SYLLABUS OF MATHEMATICS

SOCIETY FOR THE
PROMOTION OF ENGINEERING EDUCATION

1912

SYLLABUS OF MATHEMATICS

A SYMPOSIUM COMPILED BY THE COMMITTEE ON THE TEACHING OF MATHEMATICS TO STUDENTS OF ENGINEERING

ACCEPTED BY THE
SOCIETY FOR THE PROMOTION OF ENGINEERING EDUCATION
AT THE NINETEENTH ANNUAL MEETING HELD AT
PITTSBURGH, PA., JUNE, 1911

OFFICE OF THE SECRETARY
ITHACA, N. Y.
1912

PRESS OF
THE NEW ERA PRINTING COMPANY
LANCASTER, PA.

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REPORT OF THE COMMITTEE ON THE TEACHING OF MATHEMATICS TO STUDENTS OF ENGINEERING.

To the Society for the Promotion of Engineering Education:

The committee was appointed at a joint meeting of mathematicians and engineers held in Chicago, December 30–31, 1907, under the auspices of the Chicago Section of the American Mathematical Society, and Sections A and D of the American Association for the Advancement of Science,* and on the suggestion of officers of the Society for the Promotion of Engineering Education who were there present, the committee was instructed to report to this Society.

The membership of the committee is as follows:

ALGER, Philip R., professor of mathematics, U. S. Navy, Annapolis, Md.

CAMPBELL, Donald F., professor of mathematics, Armour Institute of Technology, Chicago, Ill.

ENGLER, Edmund A., president of the Worcester Polytechnic Institute, Worcester, Mass.

HASKINS, Charles N., assistant professor of mathematics, Dartmouth College, Hanover, N. H.

HOWE, Charles S., president, Case School of Applied Science, Cleveland, Ohio.

KUICHLING, Emil, consulting civil engineer, New York City.

MAGRUDER, William T., professor of mechanical engineering, Ohio State University, Columbus, Ohio.

MODJESKI, Ralph, civil engineer, Chicago, Ill.

OSGOOD, William F., professor of mathematics, Harvard University, Cambridge, Mass.

SLICHTER, Charles S., consulting engineer of the U. S. Reclamation Service, professor of applied mathematics, University of Wisconsin, Madison, Wis.

* For an account of the Chicago meeting, see *Science* for 1908 (July 12, 24, and 31; August 7 and 28; and September 4).

Steinmetz, Charles P., consulting engineer of the General Electric Company, professor of electrical engineering, Union University, Schenectady, N. Y.

Swain, George F., consulting engineer, professor of civil engineering, Harvard University, Cambridge, Mass.

Townsend, Edgar J., dean of the College of Science and professor of mathematics, University of Illinois, Urbana, Ill.

Turneaure, Frederick E., dean of the College of Mechanics and Engineering, University of Wisconsin, Madison, Wis.

Waldo, Clarence A., head professor of mathematics, Washington University, St. Louis, Mo.

Williams, Gardner S., consulting engineer, professor of civil, hydraulic and sanitary engineering, University of Michigan, Ann Arbor, Mich.

Woodward, Calvin M., dean of the School of Engineering and Architecture and professor of mathematics and applied mechanics, Washington University, St. Louis, Mo.

Woodward, Robert S., president of the Carnegie Institution of Washington, Washington, D. C.

Ziwet, Alexander, professor of mathematics, University of Michigan, Ann Arbor, Mich.

Huntington, Edward V., chairman, assistant professor of mathematics, Harvard University, Cambridge, Mass.

After deliberation, the committee decided that it could best carry out the purpose for which it was appointed by preparing *a synopsis of those fundamental principles and methods of mathematics which, in the opinion of the committee, should constitute the minimum mathematical equipment of the student of engineering.*

This synopsis, as finally adopted, consists of five parts:

1. A Syllabus of Elementary Algebra;
2. A Syllabus of Elementary Geometry and Mensuration;
3. A Syllabus of Plane Trigonometry;
4. A Syllabus of Analytic Geometry;
5. A Syllabus of Differential and Integral Calculus.

Two other syllabi, on Numerical Computation and on Elementary Dynamics, were contemplated in the original plan, but were not completed.

It is hoped that this report may be serviceable in two ways: first, to the teacher, as an indication of where the emphasis should be laid; and secondly, to the student, as a syllabus of facts and methods which are to be his working tools. *It does not include data for which the student would properly refer to an engineers' hand-book; it includes rather just those things for which he ought never to be obliged to refer to any book—the things which he should have constantly at his fingers' ends.*

The teacher of mathematics should see to it that at least these facts are perfectly familiar to all his students, so that the professor of engineering may presuppose, with confidence, at least this much mathematical knowledge on the part of his students. On the other hand, if the professor of engineering needs to use, at any point, more advanced mathematical methods than those here mentioned, he should be careful to explain them to his class.

The committee has not found it possible to propose a detailed course of study. The order in which these topics should be taken up must be left largely to the discretion of the individual teacher. The committee is firmly of the opinion, however, that whatever order is adopted, the principal part of the course should be *problems worked by the students,* and that all these problems should be *solved on the basis of a small number of fundamental principles and methods, such as are here suggested.*

The defects in the mathematical training of the student of engineering appear to be largely in knowledge and grasp of fundamental principles, and the constant effort of the teacher should be to ground the student thoroughly in these fundamentals, which are too often lost sight of in a mass of details.

A pressing need at the present time is a series of synoptical text-books, which shall present, (1) the fundamental principles of the science in compact form, and (2) a classified and graded collection of problems (which would naturally be subject to continual change and expansion). It is the hope of the committee that this report, which is confined to the first part of the desired text-book, will stimulate throughout the country practical contributions toward the second.

In the early part of its investigation the committee collected a large amount of information in regard to the present status of mathematical instruction for engineering students. Since that time, however, a much more inclusive inquiry has been undertaken by the International Commission on the Teaching of Mathematics, of which the American Commissioners are Professors D. E. Smith, J. W. A. Young and W. F. Osgood. In order to avoid unnecessary duplication, this committee voted to turn over all the results of its own inquiry in this field to the larger commission, to be worked up in accordance with the general scheme adopted by that commission, and to be incorporated in their report. This material is therefore not included in the present report.

Respectfully submitted,

EDWARD V. HUNTINGTON,

Chairman.

June, 1911.

A SYLLABUS OF THE FORMAL PART OF ELEMENTARY ALGEBRA.

This syllabus is intended to include those facts and methods of elementary algebra which a student who has completed a course in that subject should be expected to "know by heart"—that is, those fundamental principles which he ought to have made so completely a permanent part of his mental equipment that he will never need to "look them up in a book."

It is not intended as a program of study for beginners, and no attempt has been made to arrange the topics in the order in which they should be taught. In reviewing the subject, however, either at the end of the course in algebra, or at the beginning of any later course, such a syllabus will be found serviceable to both teacher and student; and in the hands of a skillful teacher, and supplemented by an adequate collection of problems, it might well be made the basis of a course of study conducted by the "syllabus method."

One of the chief defects in the present-day teaching of algebra is the multiplicity of detached rules with which the student's mind is burdened;* and every successful attempt to knit together a number of these detached rules into a single general principle (provided this principle is simple and easily applied) should conduce to economy of mental effort, and diminish the liability to error.

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* For example, in a recent prominent text-book there are no less than *fifty* italicized rules in the part of the book preceding quadratic equations!

CHAPTER I.

TRANSFORMATION OF ALGEBRAIC EXPRESSIONS.

1. The ordinary operations of transforming and simplifying algebraic expressions should be so familiar to the student that he performs them almost instinctively; at the same time he should be able, whenever called upon, to justify each step of his work by reference to some one or more of a small number of well established principles.

For example, if the student is asked *by what authority* he replaces $\frac{a+x}{b+x}$ by $\frac{a}{b}$, or $\sqrt{a^2+b^2}$ by $a+b$ (to mention only two of the commonest blunders), he will be forced to recognize either that he is making use of methods that he has never proved, and that are in fact erroneous, or else (which is more likely) that he is working altogether in the dark, without any conscious reason for the steps he has taken.

The following list of such principles, while making no pretense at logical completeness, will be sufficient for all practical purposes.

2. General laws of addition and multiplication.

$a+b=b+a.$ $\qquad ab=ba.$ $\qquad$ (Commutative laws.)

$(a+b)+c=a+(b+c).$ $\qquad (ab)c=a(bc).$
(Associative laws.)

$a(b+c)=ab+ac.$ $\qquad$ (Distributive law.)

$a+0=a.$ $\qquad a\times 1=a.$ $\qquad a\times 0=0.$

These laws hold when a, b, c are any of the quantities that occur in ordinary algebra, whether "real" or "complex."* *The student should be constantly encouraged to test general algebraic statements by substituting concrete numerical values.*

3. Type-forms of multiplication (Factoring).

The following type-forms of multiplication are the ones that are most important to remember:

* This syllabus is confined chiefly to the algebra of real quantities; the algebra of complex quantities will be treated only incidentally.

$$a^2 - b^2 = (a - b)(a + b),$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2),$$

and so on; the general case is best remembered in the form

$$1 - x^n = (1 - x)(1 + x + x^2 + x^3 + \cdots + x^{n-1}).$$

Note also that in the algebra of real quantities, $a^n + b^n$ is divisible by $a + b$ when and only when n is *odd*. Thus:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

Further: $(x + a)(x + b) = x^2 + (a + b)x + ab,$

and the "binomial theorem":

$$(a + b)^2 = a^2 + 2ab + b^2, \quad (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

$$(a - b)^2 = a^2 - 2ab + b^2, \quad (a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3;$$

$$(a + b)^n =$$

$$a^n + na^{n-1}b + \frac{n(n-1)}{2\,!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3\,!}a^{n-3}b^3 + \cdots$$

$$\cdots + \frac{n(n-1)(n-2)}{3\,!}a^3b^{n-3} + \frac{n(n-1)}{2\,!}a^2b^{n-2} + nab^{n-1} + b^n,$$

where $k! =$ "k factorial" $= 1 \times 2 \times 3 \times \cdots \times k$.

4. Fractions.

Def. If $bx = a$, then and only then we write $x = \frac{a}{b}$ (or a/b, or $a \div b$).

Here a is called the numerator and b the denominator of the fraction. A fraction with a zero denominator, as $a/0$, does not represent any definite quantity. For, if a is not zero, there is *no* quantity x such that $0 \times x = a$; and if $a = 0$, then *every* quantity x will have this property. Hence, *the denominator of a fraction must always be different from zero.*

From the definition, $a/1 = a$; also

$$\frac{a}{a} = 1, \quad \frac{0}{a} = 0, \qquad (a \neq 0).*$$

* The symbol $\neq$ means "not equal to."

To add two fractions with common denominator:

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}.$$

To multiply two fractions:

$$\frac{a}{b}\times\frac{x}{y}=\frac{ax}{by}.$$

To divide by a fraction, "invert the divisor and multiply":

$$\frac{a}{b}\div\frac{x}{y}=\frac{a}{b}\times\frac{y}{x}=\frac{ay}{bx}.$$

The value of a fraction is not changed if we multiply (or divide) both the numerator and the denominator by any quantity not zero:

$$\frac{a}{b}=\frac{ma}{mb} \qquad (m \neq 0).$$

This is the most important principle concerning fractions.

For example, *to reduce two fractions to a common denominator,* we have merely to multiply numerator and denominator of each fraction by a suitable factor.

Again, *to simplify a complex fraction,* we multiply the whole numerator and the whole denominator by any quantity which will "absorb" all the subsidiary denominators. Thus, by multiplying by xyz, we have

$$\frac{\frac{a}{x}+\frac{b}{y}}{\frac{c+d}{z}}=\frac{ayz+bxz}{(c+d)xy},$$

at once, by a single mental process. (The common practice of reducing the numerator and denominator separately, and then inverting the denominator and multiplying, is tedious and clumsy.)

Def. If $bx=1$, then $x=1/b$, which is called the *reciprocal* of b. To divide by $b\,(b\neq 0)$ is the same as to multiply by the reciprocal of b.

5. Negatives.

Def. If $a+x=0$, then and only then we write $x=-a$. In particular, $-(-a)=a$.

If a is not zero, $-a$ is always *opposite* to a; that is, if a is positive, $-a$ is negative, and if a is negative, $-a$ is positive.

Thus, if $a = -3$, which is a negative quantity, then $-a = 3$, which is positive.

The notation $|a|$, which is coming into use more and more widely, means the *absolute value of a,* that is, the numerical value of a regardless of sign; thus, $|5| = 5$, $|-5| = 5$.

The laws of operation with the minus sign are best remembered by regarding $-a$ as the product of a and -1:

$$(-1) \times a = -a,$$

whence, in particular (putting $a = -1$),

$$(-1) \times (-1) = 1.$$

When this is done, the customary formulas:

$$(-a)(-b) = ab, \quad (-a)(b) = -ab, \quad \frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}, \quad \frac{-a}{-b} = \frac{a}{b},$$

become immediate consequences of the general laws of multiplication and division, and therefore need not be separately memorized; and the same is true of the formula

$$a - (x + y - z) = a - x - y + z,$$

which, when remembered in the following form, becomes an immediate application of the distributive law: "a minus sign in front of a parenthesis must be 'distributed' through every term within, if the parentheses are to be taken away."

By knitting together in this way the rules for negatives with the general rules of operation, the total number of processes to be remembered and applied, and hence the liability to error, is materially reduced.

Def. If $a + x = b$, then and only then we write $x = b - a$. It is easily shown that $b - a = b + (-a)$; that is, *subtracting any quantity a is the same as adding the opposite of a.*

6. Radicals.

Def. If a is *positive,* and n is any positive integer, there will always be one positive value of x such that $x^n = a$. This value x is denoted by $\sqrt[n]{a}$, and is called the (principal) *nth root of a.*

It should be noticed that while there are (for example) two square roots of 9, namely 3 and -3, it is only the positive one of these two values that is denoted by $\sqrt{9}$; that is, the mark $\sqrt{9}$ means 3 and not -3.

The radical sign, except in the case of square roots, and sometimes in the case of cube roots, should always be replaced by *fractional exponents* (see below) when it is desired to compute with these quantities; this done, no special rules for the manipulation of radicals need then be remembered beyond the general laws of exponents.

Square roots. If a and b are positive,

$$\sqrt{a^2b} = a\sqrt{b}, \quad \text{and} \quad \sqrt{a}\sqrt{b} = \sqrt{ab}.$$

Note also the process called "rationalizing the denominator (or numerator) of a fraction"; for example,

$$\frac{c}{\sqrt{a}+\sqrt{b}} = \frac{c}{\sqrt{a}+\sqrt{b}} \times \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{c(\sqrt{a}-\sqrt{b})}{a-b};$$

$$\frac{\sqrt{1-x}}{\sqrt{1+x}} = \frac{\sqrt{1-x}}{\sqrt{1+x}} \times \frac{\sqrt{1-x}}{\sqrt{1-x}} = \frac{1-x}{\sqrt{1-x^2}}.$$

Def. If a is *negative,* and n is *odd,* there will always be one negative value of x such that $x^n = a$; this value is denoted by $\sqrt[n]{a}$, and is called the (principal) nth root of a.

Thus $$\sqrt[3]{-8} = -2.$$

7. Imaginaries.

If a is *negative,* and n is *even,* then there is *no* positive or negative *nth* root of a. Hence, such quantities do not occur in the algebra of positive and negative quantities. They occur only in the more general *algebra of complex quantities;* in this algebra every quantity a (except zero) has *n distinct nth roots,* the notation $\sqrt[n]{a}$ being applied, as occasion requires, to any one of these n values. The detailed study of this general algebra is probably too difficult for a first course; for such applications as occur in elementary work, the following working rules are sufficient:

1) In manipulating a complex quantity of the form $\sqrt{-b}$, where b is positive, write $\sqrt{-b} = \sqrt{-1}\,\sqrt{b} = i\sqrt{b}$, and treat i like any other letter; then simplify the result by the relation $i^2 = -1$.

2) Every complex quantity can be written in the form $a + ib$, where a and b are "real" (that is, positive, negative, or zero); and if $a + ib = a' + ib'$, then $a = a'$ and $b = b'$.

In electrical engineering the letter i is used to denote current, and $\sqrt{-1}$ is denoted by j.

8. Exponents.

The subject of negative and fractional exponents is a part of algebra in which the preparation of the student is apt to be especially unsatisfactory.

Definition of negative and fractional exponents. If a is positive, and p and q are any positive integers, then

$$a^0 = 1, \qquad a^{-p} = \frac{1}{a^p}, \qquad a^{1/q} = \sqrt[q]{a}, \qquad a^{p/q} = \sqrt[q]{a^p}.$$

9. Laws of operation with exponents.

If a and b are positive, then:

$$a^{m+n} = a^m a^n, \quad a^{mn} = (a^m)^n, \quad (ab)^m = a^m b^m,$$

$$a^0 = 1, \qquad a^{-m} = \frac{1}{a^m}, \qquad a^{1/m} = \sqrt[m]{a}.$$

All these laws hold for any values of m and n; the three fundamental ones can readily be recalled to mind through simple special cases, such as a^3a^2, $(a^3)^2$, and $(ab)^3$.

The three other laws commonly mentioned, namely

$$a^{m-n} = a^m/a^n, \quad a^{m/n} = \sqrt[n]{a^m}, \quad (a/b)^m = a^m/b^m,$$

are immediate corollaries of those just mentioned.

If a is *negative*, and m not an integer, a^m will, in general, be a complex quantity. In such cases, let $a' = -a$, so that a' is positive, and write $a^m = (-1)^m a'^m$, where $(-1)^m$ must then be handled according to the rules of operation in the algebra of complex quantities.

10. Logarithms.

The subject of logarithms should be taught in logical connection with the subject of exponents. The common practice of separating these subjects, and treating logarithms as a part of trigonometry, is unfortunate. Numerous applications of logarithms can be found that have nothing to do with trigonometry; moreover, the training in the use of logarithms which a student gets in trigonometry is usually quite inadequate as a preparation for the applications of logarithms in any of his later work outside of surveying.

Def. The *logarithm* of a (positive) number, to any (positive) base, is the *exponent* of the power to which the base must be raised to produce that number.

Thus, the notation

$$x = \log_b N$$

means

$$b^x = N.$$

Note that negative numbers in general have no logarithms in the algebra of real quantities.

From the laws of exponents we have, whatever the base may be:

$$\log(ab) = \log a + \log b, \quad \log\left(\frac{a}{b}\right) = \log a - \log b,$$

$$\log(a^n) = n \log a, \qquad \log \sqrt[n]{a} = \frac{1}{n} \log a,$$

$$\log 1 = 0, \quad \log(base) = 1.$$

Only two bases are in common use. For purposes of numerical computation, the base chosen is 10, and in this system $\log(10^n) = n.$

In higher mathematics, the base $e = 2.718 \cdots$ is used, for the reason that the use of this base simplifies certain formulas in the calculus; in this system $\log(e^n) = n$.

Change of base. To find $\log_e N$ when $\log_{10} N$ is known, let $x = \log_e N$, that is, $e^x = N$. Then take the logarithm of both sides of this equation to base 10, and solve for x.

The resulting formula, $\log_e N = (\log_{10} N)/(\log_{10} e)$, is so easily obtained in this way that it is not worth while to remember it separately. The approximate values

$$\log_{10} e = .4343, \quad \text{and} \quad \log_e N = (2.3026) \log_{10} N,$$

however, are useful to remember.

CHAPTER II.

Solution of Equations.

11. Legitimate operations on equations. If a given equation is true, it will still be true if we

(*a*) *add* any quantity we please to both sides;

(*b*) *subtract* any quantity we please from both sides;

(*c*) *multiply* both sides by any quantity we please;

(*d*) *divide* both sides by any quantity we please *except zero,*

(*e*) raise both sides to any *positive integral power;*

(*f*) *extract any *positive integral root* of both sides, except that if an *even* root is extracted, the double sign $\pm$ must be used;

(*g*) *take the *logarithm* of both sides (provided both sides are positive).

In regard to (*d*), we must never divide both sides by an unknown quantity without first excluding the possibility that that quantity is zero.

In (*f*), the restriction stated means, for example, that from $A^2 = B$ we can infer merely that $A = \pm\sqrt{B}$; that is, that *either* $A = \sqrt{B}$, or $A = -\sqrt{B}$; but we cannot tell which.

12. To solve a single equation in x, means to find all the values of x that satisfy the equation, or to show that none such exist.

Any value of x that satisfies the equation is called a *root* of the equation.

In testing a root, the only safe method is to substitute the given value in each side of the equation separately, and see whether the results, when reduced, are equal. Thus, we should find that $x = -2$ is a root of the equation $x = 2 - \sqrt{12 - 2x}$, and that $x = 4$ is not a root.

In this connection it should be noticed that if we square both sides of a given equation, the new equation will, in general, have more roots than the given equation. Thus (to use the same example), by squaring $x - 2 = -\sqrt{12 - 2x}$ we have $x^2 - 2x - 8 = 0$. This equation has of course the root -2, since $x = -2$ satisfies the original equation from

* In the algebra of complex quantities (*f*) and (*g*) are not applicable.

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which this was derived; but it has also the root 4, which was not a root of the original equation.

The formal process usually called "solving the equation" means merely transforming the equation, by a judicious choice of the legitimate operations, into a form in which the solutions are obvious.

If this is not possible, we must have recourse to the *method of trial and error* which, while often laborious, is always applicable in numerical cases.

If an equation is given in the factored form:

$$(x-\alpha)(x-\beta)(x-\gamma)\cdots=0,$$

then the roots are obviously $x=\alpha$, $x=\beta$, $x=\gamma, \cdots$. Thus, the roots of $x(x+2)=0$ are 0 and -2.

13. Quadratic equations. To solve the quadratic equation

$$ax^2+bx+c=0,$$

we may divide through by a:

$$x^2+\frac{b}{a}x = -\frac{c}{a}$$

and then "complete the square":

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2};$$

whence,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a};$$

or, we may use the general result just obtained as a formula.

The quantity which must be added to both sides in "completing the square" is obvious by analogy with $x^2+2mx+m^2$, so that this method requires less effort of the memory than the method of solution by formula.

The "method of factoring" is very convenient in certain special cases, when the factors can be obtained by inspection.

The method still sometimes used, of first multiplying through by $4a$ to avoid fractions, is apt to lead to confusion, and should be discouraged.

From the formula it is evident that the *sum of the roots* is

$x_1 + x_2 = -b/a$, and the *product of the roots* is $x_1 x_2 = c/a$; also, if the coefficients, a, b, c, are real, the roots will be real-and-distinct, real-and-coincident, or imaginary, according as $b^2 - 4ac$ is positive, zero, or negative.

14. Exponential equations. To solve an equation of the form $a^x = b$, when a and b are positive, *take the logarithm of both sides:* $x \log a = \log b$; and then solve for x.

15. To solve a set of simultaneous equation in x, y, z $\cdots$ means to find all the *sets of values* of $x, y, z, \cdots$, that satisfy all the equations at once, or show that none such exist.

Two simultaneous equations of the first degree, as $ax + by = c$ and $Ax + By = C$, can always be solved in a couple of lines, if the work is arranged as follows:

$7x - 6y = 1$	-5	-2
$14x - 10y = 3$	3	1
$(-35 + 42)x = -5 + 9$		
$(12 - 10)y = -2 + 3$		

whence the values of x and y are obvious, provided $aB - bA$ *is not zero.* (If $aB - bA = 0$, there is either no pair of values x, y that satisfies both the equations, or else there are an infinite number of pairs of values that do so; in this latter case, the equations are not independent, that is, either of them can be derived from the other.)

The theory of simultaneous equations, and sometimes the numerical computation, is facilitated by the use of determinants.

In general, n independent equations will suffice to determine n unknown quantities.

CHAPTER III.

Miscellaneous Topics.

16. Ratio and Proportion. The "ratio of a to b" means simply the fraction a/b; and a "proportion" is simply an equation between two ratios.

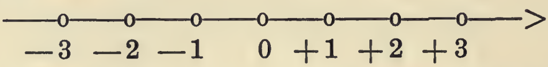
The notation $a:b::c:d$ should be replaced by the equation $a/b = c/d$; and all special terminology, such as "taking a proportion by alternation," "by composition," etc., should be dropped in favor of the ordinary language of the equation.

17. Variation. The statement "y varies as x," or "y varies directly as x," or "y is proportional to x," means $y = kx$, where k is some constant. Similarly, "y varies inversely as x," means $y = k/x$; "y varies inversely as the square of x," means $y = k/x^2$. The constant k can always be determined if we know any pair of values of x and y that belong together.

The statement "y varies as *u and v*," means y varies as the *product* of u and v, that is, $y = kuv$.

18. Inequalities. The notions of "greater and less" are thoroughly familiar when we are dealing only with positive quantities, but the extension of these terms to the algebra of all real quantities (positive, negative, and zero) is apt to cause some confusion.

(*a*) All *real* quantities (positive, negative, and zero) may be represented by the points of a directed line (running, say, from left to right):



and the notation $a < b$ (read: "a algebraically less than b") means simply that a *precedes* b, or a lies on the left of b, along this line.

Similarly, $a > b$ (read: "a algebraically greater than b") means that *a comes after b*, or lies on the right of b, along the line. (The idea that a negative quantity is a magnitude whose *size* is in some way "less than nothing" should be carefully avoided.)

Obviously, if a and b are any real quantities, one and only one of the three relations: $a=b$, $a<b$, and $a>b$, will hold between them; further, if $a<b$ and $b<c$, then $a<c$.

(*b*) *Complex* quantities require for their representation the points of a plane instead of the points of a line, and the symbols $<$ and $>$ are not used in connection with these quantities.

Legitimate operations on inequalities. If a given inequality is true, it will still be true if we

(*a*) *add* any quantity we please to both sides;

(*b*) *subtract* any quantity we please from both sides;

(*c*) *multiply* both sides by any *positive* quantity;

(*d*) *divide* both sides by any *positive* quantity;

(*e*) raise both sides to any *positive power* (integral or fractional), *provided both sides are positive.*

(*f*) take the *logarithm* of both sides, *provided both sides are positive.*

If we multiply or divide both sides by any *negative* number, we must *reverse the sense of the inequality.*

The neglect of the rules for handling inequalities is the source of many common errors.

19. Arithmetical Progression.

In an arithmetical progression:

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad \cdots,$$

each term is obtained from the preceding by *adding* a constant quantity.

The nth term is obviously $l=a+(n-1)d$.

The sum of n terms is $S=\dfrac{a+l}{2}n$.

This formula is most easily remembered in the form:

$S=$ (average of the first and last terms) $\times$ (number of terms).

The *arithmetic mean* between a and b is $A=\frac{1}{2}(a+b)$.

20. Geometric Progression.

In a geometric progression:

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad \cdots,$$

each term is obtained from the preceding by *multiplying* by a constant quantity.

The nth term is obviously $l = ar^{n-1}$.

The sum of n terms is $S = \dfrac{a(1-r^n)}{1-r}$.

