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SYNOPSIS OF THE COURSE OF LECTURES IN DESCRIPTIVE GEOMETRY AS ARRANGED FOR THE SECOND YEAR

By J. Roy Cockburn, B.A.Sc.,
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Generation and Classification of Lines

Any line may be considered to be generated by the continuous motion of a point and may be considered to be made up of an infinite number of infinitely small straight or right lines.

The law which directs the motion of the point determines the nature of the line.

Lines may be classified as follows:

(1) Right or straight lines in which the direction of motion of the point is always the same.

(2) Curved lines in which the direction of motion of the generating point is changing continuously.

Curved lines may be divided into two classes:

(*a*) Curves of single curvature having all points lying in a single plane.

(*b*) Curves of double curvature having no four consecutive points lying in a single plane.

In problems in descriptive geometry, curves like other lines are usually represented by their projections.

If a curve of single curvature have its plane perpendicular to a plane of projection, its projection on that plane is a straight line.

If the plane of the curve be parallel to the plane of projection, the projection will be of exactly the same form as the curve itself.

The projection of a curve of double curvature can never be a straight line or of the same form as the curve itself.

Tangents and Normals to Curves

A straight line is tangent to another line when it contains two consecutive points.

Two curves are tangent to each other when they contain two consecutive points or have at a common point a common tangent.

If a straight line be tangent to a curve of single curvature it will be contained in the plane of the curve.

A normal to a curve is a line at right angles to the tangent.

The normal in the plane of the curve is generally considered to be the normal.

The most important lines of single curvature are the conic sections, which may be defined as follows:

A conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its distance from a fixed line. This ratio is usually denoted by the letter e.

If $e=1$ the curve is a parabola.
If $e<1$ the curve is an ellipse.
If $e>1$ the curve is a hyperbola.

To Plot the Parabola

In Fig. 1 let F be the given point and CD the given line. Draw FD at right angles to CD. Bisect FD at the point A. Take any point M to the right of A. With F as centre and radius $FP=DM$ describe a circle cutting perpendicular PMP_1 at P and P_1. P and P_1 are points on the parabola.

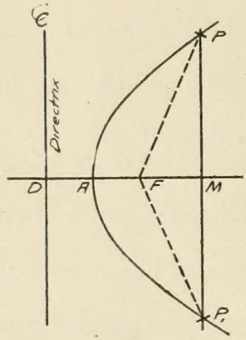


Fig. 1.

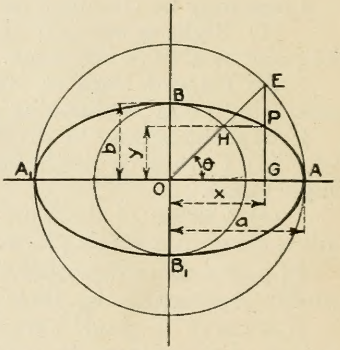


Fig. 2.

To Plot the Ellipse when given the Major and Minor Axes

In Fig. 2, AA_1 and BB_1 are the given axes of the ellipse, intersecting at the point O. With centre O and radius OB describe a circle and with the same centre and radius OA describe another circle.

Draw OHE any common radius making an angle θ with AA_1.

Draw EG perpendicular to AA_1 and HP perpendicular to EG and intersecting it at the point P. P is a point on the ellipse.

$$x = a\cos\theta \qquad y = b\sin\theta$$

$$\frac{x^2}{a^2} = \cos^2\theta \qquad \frac{y^2}{b^2} = \sin^2\theta$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Second Method of Drawing the Ellipse when given the Major and Minor Axes or a Pair of Conjugate Diameters

In Figs. 3 and 4, AA_1 and BB_1 are a pair of conjugate diameters. Upon these diameters describe a parallelogram as shown.

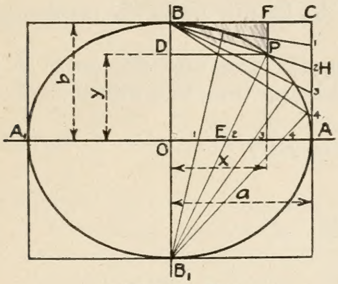


Fig. 3.

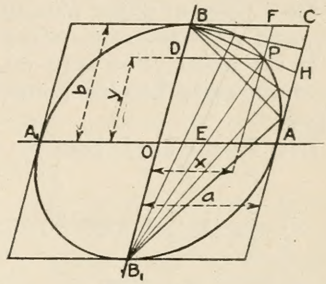


Fig. 4.

Divide the semi-diameter OA into any number of equal parts, and the side AC into the same number of equal parts.

Straight lines are drawn as shown, and their intersections are points on the required curve.

To prove that the curve is an ellipse—

OA and AC are divided at the points E and H in the same ratio.

Let this ratio be denoted by $\frac{m}{n}$

$$\text{i.e.}\ \frac{CH}{HA} = \frac{m}{n} = \frac{OE}{EA}$$

In the similar triangles BFP and BCH.

$$\frac{CH}{FP} = \frac{BC}{BF} \text{ or } \frac{\frac{m}{m+n}\,b}{b-y} = \frac{a}{x}$$

and in similar triangles B_1OE and B_1DP

$$\frac{DP}{OE} = \frac{B_1D}{B_1O} \text{ or } \frac{x}{\frac{m}{m+n}\,a} = \frac{b+y}{b}$$

Equating the values of $\frac{m}{m+n}$ we have

$$\frac{a(b-y)}{bx} = \frac{bx}{a(b+y)}$$

$$a^2(b^2-y^2) = b^2x^2$$

Dividing each side by a^2b^2 we get

$$1 - \frac{y^2}{b^2} = \frac{x^2}{a^2}$$

or $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$ The equation to an ellipse.

Third Method of Describing the Ellipse when given the Major and Minor Axes

In Fig. 5 let AA_1 be the major axis and BB_1 the minor axis intersecting it at the point O.

On a piece of paper measure off the length PR = the semi-major axis, and PQ = the semi-minor axis.

If the point Q be kept on the major axis and the point R on the minor axis, then the point P will be on the circumference of the ellipse.

Let θ = the angle between PR and BB_1

$x = a \sin\theta \qquad y = b\cos\theta$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = \sin^2\theta + \cos^2\theta = 1.$$

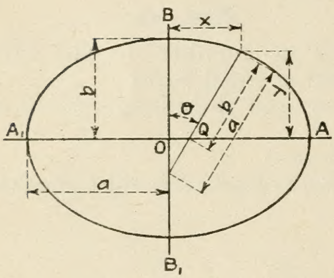


Fig. 5.

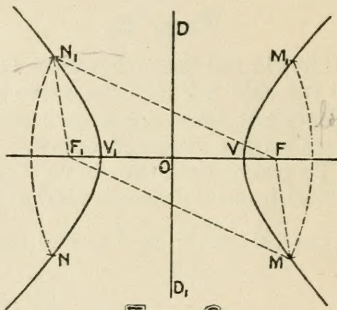


Fig. 6.

Method of Describing the Hyperbola

The hyperbola may be generated by moving a point in the same plane so that the difference of its distances from two fixed points shall be equal to a given line.

In Fig. 6 let F and F_1 be the two given points (the foci) and VV_1 the given line so placed that $FV = F_1V_1$ and in the same straight line as FF_1.

With F_1 as centre and any radius F_1M greater than F_1V describe an arc M_1M.

With F as centre and radius FM equal to $F_1M - VV_1$ describe a second arc intersecting the first arc at the points M and M_1.

M and M_1 are then points on the curve.

The Helix

A useful example of a line of double curvature is the Helix, which may be defined as follows:

The Helix is a curve generated by a point which has a uniform motion of rotation about a fixed axis and a uniform motion of translation parallel to that axis.

Note—See drawing entitled "Helix".

Generation and Classification of Surfaces

Surfaces may be classified as follows:

A. Surfaces which can be generated by straight lines:

(*a*) Planes.

(*b*) Surfaces of revolution—cylinders, cones, hyperboloid of revolution of one sheet.

(*c*) Warped surfaces.

B. Surfaces which can be generated only by curves.

Surfaces of revolution—spheroid, paraboloid and hyperboloid of revolution of two sheets.

To Prove that the Surface Generated by the Revolution of a Straight Line About Another Straight Line not in the same Plane is a Hyperboloid of Revolution of One Sheet, *i.e.*, to Prove that its Meridian Section is a Hyperbola. (Fig. 7.)

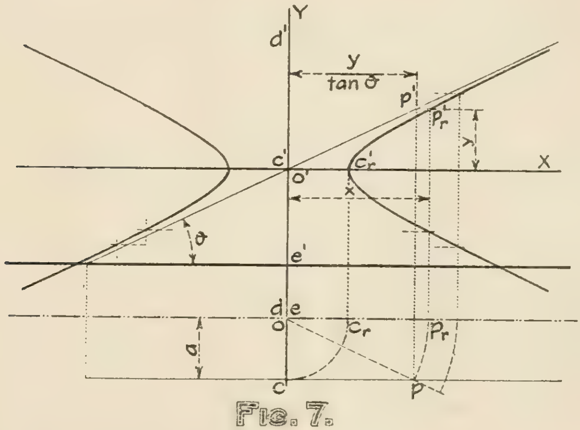


Fig. 7.

Let c^1p^1 and *cp* be the vertical and horizontal projections of the straight line *CP* which is parallel to the vertical plane and inclined at an angle θ to the horizontal plane.

Let e^1d^1 and *ed* be the vertical and horizontal projections of the line *ED* which is perpendicular to the horizontal plane.

If the line *CP* revolve about *D E* it will cut from a plane containing *DE* a pair of curves which we can prove are the two branches of a hyperbola.

Consider the point *P* on *CP*.

If *CP* be revolved to cut the plane containing *D E* and parallel to the vertical plane, the point P_r where *P* passes through the plane will be a point on the curve cut from the plane by the generating line *CP*.

Let the shortest distance between the generating line CP and the axis $DE=a$ and let $\frac{b}{a}=\tan\theta$.

Let O and C be the extremities of the shortest line between CP and DE.

The actual distance from O to P is OP; from O to $C=$a.

Let the distance from P_r to $DE=x$.

Let the vertical distance that P_r is above $O=y$.

The distance from d^1e^1 to $p^1 = \frac{y}{\tan\theta}$

Then $OP^2=a^2+\frac{y^2}{\tan^2\theta}+y^2=(OP_r)^2=x^2+y^2$

or $a^2\tan^2\theta+y^2=x^2\tan^2\theta$

Substitute $\frac{b}{a}$ for $\tan\theta$.

