A SYSTEM OF APPLIED OPTICS

A SYSTEM

OF

APPLIED OPTICS

BEING

A COMPLETE SYSTEM OF FORMULÆ OF THE SECOND ORDER, AND THE FOUNDATION OF A COMPLETE SYSTEM OF THE THIRD ORDER, WITH EXAMPLES OF THEIR PRACTICAL APPLICATION

BY

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OPTICAL MANAGER TO T. COOKE AND SONS, LTD.
OF LONDON, YORK, AND CAPE TOWN

London
MACMILLAN AND CO., LIMITED
NEW YORK: THE MACMILLAN COMPANY
1906

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NOTE AS TO THE REFERENCE LETTERS

WHILE ordinary Latin capitals and minuscules were selected as being the more suitable for the diagrams, owing to their upright character, yet the corresponding small italic letters have been employed in the text because of their greater conspicuousness. In case this should cause inconvenience to any reader of other than English nationality, who is not quite familiar with both these forms of small letters, the following list of the two sets of equivalent characters may be found useful :—

LETTERS IN DIAGRAMS	EQUIVALENTS IN TEXT	LETTERS IN DIAGRAMS	EQUIVALENTS IN TEXT
a	*a*	n	*n*
b	*b*	o	*o*
c	*c*	p	*p*
d	*d*	q	*q*
e	*e*	r	*r*
f	*f*	s	*s*
g	*g*	t	*t*
h	*h*	u	*u*
i	*i*	v	*v*
j	*j*	w	*w*
k	*k*	x	*x*
l	*l*	y	*y*
m	*m*	z	*z*

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INTRODUCTION

I TRUST that the student of Optics who casually scans the pages of this work for the first time, will not be alarmed by the complicated appearance of some of the formulæ employed in the course of working out the conclusions, and therefore infer that it is necessary to be highly trained in mathematics in order to follow the lines of reasoning employed. For such is not the case; all that is really necessary in the mathematical equipment of the student being an easy acquaintance with the ordinary manipulations of Algebra, together with a clear grasp of the Binomial Theorem, the chief propositions of Euclid, and the rudiments of the Differential Calculus. That granted, and given some instinct for the practical application of what he knows, then he will have no insuperable difficulty in following this work from cover to cover.

The greater part is easy compared to the numerous problems and theorems which the average university student is called upon to solve, and which in so many cases are treated as of purely theoretical interest. After all, is not that the truest and most fruitful teaching of mathematics which fully recognises the mutual support between theory and practice? Otherwise it is but natural if the student cleaves to the one and despises the other.

I do not wish to imply that there is no scope for the employment of the highest mathematical skill in optical science; for, on the contrary, there are numerous problems in connection with the corrections of the third order of approximation, merely glanced at in Section XI. of this work, which pre-eminently call for the elucidating and marshalling influence of some clear-headed mathematician who shall be thoroughly familiar with the properties of lenses from practical acquaintance, and not only from the theoretical point of view. The closer approach to perfection in the optical combinations of the future will lie in the more thorough elimination of the corrections of the third order, and in some cases of the fourth order, and the most highly trained mathematical skill, if it should ever deign to busy itself in this country with the higher practical requirements of optical science, would doubtless be able to evolve corollaries of the greatest importance bearing upon this question.

My chief object in working out the scheme of Applied Optics herein explained, has been to arrive at a complete system of algebraic formulæ of the second order which can be applied to any optical system likely to occur in practice with results which in general very closely approach to accuracy. I have therefore confined myself for the most part to the attainment of those practical conditions which have to be fulfilled by the best optical constructions—conditions which include, and run closely parallel to, Von Seidel's five well-recognised conditions.

As far as I know, there is only one work in the English language professing to give a sketch of Von Seidel's methods, and that is Professor Silvanus Thompson's *Contributions to Photographic Optics*,[1] after Otto Lummer, while there are numerous accounts of his methods published in German works, and several treatises built upon them, such as Steinheil and Voit's *Handbuch der Angewandten Optik*,[2] 1891, and Von Rohr's *Theorie und Geschichte des photographischen Objectivs*,[3] the latter a most instructive and valuable work; and last, but not least, Dr. Siegfried Czapski's new edition of *Der Theorie des optischen Instrumenten*,[4] 1904. This last work is a philosophical, broad, and general survey of the various problems which have to be faced, and if possible solved, by the optical designer who would rise superior to mere rule of thumb. But its perusal requires in many respects a higher level of mathematical training than is necessary for the understanding of this treatise.

In the German language there exists quite a mine of optical literature written by men who are practical opticians as well as mathematical experts, while we have scarcely anything of a corresponding nature in the English tongue.

The fact that such works as I have just mentioned have been published in Germany (as first editions, at any rate) for so many years, and yet no demand has ever arisen for English translations, is only too painful evidence of the apathy with which the Science of Optics has been regarded in this country.

There are, of course, various works on geometrical optics which have more or less recently emanated from our universities, such as Heath's *Geometrical Optics*,[5] Parkinson's *Optics*,[6] Pendlebury's *Lenses and*

[1] Macmillan and Co., 1900.
[2] Teubner, Leipzig.
[3] Julius Springer, Berlin, 1899.
[4] Barth, Leipzig.
[5] Cambridge University Press, 1895.
[6] Macmillan and Co., 1900.

Systems of Lenses,[1] Perceval's *Optics*, etc., which are excellent as furnishing material for purely mathematical students working up for examinations; but the manner in which the various problems are dealt with is in many cases ill adapted for application in practice, while certain matters of the highest importance are ignored altogether.

As a matter of fact there is not an English work on geometrical optics extant by whose guidance an ordinary photographic lens could be worked out in all particulars. Professor Silvanus Thompson's account of Von Seidel's system does not, however, give the impression that the latter's methods and notation are at all easy to comprehend, but certain it is that his system has been successfully employed for very many years by numerous mathematicians and opticians of the highest rank on the Continent, while the foundation-stone of English optical science has been left unbuilt upon.

I here allude to the all-important work which was done about thirty years before that of Von Seidel by Sir George Airy, and still more by Henry Coddington. Sir G. Airy published some highly important papers in 1827 in the *Cambridge Philosophical Transactions* on "The Spherical Aberration of Eye-pieces of Telescopes," and another paper on the Achromatism of the same.

Then Henry Coddington took up the work, and by the aid of some very ingenious devices of his own contrivance greatly added to the simplicity and universality of the formulæ arrived at by Airy. In 1829 he published his labours under the title, *A Treatise on the Reflection and Refraction of Light*, which, although still the best work on geometrical optics from the practical optician's point of view, nevertheless contains many shortcomings, which I attribute chiefly to the fact that he had not had very much practical acquaintance with lenses and their properties. It is therefore with much diffidence that I venture to criticise and to supplement many of his methods and formulæ, especially when I feel sure that had it not been for his labours this treatise would never have been undertaken.

Another very important work on geometrical optics, now very little known, was Richard Potter's *Elementary Treatise on Optics*, Part II. of which, published in 1851, contains certain formulæ for spherical aberration of the third approximation.

I may here state that the invention of the "Cooke" lenses for photography was not of a haphazard nature, but occurred in this way. I had been studying Coddington's work very carefully and did not feel quite satisfied with his method of working out the curvature

[1] Deighton, Bell and Co., Cambridge, 1884.

of the image formed by a lens, in the cases of both central and eccentric oblique refractions. He assumed the aperture of the pencil of rays in question to be infinitely narrow, and got at his results by the employment of the differential calculus. I saw that while this would be quite valid for such infinitely narrow pencils, still, as considerably broad pencils generally occur in practice, it struck me it might be worth while trying to devise a method *not* dependent upon the calculus, whereby the foci of broad oblique and eccentric pencils could be elucidated, when possibly some new results of practical importance might be forthcoming. About the year 1890 I undertook that task, and after meeting with many difficulties which almost compelled me to give up the investigation as hopeless, I at last succeeded in arriving at the results embodied in Sections V., VI., and VII. of this volume, and in so doing was fortunate enough to bring to light the formula relating to coma, a phenomenon that appears, strange as the fact may seem, never to have been noticed by Coddington. I then saw that the formulæ I thus arrived at implied corollaries of the greatest practical importance, and I was led almost directly to the conception of the Cooke lens, that is, of the older complex Cooke lens built up of two achromatic positive lenses and one achromatic negative lens. The simple Cooke lens was of later conception. Thus the theory preceded the practice, although I should say that there are certain other features of the Cooke lens, such as distortion and oblique achromatism more especially, whose theory I did not arrive at until a few years later, so that in that respect the practice preceded the theory.