This formula is best remembered in connection with the rule for factoring:

$$1 - r^n = (1-r)(1+r+r^2+r^3+\cdots+r^{n-1}).$$

The *geometric mean* between a and b is $G = \sqrt{ab}$.

The geometric mean is also called the *mean proportional.*

Infinite geometric progression. If $|r| < 1$, the sum of n terms approaches the limit

$$\frac{a}{1-r}$$

as n increases indefinitely (since, in the expression for S, if $|r| < 1$, r^n approaches zero).

21. Harmonic Progression.

A harmonic progression is a series of terms whose reciprocals are in arithmetical progression. (The harmonic progression is not of great importance.)

The *harmonic mean* between a and b is $H = \dfrac{2ab}{a+b}$.

A SYLLABUS OF ELEMENTARY GEOMETRY AND MENSURATION.

This syllabus is intended to include those facts and methods of elementary geometry which a student should have so firmly fixed in his memory that he will never think of looking them up in a book.

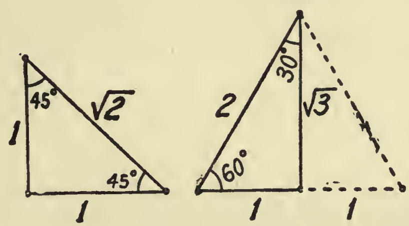
1. Right Triangles.

In a right triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides (Pythagoras, 580–501 B.C.); and the sum of the acute angles is 90°.

Examples of right triangles with integral sides: 3, 4, 5; 5, 12, 13.

Two right triangles are congruent when they agree with respect to (*a*) any side and an acute angle; or (*b*) any two sides.

In the "45° triangle" and the "30–60° triangle," the ratios of the sides are as indicated in the figure.



2. Oblique Triangles.

In any plane triangle, the sum of the angles is 180°. Hence, an exterior angle of a triangle equals the sum of the opposite interior angles.

Of two unequal sides in a triangle, the greater is opposite the greater angle.

A plane triangle is, in general, wholly determined when any three of its parts (not all angles) are given.

There are four cases:

(*a*) two angles (provided their sum is less than 180°) and one side;

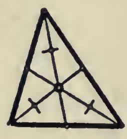
(*b*) two sides and the included angle;

(*c*) the three sides (provided the largest is less than the sum of the other two);

(*d*) two sides and the angle opposite one of them (the "ambiguous case," in which we may have two solutions, or one, or none).

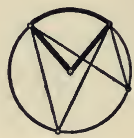
Hence the usual rules for testing the equality of two plane triangles.

The center of gravity of a plane triangle is the intersection of the three medians, and is two thirds of the way from any vertex to the middle point of the opposite side.



3. Angles in a Circle.

An angle inscribed in a semicircle is a right angle.



An angle subtended by an arc of a circle at any point of the circumference is equal to half the angle subtended by the same arc at the center.

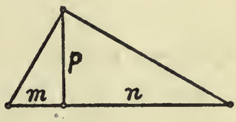
4. Similar Figures. Proportion.

If any two lines are cut by a set of parallels, the corresponding segments are proportional. (Hence the usual rule for dividing a given line into any number of equal parts.)

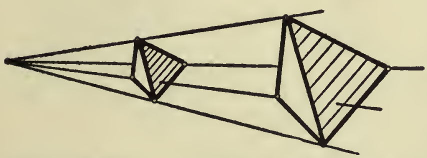
In all problems in proportion, the notation $a:b::c:d$, and all special terminology, such as "taking a proportion by alternation," "by composition,'' etc., should be abandoned in favor of the ordinary language of the equation. For example, if $a/b = c/d$, then, by adding 1 to both sides, $(a+b)/b = (c+d)/d$; and by subtracting 1 from both sides, $(a-b)/b = (c-d)/d$; etc.

If two plane triangles are similar, their corresponding sides are proportional.

In a right triangle, the perpendicular from the vertex of the right angle to the hypotenuse is a mean proportional between the segments of the hypotenuse:



$$p^2 = mn.$$

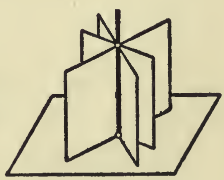


Any two similar figures, in the plane or in space, can be placed in "perspective," that is, so that lines joining corresponding points of the two figures will pass through a common point. In other words, of two similar figures, one is merely an enlargement of the other.

In two similar figures, if k is the factor of proportionality, any *length* in one $= k \times$ (the corresponding *length* in the other); any *area* in one $= k^2 \times$ (the corresponding *area* in the other); any *volume* in one $= k^3 \times$ (the corresponding *volume* in the other).

5. Lines and Planes.

If a line is perpendicular to a plane, every plane containing that line is perpendicular to the plane.



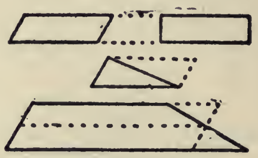
A dihedral angle is measured by a plane angle formed by two lines, one in each face, perpendicular to the edge.

6. Plane Areas.

Area of parallelogram
$= \text{base} \times \text{altitude}.$

Area of triangle
$= \frac{1}{2}\,\text{base} \times \text{altitude}.$

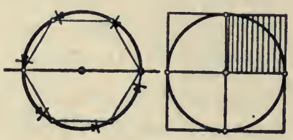
Area of trapezoid
$= \frac{1}{2}\,\text{sum of} \parallel \text{sides} \times \text{alt.}$
$= \text{mid-section} \times \text{altitude}.$



7. The Circle. ($\pi = 3.1416 \cdots = 22/7$, approximately.)

Circumference of circle $= 2\pi r$.

(Proved by regarding the circle as the limit of an inscribed or circumscribed polygon; proof rather long.)



Area of circle $= \pi r^2$.

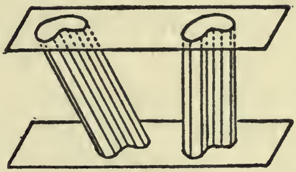
(Proof by regarding circle as limit of sum of triangles radiating out from the center, the altitude of each triangle being the radius of the circle; hence, area of circle $= \frac{1}{2}$ circumference $\times$ radius.)

$$\frac{\text{Area of sector}}{\text{area of circle}} = \frac{\text{angle of sector}}{\text{four right angles}}\text{; hence,}$$

Area of sector $= \frac{1}{2} r^2 \theta$, where θ is the angle in radians.

For area of segment, subtract triangle from sector.

8. The Cylinder.

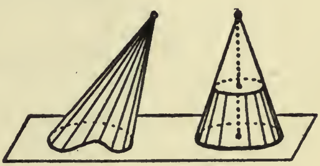


Volume of any cylinder (or prism) = base × altitude.

Area of curved surface of any *right* cylinder (or *right* prism) = perimeter of base × altitude.

(Proof by regarding the area as the limit of a sum of rectangles whose common altitude is the altitude of the cylinder; or, by slitting the cylinder along an "element" and rolling the surface out into a rectangle.)

9. The Cone.



Volume of any cone (or pyramid) = 1/3 base × altitude.

(Proof by dissecting a triangular prism; or, more simply, by the integral calculus.)

Area of curved surface of a *right* circular cone (or a *regular* pyramid) = 1/2 perimeter of base × slant height.

(Proof by regarding the area as the limit of a sum of triangles whose common altitude is the slant height of the cone.)

Area of frustum of a right circular cone (or of a regular pyramid)

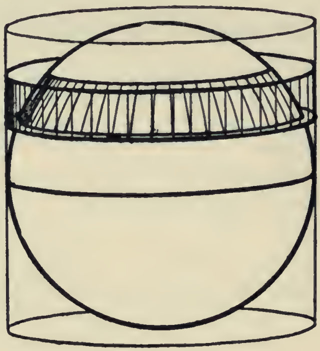
= 1/2 sum of perimeters of bases × slant height.
= perimeter of mid-section × slant height.

(Proof by regarding the area as the limit of the trapezoids whose common altitude is the slant height of the frustum.)

10. The Sphere.

Area of a zone = circumference of great circle $\times$ altitude of zone.

In other words, the area of the sphere cut out by two parallel planes is equal to the area of the portion of the circumscribing cylinder intercepted between the same pair of parallel planes. (Proof by regarding the zone as the limit of a sum of conical frustums.) Hence,



Area of sphere $= 4\pi r^2$

$= \text{area of four great circles of the sphere.}$

In other words, the *area of the sphere is equal to the area of the curved surface of the circumscribing cylinder.*

Volume of sphere $= \frac{4}{3}\pi r^3$.

(Proof by regarding sphere as limit of a sum of pyramids radiating out from the center, the altitude of each pyramid being the radius of the sphere; hence, volume of sphere $= \frac{1}{3}$ area of sphere $\times$ radius.)

$$\frac{\text{Area of a lune}}{\text{area of sphere}} = \frac{\text{angle of lune}}{\text{four right angles}}.$$

Area of spherical triangle is proportional to its spherical excess (that is, the excess of the sum of its angles over 180°).

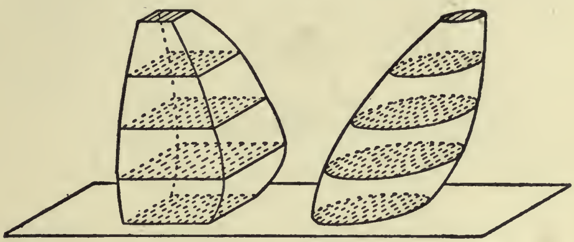
(Proof by considering three lunes which have the given triangle in common.)

The following further theorems, the proof of which involves the integral calculus, are mentioned here also, because they are easy to remember and are often serviceable in elementary work.

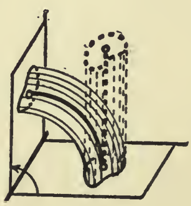
11. Cavalieri's Theorem (1598–1647).

Suppose two solids have their bases in the same plane, and let the sections made in each solid by any plane parallel to the base be called "corresponding sections." If then the corresponding sections of the two solids are always equal, the volumes of the solids will be equal.

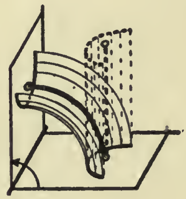
(Proof by regarding each of the solids as the limit of a pile of thin slabs.)



12. Theorems of Guldin (1577–1643), **or of Pappus** (about 290 A.D.).



1. Suppose a plane figure revolves about an axis in its plane but not cutting it. Then the volume of the solid thus generated is equal to the area of the given figure times the length of the path traced by its center of gravity.



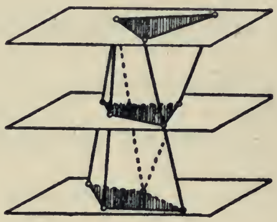
2. Suppose a plane curve revolves about an axis in its plane but not cutting it. Then the area of the surface thus generated is equal to the length of the given curve times the length of the path traced by its center of gravity.

13. The Prismoidal Formula.

The prismoidal formula holds for any solid lying between two parallel planes and such that the area of a section at distance x from the base is expressible as a polynomial of the second (or third) degree in x.

If A, B = areas of the bases, M = area of a plane section midway between the bases, and h = altitude, then

$$\text{Volume of prismoid} = \frac{h}{6}(A + B + 4M).$$



A SYLLABUS OF PLANE TRIGONOMETRY.

This syllabus is intended to include those facts and methods of plane trigonometry which a student should have so firmly fixed in his memory that he will never think of looking them up in a book.

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Definitions of sine, cosine, and tangent of an acute angle as ratios between the sides of a right triangle:

$$\sin x = \text{opp/hyp}; \ \cos x = \text{adj/hyp}; \ \tan x = \text{opp/adj}.$$

To trace the changes in these functions, as the angle changes from 0° to 90° (circle of reference).

Use of tables. Exact values of functions of 30°, 45°, and 60°.

To find remaining functions of an angle when one function is given (draw right triangle). To construct an angle from its tangent.

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To trace the changes in these functions, as the angle changes from 0° to 360° (circle of reference).

Definitions of cotangent, secant, and cosecant:

$$\cot x = 1/\tan x, \quad \sec x = 1/\cos x, \quad \csc x = 1/\sin x.$$

Definitions of versed sine and coversed sine:

$$\text{vers } x = 1 - \cos x, \quad \text{covers } x = 1 - \sin x.$$

Use of the tables: reduction to first quadrant.

Solution of oblique triangles.

Law of sines: $a/b = \sin A/\sin B$.

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CHAPTER III. GENERAL PROPERTIES OF THE TRIGONOMETRIC FUNCTIONS.

Relations between the functions of a single angle.

Functions of $(-x)$. Functions of $(x \pm n90°)$, etc.

Functions of the sum and difference of two angles:

$$\sin (x + y) = \sin x \cos y + \cos x \sin y,$$
$$\cos (x + y) = \cos x \cos y - \sin x \sin y.$$

Functions of twice an angle, and of half an angle.

The inverse functions, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, etc.

Solution of trigonometric equations.

CHAPTER I.

Sine, Cosine, and Tangent of Acute Angles.

1. *Definition of sine, cosine, and tangent of an acute angle x.*—In any right triangle, if x is one of the acute angles, the sine, cosine and tangent of x are defined as ratios between the sides of the triangle, as follows:

$$\sin x = \frac{\text{side opp.}}{\text{hypot.}} \qquad \cos x = \frac{\text{side adj.}}{\text{hypot.}}$$

$$\tan x = \frac{\text{side opp.}}{\text{side adj.}}$$

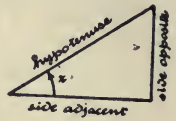


Fig. 1.

These ratios are pure numbers, depending only on the size of the angle.

2. To trace the changes in these numbers when the angle changes from 0° to 90°, draw the figure so that the denominator of the ratio is kept constant, say equal to 1 inch, and trace the changes in the numerator. Thus, from Fig. 2, when x goes from 0° to 90°, sin x goes from 0 to 1, and cos x goes from 1 to 0; from Fig 3, when x goes from 0° to 90°, tan x goes from 0 to infinity.

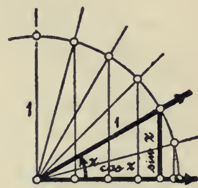


Fig. 2.

3. *Tables.*—The ratios thus defined are called "trigonometric functions" of the angle, and their values have been tabulated, to 4, 5, or 6 places of decimals, in the "tables of trigonometric functions." Before using the printed tables, the student should make his own table, for a few angles, by graphical construction, with a protractor, to two places of decimals.*

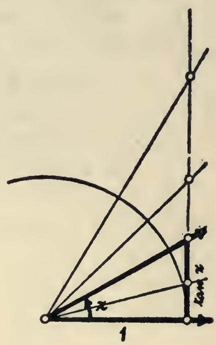


Fig. 3.

* It is clear from the figure that the values of cos x from 0° to 90° are the same as the values of sin x in reverse order; note how this fact is made use of to save space in the tables.

4. The functions of 30°, 45°, and 60° can be found exactly, without the use of the table. Thus, in the triangles which occur in Fig. 4, it is readily proved by the Pythagorean theorem that if the hypotenuse is 1 inch, the shortest side is $\frac{1}{2}$ in., the longest side is $\frac{1}{2}\sqrt{3}$ in., and the middle-sized side $\frac{1}{2}\sqrt{2}$ in. Hence any function of 30°, 45°, or 60° can be read off the figure by inspection. For example,

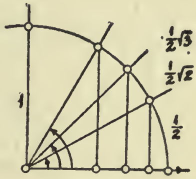


FIG. 4.

$$\sin 30° = \tfrac{1}{2}, \quad \tan 45° = 1, \quad \tan 60° = \sqrt{3}; \quad \text{etc.}$$

5. It is frequently required *to find the remaining functions of an angle when any one function is given.* To do this, draw a right triangle, mark one of the angles, and mark two sides to correspond to the given function. Then compute the remaining side by the Pythagorean theorem, and read off any desired function from the completed figure. For example,

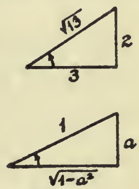


FIG. 5.

Given, $\tan x = \frac{2}{3}$. From the figure, $\sin x = 2/\sqrt{13}$; etc.

Given, $\sin x = a$. From the figure, $\tan x = a/\sqrt{1 - a^2}$; etc.

To construct an angle when any one of its functions is given, first find the *tangent* of the angle; when the tangent is known, the construction of the angle is obvious.

6. The notation $\sin^2 x$, etc., is used as an abbreviation for $(\sin x)^2$; etc.

The following fundamental relations are easily proved and remembered from the figure: for any angle x,

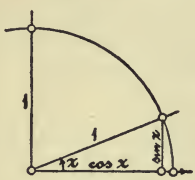


FIG. 7.

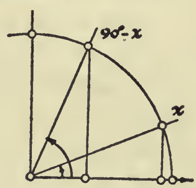


FIG. 8.

$$\sin^2 x + \cos^2 x = 1, \quad \tan x = \frac{\sin x}{\cos x}, \quad \begin{array}{l} \sin(90° - x) = \cos x. \\ \cos(90° - x) = \sin x. \end{array}$$

7. The student should be thoroughly drilled in the definitions of the sine, cosine and tangent, in right triangles in all possible positions in the plane regardless of lettering. Thus, the mental process should be as follows: pointing at the figure, "the tangent of *this* angle is *this* side, divided by *this* side"; etc.

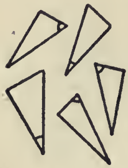


FIG. 9.

The following forms of the original equations are especially useful, and should be emphasized:

$$\text{side } opp. = \text{hypot.} \times sine; \quad \text{side } adj. = \text{hypot.} \times cosine.$$

SOLUTION OF RIGHT TRIANGLES.

8. We recall that in any right triangle, the sum of the squares on the two legs is equal to the square on the hypotenuse, and the sum of the acute angles is 90°. Hence, when either acute angle is known, the other may be found; and the sine of either acute angle is the cosine of the other:

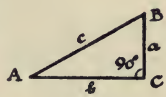


FIG. 10.

$$c^2 = a^2 + b^2, \quad \sin A = \cos B.$$

9. By the aid of a table of sines, cosines and tangents, *when any two parts of a right triangle, besides the right angle, are given, the remaining parts may be found* (except in the case where the given parts are the two acute angles, in which case the triangle is not determined).

For, we have merely to remember the definitions of the functions, selecting the equations so that only one unknown appears in each equation; then solve for the unknown quantity, and compute by the aid of the tables. The results should be checked by substituting in some relation not used in the direct computation.*

* This computation, like many other numerical computations, can often be shortened by the use of the slide rule, or by the use of logarithms; in fact, tables are provided which give the logarithms of the trigonometric functions directly in terms of the angles; but the student should thoroughly understand the use of the functions themselves before he begins to use the logarithmic tables.

10. Numerous problems involving right triangles: isosceles triangles, polygons, oblique triangles solved by means of right triangles, heights and distances, surveying problems, etc.

ORTHOGONAL PROJECTION. COMPONENTS OF FORCES, ETC.

11. The *projection of a length AB* on any line *is the given length times the cosine of the angle* between the lines. (Proof from the definition of cosine.)

The *projection of a plane area* upon any fixed plane is *the given area times the cosine of the angle* between the planes. (Proof by the theorem of limits.)

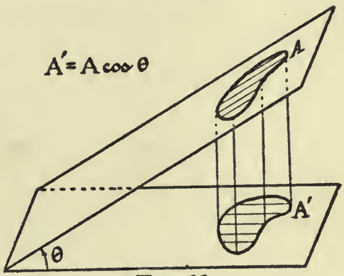


FIG. 11.

12. *The component of a force* along any fixed axis is the *magnitude of the force times the cosine of the angle* between the force and the axis.

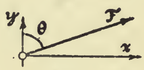


FIG. 12.

Since we usually require the components along two rectangular axes, it is important to remember that $\cos(90° - x) = \sin x$. The mental process should be as follows:

FIG. 13.

In Fig. 12, the component of F along the y-axis is F times the cosine of θ; the component of F along the x-axis is F times the cosine of the other angle, which is F times the *sine* of θ; that is, $F_y = F\cos\theta$; $F_x = F\sin\theta$. Similarly, in Fig. 13, $F_x = F\cos\phi$; $F_y = -F\sin\phi$ (minus, because it pulls backward along that line).

The components of velocities, accelerations, or any other vector quantities are to be handled in the same way.

13. Every problem should be accompanied by a *sketch* or *diagram*, to show that the student understands the meaning of each step of his work. And in many cases, an accurate graphical solution on a drawing board may be used as a valuable check on the correctness of the numerical computation.

14. Note. *That portion of trigonometry which has been outlined up to this point is so elementary in character, and so readily understood and appreciated by the student, that it may well be introduced much earlier in the course than is usually done—perhaps even as early as the elementary course in plane geometry.*

CHAPTER II.

THE TRIGONOMETRIC FUNCTIONS OF ANY ANGLE.

15. *Angles in general.*—An angle, as the term is used in applied mathematics, is the amount of rotation of a moving radius OP about a fixed point O, measured from a fixed line

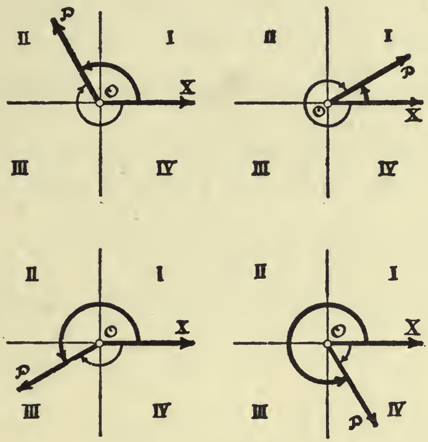


FIG. 14.

OX. Here OX is called the *initial line* and OP the *terminal line* of the angle. Counterclockwise rotation is positive, and angles are added and subtracted as algebraic quantities. The quadrants are numbered as in the figure; an "angle in quadrant *II*" for example, means an angle whose terminal line lies in quadrant *II*.

16. *Congruent angles* are angles differing by any multiple of 360°.

17. *Complementary angles* are angles whose sum is 90°; *supplementary angles* are angles whose sum is 180°.

18. Units of angular measurement are: the *degree*, subdivided into minutes and seconds, or decimally; the *grade*, subdivided decimally; and the *radian*, subdivided decimally.

1 degree = 1° = 1/90th of a right angle;

1 grade = 1/100th of a right angle (used in France);

1 radian = angle subtended by an arc equal to the radius.

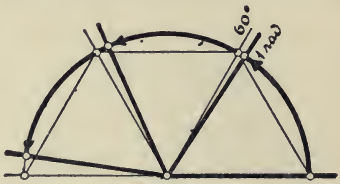


FIG. 15.

Since ratio of semi-circumference to radius $=\pi$ (where $\pi=3.1416\cdots=3\frac{1}{7}$ approximately), we have

$$\pi \text{ radians} = 180°, \text{ and hence } 1 \text{ radian} = \text{about } 57.3°.$$

19. The *radian* is especially important in problems concerning the motion of a particle in a circular path. Thus, if

r ft. = radius of the circle,
s ft. = length of arc traversed, and
θ radians = angle swept over by the moving radius, then

$$s=r\theta.$$

This important equation is not true unless the angle is measured in radians. Again, if
v ft. per sec. = linear velocity of the particle in its path, and
ω radians per sec. = its angular velocity, then

$$v=r\omega.$$

Further, if the angular velocity = ω radians per sec. = N rev. per min., then the relation between the numbers ω and N is given by

$$\omega=\frac{\pi N}{30}.$$

In all higher mathematics, when a letter is used for an angle, without designating the unit, it is understood that the letter means the number of radians in the angle.

20. *Definition of sine, cosine, and tangent of any angle.*— Let x be any angle, swept over by a moving radius revolving from OX to OL, and suppose for convenience of language that OX extends horizontally to the right. Assume, for the moment that OX and OL are not perpendicular. From any point P of the moving radius drop a perpendicular on the initial line (or the initial line produced), thus forming a right tri-

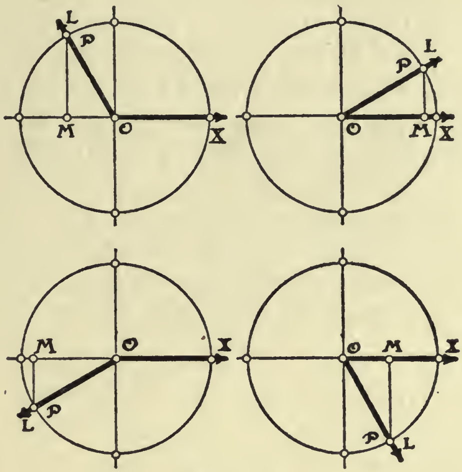


FIG. 16.

angle, called the *triangle of reference* for the given angle x. In this triangle, the perpendicular MP is called the *side opposite O,* and is positive if it runs up, negative if it runs down; the base OM is called the *side adjacent* to O, and is positive if it runs to the right, negative if it runs to the left, and the radius OP is called the *hypotenuse* of the triangle and may always be taken as positive. The *sine, cosine* and *tangent* of the angle x are then defined as follows:

$$\sin x = \frac{\text{side opp.}}{\text{hypot.}}, \quad \cos x = \frac{\text{side adj.}}{\text{hypot.}}, \quad \tan x = \frac{\text{side opp.}}{\text{side adj.}} = \frac{\sin x}{\cos x}$$

These ratios are positive or negative numbers, depending only on the position of the terminal side of the angle x, and are called trigonometric functions of x. The functions of any angle congruent to x are the same as the functions of x, so that we need consider only the angles in "the first revolution," that is, angles between 0° and 360°.

21. To trace the changes in each function as the angle changes from 0° to 360°, draw the figure so that the denominator of the ratio is kept constant, say equal to 1 inch, and trace the changes in the numerator (Fig. 17 for the sine and cosine; Fig. 18 for the tangent). Obviously, the sine will be positive for angles in the upper quadrants; the cosine will be positive for angles in the right hand quadrants; and the tangent will be positive in quadrants *I* and *III*.

The definitions of the functions of 0°, 90°, 180°, and 270°, which were not included above, can now be readily obtained by noting what becomes of the function of a variable angle x when x approaches one of these values as a limit.

In using the "circle of reference" be careful to have every angle start from the initial line that extends horizontally to the right.

Other Trigonometric Functions.

22. *Definition of other trigonometric functions.*—Besides the sine, cosine, and tangent, other functions in common use are the *cotangent*, the *secant*, and the *cosecant*, which are most conveniently defined thus:

$$\cot x = \frac{1}{\tan x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

Less important, but often convenient, are the *versed sine* and the *coversed sine:*

$$\text{vers}\, x = 1 - \cos x, \quad \text{covers}\, x = 1 - \sin x.$$

23. It is worth remembering that the sine and cosine are always less than (or equal to) 1, in absolute value; their reciprocals, the secant and cosecant, are always greater than (or equal to) 1, in absolute value; the tangent and cotangent may have any value, positive or negative; while the versed sine and coversed sine are always positive, ranging from 0 to 2.