In $a^2\tan^2\theta+y^2=x^2\tan^2\theta$

$$a^2b^2+a^2y^2=b^2x^2$$

$$1+\frac{y^2}{b^2}=\frac{x^2}{a^2} \text{ or } \frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Tangent Planes to Curved Surfaces

Problem 1

To find the projections of a right circular cone when given its dimensions and its position relative to the planes of projection. (*Fig.* 8.)

Let the position of the centre of the base be given by its projections o^1 and o and the inclination of the axis of the cone to the vertical plane be α and the inclination to the horizontal plane be β.

First, find the projections of the axis $o^1v_1^1$ and ov_1 when it makes an angle of α with the vertical plane and is parallel to the horizontal plane. Also find the projections of two diameters of the base, one parallel to the horizontal plane, $a_1^1b_1^1$ and a_1b_1 being the required projections, and the other perpendicular to the horizontal plane $c_1^1d_1^1$ and c_1d_1 being the required projections.

Next, find the projections of the axis $o^1v_2^1$ and ov_2 when it makes an angle of β with the horizontal plane and is parallel to the vertical.

With centre o^1 and radius $o^1v_1^1$ describe the arc $v_1^1v^1$ and through the point v_2^1 draw the line $v_2^1v^1$ parallel to the ground line, to intersect the arc $v_1^1v^1$ at the point v^1. The point v^1 is the vertical projection of a vertex in the required position. Its horizontal projection v is found in the same way.

Next, find the vertical projections, a^1, b^1, c^1 and d^1 of the extremities of the two diameters AB and CD and their horizontal projections a, b, c and d.

a^1b^1 and c^1d^1 are conjugate diameters of the ellipse which is the vertical projection of the base and *ab* and *cd* of the ellipse which is the horizontal projection of the base.

Describe the ellipses by the method given previously.

Draw tangents from the projections of the vertex to the corresponding projections of the base obtaining the outlines of the projections of the cone.

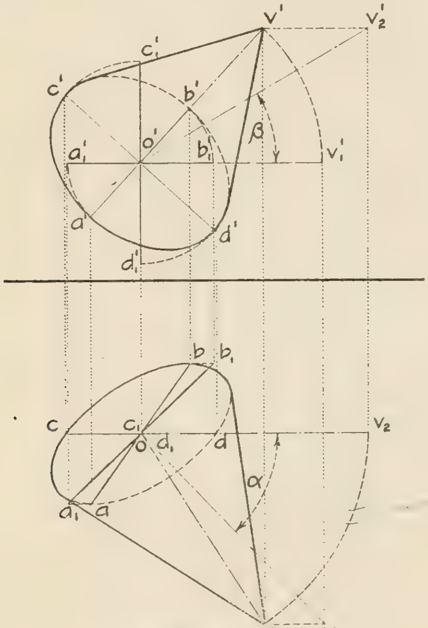


Fig. 8.

Problem 2

To find the projections of a right circular cylinder when given its dimensions and its position relative to the planes of projection. (*Fig. 9*).

Find the projections of the base and axis of the cylinder by the same construction as used in Problem 1 for the cone.

Complete the projections of the cylinder as illustrated in Fig. 9.

Problem 3

To find the traces of a plane tangent to the surface of a given cone and passing through a given point on the surface of the cone. (*Fig. 10.*)

Let the projections of the cone be as found in Problem 1.

To assume the projections of a point on the surface, first assume one projection, in this case the vertical projection, (p^1), and then find the horizontal projection, as follows:

Join the vertical projection p^1 to the vertical projection of the vertex v^1, obtaining the vertical projection of the element PV, through the point P, and find the vertical projection q^1 of the point Q where this element intersects the base.

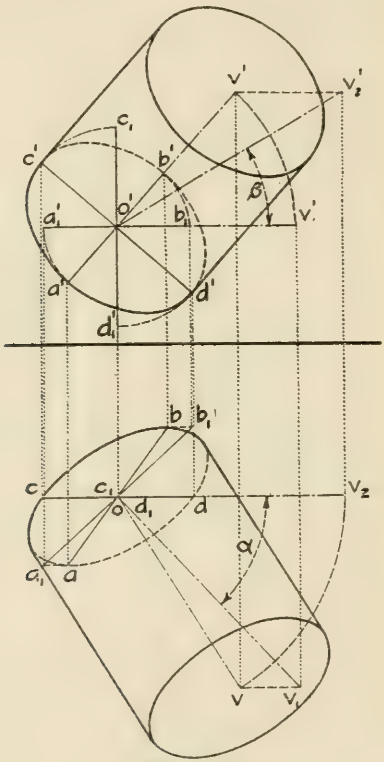


FIG. 9.

Find q the horizontal projection of point Q, and join q with V, obtaining the horizontal projection of the element through P. The horizontal projection of P is on this line qv where it is intersected by a line through p^1 perpendicular to the ground line.

To find the traces of a plane containing the point P and tangent to the cone, proceed as follows:

Find the projections of a line QR tangent to the base at the point Q. That is, draw q^1r^1 tangent to the vertical projection of the base of cone and qr tangent to the horizontal projection of base.

Find the traces of the plane containing the lines PV and QR and they will be the traces of the required plane.

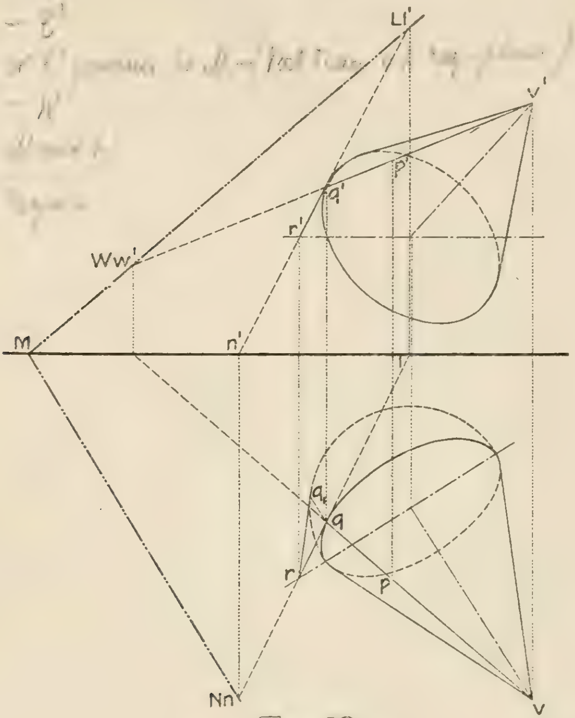


Fig. 10.

Problem 4

To find the traces of a plane tangent to the surface of a given cylinder and passing through a given point on the surface of the cylinder. (*Fig. 11.*)

Let the projections of the cylinder be as found in Problem 2.

To assume the projections of a point on the surface, first assume one projection, in this case the vertical projection (p^1) and then find the horizontal projection as follows:

Draw a straight line p^1q^1 through p^1 parallel to o^1v^1 obtaining the vertical projections of the element PQ and find the vertical projection q^1 of the point Q where this element intersects the base.

Find q the horizontal projection of Q and draw pq through q parallel to ov obtaining the horizontal projection of the element PQ.

The horizontal projection of "P" is on this line at p.

To find the traces of a plane containing the point P and tangent to the surface of the cylinder, proceed as follows:

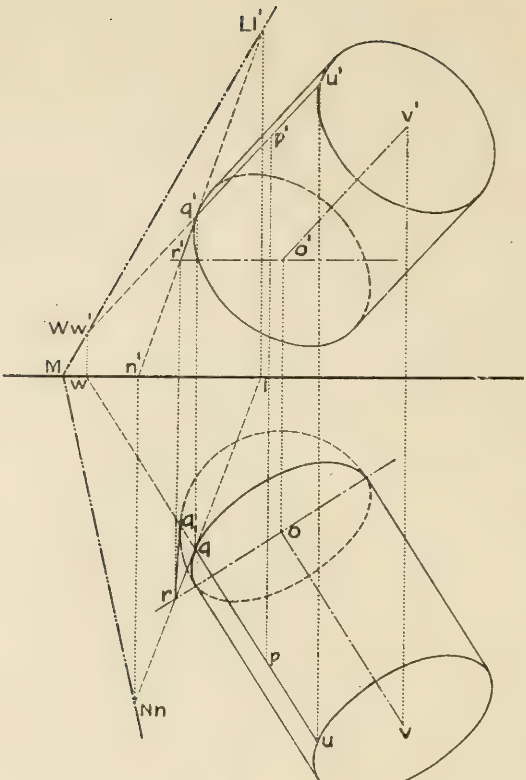


Fig. 11.

Find the projections of a line QR tangent to the base at the point Q. That is, draw q^1r^1 tangent to the vertical projection of the base of cylinder and qr tangent to the horizontal projection of the base.

Find the traces of the plane containing the lines PV and QR, viz., LM and MN, and they will be the traces of the required plane.

Problem 5

To find the traces of a plane tangent to the surface of a given cone and passing through a given point without the surface of the cone. (Fig. 12.)

Let the projections of the cone be as found in Problem 1. Let x^1 and x be the projections of the given point through which the plane must pass. Join x^1 to v^1 and x with v, obtaining the projections of the line XV, which joins the point X to the vertex of the cone. Find the traces of the plane of the base of the cone, viz., *NW* and *LS*. Find the projections r^1 and r of the point *R* where the line *XV* intersects the plane *NWLS* (the plane of the base). Find the projections q^1n^1 and qn of the line *QN*, which passes through the point *R* and is tangent to the base of the cone at the point *Q*. Find the traces of the plane containing the lines *XV* and *QN*, viz., *LM* and *YN*. These are the traces of the required plane.

