SYMPOSIUM ON MATHEMATICS FOR ENGINEERING STUDENTS

BEING THE

PROCEEDINGS OF THE JOINT SESSIONS

OF THE

CHICAGO SECTION OF THE AMERICAN MATHEMATICAL SOCIETY

AND

SECTION A, MATHEMATICS, AND SECTION D, MECHANICAL SCIENCE AND ENGINEERING

OF THE

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE

HELD AT

THE UNIVERSITY OF CHICAGO

DECEMBER 30 AND 31, 1907

This symposium was arranged for by a committee of the Chicago Section of the American Mathematical Society appointed at its meeting of December 28, 1906, and consisting of E. B. VanVleck, University of Wisconsin, *Chairman;* H. E. Slaught, University of Chicago, *Secretary;* E. J. Townsend, University of Illinois; Alexander Ziwet, University of Michigan; E. B. Skinner, University of Wisconsin; A. G. Hall, Miami University; H. L. Rietz, University of Illinois; together with the co-operation of Wm. T. Magruder, Ohio State University, Secretary of Section D, Mechanical Science and Engineering, of the American Association for the Advancement of Science.

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THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE

SECTION D—MECHANICAL SCIENCE AND ENGINEERING

ENGINEERING-MATHEMATICS SYMPOSIUM

THE meeting of the section for organization was held in Cobb Hall of the University of Chicago on December 30 and 31, 1907, and January 1, 1908. The vice-president of the section, Olin H. Landreth, professor of civil engineering, Union University, acted as chairman of the section.

At the meeting of the general committee on January 2, 1908, on the recommendation of the sectional committee, Dr. George F. Swain, professor of civil engineering, Massachusetts Institute of Technology, was elected vice-president and chairman of the section for the ensuing year; and Mr. George W. Bissell, dean of engineering and professor of mechanical engineering, Michigan Agricultural College, was elected secretary for the next five years.

The promotion of acquaintance and personal knowledge was an important factor in the success of the meeting, which was in large part due to the labors and foresight of Professor H. E. Slaught, of the department of mathematics of the University of Chicago, and Secretary of the Chicago Section of the American Mathematical Society.

A subscription dinner for engineers and mathematicians and their friends brought about one hundred persons together at Hotel Del Prado on Monday evening, December 30. The speakers at the dinner were introduced by E. B. Van Vleck, professor of mathematics, University of Wisconsin, Chairman of the Chicago Section of the American Mathematical Society. They were Calvin M. Woodward, dean of the School of Engineering and Architecture, Washington University, St. Louis, Mo.; Charles F. Scott, consulting engineer of the Westinghouse Electric & Manufacturing Co., Pittsburg, Pa.; George F. Swain, professor of civil engineering, Massachusetts Institute of Technology, Boston, Mass.; and Edward V. Huntington, assistant professor of mathematics, Harvard University, Cambridge, Mass.

The first session of the engineering-mathematics symposium was held on Monday afternoon, December 30. Professor Van Vleck acted as chairman. Four papers were presented, as follows:

The Present Condition of Mathematical Instruction for Engineers in American Colleges: EDGAR J. TOWNSEND, professor of mathematics, University of Illinois.

The Teaching of Mathematics to Engineering Students in Foreign Countries: ALEXANDER ZIWET, professor of mathematics, University of Michigan.

The Teaching of Mathematics for Engineers: CHARLES F. SCOTT, consulting engineer, Westinghouse Electric and Manufacturing Co.

The Point of View in Teaching Engineering-Mathematics: ROBERT S. WOODWARD, president of the Carnegie Institution of Washington.

The two sessions, held on the morning

and afternoon of December 31, were devoted to a symposium on the question: "What is needed in the Teaching of Mathematics to Students of Engineering? (a) Range of Subjects; (b) Extent in the Various Subjects; (c) Methods of Preparation; (d) Chief Aims." The speakers represented three phases of the subject, namely: (a) From the standpoint of the practising engineer; (b) from the standpoint of the professor of engineering; (c) from the standpoint of the professor of mathematics in the engineering college.

Professor Landreth and Professor Slaught were the chairmen of the two sessions. The speakers were as follows: Ralph Modjeski, consulting engineer, Chicago, Ill.; J. A. L. Waddell, consulting bridge engineer, Kansas City, Mo.; Gardner S. Williams, professor of civil, hydraulic, and sanitary engineering, University of Michigan; Arthur N. Talbot, professor of municipal and sanitary engineering, University of Illinois; George F. Swain, professor of civil engineering, Massachusetts Institute of Technology; Charles S. Slichter, consulting engineer, U. S. Reclamation Service, and professor of applied mathematics, University of Wisconsin; Frederick S. Woods, professor of mathematics, Massachusetts Institute of Technology; and Fred W. McNair, president of the Michigan College of Mines.

Following the presentation of the four formal papers, and of the eight prepared discussions above recorded, a general discussion was held on the entire subject. The following persons took part in this general discussion: Calvin M. Woodward, professor of mathematics and applied mechanics, Washington University; Benjamin F. Groat, professor of mechanics and mathematics, School of Mines, University of Minnesota; Charles S. Howe, president, Case School of Applied Science; Clarence A. Waldo, professor of mathematics, Purdue University; Clarke B. Williams, professor of mathematics, Kalamazoo College; J. Burkitt Webb, late professor of mathematics and mechanics, Stevens Institute; Henry T. Eddy, professor of mathematics and mechanics, College of Engineering, University of Minnesota; Arthur E. Haynes, professor of engineering-mathematics, University of Minnesota; Arthur S. Hathaway, professor of mathematics, Rose Polytechnic Institute; Edward V. Huntington, assistant professor of mathematics, Harvard University; and Donald F. Campbell, professor of mathematics, Armour Institute of Technology.

On motion of Professor Campbell, the chairman was authorized to appoint a committee of three persons, they to increase their number to fifteen, to be chosen from among teachers of mathematics and engineering and from the practising engineers of the country; and this committee of fifteen was authorized by the meeting to take into consideration the whole subject of the mathematical curriculum in the engineering and technical departments of colleges and universities, and to report to the Chicago Section of the American Mathematical Society. On motion of Wm. T. Magruder, ex-secretary of the Society for the Promotion of Engineering Education and professor of mechanical engineering, Ohio State University, the motion was amended that the committee of fifteen shall submit its report to the Society for the Promotion of Engineering Education at its annual meeting in the summer of 1909. The motion as amended was unanimously adopted by those present. It is hoped that at the meeting of the society in 1909, a second engineering-mathematics symposium may be held.

The selection of this important committee was entrusted to Professor Edward V. Huntington, Harvard University, Professor Gardner S. Williams, University of Michigan, and Professor Edgar J. Townsend, University of Illinois. They will select the remaining members of the committee, choose a chairman and secretary, and determine the scope of the investigation that they will make.

The papers will be printed in SCIENCE in the next few weeks. They will prove to be interesting reading to those engaged in either mathematical or engineering work and will show the tendencies of the thought of the meeting. The key-note of all the discussions was that we need more sympathy and knowledge of the ideals, aims and work of the other fellow.

The meeting was without doubt the best attended that the sections have held for many years, the interest never seemed to flag and, while no wonderful contributions were made to scientific knowledge, every one went away feeling either that he had gained much information as to the other man's point of view concerning scientifically instructing engineering students in mathematics and of the wishes and needs of the engineering instructor, or that he appreciated more the quality of work that was now being done by teachers of mathematics in engineering colleges.

WM. T. MAGRUDER,
Secretary, Section D

PRESENT CONDITION OF MATHEMATICAL INSTRUCTION FOR ENGINEERS IN AMERICAN COLLEGES[1]

OUR country has witnessed in recent years a most marvelous industrial expansion and development. Along with this movement has come a rapidly increasing demand for trained men, equipped with all that science can contribute, to direct and carry forward this development of our natural resources and our industrial power. In meeting this demand our technical schools have experienced a remarkable growth, and not a little of the educational thought and activity of the country is being directed toward the problems connected with technical instruction. Well-equipped engineering schools have grown up in the larger centers of population and most of the larger state universities now include strong engineering departments. Mathematics is so fundamental to all of this work, and so large a proportion of the students now receiving mathematical instruction in this country anticipate making use of it later in connection with engineering work, that it has been thought best by the Chicago Section of the American Mathematical Society to invite to a joint discussion of the "Mathematical Training of Engineering Students," representatives from some of the leading engineering schools and some of those consulting engineers whose wide experience has brought them into contact with demands of actual practise.

That we may all know what the actual conditions are with respect to this instruction and consequently have some common basis for our discussion and our conclusions, I have been asked to present a statement of the work in mathematics which is now being given to engineering students.

As the basis of our consideration, I have selected seventeen institutions where engineering work is an important feature. Of these, eight give their attention largely or exclusively to technical work, and the remaining institutions have strong engineering departments; so that the mathematical work given in these institutions may be said to fairly represent the preparation in this subject for engineering students in American institutions.

The three most important factors enter-

[1] Opening address before the joint meeting of Sections A and D of the American Association for the Advancement of Science with the Chicago Section of the American Mathematical Society for the discussion of the topic "Mathematical Training for Engineers."

TABLE I

	Armour	Case	Cornell	Illinois	Lehigh	Mass. Institute	Michigan	Minnesota	Missouri	Nebraska	Purdue	Rensselaer	Rose	Sheffield	Stevens	Wisconsin	Worcester
Algebra.........	p. L.		Coll. Alg.	q	Adv Alg.	p	q	p	q. L.	L	q	p	q	Coll. Alg.	p	p L.	p
Plane Geometry	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"
Solid Geometry	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"	"
Plane Trigonometry.....			"		"		"		e	e				"	"		
Spherical trigonometry.....			"														

p = through progressions, q = through quadratics, L = logarithms, e = elective.

ing into the consideration of our topic are: the entrance requirements, the requirements for graduation, and the qualifications of the instructional force.

As will be seen from Table I., all of these seventeen institutions require for entrance algebra through quadratics, together with plane and solid geometry. Five of the institutions require plane trigonometry, while at several others it may be counted for entrance if the student so elects. It will be observed that four institutions require elementary algebra through progressions, four require the subject of logarithms, and two, Sheffield and Cornell, require the whole of college algebra.

There is a general tendency over the country to increase rather than diminish the entrance requirements in mathematics. Several institutions have recently done so, and at a number of others there is a feeling that both trigonometry and college algebra should be required. This disposition to increase the entrance requirements has come about not so much because of a feeling that these subjects can be as well or better taught in the secondary schools, but because of a feeling on the part of the technical schools that the entrance requirements should be made as high as possible in order to give room in the curriculum for those professional and technical branches which are now deemed essential. It may well be questioned whether we are not in some danger of going too far in increasing the requirements. I am sure that we should all agree that the guiding principles should be the limitations of the secondary school program and the ability of the pupil at that stage of his maturity to readily grasp in a comprehensive manner the subjects presented. For example, the advisability of adding college algebra to the entrance requirements is certainly open to the objection that portions of it are clearly beyond the maturity of the average high school pupil, and the introduction of plane trigonometry would seem inadvisable in the average high school on the accredited list of the state universities of the Mississippi Valley. When either of the fundamental principles mentioned is violated, we shall have coming to our freshman class, students with a decided and a justifiable dislike for anything mathematical. Rather than to encounter this danger, it would be far better to extend the engineering course over five years or to require a year of college work in science and mathematics before the student enters upon his technical course. In this connection, it is interesting to note that the University of Minnesota has recently extended its course to five years for students in civil, mechanical and electrical engineering, distributing the required work in mathematics throughout the first four years.

The writer does not share with some the feeling that a greater uniformity in entrance requirements is either desirable or of any particular consequence. Each institution, and especially the state institutions, must take into consideration what the secondary schools contributary to it can do satisfactorily and then shape its work accordingly. The size of the city, the general interest in educational affairs, the trend which local interests give to the public-school curriculum, all tend to make it possible to accomplish in one community, or in one section of the country, what would be quite impossible in another. We must accept our students with such preparation as our normal constituency can give, stimulated, to be sure, and to a certain extent guided by the higher institution of learning, and build our technical courses upon that preparation as best we may.

More general dissatisfaction is expressed with reference to the preparation of our students in algebra than in any other subject. This comes from both eastern and western institutions as well as from those of the Mississippi Valley. At the University of Illinois last year forty per cent. of the freshman class failed to pass a quiz covering the main points of elementary algebra and that after a two weeks' review of the subject, and twenty-three per cent. of the class failed on a second examination some weeks later. Of the one hundred and ninety students who failed on the first test, seventy-four per cent. entered the university without conditions from schools where the work had been examined and approved by the high-school visitor. The poor results which we get in algebra are not due, in my estimation, exclusively to poor instruction in the subject or to the lack of attention in the high school. It is the one subject in mathematics which is begun in the high school and completed in the college course. Often the high-school algebra is completed in the sophomore year and then not taken up again until the student enters upon his technical course. All know how difficult it is to retain the details of any course of study during an interval of several years in which the subject has been but little used. That this lapse of time between the completion of the high-school work and the beginning of the college work is an important element in the case is shown by the fact that of the one hundred and ninety failures mentioned over fifty per cent. had not had algebra for at least four years, and only ten per cent. had studied the subject the year before.

A substantial gain would be made if we should urge upon the high schools the desirability of putting the last half year devoted to algebra in the senior year of the high-school curriculum and include in that work the more difficult parts of the subject as well as a general review of the parts presented earlier. This arrangement has become quite common in Illinois, and the best argument that can be presented in favor of such an arrangement is that of the one hundred and ninety cases of failure cited over sixty-three per cent. had completed the work in the sophomore year and less than eight per cent. had had any work in algebra in the senior year. Similar records have been kept at Illinois for the past seven or eight years and the data given are typical of the other years.

Unfortunately, we can have no assurance that when a student has once mastered a subject, he will forever afterwards retain it. Neither can we hope that algebra will ever be anything other than the weakest place in the preparation of our students so long as the present division of the subject so largely prevails. It is a situation which we must accept, and the only thing we can do is to make such recommendations as will tend to reduce the number of fatalities as the boy passes from his secondary school to his technical course. The technical school must expect to commence its course in college algebra by a brief review of the important points covered in the high school, by taking a back-stitch, so to speak, into the work already done. Most of the western schools admit by certificate to the freshman class, and when a pupil is once graduated from an accredited school, he has earned the right to commence upon his technical course. At the University of Illinois, the problem has been solved by saying to the freshmen in mathematics that while there is no disposition to deprive them of their entrance credit, the department of mathematics may nevertheless determine the conditions under which credit in college algebra can be secured. Accordingly, those students who fail to pass the review quiz are required to take two additional hours per week in the subject for the remainder of the semester in order to earn the same credit that is given to others at the close of the course. This has the advantage of placing all of the students practically upon the same basis, so far as attainments in algebra are concerned, when they enter upon the second semester's work.

A somewhat similar plan as that outlined here is followed also at the University of Wisconsin, and perhaps at other institutions. It will be seen from Table I. that a large number of technical schools are now requiring work in logarithms for entrance. This might very well be introduced in connection with theory of exponents and used with advantage in high-school physics. It is also gratifying to observe that the more recent texts on algebra provide work in the use of the graph and in the plotting of curves. It is very desirable that the work in elementary algebra, including the work of curve-plotting, should also include applications to some of the simpler phenomena studied in the high-school course in physics, and this again is made a feature in some of the more recent texts. Such an arrangement affords an additional reason for putting some of the work in algebra late in the high-school course in order that it may follow rather than precede the work in physics, thus making it possible to introduce a wider range of physical applications than could otherwise be done.

In Table II. is shown the number of restrictions given in each of the various mathematical subjects required of engineering students. The average number given to each subject for the seventeen institutions is approximately as follows: college algebra 50, plane trigonometry 46, analytic geometry 80, and calculus 130. In a number of the institutions named, spherical trigonometry is taught by one of the engineering departments, usually the civil-engineering department, in connection with its applications to geodesy. The number of recitations assigned to calculus usually includes also a short course in differential equations. In two cases where a course of more than usual length in the subject is given for the students of a particular engineering department, the subject has been listed separately.[2] One institution, Rose Polytechnic Institute, is unique among strictly engineering schools in offering throughout the four years of undergraduate work a rather large amount of elective mathematics, including short courses in advanced calculus, least squares,

[2] Table III. shows the number of recitations given to differential equations in each case when that subject was reported separately.

TABLE II

	Armour	Case	Cornell	Illinois	Lehigh	Mass. Institute*	Michigan	Minnesota	Missouri	Nebraska	Purdue	Rensselaer	Rose	Sheffield	Stevens	Wisconsin	Worcester
Algebra.........	65	45		55	{40 C.E.	30	36	70	36	36	90	55	72 [18]		15	{90}	64
Plane Trigonometry......		60		35		30		45	45	54	70	29	54				45
Spherical Trigonometry.....		10			{22 C.	{10 C.E.		25	10		20	11	[18]		10		3
Analytic Geometry........	55	100	60	90	80 {58 C.E.	60	108	110	90	108	72	60	54 [18]	90	69	90	64
Calculus.........	155	125	120	144	108 {96 C.E.	90	144	110	180	126	144	70	180 [72]	100	144	180 {160 C.E.	96
Least squares..		{48 C.E.											[18]				
Vector Analysis............																	{32 C.E.
Projective Geometry........													[18]				
Duaternions....													[18]				
Qifferential Equations....		{34 M.E.				{45 E.E.											

C. E. = civil engineers, M. E. = mechanical engineers, E. E. = electrical engineers, [ ] = elective.

*Massachusetts Institute of Technology now offers a course which combines the instruction in algebra, analytic geometry and calculus rather than teaching these subjects as separate branches. In tables II. and III., the distribution of time formerly given to these subjects is indicated as showing better the relative emphasis placed upon each.

projective geometry and quaternions. In all of the universities listed, and at the Massachusetts Institute of Technology the mathematical department offers a rather wide range of advanced subjects, all of which are open to engineering students so far as the demands of their technical course will permit.

