	923	930	7
588	938	945	953	960	967	975	982	989	997	o04	7
589	77012	019	026	034	041	048	056	063	070	078	7
590	085	093	100	107	115	122	129	137	144	151	7
591	159	166	173	181	188	195	203	210	217	225	7
592	232	240	247	254	262	269	276	283	291	298	7
593	305	313	320	327	335	342	349	357	364	371	7
594	379	386	393	401	408	415	422	430	437	444	7
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
595	77452	459	466	474	481	488	495	503	510	517	7
596	525	532	539	546	554	561	568	576	583	590	7
597	597	605	612	619	627	634	641	648	656	663	7
598	670	677	685	692	699	706	714	721	728	735	7
599	743	750	757	764	772	779	786	793	801	808	7
600	815	822	830	837	844	851	859	866	873	880	7
601	887	895	902	909	916	924	931	938	945	952	7
602	960	967	974	981	988	996	003	010	017	025	7
603	78032	039	046	053	061	068	075	082	089	097	7
604	104	111	118	125	132	140	147	154	161	168	7
605	176	183	190	197	204	211	219	226	233	240	7
606	247	254	262	269	276	283	290	297	305	312	7
607	319	326	333	340	347	355	362	369	376	383	7
608	390	398	405	412	419	426	433	440	447	455	7
609	462	469	476	483	490	497	504	512	519	526	7
610	533	540	547	554	561	569	576	583	590	597	7
611	604	611	618	625	633	640	647	654	661	668	7
612	675	682	689	696	704	711	718	725	732	739	7
613	746	753	760	767	774	781	789	796	803	810	7
614	817	824	831	838	845	852	859	866	873	880	7
615	888	895	902	909	916	923	930	937	944	951	7
616	958	965	972	979	986	993	000	007	014	021	7
617	79029	036	043	050	057	064	071	078	085	092	7
618	099	106	113	120	127	134	141	148	155	162	7
619	169	176	183	190	197	204	211	218	225	232	7
620	239	246	253	260	267	274	281	288	295	302	7
621	309	316	323	330	337	344	351	358	365	372	7
622	379	386	393	400	407	414	421	428	435	442	7
623	449	456	463	470	477	484	491	498	505	511	7
624	518	525	532	539	546	553	560	567	574	581	7
625	588	595	602	609	616	623	630	637	644	650	7
626	657	664	671	678	685	692	699	706	713	720	7
627	727	734	741	748	754	761	768	775	782	789	7
628	796	803	810	817	824	831	837	844	851	858	7
629	865	872	879	886	893	900	906	913	920	927	7
630	934	941	948	955	962	969	975	982	989	996	7
631	80003	010	017	024	030	037	044	051	058	065	7
632	072	079	085	092	099	106	113	120	127	134	7
633	140	147	154	161	168	175	182	188	195	202	7
634	209	216	223	229	236	243	250	257	264	271	7
635	277	284	291	298	305	312	318	325	332	339	7
636	346	353	359	366	373	380	387	393	400	407	7
637	414	421	428	434	441	448	455	462	468	475	7
638	482	489	496	502	509	516	523	530	536	543	7
639	550	557	564	570	577	584	591	598	604	611	7
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
640	80618	625	632	638	645	652	659	665	672	679	7
641	686	693	699	706	713	720	726	733	740	747	7
642	754	760	767	774	781	787	794	801	808	814	7
643	821	828	835	841	848	855	862	868	875	882	7
644	889	895	902	909	916	922	929	936	943	949	7
645	956	963	969	976	983	990	996	003	010	017	7
646	81023	030	037	043	050	057	064	070	077	084	7
647	090	097	104	111	117	124	131	137	144	151	7
648	158	164	171	178	184	191	198	204	211	218	7
649	224	231	238	245	251	258	265	271	278	285	7
650	291	298	305	311	318	325	331	338	345	351	7
651	358	365	371	378	385	391	398	405	411	418	7
652	425	431	438	445	451	458	465	471	478	485	7
653	491	498	505	511	518	525	531	538	544	551	7
654	558	564	571	578	584	591	598	604	611	617	7
655	624	631	637	644	651	657	664	671	677	684	7
656	690	697	704	710	717	723	730	737	743	750	7
657	757	763	770	776	783	790	796	803	809	816	7
658	823	829	836	842	849	856	862	869	875	882	7
659	889	895	902	908	915	921	928	935	941	948	7
660	954	961	968	974	981	987	994	000	007	014	7
661	82020	027	033	040	046	053	060	066	073	079	7
662	086	092	099	105	112	119	125	132	138	145	7
663	151	158	164	171	178	184	191	197	204	210	7
664	217	223	230	236	243	249	256	263	269	276	7
665	282	289	295	302	308	315	321	328	334	341	7
666	347	354	360	367	373	380	387	393	400	406	7
667	413	419	426	432	439	445	452	458	465	471	7
668	478	484	491	497	504	510	517	523	530	536	7
669	543	549	556	562	569	575	582	588	595	601	7
670	607	614	620	627	633	640	646	653	659	666	7
671	672	679	685	692	698	705	711	718	724	730	6
672	737	743	750	756	763	769	776	782	789	795	6
673	802	808	814	821	827	834	840	847	853	860	6
674	866	872	879	885	892	898	905	911	918	924	6
675	930	937	943	950	956	963	969	975	982	988	6
676	995	001	008	014	020	027	033	040	046	052	6
677	83059	065	072	078	085	091	097	104	110	117	6
678	123	129	136	142	149	155	161	168	174	181	6
679	187	193	200	206	213	219	225	232	238	245	6
680	251	257	264	270	276	283	289	296	302	308	6
681	315	321	327	334	340	347	353	359	366	372	6
682	378	385	391	398	404	410	417	423	429	436	6
683	442	448	455	461	467	474	480	487	493	499	6
684	506	512	518	525	531	537	544	550	556	563	6
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
685	83569	575	582	588	594	601	607	613	620	626	6
686	632	639	645	651	658	664	670	677	683	689	6
687	696	702	708	715	721	727	734	740	746	753	6
688	759	765	771	778	784	790	797	803	809	816	6
689	822	828	835	841	847	853	860	866	872	879	6
690	885	891	897	904	910	916	923	929	935	942	6
691	948	954	960	967	973	979	985	992	998	004	6
692	84011	017	023	029	036	042	048	055	061	067	6
693	073	080	086	092	098	105	111	117	123	130	6
694	136	142	148	155	161	167	173	180	186	192	6
695	198	205	211	217	223	230	236	242	248	255	6
696	261	267	273	280	286	292	298	305	311	317	6
697	323	330	336	342	348	354	361	367	373	379	6
698	386	392	398	404	410	417	423	429	435	442	6
699	448	454	460	466	473	479	485	491	497	504	6
700	510	516	522	528	535	541	547	553	559	566	6
701	572	578	584	590	597	603	609	615	621	628	6
702	634	640	646	652	658	665	671	677	683	689	6
703	696	702	708	714	720	726	733	739	745	751	6
704	757	763	770	776	782	788	794	800	807	813	6
705	819	825	831	837	844	850	856	862	868	874	6
706	880	887	893	899	905	911	917	924	930	936	6
707	942	948	954	960	967	973	979	985	991	997	6
708	85003	009	016	022	028	034	040	046	052	058	6
709	065	071	077	083	089	095	101	107	114	120	6
710	126	132	138	144	150	156	163	169	175	181	6
711	187	193	199	205	211	217	224	230	236	242	6
712	248	254	260	266	272	278	285	291	297	303	6
713	309	315	321	327	333	339	345	352	358	364	6
714	370	376	382	388	394	400	406	412	418	425	6
715	431	437	443	449	455	461	467	473	479	485	6
716	491	497	503	509	516	522	528	534	540	546	6
717	552	558	564	570	576	582	588	594	600	606	6
718	612	618	625	631	637	643	649	655	661	667	6
719	673	679	685	691	697	703	709	715	721	727	6
720	733	739	745	751	757	763	769	775	781	788	6
721	794	800	806	812	818	824	830	836	842	848	6
722	854	860	866	872	878	884	890	896	902	908	6
723	914	920	926	932	938	944	950	956	962	968	6
724	974	980	986	992	998	004	010	016	022	028	6
725	86034	040	046	052	058	064	070	076	082	088	6
726	094	100	106	112	118	124	130	136	141	147	6
727	153	159	165	171	177	183	189	195	201	207	6
728	213	219	225	231	237	243	249	255	261	267	6
729	273	279	285	291	297	303	308	314	320	326	6
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
730	86332	338	344	350	356	362	368	374	380	386	6
731	392	398	404	410	415	421	427	433	439	445	6
732	451	457	463	469	475	481	487	493	499	504	6
733	510	516	522	528	534	540	546	552	558	564	6
734	570	576	581	587	593	599	605	611	617	623	6
735	629	635	641	646	652	658	664	670	676	682	6
736	688	694	700	705	711	717	723	729	735	741	6
737	747	753	759	764	770	776	782	788	794	800	6
738	806	812	817	823	829	835	841	847	853	859	6
739	864	870	876	882	888	894	900	906	911	917	6
740	923	929	935	941	947	953	958	964	970	976	6
741	982	988	994	999	005	011	017	023	029	035	6
742	87040	046	052	058	064	070	075	081	087	093	6
743	099	105	111	116	122	128	134	140	146	151	6
744	157	163	169	175	181	186	192	198	204	210	6
745	216	221	227	233	239	245	251	256	262	268	6
746	274	280	286	291	297	303	309	315	320	326	6
747	332	338	344	349	355	361	367	373	379	384	6
748	390	396	402	408	413	419	425	431	437	442	6
749	448	454	460	466	471	477	483	489	495	500	6
750	506	512	518	523	529	535	541	547	552	558	6
751	564	570	576	581	587	593	599	604	610	616	6
752	622	628	633	639	645	651	656	662	668	674	6
753	679	685	691	697	703	708	714	720	726	731	6
754	737	743	749	754	760	766	772	777	783	789	6
755	795	800	806	812	818	823	829	835	841	846	6
756	852	858	864	869	875	881	887	892	898	904	6
757	910	915	921	927	933	938	944	950	955	961	6
758	967	973	978	984	990	996	001	007	013	018	6
759	88024	030	036	041	047	053	058	064	070	076	6
760	081	087	093	098	104	110	116	121	127	133	6
761	138	144	150	156	161	167	173	178	184	190	6
762	195	201	207	213	218	224	230	235	241	247	6
763	252	258	264	270	275	281	287	292	298	304	6
764	309	315	321	326	332	338	343	349	355	360	6
765	366	372	377	383	389	395	400	406	412	417	6
766	423	429	434	440	446	451	457	463	468	474	6
767	480	485	491	497	502	508	513	519	525	530	6
768	536	542	547	553	559	564	570	576	581	587	6
769	593	598	604	610	615	621	627	632	638	643	6
770	649	655	660	666	672	677	683	689	694	700	6
771	705	711	717	722	728	734	739	745	750	756	6
772	762	767	773	779	784	790	795	801	807	812	6
773	818	824	829	835	840	846	852	857	863	868	6
774	874	880	885	891	897	902	908	913	919	925	6
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
775	88930	936	941	947	953	958	964	969	975	981	6
776	986	992	997	003	009	014	020	025	031	037	6
777	89042	048	053	059	064	070	076	081	087	092	6
778	098	104	109	115	120	126	131	137	143	148	6
779	154	159	165	170	176	182	187	193	198	204	6
780	209	215	221	226	232	237	243	248	254	260	6
781	265	271	276	282	287	293	298	304	310	315	6
782	321	326	332	337	343	348	354	360	365	371	6
783	376	382	387	393	398	404	409	415	421	426	6
784	432	437	443	448	454	459	465	470	476	481	6
785	487	492	498	504	509	515	520	526	531	537	6
786	542	548	553	559	564	570	575	581	586	592	6
787	597	603	609	614	620	625	631	636	642	647	6
788	653	658	664	669	675	680	686	691	697	702	6
789	708	713	719	724	730	735	741	746	752	757	6
790	763	768	774	779	785	790	796	801	807	812	5
791	818	823	829	834	840	845	851	856	862	867	5
792	873	878	883	889	894	900	905	911	916	922	5
793	927	933	938	944	949	955	960	966	971	977	5
794	982	988	993	998	004	009	015	020	026	031	5
795	90037	042	048	053	059	064	069	075	080	086	5
796	091	097	102	108	113	119	124	129	135	140	5
797	146	151	157	162	168	173	179	184	189	195	5
798	200	206	211	217	222	227	233	238	244	249	5
799	255	260	266	271	276	282	287	293	298	304	5
800	309	314	320	325	331	336	342	347	352	358	5
801	363	369	374	380	385	390	396	401	407	412	5
802	417	423	428	434	439	445	450	455	461	466	5
803	472	477	482	488	493	499	504	509	515	520	5
804	526	531	536	542	547	553	558	563	569	574	5
805	580	585	590	596	601	607	612	617	623	628	5
806	634	639	644	650	655	660	666	671	677	682	5
807	687	693	698	703	709	714	720	725	730	736	5
808	741	747	752	757	763	768	773	779	784	789	5
809	795	800	806	811	816	822	827	832	838	843	5
810	849	854	859	865	870	875	881	886	891	897	5
811	902	907	913	918	924	929	934	940	945	950	5
812	956	961	966	972	977	982	988	993	998	004	5
813	91009	014	020	025	030	036	041	046	052	057	5
814	062	068	073	078	084	089	094	100	105	110	5
815	116	121	126	132	137	142	148	153	158	164	5
816	169	174	180	185	190	196	201	206	212	217	5
817	222	228	233	238	243	249	254	259	265	270	5
818	275	281	286	291	297	302	307	312	318	323	5
819	328	334	339	344	350	355	360	365	371	376	5
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
820	91381	387	392	397	403	408	413	418	424	429	5
821	434	440	445	450	455	461	466	471	477	482	5
822	487	492	498	503	508	514	519	524	529	535	5
823	540	545	551	556	561	566	572	577	582	587	5
824	593	598	603	609	614	619	624	630	635	640	5
825	645	651	656	661	666	672	677	682	687	693	5
826	698	703	709	714	719	724	730	735	740	745	5
827	751	756	761	766	772	777	782	787	793	798	5
828	803	808	814	819	824	829	834	840	845	850	5
829	855	861	866	871	876	882	887	892	897	903	5
830	908	913	918	924	929	934	939	944	950	955	5
831	960	965	971	976	981	986	991	997	002	007	5
832	92012	018	023	028	033	038	044	049	054	059	5
833	065	070	075	080	085	091	096	101	106	111	5
834	117	122	127	132	137	143	148	153	158	163	5
835	169	174	179	184	189	195	200	205	210	215	5
836	221	226	231	236	241	247	252	257	262	267	5
837	273	278	283	288	293	298	304	309	314	319	5
838	324	330	335	340	345	350	355	361	366	371	5
839	376	381	387	392	397	402	407	412	418	423	5
840	428	433	438	443	449	454	459	464	469	474	5
841	480	485	490	495	500	505	511	516	521	526	5
842	531	536	542	547	552	557	562	567	572	578	5
843	583	588	593	598	603	609	614	619	624	629	5
844	634	639	645	650	655	660	665	670	675	681	5
845	686	691	696	701	706	711	716	722	727	732	5
846	737	742	747	752	758	763	768	773	778	783	5
847	788	793	799	804	809	814	819	824	829	834	5
848	840	845	850	855	860	865	870	875	881	886	5
849	891	896	901	906	911	916	921	927	932	937	5
850	942	947	952	957	962	967	973	978	983	988	5
851	993	998	003	008	013	018	024	029	034	039	5
852	93044	049	054	059	064	069	075	080	085	090	5
853	095	100	105	110	115	120	125	131	136	141	5
854	146	151	156	161	166	171	176	181	186	192	5
855	197	202	207	212	217	222	227	232	237	242	5
856	247	252	258	263	268	273	278	283	288	293	5
857	298	303	308	313	318	323	328	334	339	344	5
858	349	354	359	364	369	374	379	384	389	394	5
859	399	404	409	414	420	425	430	435	440	445	5
860	450	455	460	465	470	475	480	485	490	495	5
861	500	505	510	515	520	526	531	536	541	546	5
862	551	556	561	566	571	576	581	586	591	596	5
863	601	606	611	616	621	626	631	636	641	646	5
864	651	656	661	666	671	676	682	687	692	697	5
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
865	93702	707	712	717	722	727	732	737	742	747	5
866	752	757	762	767	772	777	782	787	792	797	5
867	802	807	812	817	822	827	832	837	842	847	5
868	852	857	862	867	872	877	882	887	892	897	5
869	902	907	912	917	922	927	932	937	942	947	5
870	952	957	962	967	972	977	982	987	992	997	5
871	94002	007	012	017	022	027	032	037	042	047	5
872	052	057	062	067	072	077	082	086	091	096	5
873	101	106	111	116	121	126	131	136	141	146	5
874	151	156	161	166	171	176	181	186	191	196	5
875	201	206	211	216	221	226	231	236	240	245	5
876	250	255	260	265	270	275	280	285	290	295	5
877	300	305	310	315	320	325	330	335	340	345	5
878	349	354	359	364	369	374	379	384	389	394	5
879	399	404	409	414	419	424	429	433	438	443	5
880	448	453	458	463	468	473	478	483	488	493	5
881	498	503	507	512	517	522	527	532	537	542	5
882	547	552	557	562	567	571	576	581	586	591	5
883	596	601	606	611	616	621	626	630	635	640	5
884	645	650	655	660	665	670	675	680	685	689	5
885	694	699	704	709	714	719	724	729	734	738	5
886	743	748	753	758	763	768	773	778	783	787	5
887	792	797	802	807	812	817	822	827	832	836	5
888	841	846	851	856	861	866	871	876	880	885	5
889	890	895	900	905	910	915	919	924	929	934	5
890	939	944	949	954	959	963	968	973	978	983	5
891	988	993	998	o02	o07	o12	o17	o22	o27	o32	5
892	95036	041	046	051	056	061	066	071	075	080	5
893	085	090	095	100	105	109	114	119	124	129	5
894	134	139	143	148	153	158	163	168	173	177	5
895	182	187	192	197	202	207	211	216	221	226	5
896	231	236	240	245	250	255	260	265	270	274	5
897	279	284	289	294	299	303	308	313	318	323	5
898	328	332	337	342	347	352	357	361	366	371	5
899	376	381	386	390	395	400	405	410	415	419	5
900	424	429	434	439	444	448	453	458	463	468	5
901	472	477	482	487	492	497	501	506	511	516	5
902	521	525	530	535	540	545	550	554	559	564	5
903	569	574	578	583	588	593	598	602	607	612	5
904	617	622	626	631	636	641	646	650	655	660	5
905	665	670	674	679	684	689	694	698	703	708	5
906	713	718	722	727	732	737	742	746	751	756	5
907	761	766	770	775	780	785	789	794	799	804	5
908	809	813	818	823	828	832	837	842	847	852	5
909	856	861	866	871	875	880	885	890	895	899	5
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
910	95904	909	914	918	923	928	933	938	942	947	5
911	952	957	961	966	971	976	980	985	990	995	5
912	999	o04	o09	o14	o19	o23	o28	o33	o38	o42	5
913	96047	052	057	061	066	071	076	080	085	090	5
914	095	099	104	109	114	118	123	128	133	137	5
915	142	147	152	156	161	166	171	175	180	185	5
916	190	194	199	204	209	213	218	223	227	232	5
917	237	242	246	251	256	261	265	270	275	280	5
918	284	289	294	298	303	308	313	317	322	327	5
919	332	336	341	346	350	355	360	365	369	374	5
920	379	384	388	393	398	402	407	412	417	421	5
921	426	431	435	440	445	450	454	459	464	468	5
922	473	478	483	487	492	497	501	506	511	515	5
923	520	525	530	534	539	544	548	553	558	562	5
924	567	572	577	581	586	591	595	600	605	609	5
925	614	619	624	628	633	638	642	647	652	656	5
926	661	666	670	675	680	685	689	694	699	703	5
927	708	713	717	722	727	731	736	741	745	750	5
928	755	759	764	769	774	778	783	788	792	797	5
929	802	806	811	816	820	825	830	834	839	844	5
930	848	853	858	862	867	872	876	881	886	890	5
931	895	900	904	909	914	918	923	928	932	937	5
932	942	946	951	956	960	965	970	974	979	984	5
933	988	993	997	o02	o07	o11	o16	o21	o25	o30	5
934	97035	039	044	049	053	058	063	067	072	077	5
935	081	086	090	095	100	104	109	114	118	123	5
936	128	132	137	142	146	151	155	160	165	169	5
937	174	179	183	188	192	197	202	206	211	216	5
938	220	225	230	234	239	243	248	253	257	262	5
939	267	271	276	280	285	290	294	299	304	308	5
940	313	317	322	327	331	336	340	345	350	354	5
941	359	364	368	373	377	382	387	391	396	400	5
942	405	410	414	419	424	428	433	437	442	447	5
943	451	456	460	465	470	474	479	483	488	493	5
944	497	502	506	511	516	520	525	529	534	539	5
945	543	548	552	557	562	566	571	575	580	585	5
946	589	594	598	603	607	612	617	621	626	630	5
947	635	640	644	649	653	658	663	667	672	676	5
948	681	685	690	695	699	704	708	713	717	722	5
949	727	731	736	740	745	749	754	759	763	768	5
950	772	777	782	786	791	795	800	804	809	813	5
951	818	823	827	832	836	841	845	850	855	859	5
952	864	868	873	877	882	886	891	896	900	905	5
953	909	914	918	923	928	932	937	941	946	950	5
954	955	959	964	968	973	978	982	987	991	996	5
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
955	98000	005	009	014	019	023	028	032	037	041	5
956	046	050	055	059	064	068	073	078	082	087	5
957	091	096	100	105	109	114	118	123	127	132	5
958	137	141	146	150	155	159	164	168	173	177	5
959	182	186	191	195	200	204	209	214	218	223	5
960	227	232	236	241	245	250	254	259	263	268	5
961	272	277	281	286	290	295	299	304	308	313	5
962	318	322	327	331	336	340	345	349	354	358	5
963	363	367	372	376	381	385	390	394	399	403	5
964	408	412	417	421	426	430	435	439	444	448	5
965	453	457	462	466	471	475	480	484	489	493	4
966	498	502	507	511	516	520	525	529	534	538	4
967	543	547	552	556	561	565	570	574	579	583	4
968	588	592	597	601	605	610	614	619	623	628	4
969	632	637	641	646	650	655	659	664	668	673	4
970	677	682	686	691	695	700	704	709	713	717	4
971	722	726	731	735	740	744	749	753	758	762	4
972	767	771	776	780	784	789	793	798	802	807	4
973	811	816	820	825	829	834	838	843	847	851	4
974	856	860	865	869	874	878	883	887	892	896	4
975	900	905	909	914	918	923	927	932	936	941	4
976	945	949	954	958	963	967	972	976	981	985	4
977	989	994	998	003	007	012	016	021	025	029	4
978	99034	038	043	047	052	056	061	065	069	074	4
979	078	083	087	092	096	100	105	109	114	118	4
980	123	127	131	136	140	145	149	154	158	162	4
981	167	171	176	180	185	189	193	198	202	207	4
982	211	216	220	224	229	233	238	242	247	251	4
983	255	260	264	269	273	277	282	286	291	295	4
984	300	304	308	313	317	322	326	330	335	339	4
985	344	348	352	357	361	366	370	374	379	383	4
986	388	392	396	401	405	410	414	419	423	427	4
987	432	436	441	445	449	454	458	463	467	471	4
988	476	480	484	489	493	498	502	506	511	515	4
989	520	524	528	533	537	542	546	550	555	559	4
990	564	568	572	577	581	585	590	594	599	603	4
991	607	612	616	621	625	629	634	638	642	647	4
992	651	656	660	664	669	673	977	682	686	691	4
993	695	699	704	708	712	717	721	726	730	734	4
994	739	743	747	752	756	760	765	769	774	778	4
995	782	787	791	795	800	804	808	813	817	822	4
996	826	830	835	839	843	848	852	856	861	865	4
997	870	874	878	883	887	891	896	900	904	909	4
998	913	917	922	926	930	935	939	944	948	952	4
999	957	961	965	970	974	978	983	987	991	996	4
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
1000	00000	004	009	013	017	022	026	030	035	039	4
1001	043	048	052	056	061	065	069	074	078	082	4
1002	087	091	095	100	104	108	113	117	121	126	4
1003	130	134	139	143	147	152	156	160	165	169	4
1004	173	178	182	186	191	195	199	204	208	212	4
1005	217	221	225	230	234	238	243	247	251	255	4
1006	260	264	268	273	277	281	286	290	294	299	4
1007	303	307	312	316	320	325	329	333	337	342	4
1008	346	350	355	359	363	368	372	376	381	385	4
1009	389	393	398	402	406	411	415	419	424	428	4
1010	432	436	441	445	449	454	458	462	467	471	4
1011	475	479	484	488	492	497	501	505	509	514	4
1012	518	522	527	531	535	540	544	548	552	557	4
1013	561	565	570	574	578	582	587	591	595	600	4
1014	604	608	612	617	621	625	629	634	638	642	4
1015	647	651	655	659	664	668	672	677	681	685	4
1016	689	694	698	702	706	711	715	719	724	728	4
1017	732	736	741	745	749	753	758	762	766	771	4
1018	775	779	783	788	792	796	800	805	809	813	4
1019	817	822	826	830	834	839	843	847	852	856	4
1020	860	864	869	873	877	881	886	890	894	898	4
1021	903	907	911	915	920	924	928	932	937	941	4
1022	945	949	954	958	962	966	971	975	979	983	4
1023	988	992	996	o00	o05	o09	o13	o17	o22	o26	4
1024	01030	034	038	043	047	051	055	060	064	068	4
1025	072	077	081	085	089	094	098	102	106	111	4
1026	115	119	123	127	132	136	140	144	149	153	4
1027	157	161	166	170	174	178	182	187	191	195	4
1028	199	204	208	212	216	220	225	229	233	237	4
1029	242	246	250	254	258	263	267	271	275	280	4
1030	284	288	292	296	301	305	309	313	317	322	4
1031	326	330	334	339	343	347	351	355	360	364	4
1032	368	372	376	381	385	389	393	397	402	406	4
1033	410	414	418	423	427	431	435	439	444	448	4
1034	452	456	460	465	469	473	477	481	486	490	4
1035	494	498	502	507	511	515	519	523	528	532	4
1036	536	540	544	549	553	557	561	565	569	574	4
1037	578	582	586	590	595	599	603	607	611	616	4
1038	620	624	628	632	636	641	645	649	653	657	4
1039	662	666	670	674	678	682	687	691	695	699	4
1040	703	708	712	716	720	724	728	733	737	741	4
1041	745	749	753	758	762	766	770	774	778	783	4
1042	787	791	795	799	803	808	812	816	820	824	4
1043	828	833	837	841	845	849	853	858	862	866	4
1044	870	874	878	883	887	891	895	899	903	907	4
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
1045	01912	916	920	924	928	932	937	941	945	949	4
1046	953	957	961	966	970	974	978	982	986	991	4
1047	995	999	003	007	011	015	020	024	028	032	4
1048	02036	040	044	049	053	057	061	065	069	073	4
1049	078	082	086	090	094	098	102	107	111	115	4
1050	119	123	127	131	135	140	144	148	152	156	4
1051	160	164	169	173	177	181	185	189	193	197	4
1052	202	206	210	214	218	222	226	230	235	239	4
1053	243	247	251	255	259	263	268	272	276	280	4
1054	284	288	292	296	301	305	309	313	317	321	4
1055	325	329	333	338	342	346	350	354	358	362	4
1056	366	371	375	379	383	387	391	395	399	403	4
1057	407	412	416	420	424	428	432	436	440	444	4
1058	449	453	457	461	465	469	473	477	481	485	4
1059	490	494	498	502	506	510	514	518	522	526	4
1060	531	535	539	543	547	551	555	559	563	567	4
1061	572	576	580	584	588	592	596	600	604	608	4
1062	612	617	621	625	629	633	637	641	645	649	4
1063	653	657	661	666	670	674	678	682	686	690	4
1064	694	698	702	706	710	715	719	723	727	731	4
1065	735	739	743	747	751	755	759	763	768	772	4
1066	776	780	784	788	792	796	800	804	808	812	4
1067	816	821	825	829	833	837	841	845	849	853	4
1068	857	861	865	869	873	877	882	886	890	894	4
1069	898	902	906	910	914	918	922	926	930	934	4
1070	938	942	946	951	955	959	963	967	971	975	4
1071	979	983	987	991	995	999	003	007	011	015	4
1072	03019	024	028	032	036	040	044	048	052	056	4
1073	060	064	068	072	076	080	084	088	092	096	4
1074	100	104	109	113	117	121	125	129	133	137	4
1075	141	145	149	153	157	161	165	169	173	177	4
1076	181	185	189	193	197	201	205	209	214	218	4
1077	222	226	230	234	238	242	246	250	254	258	4
1078	262	266	270	274	278	282	286	290	294	298	4
1079	302	306	310	314	318	322	326	330	334	338	4
1080	342	346	350	354	358	362	366	371	375	379	4
1081	383	387	391	395	399	403	407	411	415	419	4
1082	423	427	431	435	439	443	447	451	455	459	4
1083	463	467	471	475	479	483	487	491	495	499	4
1084	503	507	511	515	519	523	527	531	535	539	4
1085	543	547	551	555	559	563	567	571	575	579	4
1086	583	587	591	595	599	603	607	611	615	619	4
1087	623	627	631	635	639	643	647	651	655	659	4
1088	663	667	671	675	679	683	687	691	695	699	4
1089	703	707	711	715	719	723	727	731	735	739	4
N.	0	1	2	3	4	5	6	7	8	9	D.

II. NATURAL SINES.

Deg.	0′	10′	20′	30′	40′	50′		Deg.
0	00000	00291	00582	00873	01164	01454	01745	89
1	01745	02036	02327	02618	02908	03199	03490	88
2	03490	03781	04071	04362	04653	04943	05234	87
3	05234	05524	05814	06105	06395	06685	06976	86
4	06976	07266	07556	07846	08136	08426	08716	85
5	08716	09005	09295	09585	09874	10164	10453	84
6	10453	10742	11031	11320	11609	11898	12187	83
7	12187	12476	12764	13053	13341	13629	13917	82
8	13917	14205	14493	14781	15069	15356	15643	81
9	15643	15931	16218	16505	16792	17078	17365	80
10	17365	17651	17937	18224	18509	18795	19081	79
11	19081	19366	19652	19937	20222	20507	20791	78
12	20791	21076	21360	21644	21928	22212	22495	77
13	22495	22778	23062	23345	23627	23910	24192	76
14	24192	24474	24756	25038	25320	25601	25882	75
15	25882	26163	26443	26724	27004	27284	27564	74
16	27564	27843	28123	28402	28680	28959	29237	73
17	29237	29515	29793	30071	30348	30625	30902	72
18	30902	31178	31454	31730	32006	32282	32557	71
19	32557	32832	33106	33381	33655	33929	34202	70
20	34202	34475	34748	35021	35293	35565	35837	69
21	35837	36108	36379	36650	36921	37191	37461	68
22	37461	37730	37999	38268	38537	38805	39073	67
23	39073	39341	39608	39875	40141	40408	40674	66
24	40674	40939	41204	41469	41734	41998	42262	65
25	42262	42525	42788	43051	43313	43575	43837	64
26	43837	44098	44359	44620	44880	45140	45399	63
27	45399	45658	45917	46175	46433	46690	46947	62
28	46947	47204	47460	47716	47971	48226	48481	61
29	48481	48735	48989	49242	49495	49748	50000	60
30	50000	50252	50503	50754	51004	51254	51504	59
31	51504	51753	52002	52250	52498	52745	52992	58
32	52992	53238	53484	53730	53975	54220	54464	57
33	54464	54708	54951	55194	55436	55678	55919	56
34	55919	56160	56401	56641	56880	57119	57358	55
35	57358	57596	57833	58070	58307	58543	58779	54
36	58779	59014	59248	59482	59716	59949	60182	53
37	60182	60414	60645	60876	61107	61337	61566	52
38	61566	61795	62024	62251	62479	62706	62932	51
39	62932	63158	63383	63608	63832	64056	64279	50
40	64279	64501	64723	64945	65166	65386	65606	49
41	65606	65825	66044	66262	66480	66697	66913	48
42	66913	67129	67344	67559	67773	67987	68200	47
43	68200	68412	68624	68835	69046	69256	69466	46
44	69466	69675	69883	70091	70298	70505	70711	45
Deg.		50′	40′	30′	20′	10′	0′	Deg.

NATURAL COSINES.

II. NATURAL SINES.

Deg.	0′	10′	20′	30′	40′	50′		Deg.
45	70711	70916	71121	71325	71529	71732	71934	44
46	71934	72136	72337	72537	72737	72937	73135	43
47	73135	73333	73531	73728	73924	74120	74314	42
48	74314	74509	74703	74896	75088	75280	75471	41
49	75471	75661	75851	76041	76229	76417	76604	40
50	76604	76791	76977	77162	77347	77531	77715	39
51	77715	77897	78079	78261	78442	78622	78801	38
52	78801	78980	79158	79335	79512	79688	79864	37
53	79864	80038	80212	80386	80558	80730	80902	36
54	80902	81072	81242	81412	81580	81748	81915	35
55	81915	82082	82248	82413	82577	82741	82904	34
56	82904	83066	83228	83389	83549	83708	83867	33
57	83867	84025	84182	84339	84495	84650	84805	32
58	84805	84959	85112	85264	85416	85567	85717	31
59	85717	85866	86015	86163	86310	86457	86603	30
60	86603	86748	86892	87036	87178	87321	87462	29
61	87462	87603	87743	87882	88020	88158	88295	28
62	88295	88431	88566	88701	88835	88968	89101	27
63	89101	89232	89363	89493	89623	89752	89879	26
64	89879	90007	90133	90259	90383	90507	90631	25
65	90631	90753	90875	90996	91116	91236	91355	24
66	91355	91472	91590	91706	91822	91936	92050	23
67	92050	92164	92276	92388	92499	92609	92718	22
68	92718	92827	92935	93042	93148	93253	93358	21
69	93358	93462	93565	93667	93769	93869	93969	20
70	93969	94068	94167	94264	94361	94457	94552	19
71	94552	94646	94740	94832	94924	95015	95106	18
72	95106	95195	95284	95372	95459	95545	95630	17
73	95630	95715	95799	95882	95964	96046	96126	16
74	96126	96206	96285	96363	96440	96517	96593	15
75	96593	96667	96742	96815	96887	96959	97030	14
76	97030	97100	97169	97237	97304	97371	97437	13
77	97437	97502	97566	97630	97692	97754	97815	12
78	97815	97875	97934	97992	98050	98107	98163	11
79	98163	98218	98272	98325	98378	98430	98481	10
80	98481	98531	98580	98629	98676	98723	98769	9
81	98769	98814	98858	98902	98944	98986	99027	8
82	99027	99067	99106	99144	99182	99219	99255	7
83	99255	99290	99324	99357	99390	99421	99452	6
84	99452	99482	99511	99540	99567	99594	99619	5
85	99619	99644	99668	99692	99714	99736	99756	4
86	99756	99776	99795	99813	99831	99847	99863	3
87	99863	99878	99892	99905	99917	99929	99939	2
88	99939	99949	99958	99966	99973	99979	99985	1
89	99985	99989	99993	99996	99998	99999	1.0000	0
Deg.		50′	40′	30′	20′	10′	0′	Deg.

NATURAL COSINES.

III.—NATURAL TANGENTS.

Deg.	0′	10′	20′	30′	40′	50′		Deg.
0	00000	00291	00582	00873	01164	01455	01746	89
1	01746	02036	02328	02619	02910	03201	03492	88
2	03492	03783	04075	04366	04658	04949	05241	87
3	05241	05533	05824	06116	06408	06700	06993	86
4	06993	07285	07578	07870	08163	08456	08749	85
5	08749	09042	09335	09629	09923	10216	10510	84
6	10510	10805	11099	11394	11688	11983	12278	83
7	12278	12574	12869	13165	13461	13758	14054	82
8	14054	14351	14648	14945	15243	15540	15838	81
9	15838	16137	16435	16734	17033	17333	17633	80
10	17633	17933	18233	18534	18835	19136	19438	79
11	19438	19740	20042	20345	20648	20952	21256	78
12	21256	21560	21864	22169	22475	22781	23087	77
13	23087	23393	23700	24008	24316	24624	24933	76
14	24933	25242	25552	25862	26172	26483	26795	75
15	26795	27107	27419	27732	28046	28360	28675	74
16	28675	28990	29305	29621	29938	30255	30573	73
17	30573	30891	31210	31530	31850	32171	32492	72
18	32492	32814	33136	33460	33783	34108	34433	71
19	34433	34758	35085	35412	35740	36068	36397	70
20	36397	36727	37057	37388	37720	38053	38386	69
21	38386	38721	39055	39391	39727	40065	40403	68
22	40403	40741	41081	41421	41763	42105	42447	67
23	42447	42791	43136	43481	43828	44175	44523	66
24	44523	44872	45222	45573	45924	46277	46631	65
25	46631	46985	47341	47698	48055	48414	48773	64
26	48773	49134	49495	49858	50222	50587	50953	63
27	50953	51319	51688	52057	52427	52798	53171	62
28	53171	53545	53920	54296	54673	55051	55431	61
29	55431	55812	56194	56577	56962	57348	57735	60
30	57735	58124	58513	58905	59297	59691	60086	59
31	60086	60483	60881	61280	61681	62083	62487	58
32	62487	62892	63299	63707	64117	64528	64941	57
33	64941	65355	65771	66189	66608	67028	67451	56
34	67451	67875	68301	68728	69157	69588	70021	55
35	70021	70455	70891	71329	71769	72211	72654	54
36	72654	73100	73547	73996	74447	74900	75355	53
37	75355	75812	76272	76733	77196	77661	78129	52
38	78129	78598	79070	79544	80020	80498	80978	51
39	80978	81461	81946	82434	82923	83415	83910	50
40	83910	84407	84906	85408	85912	86419	86929	49
41	86929	87441	87955	88473	88992	89515	90040	48
42	90040	90569	91099	91633	92170	92709	93252	47
43	93252	93797	94345	94896	95451	96008	96569	46
44	96569	97133	97700	98270	98843	99420	1.00000	45
Deg.		50′	40′	30′	20′	10′	0′	Deg.

NATURAL COTANGENTS.

III.—NATURAL TANGENTS.

Deg.	0′	10′	20′	30′	40′	50′		Deg.
45	1.00000	1.00583	1.01170	1.01761	1.02355	1.02952	1.03553	44
46	1.03553	1.04158	1.04766	1.05378	1.05994	1.06613	1.07237	43
47	1.07237	1.07864	1.08496	1.09131	1.09770	1.10414	1.11061	42
48	1.11061	1.11713	1.12369	1.13029	1.13694	1.14363	1.15037	41
49	1.15037	1.15715	1.16398	1.17085	1.17777	1.18474	1.19175	40
50	1.19175	1.19882	1.20593	1.21310	1.22031	1.22758	1.23490	39
51	1.23490	1.24227	1.24969	1.25717	1.26471	1.27230	1.27994	38
52	1.27994	1.28764	1.29541	1.30323	1.31110	1.31904	1.32704	37
53	1.32704	1.33511	1.34323	1.35142	1.35968	1.36800	1.37638	36
54	1.37638	1.38484	1.39336	1.40195	1.41061	1.41934	1.42815	35
55	1.42815	1.43703	1.44598	1.45501	1.46411	1.47330	1.48256	34
56	1.48256	1.49190	1.50133	1.51084	1.52043	1.53010	1.53987	33
57	1.53987	1.54972	1.55966	1.56969	1.57981	1.59002	1.60033	32
58	1.60033	1.61074	1.62125	1.63185	1.64256	1.65337	1.66428	31
59	1.66428	1.67530	1.68643	1.69766	1.70901	1.72047	1.73205	30
60	1.73205	1.74375	1.75556	1.76749	1.77955	1.79174	1.80405	29
61	1.80405	1.81649	1.82906	1.84177	1.85462	1.86760	1.88073	28
62	1.88073	1.89400	1.90741	1.92098	1.93470	1.94858	1.96261	27
63	1.96261	1.97680	1.99116	2.00569	2.02039	2.03526	2.05030	26
64	2.05030	2.06553	2.08094	2.09654	2.11233	2.12832	2.14451	25
65	2.14451	2.16090	2.17749	2.19430	2.21132	2.22857	2.24604	24
66	2.24604	2.26374	2.28167	2.29984	2.31826	2.33693	2.35585	23
67	2.35585	2.37504	2.39449	2.41421	2.43422	2.45451	2.47509	22
68	2.47509	2.49597	2.51715	2.53865	2.56046	2.58261	2.60509	21
69	2.60509	2.62791	2.65109	2.67462	2.69853	2.72281	2.74748	20
70	2.74748	2.77254	2.79802	2.82391	2.85023	2.87700	2.90421	19
71	2.90421	2.93189	2.96004	2.98868	3.01783	3.04749	3.07768	18
72	3.07768	3.10842	3.13972	3.17159	3.20406	3.23714	3.27085	17
73	3.27085	3.30521	3.34023	3.37594	3.41236	3.44951	3.48741	16
74	3.48741	3.52609	3.56557	3.60588	3.64705	3.68909	3.73205	15
75	3.73205	3.77595	3.82083	3.86671	3.91364	3.96165	4.01078	14
76	4.01078	4.06107	4.11256	4.16530	4.21933	4.27471	4.33148	13
77	4.33148	4.38969	4.44942	4.51071	4.57363	4.63825	4.70463	12
78	4.70463	4.77286	4.84300	4.91516	4.98940	5.06584	5.14455	11
79	5.14455	5.22566	5.30928	5.39552	5.48451	5.57638	6.67128	10
80	5.67128	5.76937	5.87080	5.97576	6.08444	6.19703	6.31375	9
81	6.31375	6.43484	6.56055	6.69116	6.82694	6.96823	7.11537	8
82	7.11537	7.26873	7.42871	7.59575	7.77035	7.95302	8.14435	7
83	8.14435	8.34496	8.55555	8.77689	9.00983	9.25530	9.51436	6
84	9.51436	9.78817	10.0780	10.3854	10.7119	11.0594	11.4301	5
85	11.4301	11.8262	12.2505	12.7062	13.1969	13.7267	14.3007	4
86	14.3007	14.9244	15.6048	16.3499	17.1693	18.0750	19.0811	3
87	19.0811	20.2056	21.4704	22.9038	24.5418	26.4316	28.6363	2
88	28.6363	31.2416	34.3678	38.1885	42.9641	49.1039	57.2900	1
89	57.2900	68.7501	85.9398	114.589	171.885	343.774	∞	0
Deg.		50′	40′	30′	20′	10′	0′	Deg.