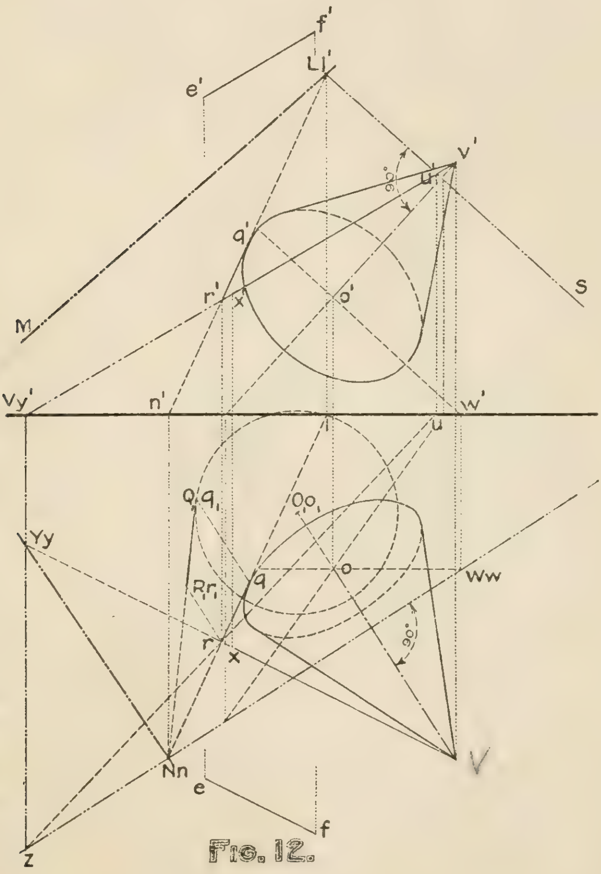


FIG. 12.

NOTE—A cylinder may be considered to be a particular case of a cone in which the vertex is at an infinite distance from the base. Therefore, the solutions given for Problems 1, 3 and 5 apply equally well to the solutions of the corresponding problems relating to the cylinder.

Problem 6

To find the traces of a plane tangent to the surface of a given cylinder and passing through a point without the surface of the cylinder. (Fig. 13.)

Let the projections of the cylinder be as found in Problem 2.

Let x^1 and x be the projections of the given point through which the plane must pass.

Draw x^1u^1 parallel to o^1v^1 and xu parallel to ov obtaining the

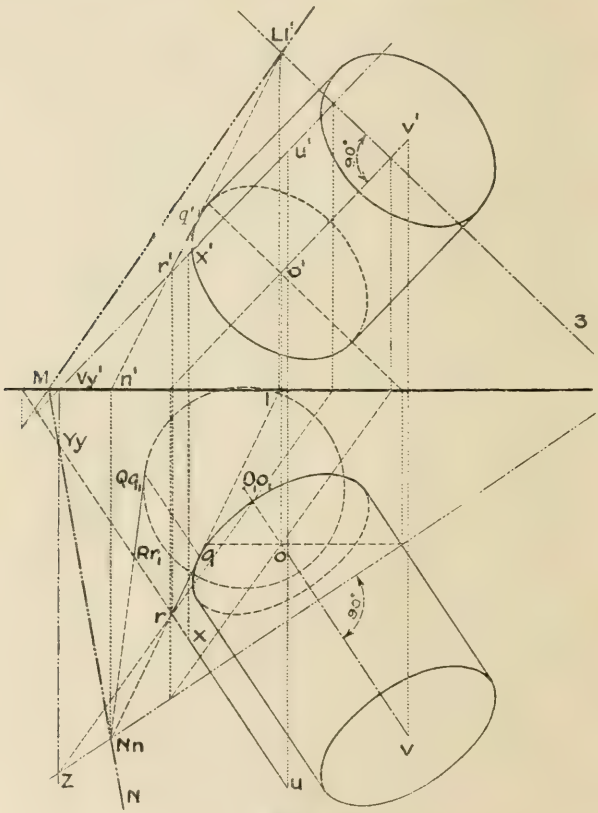


FIG. 13.

projections of the line XU which is parallel to the axis OV.

Find the traces of the plane of the base of the cylinder, viz., NZ and LS.

Find the projections r^1 and r of the point R where the line XU intersects the plane $ZNLS$ (the plane of the base).

Find the projections q^1n^1 and qn of the line QN which passes
rough the point R and is tangent to the base of the cylinder at
e point Q.

Find the traces of the plane containing the lines XU and QN,
z., LM and YM.

Problem 7

*To find the traces of a plane tangent to the surface of a given cone
d parallel to a given straight line.* (*Fig. 12.*)

Let ef and e^1f^1 be the projections of the given line EF.

Find the projections vx and v^1x^1 of a straight line VX passing
rough the vertex of the cone and parallel to the given straight
ne EF.

Find the traces of the plane containing VX and tangent to the
rface of the cone as in Problem 5.

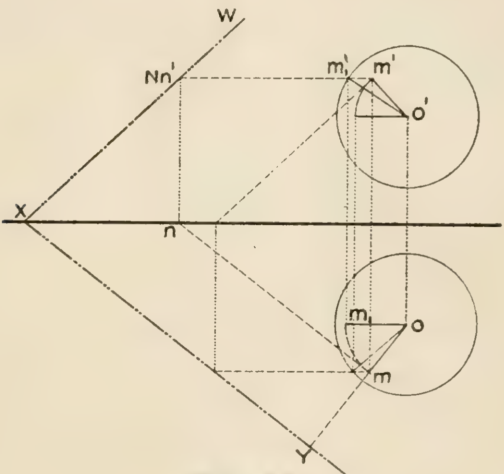


Fig. 14.

Problem 8

*To find the traces of a plane tangent to the surface of a given
ylinder and parallel to a given straight line.* (*No figure given for this
roblem.*)

Find the traces of a plane containing the given line and parallel
the axis of the given cylinder.

Find the projections of the line of intersection of this plane
ith the plane of the base of the cylinder.

Find the projections of a line parallel to this line and tangent
the base of the cylinder.

Find the traces of a plane containing this tangent and an element
f the cylinder as in Problem 6, Fig. 13.

Problem 9

To find the traces of a plane which is tangent to the surface of a given sphere and passes through a given point on the surface of the sphere. (*Fig. 14.*)

Let the centre of the sphere be given by its projection o^1 and o. In order to assume the projections of a point on the surface, proceed as follows:

Assume m^1 the vertical projection of a point M on the surface of the sphere. To find the horizontal projection of the point M, join m^1o^1, obtaining the vertical projection of the radius MO through the point M. Revolve the sphere about a vertical line through O until MO becomes parallel to the vertical plane. $m_1{}^1$ is the vertical projection of M in this position. The horizontal projection of MO is then the line m_1o parallel to the ground line, the point m being on a line through $m_1{}^1$ and perpendicular to the ground line. Revolve the sphere back to its original position, finding the horizontal projection m.

Find the traces WX and XV of the plane passing through the point M and perpendicular to the line MO and these will be traces of the required plane.

Problem 10

To find the projections of the circle of contact of a cone which envelops a given sphere and has its vertex at a given point. (*Fig. 15.*)

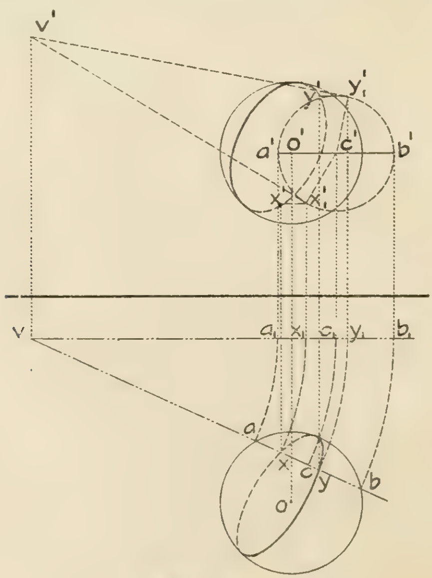


Fig. 15.

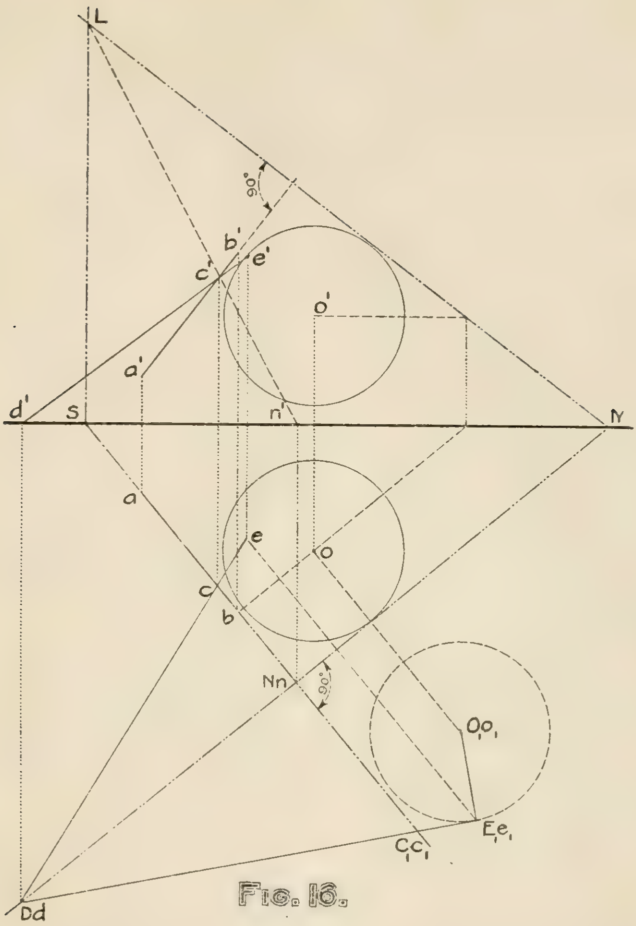


Fig. 16.

Let v^1 and v be the projections of the vertex (V) of the cone and o^1 and o of the centre (O) of the sphere.

Pass a vertical plane through the vertex V so as to cut the sphere in the circle whose horizontal projection is the straight line ab.

Revolve this plane until parallel to the vertical plane of projection. The new horizontal projection of the circle cut from the sphere is the line a_1b_1 and the point c_1 midway between a_1 and b_1 is the horizontal projection of the centre of the circle.

The vertical projection of this circle is the circle described about $c_1{}^1$ as centre and with a radius equal to c_1a_1.

The tangent $v^1x_1{}^1$ and $v^1y_1{}^1$ to this vertical projection of the circle are the vertical projections of two elements on the required tangent cone.

The points x_1^1 and y_1^1 are the vertical projections of two points on the required circle of contact.

The horizontal projections of these points are x_1 and y_1.

Revolve the sphere back to its original position obtaining the horizontal projections x and y and the vertical projections x^1 and y^1.