Having subsequently worked out a complete system of formulæ, which I have proved and tested and found reliable in all manner of ways, and recognising the great importance of theory and practice working loyally together for future improvements, I thought that as soon as I had time enough at my disposal I would gather together and arrange what has been the interrupted labour of many years, with a view to publication, if by so doing I could, even in a humble degree, forward the development of optical science in this country, wherein it has lain so long neglected, or perhaps furnish some raw material on which some far abler heads than mine should at some future time found important corollaries not yet dreamed of.

Considerations of space have compelled me to confine myself to theorems and formulæ that I consider to be of the greatest practical value, and to leave out many corollaries of minor importance that might be dealt with in a future edition, were it ever called for.

There are also many problems and theorems untouched upon, which are only of theoretical importance or of interest from a mathematical point of view, and of little value to the practical optician, such as, for instance, the theory of caustics, planes of unit magnification, etc., about which the more mathematical student can obtain full information from various contemporary works of well-known repute, such as those mentioned above, as well as Coddington's work, which, however, is now out of print and often difficult to procure.

It will be observed that I have not given the lines of reasoning by which the formulæ of the first approximation are arrived at; for I have assumed that the student will bring with him to the study of this work a knowledge of such elementary optical formulæ. For those who wish to enter upon it without that knowledge I do not know a better book to recommend as a clearly written first guide to the formulæ of the first approximation than Todhunter's *Optics* (in Part II. of his *Natural Philosophy for Beginners,* 1877, which I believe is also out of print) or Lardner's *Optics,* and the series of articles on "Applied Optics" by Dr. Drysdale in the *British Optical Journal.*

I think it must be conceded that, while the method of investigating the foci of oblique and eccentric pencils of finite or large aperture explained in this work leads to novel and highly important formulæ of the second approximation, and some others which are novel in many respects, it also opens out possibilities of working out formulæ of the third and in some cases the fourth approximations, which in the hands of a skilful mathematician may lead to new and useful results of great importance; while the application of the differential method of Coddington and other workers to infinitely narrow pencils is exceedingly limited in its scope and results, as I shall show.

At first sight it seems a remarkable thing that a system of surfaces bound by the simplest of all known curves, namely, the circle, with their centres on a common axis, should give rise to problems which, *if solved to a high degree of exactitude,* are of such extraordinary complexity.

I gladly take the present opportunity of expressing my thanks to Sir W. de W. Abney and Professor Silvanus P. Thompson for much kind encouragement and valuable help; and also to Dr. Moritz von Rohr for allowing me to reproduce some of his diagrams on Plate XXIV.

In conclusion, I shall be only too glad if any technical errors or obscurities, which must, in spite of all care, exist in a work of this kind, are pointed out to me.

H. DENNIS TAYLOR.

SECTION I

A RECAPITULATION

We will first of all recapitulate those well-known formulæ of the first approximation relating to ultimate axial rays constituting direct or axial pencils, or, in other words, extremely narrow pencils whose central or principal ray coincides with the axis or straight line joining the origin or apex of the pencil to the centre of curvature of the spherical surface. Spherical aberration is in such cases a vanishing quantity and is therefore not regarded. Throughout this work it is assumed that all reflecting and refracting surfaces are either plane or spherical.

Case of a Plane or Curved Reflector

Throughout the diagrams in this book light is supposed to be travelling from left to right.

Plane reflector.—Here if Q (Plate I.) be the origin and Q . . A, the principal ray, be perpendicular to the reflecting surface R . . R, then after reflection the rays will proceed backwards as if originating from a virtual point q situated on Q . . A projected and at a distance A . . q from the surface equal to A . . Q. On the contrary, if the incident pencil is of rays converging to the apex q, then they will be reflected back to a real point Q such that A . . Q = A . . q and Q . . q is normal to R . . R.

Law connecting conjugate focal distances for plane reflector.

If the reflecting surface be curved spherically as r . . r, Figs. 2a, 2b, 2c, and 2d, c being the centre of curvature and Q the origin or apex of the incident pencil, then the formula

Formula connecting conjugate focal distances for spherical reflector.

$$\frac{1}{A..q} = \frac{2}{A..C} - \frac{1}{A..Q} \quad \text{or} \quad = \frac{1}{F} - \frac{1}{A..Q} \qquad \text{I.}$$

universally applies and interprets itself in all cases if the following conventions are strictly adhered to, viz.—

PLATE. I.

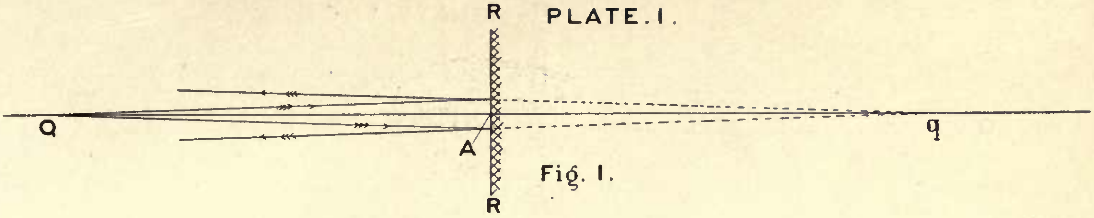


Fig. 1.

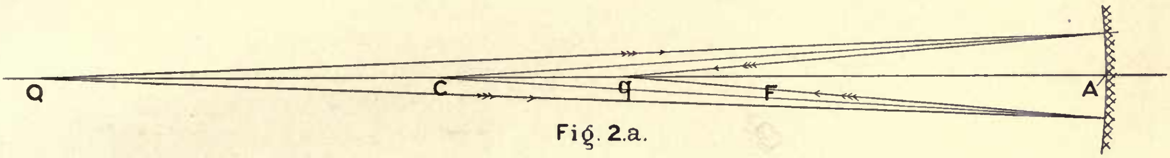


Fig. 2.a.

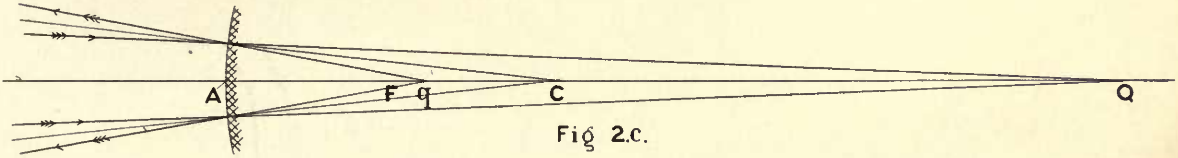


Fig 2.c.

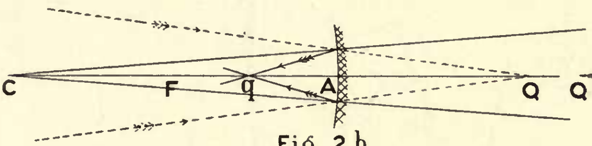


Fig. 2.b.

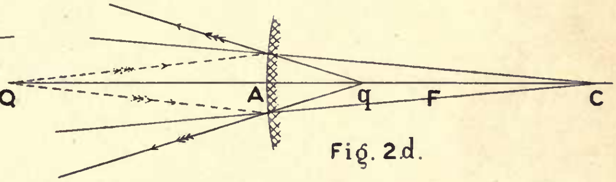


Fig. 2.d.

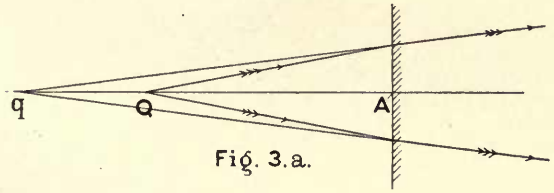


Fig. 3.a.