NATURAL COTANGENTS.

M.	Sine.	Dl″	Tang.	Dl″	M.
0	— ∞		— ∞		60
1	6.46373	502	6.46373	502	59
2	76476	293	76476	293	58
3	94085	208	94085	208	57
4	7.06579	162	7.06579	162	56
5	16270	132	16270	132	55
6	24188	112	24188	112	54
7	30882	96.7	30882	96.7	53
8	36682	85.2	36682	85.2	52
9	41797	76.3	41797	76.3	51
10	46373	69.0	46373	69.0	50
11	7.50512	63.0	7.50512	63.0	49
12	54291	57.9	54291	57.9	48
13	57767	53.6	57767	53.6	47
14	60985	49.9	60986	49.9	46
15	63982	46.7	63982	46.7	45
16	66784	43.9	66785	43.9	44
17	69417	41.4	69418	41.4	43
18	71900	39.1	71900	39.1	42
19	74248	37.1	74248	37.1	41
20	76475	35.3	76476	35.3	40
21	7.78594	33.7	7.78595	33.7	39
22	80615	32.2	80615	32.2	38
23	82545	30.8	82546	30.8	37
24	84393	29.5	84394	29.5	36
25	86166	28.4	86167	28.4	35
26	87870	27.3	87871	27.3	34
27	89509	26.3	89510	26.3	33
28	91088	25.4	91089	25.4	32
29	92612	24.5	92613	24.5	31
30	94084	23.7	94086	23.7	30
31	7.95508	23.0	7.95510	23.0	29
32	96887	22.3	96889	22.3	28
33	98223	21.6	98225	21.6	27
34	99520	21.0	99522	21.0	26
35	8.00779	20.4	8.00781	20.4	25
36	02002	19.8	02004	19.8	24
37	03192	19.3	03194	19.3	23
38	04350	18.8	04353	18.8	22
39	05478	18.3	05481	18.3	21
40	06578	17.9	06581	17.9	20
41	8.07650	17.4	8.07653	17.4	19
42	08696	17.0	08700	17.0	18
43	09718	16.6	09722	16.6	17
44	10717	16.3	10720	16.3	16
45	11693	15.9	11696	15.9	15
46	12647	15.6	12651	15.6	14
47	13581	15.2	13585	15.2	13
48	14495	14.9	14500	14.9	12
49	15391	14.6	15395	14.6	11
50	16268	14.3	16273	14.3	10
51	8.17128	14.0	8.17133	14.1	9
52	17971	13.8	17976	13.8	8
53	18798	13.5	18804	13.5	7
54	19610	13.3	19616	13.3	6
55	20407	13.0	20413	13.0	5
56	21189	12.8	21195	12.8	4
57	21958	12.6	21964	12.6	3
58	22713	12.4	22720	12.4	2
59	23456	12.2	23462	12.2	1
60	24186		24192		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.
0	8.24186	12.0	8.24192	12.0	60
1	24903	11.8	24910	11.8	59
2	25609	11.6	25616	11.6	58
3	26304	11.4	26312	11.4	57
4	26988	11.2	26996	11.2	56
5	27661	11.0	27669	11.0	55
6	28324	10.9	28332	10.9	54
7	28977	10.7	28986	10.7	53
8	29621	10.6	29629	10.6	52
9	30255	10.4	30263	10.4	51
10	30879	10.3	30888	10.3	50
11	8.31495	10.1	8.31505	10.1	49
12	32103	9.98	32112	9.99	48
13	32702	9.85	32711	9.85	47
14	33292	9.71	33302	9.72	46
15	33875	9.59	33886	9.59	45
16	34450	9.46	34461	9.47	44
17	35018	9.34	35029	9.35	43
18	35578	9.22	35590	9.22	42
19	36131	9.10	36143	9.11	41
20	36678	8.99	36689	9.00	40
21	8.37217	8.88	8.37229	8.88	39
22	37750	8.77	37762	8.78	38
23	38276	8.67	38289	8.67	37
24	38796	8.56	38809	8.57	36
25	39310	8.46	39323	8.47	35
26	39818	8.37	39832	8.37	34
27	40320	8.27	40334	8.28	33
28	40816	8.18	40830	8.18	32
29	41307	8.09	41321	8.09	31
30	41792	8.00	41807	8.00	30
31	8.42272	7.91	8.42287	7.91	29
32	42746	7.82	42762	7.83	28
33	43216	7.74	43232	7.75	27
34	43680	7.66	43696	7.66	26
35	44139	7.58	44156	7.58	25
36	44594	7.50	44611	7.50	24
37	45044	7.42	45061	7.43	23
38	45489	7.35	45507	7.35	22
39	45930	7.27	45948	7.28	21
40	46366	7.20	46385	7.21	20
41	8.46799	7.13	8.46817	7.13	19
42	47226	7.06	47245	7.07	18
43	47650	6.99	47669	7.00	17
44	48069	6.92	48089	6.93	16
45	48485	6.86	48505	6.87	15
46	48896	6.79	48917	6.80	14
47	49304	6.73	49325	6.74	13
48	49708	6.67	49729	6.68	12
49	50108	6.61	50130	6.62	11
50	50504	6.55	50527	6.55	10
51	8.50897	6.49	8.50920	6.50	9
52	51287	6.43	51310	6.44	8
53	51673	6.37	51696	6.38	7
54	52055	6.32	52079	6.33	6
55	52434	6.26	52459	6.27	5
56	52810	6.21	52835	6.22	4
57	53183	6.16	53208	6.17	3
58	53552	6.11	53578	6.11	2
59	53919	6.05	53945	6.06	1
60	54282		54308		0
M.	Cosine.	Dl″	Cotang.	Dl″	M