In the same way, by taking other auxiliary planes, a sufficient number of points on the circle of contact can be found.

Problem 11

To find the traces of a plane containing a given line and tangent to the surface of a given sphere. (*Fig. 16.*)

Let a^1b^1 and ab be the projections of the given line AB, and o^1 and o the projections of the centre O of the given sphere.

Find the traces LM and MN of a plane LMN containing the point O and perpendicular to the line AB.

Find the projections c^1 and c of the point C where AB intersects the plane LMN.

Revolve the plane LMN about its horizontal trace MN until it coincides with the horizontal plane. The point C falls at C_1 and O at O_1. With centre O_1 and radius equal to the radius of the sphere describe a circle obtaining the line cut from the sphere by the plane LMN.

Draw DC_1, E_1 tangent to this circle.

Revolve the sphere back to the original position. The point D being on the axis of revolution does not change its position. Its vertical projection is the point d^1 on the ground line.

The line c^1d^1 is the vertical projection of the line CD and cd the horizontal projection.

Find the traces of the plane containing the lines AB and CD. This last step is not shown in Fig. 16.

Intersections of Surfaces by Straight Lines

Problem 12

To find the projections of the points in which the surface of a given cone is intersected by a given straight line. (*Fig. 17.*)

Let the projections of the cone be found as in Problem 1 and let $w^1x^1$and wx be the projections of the given line WX.

Find the traces of the plane of the base of the cone, viz., ML and LS.

Find the traces of a plane containing line WX and the vertex of the cone, viz., PR and PQ.

Find the projections of the lines of intersection of the two planes MLS and RPQ, viz., q^1r^1 and qr. The line QR intersects the circumference of the base of the cone in the points Y and Z, of which the projections are y^1, z^1 amd y, z.

The lines joining the points Y and Z to the vertex of the cone are the elements cut from the cone by the plane QPR. The points E and F where the given line WX intersects these elements are the required points.

The projections of the points E and F are the points e^1, f^1 and e, f.

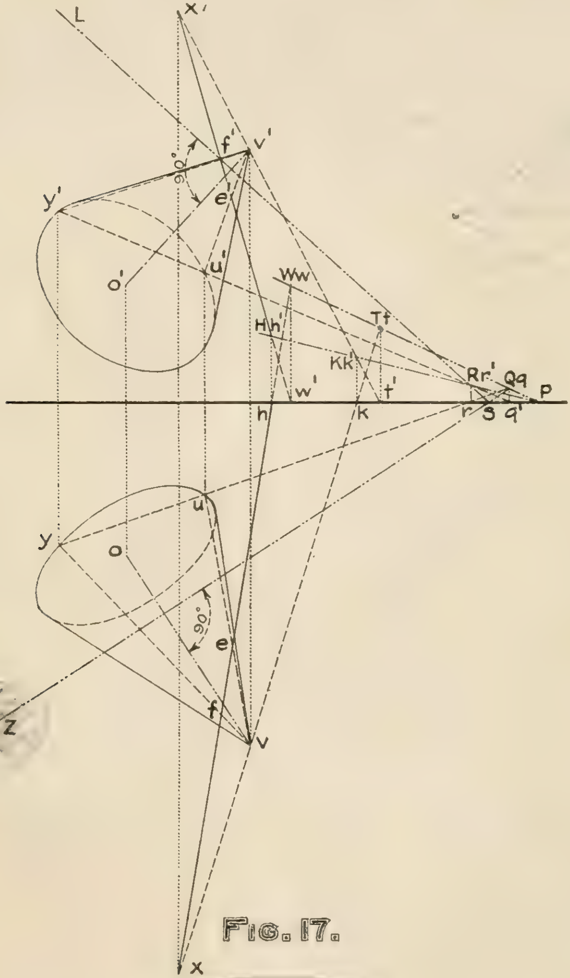


Fig. 17.

Problem 13

To find the projections of the points in which the surface of a given sphere is intersected by the given straight line. (*Fig. 18.*)

Let o^1 and o be the projections of the centre O of the given sphere, and let a^1b^1 and ab be the projections of the given line AB.

Find the traces LM and MN of the horizontal projecting plane of the line AB.

This plane LMN cuts the sphere in a circle whose horizontal projection is the line qp (part of MN).

Revolve the sphere and plane LMN about the horizontal trace MN until the plane coincides with the horizontal plane of projection. C, the centre of the circle cut from the sphere by LMN falls at C_1. The line AB falls at AB_1 and the points W_1 and X_1 where AB_1 cuts the circumference of the circle cut from the sphere by LMN are the required points. Revolve the sphere and plane back to the original position finding the projections w^1, x^1 and w, x of the points W, X where the line AB intersects the surface of the given sphere.

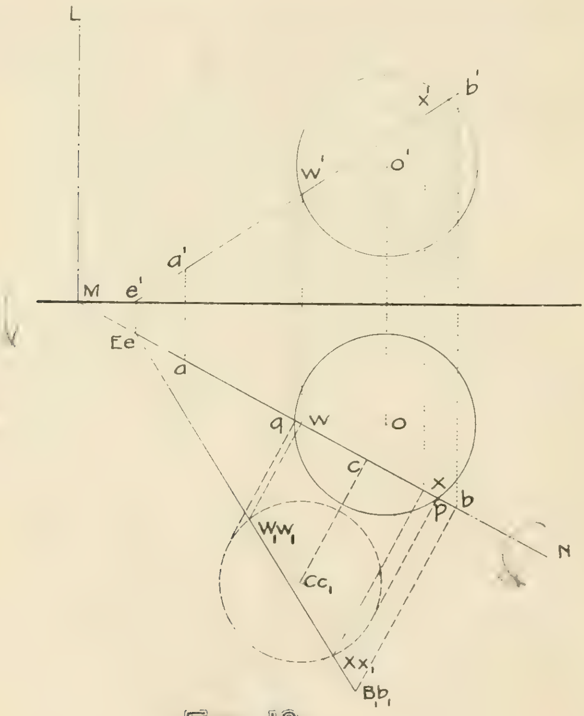


FIG. 18.

INTERSECTION OF CURVED SURFACES BY PLANES

GENERAL METHOD OF SOLUTION

Both the given curved surface and given plane are intersected by a number of auxiliary planes.

Each plane cuts each surface in a line or lines.

The points of intersection of these lines are points on the required line of intersection of the surfaces.

The auxiliary planes are so chosen that the lines cut from the given curved surface are as simple as possible.

In the case of the cylinder, the auxiliary planes should be taken either parallel to the elements of the cylinder, in which case they would cut the surface along straight lines; or parallel to the base, in which case they would cut the surface in lines similar and parallel to the circumference of the base.

In the case of the cone, the auxiliary planes should be taken either passing through the vertex, in which case they would cut the surface along straight lines; or parallel to the base, in which case they would cut the surface in lines similar and parallel to the base.

Problem 14

To find the projections of the line of intersection of a right circular cone by a plane, and to determine the exact form of the line of intersection. (*Fig. 19.*)

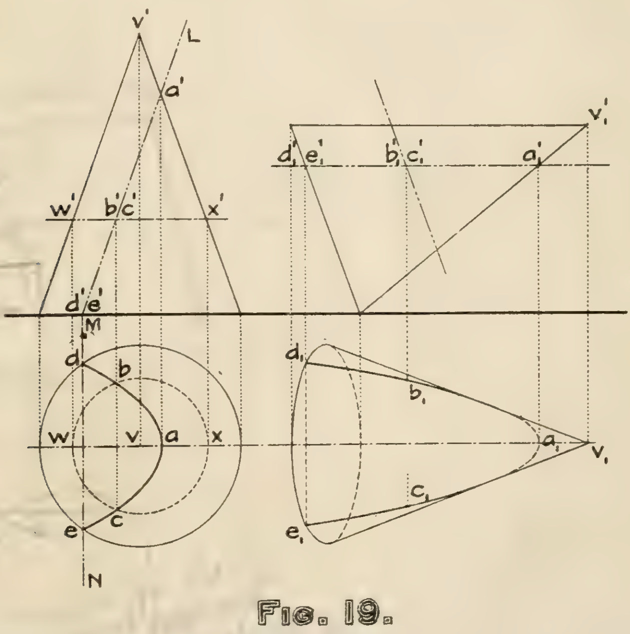


Fig. 19.

Let the projections of the cone and traces *LM* and *MN* of the plane be as given in Fig. 19.

Note—The given plane is in this case perpendicular to the vertical plane of projection. If it were not in such a position, it and the cone could be revolved about the axis of the cone until the plane would become perpendicular to the vertical plane of projection.

Solution of Problem

Take a plane parallel to the base of the cone and intersecting the surface in the circle $WXYZ$ of which the projections are the straight line w^1x^1 and the circle w, bxc.

The vertical projection of the required line of intersection is the line $a^1d^1e^1$ (the same line as the vertical trace of the plane). The vertical projection of the line cut from the cone by the auxiliary plane WX is the line w^1x^1 and the vertical projection of the line cut from the plane LMN by the auxiliary plane is the line b^1c^1 (its vertical projection being a point), and its horizontal projection is the line bc.

The points b and c are points on the horizontal projection of the required line of intersection.

By taking a sufficient number of auxiliary planes, a sufficient number of points on the horizontal projection of the line of intersection may be found.

In order to find the form of the line of intersection, the cone is revolved about some convenient axis until the given plane LMN becomes parallel to the horizontal plane of projection.

To prove that the intersection of a right circular cone by a plane is the locus of a point whose distance from a fixed point is in a constant ratio to its distance from a fixed line. (*Fig. 20.*)

Let the line $LABM$ be the line cut from the given cone by the given plane and let the line DAH be a line cut from the given plane by a plane through the axis of the cone and perpendicular to the given plane.

Consider a small sphere to be placed within the surface of the cone touching it along the circumference of the circle through the points G and E and touching the given plane at the point F.

Let DE be a line in the plane of the circle GE and in the plane containing the axis of the cone and line DAH. It intersects the line DAH in point D.

Let DC be a line through point D and perpendicular to both DE and DA.

Let point B be any point on the line $LABM$.

Let BK be the circumference of a circle through B, whose plane is perpendicular to the axis of the cone. Its plane intersects the given plane in the line BH.

Let BC be a line through B parallel to DAH.

VK is the element through points E and A.

VB is the element through point B.