By a study of Table II., it will be seen that a considerable difference exists in the amount of attention given to the various subjects. In making comparisons in algebra and trigonometry, however, the difference in entrance conditions must be taken into consideration. The amount of work given in algebra ranges from fifteen recitations at Stevens Institute, with an entrance requirement of elementary algebra through progressions, to ninety recitations at Purdue with a requirement of elementary algebra through quadratics for entrance. Likewise the work in plane trigonometry ranges from thirty recitations at the Massachusetts Institute to seventy at Purdue. Analytic geometry and calculus are naturally the most important subjects for engineers in the mathematical curriculum. One would naturally expect to find a greater uniformity here. This, however, is not the case. In analytic geometry, it will be noticed that Armour Institute requires but fifty-five recitations, while the University of Minnesota gives one hundred and ten recitations to the subject. Again in calculus the work varies from seventy recitations at Rensselaer to a maximum of one hundred and eighty at Missouri, Wisconsin and Rose. A word should be said, perhaps, concerning the number of recitations recorded in the case of Rensselaer. The department of mathematics of that institution reports that the recitations are from an hour and a quarter to an hour and a half in length and that the efficiency of the work is still further increased by the fact that but two academic studies are carried simultaneously.

It will be of interest also to compare the total amount of time spent upon mathematics at these various institutions. As will be seen from Table III., this ranges from one hundred and eighty recitations at Cornell to three hundred and ninety-six at Purdue. In making this comparison, we should again take into consideration the difference in entrance requirements. When this is done, the difference is more apparent than real. For example, if we add to the number of recitations given at Cornell the number of recitations given at Purdue to college algebra and trigonometry, which are required for entrance at Cornell as compared with three hundred and ninety-six at Purdue.

It would seem that the technical schools generally might well afford to make more ample provision for elective mathematics. Such courses as spherical trigonometry, least squares, differential equations, might well be placed in such a list. In this way certain subjects which are desirable for some branches of engineering, but not so essential for others, could be taken by those students interested. Sheffield offers as an elective another course which might be given with advantage at other technical schools, namely, a course in scientific computation in which the use of modern calculating machines of various kinds is explained and made use of. It would also be well if the stronger institutions could go still farther and introduce elective courses in spherical harmonics, vector analysis, theory of functions and the mathematical theory of heat, electricity, etc., to the end that the student with exceptional mathematical ability might lay a broader foundation for the theoretical side of engineering. In this connection, it may well be questioned whether the technical schools of this country are in general offering sufficient opportunity for that training which has made it possible for such men as Steinmetz, Osborne Reynolds and Stodola to accomplish the work which has made them famous.

Table III. shows also the sequence and the distribution by years of the required work in mathematics. We are quite as much interested, however, in the character as in the amount and distribution of the mathematical instruction given to engineering students. The close observer will have noticed the change which has been made and is now being made in this respect. In recent years there has swept over the country a wave of enthusiastic discussion concerning a closer and better correlation of mathematics with the physical sciences. This has been due for the most part to the influence felt in this country of the Perry movement in England. Much is to be learned from this movement, and still more is to be avoided. The discussions which have arisen from it have on the whole had a beneficial effect upon the teaching of mathematics both in America and in England.

It has first of all led to the introduction into our text-books, and still more generally into our teaching, of a very much better selection of problems—problems which widen the student's fund of information of physical phenomena and apply the mathematical principles which he is acquiring more extensively than was formerly the case to the physical laws with which he is familiar. Such problems as the following, taken from a recent number of an educational journal purporting to serve the interests of mathematical teachers in the secondary schools, is no longer

TABLE III

Institution	Freshman	Sophomore	Junior	Senior	Total
Armour	Al 65; An 55; C 50	C 85; Diff. eqs. 20			275
Case	Al 45; Tr 70; An 55	An 45; C 125	Diff. eqs. 34 (E.E.)	Least squares 48 (C.E.)	M.E.; 340 E.E.; 374 C.E.; 388
Cornell	An 60; C 120				180
Illinois	Al 55; Tr 35; An 90	C 144			324
Lehigh	An 80; C 108 Al 40; Tr 22; M 25	(C.E.) An 58; C 96			E.E.; M.E.; 188 C.E.; 241
Mass. Inst.	Al 30; Tr 30; An 60	C 90; Sph Tr 10 (C.E.)	Diff. eqs. 45 (E.E.)		M.E.; 210 E.E.; 255 C.E.; 220
Michigan	Al 36; An 108	C 126	Diff. eqs. 18		288
Minnesota	Al 70; Tr 70; An 40	An 70; C 110			360
Missouri	Al 36; Tr 55; An 90	C 180			360
Nebraska	Al 36; Tr 54; An 72; C 18	An 36; C 108			324
Purdue	Al 90; Tr 90	An 72; C 72	C 72		396
Rensselaer	Al 45; Tr 40; An 39	An 21; C 70			225
Rose	Al 72; Tr 54; C 36 [S Tr 18]; [Proj. Geom. 18] [Al 18]	An 54; C 90 [Quat 18] [An 18]	An Dyn 54 (calculus) [C 72] [Least squares 18]		360 [180]
Sheffield	An 90	C 100		[Probs. and computing]	190 [Probs.]
Stevens	Log 15; Sph Tr 10; An 43; C 30	An 26; C 86	Diff. eqs. 28		238
Wisconsin	Al and Tr 90; An 90	C 160; Diff. eqs. 20 (M.E.; E.E.)			M.E.; E.E.; 360 C.E.; 340
Worcester	Al 64; Tr 48	An 64; C 96	Vect. and 32 (E.E.)		272 E.E.; 304

Al = algebra, An = analytic geometry, C = calculus, Tr = trigonometry, M = mensuration, Quat = quaternions, An Dyn = analytical dynamics, [ ] = elective.

thought to be in good form by our best instructors:

"I bought 674,867 sheep at less than $10 per head; I paid for them in ten-dollar bills and received back in change $7.39. How many bills did I give?"

Need I call attention to the absurdity of putting such problems into the hands of pupils? How many farmers in any wool-producing state of the country ever even saw that many sheep in his entire life, and, should he have occasion to buy them, would for a moment think of paying for them by counting out 663,395 ten-dollar bills. So long as such problems are given out for the consideration of pupils, just so long we may expect even the best of them to ask the question so often heard in our algebra classes: "What is all of this 'stuff' good for, anyway?"

Contrast with this problem the following, taken at random from an algebra recently published:

"Two boys, *A* and *B*, having a 30-lb. weight and a teeter board, proceed to determine their respective weights as follows: They find that they balance when *B* is 6 feet and *A* 5 feet from the fulcrum. If *B* places the 30-lb. weight on the board beside him, they balance when *B* is 4 and

A is 5 feet from the fulcrum. How heavy is each boy?"

In solving this problem the boy has learned just as much mathematics as in solving the first. In addition, his mathematics has been brought into contact with a fundamental physical law, and incidentally he is made to feel that, after all, his mathematics is of consequence to him in solving the sort of questions in which he is interested or is likely to have experience with in the future.

As has been pointed out, a change in the character of the problems is gradually taking place in our mathematical texts. Perhaps a word of caution should be given lest we go too far in the opposite direction, by introducing problems which require a technical knowledge and experience beyond the comprehension of our students. Perry's calculus is a conspicuous illustration of this danger. The subjects discussed in that book would form a good sequel to a certain work in engineering, but the book seems to be hardly suited to meet the needs of American schools as a preparation for engineering study. We should aim to make the mathematical work practical and in harmony with engineering practise, but without making it at the same time technical in its applications, or without going too far afield by teaching mathematical physics.

Another improvement which has recently become noticeable in the teaching of mathematics in this country is the breaking down of the traditional barriers between the different branches and a corresponding closer correlation of the different subjects in the mathematical curriculum. In several of our institutions the sharp division of freshman work into algebra, trigonometry, and analytic geometry is being more or less disregarded and these subjects taught as a single unit. It is thought that the student is thus enabled to grasp more readily these subjects as a whole, and that the instructor can introduce much earlier the principles of analytic geometry and of calculus and postpone to the later part of the course those topics which are relatively difficult and not so essential to the elementary work of the course.

This plan is now being followed somewhat closely at the University of Wisconsin. In the first semester fifteen or twenty recitations are devoted to the elementary portions of trigonometry. This is followed by work in algebra, including the theory of complex numbers, using trigonometry and a large amount of graphic work, and the elementary principles of analytic geometry. In this work trigonometric computation and the use of the slide rule form an important part. In the second semester the algebra and trigonometry are continued and combined with the essentials of analytic geometry.

This correlation of the work of the freshman year seems to have been most thoroughly worked out at the Massachusetts Institute of Technology, where Professors Woods and Bailey have recently prepared a text covering the work given there in the freshman year, excluding, however, trigonometry. The indications are that other institutions are also contemplating a revision and better correlation of the work of the first year.

In some of the recent books, the sharp division of the calculus into differential and integral calculus is done away with, thus making it possible to introduce the student to a wide range of easy applications at an early point in the course and to relegate to its proper place some of the more difficult parts of the differential calculus. There is a tendency also to introduce the methods of the calculus earlier and make them the basis of portions of the analytic geometry. For example, Rose Polytechnic Institute gives a short course of thirty-six recitations in the subject before analytic geometry is taken, and what is accomplished there in this formal way is undertaken at other institutions by introducing into the analytics the elementary notion of derivatives or by teaching the two subjects simultaneously.

While all are agreed that for engineers mechanics should stand in a close and vital relation to the calculus, that in fact it is the principal reason for teaching calculus, not all are agreed, however, as to the best method of accomplishing this purpose. Some would maintain that it should be taught by the mathematical department and in connection with calculus; others and perhaps the larger number feel that it should be given by the engineering departments and made to follow and supplement the calculus, giving the student his first real introduction into the applications of his mathematics to the fundamental principles underlying all engineering courses. However this may be, there is little doubt that more applications to mechanics should be introduced into the course in calculus than is now usually the case, even to the exclusion, if need be, of some of the applications to geometry frequently given. Problems in work, energy and stress form just as legitimately an integral part of a course in calculus as problems in order of contact, asymptotes or envelopes. The applications to geometry and to mechanics should be given about the same relative importance in a well-balanced course in calculus.

Descriptive geometry is another subject in the engineering course which might well be revised and made more mathematical in its treatment. It is to be regretted that the subject has in this country degenerated into little more than mechanical drawing. It would be greatly improved for engineers, as well as the general student, if we should inject into it something of the scientific spirit given it in European schools.

No presentation of the subject under discussion would be complete without some consideration of the preparation which the teacher of mathematics has, or should have, who is to teach the subject to engineering students. There is a strong feeling in some quarters that such an instructor should be a trained engineer in order that he may the better appreciate the kind of applications which are best suited to the training of an engineer and to make sure that the proper emphasis be placed upon those topics considered essential in such training. Some would go still farther and insist that even in the elementary courses in mathematics usually given in the first two years, the purpose and aim of the prospective engineer is so radically different from that of the general student that the content of the course itself should be very different from what is best suited to the student who elects mathematics as a part in a general education.

It goes without saying that we should eliminate from the courses for engineering students that which is non-essential, and we should make them as practical as we may by the generous use of those physical applications which will give the students both skill and facility in applying mathematics to such concrete cases as may arise later in his experience. On the other hand, it would be disastrous to go to the extent of teaching any of the principles of mathematics empirically or of permitting students to assume as already established formulas which he has merely to learn how to apply. We should avoid the danger of going too far in allowing the student to disregard the necessity of a formal demonstration and to regard lightly the logic and the philosophy of mathematics. What is needed first of all is the ability on the part of the student to think mathematically and to have not only a ready but an intelligent command of the fundamental principles of the subject. We should introduce the applications of mathematics not for the sole purpose of giving the student a foretaste of the things which are in store for him, but because such applications give him additional opportunity for gaining a clearer comprehension of mathematical processes and principles which might otherwise be hazy; and I wish to add that this is more essential for the sound training of the special student of mathematics with his limited opportunity for the application of his subject to physical phenomena than it is for the engineering student who in the future is to have opened up to him that wide range of applications which his technical studies provide. In other words, what is essential in the way of applications for the engineering student in the first two years of his mathematical work gives the very best training for the student who is taking mathematics as an element in a liberal education. The proper place for differentiation, so far as the content of the course is concerned, would seem to be after the completion of the course in calculus rather than before. I present this as a plea for the general student, that he should have more of the applications of his mathematics rather than that the engineering student should have less. Both should have thorough drill in the fundamental principles of the subject and in addition all of the applications of those principles which their limited experience and knowledge of physical phenomena will permit. No student, enginering or otherwise, should be led to regard his mathematical work in the same light in which a carpenter may properly regard his jack-plane, a mere tool with which to accomplish certain results; neither should the instructor teach mathematics in the spirit in which a skilled operator might regard a finely-equipped machine shop whose sole purpose is to make more machines. Both extremes are to be avoided in the early courses in mathematics. The opportunity for specialization and differentiation should come later; and any student who is not capable of grasping the fundamental principles of the mathematics usually required in an engineering course should not aspire to a bachelor's degree from a large university or technical school.

What training is essential or desirable, then, on the part of the mathematical instructor of engineering students to best accomplish the general results here set forth? There is no doubt that the ideal thing would be to take men who have completed an engineering course and later supplemented it by special work in mathematics. This, however, does not seem feasible because of the few who could be induced to take such a course of training. It would be quite impossible to induce a sufficient number of engineers to take up the teaching of mathematics to meet the demand, even if that seemed desirable. In most cases the boy enters the engineering course with the view of practising his profession when he has completed the course. As a rule he has little taste or inclination for teaching, and those few who can be induced to enter the less remunerative profession of teaching are absorbed, as indeed they should be, by the engineering departments of our technical schools. To put an engineering graduate at teaching mathematics without first having had special training in mathematics would be wholly undesirable. Such an instructor knows but little about pure mathematics beyond the elementary courses which he is presenting, and, what is even worse, often has but little interest. If he can be induced to take up in a serious way the study of mathematics, he is in a fair way of becoming a good teacher of the subject. I am thoroughly convinced that mathematics should be taught by mathematicians just as engineering should be taught by trained engineers; but the mathematical instructor who wishes to teach engineers should be familiar with the general field of applied mathematics—mechanics, strength of materials, thermodynamics, and in addition so much of the broader field of mathematical physics as possible.

While the mathematical instructor should have some knowledge of its applications, it is equally desirable that the teacher of engineering should from time to time both refresh and revise his knowledge of the fundamental things in mathematics, to the end that he may keep his methods up to date and adapt his teaching to the kind of mathematical instruction which his students have had and avoid those methods and those forms of expression which have long been out of use.

In closing, I wish to add that the rapid increase in engineering students has so greatly increased the demand for mathematical instructors having some knowledge of engineering that it would be highly desirable if more attention should be paid to the preparation of men for such positions. This can best be accomplished, perhaps, in those universities having large engineering departments by a closer correlating of the work of the mathematical department with theoretical work in engineering and mathematical physics. It is to be regretted that so little attention in this country is now being given to these two fields of mathematical activity. Institutions so situated as to undertake it should offer to its students graduate work in these lines in every respect worthy of a doctor's degree, and likewise to its instructors both opportunity and encouragement to do research work in this broad and fruitful field of human endeavor.

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THE TEACHING OF MATHEMATICS TO ENGINEERING STUDENTS IN FOREIGN COUNTRIES[1]

YOUR committee has asked me to speak of the teaching of mathematics in foreign engineering colleges. My remarks will have reference almost exclusively to the German colleges and schools, partly because I am most familiar with the conditions existing in Germany and partly on account of the rather instructive campaign for reforming the whole teaching of mathematics, recently inaugurated in Germany.

As regards other countries I will only say that the situation in England and Scotland where, during the last quarter of a century, technical education has rapidly developed on quite characteristic and individual lines, deserves careful attention. But I am not sufficiently well acquainted with the facts to discuss this educational movement. In France, it is well known that the theoretical training given to engineers is on a very high level, higher even than in Germany, I believe. Thus, the requirements for admission to the École Polytechnique, or even to the École Centrale, include in mathematics almost as much as our engineering students get in their college course. On the top of this preparation, the student receives in the École Polytechnique an excellent two years' course in higher analysis and theoretical mechanics, and then only is he allowed to enter upon his special technical work. It must also be taken into account that admission to the École Polytechnique is by competitive examinations held

[1] Read before Sections A and D, American Association for the Advancement of Science, and the Chicago Section of the American Mathematical Society, Chicago meeting, December 30, 1907.

throughout France, so that this institution, receiving as it does the pick of students from the whole country, can maintain a high level of theoretical excellency. The École des Ponts et Chaussées and the École des Mines to which the student passes from the École Polytechnique, are thus what we might call graduate schools of the highest rank.

Turning now to the German engineering colleges, a comparison with our own best engineering colleges shows apparently but little difference, both as regards requirements for admission and as to the schedule of courses offered in the schools themselves. Nevertheless, I believe that the scientific standard is decidedly higher in the German than in the American engineering college. I am not here concerned with the question whether such a high standard of theoretical knowledge is essential, or even desirable, for the engineer; I merely state the fact. Moreover, it is quite possible that ultimately the average German engineer knows no more mathematics than the average American engineer. All I wish to maintain is that, in my opinion, an able German student, in his Technische Hochschule, or engineering university, can gain a more thorough scientific equipment than an equally able American student in his alma mater.

The mathematical requirements for admission are about the same in Germany as with us: algebra, geometry, trigonometry. Not a few students now enter the German engineering college with some knowledge of analytic geometry and even of calculus, but many still come without this knowledge. The important point is that the preparatory training in mathematics (including arithmetic) is distributed systematically and continuously over a period of nine years. The same is true of other preparatory studies. It is obviously quite impossible to attain in a four-year high-school course the results attained in the nine-year course of a German Gymnasium, Realgymnasium, or Oberrealschule. This difference in preparation must always be kept in mind in making comparisons between German and American universities.

