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NO. 2.

SYNTHETIC
PROJECTIVE GEOMETRY.

BY

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EDITORS' PREFACE.

The volume called Higher Mathematics, the first edition of which was published in 1896, contained eleven chapters by eleven authors, each chapter being independent of the others, but all supposing the reader to have at least a mathematical training equivalent to that given in classical and engineering colleges. The publication of that volume is now discontinued and the chapters are issued in separate form. In these reissues it will generally be found that the monographs are enlarged by additional articles or appendices which either amplify the former presentation or record recent advances. This plan of publication has been arranged in order to meet the demand of teachers and the convenience of classes, but it is also thought that it may prove advantageous to readers in special lines of mathematical literature.

It is the intention of the publishers and editors to add other monographs to the series from time to time, if the call for the same seems to warrant it. Among the topics which are under consideration are those of elliptic functions, the theory of numbers, the group theory, the calculus of variations, and non-Euclidean geometry; possibly also monographs on branches of astronomy, mechanics, and mathematical physics may be included. It is the hope of the editors that this form of publication may tend to promote mathematical study and research over a wider field than that which the former volume has occupied.

December, 1905.

AUTHOR'S PREFACE.

MAN, imprisoned in a little body with short-arm hands instead of wings, created for his guidance a mole geometry, a tactile space, codified by Euclid in his immortal Elements, whose basal principle is congruence, measurement.

Yet man is no mole. Infinite feelers radiate from the windows of his soul, whose wings touch the fixed stars. The angel of light in him created for the guidance of eye-life an independent system, a radiant geometry, a visual space, codified in 1847 by a new Euclid, by the Erlangen professor, Georg von Staudt, in his immortal *Geometrie der Lage* published in the quaint and ancient Nürnberg of Albrecht Dürer.

Born on the 24th of January, 1798, at Rothenburg ob der Tauber, von Staudt was an aristocrat, issue of the union of two of the few *regierenden* families of the then still free Reichsstadt, which four years later closed the 630 years of its renowned existence as an independent republic.

This creation of a geometry of position disembarrassed of all quantity, wholly non-metric, neither positively nor negatively quantitative, resting exclusively on relations of situation, takes as point of departure the since-famous quadrilateral construction. To-day it must be reckoned with from the abstractest domains of philosophy to the bread-winning marts of applied science. Thus Darboux says of it: "It seems to us that under the form first given it by von Staudt, projective geometry must become the necessary companion of descriptive geometry, that it is called to renovate this geometry in its spirit, its procedures, its applications."

KENYON COLLEGE, GAMBIER, OHIO,
December, 1905.

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SYNTHETIC PROJECTIVE GEOMETRY.

INTRODUCTION.

Assumption. (*a*) The aggregate of all proper points on a straight line or 'straight' is closed or made compendent by one point at infinity or figurative point.

(*b*) With regard to a pair of different points of those on a straight all remaining fall into two classes, such that every point belongs to one and only one.

(*c*) If two points belong to different classes with regard to a pair of points, then also the latter two belong to different classes with regard to the first two. Two such point pairs are said to 'separate each other.'

(*d*) Four different points on a straight can always be partitioned in one and only one way into two pairs separating each other.

(*e*) Such separation is projective, that is, is carried on over into ejects and cuts, using the words in the sense explained in Art. 2.

Definition. (*f*) The points A, B, C, D on a straight are in the sequence $ABCD$ if AC and BD are separated point pairs. Consequently this sequence is identical with the following $DABC$, $CDAB$, $BCDA$, where each letter is substituted for the one following it and the last for the first. This procedure is called cyclic permutation. Each sequence again is identical with the outcome of its own reversal, giving $DCBA$, $CBAD$, $BADC$, $ADCB$.

Theorem. (*g*) From any two such of the five sequences $ABCD$, $ABCE$, $ABDE$, $ACDE$, $BCDE$, as come from dropping each one of two consecutive elements of $ABCDE$, the other three follow.

Definition. (*h*) The sequences *ABCD, ABCE, ABDE, ACDE, BCDE* give the sequence *ABCDE.*

Assumption. (*i*) The points on a straight can be thought in a sequence in one sense or the opposite and so that: I. If any one point *A* be given, there is a sequence having the chosen sense and *A* as first point, in which 1) of two points *B* and *C* always one, say *B*, precedes the other (and then *C* follows *B*); 2) if *B* precedes *C* and *C* precedes *D*, always *B* precedes *D*; 3) indefinitely many points follow *B* and precede *C*; 4) there is no last point.

II. Both sequences having the same first point and opposite senses are reversals of one another.

III. Two sequences having the same sense and different first points, say *A* and *B*, follow one from the other by that cyclic interchange which brings *A* into the place of *B*.

Art. 1. The Elements and Primal Forms.

1. A line determined by two points on it is called a 'straight.'

2. On any two points can be put one, but only one, straight, their 'join.'

3. A surface determined by three non-costraight points on it is called a 'plane.'

4. Any three points, not costraight, lie all on one and only one plane, their 'junction.'

5. If two points lie on a plane, so does their join.

6. The plane, the straight, and the point are the elements in projective geometry.

7. A straight is not to be considered as an aggregate of points. It is a monad, an atom, a simple positional concept as primal as the point. It is the 'bearer' of any points on it. It is the bearer of any planes on it.

8. Just so the plane is an element coeval with the point. It is the bearer of any points on it, or any straights on it.

9. A point is the bearer of any straight on it or any plane on it.

10. A point which is on each of two straights is called their 'cross.'

11. Planes all on the same point, or straights all with the same cross, are called 'copunctal.'

12. Any two planes lie both on one and only one straight, their 'meet.'

13. Like points with the same join, planes with the same meet are called costraight.

14. A plane and a straight not on it have one and only one point in common, their 'pass.'

15. Any three planes not costraight are copunctal on one and only one point, their 'apex.'

16. While these elements, namely, the plane, the straight, and the point, retain their atomic character, they can be united into compound figures, of which the primal class consists of three forms, the 'range,' the 'flat-pencil,' the 'axial-pencil.'

17. The aggregate of all points on a straight is called a 'point-row,' or 'range.' If a point be common to two ranges, it is called their 'intersection.'

18. A piece of a range bounded by two points is called a 'sect.'

19. The aggregate of all coplanar, copunctal straights is called a 'flat-pencil.' The common cross is called the 'pencil-point.' The common plane is called the 'pencil-plane.'

20. A piece of a flat-pencil bounded by two of the straights, as 'sides,' is called an 'angle.'

21. The aggregate of all planes on a straight is called an 'axial-pencil,' or 'axial.' Their common meet, the 'axis,' is their bearer.

22. A piece of the axial bounded by two of its planes, as sides, is called an 'axial angle.'

23. Angles are always pieces of the figure, not rotations.

24. No use is made of motion. If a moving point is spoken of, it is to be interpreted as the mind shifting its attention.

25. When there can be no ambiguity of meaning, a figure in a pencil, though consisting only of some single elements of the complete pencil, may yet itself be called a pencil. Just so, certain separate costraight points may be called a range.

ART. 2. PROJECTING AND CUTTING.

26. To 'project' from a fixed point M (the 'projection-vertex') a figure, the 'original,' composed of points B, C, D, etc., and straights b, c, d, etc., is to construct the 'projecting straights' $\overline{MB}, \overline{MC}, \overline{MD}$, and the 'projecting planes' $\underline{\overline{Mb}}, \underline{\overline{Mc}}, \underline{\overline{Md}}$. Thus is obtained a new figure composed of straights and planes, all on M, and called an 'eject' of the original.

27. To 'cut' by a fixed plane μ (the 'picture-plane') a figure, the 'subject,' made up of planes β, γ, δ, etc., and straights b, c, d, etc., is to construct the meets $\overline{\mu\beta}, \overline{\mu\gamma}, \overline{\mu\delta}$, and the passes $\dot{\mu}b, \mu\dot{c}, \mu\dot{d}$. Thus is obtained a new figure composed of straights and points, all on μ, and called a 'cut' of the subject. If the subject is an eject of an original, the cut of the subject is an 'image' of the original.

28. Axial projection. To project from a fixed straight m (the 'projection-axis'), an original composed of points B, C, D, etc., is to construct the projecting planes $\underline{\overline{mB}}, \underline{\overline{mC}}, \underline{\overline{mD}}$. Thus is obtained a new figure composed of planes all on the axis m, and called an 'axial-eject' of the original.

29. To cut by a fixed straight m (to 'transfix') a subject composed of planes β, γ, δ, etc., is to construct the passes $\dot{m}\beta, \dot{m}\gamma, \dot{m}\delta$. The cut obtained by transfixion is a range on the 'transversal' m.

30. Any two fixed primal figures are called 'projective'* $(\overline{\wedge})$ when one can be derived from the other by any finite number of projectings and cuttings.

ART. 3. ELEMENTS AT INFINITY.

31. It is assumed that for every element in either of the three primal figures there is always an element in each of the others.

*Pascal (1625–62) and Desargues (1593–1662) seem to have been the first to derive properties of conics from the properties of the circle by considering the fact that these curves lie in perspective on the surface of the cone.

32. On each straight is one and only one point 'at infinity,' or 'figurative' point. The others are 'proper' points. Any point going either way (moving in either 'sense') ever forward on a straight is at the same time approaching and receding from its point at infinity. The straight is thus a closed line compendent through its point at infinity.

33. 'Parallels' are straights on a common point at infinity.

34. Two proper points in it divide a range into a finite sect and a sect through the infinite. Its figurative point and a proper point in it divide a range into two sects to the infinite ('rays').

35. All the straights parallel to each other on a plane are on the same point at infinity, and so form a flat-pencil whose pencil-point is figurative. Such a pencil is called a 'parallel-flat-pencil.'

36. All points at infinity on a plane lie on one straight at infinity or figurative straight.* Its cross with any proper straight on the plane is the point at infinity on the proper straight.

37. Parallel-flat-pencils on the same plane have all a straight in common, namely, the straight at infinity on which are the figurative pencil-points of all these pencils.

38. Two planes whose meet is a straight at infinity are called parallel.

39. All the planes parallel to each other are on the same figurative straight, and so form an axial pencil whose axis is at infinity. Such an axial is called a parallel-axial.

40. All points at infinity and all straights at infinity lie on a plane at infinity or figurative plane. This plane at infinity is common to all parallel-axials, since it is on the axis of each.

Prob. 1. From each of the three primal figures generate the other two by projecting and cutting.

* This statement should not be interpreted as descriptive of the nature of infinity. In the Function Theory it is expedient to consider all points in a plane at infinity as coincident.

Art. 4. Correlation and Duality.

41. Two figures are called 'correlated' when every element of each is paired with one and only one element of the other. Correlation is a one-to-one correspondence of elements. The paired elements are called 'mates.'

42. Two figures correlated to a third are correlated to each other. For each element of the third has just one mate in each of the others, and these two are thus so paired as to be themselves mates.

43. On a plane, any theorem of configuration and determination, with its proof, gives also a like theorem with its proof, by simply interchanging point with straight, join with cross, sect with angle.*

This correlation of points with straights on a plane is termed a 'principle of duality.' Each of two figures or theorems so related is called the 'dual' of the other.†

Prob. 2. When two coplanar ranges m_1 and m' are correlated as cuts of a flat-pencil M, show that the figurative point P_1, or Q', of the one is mated, in general, to a proper point P', or Q_1, of the other.

Prob. 3. Give the duals of the following:

1'. Two coplanar straights determine a flat-pencil on their cross.

2'. Two coplanar flat-pencils determine a straight, their 'concur.'

3_1. Two points bound two 'explemental' sects.

Prob. 4. To draw a straight crossing three given straights, join the passes of two with a plane on the third.