The triangles AED and AKH are evidently similar and the side AH is to the side AK as the side AD is to the side AE.

that is $\frac{AH}{AK} = \frac{AD}{AE} = \frac{DH}{EK}$ (*a*)

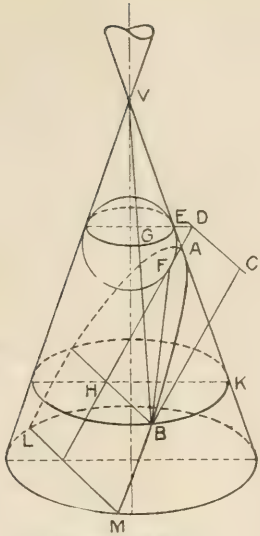


FIG. 20.

$DH = CB$

and $VB = VK$ and $VE = VG$

. : . $EK = BG$, and $BG = BF$, because both are tangent to the same sphere at the points G and F

. : . $EK = BF$ and $AE = AF$.

Substitute in equation (a).

CB for DH, BF for EK and AF for AE, obtaining the equation

$$\frac{CB}{BF} = \frac{AD}{AF}$$

But AD and AF are both constant for all positions of B on the line $LABM$.

. : . the ratio of BF to CB is constant for all positions of B, that is, the curve $LABM$ is the locus of a point whose distance from the fixed point F is in a constant ratio to its distance from the fixed line DC.

If the given plane be parallel to an element of the cone, the line BF equals BC and the ratio is therefore equal to unity, and the curve is a parabola.

If the given plane cut all the elements of the cone, the line BF is evidently less than BC and the ratio is then less than unity and the curve is an ellipse.

If the given plane is not parallel to an element and does not cut all the elements, BF is evidently greater than BC and the ratio is greater than unity and the curve is a hyperbola.

Problem 15

To find the projections of the line of intersection of a given sphere by a given plane. (*Fig. 21.*)

Let o^1 and o be the projections of the centre O of the given sphere and LM and MN the traces of the given plane LMN.

Find the projections p^1 and p of the point P, where a vertical line through centre O intersects the plane LMN.

To find the projections of P, take a plane containing the point O and parallel to the vertical plane of projection. It intersects the plane LMN in the line PQ of which pq (parallel to the ground line) is the horizontal projection, and p^1q^1 (parallel to the vertical trace LM) is the vertical projection.

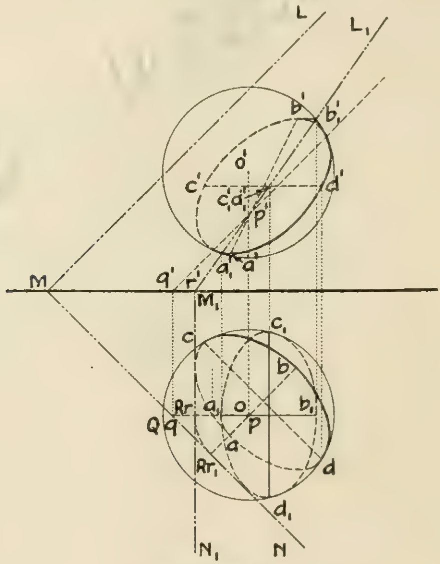


Fig. 21.

The projections p^1 and p of the point where PQ intersects the vertical line through O are then readily found.

Revolve the sphere and the plane LMN about the vertical line OP until it becomes perpendicular to the vertical plane of projection. In this position, their line of intersection is vertically projected in the straight line $a_1{}^1b_1{}^1$ and horizontally projected in the ellipse $a_1b_1c_1d_1$.

Revolve the sphere and plane back to the original position obtaining the projections of the points A, B, C and D, viz., a^1, b^1, c^1 and d^1 and a, b, c and d.

The ellipses passing through these points are the required projections of the line of intersection.

Intersection of Curved Surfaces

General Solution

Both surfaces are intersected by a system of auxiliary planes. Each auxiliary plane will cut from the given surfaces, lines the intersection of which will be points on the required line of intersection.

The auxiliary planes should be so chosen as to cut from the given surfaces the simplest lines; straight lines if possible or the circumferences of circles, or other regular curves.

Problem 16

To find the projections of the line of intersection of two cylinders. (Fig. 22.)

Let l^1m^1 and lm be the projections of the axis LM of one cylinder, and n^1o^1 and no the projections of the axis NO of the other cylinder.

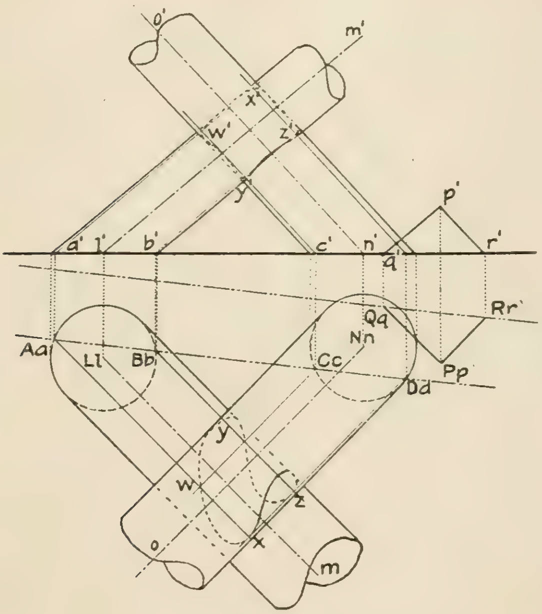


Fig. 22.

Find the traces on the planes of the bases of the two cylinders of a plane containing any point P and parallel to both axes.

Note—The planes of the bases of the cylinders in this case both coincide with the horizontal plane of projection; therefore, the line QR is the required trace upon the plane of the bases.

QR is the horizontal trace of a plane containing any point *P* and parallel to the axes of both cylinders; that is, it contains the lines *PQ* and *PR* which are respectively parallel to the axes of the cylinders.

Any other plane parallel to the axes of the cylinders will have its horizontal trace parallel to *QR*. The horizontal trace of such a plane is *ABCD* cutting the circumferences of the bases in the points *A*, *B*, *C*, *D*, and the plane must cut the cylinders in the elements which pass through these points, viz., in the elements *AWX*, *BYZ*, *CYW* and *DZX*.

The points of intersection of these elements *WXY* and *Z* are points on the required line of intersection. Their projections are the points w^1, x^1, y^1 and z^1 and w, x, y and z.

The projections of a sufficient number of points on the required line being found, the curve may be plotted as shown in the figure.

Problem 17

To find the projections of the lines of intersection of two cones. (Fig. 23.)

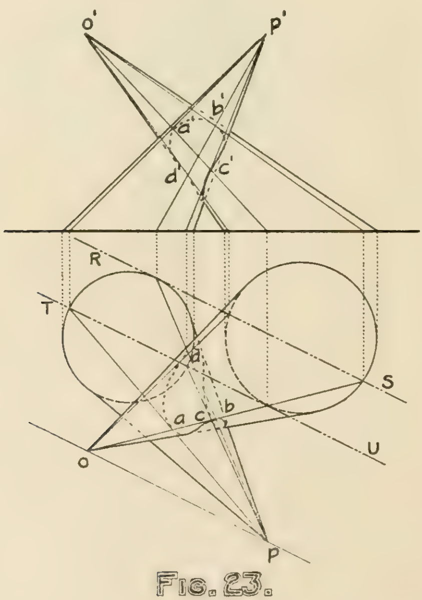


Fig. 23.

The solution of this problem is similar to that of Problem 12, with this one exception: the auxiliary planes must all pass through the vertices of the two cones instead of being parallel to the axes.

Problem 18

To find the projections of the lines of intersection of a cylinder and a cone. (Fig. 24.)

The planes are taken parallel to the axis of the cylinder and passing through the vertex of the cone; that is, they must contain a straight line through the vertex parallel to the axis of the cylinder.

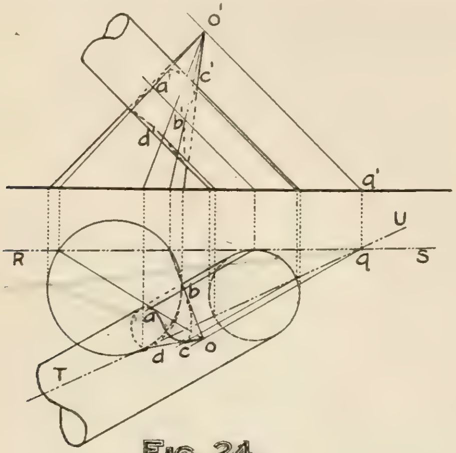


Fig. 24.

Development of Curved Surfaces

If a single curved surface be rolled over on any tangent plane until each of its elements has come into this plane, the portion of the plane thus touched by the surface and limited by the extreme elements will be a plane surface equal to the given surface and is the development of the surface.

A warped surface, or surface of double curvature, cannot be developed.

In order to determine the position of the different rectilinear elements of a single curved surface as they come into the tangent plane or plane of development, it will be necessary to find some curve upon the surface which will develop into a straight line, or a circle or some simple known curve upon which the distances between these elements can be laid off.

For illustrations of the above principles, refer to the drawings, "Intersection and Development of Cylinder and Cone" and "Intersection and Development of Pipes".

Shades and Shadows

Light is transmitted along straight lines radiating in every direction from each point of a luminous body.

If, therefore, an opaque body be placed between a source of light and a surface, it will cut off a portion of the light and a shadow will be cast on the surface and the part of the body hidden from the source of light will be in shadow.

The line of shade on a body is the line separating the illuminated part from the part in shade.

The line of shadow is the outline of the shadow.

Rays of sunlight are sensibly parallel, because the sun is a great distance away.

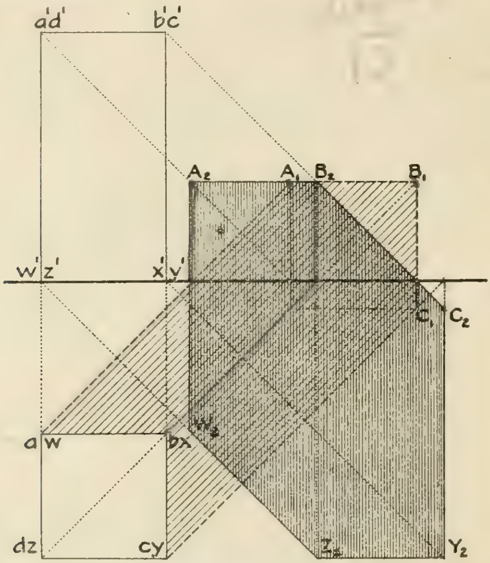


FIG. 25.