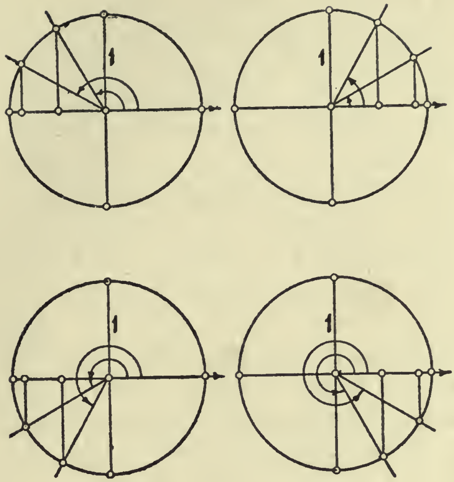


Fig. 17.

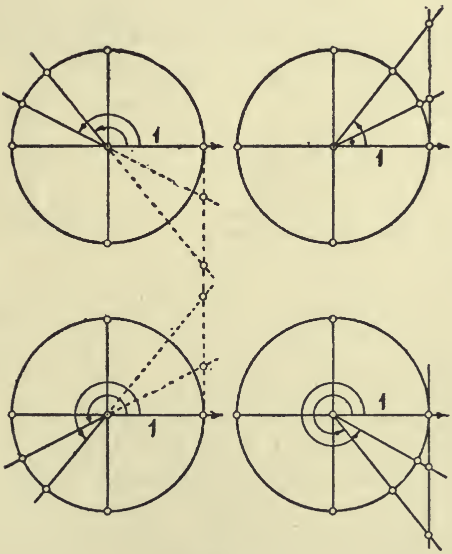


Fig. 18.

24. *Use of the tables: reduction to the first quadrant.*—The tables in common use give the values of the functions only for angles between 0° and 90°, that is, only for angles in the first quadrant. To find the functions of an angle x in one of the other quadrants, find first the "reduced angle" in quadrant *I* (that is, $x-90°$, or $x-180°$, or $x-270°$), and then proceed as in the following examples:*

(a) To find $\cos x$, when x is in quadrant *II*. Draw any angle in quadrant *II* to represent the angle x (avoiding, however, lines near the middle of the quadrant) and draw the "reduced angle" $x-90°$ in quadrant *I*. Then, pointing at the figure, $\cos x$ is *this* line (VVV) [divided by the radius], which is the same in length as *this* line (≶) [divided by the radius], which is the *sine* of $x - 90°$; but the first line is negative; hence

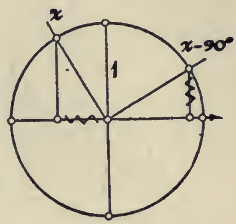


FIG. 19.

$$\cos x = -\sin (x - 90°),$$

where $\sin (x - 90°)$, of course, can be found in the table.

(b) To find $\tan x$, when x is in quadrant *II*. Pointing at the figure, $\tan x$ is this line (≶) divided by this line (|||), which is the same as this line (VVV) divided by this line (≡), which is the cotangent of $(x-90°)$; but the signs are unlike; hence

$$\tan x = -\cot (x - 90°),$$

where $\cot (x-90°)$ can be found from the table.

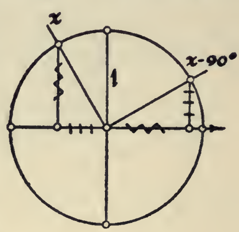


FIG. 20.

Similarly for any other case.

25. The converse problem of finding the angle corresponding to any given function is complicated by the fact that there will be (in general) *two* angles between 0° and 360° corresponding to any given function. The most satisfactory way

* The given angle is supposed to be already reduced to an angle between 0° and 360°.

to find these two angles, in any numerical case, is to draw the figure, and proceed as in the examples below, in which x_0 in each case represents an angle in the first quadrant which can be found in the table.

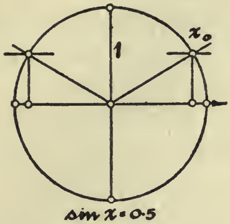


FIG. 21.

Given $\sin x = 0.5$;
$x = x_0$ or $180° - x_0$.

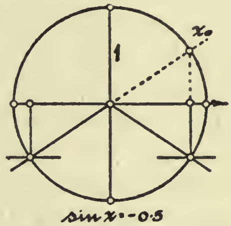


FIG. 22.

Given $\sin x = -0.5$;
$x = 180° + x_0$ or $360° - x_0$.

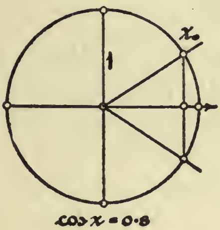


FIG. 23.

Given $\cos x = 0.8$;
$x = x_0$ or $360° - x_0$.

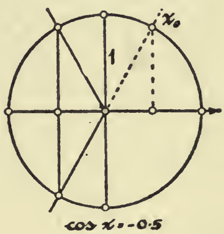


FIG. 24.

Given $\cos x = -0.5$;
$x = 180° - x_0$ or $180° + x_0$.

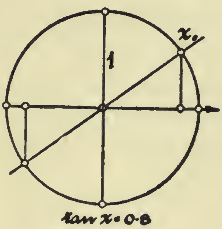


FIG. 25.

Given $\tan x = 0.8$;
$x = x_0$ or $180° + x_0$.

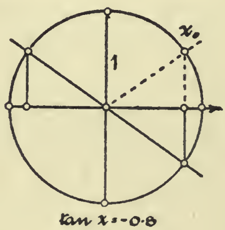


FIG. 26.

Given $\tan x = -0.8$;
$x = 180° - x_0$ or $360° - x_0$.

These results are not formulæ to be memorized; it is much safer, and more intelligent, to draw the appropriate figure, or to visualize it in the mind, for each case as it arises. The student should be thoroughly drilled in numerical cases, especially for angles in the second quadrant.

Notice that an angle is completely determined when we know the value of any one of its functions, and the *sign* of any other function (not the reciprocal of the first).

It we restrict ourselves to angles between 0° and 180°, as in the case of angles in a triangle, then an angle is wholly determined by either its cosine or its tangent; but there will be two angles, x and $180° - x$, corresponding to a given sine.

26. The functions of certain angles in the later quadrants, corresponding to 30°, 45°, and 60° in quadrant *I*, may be found exactly, without the use of the tables, by inspection of the figure (see § 4).

For example, $\cos 120° = -\frac{1}{2}$.

27. If it is required to find the remaining functions of an angle when one function is given, draw a right triangle and proceed as in § 5, considering only the absolute values of the quantities, without regard to sign; then adjust the sign of the answer according to the quadrant in which the angle lies. Or, the angle may be drawn at once in the proper quadrant.

Solution of Oblique Triangles.

28. In any plane triangle the following theorems are easily proved from a figure:

(1) *The "Law of Sines."*—Any side is to any other side as the sine of the angle opposite the first side is to the sine of the angle opposite the other side; in the usual notation:

$$\frac{a}{b} = \frac{\sin A}{\sin B},$$

with two analogous formulæ obtained by "advancing the letters."

(2) *The "Law of Cosines."*—The square of any side is equal to the sum of the squares of the other two sides, minus twice their product times the cosine of the included angle:

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

with two analogous formulæ obtained by "advancing the letters."

These two laws, with the fact that the sum of the angles is 180°, suffice to "solve" any plane triangle, and are important in many theoretical considerations.

The following formulæ which are especially adapted to logarithmic computation, give the tangents of the half-angles in terms of the sides, and are included here for reference:

$$\tan\frac{A}{2} = \frac{r}{s-a}, \qquad \tan\frac{B}{2} = \frac{r}{s-b}, \qquad \tan\frac{C}{2} = \frac{r}{s-c}$$

where

$$s = \frac{a+b+c}{2}$$

and

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \text{radius of inscribed circle.}$$

From these formulæ we have at once,

$$\text{Area} = rs = \sqrt{s(s-a)(s-b)(s-c)}.$$

29. The only case which is likely to give any difficulty, is the "ambiguous case" in which the given parts are two sides and the angle opposite one of them. Here we must remember, at a certain point in the work, that when the sine of an angle is given, there will be, in general, two angles corresponding to that sine, one the supplement of the other; so that from that point on, the problem breaks up into two separate problems. But if the sine of an angle is 1, then the only value for the angle is 90°; and if the sine is greater than 1, there is no corresponding angle, and the problem is impossible. It is advisable to construct a fairly accurate figure.

30. *Problems in oblique triangles, triangulation, etc.*

In every case at least a rough sketch should be drawn on which the known parts are clearly marked, and a "blank form" for the computation should be made out for the entire problem, before any of the quantities are looked up in the table.

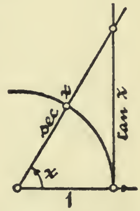
CHAPTER III.

General Properties of the Trigonometric Functions.

31. *Relations between the functions of a single angle.*—The student should convince himself that the following important relations will hold for any angle x:

$$\sin^2 x + \cos^2 x = 1, \quad \tan x = \frac{\sin x}{\cos x},$$

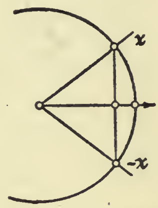
$$\sec^2 x = 1 + \tan^2 x.$$



All these relations are easily recalled by the aid of the figures.

Somewhat less important is the following:

$$\csc^2 x = 1 + \cot^2 x.$$

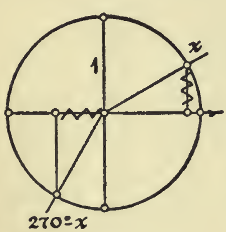


32. *Functions of* $(-x)$. From the figure,

$$\sin(-x) = -\sin x,$$
$$\cos(-x) = \cos x,$$
$$\tan(-x) = -\tan x.$$

33. *Functions of* $(90° + x)$, $(x + 180°)$, *etc.*—Any function of a combination like $(x \pm n90°)$ or $(n90° \pm x)$ can be expressed in terms of a function of x by the use of the figure.

For example, find $\sec(270° - x)$. Take as x any small angle in the first quadrant, and draw the angle $270° - x$.



Then, $\sec(270° - x)$ is 1 over the *cosine* of $(270° - x)$, which, pointing at the figure, is the radius over *this* line (VVV), which is the same, in length, as the radius over *this* line (⋛), which is 1 over the *sine* of x, or $\csc x$. But the signs are opposite; therefore,

$$\sec(270° - x) = -\csc x.$$

This method requires the memorizing of no rules or formulæ, besides the definitions of the functions; a very little practice will develop all the speed and accuracy that can be desired, and the method is one which is readily recalled to mind after long disuse. The special case of complementary angles, however, is worth remembering as a separate formula: Any function of $(90° - x) =$ the co-named function of x.

Formulas for the Sum of Two Angles, Etc.

34. In simplifying trigonometric expressions which occur in calculus, mechanics, etc., the following formulæ are so frequently required that they should be thoroughly memorized. The ability to recognize those relations readily, regardless of the special lettering employed, is a necessary condition for rapid progress in almost any branch of analysis, but *it is highly undesirable to extend the list beyond the limits here given.*

The fundamental formulæ from which all others are derived are these two, the proof of which is obtained from a figure:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y, \tag{1}$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y. \tag{2}$$

These and the following formulæ should be memorized in words, not in letters: thus, "the sine of the sum of two angles is the sine of the first times the cosine of the second, plus the cosine of the first times the sine of the second," etc.

Dividing (1) by (2) and then dividing numerator and denominator by the product of the cosines, we have

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}. \tag{3}$$

Changing the sign of y in these three formulæ, and remembering the relations for negative angles, we have the corresponding formulæ for $\sin(x - y)$, $\cos(x - y)$, $\tan(x - y)$, which will be exactly the same as (1), (2), and (3) with all the connecting signs reversed:

(4) $$\sin(x-y)=\sin x\cos y-\cos x\sin y,$$

(5) $$\cos(x-y)=\cos x\cos y+\sin x\sin y,$$

(6) $$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\ \tan y}.$$

Putting $x=y$ in (1), (2), and (3) we have at once

(7) $$\sin 2x=2\sin x\cos x,$$

(8) $$\cos 2x=\cos^2 x-\sin^2 x$$
$$=1-2\sin^2 x=2\cos^2 x-1,$$

(9) $$\tan 2x=\frac{2\tan x}{1-\tan^2 x}.$$

Solving (8) first for $\sin x$ and then for $\cos x$, and putting $2x=y$, or $x=y/2$, we find

(10) $$\sin\frac{y}{2}=\pm\sqrt{\frac{1-\cos y}{2}},*$$

(11) $$\cos\frac{y}{2}=\pm\sqrt{\frac{1+\cos y}{2}},$$

whence,

(12) $$\tan\frac{y}{2}=\pm\sqrt{\frac{1-\cos y}{1+\cos y}}.$$

This last formula may be transformed, by rationalizing numerator or denominator, into

$$\tan\frac{y}{2}=\frac{1-\cos y}{\sin y}=\frac{\sin y}{1+\cos y}.$$

Other formulas, useful for special purposes, should not be memorized, but should be derived as needed.

35. In proving the identity of two trigonometric expressions, it is best to reduce each expression separately to its simplest form.

* The *plus* sign is to be used when $\sin\frac{1}{2}y$ is positive, the *minus* sign when $\sin\frac{1}{2}y$ is negative. Similarly in the next two formulas.

The fallacy of supposing that because a true relation can be deduced from a given equation, the given equation is therefore necessarily true, should be carefully explained.

For example, from the false equation $3 = -3$ we can obtain the true equation $9 = 9$ by squaring both sides; or, from the false equation $30° = 150°$ we can obtain the true equation $\frac{1}{2} = \frac{1}{2}$ by taking the sine of both sides; but in each of these cases the step taken is not reversible.

36. The following device for transforming an expression of the form $a \cos x + b \sin x$ is often useful:

$$a \cos x + b \sin x = \sqrt{a^2 + b^2}\left[\frac{a}{\sqrt{(a^2 + b^2)}} \cos x + \frac{b}{\sqrt{(a^2+b^2)}} \sin x\right]$$
$$= A \cos (x - B),$$

where $$A = \sqrt{(a^2 + b^2)} \text{ and } \tan B = \frac{b}{a}.$$

37. *The inverse functions.*

The angle between $-90°$ and $+90°$ whose sine is x is denoted by $\sin^{-1} x$.*

The angle between $0°$ and $180°$ whose cosine is x is denoted by $\cos^{-1} x$.

The angle between $-90°$ and $+90°$ whose tangent is x is denoted by $\tan^{-1} x$.

In simplifying expressions involving these "inverse functions," it is well to take a single letter to stand for each inverse function; as, $y = \sin^{-1} x$, whence, by definition, $\sin y = x$; etc.

38. *Solution of trigonometric equations.* Many trigonometric equations can be solved only by the "method of trial and error." In other cases, however, it is possible, by the use of the formulas given above, to transform the given equation into a form involving only a single function of a single angle; if this equation can be solved for the function in question, then the required value (or values) of the angle can be found from the tables or it can be shown that no solution exists.

* The symbol $\sin^{-1} x$ (or arc sin x) is often defined as simply "the angle whose sine is x"; but since there are many such angles, it is necessary to specify which one is to be taken as "the" angle, if the symbol is to have any definite meaning.

A SYLLABUS OF ANALYTIC GEOMETRY.

This syllabus is intended to include those facts and methods of analytic geometry which a student who has completed an elementary course in that subject should have so firmly fixed in his memory that he will never think of looking them up in a book.

A course of study in analytic geometry should consist chiefly of *problems* solved by the students, each problem being solved on the basis of a small number of fundamental formulas such as are here mentioned.

This syllabus is confined mainly to the conic sections; but a satisfactory course in analytic geometry should include also the study of many other curves, both in rectangular and in polar coordinates. The syllabus takes up only those properties of curves which can be readily investigated without the aid of the calculus; but the present tendency to introduce the elements of the calculus *before* any elaborate study of geometry is attempted is to be much encouraged.

TABLE OF CONTENTS.

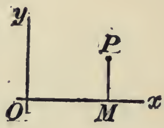
CHAPTER I.

RECTANGULAR COORDINATES.

1. In many geometrical problems it is convenient to describe the position of a point in a plane by giving its distances from two fixed (perpendicular) lines in the plane.*

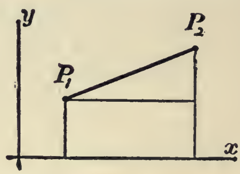
For example, on a map, the distance of a point to the east or west from a fixed meridian is called the longitude of the point, and its distance north or south from the equator is called its latitude.

So in general, in any plane, the distance of a point to the right or left from a fixed vertical axis is called the *abscissa*, x, of the point, and its distance up or down from a fixed horizontal axis is called its *ordinate*, y. The x and y together are called the *coordinates* of the point.



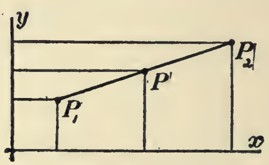
The value of $x\,(=OM)$ will be positive to the right, negative to the left; the value of $y\,(=MP)$ will be positive upward, negative downward. The point for which $x = x_1$ and $y = y_1$ is denoted by P_1, or (x_1, y_1).

2. To express *the distance between two points* in terms of their coordinates:



$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

3. To find the coordinates of *the point half way between two given points:*

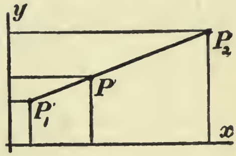


$$x = \tfrac{1}{2}(x_1 + x_2),$$

$$y = \tfrac{1}{2}(y_1 + y_2).$$

* We restrict ourselves here to *rectangular* axes; oblique axes are, however, occasionally useful.

4. *To find the coordinates of a point P on the line through two fixed points,* and such that its distance from the first point is n times the distance between the two points ($\overline{P_1P} = n\overline{P_1P_2}$):

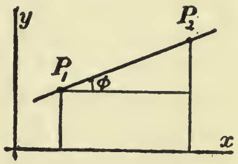


$$x = x_1 + n(x_2 - x_1),$$
$$y = y_1 + n(y_2 - y_1).$$

Here n may be any real number (positive, negative, or zero).

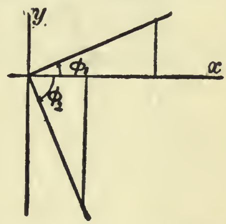
5. *To find the slope of a line through two given points:*

$$m = \tan\phi = \frac{y_2 - y_1}{x_2 - x_1}.$$



The angle ϕ is called the *inclination* of the line; $\tan\phi$ is the *slope.*

6. If two lines are *parallel,* their slopes are equal: $m_1 = m_2$.



If two lines are *perpendicular,* the product of their slopes is minus one: $m_1 m_2 = -1$.

7. To express *the areas of triangles and polygons* in terms of the coordinates of the vertices, consider the trapezoids formed by the ordinates drawn to the vertices.

8. In any problem involving an unknown point, remember that *two conditions* are necessary to determine the coordinates of the point (simultaneous equations in two unknown quantities).

CHAPTER II.

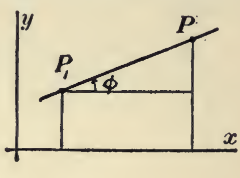
THE STRAIGHT LINE: EQUATIONS OF THE FORM $Ax + By + C = 0$.

9. We have seen that if two conditions are imposed on x and y, the position of the point (x, y) is wholly determined. If only one condition is imposed, the position of the point is only partially restricted. (Examples: $x = 5$, $x^2 + y^2 = 25$, etc.)

The collection of all points which satisfy a given condition imposed on x and y is called the *locus* of that condition; and the condition itself, expressed in algebraic language, is called the *equation of the locus.* Thus, the *equation of a straight line* is the algebraic expression of the *condition* which x and y must satisfy in order that the point (x, y) shall lie on the line; in other words, *the equation of a line is an equation which is* TRUE *when the coordinates of any point on the line are substituted for x and y, and* FALSE *when the coordinates of any point off the line are substituted for x and y;* and so in general for the equation of any locus.

10. To find the coordinates of the points of intersection of two loci whose equations are given, we have simply to find the pairs of values of x and y (if any) which satisfy both the equations at once (simultaneous equations in x and y).

11. *To find the equation of a line* (not perpendicular to either axis), *when its slope, m, and the coordinates of one of its points (x_1, y_1), are given:*



$$y - y_1 = m(x - x_1).$$

The equation of a line perpendicular to the x-axis (or the y-axis) is, by inspection, $x = a$ (or $y = b$).

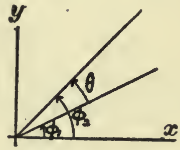
The equation of any straight line is of the form $Ax + By + C = 0$, and the locus of every equation of the form $Ax + By + C = 0$ is a straight line. Hence, to plot the locus of such an equation, it is sufficient to find any two of its points.

12. *To find the slope of a line whose equation is given* (the line being not perpendicular to an axis), write the equation in the form $y = (\ )\,x + (\ )$; then the coefficient of x will be the slope.

13. To find the equation of a line *parallel* or *perpendicular* to a given line and through a given point, remember that $m_1 = m_2$ for parallel lines, and $m_1 m_2 = -1$ for perpendicular lines (see § 6).

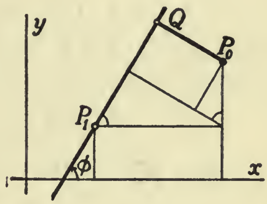
Special method: if the given line is $Ax + By + C = 0$, then the parallel is $Ax + By = k$ and the perpendicular is $Bx - Ay = K$, where k and K are to be determined.

14. *To find the angle θ between two lines whose slopes are given:*



$$\tan \theta = \frac{m_2 - m_1}{1 + m_2 m_1}.*$$

15. *To find the distance between a given point (x_0, y_0), and a given line:*



(*a*) When the inclination of the line, ϕ, and the coordinates of one of its points, (x_1, y_1), are given, we have from the figure:

$$\overline{QP_0} = (x_0 - x_1) \sin \phi - (y_0 - y_1) \cos \phi,$$

(*b*) When the equation of the line is given in the form $Ax + By + C = 0$, use the following formula:†

$$D = \left| \frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}} \right|.$$

Here the vertical bars mean "the absolute value of."

---

* Proof: By trigonometry, $\tan(\phi_2 - \phi_1) = \dfrac{\tan \phi_2 - \tan \phi_1}{1 + \tan \phi_2 \tan \phi_1}$.

† Proof: Show that the foot of the perpendicular from P_0 to the line $Ax + By + C = 0$ has the coordinates $x_2 = x_0 - hA$, $y_2 = y_0 - hB$, where $h = (Ax_0 + By_0 + C) / (A^2 + B^2)$.

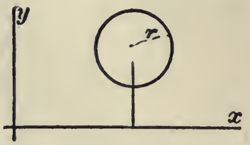
CHAPTER III.

THE CIRCLE: EQUATIONS OF THE FORM $x^2 + y^2 + Dx + Ey + F = 0$.

16. The *equation of a circle* is the algebraic expression of the *condition* which x and y must satisfy in order that the point (x,y) shall lie on the circle (see § 9 and § 10).

17. *To find the equation of a circle when its radius, r, and the coordinates* (α, β) *of its centre are given:*

$$(x - \alpha)^2 + (y - \beta)^2 = r^2.$$



When the centre is at the origin (0, 0), this equation becomes

$$x^2 + y^2 = r^2.$$

18. The equation of any circle is of the form $x^2+y^2+Dx+Ey+F=0$. Conversely, every equation of the form $x^2 + y^2 + Dx + Ey + F = 0$ can be reduced to the form $(x + \frac{D}{2})^2 + (y + \frac{E}{2})^2 = \frac{1}{4}(D^2 + E^2 - 4F)$, and therefore represents a circle with centre at $(-D/2, -E/2)$, or a single point, or no locus, according as $D^2 + E^2 - 4F$ is positive, zero, or negative. When we say, in brief, that the locus of any equation of the form $x^2 + y^2 + Dx + Ey + F = 0$ is a "circle," we must understand that the "circle" may be "real," "null," or "imaginary."

19. *To find the centre and radius of a circle whose equation is given,* do not use a formula, but "complete the squares" of the terms in x and y in each case, and compare with the standard equation in the manner just indicated.

20. *In problems concerning tangents to a circle,* use the fact that the tangent is perpendicular to the radius drawn to the point of contact.

CHAPTER IV.

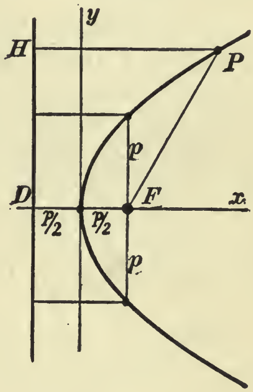
THE PARABOLA: $y^2 = 2px$.

21. DEFINITION: The locus of a point which moves so that

$$\frac{\text{its distance from a fixed point}}{\text{its distance from a fixed line}} = 1$$

is called a *parabola*.

The fixed point is called the *focus* and the fixed line the *directrix.* The line perpendicular to the directrix through the focus is called the *principal axis*. There is evidently only one point of the principal axis which is also a point of the curve, namely the point half way between the focus and the directrix; this point is called the *vertex*.



22. If we take the vertex as the origin and the principal axis as the axis of x, the *equation of the parabola* is

$$y^2 = 2px,$$

where $p =$ the distance between focus and directrix.*

23. The form of the curve is therefore that shown in the figure.† By definition $PF = PH$ for every point P on the curve.

The breadth of the curve at the focus is called the *latus rectum*, and is equal to $2p$.

---

* Proof: If (x, y) is any point on the curve, then

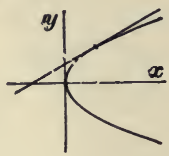
$$\sqrt{(x - \tfrac{1}{2}p)^2 + (y - 0)^2} = x + \tfrac{1}{2}p.$$

Many British authors write the equation in the form $y^2 = 4ax$, to avoid fractions. Other writers use $y^2 = 4px$ for the same purpose; this latter form, however, is unfortunate, since $2p$ is a fairly well-established notation for the latus rectum in each of the conics.

† Thus when x is 0, y is 0. When x increases, y increases, plus and minus; the curve is symmetrical with respect to the x-axis. When x is negative, y is imaginary. When $x = p/2$, $y = \pm\, p$; when $x = 2p$, $y = \pm\, 2p$.

24. To find the *equation of a tangent* to the parabola $y^2 = 2px$, use one of the following formulas:

(*a*) When the point of contact, (x_1, y_1), is given:*



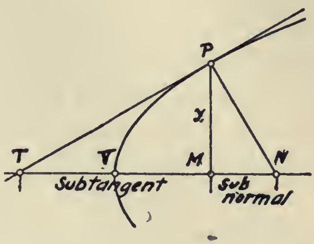
$$y_1 y = p(x + x_1);$$

(*b*) When the slope, m, is given:†

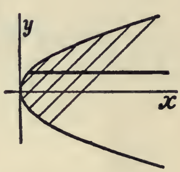
$$y = mx + \frac{p}{2m}.$$

A line perpendicular to a tangent at the point of contact is called a *normal.*

If the tangent and normal at any point P meet the principal axis at T and N, the projections of PT and PN on the principal axis are called the *subtangent* and *subnormal,* respectively. The subtangent is bisected by the vertex. The subnormal is constant, equal to the semi-latus rectum, p.