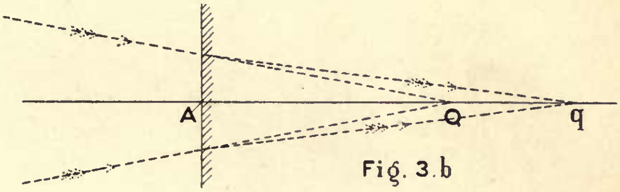


Fig. 3.b

Fig. 4.a.

Fig. 4.b.

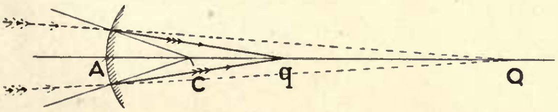


Fig. 4.c.

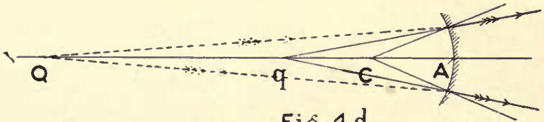


Fig. 4.d.

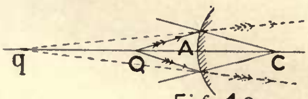


Fig. 4.e.

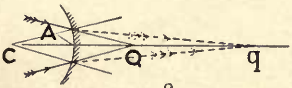


Fig. 4.f.

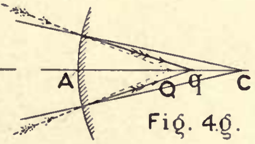


Fig. 4.g.

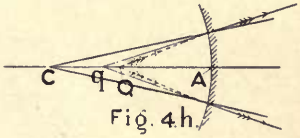


Fig. 4.h.

PLATE. I.

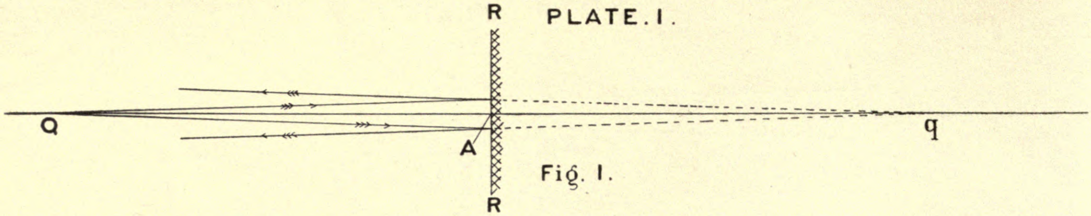


Fig. 1.

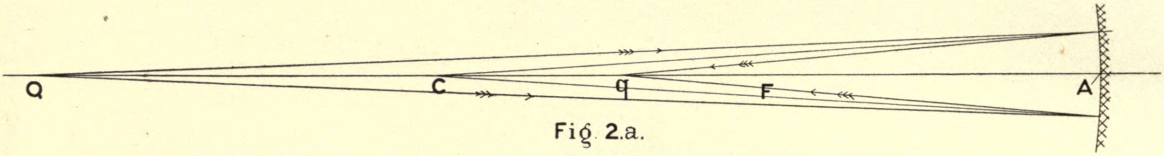
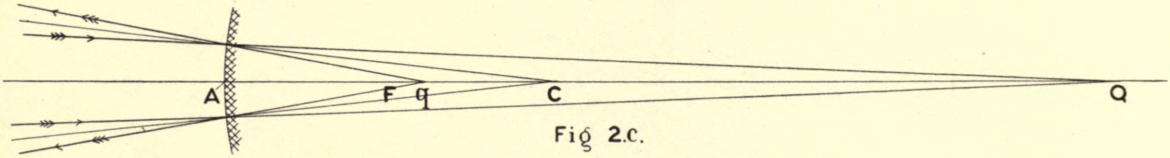


Fig. 2.a.



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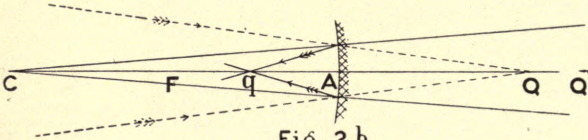

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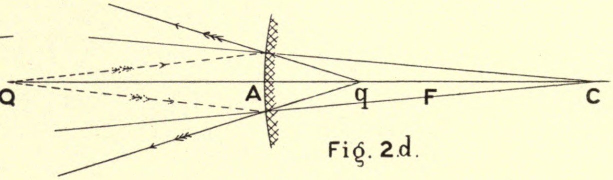

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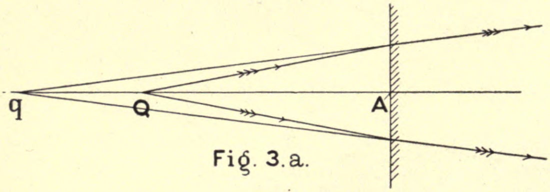


Fig. 3.a.

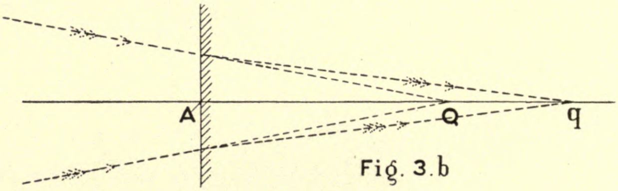


Fig. 3.b

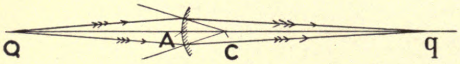


Fig. 4.a.

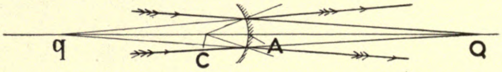


Fig. 4.b.

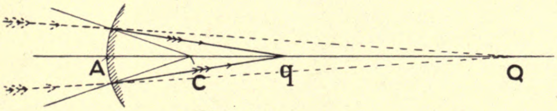


Fig. 4.c.

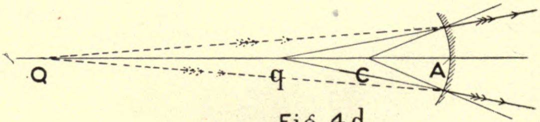
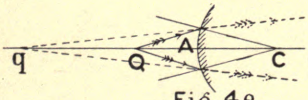


Fig. 4.d.



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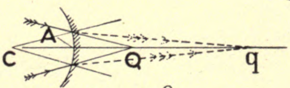


Fig 4.f.

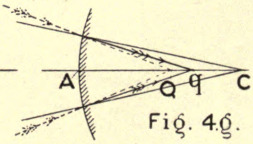


Fig. 4.g.

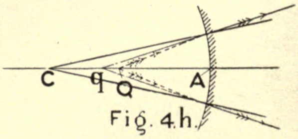


Fig. 4.h.

The radius of curvature A..C is to be considered as an intrinsically positive quantity whether the surface be convex or concave; and then—

For concave reflector—

If rays of the incident pencil are divergent, then Q . . A is positive.
If rays of incident pencil are convergent, then A . . Q is negative.
If rays of reflected pencil are convergent, then A . . q is positive.
If rays of reflected pencil are divergent, then q . . A is negative.

Reflector Conventions as to signs.

And for convex reflector—

If rays of incident pencil are convergent, then A . . Q is positive.
If rays of incident pencil are divergent, then Q . . A is negative.
If rays of reflected pencil are divergent, then q . . A is positive.
If rays of reflected pencil are convergent, then A . . q is negative.

For instance, in the case of Fig. 2*d* we have $\frac{1}{A..q} = \frac{1}{F} - \frac{1}{A..Q}$, but by convention A . . Q is a negative quantity, therefore the formula is $\frac{1}{A..q} = \frac{1}{F} - \frac{1}{-A..Q}$ or $\frac{1}{F} + \frac{1}{A..Q}$, therefore A . . Q comes out divergent and positive.

Instances of applications of signs to reflected pencils.