In descriptive geometry, the rays of light are generally assumed to take such a direction that their projections on either plane make angles of 45° with the ground line.

The line of shade on a body may be assumed to be traced out by passing a tangent ray around the body, keeping its direction constant and always in contact with the body. The locus of the point of contact is the line of shade.

The line of shade having been traced out, the shadow of the body may then be found by passing rays of light through its various points and finding their intersections with the surface upon which the shadow falls.

The line of shadow is thus an oblique projection of the line of shade.

In Fig. 25, the lines joining the points $A_2B_2C_2Y_2Z_2W_2$ are the lines of shadow cast upon the vertical plane of projection by the rectangular prism $ABCDWXYZ$ and the lines joining the points $WA_1B_1C_1YZ$ are the lines of shadow cast upon the horizontal plane of projection.

The lines of shade would be the lines joining A, B, C, Y, Z, W.

In Fig. 26, the circle $ABCD$ has its plane parallel to the horizontal plane of projection. Its shadow on the horizontal plane is therefore the equal circle $A_1B_1C_1D_1$ and its shadow on the vertical plane, the ellipse $A_2B_2C_2D_2$.

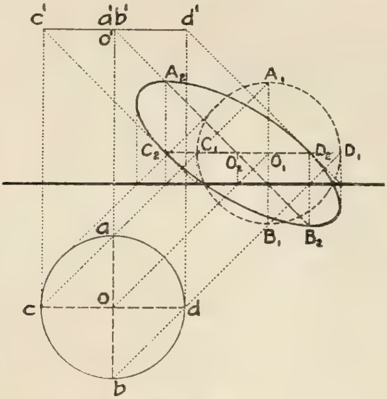


FIG. 26.

PERSPECTIVE

The perspective of an object is its projection upon a plane (usually a vertical plane) when the point of sight is at a finite distance from the plane of projection which is commonly called the picture plane.

The picture plane is usually taken between the object and the point of sight.

The point of sight is the point from which the object is viewed and is the vertex of a cone from which all projecting lines or rays to different points on the object radiate. Hence, the perspective of an object is said to be a conical projection.

The orthographic projection of the points of sight S on the picture plane is the *centre of the picture*.

A horizontal line through the centre of the picture is the *horizon*. It is the trace on the picture plane of a horizontal plane through the point of sight.

The vanishing point of a line is the perspective of a point at infinity on the line, and is found by drawing a line through the point of sight S parallel to the line and finding where it intersects the picture plane. This last point is the vanishing point.

The measuring point for a given line is the vanishing point of another line equally inclined to the given line and to the picture plane.

A line perpendicular to the picture plane is called a *perpendicular.* Another definition for the centre of the picture is this: It is the vanishing point for all perpendiculars. The term diagonal is applied to a horizontal line inclined at an angle of 45° to the picture plane.

The distance point or point of distance is a point on the horizon distant from the centre of the picture a length equal to the distance of the point of sight (S) from the picture plane. It is the vanishing point for all diagonals.

In Fig. 27, a method of drawing a perspective is shown.

The rectangular prism is given by its orthographic projection on the horizontal and vertical planes, and it is required to find its perspective on the plane PP, which is perpendicular to the ground line, the point of sight S being given by its horizontal and vertical projections.

Join the various corners of the prism to the point of sight S by straight lines, and find where these straight lines intersect the plane PP_1.

The plane PP is then revolved about its vertical trace until it coincides with the vertical plane of projection. The figure $a_1b_1c_1d_1$ $w_1x_1y_1z_1$ is the perspective of the given prism.

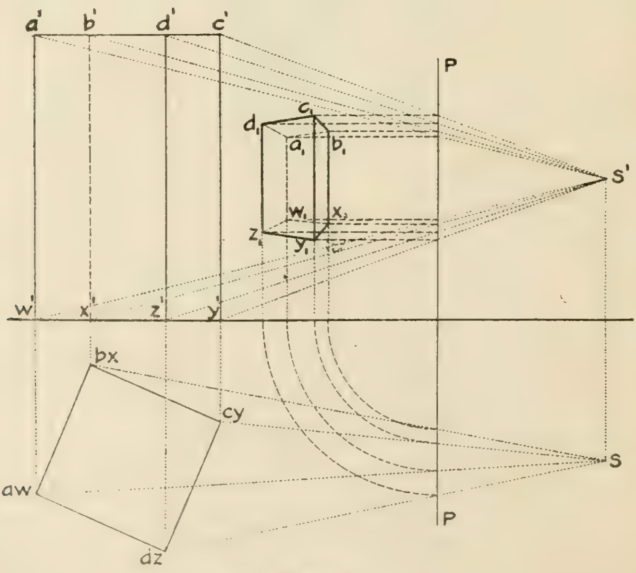


FIG. 27.

The general method of drawing perspectives of objects is to find the perspectives of lines rather than of isolated points.

The general method of drawing the perspective of a given line is illustrated in Fig. 28.

Let a^1b^1 and ab be the orthographic projections on the vertical and horizontal planes of projection, and s^1 and s the orthographic projections of the point of sight on the same planes.

We might find the perspective of the line AB on the vertical plane, by finding the perspectives of its extermities A and B, simply by joining A and B to the point of sight and finding where the lines this found intersect the vertical plane.

The following method is, however, the one generally used.

If a line SD be drawn through the point of sight S and a point at infinity on the given line AB, it will be a line parallel to the given line and the point where this pierces the picture plane, viz., D will be the perspective of the point at infinity on the line and is known as the vanishing point for AB and, consequently, the vanishing point for all lines parallel to AB.

If the point C where the given line AB intersects the vertical plane be found, the point C will be its own perspective, and the line joining C and D will be the perspective of the portion of the line AB reaching from the point C to infinity.

CD is known as the indefinite perspective of the given line AB.

To find the perspective of the point A on the given line AB, assume the horizontal and vertical projections of any line AE passing through A; then find the indefinite perspective of AE, viz., EG. The point a_1 where the lines CD and EG intersect, is the perspective of the point A. In the same way, the perspective b_1 of the point B is found.

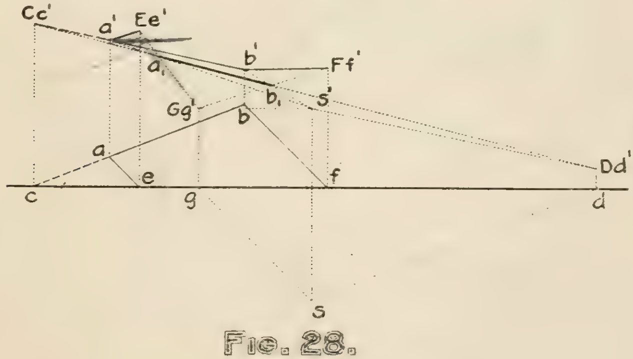


FIG. 28.

The measuring point is the vanishing point for lines equally inclined to the picture plane and the given line; that is, if the auxiliary line AE made the same angle with the picture plane (in this case the vertical plane of projection) as with the given line AB, its vanishing point Q would be the measuring point for AB.

In Fig. 29, let S and S^1 be the horizontal and vertical projection of the point of sight.

Let $ABCD$ ($wxyz$) be the horizontal projection of a rectangular prism, the base $ABCD$ resting on the horizontal plane of projection. The horizontal trace of a plane containing the vertical edge BX and the point of sight is the line Bs. The vertical trace of the plane is b_1, x_1, ∴ b_1, x_1, is the perspective of the edge BX. The perspective of the edge AB may be found as follows.

Produce AB to intersect the vertical plane at point E. If the point of sight S be joined to a point at infinity on AB by a straight line, the straight line will be parallel to AB.

Therefore, find the projection of a straight line passing through S and parallel to AB. It intersects the vertical plane at point R. The point R is the perspective of a point on AB at infinity, and is known as the vanishing point for the line.

The line joining E and R is the indefinite perspective of the line AB.

The perspective of point B is the point b_1, the intersection of b_1, x_1, and ER.

BK is a line equally inclined to the given line AB and the vertical plane ∴ $EK = EB$. The indefinite perspective of BK,

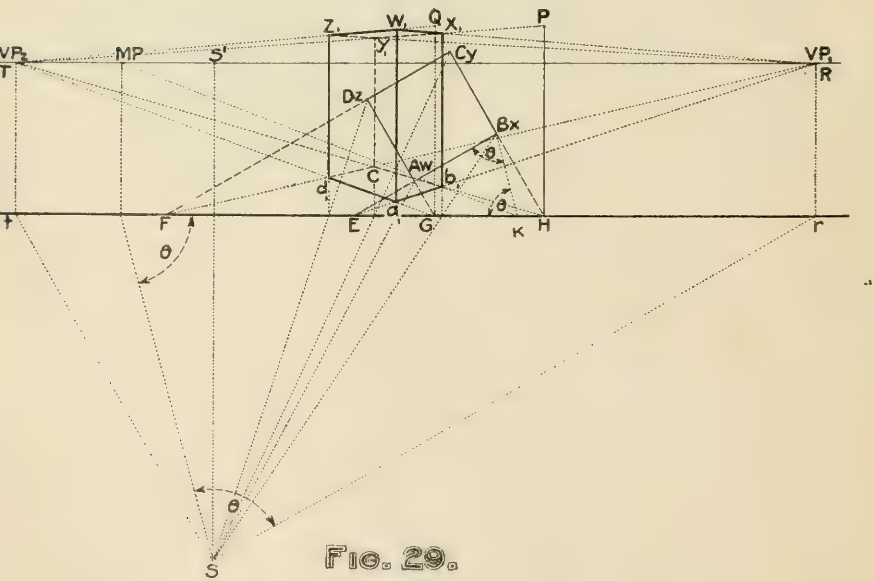


Fig. 29.

viz., BMP, is found by obtaining its vanishing point (called the measuring point for AB) and joining K and MP.

b_1 is the intersection of ab and KMP. The indefinite perspective of BC, viz., b_1, c_1, may be found by obtaining its vanishing point

VP_2 and joining the point H to VP_2. The perspective of the point B may be found by the intersection of the perspectives of any two of the lines BX, AB, CB, or BK.

Shadows in Perspective

A convenient method of obtaining shadows in perspective is illustrated in Fig. 30.