The mathematical courses offered in the German engineering colleges and required for a degree cover plane and solid analytic geometry, differential and integral calculus and differential equations—*i. e.*, about the same subjects that are required in this country. The subject of theoretical mechanics, which is treated rather differently in different schools, and even in the same school for different degrees, I shall here leave out of consideration, for the sake of simplicity. The amount of time devoted to the higher mathematics, not including mechanics, appears roughly from the following table, in which the first figure in each case gives the number of hours per week devoted to lectures, the second the

	First Semester	Second Semester	Third Semester	Fourth Semester	Total
Karlsruhe.........	6+2	6+2	3	2	17+4
Stuttgart..........	7+3	6+4	3+1	......	16+8
Munich...........	6+3	6+2	5+2	2+2	19+9
Hannover.........	8+1	6+2	......	......	14+3
Danzig............	6	5	4+1	3+1	18+2
Braunschweig ...	8+2	6+2	2	......	16+4
Zürich	8+4	8+4	4+1	.....	20+9

number of hours devoted to "exercises." These exercises are a comparatively recent innovation. In my time the student had nothing but lectures; to gain a working knowledge of the subject he had to take a text-book and work for himself. Even now, these exercises are optional; they probably exist everywhere, although the table may not show them. There are no periodic examinations such as we have at the end of each semester; but most students take at the end of their course the

Staatsexamen, or if particularly ambitious, the Diplomexamen. The lectures in mathematics are rather more advanced and more complete than those in our engineering colleges. But the requirements in the final examinations are not very high.

In addition to the more thorough preparation of the German student and to the somewhat higher standard of the lectures on pure mathematics, and largely owing to these circumstances, the treatment of applied mathematics is, I believe, on a higher level in Germany than in this country. The student is better prepared; no time is lost in "recitations," *i. e.,* in trying to find out whether the student has committed things to memory; the professor is thus enabled to treat scientific questions scientifically. Besides, on an average, the German professor of an engineering subject has himself a higher degree of scientific training and is more interested in the mathematical, and in general the scientific, aspects of his subject than his American colleague.

It is of course always hazardous and, moreover, of little use to make such general statements and comparisons; and I do not wish to attach any great importance to them. Neither the German nor the American engineering college is as good as it might be or should be; no institution ever is; an institution is good only in so far as it is continually changing, developing, rising. The above comparisons are, therefore, given merely as a basis for better understanding the efforts that are now made in Germany for the improvement of mathematical teaching in all its phases. To these efforts I wish to call your special attention.

The German movement for the reform of the teaching of mathematics is of a somewhat complex nature; at least three different movements may be distinguished. One of these, originating with the German association of engineers (Verein Deutscher Ingenieure) had as its direct object the improvement of the mathematical instruction in the engineering colleges, with a view to making the instruction less abstract and theoretical and more practically useful to the engineer. To a certain extent, this object has been attained. Practical exercises for acquiring a working knowledge of mathematics have been introduced everywhere, and the lectures on pure mathematics have become less theoretical. Some of the originators of this movement, especially Professor Riedler, of the Charlottenburg College, went so far as to demand that in engineering colleges mathematics should be taught by engineers. Whether or not this was meant as more than a threat I do not undertake to say; certainly, as far as my knowledge goes, no attempt has ever been made in a German engineering college to put the teaching of mathematics in the hands of any one but a trained mathematician. But I believe that in the selection of men for such positions more attention has been paid in recent years to the qualifications of the aspirants; mathematicians with a bent towards applied science being given the preference for positions in engineering colleges.

The second of the three movements referred to above has for its object the reform of the teaching of mathematics in the universities. It is the oldest of these movements, and has borne fruit in a variety of ways. But I can here only advert to it very briefly. The tremendous creative mathematical activity that characterized the last three quarters of the nineteenth century in Germany led to a condition in the universities that was injurious to the preparation of teachers for the secondary schools (Gymnasium, Realgymnasium, Oberrealschule). Too much stress was laid on leading the student as fast as possible to original research in some special line. The system has been described as a system, not of double entry, but of double forgetting; upon entering the university the students, most of whom are fitting for teaching in the secondary schools, are made to forget and almost despise the more elementary mathematics, and when beginning their professional teaching career they are again compelled to forget as fast as possible all the higher and highest mathematics to which they had devoted most of their time at the university. The remarkable development of mathematical activity in our country during the last fifteen or twenty years may bring about a similar situation. Fortunately, the leaders of American mathematics are well aware of the danger of losing the healthy contact with the more elementary mathematics and with applied science. Of course, it is, and always will be, the chief object of a real university to foster original research and productive scholarship. But it is well even for the most advanced specialist not to burn the bridges behind him, but to keep in mind the connection of his specialty with the foundations of knowledge, on the one hand, and with kindred branches of science on the other. As Sir Isaac Newton expressed it in his quaint way in a letter to Dr. Lord: "He that in ye mine of knowledge deepest diggeth, hath, like every other miner, ye least breathing time, and must sometimes at least come to terr, alt for air."

The desire to make the university teaching of mathematics more practically useful and bring it into live contact, as far as possible, with the whole tendency of modern scientific thought led, on the one hand, to a strengthening of all branches of applied mathematics, not only by courses offered in the universities, but also by such publications as the Encyklopädie, which includes applied mathematics in the widest application of the term; on the other, it led to reforms in the courses offered to future teachers of mathematics, and ultimately to a thorough investigation of the teaching of elementary mathematics in the secondary schools of Germany.

The improvement of the teaching of elementary mathematics is the aim of the third and most recent mathematical reform movement in Germany. The reforms proposed in this connection by the committee of the German Association of Naturforscher und Aerzte, at the Meran meeting, in 1905, appear to me to deserve very careful consideration. They would apply, in this country, to the teaching of mathematics not only in the high schools, but just as much in the engineering colleges. For, with the preparation that our students actually have, I am convinced that the best method of imparting a good working knowledge of the elements of analytic geometry and calculus is not through lectures, but through actual teaching based mainly on solving problems, that is, by the methods not of the German university, but of the German secondary school.

The proposals of the committee[2] do not change very essentially the number of hours required for mathematics. These are to be: in the Gymnasium as well as in the Realgymnasium, four hours per week in each of the nine years; in the Oberrealschule generally four hours per week, in the third and fourth years six hours. The first three years are devoted to common arithmetic and intuitional geometry, the next three years to algebra and geometry carried along together, the last three years to advanced algebra, trigonometry, advanced geometry, conic sections (treated synthetically and analytically) and, in the

[2] See *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, Vol. 36 (1905), pp. 533–580.

Oberrealschule, the elements of the calculus. Apart from matters of detail this distribution does not vary very much from the practise now followed in the best Prussian schools.

While thus the general program can not be said to constitute a radical departure from existing conditions, the statement of what should be the principal aim of mathematical teaching and the indications given for carrying out this aim throughout the whole course[3] appear to me as the most important features of the report. In addition to the well-recognized object of mathematical teaching to train the mind in rigorous logical reasoning the report insists particularly on the training of geometrical intuition and on acquiring the habit of functional thinking. The carefully prepared explanations accompanying the detailed program for the nine-year course show how these aims should guide the instruction at every step. The insistence on the idea of the functional relation can not be recommended too strongly to our writers of college text-books, from trigonometry to differential equations. But, as this report demands, it should even enter into the very elements of algebra and geometry.

It should be observed that the committee that prepared this report was not composed of mathematicians only; all branches of science taught in the secondary schools were represented in it; and all these branches received equally careful attention. While the portion of the report devoted to mathematics covers almost the whole range of the subject, from arithmetic to the elements of the calculus, required of our engineering students, there is nowhere any reference to students of engineering or to any other special class of students. I might, therefore, appear out of order in speaking of this report at the present occasion. But I wish to say most emphatically that, in my opinion, there is no special "mathematics for engineers"; nor is there any method of teaching mathematics, specially adapted to engineering students. If it is wrong to present mathematics in a form so abstract as to make it unintelligible to the student, it is just as wrong to present the results of mathematics in a form so concrete as to reduce the science to a mere art of performing certain mechanical operations, to make it, as the saying goes, a mere tool, and not a habit of thinking.

[3] *Loc. cit.*, pp. 543–545, 550–553.

In conclusion allow me to say that I should be the last to advocate a remodeling of our institutions of learning on the German plan, or the French plan, or any other existing plan. But I believe that the time has come in this country when one or two years of general college study can be demanded as preparation for the professional engineering course, at least for those more able students who wish to obtain a thoroughly scientific preparation for their professional career. An opportunity should then be offered to students of engineering of scientific ability to extend their knowledge on the theoretical side.

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THE TEACHING OF MATHEMATICS FOR ENGINEERS [1]

MATHEMATICS, from the standpoint of the engineer, is a means, and not an end. It is an instrument or tool by which he may determine the value and relations of forces and materials.

The usefulness of tools depends upon the sort of work which is to be done, upon

[1] Read before Sections A and D of the American Association for the Advancement of Science and the Chicago Section of the American Mathematical Society, at the Chicago meeting, December 30, 1907.

the kinds of tools which are available and upon the skill of the man who uses them. We may inquire, therefore, what are the uses to which the engineer may apply mathematics? What kind of mathematics does he need? And what skill should he possess in their use?

First, then, what work is to be done by the young men who are now taking engineering courses? A few—and only a few—will be original investigators or designers who will need mathematics as an instrument of research. A considerable number will regularly employ elementary mathematics in more or less routine calculations. Many will have little use for mathematics, as engineering courses are recognized as affording excellent training for various business, executive and other non-technical positions, particularly in connection with manufacturing and operating companies. It has been stated by the vice-president of a large electric manufacturing company that not over ten per cent. of the technical graduates employed by that company are fitted by temperament or by education to take up with success the work of pure engineering. A recent classification of the graduates of Sibley College, Cornell University, shows that about half are in occupations which require no advanced mathematics and it is probable that many of the 36 per cent. classed as mechanical and electrical engineers seldom go beyond the rules of arithmetic. Hence a goodly proportion of engineering graduates do not need to be mathematical experts. Their mathematical studies need not aim to produce experts, but should have as a principal object the mathematical training which is a most efficient kind of training in an engineering course. On the other hand, the engineers who will have practical use for the higher mathematics will find their ability as engineers is in a large measure determined by their ability as mathematicians.

Second, the question, what kinds of mathematics does the engineer need? is closely related to the class of work he is to do. In general a great deal of engineering work is done with much less use of higher mathematics than most professors probably imagine; and, furthermore, it may be remarked, with much less than could profitably be employed. Engineers are apt to use ordinarily the mathematical methods with which they are most familiar and which will bring the result with the least effort. One man employs calculus, another draws a diagram, another writes out formulæ, while another gets his results by mental arithmetic. The object is to get the result.

The fundamental idea that mathematics is something for the engineer to use finds many illustrative analogies in ordinary tools. Adaptation is the first requisite. Tools should be suited to the work to be done. An expensive machine tool with its refined adjustments is quite unnecessary for executing a piece of work which can be done with sufficient accuracy by a few minutes' application of a file. An ordinary calculating slide rule is infinitely better than a table of seven-piece logarithms in every-day work.

On the other hand, it is particularly wasteful to attempt to execute a difficult and intricate piece of work with inadequate tools. But more important than the tool is the skill of the man who uses it. A skillful workman can accomplish results with a few simple tools which others can not get with the most elaborate special equipment.

Third, therefore, skill in the use of mathematics is the really essential thing. A judicious use of arithmetic with a little algebra or a simple diagram often leads to more satisfactory results than others secure through elaborate processes involving lengthy equations and complicated operations. In the latter, errors are liable to occur, the common-sense import of the problem is apt to be overlooked, assumptions may be made to facilitate calculations which are physically unwarranted as one loses sight of the physical problem in the intricacy of the mathematical solution. Abstract mathematical studies, if pursued as a kind of intellectual calisthenics, may produce a pure mathematician, but they may unfit a man for practical engineering. A mathematician is not necessarily an engineer; nor is an elocutionist necessarily a good lecturer, nor is a tool expert a successful manufacturer.

Mathematics is used in engineering to express the quantitative relations of natural phenomena. The mathematician delights in the relations: he divorces them from the phenomena and gives them abstract expression, while the engineer is concerned with the natural phenomena; he demands the physical conception; the medium of expressing these relations is of secondary consequence.

The mathematician evolves the equation for a parabola and finds a convenient illustration in the law of projectiles. The engineer finds that a physical result follows from the application of certain forces, and uses the formula merely as a convenient method of expressing the law. The analogue in the case of mechanical tools is found by regarding a set of drawing instruments or a transit or a lathe, as something intelligently designed, properly proportioned, accurately made and finely finished, the merit of which lies in its own inherent excellence; or, on the other hand, by considering them as tools adapted for doing a certain range and character of work with a sufficient degree of accuracy and at low cost.

A manual-training school gives familiarity with mechanical tools and mathematical study gives familiarity with intellectual tools. In work with the manual tool the boy uses it for making something —he learns the principle on which it operates and the way to use it, by making something; if it is something useful it awakens a higher interest than does some fancy device. Likewise training of engineers in mathematics should be by doing something, by the solving of problems, by dealing with real rather than abstract conditions. Let this training be secured while applying mathematics to its normal and legitimate purpose as an auxiliary in the study of other branches.

In the teaching of mathematics for its own sake stress is apt to be laid upon the processes of deriving results rather than the real meaning of the results themselves. An engineer who uses logarithms has no more concern regarding their derivation than the ordinary user of the dictionary for finding the pronunciation of words has in their etymological derivation. The ability to reproduce demonstrations in higher mathematics from memory with the book shut is often not as important as it is to understand them with the book open. In general an engineer, who has occasion to use higher mathematics, will not be interested in evolving difficult equations, nor will he appeal to his memory, but with text-book or reference before him he will seek the things he wants to use. He should know where to find them and how to use them.

In emphasizing what a skilled mechanic can make with very ordinary tools, or the true engineer can accomplish with the parallelogram of forces and the rule of three, there is no intention of discrediting the value of fine equipments, either mechanical or mathematical, if there be the ability to use them.

Possibly the practical utility of mathematics may appear to be urged too strongly, particularly as the writer really believes in thorough mathematical training, but he has seen so many cases in which mathematical instruction has never been digested and assimilated, he has seen simple problems confused by unnecessary mathematical complications, he has seen men satisfied with results which are absurd because of some mathematical equations—sometimes quite unnecessary—which seem to obliterate common-sense perspective, and he recalls the new insight into mathematics which came through "Analytic Mechanics" under Professor S. W. Robinson at the Ohio State University, and "Problems in Mechanics," under Dr. Fabian Franklin at Johns Hopkins University, that he feels there is little danger in over-emphasizing the importance of concrete training in mathematical study.[2]

The practical questions which the discussion of this subject presents are these:

What mathematical subject-matter should be covered? And,

How should it be taught?

The first difficulty is that there is not, and can not be, a differentiation in technical education which is at all comparable with the wide range of occupations into which graduates will enter. We may assume, therefore, that we are considering the case of the average engineering student, taking for granted that options may be used by the best students for enabling them to take up the more advanced and difficult mathematics. Obviously the student should have enough mathematics to enable him to demonstrate the important engineering laws and formulas and to read intelligently mathematically written engineering literature. While only the relatively simple mathematics is commonly used by engineers, yet the ability to handle new problems with confidence requires a thorough understanding and appreciation of the significance of the mathematical and physical basis of the laws and phenomena he is to use. A man who is a thorough mathematician and knows how to apply his knowledge has a great advantage over the pure mathematician or the man without mathematical equipment. The better knowledge one has of the complex, the more certainty he has in applying the simple. A student should understand something of the power of the advanced mathematics and the field of its efficient application. Although he may not be expert in using it himself, he will know when to call for a mathematical expert.

An engineer of fairly wide experience remarked a short time ago: "The ordinary engineer does not use higher mathematics because he doesn't know how. He does not have the proper conception of the fundamental principles of the calculus because the subject has been taught by men whose ideals are those of pure mathematics."

If mathematics is something for engineers to use, let its use be taught to engineering students. After the fundamentals are learned, the students should attack the engineering problem at once and bring in mathematics as a means of solving it. Mathematics is often advocated for developing the reasoning powers and the ability to reason from cause to effect. There is danger, however, that mathematical machinery may make the mere process obscure the cause and the effect. Let them be foremost, with the process secondary or auxiliary to them.

The way mathematics is brought to bear on some engineering problems reminds one of the story of the old lady who greatly

[2] Both of these teachers of mathematics had been trained as engineers and had practised the profession.

admired her preacher because he could take a simple text and make it so very complicated.

Old traditions have not wholly disappeared, the fear of degrading the pure science of mathematics by applying it to useful things still lingers—in influence, if not in precept. We must go further and adapt mathematics to engineering, not only in subject matter, but in method. A mathematical teacher with no patience for anything except mathematics will probably teach a kind of mathematics which has no connection with anything except mathematics. Engineering mathematics may be better taught as a part of engineering by an engineer, than as a part of mathematics by a pure mathematician. The marker of levels and transits who is expert in the construction of the instruments and an enthusiast over the accuracy of the surfaces, the excellence of the bearings, the near approach to perfection in the graduation and the general refinement and beauty of workmanship, may make a good instructor on instruments, but a poor teacher of civil engineering.

After all, it is not so much abstract courses as it is personal men with which we have to do, it is not mere knowledge of facts or facility in mathematical manipulation, but it is training. The young man is to be developed, his native individuality is to be the basis, he is to increase not only his knowledge, but his powers and the ability to use them. It is not mathematical skill so much as a mathematical sense, or mathematical common-sense, which is wanted. With pure mathematics as a science we have no quarrel—and little affiliation.

If you ask men who use engineering graduates what qualities they should possess, you will find that special prominence is given to "common-sense" and "the ability to do things." In mathematical training it is quality rather than quantity which is of first consequence. It should develop the facility for systematic and logical reasoning, thus furnishing a general method as well as a specific means of getting results.

We are concerned with applied mathematics. The ability to state a problem; to recognize the elements which enter into it; to see the whole problem without overlooking some important factor; to use good judgment as to the reliability or accuracy of the data or measurements which are involved; and, on the other hand, the ability to interpret the result; to recognize its physical significance; to get a common-sense perspective view of its meaning and the consequences which may follow; to note the bearing of the various data upon the final result; to determine what changes in original conditions may change a bad result into one which is practical and efficient—such abilities as these are of a higher order than the ability to take a stated problem and work out the answer. It may be urged that all this is not strictly mathematics. But it is just this sort of judgment and insight which makes mathematics really useful, and without them there is danger that they may be neither safe nor sane.