Art. 5. Polystims and Polygrams.

44_1. A 'polystim' is a system of n coplanar points ('dots'), with all the ranges they determine ('connectors'). Assume that no three dots are costraight.

44'. A 'polygram' is a system of n coplanar straights ('sides'), with all the flat-pencils they determine ('fans'). Assume that no three sides are copunctal.

* Culmann's Graphic Statics (Zürich, 1864) made extensive use of duality. Reye's Geometrie der Lage (Hannover, 1866) was issued as a consequence of the Graphic Statics of Culmann.

† In Analytic Geometry the principle of duality consists in the interpretation of the same equation in different kinds of coordinates—point and line or point and plane coordinates.

In each dot intersect $(n-1)$ connectors, going through the remaining $(n-1)$ dots. So there are $n(n-1)/2$ connectors.

45_1. For n greater than 3, the connectors will intersect in points other than the dots. Such intersections are called 'codots.'

46_1. There are $n(n-1)(n-2)(n-3)/8$ codots.

In each side concur $(n-1)$ fans, going through the remaining $(n-1)$ sides. So there are $n(n-1)/2$ fans.

$45'$. For n greater than 3, the fans will concur in straights other than the sides. Such concurs are called 'diagonals.'

$46'$. There are $n(n-1)(n-2)(n-3)/8$ diagonals.

Proof of 46_1. In a polystim of n dots there are $n(n-1)/2$ connectors. These connectors intersect in

$$[n(n-1)/2][n(n-1)/2-1]/2 = n(n-1)(n^2-n-2)/8$$

points; i.e., the number of different combinations of $n(n-1)/2$ things, two at a time.

But some of these intersections are dots, and the remaining ones are codots. Now $(n-1)$ of these connectors meet at each dot. Therefore each dot is repeated $(n-1)(n-2)/2$ times; or the number of times the connectors intersect in points not codots, i.e. in dots, is $n(n-1)(n-2)/2$.

Therefore the number of codots is

$$\begin{aligned} &n(n-1)(n^2-n-2)/8 - n(n-1)(n-2)/2 \\ &= [n(n-1)/8][n^2-n-2-4(n-2)] \\ &= n(n-1)(n-2)(n-3)/8. \end{aligned}$$

47_1. A set of n connectors may be selected in several ways so that two and only two contain each one of the n dots. Such a set of connectors is called a 'complete set' of connectors.

48_1. There are $(n-1)!/2$ complete sets of connectors.

$47'$. A set of n fans may be selected in several ways so that two and only two contain each one of the n sides. Such a set of fans is called a 'complete set' of fans.

$48'$. There are $(n-1)!/2$ complete sets of fans.

Proof of 48_1. In a polystim of n dots there are through any single dot $(n-1)$ connectors, and hence $(n-1)(n-2)/2$ pairs of connectors. Consider one such pair, as BC and BE.

The number of different sets (each of $n-2$ connectors) from C to E through A, D, F, G, etc. [there being $(n-3)$ such dots], is $(n-3)!$, i.e. the number of permutations of $(n-3)$ things. Hence the number of complete sets of connectors having the pair BC and BE is $(n-3)!$ Therefore the whole number of complete sets of connectors is

$$(n-1)(n-2)[(n-3)!]/2 = (n-1)!/2.$$

49_1. In any complete set of connectors, when n is even, the first and the $(n/2+1)$th are called 'opposite'.

$49'$. In any complete set of fans, when n is even, the first and the $(n/2+1)$th are called 'opposite.'

50_1. A 'tetrastim' is a system of four dots with their six connectors. Each pair of opposite connectors intersect in a codot. These three codots determine the 'codot-tristim' of the tetrastim.

$50'$. A 'tetragram' is a system of four straights with their six fans. Each pair of opposite fans concur in a diagonal. These three diagonals determine the 'diagonal-trigram' of the tetragram.

51. Two correlated polystims whose paired dots and codots have their joins copunctal are called 'copolar.'

52. Two correlated polystims whose paired connectors intersect and have their intersections costraight are called 'coaxal.'

53. If two non-coplanar tristims be copolar, they are coaxal. For since AA' crosses BB', therefore AB and $A'B'$ intersect on the meet of the planes of the tristims.

54. If two non-coplanar tristims be coaxal, they are copolar. For since AB intersects $A'B'$, these four points are coplanar. The three planes $ABA'B'$, $ACA'C'$, $BCB'C'$ are copunctal. Hence so are their meets AA', BB', CC'.

55. By taking the angle between the planes evanescent, is seen that coplanar coaxal tristims are copolar; and then by reductio ad absurdum that coplanar copolar tristims are coaxal.

56. If two coplanar polystims are copolar and coaxal they are said to be 'complete plane perspectives.' Their pole and

axis are called the 'center of perspective' and the 'axis of perspective.'

57. If two coplanar tristims are copolar or coaxal, they are complete plane perspectives.

58. If two coplanar polystims are images of the same polystim from different projection vertices V_1, V_2, they are complete plane perspectives. For the joins of pairs of correlated points are all copunctal (on the pass of the straight $V_1 V_2$ with the picture plane), and the intersections of paired connectors are all costraight (on the meet of the picture plane and the plane of the original).

Prob. 5. In a hexastim there are 15 connectors and 45 codots. In a hexagram there are 15 fans and 45 diagonals.

Prob. 6. If the vertices of three coplanar angles are costraight, their sides make three tetragrams whose other diagonals are copunctal by threes four times. [Prove and give dual.]

Prob. 7. The corresponding sides of any two funiculars of a given system of forces cross on a straight parallel to the join of the poles of the two funiculars.

ART. 6. HARMONIC ELEMENTS.

59. Fundamental Theorem.—If two correlated tetrastims lie on different planes whose meet is on no one of the eight dots, and if five connectors of the one intersect their mates, then the tetrastims are coaxal. For the two pairs of tristims fixed by the five pairs of intersecting connectors being coaxal are copolar. Hence the sixth pair of connectors are coplanar.

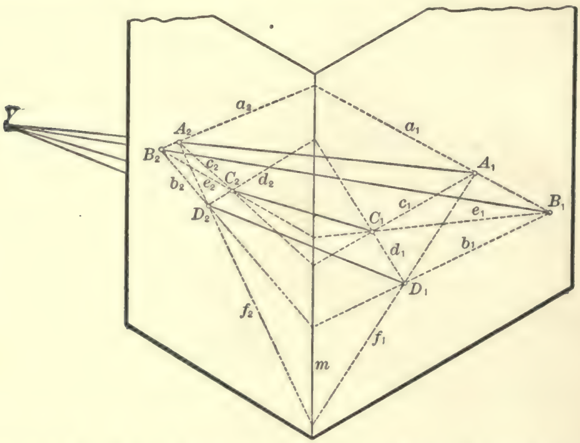
60. If the tetrastims be coplanar, and if five intersections of pairs of correlated connectors are costraight, this the coplanar case can be made to depend upon the other by substituting for one of the tetrastims its image on a second plane meeting the first on the bearer of the five intersections.

61. If the axis m is a figurative straight, the theorem reads: If of two correlated tetrastims five pairs of mated connectors are parallel, so are the remaining pair.

62. Four costraight points are called 'harmonic points,' or

a 'harmonic range,' if the first and third are codots of a tetrastim while the other two are on the connectors through the third codot.

63. By three costraight points and their order the fourth harmonic point is uniquely determined. For if the three points



in order are A, B, C, draw any two straights through A, and a third through B to cross these at K and M respectively. Join CK, crossing AM at N. Join CM, crossing AK at L. Then the join LN crosses the straight ABC, always at the same point D, the fourth harmonic to ABC separated from B.

64. In projecting from a point not coplanar with it a tetrastim defining a harmonic range, the four harmonic points are projected by four coplanar straights, called 'harmonic straights' or a 'harmonic flat-pencil.'

65. The four planes projecting harmonic points from an axis not coplanar with their bearer are called 'harmonic planes,' or a 'harmonic axial-pencil.'

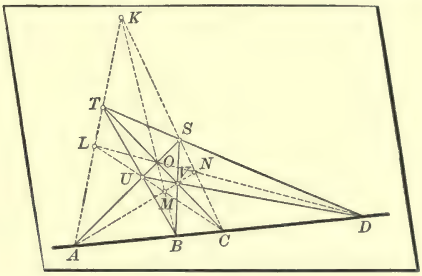
66. Projecting or cutting a harmonic primal figure gives always again a harmonic primal figure.

67. By three elements of a primal figure, given which is the second, the fourth harmonic is completely determined.

68. Defining harmonic points by the tetrastim distinguishes

two points made codots from the other two. Yet it may be shown that the two pairs of points play identically the same rôle.

First, from the definition of four harmonic points each separated two may be interchanged without the points ceasing to



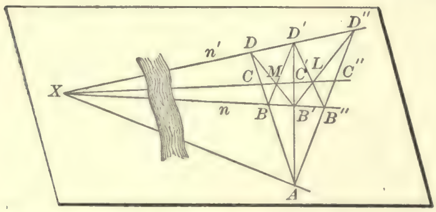
be harmonic [or, if $ABCD$ is a harmonic range, so is also $ADCB$, $CBAD$, and $CDAB$]. For the first and third remain codots.

Second, to prove that in a harmonic range the two pairs of separated points may be interchanged without the four points ceasing to be harmonic [or, if $ABCD$ is a harmonic range (and therefore $ADCB$, $CBAD$, and $CDAB$), then also is $BADC$, $DABC$, $BCDA$, and $DCBA$] : Through the third codot O draw the joins AO and CO. These determine on the connectors NK, KL, LM, and MN four new points, S, T, U, V, respectively. The tetrastim $KTOS$ has for two codots A and C, and has a connector though B; hence its remaining connector TS must pass though D. In like manner, the connector UV of the tetrastim $MVOU$ must pass through D, and a connector of each of the tetrastims $LUOT$ and $VNSO$ through B. Therefore B and D are codots of a tetrastim $STUV$ with the remaining connectors, one through A, one through C.

69. The separated points A and C are called 'conjugate points,' as also are B and D. Either two are said to be 'harmonic conjugates' with respect to the other two.

Prob. 8. To determine the join of a given point M with the inaccessible cross X of two given straights n and n'.

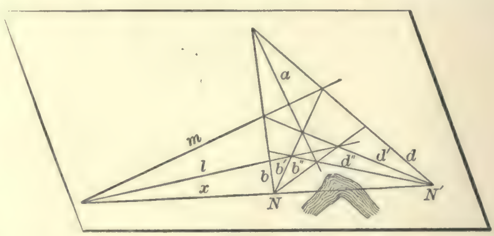
Through M draw any two straights crossing n at B and B', and n' at D and D'. Join DB and $D'B'$, crossing on A. Through A draw any third straight crossing n at B'' and n' at D''. Join $B'D''$ and $D'B''$, crossing at L. Then LM is the join required.



Proof. The tetrastim $XBMD$ makes $AB'C'D'$ a harmonic range, as $XB'LD'$ does $AB''C''D''$. But projecting $AB''C''D''$ from X, and cutting the eject by $AB'D'$ gives a harmonic range. Therefore C'', C', and X are costraight.*

Prob. 9. Through a given point to draw with the straight-edge a straight parallel to two given parallels.

Prob. 10. To determine the cross of a given straight m with the inconstructible join x of two given points N and N'. Join any two points on m with N and N', giving b and b' on N, d and d' on N'. Join the crosses db and $d'b'$ by a. On a take any third point joining with N in b'' and with N' in d''. Join the crosses $b'd''$ and $d'b''$ by l. Then lm is the cross required. [From Prob. 8, by duality.]



Prob. 11. Cut four coplanar non-copunctal straights in a harmonic range.