25. The locus of the middle points of a set of parallel chords in the parabola is a straight line parallel to the principal axis; such a line is called a *diameter.* In the parabola $y^2 = 2px$, if the slope of the parallel chords is m, then the equation of the diameter is $y = p/m$.‡



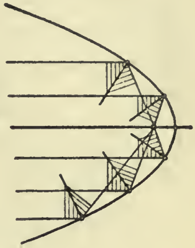
* Proof: Let $P_2 = (x_1 + h, y_1 + k)$ be a second point on the curve, near P_1; then the *slope of the tangent* at P_1 will be the limit of k/h as P_2 approaches P_1 along the curve, namely $m = p/y_1$. Then use § 11. The slope of the curve may also be found by the general method of the differential calculus.

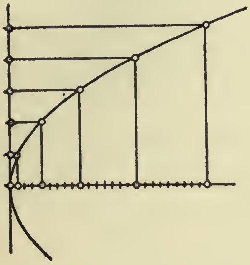
† Proof: Determine β so that the line $y = mx + \beta$ shall have only one point in common with the curve. [Remember that a quadratic equation $Ax^2 + Bx + C = 0$ will have equal roots if $B^2 - 4AC = 0$.]

‡ Proof: Let (x_0, y_0) be any point of the required locus; find the points of intersection of the curve and a line through (x_0, y_0) with slope m; then express the condition that (x_0, y_0) shall be the middle point between these two points. [Remember that the sum of the roots of a quadratic equation $Ax^2 + Bx + C = 0$ is $-B/A$.]

25a. Among the many properties of the parabola which should be worked out as problems, the following may be mentioned as especially important, and easy to remember:

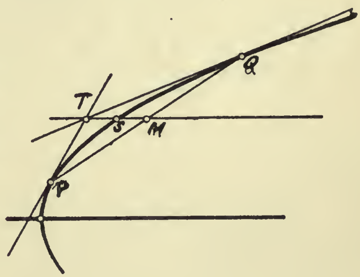
1. The normal at any point P bisects the angle formed by the line from P to the focus and the line through P parallel to the principal axis (parabolic mirror).





2. If P_1, P_2, . . . are any points on a parabola, the distances of these points from the principal axis are proportional to the squares of their distances from the tangent at the vertex.

3. If the tangents at P and Q intersect at T, and if M is the middle point of the chord PQ, then the line through T and M is a diameter, and the segment TM is bisected by its point of intersection with the curve.



4. The locus of the foot of the perpendicular from the focus on a moving tangent is the tangent at the vertex.

5. The locus of the point of intersection of perpendicular tangents is the directrix.

Note. The usual methods for constructing a parabola should also be given.

CHAPTER V.

THE ELLIPSE: $b^2x^2 + a^2y^2 = a^2b^2$.

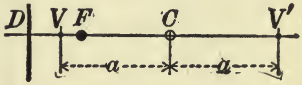
26. Definition: The locus of a point which moves so that

$$\frac{\text{its distance from a fixed point}}{\text{its distance from a fixed line}} = e < 1$$

(where e is a constant less than 1), is called an *ellipse.*

The fixed point is called the *focus*, the fixed line the *directrix*, and the constant, e, the *eccentricity*. The line perpendicular to the directrix through the focus is called the *principal axis*. There are evidently two points of the principal axis which are also points of the curve; these two points are called the *vertices*, and the point half way between them is called the *centre*.

27. If we let $2a =$ the distance between the vertices, then:*



the distance between the *centre* and either *vertex* is a;
the distance between the *centre* and the *focus* is ae;
the distance between the *centre* and the *directrix* is a/e.

28. If we take the centre as the origin and the principal axis as the axis of x, the *equation of the ellipse* is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where b is an abbreviation for $a\sqrt{1 - e^2}$.† Note that $b < a$.

---

* Proof: Since the vertices, V and V', are points of the curve, $VF/VD = e$ and $V'F/V'D = e$; that is,

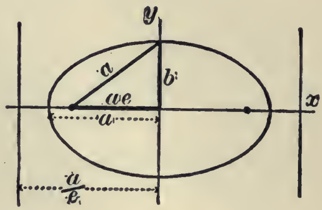
$$\frac{a - CF}{CD - a} = e \quad \text{and} \quad \frac{a + CF}{a + CD} = e,$$

whence $CF = ae$ and $CD = a/e$.

† Proof: If (x, y) is any point on the curve, then

$$\frac{\sqrt{(x + ae)^2 + (y - 0)^2}}{x + \frac{a}{e}} = e.$$

29. The form of the curve is therefore that shown in the figure.* The right triangle enables us to find any one of the three quantities a, b, and e, when the other two are given.



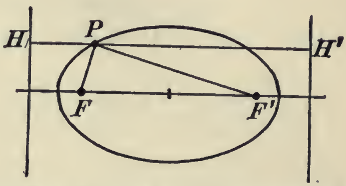
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The symmetry of the equation shows that the curve might equally well have been obtained, with the same eccentricity, e, from a *second focus and directrix*, shown on the right.

The breadth of the curve at either focus is called the *latus rectum*, and is equal to $2a(1 - e^2)$, or $2b^2/a$.

30. Let P be any point of the ellipse, F and F' the foci, and PH and PH' the perpendiculars from P to the directrices; then

(*a*) $PF/PH = e$ and $PF'/PH' = e$,

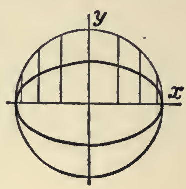


by definition of the curve. Furthermore:†

(*b*) $PF + PF' = 2a.$

In fact, the ellipse is often defined as *the locus of a point which moves so that the sum of its distances from two fixed points is constant.*

31. If a circle be described upon the major axis of an ellipse as diameter, each ordinate in the ellipse is to the corresponding ordinate in the circle as b is to a.‡ In fact, the ellipse is often defined as *the locus of points dividing the ordinates of a circle in a constant ratio.*



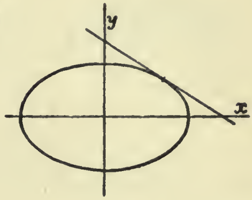
From this property it follows that the *area of an ellipse* is πab.

* Thus, when $y = 0$, $x = \pm a$; when $x = 0$, $y = \pm b$. The curve is symmetrical with respect to both axes. In first quadrant, as x increases, y decreases (slowly when x is small, and rapidly when x approaches a).

† Proof: $PF = e(PH)$ and $PF' = e(PH')$, so that $PF + PF' = e(HH') = e(2a/e) = 2a$.

‡Proof: In the ellipse, $y = \frac{b}{a}\sqrt{a^2 - x^2}$; in the circle, $y = \sqrt{a^2 - x^2}$.

32. To find the *equation of a tangent* to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, use one of the following formulas:*



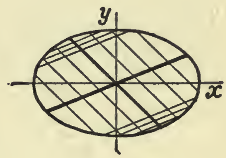
(*a*) When the point of contact, (x_1, y_1), is given:

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1;$$

(*b*) When the slope, m, is given:

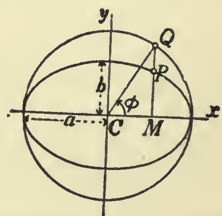
$$y = mx \pm \sqrt{a^2m^2 + b^2}.$$

33. The locus of the middle points of a set of parallel chords in the ellipse is a straight line through the centre; such a line is called a *diameter*. In the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if the slope of the parallel chords is m, then the slope of the diameter is $-\frac{b^2}{a^2 m}$.*



Any two lines through the centre, such that the product of their slopes is $-b^2/a^2$, are called a pair of *conjugate diameters*, because each bisects all chords parallel to the other.

34. The circle described in § 31 is called the *auxiliary circle*. If P is any point on the ellipse, and Q the corresponding point on the auxiliary circle (see figure), then the angle ϕ which CQ makes with the axis is called the *eccentric angle* of the point P. From the figure, and § 31,



$$x = a \cos\phi \quad \text{and} \quad y = b \sin\phi,$$

where x, y are the coordinates of P.

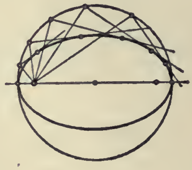
The eccentric angles of the ends of two conjugate diameters differ by 90°.

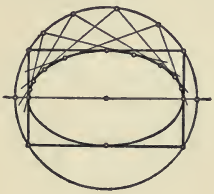
* Proof as in the case of the parabola.

4a. Among the many properties of the ellipse that should be worked out as problems, the following are especially easy to remember:

1. The normal at any point P bisects the angle formed by the lines joining P with the foci.

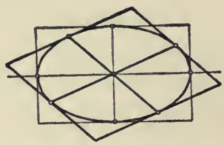
2. The locus of the foot of the perpendicular from the focus on a moving tangent is the circle on the major axis as diameter.





3. The locus of the point of intersection of perpendicular tangents is a circle with radius $\sqrt{a^2 + b^2}$.

4. The area of a parallelogram bounded by tangents parallel to conjugate diameters is constant.



Note. The usual methods for constructing an ellipse should also be given.

CHAPTER VI.

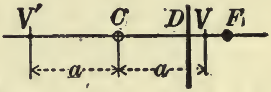
THE HYPERBOLA: $b^2x^2 - a^2y^2 = a^2b^2$.

35. Definition: The locus of a point which moves so that

$$\frac{\text{its distance from a fixed point}}{\text{its distance from a fixed line}} = e > 1$$

(where e is a constant greater than 1), is called a *hyperbola.*

The fixed point is called the *focus*, the fixed line the *directrix*, and the constant, e, the *eccentricity.* The line perpendicular to the directrix through the focus is called the *principal axis.* There are evidently two points of the principal axis which are also points of the curve; these two points are called the *vertices*, and the point half way between them is called the *centre.*



36. If we let $2a =$ the distance between the vertices, then:*

the distance between the *centre* and either *vertex* is a;
the distance between the *centre* and the *focus* is ae;
the distance between the *centre* and the *directrix* is a/e.

37. If we take the centre as the origin and the principal axis as the axis of x, the *equation of the hyperbola* is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where b is an abbreviation for $a\sqrt{e^2 - 1}$.† Note that b may be greater or less than a, or equal to a.

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* Proof: Since the vertices, V and V', are points of the curve, $VF/VD = e$ and $V'F/V'D = e$; that is,

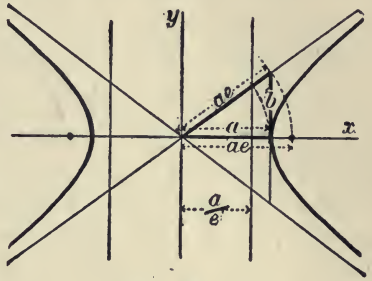
$$\frac{CF - a}{a - CD} = e \quad \text{and} \quad \frac{a + CF}{a + CD} = e,$$

whence $CF = ae$ and $CD = a/e$.

† Proof: If (x, y) is any point on the curve, then

$$\frac{\sqrt{(x - ae)^2 + (y - 0)^2}}{x - \dfrac{a}{e}} = e.$$

38. The form of the curve is therefore that shown in the figure.* The two lines through the centre with slopes $\pm b/a$ are called the *asymptotes* of the hyperbola; the two branches of the curve approach these lines more and more nearly as they recede from the centre.† The right triangle enables us to find any one of the three quantities, a, b, and e, when the other two are given.

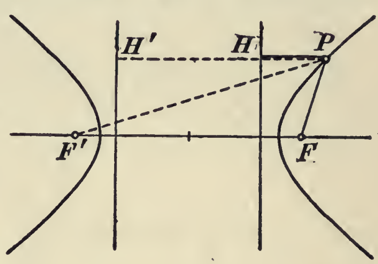


$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The symmetry of the equation shows that the curve might equally well have been obtained, with the same eccentricity, e, from a *second focus and directrix*, shown on the left.

The breadth of the curve at either focus is called the *latus rectum*, and is equal to $2a(e^2 - 1)$, or $2b^2/a$.

39. Let P be any point of the hyperbola, F and F' the foci, and PH and PH' the perpendiculars from P to the directrices; then



(*a*) $PF/PH = e$ and $PF'/PH' = e$,

by the definition of the curve. Furthermore:‡

(*b*) $| PF - PF' | = 2a.$

In fact, the hyperbola is often defined as the *locus of a point which moves so that the difference of its distances from two fixed points is constant.*

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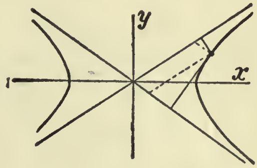
* Thus, when $y = 0, x = \pm a$; when $x = 0$, or $x < a$, y is imaginary; when x increases beyond a, y increases, plus and minus (most rapidly when x is little greater than a). The curve is symmetrical with respect to both axes.

† For, the slope $\frac{y}{x} = \frac{b}{a}\sqrt{1 - \frac{a^2}{x^2}}$ approaches $\frac{b}{a}$ as x increases; moreover, if y_1 is the ordinate of any point on the curve, and y_2 the ordinate of the corresponding point on the asymptote, then the difference $y_2 - y_1$ approaches zero; for $y_2^2 - y_1^2 = b^2$, and therefore $y_2 - y_1 = b^2/(y_2 + y_1)$.

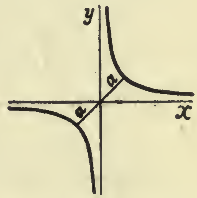
‡ Proof: $PF = e(PH)$ and $PF = e(PH')$, so that $|PF - PF'| = e(HH') = e(2a/e) = 2a$.

40. The product of the distances from any point of a hyperbola to the asymptotes is constant. Hence, the hyperbola is often defined as *the locus of a point which moves so that the product of its distances from two fixed lines is constant.* (The distances here may be the perpendicular distances; or, the distance to each line may be measured parallel to the other.)



41. An important special case is that of the "rectangular" hyperbola, whose asymptotes are perpendicular ($a = b$); the *equation of the rectangular hyperbola referred to its asymptotes as axes* is (by § 40)

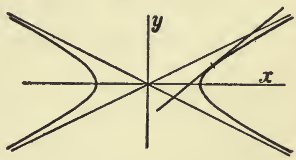
$$xy = \frac{a^2}{2}.$$



42. To find the *equation of a tangent* to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, use one of the following formulas:*

(*a*) When the point of contact, (x_1, y_1), is given:

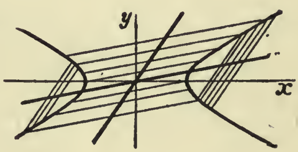
$$\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1;$$



(*b*) When the slope, m, is given:

$$y = mx \pm \sqrt{a^2m^2 - b^2}.$$

43. The locus of the middle points of a set of parallel chords in the hyperbola is a straight line through the centre; such a line is called a *diameter*. In the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if the slope of the parallel chords is m, the slope of the diameter will be $\frac{b^2}{a^2m}$.*



Any two lines through the centre, such that the product of their slopes is b^2/a^2, are called a pair of *conjugate diameters*, because each bisects all chords parallel to the other.

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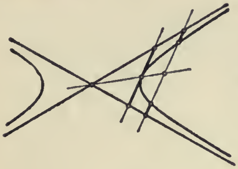
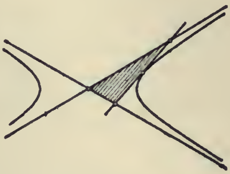
* Proof as in the case of the parabola.

43a. Among the properties of the hyperbola, the following are easy to remember:



1. If a line cuts the hyperbola and its asymptotes, the parts of the line intercepted between the curve and the asymptotes are equal. In particular, the portion of any tangent intercepted between the asymptotes is bisected by the point of tangency.

2. The area of the triangle bounded by any tangent and the asymptotes is constant.



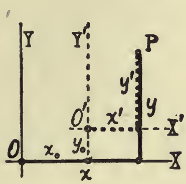
Note. The usual methods of constructing a hyperbola—especially the rectangular hyperbola—should be given.

CHAPTER VII.

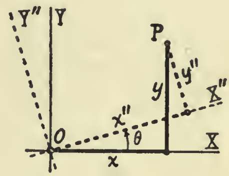
TRANSFORMATION OF COORDINATES.*

44. The equation of a curve can often be simplified by a "change of axes," either changing to a new origin (x_0, y_0), or turning the axes through an angle θ, or both.

If (x, y), (x', y'), (x'', y''), are the coordinates of the same point P, referred to three different sets of axes, as in the figures, then



$$x = x_0 + x'$$
$$y = y_0 + y'$$



$$x = x'' \cos\theta - y'' \sin\theta \dagger$$
$$y = x'' \sin\theta + y'' \cos\theta$$

Suppose now that the point P is allowed to move under certain conditions given by an equation in x and y. The same condition can be expressed in terms of x' and y' or in terms of x'' and y'' by substituting in the given equation the values of x and y just found. This process is called a transformation of coordinates, from the axes x, y to the axes x', y', or to the axes x'', y''; and the new equation can often be made simpler than the given equation by a suitable choice of x_0 and y_0, or θ.

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* See also the chapter on polar co-ordinates.

† These last formulas are most easily remembered as follows:

$x =$ easterly displacement of P,
$=$ (easterly component of x'') $+$ (easterly component of y'')
$= x'' \cos\theta - y'' \sin\theta'$,
$y =$ northerly displacement of P
$=$ (northerly component of x'') $+$ (northerly component of y'')
$= x'' \sin\theta + y'' \cos\theta$.

CHAPTER VIII.

GENERAL EQUATION OF THE SECOND DEGREE IN x AND y.

45. The general equation of the second degree in x and y is of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

By a suitable transformation of coördinates this equation can always be brought into one or other of the following forms:

$$A'x^2 + C'y^2 + F' = 0, \quad C''y^2 + D''x = 0, \quad C''y^2 + F'' = 0,$$

and hence can be shown to represent a conic section, using this term in a general sense to include (1) an ellipse, which may be real, null, or imaginary; (2) a hyperbola, or a pair of intersecting lines; (3) a parabola, or a pair of parallel lines (distinct, coincident, or imaginary).*

The student should be able to plot readily the locus of an equation of the second degree in any of the simple cases mentioned below—these being the cases which occur most often in practice.

46. To plot $Ax^2 + Cy^2 + F = 0$, where A and C have the same sign. Find the intercepts on the axes (by putting $x = 0$ and $y = 0$); if both are real, we have an ellipse in which $a =$ the larger of the two intercepts, and $b =$ the smaller; or if $A = C$, the ellipse becomes a circle. If both intercepts are zero, or imaginary, the locus is a single point, or imaginary.

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* Proof: If $B^2 - 4AC$ is not zero, transform to parallel axes with origin at (x_0, y_0), and choose x_0 and y_0 so that the terms of the first degree in the new equation shall vanish; then turn the axes through an angle θ, and choose θ so that the term in xy shall vanish. If $B^2 - 4AC = 0$, turn the axes through an angle θ, and choose θ so that the term in xy shall vanish; then transform to a new origin (x_0, y_0), and choose x_0 and y_0 so that the constant term and one of the terms of the first degree, or so that both the terms of the first degree shall vanish. For special methods of abbreviating the computation in numerical cases see § 55, note.

47. To plot $Ax^2 + Cy^2 + F = 0$, where A and C have opposite signs. Unless $F = 0$, one of the intercepts will be real and the other imaginary, and the curve will be a hyperbola whose principal axis is the axis on which the intercepts are real. To find the slopes of the asymptotes, divide by x^2 and find the limit of y/x as x increases indefinitely. If $F = 0$, the locus is a pair of intersecting lines.

48. To plot $Cy^2 + Dx + F = 0$. Write this as

$$y^2 = -\frac{D}{C}\left(x + \frac{F}{D}\right), \quad \text{or} \quad y^2 = -\frac{D}{C}x'.$$

This is a parabola with vertex at $x_0 = -F/D$, and running out along the positive or negative axis of x. Plotting one or two points will fix the direction, and comparison with the equation $y^2 = 2px$ will give the semi-latus rectum, p.

49. To plot $Ax^2 + Cy^2 + Dx + Ey + F = 0$. Write this in the form

$$A\left(x^2 + \frac{D}{A}x + \quad\right) + C\left(y^2 + \frac{E}{C}y + \quad\right) = -F,$$

and "complete the squares"; then reduce to the form $Ax'^2 + Cy'^2 + F = 0$ by an obvious change of origin.

50. To plot $Cy^2 + Dx + Ey + F = 0$. Complete the square of the terms in y and reduce to the form $Cy^2 + Dx + F = 0$ by an obvious change of origin.

51. To plot $Bxy + F = 0$. This is a rectangular hyberbola referred to its asymptotes (see § 41). The equation $Bxy + Dx + Ey + F = 0$ can be reduced to this form by moving the origin to $x_0 = -E/B$, $y_0 = -D/B$.

52. If the equation to be plotted does not come under any of the forms just considered, a fair idea of the position of the curve may be found by the following very elementary method. Solving the equation for y in terms of x, we have, if C is not zero,

$$y = -\frac{Bx + E}{2C} \pm \frac{1}{2C}\sqrt{X},$$

where X is an expression involving x alone. Finding the values of $-(Bx+E)/2C$, and adding and subtracting the values of $\sqrt{X}/2C$, for various values of x, we can find as many points (x, y) on the curve as we please. Or, again, solving for x in terms of y, we have, if A is not zero,

$$x = -\frac{By+D}{2A} \pm \frac{1}{2A}\sqrt{Y},$$

where Y is an expression involving y alone. From this equation we can find values of x corresponding to as many values of y as we please.

This method is very easy to remember, but does not give readily the exact dimensions of the curve.

53. The center of the curve will be the point of intersection of the two lines

$$2Ax + By + D = 0, \ *$$
$$Bx + 2Cy + E = 0,$$

except when $B^2 - 4AC = 0$, in which case these lines will be parallel, and the curve has no center.

54. The slopes of the lines joining the origin with the infinitely distant points of the curve (if any) are given by writing the terms of the second degree equal to zero:

$$Ax^2 + Bxy + Cy^2 = 0,$$

dividing through by x^2 (or y^2), and solving for y/x (or x/y).

55. If a more detailed discussion of the curve is required, it is best to follow the special methods of reduction given in the text-books (compare foot-note in § 45).†

56. The student should be familiar with the geometric proof that all the "conic sections" can be obtained as plane

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* The student of the calculus will recognize these equations as

$$\partial F/\partial x = 0 \quad \text{and} \quad \partial F/\partial y = 0,$$

where

$$F(x, y) \equiv Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is the equation of the curve.

† The resulting formulæ are given here for reference, although the problem is not one of common occurrence.

Required, to plot the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

sections of a right circular cone. It is a profitable exercise to construct a cone, given the vertex and a hyperbolic section. It should also be made thoroughly clear why an elliptic section is a symmetrical figure instead of egg-shaped.

Case I. *Central conic.* If B^2-4AC is not zero, transform to the center as a new origin:

$x_0=(2CD-BE)/(B^2-4AC)$, $y_0=(2AE-BD)/(B^2-4AC)$;
then turn the axes through a positive acute angle θ given by

$$\tan 2\theta = B/(A-C).$$

The transformed equation will be

$$A'x^2+C'y^2+F'=0,$$

where $F'=Dx_0/2+Ey_0/2+F$, while A' and C' are found by solving the equations $A'+C'=A+C$, $A'-C'=\pm\sqrt{(A-C)^2+B^2}$, where the sign before the radical is to be + or — acording as B is positive or negative. The reduced equation can be plotted as in §§ 46, 47.

Case II. *Parabolic type.* If $B^2-4AC=0$, the equation may be written in the form $(ax+cy)^2+Dx+Ey+F=0$, where $a=\sqrt{A}$ while $c=\sqrt{C}$ or $c=-\sqrt{C}$ according as B is positive or negative. The locus will be of the parabolic type. Take as a new axis of x' the line

$$ax+cy+m=0,$$

where $m=(aD+cE)/2(A+C)$, and choose the positive direction along this line so that it shall make a (positive or negative) acute angle with the axis of x. This line will be the principal axis of the curve.

Two subcases may now occur.

(*a*) If a/c is not equal to D/E, take as axis of y' the line

$$cx-ay+n=0,$$

where $n=(A+C)\ (m^2-F)/(aE-cD)$. This line will be the tangent at the vertex, and the transformed equation will be

$$y'^2=2px',$$

where $2p=(cD-aE)/\sqrt{(A+C)^3}$. The locus is a true parabola.

(*b*) If $a/c=D/E$, the equation referred to the axis of x' will be

$$y'^2=m^2-F,$$

which represents a pair of distinct, coincident, or imaginary parallel lines.

CHAPTER IX.

SYSTEMS OF CONICS.

57. If U and V are expressions of the second degree in x and y, the equations $U = 0$ and $V = 0$ will represent conics; then (a) the equation $U + kV = 0$, where k is any constant, will represent another conic passing through all the points of intersection of the first two, and having no other points in common with either of them; and (b) the equation $UV = 0$ will represent a curve made up of the two conics $U = 0$ and $V = 0$ taken together. Corresponding theorems hold good if U and V are any expressions in x and y (not necessarily of the second degree).

58. To find the equation of a conic through five points, let $u = 0$ and $v = 0$ be the equations of the lines P_1P_2 and P_3P_4, and let $u' = 0$ and $v' = 0$ be the equations of the lines P_1P_3 and P_2P_4. Then $uv + ku'v' = 0$, where k is any constant, will be the equation of a conic through these four points. It remains to determine k so that this conic shall pass through P_5.

59. The equation

$$\frac{x^2}{a^2 + k} + \frac{y^2}{b^2 + k} = 1,$$

where k is an arbitrary constant, represents a family of confocal ellipses and hyperbolas, which intersect at right angles.

CHAPTER X.

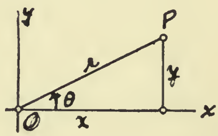
POLAR COORDINATES.

This chapter, placed here for convenience of reference, may well be introduced, in teaching, much earlier in the course.

60. It is often convenient to represent the position of a point P by giving the angle, ϕ, which the line through O and P makes with the x-axis, and the distance, r, from O to P along this line. The angle ϕ is called the *vectorial angle,* or simply the *angle,* of the point P, and is measured from the positive direction of the axis of x to the positive direction of the line through O and P. The distance $r = OP$ is called the *radius vector* of the point P, and is positive or negative according as it runs forward or backward along the line through O and P.

It is customary to take r positive, and let ϕ range from 0° to 360°.

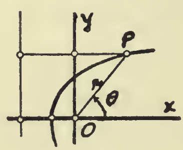
61. From the figure,



$$x = r\cos\phi, \quad y = r\sin\phi,$$
$$x^2 + y^2 = r^2, \quad y/x = \tan\phi.$$

By the aid of these relations, we can transform any equation from rectangular to polar coordinates, and vice versa.

62. The polar equation of a conic, referred to the focus as origin, and the principal axis as axis of x (see figure) is



$$r = \frac{p}{1 - e\cos\theta},$$

where p is the semi-latus rectum, and e the eccentricity.