The trend in education is to a closer relation to the affairs of life. Science and applied science, scientific and engineering laboratories, are overcoming old ideas and prejudices. Modern engineering development brings its transforming influence to bear upon education as well as the utilities of modern life. The engineering school has had a phenomenal growth within the lifetime of the recent graduate—a growth in ideals and methods as well as students and equipment. It has raised and agitated broad questions as to what constitutes efficient education for producing effective men. It has aimed to combine not only the abstract with the concrete, the lecture room with the laboratory, and the scientific experiment with the practical test; but it has sought by various means to bring the work of the school into close relation with active professional and commercial practise. It has a definiteness of aim and purpose which other educational courses are apt to lack. It sets out to produce men who can deal with forces and materials according to scientific principles. It develops men whose contact with physical facts and natural laws are first hand and whose ability to reason logically fit them for dealing with new problems. The training which fits men for handling engineering problems is the kind that is needed for dealing with the organization and directing of men. The sphere of the engineer is one the scope of which will continue to increase as engineering education and training produce men whose contact with natural phenomena gives them an inherent respect for facts as their premises, who are able to think straight to logical and common-sense conclusions, who have an equipment of technical knowledge and who can produce results.

In discussing the teaching of mathematics to engineers, we should emphasize not the mathematics nor the engineers, but the teaching. Aside from the imparting of knowledge and technical ability, the teaching of mathematics gives opportunity for training in the use of logical methods and in the drawing of intelligent conclusions from unorganized data which will make efficient men, whether they follow pure engineering, or semi-technical, or business pursuits. Such teaching does not come from the text-book; it must be personal—it comes from the teacher. He must be in sympathy with engineering work and have a just appreciation of its problems and its methods. He must be imbued with the spirit and the ideals of the engineer.

CHAS. F. SCOTT

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THE POINT OF VIEW IN TEACHING ENGINEERING MATHEMATICS[1]

I HARDLY know why I should have been asked to address you at this conference. Possibly, however, the fact that I am a civil engineer by profession, without having been permitted ever to practise this profession, and the additional fact that I have been a professional teacher of mathematical physics, without having been permitted to continue in this work, have led your committee to think that I might furnish a conspicuous illustration of the failures to which colleges and universities may lead in these lines of endeavor.

Having listened attentively to the three formal papers just read, I find it essential to revise my program and instead of following similar lines to those of the preceding speakers, it seems essential to take direct issue with them. This I am disposed to do, not so much because I differ wholly from the views they have set forth, as because it seems necessary to have other sides of the questions they have discussed represented. The preceding speakers appear to me to have taken themselves somewhat too seriously. This is a general fault of both theoretical and practical educationalists. My own experience leads me to conclude that in educational affairs the teacher, the school, the college and the university play a much less important rôle than we commonly suppose. In fact, I have reached the provisional conclusion that the majority of our students turn out fairly well in the world not so much by

[1] Extempore remarks before Sections A and D of the American Association for the Advancement of Science and the Chicago Section of the American Mathematical Society, at the Chicago meeting, December 30, 1907.

reason of the academic instruction they receive as in spite of it.

My impression also is that in taking ourselves too seriously as teachers of one subject or another, we have, as a rule, quite underestimated the magnitude and the difficulty of the psychological problems with which we have to deal. We have, as a rule, quite overestimated the capacity of our average student, and have thus usually expected too much from him. It is, of course, desirable to set our ideal high and try to rise to an elevated intellectual level; but in doing so we have commonly neglected the influence of heredity as well as of environment. I am inclined to think Dr. Holmes was right when he said that it is essential in the generation of a gentleman to begin four hundred years before he is born. So also is it necessary, if we wish to develop a student into a first-class scholar, to begin back some generations before we take up the formal work of training in our colleges or schools of engineering. It is an important fact, also too commonly overlooked, that the fundamental ideas involved in the mathematics and in the mathematical physics essential to the preliminary training of a prospective engineer are far more difficult of comprehension than we are wont to suppose. As a rule, I think we begin our elementary mathematics somewhat too early for the average mind. The result is that our students acquire a mere literary knowledge of the subject without grasping the basic ideas essential to clear thought and especially essential to applications. I am going to give you some illustrations of this fact. They will show how difficult it is for the average mind to attain a proper understanding of mathematico-physical concepts. The difficulties here are much the same as the difficulties of grammar. As you know, children learn to speak, and often speak very well, long before they know anything of formal grammar, and this is the natural mode of development, for the logic and subtleties of grammar can be appreciated only by rather mature minds.

But if the concepts which belong to the study of language and of grammar are rather formidable, those which belong to the higher mathematics and mathematical physics are profoundly more difficult of adequate comprehension. Let me illustrate this point by a citation from experience furnished by the case of a graduate from one of our universities who presented himself to me a few years ago, while I was dean of a graduate school of Columbia University, as a candidate for a higher degree in mathematical physics. This student had studied mechanics and had attained a degree in engineering. In order to learn something of the breadth and depth of his knowledge, I asked him what it is that makes the trolley car run after the current is cut off. He answered, "It is the force of the momentum of the power of the energy of the car." There is no reason to suppose that he had not received good mathematical and physical training, and yet it is plain from the answer he gave me that he knew next to nothing of the meaning of the terms he used. I may cite another case of a successful practising engineer, who was a pupil of no less authorities in mechanics and engineering than Lord Kelvin and Rankine. This man wrote me a letter in which he sought to convince me that Newton and his followers are all wrong with regard to the parallelogram of impulses. "Thus," he said in his letter, "if a particle starts out from a given point under the simultaneous action of two impulses, it will not move in the parallelogram of the impulses, but it will move in a tautochronous, brachistochronic, plane catenary curve of a resilient character."

These illustrations show how extremely difficult it is to master the fundamental ideas which belong to a great science; and the difficulties are so great that I am disposed to excuse, or at any rate palliate, the blunders made by our average student. He is, in fact, with all his blunders, not very far behind many of his teachers, for it is not uncommon for them to use in their lectures and text-books words not at all free from ambiguity. Witness, in fact, the loose use of such words as force, power, pressure, stress, and strain in some of the best text-books and treatises of the nineteenth century. The word "power," for example, is often used in two radically different senses in the same sentence.

These difficulties and ambiguities lead me to suggest, in opposition to the precepts laid down by a previous speaker, that we may well consider the desirability of printing mathematical books free from demonstrations but containing plain statements of facts. I have used such books myself and am disposed to think they are amongst the best books we may place in the hands of a student. The simple fact is that we do not follow a logical order of development in acquiring knowledge. We proceed rather by the method of "trial and error," and we often find out the facts with regard to an item of learning long before we become aware of the principle involved.

Hence I think the reason why few of our engineers know much about the formalities of mathematics and mathematical physics after they get through college is plain enough. They are driven over so many subjects during the four years of their college life that they have little or no time for reflection. This latter must come later in life when the mind has developed a sufficient degree of maturity to appreciate the more recondite principles which lie at the foundation of all the higher learning. This fact is well illustrated also by the case of our friends, the humanists, who have, as you know, for a long time proposed the study of geometry for "mental discipline." As a matter of fact, those who have acquired anything like a grasp of geometrical principles known that very few students of Euclidean geometry acquire anything like an adequate appreciation of the ideas involved, and it is only in the rarest instances that these students pursue the subject after leaving college.

I have not much sympathy with the engineers who would like to have their own kind of mathematics, and I am not disposed to commend very highly the works on calculus and other branches of pure mathematics designed especially for engineers. On the other hand, our modern mathematicians have generally failed to understand the needs of the engineer. Our more recent type of mathematician has devoted himself too largely to the refined questions of convergence and divergence of series and of existence theorems to properly equip him for the numerous and important applications which the ideal engineer should be able to make of his mathematical knowledge. The modern mathematician seems prone to make the engineer with some degree of mathematical talent afraid of himself. I have met some students whose early training had filled them with caution to such a degree that they would not use infinite series for fear that a divergent one might be encountered. It is known, however, as a matter of fact, that most series essential in the applications of mathematics to mathematical physics are safe in this regard, and one of the best ways for the elementary student to learn of the degree of convergence is to apply numerical computation to these series.

This leads me to say a few words concerning numerical computations, in which very few engineers and still fewer mathematicians show any degree of proficiency. It seems to me this is one of the most lamentable defects of our elementary teaching in mathematics, though here as elsewhere the intrinsic difficulties are much greater than we commonly suppose. This fact is in evidence at almost every meeting of our scientific societies, for it oftenest happens that the author of a paper involving numerical calculation will talk of the decimals involved instead of the significant figures. Thus, he will say, "this result is correct to five places of decimals," when he should say, "this result is correct to a specified number of significant figures," the latter form of expression being requisite to indicate the degree of precision attained. There is a grave defect in our elementary teaching in these matters; but it arises from the fact that almost none of our teachers of elementary mathematics are qualified to understand the refinements and the difficulties of precision in computation. Thus, it often happens that students will give results to five or seven significant figures when the data do not justify any such apparent precision.

To correct these evils we must have a convention of mathematicians, engineers and professional computers who will show authors how to produce elementary textbooks giving adequate attention to these matters.

As regards numerical computation, there is in general need of more practise, since it is through the concrete that we learn of the abstract and the fundamental. No important formula in any text-book or treatise should go without an appropriate illustrative numerical example.

I would like to take advantage of this occasion to express a hope with regard to the future of our country and to the possibility of development which may come through suitable cooperation between mathematicians and engineers. Nothing delights me more than to attend a meeting of this kind where mathematicians and engineers have come together. It is an auspicious sign of the times. It is one of the results I have been looking forward to for the past thirty or forty years. Some of us here are old enough to have lived in two epochs, namely, the pre-scientific and the present epoch. We can remember a time when engineers could not have got a hearing such as they have to-day. The history of their rise and development, at least in this country, is well known to some of us. It dates back to a time only about forty years ago. During this time the engineers have fought their way forward to the position now accorded them in contemporary society. They have won a place in public esteem without which it would have been impossible to hold such a conference as we are holding to-day. This esteem has been won in spite of much opposition, coming especially from the older academic institutions; but now having attained adequate recognition especially as practising engineers, we have a much higher duty to perform, and this I trust we shall be able to meet adequately through cooperation with our friends the pure mathematicians. I know of no work more important to the general advancement of mathematico-physical science than that which may lead to the development of mathematical physicists, men who possess at once good mathematical knowledge and correspondingly adequate equipment in physical science. Here is a field greatly in need of concentrated effort and of adequate appreciation. It is a lamentable fact that while we can easily develop pure mathematicians of a high order and experimental physicists of an equally high order, it seems very difficult for us to develop minds possessing both qualities. To a large extent I thing the development of pure mathematics in the future will depend, as in the past, on the stimulus furnished by mathematico-physical ideas; and in like manner success in the development of mathematical physics will depend equally in the future on mathematical ability of the highest order. In this line of work we Americans have not done our full duty, and it behooves us as mathematicians and engineers, now that we have got together on the plane of mutual interest, to give attention to this important field of work.

The French engineers led by Navier and followed by Lamé, Clapyron, and especially by the "dean of elasticians," Barré de Saint-Venant, have contributed to science the most important branch of mathematical physics, namely, what is commonly called the theory of elasticity. This is superbly difficult in its purely mathematical aspects and exquisitely beautiful in its physical aspects, and it stands as a splendid example of the possibilities which may result from adequate cooperation between mathematicians and engineers.

The chief difficulty in the way of developing mathematical physicists appears to lie in the inadequate appreciation of this type of work by contemporary society. Pure mathematics has a prestige of more than twenty centuries behind it, and the practical work of the engineer appeals even to the dullest of intellects; but we have failed thus far, in this country especially, to adequately esteem the worker in the intermediate field. We must look to it that more attention is given to this field in our colleges and universities. Every university should have two or three men eminent in mathematical physics as well as two or three men eminent in pure mathematics. Thus, while I would not advocate the pursuit of pure mathematics or the pursuit of practical engineering less, I would urge the pursuit of mathematical physics more. It is only by the cultivation of this branch of study and investigation that we can keep alive the sources of engineering knowledge. Important and indispensable as the practical work of the engineer is, the cultivation of investigation and discovery in his science is still more important and indispensable. Hence I would urge that when the more pressing questions of elementary instruction in mathematics and engineering have been adjusted, we give attention to the more inspiring and more important questions of the clarification and enlargement of the fundamental ideas of our sciences.

R. S. Woodward
Washington, D. C.

THE TEACHING OF MATHEMATICS TO STUDENTS OF ENGINEERING[1]

FROM THE STANDPOINT OF THE PRACTISING ENGINEER

I am honored by being asked to say a few words to you about the results of my experience as to the needs of the teaching of mathematics to students of engineering from the point of view of a practical engineer. I have had the good fortune of re-

[1] What is Needed in the Teaching of Mathematics to Students of Engineering? (*a*) Range of Subjects; (*b*) Extent in the Various Subjects; (*c*) Methods of Presentation; (*d*) Chief Aims. A series of prepared discussions following the formal presentation of the subject by Professor Edgar J. Townsend, Professor Alexander Ziwet, Mr. Charles F. Scott and President Robert S. Woodward. (See Science, July 17, 1908, pp. 69–79; July 24, 1908, pp. 109–113, and July 31, 1908, pp. 129–138.) Presented before Sections D and A of the American Association for the Advancement of Science and the Chicago Section of the American Mathematical Society, at the Chicago meeting, December 31, 1907.

ceiving quite a thorough mathematical training in the École des Ponts et Chaussèes of France, and I have also had the good fortune of developing into a fairly practical engineer; my remarks will therefore be backed by actual experience.

Mathematics is to an engineer what anatomy is to a surgeon, what chemistry is to an apothecary, what the drill is to an army officer. It is indispensable. I think we all agree on this point.

There is a considerable agitation at this time in France and Germany, especially the former, favoring the limitation of the present mathematical program of the engineering schools on the ground that it is unnecessarily extensive. From personal observation, I can say that the program there covers a considerably wider range than in the average American college. In the first place, a student entering an engineering college on the European continent must already know the analytical geometry, the descriptive geometry, the rudiments of differential and integral calculus, none of which are taught here until the student enters college. The average length of a college engineering course abroad is four years, one of the exceptions being the École Centrale, of Paris, France, where the course is only three years, but where the entering examinations are of a comparatively high standard and the students must be above the average in ability and application in order to hold their own during the college course. It is obvious, therefore, that in American colleges, time is spent on pure mathematics which could be devoted to practical study. I believe the time will come when only applied mathematics will be taught in colleges, and all necessary abstract mathematics will form a part of the conditions for entering.

As time goes on, every profession tends more and more towards specialization. This tendency is quite marked in the engineering profession. It would take too long to enumerate all of these special branches of engineering, but nearly every branch demands a somewhat different mathematical training. The time may come when this specialization will extend over the study of abstract mathematics, differing with each student according to the branch of engineering he intends to follow. For instance, a railway engineer who may aspire to become a railroad official requires less knowledge of calculus than an electrical or a bridge engineer; on the other hand, he requires a greater knowledge of geology than the electrical engineer, and a greater knowledge of common law than the bridge engineer. As my remarks are merely intended to furnish topics for discussion, I will put the following question: In view of the fact of the steadily growing scope of special education will it be desirable and possible to specialize mathematical courses in colleges and adapt them to each branch of engineering? This, as I understand, is done at present only to a small extent in applied mathematics.

Bridge engineering, of which I have made a specialty, requires probably as high a mathematical training as any other branch of the profession, and yet, I find that part of the higher mathematics which I have studied in college, apart from the drilling features of such studies, has been entirely useless; for instance, the theory of differential equations. The time I spent on it, though considerable, was not sufficient to make me understand it thoroughly, and would have been better employed in the study of the methods of least work, for instance, which no bridge engineer should neglect to study.

On perusing the elementary books used in high schools, I have been often struck with the dry, uninteresting manner in which the various subjects are being treated. The examples are mostly abstract, very few practical problems to work out. Unless the student is very intelligent, his mind retains nothing beyond a chaos of formulæ hard to remember and a few mechanical means of solving abstract problems. He is incapable of applying an equation to a practical problem. The methods of presentation should, therefore, be such that the student knows the why and wherefore of each operation—in other words, that he learns to *think mathematically.* This training in mathematical thinking should also be the chief aim: one does not know a foreign language unless one is able to think in that language; one does not know mathematics unless one is able to think mathematically. It is not necessary for that to go up into the highest mathematics, but it is necessary to be thoroughly drilled in elementary principles of each subject. These elementary principles should become a second nature to the student, just as a language becomes a second nature when it is thoroughly acquired. Problems arise every day in the practise of an engineer, which a mathematical mind can solve without going into calculations, such principles as those of maxima and minima, those of least work, of cumulative effect of forces and others are invaluable in assisting to arrive at a logical solution of many problems without the use of a scrap of paper; but in order that they may be applied, one has to be able to think mathematically. With a proper foundation, the engineer's mind becomes so trained that he applies those fundamental principles unconsciously; they direct his line of thought automatically, so to speak. How to secure such a foundation in a student must be left to those who make a life-study of teaching.

RALPH MODJESKI

CHICAGO, ILL.

The methods of teaching mathematics to engineering students in vogue twenty years or more ago, while often sufficiently strenuous, were invariably far from satisfactory, in that they failed to show the application of the subjects to engineering practise and to explain that mathematical quantities represent something real and tangible, not merely abstractions. Possibly methods have changed of late years; but nothing that the writer has seen or heard indicates to him that any fundamental improvement has been effected. Most people continue to believe that mathematical subjects are taught mainly for the purpose of training the mind, and that the manipulations involved in this branch of science are simply mental gymnastics. Moreover, even among engineers and professors, only a few recognize adequately the great importance of mathematics in engineering and that it is something real and substantial instead of fictitious and imaginary. It is true that higher powers than the third are not conceivable entities; but the mathematician recognizes them as temporary multiples for future reduction to entities.

