Syllabus of a course in Analytical Geometry of Three Dimensions

by

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SYLLABUS

OF A

OURSE IN ANALYTICAL GEOMETRY

OF

THREE DIMENSIONS.

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SYLLABUS

OF A

Course in Analytic Geometry of Three Dimensions.

1. Explain the method of denoting the position of a point in space by *Cartesian Coördinates.*

State the convention concerning the signs of coördinates.

2. State the convention concerning positive rotation about any axis.

3. Explain *Polar Coördinates* in space.

Obtain formulas for transforming from rectangular to polar coördinates.

$$[1] \qquad \begin{aligned} x &= r\cos\phi, \\ y &= r\sin\phi\cos\theta, \\ z &= r\sin\phi\sin\theta. \end{aligned}$$

4. Problem. To find the distance between two points whose coördinates are given.

$$[2] \qquad D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

5. Problem. To divide a line in any given ratio, $m_1 : m_2$.

$$[3] \quad x = \frac{m_2 x_1 + m_1 x_2}{m_2 + m_1}, \quad y = \frac{m_2 y_1 + m_1 y_2}{m_2 + m_1}, \quad z = \frac{m_2 z_1 + m_1 z_2}{m_2 + m_1}.$$

If the line is bisected,

$$[4] \qquad x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}, \quad z = \frac{z_1 + z_2}{2}.$$

6. Define the projection of a point on a plane; of a line on a plane; of a point on a line; of a line on a line.

Prove that the projection of a line l on a line p is

[5] $$l_p = l \cos \alpha,$$

where α is the angle between the two lines.

7. Show that the rectangular coördinates of any point are the projections of the radius vector of the point on the three axes; so that if α, β, and γ are the angles made by r with the axes of X, Y, and Z respectively, we have

[6] $$x = r \cos \alpha, \quad y = r \cos \beta, \quad z = r \cos \gamma,$$

and

[7] $$r^2 = x^2 + y^2 + z^2.$$

α, β, and γ are called the *Direction Angles* of r.

8. Define the *Direction Cosines* of a line, and prove that the sum of their squares is unity.

[8] $$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

The radius vector of a point and its direction angles may be used as a set of polar coördinates.

9. Problem. To find the angle between two lines when their direction cosines are given.

[9] $$\cos \theta = \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2.$$

The lines are perpendicular if

[10] $$0 = \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2.$$

They are parallel if

[11] $$\alpha_1 = \alpha_2, \quad \beta_1 = \beta_2, \quad \gamma_1 = \gamma_2.$$

Transformation of Coördinates.

10. Problem. To transform to a new set of axes parallel to the old.

$$[12] \qquad x = x_0 + x', \quad y = y_0 + y', \quad z = z_0 + z'.$$

11. To transform from one set of axes to a second set having the same origin.

$$[13] \qquad \begin{aligned} x &= x' \cos\alpha_1 + y' \cos\alpha_2 + z' \cos\alpha_3, \\ y &= x' \cos\beta_1 + y' \cos\beta_2 + z' \cos\beta_3, \\ z &= x' \cos\gamma_1 + y' \cos\gamma_2 + z' \cos\gamma_3. \end{aligned}$$

Show that the degree of an equation cannot be altered by either of these transformations.

12. Explain what is meant in space by the *locus* of an equation or pair of equations.

Show that a single equation between x, y, and z represents a surface. That a pair of such equations represent a line.

Show how the form of a surface whose equation is given may be investigated by means of its plane sections.

An equation containing only two variables represents a *cylindrical* surface.

Show how to obtain the equation of a surface formed by revolving a plane curve about one of the axes.

The Plane.

13. Problem. To find the equation of a plane in terms of the perpendicular from the origin and its direction cosines.

$$[14] \qquad x \cos\alpha + y \cos\beta + z \cos\gamma = p.$$

14. Prove that every equation of the first degree,

$$Ax + By + Cz + D = 0,$$

represents a plane, and show how to reduce it to the form [14].

15. Express the equation of a plane in terms of its *intercepts* on the axes.

[15] $$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

16. Problem. To find the equation of a plane through three given points.

17. Problem. To find the distance from a given point to a given plane.

[16] $$D = x_1 \cos\alpha + y_1 \cos\beta + z_1 \cos\gamma - p.$$

18. Problem. To find the angle between two planes.

[17] $$\cos\theta = \frac{A_1 A_2 + B_1 B_2 + C_1 C_2}{\sqrt{(A_1^2 + B_1^2 + C_1^2)(A_2^2 + B_2^2 + C_2^2)}}.$$

They are perpendicular if

[18] $$A_1 A_2 + B_1 B_2 + C_1 C_2 = 0.$$

They are parallel if

[19] $$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}.$$

19. Problem. To find the equation of a plane passing through a given point and parallel to a given plane; passing through two given points and perpendicular to a given plane.

The Straight Line.

20. Show that the equations of a line may always be thrown into the form

[20]
$$x = mz + a,$$
$$y = nz + b.$$

21. Find the equations of a line in terms of its direction cosines, and the coördinates of a point through which it passes.