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	8.54282	6.00	8.54308	6.01	60	0	8.71880	4.01	8.71940	4.02	0
1	54642	5.95	54669	5.96	59	1	72120	3.98	72181	3.99	59
2	54999	5.91	55027	5.91	58	2	72359	3.96	72420	3.97	58
3	55354	5.86	55382	5.87	57	3	72597	3.94	72659	3.95	57
4	55705	5.81	55734	5.82	56	4	72834	3.92	72896	3.93	56
5	56054	5.77	56083	5.77	55	5	73069	3.90	73132	3.91	55
6	56400	5.72	56429	5.73	54	6	73303	3.88	73366	3.89	54
7	56743	5.67	56773	5.68	53	7	73535	3.86	73600	3.87	53
8	57084	5.63	57114	5.64	52	8	73767	3.84	73832	3.85	52
9	57421	5.59	57452	5.60	51	9	73997	3.82	74063	3.83	51
10	57757	5.54	57788	5.55	50	10	74226	3.80	74292	3.81	50
11	8.58089	5.50	8.58121	5.51	49	11	8.74454	3.78	8.74521	3.79	49
12	58419	5.46	58451	5.47	48	12	74680	3.76	74748	3.77	48
13	58747	5.42	58779	5.43	47	13	74906	3.74	74974	3.75	47
14	59072	5.38	59105	5.39	46	14	75130	3.72	75199	3.73	46
15	59395	5.34	59428	5.35	45	15	75353	3.70	75423	3.71	45
16	59715	5.30	59749	5.31	44	16	75575	3.68	75645	3.69	44
17	60033	5.26	60068	5.27	43	17	75795	3.66	75867	3.67	43
18	60349	5.22	60384	5.23	42	18	76015	3.64	76087	3.65	42
19	60662	5.19	60698	5.19	41	19	76234	3.62	76306	3.64	41
20	60973	5.15	61009	5.16	40	20	76451	3.61	76525	3.62	40
21	8.61282	5.11	8.61319	5.12	39	21	8.76667	3.59	8.76742	3.60	39
22	61589	5.08	61626	5.08	38	22	76883	3.57	76958	3.58	38
23	61894	5.04	61931	5.05	37	23	77097	3.55	77173	3.57	37
24	62196	5.01	62234	5.02	36	24	77310	3.53	77387	3.55	36
25	62497	4.97	62535	4.98	35	25	77522	3.52	77600	3.53	35
26	62795	4.94	62834	4.95	34	26	77733	3.50	77811	3.51	34
27	63091	4.90	63131	4.91	33	27	77943	3.48	78022	3.50	33
28	63385	4.87	63426	4.88	32	28	78152	3.47	78232	3.48	32
29	63678	4.84	63718	4.85	31	29	78360	3.45	78441	3.46	31
30	63968	4.81	64009	4.82	30	30	78568	3.43	78649	3.45	30
31	8.64256	4.78	8.64298	4.78	29	31	8.78774	3.42	8.78855	3.43	29
32	64543	4.74	64585	4.75	28	32	78979	3.40	79061	3.42	28
33	64827	4.71	64870	4.72	27	33	79183	3.39	79266	3.40	27
34	65110	4.68	65154	4.69	26	34	79386	3.37	79470	3.38	26
35	65391	4.65	65435	4.66	25	35	79588	3.35	79673	3.37	25
36	65670	4.62	65715	4.63	24	36	79789	3.34	79875	3.35	24
37	65947	4.59	65993	4.60	23	37	79990	3.32	80076	3.34	23
38	66223	4.56	66269	4.57	22	38	80189	3.31	80277	3.32	22
39	66497	4.53	66543	4.54	21	39	80388	3.29	80476	3.31	21
40	66769	4.51	66816	4.52	20	40	80585	3.28	80674	3.29	20
41	8.67039	4.48	8.67087	4.49	19	41	8.80782	3.26	8.80872	3.28	19
42	67308	4.45	67356	4.46	18	42	80978	3.25	81068	3.26	18
43	67575	4.42	67624	4.43	17	43	81173	3.23	81264	3.25	17
44	67841	4.40	67890	4.41	16	44	81367	3.22	81459	3.23	16
45	68104	4.37	68154	4.38	15	45	81560	3.20	81653	3.22	15
46	68367	4.34	68417	4.35	14	46	81752	3.19	81846	3.20	14
47	68627	4.32	68678	4.33	13	47	81944	3.18	82038	3.19	13
48	68886	4.29	68938	4.30	12	48	82134	3.16	82230	3.18	12
49	69144	4.27	69196	4.28	11	49	82324	3.15	82420	3.16	11
50	69400	4.24	69453	4.25	10	50	82513	3.13	82610	3.15	10
51	8.69654	4.22	8.69708	4.23	9	51	8.82701	3.12	8.82799	3.14	9
52	69907	4.19	69962	4.20	8	52	82888	3.11	82987	3.12	8
53	70159	4.17	70214	4.18	7	53	83075	3.10	83175	3.11	7
54	70409	4.14	70465	4.15	6	54	83261	3.08	83361	3.10	6
55	70658	4.12	70714	4.13	5	55	83446	3.07	83547	3.08	5
56	70905	4.10	70962	4.11	4	56	83630	3.06	83732	3.07	4
57	71151	4.07	71208	4.08	3	57	83813	3.04	83916	3.06	3
58	71395	4.05	71453	4.06	2	58	83996	3.03	84100	3.04	2
59	71638	4.03	71697	4.04	1	59	84177	3.02	84282	3.03	1
60	71880		71940		0	60	84358		84464		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	8.84358	3.01	8.84464	3.02	60	0	8.94030	2.40	8.94195	2.42	60
1	84539	2.99	84646	3.01	59	1	94174	2.39	94340	2.41	59
2	84718	2.98	84826	3.00	58	2	94317	2.39	94485	2.41	58
3	84897	2.97	85006	2.98	57	3	94461	2.38	94630	2.40	57
4	85075	2.95	85185	2.97	56	4	94603	2.37	94773	2.39	56
5	85252	2.94	85363	2.96	55	5	94746	2.36	94917	2.38	55
6	85429	2.93	85540	2.95	54	6	94887	2.36	95060	2.37	54
7	85605	2.92	85717	2.93	53	7	95029	2.35	95202	2.37	53
8	85780	2.91	85893	2.92	52	8	95170	2.34	95344	2.36	52
9	85955	2.90	86069	2.91	51	9	95310	2.33	95486	2.35	51
10	86128	2.88	86243	2.90	50	10	95450	2.33	95627	2.34	50
11	8.86301	2.87	8.86417	2.89	49	11	8.95589	2.32	8.95767	2.34	49
12	86474	2.86	86591	2.88	48	12	95728	2.31	95908	2.33	48
13	86645	2.85	86763	2.87	47	13	95867	2.30	96047	2.32	47
14	86816	2.84	86935	2.85	46	14	96005	2.30	96187	2.31	46
15	86987	2.83	87106	2.84	45	15	96143	2.29	96325	2.31	45
16	87156	2.82	87277	2.83	44	16	96280	2.28	96464	2.30	44
17	87325	2.81	87447	2.82	43	17	96417	2.27	96602	2.29	43
18	87494	2.79	87616	2.81	42	18	96553	2.27	96739	2.29	42
19	87661	2.78	87785	2.80	41	19	96689	2.26	96877	2.28	41
20	87829	2.77	87953	2.79	40	20	96825	2.25	97013	2.27	40
21	8.87995	2.76	8.88120	2.78	39	21	8.96960	2.25	8.97150	2.26	39
22	88161	2.75	88287	2.77	38	22	97095	2.24	97285	2.26	38
23	88326	2.74	88453	2.76	37	23	97229	2.23	97421	2.25	37
24	88490	2.73	88618	2.75	36	24	97363	2.22	97556	2.24	36
25	88654	2.72	88783	2.74	35	25	97496	2.22	97691	2.24	35
26	88817	2.71	88948	2.73	34	26	97629	2.21	97825	2.23	34
27	88980	2.70	89111	2.72	33	27	97762	2.21	97959	2.22	33
28	89142	2.69	89274	2.71	32	28	97894	2.20	98092	2.22	32
29	89304	2.68	89437	2.70	31	29	98026	2.19	98225	2.21	31
30	89464	2.67	89598	2.69	30	30	98157	2.18	98358	2.20	30
31	8.89625	2.66	8.89760	2.68	29	31	8.98288	2.18	8.98490	2.20	29
32	89784	2.65	89920	2.67	28	32	98419	2.17	98622	2.19	28
33	89943	2.64	90080	2.66	27	33	98549	2.16	98753	2.18	27
34	90102	2.63	90240	2.65	26	34	98679	2.16	98884	2.18	26
35	90260	2.62	90399	2.64	25	35	98808	2.15	99015	2.17	25
36	90417	2.61	90557	2.63	24	36	98937	2.14	99145	2.16	24
37	90574	2.60	90715	2.62	23	37	99066	2.14	99275	2.16	23
38	90730	2.59	90872	2.61	22	38	99194	2.13	99405	2.15	22
39	90885	2.58	91029	2.60	21	39	99322	2.13	99534	2.15	21
40	91040	2.58	91185	2.59	20	40	99450	2.12	99662	2.14	20
41	8.91195	2.57	8.91340	2.58	19	41	8.99577	2.11	8.99791	2.13	19
42	91349	2.56	91495	2.57	18	42	99704	2.11	99919	2.13	18
43	91502	2.55	91650	2.56	17	43	99830	2.10	9.00046	2.12	17
44	91655	2.54	91803	2.56	16	44	99956	2.09	00174	2.12	16
45	91807	2.53	91957	2.55	15	45	9.00082	2.09	00301	2.11	15
46	91959	2.52	92110	2.54	14	46	00207	2.08	00427	2.10	14
47	92110	2.51	92262	2.53	13	47	00332	2.08	00553	2.10	13
48	92261	2.50	92414	2.52	12	48	00456	2.07	00679	2.09	12
49	92411	2.49	92565	2.51	11	49	00581	2.06	00805	2.08	11
50	92561	2.49	92716	2.50	10	50	00704	2.06	00930	2.08	10
51	8.92710	2.48	8.92866	2.49	9	51	9.00828	2.05	9.01055	2.07	9
52	92859	2.47	93016	2.49	8	52	00951	2.05	01179	2.07	8
53	93007	2.46	93165	2.48	7	53	01074	2.04	01303	2.06	7
54	93154	2.45	93313	2.47	6	54	01196	2.03	01427	2.06	6
55	93301	2.44	93462	2.46	5	55	01318	2.03	01550	2.05	5
56	93448	2.43	93609	2.45	4	56	01440	2.02	01673	2.05	4
57	93594	2.43	93756	2.44	3	57	01561	2.02	01796	2.04	3
58	93740	2.42	93903	2.44	2	58	01682	2.01	01918	2.03	2
59	93885	2.41	94049	2.43	1	59	01803	2.01	02040	2.03	1
60	94030		94195		0	60	01923		02162		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.01923	2.00	9.02162	2.02	60	0	9.08589	1.71	9.08914	1.74	60
1	02043	2.00	02283	2.02	59	1	08692	1.71	09019	1.73	59
2	02163	1.99	02404	2.01	58	2	08795	1.70	09123	1.73	58
3	02283	1.98	02525	2.01	57	3	08897	1.70	09227	1.73	57
4	02402	1.98	02645	2.00	56	4	08999	1.70	09330	1.72	56
5	02520	1.97	02766	1.99	55	5	09101	1.69	09434	1.72	55
6	02639	1.97	02885	1.99	54	6	09202	1.69	09537	1.72	54
7	02757	1.96	03005	1.98	53	7	09304	1.68	09640	1.71	53
8	02874	1.96	03124	1.98	52	8	09405	1.68	09742	1.71	52
9	02992	1.95	03242	1.97	51	9	09506	1.68	09845	1.70	51
10	03109	1.95	03361	1.97	50	10	09606	1.67	09947	1.70	50
11	9.03226	1.94	9.03479	1.96	49	11	9.09707	1.67	9.10049	1.69	49
12	03342	1.94	03597	1.96	48	12	09807	1.67	10150	1.69	48
13	03458	1.93	03714	1.95	47	13	09907	1.66	10252	1.69	47
14	03574	1.93	03832	1.95	46	14	10006	1.66	10353	1.68	46
15	03690	1.92	03948	1.94	45	15	10106	1.65	10454	1.68	45
16	03805	1.92	04065	1.94	44	16	10205	1.65	10555	1.68	44
17	03920	1.91	04181	1.93	43	17	10304	1.64	10656	1.67	43
18	04034	1.91	04297	1.93	42	18	10402	1.64	10756	1.67	42
19	04149	1.90	04413	1.92	41	19	10501	1.64	10856	1.67	41
20	04262	1.89	04528	1.92	40	20	10599	1.63	10956	1.66	40
21	9.04376	1.89	9.04643	1.91	39	21	9.10697	1.63	9.11056	1.66	39
22	04490	1.88	04758	1.91	38	22	10795	1.63	11155	1.65	38
23	04603	1.88	04873	1.90	37	23	10893	1.62	11254	1.65	37
24	04715	1.87	04987	1.90	36	24	10990	1.62	11353	1.65	36
25	04828	1.87	05101	1.89	35	25	11087	1.62	11452	1.64	35
26	04940	1.87	05214	1.89	34	26	11184	1.61	11551	1.64	34
27	05052	1.86	05328	1.88	33	27	11281	1.61	11649	1.64	33
28	05164	1.86	05441	1.88	32	28	11377	1.61	11747	1.63	32
29	05275	1.85	05553	1.87	31	29	11474	1.60	11845	1.63	31
30	05386	1.85	05666	1.87	30	30	11570	1.60	11943	1.62	30
31	9.05497	1.84	9.05778	1.87	29	31	9.11666	1.59	9.12040	1.62	29
32	05607	1.84	05890	1.86	28	32	11761	1.59	12138	1.62	28
33	05717	1.83	06002	1.85	27	33	11857	1.59	12235	1.62	27
34	05827	1.83	06113	1.85	26	34	11952	1.58	12332	1.61	26
35	05937	1.82	06224	1.85	25	35	12047	1.58	12428	1.61	25
36	06046	1.82	06335	1.84	24	36	12142	1.58	12525	1.60	24
37	06155	1.81	06445	1.84	23	37	12236	1.57	12621	1.60	23
38	06264	1.81	06556	1.83	22	38	12331	1.57	12717	1.60	22
39	06372	1.80	06666	1.83	21	39	12425	1.57	12813	1.59	21
40	06481	1.80	06775	1.82	20	40	12519	1.56	12909	1.59	20
41	9.06589	1.79	9.06885	1.82	19	41	9.12612	1.56	9.13004	1.59	19
42	06696	1.79	06994	1.82	18	42	12706	1.56	13099	1.58	18
43	06804	1.79	07103	1.81	17	43	12799	1.55	13194	1.58	17
44	06911	1.78	07211	1.81	16	44	12892	1.55	13289	1.58	16
45	07018	1.78	07320	1.80	15	45	12985	1.55	13384	1.57	15
46	07124	1.77	07428	1.80	14	46	13078	1.54	13478	1.57	14
47	07231	1.77	07536	1.79	13	47	13171	1.54	13573	1.57	13
48	07337	1.76	07643	1.79	12	48	13263	1.53	13667	1.56	12
49	07442	1.76	07751	1.78	11	49	13355	1.53	13761	1.56	11
50	07548	1.75	07858	1.78	10	50	13447	1.53	13854	1.56	10
51	9.07653	1.75	9.07964	1.77	9	51	9.13539	1.52	9.13948	1.55	9
52	07758	1.75	08071	1.77	8	52	13630	1.52	14041	1.55	8
53	07863	1.74	08177	1.77	7	53	13722	1.52	14134	1.55	7
54	07968	1.74	08283	1.76	6	54	13813	1.52	14227	1.55	6
55	08072	1.73	08389	1.76	5	55	13904	1.51	14320	1.54	5
56	08176	1.73	08495	1.75	4	56	13994	1.51	14412	1.54	4
57	08280	1.72	08600	1.75	3	57	14085	1.51	14504	1.54	3
58	08383	1.72	08705	1.75	2	58	14175	1.50	14597	1.53	2
59	08486	1.72	08810	1.74	1	59	14266	1.50	14688	1.53	1
60	08589		08914		0	60	14356		14780		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.14356	1.50	9.14780	1.53	60	0	9.19433	1.33	9.19971	1.36	60
1	14445	1.49	14872	1.52	59	1	19513	1.33	20053	1.36	59
2	14535	1.49	14963	1.52	58	2	19592	1.32	20134	1.36	58
3	14624	1.49	15054	1.52	57	3	19672	1.32	20216	1.35	57
4	14714	1.48	15145	1.51	56	4	19751	1.32	20297	1.35	56
5	14803	1.48	15236	1.51	55	5	19830	1.32	20378	1.35	55
6	14891	1.48	15327	1.51	54	6	19909	1.31	20459	1.35	54
7	14980	1.48	15417	1.50	53	7	19988	1.31	20540	1.35	53
8	15069	1.47	15508	1.50	52	8	20067	1.31	20621	1.34	52
9	15157	1.47	15598	1.50	51	9	20145	1.31	20701	1.34	51
10	15245	1.47	15688	1.50	50	10	20223	1.30	20782	1.34	50
11	9.15333	1.46	9.15777	1.49	49	11	9.20302	1.30	9.20862	1.33	49
12	15421	1.46	15867	1.49	48	12	20380	1.30	20942	1.33	48
13	15508	1.46	15956	1.49	47	13	20458	1.30	21022	1.33	47
14	15596	1.45	16046	1.48	46	14	20535	1.29	21102	1.33	46
15	15683	1.45	16135	1.48	45	15	20613	1.29	21182	1.33	45
16	15770	1.45	16224	1.48	44	16	20691	1.29	21261	1.32	44
17	15857	1.45	16312	1.48	43	17	20768	1.29	21341	1.32	43
18	15944	1.44	16401	1.47	42	18	20845	1.28	21420	1.32	42
19	16030	1.44	16489	1.47	41	19	20922	1.28	21499	1.32	41
20	16116	1.44	16577	1.47	40	20	20999	1.28	21578	1.32	40
21	9.16203	1.43	9.16665	1.46	39	21	9.21076	1.28	9.21657	1.31	39
22	16289	1.43	16753	1.46	38	22	21153	1.27	21736	1.31	38
23	16374	1.43	16841	1.46	37	23	21229	1.27	21814	1.31	37
24	16460	1.42	16928	1.46	36	24	21306	1.27	21893	1.31	36
25	16545	1.42	17016	1.45	35	25	21382	1.27	21971	1.30	35
26	16631	1.42	17103	1.45	34	26	21458	1.27	22049	1.30	34
27	16716	1.42	17190	1.45	33	27	21534	1.26	22127	1.30	33
28	16801	1.41	17277	1.44	32	28	21610	1.26	22205	1.30	32
29	16886	1.41	17363	1.44	31	29	21685	1.26	22283	1.29	31
30	16970	1.41	17450	1.44	30	30	21761	1.26	22361	1.29	30
31	9.17055	1.40	9.17536	1.44	29	31	9.21836	1.25	9.22438	1.29	29
32	17139	1.40	17622	1.43	28	32	21912	1.25	22516	1.29	28
33	17223	1.40	17708	1.43	27	33	21987	1.25	22593	1.29	27
34	17307	1.40	17794	1.43	26	34	22062	1.25	22670	1.28	26
35	17391	1.39	17880	1.42	25	35	22137	1.25	22747	1.28	25
36	17474	1.39	17965	1.42	24	36	22211	1.24	22824	1.28	24
37	17558	1.39	18051	1.42	23	37	22286	1.24	22901	1.28	23
38	17641	1.39	18136	1.42	22	38	22361	1.24	22977	1.28	22
39	17724	1.38	18221	1.42	21	39	22435	1.24	23054	1.27	21
40	17807	1.38	18306	1.41	20	40	22509	1.24	23130	1.27	20
41	9.17890	1.38	9.18391	1.41	19	41	9.22583	1.23	9.23206	1.27	19
42	17973	1.37	18475	1.41	18	42	22657	1.23	23283	1.27	18
43	18055	1.37	18560	1.40	17	43	22731	1.23	23359	1.27	17
44	18137	1.37	18644	1.40	16	44	22805	1.23	23435	1.26	16
45	18220	1.37	18728	1.40	15	45	22878	1.22	23510	1.26	15
46	18302	1.36	18812	1.40	14	46	22952	1.22	23586	1.26	14
47	18383	1.36	18896	1.39	13	47	23025	1.22	23661	1.26	13
48	18465	1.36	18979	1.39	12	48	23098	1.22	23737	1.25	12
49	18547	1.36	19063	1.39	11	49	23171	1.22	23812	1.25	11
50	18628	1.35	19146	1.39	10	50	23244	1.21	23887	1.25	10
51	9.18709	1.35	9.19229	1.38	9	51	9.23317	1.21	9.23962	1.25	9
52	18790	1.35	19312	1.38	8	52	23390	1.21	24037	1.25	8
53	18871	1.35	19395	1.38	7	53	23462	1.21	24112	1.24	7
54	18952	1.34	19478	1.38	6	54	23535	1.20	24186	1.24	6
55	19033	1.34	19561	1.37	5	55	23607	1.20	24261	1.24	5
56	19113	1.34	19643	1.37	4	56	23679	1.20	24335	1.24	4
57	19193	1.34	19725	1.37	3	57	23752	1.20	24410	1.24	3
58	19273	1.33	19807	1.37	2	58	23823	1.20	24484	1.23	2
59	19353	1.33	19889	1.36	1	59	23895	1.20	24558	1.23	1
60	19433		19971		0	60	23967		24632		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.23967	1.19	9.24632	1.23	60	0	9.28060	1.08	9.28865	1.12	60
1	24039	1.19	24706	1.23	59	1	28125	1.08	28933	1.12	59
2	24110	1.19	24779	1.23	58	2	28190	1.08	29000	1.12	58
3	24181	1.19	24853	1.22	57	3	28254	1.08	29067	1.12	57
4	24253	1.18	24926	1.22	56	4	28319	1.08	29134	1.12	56
5	24324	1.18	25000	1.22	55	5	28384	1.07	29201	1.12	55
6	24395	1.18	25073	1.22	54	6	28448	1.07	29268	1.11	54
7	24466	1.18	25146	1.22	53	7	28512	1.07	29335	1.11	53
8	24536	1.18	25219	1.22	52	8	28577	1.07	29402	1.11	52
9	24607	1.17	25292	1.21	51	9	28641	1.07	29468	1.11	51
10	24677	1.17	25365	1.21	50	10	28705	1.07	29535	1.11	50
11	9.24748	1.17	9.25437	1.21	49	11	9.28769	1.06	9.29601	1.11	49
12	24818	1.17	25510	1.21	48	12	28833	1.06	29668	1.10	48
13	24888	1.17	25582	1.20	47	13	28896	1.06	29734	1.10	47
14	24958	1.17	25655	1.20	46	14	28960	1.06	29800	1.10	46
15	25028	1.16	25727	1.20	45	15	29024	1.06	29866	1.10	45
16	25098	1.16	25799	1.20	44	16	29087	1.06	29932	1.10	44
17	25168	1.16	25871	1.20	43	17	29150	1.05	29998	1.10	43
18	25237	1.16	25943	1.20	42	18	29214	1.05	30064	1.10	42
19	25307	1.16	26015	1.19	41	19	29277	1.05	30130	1.09	41
20	25376	1.15	26086	1.19	40	20	29340	1.05	30195	1.09	40
21	9.25445	1.15	9.26158	1.19	39	21	9.29403	1.05	9.30261	1.09	39
22	25514	1.15	26229	1.19	38	22	29466	1.05	30326	1.09	38
23	25583	1.15	26301	1.19	37	23	29529	1.04	30391	1.09	37
24	25652	1.15	26372	1.18	36	24	29591	1.04	30457	1.09	36
25	25721	1.14	26443	1.18	35	25	29654	1.04	30522	1.08	35
26	25790	1.14	26514	1.18	34	26	29716	1.04	30587	1.08	34
27	25858	1.14	26585	1.18	33	27	29779	1.04	30652	1.08	33
28	25927	1.14	26655	1.18	32	28	29841	1.04	30717	1.08	32
29	25995	1.14	26726	1.18	31	29	29903	1.04	30782	1.08	31
30	26063	1.13	26797	1.17	30	30	29966	1.03	30846	1.08	30
31	9.26131	1.13	9.26867	1.17	29	31	9.30028	1.03	9.30911	1.07	29
32	26199	1.13	26937	1.17	28	32	30090	1.03	30975	1.07	28
33	26267	1.13	27008	1.17	27	33	30151	1.03	31040	1.07	27
34	26335	1.13	27078	1.17	26	34	30213	1.03	31104	1.07	26
35	26403	1.13	27148	1.17	25	35	30275	1.03	31168	1.07	25
36	26470	1.12	27218	1.16	24	36	30336	1.03	31233	1.07	24
37	26538	1.12	27288	1.16	23	37	30398	1.02	31297	1.07	23
38	26605	1.12	27357	1.16	22	38	30459	1.02	31361	1.07	22
39	26672	1.12	27427	1.16	21	39	30521	1.02	31425	1.06	21
40	26739	1.12	27496	1.16	20	40	30582	1.02	31489	1.06	20
41	9.26806	1.12	9.27566	1.15	19	41	9.30643	1.02	9.31552	1.06	19
42	26873	1.11	27635	1.15	18	42	30704	1.02	31616	1.06	18
43	26940	1.11	27704	1.15	17	43	30765	1.01	31679	1.06	17
44	27007	1.11	27773	1.15	16	44	30826	1.01	31743	1.06	16
45	27073	1.11	27842	1.15	15	45	30887	1.01	31806	1.05	15
46	27140	1.11	27911	1.15	14	46	30947	1.01	31870	1.05	14
47	27206	1.10	27980	1.15	13	47	31008	1.01	31933	1.05	13
48	27273	1.10	28049	1.14	12	48	31068	1.01	31996	1.05	12
49	27339	1.10	28117	1.14	11	49	31129	1.01	32059	1.05	11
50	27405	1.10	28186	1.14	10	50	31189	1.00	32122	1.05	10
51	9.27471	1.10	9.28254	1.14	9	51	9.31250	1.00	9.32185	1.05	9
52	27537	1.10	28323	1.14	8	52	31310	1.00	32248	1.05	8
53	27602	1.09	28391	1.13	7	53	31370	1.00	32311	1.04	7
54	27668	1.09	28459	1.13	6	54	31430	1.00	32373	1.04	6
55	27734	1.09	28527	1.13	5	55	31490	1.00	32436	1.04	5
56	27799	1.09	28595	1.13	4	56	31549	1.00	32498	1.04	4
57	27864	1.09	28662	1.13	3	57	31609	.99	32561	1.04	3
58	27930	1.09	28730	1.13	2	58	31669	.99	32623	1.04	2
59	27995	1.08	28798	1.12	1	59	31728	.99	32685	1.04	1
60	28060		28865		0	60	31788		32747		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.31788	0.99	9.32747	1.03	60	0	9.35209	0.91	9.36336	0.96	60
1	31847	.99	32810	1.03	59	1	35263	.91	36394	.96	59
2	31907	.99	32872	1.03	58	2	35318	.91	36452	.96	58
3	31966	.99	32933	1.03	57	3	35373	.91	36509	.96	57
4	32025	.98	32995	1.03	56	4	35427	.91	36566	.96	56
5	32084	.98	33057	1.03	55	5	35481	.91	36624	.95	55
6	32143	.98	33119	1.03	54	6	35536	.90	36681	.95	54
7	32202	.98	33180	1.03	53	7	35590	.90	36738	.95	53
8	32261	.98	33242	1.02	52	8	35644	.90	36795	.95	52
9	32319	.98	33303	1.02	51	9	35698	.90	36852	.95	51
10	32378	.98	33365	1.02	50	10	35752	.90	36909	.95	50
11	9.32437	.97	9.33426	1.02	49	11	9.35806	.90	9.36966	.95	49
12	32495	.97	33487	1.02	48	12	35860	.90	37023	.95	48
13	32553	.97	33548	1.02	47	13	35914	.90	37080	.95	47
14	32612	.97	33609	1.02	46	14	35968	.89	37137	.94	46
15	32670	.97	33670	1.01	45	15	36022	.89	37193	.94	45
16	32728	.97	33731	1.01	44	16	36075	.89	37250	.94	44
17	32786	.97	33792	1.01	43	17	36129	.89	37306	.94	43
18	32844	.97	33853	1.01	42	18	36182	.89	37363	.94	42
19	32902	.96	33913	1.01	41	19	36236	.89	37419	.94	41
20	32960	.96	33974	1.01	40	20	36289	.89	37476	.94	40
21	9.33018	.96	9.34034	1.01	39	21	9.36342	.89	9.37532	.94	39
22	33075	.96	34095	1.01	38	22	36395	.89	37588	.94	38
23	33133	.96	34155	1.00	37	23	36449	.88	37644	.93	37
24	33190	.96	34215	1.00	36	24	36502	.88	37700	.93	36
25	33248	.96	34276	1.00	35	25	36555	.88	37756	.93	35
26	33305	.95	34336	1.00	34	26	36608	.88	37812	.93	34
27	33362	.95	34396	1.00	33	27	36660	.88	37868	.93	33
28	33420	.95	34456	1.00	32	28	36713	.88	37924	.93	32
29	33477	.95	34516	1.00	31	29	36766	.88	37980	.93	31
30	33534	.95	34576	1.00	30	30	36819	.88	38035	.93	30
31	9.33591	.95	9.34635	.99	29	31	9.36871	.88	9.38091	.93	29
32	33647	.95	34695	.99	28	32	36924	.87	38147	.92	28
33	33704	.95	34755	.99	27	33	36976	.87	38202	.92	27
34	33761	.94	34814	.99	26	34	37028	.87	38257	.92	26
35	33818	.94	34874	.99	25	35	37081	.87	38313	.92	25
36	33874	.94	34933	.99	24	36	37133	.87	38368	.92	24
37	33931	.94	34992	.99	23	37	37185	.87	38423	.92	23
38	33987	.94	35051	.99	22	38	37237	.87	38479	.92	22
39	34043	.94	35111	.98	21	39	37289	.87	38534	.92	21
40	34100	.94	35170	.98	20	40	37341	.87	38589	.92	20
41	9.34156	.93	9.35229	.98	19	41	9.37393	.86	9.38644	.92	19
42	34212	.93	35288	.98	18	42	37445	.86	38699	.91	18
43	34268	.93	35347	.98	17	43	37497	.86	38754	.91	17
44	34324	.93	35405	.98	16	44	37549	.86	38808	.91	16
45	34380	.93	35464	.98	15	45	37600	.86	38863	.91	15
46	34436	.93	35523	.98	14	46	37652	.86	38918	.91	14
47	34491	.93	35581	.98	13	47	37703	.86	38972	.91	13
48	34547	.93	35640	.97	12	48	37755	.86	39027	.91	12
49	34602	.92	35698	.97	11	49	37806	.86	39082	.91	11
50	34658	.92	35757	.97	10	50	37858	.85	39136	.91	10
51	9.34713	.92	9.35815	.97	9	51	9.37909	.85	9.39190	.91	9
52	34769	.92	35873	.97	8	52	37960	.85	39245	.90	8
53	34824	.92	35931	.97	7	53	38011	.85	39299	.90	7
54	34879	.92	35989	.97	6	54	38062	.85	39353	.90	6
55	34934	.92	36047	.97	5	55	38113	.85	39407	.90	5
56	34989	.92	36105	.96	4	56	38164	.85	39461	.90	4
57	35044	.92	36163	.96	3	57	38215	.85	39515	.90	3
58	35099	.91	36221	.96	2	58	38266	.85	39569	.90	2
59	35154	.91	36279	.96	1	59	38317	.85	39623	.90	1
60	35209		36336		0	60	38368		39677		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.38368	0.84	9.39677	0.90	60	0	9.41300	0.79	9.42805	0.84	60
1	38418	.84	39731	.90	59	1	41347	.78	42856	.84	59
2	38469	.84	39785	.89	58	2	41394	.78	42906	.84	58
3	38519	.84	39838	.89	57	3	41441	.78	42957	.84	57
4	38570	.84	39892	.89	56	4	41488	.78	43007	.84	56
5	38620	.84	39945	.89	55	5	41535	.78	43057	.84	55
6	38670	.84	39999	.89	54	6	41582	.78	43108	.84	54
7	38721	.84	40052	.89	53	7	41628	.78	43158	.84	53
8	38771	.84	40106	.89	52	8	41675	.78	43208	.83	52
9	38821	.83	40159	.89	51	9	41722	.78	43258	.83	51
10	38871	.83	40212	.89	50	10	41768	.78	43308	.83	50
11	9.38921	.83	9.40266	.89	49	11	9.41815	.78	9.43358	.83	49
12	38971	.83	40319	.88	48	12	41861	.77	43408	.83	48
13	39021	.83	40372	.88	47	13	41908	.77	43458	.83	47
14	39071	.83	40425	.88	46	14	41954	.77	43508	.83	46
15	39121	.83	40478	.88	45	15	42001	.77	43558	.83	45
16	39170	.83	40531	.88	44	16	42047	.77	43607	.83	44
17	39220	.83	40584	.88	43	17	42093	.77	43657	.83	43
18	39270	.83	40636	.88	42	18	42140	.77	43707	.83	42
19	39319	.82	40689	.88	41	19	42186	.77	43756	.83	41
20	39369	.82	40742	.88	40	20	42232	.77	43806	.83	40
21	9.39418	.82	9.40795	.88	39	21	9.42278	.77	9.43855	.82	39
22	39467	.82	40847	.88	38	22	42324	.77	43905	.82	38
23	39517	.82	40900	.87	37	23	42370	.76	43954	.82	37
24	39566	.82	40952	.87	36	24	42416	.76	44004	.82	36
25	39615	.82	41005	.87	35	25	42461	.76	44053	.82	35
26	39664	.82	41057	.87	34	26	42507	.76	44102	.82	34
27	39713	.82	41109	.87	33	27	42553	.76	44151	.82	33
28	39762	.82	41161	.87	32	28	42599	.76	44201	.82	32
29	39811	.81	41214	.87	31	29	42644	.76	44250	.82	31
30	39860	.81	41266	.87	30	30	42690	.76	44299	.82	30
31	9.39909	.81	9.41318	.87	29	31	9.42735	.76	9.44348	.82	29
32	39958	.81	41370	.87	28	32	42781	.76	44397	.82	28
33	40006	.81	41422	.87	27	33	42826	.76	44446	.81	27
34	40055	.81	41474	.86	26	34	42872	.76	44495	.81	26
35	40103	.81	41526	.86	25	35	42917	.75	44544	.81	25
36	40152	.81	41578	.86	24	36	42962	.75	44592	.81	24
37	40200	.81	41629	.86	23	37	43008	.75	44641	.81	23
38	40249	.81	41681	.86	22	38	43053	.75	44690	.81	22
39	40297	.81	41733	.86	21	39	43098	.75	44738	.81	21
40	40346	.80	41784	.86	20	40	43143	.75	44787	.81	20
41	9.40394	.80	9.41836	.86	19	41	9.43188	.75	9.44836	.81	19
42	40442	.80	41887	.86	18	42	43233	.75	44884	.81	18
43	40490	.80	41939	.86	17	43	43278	.75	44933	.81	17
44	40538	.80	41990	.86	16	44	43323	.75	44981	.81	16
45	40586	.80	42041	.85	15	45	43367	.75	45029	.81	15
46	40634	.80	42093	.85	14	46	43412	.75	45078	.80	14
47	40682	.80	42144	.85	13	47	43457	.74	45126	.80	13
48	40730	.80	42195	.85	12	48	43502	.74	45174	.80	12
49	40778	.80	42246	.85	11	49	43546	.74	45222	.80	11
50	40825	.79	42297	.85	10	50	43591	.74	45271	.80	10
51	9.40873	.79	9.42348	.85	9	51	9.43635	.74	9.45319	.80	9
52	40921	.79	42399	.85	8	52	43680	.74	45367	.80	8
53	40968	.79	42450	.85	7	53	43724	.74	45415	.80	7
54	41016	.79	42501	.85	6	54	43769	.74	45463	.80	6
55	41063	.79	42552	.85	5	55	43813	.74	45511	.80	5
56	41111	.79	42603	.85	4	56	43857	.74	45559	.80	4
57	41158	.79	42653	.84	3	57	43901	.74	45606	.80	3
58	41205	.79	42704	.84	2	58	43946	.74	45654	.80	2
59	41252	.79	42755	.84	1	59	43990	.73	45702	.80	1
60	41300		42805		0	60	44034		45750		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.44034	0.73	9.45750	0.79	60	0	9.46594	0.69	9.48534	0.75	60
1	44078	.73	45797	.79	59	1	46635	.69	48579	.75	59
2	44122	.73	45845	.79	58	2	46676	.69	48624	.75	58
3	44166	.73	45892	.79	57	3	46717	.69	48669	.75	57
4	44210	.73	45940	.79	56	4	46758	.69	48714	.75	56
5	44253	.73	45987	.79	55	5	46800	.68	48759	.75	55
6	44297	.73	46035	.79	54	6	46841	.68	48804	.75	54
7	44341	.73	46082	.79	53	7	46882	.68	48849	.75	53
8	44385	.73	46130	.79	52	8	46923	.68	48894	.75	52
9	44428	.73	46177	.79	51	9	46964	.68	48939	.75	51
10	44472	.73	46224	.79	50	10	47005	.68	48984	.75	50
11	9.44516	.73	9.46271	.79	49	11	9.47045	.68	9.49029	.75	49
12	44559	.72	46319	.79	48	12	47086	.68	49073	.74	48
13	44602	.72	46366	.78	47	13	47127	.68	49118	.74	47
14	44646	.72	46413	.78	46	14	47168	.68	49163	.74	46
15	44689	.72	46460	.78	45	15	47209	.68	49207	.74	45
16	44733	.72	46507	.78	44	16	47249	.68	49252	.74	44
17	44776	.72	46554	.78	43	17	47290	.68	49296	.74	43
18	44819	.72	46601	.78	42	18	47330	.68	49341	.74	42
19	44862	.72	46648	.78	41	19	47371	.67	49385	.74	41
20	44905	.72	46694	.78	40	20	47411	.67	49430	.74	40
21	9.44948	.72	9.46741	.78	39	21	9.47452	.67	9.49474	.74	39
22	44992	.72	46788	.78	38	22	47492	.67	49519	.74	38
23	45035	.72	46835	.78	37	23	47533	.67	49563	.74	37
24	45077	.71	46881	.78	36	24	47573	.67	49607	.74	36
25	45120	.71	46928	.78	35	25	47613	.67	49652	.74	35
26	45163	.71	46975	.78	34	26	47654	.67	49696	.74	34
27	45206	.71	47021	.77	33	27	47694	.67	49740	.74	33
28	45249	.71	47068	.77	32	28	47734	.67	49784	.73	32
29	45292	.71	47114	.77	31	29	47774	.67	49828	.73	31
30	45334	.71	47160	.77	30	30	47814	.67	49872	.73	30
31	9.45377	.71	9.47207	.77	29	31	9.47854	.67	9.49916	.73	29
32	45419	.71	47253	.77	28	32	47894	.67	49960	.73	28
33	45462	.71	47299	.77	27	33	47934	.67	50004	.73	27
34	45504	.71	47346	.77	26	34	47974	.66	50048	.73	26
35	45547	.71	47392	.77	25	35	48014	.66	50092	.73	25
36	45589	.71	47438	.77	24	36	48054	.66	50136	.73	24
37	45632	.71	47484	.77	23	37	48094	.66	50180	.73	23
38	45674	.70	47530	.77	22	38	48133	.66	50223	.73	22
39	45716	.70	47576	.77	21	39	48173	.66	50267	.73	21
40	45758	.70	47622	.77	20	40	48213	.66	50311	.73	20
41	9.45801	.70	9.47668	.77	19	41	9.48252	.66	9.50355	.73	19
42	45843	.70	47714	.76	18	42	48292	.66	50398	.73	18
43	45885	.70	47760	.76	17	43	48332	.66	50442	.73	17
44	45927	.70	47806	.76	16	44	48371	.66	50485	.73	16
45	45969	.70	47852	.76	15	45	48411	.66	50529	.72	15
46	46011	.70	47897	.76	14	46	48450	.66	50572	.72	14
47	46053	.70	47943	.76	13	47	48490	.66	50616	.72	13
48	46095	.70	47989	.76	12	48	48529	.66	50659	.72	12
49	46136	.70	48035	.76	11	49	48568	.65	50703	.72	11
50	46178	.70	48080	.76	10	50	48607	.65	50746	.72	10
51	9.46220	.69	9.48126	.76	9	51	9.48647	.65	9.50789	.72	9
52	46262	.69	48171	.76	8	52	48686	.65	50833	.72	8
53	46303	.69	48217	.76	7	53	48725	.65	50876	.72	7
54	46345	.69	48262	.76	6	54	48764	.65	50919	.72	6
55	46386	.69	48307	.76	5	55	48803	.65	50962	.72	5
56	46428	.69	48353	.76	4	56	48842	.65	51005	.72	4
57	46469	.69	48398	.75	3	57	48881	.65	51048	.72	3
58	46511	.69	48443	.75	2	58	48920	.65	51092	.72	2
59	46552	.69	48489	.75	1	59	48959	.65	51135	.72	1
60	46594		48534		0	60	48998		51178		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.48998	0.65	9.51178	0.72	60	0	9.51264	0.61	9.53697	0.68	60
1	49037	.65	51221	.72	59	1	51301	.61	53738	.68	59
2	49076	.65	51264	.71	58	2	51338	.61	53779	.68	58
3	49115	.65	51306	.71	57	3	51374	.61	53820	.68	57
4	49153	.65	51349	.71	56	4	51411	.61	53861	.68	56
5	9.49192	.64	51392	.71	55	5	51447	.61	53902	.68	55
6	49231	.64	51435	.71	54	6	51484	.61	53943	.68	54
7	49269	.64	51478	.71	53	7	51520	.61	53984	.68	53
8	49308	.64	51520	.71	52	8	51557	.61	54025	.68	52
9	49347	.64	51563	.71	51	9	51593	.61	54065	.68	51
10	49385	.64	51606	.71	50	10	51629	.61	54106	.68	50
11	9.49424	.64	9.51648	.71	49	11	9.51666	.60	9.54147	.68	49
12	49462	.64	51691	.71	48	12	51702	.60	54187	.68	48
13	49500	.64	51734	.71	47	13	51738	.60	54228	.68	47
14	49539	.64	51776	.71	46	14	51774	.60	54269	.68	46
15	49577	.64	51819	.71	45	15	51811	.60	54309	.68	45
16	49615	.64	51861	.71	44	16	51847	.60	54350	.68	44
17	49654	.64	51903	.71	43	17	51883	.60	54390	.68	43
18	49692	.64	51946	.71	42	18	51919	.60	54431	.67	42
19	49730	.64	51988	.71	41	19	51955	.60	54471	.67	41
20	49768	.64	52031	.70	40	20	51991	.60	54512	.67	40
21	9.49806	.63	9.52073	.70	39	21	9.52027	.60	9.54552	.67	39
22	49844	.63	52115	.70	38	22	52063	.60	54593	.67	38
23	49882	.63	52157	.70	37	23	52099	.60	54633	.67	37
24	49920	.63	52200	.70	36	24	52135	.60	54673	.67	36
25	49958	.63	52242	.70	35	25	52171	.60	54714	.67	35
26	49996	.63	52284	.70	34	26	52207	.60	54754	.67	34
27	50034	.63	52326	.70	33	27	52242	.60	54794	.67	33
28	50072	.63	52368	.70	32	28	52278	.60	54835	.67	32
29	50110	.63	52410	.70	31	29	52314	.59	54875	.67	31
30	50148	.63	52452	.70	30	30	52350	.59	54915	.67	30
31	9.50185	.63	9.52494	.70	29	31	9.52385	.59	9.54955	.67	29
32	50223	.63	52536	.70	28	32	52421	.59	54995	.67	28
33	50261	.63	52578	.70	27	33	52456	.59	55035	.67	27
34	50298	.63	52620	.70	26	34	52492	.59	55075	.67	26
35	50336	.63	52661	.70	25	35	52527	.59	55115	.67	25
36	50374	.63	52703	.70	24	36	52563	.59	55155	.67	24
37	50411	.63	52745	.70	23	37	52598	.59	55195	.67	23
38	50449	.62	52787	.70	22	38	52634	.59	55235	.66	22
39	50486	.62	52829	.69	21	39	52669	.59	55275	.66	21
40	50523	.62	52870	.69	20	40	52705	.59	55315	.66	20
41	9.50561	.62	9.52912	.69	19	41	9.52740	.59	9.55355	.66	19
42	50598	.62	52953	.69	18	42	52775	.59	55395	.66	18
43	50635	.62	52995	.69	17	43	52811	.59	55434	.66	17
44	50673	.62	53037	.69	16	44	52846	.59	55474	.66	16
45	50710	.62	53078	.69	15	45	52881	.59	55514	.66	15
46	50747	.62	53120	.69	14	46	52916	.59	55554	.66	14
47	50784	.62	53161	.69	13	47	52951	.58	55593	.66	13
48	50821	.62	53202	.69	12	48	52986	.58	55633	.66	12
49	50858	.62	53244	.69	11	49	53021	.58	55673	.66	11
50	50896	.62	53285	.69	10	50	53056	.58	55712	.66	10
51	9.50933	.62	9.53327	.69	9	51	9.53092	.58	9.55752	.66	9
52	50970	.62	53368	.69	8	52	53126	.58	55791	.66	8
53	51007	.62	53409	.69	7	53	53161	.58	55831	.66	7
54	51043	.61	53450	.69	6	54	53196	.58	55870	.66	6
55	51080	.61	53492	.69	5	55	53231	.58	55910	.66	5
56	51117	.61	53533	.69	4	56	53266	.58	55949	.66	4
57	51154	.61	53574	.69	3	57	53301	.58	55989	.66	3
58	51191	.61	53615	.68	2	58	53336	.58	56028	.66	2
59	51227	.61	53656	.68	1	59	53370	.58	56067	.66	1
60	51264		53697		0	60	53405		56107		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.53405	0.58	9.56107	0.65	60	0	9.55433	0.55	9.58418	0.63	60
1	53440	.58	56146	.65	59	1	55466	.55	58455	.63	59
2	53475	.58	56185	.65	58	2	55499	.55	58493	.63	58
3	53509	.58	56224	.65	57	3	55532	.55	58531	.63	57
4	53544	.58	56264	.65	56	4	55564	.55	58569	.63	56
5	53578	.58	56303	.65	55	5	55597	.55	58606	.63	55
6	53613	.58	56342	.65	54	6	55630	.55	58644	.63	54
7	53647	.57	56381	.65	53	7	55663	.55	58681	.63	53
8	53682	.57	56420	.65	52	8	55695	.54	58719	.63	52
9	53716	.57	56459	.65	51	9	55728	.54	58757	.63	51
10	53751	.57	56498	.65	50	10	55761	.54	58794	.63	50
11	9.53785	.57	9.56537	.65	49	11	9.55793	.54	9.58832	.62	49
12	53819	.57	56576	.65	48	12	55826	.54	58869	.62	48
13	53854	.57	56615	.65	47	13	55858	.54	58907	.62	47
14	53888	.57	56654	.65	46	14	55891	.54	58944	.62	46
15	53922	.57	56693	.65	45	15	55923	.54	58981	.62	45
16	53957	.57	56732	.65	44	16	55956	.54	59019	.62	44
17	53991	.57	56771	.65	43	17	55988	.54	59056	.62	43
18	54025	.57	56810	.65	42	18	56021	.54	59094	.62	42
19	54059	.57	56849	.65	41	19	56053	.54	59131	.62	41
20	54093	.57	56887	.65	40	20	56085	.54	59168	.62	40
21	9.54127	.57	9.56926	.65	39	21	9.56118	.54	9.59205	.62	39
22	54161	.57	56965	.65	38	22	56150	.54	59243	.62	38
23	54195	.57	57004	.64	37	23	56182	.54	59280	.62	37
24	54229	.57	57042	.64	36	24	56215	.54	59317	.62	36
25	54263	.57	57081	.64	35	25	56247	.54	59354	.62	35
26	54297	.56	57120	.64	34	26	56279	.54	59391	.62	34
27	54331	.56	57158	.64	33	27	56311	.54	59429	.62	33
28	54365	.56	57197	.64	32	28	56343	.54	59466	.62	32
29	54399	.56	57235	.64	31	29	56375	.53	59503	.62	31
30	54433	.56	57274	.64	30	30	56408	.53	59540	.62	30
31	9.54466	.56	9.57312	.64	29	31	9.56440	.53	9.59577	.62	29
32	54500	.56	57351	.64	28	32	56472	.53	59614	.62	28
33	54534	.56	57389	.64	27	33	56504	.53	59651	.62	27
34	54567	.56	57428	.64	26	34	56536	.53	59688	.62	26
35	54601	.56	57466	.64	25	35	56568	.53	59725	.62	25
36	54635	.56	57504	.64	24	36	56599	.53	59762	.62	24
37	54668	.56	57543	.64	23	37	56631	.53	59799	.61	23
38	54702	.56	57581	.64	22	38	56663	.53	59835	.61	22
39	54735	.56	57619	.64	21	39	56695	.53	59872	.61	21
40	54769	.56	57658	.64	20	40	56727	.53	59909	.61	20
41	9.54802	.56	9.57696	.64	19	41	9.56759	.53	9.59946	.61	19
42	54836	.56	57734	.64	18	42	56790	.53	59983	.61	18
43	54869	.56	57772	.64	17	43	56822	.53	60019	.61	17
44	54903	.56	57810	.64	16	44	56854	.53	60056	.61	16
45	54936	.56	57849	.64	15	45	56886	.53	60093	.61	15
46	54969	.55	57887	.63	14	46	56917	.53	60130	.61	14
47	55003	.55	57925	.63	13	47	56949	.53	60166	.61	13
48	55036	.55	57963	.63	12	48	56980	.53	60203	.61	12
49	55069	.55	58001	.63	11	49	57012	.53	60240	.61	11
50	55102	.55	58039	.63	10	50	57044	.53	60276	.61	10
51	9.55136	.55	9.58077	.63	9	51	9.57075	.52	9.60313	.61	9
52	55169	.55	58115	.63	8	52	57107	.52	60349	.61	8
53	55202	.55	58153	.63	7	53	57138	.52	60386	.61	7
54	55235	.55	58191	.63	6	54	57169	.52	60422	.61	6
55	55268	.55	58229	.63	5	55	57201	.52	60459	.61	5
56	55301	.55	58267	.63	4	56	57232	.52	60495	.61	4
57	55334	.55	58304	.63	3	57	57264	.52	60532	.61	3
58	55367	.55	58342	.63	2	58	57295	.52	60568	.61	2
59	55400	.55	58380	.63	1	59	57326	.52	60605	.61	1
60	55433		58418		0	60	57358		60641		0
M.	Cosine.	D″l	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	D.″	M.
0	9.57358	0.52	9.60641	0.61	60	0	9.59188	0.50	9.62785	0.59	60
1	57389	.52	60677	.61	59	1	59218	.50	62820	.58	59
2	57420	.52	60714	.61	58	2	59247	.49	62855	.58	58
3	57451	.52	60750	.60	57	3	59277	.49	62890	.58	57
4	57482	.52	60786	.60	56	4	59307	.49	62926	.58	56
5	57514	.52	60823	.60	55	5	59336	.49	62961	.58	55
6	57545	.52	60859	.60	54	6	59366	.49	62996	.58	54
7	57576	.52	60895	.60	53	7	59396	.49	63031	.58	53
8	57607	.52	60931	.60	52	8	59425	.49	63066	.58	52
9	57638	.52	60967	.60	51	9	59455	.49	63101	.58	51
10	57669	.52	61004	.60	50	10	59484	.49	63135	.58	50
11	9.57700	.52	9.61040	.60	49	11	9.59514	.49	9.63170	.58	49
12	57731	.52	61076	.60	48	12	59543	.49	63205	.58	48
13	57762	.52	61112	.60	47	13	59573	.49	63240	.58	47
14	57793	.51	61148	.60	46	14	59602	.49	63275	.58	46
15	57824	.51	61184	.60	45	15	59632	.49	63310	.58	45
16	57855	.51	61220	.60	44	16	59661	.49	63345	.58	44
17	57885	.51	61256	.60	43	17	59690	.49	63379	.58	43
18	57916	.51	61292	.60	42	18	59720	.49	63414	.58	42
19	57947	.51	61328	.60	41	19	59749	.49	63449	.58	41
20	57978	.51	61364	.60	40	20	59778	.49	63484	.58	40
21	9.58008	.51	9.61400	.60	39	21	9.59808	.49	9.63519	.58	39
22	58039	.51	61436	.60	38	22	59837	.49	63553	.58	38
23	58070	.51	61472	.60	37	23	59866	.49	63588	.58	37
24	58101	.51	61508	.60	36	24	59895	.49	63623	.58	36
25	58131	.51	61544	.60	35	25	59924	.49	63657	.58	35
26	58162	.51	61579	.60	34	26	59954	.49	63692	.58	34
27	58192	.51	61615	.60	33	27	59983	.49	63726	.58	33
28	58223	.51	61651	.60	32	28	60012	.48	63761	.58	32
29	58253	.51	61687	.60	31	29	60041	.48	63796	.58	31
30	58284	.51	61722	.60	30	30	60070	.48	63830	.58	30
31	9.58314	.51	9.61758	.60	29	31	9.60099	.48	9.63865	.58	29
32	58345	.51	61794	.59	28	32	60128	.48	63899	.57	28
33	58375	.51	61830	.59	27	33	60157	.48	63934	.57	27
34	58406	.51	61865	.59	26	34	60186	.48	63968	.57	26
35	58436	.51	61901	.59	25	35	60215	.48	64003	.57	25
36	58467	.51	61936	.59	24	36	60244	.48	64037	.57	24
37	58497	.51	61972	.59	23	37	60273	.48	64072	.57	23
38	58527	.50	62008	.59	22	38	60302	.48	64106	.57	22
39	58557	.50	62043	.59	21	39	60331	.48	64140	.57	21
40	58588	.50	62079	.59	20	40	60359	.48	64175	.57	20
41	9.58618	.50	9.62114	.59	19	41	9.60388	.48	9.64209	.57	19
42	58648	.50	62150	.59	18	42	60417	.48	64243	.57	18
43	58678	.50	62185	.59	17	43	60446	.48	64278	.57	17
44	58709	.50	62221	.59	16	44	60474	.48	64312	.57	16
45	58739	.50	62256	.59	15	45	60503	.48	64346	.57	15
46	58769	.50	62292	.59	14	46	60532	.48	64381	.57	14
47	58799	.50	62327	.59	13	47	60561	.48	64415	.57	13
48	58829	.50	62362	.59	12	48	60589	.48	64449	.57	12
49	58859	.50	62398	.59	11	49	60618	.48	64483	.57	11
50	58889	.50	62433	.59	10	50	60646	.48	64517	.57	10
51	9.58919	.50	9.62468	.59	9	51	9.60675	.48	9.64552	.57	9
52	58949	.50	62504	.59	8	52	60704	.48	64586	.57	8
53	58979	.50	62539	.59	7	53	60732	.48	64620	.57	7
54	59009	.50	62574	.59	6	54	60761	.48	64654	.57	6
55	59039	.50	62609	.59	5	55	60789	.47	64688	.57	5
56	59069	.50	62645	.59	4	56	60818	.47	64722	.57	4
57	59098	.50	62680	.59	3	57	60846	.47	64756	.57	3
58	59128	.50	62715	.59	2	58	60875	.47	64790	.57	2
59	59158	.50	62750	.59	1	59	60903	.47	64824	.57	1
60	59188		62785		0	60	60931		64858		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.60931	0.47	9.64858	0.57	60	0	9.62595	0.45	9.66867	0.55	60
1	60960	.47	64892	.57	59	1	62622	.45	66900	.55	59
2	60988	.47	64926	.57	58	2	62649	.45	66933	.55	58
3	61016	.47	64960	.57	57	3	62676	.45	66966	.55	57
4	61045	.47	64994	.57	56	4	62703	.45	66999	.55	56
5	61073	.47	65028	.57	55	5	62730	.45	67032	.55	55
6	61101	.47	65062	.56	54	6	62757	.45	67065	.55	54
7	61129	.47	65096	.56	53	7	62784	.45	67098	.55	53
8	61158	.47	65130	.56	52	8	62811	.45	67131	.55	52
9	61186	.47	65164	.56	51	9	62838	.45	67163	.55	51
10	61214	.47	65197	.56	50	10	62865	.45	67196	.55	50
11	9.61242	.47	9.65231	.56	49	11	9.62892	.45	9.67229	.55	49
12	61270	.47	65265	.56	48	12	62918	.45	67262	.55	48
13	61298	.47	65299	.56	47	13	62945	.45	67295	.55	47
14	61326	.47	65333	.56	46	14	62972	.45	67327	.55	46
15	61354	.47	65366	.56	45	15	62999	.45	67360	.55	45
16	61382	.47	65400	.56	44	16	63026	.45	67393	.55	44
17	61411	.47	65434	.56	43	17	63052	.45	67426	.55	43
18	61438	.47	65467	.56	42	18	63079	.45	67458	.54	42
19	61466	.47	65501	.56	41	19	63106	.44	67491	.54	41
20	61494	.47	65535	.56	40	20	63133	.44	67524	.54	40
21	9.61522	.47	9.65568	.56	39	21	9.63159	.44	9.67556	.54	39
22	61550	.46	65602	.56	38	22	63186	.44	67589	.54	38
23	61578	.46	65636	.56	37	23	63213	.44	67622	.54	37
24	61606	.46	65669	.56	36	24	63239	.44	67654	.54	36
25	61634	.46	65703	.56	35	25	63266	.44	67687	.54	35
26	61662	.46	65736	.56	34	26	63292	.44	67719	.54	34
27	61689	.46	65770	.56	33	27	63319	.44	67752	.54	33
28	61717	.46	65803	.56	32	28	63345	.44	67785	.54	32
29	61745	.46	65837	.56	31	29	63372	.44	67817	.54	31
30	61773	.46	65870	.56	30	30	63398	.44	67850	.54	30
31	9.61800	.46	9.65904	.56	29	31	9.63425	.44	9.67882	.54	29
32	61828	.46	65937	.56	28	32	63451	.44	67915	.54	28
33	61856	.46	65971	.56	27	33	63478	.44	67947	.54	27
34	61883	.46	66004	.56	26	34	63504	.44	67980	.54	26
35	61911	.46	66038	.56	25	35	63531	.44	68012	.54	25
36	61939	.46	66071	.56	24	36	63557	.44	68044	.54	24
37	61966	.46	66104	.56	23	37	63583	.44	68077	.54	23
38	61994	.46	66138	.56	22	38	63610	.44	68109	.54	22
39	62021	.46	66171	.56	21	39	63636	.44	68142	.54	21
40	62049	.46	66204	.56	20	40	63662	.44	68174	.54	20
41	9.62076	.46	9.66238	.55	19	41	9.63689	.44	9.68206	.54	19
42	62104	.46	66271	.55	18	42	63715	.44	68239	.54	18
43	62131	.46	66304	.55	17	43	63741	.44	68271	.54	17
44	62159	.46	66337	.55	16	44	63767	.44	68303	.54	16
45	62186	.46	66371	.55	15	45	63794	.44	68336	.54	15
46	62214	.46	66404	.55	14	46	63820	.44	68368	.54	14
47	62241	.46	66437	.55	13	47	63846	.44	68400	.54	13
48	62268	.46	66470	.55	12	48	63872	.44	68432	.54	12
49	62296	.46	66503	.55	11	49	63898	.44	68465	.54	11
50	62323	.45	66537	.55	10	50	63924	.43	68497	.54	10
51	9.62350	.45	9.66570	.55	9	51	9.63950	.43	9.68529	.54	9
52	62377	.45	66603	.55	8	52	63976	.43	68561	.54	8
53	62405	.45	66636	.55	7	53	64002	.43	68593	.54	7
54	62432	.45	66669	.55	6	54	64028	.43	68626	.54	6
55	62459	.45	66702	.55	5	55	64054	.43	68658	.54	5
56	62486	.45	66735	.55	4	56	64080	.43	68690	.54	4
57	62513	.45	66768	.55	3	57	64106	.43	68722	.53	3
58	62541	.45	66801	.55	2	58	64132	.43	68754	.53	2
59	62568	.45	66834	.55	1	59	64158	.43	68786	.53	1
60	62595		66867		0	60	64184		68818		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.64184	0.43	9.68818	0.53	60	0	9.65705	0.41	9.70717	0.52	60
1	64210	.43	68850	.53	59	1	65729	.41	70748	.52	59
2	64236	.43	68882	.53	58	2	65754	.41	70779	.52	58
3	64262	.43	68914	.53	57	3	65779	.41	70810	.52	57
4	64288	.43	68946	.53	56	4	65804	.41	70841	.52	56
5	64313	.43	68978	.53	55	5	65828	.41	70873	.52	55
6	64339	.43	69010	.53	54	6	65853	.41	70904	.52	54
7	64365	.43	69042	.53	53	7	65878	.41	70935	.52	53
8	64391	.43	69074	.53	52	8	65902	.41	70966	.52	52
9	64417	.43	69106	.53	51	9	65927	.41	70997	.52	51
10	64442	.43	69138	.53	50	10	65952	.41	71028	.52	50
11	9.64468	.43	9.69170	.53	49	11	9.65976	.41	9.71059	.52	49
12	64494	.43	69202	.53	48	12	66001	.41	71090	.52	48
13	64519	.43	69234	.53	47	13	66025	.41	71121	.52	47
14	64545	.43	69266	.53	46	14	66050	.41	71153	.52	46
15	64571	.43	69298	.53	45	15	66075	.41	71184	.52	45
16	64596	.43	69329	.53	44	16	66099	.41	71215	.52	44
17	64622	.43	69361	.53	43	17	66124	.41	71246	.52	43
18	64647	.43	69393	.53	42	18	66148	.41	71277	.52	42
19	64673	.43	69425	.53	41	19	66173	.41	71308	.52	41
20	64698	.42	69457	.53	40	20	66197	.41	71339	.52	40
21	9.64724	.42	9.69488	.53	39	21	9.66221	.41	9.71370	.52	39
22	64749	.42	69520	.53	38	22	66246	.41	71401	.52	38
23	64775	.42	69552	.53	37	23	66270	.41	71431	.52	37
24	64800	.42	69584	.53	36	24	66295	.41	71462	.52	36
25	64826	.42	69615	.53	35	25	66319	.41	71493	.51	35
26	64851	.42	69647	.53	34	26	66343	.41	71524	.51	34
27	64877	.42	69679	.53	33	27	66368	.41	71555	.51	33
28	64902	.42	69710	.53	32	28	66392	.40	71586	.51	32
29	64927	.42	69742	.53	31	29	66416	.40	71617	.51	31
30	64953	.42	69774	.53	30	30	66441	.40	71648	.51	30
31	9.64978	.42	9.69805	.53	29	31	9.66465	.40	9.71679	.51	29
32	65003	.42	69837	.53	28	32	66489	.40	71709	.51	28
33	65029	.42	69868	.53	27	33	66513	.40	71740	.51	27
34	65054	.42	69900	.53	26	34	66537	.40	71771	.51	26
35	65079	.42	69932	.53	25	35	66562	.40	71802	.51	25
36	65104	.42	69963	.53	24	36	66586	.40	71833	.51	24
37	65130	.42	69995	.53	23	37	66610	.40	71863	.51	23
38	65155	.42	70026	.53	22	38	66634	.40	71894	.51	22
39	65180	.42	70058	.52	21	39	66658	.40	71925	.51	21
40	65205	.42	70089	.52	20	40	66682	.40	71955	.51	20
41	9.65230	.42	9.70121	.52	19	41	9.66706	.40	9.71986	.51	19
42	65255	.42	70152	.52	18	42	66731	.40	72017	.51	18
43	65281	.42	70184	.52	17	43	66755	.40	72048	.51	17
44	65306	.42	70215	.52	16	44	66779	.40	72078	.51	16
45	65331	.42	70247	.52	15	45	66803	.40	72109	.51	15
46	65356	.42	70278	.52	14	46	66827	.40	72140	.51	14
47	65381	.42	70309	.52	13	47	66851	.40	72170	.51	13
48	65406	.42	70341	.52	12	48	66875	.40	72201	.51	12
49	65431	.42	70372	.52	11	49	66899	.40	72231	.51	11
50	65456	.42	70404	.52	10	50	66922	.40	72262	.51	10
51	9.65481	.42	9.70435	.52	9	51	9.66946	.40	9.72293	.51	9
52	65506	.42	70466	.52	8	52	66970	.40	72323	.51	8
53	65531	.42	70498	.52	7	53	66994	.40	72354	.51	7
54	65556	.41	70529	.52	6	54	67018	.40	72384	.51	6
55	65580	.41	70560	.52	5	55	67042	.40	72415	.51	5
56	65605	.41	70592	.52	4	56	67066	.40	72445	.51	4
57	65630	.41	70623	.52	3	57	67090	.40	72476	.51	3
58	65655	.41	70654	.52	2	58	67113	.40	72506	.51	2
59	65680	.41	70685	.52	1	59	67137	.40	72537	.51	1
60	65705		70717		0	60	67161		72567		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.67161	0.40	9.72567	0.51	60	0	9.68557	0.38	9.74375	0.50	60
1	67185	.40	72598	.51	59	1	68580	.38	74405	.50	59
2	67208	.40	72628	.51	58	2	68603	.38	74435	.50	58
3	67232	.39	72659	.51	57	3	68625	.38	74465	.50	57
4	67256	.39	72689	.51	56	4	68648	.38	74494	.50	56
5	67280	.39	72720	.51	55	5	68671	.38	74524	.50	55
6	67303	.39	72750	.51	54	6	68694	.38	74554	.50	54
7	67327	.39	72780	.51	53	7	68716	.38	74583	.50	53
8	67350	.39	72811	.51	52	8	68739	.38	74613	.50	52
9	67374	.39	72841	.51	51	9	68762	.38	74643	.49	51
10	67398	.39	72872	.51	50	10	68784	.38	74673	.49	50
11	9.67421	.39	9.72902	.51	49	11	9.68807	.38	9.74702	.49	49
12	67445	.39	72932	.51	48	12	68829	.38	74732	.49	48
13	67468	.39	72963	.51	47	13	68852	.38	74762	.49	47
14	67492	.39	72993	.51	46	14	68875	.38	74791	.49	46
15	67515	.39	73023	.50	45	15	68897	.38	74821	.49	45
16	67539	.39	73054	.50	44	16	68920	.38	74851	.49	44
17	67562	.39	73084	.50	43	17	68942	.38	74880	.49	43
18	67586	.39	73114	.50	42	18	68965	.38	74910	.49	42
19	67609	.39	73144	.50	41	19	68987	.37	74939	.49	41
20	67633	.39	73175	.50	40	20	69010	.37	74969	.49	40
21	9.67656	.39	9.73205	.50	39	21	9.69032	.37	9.74998	.49	39
22	67680	.39	73235	.50	38	22	69055	.37	75028	.49	38
23	67703	.39	73265	.50	37	23	69077	.37	75058	.49	37
24	67726	.39	73295	.50	36	24	69100	.37	75087	.49	36
25	67750	.39	73326	.50	35	25	69122	.37	75117	.49	35
26	67773	.39	73356	.50	34	26	69144	.37	75146	.49	34
27	67796	.39	73386	.50	33	27	69167	.37	75176	.49	33
28	67820	.39	73416	.50	32	28	69189	.37	75205	.49	32
29	67843	.39	73446	.50	31	29	69212	.37	75235	.49	31
30	67866	.39	73476	.50	30	30	69234	.37	75264	.49	30
31	9.67890	.39	9.73507	.50	29	31	9.69256	.37	9.75294	.49	29
32	67913	.39	73537	.50	28	32	69279	.37	75323	.49	28
33	67936	.39	73567	.50	27	33	69301	.37	75353	.49	27
34	67959	.39	73597	.50	26	34	69323	.37	75382	.49	26
35	67982	.39	73627	.50	25	35	69345	.37	75411	.49	25
36	68006	.39	73657	.50	24	36	69368	.37	75441	.49	24
37	68029	.39	73687	.50	23	37	69390	.37	75470	.49	23
38	68052	.39	73717	.50	22	38	69412	.37	75500	.49	22
39	68075	.39	73747	.50	21	39	69434	.37	75529	.49	21
40	68098	.38	73777	.50	20	40	69456	.37	75558	.49	20
41	9.68121	.38	9.73807	.50	19	41	9.69479	.37	9.75588	.49	19
42	68144	.38	73837	.50	18	42	69501	.37	75617	.49	18
43	68167	.38	73867	.50	17	43	69523	.37	75647	.49	17
44	68190	.38	73897	.50	16	44	69545	.37	75676	.49	16
45	68213	.38	73927	.50	15	45	69567	.37	75705	.49	15
46	68237	.38	73957	.50	14	46	69589	.37	75735	.49	14
47	68260	.38	73987	.50	13	47	69611	.37	75764	.49	13
48	68283	.38	74017	.50	12	48	69633	.37	75793	.49	12
49	68305	.38	74047	.50	11	49	69655	.37	75822	.49	11
50	68328	.38	74077	.50	10	50	69677	.37	75852	.49	10
51	9.68351	.38	9.74107	.50	9	51	9.69699	.37	9.75881	.49	9
52	68374	.38	74137	.50	8	52	69721	.37	75910	.49	8
53	68397	.38	74166	.50	7	53	69743	.37	75939	.49	7
54	68420	.38	74196	.50	6	54	69765	.37	75969	.49	6
55	68443	.38	74226	.50	5	55	69787	.37	75998	.49	5
56	68466	.38	74256	.50	4	56	69809	.37	76027	.49	4
57	68489	.38	74286	.50	3	57	69831	.37	76056	.49	3
58	68512	.38	74316	.50	2	58	69853	.37	76086	.49	2
59	68534	.38	74345	.50	1	59	69875	.36	76115	.49	1
60	68557		74375		0	60	69897		76144		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.69897	0.36	9.76144	0.49	60	0	9.71184	0.35	9.77877	0.48	60
1	69919	.36	76173	.49	59	1	71205	.35	77906	.48	59
2	69941	.36	76202	.49	58	2	71226	.35	77935	.48	58
3	69963	.36	76231	.49	57	3	71247	.35	77963	.48	57
4	69984	.36	76261	.49	56	4	71268	.35	77992	.48	56
5	70006	.36	76290	.49	55	5	71289	.35	78020	.48	55
6	70028	.36	76319	.49	54	6	71310	.35	78049	.48	54
7	70050	.36	76348	.48	53	7	71331	.35	78077	.48	53
8	70072	.36	76377	.48	52	8	71352	.35	78106	.48	52
9	70093	.36	76406	.48	51	9	71373	.35	78135	.48	51
10	70115	.36	76435	.48	50	10	71393	.35	78163	.48	50
11	9.70137	.36	9.76464	.48	49	11	9.71414	.35	9.78192	.48	49
12	70159	.36	76493	.48	48	12	71435	.35	78220	.48	48
13	70180	.36	76522	.48	47	13	71456	.35	78249	.47	47
14	70202	.36	76551	.48	46	14	71477	.35	78277	.47	46
15	70224	.36	76580	.48	45	15	71498	.35	78306	.47	45
16	70245	.36	76609	.48	44	16	71519	.35	78334	.47	44
17	70267	.36	76639	.48	43	17	71539	.35	78363	.47	43
18	70288	.36	76668	.48	42	18	71560	.35	78391	.47	42
19	70310	.36	76697	.48	41	19	71581	.35	78419	.47	41
20	70332	.36	76725	.48	40	20	71602	.35	78448	.47	40
21	9.70353	.36	9.76754	.48	39	21	9.71622	.35	9.78476	.47	39
22	70375	.36	76783	.48	38	22	71643	.35	78505	.47	38
23	70396	.36	76812	.48	37	23	71664	.35	78533	.47	37
24	70418	.36	76841	.48	36	24	71685	.34	78562	.47	36
25	70439	.36	76870	.48	35	25	71705	.34	78590	.47	35
26	70461	.36	76899	.48	34	26	71726	.34	78618	.47	34
27	70482	.36	76928	.48	33	27	71747	.34	78647	.47	33
28	70504	.36	76957	.48	32	28	71767	.34	78675	.47	32
29	70525	.36	76986	.48	31	29	71788	.34	78704	.47	31
30	70547	.36	77015	.48	30	30	71809	.34	78732	.47	30
31	9.70568	.36	9.77044	.48	29	31	9.71829	.34	9.78760	.47	29
32	70590	.36	77073	.48	28	32	71850	.34	78789	.47	28
33	70611	.36	77101	.48	27	33	71870	.34	78817	.47	27
34	70633	.36	77130	.48	26	34	71891	.34	78845	.47	26
35	70654	.36	77159	.48	25	35	71911	.34	78874	.47	25
36	70675	.36	77188	.48	24	36	71932	.34	78902	.47	24
37	70697	.36	77217	.48	23	37	71952	.34	78930	.47	23
38	70718	.36	77246	.48	22	38	71973	.34	78959	.47	22
39	70739	.36	77274	.48	21	39	71994	.34	78987	.47	21
40	70761	.35	77303	.48	20	40	72014	.34	79015	.47	20
41	9.70782	.35	9.77332	.48	19	41	9.72034	.34	9.79043	.47	19
42	70803	.35	77361	.48	18	42	72055	.34	79072	.47	18
43	70824	.35	77390	.48	17	43	72075	.34	79100	.47	17
44	70846	.35	77418	.48	16	44	72096	.34	79128	.47	16
45	70867	.35	77447	.48	15	45	72116	.34	79156	.47	15
46	70888	.35	77476	.48	14	46	72137	.34	79185	.47	14
47	70909	.35	77505	.48	13	47	72157	.34	79213	.47	13
48	70931	.35	77533	.48	12	48	72177	.34	79241	.47	12
49	70952	.35	77562	.48	11	49	72198	.34	79269	.47	11
50	70973	.35	77591	.48	10	50	72218	.34	79297	.47	10
51	9.70994	.35	9.77619	.48	9	51	9.72238	.34	9.79326	.47	9
52	71015	.35	77648	.48	8	52	72259	.34	79354	.47	8
53	71036	.35	77677	.48	7	53	72279	.34	79382	.47	7
54	71058	.35	77706	.48	6	54	72299	.34	79410	.47	6
55	71079	.35	77734	.48	5	55	72320	.34	79438	.47	5
56	71100	.35	77763	.48	4	56	72340	.34	79466	.47	4
57	71121	.35	77791	.48	3	57	72360	.34	79495	.47	3
58	71142	.35	77820	.48	2	58	72381	.34	79523	.47	2
59	71163	.35	77849	.48	1	59	72401	.34	79551	.47	1
60	71184		77877		0	60	72421		79579		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.72421	0.34	9.79579	0.47	60	0	9.73611	0.32	9.81252	0.46	60
1	72441	.34	79607	.47	59	1	73630	.32	81279	.46	59
2	72461	.34	79635	.47	58	2	73650	.32	81307	.46	58
3	72482	.34	79663	.47	57	3	73669	.32	81335	.46	57
4	72502	.34	79691	.47	56	4	73689	.32	81362	.46	56
5	72522	.34	79719	.47	55	5	73708	.32	81390	.46	55
6	72542	.34	79747	.47	54	6	73727	.32	81418	.46	54
7	72562	.34	79776	.47	53	7	73747	.32	81445	.46	53
8	72582	.34	79804	.47	52	8	73766	.32	81473	.46	52
9	72602	.33	79832	.47	51	9	73785	.32	81500	.46	51
10	72622	.33	79860	.47	50	10	73805	.32	81528	.46	50
11	9.72643	.33	9.79888	.47	49	11	9.73824	.32	9.81556	.46	49
12	72663	.33	79916	.47	48	12	73843	.32	81583	.46	48
13	72683	.33	79944	.47	47	13	73863	.32	81611	.46	47
14	72703	.33	79972	.47	46	14	73882	.32	81638	.46	46
15	72723	.33	80000	.47	45	15	73901	.32	81666	.46	45
16	72743	.33	80028	.47	44	16	73921	.32	81693	.46	44
17	72763	.33	80056	.47	43	17	73940	.32	81721	.46	43
18	72783	.33	80084	.47	42	18	73959	.32	81748	.46	42
19	72803	.33	80112	.47	41	19	73978	.32	81776	.46	41
20	72823	.33	80140	.47	40	20	73997	.32	81803	.46	40
21	9.72843	.33	9.80168	.47	39	21	9.74017	.32	9.81831	.46	39
22	72863	.33	80195	.47	38	22	74036	.32	81858	.46	38
23	72883	.33	80223	.47	37	23	74055	.32	81886	.46	37
24	72902	.33	80251	.47	36	24	74074	.32	81913	.46	36
25	72922	.33	80279	.47	35	25	74093	.32	81941	.46	35
26	72942	.33	80307	.47	34	26	74113	.32	81968	.46	34
27	72962	.33	80335	.46	33	27	74132	.32	81996	.46	33
28	72982	.33	80363	.46	32	28	74151	.32	82023	.46	32
29	73002	.33	80391	.46	31	29	74170	.32	82051	.46	31
30	73022	.33	80419	.46	30	30	74189	.32	82078	.46	30
31	9.73041	.33	9.80447	.46	29	31	9.74208	.32	9.82106	.46	29
32	73061	.33	80474	.46	28	32	74227	.32	82133	.46	28
33	73081	.33	80502	.46	27	33	74246	.32	82161	.46	27
34	73101	.33	80530	.46	26	34	74265	.32	82188	.46	26
35	73121	.33	80558	.46	25	35	74284	.32	82215	.46	25
36	73140	.33	80586	.46	24	36	74303	.32	82243	.46	24
37	73160	.33	80614	.46	23	37	74322	.32	82270	.46	23
38	73180	.33	80642	.46	22	38	74341	.32	82298	.46	22
39	73200	.33	80669	.46	21	39	74360	.32	82325	.46	21
40	73219	.33	80697	.46	20	40	74379	.32	82352	.46	20
41	9.73239	.33	9.80725	.46	19	41	9.74398	.32	9.82380	.46	19
42	73259	.33	80753	.46	18	42	74417	.32	82407	.46	18
43	73278	.33	80781	.46	17	43	74436	.32	82435	.46	17
44	73298	.33	80808	.46	16	44	74455	.32	82462	.46	16
45	73318	.33	80836	.46	15	45	74474	.32	82489	.46	15
46	73337	.33	80864	.46	14	46	74493	.31	82517	.46	14
47	73357	.33	80892	.46	13	47	74512	.31	82544	.46	13
48	73377	.33	80919	.46	12	48	74531	.31	82571	.46	12
49	73396	.33	80947	.46	11	49	74549	.31	82599	.46	11
50	73416	.33	80975	.46	10	50	74568	.31	82626	.46	10
51	9.73435	.33	9.81003	.46	9	51	9.74587	.31	9.82653	.46	9
52	73455	.33	81030	.46	8	52	74606	.31	82681	.45	8
53	73474	.33	81058	.46	7	53	74625	.31	82708	.45	7
54	73494	.33	81086	.46	6	54	74644	.31	82735	.45	6
55	73513	.33	81113	.46	5	55	74662	.31	82762	.45	5
56	73533	.32	81141	.46	4	56	74681	.31	82790	.45	4
57	73552	.32	81169	.46	3	57	74700	.31	82817	.45	3
58	73572	.32	81196	.46	2	58	74719	.31	82844	.45	2
59	73591	.32	81224	.46	1	59	74737	.31	82871	.45	1
60	73611		81252		0	60	74756		82899		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.74756	0.31	9.82899	0.45	60	0	9.75859	0.30	9.84523	0.45	60
1	74775	.31	82926	.45	59	1	75877	.30	84550	.45	59
2	74794	.31	82953	.45	58	2	75895	.30	84576	.45	58
3	74812	.31	82980	.45	57	3	75913	.30	84603	.45	57
4	74831	.31	83008	.45	56	4	75931	.30	84630	.45	56
5	74850	.31	83035	.45	55	5	75949	.30	84657	.45	55
6	74868	.31	83062	.45	54	6	75967	.30	84684	.45	54
7	74887	.31	83089	.45	53	7	75985	.30	84711	.45	53
8	74906	.31	83117	.45	52	8	76003	.30	84738	.45	52
9	74924	.31	83144	.45	51	9	76021	.30	84764	.45	51
10	74943	.31	83171	.45	50	10	76039	.30	84791	.45	50
11	9.74961	.31	9.83198	.45	49	11	9.76057	.30	9.84818	.45	49
12	74980	.31	83225	.45	48	12	76075	.30	84845	.45	48
13	74999	.31	83252	.45	47	13	76093	.30	84872	.45	47
14	75017	.31	83280	.45	46	14	76111	.30	84899	.45	46
15	75036	.31	83307	.45	45	15	76129	.30	84925	.45	45
16	75054	.31	83334	.45	44	16	76146	.30	84952	.45	44
17	75073	.31	83361	.45	43	17	76164	.30	84979	.45	43
18	75091	.31	83388	.45	42	18	76182	.30	85006	.45	42
19	75110	.31	83415	.45	41	19	76200	.30	85033	.45	41
20	75128	.31	83442	.45	40	20	76218	.30	85059	.45	40
21	9.75147	.31	9.83470	.45	39	21	9.76236	.30	9.85086	.45	39
22	75165	.31	83497	.45	38	22	76253	.30	85113	.45	38
23	75184	.31	83524	.45	37	23	76271	.30	85140	.45	37
24	75202	.31	83551	.45	36	24	76289	.30	85166	.45	36
25	75221	.31	83578	.45	35	25	76307	.30	85193	.45	35
26	75239	.31	83605	.45	34	26	76324	.30	85220	.45	34
27	75258	.31	83632	.45	33	27	76342	.30	85247	.45	33
28	75276	.31	83659	.45	32	28	76360	.30	85273	.45	32
29	75294	.31	83686	.45	31	29	76378	.30	85300	.45	31
30	75313	.31	83713	.45	30	30	76395	.30	85327	.45	30
31	9.75331	.31	9.83740	.45	29	31	9.76413	.29	9.85354	.45	29
32	75350	.31	83768	.45	28	32	76431	.29	85380	.45	28
33	75368	.31	83795	.45	27	33	76448	.29	85407	.45	27
34	75386	.31	83822	.45	26	34	76466	.29	85434	.44	26
35	75405	.31	83849	.45	25	35	76484	.29	85460	.44	25
36	75423	.31	83876	.45	24	36	76501	.29	85487	.44	24
37	75441	.30	83903	.45	23	37	76519	.29	85514	.44	23
38	75459	.30	83930	.45	22	38	76537	.29	85540	.44	22
39	75478	.30	83957	.45	21	39	76554	.29	85567	.44	21
40	75496	.30	83984	.45	20	40	76572	.29	85594	.44	20
41	9.75514	.30	9.84011	.45	19	41	9.76590	.29	9.85620	.44	19
42	75533	.30	84038	.45	18	42	76607	.29	85647	.44	18
43	75551	.30	84065	.45	17	43	76625	.29	85674	.44	17
44	75569	.30	84092	.45	16	44	76642	.29	85700	.44	16
45	75587	.30	84119	.45	15	45	76660	.29	85727	.44	15
46	75605	.30	84146	.45	14	46	76677	.29	85754	.44	14
47	75624	.30	84173	.45	13	47	76695	.29	85780	.44	13
48	75642	.30	84200	.45	12	48	76712	.29	85807	.44	12
49	75660	.30	84227	.45	11	49	76730	.29	85834	.44	11
50	75678	.30	84254	.45	10	50	76747	.29	85860	.44	10
51	9.75696	.30	9.84280	.45	9	51	9.76765	.29	9.85887	.44	9
52	75714	.30	84307	.45	8	52	76782	.29	85913	.44	8
53	75733	.30	84334	.45	7	53	76800	.29	85940	.44	7
54	75751	.30	84361	.45	6	54	76817	.29	85967	.44	6
55	75769	.30	84388	.45	5	55	76835	.29	85993	.44	5
56	75787	.30	84415	.45	4	56	76852	.29	86020	.44	4
57	75805	.30	84442	.45	3	57	76870	.29	86046	.44	3
58	75823	.30	84469	.45	2	58	76887	.29	86073	.44	2
59	75841	.30	84496	.45	1	59	76904	.29	86100	.44	1
60	75859		84523		0	60	76922		86126		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.76922	0.29	9.86126	0.44	60	0	9.77946	0.28	9.87711	0.44	60
1	76939	.29	86153	.44	59	1	77963	.28	87738	.44	59
2	76957	.29	86179	.44	58	2	77980	.28	87764	.44	58
3	76974	.29	86206	.44	57	3	77997	.28	87790	.44	57
4	76991	.29	86232	.44	56	4	78013	.28	87817	.44	56
5	77009	.29	86259	.44	55	5	78030	.28	87843	.44	55
6	77026	.29	86285	.44	54	6	78047	.28	87869	.44	54
7	77043	.29	86312	.44	53	7	78063	.28	87895	.44	53
8	77061	.29	86338	.44	52	8	78080	.28	87922	.44	52
9	77078	.29	86365	.44	51	9	78097	.28	87948	.44	51
10	77095	.29	86392	.44	50	10	78113	.28	87974	.44	50
11	9.77112	.29	9.86418	.44	49	11	9.78130	.28	9.88000	.44	49
12	77130	.29	86445	.44	48	12	78147	.28	88027	.44	48
13	77147	.29	86471	.44	47	13	78163	.28	88053	.44	47
14	77164	.29	86498	.44	46	14	78180	.28	88079	.44	46
15	77181	.29	86524	.44	45	15	78197	.28	88105	.44	45
16	77199	.29	86551	.44	44	16	78213	.28	88131	.44	44
17	77216	.29	86577	.44	43	17	78230	.28	88158	.44	43
18	77233	.29	86603	.44	42	18	78246	.28	88184	.44	42
19	77250	.29	86630	.44	41	19	78263	.28	88210	.44	41
20	77268	.29	86656	.44	40	20	78280	.28	88236	.44	40
21	9.77285	.29	9.86683	.44	39	21	9.78296	.28	9.88262	.44	39
22	77302	.29	86709	.44	38	22	78313	.28	88289	.44	38
23	77319	.29	86736	.44	37	23	78329	.28	88315	.44	37
24	77336	.29	86762	.44	36	24	78346	.28	88341	.44	36
25	77353	.29	86789	.44	35	25	78362	.28	88367	.44	35
26	77370	.29	86815	.44	34	26	78379	.27	88393	.44	34
27	77387	.28	86842	.44	33	27	78395	.27	88420	.44	33
28	77405	.28	86868	.44	32	28	78412	.27	88446	.44	32
29	77422	.28	86894	.44	31	29	78428	.27	88472	.44	31
30	77439	.28	86921	.44	30	30	78445	.27	88498	.44	30
31	9.77456	.28	9.86947	.44	29	31	9.78461	.27	9.88524	.44	29
32	77473	.28	86974	.44	28	32	78478	.27	88550	.44	28
33	77490	.28	87000	.44	27	33	78494	.27	88577	.44	27
34	77507	.28	87027	.44	26	34	78510	.27	88603	.44	26
35	77524	.28	87053	.44	25	35	78527	.27	88629	.44	25
36	77541	.28	87079	.44	24	36	78543	.27	88655	.44	24
37	77558	.28	87106	.44	23	37	78560	.27	88681	.44	23
38	77575	.28	87132	.44	22	38	78576	.27	88707	.44	22
39	77592	.28	87158	.44	21	39	78592	.27	88733	.44	21
40	77609	.28	87185	.44	20	40	78609	.27	88759	.44	20
41	9.77626	.28	9.87211	.44	19	41	9.78625	.27	9.88786	.44	19
42	77643	.28	87238	.44	18	42	78642	.27	88812	.44	18
43	77660	.28	87264	.44	17	43	78658	.27	88838	.44	17
44	77677	.28	87290	.44	16	44	78674	.27	88864	.43	16
45	77694	.28	87317	.44	15	45	78691	.27	88890	.43	15
46	77711	.28	87343	.44	14	46	78707	.27	88916	.43	14
47	77728	.28	87369	.44	13	47	78723	.27	88942	.43	13
48	77744	.28	87396	.44	12	48	78739	.27	88968	.43	12
49	77761	.28	87422	.44	11	49	78756	.27	88994	.43	11
50	77778	.28	87448	.44	10	50	78772	.27	89020	.43	10
51	9.77795	.28	9.87475	.44	9	51	9.78788	.27	9.89046	.43	9
52	77812	.28	87501	.44	8	52	78805	.27	89073	.43	8
53	77829	.28	87527	.44	7	53	78821	.27	89099	.43	7
54	77846	.28	87554	.44	6	54	78837	.27	89125	.43	6
55	77862	.28	87580	.44	5	55	78853	.27	89151	.43	5
56	77879	.28	87606	.44	4	56	78869	.27	89177	.43	4
57	77896	.28	87633	.44	3	57	78886	.27	89203	.43	3
58	77913	.28	87659	.44	2	58	78902	.27	89229	.43	2
59	77930	.28	87685	.44	1	59	78918	.27	89255	.43	1
60	77946		87711		0	60	78934		89281		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.78934	0.27	9.89281	0.43	60	0	9.79887	0.26	9.90837	0.43	60
1	78950	.27	89307	.43	59	1	79903	.26	90863	.43	59
2	78967	.27	89333	.43	58	2	79918	.26	90889	.43	58
3	78983	.27	89359	.43	57	3	79934	.26	90914	.43	57
4	78999	.27	89385	.43	56	4	79950	.26	90940	.43	56
5	79015	.27	89411	.43	55	5	79965	.26	90966	.43	55
6	79031	.27	89437	.43	54	6	79981	.26	90992	.43	54
7	79047	.27	89463	.43	53	7	79996	.26	91018	.43	53
8	79063	.27	89489	.43	52	8	80012	.26	91043	.43	52
9	79079	.27	89515	.43	51	9	80027	.26	91069	.43	51
10	79095	.27	89541	.43	50	10	80043	.26	91095	.43	50
11	9.79111	.27	9.89567	.43	49	11	9.80058	.26	9.91121	.43	49
12	79128	.27	89593	.43	48	12	80074	.26	91147	.43	48
13	79144	.27	89619	.43	47	13	80089	.26	91172	.43	47
14	79160	.27	89645	.43	46	14	80105	.26	91198	.43	46
15	79176	.27	89671	.43	45	15	80120	.26	91224	.43	45
16	79192	.27	89697	.43	44	16	80136	.26	91250	.43	44
17	79208	.27	89723	.43	43	17	80151	.26	91276	.43	43
18	79224	.27	89749	.43	42	18	80166	.26	91301	.43	42
19	79240	.27	89775	.43	41	19	80182	.26	91327	.43	41
20	79256	.27	89801	.43	40	20	80197	.26	91353	.43	40
21	9.79272	.27	9.89827	.43	39	21	9.80213	.26	9.91379	.43	39
22	79288	.27	89853	.43	38	22	80228	.26	91404	.43	38
23	79304	.27	89879	.43	37	23	80244	.26	91430	.43	37
24	79319	.27	89905	.43	36	24	80259	.26	91456	.43	36
25	79335	.27	89931	.43	35	25	80274	.26	91482	.43	35
26	79351	.27	89957	.43	34	26	80290	.26	91507	.43	34
27	79367	.26	89983	.43	33	27	80305	.26	91533	.43	33
28	79383	.26	90009	.43	32	28	80320	.26	91559	.43	32
29	79399	.26	90035	.43	31	29	80336	.26	91585	.43	31
30	79415	.26	90061	.43	30	30	80351	.26	91610	.43	30
31	9.79431	.26	9.90086	.43	29	31	9.80366	.26	9.91636	.43	29
32	79447	.26	90112	.43	28	32	80382	.25	91662	.43	28
33	79463	.26	90138	.43	27	33	80397	.25	91688	.43	27
34	79478	.26	90164	.43	26	34	80412	.25	91713	.43	26
35	79494	.26	90190	.43	25	35	80428	.25	91739	.43	25
36	79510	.26	90216	.43	24	36	80443	.25	91765	.43	24
37	79526	.26	90242	.43	23	37	80458	.25	91791	.43	23
38	79542	.26	90268	.43	22	38	80473	.25	91816	.43	22
39	79558	.26	90294	.43	21	39	80489	.25	91842	.43	21
40	79573	.26	90320	.43	20	40	80504	.25	91868	.43	20
41	9.79589	.26	9.90346	.43	19	41	9.80519	.25	9.91893	.43	19
42	79605	.26	90371	.43	18	42	80534	.25	91919	.43	18
43	79621	.26	90397	.43	17	43	80550	.25	91945	.43	17
44	79636	.26	90423	.43	16	44	80565	.25	91971	.43	16
45	79652	.26	90449	.43	15	45	80580	.25	91996	.43	15
46	79668	.26	90475	.43	14	46	80595	.25	92022	.43	14
47	79684	.26	90501	.43	13	47	80610	.25	92048	.43	13
48	79699	.26	90527	.43	12	48	80625	.25	92073	.43	12
49	79715	.26	90553	.43	11	49	80641	.25	92099	.43	11
50	79731	.26	90578	.43	10	50	80656	.25	92125	.43	10
51	9.79746	.26	9.90604	.43	9	51	9.80671	.25	9.92150	.43	9
52	79762	.26	90630	.43	8	52	80686	.25	92176	.43	8
53	79778	.26	90656	.43	7	53	80701	.25	92202	.43	7
54	79793	.26	90682	.43	6	54	80716	.25	92227	.43	6
55	79809	.26	90708	.43	5	55	80731	.25	92253	.43	5
56	79825	.26	90734	.43	4	56	80746	.25	92279	.43	4
57	79840	.26	90759	.43	3	57	80762	.25	92304	.43	3
58	79856	.26	90785	.43	2	58	80777	.25	92330	.43	2
59	79872	.26	90811	.43	1	59	80792	.25	92356	.43	1
60	79887		90837		0	60	80807		92381		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.80807	0.25	9.92381	0.43	60	0	9.81694	0.24	9.93916	0.43	60
1	80822	.25	92407	.43	59	1	81709	.24	93942	.43	59
2	80837	.25	92433	.43	58	2	81723	.24	93967	.43	58
3	80852	.25	92458	.43	57	3	81738	.24	93993	.43	57
4	80867	.25	92484	.43	56	4	81752	.24	94018	.43	56
5	80882	.25	92510	.43	55	5	81767	.24	94044	.43	55
6	80897	.25	92535	.43	54	6	81781	.24	94069	.43	54
7	80912	.25	92561	.43	53	7	81796	.24	94095	.42	53
8	80927	.25	92587	.43	52	8	81810	.24	94120	.42	52
9	80942	.25	92612	.43	51	9	81825	.24	94146	.42	51
10	80957	.25	92638	.43	50	10	81839	.24	94171	.42	50
11	9.80972	.25	9.92663	.43	49	11	9.81854	.24	9.94197	.42	49
12	80987	.25	92689	.43	48	12	81868	.24	94222	.42	48
13	81002	.25	92715	.43	47	13	81882	.24	94248	.42	47
14	81017	.25	92740	.43	46	14	81897	.24	94273	.42	46
15	81032	.25	92766	.43	45	15	81911	.24	94299	.42	45
16	81047	.25	92792	.43	44	16	81926	.24	94324	.42	44
17	81061	.25	92817	.43	43	17	81940	.24	94350	.42	43
18	81076	.25	92843	.43	42	18	81955	.24	94375	.42	42
19	81091	.25	92868	.43	41	19	81969	.24	94401	.42	41
20	81106	.25	92894	.43	40	20	81983	.24	94426	.42	40
21	9.81121	.25	9.92920	.43	39	21	9.81998	.24	9.94452	.42	39
22	81136	.25	92945	.43	38	22	82012	.24	94477	.42	38
23	81151	.25	92971	.43	37	23	82026	.24	94503	.42	37
24	81166	.25	92996	.43	36	24	82041	.24	94528	.42	36
25	81180	.25	93022	.43	35	25	82055	.24	94554	.42	35
26	81195	.25	93048	.43	34	26	82069	.24	94579	.42	34
27	81210	.25	93073	.43	33	27	82084	.24	94604	.42	33
28	81225	.25	93099	.43	32	28	82098	.24	94630	.42	32
29	81240	.25	93124	.43	31	29	82112	.24	94655	.42	31
30	81254	.25	93150	.43	30	30	82126	.24	94681	.42	30
31	9.81269	.25	9.93175	.43	29	31	9.82141	.24	9.94706	.42	29
32	81284	.25	93201	.43	28	32	82155	.24	94732	.42	28
33	81299	.25	93227	.43	27	33	82169	.24	94757	.42	27
34	81314	.25	93252	.43	26	34	82184	.24	94783	.42	26
35	81328	.25	93278	.43	25	35	82198	.24	94808	.42	25
36	81343	.25	93303	.43	24	36	82212	.24	94834	.42	24
37	81358	.25	93329	.43	23	37	82226	.24	94859	.42	23
38	81372	.25	93354	.43	22	38	82240	.24	94884	.42	22
39	81387	.25	93380	.43	21	39	82255	.24	94910	.42	21
40	81402	.25	93406	.43	20	40	82269	.24	94935	.42	20
41	9.81417	.24	9.93431	.43	19	41	9.82283	.24	9.94961	.42	19
42	81431	.24	93457	.43	18	42	82297	.24	94986	.42	18
43	81446	.24	93482	.43	17	43	82311	.24	95012	.42	17
44	81461	.24	93508	.43	16	44	82326	.24	95037	.42	16
45	81475	.24	93533	.43	15	45	82340	.24	95062	.42	15
46	81490	.24	93559	.43	14	46	82354	.24	95088	.42	14
47	81505	.24	93584	.43	13	47	82368	.24	95113	.42	13
48	81519	.24	93610	.43	12	48	82382	.24	95139	.42	12
49	81534	.24	93636	.43	11	49	82396	.24	95164	.42	11
50	81549	.24	93661	.43	10	50	82410	.24	95190	.42	10
51	9.81563	.24	9.93687	.43	9	51	9.82424	.23	9.95215	.42	9
52	81578	.24	93712	.43	8	52	82439	.23	95240	.42	8
53	81592	.24	93738	.43	7	53	82453	.23	95266	.42	7
54	81607	.24	93763	.43	6	54	82467	.23	95291	.42	6
55	81622	.24	93789	.43	5	55	82481	.23	95317	.42	5
56	81636	.24	93814	.43	4	56	82495	.23	95342	.42	4
57	81651	.24	93840	.43	3	57	82509	.23	95368	.42	3
58	81665	.24	93865	.43	2	58	82523	.23	95393	.42	2
59	81680	.24	93891	.43	1	59	82537	.23	95418	.42	1
60	81694		93916		0	60	82551		95444		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.82551	0.23	9.95444	0.42	60	0	9.83378	0.23	9.96966	0.42	60
1	82565	.23	95469	.42	59	1	83392	.23	96991	.42	59
2	82579	.23	95495	.42	58	2	83405	.23	97016	.42	58
3	82593	.23	95520	.42	57	3	83419	.23	97042	.42	57
4	82607	.23	95545	.42	56	4	83432	.23	97067	.42	56
5	82621	.23	95571	.42	55	5	83446	.23	97092	.42	55
6	82635	.23	95596	.42	54	6	83459	.22	97118	.42	54
7	82649	.23	95622	.42	53	7	83473	.22	97143	.42	53
8	82663	.23	95647	.42	52	8	83486	.22	97168	.42	52
9	82677	.23	95672	.42	51	9	83500	.22	97193	.42	51
10	82691	.23	95698	.42	50	10	83513	.22	97219	.42	50
11	9.82705	.23	9.95723	.42	49	11	9.83527	.22	9.97244	.42	49
12	82719	.23	95748	.42	48	12	83540	.22	97269	.42	48
13	82733	.23	95774	.42	47	13	83554	.22	97295	.42	47
14	82747	.23	95799	.42	46	14	83567	.22	97320	.42	46
15	82761	.23	95825	.42	45	15	83581	.22	97345	.42	45
16	82775	.23	95850	.42	44	16	83594	.22	97371	.42	44
17	82788	.23	95875	.42	43	17	83608	.22	97396	.42	43
18	82802	.23	95901	.42	42	18	83621	.22	97421	.42	42
19	82816	.23	95926	.42	41	19	83634	.22	97447	.42	41
20	82830	.23	95952	.42	40	20	83648	.22	97472	.42	40
21	9.82844	.23	9.95977	.42	39	21	9.83661	.22	9.97497	.42	39
22	82858	.23	96002	.42	38	22	83674	.22	97523	.42	38
23	82872	.23	96028	.42	37	23	83688	.22	97548	.42	37
24	82885	.23	96053	.42	36	24	83701	.22	97573	.42	36
25	82899	.23	96078	.42	35	25	83715	.22	97598	.42	35
26	82913	.23	96104	.42	34	26	83728	.22	97624	.42	34
27	82927	.23	96129	.42	33	27	83741	.22	97649	.42	33
28	82941	.23	96155	.42	32	28	83755	.22	97674	.42	32
29	82955	.23	96180	.42	31	29	83768	.22	97700	.42	31
30	82968	.23	96205	.42	30	30	83781	.22	97725	.42	30
31	9.82982	.23	9.96231	.42	29	31	9.83795	.22	9.97750	.42	29
32	82996	.23	96256	.42	28	32	83808	.22	97776	.42	28
33	83010	.23	96281	.42	27	33	83821	.22	97801	.42	27
34	83023	.23	96307	.42	26	34	83834	.22	97826	.42	26
35	83037	.23	96332	.42	25	35	83848	.22	97851	.42	25
36	83051	.23	96357	.42	24	36	83861	.22	97877	.42	24
37	83065	.23	96383	.42	23	37	83874	.22	97902	.42	23
38	83078	.23	96408	.42	22	38	83887	.22	97927	.42	22
39	83092	.23	96433	.42	21	39	83901	.22	97953	.42	21
40	83106	.23	96459	.42	20	40	83914	.22	97978	.42	20
41	9.83120	.23	9.96484	.42	19	41	9.83927	.22	9.98003	.42	19
42	83133	.23	96510	.42	18	42	83940	.22	98029	.42	18
43	83147	.23	96535	.42	17	43	83954	.22	98054	.42	17
44	83161	.23	96560	.42	16	44	83967	.22	98079	.42	16
45	83174	.23	96586	.42	15	45	83980	.22	98104	.42	15
46	83188	.23	96611	.42	14	46	83993	.22	98130	.42	14
47	83202	.23	96636	.42	13	47	84006	.22	98155	.42	13
48	83215	.23	96662	.42	12	48	84020	.22	98180	.42	12
49	83229	.23	96687	.42	11	49	84033	.22	98206	.42	11
50	83242	.23	96712	.42	10	50	84046	.22	98231	.42	10
51	9.83256	.23	9.96738	.42	9	51	9.84059	.22	9.98256	.42	9
52	83270	.23	96763	.42	8	52	84072	.22	98281	.42	8
53	83283	.23	96788	.42	7	53	84085	.22	98307	.42	7
54	83297	.23	96814	.42	6	54	84098	.22	98332	.42	6
55	83310	.23	96839	.42	5	55	84112	.22	98357	.42	5
56	83324	.23	96864	.42	4	56	84125	.22	98383	.42	4
57	83338	.23	96890	.42	3	57	84138	.22	98408	.42	3
58	83351	.23	96915	.42	2	58	84151	.22	98433	.42	2
59	83365	.23	96940	.42	1	59	84164	.22	98458	.42	1
60	83378		96966		0	60	84177		98484		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.84177	0.22	9.98484	0.42	60	0	9.84949	0.21	10.00000	0.42	60
1	84190	.22	98509	.42	59	1	84961	.21	00025	.42	59
2	84203	.22	98534	.42	58	2	84974	.21	00051	.42	58
3	84216	.22	98560	.42	57	3	84986	.21	00076	.42	57
4	84229	.22	98585	.42	56	4	84999	.21	00101	.42	56
5	84242	.22	98610	.42	55	5	85012	.21	00126	.42	55
6	84255	.22	98635	.42	54	6	85024	.21	00152	.42	54
7	84269	.22	98661	.42	53	7	85037	.21	00177	.42	53
8	84282	.22	98686	.42	52	8	85049	.21	00202	.42	52
9	84295	.22	98711	.42	51	9	85062	.21	00227	.42	51
10	84308	.22	98737	.42	50	10	85074	.21	00253	.42	50
11	9.84321	.22	9.98762	.42	49	11	9.85087	.21	10.00278	.42	49
12	84334	.22	98787	.42	48	12	85100	.21	00303	.42	48
13	84347	.22	98812	.42	47	13	85112	.21	00328	.42	47
14	84360	.22	98838	.42	46	14	85125	.21	00354	.42	46
15	84373	.22	98863	.42	45	15	85137	.21	00379	.42	45
16	84385	.22	98888	.42	44	16	85150	.21	00404	.42	44
17	84398	.22	98913	.42	43	17	85162	.21	00430	.42	43
18	84411	.22	98939	.42	42	18	85175	.21	00455	.42	42
19	84424	.22	98964	.42	41	19	85187	.21	00480	.42	41
20	84437	.22	98989	.42	40	20	85200	.21	00505	.42	40
21	9.84450	.22	9.99015	.42	39	21	9.85212	.21	10.00531	.42	39
22	84463	.22	99040	.42	38	22	85225	.21	00556	.42	38
23	84476	.21	99065	.42	37	23	85237	.21	00581	.42	37
24	84489	.21	99090	.42	36	24	85250	.21	00606	.42	36
25	84502	.21	99116	.42	35	25	85262	.21	00632	.42	35
26	84515	.21	99141	.42	34	26	85274	.21	00657	.42	34
27	84528	.21	99166	.42	33	27	85287	.21	00682	.42	33
28	84540	.21	99191	.42	32	28	85299	.21	00707	.42	32
29	84553	.21	99217	.42	31	29	85312	.21	00733	.42	31
30	84566	.21	99242	.42	30	30	85324	.21	00758	.42	30
31	9.84579	.21	9.99267	.42	29	31	9.85337	.21	10.00783	.42	29
32	84592	.21	99293	.42	28	32	85349	.21	00809	.42	28
33	84605	.21	99318	.42	27	33	85361	.21	00834	.42	27
34	84618	.21	99343	.42	26	34	85374	.21	00859	.42	26
35	84630	.21	99368	.42	25	35	85386	.21	00884	.42	25
36	84643	.21	99394	.42	24	36	85399	.21	00910	.42	24
37	84656	.21	99419	.42	23	37	85411	.21	00935	.42	23
38	84669	.21	99444	.42	22	38	85423	.21	00960	.42	22
39	84682	.21	99469	.42	21	39	85436	.21	00985	.42	21
40	84694	.21	99495	.42	20	40	85448	.21	01011	.42	20
41	9.84707	.21	9.99520	.42	19	41	9.85460	.21	10.01036	.42	19
42	84720	.21	99545	.42	18	42	85473	.21	01061	.42	18
43	84733	.21	99570	.42	17	43	85485	.21	01087	.42	17
44	84745	.21	99596	.42	16	44	85497	.21	01112	.42	16
45	84758	.21	99621	.42	15	45	85510	.21	01137	.42	15
46	84771	.21	99646	.42	14	46	85522	.20	01162	.42	14
47	84784	.21	99672	.42	13	47	85534	.20	01188	.42	13
48	84796	.21	99697	.42	12	48	85547	.20	01213	.42	12
49	84809	.21	99722	.42	11	49	85559	.20	01238	.42	11
50	84822	.21	99747	.42	10	50	85571	.20	01263	.42	10
51	9.84835	.21	9.99773	.42	9	51	9.85583	.20	10.01289	.42	9
52	84847	.21	99798	.42	8	52	85596	.20	01314	.42	8
53	84860	.21	99823	.42	7	53	85608	.20	01339	.42	7
54	84873	.21	99848	.42	6	54	85620	.20	01365	.42	6
55	84885	.21	99874	.42	5	55	85632	.20	01390	.42	5
56	84898	.21	99899	.42	4	56	85645	.20	01415	.42	4
57	84911	.21	99924	.42	3	57	85657	.20	01440	.42	3
58	84923	.21	99949	.42	2	58	85669	.20	01466	.42	2
59	84936	.21	99975	.42	1	59	85681	.20	01491	.42	1
60	84949		10.00000		0	60	85693		01516		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M
0	9.85693	0.20	10.01516	0.42	60	0	9.86413	0.20	10.03034	0.42	60
1	85706	.20	01542	.42	59	1	86425	.20	03060	.42	59
2	85718	.20	01567	.42	58	2	86436	.20	03085	.42	58
3	85730	.20	01592	.42	57	3	86448	.20	03110	.42	57
4	85742	.20	01617	.42	56	4	86460	.20	03136	.42	56
5	85754	.20	01643	.42	55	5	86472	.20	03161	.42	55
6	85766	.20	01668	.42	54	6	86483	.20	03186	.42	54
7	85779	.20	01693	.42	53	7	86495	.20	03212	.42	53
8	85791	.20	01719	.42	52	8	86507	.20	03237	.42	52
9	85803	.20	01744	.42	51	9	86518	.20	03262	.42	51
10	85815	.20	01769	.42	50	10	86530	.20	03288	.42	50
11	9.85827	.20	10.01794	.42	49	11	9.86542	.20	10.03313	.42	49
12	85839	.20	01820	.42	48	12	86554	.19	03338	.42	48
13	85851	.20	01845	.42	47	13	86565	.19	03364	.42	47
14	85864	.20	01870	.42	46	14	86577	.19	03389	.42	46
15	85876	.20	01896	.42	45	15	86589	.19	03414	.42	45
16	85888	.20	01921	.42	44	16	86600	.19	03440	.42	44
17	85900	.20	01946	.42	43	17	86612	.19	03465	.42	43
18	85912	.20	01971	.42	42	18	86624	.19	03490	.42	42
19	85924	.20	01997	.42	41	19	86635	.19	03516	.42	41
20	85936	.20	02022	.42	40	20	86647	.19	03541	.42	40
21	9.85948	.20	10.02047	.42	39	21	9.86659	.19	10.03567	.42	39
22	85960	.20	02073	.42	38	22	86670	.19	03592	.42	38
23	85972	.20	02098	.42	37	23	86682	.19	03617	.42	37
24	85984	.20	02123	.42	36	24	86694	.19	03643	.42	36
25	85996	.20	02149	.42	35	25	86705	.19	03668	.42	35
26	86008	.20	02174	.42	34	26	86717	.19	03693	.42	34
27	86020	.20	02199	.42	33	27	86728	.19	03719	.42	33
28	86032	.20	02224	.42	32	28	86740	.19	03744	.42	32
29	86044	.20	02250	.42	31	29	86752	.19	03769	.42	31
30	86056	.20	02275	.42	30	30	86763	.19	03795	.42	30
31	9.86068	.20	10.02300	.42	29	31	9.86775	.19	10.03820	.42	29
32	86080	.20	02326	.42	28	32	86786	.19	03845	.42	28
33	86092	.20	02351	.42	27	33	86798	.19	03871	.42	27
34	86104	.20	02376	.42	26	34	86809	.19	03896	.42	26
35	86116	.20	02402	.42	25	35	86821	.19	03922	.42	25
36	86128	.20	02427	.42	24	36	86832	.19	03947	.42	24
37	86140	.20	02452	.42	23	37	86844	.19	03972	.42	23
38	86152	.20	02477	.42	22	38	86855	.19	03998	.42	22
39	86164	.20	02503	.42	21	39	86867	.19	04023	.42	21
40	86176	.20	02528	.42	20	40	86879	.19	04048	.42	20
41	9.86188	.20	10.02553	.42	19	41	9.86890	.19	10.04074	.42	19
42	86200	.20	02579	.42	18	42	86902	.19	04099	.42	18
43	86211	.20	02604	.42	17	43	86913	.19	04125	.42	17
44	86223	.20	02629	.42	16	44	86924	.19	04150	.42	16
45	86235	.20	02655	.42	15	45	86936	.19	04175	.42	15
46	86247	.20	02680	.42	14	46	86947	.19	04201	.42	14
47	86259	.20	02705	.42	13	47	86959	.19	04226	.42	13
48	86271	.20	02731	.42	12	48	86970	.19	04252	.42	12
49	86283	.20	02756	.42	11	49	86982	.19	04277	.42	11
50	86295	.20	02781	.42	10	50	86993	.19	04302	.42	10
51	9.86306	.20	10.02807	.42	9	51	9.87005	.19	10.04328	.42	9
52	86318	.20	02832	.42	8	52	87016	.19	04353	.42	8
53	86330	.20	02857	.42	7	53	87028	.19	04378	.42	7
54	86342	.20	02882	.42	6	54	87039	.19	04404	.42	6
55	86354	.20	02908	.42	5	55	87050	.19	04429	.42	5
56	86366	.20	02933	.42	4	56	87062	.19	04455	.42	4
57	86377	.20	02958	.42	3	57	87073	.19	04480	.42	3
58	86389	.20	02984	.42	2	58	87085	.19	04505	.42	2
59	86401	.20	03009	.42	1	59	87096	.19	04531	.42	1
60	86413		03034		0	60	87107		04556		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.87107	0.19	10.04556	0.42	60	0	9.87778	0.18	10.06084	0.43	60
1	87119	.19	04582	.42	59	1	87789	.18	06109	.43	59
2	87130	.19	04607	.42	58	2	87800	.18	06135	.43	58
3	87141	.19	04632	.42	57	3	87811	.18	06160	.43	57
4	87153	.19	04658	.42	56	4	87822	.18	06186	.43	56
5	87164	.19	04683	.42	55	5	87833	.18	06211	.43	55
6	87175	.19	04709	.42	54	6	87844	.18	06237	.43	54
7	87187	.19	04734	.42	53	7	87855	.18	06262	.43	53
8	87198	.19	04760	.42	52	8	87866	.18	06288	.43	52
9	87209	.19	04785	.42	51	9	87877	.18	06313	.43	51
10	87221	.19	04810	.42	50	10	87887	.18	06339	.43	50
11	9.87232	.19	10.04836	.42	49	11	9.87898	.18	10.06364	.43	49
12	87243	.19	04861	.42	48	12	87909	.18	06390	.43	48
13	87255	.19	04887	.42	47	13	87920	.18	06416	.43	47
14	87266	.19	04912	.42	46	14	87931	.18	06441	.43	46
15	87277	.19	04938	.42	45	15	87942	.18	06467	.43	45
16	87288	.19	04963	.42	44	16	87953	.18	06492	.43	44
17	87300	.19	04988	.42	43	17	87964	.18	06518	.43	43
18	87311	.19	05014	.42	42	18	87975	.18	06543	.43	42
19	87322	.19	05039	.42	41	19	87985	.18	06569	.43	41
20	87334	.19	05065	.42	40	20	87996	.18	06594	.43	40
21	9.87345	.19	10.05090	.42	39	21	9.88007	.18	10.06620	.43	39
22	87356	.19	05116	.42	38	22	88018	.18	06646	.43	38
23	87367	.19	05141	.42	37	23	88029	.18	06671	.43	37
24	87378	.19	05166	.42	36	24	88040	.18	06697	.43	36
25	87390	.19	05192	.42	35	25	88051	.18	06722	.43	35
26	87401	.19	05217	.42	34	26	88061	.18	06748	.43	34
27	87412	.19	05243	.42	33	27	88072	.18	06773	.43	33
28	87423	.19	05268	.42	32	28	88083	.18	06799	.43	32
29	87434	.19	05294	.42	31	29	88094	.18	06825	.43	31
30	87446	.19	05319	.42	30	30	88105	.18	06850	.43	30
31	9.87457	.19	10.05345	.42	29	31	9.88115	.18	10.06876	.43	29
32	87468	.19	05370	.42	28	32	88126	.18	06901	.43	28
33	87479	.19	05396	.42	27	33	88137	.18	06927	.43	27
34	87490	.19	05421	.42	26	34	88148	.18	06952	.43	26
35	87501	.19	05446	.42	25	35	88158	.18	06978	.43	25
36	87513	.19	05472	.42	24	36	88169	.18	07004	.43	24
37	87524	.19	05497	.42	23	37	88180	.18	07029	.43	23
38	87535	.19	05523	.42	22	38	88191	.18	07055	.43	22
39	87546	.19	05548	.42	21	39	88201	.18	07080	.43	21
40	87557	.19	05574	.42	20	40	88212	.18	07106	.43	20
41	9.87568	.19	10.05599	.42	19	41	9.88223	.18	10.07132	.43	19
42	87579	.18	05625	.42	18	42	88234	.18	07157	.43	18
43	87590	.18	05650	.42	17	43	88244	.18	07183	.43	17
44	87601	.18	05676	.42	16	44	88255	.18	07208	.43	16
45	87613	.18	05701	.42	15	45	88266	.18	07234	.43	15
46	87624	.18	05727	.42	14	46	88276	.18	07260	.43	14
47	87635	.18	05752	.42	13	47	88287	.18	07285	.43	13
48	87646	.18	05778	.42	12	48	88298	.18	07311	.43	12
49	87657	.18	05803	.42	11	49	88308	.18	07337	.43	11
50	87668	.18	05829	.42	10	50	88319	.18	07362	.43	10
51	9.87679	.18	10.05854	.42	9	51	9.88330	.18	10.07388	.43	9
52	87690	.18	05880	.43	8	52	88340	.18	07413	.43	8
53	87701	.18	05905	.43	7	53	88351	.18	07439	.43	7
54	87712	.18	05931	.43	6	54	88362	.18	07465	.43	6
55	87723	.18	05956	.43	5	55	88372	.18	07490	.43	5
56	87734	.18	05982	.43	4	56	88383	.18	07516	.43	4
57	87745	.18	06007	.43	3	57	88394	.18	07542	.43	3
58	87756	.18	06033	.43	2	58	88404	.18	07567	.43	2
59	87767	.18	06058	.43	1	59	88415	.18	07593	.43	1
60	87778		06084		0	60	88425		07619		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl"	Tang.	Dl"	M.	M.	Sine.	Dl"	Tang.	Dl"	M.
0	9.88425	0.18	10.07619	0.43	60	0	9.89050	0.17	10.09163	0.43	60
1	88436	.18	07644	.43	59	1	89060	.17	09189	.43	59
2	88447	.18	07670	.43	58	2	89071	.17	09215	.43	58
3	88457	.18	07696	.43	57	3	89081	.17	09241	.43	57
4	88468	.18	07721	.43	56	4	89091	.17	09266	.43	56
5	88478	.18	07747	.43	55	5	89101	.17	09292	.43	55
6	88489	.18	07773	.43	54	6	89112	.17	09318	.43	54
7	88499	.18	07798	.43	53	7	89122	.17	09344	.43	53
8	88510	.18	07824	.43	52	8	89132	.17	09370	.43	52
9	88521	.18	07850	.43	51	9	89142	.17	09396	.43	51
10	88531	.18	07875	.43	50	10	89152	.17	09422	.43	50
11	9.88542	.18	10.07901	.43	49	11	9.89162	.17	10.09447	.43	49
12	88552	.18	07927	.43	48	12	89173	.17	09473	.43	48
13	88563	.18	07952	.43	47	13	89183	.17	09499	.43	47
14	88573	.18	07978	.43	46	14	89193	.17	09525	.43	46
15	88584	.17	08004	.43	45	15	89203	.17	09551	.43	45
16	88594	.17	08029	.43	44	16	89213	.17	09577	.43	44
17	88605	.17	08055	.43	43	17	89223	.17	09603	.43	43
18	88615	.17	08081	.43	42	18	89233	.17	09629	.43	42
19	88626	.17	08107	.43	41	19	89244	.17	09654	.43	41
20	88636	.17	08132	.43	40	20	89254	.17	09680	.43	40
21	9.88647	.17	10.08158	.43	39	21	9.89264	.17	10.09706	.43	39
22	88657	.17	08184	.43	38	22	89274	.17	09732	.43	38
23	88668	.17	08209	.43	37	23	89284	.17	09758	.43	37
24	88678	.17	08235	.43	36	24	89294	.17	09784	.43	36
25	88688	.17	08261	.43	35	25	89304	.17	09810	.43	35
26	88699	.17	08287	.43	34	26	89314	.17	09836	.43	34
27	88709	.17	08312	.43	33	27	89324	.17	09862	.43	33
28	88720	.17	08338	.43	32	28	89334	.17	09888	.43	32
29	88730	.17	08364	.43	31	29	89344	.17	09914	.43	31
30	88741	.17	08390	.43	30	30	89354	.17	09939	.43	30
31	9.88751	.17	10.08415	.43	29	31	9.89364	.17	10.09965	.43	29
32	88761	.17	08441	.43	28	32	89375	.17	09991	.43	28
33	88772	.17	08467	.43	27	33	89385	.17	10017	.43	27
34	88782	.17	08493	.43	26	34	89395	.17	10043	.43	26
35	88793	.17	08518	.43	25	35	89405	.17	10069	.43	25
36	88803	.17	08544	.43	24	36	89415	.17	10095	.43	24
37	88813	.17	08570	.43	23	37	89425	.17	10121	.43	23
38	88824	.17	08596	.43	22	38	89435	.17	10147	.43	22
39	88834	.17	08621	.43	21	39	89445	.17	10173	.43	21
40	88844	.17	08647	.43	20	40	89455	.17	10199	.43	20
41	9.88855	.17	10.08673	.43	19	41	9.89465	.17	10.10225	.43	19
42	88865	.17	08699	.43	18	42	89475	.17	10251	.43	18
43	88875	.17	08724	.43	17	43	89485	.17	10277	.43	17
44	88886	.17	08750	.43	16	44	89495	.17	10303	.43	16
45	88896	.17	08776	.43	15	45	89504	.17	10329	.43	15
46	88906	.17	08802	.43	14	46	89514	.17	10355	.43	14
47	88917	.17	08828	.43	13	47	89524	.17	10381	.43	13
48	88927	.17	08853	.43	12	48	89534	.17	10407	.43	12
49	88937	.17	08879	.43	11	49	89544	.17	10433	.43	11
50	88948	.17	08905	.43	10	50	89554	.17	10459	.43	10
51	9.88958	.17	10.08931	.43	9	51	9.89564	.17	10.10485	.43	9
52	88968	.17	08957	.43	8	52	89574	.17	10511	.43	8
53	88978	.17	08982	.43	7	53	89584	.17	10537	.43	7
54	88989	.17	09008	.43	6	54	89594	.17	10563	.43	6
55	88999	.17	09034	.43	5	55	89604	.16	10589	.43	5
56	89009	.17	09060	.43	4	56	89614	.16	10615	.43	4
57	89020	.17	09086	.43	3	57	89624	.16	10641	.43	3
58	89030	.17	09111	.43	2	58	89633	.16	10667	.43	2
59	89040	.17	09137	.43	1	59	89643	.16	10693	.43	1
60	89050		09163		0	60	89653		10719		0
M.	Cosine.	Dl"	Cotang.	Dl"	M.	M.	Cosine.	Dl"	Cotang.	Dl"	M.