63. Plotting curves in polar coordinates is an excellent exercise in reviewing the trigonometric functions. The work should be so arranged that no critical value of the function occurs between two successive assigned values of θ.

CHAPTER XI.

COORDINATES IN SPACE.

64. Four methods are in use for representing numerically the position of a point in space. If Ox, Oy, Oz are three mutually perpendicular axes, the position of any point P may be determined by:

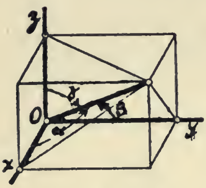
(1) Rectangular coordinates, x, y, z;

(2) Polar coordinates in space, r, α, β, γ, where the angles α, β, γ are subject to the restriction $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$;

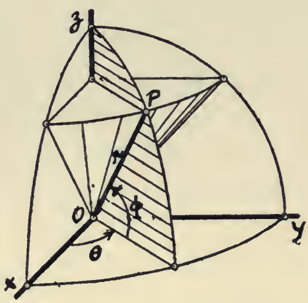
(3) Spherical coordinates, r, ϕ, θ, where $\phi =$ the latitude of P, and θ its longitude:

(4) Cylindrical coordinates ρ, θ, z.

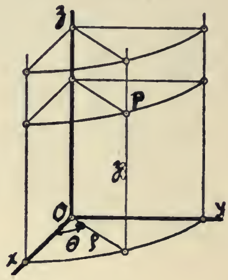
The relations between the various sets of coordinates are as follows:



$$x = r\cos\alpha,$$
$$y = r\cos\beta,$$
$$z = r\cos\gamma.$$



$$x = r\cos\phi\cos\theta$$
$$y = r\cos\phi\sin\theta$$
$$z = r\sin\phi$$

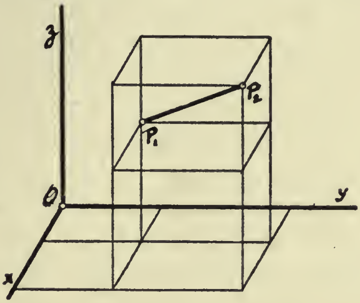


$$\rho = r\cos\phi,$$
$$\rho^2 = x^2 + y^2,$$
$$x^2 + y^2 + z^2 = r^2.$$

As there is no well-established uniformity in the use of the letters in spherical coordinates, or in the choice of the positive directions along the axes, it is important, in reading any author, to note, on a figure, the exact meanings of the letters he employs.

65. Distance between two points, in terms of their coordinates:

$$P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$



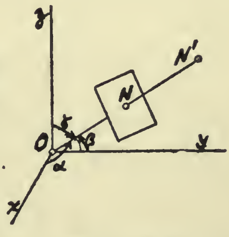
66. Angle ψ between two lines whose direction cosines are given:

$$\cos\psi = l_1 l_2 + m_1 m_2 + n_1 n_2,$$

where $l_1 = \cos\alpha_1$, $m_1 = \cos\beta_1$, $n_1 = \cos\gamma_1$, etc.*

67. Equation of a plane:

$$lx + my + nz = p,$$



where p = perpendicular distance from the origin, and l, m, n = the direction cosines of the normal to the plane.†

Every equation of the form $Ax + By + Cz + D = 0$ represents a plane; for, it can be thrown into the form $lx + my + nz = p$ by dividing through by $\sqrt{A^2 + B^2 + C^2}$.

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* Proof: let (1) and (2) be lines through the origin, parallel to the given lines; on these lines take points P_1, P_2 at a distance r from the origin; then

$$P_1P_2^2 = r^2 + r^2 - 2rr\cos\psi = (rl_1 - rl_2)^2 + (rm_1 - rm_2)^2 + (rn_1 - rn_2)^2.$$

† Proof: The foot of the perpendicular is $N = (pl, pm, pn)$; take $N' = (2pl, 2pm, 2pn)$ and express the condition that the point (x, y, z) shall be equidistant from O and N'.

68. Equation of sphere with center at the origin:

$$x^2 + y^2 + z^2 = r^2.$$

69. Equation of ellipsoid, with center at the origin:

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$$

70. Any equation in x, y, z will represent a surface (real or imaginary), the form of which can be investigated by the *method of plane sections.* Thus, putting $x = x_1$, the equation becomes an equation in y and z, which represents a curve in the plane $x = x_1$; similarly for $y = y_1$ and $z = z_1$.

71. Any equation of the second degree in x, y, z represents a (real or imaginary) surface of the second degree, or *conicoid.* The types of real conicoids are as follows:

(1) *Ellipsoid,* with semi-axes a, b, c. Special case: ellipsoid of revolution, generated by rotating an ellipse about its major axis (prolate spheroid) or about its minor axis (oblate spheroid).

(2) *Hyperboloid of two sheets.* Special case: generated by rotating a hyperbola about its principal axis.

(3) *Hyperboloid of one sheet,* or *ruled hyperboloid.* Special case: generated by rotating a hyperbola about its conjugate axis. Two sets of straight lines can be drawn on this surface.

(4) *Elliptic paraboloid.* Special case: generated by rotating a parabola about its principal axis.

(5) *Hyperbolic paraboloid,* or *ruled paraboloid.* A saddle-shaped figure, on which two sets of straight lines can be drawn.

(6) *Cone,* generated by a straight line always passing through a fixed point called the vertex, and always touching a fixed conic, called the directrix. If the directrix is a circle, the cone is a circular cone (right or oblique). If the vertex recedes to infinity, the cone becomes a cylinder. On any cone a single set of straight lines can be drawn.

The student should become familiar with at least the shapes of these surfaces, through diagrams or models.

Any plane section of any surface of the second degree is a conic.

A SYLLABUS OF DIFFERENTIAL AND INTEGRAL CALCULUS.

This syllabus is intended to include those facts and methods of the calculus which every student who has completed an elementary course in the subject should have so firmly fixed in his memory that he will never think of looking them up in a book. The topics here mentioned are therefore not by any means the only topics that should be included in a course of study, nor does the arrangement of these topics, as classified in the following table of contents, necessarily indicate the order in which they should be presented to a beginner.

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* In preparation.

CHAPTER I.

FUNCTIONS AND THEIR GRAPHICAL REPRESENTATION.

1. Function and argument.—In many problems in practical life we have to deal with the relation between two variable quantities, one of which depends on the other for its value.

For example, the temperature of a fever patient depends on the time; the velocity acquired by a falling body depends on the distance fallen; the weight of an iron ball depends on its diameter, etc.

In general, if any quantity y depends on another quantity x, then y is called a *function* of x, written, for brevity, $y=f(x)$, and the independent variable x is called the *argument* of the function. More precisely stated, the notation $y=f(x)$ means that to every value of the argument x (within the range considered), there corresponds some definite value of the function, y; the value of y, or $f(x)$, corresponding to any particular value $x=a$ is denoted by $f(a)$.

If several values of y correspond to each value of x, we have what is called a "multiple valued function of x," which is really a collection of several distinct functions. For example, if $y^2=x$, then $y=\pm\sqrt{x}$, which is a double valued function of x.

Any mathematical expression involving a variable x is a function of x; but there are many important functional relations which cannot be expressed in any simple mathematical form.

2. A function is said to be *tabulated* when values of the argument (as many as we please, preferably at regular intervals) are set down in one column, and the corresponding values of the function are set down in another column, opposite the first. For example, in a table of sines, the angle is the argument, and the sine of the angle is the function.

3. A function may also be exhibited *graphically,* as follows: Lay off the values of the argument as abscissas along a (horizontal) axis, Ox, and at each point of the axis erect an ordinate, y, whose length shall indicate the value of the function at that point; a curve drawn through the tops of these ordinates is called the *curve*, or the *graph*, of the function. It should be clearly understood, however, that it is the height of the ordinate up to the curve, rather than the curve itself, that represents the function.

In plotting the curve for any function, it is important to indicate on each axis the scale which is used on that axis, and the name of the unit. For example, if we plot distance as a function of the time, the units on the y-axis may represent feet, and those on the x-axis, seconds.*

The obvious method of obtaining the graph of the sum or difference of two functions directly from the graphs of those functions should be noted.

4. The elementary mathematical functions.—In many important cases, the relation between the function and the argument can be expressed by a simple mathematical formula. For example, if $s =$ the distance fallen from rest in the time t, then $s = \frac{1}{2}gt^2$. In such cases, the value of the function for any given value of the argument can be found by simple substitution in the formula.

The most important elementary mathematical functions are the following:

Algebraic functions: cx, c/x; x^2, x^3; $\sqrt{x}$ (x positive).

Here $\sqrt{x} =$ the positive value of y for which $y^2 = x$.

Trigonometric functions: $\sin x$, $\cos x$, $\tan x$ (x in radians).

Exponential function: e^x ($e = 2.718 \ldots$).

Logarithmic function: $\log_e x$ (x positive).

The student should be thoroughly familiar with the curves of each of these functions, so as to be able to sketch them, or visualize them, at any moment; many of the essential properties of the functions can be obtained by inspection of the curve.

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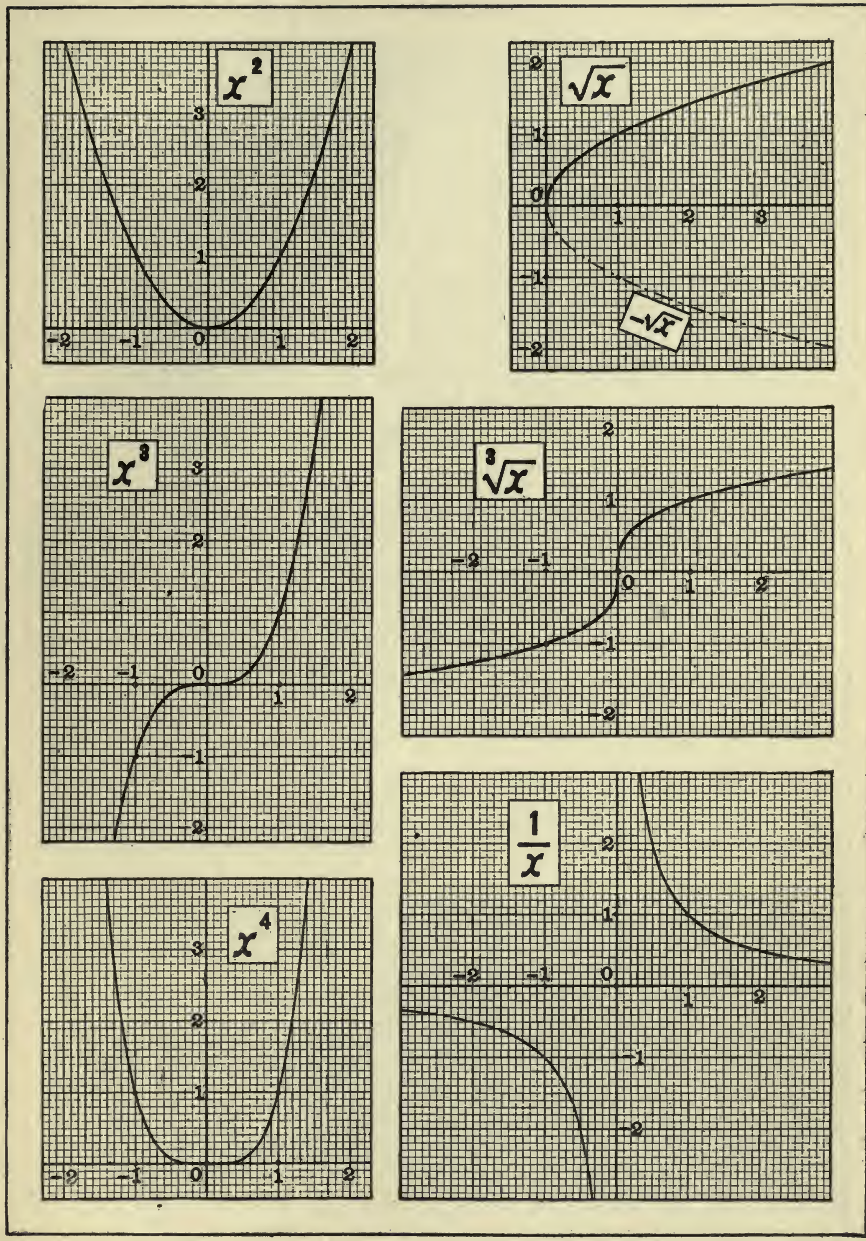
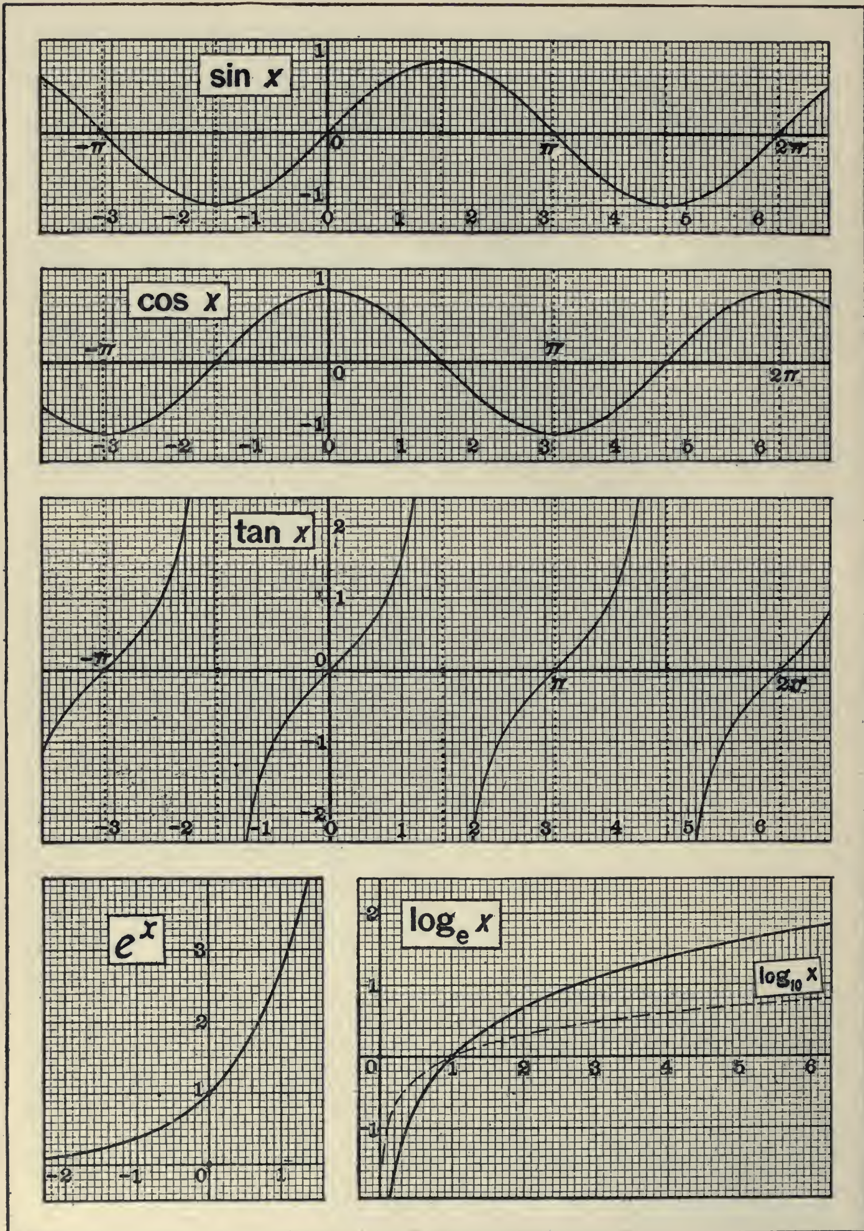
* It is not necessary that the lengths representing the units of x and y shall be equal; scales should be so chosen that the completed graph is of convenient size to fit the paper. In applications to geometry, however (see Chapter VI), the scales must be equal.


x^2
$\sqrt{x}$
$-\sqrt{x}$
x^3
$\sqrt[3]{x}$
$\frac{1}{x}$
x^4


sin x
cos x
tan x
e^x
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π
2π
−3
−2
−1
0
1
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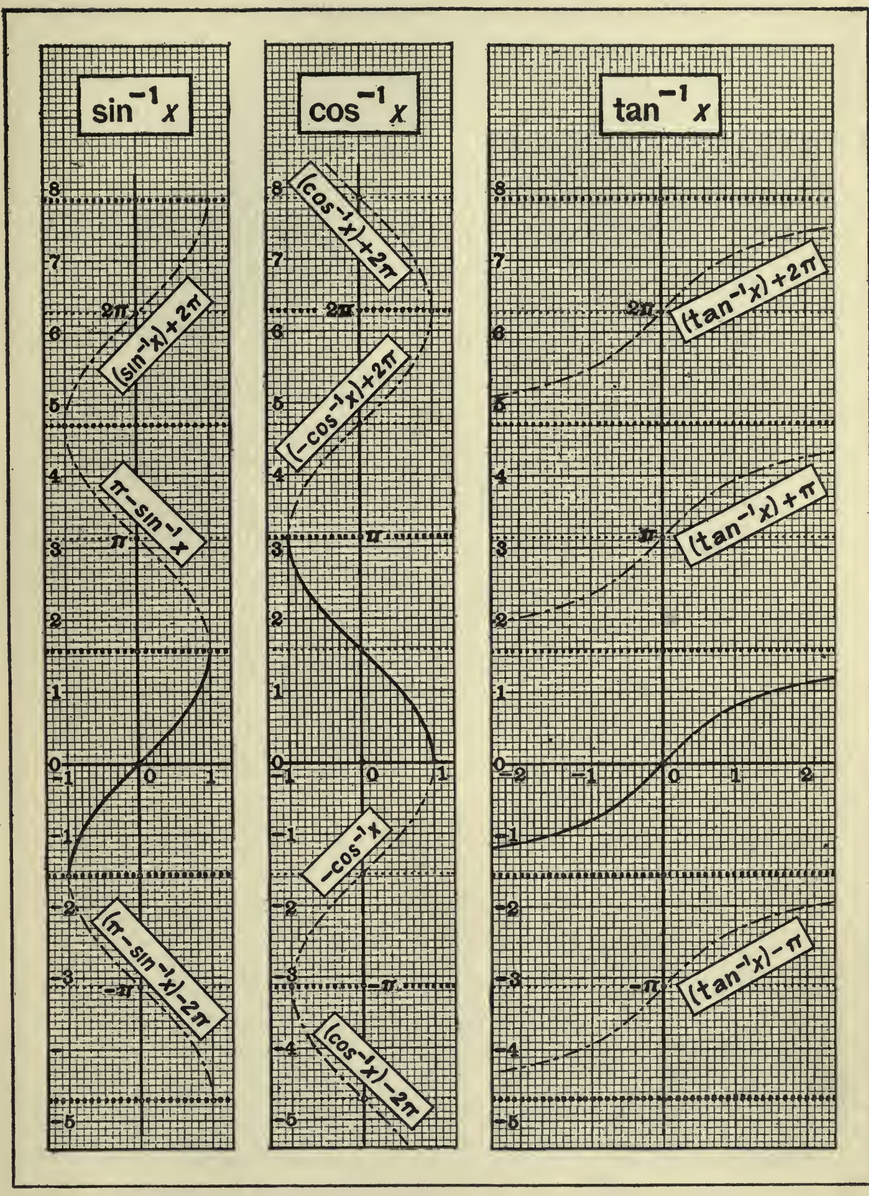
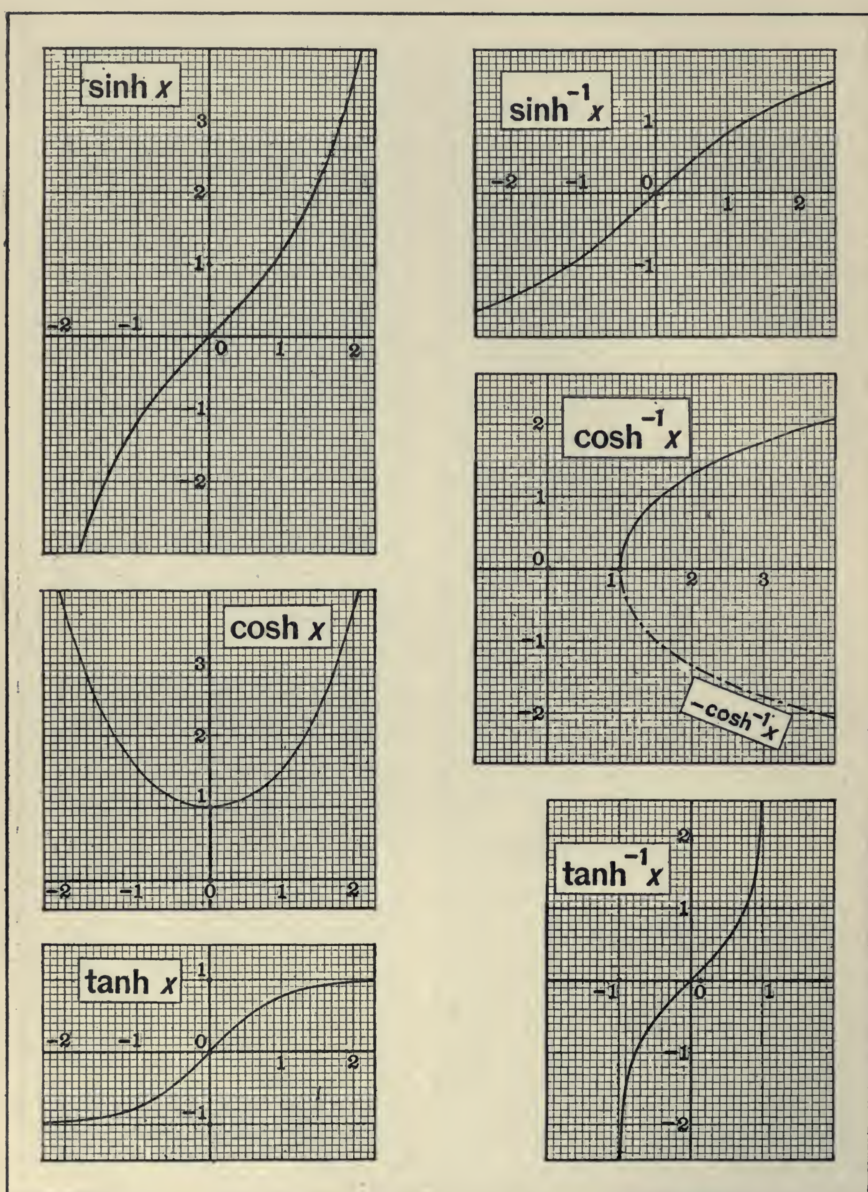
4
5
6


sin⁻¹x
cos⁻¹x
tan⁻¹x
(sin⁻¹x)+2π
π−sin⁻¹x
(π−sin⁻¹x)−2π
(cos⁻¹x)+2π
(−cos⁻¹x)+2π
−cos⁻¹x
(cos⁻¹x)−2π
(tan⁻¹x)+2π
(tan⁻¹x)+π
(tan⁻¹x)−π


sinh x
sinh^{-1}x
cosh x
cosh^{-1}x
−cosh^{-1}x
tanh x
tanh^{-1}x

He should also be familiar with the formulas necessary for handling expressions involving these functions. The better drilled the student is in this formal algebraic work, the more rapid progress can he make in the really vital parts of the subject. (See chapters of this report on algebra and trigonometry.)

5. Next in importance are the following: the *hyperbolic functions,* which are coming more and more into use:

$$\sinh x = (e^x - e^{-x})/2, \quad \cosh x = (e^x + e^{-x})/2,$$
$$\tanh x = (e^x - e^{-x})/(e^x + e^{-x});$$

the *inverse trigonometric functions:*

$\sin^{-1} x =$ the angle between $-\pi/2$ and $+\pi/2$ radians (inclusive) whose sine is x;*

$\cos^{-1} x =$ the angle between 0 and π (inclusive) whose cosine is x;

$\tan^{-1} x =$ the angle between $-\pi/2$ and $+\pi/2$ (inclusive) whose tangent is x;

and the *inverse hyperbolic functions:*

$\sinh^{-1} x =$ the value of y for which $\sinh y = x$;

$\cosh^{-1} x =$ the positive value of y for which $\cosh y = x$;

$\tanh^{-1} x =$ the value of y for which $\tanh y = x$.

It should be noticed that the curves for the inverse functions can be obtained from the curves for the direct functions by rotating the plane through 180° about the line bisecting the first quadrant.

Formulas for the hyperbolic functions resemble those for the trigonometric functions, but the differences are so

* The symbol $\sin^{-1} x$ is often defined as simply "the angle whose sine is x"; but since there are many such angles, it is necessary to specify which one is to be taken as "the" angle, if the symbol is to have any definite meaning. Thus, if $\sin x = \frac{1}{2}$, x may equal $\pi/6$, or $5\pi/6$, etc.; but only one of these values, namely $\pi/6$, is properly denoted by the symbol $\sin^{-1} \frac{1}{2}$. Similarly for $\cos^{-1} x$ and $\tan^{-1} x$; and also for $\cosh^{-1} x$, which is like $\sqrt{x}$ in this respect. The conventions adopted to avoid ambiguity may be readily recalled from the figure, if we note that in each case the complete curve consists of two or more "branches," and that that one is taken as the "principal branch" which passes through the origin, or which lies nearest the origin on the positive side of the x-axis.

confusing that it is better not to try to memorize any formulas for the hyperbolic functions, but to look them up whenever they are needed. (The list in B. O. Peirce's Table of Integrals, for example, is entirely adequate.)

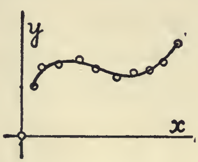
6. Continuity. A function $y=f(x)$ is said to be *continuous* at a given point $x=a$, if a small change in x produces only a small change in y; or more precisely, if $f(x)$ always approaches $f(a)$ as a limit when x approaches a in any manner.

A function may be *dis*continuous at a given point in three ways: (1) it may become infinite at that point, as $y=1/x$ at $x=0$; or (2) it may make a finite jump, as $y=\tan^{-1}(1/x)$ at $x=0$;* or (3) the limit $\underset{x \doteq a}{\mathrm{L}} f(x)$ may fail to exist because of the oscillation of the function in the neighborhood of $x=a$, as $y=\sin 1/x$ at $x=0$. In each of these cases, the function is, properly speaking, not defined at the point in question.

A good example of a discontinuous function is the velocity of a shadow cast by a moving object on a zig-zag fence.

In what follows, we shall confine our attention to functions that are continuous, or that have only isolated points of discontinuity.