Prob. 12. On a given straight determine a point from which the ejects of three given points form with the given straight a harmonic pencil.

Art. 7. Projectivity.

70. Two primal figures of three elements are always projective.—If one be a pencil, take its cut by a transversal. If the bearers of ABC and $A'B'C'$ be not coplanar, join AA', BB', CC', and cut these joins by a transversal, m. Then ABC and $A'B'C'$ are two cuts of the axial mAA', mBB', mCC'.

* Numerous problems in Surveying may be solved by the application of the preceding principles, but such application has not been found advantageous in practice. See Gillespie's Treatise on Land Surveying, New York, 1872.

If the bearers are coplanar, take on the join AA' any two projection vertices M and M'. Join MB and $M'B'$, crossing at B''; join MC and $M'C'$, crossing at C''. Join $B''C''$ crossing AA' at A''. Then ABC and $A'B'C'$ are images of $A''B''C''$.

71. If any four harmonic elements are taken in one of two projective figures, the four elements correlated to these are also harmonic. For both ejects and cuts of harmonic figures are themselves harmonic.

72. Two primal figures are projective if they are so correlated that to every four harmonic elements of the one are correlated always four harmonic elements of the other. For the same projectings and cuttings which derive $A'B'C'$ from ABC will give D_1 from D. Therefore $A'B'C'D_1$ is harmonic. But by hypothesis $A'B'C'D'$ is harmonic. Therefore D_1 is D'.

73. If two primal figures are projective, then to every consecutive order of elements of the one on a bearer corresponds a consecutive order of the correlated elements of the other on a bearer.

74. Two projective primal figures having three elements self-correlated are identical. For two self-correlated elements cannot bound an interval containing no such element, since they must harmonically separate one without it from one within.

75. Two ranges are called 'perspective' if cuts of the same flat pencil.

Two flat pencils are perspective if cuts of the same axial pencil, or ejects of the same range. Two axials are perspective if ejects of the same flat pencil.

A range and a flat pencil, a range and an axial pencil, or a flat pencil and an axial are perspective if the first is a cut of the second.

76_1. If two projective ranges not costraight have a self-correlated point A, they are perspective.

$76'$. If two coplanar projective flat pencils not copunctal have a self-correlated straight a, they are perspective.

Let the join of any pair of correlated points BB' cross the join of any other pair CC' at V.

Projecting the two given ranges from V, their ejects are identical, since they are projective and have the three straights VA, VBB', VCC' self-correlated.

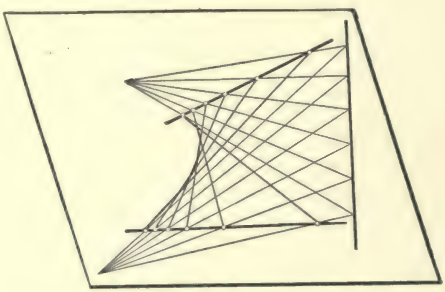
Let the cross of any pair of correlated straights bb' join the cross of any other pair cc' by m.

Cutting the two given flat pencils by m, their cuts are identical, since they are projective and have the three points ma, mbb', mcc' self-correlated.

Art. 8. Curves of the Second Degree.

77_1. If two coplanar noncopunctal flat pencils are projective but not perspective, the crosses of correlated straights form a 'range of the second degree,' or 'conic range.'

$77'$. If two coplanar noncostraight ranges are projective but not perspective, the joins of correlated points form a 'pencil of the second class,' or 'conic pencil.'



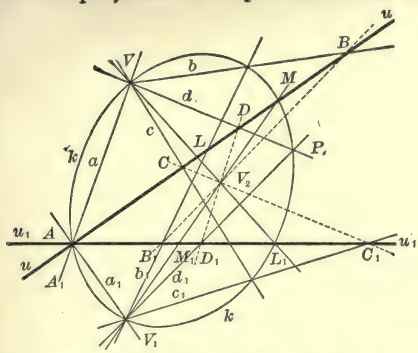
78_1. If two copunctual noncostraight axial pencils are projective but not perspective, the meets of correlated planes form a 'conic surface of the second order,' or 'cone.'

$78'$. If two copunctal noncoplanar flat pencils are projective but not perspective, the planes of correlated straights form a 'pencil of planes of the second class,' or 'cone of planes.'

79. All results obtained for the conic range or the conic pencil are interpretable for the cone or cone of planes, since the eject of a conic is a cone and the cut of a cone is a conic.

80_1. On the cross A of any pair of correlated straights a and a_1

$80'$. On the join a of any pair of correlated points A and A_1 of of the projective flat pencils V



and V_1 draw two straights u and u_1.

The cuts ABC and $A_1B_1C_1$ being projective and having a pair of correlated points A, A_1 coincident, are perspective, both being cuts of the pencil on V_2, the cross of the joins BB_1 and CC_1.

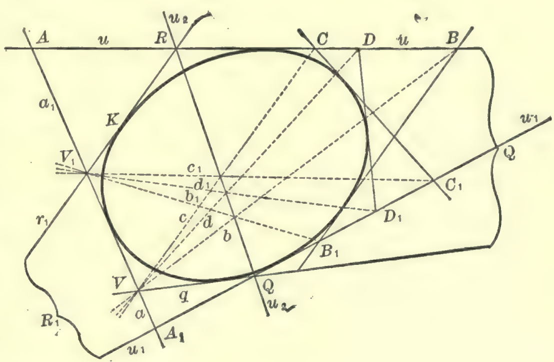
Any straight d of V, crossing u at D, is then correlated to the join of V_1 with the cross D_1 of u_1 and the join DV_2. Any d crosses its d_1 so determined, at P, a point of the conic range k.

the projective ranges u and u_1 take two points V and V_1.

The ejects abc and $a_1b_1c_1$ being projective and having a pair of correlated straights a, a_1 coincident, are perspective, both being ejects of the range on u_2, the join of the crosses bb_1 and cc_1.

Any point D of u, joined with V by d, is then correlated to the cross of u_1 with the join d_1 of V_1 and the cross du_2.

Any D joined to its D_1 so determined, gives p a straight of the conic pencil K.



81_1. The pencil-points V, V_1 of the generating pencils pertain to the conic, since their join VV_1 is crossed by the element correlated to it in either pencil at its pencil-point.

$81'$. The bearers u, u_1 of the generating ranges pertain to the conic, since their cross uu_1 is joined to the element correlated to it in either range by its bearer.

82_1. The straight on V correlated to V_1V is called the 'tangent' at V. Every other straight on V is its join with a second point of the conic.

83_1. On any straight, as u, on any point A of the conic, its second element is its cross M with the join V_1V_2.

84_1. From the five given points VV_1AML_1 of k construct a sixth, P. The cross D of u with the join VP, and the cross D_1 of u_1 with the join V_1P are costraight with V_2. Therefore* the three opposite pairs in every complete set of connectors of a hexastim whose dots are in a conic intersect in three costraight codots whose bearer is called a 'Pascal straight.'

This hexastim has sixty Pascal straights, since it has sixty complete sets of connectors.

85_1. The ejects of the points of a conic from any two are projective.

86_1. By five of its points a conic is completely determined.

87_1. Instead of five points may be given the two pencil-points and three pairs of correlated straights. If one given straight is the join of the pencil-points, then four points and a tangent at one of them are given.

Thus by four of its points and the tangent at one of them a

* Pascal, 1640.

$82'$. The point on u correlated to u_1u is called the 'contact' on u. Every other point on u is its cross with a second straight of the conic.

$83'$. On any point, as V, on any straight a of the conic, its second element is its join q with the cross u_1u_2.

$84'$. From the five given straights u, u_1, a, q, r_1, of K construct a sixth DD_1 or p. The join d of V with the cross up, and the join d_1 of V_1 with the cross u_1p are copunctal with u_2. Therefore † the three opposite pairs in every complete set of fans of a hexagram whose sides are in a conic concur in three copunctal diagonals whose bearer is called a 'Brianchon point.'

This hexagram has sixty Brianchon points, since it has sixty complete sets of fans.

$85'$. The cuts of the straights of a conic by any two are projective.

$86'$. By five of its straights a conic is completely determined.

$87'$. Instead of five straights may be given the two bearers and three pairs of correlated points.

If one given point is the cross of the bearers, then four straights and a contact point on one of them are given.

Thus by four of its straights and a contact-point on one of

† Brianchon, 1806.

conic is completely determined.

88_1. By three of its points and the tangents at two of them the conic is completely determined.

89_1. Interpreting a pentastim as a hexastim with two dots coinciding gives: In every complete set of connectors of a pentastim whose dots are in a conic, two pairs of non-consecutive connectors determine by their two intersections a straight on which is the cross of the fifth connector with the tangent at the opposite dot.

them a conic is completely determined.

88′. By three of its straights and the contact-points on two of them the conic is completely determined.

89′. Interpreting a pentagram as a hexagram with two sides coinciding gives: In every complete set of fans of a pentagram whose sides are in a conic, two pairs of non-consecutive fans determine by their two concurs a point on which is the join of the fifth fan-point with the contact-point on the opposite side.

Thence follows the solution of the problems:

90_1. Given five points of a conic, to construct tangents at the points, using the ruler only.

91_1.* The hexastim with a pair of opposite connectors replaced by tangents gives: The intersections of the two opposite pairs in every complete set of connectors of a tetrastim with dots in a conic are both costraight with the crosses of the two pairs of tangents at opposite dots.

Or: A tetrastim with dots in a conic has each pair of codots costraight with a pair of fan-points of the tetragram of tangents at the dots.

90′. Given five straights of a conic, to find contact-points on the straights, using the ruler only.

91′. The hexagram with a pair of opposite fans replaced by contact-points gives: The concurs of the two opposite pairs in every complete set of fans of a tetragram with sides in a conic are both copunctal with the joins of the two pairs of contact-points on opposite sides.

Or: A tetragram with sides in a conic has each pair of diagonals copunctal with a pair of connectors of the tetrastim of contacts on the sides.

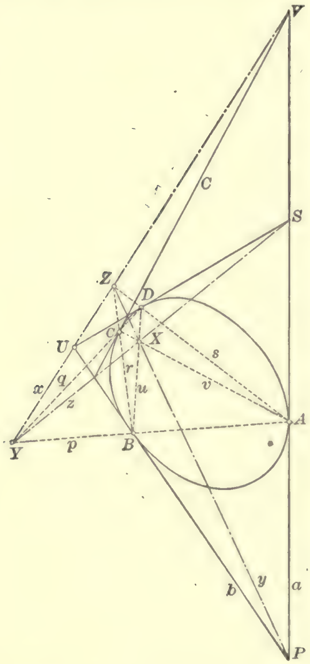
The figure for 91_1 and that for 91′ are identical, and called Maclaurin's Configuration. (See page 86.)

92_1. The tangents of a conic range are a conic pencil.

92′. The contact-points of a conic pencil are a conic range.

* Due to Maclaurin, 1748.

93. The points of a conic range may now be conceived as all on a curve, a 'conic curve,' their bearer. The straights of the corresponding conic pencil, tangents of this conic range, may now also be conceived as all on this same conic curve on which are their contact-points. Consequently the conic curve is dual to itself, and so the principle of duality on a plane receives an important extension.