The engineering student in his pure-mathematical classes is not taught what equations really mean, nor what are their denominations or those of their component parts. All that he learns is how to juggle with quantities in order to produce certain results. It is left to the professor of rational mechanics to teach engineering students the reality of mathematics; and too often he fails to do so, sometimes, perhaps, because his own conception thereof is rather vague.

Concerning the teaching of pure mathematics by the professor of rational mechanics the writer speaks from personal experience; for more than a quarter of a century ago he taught that branch of engineering education in one of America's leading technical schools. Notwithstanding the fact that the courses in pure mathematics then given there were rigid and even severe, the students, as a rule, had no idea of how properly to apply the knowledge they had accumulated; nor did they know what the mathematical terms employed really meant. It was necessary for the writer not only to teach his own branch, but also to supplement the students' knowledge of pure mathematics by explaining such things as limits, differential coefficients, total and partial differentials, and maxima and minima.

Throughout the entire course in rational mechanics the writer either demanded from the students or gave them demonstrations of all difficult or important formulæ; and the students in explaining their blackboard work were repeatedly asked to state the denominations, not only of the equations as a whole, but also of their factors and component parts. The answers to such questions evidenced clearly whether the student had a true conception of the mathematical work he was doing, or whether he had merely memorized certain manipulations of quantities.

It was the writer's custom also to supplement as much as possible all analytical work by graphical demonstrations; and if he were to resume the teaching of mechanics, he would adhere to this method.

In teaching technical mechanics the writer followed only to a certain extent the manner of instruction just described; for by the time his students had reached the technical studies, they were so well drilled and weeded out that constant quizzing on fundamentals was no longer necessary; nevertheless the question, "what is the denomination of that equation or of that quantity," was one that was very likely to be asked any student who gave his demonstrations haltingly or who evidenced at all a lack of conception of the principles involved.

In the writer's opinion, the manner of teaching pure mathematics to engineering students should differ materially from that usually employed in academic courses; for while in the latter case it suffices if the instructors be good mathematicians, in the former they should also be engineers, and should have taught, or at least should have studied specially, both rational and technical mechanics.

Some institutions still adhere to the antiquated custom of teaching pure mathematics by lectures. This method has always appeared to the writer to be perfectly absurd; for the primary benefit to be obtained from the study of mathematics is mental training; and the student can get this only by severe effort, and not by having another man's mind do the reasoning for him. Midnight oil and the damp towel are for most students necessary accessories to the courses in pure mathematics.

The writer believes that the only legitimate lectures in pure-mathematical courses for engineering students are as follows:

First: A short opening lecture to outline the work that is to be covered in the course and to explain how best to study the subject.

Second: Frequent informal talks to indicate the application of the mathematics studied to engineering practise, to explain clearly the meaning of all equations, factors and terms, and to show the true *raison d'être* of all that is being done.

Third: A concluding lecture in the nature of a résumé to call attention to what has been accomplished during the entire course and to the importance thereof.

Fourth: Personal and forcible lectures to lazy students so as to give them clearly to understand that they must either study harder or drop out of the class.

All mathematical work done by engineering students should be so thorough and complete that the subject shall be almost as much at command as the English language or the four simple rules of arithmetic. Only such thorough knowledge will enable the engineer to use mathematics readily as a tool, rather than as a final resource to be employed solely in extreme need.

Analytical geometry should be taught graphically as well as analytically in order that the student shall comprehend it fully and shall realize that the work is real and tangible and that the equations represent lines, surfaces, and volumes, and are not the results of mere gymnastics. A knowledge of the graphics of analytical geometry is especially valuable in mechanical work, in the investigation of earth pressures, in suspension, bridge work, and in many other lines of engineering.

The proper conception of the meaning of the calculus is rarely carried away by the student. He knows the rules and can perform the operations, but their significance is beyond him; consequently he does haltingly and bunglingly the original work which facility in the use of the calculus should enable him to perform easily and well. This state of affairs is a crying evil which should be corrected in all schools that aim to give first class engineering courses.

Descriptive geometry is of very large value in the preparation of drawings; but, in addition, a thorough knowledge of it greatly aids in the conception of an object in space, and, consequently, is of large assistance in the evolution of original designs. A knowledge of it prior to the study of the courses in pure mathematics assists materially in the conception of what the latter really mean; consequently descriptive geometry should be one of the earliest courses in an engineering curriculum.

A sound knowledge of mechanics, the foundation of engineering, is impossible without a thorough understanding of mathematics. It is true that mechanics may be learned by rote or by so-called common-sense methods; but the "rule of thumb" or "pocket-book" engineer never rises to noticeable heights. Such an engineer almost invariably fails at the critical moment, when a decision must be supported by fundamental principles. It is true that the actual use of analytical geometry, calculus, least squares, or even higher algebra and spherical trigonometry, is rare in the practise of most engineers; but an engineer's grasp of technical work depends upon his knowledge of these subjects; and it is generally conceded that a heavy structure can not be continuously supported on a weak foundation.

Mathematics higher than the calculus is of small value to the engineer, except possibly as a training for the mind; but the writer is of the opinion that any such further study of mathematics is a detriment rather than a help, in that it tends to a desire to reduce all work to mathematical calculation and thus to weaken the judgment. In other words, excess of mathematical development sometimes produces an unpractical engineer.

Most graduate engineers immediately after leaving their *alma mater* drop forever the study of mathematics, both pure and applied, except in so far as they are forced to use them by their professional work. No greater mistake than this can be made, for it takes very few years of nonuse of these subjects to cause one to forget them utterly. Every young engineer should make it a point to devote a certain portion of his time to the reviewing of the mathematical studies of his technical course so as never to become rusty in them; and the writer believes that it is the duty of every professor of mathematics and mechanics to impress this fact continually upon the minds of his students, even up to the very day of their graduation.

KANSAS CITY, MO. J. A. L. WADDELL

FROM THE STANDPOINT OF THE PROFESSOR OF ENGINEERING

When I come to think of what the Mathematical Society has brought upon itself, I fear that it may feel something like the football when it is kicked back and forth upon the field. On the one hand we have the trade-school element demanding more knowledge of rules and, on the other, the engineer demanding more knowledge of principles. No fair discussion of this subject can be had without considering for a moment the conditions and definition of engineering itself. The most common definition was promulgated more than half a century ago by Thomas Tredgold, to the effect that civil engineering, which was the only branch of engineering then known, so the definition may be considered as being general, that "civil engineering is the art of directing the great sources of power in nature to the use and convenience of man." I should say that "civil engineering to-day is the art *and science* of directing the great sources of power in nature to the use and convenience of man," and from that standpoint I am willing to discuss the question as to how much and how far mathematical instruction should enter.

If engineering is merely an art, then mathematics as a science has no place in the training of the engineer, but if engineering is a science, then mathematics has a place. Engineering stands to-day in the act of rising to the status of a science, but is still hampered by the tradesman. On the one hand, we have the demand that the student's training be such as primarily to make him useful to some one to-morrow; and, on the other side, that it make him useful to the world perhaps ten years hence. The two requirements are inconsistent and do not belong together. One is that of the trade school, and many should not go farther than that because they have not the mental capacity, and the other is the demand of the profession into which a smaller number are qualified to enter. The trade school has caused most of the trouble with the teaching of mathematics because those who are products of the trade school have no use for mathematics as a science. The complaint about the teaching of mathematics does not come from engineers; they are ready to use mathematics as a science. In civil engineering it is fortunate that the profession has developed along lines laid down by Rankine rather than by Trautwine. Both have had their use, but one of them produced the scientist and the other produced the tradesman.

It is maintained in the institution which I have the honor to represent that they who would teach engineering must practise it, and by analogy we might say that those who teach mathematics to engineers should themselves be engineers. It seems to me that a time may come when such a condition will be desirable, but let me say now that there are few engineers to-day who have had sufficient training in mathematics to teach it themselves, much less to tell mathematicians how it should be taught. We can perhaps judge of the deficiency of the student who comes to us, but my feeling is that the remedy is not a question of *what,* but of *how.* Men in my institution are sending us students well prepared in mathematics. Others do not seem to be so fortunate. Both are teaching the same subjects. We have to realize that the student himself is a factor in this question. Some students become mathematicians under any *one;* others would not under *any* one. To be taught mathematics properly, the point at which engineering minds must begin, is a long way back. I am inclined to think they must begin some generations before birth. The mathematics of grammar schools needs overhauling more than the mathematics of any other part of our educational system, and probably the mathematics of high schools stands next. The essential thing that we ask of mathematics is that it should develop the quantitative reasoning power, and the student must be able to think mathematically. If he has not acquired that, then he should drop out of engineering and take up a trade. It was mentioned by a previous speaker that a relatively small percentage of the graduates from a certain engineering school were engaged in occupations in which mathematics was of importance. From a somewhat intimate acquaintance with the graduates of that institution, I may add that a much less proportion had sufficient mathematical training to take positions in which mathematics was an important requirement. Until recently, that college has stood for hardly more than a highly developed trade school, and it is not fair to cite its statistics as showing conditions of *engineering schools*. The director of that institution stated many years ago that he did not consider descriptive geometry necessary for mechanical engineers, and his students, having had their course in machine design in the junior year were frequently found taking their only course of descriptive geometry when seniors.

The question has been raised as to the increase of mathematics for entrance to engineering schools. My view of that is that it would not be wise to raise the requirements at this time. Cornell has, it is true, increased the requirements, but at the sacrifice of both physics and chemistry, and to my mind it is best that physics and chemistry be taught at the age of high school students, rather than analytics and trigonometry. If you can not do both it is better that the young mind have impressed upon it some physical science rather than encounter the more abstract demands of mathematics. In the training of students in mathematics I would wipe out formulæ. We want principles. There is generally taught too much of the formula, as that is what the trade school has demanded. Some have objected to the statement that mathematics should be a tool. To my mind it is certainly an instrument. It is one of the things that the engineer must use, and in order that he may use it, he must be sufficiently familiar with it, so that it will respond to his use when he desires it. The question of election in mathematics has been suggested. I am certainly favorable to elections in that subject, but I question the advisability of such opportunity in any subject for the ordinary student, before the fourth year. My own observation leads me to conclude that very few students are able to elect intelligently before that time. The remarks relative to the employment of inexperienced instructors instead of competent professors show a fault to lie with the heads of the various departments themselves. If they are willing to accept, for the purpose of instructing students, the men who have been unable to find positions elsewhere, and employ only such as will work for seven to nine hundred dollars per year, the unsatisfactory results are their own fault. The responsible parties, the trustees and regents of educational institutions, will furnish what is shown to be necessary. If it is necessary that you have better men, then say so and get them, but if you are satisfied with what you now have, then you can expect to see decorative cornices and stained glass windows, rather than intellect and culture, the characteristics of our universities.

GARDNER S. WILLIAMS
UNIVERSITY OF MICHIGAN

It may save time to state briefly at the beginning my thought on what is needed in the teaching of mathematics to engineering students. It seems to me that, outside of the general cultural and developmental purpose of the study of mathematics, the instruction of engineering students may be discussed under three different phases, which for want of better terms may be named: (1) theory, (2) practise, (3) philosophy; that successful teaching of mathematics to engineering students depends upon giving the right relative proportion or emphasis to these three phases of instruction; that the content of the instruction, within the limits of present usage in engineering schools, is of minor importance; that thoroughness is essential, and that it is better to cut down the extent of the matter gone over if thereby a more thorough grasp of the subject is secured; and that the instructor must always keep in mind that he is training an average boy of average preparation with a view to using mathematical principles and methods of attack and mathematical operations and conceptions in the mastery of his engineering studies and in the treatment of the varied problems which will arise in his later engineering experience.

The great mass of our engineering students, like the great mass of our engineers, are not mathematical geniuses. In the discussion of the subject we must keep ever in mind that the average engineering student is not of strong mathematical bent. Many of those with only mediocre mathematical ability make successful engineers, and the student of strong mathematical turn may lack in some direction or may have a disproportionate measure of the importance of his analytical powers and drop behind his less mathematical classmate. I want to make a plea for the average student; the boy whose analytical powers have to be encouraged and developed. The methods of presentation must be made elastic enough to include this great class of students, or we shall fail to do our duty as teachers.

I have mentioned three phases in the presentation of mathematical subjects. These may be considered in order. It must be understood that these phases are not mutually exclusive.

1. *Theory.*—Analysis, demonstration and the general derivation and presentation of mathematical principles. The derivation and exposition of mathematical principles and operations and the appreciation of mathematical concepts are universally accepted as important elements in the education of an engineer. The use of mathematical forms of attack, the training in processes of reasoning, the formation of logical habits of thought, are hardly secondary in importance. And yet much less emphasis is placed on formal demonstration and reasoning than formerly—frequently this element is overlooked or treated in a slipshod way. The student comes to feel that he is after facts and that the derivation and proof of principles involves useless effort—he is willing to accept their authenticity. It may be that years ago our instructional methods carried formal processes to an extreme and that as a result mathematical work became meaningless lingo or memorized facts to many students. This does not furnish argument for the abandonment of training in formal reasoning. For the young mind, practise in analysis, in formal demonstration is illuminating and developing. Even the repetitive forms of analysis in the old-time mental arithmetic had great mathematical educational value. The speaker feels that in the effort to avoid barren formalism the pendulum has swung too far the other way, and that both in high school and in technical school, and in the applied engineering subjects as well, the training in analytical methods and formal processes is weak. He believes that good results would follow putting greater emphasis on this

phase of instruction than now seems to be the trend.

2. *Practise.*—The use and applicability of mathematical principles and processes in the solution of problems, drill on these principles, and the acquisition of facility in their use. To the average student the working of examples is illuminating. Without it the concept is but vaguely comprehended, the derivation only faintly understood, the process may seem merely verbal legerdemain. Properly used, this phase of mathematical instruction is of great advantage to the student of average mathematical ability. It opens up the view; it clears away uncertainties; it fixes principles and concepts; it gives life to the subject. The problems used should be within the field of the students' experience and comprehension and may well bear some relation to his future work, both in the engineering class-room and beyond. And the second part of this heading is not less important. Mathematics is a tool for the engineering student, and he must acquire facility in its use. This does not mean that the instructor should attempt to make him a finished calculator or an expert workman—time is too short—but mathematical principles and processes must be more to the student than a vague something which he recognizes when his attention is directed thereto. Instead, he must have a mastery of at least the fundamentals and he must be able to use such principles and processes in his later studies without having to divert his attention and energy too much from the engineering features involved. To acquire this facility requires drill and repetition, and this drill must constitute a part of the mathematical training of the engineering student. The multiplication table had to be learned, and many other important things have to be acquired in the same way.

But it seems that this important side of instruction may be abused. The student who thinks that to accept facts and work problems is sufficient and the instructor who thinks that illustrations and practise work alone constitute mathematical training or that mere laboratory methods suffice are greatly mistaken. The mere substitution in formulas is only rule-of-thumb work, so much decried in engineering; and the mechanic who knows how to use tools, and no more, is not an engineer. There must be a direct connection with the theory and the philosophy of the subject to make the practise side serve its proper purpose. In teaching mathematics years ago, expressions of approval came to me because I was so "practical," but the underlying purpose of the practical part was not always understood, though this lack of understanding did not affect the results of the method. Inside the "sugar coating" there should always be a principle to fix, a concept to illumine, a process to exemplify, a derivation to expound. There seems to be a tendency among some to overdo this side of the work to the detriment of the first side. While the practise feature is a valuable auxiliary in mathematical instruction, it should never be the leading motive. Student and instructor alike should recognize this.

3. *Philosophy of the Subject.*—The basis on which the science rests, the underlying meaning of the mathematical processes used, a philosophical study of the method of treatment and of the concepts used, their connection with related things. This is difficult to discuss in a general way, and of course this phase is intimately connected with the first and second. To my mind this phase should not be neglected. It must be apportioned according to the ability of the student. An understanding of the philosophy of the subject will widen his field of view and lessen the chances of error. The better grasp of the meaning

will be advantageous. Its presentation involves difficulties, and text-books generally disregard it. It must not be overemphasized, as is illustrated by the treatment in a recent text-book in applied mathematics, where it is used largely to the exclusion of analysis and demonstration.

Effective methods in mathematical subjects involve, then, the skillful selection in proper proportion from these three phases, and the best teacher will make for himself the best selection. The derivation and elucidation of mathematical principles, facility in their use and application, and an understanding of the basis on which principles and methods rest are all essential. A good text-book—one properly proportioned—aids greatly in the work of instruction. However, it is the teacher on whom reliance is placed in the end, and for the student of average mathematical ability the teacher's influence constitutes a large element. It is highly advantageous for the teacher to have a fair knowledge of the applications of mathematics which the student will make in later work and to have sympathy and interest in such work. Let us also emphasize the importance of having the best of teachers for mathematical instruction.

Let me add to this that it is my belief, growing stronger after many years of observation, that the average engineering student gets relatively little from lectures on mathematical subjects; that many instructors talk too much themselves; that the student must have the opportunity to express himself and must be required to use the mathematical language and to try his own skill, and this in other than formal quizzes; and that recitation and drill work are essential factors in giving training to this average student.

Little can be said in the time at my disposal on the ground which should be covered in mathematical instruction. Two classes of matter are studied: (1) fundamental principles forming the skeleton of the work, and (2) the more complicated topics, involving further detail and insight. There will be little difference of opinion on the first class. There will be more on the second. I have found in the teaching of mechanics and of various engineering subjects that certain topics and methods not ordinarily given in mathematical instruction may advantageously be used in the presentation of the work. The teacher of thermo-dynamics or of electro-dynamics has other topics to suggest, and still other topics will come from other sources. Not all of these may be allowed. In fact, it makes little difference what particular topics are included so long as the student has thorough training in some of the more complex work. The difficulty of giving instruction in complex work lies not so much in the time required, as in the obstacle that the concepts lie beyond the student's experience and that he is not ready to comprehend their meaning. If he had the opportunity to study these topics after he has reached the subject in which they are to be used, or if he could go back over a part of mathematics after his study has taken him into their field of application, as indeed his instructor has done for himself, the result would be more satisfactory. All these limitations must be considered in choosing the ground to be covered in mathematical instruction.