Should Q . . A or A . . Q be infinite or the rays of the incident pencil be parallel, then of course $\frac{1}{A..Q}$ becomes zero, and $\frac{1}{A..q}$ becomes $\frac{2}{A..C}$ or $\frac{1}{F}$, and the rays converge to or diverge from the principal focus of the mirror.

The dotted lines in the figures indicate negative distances, and the full lines the positive distances.

Plane Refracting Surfaces

In the case of normal or perpendicular incidence of small pencils at a plane refracting surface bounding a transparent substance whose refractive index $= \mu$, while that of the left-hand medium $= o$, the simple relationship A . . $q = \mu$(A . . Q) holds good. See Figs. 3*a* and 3*b*. 1

μ = refractive index.

Spherical Refracting Surfaces

In the case of direct refraction of normal pencils by spherical surfaces, as in Figs. 4*a*, *b*, *c*, *d*, *e*, *f*, *g*, and *h*, the formula

$$\frac{\mu}{Aq} = \frac{\mu - 1}{A..C} - \frac{1}{A..Q}$$

or

$$\frac{\mu}{u} = \frac{\mu - 1}{r} - \frac{1}{u}, \qquad \text{II.}$$

Formula connecting focal distances in case of refraction at single surface.

holds good if we put u for A..Q, r for the radius A..C, and $\grave{u}$ for A..q, and this formula interprets itself for all cases, provided the following conventions are strictly adhered to, viz.:—

Convention as to signs of focal distances.

The radii of all surfaces, whether convex or concave, to be considered intrinsically positive with respect to the conjugate distances whose signs are to be assessed.

Then for convex surfaces—

Rays of incident pencil diverging, then Q..A or u is positive. Figs. 4a and 4e.

Rays of incident pencil converging, then A..Q or u is negative. Figs. 4c and 4g.

Rays of refracted pencil converging, then A..q or $\grave{u}$ is positive. Figs. 4a, 4c, and 4g.

Rays of refracted pencil diverging, then q..A or $\grave{u}$ is negative. Fig. 4e.

And for concave surfaces—

Rays of incident pencil converging, then A..Q or u is positive. Figs. 4b and 4f.

Rays of incident pencil diverging, then Q..A or u is negative. Figs. 4d and 4h.

Rays of refracted pencil diverging, then q..A or $\grave{u}$ is positive. Figs. 4b, 4d, and 4h.

Rays of refracted pencil converging, then A..q or $\grave{u}$ is negative. Fig. 4f.

Thus, in the case of Fig. 4c, A..Q is convergent and therefore u is negative, and

$$\frac{\mu}{\grave{u}}=\frac{\mu-1}{r}-\frac{1}{u}$$

becomes

Rays entering convex surface convergent.

$$\frac{\mu}{\grave{u}}=\frac{\mu-1}{r}+\frac{1}{u}.$$

And, again, in a case where Q..A in Fig. 4a becomes less than $\frac{r}{\mu-1}$, then of course $\frac{\mu}{\grave{u}}=\frac{\mu-1}{r}-\frac{1}{u}$ gives a negative result, and the refracted pencil is shown to be divergent, as in Fig. 4e.

Rays leaving convex surface divergent.

If the rays of the incident pencil are parallel and therefore

$$\frac{1}{Q..A}=\text{zero},$$

therefore

$$\frac{\mu}{\grave{u}}=\frac{\mu-1}{r},$$

Focal distance when entering rays are parallel.

and

$$\grave{u}=r\,\frac{\mu}{\mu-1}.$$

If, on the other hand, $QA=\frac{r}{\mu-1}$, then $\frac{\mu}{\grave{u}}=0$, and the rays of the refracted pencil are parallel.

Refracted rays parallel.

We are now in a position to consider the cases of two spherical surfaces in succession enclosing glass between them and forming a lens. We will assume the axial thicknesses of such lenses to be negligible, the two spherical surfaces being brought to a sharp edge in the case of collective lenses and the diameter or aperture being very small compared to the principal focal length, while in the case of dispersive lenses the two spherical surfaces may be supposed to touch one another on the lens axis, the axial thickness being zero. Let us take a case like Fig. 4*a*, wherein the rays after refraction at the first surface are convergent and $\grave{u}$ is positive. Let these convergent rays proceed through a second convex surface, as shown in Fig. 5*a*.

Two closely following surfaces constitute a lens.

So far, lenses assumed to have no central thickness.

We saw that in the case of Fig. 4*a* the distance $A\,..\,q$ or $\grave{u}$ was given by the equation $\frac{\mu}{\grave{u}}=\frac{\mu-1}{r}-\frac{1}{u}$, from which we get $\frac{1}{u}=\frac{\mu-1}{r}-\frac{\mu}{\grave{u}}$. We can apply this equation to the refraction, taken in the reverse direction, at the second surface, as shown in Fig. 5*a*, Plate II., wherein $A_2\,..\,Q_2$ corresponds to u, and $A_2\,..\,q=\grave{u}$; only in this case $A_2\,..\,Q_2$ may be better expressed as $=v$, and the radius of curvature as s, so that we get

$$\frac{\mu}{\grave{u}}=\frac{\mu-1}{s}-\frac{1}{v}$$

Course of rays at second surface considered reversed.

and

$$\frac{1}{v}=\frac{\mu-1}{s}-\frac{\mu}{\grave{u}}.$$

But as the rays of the pencil are converging (left to right) into the second surface, and the distance $\grave{u}$ becomes, relatively to the second surface, negative, therefore the above equation becomes

$$\frac{1}{v}=\frac{\mu-1}{s}+\frac{\mu}{\grave{u}}.$$

But $\frac{\mu}{\grave{u}}$ by the refraction at the first surface was shown to be $\frac{\mu-1}{r}-\frac{1}{u}$. Substituting this value in the above equation we get

$$\frac{1}{v}=\frac{\mu-1}{s}+\frac{\mu-1}{r}-\frac{1}{u};$$

Formula connecting conjugate focal distances in the case of a lens.

or

$$\frac{1}{v} = (\mu - 1)\left(\frac{1}{r} + \frac{1}{s}\right) - \frac{1}{u}, \qquad \text{III.}$$

which well-known formula applies to all thin lenses whatsoever under the following conventions.

Collective Lenses

Conventions as to signs of radii.

The focal length of a collective lens must be considered a positive quantity with respect to the conjugate focal distances. The radii of all convex surfaces are considered intrinsically positive, while the radii of all concave surfaces are considered intrinsically negative, their radii, of course, being always numerically greater than the radii of the convex surfaces in the same lenses, so that the deeper curved surface determines the character of the lens.

Conventions as to signs of conjugate focal distances.

If rays of incident pencil are diverging, u is real and $+$. Figs. 6*a* and 6*e*.

If rays of incident pencil are converging, u is virtual and $-$. Fig. 6*c*.

If rays of emergent pencil are converging, v is real and $+$. Figs. 6*a* and 6*c*.

If rays of emergent pencil are diverging, v is virtual and $-$. Fig. 6*e*.

Dispersive Lenses

Conventions as to signs of radii.

The focal length of a dispersive lens is also to be considered a positive quantity with respect to the conjugate focal distances. The radii of all concave surfaces are considered intrinsically positive, while the radii of all convex surfaces are considered intrinsically negative, their radii, of course, being always numerically greater than the radii of the concave surfaces in the same lenses, the deeper curved surface again determining the character of the lens.

Conventions as to signs of conjugate focal distances.

If rays of incident pencil are converging, u is virtual and $+$. Figs. 6*b* and 6*f*.

If rays of incident pencil are diverging, u is real and $-$. Fig. 6*d*.

If rays of emergent pencil are diverging, v is virtual and $+$. Figs. 6*b* and 6*d*.

If rays of emergent pencil are converging, v is real and $-$. Fig. 6*f*.

Illustrations. Meaning of full and dotted lines.

Figs. 6*a*, *b*, *c*, *d*, *e*, and *f* are illustrations of these conventions. As in Fig. 4, and generally throughout this book, all intrinsically positive distances are drawn in full lines, drawn thinner where

PLATE. II.