7. To find a mathematical function to represent an empirically given curve.—In many cases the form of the function is given only empirically; that is, the values of the function for certain special values of the argument are given by experiment, and the intermediate values are not accurately



known (for example, the temperature of a fever patient, taken every hour). In such cases, the methods of the calculus are not of much assistance, unless some simple mathematical law can be found which represents the function sufficiently accu-

* This function approaches $\pi/2$ when x approaches 0 from above, and $-\pi/2$ when x approaches 0 from below.

rately.* *This problem of finding a mathematical function whose graph shall pass through a series of empirically given points* is a very important one, which is much neglected in the current text-books. The complete discussion of the problem involves, it is true, the theory of least squares, which would undoubtedly be out of place in a first course in the calculus; but an elementary treatment of the problem in simple cases would be very desirable.†

The curves which are most likely to be worth trying, in any given case, are these:

$y = a + bx$ (straight line);

$y = a + bx + cx^2$ (parabola);

$y = a + c/(x + b)$ (hyperbola);

$y = a \sin(bx + c)$ (sine curve); and

$y = ax^m$.

In testing this last curve, put $y' = \log y$, $x' = \log x$, and $a' = \log a$, and see whether y' and x' satisfy the straight line relation $y' = a' + mx'$; the use of "logarithmic squared paper" greatly facilitates the process.

The student should be familiar with all the possible forms of these curves, for various values of the constants a, b, c, and m.

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* If no simple law can be found to represent the entire curve, it is sometimes possible to break up the curve into parts, and find a separate law for each part.

† Numerous examples may be found in John Perry's "Practical Mathematics," and in F. M. Saxelby's "Practical Mathematics" (Longmans, 1905).

CHAPTER II.

Differentiation. Rate of Change of a Function.

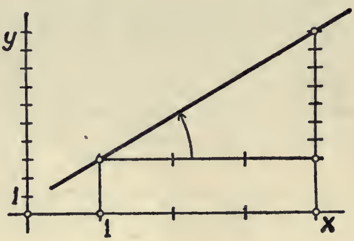
For the sake of clearness, this chapter is divided into four parts, A, B, C, D.

A. Definitions and Notation.

8. Rate of change of function; slope of curve.—Given a function, $y = f(x)$, one of the most important questions we can ask about it is, what is the *rate of change* of the function at a given instant?

For example, the distance of a railroad train from the starting point is a function of the time elapsed, and we may ask, what is the rate of change of this distance? The answer is, so-and-so many *miles per hour*. Again, the volume of a metal sphere is a function of the temperature, and we may ask, what is the rate of change of this volume? The answer is, so-and-so many *cubic inches per degree.*

If the graph of the function is a straight line, then clearly the rate of change of the function will be constant; for, at



any instant, (change in y)/(change in x) = the slope of the line.

If the scales along x and y are the same, the *slope of the line* = tan ϕ, where ϕ is the angle which the line makes with the x axis. If the scales are not the same, the slope of the line may still be interpreted as the ratio of the "side opposite" to the "side adjacent" in the triangle of reference for ϕ, provided each side is measured in the proper units. For example, in the figure, slope = 7/3.

If the graph is not a straight line, the meaning of "rate of change" at a given instant must be made more precise, as follows: Consider a particular value, $x = x_0$; give x an arbitrary change, Δx, and compute the corresponding change in y, namely, $\Delta y = f(x_0 + \Delta x) - f(x_0)$. Then the ratio $\Delta y/\Delta x$ may be called the AVERAGE *rate of change of the function during the interval from* $x = x_0$ *to* $x = x_0 + \Delta x$. (Geometrically, $\Delta y/\Delta x$ is the slope of the secant PQ in the figure.) Now let Δx approach zero, so that the interval in question closes down about the point $x = x_0$. Then *the ratio* $\Delta y/\Delta x$ *will in general approach a definite limit, and this limit is called the* ACTUAL *rate of change at the point* $x = x_0$. (Geometrically, the limit of $\Delta y/\Delta x$ is the slope of the tangent at P.*)

9. Derivatives. The *rate of change* of a function $y = f(x)$ at any point, or the *slope of the curve* at that point, is called the **derivative** of the function at that point, and is denoted by $f'(x)$, or $D_x y$, or y'.

The notation $\dot{y}$ is also used, but only when the independent variable is the time.

This definition of the derivative of a function as the limit of $\Delta y/\Delta x$ is the fundamental concept of the differential calculus. It is desirable that the meaning of the definition be made perfectly clear, by numerous and varied illustrations, before any formal work in differentiation is taken up.

10. Increments and Differentials.—The value Δx is called the *increment* given to x, and Δy the corresponding *increment*

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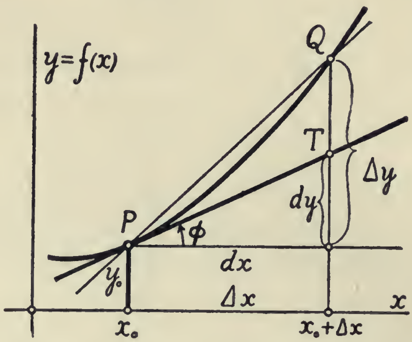
* The sense in which the tangent line is the "limit" of the secant lines should be made thoroughly clear. First, the tangent is a *fixed* line; secondly, the secant is a *variable* line, depending on the value given to Δx (that is, for every value of Δx, except the value 0, there is a corresponding position of the secant); thirdly, the angle between the tangent and the secant can be made to become *and remain* as small as we please by taking Δx sufficiently small. The tangent line itself does not in general belong to the series of secant lines; it is not in any sense the "last one" of the secants; it is a separate line, which bears a special relation to the series of secant lines, as described. The student may readily convince himself that the tangent is the *only* line through P that has the property just stated.

produced in y. The value that Δy *would have* if the curve coincided with its tangent (see figure) is called the **differential of** y and is denoted by dy.



In case of the independent variable x, the differential of x is, by definition, the same as the increment: $dx = \Delta x$.

The use of differentials gives us a *new notation for the derivative,*

$$f'(x) = \frac{dy}{dx}.$$

Both these notations are in common use.

Notice that Δy and dy are both variables which approach zero when we make Δx approach zero; dy/dx is a constant, equal to $\tan\phi$; $\Delta y/\Delta x$ is a variable, approaching $\tan\phi$ as a limit. Hence we may write:

$$\lim_{\Delta x \doteq 0} \frac{\Delta \mathbf{y}}{\Delta \mathbf{x}} = \mathbf{f}'(\mathbf{x}) = \mathbf{D}_x\mathbf{y} = \frac{\mathbf{dy}}{\mathbf{dx}} = \tan\phi,$$

and

$$\mathbf{dy} = \mathbf{f}'(\mathbf{x})\mathbf{dx}.$$

These relations between increments, differentials, and derivatives should be thoroughly mastered; they are readily recalled by the figure. Note especially that Δx and dx are quantities measured in the same unit as x; and Δy and dy in the same unit as y; while the derivative, dy/dx, that is, the slope, is (in general) measured in a compound unit (like miles per hour).

If the lengths representing the units of x and y are not equal, the slope of the curve, or tan ϕ, must be understood in the generalized sense explained above.

The process of finding the derivative, or the equivalent process of finding the differential of the function in terms of the differential of the argument, is called *differentiation.*

11. Higher derivatives. Since the slope of the curve varies, in general, from point to point, the derivative, $f'(x)$, is itself a function of x (often called the *derived* function); the derivative of $f'(x)$ is called the *second derivative* of the given function, and is denoted by $f''(x)$, or $D_x{}^2y$, or y'' (or by $\ddot{y}$ in case the independent variable is the time); and so on for the higher derivatives.

It is also easy to define second, third, . . . *differentials,* but they are not of great importance. One matter of notation, however, should be carefully noticed, namely that d^2y/dx^2 is commonly used to denote $f''(x)$, that is $\dfrac{d(dy/dx)}{dx}$, and not, as one might expect $\dfrac{d(dy)}{(dx)^2}$.

As an example where the distinction is important, consider

$$x = \theta - \sin \theta \text{ and } y = 1 - \cos 6,$$

where θ is the independent variable.

B. To Find the Derivative when the Function is Given.

12. Formal work in differentiation. The student should be thoroughly familiar with the results of differentiating all the elementary functions. A list of the formulas which should be memorized is given below; any other formulas should be worked out as needed, or looked up in a book.

To establish these formulas, first prove the following important limits:

$$\lim_{\Delta u \doteq 0} \frac{\sin \Delta u}{\Delta u} = 1, \quad \text{and} \quad \lim_{\Delta u \doteq 0} \frac{1 - \cos \Delta u}{\Delta u} = 0,$$

provided u is in radians; and

$$\lim_{n=\infty} \left(1 + \frac{1}{n}\right)^n = e = 2.718 \cdots ;*$$

and hence prove the formulas for differentiating the *sine* and the *logarithm.*

The proofs of the other formulas present no difficulty.

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* These limits having been established, it can then be shown that

$$\lim_{\Delta u \doteq 0} \frac{\sin(u + \Delta u) - \sin u}{\Delta u} = \frac{\pi}{180} \cos u, \text{ if } u \text{ is measured in degrees,}$$

$$= \cos u, \text{ if } u \text{ is measured in radians;}$$

$$\lim_{\Delta u \doteq 0} \frac{\log(u + \Delta u) - \log u}{\Delta u} = (0.4343 \cdots) \frac{1}{u}, \text{ if the base is 10,}$$

$$= \frac{1}{u}, \text{ if the base is } e = 2.718 \cdots.$$

The reason for choosing the radian as the unit angle, and e as the base of the "natural" system of logarithms is the simplification in the formulas for the derivatives of the sine and the logarithm which results from this choice.

Rules for Differentiating the Elementary Functions of a Single Variable.*

(The first four of these rules are the fundamental ones, from which all the others can be derived.)

The differential of a constant is zero: —

$$dk = 0.$$

The differential of the LOGARITHM to the base e of any function is one over that function, times the differential of the function: —

$$d(\log_e x) = \frac{1}{x}\,dx \qquad (e = 2.718\ldots).$$

The differential of the SINE of any function (in radians) is the cosine of that function, times the differential of the function:

$$d(\sin x) = \cos x\,dx.$$

The differential of the sum [or difference] of two functions is the differential of the first plus [or minus] the differential of the second: —

$$d(u \pm v) = du \pm dv.$$

The differential of a constant times any function is the constant times the differential of the function: —

$$d(kx) = k\,dx.\dagger$$

The differential of a function to any constant power is the exponent of the power, times the function to the power one less, times the differential of the function: —

$$d(x^n) = nx^{n-1}dx.$$

Useful special cases of this rule are: —

$$d\sqrt{x} = \frac{1}{2\sqrt{x}}\,dx; \qquad d\left(\frac{1}{x}\right) = -\frac{1}{x^2}\,dx.$$

The differential of e with a variable exponent is e with the same exponent, times the differential of the exponent: —

$$d(e^x) = e^x\,dx \qquad (e = 2.718\ldots).$$

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* All these rules remain valid when the word "derivative" is put in place of "differential," and the symbol "D" in place of "d."

† To prove this and the next five rules, let $y =$ the function, and take the logarithm of both sides before differentiating.

The differential of the product of two functions is the first times the differential of the second, plus the second times the differential of the first: —

$$d(uv) = u\,dv + v\,du.$$

The differential of the quotient of two functions is the denominator times the differential of the numerator, minus the numerator times the differential of the denominator, all divided by the denominator squared: —

$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$$

The differential of the cosine of any function is minus the sine of that function, times the differential of the function: —

$$d(\cos x) = -\sin x\,dx.\text{*}$$

The differential of the tangent of any function is the secant-square of that function, times the differential of the function: —

$$d(\tan x) = \sec^2 x\,dx.\dagger$$

The differentials of the inverse sine, the inverse cosine, and the inverse tangent, of any function, are given by the following formulas, which the student may put into words for himself: —

$$d(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}\,dx,\ddagger \qquad (-\tfrac{1}{2}\pi \leqq \sin^{-1}x \leqq \tfrac{1}{2}\pi)$$

$$d(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}\,dx, \qquad (0 \leqq \cos^{-1}x \leqq \pi)$$

$$d(\tan^{-1} x) = \frac{1}{1 + x^2}\,dx. \qquad (-\tfrac{1}{2}\pi \leqq \tan^{-1}x \leqq \tfrac{1}{2}\pi)$$

[To find the differential of u to the vth power, where u and v are any functions, let

$$y = u^v,$$

and take the logarithm to base e of both sides before differentiating.— Similarly, to find the differential of the logarithm of u to any base v, let

$$y = \log_v u, \text{ whence } v^y = u;$$

then differentiate both sides.]

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* Proof: $\cos x = \sin(\tfrac{1}{2}\pi - x)$. † Proof: $\tan x = \sin x / \cos x$.

‡ Proof: Let $y = \sin^{-1}x$, that is, $\sin y = x$; then differentiate both sides.—Similarly for the next two formulas.

The rules on these two pages suffice for the differentiation of any elementary function; they should be carefully memorized.

The differentials of the hyperbolic functions are given by the following formulas, which are also worth remembering:

$$d \sinh x = \cosh x\, dx; \quad d \cosh x = \sinh x\, dx;$$
$$d \tanh x = \operatorname{sech}^2 x\, dx;$$

hence,

$$d \sinh^{-1} x = \frac{dx}{\sqrt{x^2+1}}, \quad d \cosh^{-1} x = \frac{dx}{\sqrt{x^2-1}}, \quad d \tanh^{-1} x = \frac{dx}{1-x^2}.$$

13. Differentiation of implicit functions, and of functions expressed in terms of a parameter.

(*a*) Suppose we have an equation connecting x and y, but not giving y explicitly as a function of x; as, for example, $9x^2 + 4y^2 = 36$. In finding dy/dx in cases of this kind, instead of first solving the equation for y in terms of x, and then differentiating, it is usually better to *differentiate the equation through as it stands* (remembering that both x and y are variables); thus, in the present example we have

$$18xdx + 8ydy = 0, \text{ whence, } dy/dx = -9x/4y.$$

This result can then, if desired, be expressed wholly in terms of x, by aid of the original equation.

(*b*) Again, suppose y is given as a function of u and v, where u and v are both functions of x; as, for example, $y = u^2 + v \sin u$. Differentiating both sides by the regular rules, we have $dy = 2udu + v \cos u\, du + \sin u\, dv$, whence, collecting the terms in du and dv, and dividing by dx,

$$\frac{dy}{dx} = (2u + v \cos u)\frac{du}{dx} + (\sin u)\frac{dv}{dx}.$$

This result shows how the rate of change of y depends on the rates of change of u and v, which are supposed to be known.

(*c*) Finally, both x and y may be given as functions of a third variable, t; as, $x = F(t)$, $y = f(t)$. To every value of this auxiliary variable, or "parameter," t, there corresponds a *pair* of values of x and y, so that here again y is indirectly determined as a function of x. Of course if we can eliminate t

we shall have a single equation connecting x and y; but it is often more convenient to *keep the equations in the parameter form.* Thus, to find dy/dx, we have merely to differentiate both of the given equations: $dx = F'(t)\,dt$, $dy = f'(t)\,dt$; and then divide the second result by the first: $dy/dx = f'(t)/F'(t)$.

C. To Find the Derivative when the Function Itself is not Given; Setting up a Differential Equation.

14. In many cases it is required to find the rate of change of a function when the function itself is not directly given; in fact it is often easier to find the derivative of a function than it is to find the function itself.

For example, a hemispherical bowl of radius r, full of water, is being emptied through a hole in the bottom; find the rate of change of the volume of water drawn off, regarded as a function of the distance, y, between the level of the water and the center of the bowl. To compute this value directly from the definition, we notice first that the increment ΔV produced in V by an increment Δy given to y will have a value between $\pi(r^2 - y^2)\Delta y$ and $\pi[r^2 - (y + \Delta y)^2]\Delta y$; dividing either of these values by Δy, and taking the limit of the ratio $\Delta V/\Delta y$, we find at once $dV/dy = \pi\ (r^2 - y^2)$, which gives the required value of dV/dy for any value of y from $y = 0$ to $y = r$.

This process of finding the derivative directly from first principles, as the limit of the ratio of the increments, when the function itself is not given, is called "setting up a differential equation," since the result of the process is an equation between the differentials of the function and of the argument.*

Every problem of this kind is a problem in finding the *limit of the ratio of two variable quantities, each of which is approaching zero;* and in this connection the following theorems on infinitesimals are extremely useful, if not indispensable.

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* The problem of finding the relation between the quantities themselves when the relation between their differentials is known will be discussed in the next chapter.

15. Theorems on infinitesimals.

Def. Any variable quantity that approaches 0 as a limit is called an *infinitesimal.* For example, Δx, Δy, dx, dy, are infinitesimals.

The erroneous notion that an infinitesimal is a *constant* quantity which is "smaller than any other quantity, however small, and yet not zero" should be carefully avoided.

Notation. The notation lim $x = a$, or $x \rightarrow a$ (read: "x approaches a as a limit"), means that $x = a + \epsilon$, where ϵ is a variable approaching zero. Thus a statement expressed in terms of "lim" or "$\rightarrow$" can always be translated into an *equation,* which can then be handled by the ordinary rules of algebra. The symbol $\rightarrow$ is preferable to $\doteq$ and seems likely to replace it.

Def. If α and β are infinitesimals, and lim $(\alpha/\beta) = 0$, then α is said to be an *infinitesimal of higher order* than β.

For example, if $\Delta u = \epsilon \cdot \Delta v$, where ϵ itself approaches 0, then Δu is of higher order than Δv. Again, $1 - \cos \Delta\theta$ is of higher order than $\Delta\theta$.

If the difference between two infinitesimals is of higher order than either, then their ratio approaches 1 as a limit; and conversely, if the ratio of two infinitesimals approaches 1, then their difference is of higher order than either. Two infinitesimals having this relation may be called "**similar**" or "**equivalent**" infinitesimals.

Important examples are the following: a convex arc of a curve, and the chord of that arc, are "similar" infinitesimals. Again, $\sin \Delta x$ and $\tan \Delta x$ are both "similar" to Δx, provided Δx is in radians.

First Replacement Theorem for Infinitesimals. *In finding the limit of the ratio of two infinitesimals, either of them may be replaced by a "similar" infinitesimal, without affecting the value of the limit.*

As explained above, two infinitesimals are *"similar":* (1) if the *difference between them is of higher order than either;* or (2) if the *limit of their ratio is 1.* (Sometimes the first test is more convenient, sometimes the second.)

This theorem frequently enables us to replace a complicated infinitesimal, like $\pi(r+\Delta r)^2\Delta x$, by a simpler one, as $\pi r^2\Delta x$; *but it justifies this replacement only in the case expressly stated in the hypothesis of the theorem,* namely the case in which we are *finding the limit of a ratio.** (The fallacy that "infinitesimals of higher order can *always* be neglected" should be carefully guarded against.)

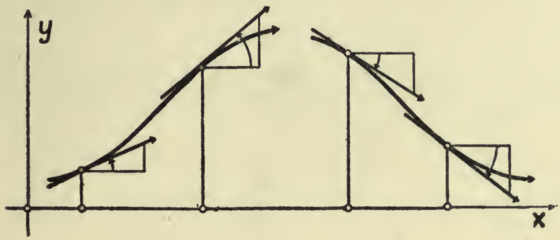
* A second replacement theorem for infinitesimals will be given in the chapter on Definite Integrals.

D. Applications of Differentiation in Studying the Properties of a Given Function.

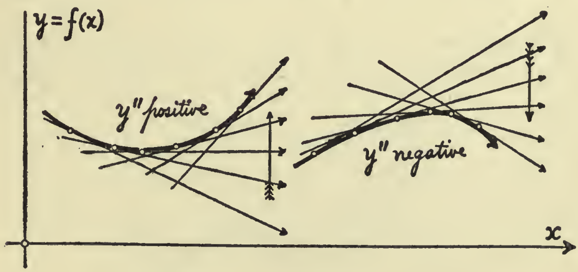
16. That a knowledge of differentiation is of fundamental importance in studying the variation of a given function is evident from the following theorems.

Let the given function be $y = f(x)$.

I. *The value of the derivative at any point shows the slope of the curve at that point.*



Hence, *if the derivative is positive* at any point, the curve is rising at that point (as we move in the positive direction along the axis); that is, *the function is increasing.* And *if the derivative is negative* at any point, the curve is falling at that point; that is, *the function is decreasing.*

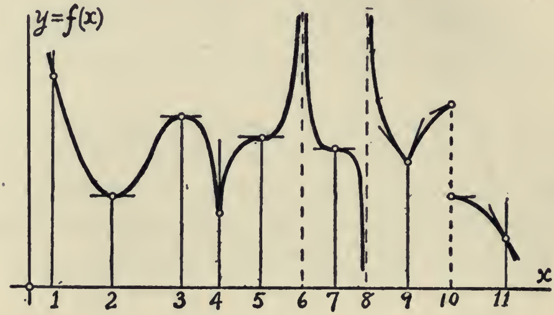


II. *If the second derivative is positive* at any point, the slope is increasing at that point, and hence *the curve is concave upward;* and *if the second derivative is negative* at any point, the slope is decreasing at that point, and hence *the curve is concave downward.*

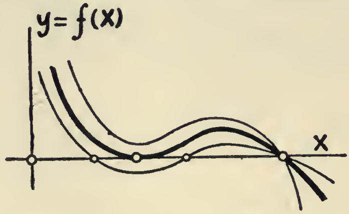
A point where the concavity changes sign is called a *point of inflexion;* at every such point, the second derivative is 0 or ∞.*

17. Maxima and minima.—The application to problems in maxima and minima is immediate. In seeking the largest or smallest value of a given function in a given interval, we need consider only (1) the points where the slope is zero; (2) the points where the slope is infinite (or otherwise discontinuous); and (3) the end-points of the interval; for among these points the desired point will certainly be found. In most practical cases it will be a point where the slope is zero.

The conditions of the problem will usually show clearly which of these points, if any, is a maximum (or a minimum).



18. Multiple roots.—The roots, or the zeros, of a function, are the values of the argument for which the function becomes



zero. An inspection of the figure will show that any value of x for which $f(x)$ and $f'(x)$ are both zero simultaneously, will count as at least a double root.

* But the second derivative may be zero at points which are not points of inflexion; for example, $y = x^4$ at $x = 0$.

19. Small errors.—The following theorem is very useful in discussing the effect, on a computed value, of small errors in the data:

III. If dx is small, dy and Δy *are nearly equal.*

That is, the difference between dy and Δy can be made as small as we please, in comparison with dx, by making dx sufficiently small (except at points where dy/dx does not have a finite value).

Thus, if we wish to find approximately the error Δy produced by a small error in x, it will usually be sufficiently accurate to compute, instead of Δy, the simpler value, dy.

In problems concerning the *relative* error, dy/y, or dx/x, it is often convenient to take the logarithm of both sides of the given equation $y = f(x)$ before differentiating.

This class of problems is of great practical value.

CHAPTER III.

INTEGRATION AS THE INVERSE OF DIFFERENTIATION. SIMPLE DIFFERENTIAL EQUATIONS.

20. In many problems in pure and applied mathematics, we have given the derivative [or differential] of a function, and are required to find the function itself.

Suppose $f(x)$ [or $f(x)dx$] is the given derivative [or differential]; it is required to find a function $F(x)$ which, when differentiated, will give $f(x)$ [or $f(x)dx$]. Clearly, if one such function $F(x)$ has been found, then any function of the form $F(x) + C$, where C is any constant, will have the same property.

DEFINITION.—*Any function $F(x)$ whose differential is $f(x)dx$ is denoted by*

$$\int f(x)dx,$$

read: **an integral** *of $f(x)dx$.* The process of finding an integral of a function is called *integration,* or the *inverse of differentiation.*

If $F(x)$ is any particular integral of $f(x)dx$, then every integral of $f(x)dx$ can be expressed in the form $F(x) + C$, where C is a constant, called the *constant of integration.*

It can be shown that every continuous function has an integral; but this integral may not (in general, will not) be expressible in terms of the elementary functions.*

Most of the functions that occur in practice can, however, be integrated in terms of elementary functions, by the aid of a table of integrals, such as B. O. Peirce's well-known table of integrals. The entries in such a table can be verified by differentiation.

21. Formal work in integration.—The time devoted to the formal work of integration should not be longer than is nec-

* In such cases, an approximate expression for the integral may be obtained by infinite series.

essary to give the student a reasonable degree of expertness in the use of the tables.

The following integration formulas should be memorized; they are derived immediately from the corresponding formulas for differentiation.

$$\int cudx = c\int udx; \quad \int (u + v + \cdots)dx = \int udx + \int vdx + \cdots;$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} \text{ (provided } n \neq -1\text{)};$$

(in words: an integral of any function raised to a constant power, $\neq -1$, times the differential of that function, is equal to the function raised to a power one greater, divided by the new exponent);

$$\int \frac{dx}{x} = \log_e x; \qquad \int e^x dx = e^x;$$

$$\int \sin x\, dx = -\cos x; \qquad \int \cos x\, dx = \sin x; \qquad \int \sec^2 x\, dx = \tan x;$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x, \text{ or } -\cos^{-1} x; \qquad \int \frac{dx}{1+x^2} = \tan^{-1} x.$$

The constant of integration must be supplied in each case.

A large number of integrals can be brought under the form $\int x^n dx$ by a simple transformation. For example,

$$\int \cos^3 x\, dx = \int \cos^2 x \cos x\, dx = \int (1 - \sin^2 x) \cos x\, dx$$
$$= \int \cos x\, dx - \int \sin^2 x \cos x\, dx = \int \cos x\, dx - \int (\sin x)^2 d(\sin x)$$
$$= \sin x - (\sin x)^3/3.$$

Similarly for any odd power of the sine or cosine.

The following integrals are also important, though it is not worth while to memorize them when a table is at hand:

$$\int \sin^2 x dx = \tfrac{1}{2}(x - \sin x \cos x); \quad \int \cos^2 x dx = \tfrac{1}{2}(x + \sin x \cos x);$$

$$\int \frac{dx}{\cos x} = \log_e \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \tfrac{1}{2} \log_e \frac{1+\sin x}{1-\sin x}; \quad \int \frac{dx}{\sin x} = \log_e \tan \frac{x}{2};$$

$$\int \sinh x\, dx = \cosh x; \quad \int \cosh x\, dx = \sinh x; \quad \int \operatorname{sech}^2 x dx = \tanh x.$$

22. Among the other formulas of integration, the following are perhaps the ones that occur most often in practice; they are inserted here for reference, and especially to illustrate the usefulness of the hyperbolic functions.