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THE TEACHING OF MATHEMATICS TO STUDENTS OF ENGINEERING[1]

FROM THE STANDPOINT OF THE PROFESSOR OF ENGINEERING

I feel that in this discussion we engineers occupy rather an unfortunate position, on account of the fact that we are

[1] Continued from the issue of August 7.

compelled to assume the position of critics. The student comes to us from the teachers of mathematics, presumably equipped with a knowledge of that subject, and it becomes our duty to teach him subjects in which he makes use of this preparation, and to find out whether he has learned to use mathematics as a tool. However, I believe that only by friendly criticism can progress be made, and that every one ought to be willing to accept such criticism when given in the proper spirit. I had much rather be criticized than criticize others, and we teachers of engineering hope that we are always ready to receive suggestions, not only from other teachers, but from practising engineers.

I must first insist that for the engineer mathematics is to be regarded as a tool—not as something which is studied simply for the development of some mental powers, but for the ability which it ought to give a man to *do* something—to use the results and methods which he has been taught in solving the problems of his profession.

There has been a good deal of discussion in the past as to the value of mathematics simply as a means of mental training, without reference to its use, and perhaps most of us remember the paper by Sir William Hamilton written seventy-six years ago, in which he maintains that there is no one of the subjects in the curriculum which develops a smaller number of mental faculties or develops them in a more imperfect and inadequate manner than mathematics. I have never seen what has seemed to me a conclusive refutation of Sir William Hamilton's main arguments, and for my part I am disposed to agree with him in general, and to assign a comparatively low value to mathematics simply as a training, aside from its applications. I have not observed that students trained in this subject are able *to reason* any better than students who have ignored mathematics; indeed, I believe that many non-mathematical subjects afford a better training in reasoning than the study of mathematics. This view may perhaps be justified by remembering that mathematics, aside from geometry, deals with questions of quantity and number, but not with questions of quality. The student puts certain fixed data into his mathematical machine and grinds out the result. He does not learn to observe and to discover the finer and more elusive, but equally important, sources of error likely to occur in the ordinary questions of daily life, because he is dealing with a rigid, unyielding, logical machine. In this way his mind may become hardened—he deals with rigid demonstrations and is unwilling or unable to appreciate or submit to a less rigid method, which is often the only possible one. The best student of mathematics is frequently one of the poorest of engineers. Give him fixed data and he will get the proper result, but he may be entirely incapable of attacking a practical problem, or of deciding what the proper data are.

I have not observed that students of mathematics are, as a rule, more *accurate* than other students, or that a training in the branches of mathematics above arithmetic leads to accuracy. Indeed, it more often appears to pervert the sense of perspective, and to lead students to work out a result to several figures in cases where a smaller number only may be significant. Mathematics does not train the *observation*, neither does it train the *imagination*, except in the geometrical branches, which are now comparatively neglected since the powerful modern methods in analysis have been introduced.

Hamilton only allowed, as I remember, that mathematics adequately trained one faculty, namely, that of *continuous attention:* but I fail to see that this is trained any better by the study of mathematics than by that of language, chemistry or by other natural sciences. Unfortunately, as at present taught it does train the memory, in a way that it ought not to do. The ordinary student of mathematics subordinates *perception* to a *memorization* of formulæ and rules.

I believe, therefore, that from the point of view of the engineer, mathematics should be taught with the object of giving the student power to use it as a tool. With reference to this I think it is fair to say that the consensus of opinion among engineering teachers and practitioners is that the results of the present mathematical training are very poor. The average student who has completed his mathematical course is frequently quite helpless when called upon to attack a concrete engineering problem, and it is a common remark by civil engineering students that they did not really learn any mathematics until they studied mechanics or the theory of structures. The results seem to be almost equally poor no matter what institution the student comes from, for in my classes there have been students from most of the principal universities and technical schools in the country and I have failed to notice any great difference in them in that respect. They very generally lack the power to *do anything* with the mathematics which they have been taught.

With reference to the reasons for this state of things, I venture to state what seem to me to be some of them, and the suggestions which have occurred to me by which possibly the results might be improved.

1. In one of the previous papers a statement was made that many students who studied advanced algebra in the technical schools had not studied algebra in the preparatory schools for the two years previous. This illustrates what I believe to be one failing in our so-called system of education, namely, the lack of continuity. The remedy is to reform and simplify the curriculum, and to unify and simplify the entrance examinations to our colleges and technical schools. So long as these entrance examinations are so extended and cover so large a range of subjects, our preparatory schools will be unable to carry out their true purpose, which is, as it seems to me, no less and no more than that of all education, namely, to train a man *thoroughly* in a few things and to give him the power to do some little thinking for himself and to take up new subjects without assistance.

2. The great inherent difficulty which teachers of mathematics as well as teachers of every other subject meet with is the attitude of the student, and his inability to realize the seriousness and the importance of his work. I am fond of expressing my view in regard to this by the statement that the school is not a restaurant, but a gymnasium; not a place where a student comes to be filled up, but a place where he finds apparatus and the instruction, by making use of which he may strengthen his mental muscles.

The manufacturer can take his raw material and shape it into the form which he desires. The raw material of the teacher is the student, but the teacher can not take this material and shape it; he can only show it how it can shape itself. I believe, however, that much may be done in impressing upon students the proper attitude which they should take toward their work, and by a proper cooperation between teachers and parents, which is unfortunately lacking as a rule in this country, and the responsibility for which must largely fall upon the parents.

3. I believe that one cause of the poor results in mathematical teaching is that too great a stress is laid upon *analysis*. Mathematics is, of course, divided into geometry

and calculus, using the words in their widest sense. Geometry is concrete; and the mind perceives the steps in a geometrical demonstration. This branch, the oldest branch of mathematics, however, has been largely supplanted by the modern analytic methods which have been developed during the past three centuries, largely to the detriment, it seems to me, of the educational results obtained. Analysis is abstract—it is a powerful machine, an invention for doing certain things. Into one end of the machine we put the data; we turn the crank, and the result comes out with absolute correctness so far as is warranted by the data. Now I believe that too much stress is laid on these analytical processes; that the student is not urged to visualize his results, to express them geometrically and to interpret his equations. I warmly second the remarks of Professor Ziwet with reference to descriptive geometry, which I believe should be treated as a branch of mathematics and taught more thoroughly, as it is taught in Germany. For my part, I derived as much benefit from my study of descriptive geometry, and afterward from the study of projective geometry, as from any other mathematical studies. These studies train the imagination, which analysis does not do. But in the use of analysis, the first step, namely, the formulation of a problem, is really concrete. This, too, is neglected in our usual courses. Our examination papers are full of questions which involve simply the analytic processes—the differentiation, the integration, the twisting and turning of equations, while much less attention is paid to the formulation in mathematical language of practical problems. Our students, therefore, when they meet a practical problem, are unable to select or judge of the correctness of the data, and even if they can do this, are unable to formulate the data as a preliminary to the solving of the problem by the use of the mathematical machine.

One of the great defects which I find in students of mathematics is one already referred to, namely, that they do not *interpret their equations*. The average student who has completed his mathematical course, for instance, has not the slightest conception of what a parabola is. I make this statement advisedly, because I have tested it again and again for years. If he could tell you what a parabola really is in his mind, he would probably tell you that it was a curve of more or less beauty represented by letters. Perhaps he could tell you what the letters are, but give him a concrete problem and he would convince you immediately that he did not *know* what the letters mean.

4. Another defect, as it seems to me, in our present methods, is the lack of training in mental operations. In the good old days *mental* arithmetic was taught, but that seems to have gone out of fashion, with so many of the other good old methods. Ask the ordinary graduate of our mathematical courses to tell you the square of 20.75 without using pencil or paper and he will look at you open-mouthed with astonishment, but if he had really grasped the meaning of the binomial theorem and had learned to do a few "sums" in his head, any grammar-school boy would, of course, be able to give the result immediately.

5. Another reason for poor results is, I believe, inadequate class-room methods, and especially the use of the lecture system. In Germany, where the students in the universities have had the advantage of a thorough preliminary training, they may be able to appreciate lectures on mathematical subjects, although I doubt even this in the case of the average student. For students in our American universities, however, I believe that lectures in mathe-

matics are almost useless, except for a very small number of students; and yet, I am told that even in some of our high schools mathematics is taught to a considerable extent by lectures. The lecture system is easy for the teacher. It involves no cross-questioning, no endeavor to discern what is going on in the student's mind, no adaptation of question with the object of putting him on the right track.

Again, some mathematical exercises are conducted by sending the students to the board, each with a problem to solve, and then marking that on the correctness of their work. Occasionally a formal explanation of his problem is required of the student. This, again, seems to me to be a mistaken method. Many a student can go through a demonstration of a principle, or solve a problem by substitution in a formula, while knowing nothing of the real meaning of the subject. In my opinion class-room instruction should be conducted by the Socratic method—by question and answer—the teacher endeavoring to put and keep the student upon the right track by showing him what he can do for himself if he will only learn how.

6. Reference has been made to the kind of teachers of mathematics. Personally I believe that in teaching the subject to engineering students the best results would be obtained if the teachers were engineers, or at least if they were near enough to being engineers to take an interest *in the concrete problems themselves* as distinct from their solution. If I am correct in the belief that mathematics should be taught as a tool, then it can be taught best by those who know how to use it as a tool. Unfortunately, however, it is difficult to get engineers who are sufficiently interested in mathematics and sufficiently masters of that subject, who are willing to devote themselves to teaching. The men who are interested in the problems prefer to devote themselves to those problems, and to go into practical work. It is not necessary, however, as suggested above, that the teachers of mathematics should be engineers if only they will take an interest in the problems themselves, and in the point of view which the student should take. They can do this by cooperation with the engineering teachers, by attending engineering courses, and, perhaps, by a little more realization than they now have that their work is preliminary to other and more important work, and that as a matter of fact if the engineering student does not learn to use his mathematics as a tool it is practically of no value to him. For the engineer, mathematics is the servant, and the mathematical teacher should aim to teach the subject in such a way as to obtain as nearly as possible the results which intelligent engineering teachers and practitioners desire to have obtained.

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FROM THE STANDPOINT OF THE PROFESSOR OF MATHEMATICS IN THE ENGINEERING COLLEGE

We must not take too seriously what engineers have to say in an educational discussion, nor take too much to heart their views on the mathematical curriculum. Practising engineers are not in the habit of thinking very continuously on any educational question, although, of course, they must not confess inability to respond when they are called upon for pedagogical opinions. Every practitioner in the law would doubtless express views concerning legal education if summoned to do so, but he would be a rash educator who would attempt to follow their advice without much circumspection. I, myself, prefer to judge of the engineer's views upon educational matters by studying his actions rather than his words. The things engineers "do" may be taken as a true expression of their deliberate judgment—what they "say" is often ill thought out and in contradiction to their deeds. I therefore prefer to judge of the present needs in the mathematical instruction for engineers by the actual tendencies that I observe in the evolution of technology itself.

What are the great changes that the engineering profession has made in technical science in this country in the last quarter of a century? The changes are quite obvious and not difficult to state. In former days engineering technology was founded chiefly upon current practise rather than upon established principles; it was more closely allied to the crafts than to science. Not only is that day past, but it is no longer the case that technical science looks entirely to pure science for its fundamental material. It has so grown that it is investigating for itself and, in greater and greater measure, developing the basal principles for its own needs. There are very few American treatises in pure science which will compare in scientific thoroughness with several treatises which have lately issued from the engineering press. This is a very hopeful sign in the growth of knowledge—to see applied science and pure science approaching each other at numerous points, so that it is increasingly difficult to distinguish any line of demarcation between them. In this change, *science is not sacrificing any of its strength nor compromising its ideals.* It is *technology* that is changing—that is becoming less empirical, more systematic, more quantitative, more scientific.

With these well recognized changes in applied science before us, what should be our attitude toward the mathematical science that is necessarily associated with engineering education? What is technology really requiring of the basal sciences? Judging the engineers by their acts and not by their words, what is the real demand that they are making of the physicist, of the chemist or of the mathematician? Is the demand to teach physics or chemistry in this or that particular way, or is the demand of a profounder and more radical sort? The most superficial observation shows that the demand is of the latter kind. The engineer in this twentieth century is saying to the physicist, and chemist, and mathematician: "Know more science. Discover more facts in electricity—in light—in all properties of matter. Give to the world more men like Kelvin, Hertz, Helmholtz. Fill the shelves with ten times the knowledge we now have." These words more truly express the real pressure that engineers are putting upon workers in pure science, than do the words they have uttered in this discussion. As a single example, note that the great electrical and other manufacturing companies are impatient at the rate at which pure science grows, and large sums are spent by them each year in the search for new truth and in filling up the gaps in existing knowledge.

The real demand of the engineer is not for better instruments or tools with which to do his work, nor is the demand for more difficult projects to test his skill, nor even for more capital with which to construct them. The real demand is for more knowledge, more science, and for more of the spirit of science in technology and in technical education. I take as my text a saying of Ostwald: *"Science is the best technology."* If we teach a trade and not a science the time is largely wasted. If we teach *dyeing* and not *chemistry,* the graduate is already out of date when he begins his career, and he has not the fundamental principles wherewith to bring himself abreast of the times. I therefore regard it of greatest importance that mathematics be taught to engineering students with real enthusiasm for the science itself. It should be taught by men who themselves are actively contributing to the growth of mathematical science. The present spirit of engineering science is such that no instructor in any of the basal sciences is satisfactory who does not see that it is his duty not only to teach what is old, but to be interested in and to take an active part in the development of what is new.

I regard of secondary importance the particular things we do in the mathematical course in the engineering school. Different instructors, equally successful, will have different opinions. Various changes and improvements have been tried at various institutions. At the University of Wisconsin we have made innovations whenever we thought it best, but I regard them all of secondary importance to the first requirement of all, namely, that we demand the right sort of teachers, and that the teaching be done in the right sort of scientific spirit.

The only imperative requirement put upon the mathematics in engineering schools that does not rest as heavily upon the mathematics of the ordinary college course is the demand for compactness. It is possible that there is some room in the courses in colleges of pure science for the whims and fads of the various instructors, for at some later place in the course the balance may be restored. This, however, is not true in a school of engineering. There is very little room for the practise of fads and new schemes. It is easy to exaggerate the need of a special sort of subject matter in mathematics and a special class of problems for engineering students. We are apt to make some very foolish mistakes, if we undertake to change too freely the scientific material that is presented to engineering students. A good engineer is worthy of the best science and the best instruction that can be brought to him—he himself would be the first to object if a different program were carried out.

I have had a little experience in employing engineering graduates in engineering work. In the past ten years I have given employment, in various capacities, to about one hundred and thirty engineering graduates. This work has been scattered over quite a wide territory and the men have come from the institutions of the east, from the Pacific Coast, from the Mississippi Valley and from the south. I have been able to judge within the limits of my experience what the young engineering graduates know, and what they have forgotten. I find it true that the boys have forgotten a great deal of the material they had in college, and that they have remembered other things. They remember the manual and the mechanical things—how to swim, how to ride a horse, how to fish, how to play ball, how to run the level, how to work the plane table, and how to do stadia work. Now what have they forgotten? The men have forgotten the intellectual things—hydraulics, electrical science, thermodynamics, etc. The human mind possesses an unlimited capacity for forgetting. But my experience shows that the young men forget their hydraulics just as quickly as they forget their mathematics or their mechanics. The enginee in the field observes that a boy remembers the right end of an instrument and seems to be amazed that the same man does not know the right end of an integral sign. He therefore concludes that the mathematics has not been "taught right." If he will compare intellectual things with intellectual things he will find that a miscellaneous group of engineers will pass as good an examination in mathematics ten years after graduation as they would pass in thermodynamics or hydraulics.

It grates on me to hear mathematics

spoken of as a tool. Mathematics is to the engineer a *basal science* and not a tool. The spirit of that science is of more value to the engineer than the particular things that can be accomplished. The engineer need not be a mathematician, but he needs to think mathematically, and, to my mind, he needs the power of mathematical thought more than skill in manipulating a few mathematical tools in mechanical fashion. There are already too many factory-made products turned over to the college by the secondary schools. I make a fundamental contrast between the engineer with his mind endowed with the power of creative and rational design, and the artisan with his hands equipped with tools for physical construction. A great engineer must be trained in correct seeing and thinking, and must have the power of reasoning concerning some of the highest abstractions of the human mind. In this aspect mathematics is not a tool—it is a basal science.

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At the close of Professor Townsend's address he urged the desirability of technical schools offering more elective advanced work in mathematics. It may not be out of place, therefore, for me to call attention to the fact that in the Massachusetts Institute of Technology we have offered and given, among others, the following courses: advanced calculus, vector analysis, fourier series, least squares, theory of surfaces, theory of functions, elliptic functions, hydrodynamics and differential equations of mechanics and physics. Some of these subjects are required in one or more of our courses, but not in any one of the larger engineering courses, which are taken as the basis of Professor Townsend's tables. This elective work, therefore, while valuable in many respects, is not the main work of the mathematical department.

The mathematical teacher is in the engineering school primarily to teach to students of engineering the amount of mathematics which is necessary to them for the proper understanding and practise of their profession. The object is to give the student a grasp of mathematical concepts and processes through their use, as one learns grammar by speaking a language. Hence there is no place in the required mathematics of a technical school, nor indeed in the first courses in a college of liberal arts, for the refinements of modern "rigor." At the same time there should be no patience with a loose or unscientific presentation of first principles. The teacher himself must be thoroughly conversant with modern thought, else he will teach falsehood for truth, and must be enthusiastic in his interest in his subject, else he will fail to inspire his pupils. Hence the teacher of mathematics should be primarily a mathematician and not an engineer. It is hard to find an engineer who has any knowledge of mathematics other than a small fragment which he habitually uses, and any elementary teacher whose instruction goes to the very limits of his knowledge is sure of failure. It may, of course, be possible to superimpose a mathematical training upon an engineering one, but in that case the engineer becomes a mathematician and my contention that mathematics should be taught by a mathematician is not invalidated.

