SYLLABUS

OF A

COURSE IN THE THEORY OF EQUATIONS.

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COURSE IN THE THEORY OF EQUATIONS.

1. Define a *root* of an equation. Explain a short method of substituting any given number for x in a numerical equation, using only the detached coefficients.

2. If a is a root of the equation, the first member is divisible by $x-a$. Give a short formal proof. Give a second proof showing the form of the quotient; *i.e.*, substitute a for x in the first member of the equation, and then subtract the result from the given first member, and the equation will be in a form where every term is obviously divisible by $x-a$.

3. Prove the converse of the theorem in § 2. If $x-a$ will divide the first member of the equation, a is a root.

4. Assuming that every equation has at least one root, prove that every equation of the nth degree has n roots and can be thrown into the form

$$A(x-a)(x-b)(x-c)\cdots=0.$$

5. Show that the coefficients of the equation are simple functions of the negatives of the roots.

6. Show that, if, on substituting two different values for x in turn in the first member of an equation, the results are of opposite signs, there must be an odd number of roots between the values substituted.

Solution of Numerical Equations.

Commensurable Roots.

7. Prove that if the coefficients of an equation are whole numbers, and the coefficient of the first term is unity, the equation cannot have a fractional root. Any commensurable root of such an equation must then be an exact divisor of the constant term by § 5.

8. After a root a is found, the degree of the equation may be lowered by dividing by $x-a$. Explain abbreviated methods of division: first, by detached coefficients; second, by synthetic division.

9. Prove the theorem: If a is a commensurable root it will exactly divide the constant term, the quotient thus obtained plus the coefficient of the preceding term, this quotient plus the preceding coefficient, and so on, and the last quotient will be -1.

Prove the converse of this theorem.

Give the working method of finding all the commensurable roots of the equation described in § 7.

10. If the coefficients of an equation are whole numbers, and the coefficient of the first term is not unity, show that the equation may easily be transformed into one where the coefficients are whole numbers and the coefficient of the first term is unity, and may then be treated by § 9.

11. *Descartes' Rule of Signs.* Explain what is meant by a *permanence* of sign; a *variation* of sign.

Prove that the number of positive roots of an equation, complete or incomplete, cannot exceed the number of variations of sign. Show that, by reversing the signs of the terms of odd degree, the equation may be transformed into one whose roots are the negatives of the roots of the given equation, and that by applying Descartes' Rule to the transformed equation, information may be obtained concerning the negative roots of the given equation.

Incommensurable Roots. Methods of Approximation.

12. Explain the rough laborious method of approximating to the value of an incommensurable root based upon § 6, and show that it is theoretically capable of any desired degree of accuracy, and as applicable to a transcendental as to an algebraic equation.

13. Explain briefly *Newton's Method; i.e.*, find a portion a of the root by § 12; let $x = a + h$, and substitute this value for x in the equation, neglecting higher powers of h than the first, thus obtaining an equation of the first degree to determine an approximate value of h. The result is reasonably accurate if h is small.

14. *Horner's Method.* Find by § 12 a portion a of the root. Transform the equation into one whose roots are less by a than those of the original equation. Treat the resulting equation by § 12.

Show that the coefficients of the transformed equation will be the remainders obtained in dividing the original equation repeatedly by $x - a$.

Newton's Method, § 13, shows that an approximate value for the rest of the root may be found by dividing the constant term of the transformed equation by the preceding coefficient.

Describe *Horner's Method* in its practical abbreviated working form. Show how to deal with negative roots.

General Methods and Theorems.

15. Explain the method of finding equal roots by obtaining the greatest common divisor of the first member of the equation and its derivative with respect to x.

16. Describe *Sturm's Functions.* Prove Sturm's Theorem.

17. In dealing with a given numerical equation of high degree: first, test for commensurable roots, and lower the degree of the equation by their aid if any are found; second, test for equal roots; third, use Horner's Method in finding approximately the incommensurable roots, employing Sturm's Theorem as an auxiliary if it proves absolutely necessary.

IMAGINARIES.

18. The treatment and use of *imaginaries* is purely arbitrary and conventional. Define the square root of -1 as a symbol of operation, and state the conventions adopted to govern its use.

$$\sqrt{-a^2} = a\sqrt{-1},$$

$$(a+b)\sqrt{-1} = a\sqrt{-1} + b\sqrt{-1},$$

$$a\sqrt{-1} = \sqrt{-1} \, . \, a.$$

Interpret the powers of $\sqrt{-1}$ by the aid of the definition and these conventions.