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a},$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log_e \frac{a + x}{a - x} = \frac{1}{a} \tanh^{-1} \frac{x}{a},$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log_e \frac{x - a}{x + a} = -\frac{1}{a} \coth^{-1} \frac{x}{a},$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}, \quad \text{or} \quad = -\cos^{-1} \frac{x}{a},$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log_e (x + \sqrt{x^2 + a^2}), \quad \text{or} \quad = \sinh^{-1} \frac{x}{a},$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log_e (x + \sqrt{x^2 - a^2}), \quad \text{or} \quad = \cosh^{-1} \frac{x}{a},$$

$$\int \sqrt{a^2 - x^2}\, dx = \tfrac{1}{2} \left[x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right],$$

$$\int \sqrt{x^2 + a^2}\, dx = \tfrac{1}{2} [x \sqrt{x^2 + a^2} + a^2 \log_e (x + \sqrt{x^2 + a^2})],$$

$$\text{or} = \frac{1}{2} \left[x \sqrt{x^2 + a^2} + a^2 \sinh^{-1} \frac{x}{a} \right],$$

$$\int \sqrt{x^2 - a^2}\, dx = \tfrac{1}{2} [x \sqrt{x^2 - a^2} - a^2 \log_e (x + \sqrt{x^2 - a^2})],$$

$$\text{or} = \frac{1}{2} \left[x \sqrt{x^2 - a^2} - a^2 \cosh^{-1} \frac{x}{a} \right].$$

23. Methods of Integration. Among the methods by which a given integral may be reduced to a form in the tables (or an integral in the table to one of the fundamental forms), the most important are (1) the method of substitution and (2) the method of integration by parts.

In the *method of substitution*, the given integral, $\int f(x)dx$, is expressed wholly in terms of some new variable y (and dy), in the hope that the new integral may be easier to handle than the old one. The substitutions which are most likely to be useful are the following:

(*a*) $y =$ any part of the given expression whose differential occurs as a factor; $y = x^k$; $y = 1/x$; $y = \sin x$; $y = \cos x$; $y = \tan(x/2)$.

(*b*) $x = a \sin y$, or $= a \tan y$, or $= a \sec y$, in expressions involving $\sqrt{a^2 - x^2}$, or $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$, respectively.

But much can be done without formal substitution of a new letter, if one remembers that the "x" in the formulas of integration may stand for any function.

The *method of integration by parts* is an application of the formula

$$\int u dv = uv - \int v du.$$

Take as dv a part of the given expression which can be readily integrated; on applying the formula, the new integral may be simpler than the old one.

The student should be practiced in both of these methods.

24. Simple differential equations. In a large number of problems in pure and applied mathematics, it is possible to write down an expression involving the rate of change of a desired function more readily than to write down the expression for the function itself. (Compare Chap. II, B.) In other words, it is often easier to write down a relation between the differentials of two variables than to write down the relation between the variables themselves. Such a relation connecting the differentials of two or more quantities, is called a *differential equation,* and any function which satisfies the equation, when substituted therein, is called a *solution* of the equation.

Every such problem, then, breaks up into two parts: (1) setting up the differential equation; (2) solving that equation.

The first part of the problem has already been treated in Chap. II, B. This part of the problem is too apt to be neglected in elementary courses; there is scarcely anything that develops real appreciation of the power of the calculus more effectively than practice in setting up for one's self the differential equations for various physical phenomena.

As to the second part of the problem, namely, the *solution* of the differential equation, the general plan is to reduce the given equation, by more or less ingenious devices, to the form $dy = f(x)\,dx$, or $y = \int f(x)\,dx$, and then to complete the solution, if possible, by the aid of a table of integrals. In a technical sense, the differential equation is said to be "solved" when it is thus reduced to a simple "quadrature"; that is, to a single integration.

The solution of a differential equation of the nth order, that is, an equation involving the nth derivative, will contain n arbitrary constants; to determine these constants, n conditions connecting x, y, y' . . ., $y^{(n)}$ must be known (the "initial" or "auxiliary" conditions of the problem).

25. The general discussion of differential equations is too large and too difficult a topic to find a place in a first course in the calculus, but two, at least, of the simpler equations are so important that their solution should be given, as an exercise in integration.

These equations are the following:

$$(1) \qquad \frac{dy'}{dt} + n^2 y = 0, \text{ where } y' = \frac{dy}{dt}.$$

The solution is

$$y = C_1 \sin(nt + C_2) \text{ or, } y = C_3 \sin nt + C_4 \cos nt,$$

where the C's are arbitrary constants.

$$(2) \qquad \frac{dy'}{dt} - n^2 y = 0, \text{ where } y' = \frac{dy}{dt}.$$

The solution is

$$y = C_1 \sinh(nt + C_2), \text{ or, } y = C_3 e^{nt} + C_4 e^{-nt},$$

where the C's are arbitrary constants.

The method of obtaining these results, rather than the results themselves, should be remembered: namely, multiply through by dy, noting that $dy/dt = y'$, and integrate each term, getting $\frac{1}{2}y'^2 + \frac{1}{2}n^2y^2 = C$; then replace y' by dy/dt, "separate the variables," and integrate again. By a similar method, any equation of the form $dy'/dt + f(y) = 0$ can be solved, if we can integrate $f(y)\,dy$.

26. Another very important differential equation is the equation for " damped vibration " :

$$\frac{dy'}{dt} + 2b\frac{dy}{dt} + a^2y = 0, \text{ where } y' = \frac{dy}{dt}.$$

The solution is given here for reference:

Case 1. If $a^2 - b^2 > 0$, let $m = \sqrt{a^2 - b^2}$; then

$$y = C_1e^{-bt}\sin(mt + C_2),$$

$$\text{or } y = [C_3\sin(mt) + C_4\cos(mt)]e^{-bt}.$$

Case 2. If $a^2 - b^2 = 0$,

$$y = e^{-bt}(C_1 + C_2t).$$

Case 3. If $a^2 - b^2 < 0$, let $n = \sqrt{b^2 - a^2}$; then

$$y = C_1e^{-bt}\sinh(nt + C_2),$$

$$\text{or } y = C_3e^{-(b+n)t} + C_4e^{-(b-n)t}.$$

26a. Another important case is the linear differential equation of the first order:

$$\frac{dy}{dx} + Py = Q,$$

where P and Q are functions of x (or constants), but do not contain y. The solution is given here for reference:

$$ye^F = \int Qe^F dx + \text{const.},$$

where

$$F = \int P dx.$$

CHAPTER IV.

INTEGRATION AS THE LIMIT OF A SUM. DEFINITE INTEGRALS.

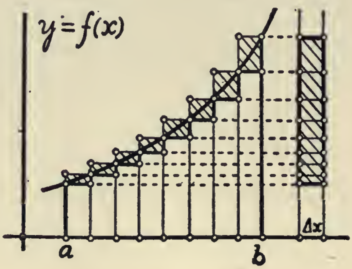
27. The limit of a sum. Many problems in pure and applied mathematics can be brought under the following general form:

Given, a continuous function, $y=f(x)$, from $x=a$ to $x=b$. Divide the interval from $x=a$ to $x=b$ into n equal parts, of length $\Delta x=(b-a)/n$. Let $x_1, x_2, x_3, \ldots x_n$ be values of x, one in each interval; take the value of the function at each of these points, and multiply by Δx; then form the sum:*

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

Required, the limit of this sum, as n increases indefinitely, and $\Delta x \doteq 0$.

This problem may be interpreted geometrically as the problem of finding the area under the curve $y=f(x)$, between the ordinates $x=a$ and $x=b$; each term of the sum represents



the area of a rectangle whose base is Δx and whose altitude is the height of the curve at one of the points selected. It is easily seen that the difference between the sum of the rectangles and the area of the curve is less than a rectangle

* It is not necessary that the parts be equal, provided the largest of them approaches zero when n is made to increase indefinitely.

whose base is Δx and whose altitude is constant. This difference approaches zero as $\Delta x \doteq 0$; therefore the sum of the rectangles approaches the area of the curve as a limit.

In this way, or by an analytic proof, it is shown that the limit of the sum in question always exists. The problem then is, to find the value of this limit.

The value of the limit can always be obtained by the following fundamental theorem, whenever an integral of the given function $f(x)$ can be found.

FUNDAMENTAL THEOREM OF SUMMATION. *If $x_1, x_2, \cdots x_n$ are values of x ranging from $x = a$ to $x = b$, as in the statement of the general problem above, then*

$$\lim_{\Delta x \doteq 0} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x] = F(b) - F(a),$$

where

$$F(x) = \int f(x)dx$$

is any function whose derivative is the given function $f(x)$.

The proof of this remarkable theorem is best given by showing that the right hand side of the equation, as well as the left, is equal to the area under the curve from $x = a$ to $x = b$; to do this, consider the area from $x = a$ to a variable point $x = x$, and find the rate of change of this area regarded as a function of x; hence find the area itself as a function of x, determine the constant of integration in the usual way, and then put $x = b$ in the result.

DEFINITION. *The limit of a sum of the kind described above is called the* **definite integral** *of $f(x)dx$ from $x = a$ to $x = b$, and is denoted by*

$$\lim_{n \doteq \infty} \sum_{i=0}^{i=n} f(x_i)\Delta x, \quad \text{or} \int_{x=a}^{x=b} f(x)dx.$$

The function obtained by the inverse of differentiation is called, for distinction, an *indefinite* integral. By the fundamental theorem just stated, the definite integral is equal to the difference between two values of the indefinite integral:

$$\int_{x=a}^{x=b} f(x)dx = \left[\int f(x)dx\right]_{x=b} - \left[\int f(x)dx\right]_{x=a}.$$

The double use of the term "integration"—meaning in one case anti-differentiation, and in the other case finding the limit of a sum—and the fundamental theorem connecting these two distinct concepts, should be made thoroughly clear.*

The concept of the definite integral is the most useful concept in the application of the calculus, and the study of problems which can be formulated as definite integrals may well occupy one third of the time of a first course.

For example, problems in areas, volumes, surfaces, length of arc, center of gravity, moments of inertia, center of fluid pressure, etc. *Many of these problems require two applications of the fundamental theorem.*

28. Properties of definite integrals. From the definition of the definite integral we have at once:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx;$$

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx;$$

and, by the aid of a figure, the *Mean Value theorem:*

$$\int_a^b F(x)f(x)dx = F(X)\int_a^b f(x)dx,$$

where X is some (unknown) value of x between a and b, and $F(x)$ and $f(x)$ are any continuous functions, provided $f(x)$ *does not change sign* from $x=a$ to $x=b$.

We have also the following important theorem on *change of variable:*

In evaluating the integral

$$\int_{x=a}^{x=b} f(x)dx,$$

if x is a function of a new variable t, we may replace $f(x)dx$ by its value in terms of t and dt, and replace $x=a$ and $x=b$

* The use of the term in the sense of summation was historically the earlier, and the symbol $\int$ is the old English "long s," the first letter of "sum."

by the corresponding values $t = \alpha$ and $t = \beta$, without altering the value of the integral, provided that thoroughout the interval considered there is one and only one value of x for every value of t, and one and only one value of t for every value of x.

29. All problems leading to a definite integral are problems in finding the limit of a sum, each term of which is approaching zero, while the number of terms is increasing indefinitely. Whenever a function $f(x)$ can be found, such that all terms of the sum are obtained by substituting successively x_1, x_2, etc., in the expression $f(x)dx$, then the formulation of the problem as a definite integral is immediately obvious. The separate terms of the sum, of which $f(x_k)dx$ is a type, are called *elements*.

Thus, in finding the area under a curve, an obvious element of area is the rectangle ydx; if the curve revolves about the x-axis, the element of volume of the solid thus generated is the cylinder $\pi y^2 dx$. Here y must be expressed as a function of x before the integration can be completed. Again, in polar coördinates, the element of area is the sector, $\frac{1}{2}r^2 d\theta$, where r must be a known function of θ.

In many cases, however, the proper function is not so immediately obvious. In such cases, the following theorem is of great service:

SECOND REPLACEMENT THEOREM FOR INFINITESMALS (THEOREM OF DUHAMEL). *In finding the limit of a sum of positive terms, each of which approaches zero while the number of terms increases indefinitely, any term may be replaced by a "similar" term without affecting the value of the limit. Two variables α and β are called "similar" if*

$$(1) \qquad \lim \frac{\alpha}{\beta} = 1, \quad \textit{or} \quad \textit{if}\ (2)\ \lim \frac{\alpha - \beta}{\alpha} = 0.$$

For example, let us find the weight of a rod whose density, w, and cross-section, A, are both functions of x. The "true element" of weight, ΔW, corresponding to a given length Δx, will certainly lie between the values $w'A'\Delta x$ and $w''A''\Delta x$, where

w', A' are the smallest values, and w'', A'' the largest values of w and A within the interval from $x = x$ to $x = x + \Delta x$; but either of these extreme values may be replaced by the simpler value $wA\Delta x$, where w, A are the values of w and A at the *beginning* of the interval, for,

$$\lim \frac{w'A'\Delta x}{wA\Delta x} = \lim \frac{w''A''\Delta x}{wA\Delta x} = 1.$$

Hence, ΔW itself, which lies between these extremes, can be replaced by $wA\Delta x$, which is therefore the required "differential element" of weight.* The total weight of the rod, from $x = a$ to $x = b$, is then equal to the definite integral

$$\int_{x=a}^{x=b} wA dx;$$

where w and A must of course be expressed as functions of x before the integration can be completed.

In justifying replacements of this kind by Duhamel's theorem, sometimes the first test is more convenient, sometimes the second. When once the common replacements have been justified, the use of the theorem in practice rapidly becomes almost intuitive.

30. Approximate methods of integration.—If the function $f(x)$ is given only empirically, the theorem on evaluating the definite integral by purely mathematical means cannot be applied. In such cases, an approximate value of the definite integral

$$\int_{x=a}^{x=b} f(x) dx$$

may be found by plotting the curve $y = f(x)$ on squared paper, and estimating the area by *counting squares* (and fractions of squares).

Another method of approximation is by *Simpson's Rule:*

* When x is the independent variable, it is immaterial whether we write Δx or dx.

Divide the area into n panels, where n is *even*, and number the ordinates from 1 to $n+1$; then, if Δx is the width of each panel,

Area $= \frac{1}{3}\Delta x$ (first ordinate + last ordinate
+ twice the sum of the other odd ordinates
+ four times the sum of the even ordinates).

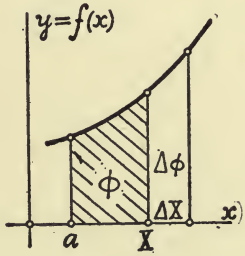
The instrument known as a *planimeter* provides a mechanical means of integration, used especially in measuring the areas of indicator cards.

Another and very important method of approximation is by the use of *series;* see the next chapter.

31. Definite Integral as a function of its upper limit.—If X is a variable, the definite integral

$$\int_{x=a}^{x=X} f(x)dx$$

represents the area under the curve $y=f(x)$ from $x=a$ to the variable ordinate $x=X$, and is therefore a function of X,



say $\phi(X)$. By applying the definition of derivative to this function, it is easy to see from the figure that $\phi'(X)=f(X)$:

$$\frac{d}{dX}\int_{x=a}^{x=X} f(x)dx = f(X).$$

Thus $\phi(X)$ is one of the indefinite integrals of $f(X)$.

Any indefinite integral which cannot be expressed in terms of known functions can always be written as a definite integral regarded as a function of its upper limit, and its value, for any given value of the argument, can then be found by one of the methods of approximate integration.

The elliptic integrals, the most important of which are

$$\int_{\theta=0}^{\theta=\phi} \frac{d\theta}{\sqrt{1-(k^2)\sin^2\theta}} \quad \text{and} \quad \int_{\theta=0}^{\theta=\phi} \sqrt{1-(k^2)\sin^2\theta}\, d\theta,$$

are handled in this way, by the method of expansion in series. The student should be made familiar with the construction and use of tables of the elliptic integrals.

In such tables, k^2 is usually expressed in the form $\sin^2 \alpha$, which emphasizes the fact that $k^2 \leqq 1$.

CHAPTER V.

APPLICATIONS TO ALGEBRA: EXPANSION IN SERIES; INDETERMINATE FORMS.

Note.—This chapter may be taken, if preferred, immediately after the chapter on differentiation. It is in reality an extension of the "formal work" of that chapter, since it deals with changes in the *form* of algebraic expressions.

32. Taylor's theorem.—It is often desirable to obtain an approximate expression for a given function, in the neighborhood of a given point $x = a$, in the form of a series arranged according to ascending powers of $x - a$, with constant coefficients. For values of x near to a, the higher powers of $x - a$ will then become negligible.

The most convenient theorem for this purpose is the following:

TAYLOR'S THEOREM. *If $f(x)$ is continuous, and has derivatives through the $(n+1)$st, in the neighborhood of a given point $x = a$, then, for any value of x in this neighborhood,*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \frac{f^{n+1}(X)}{(n+1)!}(x-a)^{n+1},$$

where X is some unknown quantity between a and x. The last term,

$$R = \frac{f^{n+1}(X)}{(n+1)!}(x-a)^{n+1},$$

is the *error* committed if we stop the series with the term in $(x-a)^n$, and the formula is useful only when this error becomes smaller and smaller as we increase the number of terms.

This form for the "remainder" R is easily remembered since it differs from the general term of the series only by the fact that the derivative in the coefficient of the power of $(x-a)$ is taken for $x=X$ instead of for $x=a$.* (There are also other forms of the remainder which are sometimes useful.)

33. The special case where $a=0$ is called **Maclaurin's Theorem:**

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^n(0)}{n!}x^n+\frac{f^{n+1}(X)}{(n+1)!}x^{n+1},$$

where X is some unknown quantity between 0 and x.

34. Another special case, obtained by putting $n=0$, gives

$$f(x)-f(a)=f'(X)(x-a),$$

where again X is some unknown quantity between a and x. This theorem is called the **Law of the Mean**, and is of great importance in the theoretical development of the subject.

35. If the error-term in Taylor's Theorem approaches zero as n increases, the formula becomes a convergent infinite series, called the **Taylor's series** for the given function, about the given point $x=a$.

The series with which the student should be especially familiar are the following:

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* The simplest proof of this theorem is by means of integration. For example, for the case $n=2$, we have

$$\int_a^x f'''(t)dt=f'''(X)(x-a),$$

where X is some (unknown) constant between a and x (as is evident from a figure); but also

$$\int_a^x f'''(t)dt=f''(x)-f''(a),$$

by the fundamental theorem; so that

$$f''(x)-f''(a)=f'''(X)(x-a).$$

Integrating this equation twice between the limits $x=a$ and $x=x$, remembering that $f''(a)$ and $f'''(X)$ are constants, we have at once:

$$f'(x)-f'(a)-f''(a)(x-a)=f'''(X)\tfrac{1}{2}(x-a)^2,$$
$$f(x)-f(a)-f'(a)(x-a)-f''(a)\tfrac{1}{2}(x-a)^2=f'''(X)\tfrac{1}{2}\tfrac{1}{3}(x-a)^3.$$

Binomial series:

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots,$$

provided $|x| < 1$.

Sine series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad (x \text{ in radians}).$$

Cosine series:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad (x \text{ in radians}).$$

Exponential series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

Next in importance are the series for $\log(1+x)$, $\tan^{-1} x$, $\sinh x$, and $\cosh x$.

From these series we have the following important approximations, when x is small:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \cdots, \quad \frac{1}{1+x} = 1 - x + \cdots, \quad \frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \cdots,$$

$$\sin x = x - \cdots, \quad \cos x = 1 - \cdots, \text{ etc.}$$

An important special case of the binomial series is the geometric series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \quad \text{provided } |x| < 1.$$

36. The student should also understand the comparison test, and the test-ratio test, for the convergence of an infinite series, and the following theorem on *alternating* series: If the terms of a series are alternately positive and negative, each being numerically less than or equal to the preceding, and if the nth term approaches zero as n increases, then the series is convergent, and the error made by breaking off the series at any given term does not exceed numerically the value of the last term retained.

Further, *a power series can be differentiated or integrated term by term, within the interval of convergence.*

37. Indeterminate forms.—The evaluation of indeterminate forms can often be facilitated by the use of the following theorem, in which $f(x)$ and $F(x)$ are functions which possess derivatives at a given point $x=a$.

Theorem of indeterminate forms. If $f(x)$ and $F(x)$ both approach zero, or both become infinite, when x approaches a, then

$$\lim_{x \doteq a} \left[\frac{f(x)}{F(x)} \right] = \lim_{x \doteq a} \left[\frac{f'(x)}{F'(x)} \right].$$

The second limit may often be easier to evaluate than the first.

The student should thoroughly understand the meaning of indeterminate forms, for which the common symbols $\frac{0}{0}$, 1^∞, etc., are merely a suggestive short-hand notation.

Thus, "0/0" means that we are asked to find the limit of a function $y=f(x)/F(x)$, when $f(x)$ and $F(x)$ both approach zero. Now the change in $f(x)$ alone would tend to decrease y numerically, while the change in $F(x)$ alone would tend to increase y; hence we cannot tell, without further investigation, what the combined effect of both changes, taking place simultaneously, will be.

Again, the symbol 1^∞ means that we are asked to find the limit of a function $y=f(x)^{F(x)}$, when $f(x)$ approaches 1 and $F(x)$ becomes infinite. Now the change in $f(x)$ alone would tend to make y approach 1, while the change in $F(x)$ alone would tend to make y recede from 1; hence we cannot tell, without further investigation, what the combined effect will be.

The student should thoroughly master in this way the meaning of all the seven types of indeterminate forms, namely,

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ 0^0,\ 1^\infty,\ \infty^0,\ \infty-\infty.$$

The cases involving exponents are best treated by first finding the limit of the logarithm of y, from which the limit of y

can then be obtained. The form $0 \cdot \infty$, or $y = f(x) \cdot F(x)$, can be written as $y = \frac{f(x)}{1/F(x)}$, or $y = \frac{F(x)}{1/f(x)}$, which then comes under one of the first two forms. The last form, $\infty - \infty$, is usually best handled by the method of series.

Before applying the theorem of indeterminate forms, one should, of course, try first to find the required limit by a simple algebraic transformation, if possible.

CHAPTER VI.

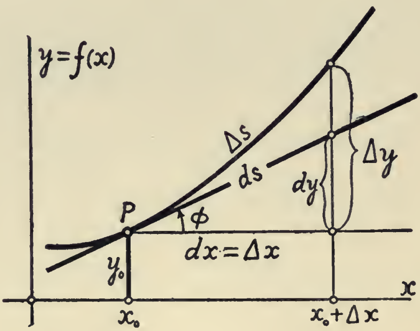
APPLICATIONS TO GEOMETRY AND MECHANICS.

In all applications to geometry, in which a curve is represented by an equation connecting x and y, the *scales on the x and y axes must be equal* (compare §3, footnote).

38. Tangent and normal.—The equation of the tangent at any point (and hence the equation of the normal) can be written down at once when we know the slope and the coördinates of the point of contact.

Again, to find the *subtangent* or *subnormal* at any point, we have simply to find the ordinate and the slope at that point, and then solve a right triangle.

39. Differential of arc. If $s =$ length of arc of the curve $y = f(x)$, measured from some fixed point A of the curve, then s, like y, is a function of x, and we may ask what is the



rate of change of s with respect to x, that is, what is the value of ds/dx. Now $ds/dx = \lim\ (\Delta s/\Delta x)$, and in finding this limit we may replace the arc Δs by its chord, $\sqrt{(\Delta x)^2 + (\Delta y)^2}$; hence $ds/dx = \lim \sqrt{1 + (\Delta y/\Delta x)^2} = \sqrt{1 + (dy/dx)^2}$, or

$$ds = \sqrt{(dx)^2 + (dy)^2},$$

as indicated in the figure. This formula, and the corresponding relations

$$dx = ds \cos \phi, \quad dy = ds \sin \phi,$$

are important, and are readily recalled to mind by the figure.

In the case of a circle of radius r, if $d\theta =$ the angle at the center, subtended by the arc ds, then

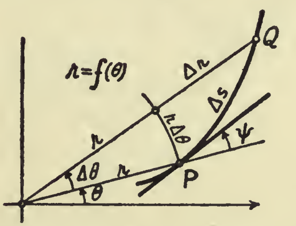
$$ds = r d\theta,$$

provided the angle is measured in radians.

40. Again, in case of a curve whose equation is given in polar coordinates, $r = f(\theta)$, we see at once from the figure, by the aid of the replacement theorem, that

$$ds = \sqrt{(dr)^2 + (rd\theta)^2} \quad \text{and} \quad \tan \psi = \frac{rd\theta}{dr},$$

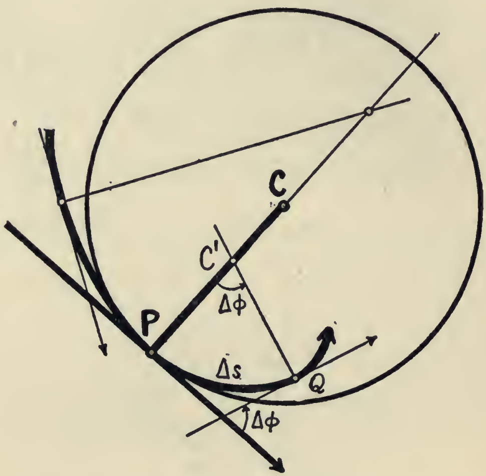
where ψ is the angle which the tangent makes with the radius vector produced.



41. Radius of Curvature.—Consider the normal to a given curve at a given point, P, and also the normal at a neighboring point, Q. These two normals will intersect at some point C' on the concave side of the curve; and as Q approaches P, along the curve, this point C' will (in general) approach a definite position C as a limit. The circle described with a center at this point C and radius equal to CP will fit the given curve more closely, in the neighborhood of the point P, than does any other circle. This circle is called the *osculating circle,* or the circle of curvature, at the point P; its center C is called the *center of curvature,* and its radius CP is called the *radius of curvature,* at the point P.

The radius of curvature may thus be taken as a measure of the flatness or sharpness of the curve; the smaller the radius of curvature, the sharper the curve.

The length of the radius of curvature, R, at any point P is most readily found as follows: In the triangle $PC'Q$, we have $C'P/PQ = \sin Q / \sin \Delta\phi$, where $\Delta\phi$ is the angle between the normals (or between the tangents) at P and Q. Therefore $R = \lim C'P = \lim$ (chord $PQ/\sin \Delta\phi$) $\sin Q$; or, replacing



the chord by the arc Δs, and $\sin \Delta\phi$ by $\Delta\phi$, and noticing that Q is approaching 90°, so that $\lim \sin Q = 1$, we have $R = \lim (\Delta s/\Delta\phi)$, or,

$$R = \frac{ds}{d\phi}.$$

This important formula is readily recalled to mind from the figure, if one thinks of the arc Δs as approximately a circular arc.

To express R in terms of x and y, we have only to remember that $ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + y'^2}dx$, and $\tan \phi = dy/dx = y'$, whence $d\phi = y''dx/(1 + y'^2)$; then

$$R = \frac{(1 + y'^2)^{\frac{3}{2}}}{y''}$$

Def. The *curvature* of a curve at a point is defined as the rate at which the angle ϕ is changing with respect to the length of arc s; that is,

$$\text{curvature} = \frac{d\phi}{ds} = \frac{1}{R}.$$

If the slope of the curve is small, the curvature is approximately equal to y''.

Def. The locus of the center of curvature is called the *evolute* of the curve.

The normals to the given curve are tangent to the evolute, and the given curve may be traced by unwinding a string from the evolute.

42. Velocity and acceleration.—Consider a particle moving along a *straight line.* Its distance from the origin is a function of the time:

$$x = F(t).$$

The *velocity* of the particle is the *rate of change of its distance:*

$$v = dx/dt = F'(t) = x'.$$

The velocity will be positive or negative, according as the particle is moving forward or backward along the line.