[21]
$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}.$$

22. Problem. To throw the equations of any line into the form [21]

[22]
$$\frac{\frac{x-a}{m}}{\sqrt{1 + m^2 + n^2}} = \frac{\frac{y-b}{n}}{\sqrt{1 + m^2 + n^2}} = \frac{\frac{z-c}{1}}{\sqrt{1 + m^2 + n^2}}.$$

23. Problem. To find the equations of a line passing through two given points.

[23]
$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

24. Problems concerning the angles between lines, or between lines and planes, can be readily solved by the use of the direction cosines of the lines and those of the normals to the planes.

The Sphere.

25. Problem. To find the equation of a sphere in terms of the coördinates of its centre and the length of its radius.

[24]
$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

When the centre is at the origin, this becomes

[25]
$$x^2 + y^2 + z^2 = r^2.$$

26. Show that the most general form of the equation of a sphere is

$$[26] \qquad x^2 + y^2 + z^2 + Gx + Hy + Iz + K = 0,$$

and show how to reduce any equation of this form to the form [24], and thus to determine its centre and radius.

27. Problem. To find the equation of a sphere passing through four given points.

28. Show that any two spheres intersect in a circle.

29. Find the equation of the tangent plane at a given point on the surface of the sphere, $x^2 + y^2 + z^2 = r^2$.

$$[27] \qquad x_1 x + y_1 y + z_1 z = r^2.$$

30. Problem. To find the equations of the normal at any point of the sphere.

$$[28] \qquad \frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1}.$$

Prove that every normal is a radius.

31. Problem. To find the locus of points dividing harmonically secants drawn from a given point to a sphere.

$$[29] \qquad x_1 x + y_1 y + z_1 z = r^2.$$

This is called the *polar plane* of the given point, and passes through the points of contact of all the tangents that can be drawn from the given point to the sphere.

32. Prove that if several points lie in a plane, their polar planes pass through the pole of the given plane; and conversely, that if several planes pass through a point, their poles lie on the polar plane of that point.

33. Prove that the polar plane of a point is perpendicular to the line joining the point with the centre of the sphere, and that the product of the distance of the pole from the centre and the distance of the polar plane from the centre is equal to the square of the radius.

34. PROBLEM. To find the locus of the middle points of a set of parallel chords.

[30] $$x\cos\alpha + y\cos\beta + z\cos\gamma = 0.$$

Such a locus is a *diametral plane.*

Define *diameter; conjugate diameters.*

THE CENTRAL QUADRICS.

35. The central quadrics are the *ellipsoid*, the *bi-parted hyperboloid*, the *un-parted hyperboloid*, and the *cone.*

[31] $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\quad \frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$$

Investigate their forms.

36. Find the equation of the tangent plane to a central quadric.

[32] $$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}+\frac{z_1z}{c^2}=1.$$

Of the normal line.

[33] $$\frac{a^2}{x_1}(x-x_1)=\frac{b^2}{y_1}(y-y_1)=\frac{c^2}{z_1}(z-z_1).$$

37. Find the equation of the polar plane of a point with respect to a central quadric,

[34] $$\frac{x_1 x}{a^2}+\frac{y_1 y}{b^2}+\frac{z_1 z}{c^2}=1,$$

and prove that sections 31 and 32 apply to any central quadric as well as to the sphere.

38. Find the equation of the diametral plane conjugate to a given chord of a central quadric.

[35] $$\frac{x\cos\alpha}{a^2}+\frac{y\cos\beta}{b^2}+\frac{z\cos\gamma}{c^2}=0.$$

39. The diametral plane conjugate to the diameter through (x_1, y_1, z_1) is

[36] $$\frac{x_1 x}{a^2}+\frac{y_1 y}{b^2}+\frac{z_1 z}{c^2}=0.$$

40. Show that when two diameters are conjugate, their direction cosines are connected by the relation

[37] $$\frac{\cos\alpha_1\cos\alpha_2}{a^2}+\frac{\cos\beta_1\cos\beta_2}{b^2}+\frac{\cos\gamma_1\cos\gamma_2}{c^2}=0.$$

41. Show that the coördinates of any point of a central quadric can be expressed in the form

[38] $$x=a\cos\lambda,\quad y=b\cos\mu,\quad z=c\cos\nu,$$

where λ, μ, and ν are the direction angles of an auxiliary line.

42. Show that if two diameters are conjugate, the auxiliary lines corresponding to their extremities are mutually perpendicular.

43. Show that the sum of the squares of three conjugate diameters is constant.

44. Prove that the parallelopiped whose edges are three conjugate diameters has a constant volume.

Circular Sections.

45. Prove that through the centre of every central quadric two planes can be drawn, each of which will cut the quadric in a circle.

46. Show that every plane section of a quadric parallel to a circular section is a circle.

47. Define the *umbilics* of an ellipsoid, and show how to find them.

48. Show that the circular sections of an hyperboloid and of its asymptotic cone are the same.

Ruled Surfaces.

49. Show that on the un-parted hyperboloid two sets of right lines can be drawn, lying wholly in the surface of the hyperboloid; and that through every point of the hyperboloid one line of each set will pass.

50. Prove that the two lines passing through a given point of the hyperboloid lie in the tangent plane drawn at the point in question.

51. Show that each line of one system meets all the lines of the other system, and none of the lines of its own system.

52. Prove that an un-parted hyperboloid may be generated by the motion of a line which always touches three given lines, no two of which are in the same plane.

53. Show that if a line revolve about another line not in the same plane, it will generate a *ruled hyperboloid.*

54. Investigate the properties of the *ruled paraboloid.*

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