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## Systematic Risk, Leverage, and Default Risk

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Abstract

The purpose of this note is to investigate the theoretical relationship between the systematic risk of equity, the systematic risk of debt, the systematic risk of the unlevered firm, and leverage in the presence of default risk. The cash-flow approach is adopted in contrast to the literature. The analysis demonstrates that a truncation factor (or survival probability) exists in addition to Hamada and Rubinstein's traditional formulation. Hence, the result derived here is more general.

# SYSTEMATIC RISK, LEVERAGE, AND DEFAULT RISK

In their classical paper, Modigliani and Miller (M&M) [15, 16], based upon the risk-class assumption and the arbitrage argument, have shown the famous propositions I and II. By integrating M&M's proposition I with the mean-variance, Hamada [9] and Rubinstein [18] have shown that the systematic risk of a firm's equity should be positively correlated with the firm's leverage. Numerous subsequent studies have empirically and theoretically investigated the effect of financial leverage on the systematic risk of equity [3, 4, 7, 8, 10, 14]. However, only few of them have incorporated default risk in the analysis [7, 8].

The purpose of this note is to investigate the theoretical relationship between the systematic risk of equity and leverage in the presence of default risk within a framework of one-period Capital Asset Pricing Model (CAPM) under uncertainty. We adopt cash-flow approach which distinguishes from [2] and [8] with option-pricing approach and [7] with expected-rate-of-return approach. In Section I, we discuss the pricing of market values of different claims, which is borrowed from Chen [6]. In Section II, we derive the relationship between systematic risks and leverage. Section III presents the conclusion.

## I. Market Values of Different Claims

Sharpe [19], Lintner [12], and Mossin [17] have derived the following two-parameter equilibrium valuation moidel, referred to as the Capital Asset Pricing Model, in a hypothetical world with three key assumptions.[1]

$$V_j = (R)^{-1}[E(\tilde{Y}_j) - \lambda \mathrm{Cov}(\tilde{Y}_j, \tilde{R}_m)], \tag{1}$$

where

$V_j$ = the equilibrium value of asset j;

$E(\tilde{Y}_j)$ = the expected value of the end-of-period cash flows to the owners of asset j;

$R = 1 + R_f$, where $R_f$ is the risk-free interest rate;

$\mathrm{Cov}(\tilde{Y}_j, \tilde{R}_m)$ = the covariance between the total cash flows of asset j and the return on the market portfolio;

$\lambda$ = the market price of risk.

Equation (1) states that in equilibrium the value of asset j is the present value of the certainty-equivalent (CEQ) of the asset's random cash flow.

A. The Market Value of All-Equity Firm

Denote X as the firm's operating income which is assumed to be jointly normally distributed with the return on the market portfolio so that

$$X = N(\overline{X}, \sigma_X^2)$$

for any given assessment of $\overline{R}_m$ and $\sigma_m^2$. The after-tax cash flows to the owners of the unlevered firm are

$$\tilde{Y}_u = \begin{cases} \tilde{X}(1-\tau) & \text{if } \tilde{X} > 0 \\ 0 & \text{if } \tilde{X} \le 0 \end{cases} \tag{2}$$

where $\tau$ is the proportional corporate income tax. Therefore, the market value of the unlevered firm is given by[2]

$$V_u = (1-\tau)[E_0(\tilde{X}) - \lambda Cov_0(\tilde{X},\tilde{R}_m)](R)^{-1}. \tag{3}$$

where

$$E_0(\tilde{X}) = \int_0^{\infty} \tilde{X}f(\tilde{X})d\tilde{X}; \; Cov_0(\tilde{X},\tilde{R}_m) = E\{[\tilde{X}_0-E_0(\tilde{X})][\tilde{R}_m-E(\tilde{R}_m)]\},$$

the partial covariance between $\tilde{X}$ truncated from 0 upward and $\tilde{R}_m$.[3]