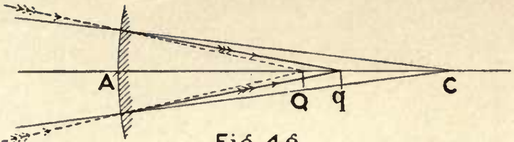


Fig. 4.g.

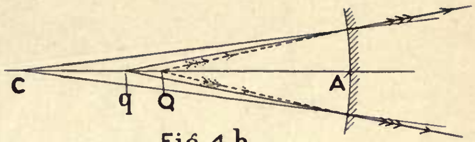


Fig. 4.h.

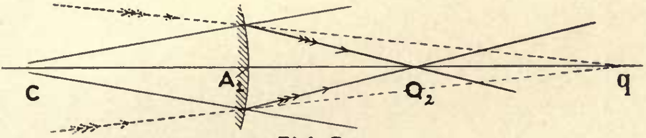


Fig. 5.a.

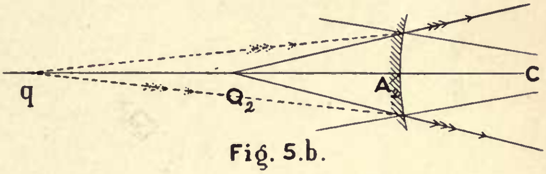


Fig. 5.b.

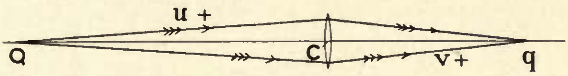


Fig. 6.a

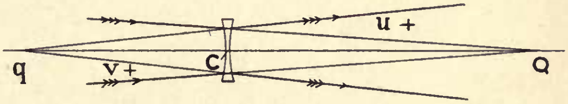


Fig. 6.b.

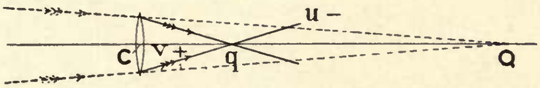


Fig. 6.c.

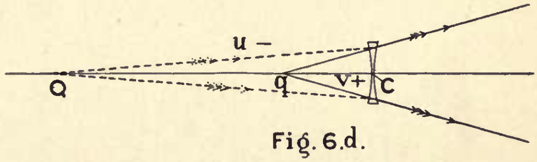


Fig. 6.d.

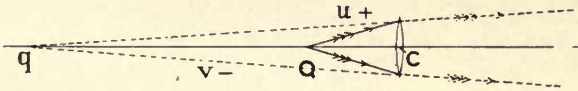


Fig. 6.e.

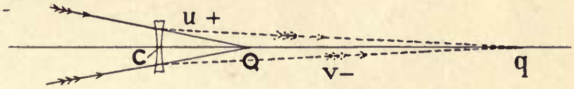


Fig. 6.f.

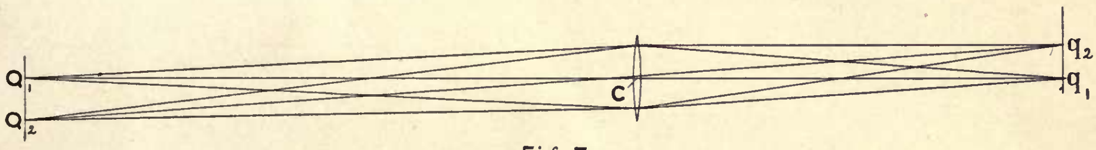


Fig. 7.a.

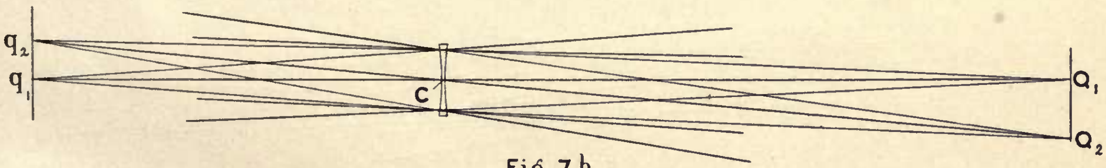


Fig. 7.b.

PLATE. II.

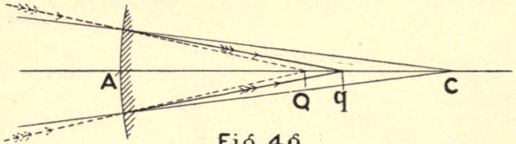


Fig. 4.g.

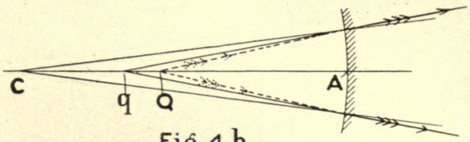


Fig. 4.h.

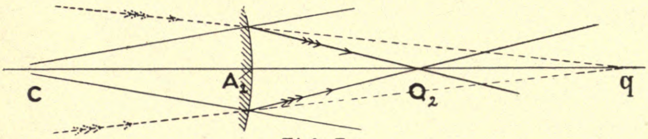


Fig. 5.a.

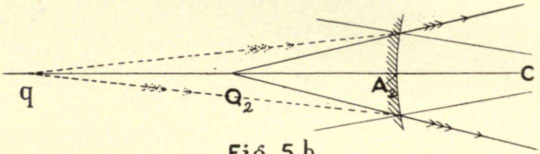


Fig. 5.b.

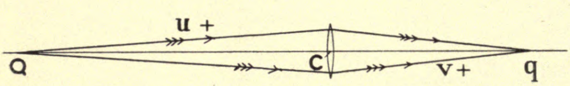


Fig. 6.a

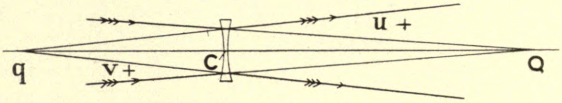


Fig. 6.b.

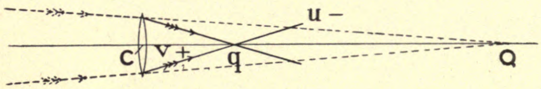


Fig. 6.c.

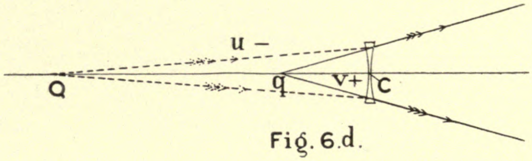


Fig. 6.d.

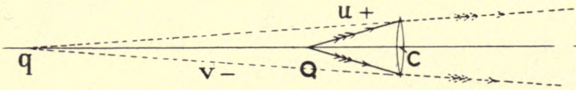


Fig. 6.e.

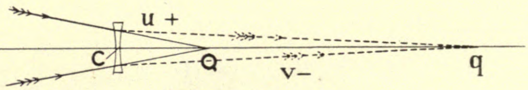


Fig. 6.f.

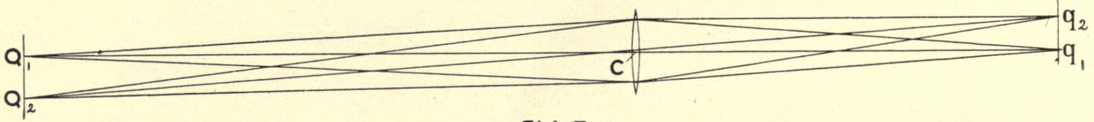


Fig. 7.a.

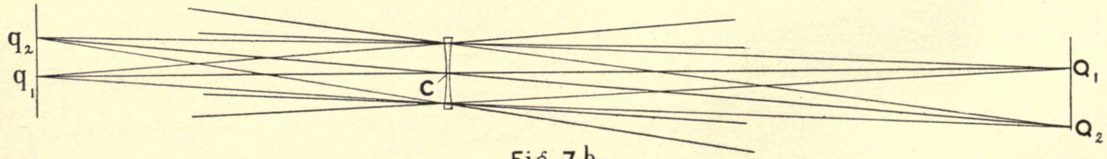


Fig. 7.b.

virtual; and all intrinsically negative distances are drawn in dotted lines, with their virtual extensions drawn lighter.

Theorem of Central Projection

Having now the formulæ relating to axial pencils of rays, we may next consider the case like that shown in Figs. 7*a* and *b*.