94. It follows immediately from their generation that all conics are closed curves, though they may be compendent through one or two points at infinity. With two points at infinity the curve is called 'hyperbola;' with one, 'parabola;' with none, 'ellipse.' *

95. If a point has on it tangents to the curve, it is called 'without' the curve; if none, 'within' the curve. The contact-point on a tangent is 'on' the curve; all other points on a tan-

* The generation shows that a straight cuts the curves in two points and that from any point two tangents to the curves may be drawn. Hence the curves are of the second order and of the second class, that is they are identical with the conics of analytic geometry. Analytically the equations $P+\lambda Q=0$, $P'+\lambda Q'=0$, where P, Q, P', Q' are linear functions of point coordinates, represent two projective pencils, the correlated rays corresponding to the same value of λ. Hence the locus of the intersection of correlated rays is represented by $PQ'-P'Q=0$, a second-degree point equation. Projective ranges are represented by $R+\lambda S=0$, $R'+\lambda S'=0$, where R, S, R', S' are linear functions of line coordinates. The envelope of the joins of correlated points is represented by $RS'-R'S=0$, a second-degree line equation.

The projective generation of conics is developed synthetically in Steiner's Theorie der Kegelschnitte, 1866, and in Chasles' Géométrie supérieure, 1852. For the analytic treatment see Clebsch, Geometrie, vol. 1, 1876.

gent are without the curve. Every straight in its plane contains innumerable points without the curve, since the straight crosses every tangent.

Prob. 13. Given four points on a conic and the tangent at one of them, draw the tangent at another.

Prob. 14. If the n sides of a polygram rotate respectively about n fixed points not costraight, while $(n-1)$ of a complete set of fan-points glide respectively on $(n-1)$ fixed straights, then every remaining fan-point describes a conic.*

Prob. 15. In any tristim with dots on a conic the three crosses of the connectors with the tangents at the opposite dots are costraight.†

Prob. 16. If two given angles rotate about their fixed vertices so that one cross of their sides is on a straight, either of the other three crosses describes a conic.‡

Prob. 17. Construct a hyperbola from three given points, and straights on its figurative points.

Art. 9. Pole and Polar.

96. Taking every tangent to a conic as the dual to its own contact-point fixes as dual to any given point in the plane one particular straight, its 'polar,' of which the point is the 'pole.'

97. With reference to any given conic, to construct the polar of any given point in its plane. Put on the given point Z two secants crossing the curve, one at A and D, the other at B and C. The join of the other codots X and Y of $ABCD$ is the polar of Z. Varying either secant, as ZBC, does not change this polar, since on it must always be the cross S of the tangents at A and D, and also the point which D and A harmonically separate from Z (given by each of the variable tetrastims $BXCY$).

98. The join of any two codots of a tetrastim with dots on a conic is the polar of the third codot with respect to that

* Due to Braikenridge, 1735.

† From Pascal; dual from Brianchon.

‡ Given by Newton in Principia, Book I, lemma xxi, under the name of "the organic description" of a conic.

conic, and either codot is the pole of the join of the other two. Any point is harmonically separated from its polar by the conic.

99. To draw with ruler only the tangents to a conic from a point without, join it to the crosses of its polar with the conic.

100_1. Two points are called 'conjugate' with reference to a conic if one (and so each) is on the polar of the other.

$100'$. Two straights are called 'conjugate' with reference to a conic if one (and so each) is on the pole of the other.

101_1. All points on a tangent are conjugate to its contact-point.

$101'$. All straights on a contact-point are conjugate to its tangent.

102_1. The points of a range are projective to their conjugates on its bearer.

$102'$. The straights of a flat pencil are projective to their conjugates on its bearer.

103_1. With reference to a given conic, the 'kerncurve,' the polars of all points on a second conic make a conic pencil, whose bearer is the 'polarcurve' of the second conic.

$103'$. With reference to a given conic, the 'kerncurve,' the poles of all tangents on a second conic make a conic range, whose bearer is the 'polarcurve' of the second conic.

Prob. 18. Either diagonal of a circumscribed tetragram is the polar of the cross of the others.

Prob. 19. A pair of tangents from any point on a polar harmonically separate it from its pole.

Prob. 20. A pair of tangents are harmonic conjugates with respect to any pair of straights on their cross which are conjugate with respect to the conic.

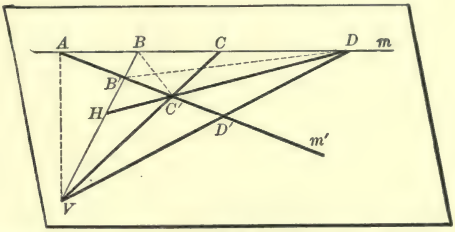
ART. 10. INVOLUTION.

104. If in a primal figure of four elements (a 'throw') first any two be interchanged, then the other two, the result is projective to the original.

[That is, $ABCD \barwedge BADC \barwedge CDAB \barwedge DCBA$.]

Let $ABCD$ be a throw on m. Project it from V. Cut this eject by a straight (m') on A. The cut is $AB'C'D'$. Now project $ABCD$ from C'. The cut of this latter eject by VB is

$B'BVH$. Project $B'BVH$ from D and cut the eject by m'. The cut is $B'AD'C'$, which is perspective to $BADC$.



105. Two projective primal figures of the same kind of elements and both on the same bearer are called 'conjective.' When in two conjective primal figures one particular element has the same mate to whichever figure it be regarded as belonging, then every element has this property.

If $AA'BB'$ is projective to $A'AB'X$, then by § 104, $AA'BB'$ is projective to $AA'XB'$, and having three elements self-correlated, they are identical.

106. Two conjective figures such that the elements are mutually paired (' coupled ') form an 'Involution.' For example, the points of a range, and, on the same bearer, their conjugates with respect to a conic, form an involution. Every eject and every cut of an involution is an involution.

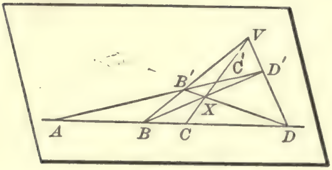
107. When two ranges are projective, the point at infinity of either one is correlated to a point of the other called its 'vanishing point.'

108. When two conjective ranges form an involution the two vanishing points coincide in a point called the 'center' of the involution.

109. If two figures forming an involution have self-correlated elements, these are called the 'double' elements of the involution. An involution has at most two double elements; for were three self-correlated, all would be self-correlated.

110. If a primal figure of four elements is projective with a second made by interchanging two of these elements, they harmonically separate the other two.

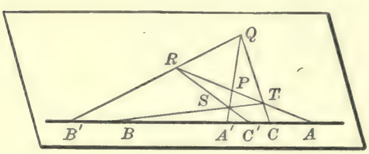
For project the range $ABCD$ from V and cut the eject by a straight on A. The cut $AB'C'D'$ is projective to $ABCD$, which by hypothesis is projective to $ADCB$. Therefore $ADCB$ is perspective to $AB'C'D'$. So $VC'C$ is on the cross X of the joins DB' and BD'. So B and D are codots of the tetrastim $VD'XB'$, while A and C are on the connectors through C', the third codot.



111. If an involution has two double elements these separate harmonically any two coupled elements. Let A and C be the double elements. Then $ABCB'$ is projective to $AB'CB$; therefore by § 110 $ABCB'$ is harmonic.

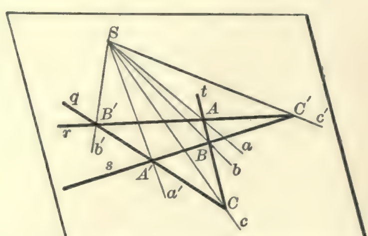
112. An involution is completely determined by two couples. For the projective correspondence $AA'B \ldots \barwedge A'AB' \ldots$ is completely determined by the three given pairs of correlated elements, and since among them is one couple, so are all correlated elements couples.

113. When there are double elements, then the elements of no couple are separated by those of another couple. Inversely, when the elements of one couple separate those of another, then the elements of every couple are separated by those of every other, and there are no double elements.



114'. The three pairs of opposite fan-points of a tetragram are projected from any projection-vertex by three couples of an involution of straights.

114₁. The three pairs of opposite connectors of a tetrastim are cut by any transversal in three couples of a point involution.*



* Due to Desargues, 1639.

Let $QRST$ be a tetrastim of which the pairs of opposite connectors RT and QS, ST and QR, QT and RS are cut by any transversal respectively in A and A', B and B', C and C'. From the projection-vertex Q, the ranges $ATPR$ and $ACA'B'$ are perspective. But $ATPR$ and $ABA'C'$ are perspective from S. Therefore $ACA'B'$ is projective to $ABA'C'$, and therefore to $A'C'AB$ (§ 104). Since thus A and A' are coupled, so (§ 105) are B and B', and C and C'.

115. To construct the sixth point C' of an involution of which five points are given, draw through C any straight, on which take any two points Q and T. Join AT, $B'Q$ crossing at R. Join BT, $A'Q$ crossing at S. The join RS cuts the bearer of the involution in C'.

Prob. 21. Find the center O of a point involution of which two couples $AA'BB'$ are given.

Prob. 22. If two points M and N on m are harmonically separated by *two* pairs of opposite connectors of a tetrastim, then so are they by the third pair.

Prob. 23. To construct a conic which shall be on three given points, and with regard to which the couples of points of an involution on a given straight shall be conjugate points.

ART. 11. PROJECTIVE CONIC RANGES.

116. Four points on a conic are called harmonic if they are projected from any (and so every) fifth point on the conic by four harmonic straights.

117. A conic and a primal figure or two conics are called projective when so correlated that every four harmonic elements of the one correspond to four harmonic elements of the other.

118. If a conic range and a flat pencil are projective, and every element of the one is on the correlated element of the other, they are called perspective. A conic is projected from every point on it by a flat pencil perspective to it. Inversely the pencil-point of every flat pencil perspective to a conic is on the conic.

119. Two conics are projective if flat pencils respectively perspective to them are projective. Therefore any three elements in one can be correlated to any three elements in the other, but this completely pairs all the elements.

120. Two different conic ranges on the same bearer have at most two self-correlated elements.

121. Two different coplanar conic ranges with a point V in common are projective if every two points costraight with V are correlated. For both are then perspective to the flat pencil on V. Every common point other than V is self-correlated; but V only when they have there a common tangent. They can have at most three self-correlated points.

122. If a flat pencil V and conic range k are coplanar and projective but not perspective, then at most three straights of the pencil are on their correlated points of the conic; but at least one.

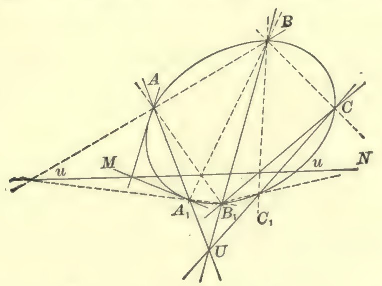
For any flat pencil M perspective to k is projective to V, and with it determines in general a second conic range which must have in common with k every point which lies on its correlated straight of V. So if more than three straights of V were on their correlated points of k, the conics would be identical and V perspective to k.

Again, since every conic is compendent, and so divides its plane into two severed pieces, therefore the two different conics if they cross at their common point M must cross again, say at P. In this case the straights VP and MP are correlated, and so VP is on the point P correlated to it on k.

In case they do not cross at their common point M, the straight VM corresponds to the common tangent at M, and so to the point M correlated to it on k.

123. Two projective conic ranges on the same curve form an involution if a pair of points are doubly correlated. Besides the couple AA_1, let B and B_1 be any other two correlated points, so that AA_1B corresponds to A_1AB_1. The cross of AA_1 and BB_1 call U, and its polar u. Project AA_1B from B_1.

Project A_1AB_1 from B. The ejects $B_1(AA_1B)$ and $B(A_1AB_1)$ are projective, and having the straight B_1B (or BB_1) self-correlated, so are perspective. The crosses of their correlated elements are therefore costraight. But the cross of B_1A with its correlated straight BA_1 is known to be on u, the polar of U, the



cross of AA_1 with BB_1. Likewise the cross of B_1A_1 with BA is on u. Therefore the point C_1 correlated to C is the cross of CU with the curve. So C and C_1 are coupled.