B. The Market Value of Debt

For simplicity, we assume that the total promised payment to bondholders is tax deductible. Bondholders receive their contractual claims of D at the end of the period if the firm is solvent, and the entire value of the firm if the firm is declared bankrupt.[4] Hence, the total cash flows to bondholders at the end of the period are[5]

$$\tilde{Y}_D = \begin{cases} D & \text{if } \tilde{X} \geq D \\ X & \text{if } 0 < \tilde{X} < D \\ 0 & \text{if } \tilde{X} \leq 0. \end{cases} \tag{4}$$

The market value of debt can be expressed as

$$V_D = \{D[1-F(D)] + [E_0^D(\tilde{X}) - \lambda Cov_0^D(\tilde{X},\tilde{R}_m)]\} \; (R)^{-1} \tag{5}$$

where $F(D) = \int_{-\infty}^{D} f(\tilde{X})d\tilde{X}$, the probability that the firm is declared bankrupt.

C. The Market Value of Equity

At the end of the period shareholders receive the after-tax residual value of the firm if it remains solvent, and they receive nothing if the firm goes bankrupt. Therefore, the end-of-period cash flows to shareholders are

$$\tilde{Y}_E = \begin{cases} (1-\tau)(\tilde{X}-D) & \text{if } \tilde{X} > D \\ 0 & \text{if } \tilde{X} \leq D. \end{cases} \tag{6}$$

The market value of equity can be expressed as

$$V_E = (1-\tau)\{E_D(\tilde{X}) - \lambda \mathrm{Cov}_D(\tilde{X},\tilde{R}_m) - D[1-F(D)]\} (R)^{-1}. \tag{7}$$

D. The Market Value of the Levered Firm

The market value of the levered firm is simply the sum of market values of its debt and equity. Adding $V_D$ in (5) and $V_E$ in (7), we get

$$V = \{(1-\tau)[E_0(\tilde{X}) - \lambda \mathrm{Cov}_0(\tilde{X},\tilde{R}_m)] + \tau[E_0^D(\tilde{X}) - \lambda \mathrm{Cov}_0^D(\tilde{X},\tilde{R}_m)]$$
$$+ \tau D[1-F(D)]\} (R)^{-1} \tag{8}$$

Then, substituting $V_u$ in (3) and $V_D$ in (5) into (8), the market value of the levered firm can be expressed as

$$V = V_E + V_D = V_u + \tau V_D. \tag{9}$$

Equation (9) shows that the market value of the levered firm is the sum of the market value of the unlevered firm plus the tax subsidy on debt. This result is consistent with M&M [16] within a framework of risky debt.

## II. Systematic Risks and Leverage

In this section we are trying to develop the theoretical linkage between systematic risks of equity, debt, and the unlevered firm and leverage.

### A. The Systematic Risk of Equity, the Systematic Risk of the Unlevered Firm, and Leverage

From Sharpe-Lintner-Mossin's CAPM, the systematic risk of equity is defined as

$$\beta^S = \frac{Cov(\tilde{Y}_E, \tilde{R}_m)}{V_E\ \sigma_m^2} \tag{10}$$

where $\sigma_m^2$ is the variance of market portfolio's returns. Substituting (6) into this definition yields

$$\beta^S = \frac{(1-\tau)Cov(\tilde{X}, \tilde{R}_m)}{V_E\ \sigma_m^2} \cdot [1-F(D)] \tag{11}$$

By the same token, substituting (2) into the definition of the systematic risk of the unlevered firm yields

$$\beta^u = \frac{Cov(\tilde{Y}_u, \tilde{R}_m)}{V_u\ \sigma_m^2}$$

$$= \frac{(1-\tau)Cov(\tilde{X}, \tilde{R}_m)}{V_u\ \sigma_m^2} \cdot [1-F(0)] \tag{12}$$

By solving (11) and (12) for $(1-\tau)Cov(\tilde{X}, \tilde{R}_m)/\sigma_m^2$, we can derive the relationship between the systematic risk of equity and the systematic risk of the unlevered firm as follows

$$\beta^S = \beta^u \left(\frac{V_u}{V_E}\right) \left[\frac{1-F(D)}{1-F(0)}\right]. \tag{13}$$