On the other hand, the mathematician should know something of the uses to which an engineer wishes to put mathematics. For that reason such meetings as this are helpful, but I must confess to feeling a little disappointment in not obtaining from the engineers any new light on the concrete problem which confronts the teacher of mathematics in an engineering school. I have met the same disappointment elsewhere in similar meetings. It has happened, elsewhere if not here, that engineers will tell the mathematicians what and how they should teach, in apparently total ignorance of the fact that what the engineer promulgates as a new gospel has been the commonplace thought of the mathematician for years. This ignorance may be due to the fact that the engineer remembers his own training of twenty or thirty years ago and does not know that improvements have taken place. That such is the case may be seen by a comparison of modern with older text-books. Such criticism from the engineers is amusing, but another kind of criticism is not. I refer to the kind which seizes upon the failure of a student to have learned mathematics thoroughly as evidence of poor aims and inefficient teaching of the mathematical instructor. We all know that students pass through our classes and graduate from our schools whose attainments are not what we wish, but while the mathematical teacher delivers his product to the engineering departments and hears of his comparative failures, the engineering professor delivers his product to the world and rarely hears of the specific blunders of his students. Another unfair criticism is sometimes heard from the professor of engineering who says that students can not use their mathematics, when the truth is they have simply forgotten some particular fact, formula, or process, which is a fad of that professor. It is unfair to test mathematical training by tenacity of memory or mere quickness in reasoning.

I have said that we must teach our students to use their mathematics. Now in the application of mathematics to a concrete problem there may be distinguished three steps:

1. The interpretation of the data of the problem into mathematical language.

2. The formal operations upon the expression or equations thus obtained.

3. The interpretation of the results back into the terms of the original problem.

The first and third of these steps are really the most important, but there seems to be a popular impression that the second comprises the whole of mathematics. This impression is doubtless responsible for some criticisms of the educative value of mathematics. It is true that relatively a great amount of time must be spent in the classroom in teaching the mechanical processes involved in the second step, and many students in school and college get no farther. To object to the amount of time spent in this way and to demand, as some do, that we confine our time to teaching general principles and applications is to talk as sensibly as a fond mother who objects to a child beginning his musical education by playing finger exercises instead of tunes. The technique of mathematics must be learned first, but the student who never gets beyond the technique has not learned mathematics.

The teacher of mathematics should, then, use all possible means of teaching the first and third of the above steps and should bring his pupils to think of them as the real thing. For that purpose he should seek for applications and illustrations from as wide a range of subjects as possible. He will find himself handicapped, however, in using many problems of real scientific or engineering importance because of the ignorance of his pupils, especially in the first year in the technical school. To illustrate a new mathematical principle by an application to a science with which a student is not familiar is to befog and not illumine the subject. Hence there is something to be said in favor of some of the much-criticized problems of the older textbooks. To my mind a problem is successful if it causes the student to take the three steps just enumerated and is couched in terms familiar to the student, even though it may not be "practical." On the other hand, a type of problem lately coming into use, in which the student is given some formula from a science of which he knows nothing, and is asked to find, say, a maximum value, is as fruitless as if the problem were stated in terms of x, y and z, unless it may serve to convince a sceptical student that the matter he is studying has some practical application.

And this leads me to the most important thing I have to say, and that is that after the mathematical professor has done his utmost to teach the use of mathematics the engineering professor must take up and complete his work. I doubt if any one really learned the use of mathematics in a first course. Facility in using mathematics comes from actual use and not from the solution of illustrative examples. In the course in mathematics the student expects his problem to be solved mathematically and has his mind alert to find the solution, and that too with mathematical principles fresh in his mind. In a course in engineering, his point of view has widely changed. The practical problem has now his main interest, mathematical concepts are in the background, and he often fails to see the possibility of using mathematical principles until he is trained to do so by the professor of engineering. If the professor, through lack of knowledge or lack of interest, avoids the use of mathematics, the student will soon lose the little he has learned.

In other words, the mathematical training of a student is not complete when he leaves the department of mathematics. It is possible that better results could be obtained if the mathematical department had more time, say for a course in applications of mathematics to miscellaneous problems. But, as a rule, in our technical schools the department of mathematics is allowed barely time to teach the necessary technique with what illustrations and applications can be squeezed in. Hence the mathematical department delivers to the engineering department an unfinished product and it is the engineer's duty to teach the student to use the mathematics he has learned. Unfortunately, the professor of engineering is too often a poor mathematician and avoids this duty.

One of the hardest things a student has to do is to combine two different domains of knowledge, each somewhat unfamiliar, so that he may work freely in both at once, using each as a help in the other. It is this difficulty which makes analytical geometry traditionally hard, and which the student meets again when he studies any form of applied mathematics. It is partly to help overcome this difficulty that we have just made a rearrangement of our mathematical instruction in the Massachusetts Institute of Technology. We no longer have courses in algebra, analytic geometry and differential and integral calculus, but have combined these into one "course in mathematics" extending through two years. Into this course the elements of analytic geometry and of calculus are introduced early and continued late. We hope thus to give these principles more time to become completely domiciled in the student's mind. We have also been enabled to carry out two principles: the first is to introduce no subject until some use is to be made of it, and the second to handle each problem by the method best adapted to it, rather than by the methods of the particular branch of mathematics which one might at the moment be studying under the old classification. We hope in this way to increase the efficiency of our mathematical teaching.

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Massachusetts Institute of Technology

The program shows three standpoints from which discussion is to emanate. I occupy no one of them. It is true I have had some engineering practise, but I can not be termed a practising engineer. I have had charge of mathematics for engineering students in two engineering colleges, but for nearly a decade now I have not met students in mathematics; and, indeed, I have taught, all told, but an insignificant amount. I am in somewhat close touch with engineering students, but they belong to a particular field, namely, mining, which is possibly less dependent on mathematics than are other branches of engineering. My view-point is, therefore, somewhat of a compromise or average of the three specified in the announcement.

The present discussion seems to me significant. It may bring forth results. In fact it seems to have had some immediate consequences. Last evening after the dinner I heard a very clever mathematician admit that he felt really humble, and I heard a well-known engineer say that to his great surprise some mathematicians had a human side. I asked a pure mathematician sitting near me to show me his human side, but he only shrugged his shoulders. Perhaps he was not yet sufficiently humbled.

This occasion appears to me to be significant, but as showing conditions which exist rather than as forecasting future changes. It is a symptom of the approach —the arrival, perhaps—of healthful conditions rather than a cause. It may, of course, in its turn become a cause, and operate toward good results. That is not so certain. At the moment it indicates conditions surrounding the teaching of mathematics to engineering students, including the relations between the teachers of mathematics and those of engineering which have been the growth of many years. Those young and virile gentlemen whom we all delight to honor, the Woodwards, have been striving for decades to bring about a closer relation between the teaching of mathematics and the subsequent study of practise of engineering. Ten years ago at the Toronto meeting of the Society for Promotion of Engineering Education I presented a paper looking to this end.[2] There are gentlemen here present who discussed that paper and who may perhaps recall the remarkable unanimity between the teachers of mathematics and those of engineering as to the results most to be desired in teaching mathematics to engineering students, and, indeed, as to the best available methods for producing such results. This movement is old. Most of the ideas which have been brought out here were first conceived a long time since. Nevertheless, it is good to get together and talk them over, and such discussions may result in help to the individual teacher.

We have heard here much of the ideal which the engineering school should set before itself, but it might well be asked what problem is presented first to the school as a matter of fact? President Woodward put it in part when he spoke of the difficulty of getting the right men in the schools when operators are so eager for good men and are competing on the basis of "so much per month." And what do the employers demand? They call for men who can do something, men who can think in a logical and common-sense way, but, withal, when they leave the school can be put to some immediate use. The first problem confronting the engineering college is how to meet this demand, for the demand must be met in some degree at least or the college will cease to train men.

It is inevitable that the character of this demand shall influence largely what the school must do. The call is not for men

[2] See *Proceedings* of Society for Promotion of Engineering Education, Vol. V., 1897, p. 139.

highly trained in mathematics, however much we may feel it ought to be. It is for men who know well a little mathematics, and who can do something with it, who can use it "as a tool." And, however obnoxious that expression may be to a mathematical teacher, he who forgets or disregards the fact which lies behind it will surely weaken his instruction of engineering students.

I do not defend the specification of the employer, I point to the fact with which we must deal. Personally I am inclined to find fault with it, but the matter rests largely in the hands of the practising engineer. He, though he often objects to the college product, is to a great extent responsible for its general make-up. In the long run and within reasonable limits he can have what he wants. Sometimes he is inclined to require too much technical knowledge on the part of the graduate. His brother teaching in the college in order to meet his requirement says to the teacher of mathematics I must have those students ready earlier with their mathematics. This fact, together with the general tendency in the colleges to raise the standards, causes the mathematical training to be crowded into the first year and a half or two years, when the student is least mature. More of it is being pushed back to the secondary school, and, in turn, into the grades. Mathematical concepts are difficult, and with President Woodward I am inclined to think we are demanding too much, and calling for it too soon. Covering less ground and at a slower pace will help to make better engineers.

The student comes to the engineering school with the notion that he is to be filled up with a lot of technical knowledge, the items of which will be used by him when he is a practising engineer. He seems unable to comprehend that he is in college to acquire mastery over his own powers. He is eager for useful facts and of course he forgets most of those he learns not a great while after leaving college. The forgetting is to be assumed. Under such conditions the task before the teacher of mathematics, and quite as well before the teacher of engineering, is to do his utmost to train his student to think logically and accurately about things. To this end there seems to me nothing so efficient as the solution of a large number of carefully chosen problems. Indeed what is one's life, if it be active, except meeting a never ending succession of problems which must be solved if success is to be gained? If you can teach your student to take vigorous hold of a problem, to first assemble all the facts which bear on the question, then from the facts to reason logically to a sound and safe conclusion, you have started him well whether his aim be engineering or otherwise.

Of transcendent importance is the teacher, his personality, his attitude toward his work, his knowledge of his students, not as a class, but of each as a human being. If we can procure the teacher who can idealize his work, who can show sustained enthusiasm for it and perform cheerfully the drudgery we heard mentioned a few minutes ago, we can safely leave detailed methods to him. Whatever methods such a man adopts in the classroom are likely to be effective.

FRED W. McNAIR
MICHIGAN COLLEGE OF MINES

THE TEACHING OF MATHEMATICS TO STUDENTS OF ENGINEERING[1]

WHAT IS NEEDED IN THE TEACHING OF MATHEMATICS TO STUDENTS OF ENGINEERING? (*a*) RANGE OF SUBJECTS; (*b*) EXTENT IN THE VARIOUS SUBJECTS; (*c*) METHODS OF PRESENTATION; (*d*) CHIEF AIMS.

By CALVIN M. WOODWARD, Professor of Mathematics and Applied Mechanics, and Dean of the School of Engineering and Architecture, Washington University.

I want to emphasize the point which Mr. Scott has just touched on, and that is that we often attempt too early to teach the subjects that require mature and reflecting minds. I want to tell you a story, a true biography of some one you all know of. He went through, in the city of New York, the whole range of mathematics, including analytic geometry and calculus. He learned his formulæ and definitions and "passed" in some manner, but, he told me, he did not know anything about them. He believed he was a dunce, and whenever he was required to make an intelligible demonstration, he could not do it; his teachers and his parents concluded that he was a dunce in mathematics, and could never do anything in it. He would have gone through life with that notion, if some one had not offered him an appointment to West Point. He doubted his ability to pass the entrance examination in arithmetic; but his friends advised him to get an arithmetic and study. He bought a book and sat down and read the book through, and to his astonishment he found it easy. He passed his examination with flying colors. He entered West Point and graduated at the head of his class in mathematics, and is now at the head of a high grade technical school. If it had not been for the opportunity of going again over his whole course of mathematics, he would have gone to his grave thinking he had no capacity for mathematical analysis. That comes from poor or premature teaching.

I am opposed to putting college mathematics in high schools. Those young people may get a glimmer of it, but they get false impressions from it which are hard to remove. I have been teaching mathematics for forty years or more, and have been teaching applied mechanics for the same time. I taught Rankine for twenty-five years. It has always been my duty and my privilege to make my students see what mathematics was good for. And I want to defend the teachers of high school and freshman mathematics from what I think is unjust criticism. It is charged they do not make their students understand what mathematics is good for. It is simply impossible for them to do so, as I can do in mechanics. A man is very fortunate who can teach mathematics and then show what it is good for. I am old enough to quote a little of my early experience. I am led to it by something Professor Swain said in regard to mental processes. There is nothing so valuable to mathematical success as a clear grasp of fundamental principles. When I was preparing for college I gave all my time to Latin and Greek. I had done all my freshman mathematics and was reputed to be strong on that branch, when a new teacher came into the school who said, "Here's a new book in intellectual arithmetic, and I would like to have every student in the school go through it." It was

[1] General discussion following the presentation of four formal papers (see SCIENCE, July 17, 26, 31, 1908), and of the eight prepared discussions (see SCIENCE, August 7 and 28, 1908). Presented before Sections D and A of the American Association for the Advancement of Science and the Chicago Section of the American Mathematical Society, at the Chicago meeting, December 31, 1907.

fun for me, of course, but I went through the book from A to Z; no other mathematics that I ever studied did me so much good. The teacher's maxim was, "Take hold of the thread at the right end." That was the secret of his splendid teaching. I have applied that maxim to every branch of mathematics I have ever studied or taught. I have learned to take hold of mathematics at the right end, and in a measure I have taught my students to do so.

By B. F. Groat, Professor of Mechanics and Mathematics, School of Mines, University of Minnesota.

Most of the speakers have stated that what they were about to say had already been said by preceding speakers. I am going to try to state a general principle I have not heard clearly put since I came here. During the lunch hour Professor Slaught said that he had not heard a single general pedagogical principle brought out. I am going to take the honor to myself, to give expression to what seems to me to be a general educational principle.

Mathematics is mathematics and engineering is engineering. There is just as much art, science or principle in the teaching of mathematics as there is in the teaching of engineering and these two subjects should be distinguished, separated and kept separate. If you are going to teach engineering you must teach the pure principles. If you are going to teach mathematics you have got to teach pure mathematics. Let it be pure or applied mathematics, it is the *principle involved* which must be taught. If this rule is not adhered to we shall find ourselves teaching something different from that which it was intended to teach.

The principle is that the technical courses in our engineering schools must be separated from our general educational courses. The technical courses are for the purpose of fitting the man for a special life work which is to come later on. The general education which he should have, by way of preparation, should precede his technical course as far as possible.

The straight technical course should be given as a course of two years extent, while the general and preparatory subjects should precede in a three- or four-year course.

The University of Minnesota has adopted a five-year engineering course. This is along the lines I am recommending and I prophesy that it will soon be extended to other schools and separated into two parts.

Let your professor of engineering teach engineering and your professor of mathematics teach mathematics. That is the general pedagogical principle I want to announce.

By C. S. Howe, President, Case School of Applied Science.

I have been very much interested in the discussion of this subject because for thirteen years I was a professor of mathematics in an engineering school and during the past five years I have been endeavoring to reconcile the differences between professors of mathematics and professors of engineering. One thing in this discussion which strikes me as very peculiar is the sad lack of knowledge displayed by the engineering professors as to what is being done in mathematics in their own schools. I believe from my experience and from what I have seen in other institutions that the professors of mathematics are teaching mathematics most admirably as mathematics, but they are not teaching mathematics as a department of engineering. I do not believe that mathematics should be taught as a department of engineering. Mathematics is a science in itself and should be taught by specialists in that science if our students are to be trained in the proper way. The professor of mathematics has two duties to perform. One is to teach his students the principles of mathematics—that is, to teach them to reason and to understand why certain processes are right and why others are wrong. The student must also be taught how to use his mathematics so that he can solve any problem as soon as that problem is expressed in mathematical terms. Another duty of the mathematician is to teach the student to be exact. Unless the engineer is exact, unless he can obtain definite and reliable results in his engineering work, he can not succeed in his profession. This accuracy must be very largely taught in the mathematical department and much of the time and care bestowed upon classes is for the purpose of accomplishing this result.

I believe also that the professors of engineering are teaching engineering thoroughly and well. The difficulty which we are discussing to-day is not in the teaching of mathematics alone nor in the teaching of engineering alone, but in the connection between the two. The technical student is, I believe, taught pure mathematics well, but when he enters the class in engineering he finds that he has to deal with mathematics under a new form—that is, the particular engineering subject he is studying must be translated into mathematical terms and this is where he frequently meets with great difficulty. The student in algebra who has learned to solve equations of the first degree may have great difficulty with problems involving equations of the first degree because he has not learned to state the problems in mathematical language. So the student who begins electrical work finds certain problems containing known and unknown quantities, but not yet expressed in mathematical terms. Now I can not believe that it is the duty of the professor of mathematics to teach the student to express problems in the various branches of engineering in the form of equations or other mathematical terms. In order to do this it would be necessary for him to understand all the various branches of engineering and it is manifestly impossible for him to do this. The professor of civil engineering understands the problems of that subject and he should show the student in his department how to express these problems in such terms that the student can deal with them mathematically. The same may be said of each of the departments of engineering. When the professors of engineering have taught their students to state the problems of their own departments in mathematical language, then the student who has had the course in mathematics ought to be able to solve the problems, and if he can not he has not been taught his mathematics thoroughly or so much time has elapsed since he studied the subject that he has forgotten some parts of it.

Again, I believe that the professor of engineering should ascertain in a general way how mathematics is being taught in his institution and in just what form the student is using certain terms so that he may express his own problems in a way familiar to the student. If, for instance, in calculus the mathematical department has been using derivations, the professors of engineering in writing their problems should use differential coefficients and not attempt to express problems in terms of differentials. I know from experience that many professors of engineering do not do this and their students are confused by a difference of terms and not by a lack of knowledge of the subject. It is evident that the professors of engineering must conform to the methods of the department of mathematics because the department of mathematics can use but one method while the five or more departments of engineering might have several different methods. It is obvious, then, that for the sake of simplicity one method must be used and that method must be the method of the department of mathematics.