124. If two conic ranges form an involution, the joins of coupled points are all copunctal on the 'involutioncenter.'

125. Calling projective the conic pencils dual to projective conic ranges, if these ranges form an involution, so do the pencils, and the crosses of coupled tangents are all costraight on the 'involutionaxis.'

So two conic pencils forming an involution are cut by each of their straights in two ranges forming an involution. Two conic ranges forming an involution are projected from each of their points in two flat pencils, forming an involution.

126. If the involutioncenter lies without the conic bearer of an involution, it has two double elements where it is cut by the involutionaxis.

127. To construct the self-correlated points of two projective conic ranges on the same conic.—Let A, B, C be any three points of k, and A_1, B_1, C_1 their correlated points of k_1. The projective flat pencils $A(A_1B_1C_1)$ and $A_1(ABC)$ have AA_1 self-corresponding, hence they are perspective to a range on the join u of the cross of AB_1 and A_1B with the cross of AC_1

and A_1C. The crosses of the conic and this join u are the self-correlated points of k and k_1.

128. If the dots of a tetrastim are on a conic, the six points where a straight not on a dot cuts the conic and two pairs of opposite connectors form an involution.

For the two flat pencils in which the two crosses of m with the conic, P, P_1, and two opposite dots R, T, are projected from the other two dots Q, S, are projective, and consequently so are the cuts of these flat pencils by m; that is, $PBP_1A \barwedge PA_1P_1B_1$. But $PA_1P_1B_1 \barwedge P_1B_1PA_1$. Therefore $PBP_1A \barwedge P_1B_1PA_1$.

129_1. Conics on which are the dots of a tetrastim are cut by a transversal in points of an involution. At its double points the transversal is tangent to two of those conics.

129′. Copunctal tangents to conics on which are the sides of a tetragram form an involution. The double straights touch two of those conics at the pencil-point.

Prob. 24. The pairs of points in which a conic is cut by the straights of a pencil whose pencil-point is not on the conic form an involution.

Art. 12. Center and Diameter.

130. The harmonic conjugate of a point at infinity with respect to the end points of a finite sect is the 'center' of that sect.

131. The pole of a straight at infinity with respect to a certain conic is the 'center' of the conic.

132. The polar of any figurative point is on the centre of the conic, and is called a 'diameter.'

133. If a straight crosses a conic the sect between the crosses is called a 'chord.'

The center of a conic is the center of all chords on it.

134. The centers of chords on straights conjugate to a diameter are all on the diameter.

135. Two diameters are conjugate when each is the polar of the figurative point on the other.

136. The tangents at the crosses of a straight with a conic cross on the diameter which is a conjugate to that straight.

137. The joins of any point on the conic to the crosses of a diameter with the conic are parallel to two conjugate diameters.

138. Of two conjugate diameters, each is on the centers of the chords parallel to the other; and if one crosses the conic, the tangents at its crosses are parallel to the other diameter.

139. The center of an ellipse is within it, for its polar does not meet the curve, and so there are no tangents from it to the curve. The centre of a parabola is the contact point of the figurative straight. The centre of a hyperbola lies without the curve, since the figurative straight crosses the curve. The tangents from the center to the hyperbola are called 'asymptotes.' Their contact-points are the two points at infinity on the curve.

140. If a diameter which cuts the curve be given, the tangents at its crosses can be constructed with ruler only, and so however many chords on straights conjugate to the diameter.

141. Every flat pencil is an involution of conjugates with respect to a given conic. Hence the pairs of conjugate diameters of a conic form an involution.

If the conic is a hyperbola, the asymptotes are the double straights of the involution. Hence any two conjugate diameters of a hyperbola are harmonically separated by the asymptotes; and since the hyperbola lies wholly in one of the two explemental angles made by the asymptotes, one diameter cuts the curve, the other does not.

142. Any one pair of conjugate diameters of an ellipse is always separated by any other pair. Any one pair of conjugate diameters of a hyperbola is never separated by any other pair.

143. If a tangent to a hyperbola cuts the asymptotes at A and C, then the contact-point B is the center of the sect AC, since the tangent cuts the harmonic pencil made by the diameter through B, the conjugate diameter and the asymptotes, in the harmonic range $ABCD$ where D is at infinity. Just so the

center of any chord is the center of the costraight sect bounded by the asymptotes.

144. If a point is the center of two chords it is the center of the conic, for its polar is the figurative straight.

145. As many points as desired of a conic may be constructed by the ruler alone.

With the aid of one fixed conic all problems solvable by ruler and compasses can be solved by ruler alone, that is, by pure projective geometry. For example: Of two projective primal figures (say ranges) on the same bearer, given three pairs of correlated elements to find the self-corresponding elements, if there be any. Project the two ranges from any point V of the given conic. These ejects are cut by the conic in projective conic ranges. Of these determine the self-correlated points by § 127.

Project these from V. The ejects cut the bearer of the original ranges in the required self-correlated points.

Prob. 25. Find the crosses of a straight with a conic given only by five points.

Prob. 26. Given a conic and its center, find a point B such that for two given points A, C, the center of the sect AB shall be C.

Prob. 27. The join of the other extremities of two coinitial sects is parallel to the join of their centers.

Prob. 28. In an ellipse let A and B be crosses of conjugate diameters CA, CB with the curve. Through A' the cross of the diameter conjugate to CA with the curve draw a parallel to the join AB. Let it cut the curve again at B'. Then CB' is the diameter conjugate to CB.

ART. 13. PLANE AND POINT DUALITY.

146_1. On a plane are ∞^2 points, a 'point-field.'

147_1. The ∞^1 planes of a single axial pencil have on them all the points of point-space; so there are just ∞^3 points.

Point-space is tridimensional.

$146'$. On a point are ∞^2 planes, a 'plane-sheaf.'

$147'$. The ∞^1 points of a single range have on them all the planes of plane-space; so there are just ∞^3 planes.

Plane-space is tridimensional.

148. With the straight as element, space is of four dimensions.

On a plane are ∞^2 straights, a 'straight-field.'

On a straight are ∞^1 planes, and so ∞^3 straights.

On each of the ∞^2 points on a plane are the ∞^2 straights of a straight-sheaf; so there are just ∞^4 straights.

On a point are ∞^2 straights, a 'straight-sheaf.'

On a straight are ∞^1 points, and so ∞^3 straights.

On each of the ∞^2 planes on a point are the ∞^2 straights of a straight-field; so there are just ∞^4 straights.

149_1. Two planes determine a straight, their meet.

$149'$. Two points determine a straight, their join.

150_1. Two planes determine an axial-pencil on their meet.

$150'$. Two points determine a range on their join.

151_1. Two bounding planes determine an axial angle.

$151'$. Two bounding points determine a sect.

152_1. A plane and a straight not on it determine a point, their pass.

$152'$. A point and a straight not on it determine a plane.

153_1. An axial pencil and a plane not on its bearer determine a flat pencil.

$153'$. A range and a point not on its bearer determine a flat pencil.

154_1. Three planes determine a point, their apex.

$154'$. Three points determine a plane, their junction.

155_1. Three planes determine a plane-sheaf.

$155'$. Three points determine a point-field.

156_1. Two coplanar straights are copunctal.

$156'$. Two copunctal straights are coplanar.

157. Any figure, or the proof of any theorem of configuration and determination, gives a dual figure or proves a dual theorem by simply interchanging point with plane. Thus all the pure projective geometry on a plane may be read as geometry on a point.

Prob. 29. If of straights copunctal in pairs not all are copunctal, then all are coplanar.

Prob. 30. On a given point put a straight to cut two given straights.

Prob. 31. If two triplets of planes $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ are such that the meets $\beta\gamma$ and $\beta'\gamma'$, $\gamma\alpha$ and $\gamma'\alpha'$, $\alpha\beta$ and $\alpha'\beta'$ lie on three planes α'', β'', γ'' which are costraight, then the meets $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ are coplanar.

Prob. 32. Describe the figures in space dual to the polystim and the polygram.

Art. 14. Ruled Quadric Surfaces.

158. The joins of the correlated points of two projective ranges whose bearers are not coplanar form a 'ruled system' of straights no two coplanar. For were two coplanar, then two points on the bearer m and two on the bearer m_1 would all four be on this plane, and so m and m_1 coplanar, contrary to hypothesis.

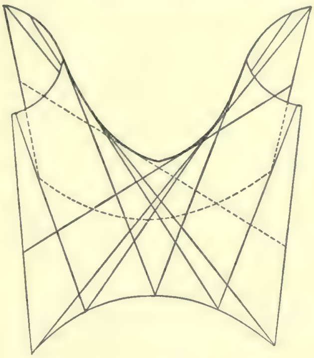
159. Let the straights n, n_1, n_2 be any three of the elements of a ruled system, and N_2 any point on n_2. Put a plane on N_2 and the straight n_1, and let its pass with n be called N. The straight NN_2 cuts n, n_1, n_2 all three. Projecting the generating ranges of the ruled system (on the bearers m and m_1) from the straight NN_2 (or m_2) as axis produces two projective axial pencils, which having three planes m_2n, m_2n_1, m_2n_2 self-corresponding, are identical. Therefore every pair of correlated points of the ranges on m and m_1 is coplanar with m_2; that is, m_2 cuts every element of the ruled system.

By varying the point N_2 ∞^1 straights are obtained, all cutting all the ∞^1 straights of the original ruled system and making on every two projective ranges. Of the straights so obtained no two cross, for that would make two of the first ruled system coplanar.

Either of these two systems may be considered as generating a 'ruled surface,' which is the bearer of both. Each of the two systems is completely determined by any three straights of the other, and therefore so is the ruled surface also. From the construction follows that the straights of either ruled system cut all the straights of the other in projective ranges. So any two straights of either system may be considered as bearers of projective ranges generating the other system, or indeed the ruled surface.

160. On each point of this ruled surface are two and only two straights lying wholly in the surface (one in each ruled

system). So a plane on one straight of the ruled surface is also on another straight of this surface.



161. If in the two generating projective ranges the point at infinity of one is correlated to the point at infinity of the other, the ruled surface is called a 'hyperbolic-paraboloid.'

The join of these figurative points is on the figurative plane. Therefore the plane at infinity cuts the surface in a straight and so has a second straight in common with the ruled surface.

That a hyperbolic-paraboloid has two straights in common with the plane at infinity may also be proved as follows:

Call the bearers of the generating ranges m and m_1, and let n, n_1 be any two elements, and f the element at infinity. By § 159 the ruled surface may be considered as generated by the straights on the three elements n, n_1, f. But all these straights must be parallel to the same plane, namely, to any plane on f. On f and each one of these straights put a plane; these planes make a parallel-axial-pencil, and cut any two of the original elements in projective ranges with the figurative points correlated. Therefore the figurative straight joining the figurative points of n and n_1 is wholly on the ruled surface.

162. From § 161 follows that all straights pertaining to the same ruled system on a hyperbolic-paraboloid are parallel to the same plane. Such planes are called 'asymptote-planes.' A hyperbolic-paraboloid is completely determined by two non-coplanar straights and an asymptote-plane cutting them. To get an element cut the two given straights by any plane parallel to the asymptote-plane, and join the meets.

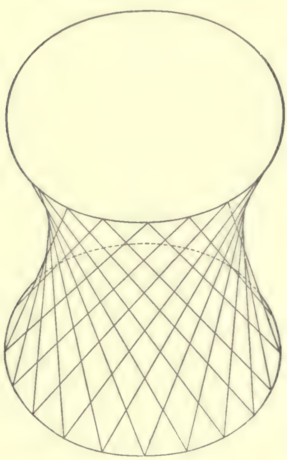
163. Three non-crossing straights, all parallel to the same plane, completely determine a hyperbolic-paraboloid. Let m, m_1, m_2 be the given straights. The passes of planes on m_2

with m and m_1 are projective ranges whose joins are a ruled system.