Besides the conjugate axial pencils $Q_1 .. q_1$, let another point of origin Q_2, in the case of the collective lens, or another apex of convergence Q_2, in the case of the dispersive lens, be taken at some small but appreciable distance away from the axis, such that Q_1 and Q_2 are on a plane perpendicular to the axis. It is evident that a ray drawn from Q_2 through the centre of the lens will pass straight on, as it is crossing two elements of surfaces which are parallel and practically touching. If a straight line from Q_2 is therefore drawn through the centre of the lens and produced until it cuts the other so-called conjugate focal plane $q_1 .. q_2$ (which is perpendicular to the axis and passes through q_1, the conjugate focus to Q_1), then the point of intersection q_2 is where the conjugate image of the point Q_2 is formed. That is, the centre of the lens is always in a straight line between any point Q_2 or Q_3 of a plane object and its conjugate image q_2 or q_3. This theorem is capable of a further extension, as shown in Figs. 8*a* and *b*, Plate III.

The optic axis departed from.

Oblique conjugate focal distances.

Here are two cases in which the pencil of rays from Q_2 (here drawn in solid lines) is eccentric; that is, none of the rays of the eccentric pencil actually pass through the centre of the lens owing to the stop *s* being interposed. But it is assumed that the rays constituting such an eccentric pencil are but a part of a larger pencil of rays filling the whole lens; and since the lens is assumed so small that all the rays refracted through it from any one point are caused to converge to or diverge from one and the same image point, therefore these eccentric rays may be regarded as coming under the same law, and the conjugate points Q_2 and q_2 may be considered to be strictly on a straight line of projection drawn through the centre of the lens. Thus the pencils of rays are assumed to be homocentric—that is, all the rays constituting each pencil are assumed to diverge from or converge to one point. From this it follows that the distance $q_1 .. q_2 = (Q_1 .. Q_2)\frac{v}{u}$, and the scale of any conjugate image formed of the plane $Q_1 .. Q_2$ is $\frac{v}{u}$ times the scale of the original. The scales of image and object are in direct ratio to their axial distances from the lens centre.

When oblique pencils are also eccentric.

Definition of homocentric pencils.

Relative scales of object and its image.

Limitations.

Although this theorem, which is a part of the larger Gauss theory, is in its nature only true for minute angles of obliquity and for exceedingly narrow pencils, which never have more than a very small degree of eccentricity, yet it is of the highest importance when we proceed to ascertain that very important function of a more or less complex combination of lenses, known as the equivalent focal length.

Corrected lens system. Theorem untrue for the parts, but true for the whole.

While the theorem is of little practical worth when applied to simple uncorrected lenses of substantial aperture, yet, for a combination of lenses yielding a flat and rectilinear image, it becomes absolutely true in the sum for the series, since the departures from its truth in any one lens are in that case neutralised by contrary departures from its truth in the other lenses.

Thick Lenses

Gauss and Listing.

We may now proceed to deal with the case of lenses of considerable thickness as measured along the axis. This subject was long ago worked out by Gauss (about 1838) and Listing (about 1868), and it will suffice to recapitulate here the most important results, although perhaps arriving at them by methods differing from theirs, but more convenient for our purpose. Let Figs. 9*a*, *b*, *c*, *d*, *e*, *f*, and *g* represent various forms of lenses, of central thicknesses $A_1 \,..\, A_2$, and radius $c_1 \,..\, r_1$ for first spherical surface, and $c_2 \,..\, r_2$ for second surface. It is obvious that if any two radii $c_1 \,..\, r_1$ and $c_2 \,..\, r_2$ are drawn parallel to one another and joined by the straight line $r_1 \,..\, r_2$, then the latter will cut the axis at the point C, so that we have two similar triangles c_1Cr_1 and c_2Cr_2, and two similar mixtilinear triangles CA_1r_1 and CA_2r_2, and the distance $C \,..\, A_1 : C \,..\, A_2 :: c_1 \,..\, r_1 : c_2 \,..\, r_2$, and moreover the straight line $r_1 \,..\, r_2$ cuts the first surface or its tangent at r_1, at exactly the same angle as it cuts the second surface or its tangent at r_2. If, therefore, $r_1 \,..\, r_2$ represents a ray of light, it will obviously, if refracted out of the surface at r_1, be deviated from the direction $r_2 \,..\, r_1$ by exactly the same angle as it would be deviated from the direction $r_1 \,..\, r_2$ if refracted outwards at the point r_2, only the deviation will be in opposite directions. Hence the ray after refraction at r_1 will pursue a course $r_1 \,..\, t_1$, and after refraction at r_2 will pursue a course $r_2 \,..\, t_2$, and these refracted rays are parallel to one another. If, then, $r_1 \,..\, t_1$ and $r_2 \,..\, t_2$ are produced backwards (if necessary) to cut the axis at two points p_1 and p_2, we then get again two similar mixtilinear triangles $r_1A_1p_1$ and $r_2A_2p_2$, and again have $A_1 \,..\, p_1 : A_2 \,..\, p_2 :: c_1 \,..\, r_1 : c_2 \,..\, r_2$. These two points p_1 and p_2 are the two principal points of the lens or nodal points (sometimes

Principal points or nodal points.

PLATE. III

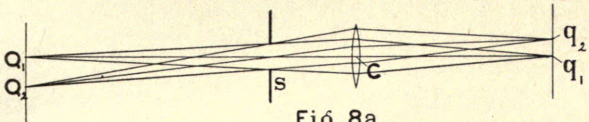


Fig. 8.a.

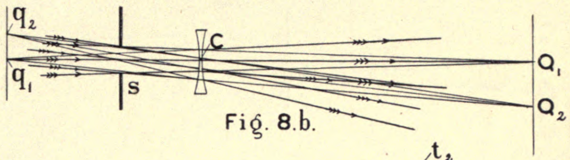


Fig. 8.b.

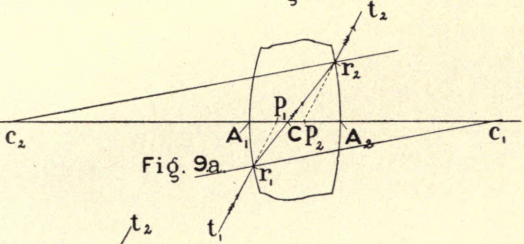


Fig. 9a.

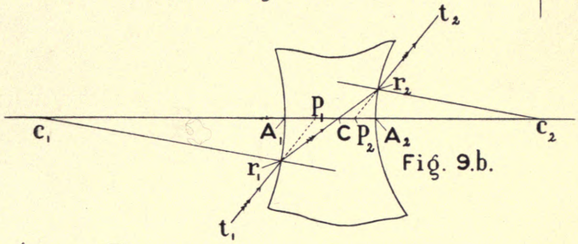


Fig. 9.b.

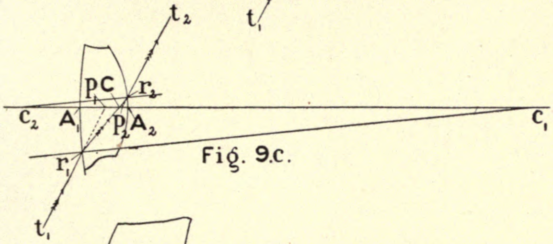


Fig. 9.c.

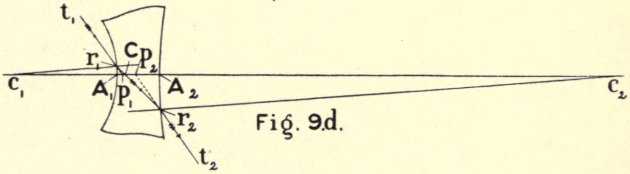


Fig. 9.d.

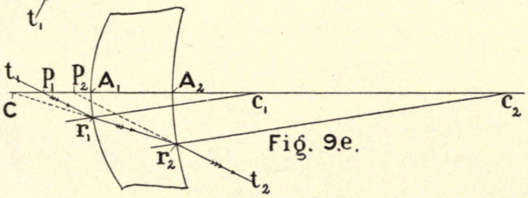


Fig. 9.e.