This result shows that the systematic risk of the levered firm is equal to the systematic risk of the unlevered firm adjusted for the difference in equity value of the two firms and the survival probability (the bracket in (13)). When no bankruptcy risk (or no truncation of the firm's operating income distribution) is assumed, (13) is identical to Hamada's [9] result. Furthermore, substituting the accounting identity in (9) for $V_u$, we derive the following expression:

$$\beta^S = \beta^u\left[1+(1-\tau)\frac{V_D}{V_E}\right]\left[\frac{1-F(D)}{1-F(0)}\right] \tag{14}$$

This result states that the systematic risk of equity is equal to the systematic risk of the same firm without leverage times one plus the leverage ratio (debt to equity) multiplied by one minus tax rate and times the survival probability. If no bankruptcy risk is assumed, the second bracket in (13) disappears and (14) is identical to what Hamada [9] and Rubinstein [18] have shown. Hence, the model we derive here is claimed to be more general.

To further study the comparative statics of (14), we use numerical analysis instead of mathematic analysis for the sake of simplicity.[6] The data for the numerical example is given in table I.

<u>Insert Table I</u>

Figure 1 illustrates the effect of leverage (debt ratio) on the systematic risk of equity. As is expected from this figure, the systematic risk of equity increases monotonically with leverage.

Insert Figure 1

Figure 2 shows the effect of the face value of debt on the systematic risk of equity. Not surprisingly, the systematic risk of equity is a positive function of the face value of debt.

Insert Figure 2

In figure 3, the impact of business risk on the systematic risk of equity is depicted, where business risk is represented by standard deviation of the firm's operating income. To isolate the leverage effect, we designate the face value of debt equal to 150,000. As is evident from this figure, the more risky the firm (the higher the standard deviation), the smaller the systematic risk of equity because stockholders profit from the probability that the value of the firm will exceed the face value of debt.

Insert Figure 3

In the option pricing literature, Black and Scholes [2] and Galai and Masulis [8] have shown that[7]

$$\beta^S = \eta_S \beta^V$$

$$= (1 + \frac{V_D}{V_E}) \frac{\partial V_E}{\partial V} \beta^V \qquad (15)$$

where

$\eta_S = \frac{\partial V_E}{\partial V} \cdot \frac{V}{V_E}$, the elasticity of equity value with respect to firm value;

$\beta^V$ = the systematic risk of the firm.

Comparing (14) with (15) without corporate tax, both equations are quite similar in the sense that the truncation factor in (14) and the partial-derivative factor in (15) both reflect the default risk, and the relationship between the systematic risk of equity and leverage is curvilinear. However, (15) with elasticity concept is not as empirically appealing as (14) with truncated distribution. The latter can be estimated in a way similar to Aharony, Jones, and Swary [1], who estimate the probability of bankruptcy from a truncated normal distribution. Omitting the truncation factor is (14) which is always less than one with positive leverage will cause the systematic risk of equity overestimated. Hence, the implication of this model stands along the same line as Hamada [9, p. 445] in the sense that it should be possible to improve the forecast of a stock's systematic risk by forecasting the total firm's systematic risk first, and then make adjustments on leverage and survival probability.

B. The Systematic Risk of Equity, the Systematic Risk of the Unlevered Firm, and the Systematic Risk of Debt

Like the systematic risk of equity, the systematic risk of debt can be defined by the CAPM as[8]

$$\beta^D = \frac{\mathrm{Cov}(\tilde{Y}_D, \tilde{R}_m)}{V_D\ \sigma_m^2}$$

$$= \frac{\mathrm{Cov}(\tilde{X}, \tilde{R}_m)}{V_D\ \sigma_m^2} [F(D) - F(0)] \qquad (16)$$

Given the result shown in (16), we can further demonstrate the linkage between the systematic risk of equity, the systematic risk of the unlevered firm, and the systematic risk of debt (the proof is shown in the Appendix).

$$\beta^S = \beta^u[1+(1-\tau)\frac{V_D}{V_E}] - \beta^D[(1-\tau)\frac{V_D}{V_E}] \tag{17}$$

This result is consistent with Conine [7] in the presence of risky corporate debt. The same result without corporate tax can be derived from the option pricing model.[9] The model says that the systematic risk of equity is a weighted average of the systematic risk of the unlevered firm and the systematic risk of debt (with negative weight), which is intuitively appealing in a portfolio sense. By using the same numerical example, figure 2 illustrates that the systematic risk of debt not only is a positive function of the face value of debt but cannot in equilibrium exceed the systematic risk of the unlevered firm. Under this kind of formulation, the truncation of the distribution due to default risk is not shown in (17), instead is embedded in $\beta^D$ and leverage. When corporate debt is riskfree, $\beta^D$ is equal to zero and (17) is identical to the traditional formulation shown by Hamada and Rubinstein.