But from the hypothesis one of these planes is parallel to both m and m_1. Therefore their points at infinity are correlated and the ruled surface is a hyperbolic-paraboloid.

164. If two non-coplanar projective ranges be each axially projected from the bearer of the other, two projective axial pencils are formed, with those planes correlated on which are the correlated points of the ranges. If A, A_1 be correlated points, then the straight AA_1 is the meet of correlated planes. Thus two projective axial pencils with axes not coplanar generate a ruled system. If the whole figure be cut by a plane, this will cut these axial pencils in two projective flat pencils, and the conic generated by these will be the cut of the ruled surface. So every plane cuts it in a conic or a pair of straights. Hence no straight not wholly on the surface can cut it in more than two points. The surface is therefore of the second degree (quadric).

If the plane at infinity cuts the ruled surface in a pair of straights, it is a hyperbolic-paraboloid. If not, it is called a 'hyperboloid of one nappe,' a figure of which is here shown.



164½. Copunctal straights parallel to the generating elements of a hyperboloid of one nappe are on a cone. Copunctal straights parallel to the generating elements of a hyperbolic-paraboloid are on a system of two planes.

For the figurative plane cuts the hyperboloid of one nappe in a conic curve, but cuts the hyperbolic-paraboloid in two straights; and each of the copunctal straights goes to a point of the figurative cut.

165. Each straight in one ruled system of a hyperboloid of

one nappe is parallel to one, but only to one, straight in the other ruled system. Of the straights on a hyperbolic-paraboloid no two are parallel. Let n and n_1, any two elements of one ruled system, be the bearers of the generating ranges R and R_1. If V is the vanishing point of R, then the straight on V parallel to n_1 is an element of the other ruled system. But for the hyperbolic-paraboloid V is itself a figurative point.

166. Any straight of one ruled system on a ruled surface is called a 'guide-straight' of the other ruled system.

167_1. A ruled system is cut by any two of its guide-straights in projective ranges.

$167'$. A ruled system is projected from any two of its guide-straights in projective axial pencils.

For if m, m_1, m_2 be any three guide-straights of the ruled system, the planes on m_2 cut m and m_1 in projective ranges the joins of whose correlated points are the elements of the ruled system. Again, if the points on m_2 be projected axially from m and m_1, the meets of the planes so correlated are the elements of the ruled system.

168. Four straights of a ruled system are called harmonic straights if they are cut in four harmonic points by one (and so by every) guide-straight. By three straights, no two coplanar, a fourth harmonic is determined lying in a ruled system with the given three and on a fourth harmonic point to any three costraight points of the given three.

169. A plane cutting the ruled surface in a straight m of one ruled system and consequently also in a straight n of the other ruled system has in common with the surface no point not on one of these straights. For any straight from such a point cutting both these straights would lie wholly on the ruled surface; and so therefore would their whole plane, which is impossible. Any third straight coplanar with m and n on their cross has no second point in common with the surface and so is a tangent, and the plane of m and n is called tangent at their cross, the point mn.

The number of planes tangent to the ruled surface and on a given straight equals the number of points the straight has in common with the ruled surface, that is two; so the ruled surface is of the second class.

170. Project the two generating ranges of a ruled system from any projection-vertex V not on it. The eject consists of two copunctal projective flat pencils. The plane of any two correlated straights is on an element of the ruled system. All such planes form a cone of planes.

The points of contact of these planes with the ruled surface are a conic range. The planes tangent to a ruled surface at the points on its cut with a plane form a cone of planes.

171. The cut of a hyperbolic-paraboloid by a plane not on an element has on it the passes of the plane with the two figurative elements, and so is a hyperbola except when their cross is on the plane, in which case it is a parabola. The figurative plane is a tangent plane.

172. The planes tangent at the figurative points of a hyperboloid of one nappe are all proper planes, copunctal and forming a cone of planes tangent to the 'asymptote-cone' of the hyperboloid. Each element to the asymptote-cone is parallel to one element of each ruled system.

Any plane not on an element of the hyperboloid of one nappe cuts it in a hyperbola, parabola, or ellipse, according as it is parallel to two elements, one, or no element of the asymptote-cone, that is, according as it has in common with the figurative conic on the hyperboloid two points, one, or no point.

173. If an axial pencil and a ruled system are projective, they generate in general a 'twisted cubic curve,' which any plane cuts in one point at least and three at most. For a plane cuts the ruled system in a conic range perspective to it, of which in general three points at most lie on the corresponding planes of the pencil.

174. The ruled quadric surface is the only surface doubly

ruled. The figure of two so united ruled systems is one of the most noteworthy discovered by the modern geometry.*

175. To find the straights crossing four given straights.— Let u_1, u_2, u_3, u_4 be the given straights. Projecting the range R_1 on u_1 from the axes u_2 and u_3 gives two axial pencils, each perspective to R_1, and consequently projective. The meets of their correlated planes are all the ∞^1 straights on u_1, u_2, u_3, and form a ruled system of which u_1, u_2, u_3 are guide-straights. The two projective axial-pencils cut the fourth straight u_4 in two 'conjective' ranges. [Two projective primal figures of the same kind and on the same bearer are called conjective.] If now a straight m of the ruled system crosses u_4, then the two correlated planes of which this straight m is the meet must cut u_4 in the same point, which consequently is a self-corresponding point of the two conjective ranges. Since there are two such (the points common to u_4 and the ruled surface), so there are two straights (real or conjugate imaginary) crossing four given straights. Their construction is shown to depend on that for the two self-correlated points of two conjective ranges.

This important problem in the four-dimensional space of straights, 'what is common to four straights?' is the analogue of the problem in the space of points, 'what is common to three points?' and its dual in the space of planes, 'what is common to three planes?'

It shows not only their fundamental diversity, but also, as compared to points-geometry and planes-geometry, the inherently quadratic character of straights-geometry.

Prob. 33. Find the straights cutting two given straights and parallel to a third.

Prob. 34. Three diagonals of a skew hexagram whose six sides are on a ruled surface are copunctal.

Prob. 35. If a flat pencil and a range not on parallel planes are projective, then straights on the points of the range parallel to the correlated straights of the pencil form one ruled system of a hyperbolic-paraboloid.

* See Monge, Journal de l'école polytechnique, Vol. I.

Prob. 36. What is the locus of a point harmonically separated from a given point by a ruled surface?

Art. 15. Cross-Ratio.

176. Lindemann has shown how every one number, whether integer, fraction, or irrational, + or −, may be correlated to one point of a straight, without making any use of measurement, without any comparison of sects by application of a unit sect.* He gets an analytic definition of the 'cross-ratio' of four copunctal straights. Then this expression is applied to four costraight points. Then is deduced that the number previously attached to a point on a straight is the same as the cross-ratio of that point with three fixed points of the straight. Thus analytic geometry and metric geometry may be founded without using ratio in its old sense, involving measurement. Thus also the non-Euclidean geometries, that of Bolyai-Lobachévski in which the straight has two points at infinity, and that of Riemann in which the straight has no point at infinity, may be treated together with the limiting case of each between them, the Euclidean geometry, wherein the straight has one but only one point at infinity.

Relinquishing for brevity this pure projective standpoint and reverting to the old metric usages where an angle is an inclination, a sect is a piece of a straight, and any ratio is a number; distinguishing the sect AC from CA as of opposite 'sense,' so that $AC = - CA$, the ratio $[AC/BC]/[AD/BD]$ is called the cross-ratio of the range $ABCD$ and is written $[\dot{A}\dot{B}CD]$ where A and B, called conjugate points of the cross-ratio, may be looked upon as the extremities of a sect divided internally or externally by C and again by D.†

* Von Staudt in Beiträge zur Geometrie der Lage, 1856–60, determines the projective definition of number, and thus makes the metric geometry a consequence of projective geometry.

† The fundamental property of cross-ratio is stated in the Mathematical Collections of Pappus, about 370 A.D. The cross-ratio is the basis of Poncelet's Traité des propriétés projectives, 1822, which distinguishes sharply the projective and metric properties of curves.

177. If on $ABCD$ respectively be the straights $abcd$ copunctal on V, then $AC/BC = \Delta AVC/\Delta BVC$

$$\text{or } AC/BC = \tfrac{1}{2}AV.VC\sin(ac)/\tfrac{1}{2}BV.VC\sin(bc).$$

$$AD/BD = \Delta AVD/\Delta BVD$$

$$= \tfrac{1}{2}AV.VD\sin(ad)/\tfrac{1}{2}BV.VD\sin(bd).$$

Therefore $\quad [\dot{A}\dot{B}CD] = [\sin(ac)/\sin(bc)]/[\sin ad/\sin(bd)].$

Thus as the cross-ratio of any flat pencil $V[abcd]$ or axial pencil $u(\alpha\beta\gamma\delta)$ may be taken the cross-ratio of the cut $ABCD$ on any transversal.

178. Two projective primal figures are 'equicross;' and inversely two equicross primal figures are projective.

179. As D approaches the point at infinity, AD/BD approaches 1. The cross-ratio $[\dot{A}\dot{B}CD]$ when D is figurative equals AC/BC.

180. Given three costraight points ABC, to find D so that $[\dot{A}\dot{B}CD]$ may equal a given number n (+ or −). On any straight on C take A' and B' such that $CA'/CB' = n$; A' and B' lying on the same side of C if n be positive, but on opposite sides if n be negative. Join AA', BB', crossing in V. The parallel to $A'B'$ on V will cut AB in the required D. For if D' be the point at infinity on $A'B'$, and $ABCD$ be projected from V, then $A'B'CD'$ is a cut of the eject; so

$$[\dot{A}\dot{B}CD] = [\dot{A}'\dot{B}'CD'] = A'C/B'C = n.$$

181. If $[\dot{A}\dot{B}CD] = [\dot{A}\dot{B}CD_1]$, then D_1 coincides with D.

182. If two figures be complete plane perspectives, four costraight points (or copunctal straights) in one are equicross with the correlated four in the other. Let O be the center of perspective. Let M and M' be any pair of correlated points of the two figures, N and N' another pair of correlated points lying on the straight OMM' whose cross with the axis of perspective is X. Then $[\dot{O}\dot{X}MN] = [\dot{O}\dot{X}M'N']$.

That is, $[OM/XM]/[ON/XN] = [OM'/XM']/[ON'/XN']$.
Therefore $[OM/XM]/[OM'/XM'] = [ON/XN]/[ON'/XN']$.
That is, $[\dot{O}\dot{X}MM'] = [\dot{O}\dot{X}NN']$; or the cross-ratio $[\dot{O}\dot{X}MM']$

is constant for all pairs of correlated points M and M' taken on a straight OX on the center of perspective.

Next let L and L' be another pair of correlated points and Y the cross of OLL' with the axis of perspective. Since LM and $L'M'$ cross on some point Z of the axis XY, therefore if $OXMM'$ be projected from Z, the cut of the eject by OY is $OYLL'$. So $[\dot{O}\dot{X}MM']=[\dot{O}\dot{Y}LL']$; or the cross-ratio $[\dot{O}\dot{X}MM']$ is constant for all pairs of correlated points.

It is called the 'parameter' of the correlation. When the parameter equals -1, the range $OXMM'$ is harmonic, and two correlated elements correspond doubly, are coupled, and the correlation is 'involutorial.'

183. When the correlation is involutorial and the center of perspective is the figurative point on a perpendicular to the axis of perspective, this is called the 'axis of symmetry,' and the complete plane perspectives are said to be 'symmetrical.'

184. When the correlation is involutorial and the axis of perspective is figurative, then the center of perspective is called the 'symcenter,' and the complete plane perspectives are said to be 'symcentral.'