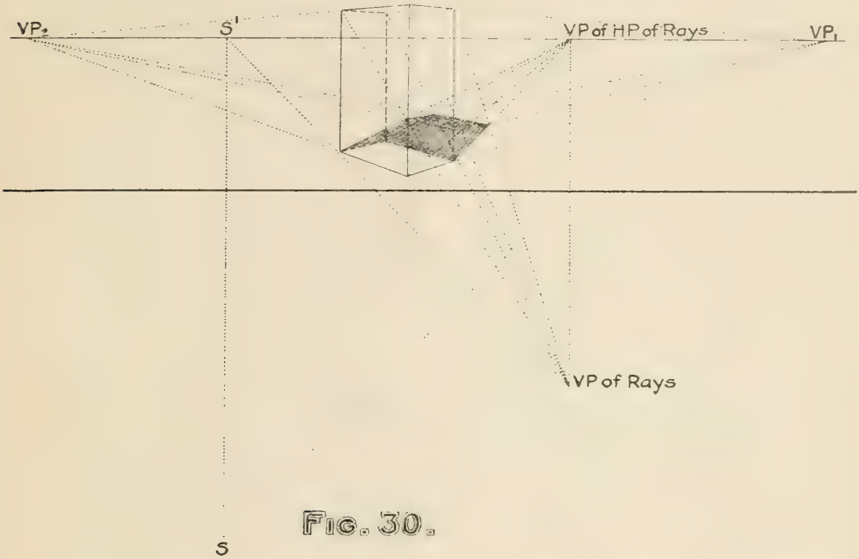


Fig. 30.

The method of obtaining the perspective of the shadow of the prism on the horizontal or ground plane is as follows.

Join the perspective of a point to the vanishing point of rays and join the perspective of the horizontal projection of the same point to the vanishing point of the horizontal projections of rays. The intersection of the two lines is the perspective of the saadow of the point on the horizontal plane.

APPENDIX

Methods of Computing Areas, Centres of Gravity, and Moments of Inertia

1. The Trapezoidal Rule.

Area $= h\left(\frac{a}{2}+b+c+d\ldots+\frac{z}{2}\right)$ where a, b, c, . . . z are the ordinates and h the common interval between them. See Fig. 1.

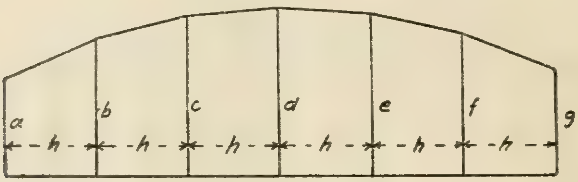


Fig. 1

2. Simpson's First Rule.

$$\text{Area} = \frac{1h}{3}(a+4b+2c+4d+2e+4f+g)$$

An approximate proof for this rule is as follows. See Fig. 2.

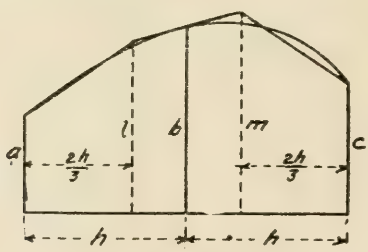


Fig. 2

The area below the curve is practically equal to the area of the trapezoid.

$$\text{Area} = \left(\frac{a+l}{2}\right)\frac{2h}{3} + \left(\frac{l+m}{2}\right)\frac{2h}{3} + \left(\frac{m+c}{2}\right)\frac{2h}{3}.$$

$$= (a+2l+2m+c)\ \frac{1h}{3}.$$

$$= (a+4b+c)\ \frac{1h}{3}.$$

A more exact proof is as follows:

Assuming the equation to the curve to be $y=a_0+a_1x+a_2x^2$. See Fig. 3.

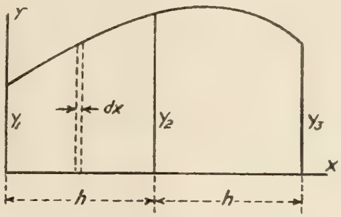


Fig. 3

Area of narrow strip $= ydx$.

$$\text{whole area} = \int_0^{2h} ydx$$

$$\text{or} \int_0^{2h} (a_0+a_1x+a_2x^2)dx.$$

$$\text{which} = \left(a_0x+\frac{1}{2}a_1x^2+\frac{1}{3}a_2x^3\right)_0^{2h}$$

which has to be evaluated between the limit $x=2h$ and $x=0$. The expression then becomes

$$a_02h+\frac{1}{2}a_14h^2+\frac{1}{3}a_28h^3. \qquad \text{(A)}$$

Now from the equation to the curve when

$x=0 : y=a_0$
$x=h : y=a_0+a_1h+a_2h^2$
$x=2h : y=a_0+2a_1h+4a_2h^2$

But calling the ordinates in the ordinary way y_1, y_2, y_3.
when $x=0$, $y=y_1$,
$x=h$, $y=y_2$,
$x=2h$, $y=y_3$,
therefore $a_0=y_1$

$a_0+a_1h+a_2h^2=y_2$
$a_0+2a_1h+4a_2h^2=y_3$

$$\therefore a_0=y_1$$

$$a_1=\frac{1}{2h}(4y_2-3y_1-y_3)$$

$$a_2=\frac{1}{2h^2}(y_3-2y_2+y_1).$$

Substituting these values of a_0, a_1 and a_2 in expression (A)

$$\therefore \text{Area} = y_1 2h + \frac{4h^2}{2}\cdot\frac{1}{2h}(4y_2 - 3y_1 - y_3) + \frac{8h^3}{3}\cdot\frac{1}{2h^2}(y_3 - 2y_2 + y_1)$$

$$= \frac{1}{3}h(y_1 + 4y_2 + y_3).$$

Indirect Proof for Simpson's First Rule.

Assuming the equation to the curve to be $y = a_0 + a_1x + a_2x^2$.

By Simpson's First Rule, Area $= \frac{h}{3}(y_1 + 4y_2 + y_3)$.

$$y_1 = a,$$
$$y_2 = a_0 + a_1h + a_2h^2,$$
$$y_3 = a_0 + 2a_1h + 4a_2h^2.$$

$$\text{Area} = \frac{h}{3}(a_0 + 4a_0 + 4a_1h + 4a_2h^2 + a_0 + 2a_1h + 4a_2h^2)$$

$$= \frac{h}{3}(6a_0 + 6a_1h + 8a_2h^2)$$

$$= 2a_0h + 2a_1h^2 + \frac{8}{3}a_2h^3. \qquad \text{(B)}$$

$$\text{Area} = \int_o^{2h} ydx = \int_o^{2h}(a_0 + a_1x + a_2x^2)dx$$

$$= \left(a_0x + \frac{1}{2}a_1x^2 + \frac{1}{3}a_2x^3\right)_o^{2h}$$

$$= 2a_0h + 2a_1h^2 + \frac{8}{3}a_2h^3$$

which is the same as expression (B).

3. Simpson's Second Rule.

$$\text{Area} = \frac{3}{8}h\,(a + 3b + 3c + 2d + 3e + 3f + g).$$

Assuming the equation to the curve to be
$y = a_0 + a_1x + a_2x^2 + a_3x^3$. See Fig. 4.

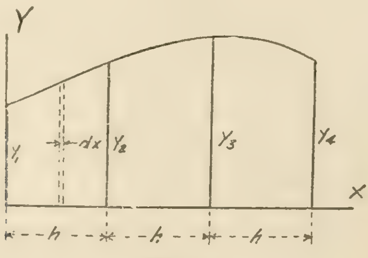


Fig. 4

Area of narrow strip ydx

whole area $= \int_o^{3h} ydx$

$$= \int_o^{3h} (a_0+a_1x+a_2x^2+a_3x^3)\,dx$$

which equals $\left(a_0 + \frac{1}{2}a_1x^2 + \frac{1}{3}a_2x^3 + \frac{1}{4}a_3x^4\right)_o^{2h}$

which has to be evaluated between the limits $x=3h$ and $x=0$.

The expression then becomes

$$a_0 3h + \frac{1}{2}a_1 9h^2 + \frac{1}{3}a_2 27h^3 + \frac{1}{4}81h^4$$

From the equation to the curve.

When

$x=0,\ y=a_0=y_1$
$x=h,\ y=a_0+a_1h+a_2h^2+a_3h^3=y_2$
$x=2h,\ y=a_0+2a_1h+4a_2h^2+8a_3h^3=y_3$
$x=3h,\ y=a_0+3a_1h+9a_2h^2+27a_3h^3=y_4$

Solve the above equations for values of a_0, a_1, a_2 and a_3 and substitute these values in the equation for the area and the result is

$$\text{area} = \frac{3}{8}h\,(y_1+3y_2+3y_3+y_4).$$

Indirect Proof for Simpson's Second Rule.

Assuming the equation to the curve to be

$y=a_0+a_1x+a_2x^2+a_3x^3.$

$$\text{Area} = \int_o^{3h} (a_0+a_1x+a_2x^2+a_3x^3)\,dx$$

$$= \left(a_0x + \frac{1}{2}a_1x^2 + \frac{1}{3}a_2x^3 + \frac{1}{4}a_3x^4\right)_o^{3h}$$

$$= 3a_0h + \frac{9}{2}a_1h^2 + 9a_2h^3 + \frac{81}{4}a_3h^4 \qquad \text{(C)}$$

By Simpson's Second Rule,

$\text{Area} = \frac{3}{8}h\,(y_1+3y_2+3y_3+y_4)$

Now $y_1=a_0,$

$y_2=a_0+a_1h+a_2h^2+a_3h^3,$
$y_3=a_0+2a_1h+4a_2h^2+8a_3h^3,$
$y_4=a_0+3a_1h+9a_2h^2+27a_3h^3.$

Substitute these values in expression

$\text{Area} = \frac{3}{8}h\,(y_1+3y_2+3y_3+y_4)$

$$= \frac{3}{8} h\,(a_0 + 3(a_0 + a_1h + a_2h^2 + a_3h^3) + 3(a_0 + 2a_1h + 4a_2h^2$$
$$+ 8a_3h) + a_0 + 3a_1h + 9a_2h^2 + 27a_3h^3)$$
$$= \frac{3}{8} h\,(8a_0 + 12a_1h + 24a_2h^2 + 54a_3h^3)$$
$$= 3a_0h + \frac{9}{2} a_1h + 9a_2h^2 + \frac{81}{4} a_3h_4$$

which is the same as expression (C).