M.	Sine.	Dl"	Tang.	Dl"	M.	M.	Sine.	Dl"	Tang.	Dl"	M.
0	9.89653	0.16	10.10719	0.43	60	0	9.90235	0.16	10.12289	0.44	60
1	89663	.16	10745	.43	59	1	90244	.16	12315	.44	59
2	89673	.16	10771	.43	58	2	90254	.16	12341	.44	58
3	89683	.16	10797	.43	57	3	90263	.16	12367	.44	57
4	89693	.16	10823	.43	56	4	90273	.16	12394	.44	56
5	89702	.16	10849	.43	55	5	90282	.16	12420	.44	55
6	89712	.16	10875	.43	54	6	90292	.16	12446	.44	54
7	89722	.16	10901	.43	53	7	90301	.16	12473	.44	53
8	89732	.16	10927	.43	52	8	90311	.16	12499	.44	52
9	89742	.16	10954	.43	51	9	90320	.16	12525	.44	51
10	89752	.16	10980	.43	50	10	90330	.16	12552	.44	50
11	9.89761	.16	10.11006	.43	49	11	9.90339	.16	10.12578	.44	49
12	89771	.16	11032	.43	48	12	90349	.16	12604	.44	48
13	89781	.16	11058	.43	47	13	90358	.16	12631	.44	47
14	89791	.16	11084	.43	46	14	90368	.16	12657	.44	46
15	89801	.16	11110	.44	45	15	90377	.16	12683	.44	45
16	89810	.16	11136	.44	44	16	90386	.16	12710	.44	44
17	89820	.16	11162	.44	43	17	90396	.16	12736	.44	43
18	89830	.16	11188	.44	42	18	90405	.16	12762	.44	42
19	89840	.16	11214	.44	41	19	90415	.16	12789	.44	41
20	89849	.16	11241	.44	40	20	90424	.16	12815	.44	40
21	9.89859	.16	10.11267	.44	39	21	9.90434	.16	10.12842	.44	39
22	89869	.16	11293	.44	38	22	90443	.16	12868	.44	38
23	89879	.16	11319	.44	37	23	90452	.16	12894	.44	37
24	89888	.16	11345	.44	36	24	90462	.16	12921	.44	36
25	89898	.16	11371	.44	35	25	90471	.16	12947	.44	35
26	89908	.16	11397	.44	34	26	90480	.16	12973	.44	34
27	89918	.16	11423	.44	33	27	90490	.16	13000	.44	33
28	89927	.16	11450	.44	32	28	90499	.16	13026	.44	32
29	89937	.16	11476	.44	31	29	90509	.16	13053	.44	31
30	89947	.16	11502	.44	30	30	90518	.16	13079	.44	30
31	9.89956	.16	10.11528	.44	29	31	9.90527	.16	10.13106	.44	29
32	89966	.16	11554	.44	28	32	90537	.16	13132	.44	28
33	89976	.16	11580	.44	27	33	90546	.16	13158	.44	27
34	89985	.16	11607	.44	26	34	90555	.16	13185	.44	26
35	89995	.16	11633	.44	25	35	90565	.16	13211	.44	25
36	90005	.16	11659	.44	24	36	90574	.16	13238	.44	24
37	90014	.16	11685	.44	23	37	90583	.16	13264	.44	23
38	90024	.16	11711	.44	22	38	90592	.15	13291	.44	22
39	90034	.16	11738	.44	21	39	90602	.15	13317	.44	21
40	90043	.16	11764	.44	20	40	90611	.15	13344	.44	20
41	9.90053	.16	10.11790	.44	19	41	9.90620	.15	10.13370	.44	19
42	90063	.16	11816	.44	18	42	90630	.15	13397	.44	18
43	90072	.16	11842	.44	17	43	90639	.15	13423	.44	17
44	90082	.16	11869	.44	16	44	90648	.15	13449	.44	16
45	90091	.16	11895	.44	15	45	90657	.15	13476	.44	15
46	90101	.16	11921	.44	14	46	90667	.15	13502	.44	14
47	90111	.16	11947	.44	13	47	90676	.15	13529	.44	13
48	90120	.16	11973	.44	12	48	90685	.15	13555	.44	12
49	90130	.16	12000	.44	11	49	90694	.15	13582	.44	11
50	90139	.16	12026	.44	10	50	90704	.15	13608	.44	10
51	9.90149	.16	10.12052	.44	9	51	9.90713	.15	10.13635	.44	9
52	90159	.16	12078	.44	8	52	90722	.15	13662	.44	8
53	90168	.16	12105	.44	7	53	90731	.15	13688	.44	7
54	90178	.16	12131	.44	6	54	90741	.15	13715	.44	6
55	90187	.16	12157	.44	5	55	90750	.15	13741	.44	5
56	90197	.16	12183	.44	4	56	90759	.15	13768	.44	4
57	90206	.16	12210	.44	3	57	90768	.15	13794	.44	3
58	90216	.16	12236	.44	2	58	90777	.15	13821	.44	2
59	90225	.16	12262	.44	1	59	90787	.15	13847	.44	1
60	90235		10.12289		0	60	90796		13874		0
M.	Cosine.	Dl"	Cotang.	Dl"	M.	M.	Cosine.	Dl"	Cotang.	Dl"	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.90796	0.15	10.13874	0.44	60	0	9.91336	0.15	10.15477	0.45	60
1	90805	.15	13900	.44	59	1	91345	.15	15504	.45	59
2	90814	.15	13927	.44	58	2	91354	.15	15531	.45	58
3	90823	.15	13954	.44	57	3	91363	.15	15558	.45	57
4	90832	.15	13980	.44	56	4	91372	.15	15585	.45	56
5	90842	.15	14007	.44	55	5	91381	.15	15612	.45	55
6	90851	.15	14033	.44	54	6	91389	.15	15639	.45	54
7	90860	.15	14060	.44	53	7	91398	.15	15666	.45	53
8	90869	.15	14087	.44	52	8	91407	.15	15693	.45	52
9	90878	.15	14113	.44	51	9	91416	.15	15720	.45	51
10	90887	.15	14140	.44	50	10	91425	.15	15746	.45	50
11	9.90896	.15	10.14166	.44	49	11	9.91433	.15	10.15773	.45	49
12	90906	.15	14193	.44	48	12	91442	.15	15800	.45	48
13	90915	.15	14220	.44	47	13	91451	.15	15827	.45	47
14	90924	.15	14246	.44	46	14	91460	.15	15854	.45	46
15	90933	.15	14273	.44	45	15	91469	.15	15881	.45	45
16	90942	.15	14300	.44	44	16	91477	.15	15908	.45	44
17	90951	.15	14326	.44	43	17	91486	.15	15935	.45	43
18	90960	.15	14353	.44	42	18	91495	.15	15962	.45	42
19	90969	.15	14380	.44	41	19	91504	.15	15989	.45	41
20	90978	.15	14406	.44	40	20	91512	.15	16016	.45	40
21	9.90987	.15	10.14433	.44	39	21	9.91521	.15	10.16043	.45	39
22	90996	.15	14460	.44	38	22	91530	.15	16070	.45	38
23	91005	.15	14486	.44	37	23	91538	.15	16097	.45	37
24	91014	.15	14513	.44	36	24	91547	.15	16124	.45	36
25	91023	.15	14540	.44	35	25	91556	.15	16151	.45	35
26	91033	.15	14566	.45	34	26	91565	.14	16178	.45	34
27	91042	.15	14593	.45	33	27	91573	.14	16205	.45	33
28	91051	.15	14620	.45	32	28	91582	.14	16232	.45	32
29	91060	.15	14646	.45	31	29	91591	.14	16260	.45	31
30	91069	.15	14673	.45	30	30	91599	.14	16287	.45	30
31	9.91078	.15	10.14700	.45	29	31	9.91608	.14	10.16314	.45	29
32	91087	.15	14727	.45	28	32	91617	.14	16341	.45	28
33	91096	.15	14753	.45	27	33	91625	.14	16368	.45	27
34	91105	.15	14780	.45	26	34	91634	.14	16395	.45	26
35	91114	.15	14807	.45	25	35	91643	.14	16422	.45	25
36	91123	.15	14834	.45	24	36	91651	.14	16449	.45	24
37	91132	.15	14860	.45	23	37	91660	.14	16476	.45	23
38	91141	.15	14887	.45	22	38	91669	.14	16503	.45	22
39	91149	.15	14914	.45	21	39	91677	.14	16530	.45	21
40	91158	.15	14941	.45	20	40	91686	.14	16558	.45	20
41	9.91167	.15	10.11967	.45	19	41	9.91695	.14	10.16585	.45	19
42	91176	.15	14994	.45	18	42	91703	.14	16612	.45	18
43	91185	.15	15021	.45	17	43	91712	.14	16639	.45	17
44	91194	.15	15048	.45	16	44	91720	.14	16666	.45	16
45	91203	.15	15075	.45	15	45	91729	.14	16693	.45	15
46	91212	.15	15101	.45	14	46	91738	.14	16720	.45	14
47	91221	.15	15128	.45	13	47	91746	.14	16748	.45	13
48	91230	.15	15155	.45	12	48	91755	.14	16775	.45	12
49	91239	.15	15182	.45	11	49	91763	.14	16802	.45	11
50	91248	.15	15209	.45	10	50	91772	.14	16829	.45	10
51	9.91257	.15	10.15236	.45	9	51	9.91781	.14	10.16856	.45	9
52	91266	.15	15262	.45	8	52	91789	.14	16883	.45	8
53	91274	.15	15289	.45	7	53	91798	.14	16911	.45	7
54	91283	.15	15316	.45	6	54	91806	.14	16938	.45	6
55	91292	.15	15343	.45	5	55	91815	.14	16965	.45	5
56	91301	.15	15370	.45	4	56	91823	.14	16992	.45	4
57	91310	.15	15397	.45	3	57	91832	.14	17020	.45	3
58	91319	.15	15424	.45	2	58	91840	.14	17047	.45	2
59	91328	.15	15450	.45	1	59	91849	.14	17074	.45	1
60	91336		15477		0	60	91857		17101		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl"	Tang.	Dl"	M.	M.	Sine.	Dl"	Tang.	Dl"	M.
0	9.91857	0.14	10.17101	0.45	60	0	9.92359	0.14	10.18748	0.46	60
1	91866	.14	17129	.45	59	1	92367	.14	18776	.46	59
2	91874	.14	17156	.45	58	2	92376	.14	18804	.46	58
3	91883	.14	17183	.45	57	3	92384	.14	18831	.46	57
4	91891	.14	17210	.45	56	4	92392	.14	18859	.46	56
5	91900	.14	17238	.45	55	5	92400	.14	18887	.46	55
6	91908	.14	17265	.45	54	6	92408	.14	18914	.46	54
7	91917	.14	17292	.45	53	7	92416	.14	18942	.46	53
8	91925	.14	17319	.46	52	8	92425	.14	18970	.46	52
9	91934	.14	17347	.46	51	9	92433	.14	18997	.46	51
10	91942	.14	17374	.46	50	10	92441	.14	19025	.46	50
11	9.91951	.14	10.17401	.46	49	11	9.92449	.14	10.19053	.46	49
12	91959	.14	17429	.46	48	12	92457	.14	19081	.46	48
13	91968	.14	17456	.46	47	13	92465	.14	19108	.46	47
14	91976	.14	17483	.46	46	14	92473	.14	19136	.46	46
15	91985	.14	17511	.46	45	15	92482	.14	19164	.46	45
16	91993	.14	17538	.46	44	16	92490	.14	19192	.46	44
17	92002	.14	17565	.46	43	17	92498	.14	19219	.46	43
18	92010	.14	17593	.46	42	18	92506	.14	19247	.46	42
19	92018	.14	17620	.46	41	19	92514	.13	19275	.46	41
20	92027	.14	17648	.46	40	20	92522	.13	19303	.46	40
21	9.92035	.14	10.17675	.46	39	21	9.92530	.13	10.19331	.46	39
22	92044	.14	17702	.46	38	22	92538	.13	19358	.46	38
23	92052	.14	17730	.46	37	23	92546	.13	19386	.46	37
24	92060	.14	17757	.46	36	24	92555	.13	19414	.46	36
25	92069	.14	17785	.46	35	25	92563	.13	19442	.46	35
26	92077	.14	17812	.46	34	26	92571	.13	19470	.46	34
27	92086	.14	17839	.46	33	27	92579	.13	19498	.46	33
28	92094	.14	17867	.46	32	28	92587	.13	19526	.46	32
29	92102	.14	17894	.46	31	29	92595	.13	19553	.46	31
30	92111	.14	17922	.46	30	30	92603	.13	19581	.46	30
31	9.92119	.14	10.17949	.46	29	31	9.92611	.13	10.19609	.46	29
32	92127	.14	17977	.46	28	32	92619	.13	19637	.47	28
33	92136	.14	18004	.46	27	33	92627	.13	19665	.47	27
34	92144	.14	18032	.46	26	34	92635	.13	19693	.47	26
35	92152	.14	18059	.46	25	35	92643	.13	19721	.47	25
36	92161	.14	18087	.46	24	36	92651	.13	19749	.47	24
37	92169	.14	18114	.46	23	37	92659	.13	19777	.47	23
38	92177	.14	18142	.46	22	38	92667	.13	19805	.47	22
39	92186	.14	18169	.46	21	39	92675	.13	19832	.47	21
40	92194	.14	18197	.46	20	40	92683	.13	19860	.47	20
41	9.92202	.14	10.18224	.46	19	41	9.92691	.13	10.19888	.47	19
42	92211	.14	18252	.46	18	42	92699	.13	19916	.47	18
43	92219	.14	18279	.46	17	43	92707	.13	19944	.47	17
44	92227	.14	18307	.46	16	44	92715	.13	19972	.47	16
45	92235	.14	18334	.46	15	45	92723	.13	20000	.47	15
46	92244	.14	18362	.46	14	46	92731	.13	20028	.47	14
47	92252	.14	18389	.46	13	47	92739	.13	20056	.47	13
48	92260	.14	18417	.46	12	48	92747	.13	20084	.47	12
49	92269	.14	18444	.46	11	49	92755	.13	20112	.47	11
50	92277	.14	18472	.46	10	50	92763	.13	20140	.47	10
51	9.92285	.14	10.18500	.46	9	51	9.92771	.13	10.20168	.47	9
52	92293	.14	18527	.46	8	52	92779	.13	20196	.47	8
53	92302	.14	18555	.46	7	53	92787	.13	20224	.47	7
54	92310	.14	18582	.46	6	54	92795	.13	20253	.47	6
55	92318	.14	18610	.46	5	55	92803	.13	20281	.47	5
56	92326	.14	18638	.46	4	56	92810	.13	20309	.47	4
57	92335	.14	18665	.46	3	57	92818	.13	20337	.47	3
58	92343	.14	18693	.46	2	58	92826	.13	20365	.47	2
59	92351	.14	18721	.46	1	59	92834	.13	20393	.47	1
60	92359		18748		0	60	92842		20421		0
M.	Cosine.	Dl"	Cotang.	Dl"	M.	M.	Cosine.	Dl"	Cotang.	Dl"	M.