Prob. 37. In a plane are given a parallelogram and any sect. With the ruler alone find the center of the sect and draw a parallel to it.

Prob. 38. The locus of a point such that its joins to four given points have a given cross-ratio is a conic on which are the points.

Prob. 39. If the sides of a trigram are tangent to a conic, the joins of two of its fan-points to any point on the polar of the third are conjugate with respect to the conic.

Art 16.* Homography and Reciprocation.

185. Two planes taken as both point fields and straight fields, are called 'homographic' (collinear) if they are so correlated that to each point on the one (without exception) one and only one point on the other corresponds, and vice versa; and so that a point and straight of the one plane which belong to one another

* This article follows the model set by Enriques.

correspond, in the other plane, to a point and straight belonging to one another. The relation between homographic planes is called 'homography' (collineation).

An homography exists between a plane original and its image [the cut of its eject from a projection-vertex not on it].

Homography is the most general transformation which transforms straights into straights.

186. Two planes, taken as both point fields and straight fields, are called 'reciprocal' when they are so correlated that to each point on the one (without exception) one and only one straight on the other corresponds and vice versa; and so that to a point and straight of the one plane which belong to one another correspond in the other plane a point and straight belonging to one another.

To costraight points in the first plane correspond copunctal straights in the second.

187. Two sheaves, taken as both plane sheaves and straight sheaves, are called homographic if to each straight of the one corresponds a straight of the other, and to each plane a plane, and vice versa; and so that if a straight and plane belong to each other in the one, so do their correlatives in the other.

188. Two sheaves, taken as both plane sheaves and straight sheaves, are reciprocal if to each straight of the one corresponds a plane of the other, and to each plane a straight, and vice versa; and so that if a straight and plane belong to each other in the one, so do their correlatives in the other.

189. A plane and sheaf are homographic when to every point of the plane corresponds a straight of the sheaf, and to every straight of the plane a plane of the sheaf; and so that if the point and straight in the plane belong to each other, so do the straight and plane of the sheaf.

190. A plane and sheaf are reciprocal when to each point of the plane corresponds a plane of the sheaf, and to every straight of the plane corresponds a straight of the sheaf; and so that if the point and straight belong to each other in the plane, the plane and straight of the sheaf belong to each other.

Homography and reciprocation are included together as cases of projectivity.

191. In two homographic planes two corresponding ranges are projective.

192. In two reciprocal planes a range is projective to the corresponding flat pencil.

193. Calling the plane taken as a point field and a straight field and the sheaf taken as a plane sheaf and a straight sheaf figures of the second class or secondary figures, then between two secondary figures there is a fixed projectivity in which two pairs of projective primal figures correspond if the common elements of the pairs are correlated.

194. Between two secondary figures there is a projectivity fixed by four pairs of corresponding elements, no three in one primal figure.

195. If two secondary figures are homographic one can be derived from the other by a finite number of projectings and cuttings.

196. To depict the homography between two planes α and α' when are given four pairs

of corresponding points, dots of two tetrastims $ABCD$ and $A'B'C'D'$.

Set up a projectivity between the connectors of corresponding dots, for example between AB and $A'B'$, in which correspond A to A', B to B' and the codot $AB \,.\, CD$ to the codot $A'B' \,.\, C'D'$.

Just so set up a projectivity between the flat pencils with pencil-points A and A' in which to the straights AB, AC, AD correspond $A'B'$, $A'C'$, $A'D'$, and so also for those with pencil-points B and B', etc.

of corresponding straights, sides of two tetragrams $abcd$ and $a'b'c'd'$.

Set up a projectivity between the fans of corresponding sides, for example between ab and $a'b'$ in which correspond a to a', b to b' and the diagonal $ab \,.\, cd$ to the diagonal $a'b' \,.\, c'd'$.

Just so set up a projectivity between the ranges on a and a' in which to the points ab, ac, ad correspond $a'b'$, $a'c'$, $a'd'$, and so also for those with bearers b and b', etc.

Now in α let any straight s be given not on one of the points A, B, C, D; then it will cut the ranges AB and CD in two points, to which the corresponding points on the ranges $A'B'$ and $C'D'$ can be determined; the straight s' in α' which joins these points will be the straight s' which is correlated to the straight s in the homography set up between α and α'.

In α if on the other hand a point P be given not in one of the connectors of the tetrastim $ABCD$, take its eject from A and its eject from B and find the straights corresponding to these in the pencils A' and B'; the cross of these straights will be the point P' which is correlated to the point P in this homography set up between α and α'.

Now in α if any point P be given not on one of the straights a, b, c, d, take its eject from the projection vertex ab and also from cd, and then determine their corresponding straights in the flat pencils $a'b'$ and $c'd'$; the cross P' of these will be the point P' which is correlated to P in the homography set up between α and α'.

In α if on the other hand a straight s be given not in a fan of the tetragram $abcd$, it will determine a flat pencil as and another, bs. Take those in α' corresponding to these. Their concur will be the straight s' correlated to s in the homography set up between α and α'.

197. To depict the reciprocation $\begin{pmatrix} ABCD \\ abcd \end{pmatrix}$ where four dots of a tetrastim in α are correlated to four sides of a tetragram in α', the flat pencils A, B, C, D are made projective to the ranges a, b, c, d, and the ranges AB, CD, etc., to the flat pencils ab, cd, etc.

If now in α a point P be given not in a connector of $ABCD$, then take its eject from A and B and determine the points corresponding to these on a and b; their join will be the straight in α' correlated to P in the reciprocation between α and α'.

In α if on the other hand a straight s be given not on a dot, determine its crosses with AB and CD and then the straights corresponding to these in the flat pencils ab and cd; their cross will be the point P' correlated to s in this reciprocation between α and α'.

198. Perspective Secondary Figures.

If two different planes are perspective, that is, correlated by projection from an outside vertex, then their meet is a self-correlated straight and bearer of a range of self-correlated points.

199. If two different sheaves are perspective, that is ejects of the same plane, then the planes on their concur are self-correlated.

200. Inversely we have the theorem:

If two different planes are homographic and their meet the bearer of a range of self-correlated points, the planes are perspective.

If two different sheaves are homographic and their common axial pencil consists wholly of self-correlated planes, the sheaves are perspective.

Proof. If α and α' are the planes and a their meet, then every straight s of α crosses its correlated straight s' of α' in the self-correlated point as.

Now let A and B be two points of α, and A' and B' the two corresponding points of α'. The straights AB and $A'B'$ are correlated and hence cross on a; thence follows that AA' and BB' are coplanar and therefore incident.

Consequently the joins of corresponding points in α and α' are every two incident, and since they are evidently not all coplanar, they must be copunctal; thus α and α' are perspective.

201. Homology. Consider a homography between two coplanar planes, that is in a plane α; call an element coinciding with the corresponding element a double element.

If four points of the plane α, no three costraight, be taken as double points, a homography is thus set up called the identical homography; in it every element is self-correlated.

So in a non-identical homography of the plane α there cannot be four double points no three costraight, nor four double straights no three copunctal.

202. In α the join of two double points is from the homography a double straight and projectively correlated to itself; if then there be on it a third double point, then are all its points

double points. So all straights of a flat pencil having three double straights are double straights. Hence follows:

If in a plane non-identical homography there be four double points, there is a range of double points; if four double straights, there is a flat pencil of double straights.

203. If in the homography there be a range n of double points, its bearer n crosses every straight in a point which, as double point, must belong to the corresponding straight, that is, any two corresponding straights cross on n.

Inversely: If in a plane homography all pairs of corresponding straights cross on a straight, its range consists of double points, since every point of it is pencil-point of a double flat pencil.

204. The necessary and sufficient condition for the existence of a flat pencil of double straights in a non-identical plane homography is that all pairs of corresponding points are costraight with a fixed point.

205. Theorem. Two coplanar homographic planes which have

three costraight double points, and hence a range, u, of double points, have also a flat pencil of double straights.	three copunctal double straights, and hence a flat pencil of double straights, have also a range of double points.

Proof. All pairs of corresponding straights a and a' cross on u. In fact au as double point must coincide with $a'u$.

Put through u a plane α_1 different from α ($\equiv\alpha'$) and project α' on α_1 from an outside vertex V. There results an homography between α_1 and α, for which u is a range of self-correlated points, hence (§ 200) a perspectivity; consequently the pairs of corresponding points MM_1, NN_1 . . . are all costraight with a fixed point U_1. Now from V project α_1 back upon α'; the joins of the pairs of homologous points (MM', NN', . . .) in the homography given between α and α' will now all be copunctal on U, the image of U_1.

So U is the pencil-point of a flat pencil of double straights in the homography between α and α'.

206. The special plane homography (between two coplanar planes) in which there is a range u of double points and a flat pencil U of double straights is called Homology with the axis u and the center U (central homography, perspective homography).

In it corresponding

straights cross on the axis of homology.

points are costraight with the center of homology.

207. 'Special homology' is where the center is on the axis.

208. Particular cases of homology are 1) affine homology (perspective affinity), where the center is a figurative point and the axis a proper straight; 2) homothety (perspective similarity), where the axis is the straight at infinity and the center a proper point; 3) translation, where both axis and center are figurative.

209. Theorem. There is a plane homology, having a given axis u, and a given center U, in which correspond

two points A and A' costraight with U (differing from it and not on the axis).

This is the homography determined by the assumption that u is self-correlated and on it exists the identical projectivity, and that the straight AA' is self-correlated and on it (as set up by the homography) exists that projectivity in which U and $C \equiv AA' . u$ are double points, and the points A and A' correspond.

The point B' corresponding to a given point B outside AA' is the cross of BU with the straight $A'O$ corresponding to AB, where $O \equiv AB . u$.

two straights a and a' crossing on u (differing from it and not on the center).

This is the homography determined by the assumption that the points U and aa' are double points and that in the flat pencil U (as set up by the homography) exists the identical projectivity, and in the flat pencil aa' that projectivity having u and $c \equiv aa' . U$ as double straights, and in which the straights a and a' correspond.

The straight b' corresponding to a given straight b not on the point aa' is the join of the point $a'o$ corresponding to ab, where $o \equiv ab . U$.

210. Involution. In a plane non-identical homography two

corresponding elements are not in general doubly correlated, that is, if to the element A corresponds the element A', then to the element A' corresponds in general an element different from A. If in a plane homography w every two corresponding elements are doubly correlated (if $w \equiv w^{-1}$), then the non-identical homography is called *involution.*

211. If in a homology the range $(AA'\ UC)$ $[C \equiv AA' \cdot u]$ (and therefore every analogous range) is assumed to be harmonic, the homology (then called harmonic) is an involution.

Inversely, considering an involution in α, the joins of two corresponding points, as A and A', are self-correlated, and so there is an infinity of double straights; just so there is also an infinity of double points as crosses of pairs of corresponding straights. But if in a non-identical homography there are more than three double elements, then three belong to a primal figure, which then consists wholly of double elements; consequently the involution in the plane α is a homology; but on each double straight not the axis the corresponding points make a hyperbolic involution; consequently the homology is harmonic.

The necessary and sufficient condition that a plane homography should be an involution is that it be a harmonic homology.

212. Plane Polarity. In general in a reciprocation between two coplanar planes two corresponding elements are not doubly correlated, that is, to a point A corresponds a straight a, and this a in the given reciprocation corresponds to a point A' different from A.

A plane reciprocation in which any two corresponding elements are doubly correlated, that is a reciprocation identical with its inverse, is called a *polarity;* a point and straight which correspond in a plane polarity are called the pole and polar of one another.

Polarity in a plane may also be defined as a unique reversible correlation between the points and straights such that if the straight (polar) corresponding to A is on B, the polar of B is on A.