The *acceleration* of the particle is the *rate of change of its velocity:*

$$A = dv/dt = F''(t) = x''.$$

The acceleration will be positive or negative according as the velocity is increasing or decreasing (algebraically).

If a particle is moving along a *plane curve,* we must consider the *components* of its motion along two fixed axes. The components of acceleration along the x- and y-axes are x'' and y''; the components of acceleration along the tangent and normal are dv/dt and v^2/R, respectively, where $v = \sqrt{x'^2 + y'^2}$ = the path velocity, and R = the radius of curvature.

It should be carefully noticed that dv/dt is not the whole acceleration, but only that component of the acceleration which lies along the tangent.

The importance of this application in problems in mechanics is obvious.

Note.—As explained in the preface of this report, these pages are intended merely to give a résumé of the working principles of the calculus with which the student should be perfectly familiar after having taken a course in this subject. *The main part of the work of such a course should be problems done by the students—each problem being solved on the basis of the small number of fundamental theorems here mentioned.*

Discussion.

Professor Chas. O. Gunther: It seems to me that in this report some mention should be made of imaginary and complex quantities. A little knowledge of these quantities can, for instance, be utilized to good advantage by applying it to that part of the calculus known as integration. In fact, integration can be simplified to the extent of eliminating the usual "reduction formulæ" and rendering the use of tables of integrals unnecessary.

As found in text-books in general, there are three cases for which the expression

$$dy = \cos^h \theta \sin^k \theta d\theta \tag{1}$$

can be easily integrated. Two of these cases include fractional values for h and k. All other cases in which h and k are integers can either directly, or by means of a single imaginary trigonometric substitution ($\tan\theta = i \sin\alpha$, in which α is an imaginary quantity), be reduced to one or more of the three cases just referred to.

The general binomial differential expression

$$dy = x^m (a + bx^n)^{p/q} dx \tag{2}$$

is only another form of (1) since $\sqrt{a + bx^n}$ can always be represented by one of the three sides of a right triangle and therefore expressed as a trigonometric function of one of the acute angles of the triangle.

To make this transformation the student must know the relation between the hypotenuse and the two sides of a right triangle, the values of the trigonometric functions of an angle in terms of the sides of a right triangle, and the rules for differentiation.

Differential expressions involving trinomial surds may be rationalized in a similar manner.

The expressions

$$\frac{d^n y}{dx^n} = e^{ax} \cos bx, \tag{3}$$

$$\frac{d^n y}{dx^n} = e^{ax} \sin bx, \tag{4}$$

may be integrated with great facility if complex quantities are employed, because $e^{ax}\cos bx$ and $e^{ax}\sin bx$ are the rectangular components of a vector whose modulus is e^{ax} and whose argument is bx. The integrals of (3) and (4) are found from the integral of

$$\frac{d^n z}{dx^n} = e^{(a+ib)x}, \tag{5}$$

in which z is a complex variable of the form $y + iy$. The integral of (5) is readily found to be

$$z = \frac{e^{(a+ib)x}}{(a+ib)^n} + \dot{C}_{n-1}x^{n-1} + \cdots + \dot{C}_1 x + \dot{C}_0, \tag{6}$$

in which $\dot{C}_{n-1}, \cdots \dot{C}_1, \dot{C}_0$, are constants of the form $\dot{C} = C + iC$. Equation (6) may be written

$$z = \frac{e^{ax}}{(a^2+b^2)^{n/2}} e^{i(bx - n\tan^{-1} b/a)} + \dot{C}_{n-1}x^{n-1} + \cdots + \dot{C}_1 x + \dot{C}_0. \tag{7}$$

The integral of (3) is the real part of (7) and the integral of (4) is the imaginary part of (7) divided by i.

Again in differential equations we find the linear equations

$$\frac{dy}{dx} + ay = b\cos nx, \tag{8}$$

$$\frac{dy}{dx} + ay = b\sin nx, \tag{9}$$

and their solutions can be obtained from the solution of the equation

$$\frac{dz}{dx} + az = be^{inx}, \tag{10}$$

in which $z = y + iy$.

The foregoing illustrates a few of the applications of complex and imaginary quantities, and includes a first treatment of hyperbolic functions as trigonometric functions of imaginary quantities.

Some little consideration should also be given to the complex and imaginary branches of certain curves, as for example, the circle, the ellipse, and the hyperbola. It should be noted that the equation of the circle $x^2 + y^2 = a^2$ is also the equation of an imaginary hyperbola for values of $x > a$ and $< -a$. This is important, since of the three forms of binomial surds $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$, the first is obtained from the equation of the circle $x^2 + y^2 = a^2$, and the latter two from the equation of the hyperbola $x^2 - y^2 = a^2$; but all three are obtained from the equation of the circle if imaginary quantities are made use of.

Professor J. E. Boyd: I want to emphasize everything Professor Gunther just said about the use of complex quantities. We cannot derive a formula for an eccentrically loaded long column without the use of them; we cannot make alternating current calculations without them. A student might as well learn how to use them. I endorse what he says about the use of integral tables in teaching calculus. Our professors in calculus last year adopted a book that advised the use of tables. This year a book of the other type was selected. We did not use the tables any more than was absolutely necessary and found the result satisfactory. The student does not need tables often, except to make use of the several transformations.

Professor F. L. Emory: The average student is vastly lacking in a knowledge of the use of logarithms. He also lacks the ability to read trigonometric formulæ from the triangle.

The tendency of the report is to include more material than can be covered in an engineering course. I would be satisfied to have a little more training in a few principles which students must know so well that they have confidence in their knowledge. One of the most serious difficulties that I encounter is with the constant of integration. This is largely the fault of the text-books. I have a grievance against the text-book writer who omits the constant in all cases, supplemented by the remark that it should always be added. We cannot expect the student to remember a footnote to be applied with each operation.

Professor J. B. Webb: I am pleased to hear what Professor Emory said about the constant of integration and his explanation of the difficulty, but I think the trouble is more in the teaching than in the text-books. In reading Mr. F. W. Taylor's book on his system, I was interested in one of the illustrations which he uses. He takes the case of loading cars with pig iron, where, by the application of his system he about tripled the amount that a man could do in a day, and at the same time enabled the man to earn more money. One of the first things he did was to examine the men that were in the gang, and he found that but one man in eight was suitable for this work. He used only those who were fitted for it. We have about the same proportion, perhaps, of the unfit in our classes, and the ones fitted for engineering could do three times the work and do it better if our classes were conducted on the Taylor system.

I have had some interesting experiences with the complex variable. Having studied the subject in Germany in 1878–1880, on my return to this country I tried to teach its use. Objections were made by those not acquainted with the subject, that it was too advanced and of little practical use, so that it proved to be harder to convince the average American teacher of its importance than to arouse the interest of intelligent students. Some of the professors were convinced, but that was where the trouble lay. If a student was conditioned because he did not get through with his mathematics, some said I taught "over his head" and gradually the standard would be forced down. The trouble with the present schools is that they want too many students and are going to hold all they have and get more if they can. They do not call out the seven and keep the one. After Dr. Steinmetz, a layman, produced his book on the treatment of alternating currents, using complex variables, there was less objection made to them. Now I say it is a disgrace that it should be necessary for a layman to show professional teachers that a certain part of mathematics is needed. What we should do is to eliminate the students who are not capable of profiting by what we know should be taught, and then hold the others to a high standard.

We expect too much of the student who takes calculus. Of a semester in calculus at least one-half is spent in reviewing previous mathematics. A course in calculus is an excellent review of geometry, algebra and especially of trigonometry, and at its close we should not expect the average student to know much more about it than he did about trigonometry at the start.

This committee was appointed to see what was the matter with the teaching of mathematics. They imply that good text-books are lacking. I cannot agree with this and would rather have one of the old-fashioned text-books than those outlined in their report. If they intend to give simply a list of subjects that students should be drilled in, well and good; but if the report intends to prescribe the methods of thought and of logical deduction, to be used in those subjects, then I think it is all wrong.

Professor Magruder: The introduction to the report states clearly the purpose of the syllabus.

Professor W. J. Risley: The suggestions that have been made here this afternoon are very good. I am in favor of a section on imaginary quantities. When I approached some of the Harvard professors of engineering subjects I found that they wanted their students to perform vector addition analytically. They said that the teachers of mathematics were teaching a lot of things of which little or no use was made later. To a great extent Professor Webb was right in stating that he had to teach the professors of engineering what they ought to teach, in order that they might understand some of the mathematics which he attempted to send to them. On the other hand, sometimes the professors of engineering have to teach the professors of mathematics some things that they don't know that their students ought to know. Neither set is to be criticized too severely unless they are unwilling to learn when the right way is pointed out.

Principal Arthur L. Williston: I was very much interested in what Professor Webb said a moment ago, referring to Mr. Taylor's work and his method of culling out one man of the eight who was especially adapted for a particular kind of work, using him intensively on that kind of work, and finding tasks for which the other seven are fitted. That idea is really at the bottom of all of our difficulty in this discussion of teaching mathematics to engineers, which we have had almost since we began trying to teach engineers. As there are few men of the naturally analytical kind that Professor Webb describes it does not make much difference what sort of methods we use with them. As a matter of fact a very small proportion of the men who form the body of eminent engineers have that type of mind. We all know the sort of fellow who thrives on complex quantities. And I am sure the majority of those here will bear me out in my statement that a very small proportion of the successful, eminent engineers of this country are of that kind. The ideal plan would be to separate those fellows from the mass and give them a course in real mathematics. They would like it and it would be a pleasure to the instructors to teach them. But let us take the other group. For the most part, the man who is going to be a successful engineer in industrial work is a practical, concrete man. He does not handle imaginary, complex, abstract quantities easily. And yet that is the very type of mind that the world wants in its important industrial activities. Those fellows, who, by the way, constitute the great majority, want mathematics not as an analytical light but simply as a necessary evil, if you please, as a tool that they must use. If in our talking and our thinking we could learn to talk of mathematics as two subjects, one thing for the first type, another for the second, it would simplify all our discussion. It is absolutely futile to attempt to teach the first kind of mathematics to three out of four young men who will be good engineers whether the colleges turn them out as fitted to be engineers or not. They are going to be engineers. As I understand it, the work of this committee has been to some extent a movement toward trying to get the teaching of mathematics for engineers differentiated from the teaching of pure mathematics. I am sorry that the difference is not more marked.

Professor E. R. Maurer: I prefer to hear a teacher of mathematics discuss this syllabus, because he can see it in the light of his experience in teaching the subject. To be sure, others have good ideas as to what knowledge and training engineers ought to have in mathematics, but they fail to appreciate the difficulties of teaching the subject. So, between two criticisms, one offered by teachers of mathematics and the other by teachers who have never taught mathematics, I place more confidence in the former. In estimating the value of mathematical instruction we are apt to forget that, in many schools, particularly the large ones, more or less inexperienced men are employed as instructors in the departments of mathematics. The results suffer on that account. In addition there is the poor quality of the working material. I try to be charitable when I judge the students that come to me from the department of mathematics on those two accounts. Many of the boys have had their training at the hands of inexperienced men and many have very little mathematical talent. I think the syllabus is good as a list of topics with which all engineering students ought to be familiar. I agree with Professor Webb in that we ought not to set this up as a subject matter for all teachers of mathematics to use and not depart from it in any particular. The teacher of mathematics, or of any subject in an engineering school ought to understand his subjects well enough to get up his own syllabus, if necessary.

The President: An informal committee of instructors in the University of Illinois, formed of a dozen men representing the department of mathematics, mechanics, civil engineering, electrical engineering and mechanical engineering, made a careful study of the report of the Mathematics Committee to see whether the syllabi covered the ground which these professors thought should be covered in class. In general I may say that they approve almost wholly of the contents and in general of the matters of emphasis as to what part should be well understood, what other, only partly known. With your permission I shall include this report in the discussion.

The committee of University of Illinois instructors selected to discuss the preliminary report of the Committee on the Teaching of Mathematics to Students of Engineering submit the following recommendations:

Since the syllabi are meant to embody the minimum equipment in mathematics of a good engineer, they have been discussed from that point of view. But it is the opinion of the committee that much could be gained by publishing a list of topics that should be included in the courses discussed, and emphasizing by a star those which are " so essential that every engineering student should have them so firmly fixed in his memory that he will never need to look them up in a book." The discussions of the committee were confined to the syllabi which are printed in the *Proceedings, i. e.,* Algebra, Trigonometry, Analytic Geometry, and Calculus. Section numbers refer to the sections as published in the syllabus.

Algebra.

1. Under factoring some mention should be made of the important cases of collecting coefficients, and of quadratic trinomials.

2. Important principles and rules should be given in translated word form as well as in symbolic form (as is done once on page 8 and in the differentiation rules in the calculus syllabus). Students often fail to get the full meaning of symbolic forms. The operations with fractions and the definitions and laws of exponents especially need statement in word form.

3. If algebra follows trigonometry, the three forms for imaginaries should be included.

4. The notions equality, identity and equation should be carefully differentiated.

5. The principles of equivalent equations should be included, for a student should know what operations introduce or take out roots.

6. More emphasis is needed on the "completing the square" process, for it is often needed later in integration and analytics when no solution is required.

7. Harmonic progression should be omitted.

Trigonometry.

1. The committee agrees that the syllabus is satisfactory and probably is complete enough for the average engineer. Some members expressed a desire for the memorization of more formulas as particularly useful to electric engineers.

2. Some members desired greater stress on the visible handling of formulæ. By visible is meant graphical so far as the expression of relationship and formulæ can be. For example the student should not so much remember the six fundamental definitions as formulæ as he should remember the defining triangle and its ratios. The same idea should be carried throughout.

Analytic Geometry.

1. The syllabus states in the introduction, "This syllabus is confined mainly to the conic sections; but a satisfactory course in analytic geometry should include also the study of many other curves." This committee believes that the syllabus would be improved by including the most important of these "many other curves" including the so-called engineering curves.

2. The equation of a straight line passing through two given points should be included.

3. The equation of a straight line should be written in such a form and taught in such a manner that *all* constants of the line are readily determined.

4. The method of treating the conic sections in the syllabus is commended. For obtaining a proper facility in handling the practical applications of these curves, it is desirable to study each form separately even at the expense of the additional time that is required when this method is employed. The properties of these curves as given are amply sufficient.

5. The geometrical construction of the conics should be included and given more than a mere reference.

6. In the transformation of coordinates the method rather than the equations should be remembered.

7. The subject matter in articles 46–54 is not that which a

student should remember, but belongs to that class of things which can easily be referred to when required.

8. Much greater emphasis should be placed upon work in polar coordinates.

9. It is desirable for the student to be familiar with cylindrical coordinates and the committee commends the inclusion of these coordinates in the syllabus.

10. Great stress should be laid upon representation with space coordinates. Any single equation in space coordinates represents some surface. If the equation is in three variables the surface may be any form, if in two variables the surface is a cylinder, if in one variable the surface is a plane or a system of planes parallel to one of the coordinate planes. Great emphasis should be placed upon the fact that it requires a pair of simultaneous equations to determine a line in space.

11. Article 71 should be omitted from the syllabus, though included in a course in Analytic Geometry.

12. In the first sentence of the second paragraph of the introduction the phrase "a course should consist *chiefly* of problems" should be changed to read "a large number of problems should supplement the treatment of general principles."

Calculus.

The committee reports very favorably on the syllabus for the first part of calculus. A subcommittee drew up a synopsis of a course in calculus before reading the syllabus as printed in the *Bulletin.* The two did not differ in many essential details. The main question that came up was whether a topic was included under "those facts and methods which every student should have so firmly fixed in his memory that he will never need to look them up in a book," or simply under "those topics included in an elementary course in calculus." These two classes are referred to below as first and second classes. The specific changes suggested in the syllabus are as follows:

1. Section 5. Hyperbolic functions should be included in the second of the above classes. Mnemonic rules for changing

a trigonometric formula to the corresponding formula in hyperbolic functions should be included.

2. Use arc sin x, arc cos x, etc., instead of $\sin^{-1} x$, $\cos^{-1}$, etc.

3. Section 21 (Formal work in integration). Tables of integrals should not be used until the student has had considerable practice in formal integration.

4. Include in section 22, integration by separation into partial fractions.

5. Much practice in differentiation and integration with respect to variables represented by symbols other than x, y, z should be given.

6. In connection with differential equations (Sections 24, 25, 26) use d^2y/dx^2 instead of dy'/dx.

7. Include linear differential equations of first order in connection with sections 25, 26.

8. Include sections 15 (Theorems on infinitesimals), 22 (Integration formulas), 35 (Theorem of Duhamel), 38 (Subtangents, subnormals, etc.), 41 (Curvature), in the second of the above classes.

9. Include angular velocity and acceleration in section 42.

10. We particularly commend sections 7, 12 (note) and 14.

Professor A. M. Buck: A good many people, and especially some who are mathematicians, forget that with the engineering student mathematics is a subject that is taken not for its own sake, but in order that problems can be solved afterwards. If we take the view-point of the students we find that they appreciate this point better than their teachers do. Students have told me that they could not get along in mathematics because they did not know what use they were going to make of it. Had it been brought to their attention that the mathematics would have some application to their engineering work they would have gone into it with good spirit and would have obtained more benefit from the work. Taking it as an abstract study they simply would not give it the necessary time. If the teacher of mathematics will look at his subject from an engineering view-point and see that those things which he teaches are to be used as tools and that the better the student has his tools in hand the better work he can do, then there will be an improvement in the teaching.

Professor H. R. Thayer: I am going to state my opinion from the view-point of the engineer. I have spent more time outside in practice than I have in teaching. All of the latter has been along the line of structural design, where I have been using the work of the mathematical department. In the first place, in my experience as a student, mathematics came fairly easy to me. I found that when an examination was imminent, I could cram up for it the night before and forget it afterwards. That is about what nineteen out of twenty students will do. Complicated notation tends to discourage the student from getting what is extremely important to get, namely, fundamental principles. I find that students know their mathematics fairly well but they don't know how to apply it. This, it seems to me, is far more important for them to learn than such extremely complicated mathematical problems as are often given them. In actual engineering experience the applications of any but these fundamental formulæ are few and far apart. In the very infrequent cases where the more complicated formulæ are used it is only necessary to refer to tables in the text-books, as the majority of successful engineers do today. In my opinion, the ideal engineer need not have an extremely mathematical training. In running a railroad, it is far less important to get the line exactly curved and mathematically accurate, than it is to run it where it will cut least into expenses, which is the main point involved. Imaginary quantities do not teach this. The student must be taught to use efficiency engineering in handling his mathematics. If this can be taught well, we shall have better engineering students than if we attempt to teach them to handle their problems by imaginary complex quantities.

Professor G. H. Morse: A previous speaker has referred to alternating currents and to lack of familiarity with the mathematics needed for this subject. I wish to emphasize the absolute necessity for a certain amount of study of complex quantities in this connection. I recently made a tour of a number of western institutions—Illinois, Purdue, Armour Institute, and Wisconsin—with the object of discovering how the professors were teaching electrical engineering. At Illinois I found Professor Berg, who spent a great many years at the General Electric works developing their many products. I learned that he has given up entirely all methods of teaching alternating currents except that involving the use of complex quantities illustrated by graphics, of course. He insists upon this method, both for himself and his assistants. The so-called trigonometrical methods have no standing with him whatever. At Purdue I found Professor Harding, and his attitude, while not as radical as that of Professor Berg, was very similar. In my own case I find that the use of complex quantities in teaching alternating currents is wonderfully elucidating in certain parts of the subject.

Some years ago I had the notion that there was mathematics for engineers to use, the kind that is a necessary evil, and that there was mathematics for mathematicians, in which they had great pleasure in soaring, and which they jealously guarded from use, preferring not to have any practical applications made of it. Since I have been associated with the mathematicians at the University of Nebraska my ideas have entirely changed. I now find that every stage of these flights in pure mathematics is a "short cut." The higher the flight the shorter and more useful the cut. If only the engineers can appreciate these flights their work will be greatly simplified.

Professor S. B. Charters, Jr.: I wish to emphasize the fact that we are dealing in engineering with two totally different classes of students. In every group, in the proportion of about one to fifteen or twenty, there is one engineer. Such a man should have and will take and enjoy the fullest mathematical training. On the other hand, the comparatively larger number are not engineers at all. They are simply men who are getting a certain amount of engineering training; and these men fill the bulk of the positions. From the colleges of the west a great many must go out into practical work as mining superintendents, superintendents of construction in the installation of plants, etc. Now that class of work absorbs the bulk of our men, and these have no use for higher mathematics whatever. It might be a help to them and it might not. Among those graduates whom I have observed, the ones who have had the best success have not been great mathematicians. The highest paid man we have among our alumni today, is one who could not pass any mathematical examination, I am reasonably certain. We have a few men who graduate every year who should be given higher mathematics. We have a feeling that, if it were possible, engineering should be divided into two courses; the longer course of, say, five or six years, with adequate mathematical training, for the man who shows special aptitude on those lines. Those men should be the leaders in the designing branch of the profession. A second class of men need not have the higher mathematics, but should have the proper training in handling men. These must do the bulk of the work. We need a certain number of men to do the designing and hand down formulæ which these other men can follow. We need more men to take those mathematical formulæ and from them get the results. That was illustrated to me by a friend who stated that in the American Bell Telephone Company there is one man who does the principal mathematical work for the system. In each division they have mathematicians to interpret this work to the rank and file. Probably twenty-five or thirty experts do the mathematical work for this large company and the rest of it is done by the engineers who need have only the ordinary mathematical training.

Professor H. S. Jacoby: Allow me to call attention to the fact that this report deals with minimum requirements, and that we should express our appreciation of the splendid work done by the committee. The report may not be perfect in every part, but it will be worth a great deal to have it adopted, printed and made available to the teachers whose work is affected by it. It may be made a starting point for definite recommendations; changes may be made later as the necessity for them appears. If in any institution the mathematical courses are of such a character as to require enlargements to conform to the recommendations, it is very likely that they will be modified in time. The report ought not to be a hindrance to any teacher of mathematics, or to any course of study which is now more extensive in its scope.

Professor Webb: It occurs to me that there is something else that can be said about the cause of the trouble between engineers and mathematicians. An engineer very often has a problem that he does not see through. He has a general idea that mathematics is a powerful instrument, which needs a mathematician to solve the problem; and he thinks that if he knew a little more mathematics he could solve it himself. As a matter of fact, the problem may be very simple as to its mathematics, and it may be only that he does not see through its practical or engineering side. A school teacher came to me with a problem a few days ago and said she had given it to different people to solve, and some advocated one solution and some another. One said that its solution needed calculus; I thought it could be solved quite simply, but she thought not. This was a problem of the so-called practical variety. A barn forty feet square has a horse tethered to one corner of it by a rope one hundred feet long. How much grass can the horse graze over without going over the same grass twice? The solution of this is very simple, but one should not expect mathematics to solve it before the problem has been thoroughly analyzed. Problems of this nature are constantly met with in engineering work. Very little mathematics may be needed after they are properly analyzed, but if this calls for more common engineering sense and ingenuity than the engineer has, one must not expect the average mathematician, much less the recruit graduate, to make good the deficiency.

SYLLABUS ON COMPLEX QUANTITIES.*

BY CHAS. O. GUNTHER,

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1. Derivation of formulæ:

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad e^{-i\theta} = \cos\theta - i\sin\theta,$$
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad i\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}.$$

2. Definition and graphical representation of a complex quantity. Polar trigonometric and polar exponential equivalents of $z = x + iy$, that is,

$$z = \rho(\cos\theta + i\sin\theta), \quad \text{polar trigonometric;}$$
$$z = \rho e^{i\theta}, \quad \text{polar exponential;}$$

in which $\rho = \sqrt{x^2 + y^2}$ is the *modulus* (the positive sign being always associated with it); and θ, given by the relation $\tan\theta = y/x$, is the *argument* of z. Any multiple of 2π may be added to the argument without altering the complex quantity.

3. Graphical addition, subtraction, multiplication and division of complex quantities. Graphical solution of the equation $x^n \pm 1 = 0$. Logarithms of complex quantities.

Applications to Integration.

4. The expression

$$dy = \tan^{2p}\theta \sec^{2r+1}\theta d\theta,$$

in which p and r are positive integers or zero, is by the substi-

* This syllabus was prepared as an appendix to the report of the Committee on the Teaching of Mathematics to Engineering Students at the request of the members of the Society present at the Pittsburgh meeting.

tution $\tan\theta = i \sin\alpha$ (α being an imaginary quantity) transformed into

$$dy = i(-1)^p \sin^{2p}\alpha \cos^{2r}\alpha d\alpha.$$

This latter expression can be integrated by doubling α as many times as necessary.

The foregoing includes the integration of the expression

$$dy = \cot^{2p}\theta \csc^{2r+1}\theta d\theta,$$

since the latter expression may be written

$$dy = -\tan^{2p}\left(\frac{\pi}{2} - \theta\right) \sec^{2r+1}\left(\frac{\pi}{2} - \theta\right) d\left(\frac{\pi}{2} - \theta\right).$$

As found in text-books, the integration of the expression

$$dy = \cos^h\theta \sin^k\theta d\theta \tag{1}$$

is readily accomplished in three cases, namely:

(a) When either h or k is an odd positive integer.

(b) When $h + k$ is an even negative integer.

(c) When both h and k are even positive integers, or zero.

The first two of these cases include *fractional* values for h and k.

By means of the substitution given above, all the other cases in which h and k are integers can be brought under one or more of the three cases just mentioned.

In the above are also included all the cases for which the general binomial differential expression

$$dy = x^m(a + bx^n)^{p/q}dx$$

can be integrated without resorting to infinite series. This expression is only another form of (1), since $\sqrt{a + bx^n}$ can always be represented by one of the three sides of a right triangle and therefore expressed as a trigonometric function of one of the acute angles of the triangle.

In determining the value of a definite integral, if the variable is changed the limits should be changed to correspond. For example, in finding the length of the arc of the parabola,

$y^2 = 4ax$, from the vertex to the point $(a, 2a)$, we have

$$S = \frac{1}{2a}\int_0^{2a} \sqrt{4a^2 + y^2}\, dy$$

$$= 2a \int_{\tan\theta=0}^{\tan\theta=1} \sec^3\theta\, d\theta$$

$$= 2ai \int_{i\sin\alpha=0}^{i\sin\alpha=1} \cos^2\alpha\, d\alpha = ai \int_{i\sin\alpha=0}^{i\sin\alpha=1} (1 + \cos 2\alpha) d\alpha$$

$$= a\log_e(\cos\alpha + i\sin\alpha) + ai\sin\alpha\cos\alpha + C\Big]_{i\sin\alpha=0}^{i\sin\alpha=1}$$

$$= a\left[\log_e(\sqrt{2} + 1) + \sqrt{2}\right].$$

Further applications of complex quantities to integration will be found in the author's discussion on p. 119.