Show that these conventions enable us to deal with imaginary roots of a quadratic, and that their treatment and properties are closely analogous to those of real roots.

19. Show why $a + b\sqrt{-1}$ is taken as the typical form of an imaginary. Explain the ordinary geometrical representation of an imaginary by the position of a point in a plane. N.B. This interpretation is entirely arbitrary, but has proved very useful in suggesting important relations which might not otherwise have been discovered.

20. Show that the *sum*, the *product*, and the *quotient* of two imaginaries are imaginaries of the typical form.

21. Give the second typical form of an imaginary suggested by the graphical construction of § 19.

$$r(\cos\phi + \sqrt{-1}\cdot\sin\phi).$$

Define the *modulus* and the *argument* of an imaginary.

State the convention concerning the sign of the modulus, and show that the argument may have an infinite number of values differing by multiples of 2π.

22. Show that the modulus of the product of two imaginaries is the product of their moduli, and that the argument of their product is the sum of their arguments. Prove the theorems concerning the modulus and argument of the quotient of two imaginaries; of a power of an imaginary; of a root of an imaginary.

23. Show that the nth root of any real or imaginary has n values, having the same modulus and arguments differing by multiples of $\frac{2\pi}{n}$.

24. Define *conjugate* imaginaries. Prove that conjugate imaginaries have a real sum and a real product.

Show that if an equation with real coefficients has an imaginary root, the conjugate of that root is also a root of the equation.

25. Give *Cardan's Solution of a Cubic* of the form

$$x^3 + qx + r = 0.$$

Consider the *irreducible case.* Give a *Trigonometric Solution.* Show that any cubic can be reduced to the form

$$x^3 + qx + r = 0.$$

Obtain the general solution of any cubic.

26. Give Descartes' and Euler's Methods of solving a biquadratic equation.

SYMMETRIC FUNCTIONS OF THE ROOTS OF AN EQUATION.

27. Define a *symmetric function* of several quantities. Show that any combination of symmetric functions is symmetric.

The coefficients of an equation are symmetric functions of the roots of the equation by § 5.

28. Explain *Newton's Method* of expressing the sums of powers of the roots of an equation in terms of the coefficients,

$$fx = (x - a)(x - b)(x - c) \cdots$$

Take the logarithm of each member and differentiate

$$\frac{f'x}{fx} = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} \cdots$$

Consider the case where the required power is less than the degree of the equation; where the required power is greater than the degree of the equation.

29. Give the short practical method of obtaining the sums of powers of the roots of a numerical equation.; divide $xf'x$ by fx, and the coefficients of x^{-1}, x^{-2}, x^{-3}, etc., in the quotient are s_1, s_2, s_3, etc. Shorten by using detached coefficients.

30. Any complicated symmetric function can be made to depend upon simpler functions so that only rational integral forms need be specially investigated.

31. Show that symmetric functions may be expressed in terms of the sums of powers of the quantities involved.

$$\Sigma a^m = s_m,$$

$$\Sigma a^m b^p = s_m s_p - s_{m+p},$$

$$\Sigma a^m b^p c^q = s_m s_p s_q - s_{m+p} s_q - s_{m+q} s_p - s_{p+q} s_m + 2 s_{m+p+q}.$$

Consider special cases.

32. Explain the method of *elimination by the aid of symmetric functions.*

Determinants.

33. Show that, if two simultaneous equations of the first degree,

$$a_1 x + b_1 y + c_1 = 0,$$

$$a_2 x + b_2 y + c_2 = 0,$$

are solved, the numerators and denominators of the values of x and y have a peculiar symmetric form.

Explain the notation adopted for writing compactly such expressions.

Describe a *Determinant*, its *rows*, *columns*, and *diagonal term*.

Give a rule for expanding a determinant. Give the *law of signs*.

Illustrate by determinants of the second and third orders.

34. Show that a determinant may be broken up into a sum of terms each involving a *sub-determinant*. Illustrate.

35. Show that an interchange of two rows or of two columns will change the sign of a determinant.

36. Show that if two rows or two columns are identical, the value of the determinant is zero.

37. Show that if each constituent of any row or column is multiplied by a given quantity, the whole determinant is multiplied by that quantity.

38. Show that if each constituent of any row or of any column is a binomial, the determinant can be broken up into the sum of two other determinants of the same order.

39. Show how to compute the value of a numerical determinant. Consider examples.

40. Explain the application of determinants to *elimination* in the case of n equations of the first degree between n unknown quantities.

41. Explain the application of determinants to elimination in the case of two equations of any degree involving two unknown quantities.

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