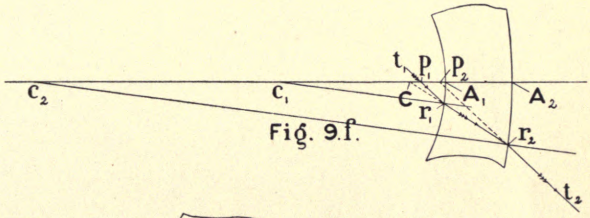


Fig. 9.f.

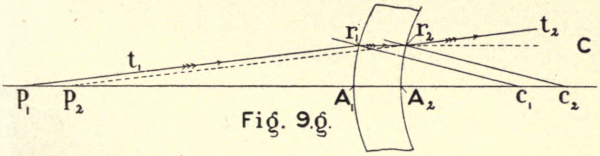


Fig. 9.g.

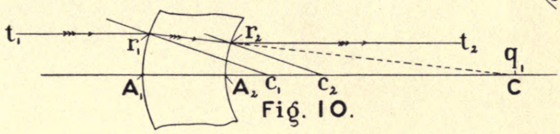


Fig. 10.

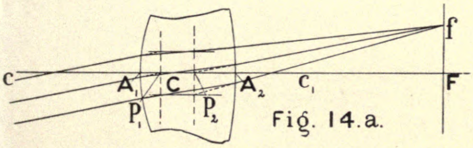


Fig. 14.a.

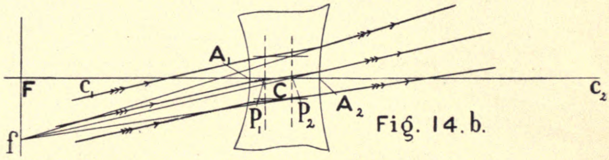
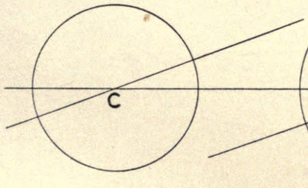
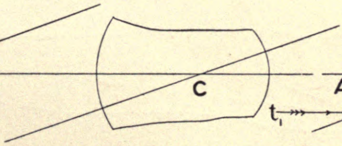


Fig. 14.b.





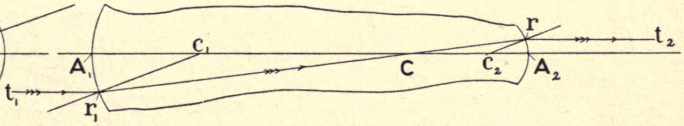


Fig. 12.

Fig. 13.

Fig. 11.

PLATE. III

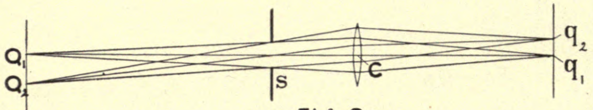


Fig. 8.a.

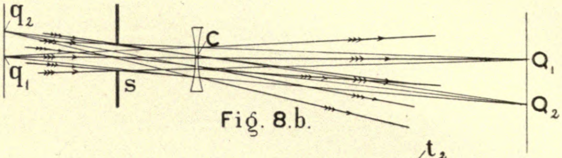


Fig. 8.b.

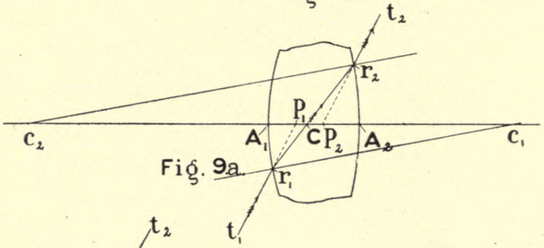


Fig. 9a.

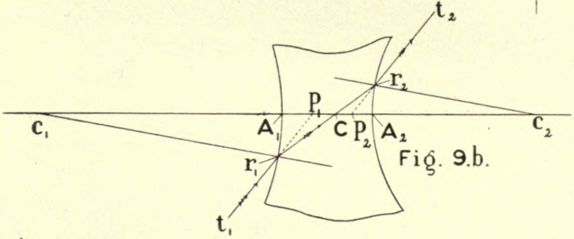


Fig. 9.b.

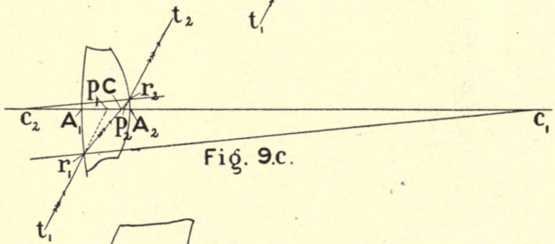


Fig. 9.c.

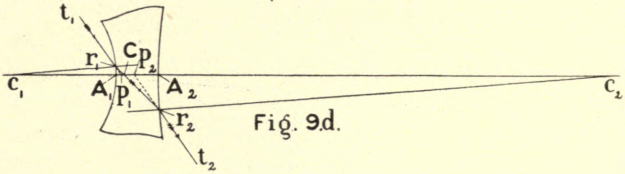


Fig. 9.d.

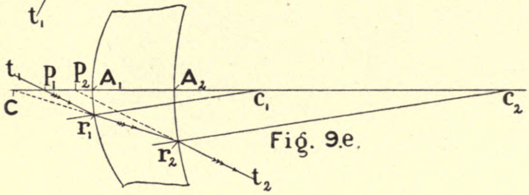


Fig. 9.e.

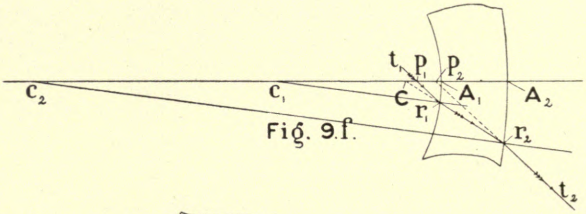


Fig. 9.f.

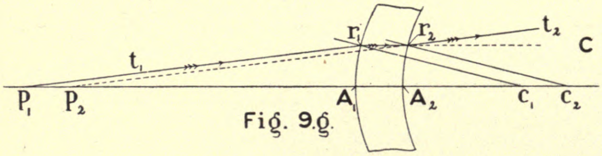


Fig. 9.g.

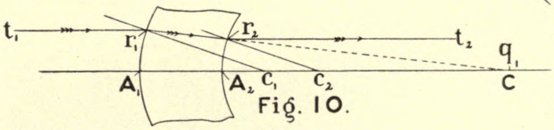


Fig. 10.

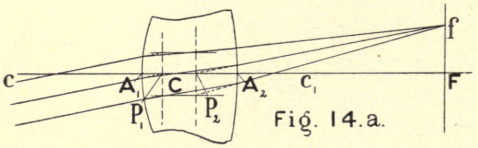


Fig. 14.a.

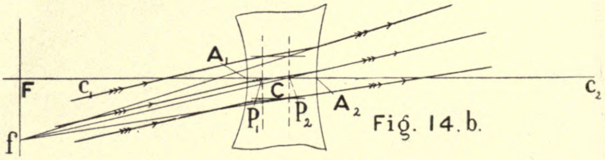


Fig. 14.b.

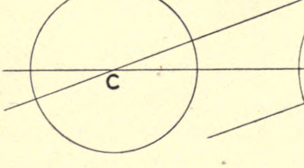


Fig. 12.

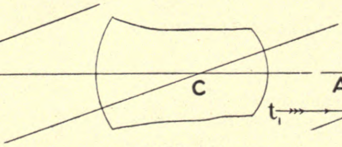


Fig. 13.

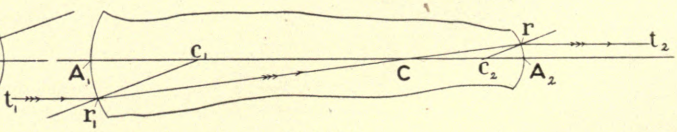


Fig. 11.

also called Gauss points), and, as we have seen, have this important property, that any ray which, while outside the lens, passes through the first principal or nodal point, will, after passage through the lens, emerge on the other side in a direction parallel to its first direction, and radiating from the second principal point; moreover, the same ray, while traversing the interior substance of the lens, passes *ex hypothesi* through the geometric centre of the lens or the point C.