213. A plane reciprocation is a polarity if there be a triangle in which each vertex corresponds to the opposite side. For

if the three vertices A, B, C correspond to the opposite sides a, b, c, then must AB correspond to ab, that is c to C, etc. Hence the vertices and sides are doubly correlated.

Now in the given projectivity the range a is projective to the flat pencil A of the corresponding straights, so that, if this pencil be cut by a, we obtain on it a projectivity; since in this projectivity the points B and C are doubly correlated, therefore it is an involution; consequently the points of a and the straights of A are doubly correlated.

The same is true of the points on b and c and the corresponding straights on B and C. As a consequence is also every point P, where two straights a' and b' on A and B cross, doubly correlated to the corresponding straight s, fixed as the join of A' and B' (on a and b) which correspond to a' and b'. Hence the reciprocation considered is a polarity.

214. In a plane polarity, triangles whose vertices are poles of the opposite sides are called self-conjugate or auto-polar (or auto-reciprocal) triangles. There is an infinity of auto-polar triangles in a plane polarity.

215. The most general mode of obtaining a polarity is to designate a triangle which shall be auto-polar, and, not on a vertex, a straight as polar of a point not on a side.

216. In a polarity two points are called *conjugate* or reciprocal if one is on the polar of the other; and so likewise for straights. A point on its own polar is called *self-conjugate;* like a straight on its own pole.

217. A triangle whose three vertices or three sides are every two conjugate is an auto-polar triangle of the polarity.

218. If a point A is on its polar a, then no other point on a is self-conjugate.

219. No straight has on it more than two self-conjugate points. It may have none.

220. In a plane polarity, the pairs of conjugate points on a non-self-conjugate straight make an involution either con-

taining no self-conjugate point or two which harmonically separate the pairs of conjugate points.

221. In a plane polarity, if ABC be an auto-polar triangle and P a point within ABC, then if p, the polar of P, lies wholly without the triangle ABC, the polarity has no self-conjugate element and is called "uniform" since in it every involution of conjugate elements is elliptic.

222. In a plane polarity, if ABC be an auto-polar triangle and P a point within ABC, then if p, the polar of P, penetrate the triangle ABC, the polarity has self-conjugate elements, and is called non-uniform, since of the three involutions of conjugate points on the three sides of an auto-polar triangle, two are hyperbolic and one elliptic.

223. In the plane, duality is valid for all visual properties. In the sheaf, duality is valid for all properties, whether visual or metric.

Prob. 40. In a given plane polarity consider as polars all the tangents of a given curve C, that is suppose a polar to envelop the given curve; then its pole will define another curve C' whose points are the poles of the tangents of C. Reciprocally the points of C are the poles of the tangents of C'.

224. Two curves C and C' such that each is the locus of the poles of the tangents of the other, and likewise the envelope of the polars of the points of the other, are called polar reciprocals one of the other with respect to the polarity P.

225. The degree or order of a curve is given by the greatest number of points in which it can be cut by any arbitrary plane (for a plane curve, by any coplanar straight).

226. The class of a plane curve is given by the greatest number of tangents which can be drawn to it from any arbitrary point in the plane.

Prob. 41. The degree and class of a curve are equal to the class and degree respectively of its polar reciprocal.

Prob. 42. The polar reciprocal of a conic is a conic.

227. Two reciprocal figures are duals which have a definite

special relation to one another with respect to their positions, while on the other hand if two figures are merely duals, there is no relation of any kind between them as regards their position.

Prob. 43. If two triangles are both auto-polar with respect to a given conic, their six vertices are on a conic, and their six sides touch another conic.

Prob. 44. If a conic C touch the sides of a triangle abc auto-polar with regard to another conic K, there is an infinity of other triangles auto-polar with regard to K which circumscribe C.

Prob. 45. Two triangles circumscribing the same conic have their vertices on another conic. If two triangles are inscribed in the same conic, their six sides touch another conic.

Prob. 46. If a conic C circumscribe a triangle auto-polar with respect to another conic K, there is an infinity of other triangles inscribed in C and auto-polar with respect to K; and the straights which cut C and K in harmonic conjugates touch a third conic C', the polar reciprocal of C with regard to K.

Prob. 47. If a triangle inscribed in one conic circumscribes another conic, then there is an infinity of such triangles.

Prob. 48. Two triangles reciprocal with respect to a conic are in homology.

Prob. 49. Two triangles in homology determine a polarity, in which the center of homology is the pole of the axis of homology and any vertex is the pole of the corresponding side of the other triangle. If no point is self-conjugate, this is not a polarity with respect to a conic.

If a point be self-conjugate, construct the conic with regard to which the triangles are reciprocal.

ART. 17. TRANSFORMATION. PENCILS AND RANGES OF CONICS.

228. If any two points be assumed to determine not only a straight but also a sect, that point-row on their straight of which they are the end points, then the sect determined by a fixed point O, the origin, and a point X, may be represented by x.

Then if two conjective ranges be correlated, this correlation is defined by the equation

$$axx' + bx + cx' + d = 0,$$

where x' is the sect OX', X$'$ corresponding to X. Hence $x = -(cx'+d)/(ax'+b)$; therefore if K, L, M, N be four costraight points and K', L', M', N', their conjective correlates,

$$[\dot{K}\dot{L}MN] = [\dot{K}'\dot{L}'M'N'].$$

Proof. $PQ = x_2 - x_1 = (x_2' - x_1')(ad-bc)/(ax_1'+b)(ax_2'+b)$;
$SR = x_3 - x_4 = (x_3' - x_4')(ad-bc)/(ax_3'+b)(ax_4'+b)$,
etc.; hence

$$[\dot{K}\dot{L}MN] = (x_1 - x_2)(x_4 - x_3)/(x_2 - x_3)(x_1 - x_4)$$
$$= (x_1' - x_2')(x_4' - x_3')/(x_2' - x_3')(x_1' - x_4') = [\dot{K}'\dot{L}'M'N'].$$

So if three pairs of points be mated the correlation is determined.

229. United Points. Making $x = x'$ we have $ax^2 + (b+c)x + d = 0$; hence in every conjective correlation the coincidence of a point with its corresponding point will occur twice, that is there are two united points, U, U' (real or imaginary).

If the origin O be the center of $UU' = u$, then $b + c = 0$; hence

$$(1) \quad axx' + b(x-x') + d = 0, \quad \text{and} \quad (2) \quad a(u/2)^2 + d = 0.$$

Combining (1) and (2), the equation of correlation becomes

$$xx' + (x-x')b/a - (u/2)^2 = 0,$$
$$\therefore\ (x+u/2)(x'-u/2) = (x'-x)(u/2+b/a).$$
$$\therefore\ u/(u/2+b/a) = (x-x')(-u)/(x'-u/2)(x+u/2)$$
$$= \mathrm{XX}'\cdot UU'/(\mathrm{X}'U\cdot \mathrm{X}U') = [\mathrm{XX}'UU'].$$

So the cross-ratio of a point, its corresponding point, and the united points is *constant*. The correlation is therefore determined if its united points and one pair of corresponding points be given.

230. Double Points and Involution. If $b = c$, then $axx' + b(x+x') + d = 0$; hence in whichever of the two ranges a point be taken, it has the same mate; hence the elements are coupled, the correlation is involutoric.

The equation may be written $a(x+b/a)(x'+b/a) = (b^2 - ad)/a$, which gives for the united points the values $-b\ a \pm (b^2-ad)^{\frac{1}{2}}/a$.

So if M be the center of UU', then $OM = -b/a$.

$MX \cdot MX' = MU^2$; hence the 'double' points U and U' of the involution separate harmonically any couple.

231. An operation which replaces a given figure by a second figure in accordance with a given law is called a 'transformation.'

If a transformation replaces the points of one figure by the points of a second, it is called a 'point transformation.'

If a point transformation replaces $X(x, y)$ by $X'(x', y')$, then the equations expressing x' and y' in terms of x and y, or inversely, are called the 'equations of the transformation.'

If the corresponding costraight points have the same cross ratio, the transformation is called 'projective.' We have seen that $x' = (mx + n)/(m_1x + n_1)$ is the equation of a projective transformation.

For homography, the general projective transformation of the plane, the equations are

$$x' = (a_1x + b_1y + c)/(a_3x + b_3y + c_3),$$
$$y' = (a_2x + b_2y + c_2)/(a_3x + b_3y + c_3).$$

232. The assemblage of conics on which are the dots A, B, C, D of a given tetrastim is called the 'pencil of conics' through the 'basal points' A, B, C, D. The three pairs of opposite connectors of the tetrastim are called the 'degenerate conics' of the pencil, and determine on any transversal an involution in which its intersection-points with any conic of the pencil are a couple. (Desargues-Sturm theorem.)

233. The assemblage of conics on which are the sides a, b, c, d of a given tetragram is called the 'range of conics' touching the 'basal straights' a, b, c, d. The three pairs of opposite fan-points of the tetragram are called the degenerate conics of the range, and determine on any external point not on a side of the tetragram an involution in which its tangents to any conic of the range are a couple.

234. All polars of a point P with respect to the conics of a pencil are copunctal [in Q, and, inversely, of Q, in P]. (Both points are called conjugate with regard to the conics of the pencil.)

Prob. 50. Pole-conic of a straight. If a point P describe a straight s, then the intersection point P' of its polars with respect to the conics of a pencil describes a conic. (This is also the locus of the poles of s with regard to the individual conics of the pencil.)

Prob. 51. Newton. There are two conics which go through four given points and touch a given straight.

Prob. 52. Through a given point there are two conics tangent to the sides of a given tetragram, or none.

Prob. 53. To determine whether there is a conic through the dots of a given tetrastim $ABCD$ and tangent to a given straight s, it suffices to determine whether the point-pairs in which s is cut by two pairs of opposite connectors of $ABCD$ separate each other.

Prob. 54. Midpoints conic. The centers of the conics of a pencil are on a conic, whose center is the mass-center of the basal points.

Prob. 55. The codots of a tetrastim and the centers of its six connectors are on one conic.

235. All poles of a straight with respect to the conics of a range are costraight.

Prob. 56. If a straight s rotates about a fixed point, then the bearer of all its poles with respect to the conics of a range envelops a conic.

236. Midpoints straight. (Newton.) The centers of the conics of a range are costraight, on the join of the centers of the diagonals of the basal tetragram.

Prob. 57. The centers of the three diagonals of a tetragram are costraight.

237. The intersections of a pencil of conics and a projective flat pencil are a curve of the third order containing the pencil-point and the basal points.

238. Every cubic is the intersection of a pencil of conics and a projective flat pencil.

239. A range of conics combined with a projective point-row gives a curve of the third class touching the bearer of the point-row and the basal straights.

240. Every curve of the third class is the envelope of the system of straights obtained from combining a range of conics with a projective point-row.

241. Definition:

Four conics of a pencil are called 'harmonic' if the polars of any point with respect to them are four harmonic straights.

Four conics of a range are called 'harmonic' if the poles of any straight with respect to them are four harmonic points.

242. Two projective pencils of conics produce a curve of the fourth order.

243. Every curve of the fourth order may be produced by two projective pencils of conics.

Prob. 58. Two projective ranges of conics produce a curve of the fourth class.

244. Every curve of the fourth class may be produced by two projective ranges of conics.

Prob. 59. The codot tristim of the basal tetrastim of a pencil of conics is auto-polar with regard to each of them. So is the diagonal trigram of the basal tetragram of a range of conics.

Prob. 60. The conics of a pencil determine on two straights through one basal point, or through two basal points, projective point-rows.

Prob. 61. An arbitrary straight of its plane is touched by only one conic of a range, but two go through an arbitrary point.

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