I also believe that the professor of mathematics should occasionally confer with the professors of engineering in order to find out from them just what mathematical subjects engineering students are weak in and what subjects it is especially desirable to have them well trained in and to see that his students are taught these things. Friendly conferences between the departments are of great value and should be encouraged by both the mathematicians and the engineers.

By Clarence A. Waldo, Professor of Mathematics, Purdue University.

In the table of hours for mathematics in the various institutions cited by Professor Townsend, the largest total stands against Purdue. Also a whole semester is assigned to trigonometry. Both of these conditions are in a measure due to the fact that we have recently passed through a transitional period in which for engineers solid geometry has been relegated to the secondary schools. The first semester was formerly divided between solid geometry and trigonometry. Now it is wholly given to the latter, while the second semester is set aside to college algebra. Experience shows that for the ordinary student college algebra is more difficult than trigonometry and this determines their order in our program.

Placing trigonometry first and giving it so much time has developed with us several interesting facts.

1. Being easy to understand and having interesting applications, it naturally follows secondary work.

2. While trigonometry is easy to understand, yet to acquire facility in its use and absolute mastery over it as a fundamental science requires close and long-continued study, yet the student, ambitious to become an engineer, quickly sees that he must have facility in this subject and mastery over it.

As a subject of study, therefore, at the beginning of a young man's college career it is well adapted to give power and to instill habits of thoroughness, application, concentration and mastery.

3. Engineers have been recommending that a generous amount of time shall be given to trigonometry, at the expense of the calculus if necessary.

4. The subject is used to review and emphasize much of the preparatory mathematics, while it is also used to clear the way for that which is to come.

Another peculiarity in which Purdue stands almost alone we are quite prepared to defend. We do not crowd the pure mathematical work into the first two years, much less into the first year, but give it an hour less in the second year, than the first, yet at the outset of the third year, with his first course of calculus fairly mastered, we have the student well prepared to begin attack upon theoretical mechanics and kindred subjects. However, with two hours a week during junior year devoted to the further exploration of the calculus carried on side by side with its application to studies of a nature more or less professional, like thermodynamics, the student is likely to come finally into living contact with calculus ideas. Through three years, then, mathematical ideas are held persistently and prominently before the mind of the student, so that at the end of that time the mental change which I call the mathematical transformation is quite complete. If you are intent upon making a physical transformation by which a weak man becomes robust and powerful, you give one, two or three years for the muscles to grow and the chest to expand through long-continued and systematic exercise. Similarly the average student does not become habitually mathematical and exact in his thinking unless you give him careful direction and devote plenty of time to his development. The man who uses his memory and copies slavishly must disappear. In his place must stand the man of trained intellect, thoughtful, persistent, rich in expedients, powerful in attack. To produce him there are on the mathematical side two indispensable requisites, thoroughness in the fundamentals, and a sufficient time to make the mathematical attack of a problem habitual and natural, and to give such a control of and power in the use of the tools of mathematics that the solution of a problem of average difficulty shall be easy and pleasurable.

In the required mathematical part of the engineering courses at Purdue these are the considerations that determine the distribution of the work in the four-year program, and all of the time we are teaching not alone the particular subject that happens to be named in the curriculum—but mathematics.

Some years ago it was my fortune to study descriptive geometry under Marx and Von Derlin in Munich. They taught their subject from the standpoint of the mathematician rather than that of the draftsman. They made their students visualize geometric form in space and by the use of that power discover methods of solving on paper synthetic problems of much difficulty. The German schools teach descriptive geometry as a mathematical subject, the American schools as a body of problems to be solved by rule on the drawing board. The former method makes descriptive geometry the finest discipline of the four years' course; from the other method little educational benefit arises. Some years ago at the Rose Polytechnic, where for a time we taught descriptive geometry in the German way, it was not unusual to meet students who declared enthusiastically that they got more real good from this subject than from anything else in their entire course.

I would ask the new committee to inquire how and by whom descriptive geometry should be taught?

By C. B. Williams, Professor of Mathematics, Kalamazoo College.

The teachers of mathematics in the small colleges of the middle west are preparing many men for work in the better technical schools. From our standpoint there is substantial agreement between the two representatives of the Massachusetts Institute of Technology (Professors Wood and Swain). They expressed themselves so differently that one might easily fail to see how closely they agree. Both want longer and stronger courses in mathematics in the secondary schools. I would like to know the college teacher of mathematics who does not agree with them. They want more mathematics taught and to have it taught better, to have longer and more consecutive mathematical courses in the secondary and primary schools. In other words, the faculties of the technical schools and colleges are working toward the same end, that is, to have more effective courses in primary and secondary mathematics so that college students can do more and better mathematical work. If we could have properly prepared students, we could turn out the kind of men the better technical schools should have.

The engineers and teachers of engineering have insisted that the most necessary qualification for a real engineer is that he should be able to realize his mathematics, to "think mathematically," as they express it. The mathematicians want the same thing. We are trying to make use of and to train the faculty of geometric intuition, to emphasize the functional notion and to develop functional thinking. There is substantial agreement that the best way to do this is through geometry, with perhaps some help from elementary mechanics. It is true that sometimes we are tempted to use too big and complicated machines for little problems, but this is only because we are attempting to develop methods powerful enough to solve big problems.

By J. B. Webb, Professor of Mathematics and Mechanics, Stevens Institute.

Every practical problem requiring mathematics for its solution consists of three parts:

(*a*) An *Analysis*, which resolves the problem into its elements, examines these in the light of natural laws, rejects unimportant ones and defines the relations existing between those upon which the solution depends. This involves the adoption or discovery of methods of measuring the elements, so that they may be expressed quantitatively by symbols, and of the reduction of the relations between them to the standard mathematical forms of expression. The result is a *mathematical statement of the problem* by one or more *equations*.

(*b*) A *solution* of the equations by which the relations sought for between the quantities are clearly expressed or the quantities put in proper form to have their values calculated.

(*c*) The *interpretation* of the result, which involves a translation of the same from the mathematical language in which it has been obtained into the original language of the problem and a discussion of the practical bearings of the same.

In conversation with a fellow mathematician at this meeting he surprised me by saying that he expected a problem to be put into mathematical language before it was submitted to him and I presume he did not feel bound to interpret his results. Now if "pure mathematicians" regard practical problems in this way, engineers and other practical men have just cause for finding fault with "pure mathematics," and to teach mathematics in this way is to render it valueless to most students. Personally I should refuse to undertake a problem unless I made the analysis and interpretation as well as the solution.

In many if not in most problems the analysis and interpretation are the main parts. They require a broad knowledge of practical conditions and of other sciences and are far more interesting than the mere solution, especially as they often bring into play a large amount of ingenuity and invention, as well as imagination and judgment. A mathematician who can not make the analysis and interpretation of a problem is not to be trusted with the solution and an engineer who is fully competent to make them had better undertake the solution himself or put the whole problem into the hands of a mathematician fully competent to undertake it.

There is no excuse for a "pure mathematician" remaining ignorant of the practical side of the problems he teaches, and his mathematics will not be interesting or trustworthy. Let him cultivate the acquaintance of the truly educated engineer, who will be only too glad to discuss problems with him and give him all the practical information he needs. But there are too many engineers who are not truly educated and who know less about mathematics than the "pure mathematician" does about practical things, and they ought to cultivate the acquaintance of the mathematician and rub off the worst parts of their ignorance before they attempt to criticize the teaching of mathematics. But it is much easier to find fault and say that they never found any use for such and such mathematical branches, when they never gave them enough attention to make them of any use.

Every mathematical teacher should teach all three parts of a problem, but the average engineering student is so indifferent to real progress and his limited time is so taken up with other things that he may get through his course knowing very little about mathematics, no matter how well it may be taught.

Students with fair ability that really want to learn a particular subject can do it even under indifferent teachers, but unless students exert themselves to learn, the best teacher can not put knowledge into them. Discuss the subject to the limit, analyze and adjust the engineering courses to a nicety, write new text-books, adopt new systems and get new teachers and the thing will remain about as it is; teachers will teach and students will expect them to, while only a few will learn, whether the teacher expects them to or not.

By H. T. Eddy, Dean of the Graduate School and Professor of Mathematics and Mechanics, College of Engineering, University of Minnesota.

Complaint has been made that in our teaching of mathematics we do not pay due attention to psychological and pedagogical principles. I want to consider for a moment the application of two of these principles.

First, it is necessary for the engineering student to have an ample undergraduate course in mathematics, and such an extended drill in and habitual acquaintance with its processes that when he has forgotten nine tenths of it, just as he will of this and all other subjects which he studies in college, what remains with him will be a sufficient equipment in this line for his professional career. In other subjects his residuum of knowledge is easily refreshed and increased. Not so in mathematics. The stock of mathematical knowledge of which he is easily master on entering his profession will practically be the end of his attainments in that direction. Restricting the course in mathematics to bare essentials is suicidal, for of it a small fraction only will remain as a permanent possession, and that fraction is likely to be smaller, the smaller the amount originally attempted.

Second, the teacher of mathematics is prone to think that a clear presentation of mathematical truth on his part, and a logical demonstration by the student, are all that is required in this subject. But important as these things assuredly are, they are insufficient to produce successful results. The question is one in which human interest is really of more importance than logic, for mathematical knowledge can not be successfully imparted unless genuine interest on the part of the student can be in some way aroused. It goes without saying, that the teacher must first of all have that interest himself or he ceases to be a fit teacher. How he will awaken interest in his pupil depends upon his own personality. Many do this by help of problems which elucidate and apply the principles. Just here lies the reason for the usual inability of professional engineers to teach mathematics. They have no interest in mathematics itself. It is the engineering problem alone that interests them. To this matter of interest, or the lack of it, may be traced the failure which is apt to attend the separation of classes into divisions according to scholarship, for in that case the divisions made up of poor students lose the impetus to be derived from the interest which the good students exhibit in their work in which all participate to some degree.

By S. M. Barton, Professor of Mathematics, University of the South.

While standing here in the heart of the modern, bustling city of Chicago, and listening to this discussion, my mind goes back to the ancient city of Tarentum and her distinguished governor, Archytas. Archytas, while an able mathematician, was too practical, as we learn, to suit the ideas of the Platonic School, who objected to his mechanical solutions of certain mathematical problems as interfering with pure reasoning. Now, while I take an immense interest in applied mathematics (what mathematician at this day would not?) yet I confess to a feeling of sympathy with Plato in his condemnation of Archytas. At any rate I wish to enter my protest against a possible tendency to degrade mathematical teaching to the memorizing of thumb-rules, and to urge the advantage of a strong backbone of pure mathematics in our engineering courses.

I read with interest a paper presented at the Ithaca meeting of the Society for the Promotion of Engineering Education, by Professor Arthur E. Haynes of the University of Minnesota, in justification of the use of the expression "engineering-mathematics." I must say I was at first somewhat shocked by the expression, for I had always believed that *mathematics is mathematics* take it when and where you will. While I would agree heartily with much that Professor Haynes said, and I do not doubt that his courses are interesting and instructive, yet I question the wisdom of drawing any sharp distinction in the college curriculum between the mathematics given to the engineering student and to any other class of students.

I find myself differing absolutely from the gentleman from the Massachusetts Institute of Technology, who apparently sees no beauty, much less utility, in the higher branches of pure mathematics. How Professor Woods, who has, by the way, written such a sound text-book on mathematics, can live amicably in the same state, much less in the same college, as his engineer-colleague, I am at a loss to understand—perhaps they have an occasional fight. But, joking aside, there is a dangerous tendency to adopt rules (slide and mental) and short-cut, approximate solution to the utter exclusion of rigid proofs. Is it wise to make a mere machine of the young engineer, even if thereby he becomes rich faster or grows poor less slowly? I freely admit, however, that too much theory would be disastrous, and that there is great room for improvement in the teaching of mathematics. The student should be taught how to use his mathematics, and the existing gap between theory and practise be bridged. While affording every possible facility to the student for making experiments, collecting data, becoming expert in handling instruments, making calculations, etc., I urge that we give them, one and all, a good rigid course in *pure* mathematics.

By Arthur E. Haynes, Professor of Engineering-Mathematics, University of Minnesota.

I have been called upon, by name, to defend the use of the term, "Engineering Mathematics." The justification of the term will be found in my paper on the subject in Volume XIV. of the Proceedings of the Society for the Promotion of Engineering Education. As the paper was not read before this association, many of the members present are not acquainted with its contents.

In brief, the reasons there given for the use of the term are:

(*a*) Because of the main object of the study of mathematics in engineering courses, viz: its use as a tool.

(*b*) Because of the proper method of teaching the mathematics of such courses.

(*c*) Because of the content of the mathematics of such courses.

It is not a degradation of mathematics to make it practical, it is rather an added glory. It is as justifiable to use this term as to use the corresponding terms agricultural chemistry, agricultural botany, engineering drawing, etc. We do not degrade chemistry or botany or drawing by the use of these terms: but their employment is justified by the objects of the study, by the methods required in teaching them and by their content, as in mathematics.

It has been suggested that a less thorough study of mathematics is advocated. In reply to this, may I quote from an article in Volume VIII. of the Proceedings of the Society for the Promotion of Engineering Education, on "The Teaching of Mathematics to Engineering Students," where in speaking of such teaching I said:

(*a*) It should be of such a character as to produce an enduring stimulating effect upon the mind of the student.

(*b*) It should give the student the power to properly interpret mathematical language, and to accurately and skillfully use it.

(*c*) To secure these results, the teaching must be based upon a proper order of studies and carried forward in a rational, intelligent manner.

By Arthur S. Hathaway, Professor of Mathematics, Rose Polytechnic Institute.

In a paper on "Pure Mathematics for Engineering Students," published in the *Bulletin* for March, 1901, I expressed opinions which coincide with those given here to-day. I then said that instruction in mathematics for engineering students should have two objects (1) to develop an engineering mind, and (2) to develop mathematics as an instrument of research for the engineer. I came to these conclusions at that time as a result of inquiries made of graduates of several institutions, who were in engineering practise, and of their employers. From the latter, I have had the statement that it is inadvisable to place a man in the higher positions in engineering who has not had a good mathematical training, especially, in the calculus, which, they assert, develops those modes of thought which are necessary to the engineer.

I wish to call your attention to the fact that the fifty-four hours of analytical dynamics credited to Rose Polytechnic Institute on this chart are spent on applied calculus. There is a regular course of one hundred and forty-four hours in Rankine not mentioned here, which is given by my colleague, Professor Gray. In applied calculus we take up problems which require the use of the calculus, such as motions in constant, elastic and central fields, the bending of beams, the twisting of shafts, problems in electricity, in chemistry, etc. We take problems gathered from all sources, text-books, magazines, engineering professors, and discuss them in the class-room, with special reference to the analysis and its mode of application.

By Edward V. Huntington, Assistant Professor of Mathematics, Harvard University.

I desire to call attention to the fact that besides the analogy of mathematics as a tool or instrument, there is also the perhaps more significant analogy of the mathematician as the discoverer of quantitative relations which already exist in the problems themselves. Logarithmic relations between varying quantities, for instance, are not dragged into the problem from some artificial tool-chest, but are already present in the problem, and are analyzed out of the problem much as the precious metal is analyzed out of the ingot by the metallurgist. The practical mathematician is simply a scientist specially trained to perceive the quantitative aspects of physical phenomena.

By DONALD F. CAMPBELL, Professor of Mathematics, Armour Institute of Technology.

We have had a number of good ideas set before us in the last two days—ideas which we ought to make an effort to crystallize. I think that the present time is the psychological moment to have a committee appointed to draw up a report on mathematics for colleges of engineering. This report perhaps might be in the nature of a symposium, but it would be especially valuable if it considered in detail the subjects which should be emphasized in a course in mathematics for engineering students. These, however, are merely suggestions. I would not hamper the committee in their deliberations by outlining any particular course which they should pursue. The only condition which I would impose is that the committee be representative enough that all of us can look towards their report with the utmost confidence.

I would move that the chairman be empowered to appoint a committee of three, these three to increase their number to fifteen, chosen from among the teachers of mathematics and engineering and the practising engineers, these fifteen to constitute a committee authorized by this meeting to make such a report on mathematics for colleges of engineering as in their opinion will be of service to teachers in such institutions, and to submit this report when completed to the Chicago Section of the American Mathematical Society.[1]

[1] Professor Campbell's motion, as amended by Professor Magruder, requires the Committee of Fifteen to report to the Society for the Promotion of Engineering Education at its meeting in the summer of 1909.

The chairman appointed Professors Huntington, Williams and Townsend as the committee of three. See pages 2 and 3 of this report.

The committee of three appointed the following persons as members of the Committee of Fifteen: E. V. Huntington, Harvard University, Cambridge, Mass., *Chairman;* Philip R. Alger, U. S. Naval Academy, Annapolis, Md.; D. F. Campbell, Armour Institute of Technology, Chicago, Ill.; E. A. Engler, Worcester Polytechnic Institute, Worcester, Mass.; C. N. Haskins, University of Illinois, Urbana, Ill.; C. S. Howe, Case School of Applied Science, Cleveland, Ohio; Emil Kuichling, New York, N. Y.; W. T. Magruder, Ohio State University, Columbus, Ohio; Ralph Modjeski, Chicago, Ill.; W. F. Osgood, Harvard University, Cambridge, Mass.; C. S. Slichter, University of Wisconsin, Madison, Wis.; C. P. Steinmetz, Schenectady, N. Y.; G. F. Swain, Massachusetts Institute of Technology, Boston, Mass.; E. J. Townsend, University of Illinois, Urbana, Ill.; F. E. Turneaure, University of Wisconsin, Madison, Wis.; C. A. Waldo, Washington University, St. Louis, Mo.; G. S. Williams, University of Michigan, Ann Arbor, Mich.; C. M. Woodward, Washington University, St. Louis, Mo.; R. S. Woodward, Carnegie Institution, Washington, D. C.; Alexander Ziwet, University of Michigan, Ann Arbor, Mich.