M.	Sine.	Dl"	Tang.	Dl"	M.	M.	Sine.	Dl"	Tang.	Dl"	M.
0	9.92842	0.13	10.20421	0.47	60	0	9.93307	0.13	10.22123	0.48	60
1	92850	.13	20449	.47	59	1	93314	.13	22151	.48	59
2	92858	.13	20477	.47	58	2	93322	.13	22180	.48	58
3	92866	.13	20505	.47	57	3	93329	.13	22209	.48	57
4	92874	.13	20534	.47	56	4	93337	.13	22237	.48	56
5	92881	.13	20562	.47	55	5	93344	.13	22266	.48	55
6	92889	.13	20590	.47	54	6	93352	.13	22294	.48	54
7	92897	.13	20618	.47	53	7	93360	.13	22323	.48	53
8	92905	.13	20646	.47	52	8	93367	.13	22352	.48	52
9	92913	.13	20674	.47	51	9	93375	.13	22381	.48	51
10	92921	.13	20703	.47	50	10	93382	.13	22409	.48	50
11	9.92929	.13	10.20731	.47	49	11	9.93390	.13	10.22438	.48	49
12	92936	.13	20759	.47	48	12	93397	.13	22467	.48	48
13	92944	.13	20787	.47	47	13	93405	.13	22495	.48	47
14	92952	.13	20815	.47	46	14	93412	.13	22524	.48	46
15	92960	.13	20844	.47	45	15	93420	.13	22553	.48	45
16	92968	.13	20872	.47	44	16	93427	.13	22582	.48	44
17	92976	.13	20900	.47	43	17	93435	.13	22610	.48	43
18	92983	.13	20928	.47	42	18	93442	.12	22639	.48	42
19	92991	.13	20957	.47	41	19	93450	.12	22668	.48	41
20	92999	.13	20985	.47	40	20	93457	.12	22697	.48	40
21	9.93007	.13	10.21013	.47	39	21	9.93465	.12	10.22726	.48	39
22	93014	.13	21041	.47	38	22	93472	.12	22754	.48	38
23	93022	.13	21070	.47	37	23	93480	.12	22783	.48	37
24	93030	.13	21098	.47	36	24	93487	.12	22812	.48	36
25	93038	.13	21126	.47	35	25	93495	.12	22841	.48	35
26	93046	.13	21155	.47	34	26	93502	.12	22870	.48	34
27	93053	.13	21183	.47	33	27	93510	.12	22899	.48	33
28	93061	.13	21211	.47	32	28	93517	.12	22927	.48	32
29	93069	.13	21240	.47	31	29	93525	.12	22956	.48	31
30	93077	.13	21268	.47	30	30	93532	.12	22985	.48	30
31	9.93084	.13	10.21296	.47	29	31	9.93539	.12	10.23014	.48	29
32	93092	.13	21325	.47	28	32	93547	.12	23043	.48	28
33	93100	.13	21353	.47	27	33	93554	.12	23072	.48	27
34	93108	.13	21382	.47	26	34	93562	.12	23101	.48	26
35	93115	.13	21410	.47	25	35	93569	.12	23130	.48	25
36	93123	.13	21438	.47	24	36	93577	.12	23159	.48	24
37	93131	.13	21467	.47	23	37	93584	.12	23188	.48	23
38	93138	.13	21495	.47	22	38	93591	.12	23217	.48	22
39	93146	.13	21524	.47	21	39	93599	.12	23246	.48	21
40	93154	.13	21552	.47	20	40	93606	.12	23275	.48	20
41	9.93161	.13	10.21581	.47	19	41	9.93614	.12	10.23303	.48	19
42	93169	.13	21609	.47	18	42	93621	.12	23332	.48	18
43	93177	.13	21637	.47	17	43	93628	.12	23361	.48	17
44	93184	.13	21666	.47	16	44	93636	.12	23391	.48	16
45	93192	.13	21694	.47	15	45	93643	.12	23420	.48	15
46	93200	.13	21723	.47	14	46	93650	.12	23449	.48	14
47	93207	.13	21751	.48	13	47	93658	.12	23478	.48	13
48	93215	.13	21780	.48	12	48	93665	.12	23507	.48	12
49	93223	.13	21808	.48	11	49	93673	.12	23536	.48	11
50	93230	.13	21837	.48	10	50	93680	.12	23565	.48	10
51	9.93238	.13	10.21865	.48	9	51	9.93687	.12	10.23594	.48	9
52	93246	.13	21894	.48	8	52	93695	.12	23623	.48	8
53	93253	.13	21923	.48	7	53	93702	.12	23652	.49	7
54	93261	.13	21951	.48	6	54	93709	.12	23681	.49	6
55	93269	.13	21980	.48	5	55	93717	.12	23710	.49	5
56	93276	.13	22008	.48	4	56	93724	.12	23739	.49	4
57	93284	.13	22037	.48	3	57	93731	.12	23769	.49	3
58	93291	.13	22065	.48	2	58	93738	.12	23798	.49	2
59	93299	.13	22094	.48	1	59	93746	.12	23827	.49	1
60	93307		22123		0	60	93753		23856		0
M.	Cosine.	Dl"	Cotang.	Dl"	M.	M.	Cosine.	Dl"	Cotang.	Dl"	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.93753	0.12	10.23856	0.49	60	0	9.94182	0.12	10.25625	0.50	60
1	93760	.12	23885	.49	59	1	94189	.12	25655	.50	59
2	93768	.12	23914	.49	58	2	94196	.12	25684	.50	58
3	93775	.12	23944	.49	57	3	94203	.12	25714	.50	57
4	93782	.12	23973	.49	56	4	94210	.12	25744	.50	56
5	93789	.12	24002	.49	55	5	94217	.12	25774	.50	55
6	93797	.12	24031	.49	54	6	94224	.12	25804	.50	54
7	93804	.12	24061	.49	53	7	94231	.12	25834	.50	53
8	93811	.12	24090	.49	52	8	94238	.12	25863	.50	52
9	93819	.12	24119	.49	51	9	94245	.12	25893	.50	51
10	93826	.12	24148	.49	50	10	94252	.12	25923	.50	50
11	9.93833	.12	10.24178	.49	49	11	9.94259	.12	10.25953	.50	49
12	93840	.12	24207	.49	48	12	94266	.12	25983	.50	48
13	93847	.12	24236	.49	47	13	94273	.12	26013	.50	47
14	93855	.12	24265	.49	46	14	94279	.12	26043	.50	46
15	93862	.12	24295	.49	45	15	94286	.12	26073	.50	45
16	93869	.12	24324	.49	44	16	94293	.12	26103	.50	44
17	93876	.12	24353	.49	43	17	94300	.12	26133	.50	43
18	93884	.12	24383	.49	42	18	94307	.12	26163	.50	42
19	93891	.12	24412	.49	41	19	94314	.12	26193	.50	41
20	93898	.12	24442	.49	40	20	94321	.12	26223	.50	40
21	9.93905	.12	10.24471	.49	39	21	9.94328	.11	10.26253	.50	39
22	93912	.12	24500	.49	38	22	94335	.11	26283	.50	38
23	93920	.12	24530	.49	37	23	94342	.11	26313	.50	37
24	93927	.12	24559	.49	36	24	94349	.11	26343	.50	36
25	93934	.12	24589	.49	35	25	94355	.11	26373	.50	35
26	93941	.12	24618	.49	34	26	94362	.11	26403	.50	34
27	93948	.12	24647	.49	33	27	94369	.11	26433	.50	33
28	93955	.12	24677	.49	32	28	94376	.11	26463	.50	32
29	93963	.12	24706	.49	31	29	94383	.11	26493	.50	31
30	93970	.12	24736	.49	30	30	94390	.11	26524	.50	30
31	9.93977	.12	10.24765	.49	29	31	9.94397	.11	10.26554	.50	29
32	93984	.12	24795	.49	28	32	94404	.11	26584	.50	28
33	93991	.12	24824	.49	27	33	94410	.11	26614	.50	27
34	93998	.12	24854	.49	26	34	94417	.11	26644	.50	26
35	94005	.12	24883	.49	25	35	94424	.11	26674	.50	25
36	94012	.12	24913	.49	24	36	94431	.11	26705	.50	24
37	94020	.12	24942	.49	23	37	94438	.11	26735	.50	23
38	94027	.12	24972	.49	22	38	94445	.11	26765	.50	22
39	94034	.12	25002	.49	21	39	94451	.11	26795	.50	21
40	94041	.12	25031	.49	20	40	94458	.11	26825	.50	20
41	9.94048	.12	10.25061	.49	19	41	9.94465	.11	10.26856	.50	19
42	94055	.12	25090	.49	18	42	94472	.11	26886	.50	18
43	94062	.12	25120	.49	17	43	94479	.11	26916	.50	17
44	94069	.12	25149	.49	16	44	94485	.11	26946	.50	16
45	94076	.12	25179	.49	15	45	94492	.11	26977	.51	15
46	94083	.12	25209	.49	14	46	94499	.11	27007	.51	14
47	94090	.12	25238	.49	13	47	94506	.11	27037	.51	13
48	94098	.12	25268	.49	12	48	94513	.11	27068	.51	12
49	94105	.12	25298	.49	11	49	94519	.11	27098	.51	11
50	94112	.12	25327	.49	10	50	94526	.11	27128	.51	10
51	9.94119	.12	10.25357	.50	9	51	9.94533	.11	10.27159	.51	9
52	94126	.12	25387	.50	8	52	94540	.11	27189	.51	8
53	94133	.12	25417	.50	7	53	94546	.11	27220	.51	7
54	94140	.12	25446	.50	6	54	94553	.11	27250	.51	6
55	94147	.12	25476	.50	5	55	94560	.11	27280	.51	5
56	94154	.12	25506	.50	4	56	94567	.11	27311	.51	4
57	94161	.12	25535	.50	3	57	94573	.11	27341	.51	3
58	94168	.12	25565	.50	2	58	94580	.11	27372	.51	2
59	94175	.12	25595	.50	1	59	94587	.11	27402	.51	1
60	94182		25625		0	60	94593		27433		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

62°

M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.94593	0.11	10.27433	0.51	60
1	94600	.11	27463	.51	59
2	94607	.11	27494	.51	58
3	94614	.11	27524	.51	57
4	94620	.11	27555	.51	56
5	94627	.11	27585	.51	55
6	94634	.11	27616	.51	54
7	94640	.11	27646	.51	53
8	94647	.11	27677	.51	52
9	94654	.11	27707	.51	51
10	94660	.11	27738	.51	50
11	9.94667	.11	10.27769	.51	49
12	94674	.11	27799	.51	48
13	94680	.11	27830	.51	47
14	94687	.11	27860	.51	46
15	94694	.11	27891	.51	45
16	94700	.11	27922	.51	44
17	94707	.11	27952	.51	43
18	94714	.11	27983	.51	42
19	94720	.11	28014	.51	41
20	94727	.11	28045	.51	40
21	9.94734	.11	10.28075	.51	39
22	94740	.11	28106	.51	38
23	94747	.11	28137	.51	37
24	94753	.11	28167	.51	36
25	94760	.11	28198	.51	35
26	94767	.11	28229	.51	34
27	94773	.11	28260	.51	33
28	94780	.11	28291	.51	32
29	94786	.11	28321	.51	31
30	94793	.11	28352	.51	30
31	9.94799	.11	10.28383	.51	29
32	94806	.11	28414	.51	28
33	94813	.11	28445	.51	27
34	94819	.11	28476	.51	26
35	94826	.11	28507	.52	25
36	94832	.11	28538	.52	24
37	94839	.11	28569	.52	23
38	94845	.11	28599	.52	22
39	94852	.11	28630	.52	21
40	94858	.11	28661	.52	20
41	9.94865	.11	10.28692	.52	19
42	94871	.11	28723	.52	18
43	94878	.11	28754	.52	17
44	94885	.11	28785	.52	16
45	94891	.11	28816	.52	15
46	94898	.11	28847	.52	14
47	94904	.11	28879	.52	13
48	94911	.11	28910	.52	12
49	94917	.11	28941	.52	11
50	94923	.11	28972	.52	10
51	9.94930	.11	10.29003	.52	9
52	94936	.11	29034	.52	8
53	94943	.11	29065	.52	7
54	94949	.11	29096	.52	6
55	94956	.11	29127	.52	5
56	94962	.11	29159	.52	4
57	94969	.11	29190	.52	3
58	94975	.11	29221	.52	2
59	94982	.11	29252	.52	1
60	94988		29283		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.