4. The Five-Eight Rule or Simpson's Third Rule.

Area $= \frac{1}{12} h\,(5y_1 + 8y_2 - y_3)$.

The proof of this rule follows directly from the proof of Simpson's first rule. See Fig. 5.

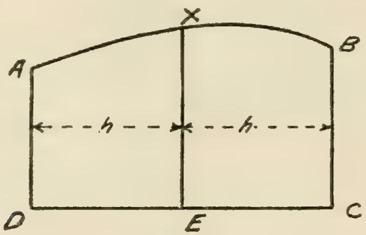


Fig. 5

By Simpson's First Rule

Area of figure $AXBCD = \frac{1}{3} h\,(AD + 4XE + BC)$.

By the five-eight rule

Area $AXED = \frac{1}{12} h \cdot (5AD + 8XE - BC)$

and area $XBCE = \frac{1}{12} h\,(5BC + 8XE - AD)$.

Adding these two areas

$$= \frac{1}{12} h\,(4AD + 16XE - AD)$$
$$= \frac{1}{3} h\,(AD + 4XE + BC).$$

It should be noted that the result found by the formula is the area between two ordinates only, although three ordinates are required to find it.

The extension of the five-eight rule is as follows:

$$\frac{1}{12}h\begin{cases}5 \quad 8-1 \\ \quad 5 \quad 8-1 \\ \quad\quad 5 \quad 8-1 \\ \quad\quad\quad 5 \quad 8-1\end{cases}$$

$$\frac{1}{12}h\,(5y_1+13y_2+12y_3+12y_4+7y_5-y_6)$$

$$\text{or} = h\left(\frac{5}{12}y_1+\frac{13}{12}y_2+y_3+y_4+\frac{7}{12}y_5-\frac{1}{12}y_6\right)$$

The trapezoidal rule is applicable when there are any number of equidistant ordinates.

Simpson's first rule is applicable only when there are an even number of equal spaces between ordinates.

Simpson's second rule is applicable only when the number of spaces between the ordinates is divisible by three.

The five-eight rule is applicable when there are any number of equidistant ordinates, provided an ordinate beyond the area required be known.

The position of the centre of gravity of a plane figure or solid may be readily found by taking moments about any convenient axis.

The following examples illustrate the method of obtaining the area and position of the centre of gravity of a plane figure bounded on one side by a curve.

1. Solution by Simpson's First Rule.

Distances between ordinates = 2′ − 0″.

No.	Length of Ordinate	Simp. Mult.	Functions of Ordinate	Levers	Products
1.........	1.45	1	1.45	0	0.0
2.........	2.65	4	10.60	1	10.60
3.........	4.35	2	8.70	2	17.40
4.........	6.45	4	25.80	3	77.40
5.........	8.50	2	17.00	4	68.00
6.........	10.40	4	41.60	5	208.00
7.........	11.85	1	11.85	6	71.10
			117.00		452.50

$$\text{Area} = \frac{1}{3}(117 \times 2) = 78 \text{ sq. ft.}$$

$$\text{Distance of } C \text{ of } G \text{ from Ordinate No. } 1 = \frac{452.50}{117} \times 2 = 7.72 \text{ ft.}$$

2. Solution by Simpson's Second Rule.

No.	Length of Ordinate	Simp. Mult.	Functions of Ordinate	Levers	Products
1	1.45	1	1.45	0	0.0
2	2.65	3	7.95	1	7.95
3	4.35	3	13.05	2	26.10
4	6.45	2	12.90	3	38.70
5	8.50	3	25.50	4	102.00
6	10.40	3	31.20	5	156.00
7	11.85	1	11.85	6	71.10
			103.90		401.80

Area $= \frac{3}{8}(103.90 \times 2) = 77.925$ sq. ft.

Distance of C of G from Ordinate No. $1 = \frac{401.80}{103.90} \times 2 = 7.71$ ft.

Moment of Inertia.

$$I = \frac{1}{3}\int y^3\,dx$$

The moment of inertia about the axis OX may be found by Simpson's Rules as follows. Substitute the cubes of the ordinates for the ordinates.

Multiply these by Simpson's multipliers in the ordinary way thus obtaining the required functions of cubes.

Multiply the sum of the functions of cubes by $\frac{1}{3}$ the common interval If using Simpson's First Rule, and by $\frac{3}{8}$ the common interval if using Simpson's Second Rule.

This gives the value of $\int y^3 dx$. Divide this result by 3 and the result is the required Moment of Inertia.

AMSER'S MECHANICAL INTEGRATOR

Areas

Area of elementary strip $= ydx$

$= l \sin . \theta dx.$

For a movement dx of Integrator, movement of wheel A (length of arc on circumference) $= dx \sin . \theta.$

$\therefore l.$ (movement of wheel A) $= l \sin . \theta dx =$ elementary area and $l.$ (total movement of wheel A) $= \int l \sin . \theta dx =$ total area, *i.e.*, if each unit length on the circumference of the wheel be divided into l equal spaces, the number of spaces through which the wheel turns when the tracing point moves around a plane figure equals the number of square units contained within the figure.

Moments

Moment of elementary strip about axis $OX = \frac{1}{2} y^2 dx = \frac{1}{2} (l \sin . \theta)^2 dx.$

Movement of wheel M (length of arc on circumference) $= dx \cos 2\theta = dx(\cos^2\theta - \sin^2\theta) = dx(1 - 2 \sin^2\theta).$

When this is integrated $\int dx = 0$ because the tracing point returns to its original position.

$\therefore$ Total movement of wheel $= \int -(2 \sin^2\theta) dx$ or

$$= -2 \int \sin^2\theta dx.$$

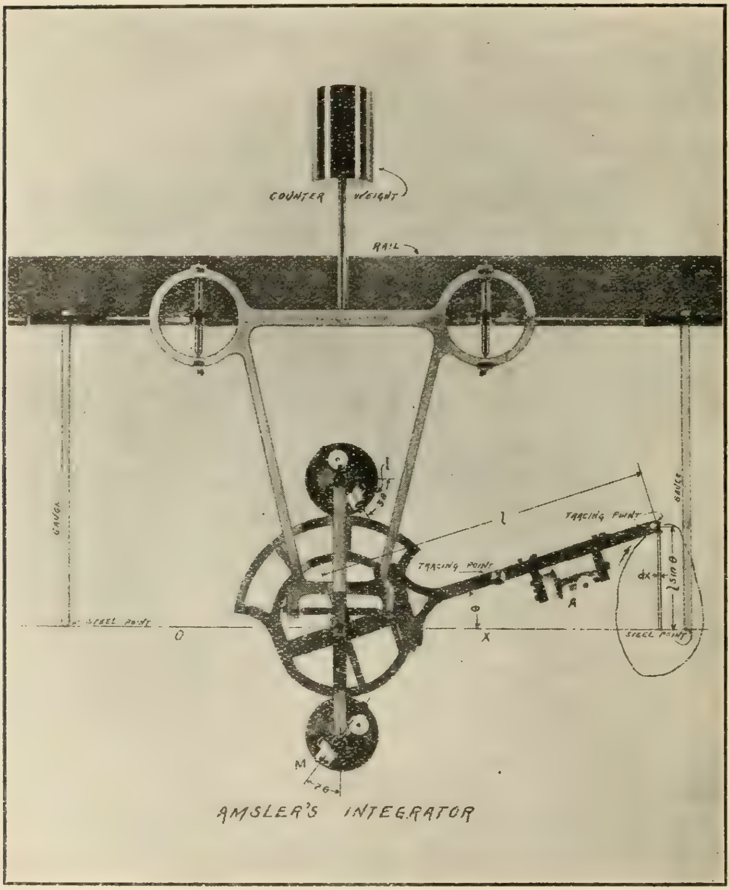
If length of divisions on wheel $= \frac{1}{l}$, reading on wheel $= -2l \sin^2\theta dx$ or $l \frac{\text{reading on wheel}}{-4} = \frac{1}{2} l^2 \int \sin^2\theta dx =$ Moment of area about axis OX.

By reversing the sense of the graduations on the wheel M the moment is given by the expression

$$\frac{l \text{ reading on wheel}}{4}.$$

$$\text{Moment of Inertia} = \frac{1}{3}\int y^3 dx = \frac{1}{3}\int l^3 . \sin^3\theta . dx.$$

because $y = l \sin \theta$



$$\text{Reading on } A \text{ Scale} = \int dx \sin \theta \text{ (constant}_1)$$

$$\text{Reading on } I \text{ Scale} = \int dx \sin 3\theta \text{ (constant}_2)$$

$$\text{Sin } 3\theta = \sin\theta\cos^2\theta - \sin^3\theta + 2\sin\theta\cos^2\theta$$
$$= 3\sin\theta\cos^2\theta - \sin^3\theta$$

$$=3\sin\theta(1-\sin^2\theta)-\sin^3\theta$$
$$=3\sin\theta-4\sin^3\theta$$

dividing both sides by 12

$$\frac{\sin 3\theta}{12}=\frac{\sin\theta}{4}-\frac{\sin^3\theta}{3}$$

$$\therefore\ \frac{\sin^3\theta}{3}=\frac{\sin\theta}{4}-\frac{\sin 3\theta}{12}$$

$$\frac{\sin^3\theta}{3}l^3=\frac{\sin\theta}{4}l^3-\frac{\sin 3\theta}{12}l^3$$

$$I=\frac{\sin^3\theta}{3}l^3dx=\frac{\sin\theta}{4}l^3dx-\frac{\sin 3\theta}{12}l^3dx$$

$$I=\sin\theta dx\left(\frac{l^3}{4}\right)-\sin 3\theta dx\left(\frac{l^3}{12}\right)$$

$$=\frac{\text{Reading on } A \text{ Scale}}{(\text{Constant}_1)}\left(\frac{l^3}{4}\right)-\frac{\text{Reading on } I \text{ Scale}}{(\text{Constant}_2)}\left(\frac{l^3}{12}\right)$$

Const. 1 = number of divisions per unit length on circumference of A wheel.

Const. 2 = number of divisions per unit length on circumference of I wheel.

No 1

No 2

No 6

No 10

No 12 / 13

Intersection of
curve surfaces
by a plane (15)

General development
of a curve surface

General on
perspective

Pay attention
the drawing
[illegible]