## III. Conclusion

The purpose of this note is to investigate the theoretical relationship between systematic risks and leverage in the presence of default risk. The cash-flow approach is used in contrast to the literature. The analysis shows that a truncation factor (or survival probability) exists in addition to Hamada [9] and Rubinstein's [18] traditional formulation. Hence, the result derived here is claimed to be more general.

Footnotes

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[1](1) There exists a fixed risk-free interest rate in perfectly competitive capital markets; (2) all investors have homogeneous expectations with respect to the probability distributions of future yields on risky assets; and (3) all investors are risk-averse and the expected utility of terminal wealth maximizers.

[2]Because of the existence of default risk, the assumption of quadratic utility is implicitly required to apply the CAPM.

[3]For discussion of truncation, refer to Lintner [13] and Chen [6].

[4]This is an agency-cost issue. A numerical example can illustrate why bondholders will not receive the entire after-tax value of the firm if the firm is declared bankrupt. Let D = \$100, $\tau$ = 50%, and X = \$99. In this case, the firm is declared bankrupt because $X < D$. If bondholders had to receive the after-tax value of the firm, \$49.5, they would be better off by making side payments of the one dollar short to stockholders to persuade them not to go bankrupt. Hence, bondholders would net \$99, which is exactly equal to X.

[5]We assume that there are no costs of voluntary liquidation or bankruptcy, e.g., court or reorganization costs.

[6]We should expect to get identical results as shown by Galai and Masuli [8] in an option pricing context.

[7]No corporate and personal taxes are assumed.

[8]Since the debt by nature is a single-period discount bond, the problem of duration on the systematic risk of debt does not arise.

[9]Black and Scholes [2] and Galai and Masulis [8] have shown that

$$\beta^S = N(d_1)\frac{V}{V_E}\,\beta^V \tag{18}$$

$$\beta^D = [1-N(d_1)]\,\frac{V}{V_D}\,\beta^V \tag{19}$$

where $N(\cdot)$ is the standardized normal cumulative probability density function. Then, multiplying (19) by $\frac{V_D}{V_E}$, adding (18), and rearranging yields

$$\beta^S = \beta^V\left(\frac{V}{V_E}\right) - \beta^D\left(\frac{V_D}{V_E}\right).$$

$$= \beta^V\left(1+\frac{V_D}{V_E}\right) - \beta^D\left(\frac{V_D}{V_E}\right).$$

Q.E.D.