63°

M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.94988	0.11	10.29283	0.52	60
1	94995	.11	29315	.52	59
2	95001	.11	29346	.52	58
3	95007	.11	29377	.52	57
4	95014	.11	29408	.52	56
5	95020	.11	29440	.52	55
6	95027	.11	29471	.52	54
7	95033	.11	29502	.52	53
8	95039	.11	29534	.52	52
9	95046	.11	29565	.52	51
10	95052	.11	29596	.52	50
11	9.95059	.11	10.29628	.52	49
12	95065	.11	29659	.52	48
13	95071	.11	29691	.52	47
14	95078	.11	29722	.52	46
15	95084	.11	29753	.52	45
16	95090	.11	29785	.52	44
17	95097	.11	29816	.52	43
18	95103	.11	29848	.52	42
19	95110	.11	29879	.52	41
20	95116	.11	29911	.53	40
21	9.95122	.11	10.29942	.53	39
22	95129	.11	29974	.53	38
23	95135	.11	30005	.53	37
24	95141	.11	30037	.53	36
25	95148	.11	30068	.53	35
26	95154	.11	30100	.53	34
27	95160	.11	30132	.53	33
28	95167	.11	30163	.53	32
29	95173	.11	30195	.53	31
30	95179	.10	30226	.53	30
31	9.95185	.10	10.30258	.53	29
32	95192	.10	30290	.53	28
33	95198	.10	30321	.53	27
34	95204	.10	30353	.53	26
35	95211	.10	30385	.53	25
36	95217	.10	30416	.53	24
37	95223	.10	30448	.53	23
38	95229	.10	30480	.53	22
39	95236	.10	30512	.53	21
40	95242	.10	30543	.53	20
41	9.95248	.10	10.30575	.53	19
42	95254	.10	30607	.53	18
43	95261	.10	30639	.53	17
44	95267	.10	30671	.53	16
45	95273	.10	30702	.53	15
46	95279	.10	30734	.53	14
47	95286	.10	30766	.53	13
48	95292	.10	30798	.53	12
49	95298	.10	30830	.53	11
50	95304	.10	30862	.53	10
51	9.95310	.10	10.30894	.53	9
52	95317	.10	30926	.53	8
53	95323	.10	30958	.53	7
54	95329	.10	30990	.53	6
55	95335	.10	31022	.53	5
56	95341	.10	31054	.53	4
57	95348	.10	31086	.53	3
58	95354	.10	31118	.53	2
59	95360	.10	31150	.53	1
60	95366		31182		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.95366	0.10	10.31182	0.53	60	0	9.95728	0.10	10.33133	0.55	60
1	95372	.10	31214	.53	59	1	95733	.10	33166	.55	59
2	95378	.10	31246	.53	58	2	95739	.10	33199	.55	58
3	95384	.10	31278	.54	57	3	95745	.10	33232	.55	57
4	95391	.10	31310	.54	56	4	95751	.10	33265	.55	56
5	95397	.10	31342	.54	55	5	95757	.10	33298	.55	55
6	95403	.10	31374	.54	54	6	95763	.10	33331	.55	54
7	95409	.10	31407	.54	53	7	95769	.10	33364	.55	53
8	95415	.10	31439	.54	52	8	95775	.10	33397	.55	52
9	95421	.10	31471	.54	51	9	95780	.10	33430	.55	51
10	95427	.10	31503	.54	50	10	95786	.10	33463	.55	50
11	9.95434	.10	10.31535	.54	49	11	9.95792	.10	10.33497	.55	49
12	95440	.10	31568	.54	48	12	95798	.10	33530	.55	48
13	95446	.10	31600	.54	47	13	95804	.10	33563	.55	47
14	95452	.10	31632	.54	46	14	95810	.10	33596	.55	46
15	95458	.10	31664	.54	45	15	95815	.10	33629	.55	45
16	95464	.10	31697	.54	44	16	95821	.10	33663	.55	44
17	95470	.10	31729	.54	43	17	95827	.10	33696	.55	43
18	95476	.10	31761	.54	42	18	95833	.10	33729	.55	42
19	95482	.10	31794	.54	41	19	95839	.10	33762	.56	41
20	95488	.10	31826	.54	40	20	95844	.10	33796	.56	40
21	9.95494	.10	10.31858	.54	39	21	9.95850	.10	10.33829	.56	39
22	95500	.10	31891	.54	38	22	95856	.10	33862	.56	38
23	95507	.10	31923	.54	37	23	95862	.10	33896	.56	37
24	95513	.10	31956	.54	36	24	95868	.10	33929	.56	36
25	95519	.10	31988	.54	35	25	95873	.10	33962	.56	35
26	95525	.10	32020	.54	34	26	95879	.10	33996	.56	34
27	95531	.10	32053	.54	33	27	95885	.10	34029	.56	33
28	95537	.10	32085	.54	32	28	95891	.10	34063	.56	32
29	95543	.10	32118	.54	31	29	95897	.10	34096	.56	31
30	95549	.10	32150	.54	30	30	95902	.10	34130	.56	30
31	9.95555	.10	10.32183	.54	29	31	9.95908	.10	10.34163	.56	29
32	95561	.10	32215	.54	28	32	95914	.10	34197	.56	28
33	95567	.10	32248	.54	27	33	95920	.10	34230	.56	27
34	95573	.10	32281	.54	26	34	95925	.10	34264	.56	26
35	95579	.10	32313	.54	25	35	95931	.10	34297	.56	25
36	95585	.10	32346	.54	24	36	95937	.10	34331	.56	24
37	95591	.10	32378	.54	23	37	95942	.10	34364	.56	23
38	95597	.10	32411	.54	22	38	95948	.10	34398	.56	22
39	95603	.10	32444	.54	21	39	95954	.10	34432	.56	21
40	95609	.10	32476	.54	20	40	95960	.10	34465	.56	20
41	9.95615	.10	10.32509	.54	19	41	9.95965	.10	10.34499	.56	19
42	95621	.10	32542	.55	18	42	95971	.09	34533	.56	18
43	95627	.10	32574	.55	17	43	95977	.09	34566	.56	17
44	95633	.10	32607	.55	16	44	95982	.09	34600	.56	16
45	95639	.10	32640	.55	15	45	95988	.09	34634	.56	15
46	95645	.10	32673	.55	14	46	95994	.09	34667	.56	14
47	95651	.10	32705	.55	13	47	96000	.09	34701	.56	13
48	95657	.10	32738	.55	12	48	96005	.09	34735	.56	12
49	95663	.10	32771	.55	11	49	96011	.09	34769	.56	11
50	95668	.10	32804	.55	10	50	96017	.09	34803	.56	10
51	9.95674	.10	10.32837	.55	9	51	9.96022	.09	10.34836	.56	9
52	95680	.10	32869	.55	8	52	96028	.09	34870	.56	8
53	95686	.10	32902	.55	7	53	96034	.09	34904	.56	7
54	95692	.10	32935	.55	6	54	96039	.09	34938	.57	6
55	95698	.10	32968	.55	5	55	96045	.09	34972	.57	5
56	95704	.10	33001	.55	4	56	96050	.09	35006	.57	4
57	95710	.10	33034	.55	3	57	96056	.09	35040	.57	3
58	95716	.10	33067	.55	2	58	96062	.09	35074	.57	2
59	95722	.10	33100	.55	1	59	96067	.09	35108	.57	1
60	95728		10.33133		0	60	96073		35142		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.96073	0.09	10.35142	0.57	60	0	9.96403	0.09	10.37215	0.59	60
1	96079	.09	35176	.57	59	1	96408	.09	37250	.59	59
2	96084	.09	35210	.57	58	2	96413	.09	37285	.59	58
3	96090	.09	35244	.57	57	3	96419	.09	37320	.59	57
4	96095	.09	35278	.57	56	4	96424	.09	37355	.59	56
5	96101	.09	35312	.57	55	5	96429	.09	37391	.59	55
6	96107	.09	35346	.57	54	6	96435	.09	37426	.59	54
7	96112	.09	35380	.57	53	7	96440	.09	37461	.59	53
8	96118	.09	35414	.57	52	8	96445	.09	37496	.59	52
9	96123	.09	35448	.57	51	9	96451	.09	37532	.59	51
10	96129	.09	35483	.57	50	10	96456	.09	37567	.59	50
11	9.96135	.09	10.35517	.57	49	11	9.96461	.09	10.37602	.59	49
12	96140	.09	35551	.57	48	12	96467	.09	37638	.59	48
13	96146	.09	35585	.57	47	13	96472	.09	37673	.59	47
14	96151	.09	35619	.57	46	14	96477	.09	37708	.59	46
15	96157	.09	35654	.57	45	15	96483	.09	37744	.59	45
16	96162	.09	35688	.57	44	16	96488	.09	37779	.59	44
17	96168	.09	35722	.57	43	17	96493	.09	37815	.59	43
18	96174	.09	35757	.57	42	18	96498	.09	37850	.59	42
19	96179	.09	35791	.57	41	19	96504	.09	37886	.59	41
20	96185	.09	35825	.57	40	20	96509	.09	37921	.59	40
21	9.96190	.09	10.35860	.57	39	21	9.96514	.09	10.37957	.59	39
22	96196	.09	35894	.57	38	22	96520	.09	37992	.59	38
23	96201	.09	35928	.57	37	23	96525	.09	38028	.59	37
24	96207	.09	35963	.57	36	24	96530	.09	38064	.59	36
25	96212	.09	35997	.57	35	25	96535	.09	38099	.59	35
26	96218	.09	36032	.57	34	26	96541	.09	38135	.59	34
27	96223	.09	36066	.58	33	27	96546	.09	38170	.59	33
28	96229	.09	36101	.58	32	28	96551	.09	38206	.60	32
29	96234	.09	36135	.58	31	29	96556	.09	38242	.60	31
30	96240	.09	36170	.58	30	30	96562	.09	38278	.60	30
31	9.96245	.09	10.36204	.58	29	31	9.96567	.09	10.38313	.60	29
32	96251	.09	36239	.58	28	32	96572	.09	38349	.60	28
33	96256	.09	36274	.58	27	33	96577	.09	38385	.60	27
34	96262	.09	36308	.58	26	34	96582	.09	38421	.60	26
35	96267	.09	36343	.58	25	35	96588	.09	38456	.60	25
36	96273	.09	56377	.58	24	36	96593	.09	38492	.60	24
37	96278	.09	36412	.58	23	37	96598	.09	38528	.60	23
38	96284	.09	36447	.58	22	38	96603	.09	38564	.60	22
39	96289	.09	36481	.58	21	39	96608	.09	38600	.60	21
40	96294	.09	36516	.58	20	40	96614	.09	38636	.60	20
41	9.96300	.09	10.36551	.58	19	41	9.96619	.09	10.38672	.60	19
42	96305	.09	36586	.58	18	42	96624	.09	38708	.60	18
43	96311	.09	36621	.58	17	43	96629	.09	38744	.60	17
44	96316	.09	36655	.58	16	44	96634	.09	38780	.60	16
45	96322	.09	36690	.58	15	45	96640	.09	38816	.60	15
46	96327	.09	36725	.58	14	46	96645	.09	38852	.60	14
47	96333	.09	36760	.58	13	47	96650	.09	38888	.60	13
48	96338	.09	36795	.58	12	48	96655	.09	38924	.60	12
49	96343	.09	36830	.58	11	49	96660	.09	38960	.60	11
50	96349	.09	36865	.58	10	50	96665	.09	38996	.60	10
51	9.96354	.09	10.36899	.58	9	51	9.96670	.09	10.39033	.60	9
52	96360	.09	36934	.58	8	52	96676	.09	39069	.60	8
53	96365	.09	36969	.58	7	53	96681	.09	39105	.60	7
54	96370	.09	37004	.58	6	54	96686	.09	39141	.60	6
55	96376	.09	37039	.58	5	55	96691	.09	39177	.60	5
56	96381	.09	37074	.58	4	56	96696	.09	39214	.60	4
57	96387	.09	37110	.58	3	57	96701	.09	39250	.61	3
58	96392	.09	37145	.58	2	58	96706	.09	39286	.61	2
59	96397	.09	37180	.59	1	59	96711	.09	39323	.61	1
60	96403		37215		0	60	96717		39359		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl"	Tang.	Dl"	M.	M.	Sine.	Dl"	Tang.	Dl"	M.
0	9.96717	0.08	10.39359	0.61	60	0	9.97015	0.08	10.41582	0.63	60
1	96722	.08	39395	.61	59	1	97020	.08	41620	.63	59
2	96727	.08	39432	.61	58	2	97025	.08	41658	.63	58
3	96732	.08	39468	.61	57	3	97030	.08	41696	.63	57
4	96737	.08	39505	.61	56	4	97035	.08	41733	.63	56
5	96742	.08	39541	.61	55	5	97039	.08	41771	.63	55
6	96747	.08	39578	.61	54	6	97044	.08	41809	.63	54
7	96752	.08	39614	.61	53	7	97049	.08	41847	.63	53
8	96757	.08	39651	.61	52	8	97054	.08	41885	.63	52
9	96762	.08	39687	.61	51	9	97059	.08	41923	.63	51
10	96767	.08	39724	.61	50	10	97063	.08	41961	.63	50
11	9.96772	.08	10.39760	.61	49	11	9.97068	.08	10.41999	.63	49
12	96778	.08	39797	.61	48	12	97073	.08	42037	.63	48
13	96783	.08	39834	.61	47	13	97078	.08	42075	.63	47
14	96788	.08	39870	.61	46	14	97083	.08	42113	.64	46
15	96793	.08	39907	.61	45	15	97087	.08	42151	.64	45
16	96798	.08	39944	.61	44	16	97092	.08	42190	.64	44
17	96803	.08	39981	.61	43	17	97097	.08	42228	.64	43
18	96808	.08	40017	.61	42	18	97102	.08	42266	.64	42
19	96813	.08	40054	.61	41	19	97107	.08	42304	.64	41
20	96818	.08	40091	.61	40	20	97111	.08	42342	.64	40
21	9.96823	.08	10.40128	.61	39	21	9.97116	.08	10.42381	.64	39
22	96828	.08	40165	.61	38	22	97121	.08	42419	.64	38
23	96833	.08	40201	.62	37	23	97126	.08	42457	.64	37
24	96838	.08	40238	.62	36	24	97130	.08	42496	.64	36
25	96843	.08	40275	.62	35	25	97135	.08	42534	.64	35
26	96848	.08	40312	.62	34	26	97140	.08	42572	.64	34
27	96853	.08	40349	.62	33	27	97145	.08	42611	.64	33
28	96858	.08	40386	.62	32	28	97149	.08	42649	.64	32
29	96863	.08	40423	.62	31	29	97154	.08	42688	.64	31
30	96868	.08	40460	.62	30	30	97159	.08	42726	.64	30
31	9.96873	.08	10.40497	.62	29	31	9.97163	.08	10.42765	.64	29
32	96878	.08	40534	.62	28	32	97168	.08	42803	.64	28
33	96883	.08	40571	.62	27	33	97173	.08	42842	.64	27
34	96888	.08	40609	.62	26	34	97178	.08	42880	.64	26
35	96893	.08	40646	.62	25	35	97182	.08	42919	.64	25
36	96898	.08	40683	.62	24	36	97187	.08	42958	.64	24
37	96903	.08	40720	.62	23	37	97192	.08	42996	.65	23
38	96907	.08	40757	.62	22	38	97196	.08	43035	.65	22
39	96912	.08	40795	.62	21	39	97201	.08	43074	.65	21
40	96917	.08	40832	.62	20	40	97206	.08	43113	.65	20
41	9.96922	.08	10.40869	.62	19	41	9.97210	.08	10.43151	.65	19
42	96927	.08	40906	.62	18	42	97215	.08	43190	.65	18
43	96932	.08	40944	.62	17	43	97220	.08	43229	.65	17
44	96937	.08	40981	.62	16	44	97224	.08	43268	.65	16
45	96942	.08	41019	.62	15	45	97229	.08	43307	.65	15
46	96947	.08	41056	.62	14	46	97234	.08	43346	.65	14
47	96952	.08	41093	.62	13	47	97238	.08	43385	.65	13
48	96957	.08	41131	.62	12	48	97243	.08	43424	.65	12
49	96962	.08	41168	.63	11	49	97248	.08	43463	.65	11
50	96966	.08	41206	.63	10	50	97252	.08	43502	.65	10
51	9.96971	.08	10.41243	.63	9	51	9.97257	.08	10.43541	.65	9
52	96976	.08	41281	.63	8	52	97262	.08	43580	.65	8
53	96981	.08	41319	.63	7	53	97266	.08	43619	.65	7
54	96986	.08	41356	.63	6	54	97271	.08	43658	.65	6
55	96991	.08	41394	.63	5	55	97276	.08	43697	.65	5
56	96996	.08	41431	.63	4	56	97280	.08	43736	.65	4
57	97001	.08	41469	.63	3	57	97285	.08	43776	.65	3
58	97005	.08	41507	.63	2	58	97289	.08	43815	.65	2
59	97010	.08	41545	.63	1	59	97294	.08	43854	.65	1
60	97015		41582		0	60	97299		43893		0
M.	Cosine.	Dl"	Cotang.	Dl"	M.	M.	Cosine.	Dl"	Cotang.	Dl"	M.

M.	Sine.	Dl"	Tang.	Dl"	M.	M.	Sine.	Dl"	Tang.	Dl"	M.
0	9.97299	0.08	10.43893	0.66	60	0	9.97567	0.07	10.46303	0.68	60
1	97303	.08	43933	.66	59	1	97571	.07	46344	.68	59
2	97308	.08	43972	.66	58	2	97576	.07	46385	.69	58
3	97312	.08	44011	.66	57	3	97580	.07	46426	.69	57
4	97317	.08	44051	.66	56	4	97584	.07	46467	.69	56
5	97322	.08	44090	.66	55	5	97589	.07	46508	.69	55
6	97326	.08	44130	.66	54	6	97593	.07	46550	.69	54
7	97331	.08	44169	.66	53	7	97597	.07	46591	.69	53
8	97335	.08	44209	.66	52	8	97602	.07	46632	.69	52
9	97340	.08	44248	.66	51	9	97606	.07	46673	.69	51
10	97344	.08	44288	.66	50	10	97610	.07	46715	.69	50
11	9.97349	.08	10.44327	.66	49	11	9.97615	.07	10.46756	.69	49
12	97353	.08	44367	.66	48	12	97619	.07	46798	.69	48
13	97358	.08	44407	.66	47	13	97623	.07	46839	.69	47
14	97363	.08	44446	.66	46	14	97628	.07	46880	.69	46
15	97367	.08	44486	.66	45	15	97632	.07	46922	.69	45
16	97372	.08	44526	.66	44	16	97636	.07	46963	.69	44
17	97376	.08	44566	.66	43	17	97640	.07	47005	.69	43
18	97381	.08	44605	.66	42	18	97645	.07	47047	.69	42
19	97385	.08	44645	.66	41	19	97649	.07	47088	.69	41
20	97390	.08	44685	.66	40	20	97653	.07	47130	.69	40
21	9.97394	.08	10.44725	.67	39	21	9.97657	.07	10.47171	.70	39
22	97399	.08	44765	.67	38	22	97662	.07	47213	.70	38
23	97403	.07	44805	.67	37	23	97666	.07	47255	.70	37
24	97408	.07	44845	.67	36	24	97670	.07	47297	.70	36
25	97412	.07	44885	.67	35	25	97674	.07	47339	.70	35
26	97417	.07	44925	.67	34	26	97679	.07	47380	.70	34
27	97421	.07	44965	.67	33	27	97683	.07	47422	.70	33
28	97426	.07	45005	.67	32	28	97687	.07	47464	.70	32
29	97430	.07	45045	.67	31	29	97691	.07	47506	.70	31
30	97435	.07	45085	.67	30	30	97696	.07	47548	.70	30
31	9.97439	.07	10.45125	.67	29	31	9.97700	.07	10.47590	.70	29
32	97444	.07	45165	.67	28	32	97704	.07	47632	.70	28
33	97448	.07	45206	.67	27	33	97708	.07	47674	.70	27
34	97453	.07	45246	.67	26	34	97713	.07	47716	.70	26
35	97457	.07	45286	.67	25	35	97717	.07	47758	.70	25
36	97461	.07	45327	.67	24	36	97721	.07	47800	.70	24
37	97466	.07	45367	.67	23	37	97725	.07	47843	.70	23
38	97470	.07	45407	.67	22	38	97729	.07	47885	.70	22
39	97475	.07	45448	.67	21	39	97734	.07	47927	.70	21
40	97479	.07	45488	.67	20	40	97738	.07	47969	.71	20
41	9.97484	.07	10.45529	.67	19	41	9.97742	.07	10.48012	.71	19
42	97488	.07	45569	.68	18	42	97746	.07	48054	.71	18
43	97492	.07	45610	.68	17	43	97750	.07	48097	.71	17
44	97497	.07	45650	.68	16	44	97754	.07	48139	.71	16
45	97501	.07	45691	.68	15	45	97759	.07	48181	.71	15
46	97506	.07	45731	.68	14	46	97763	.07	48224	.71	14
47	97510	.07	45772	.68	13	47	97767	.07	48266	.71	13
48	97515	.07	45813	.68	12	48	97771	.07	48309	.71	12
49	97519	.07	45853	.68	11	49	97775	.07	48352	.71	11
50	97523	.07	45894	.68	10	50	97779	.07	48394	.71	10
51	9.97528	.07	10.45935	.68	9	51	9.97784	.07	10.48437	.71	9
52	97532	.07	45975	.68	8	52	97788	.07	48480	.71	8
53	97536	.07	46016	.68	7	53	97792	.07	48522	.71	7
54	97541	.07	46057	.68	6	54	97796	.07	48565	.71	6
55	97545	.07	46098	.68	5	55	97800	.07	48608	.71	5
56	97550	.07	46139	.68	4	56	97804	.07	48651	.71	4
57	97554	.07	46180	.68	3	57	97808	.07	48694	.72	3
58	97558	.07	46221	.68	2	58	97812	.07	48736	.72	2
59	97563	.07	46262	.68	1	59	97817	.07	48779	.72	1
60	97567		46303		0	60	97821		48822		0
M.	Cosine.	Dl"	Cotang.	Dl"	M.	M.	Cosine.	Dl"	Cotang.	Dl"	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.97821	0.07	10.48822	0.72	60	0	9.98060	0.06	10.51466	0.75	60
1	97825	.07	48865	.72	59	1	98063	.06	51511	.75	59
2	97829	.07	48908	.72	58	2	98067	.06	51557	.75	58
3	97833	.07	48952	.72	57	3	98071	.06	51602	.76	57
4	97837	.07	48995	.72	56	4	98075	.06	51647	.76	56
5	97841	.07	49038	.72	55	5	98079	.06	51693	.76	55
6	97845	.07	49081	.72	54	6	98083	.06	51738	.76	54
7	97849	.07	49124	.72	53	7	98087	.06	51783	.76	53
8	97853	.07	49167	.72	52	8	98090	.06	51829	.76	52
9	97857	.07	49211	.72	51	9	98094	.06	51874	.76	51
10	97861	.07	49254	.72	50	10	98098	.06	51920	.76	50
11	9.97866	.07	10.49297	.72	49	11	9.98102	.06	10.51965	.76	49
12	97870	.07	49341	.72	48	12	98106	.06	52011	.76	48
13	97874	.07	49384	.72	47	13	98110	.06	52057	.76	47
14	97878	.07	49428	.72	46	14	98113	.06	52103	.76	46
15	97882	.07	49471	.73	45	15	98117	.06	52148	.76	45
16	97886	.07	49515	.73	44	16	98121	.06	52194	.76	44
17	97890	.07	49558	.73	43	17	98125	.06	52240	.76	43
18	97894	.07	49602	.73	42	18	98129	.06	52286	.77	42
19	97898	.07	49645	.73	41	19	98132	.06	52332	.77	41
20	97902	.07	49689	.73	40	20	98136	.06	52378	.77	40
21	9.97906	.07	10.49733	.73	39	21	9.98140	.06	10.52424	.77	39
22	97910	.07	49777	.73	38	22	98144	.06	52470	.77	38
23	97914	.07	49820	.73	37	23	98147	.06	52516	.77	37
24	97918	.07	49864	.73	36	24	98151	.06	52562	.77	36
25	97922	.07	49908	.73	35	25	98155	.06	52608	.77	35
26	97926	.07	49952	.73	34	26	98159	.06	52654	.77	34
27	97930	.07	49996	.73	33	27	98162	.06	52701	.77	33
28	97934	.07	50040	.73	32	28	98166	.06	52747	.77	32
29	97938	.07	50084	.73	31	29	98170	.06	52793	.77	31
30	97942	.07	50128	.73	30	30	98174	.06	52840	.77	30
31	9.97946	.07	10.50172	.74	29	31	9.98177	.06	10.52886	.77	29
32	97950	.07	50216	.74	28	32	98181	.06	52932	.77	28
33	97954	.07	50260	.74	27	33	98185	.06	52979	.78	27
34	97958	.07	50304	.74	26	34	98189	.06	53025	.78	26
35	97962	.07	50348	.74	25	35	98192	.06	53072	.78	25
36	97966	.07	50393	.74	24	36	98196	.06	53119	.78	24
37	97970	.07	50437	.74	23	37	98200	.06	53165	.78	23
38	97974	.07	50481	.74	22	38	98204	.06	53212	.78	22
39	97978	.07	50526	.74	21	39	98207	.06	53259	.78	21
40	97982	.07	50570	.74	20	40	98211	.06	53306	.78	20
41	9.97986	.07	10.50615	.74	19	41	9.98215	.06	10.53352	.78	19
42	97989	.07	50659	.74	18	42	98218	.06	53399	.78	18
43	97993	.07	50704	.74	17	43	98222	.06	53446	.78	17
44	97997	.07	50748	.74	16	44	98226	.06	53493	.78	16
45	98001	.07	50793	.74	15	45	98229	.06	53540	.78	15
46	98005	.07	50837	.74	14	46	98233	.06	53587	.78	14
47	98009	.07	50882	.75	13	47	98237	.06	53634	.79	13
48	98013	.07	50927	.75	12	48	98240	.06	53681	.79	12
49	98017	.07	50971	.75	11	49	98244	.06	53729	.79	11
50	98021	.07	51016	.75	10	50	98248	.06	53776	.79	10
51	9.98025	.06	10.51061	.75	9	51	9.98251	.06	10.53823	.79	9
52	98029	.06	51106	.75	8	52	98255	.06	53870	.79	8
53	98032	.06	51151	.75	7	53	98259	.06	53918	.79	7
54	98036	.06	51196	.75	6	54	98262	.06	53965	.79	6
55	98040	.06	51241	.75	5	55	98266	.06	54013	.79	5
56	98044	.06	51286	.75	4	56	98270	.06	54060	.79	4
57	98048	.06	51331	.75	3	57	98273	.06	54108	.79	3
58	98052	.06	51376	.75	2	58	98277	.06	54155	.79	2
59	98056	.06	51421	.75	1	59	98281	.06	54203	.79	1
60	98060		51466		0	60	98284		54250		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl"	Tang.	Dl"	M.	M.	Sine.	Dl"	Tang.	Dl"	M.
0	9.98284	0.06	10.54250	0.80	60	0	9.98494	0.06	10.57195	0.84	60
1	98288	.06	54298	.80	59	1	98498	.06	57245	.84	59
2	98291	.06	54346	.80	58	2	98501	.06	57296	.84	58
3	98295	.06	54394	.80	57	3	98505	.06	57347	.85	57
4	98299	.06	54441	.80	56	4	98508	.06	57397	.85	56
5	98302	.06	54489	.80	55	5	98511	.06	57448	.85	55
6	98306	.06	54537	.80	54	6	98515	.06	57499	.85	54
7	98309	.06	54585	.80	53	7	98518	.06	57550	.85	53
8	98313	.06	54633	.80	52	8	98521	.06	57601	.85	52
9	98317	.06	54681	.80	51	9	98525	.06	57652	.85	51
10	98320	.06	54729	.80	50	10	98528	.06	57703	.85	50
11	9.98324	.06	10.54778	.80	49	11	9.98531	.06	10.57754	.85	49
12	98327	.06	54826	.80	48	12	98535	.06	57805	.85	48
13	98331	.06	54874	.80	47	13	98538	.06	57856	.85	47
14	98334	.06	54922	.81	46	14	98541	.06	57907	.85	46
15	98338	.06	54971	.81	45	15	98545	.06	57959	.86	45
16	98342	.06	55019	.81	44	16	98548	.06	58010	.86	44
17	98345	.06	55067	.81	43	17	98551	.06	58061	.86	43
18	98349	.06	55116	.81	42	18	98555	.06	58113	.86	42
19	98352	.06	55164	.81	41	19	98558	.06	58164	.86	41
20	98356	.06	55213	.81	40	20	98561	.06	58216	.86	40
21	9.98359	.06	10.55262	.81	39	21	9.98565	.06	10.58267	.86	39
22	98363	.06	55310	.81	38	22	98568	.05	58319	.86	38
23	98366	.06	55359	.81	37	23	98571	.05	58371	.86	37
24	98370	.06	55408	.81	36	24	98574	.05	58422	.86	36
25	98373	.06	55456	.81	35	25	98578	.05	58474	.86	35
26	98377	.06	55505	.81	34	26	98581	.05	58526	.87	34
27	98381	.06	55554	.82	33	27	98584	.05	58578	.87	33
28	98384	.06	55603	.82	32	28	98588	.05	58630	.87	32
29	98388	.06	55652	.82	31	29	98591	.05	58682	.87	31
30	98391	.06	55701	.82	30	30	98594	.05	58734	.87	30
31	9.98395	.06	10.55750	.82	29	31	9.98597	.05	10.58786	.87	29
32	98398	.06	55799	.82	28	32	98601	.05	58839	.87	28
33	98402	.06	55849	.82	27	33	98604	.05	58891	.87	27
34	98405	.06	55898	.82	26	34	98607	.05	58943	.87	26
35	98409	.06	55947	.82	25	35	98610	.05	58995	.87	25
36	98412	.06	55996	.82	24	36	98614	.05	59048	.87	24
37	98415	.06	56046	.82	23	37	98617	.05	59100	.88	23
38	98419	.06	56095	.82	22	38	98620	.05	59153	.88	22
39	98422	.06	56145	.83	21	39	98623	.05	59205	.88	21
40	98426	.06	56194	.83	20	40	98627	.05	59258	.88	20
41	9.98429	.06	10.56244	.83	19	41	9.98630	.05	10.59311	.88	19
42	98433	.06	56293	.83	18	42	98633	.05	59364	.88	18
43	98436	.06	56343	.83	17	43	98636	.05	59416	.88	17
44	98440	.06	56393	.83	16	44	98640	.05	59469	.88	16
45	98443	.06	56442	.83	15	45	98643	.05	59522	.88	15
46	98447	.06	56492	.83	14	46	98646	.05	59575	.88	14
47	98450	.06	56542	.83	13	47	98649	.05	59628	.88	13
48	98453	.06	56592	.83	12	48	98652	.05	59681	.89	12
49	98457	.06	56642	.83	11	49	98656	.05	59734	.89	11
50	98460	.06	56692	.83	10	50	98659	.05	59788	.89	10
51	9.98464	.06	10.56742	.84	9	51	9.98662	.05	10.59841	.89	9
52	98467	.06	56792	.84	8	52	98665	.05	59894	.89	8
53	98471	.06	56842	.84	7	53	98668	.05	59948	.89	7
54	98474	.06	56892	.84	6	54	98671	.05	60001	.89	6
55	98477	.06	56943	.84	5	55	98675	.05	60055	.89	5
56	98481	.06	56993	.84	4	56	98678	.05	60108	.89	4
57	98484	.06	57043	.84	3	57	98681	.05	60162	.89	3
58	98488	.06	57094	.84	2	58	98684	.05	60215	.90	2
59	98491	.06	57144	.84	1	59	98687	.05	60269	.90	1
60	98494		57195		0	60	98690		60323		0
M.	Cosine.	Dl"	Cotang.	Dl"	M.	M.	Cosine.	Dl"	Cotang.	Dl"	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.98690	0.05	10.60323	0.90	60	0	9.98872	0.05	10.63664	0.96	60
1	98694	.05	60377	.90	59	1	98875	.05	63721	.96	59
2	98697	.05	60431	.90	58	2	98878	.05	63779	.96	58
3	98700	.05	60485	.90	57	3	98881	.05	63837	.96	57
4	98703	.05	60539	.90	56	4	98884	.05	63895	.97	56
5	98706	.05	60593	.90	55	5	98887	.05	63953	.97	55
6	98709	.05	60647	.90	54	6	98890	.05	64011	.97	54
7	98712	.05	60701	.90	53	7	98893	.05	64069	.97	53
8	98715	.05	60755	.91	52	8	98896	.05	64127	.97	52
9	98719	.05	60810	.91	51	9	98898	.05	64185	.97	51
10	98722	.05	60864	.91	50	10	98901	.05	64243	.97	50
11	9.98725	.05	10.60918	.91	49	11	9.98904	.05	10.64302	.97	49
12	98728	.05	60973	.91	48	12	98907	.05	64360	.98	48
13	98731	.05	61028	.91	47	13	98910	.05	64419	.98	47
14	98734	.05	61082	.91	46	14	98913	.05	64477	.98	46
15	98737	.05	61137	.91	45	15	98916	.05	64536	.98	45
16	98740	.05	61192	.91	44	16	98919	.05	64595	.98	44
17	98743	.05	61246	.91	43	17	98921	.05	64653	.98	43
18	98746	.05	61301	.92	42	18	98924	.05	64712	.98	42
19	98750	.05	61356	.92	41	19	98927	.05	64771	.98	41
20	98753	.05	61411	.92	40	20	98930	.05	64830	.98	40
21	9.98756	.05	10.61466	.92	39	21	9.98933	.05	10.64889	.99	39
22	98759	.05	61521	.92	38	22	98936	.05	64949	.99	38
23	98762	.05	61577	.92	37	23	98938	.05	65008	.99	37
24	98765	.05	61632	.92	36	24	98941	.05	65067	.99	36
25	98768	.05	61687	.92	35	25	98944	.05	65126	.99	35
26	98771	.05	61743	.92	34	26	98947	.05	65186	.99	34
27	98774	.05	61798	.93	33	27	98950	.05	65245	.99	33
28	98777	.05	61853	.93	32	28	98953	.05	65305	.99	32
29	98780	.05	61909	.93	31	29	98955	.05	65365	1.00	31
30	98783	.05	61965	.93	30	30	98958	.05	65424	1.00	30
31	9.98786	.05	10.62020	.93	29	31	9.98961	.05	10.65484	1.00	29
32	98789	.05	62076	.93	28	32	98964	.05	65544	1.00	28
33	98792	.05	62132	.93	27	33	98967	.05	65604	1.00	27
34	98795	.05	62188	.93	26	34	98969	.05	65664	1.00	26
35	98798	.05	62244	.93	25	35	98972	.05	65724	1.00	25
36	98801	.05	62300	.93	24	36	98975	.05	65785	1.00	24
37	98804	.05	62356	.94	23	37	98978	.05	65845	1.01	23
38	98807	.05	62412	.94	22	38	98980	.05	65905	1.01	22
39	98810	.05	62468	.94	21	39	98983	.05	65966	1.01	21
40	98813	.05	62524	.94	20	40	98986	.05	66026	1.01	20
41	9.98816	.05	10.62581	.94	19	41	9.98989	.05	10.66087	1.01	19
42	98819	.05	62637	.94	18	42	98991	.05	66147	1.01	18
43	98822	.05	62694	.94	17	43	98994	.05	66208	1.01	17
44	98825	.05	62750	.94	16	44	98997	.05	66269	1.01	16
45	98828	.05	62807	.94	15	45	99000	.05	66330	1.02	15
46	98831	.05	62863	.95	14	46	99002	.05	66391	1.02	14
47	98834	.05	62920	.95	13	47	99005	.05	66452	1.02	13
48	98837	.05	62977	.95	12	48	99008	.05	66513	1.02	12
49	98840	.05	63034	.95	11	49	99011	.05	66574	1.02	11
50	98843	.05	63091	.95	10	50	99013	.05	66635	1.02	10
51	9.98846	.05	10.63148	.95	9	51	9.99016	.05	10.66697	1.02	9
52	98849	.05	63205	.95	8	52	99019	.05	66758	1.03	8
53	98852	.05	63262	.95	7	53	99022	.05	66820	1.03	7
54	98855	.05	63319	.95	6	54	99024	.05	66881	1.03	6
55	98858	.05	63376	.96	5	55	99027	.05	66943	1.03	5
56	98861	.05	63434	.96	4	56	99030	.05	67005	1.03	4
57	98864	.05	63491	.96	3	57	99032	.04	67067	1.03	3
58	98867	.05	63548	.96	2	58	99035	.04	67128	1.03	2
59	98869	.05	63606	.96	1	59	99038	.04	67190	1.03	1
60	98872		63664		0	60	99040		67253		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.99040	0.04	10.67253	1.04	60	0	9.99195	0.04	10.71135	1.13	60
1	99043	.04	67315	1.04	59	1	99197	.04	71202	1.13	59
2	99046	.04	67377	1.04	58	2	99200	.04	71270	1.13	58
3	99048	.04	67439	1.04	57	3	99202	.04	71338	1.13	57
4	99051	.04	67502	1.04	56	4	99204	.04	71405	1.13	56
5	99054	.04	67564	1.04	55	5	99207	.04	71473	1.13	55
6	99056	.04	67627	1.04	54	6	99209	.04	71541	1.13	54
7	99059	.04	67689	1.05	53	7	99212	.04	71609	1.14	53
8	99062	.04	67752	1.05	52	8	99214	.04	71677	1.14	52
9	99064	.04	67815	1.05	51	9	99217	.04	71746	1.14	51
10	99067	.04	67878	1.05	50	10	99219	.04	71814	1.14	50
11	9.99070	.04	10.67941	1.05	49	11	9.99221	.04	10.71883	1.14	49
12	99072	.04	68004	1.05	48	12	99224	.04	71951	1.14	48
13	99075	.04	68067	1.05	47	13	99226	.04	72020	1.15	47
14	99078	.04	68130	1.06	46	14	99229	.04	72089	1.15	46
15	99080	.04	68194	1.06	45	15	99231	.04	72158	1.15	45
16	99083	.04	68257	1.06	44	16	99233	.04	72227	1.15	44
17	99086	.04	68321	1.06	43	17	99236	.04	72296	1.15	43
18	99088	.04	68384	1.06	42	18	99238	.04	72365	1.16	42
19	99091	.04	68448	1.06	41	19	99241	.04	72434	1.16	41
20	99093	.04	68511	1.06	40	20	99243	.04	72504	1.16	40
21	9.99096	.04	10.68575	1.07	39	21	9.99245	.04	10.72573	1.16	39
22	99099	.04	68639	1.07	38	22	99248	.04	72643	1.16	38
23	99101	.04	68703	1.07	37	23	99250	.04	72712	1.16	37
24	99104	.04	68767	1.07	36	24	99252	.04	72782	1.17	36
25	99106	.04	68832	1.07	35	25	99255	.04	72852	1.17	35
26	99109	.04	68896	1.07	34	26	99257	.04	72922	1.17	34
27	99112	.04	68960	1.07	33	27	99260	.04	72992	1.17	33
28	99114	.04	69025	1.08	32	28	99262	.04	73063	1.17	32
29	99117	.04	69089	1.08	31	29	99264	.04	73133	1.17	31
30	99119	.04	69154	1.08	30	30	99267	.04	73203	1.18	30
31	9.99122	.04	10.69218	1.08	29	31	9.99269	.04	10.73274	1.18	29
32	99124	.04	69283	1.08	28	32	99271	.04	73345	1.18	28
33	99127	.04	69348	1.08	27	33	99274	.04	73415	1.18	27
34	99130	.04	69413	1.08	26	34	99276	.04	73486	1.18	26
35	99132	.04	69478	1.09	25	35	99278	.04	73557	1.19	25
36	99135	.04	69543	1.09	24	36	99281	.04	73628	1.19	24
37	99137	.04	69609	1.09	23	37	99283	.04	73699	1.19	23
38	99140	.04	69674	1.09	22	38	99285	.04	73771	1.19	22
39	99142	.04	69739	1.09	21	39	99288	.04	73842	1.19	21
40	99145	.04	69805	1.09	20	40	99290	.04	73914	1.19	20
41	9.99147	.04	10.69870	1.10	19	41	9.99292	.04	10.73985	1.20	19
42	99150	.04	69936	1.10	18	42	99294	.04	74057	1.20	18
43	99152	.04	70002	1.10	17	43	99297	.04	74129	1.20	17
44	99155	.04	70068	1.10	16	44	99299	.04	74201	1.20	16
45	99157	.04	70134	1.10	15	45	99301	.04	74273	1.20	15
46	99160	.04	70200	1.10	14	46	99304	.04	74345	1.21	14
47	99162	.04	70266	1.10	13	47	99306	.04	74418	1.21	13
48	99165	.04	70332	1.11	12	48	99308	.04	74490	1.21	12
49	99167	.04	70399	1.11	11	49	99310	.04	74563	1.21	11
50	99170	.04	70465	1.11	10	50	99313	.04	74635	1.21	10
51	9.99172	.04	10.70532	1.11	9	51	9.99315	.04	10.74708	1.21	9
52	99175	.04	70598	1.11	8	52	99317	.04	74781	1.22	8
53	99177	.04	70665	1.11	7	53	99319	.04	74854	1.22	7
54	99180	.04	70732	1.12	6	54	99322	.04	74927	1.22	6
55	99182	.04	70799	1.12	5	55	99324	.04	75000	1.22	5
56	99185	.04	70866	1.12	4	56	99326	.04	75074	1.22	4
57	99187	.04	70933	1.12	3	57	99328	.04	75147	1.23	3
58	99190	.04	71000	1.12	2	58	99331	.04	75221	1.23	2
59	99192	.04	71067	1.12	1	59	99333	.04	75294	1.23	1
60	99195		71135		0	60	99335		75368		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.99335	0.04	10.75368	1.23	60	0	9.99462	0.03	10.80029	1.36	60
1	99337	.04	75442	1.23	59	1	99464	.03	80111	1.37	59
2	99340	.04	75516	1.24	58	2	99466	.03	80193	1.37	58
3	99342	.04	75590	1.24	57	3	99468	.03	80275	1.37	57
4	99344	.04	75665	1.24	56	4	99470	.03	80357	1.37	56
5	99346	.04	75739	1.24	55	5	99472	.03	80439	1.38	55
6	99348	.04	75814	1.24	54	6	99474	.03	80522	1.38	54
7	99351	.04	75888	1.25	53	7	99476	.03	80605	1.38	53
8	99353	.04	75963	1.25	52	8	99478	.03	80688	1.38	52
9	99355	.04	76038	1.25	51	9	99480	.03	80771	1.39	51
10	99357	.04	76113	1.25	50	10	99482	.03	80854	1.39	50
11	9.99359	.04	10.76188	1.25	49	11	9.99484	.03	10.80937	1.39	49
12	99362	.04	76263	1.26	48	12	99486	.03	81021	1.39	48
13	99364	.04	76339	1.26	47	13	99488	.03	81104	1.40	47
14	99366	.04	76414	1.26	46	14	99490	.03	81188	1.40	46
15	99368	.04	76490	1.26	45	15	99492	.03	81272	1.40	45
16	99370	.04	76565	1.26	44	16	99494	.03	81356	1.40	44
17	99372	.04	76641	1.27	43	17	99495	.03	81440	1.41	43
18	99375	.04	76717	1.27	42	18	99497	.03	81525	1.41	42
19	99377	.04	76794	1.27	41	19	99499	.03	81609	1.41	41
20	99379	.04	76870	1.27	40	20	99501	.03	81694	1.41	40
21	9.99381	.04	10.76946	1.28	39	21	9.99503	.03	10.81779	1.42	39
22	99383	.04	77023	1.28	38	22	99505	.03	81864	1.42	38
23	99385	.04	77099	1.28	37	23	99507	.03	81949	1.42	37
24	99388	.04	77176	1.28	36	24	99509	.03	82035	1.43	36
25	99390	.04	77253	1.28	35	25	99511	.03	82120	1.43	35
26	99392	.04	77330	1.29	34	26	99513	.03	82206	1.43	34
27	99394	.04	77407	1.29	33	27	99515	.03	82292	1.43	33
28	99396	.04	77484	1.29	32	28	99517	.03	82378	1.44	32
29	99398	.04	77562	1.29	31	29	99518	.03	82464	1.44	31
30	99400	.04	77639	1.29	30	30	99520	.03	82550	1.44	30
31	9.99402	.04	10.77717	1.30	29	31	9.99522	.03	10.82637	1.44	29
32	99404	.04	77795	1.30	28	32	99524	.03	82723	1.45	28
33	99407	.04	77873	1.30	27	33	99526	.03	82810	1.45	27
34	99409	.03	77951	1.30	26	34	99528	.03	82897	1.45	26
35	99411	.03	78029	1.31	25	35	99530	.03	82984	1.46	25
36	99413	.03	78107	1.31	24	36	99532	.03	83072	1.46	24
37	99415	.03	78186	1.31	23	37	99533	.03	83159	1.46	23
38	99417	.03	78264	1.31	22	38	99535	.03	83247	1.46	22
39	99419	.03	78343	1.31	21	39	99537	.03	83335	1.47	21
40	99421	.03	78422	1.32	20	40	99539	.03	83423	1.47	20
41	9.99423	.03	10.78501	1.32	19	41	9.99541	.03	10.83511	1.47	19
42	99425	.03	78580	1.32	18	42	99543	.03	83599	1.48	18
43	99427	.03	78659	1.32	17	43	99545	.03	83688	1.48	17
44	99429	.03	78739	1.33	16	44	99546	.03	83776	1.48	16
45	99432	.03	78818	1.33	15	45	99548	.03	83865	1.48	15
46	99434	.03	78898	1.33	14	46	99550	.03	83954	1.49	14
47	99436	.03	78978	1.33	13	47	99552	.03	84044	1.49	13
48	99438	.03	79058	1.34	12	48	99554	.03	84133	1.49	12
49	99440	.03	79138	1.34	11	49	99556	.03	84223	1.50	11
50	99442	.03	79218	1.34	10	50	99557	.03	84312	1.50	10
51	9.99444	.03	10.79299	1.34	9	51	9.99559	.03	10.84402	1.50	9
52	99446	.03	79379	1.34	8	52	99561	.03	84492	1.51	8
53	99448	.03	79460	1.35	7	53	99563	.03	84583	1.51	7
54	99450	.03	79541	1.35	6	54	99565	.03	84673	1.51	6
55	99452	.03	79622	1.35	5	55	99566	.03	84764	1.51	5
56	99454	.03	79703	1.35	4	56	99568	.03	84855	1.52	4
57	99456	.03	79784	1.36	3	57	99570	.03	84946	1.52	3
58	99458	.03	79866	1.36	2	58	99572	.03	85037	1.52	2
59	99460	.03	79947	1.36	1	59	99574	.03	85128	1.53	1
60	99462		80029		0	60	99575		85220		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.99575	0.03	10.85220	1.53	60	0	9.99675	0.03	10.91086	1.74	60
1	99577	.03	85312	1.53	59	1	99677	.03	91190	1.75	59
2	99579	.03	85403	1.54	58	2	99678	.03	91295	1.75	58
3	99581	.03	85496	1.54	57	3	99680	.03	91400	1.76	57
4	99582	.03	85588	1.54	56	4	99681	.03	91505	1.76	56
5	99584	.03	85680	1.55	55	5	99683	.03	91611	1.76	55
6	99586	.03	85773	1.55	54	6	99684	.03	91717	1.77	54
7	99588	.03	85866	1.55	53	7	99686	.03	91823	1.77	53
8	99589	.03	85959	1.55	52	8	99687	.03	91929	1.78	52
9	99591	.03	86052	1.56	51	9	99689	.03	92036	1.78	51
10	99593	.03	86146	1.56	50	10	99690	.03	92142	1.78	50
11	9.99595	.03	10.86239	1.56	49	11	9.99692	.03	10.92249	1.79	49
12	99596	.03	86333	1.57	48	12	99693	.03	92357	1.79	48
13	99598	.03	86427	1.57	47	13	99695	.03	92464	1.80	47
14	99600	.03	86522	1.57	46	14	99696	.02	92572	1.80	46
15	99601	.03	86616	1.58	45	15	99698	.02	92680	1.81	45
16	99603	.03	86711	1.58	44	16	99699	.02	92789	1.81	44
17	99605	.03	86806	1.58	43	17	99701	.02	92897	1.81	43
18	99607	.03	86901	1.59	42	18	99702	.02	93006	1.82	42
19	99608	.03	86996	1.59	41	19	99704	.02	93115	1.82	41
20	99610	.03	87091	1.59	40	20	99705	.02	93225	1.83	40
21	9.99612	.03	10.87187	1.60	39	21	9.99707	.02	10.93334	1.83	39
22	99613	.03	87283	1.60	38	22	99708	.02	93444	1.84	38
23	99615	.03	87379	1.60	37	23	99710	.02	93555	1.84	37
24	99617	.03	87475	1.61	36	24	99711	.02	93665	1.85	36
25	99618	.03	87572	1.61	35	25	99713	.02	93776	1.85	35
26	99620	.03	87668	1.61	34	26	99714	.02	93887	1.86	34
27	99622	.03	87765	1.62	33	27	99716	.02	93998	1.86	33
28	99624	.03	87862	1.62	32	28	99717	.02	94110	1.86	32
29	99625	.03	87960	1.63	31	29	99718	.02	94222	1.87	31
30	99627	.03	88057	1.63	30	30	99720	.02	94334	1.87	30
31	9.99629	.03	10.88155	1.63	29	31	9.99721	.02	10.94447	1.88	29
32	99630	.03	88253	1.64	28	32	99723	.02	94559	1.88	28
33	99632	.03	88351	1.64	27	33	99724	.02	94672	1.89	27
34	99633	.03	88449	1.64	26	34	99726	.02	94786	1.89	26
35	99635	.03	88548	1.65	25	35	99727	.02	94899	1.90	25
36	99637	.03	88647	1.65	24	36	99728	.02	95013	1.90	24
37	99638	.03	88746	1.65	23	37	99730	.02	95127	1.91	23
38	99640	.03	88845	1.66	22	38	99731	.02	95242	1.91	22
39	99642	.03	88944	1.66	21	39	99733	.02	95357	1.92	21
40	99643	.03	89044	1.67	20	40	99734	.02	95472	1.92	20
41	9.99645	.03	10.89144	1.67	19	41	9.99736	.02	10.95587	1.93	19
42	99647	.03	89244	1.67	18	42	99737	.02	95703	1.93	18
43	99648	.03	89344	1.68	17	43	99738	.02	95819	1.94	17
44	99650	.03	89445	1.68	16	44	99740	.02	95935	1.94	16
45	99651	.03	89546	1.68	15	45	99741	.02	96052	1.95	15
46	99653	.03	89647	1.69	14	46	99742	.02	96168	1.95	14
47	99655	.03	89748	1.69	13	47	99744	.02	96286	1.96	13
48	99656	.03	89850	1.70	12	48	99745	.02	96403	1.96	12
49	99658	.03	89951	1.70	11	49	99747	.02	96521	1.97	11
50	99659	.03	90053	1.70	10	50	99748	.02	96639	1.97	10
51	9.99661	.03	10.90155	1.71	9	51	9.99749	.02	10.96758	1.98	9
52	99663	.03	90258	1.71	8	52	99751	.02	96876	1.98	8
53	99664	.03	90360	1.71	7	53	99752	.02	96995	1.99	7
54	99666	.03	90463	1.72	6	54	99753	.02	97115	2.00	6
55	99667	.03	90566	1.72	5	55	99755	.02	97234	2.00	5
56	99669	.03	90670	1.73	4	56	99756	.02	97355	2.01	4
57	99670	.03	90773	1.73	3	57	99757	.02	97475	2.01	3
58	99672	.03	90877	1.73	2	58	99759	.02	97596	2.02	2
59	99674	.03	90981	1.74	1	59	99760	.02	97717	2.02	1
60	99675		91086		0	60	99761		97838		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.99761	0.02	10.97838	2.03	60	0	9.99834	0.02	11.05805	2.43	60
1	99763	.02	97960	2.03	59	1	99836	.02	05951	2.44	59
2	99764	.02	98082	2.04	58	2	99837	.02	06097	2.45	58
3	99765	.02	98204	2.04	57	3	99838	.02	06244	2.45	57
4	99767	.02	98327	2.05	56	4	99839	.02	06391	2.46	56
5	99768	.02	98450	2.06	55	5	99840	.02	06538	2.47	55
6	99769	.02	98573	2.06	54	6	99841	.02	06687	2.48	54
7	99771	.02	98697	2.07	53	7	99842	.02	06835	2.49	53
8	99772	.02	98821	2.07	52	8	99843	.02	06984	2.50	52
9	99773	.02	98945	2.08	51	9	99844	.02	07134	2.50	51
10	99775	.02	99070	2.09	50	10	99845	.02	07284	2.51	50
11	9.99776	.02	10.99195	2.09	49	11	9.99846	.02	11.07435	2.52	49
12	99777	.02	99321	2.10	48	12	99847	.02	07586	2.53	48
13	99778	.02	99447	2.10	47	13	99848	.02	07738	2.54	47
14	99780	.02	99573	2.11	46	14	99850	.02	07890	2.55	46
15	99781	.02	99699	2.12	45	15	99851	.02	08043	2.56	45
16	99782	.02	99826	2.12	44	16	99852	.02	08197	2.56	44
17	99783	.02	99954	2.13	43	17	99853	.02	08350	2.57	43
18	99785	.02	11.00081	2.13	42	18	99854	.02	08505	2.58	42
19	99786	.02	00209	2.14	41	19	99855	.02	08660	2.59	41
20	99787	.02	00338	2.15	40	20	99856	.02	08815	2.60	40
21	9.99788	.02	11.00466	2.15	39	21	9.99857	.02	11.08971	2.61	39
22	99790	.02	00595	2.16	38	22	99858	.02	09128	2.62	38
23	99791	.02	00725	2.16	37	23	99859	.02	09285	2.63	37
24	99792	.02	00855	2.17	36	24	99860	.02	09443	2.64	36
25	99793	.02	00985	2.18	35	25	99861	.02	09601	2.65	35
26	99795	.02	01116	2.18	34	26	99862	.02	09760	2.66	34
27	99796	.02	01247	2.19	33	27	99863	.02	09920	2.67	33
28	99797	.02	01378	2.20	32	28	99864	.02	10080	2.68	32
29	99798	.02	01510	2.20	31	29	99865	.02	10240	2.69	31
30	99800	.02	01642	2.21	30	30	99866	.02	10402	2.70	30
31	9.99801	.02	11.01775	2.22	29	31	9.99867	.02	11.10563	2.71	29
32	99802	.02	01908	2.22	28	32	99868	.02	10726	2.72	28
33	99803	.02	02041	2.23	27	33	99869	.02	10889	2.73	27
34	99804	.02	02175	2.24	26	34	99870	.02	11052	2.74	26
35	99806	.02	02309	2.24	25	35	99871	.02	11217	2.75	25
36	99807	.02	02444	2.25	24	36	99872	.02	11382	2.76	24
37	99808	.02	02579	2.26	23	37	99873	.02	11547	2.77	23
38	99809	.02	02715	2.26	22	38	99874	.02	11713	2.78	22
39	99810	.02	02850	2.27	21	39	99875	.02	11880	2.79	21
40	99812	.02	02987	2.28	20	40	99876	.02	12047	2.80	20
41	9.99813	.02	11.03123	2.29	19	41	9.99877	.02	11.12215	2.81	19
42	99814	.02	03261	2.29	18	42	99878	.02	12384	2.82	18
43	99815	.02	03398	2.30	17	43	99879	.02	12553	2.83	17
44	99816	.02	03536	2.31	16	44	99879	.02	12723	2.84	16
45	99817	.02	03675	2.31	15	45	99880	.02	12894	2.85	15
46	99819	.02	03813	2.32	14	46	99881	.02	13065	2.87	14
47	99820	.02	03953	2.33	13	47	99882	.02	13237	2.88	13
48	99821	.02	04092	2.34	12	48	99883	.02	13409	2.89	12
49	99822	.02	04233	2.34	11	49	99884	.02	13583	2.90	11
50	99823	.02	04373	2.35	10	50	99885	.02	13757	2.91	10
51	9.99824	.02	11.04514	2.36	9	51	9.99886	.02	11.13931	2.92	9
52	99825	.02	04656	2.37	8	52	99887	.02	14107	2.93	8
53	99827	.02	04798	2.37	7	53	99888	.02	14283	2.95	7
54	99828	.02	04940	2.38	6	54	99889	.02	14460	2.96	6
55	99829	.02	05083	2.39	5	55	99890	.01	14637	2.97	5
56	99830	.02	05227	2.40	4	56	99891	.01	14815	2.98	4
57	99831	.02	05370	2.41	3	57	99891	.01	14994	2.99	3
58	99832	.02	05515	2.41	2	58	99892	.01	15174	3.01	2
59	99833	.02	05660	2.42	1	59	99893	.01	15354	3.02	1
60	99834		05805		0	60	99894		15536		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.99894	0.01	11.15536	3.03	60	0	9.99940	0.01	11.28060	4.04	60
1	99895	.01	15718	3.04	59	1	99941	.01	28303	4.06	59
2	99896	.01	15900	3.06	58	2	99942	.01	28547	4.09	58
3	99897	.01	16084	3.07	57	3	99942	.01	28792	4.11	57
4	99898	.01	16268	3.08	56	4	99943	.01	29038	4.13	56
5	99898	.01	16453	3.10	55	5	99944	.01	29286	4.15	55
6	99899	.01	16639	3.11	54	6	99944	.01	29535	4.18	54
7	99900	.01	16825	3.12	53	7	99945	.01	29786	4.20	53
8	99901	.01	17013	3.14	52	8	99946	.01	30038	4.23	52
9	99902	.01	17201	3.15	51	9	99946	.01	30292	4.25	51
10	99903	.01	17390	3.16	50	10	99947	.01	30547	4.28	50
11	9.99904	.01	11.17580	3.18	49	11	9.99948	.01	11.30804	4.30	49
12	99904	.01	17770	3.19	48	12	99948	.01	31062	4.33	48
13	99905	.01	17962	3.21	47	13	99949	.01	31322	4.35	47
14	99906	.01	18154	3.22	46	14	99949	.01	31583	4.38	46
15	99907	.01	18347	3.23	45	15	99950	.01	31846	4.41	45
16	99908	.01	18541	3.25	44	16	99951	.01	32110	4.43	44
17	99909	.01	18736	3.26	43	17	99951	.01	32376	4.46	43
18	99909	.01	18932	3.28	42	18	99952	.01	32644	4.49	42
19	99910	.01	19128	3.29	41	19	99952	.01	32913	4.52	41
20	99911	.01	19326	3.31	40	20	99953	.01	33184	4.54	40
21	9.99912	.01	11.19524	3.32	39	21	9.99954	.01	11.33457	4.57	39
22	99913	.01	19723	3.34	38	22	99954	.01	33731	4.60	38
23	99913	.01	19924	3.35	37	23	99955	.01	34007	4.63	37
24	99914	.01	20125	3.37	36	24	99955	.01	34285	4.66	36
25	99915	.01	20327	3.38	35	25	99956	.01	34565	4.69	35
26	99916	.01	20530	3.40	34	26	99956	.01	34846	4.72	34
27	99917	.01	20734	3.41	33	27	99957	.01	35130	4.75	33
28	99917	.01	20939	3.43	32	28	99958	.01	35415	4.78	32
29	99918	.01	21145	3.45	31	29	99958	.01	35702	4.82	31
30	99919	.01	21351	3.46	30	30	99959	.01	35991	4.85	30
31	9.99920	.01	11.21559	3.48	29	31	9.99959	.01	11.36282	4.88	29
32	99920	.01	21768	3.50	28	32	99960	.01	36574	4.91	28
33	99921	.01	21978	3.51	27	33	99960	.01	36869	4.95	27
34	99922	.01	22189	3.53	26	34	99961	.01	37166	4.98	26
35	99923	.01	22400	3.55	25	35	99961	.01	37465	5.02	25
36	99923	.01	22613	3.57	24	36	99962	.01	37766	5.05	24
37	99924	.01	22827	3.58	23	37	99962	.01	38069	5.09	23
38	99925	.01	23042	3.60	22	38	99963	.01	38374	5.12	22
39	99926	.01	23258	3.62	21	39	99963	.01	38681	5.16	21
40	99926	.01	23475	3.64	20	40	99964	.01	38991	5.19	20
41	9.99927	.01	11.23694	3.65	19	41	9.99964	.01	11.39302	5.23	19
42	99928	.01	23913	3.67	18	42	99965	.01	39616	5.27	18
43	99929	.01	24133	3.69	17	43	99966	.01	39932	5.31	17
44	99929	.01	24355	3.71	16	44	99966	.01	40251	5.35	16
45	99930	.01	24577	3.73	15	45	99967	.01	40572	5.39	15
46	99931	.01	24801	3.75	14	46	99967	.01	40895	5.43	14
47	99932	.01	25026	3.77	13	47	99967	.01	41221	5.47	13
48	99932	.01	25252	3.79	12	48	99968	.01	41549	5.51	12
49	99933	.01	25479	3.81	11	49	99968	.01	41879	5.55	11
50	99934	.01	25708	3.83	10	50	99969	.01	42212	5.59	10
51	9.99934	.01	11.25937	3.85	9	51	9.99969	.01	11.42548	5.64	9
52	99935	.01	26168	3.87	8	52	99970	.01	42886	5.68	8
53	99936	.01	26400	3.89	7	53	99970	.01	43227	5.73	7
54	99936	.01	26634	3.91	6	54	99971	.01	43571	5.77	6
55	99937	.01	26868	3.93	5	55	99971	.01	43917	5.82	5
56	99938	.01	27104	3.95	4	56	99972	.01	44266	5.87	4
57	99938	.01	27341	3.97	3	57	99972	.01	44618	5.91	3
58	99939	.01	27580	3.40	2	58	99973	.01	44973	5.96	2
59	99940	.01	27819	4.02	1	59	99973	.01	45331	6.01	1
60	99940		28060		0	60	99974		45692		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

M.	Sine.	Dl″	Tang.	Dl″	M.	M.	Sine.	Dl″	Tang.	Dl″	M.
0	9.99974	0.01	11.45692	6.06	60	0	9.99993	.004	11.75808	12.2	60
1	99974	.01	46055	6.11	59	1	99994	.004	76538	12.4	59
2	99974	.01	46422	6.16	58	2	99994	.004	77280	12.6	58
3	99975	.01	46792	6.22	57	3	99994	.003	78036	12.8	57
4	99975	.01	47165	6.27	56	4	99994	.003	78805	13.0	56
5	99976	.01	47541	6.33	55	5	99994	.003	79587	13.3	55
6	99976	.01	47921	6.38	54	6	99995	.003	80384	13.5	54
7	99977	.01	48304	6.44	53	7	99995	.003	81196	13.8	53
8	99977	.01	48690	6.50	52	8	99995	.003	82024	14.1	52
9	99977	.01	49080	6.55	51	9	99995	.003	82867	14.3	51
10	99978	.01	49473	6.61	50	10	99995	.003	83727	14.6	50
11	9.99978	.01	11.49870	6.68	49	11	9.99996	.003	11.84605	14.9	49
12	99979	.01	50271	6.74	48	12	99996	.003	85500	15.2	48
13	99979	.01	50675	6.80	47	13	99996	.003	86415	15.6	47
14	99979	.01	51083	6.87	46	14	99996	.003	87349	15.9	46
15	99980	.01	51495	6.93	45	15	99996	.003	88304	16.3	45
16	99980	.01	51911	7.00	44	16	99996	.003	89280	16.6	44
17	99981	.01	52331	7.07	43	17	99997	.003	90278	17.0	43
18	99981	.01	52755	7.14	42	18	99997	.003	91300	17.4	42
19	99981	.01	53183	7.21	41	19	99997	.002	92347	17.9	41
20	99982	.01	53615	7.28	40	20	99997	.002	93419	18.3	40
21	9.99982	.01	11.54052	7.35	39	21	9.99997	.002	11.94519	18.8	39
22	99982	.01	54493	7.43	38	22	99997	.002	95647	19.3	38
23	99983	.01	54939	7.50	37	23	99997	.002	96806	19.8	37
24	99983	.01	55389	7.58	36	24	99998	.002	97996	20.4	36
25	99983	.01	55844	7.66	35	25	99998	.002	99219	21.0	35
26	99984	.01	56304	7.75	34	26	99998	.002	12.00478	21.6	34
27	99984	.01	56768	7.83	33	27	99998	.002	01775	22.3	33
28	99984	.01	57238	7.91	32	28	99998	.002	03111	23.0	32
29	99985	.01	57713	8.00	31	29	99998	.002	04490	23.7	31
30	99985	.01	58193	8.09	30	30	99998	.002	05914	24.5	30
31	9.99985	.01	11.58679	8.18	29	31	9.99998	.002	12.07387	25.4	29
32	99986	.01	59170	8.28	28	32	99999	.002	08911	26.3	28
33	99986	.01	59666	8.37	27	33	99999	.002	10490	27.3	27
34	99986	.01	60168	8.47	26	34	99999	.002	12129	28.4	26
35	99987	.01	60677	8.57	25	35	99999	.002	13833	29.5	25
36	99987	.01	61191	8.67	24	36	99999	.001	15606	30.8	24
37	99987	.01	61711	8.78	23	37	99999	.001	17454	32.2	23
38	99988	.005	62238	8.88	22	38	99999	.001	19385	33.7	22
39	99988	.005	62771	8.99	21	39	99999	.001	21405	35.3	21
40	99988	.005	63311	9.11	20	40	99999	.001	23524	37.1	20
41	9.99989	.005	11.63857	9.22	19	41	9.99999	.001	12.25752	39.1	19
42	99989	.005	64410	9.34	18	42	99999	.001	28100	41.4	18
43	99989	.005	64971	9.46	17	43	99999	.001	30582	43.9	17
44	99989	.005	65539	9.59	16	44	10.00000	.001	33215	46.7	16
45	99990	.005	66114	9.72	15	45	00000	.001	36018	49.9	15
46	99990	.004	66698	9.85	14	46	00000	.001	39014	53.6	14
47	99990	.004	67289	9.99	13	47	00000	.001	42233	57.9	13
48	99990	.004	67888	10.1	12	48	00000	.001	45709	63.0	12
49	99991	.004	68495	10.3	11	49	00000	.001	49488	69.0	11
50	99991	.004	69112	10.4	10	50	00000	.001	53627	76.3	10
51	9.99991	.004	11.69737	10.6	9	51	10.00000	.001	12.58203	85.3	9
52	99992	.004	70371	10.7	8	52	00000	.000	63318	96.7	8
53	99992	.004	71014	10.9	7	53	00000	.000	69118	112	7
54	99992	.004	71668	11.1	6	54	00000	.000	75812	132	6
55	99992	.004	72331	11.2	5	55	00000	.000	83730	162	5
56	99992	.004	73004	11.4	4	56	00000	.000	93421	208	4
57	99993	.004	73688	11.6	3	57	00000	.000	13.05915	294	3
58	99993	.004	74384	11.8	2	58	00000	.000	23524	502	2
59	99993	.004	75090	12.0	1	59	00000		53627		1
60	99993		75808		0	60	00000		Infinite.		0
M.	Cosine.	Dl″	Cotang.	Dl″	M.	M.	Cosine.	Dl″	Cotang.	Dl″	M.

D.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	D.
	0° 0′		0° 15′		0° 30′		0° 45′		
1	1.000	.000	1.000	.004	1.000	.009	1.000	.013	1
2	2.000	.000	2.000	.009	2.000	.018	2.000	.026	2
3	3.000	.000	3.000	.013	3.000	.026	3.000	.039	3
4	4.000	.000	4.000	.018	4.000	.035	4.000	.052	4
5	5.000	.000	5.000	.022	5.000	.044	5.000	.065	5
6	6.000	.000	6.000	.026	6.000	.052	5.999	.079	6
7	7.000	.000	7.000	.031	7.000	.061	6.999	.092	7
8	8.000	.000	8.000	.035	8.000	.070	7.999	.105	8
9	9.000	.000	9.000	.039	9.000	.079	8.999	.118	9
10	10.000	.000	10.000	.044	10.000	.087	9.999	.131	10
	90° 0′		89° 45′		89° 30′		89° 15′		
	1° 0′		1° 15′		1° 30′		1° 45′		
1	1.000	.017	1.000	.022	1.000	.026	1.000	.031	1
2	2.000	.035	2.000	.044	1.999	.052	1.999	.061	2
3	3.000	.052	2.999	.065	2.999	.079	2.999	.092	3
4	3.999	.070	3.999	.087	3.999	.105	3.998	.122	4
5	4.999	.087	4.999	.109	4.998	.131	4.998	.153	5
6	5.999	.105	5.999	.131	5.998	.157	5.997	.183	6
7	6.999	.122	6.998	.153	6.998	.183	6.997	.214	7
8	7.999	.140	7.998	.175	7.997	.209	7.996	.244	8
9	8.999	.157	8.998	.196	8.997	.236	8.996	.275	9
10	9.999	.174	9.998	.218	9.997	.262	9.995	.305	10
	89° 0′		88° 45′		88° 30′		88° 15′		
	2° 0′		2° 15′		2° 30′		2° 45′		
1	.999	.035	.999	.039	.999	.044	.999	.048	1
2	1.999	.070	1.999	.079	1.998	.087	1.998	.096	2
3	2.998	.105	2.998	.118	2.997	.131	2.997	.144	3
4	3.998	.140	3.997	.157	3.996	.174	3.995	.192	4
5	4.997	.174	4.996	.196	4.995	.218	4.994	.240	5
6	5.996	.209	5.995	.236	5.994	.262	5.993	.288	6
7	6.996	.244	6.995	.275	6.993	.305	6.992	.336	7
8	7.995	.279	7.994	.314	7.992	.349	7.991	.384	8
9	8.995	.314	8.993	.353	8.991	.393	8.990	.432	9
10	9.994	.349	9.992	.393	9.990	.436	9.988	.480	10
	88° 0′		87° 45′		87° 30′		87° 15′		
	3° 0′		3° 15′		3° 30′		3° 45′		
1	.999	.052	.998	.057	.998	.061	.998	.065	1
2	1.997	.105	1.997	.113	1.996	.122	1.996	.131	2
3	2.996	.157	2.995	.170	2.994	.183	2.994	.196	3
4	3.995	.209	3.994	.227	3.993	.244	3.991	.262	4
5	4.993	.262	4.992	.283	4.991	.305	4.989	.327	5
6	5.992	.314	5.990	.340	5.989	.366	5.987	.392	6
7	6.990	.366	6.989	.397	6.987	.427	6.985	.458	7
8	7.989	.419	7.987	.454	7.985	.488	7.983	.523	8
9	8.988	.471	8.986	.510	8.983	.549	8.981	.589	9
10	9.986	.523	9.984	.567	9.981	.610	9.979	.654	10
	87° 0′		86° 45′		86° 30′		86° 15′		
D.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	D.

D.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	D.
	4° 0′		4° 15′		4° 30′		4° 45′		
1	.998	.070	.997	.074	.997	.078	.997	.083	1
2	1.995	.140	1.995	.148	1.994	.157	1.993	.166	2
3	2.993	.209	2.992	.222	2.991	.235	2.990	.248	3
4	3.990	.279	3.989	.296	3.988	.314	3.986	.331	4
5	4.988	.349	4.986	.371	4.985	.392	4.983	.414	5
6	5.986	.418	5.984	.445	5.981	.471	5.979	.497	6
7	6.983	.488	6.981	.519	6.978	.549	6.976	.580	7
8	7.981	.558	7.978	.593	7.975	.628	7.973	.662	8
9	8.978	.628	8.975	.667	8.972	.706	8.969	.745	9
10	9.976	.698	9.973	.741	9.969	.785	9.966	.828	10
	86° 0′		85° 45′		85° 30′		85° 15′		
	5° 0′		5° 15′		5° 30′		5° 45′		
1	.996	.087	.996	.092	.995	.096	.995	.100	1
2	1.992	.174	1.992	.183	1.991	.192	1.990	.200	2
3	2.989	.261	2.987	.275	2.986	.288	2.985	.301	3
4	3.985	.349	3.983	.366	3.982	.383	3.980	.401	4
5	4.981	.436	4.979	.458	4.977	.479	4.975	.501	5
6	5.977	.523	5.975	.549	5.972	.575	5.970	.601	6
7	6.973	.610	6.971	.641	6.968	.671	6.965	.701	7
8	7.970	.697	7.966	.732	7.963	.767	7.960	.802	8
9	8.966	.784	8.962	.824	8.959	.863	8.955	.902	9
10	9.962	.872	9.958	.915	9.954	.958	9.950	1.002	10
	85° 0′		84° 45′		84° 30′		84° 15′		
	6° 0′		6° 15′		6° 30′		6° 45′		
1	.995	.105	.994	.109	.994	.113	.993	.118	1
2	1.989	.209	1.988	.218	1.987	.226	1.986	.235	2
3	2.984	.314	2.982	.327	2.981	.340	2.979	.353	3
4	3.978	.418	3.976	.435	3.974	.453	3.972	.470	4
5	4.973	.523	4.970	.544	4.968	.566	4.965	.588	5
6	5.967	.627	5.964	.653	5.961	.679	5.958	.705	6
7	6.962	.732	6.958	.762	6.955	.792	6.952	.823	7
8	7.956	.836	7.952	.871	7.949	.906	7.945	.940	8
9	8.951	.941	8.947	.980	8.942	1.019	8.938	1.058	9
10	9.945	1.045	9.941	1.089	9.936	1.132	9.931	1.175	10
	84° 0′		83° 45′		83° 30′		83° 15′		
	7° 0′		7° 15′		7° 30′		7° 45′		
1	.993	.122	.992	.126	.991	.131	.991	.135	1
2	1.985	.244	1.984	.252	1.983	.261	1.982	.270	2
3	2.978	.366	2.976	.379	2.974	.392	2.973	.405	3
4	3.970	.487	3.968	.505	3.966	.522	3.963	.539	4
5	4.963	.609	4.960	.631	4.957	.653	4.954	.674	5
6	5.955	.731	5.952	.757	5.949	.783	5.945	.809	6
7	6.948	.853	6.944	.883	6.940	.914	6.936	.944	7
8	7.940	.975	7.936	1.010	7.932	1.044	7.927	1.079	8
9	8.933	1.097	8.928	1.136	8.923	1.175	8.918	1.214	9
10	9.925	1.219	9.920	1.262	9.914	1.305	9.909	1.349	10
	83° 0′		82° 45′		82° 30′		82° 15′		
D.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	D.

D.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	D.
	8°	0′	8°	15′	8°	30′	8°	45′	
1	.990	.139	.990	.143	.989	.148	.988	.152	1
2	1.981	.278	1.979	.287	1.978	.296	1.977	.304	2
3	2.971	.418	2.969	.431	2.967	.443	2.965	.456	3
4	3.961	.557	3.959	.574	3.956	.591	3.953	.608	4
5	4.951	.696	4.948	.717	4.945	.739	4.942	.761	5
6	5.942	.835	5.938	.861	5.934	.887	5.930	.913	6
7	6.932	.974	6.928	1.004	6.923	1.035	6.919	1.065	7
8	7.922	1.113	7.917	1.148	7.912	1.182	7.907	1.217	8
9	8.912	1.253	8.907	1.291	8.901	1.330	8.895	1.369	9
10	9.903	1.392	9.897	1.435	9.890	1.478	9.884	1.521	10
	82°	0′	81°	45′	81°	30′	81°	15′	
	9°	0′	9°	15′	9°	30′	9°	45′	
1	.988	.156	.987	.161	.986	.165	.986	.169	1
2	1.975	.313	1.974	.321	1.973	.330	1.971	.339	2
3	2.963	.469	2.961	.482	2.959	.495	2.957	.508	3
4	3.951	.626	3.948	.643	3.945	.660	3.942	.677	4
5	4.938	.782	4.935	.804	4.931	.825	4.928	.847	5
6	5.926	.939	5.922	.964	5.918	.990	5.914	1.016	6
7	6.914	1.095	6.909	1.125	6.904	1.155	6.899	1.185	7
8	7.902	1.251	7.896	1.286	7.890	1.320	7.884	1.355	8
9	8.889	1.408	8.883	1.447	8.877	1.485	8.870	1.524	9
10	9.877	1.564	9.870	1.607	9.863	1.650	9.856	1.693	10
	81°	0′	80°	45′	80°	30′	80°	15′	
	10°	0′	10°	15′	10°	30′	10°	45′	
1	.985	.174	.984	.178	.983	.182	.982	.187	1
2	1.970	.347	1.968	.356	1.967	.364	1.965	.373	2
3	2.954	.521	2.952	.534	2.950	.547	2.947	.560	3
4	3.939	.695	3.936	.712	3.933	.729	3.930	.746	4
5	4.924	.868	4.920	.890	4.916	.911	4.912	.933	5
6	5.909	1.042	5.904	1.068	5.900	1.093	5.895	1.119	6
7	6.894	1.216	6.888	1.246	6.883	1.276	6.877	1.306	7
8	7.878	1.389	7.872	1.424	7.866	1.458	7.860	1.492	8
9	8.863	1.563	8.856	1.601	8.849	1.640	8.842	1.679	9
10	9.848	1.736	9.840	1.779	9.833	1.822	9.825	1.865	10
	80°	0′	79°	45′	79°	30′	79°	15′	
	11°	0′	11°	15′	11°	30′	11°	45′	
1	.982	.191	.981	.195	.980	.199	.979	.204	1
2	1.963	.382	1.962	.390	1.960	.399	1.958	.407	2
3	2.945	.572	2.942	.585	2.940	.598	2.937	.611	3
4	3.927	.763	3.923	.780	3.920	.797	3.916	.815	4
5	4.908	.954	4.904	.976	4.900	.997	4.895	1.018	5
6	5.890	1.145	5.885	1.171	5.880	1.196	5.874	1.222	6
7	6.871	1.336	6.866	1.366	6.860	1.396	6.853	1.426	7
8	7.853	1.526	7.846	1.561	7.839	1.595	7.832	1.629	8
9	8.835	1.717	8.827	1.756	8.819	1.794	8.811	1.833	9
10	9.816	1.908	9.808	1.951	9.799	1.994	9.790	2.036	10
	79°	0′	78°	45′	78°	30′	78°	15′	
D.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	D.

D.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	D.
	12° 0′		**12° 15′**		**12° 30′**		**12° 45′**		
1	.978	.208	.977	.212	.976	.216	.975	.221	1
2	1.956	.416	1.954	.424	1.953	.433	1.951	.441	2
3	2.934	.624	2.932	.637	2.929	.649	2.926	.662	3
4	3.913	.832	3.909	.849	3.906	.866	3.901	.883	4
5	4.891	1.040	4.886	1.061	4.882	1.082	4.877	1.103	5
6	5.869	1.247	5.863	1.273	5.858	1.299	5.852	1.324	6
7	6.847	1.455	6.841	1.485	6.834	1.515	6.827	1.545	7
8	7.825	1.663	7.818	1.697	7.810	1.731	7.803	1.766	8
9	8.803	1.871	8.795	1.910	8.787	1.948	8.778	1.986	9
10	9.781	2.079	9.772	2.122	9.763	2.164	9.753	2.207	10
	78° 0′		**77° 45′**		**77° 30′**		**77° 15′**		
	13° 0′		**13° 15′**		**13° 30′**		**13° 45′**		
1	.974	.225	.973	.229	.972	.233	.971	.238	1
2	1.949	.450	1.947	.458	1.945	.467	1.943	.475	2
3	2.923	.675	2.920	.688	2.917	.700	2.914	.713	3
4	3.897	.900	3.894	.917	3.889	.934	3.885	.951	4
5	4.872	1.125	4.867	1.146	4.862	1.167	4.857	1.188	5
6	5.846	1.350	5.840	1.375	5.834	1.401	5.828	1.426	6
7	6.821	1.575	6.814	1.604	6.807	1.634	6.799	1.664	7
8	7.795	1.800	7.787	1.834	7.779	1.868	7.771	1.901	8
9	8.769	2.025	8.760	2.063	8.751	2.101	8.742	2.139	9
10	9.744	2.250	9.734	2.292	9.724	2.334	9.713	2.377	10
	77° 0′		**76° 45′**		**76° 30′**		**76° 15′**		
	14° 0′		**14° 15′**		**14° 30′**		**14° 45′**		
1	.970	.242	.969	.246	.968	.250	.967	.255	1
2	1.941	.484	1.938	.492	1.936	.501	1.934	.509	2
3	2.911	.726	2.908	.738	2.904	.751	2.901	.764	3
4	3.881	.968	3.877	.985	3.873	1.002	3.868	1.018	4
5	4.851	1.210	4.846	1.231	4.841	1.252	4.835	1.273	5
6	5.822	1.452	5.815	1.477	5.809	1.502	5.802	1.528	6
7	6.792	1.693	6.785	1.723	6.777	1.753	6.769	1.782	7
8	7.762	1.935	7.754	1.969	7.745	2.003	7.736	2.037	8
9	8.733	2.177	8.723	2.215	8.713	2.253	8.703	2.291	9
10	9.703	2.419	9.692	2.462	9.681	2.504	9.670	2.546	10
	76° 0′		**75° 45′**		**75° 30′**		**75° 15′**		
	15° 0′		**15° 15′**		**15° 30′**		**15° 45′**		
1	.966	.259	.965	.263	.964	.267	.962	.271	1
2	1.932	.518	1.930	.526	1.927	.534	1.925	.543	2
3	2.898	.776	2.894	.789	2.891	.802	2.887	.814	3
4	3.864	1.035	3.859	1.052	3.855	1.069	3.850	1.086	4
5	4.830	1.294	4.824	1.315	4.818	1.336	4.812	1.357	5
6	5.796	1.553	5.789	1.578	5.782	1.603	5.775	1.629	6
7	6.761	1.812	6.754	1.841	6.745	1.871	6.737	1.900	7
8	7.727	2.071	7.718	2.104	7.709	2.138	7.700	2.172	8
9	8.693	2.329	8.683	2.367	8.673	2.405	8.662	2.443	9
10	9.659	2.588	9.648	2.630	9.636	2.672	9.625	2.714	10
	75° 0′		**74° 45′**		**74° 30′**		**74° 15′**		
D.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	D.

D.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	D.
	16° 0′		**16° 15′**		**16° 30′**		**16° 45′**		
1	.961	.276	.960	.280	.959	.284	.958	.288	1
2	1.923	.551	1.920	.560	1.918	.568	1.915	.576	2
3	2.884	.827	2.880	.839	2.876	.852	2.873	.865	3
4	3.845	1.103	3.840	1.119	3.835	1.136	3.830	1.153	4
5	4.806	1.378	4.800	1.399	4.794	1.420	4.788	1.441	5
6	5.768	1.654	5.760	1.679	5.753	1.704	5.745	1.729	6
7	6.729	1.929	6.720	1.959	6.712	1.988	6.703	2.017	7
8	7.690	2.205	7.680	2.239	7.671	2.272	7.661	2.306	8
9	8.651	2.481	8.640	2.518	8.629	2.556	8.618	2.594	9
10	9.613	2.756	9.600	2.798	9.588	2.840	9.576	2.882	10
	74° 0′		**73° 45′**		**73° 30′**		**73° 15′**		
	17° 0′		**17° 15′**		**17° 30′**		**17° 45′**		
1	.956	.292	.955	.297	.954	.301	.952	.305	1
2	1.913	.585	1.910	.593	1.907	.601	1.905	.610	2
3	2.869	.877	2.865	.890	2.861	.902	2.857	.915	3
4	3.825	1.169	3.820	1.186	3.815	1.203	3.810	1.219	4
5	4.782	1.462	4.775	1.483	4.769	1.504	4.762	1.524	5
6	5.738	1.754	5.730	1.779	5.722	1.804	5.714	1.829	6
7	6.694	2.047	6.685	2.076	6.676	2.105	6.667	2.134	7
8	7.650	2.339	7.640	2.372	7.630	2.406	7.619	2.439	8
9	8.607	2.631	8.595	2.669	8.583	2.707	8.572	2.744	9
10	9.563	2.924	9.550	2.965	9.537	3.007	9.524	3.049	10
	73° 0′		**72° 45′**		**72° 30′**		**72° 15′**		
	18° 0′		**18° 15′**		**18° 30′**		**18° 45′**		
1	.951	.309	.950	.313	.948	.317	.947	.321	1
2	1.902	.618	1.899	.626	1.897	.635	1.894	.643	2
3	2.853	.927	2.849	.939	2.845	.952	2.841	.964	3
4	3.804	1.236	3.799	1.253	3.793	1.269	3.788	1.286	4
5	4.755	1.545	4.748	1.566	4.742	1.587	4.735	1.607	5
6	5.706	1.854	5.698	1.879	5.690	1.904	5.682	1.929	6
7	6.657	2.163	6.648	2.192	6.638	2.221	6.628	2.250	7
8	7.608	2.472	7.598	2.505	7.587	2.538	7.575	2.572	8
9	8.559	2.781	8.547	2.818	8.535	2.856	8.522	2.893	9
10	9.511	3.090	9.497	3.132	9.483	3.173	9.469	3.214	10
	72° 0′		**71° 45′**		**71° 30′**		**71° 15′**		
	19° 0′		**19° 15′**		**19° 30′**		**19° 45′**		
1	.946	.326	.944	.330	.943	.334	.941	.338	1
2	1.891	.651	1.888	.659	1.885	.668	1.882	.676	2
3	2.837	.977	2.832	.989	2.828	1.001	2.824	1.014	3
4	3.782	1.302	3.776	1.319	3.771	1.335	3.765	1.352	4
5	4.728	1.628	4.720	1.648	4.713	1.669	4.706	1.690	5
6	5.673	1.953	5.665	1.978	5.656	2.003	5.647	2.027	6
7	6.619	2.279	6.609	2.308	6.598	2.337	6.588	2.365	7
8	7.564	2.605	7.553	2.638	7.541	2.670	7.529	2.703	8
9	8.510	2.930	8.497	2.967	8.484	3.004	8.471	3.041	9
10	9.455	3.256	9.441	3.297	9.426	3.338	9.412	3.379	10
	71° 0′		**70° 45′**		**70° 30′**		**70° 15′**		
D.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	D.