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Table I

Parameters for Numerical Example

Corporate tax rate ($\tau$) = 0.5

Expected market return ($\overline{R}_m$) = 0.15

One plus risk free rate (R) = 1.05

Standard deviation of market return ($\sigma_m$) = 0.2

Standard deviation of operating income ($\sigma_x$) = 80,000

Mean of operating income ($\overline{X}$) = 120,000

Correlation coefficient between the firm and the market = 0.5

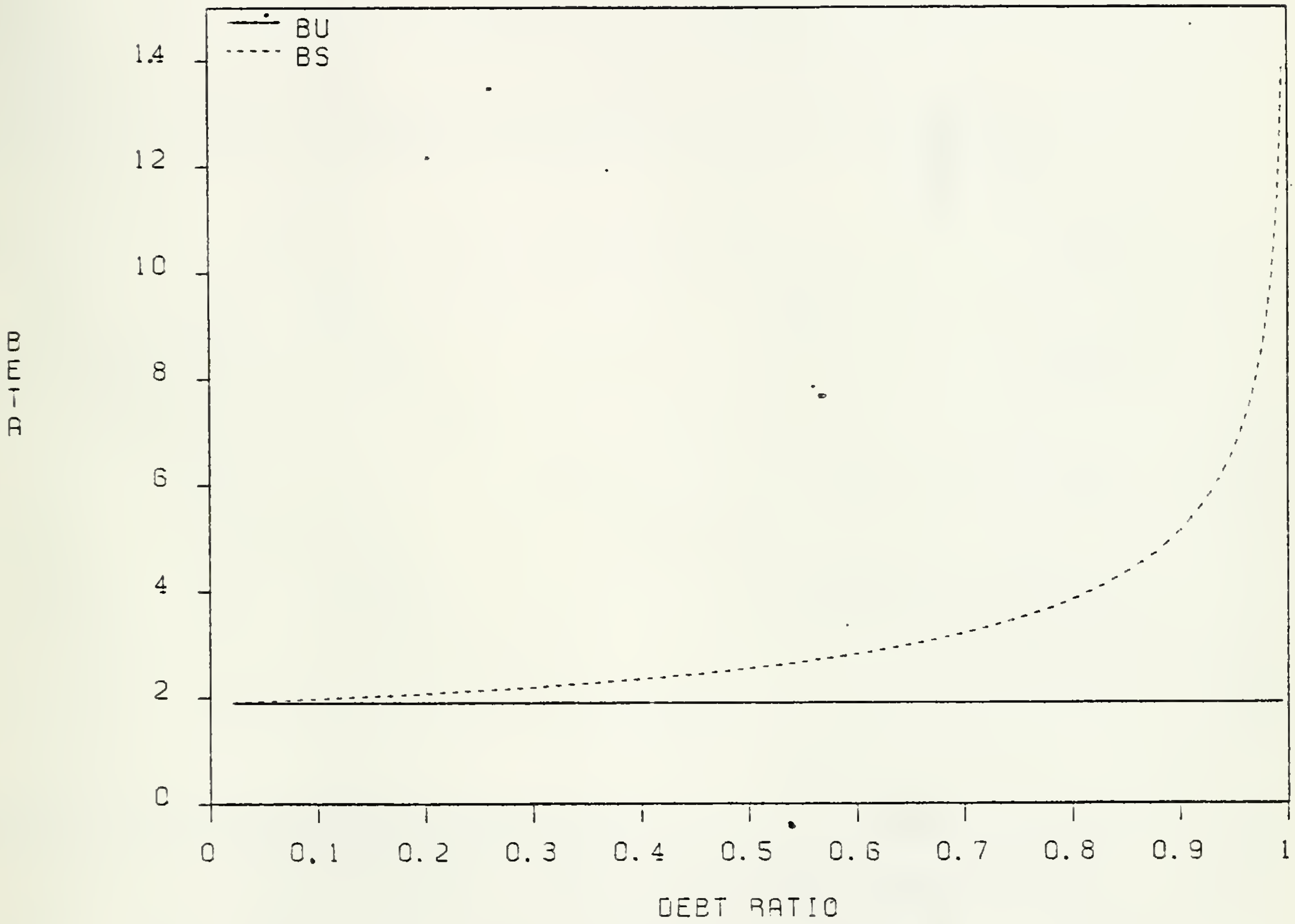


Figure 1. The relationship between systematic risks and debt ratio