D.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	D.
	20°	0′	20°	15′	20°	30′	20°	45′	
1	.940	.342	.938	.346	.937	.350	.935	.354	1
2	1.879	.684	1.876	.692	1.873	.700	1.870	.709	2
3	2.819	1.026	2.815	1.038	2.810	1.051	2.805	1.063	3
4	3.759	1.368	3.753	1.384	3.747	1.401	3.740	1.417	4
5	4.698	1.710	4.691	1.731	4.683	1.751	4.676	1.771	5
6	5.638	2.052	5.629	2.077	5.620	2.101	5.611	2.126	6
7	6.578	2.394	6.567	2.423	6.557	2.451	6.546	2.480	7
8	7.518	2.736	7.506	2.769	7.493	2.802	7.481	2.834	8
9	8.457	3.078	8.444	3.115	8.430	3.152	8.416	3.189	9
10	9.397	3.420	9.382	3.461	9.367	3.502	9.351	3.543	10
	70°	0′	69°	45′	69°	30′	69°	15′	
	21°	0′	21°	15′	21°	30′	21°	45′	
1	.934	.358	.932	.362	.930	.367	.929	.371	1
2	1.867	.717	1.864	.725	1.861	.733	1.858	.741	2
3	2.801	1.075	2.796	1.087	2.791	1.100	2.786	1.112	3
4	3.734	1.433	3.728	1.450	3.722	1.466	3.715	1.482	4
5	4.668	1.792	4.660	1.812	4.652	1.833	4.644	1.853	5
6	5.601	2.150	5.592	2.175	5.582	2.199	5.573	2.223	6
7	6.535	2.509	6.524	2.537	6.513	2.566	6.502	2.594	7
8	7.469	2.867	7.456	2.900	7.443	2.932	7.430	2.964	8
9	8.402	3.225	8.388	3.262	8.374	3.299	8.359	3.335	9
10	9.336	3.584	9.320	3.624	9.304	3.665	9.288	3.706	10
	69°	0′	68°	45′	68°	30′	68°	15′	
	22°	0′	22°	15′	22°	30′	22°	45′	
1	.927	.375	.926	.379	.924	.383	.922	.387	1
2	1.854	.749	1.851	.757	1.848	.765	1.844	.773	2
3	2.782	1.124	2.777	1.136	2.772	1.148	2.767	1.160	3
4	3.709	1.498	3.702	1.515	3.696	1.531	3.689	1.547	4
5	4.636	1.873	4.628	1.893	4.619	1.913	4.611	1.934	5
6	5.563	2.248	5.553	2.272	5.543	2.296	5.533	2.320	6
7	6.490	2.622	6.479	2.651	6.467	2.679	6.455	2.707	7
8	7.418	2.997	7.404	3.029	7.391	3.062	7.378	3.094	8
9	8.345	3.371	8.330	3.408	8.315	3.444	8.300	3.480	9
10	9.272	3.746	9.255	3.786	9.239	3.827	9.222	3.867	10
	68°	0′	67°	45′	67°	30′	67°	15′	
	23°	0′	23°	15′	23°	30′	23°	45′	
1	.921	.391	.919	.395	.917	.399	.915	.403	1
2	1.841	.781	1.838	.789	1.834	.797	1.831	.805	2
3	2.762	1.172	2.756	1.184	2.751	1.196	2.746	1.208	3
4	3.682	1.563	3.675	1.579	3.668	1.595	3.661	1.611	4
5	4.603	1.954	4.594	1.974	4.585	1.994	4.577	2.014	5
6	5.523	2.344	5.513	2.368	5.502	2.392	5.492	2.416	6
7	6.444	2.735	6.432	2.763	6.419	2.791	6.407	2.819	7
8	7.364	3.126	7.350	3.158	7.336	3.190	7.322	3.222	8
9	8.285	3.517	8.269	3.553	8.254	3.589	8.238	3.625	9
10	9.205	3.907	9.188	3.947	9.171	3.987	9.153	4.027	10
	67°	0′	66°	45′	66°	30′	66°	15′	
D.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	D.

D.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	D.
	24° 0′		**24° 15′**		**24° 30′**		**24° 45′**		
1	.914	.407	.912	.411	.910	.415	.908	.419	1
2	1.827	.813	1.824	.821	1.820	.829	1.816	.837	2
3	2.741	1.220	2.735	1.232	2.730	1.244	2.724	1.256	3
4	3.654	1.627	3.647	1.643	3.640	1.659	3.633	1.675	4
5	4.568	2.034	4.559	2.054	4.550	2.073	4.541	2.093	5
6	5.481	2.440	5.471	2.464	5.460	2.488	5.449	2.512	6
7	6.395	2.847	6.382	2.875	6.370	2.903	6.357	2.931	7
8	7.308	3.254	7.294	3.286	7.280	3.318	7.265	3.349	8
9	8.222	3.661	8.206	3.696	8.190	3.732	8.173	3.768	9
10	9.135	4.067	9.118	4.107	9.100	4.147	9.081	4.187	10
	66° 0′		**65° 45′**		**65° 30′**		**65° 15′**		
	25° 0′		**25° 15′**		**25° 30′**		**25° 45′**		
1	.906	.423	.904	.427	.903	.431	.901	.434	1
2	1.813	.845	1.809	.853	1.805	.861	1.801	.869	2
3	2.719	1.268	2.713	1.280	2.708	1.292	2.702	1.303	3
4	3.625	1.690	3.618	1.706	3.610	1.722	3.603	1.738	4
5	4.532	2.113	4.522	2.133	4.513	2.153	4.504	2.172	5
6	5.438	2.536	5.427	2.559	5.416	2.583	5.404	2.607	6
7	6.344	2.958	6.331	2.986	6.318	3.014	6.305	3.041	7
8	7.250	3.381	7.236	3.413	7.221	3.444	7.206	3.476	8
9	8.157	3.804	8.140	3.839	8.123	3.875	8.106	3.910	9
10	9.063	4.226	9.045	4.266	9.026	4.305	9.007	4.344	10
	65° 0′		**64° 45′**		**64° 30′**		**64° 15′**		
	26° 0′		**26° 15′**		**26° 30′**		**26° 45′**		
1	.899	.438	.897	.442	.895	.446	.893	.450	1
2	1.798	.877	1.794	.885	1.790	.892	1.786	.900	2
3	2.696	1.315	2.691	1.327	2.685	1.339	2.679	1.350	3
4	3.595	1.753	3.587	1.769	3.580	1.785	3.572	1.800	4
5	4.494	2.192	4.484	2.211	4.475	2.231	4.465	2.250	5
6	5.393	2.630	5.381	2.654	5.370	2.677	5.358	2.701	6
7	6.292	3.069	6.278	3.096	6.265	3.123	6.251	3.151	7
8	7.190	3.507	7.175	3.538	7.159	3.570	7.144	3.601	8
9	8.089	3.945	8.072	3.981	8.054	4.016	8.037	4.051	9
10	8.988	4.384	8.969	4.423	8.949	4.462	8.930	4.501	10
	64° 0′		**63° 45′**		**63° 30′**		**63° 15′**		
	27° 0′		**27° 15′**		**27° 30′**		**27° 45′**		
1	.891	.454	.889	.458	.887	.462	.885	.466	1
2	1.782	.908	1.778	.916	1.774	.923	1.770	.931	2
3	2.673	1.362	2.667	1.374	2.661	1.385	2.655	1.397	3
4	3.564	1.816	3.556	1.831	3.548	1.847	3.540	1.862	4
5	4.455	2.270	4.445	2.289	4.435	2.309	4.425	2.328	5
6	5.346	2.724	5.334	2.747	5.322	2.770	5.310	2.794	6
7	6.237	3.178	6.223	3.205	6.209	3.232	6.195	3.259	7
8	7.128	3.632	7.112	3.663	7.096	3.694	7.080	3.725	8
9	8.019	4.086	8.001	4.121	7.983	4.156	7.965	4.190	9
10	8.910	4.540	8.890	4.579	8.870	4.617	8.850	4.656	10
	63° 0′		**62° 45′**		**62° 30′**		**62° 15′**		
D.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	D.

D.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	D.
	28° 0′		**28° 15′**		**28° 30′**		**28° 45′**		
1	.883	.469	.881	.473	.879	.477	.877	.481	1
2	1.766	.939	1.762	.947	1.758	.954	1.753	.962	2
3	2.649	1.408	2.643	1.420	2.636	1.431	2.630	1.443	3
4	3.532	1.878	3.524	1.893	3.515	1.909	3.507	1.924	4
5	4.415	2.347	4.404	2.367	4.394	2.386	4.384	2.405	5
6	5.298	2.817	5.285	2.840	5.273	2.863	5.260	2.886	6
7	6.181	3.286	6.166	3.313	6.152	3.340	6.137	3.367	7
8	7.064	3.756	7.047	3.787	7.031	3.817	7.014	3.848	8
9	7.947	4.225	7.928	4.260	7.909	4.294	7.890	4.329	9
10	8.829	4.695	8.809	4.733	8.788	4.772	8.767	4.810	10
	62° 0′		**61° 45′**		**61° 30′**		**61° 15′**		
	29° 0′		**29° 15′**		**29° 30′**		**29° 45′**		
1	.875	.485	.872	.489	.870	.492	.868	.496	1
2	1.749	.970	1.745	.977	1.741	.985	1.736	.992	2
3	2.624	1.454	2.617	1.466	2.611	1.477	2.605	1.489	3
4	3.498	1.939	3.490	1.954	3.481	1.970	3.473	1.985	4
5	4.373	2.424	4.362	2.443	4.352	2.462	4.341	2.481	5
6	5.248	2.909	5.235	2.932	5.222	2.954	5.209	2.977	6
7	6.122	3.394	6.107	3.420	6.092	3.447	6.077	3.473	7
8	6.997	3.878	6.980	3.909	6.963	3.939	6.946	3.970	8
9	7.872	4.363	7.852	4.398	7.833	4.432	7.814	4.466	9
10	8.746	4.848	8.725	4.886	8.704	4.924	8.682	4.962	10
	61° 0′		**60° 45′**		**60° 30′**		**60° 15′**		
	30° 0′		**30° 15′**		**30° 30′**		**30° 45′**		
1	.866	.500	.864	.504	.862	.508	.859	.511	1
2	1.732	1.000	1.728	1.008	1.723	1.015	1.719	1.023	2
3	2.598	1.500	2.592	1.511	2.585	1.523	2.578	1.534	3
4	3.464	2.000	3.455	2.015	3.446	2.030	3.438	2.045	4
5	4.330	2.500	4.319	2.519	4.308	2.538	4.297	2.556	5
6	5.196	3.000	5.183	3.023	5.170	3.045	5.156	3.068	6
7	6.062	3.500	6.047	3.526	6.031	3.553	6.016	3.579	7
8	6.928	4.000	6.911	4.030	6.893	4.060	6.875	4.090	8
9	7.794	4.500	7.775	4.534	7.755	4.568	7.735	4.602	9
10	8.660	5.000	8.638	5.038	8.616	5.075	8.594	5.113	10
	60° 0′		**59° 45′**		**59° 30′**		**59° 15′**		
	31° 0′		**31° 15′**		**31° 30′**		**31° 45′**		
1	.857	.515	.855	.519	.853	.522	.850	.526	1
2	1.714	1.030	1.710	1.038	1.705	1.045	1.701	1.052	2
3	2.572	1.545	2.565	1.556	2.558	1.567	2.551	1.579	3
4	3.429	2.060	3.420	2.075	3.411	2.090	3.401	2.105	4
5	4.286	2.575	4.275	2.594	4.263	2.612	4.252	2.631	5
6	5.143	3.090	5.129	3.113	5.116	3.135	5.102	3.157	6
7	6.000	3.605	5.984	3.631	5.968	3.657	5.952	3.683	7
8	6.857	4.120	6.839	4.150	6.821	4.180	6.803	4.210	8
9	7.715	4.635	7.694	4.669	7.674	4.702	7.653	4.736	9
10	8.572	5.150	8.549	5.188	8.526	5.225	8.504	5.262	10
	59° 0′		**58° 45′**		**58° 30′**		**58° 15′**		
D.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	D.

D.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	D.
	32°	0′	32°	15′	32°	30′	32°	45′	
1	.848	.530	.846	.534	.843	.537	.841	.541	1
2	1.696	1.060	1.691	1.067	1.687	1.075	1.682	1.082	2
3	2.544	1.590	2.537	1.601	2.530	1.612	2.523	1.623	3
4	3.392	2.120	3.383	2.134	3.374	2.149	3.364	2.164	4
5	4.240	2.650	4.229	2.668	4.217	2.686	4.205	2.705	5
6	5.088	3.180	5.074	3.202	5.060	3.224	5.046	3.246	6
7	5.936	3.709	5.920	3.735	5.904	3.761	5.887	3.787	7
8	6.784	4.239	6.766	4.269	6.747	4.298	6.728	4.328	8
9	7.632	4.769	7.612	4.802	7.591	4.836	7.569	4.869	9
10	8.480	5.299	8.457	5.336	8.434	5.373	8.410	5.410	10
	58°	0′	57°	45′	57°	30′	57°	15′	
	33°	0′	33°	15′	33°	30′	33°	45′	
1	.839	.545	.836	.548	.834	.552	.831	.556	1
2	1.677	1.089	1.673	1.097	1.668	1.104	1.663	1.111	2
3	2.516	1.634	2.509	1.645	2.502	1.656	2.494	1.667	3
4	3.355	2.179	3.345	2.193	3.336	2.208	3.326	2.222	4
5	4.193	2.723	4.181	2.741	4.169	2.760	4.157	2.778	5
6	5.032	3.268	5.018	3.290	5.003	3.312	4.989	3.333	6
7	5.871	3.812	5.854	3.838	5.837	3.864	5.820	3.889	7
8	6.709	4.357	6.690	4.386	6.671	4.416	6.652	4.445	8
9	7.548	4.902	7.527	4.935	7.505	4.967	7.483	5.000	9
10	8.387	5.446	8.363	5.483	8.339	5.519	8.315	5.556	10
	57°	0′	56°	45′	56°	30′	56°	15′	
	34°	0′	34°	15′	34°	30′	34°	45′	
1	.829	.559	.827	.563	.824	.566	.822	.570	1
2	1.658	1.118	1.653	1.126	1.648	1.133	1.643	1.140	2
3	2.487	1.678	2.480	1.688	2.472	1.699	2.465	1.710	3
4	3.316	2.237	3.306	2.251	3.297	2.266	3.287	2.280	4
5	4.145	2.796	4.133	2.814	4.121	2.832	4.108	2.850	5
6	4.974	3.355	4.960	3.377	4.945	3.398	4.930	3.420	6
7	5.803	3.914	5.786	3.940	5.769	3.965	5.752	3.990	7
8	6.632	4.474	6.613	4.502	6.593	4.531	6.573	4.560	8
9	7.461	5.033	7.439	5.065	7.417	5.098	7.395	5.130	9
10	8.290	5.591	8.266	5.628	8.241	5.664	8.216	5.700	10
	56°	0′	55°	45′	55°	30′	55°	15′	
	35°	0′	35°	15′	35°	30′	35°	45′	
1	.819	.574	.817	.577	.814	.581	.812	.584	1
2	1.638	1.147	1.633	1.154	1.628	1.161	1.623	1.168	2
3	2.457	1.721	2.450	1.731	2.442	1.742	2.435	1.753	3
4	3.277	2.294	3.267	2.309	3.256	2.323	3.246	2.337	4
5	4.096	2.868	4.083	2.886	4.071	2.904	4.058	2.921	5
6	4.915	3.441	4.900	3.463	4.885	3.484	4.869	3.505	6
7	5.734	4.015	5.716	4.040	5.699	4.065	5.681	4.090	7
8	6.553	4.589	6.533	4.617	6.513	4.646	6.493	4.674	8
9	7.372	5.162	7.350	5.194	7.327	5.226	7.304	5.258	9
10	8.192	5.736	8.166	5.771	8.141	5.807	8.116	5.842	10
	55°	0′	54°	45′	54°	30′	54°	15′	
D.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	D.

D.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	D.
	36°	0′	36°	15′	36°	30′	36°	45′	
1	.809	.588	.806	.591	.804	.595	.801	.598	1
2	1.618	1.176	1.613	1.183	1.608	1.190	1.603	1.197	2
3	2.427	1.763	2.419	1.774	2.412	1.784	2.404	1.795	3
4	3.236	2.351	3.226	2.365	3.215	2.379	3.205	2.393	4
5	4.045	2.939	4.032	2.957	4.019	2.974	4.006	2.992	5
6	4.854	3.527	4.839	3.548	4.823	3.569	4.808	3.590	6
7	5.663	4.115	5.645	4.139	5.627	4.164	5.609	4.188	7
8	6.472	4.702	6.452	4.730	6.431	4.759	6.410	4.787	8
9	7.281	5.290	7.258	5.322	7.235	5.353	7.211	5.385	9
10	8.090	5.878	8.064	5.913	8.039	5.948	8.013	5.983	10
	54°	0′	53°	45′	53°	30′	53°	15′	
	37°	0′	37°	15′	37°	30′	37°	45′	
1	.799	.602	.796	.605	.793	.609	.791	.612	1
2	1.597	1.204	1.592	1.211	1.587	1.218	1.581	1.224	2
3	2.396	1.805	2.388	1.816	2.380	1.826	2.372	1.837	3
4	3.195	2.407	3.184	2.421	3.173	2.435	3.163	2.449	4
5	3.993	3.009	3.980	3.026	3.967	3.044	3.953	3.061	5
6	4.792	3.611	4.776	3.632	4.760	3.653	4.744	3.673	6
7	5.590	4.213	5.572	4.237	5.553	4.261	5.535	4.286	7
8	6.389	4.815	6.368	4.842	6.347	4.870	6.326	4.898	8
9	7.188	5.416	7.164	5.448	7.140	5.479	7.116	5.510	9
10	7.986	6.018	7.960	6.053	7.934	6.088	7.907	6.122	10
	53°	0′	52°	45′	52°	30′	52°	15′	
	38°	0′	38°	15′	38°	30′	38°	45′	
1	.788	.616	.785	.619	.783	.623	.780	.626	1
2	1.576	1.231	1.571	1.238	1.565	1.245	1.560	1.252	2
3	2.364	1.847	2.356	1.857	2.348	1.868	2.340	1.878	3
4	3.152	2.463	3.141	2.476	3.130	2.490	3.120	2.504	4
5	3.940	3.078	3.927	3.095	3.913	3.113	3.899	3.130	5
6	4.728	3.694	4.712	3.715	4.696	3.735	4.679	3.756	6
7	5.516	4.310	5.497	4.334	5.478	4.358	5.459	4.381	7
8	6.304	4.925	6.283	4.953	6.261	4.980	6.239	5.007	8
9	7.092	5.541	7.068	5.572	7.043	5.603	7.019	5.633	9
10	7.880	6.157	7.853	6.191	7.826	6.225	7.799	6.259	10
	52°	0′	51°	45′	51°	30′	51°	15′	
	39°	0′	39°	15′	39°	30′	39°	45′	
1	.777	.629	.774	.633	.772	.636	.769	.639	1
2	1.554	1.259	1.549	1.265	1.543	1.272	1.538	1.279	2
3	2.331	1.888	2.323	1.898	2.315	1.908	2.307	1.918	3
4	3.109	2.517	3.098	2.531	3.086	2.544	3.075	2.558	4
5	3.886	3.147	3.872	3.164	3.858	3.180	3.844	3.197	5
6	4.663	3.776	4.646	3.796	4.630	3.816	4.613	3.837	6
7	5.440	4.405	5.421	4.429	5.401	4.453	5.382	4.476	7
8	6.217	5.035	6.195	5.062	6.173	5.089	6.151	5.116	8
9	6.994	5.664	6.970	5.694	6.945	5.725	6.920	5.755	9
10	7.771	6.293	7.744	6.327	7.716	6.361	7.688	6.394	10
	51°	0′	50°	45′	50°	30′	50°	15′	
D.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	D.

D.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	D.
	40°	0′	40°	15′	40°	30′	40°	45′	
1	.766	.643	.763	.646	.760	.649	.758	.653	1
2	1.532	1.286	1.526	1.292	1.521	1.299	1.515	1.306	2
3	2.298	1.928	2.290	1.938	2.281	1.948	2.273	1.958	3
4	3.064	2.571	3.053	2.584	3.042	2.598	3.030	2.611	4
5	3.830	3.214	3.816	3.231	3.802	3.247	3.788	3.264	5
6	4.596	3.857	4.579	3.877	4.562	3.897	4.545	3.917	6
7	5.362	4.500	5.343	4.523	5.323	4.546	5.303	4.569	7
8	6.128	5.142	6.106	5.169	6.083	5.196	6.061	5.222	8
9	6.894	5.785	6.869	5.815	6.844	5.845	6.818	5.875	9
10	7.660	6 428	7.632	6.461	7.604	6.494	7.576	6.528	10
	50°	0′	49°	45′	49°	30′	49°	15′	
	41°	0′	41°	15′	41°	30′	41°	45′	
1	.755	.656	.752	.659	.749	.663	.746	.666	1
2	1.509	1.312	1.504	1.319	1.498	1.325	1.492	1.332	2
3	2.264	1.968	2.256	1.978	2.247	1.988	2.238	1.998	3
4	3.019	2.624	3.007	2.637	2.996	2.650	2.984	2.664	4
5	3.774	3.280	3.759	3.297	3.745	3.313	3.730	3.329	5
6	4.528	3.936	4.511	3.956	4.494	3.976	4.476	3.995	6
7	5.283	4.592	5.263	4.615	5.243	4.638	5.222	4.661	7
8	6.038	5.248	6.015	5.275	5.992	5.301	5.968	5.327	8
9	6.792	5.905	6.767	5.934	6.741	5.964	6.715	5.993	9
10	7.547	6.561	7.518	6.593	7.490	6.626	7.461	6.659	10
	49°	0′	48°	45′	48°	30′	48°	15′	
	42°	0′	42°	15′	42°	30′	42°	45′	
1	.743	.669	.740	.672	.737	.676	.734	.679	1
2	1.486	1.338	1.480	1.345	1.475	1.351	1.469	1.358	2
3	2.229	2.007	2.221	2.017	2.212	2.027	2.203	2.036	3
4	2.973	2.677	2.961	2.689	2.949	2.702	2.937	2.715	4
5	3.716	3.346	3.701	3.362	3.686	3.378	3.672	3.394	5
6	4.459	4.015	4.441	4.034	4.424	4.054	4.406	4.073	6
7	5.202	4.684	5.182	4.707	5.161	4.729	5.140	4.752	7
8	5.945	5.353	5.922	5.379	5.898	5.405	5.875	5.430	8
9	6.688	6.022	6.662	6.051	6.636	6.080	6.609	6.109	9
10	7.431	6.691	7.402	6.724	7.373	6.756	7.343	6.788	10
	48°	0′	47°	45′	47°	30′	47°	15′	
	43°	0′	43°	15′	43°	30′	43°	45′	
1	.731	.682	.728	.685	.725	.688	.722	.692	1
2	1.463	1.364	1.457	1.370	1.451	1.377	1.445	1.383	2
3	2.194	2.046	2.185	2.056	2.176	2.065	2.167	2.075	3
4	2.925	2.728	2.913	2.741	2.901	2.753	2.889	2.766	4
5	3.657	3.410	3.642	3.426	3.627	3.442	3.612	3.458	5
6	4.388	4.092	4.370	4.111	4.352	4.130	4.334	4.149	6
7	5.119	4.774	5.099	4.796	5.078	4.818	5.057	4.841	7
8	5.851	5.456	5.827	5.481	5.803	5.507	5.779	5.532	8
9	6.582	6.138	6.555	6.167	6.528	6.195	6.501	6.224	9
10	7.314	6.820	7.284	6.852	7.254	6.884	7.224	6.915	10
	47°	0′	46°	45′	46°	30′	46°	15′	
D.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	D.

D.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	D.
	44° 0′		44° 15′		44° 30′		44° 45′		
1	.719	.695	.716	.698	.713	.701	.710	.704	1
2	1.439	1.389	1.433	1.396	1.427	1.402	1.420	1.408	2
3	2.158	2.084	2.149	2.093	2.140	2.103	2.131	2.112	3
4	2.877	2.779	2.865	2.791	2.853	2.804	2.841	2.816	4
5	3.597	3.473	3.582	3.489	3.566	3.505	3.551	3.520	5
6	4.316	4.168	4.298	4.187	4.280	4.205	4.261	4.224	6
7	5.035	4.863	5.014	4.885	4.993	4.906	4.971	4.928	7
8	5.755	5.557	5.730	5.582	5.706	5.607	5.682	5.632	8
9	6.474	6.252	6.447	6.280	6.419	6.308	6.392	6.336	9
10	7.193	6.947	7.163	6.978	7.133	7.009	7.102	7.040	10
	46° 0′		45° 45′		45° 30′		45° 15′		
	45° 0′		45° 15′		45° 30′		45° 45′		
1	.707	.707	.704	.710	.701	.713	.698	.716	1
2	1.414	1.414	1.408	1.420	1.402	1.427	1.396	1.433	2
3	2.121	2.121	2.112	2.131	2.103	2.140	2.093	2.149	3
4	2.828	2.828	2.816	2.841	2.804	2.853	2.791	2.865	4
5	3.536	3.536	3.520	3.551	3.505	3.566	3.489	3.582	5
6	4.243	4.243	4.224	4.261	4.205	4.280	4.187	4.298	6
7	4.950	4.950	4.928	4.971	4.906	4.993	4.885	5.014	7
8	5.657	5.657	5.632	5.682	5.607	5.706	5.582	5.730	8
9	6.364	6.364	6.336	6.392	6.308	6.419	6.280	6.447	9
10	7.071	7.071	7.040	7.102	7.009	7.133	6.978	7.163	10
	45° 0′		44° 45′		44° 30′		44° 15′		
D.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	D.

MISCELLANEOUS TABLE.

When Diameter = 1.		Log.
Circumference of circle, π, ...	3.14159	0.49715
Area of circle, ...	.78540	9.89509–10
Contents of sphere, ...	.52360	9.71900–10
Earth's equatorial radius, in miles, . .	.3962.57	3.59798
Earth's polar radius, in miles, ...	.3949.324	3.59652
Compression, 1 ÷ 299.1528, ...	0.00334	7.52411–10

EQUIVALENTS.

1 American mile =	.86756	nautical miles, . . .	9.93830–10
1 " " =	1609.40831	meters, ...	3.20667
1 " " =	.21689	German geograph. miles,	9.33624–10
1 " " =	1.50866	Russian versts, . . .	0.17859
1 " yard =	.91444	meters, ...	9.96115–10
1 " " =	.48217	Vienna klafter, . . .	9.68320–10
1 " foot =	.30481	meters, ...	9.48403–10
1 " " =	.15639	toises, ...	9.19421–10
1 " " =	.93835	Parisian feet, ...	9.97236–10
1 " " =	.96435	Vienna feet, ...	9.98423–10
1 " " =	1.09395	Spanish feet, ...	0.03900

Deg.	0′	10′	20′	30′	40′	50′
0	0.0	9.9	19.9	29.8	39.7	49.7
1	59.6	69.5	79.5	89.4	99.3	109.3
2	119.2	129.2	139.1	149.0	159.0	168.9
3	178.9	188.8	198.8	208.7	218.7	228.6
4	238.6	248.6	258.5	268.5	278.4	288.4
5	298.4	308.4	318.3	328.3	338.3	348.3
6	358.3	368.3	378.2	388.2	398.2	408.2
7	418.3	428.3	438.3	448.3	458.3	468.3
8	478.4	488.4	498.4	508.5	518.5	528.6
9	538.6	548.7	458.8	568.8	578.9	589.0
10	599.1	609.2	619.3	629.4	639.5	649.6
11	659.7	669.8	680.0	690.1	700.2	710.4
12	720.5	730.7	740.9	751.0	761.2	771.4
13	781.6	791.8	802.0	812.2	822.5	832.7
14	842.9	853.2	863.4	873.7	884.0	894.2
15	904.5	914.8	925.1	935.4	945.7	956.1
16	966.4	976.7	987.1	997.5	1007.8	1018.2
17	1028.6	1039.0	1049.4	1059.8	1070.2	1080.7
18	1091.1	1101.6	1112.0	1122.5	1133.0	1143.5
19	1154.0	1164.5	1175.1	1185.6	1196.1	1206.7
20	1217.3	1227.9	1238.5	1249.1	1259.7	1270.3
21	1281.0	1291.6	1302.3	1313.0	1323.7	1334.4
22	1345.1	1355.8	1366.6	1377.3	1388.1	1398.9
23	1409.7	1420.5	1431.3	1442.1	1453.0	1463.8
24	1474.7	1485.6	1496.5	1507.4	1518.4	1529.3
25	1540.3	1551.3	1562.3	1573.3	1584.3	1595.4
26	1606.4	1617.5	1628.6	1639.7	1650.8	1661.9
27	1673.1	1684.3	1695.5	1706.7	1717.9	1729.1
28	1740.4	1751.7	1762.9	1774.3	1785.6	1796.9
29	1808.3	1819.7	1831.1	1842.5	1854.0	1865.4
30	1876.9	1888.4	1899.9	1911.4	1923.0	1934.6
31	1946.2	1957.8	1969.4	1981.1	1992.8	2004.5
32	2016.2	2028.0	2039.7	2051.5	2063.3	2075.2
33	2087.0	2098.9	2110.8	2122.7	2134.7	2146.7
34	2158.6	2170.7	2182.7	2194.8	2206.9	2219.0
35	2231.1	2243.3	2255.5	2267.7	2279.9	2292.2
36	2304.5	2316.8	2329.2	2341.5	2353.9	2366.4
37	2378.8	2391.3	2403.8	2416.3	2428.9	2441.5
38	2454.1	2466.8	2479 5	2492.2	2504.9	2517.7
39	2530.5	2543.3	2556.2	2569.1	2582.0	2594.9
40	2607.9	2621.0	2634.0	2647.1	2660.2	2673.3
41	2866.5	2699.7	2713.0	2726.3	2739.6	2752.9
42	2766.3	2779.8	2793.2	2806.7	2820.3	2833.8

Deg.	0′	10′	20′	30′	40′	50′
43	2847.4	2861.1	2874.8	2888.5	2902.2	2916.0
44	2929.9	2943.7	2957.6	2971.6	2985.6	2999.6
45	3013.7	3027.8	3042.0	3056.2	3070.4	3084.7
46	3099.0	3113.4	3127.8	3142.3	3156.8	3171.3
47	3185.9	3200.5	3215.2	3230.0	3244.7	3259.6
48	3274.5	3289.4	3304.3	3319.4	3334.4	3349.6
49	3364.7	3380.0	3395.2	3410.6	3425.9	3441.4
50	3456.9	3472.4	3488.0	3503.7	3519.4	3535.1
51	3550.9	3566.8	3582.8	3598.7	3614.8	3630.9
52	3647.1	3663.2	3679.6	3696.0	3712.4	3728.9
53	3745.4	3762.0	3778.7	3795.4	3812.2	3829.1
54	3846.0	3863.1	3880.1	3897.3	3914.5	3931.8
55	3949.1	3966.6	3984.1	4001.7	4019.3	4037.0
56	4054.8	4072.7	4090.7	4108.7	4126.9	4145.1
57	4163.3	4181.7	4200.2	4218.7	4237.3	4256.0
58	4274.8	4293.7	4312.7	4331.7	4350.9	4370.1
59	4389.4	4408.9	4428.4	4448.0	4467.7	4487.5
60	4507.5	4527.5	4547.6	4567.8	4588.1	4608.6
61	4629.1	4649.8	4670.5	4691.4	4712.4	4733.5
62	4754.7	4776.0	4797.5	4819.0	4840.7	4862.5
63	4884.5	4906.5	4928.7	4951.0	4973.5	4996.0
64	5018.8	5041.6	5064.6	5087.7	5111.0	5134.4
65	5158.0	5181.7	5205.5	5229.5	5253.7	5278.0
66	5302.5	5327.1	5351.9	5376.9	5402.1	5427.4
67	5452.8	5478.5	5504.3	5530.3	5556.5	5582.9
68	5609.5	5636.3	5663.2	5690.4	5717.7	5745.3
69	5773.1	5801.1	5829.3	5857.7	5886.3	5915.2
70	5944.3	5973.6	6003.2	6033.0	6063.1	6093.4
71	6124.0	6154.8	6185.9	6217.2	6248.9	6280.8
72	6313.0	6345.5	6378.2	6411.3	6444.7	6478.4
73	6512.4	6546.8	6581.5	6616.5	6651.8	6687.6
74	6723.6	6760.1	6796.9	6834.1	6871.7	6909.7
75	6948.1	6987.0	7026.2	7065.9	7106.1	7146.7
76	7187.8	7229.3	7271.4	7313.9	7357.0	7400.6
77	7444.8	7489.5	7534.8	7580.7	7627.0	7674.3
78	7722.1	7770.5	7819.6	7869.4	7919.9	7971.1
79	8023.1	8075.9	8129.5	8184.0	8239.3	8295.4
80	8352.5	8410.6	8469.6	8529.7	8590.8	8653.0
81	8716.3	8780.9	8846.6	8913.6	8981.9	9051.6
82	9122.7	9195.3	9269.4	9345.2	9422.7	9501.9
83	9583.0	9666.0	9751.1	9838.3	9927.8	10019.6
84	10114.0	10211.0	10310.8	10413.6	10519.6	10628.8
85	10741.7	10858.4	10979.2	11104.3	11234.2	11369.1

CORRECTIONS FOR MIDDLE LATITUDE.

DIFFERENCE OF LATITUDE.

Mid. Lat.	2°	3°	4°	5°	6°	7°	8°	9°	10°	11°	12°	13°	14°	15°	16°	17°	18°	19°	20°	Mid. Lat.
15°	1′	2′	3′	5′	7′	9′	12′	15′	18′	22′	26′	31′	36′	41′	47′	52′	59′	65′	72′	15°
16	1	2	3	4	6	9	11	14	18	21	25	30	34	39	44	50	56	62	69	16
17	1	2	3	4	6	8	11	14	17	20	24	28	33	38	43	48	54	60	66	17
18	1	1	3	4	6	8	10	13	16	20	23	27	32	36	41	46	52	58	64	18
19	1	1	3	4	6	8	10	13	16	19	22	26	30	35	40	45	50	56	61	19
20	1	1	2	4	5	7	10	12	15	18	22	25	29	34	38	43	48	54	60	20
21	1	1	2	4	5	7	9	12	15	18	21	25	29	33	37	42	47	52	58	21
22	1	1	2	4	5	7	9	12	14	17	21	24	28	32	36	41	46	51	56	22
23	1	1	2	3	5	7	9	11	14	17	20	23	27	31	35	40	45	50	55	23
24	1	1	2	3	5	7	9	11	14	16	20	23	27	31	35	39	44	49	54	24
25	1	1	2	3	5	7	9	11	13	16	19	23	26	30	34	39	43	48	53	25
26	1	1	2	3	5	6	8	11	13	16	19	22	26	30	34	38	42	47	52	26
27	1	1	2	3	5	6	8	11	13	16	19	22	25	29	33	37	42	47	52	27
28	1	1	2	3	5	6	8	10	13	16	18	22	25	29	33	37	41	46	51	28
29	1	1	2	3	5	6	8	10	13	15	18	21	25	28	32	37	41	46	51	29
30	1	1	2	3	5	6	8	10	13	15	18	21	25	28	32	36	41	45	50	30
31	1	1	2	3	5	6	8	10	12	15	18	21	24	28	32	36	40	45	50	31
32	0	1	2	3	4	6	8	10	12	15	18	21	24	28	32	36	40	45	50	32
33	0	1	2	3	4	6	8	10	12	15	18	21	24	28	32	36	40	45	49	33
34	0	1	2	3	4	6	8	10	12	15	18	21	24	28	32	36	40	45	49	34
35	0	1	2	3	4	6	8	10	12	15	18	21	24	28	32	36	40	45	49	35
36	1	1	2	3	4	6	8	10	12	15	18	21	24	28	32	36	40	45	49	36
37	1	1	2	3	4	6	8	10	12	15	18	21	24	28	32	36	40	45	49	37
38	1	1	2	3	4	6	8	10	12	15	18	21	24	28	32	36	40	45	50	38
39	1	1	2	3	4	6	8	10	12	15	18	21	24	28	32	36	40	45	50	39
40	1	1	2	3	5	6	8	10	13	15	18	21	25	28	32	36	41	45	50	40
41	1	1	2	3	5	6	8	10	13	15	18	21	25	28	32	37	41	46	51	41
42	1	1	2	3	5	6	8	10	13	15	18	22	25	29	33	37	41	46	51	42
43	1	1	2	3	5	6	8	10	13	16	18	22	25	29	33	37	42	46	52	43
44	1	1	2	3	5	6	8	10	13	16	19	22	25	29	33	38	42	47	52	44
45	1	1	2	3	5	6	8	11	13	16	19	22	26	30	34	38	43	48	53	45
46	1	1	2	3	5	6	8	11	13	16	19	22	26	30	34	38	43	48	53	46
47	1	1	2	3	5	7	9	11	13	16	19	23	26	30	35	39	44	49	54	47
48	1	1	2	3	5	7	9	11	14	17	20	23	27	31	35	40	44	50	55	48
49	1	1	2	3	5	7	9	11	14	17	20	23	27	31	36	40	45	50	56	49
50	1	1	2	4	5	7	9	11	14	17	20	24	28	32	36	41	46	51	57	50
51	1	1	2	4	5	7	9	12	14	17	21	24	28	32	37	42	47	52	58	51
52	1	1	2	4	5	7	9	12	15	18	21	25	29	33	38	43	48	53	59	52
53	1	1	2	4	5	7	10	12	15	18	21	25	29	34	38	43	49	54	60	53
54	1	1	2	4	5	7	10	12	15	18	22	26	30	34	39	44	50	56	62	54
55	1	1	2	4	6	8	10	13	16	19	22	26	31	35	40	45	51	57	63	55
56	1	1	3	4	6	8	10	13	16	19	23	27	31	36	41	46	52	58	65	56
57	1	1	3	4	6	8	10	13	16	20	24	28	32	37	42	48	54	60	66	57
58	1	2	3	4	6	8	11	14	17	20	24	28	33	38	43	49	55	61	68	58
59	1	2	3	4	6	8	11	14	17	21	25	29	34	39	45	50	57	63	70	59
60	1	2	3	4	6	9	11	14	18	22	26	30	35	40	46	52	58	65	72	60
61	1	2	3	5	7	9	12	15	18	22	26	31	36	42	47	53	60	67	75	61
62	1	2	3	5	7	9	12	15	19	23	27	32	37	43	49	55	62	70	77	62
63	1	2	3	5	7	10	12	16	20	24	28	33	39	44	51	57	64	72	80	63
64	1	2	3	5	7	10	13	16	20	24	29	34	40	46	52	59	67	75	83	64
65	1	2	3	5	7	10	13	17	21	25	30	36	41	48	54	62	69	78	86	65
66	1	2	3	5	8	11	14	18	22	26	32	37	43	50	57	64	72	81	90	66
67	1	2	4	6	8	11	14	18	23	28	33	39	45	52	59	67	76	85	94	67
68	1	2	4	6	8	12	15	19	24	29	34	40	47	54	62	70	79	89	99	68
69	1	2	4	6	9	12	16	20	25	30	36	42	49	57	65	74	83	93	104	69
70	1	2	4	6	9	13	16	21	26	32	38	44	52	60	68	78	88	98	110	70