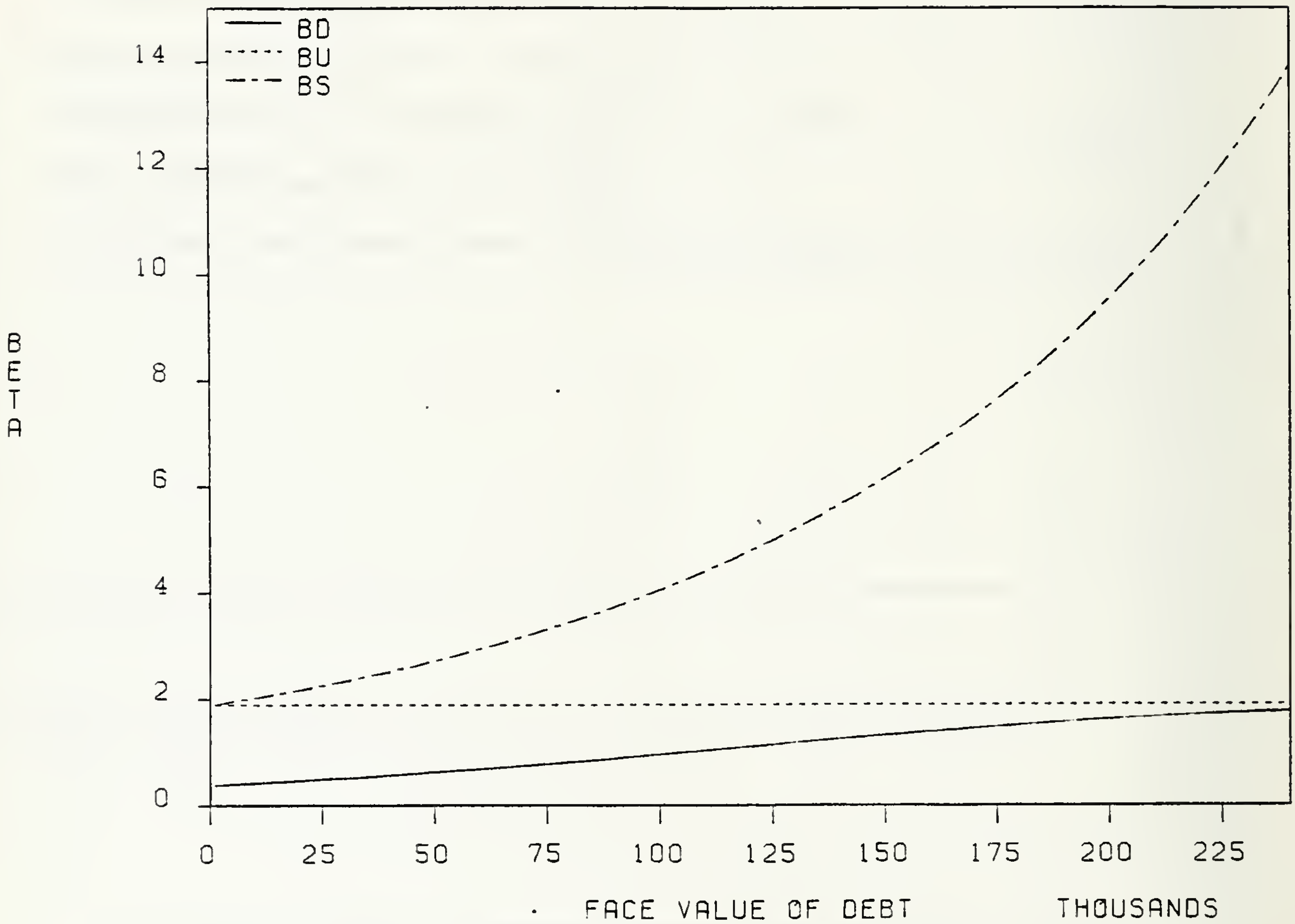


Figure 2. The relationship between systematic risks and face value of debt

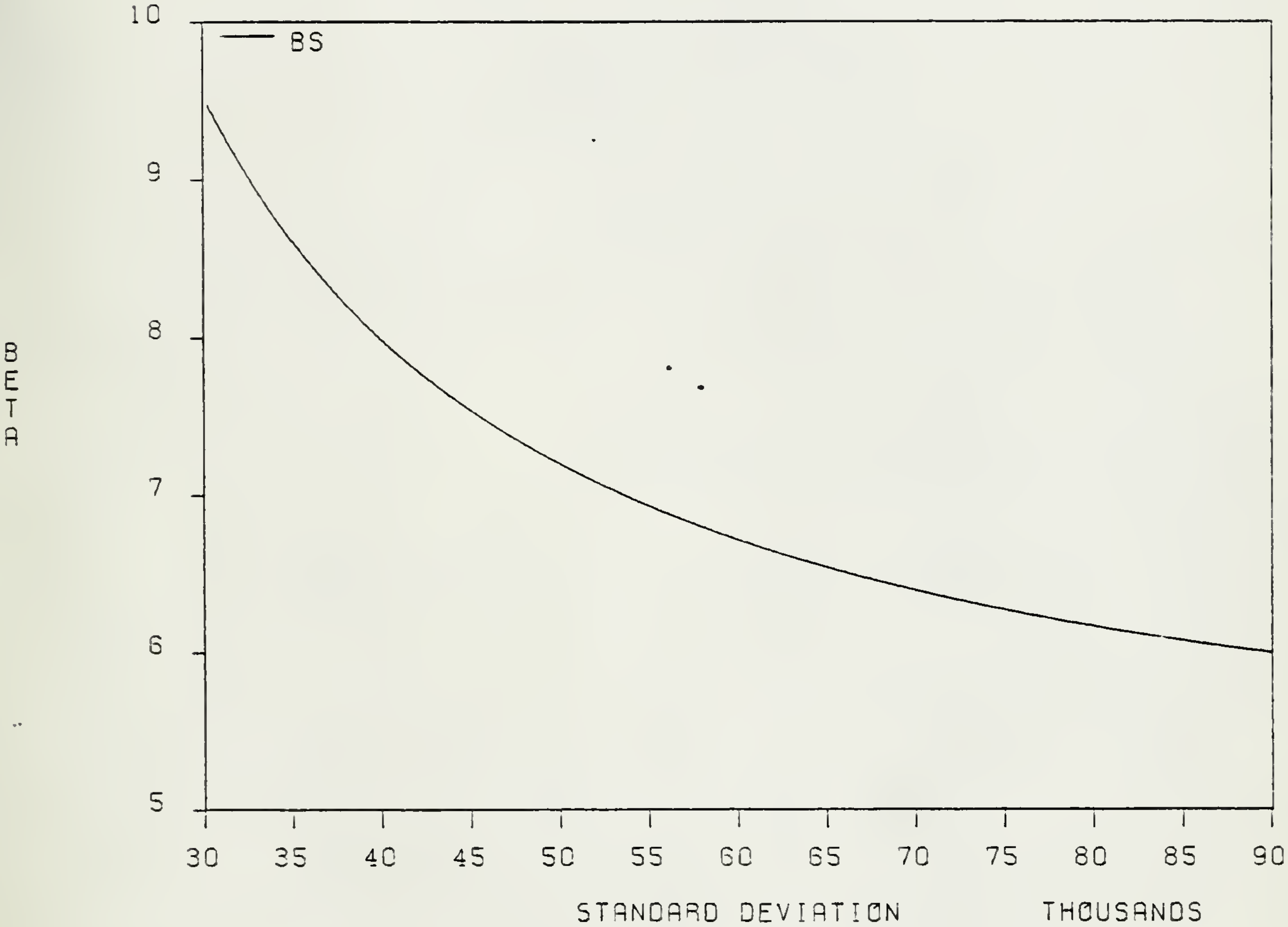


Figure 3. The relationship between systematic risk and standard deviation

## Appendix

Let's restate equation (14) as follows

$$\beta^S = \beta^u \left(\frac{V_u}{V_E}\right) \left[\frac{1-F(D)}{1-F(0)}\right]$$

$$= \beta^u \left(\frac{V_u}{V_E}\right) - \beta^u \left(\frac{V_u}{V_E}\right) \left[\frac{F(D)-F(0)}{1-F(0)}\right] \tag{18}$$

We also can derive the relationship between the systematic risk of the unlevered firm and the systematic risk of debt by solving (12) and (16) for $Cov(\tilde{X},\tilde{R}m)/\sigma_m^2$.

$$\beta^u = \beta^D (1-\tau) \frac{V_D}{V_u} \left[\frac{1-F(0)}{F(D)-F(0)}\right] \tag{19}$$

Then, substituting (19) into the second term of (18) yields

$$\beta^S = \beta^u \left(\frac{V_u}{V_E}\right) - \beta^D (1-\tau) \frac{V_D}{V_E}$$

$$= \beta^u \left[1+(1-\tau) \frac{V_D}{V_E}\right] - \beta^D \left[(1-\tau) \frac{V_D}{V_E}\right]$$

Q.E.D.