Conjugate focal distances to be measured from the principal points.

As a corollary from the above principle, it follows that if we wish to know the relative sizes or scales of conjugate images formed by thick lenses, we must then measure the focal distances of such images from the principal points of the lens. The focal distance of the first image or object, virtual or otherwise, formed by the entering rays must be measured from the first principal point p_1, and the distance of the second image formed by the emergent rays must be measured from the second principal point p_2, when the sizes of the images will be in direct ratio to those focal distances. Our theorem of central projection still holds good, with this modification, viz. that the centre of the lens presents two aspects, or two different positions, according to whether the lens is viewed from one side or the other. Regarded from the left hand the centre of the lens is practically the first principal point p_1, but regarded from the right hand the centre of the lens is practically the second principal point p_2, and these two points are but the refracted images of the geometric centre C of the lens. That is, p_1 is the conjugate image of C by refraction at the first surface, and p_2 is the conjugate image of C by refraction at the second surface. Therefore the distances A_1p_1 and A_2p_2 may be derived from the Formula II.,

A thick lens exists virtually in two positions.

Method of locating the principal points.

$$\frac{\mu}{\grave{u}} = \frac{\mu - 1}{r} - \frac{1}{u},$$

in its more special application to Figs. 4*f* and 4*g*. At the first surface we have $u = A_1 \,..\, p_1$ (Fig. 9*a*), which by convention is a minus quantity, while $A_1 \,..\, C = \grave{u}$, and is a plus quantity, and $A_1c_1 = r$. Let r and s = first and second radii of curvature respectively, and let the thickness be denoted by t, therefore

$$\frac{\mu}{A_1..C} = \frac{\mu - 1}{r} + \frac{1}{A_1..p_1},$$

and

$$\frac{1}{A_1..p_1} = \frac{\mu}{A_1..C} - \frac{\mu - 1}{r};$$

but

$$A_1..C = t\frac{r}{r+s},$$

therefore

$$\frac{1}{A_1..p_1}=\frac{\mu(r+s)}{tr}-\frac{\mu-1}{r}=\frac{\mu(r+s)-t(\mu-1)}{tr},$$

and

Distance of first principal point from first vertex.

$$A_1..p_1=\frac{tr}{\mu(r+s)-t(\mu-1)}. \qquad \text{IV.}$$

Similarly, at the second refraction we have

$$\frac{1}{A_2..p_2}=\frac{\mu}{A_2..C}-\frac{\mu-1}{s},$$

in which

$$A_2..C=t\frac{s}{r+s},$$

therefore

$$\frac{1}{A_2..p_2}=\frac{\mu(r+s)}{ts}-\frac{\mu-1}{s}=\frac{\mu(r+s)-t(\mu-1)}{ts},$$

and

Distance of second principal point from second vertex.

$$A_2..p_2=\frac{ts}{\mu(r+s)-t(\mu-1)}. \qquad \text{V.}$$

Thickness of a collective lens positive, and that of a dispersive lens negative.

These two formulæ thus give the distances from the vertices A_1 and A_2 of the two principal points of a lens. They obviously give a positive result in the case of any double convex lens, which is as it should be, since these distances are really additions to the conjugate focal distances when both, as in Fig. 6*a*, are positive. But in order to make the formulæ apply to the case of the double concave lens whose normal object and image distances are virtual, but positive, we must consider t, the thickness, to be intrinsically a negative quantity, thus making $A_1..p_1$ and $A_2..p_2$ negative quantities. For they are obviously deductions from the conjugate focal distances when both are positive, as in Fig. 6*b*. That having been settled, then the formulæ will interpret themselves correctly in all cases.

Case of collective meniscus, first p. p.

In the case of the collective meniscus (Fig. 9*e*) s must be entered as a negative quantity in the Formula IV., and being necessarily greater than r, then $r+s$ comes out negative, and we get a negative denominator in the formula. Obviously in this case $A_1..p_1$ is measured outside the lens and is a deduction from the value of u, if plus.

Case of dispersive meniscus, first p. p.

In the corresponding case of a dispersive meniscus (Fig. 9*f*) $A_1..p_1$ comes out positive, both numerator and denominator being negative.

Case of collective meniscus, second p. p.

At the second surface in Fig. 9*e* the Formula V. gives both numerator and denominator negative and the result is positive, for $A_2..p_2$ is an addition to the back focal distance v, if plus.

Case of dispersive meniscus, second p. p.

In the corresponding case of the dispersive meniscus (Fig. 9*f*) Formula V.

yields a product of two negatives for numerator and a negative denominator, and $A_2 \ldots p_2$ comes out negative, being a deduction from a positive focal distance v. Fig. $9g$ represents a special case worthy of note, a case in which the two radii of curvature are equal, but of opposite signs. Here the distance of c, the centre of the lens, from either A_1 or A_2 comes out infinity $\left(\text{or } t\frac{r}{r+s} = t\frac{r}{o}\right)$. The straight line joining the two points r_1 and r_2, where the two parallel radii cut the surfaces, is parallel to the axis, and obviously after refraction by either surface will intersect the axis at a distance from the vertex of either surface equal to $\frac{r}{\mu - 1}$ and $\frac{s}{\mu - 1}$ or $A_1 \ldots p_1$ and $A_2 \ldots p_2$, in the first case negative and in the second positive. We shall also see later on that such a lens, of watch-glass form, really possesses collective power and can form a real image. But it is easy to see that if a real object is placed at the first principal point p_1, then, after passage through the lens, a virtual image will be formed at p_2 of the same size as the original. In such case both u and $v = o$.

Two radii equal, but of opposite signs.

We have, then, here an actual and realisable example of the theorem dwelt upon by various writers on optics, Dr. Drysdale for instance, in the *British Optical Journal*, to the effect that the two planes passing through the two principal points are planes of unit magnification, or, in other words, if an original object or an image lies in the first principal plane, then an equal-sized image of it, real or virtual, will be formed in the second principal plane. We shall have occasion to refer again to this theorem in the next section.

Theorem as to principal planes.

Figs. 10 and 11 are peculiarly interesting cases, since we have the radii and thickness so related that the ray $r_1 \ldots r_2$ within the glass is, after refraction outwards, either way parallel to the axis. This condition is seen to be fulfilled when $t = r\frac{\mu}{\mu - 1} + s\frac{\mu}{\mu - 1}$, r in Fig. 10 being a negative quantity and in Fig. 11 a positive quantity. Such thick lenses as these may be said to have no principal points at all, and therefore no focal length, and their analogy to the Galilean and astronomical telescope respectively will be more fully realised later on.

Lenses without any principal points and without focal length.

In Fig. 12 we have the simplest case of all, that is, the sphere, wherein the two principal points merge in the geometric centre.

A sphere has only one principal point.

In Fig. 13 the case is extended to one in which the two radii of curvature are different, yet struck from a common centre. Here again the two principal points merge in the geometric centre.

Other lenses with only one principal point.

Figs. $14a$ and $14b$ show, for a collective lens and for a dispersive

Course of oblique pencils with respect to the principal planes.

lens respectively, the course of a complete oblique pencil of rays through the lens with respect to the principal points p_1 and p_2 and the principal planes passing through the latter.

Influence of thickness upon the principal focal length of a lens.

Having now settled the positions of the two principal points of any lens, we may proceed to ascertain what influence the axial thickness t of a lens exercises upon its equivalent principal focal length. Fig. 14a represents a double convex lens forming a real image f, in its principal focal plane F, of an object situated at an infinite distance away on the left; Fig. 14b the corresponding case of a dispersive lens. Here the principal focal length required is the distance $p_2 \,..\, \mathrm{F}$ measured from the second principal point p_2 to the principal focal plane F. It

Back focal length to be ascertained.

consists of two parts: first, the back focal distance $\mathrm{A}_2\mathrm{F}$ measured from the vertex of the second surface; and second, the distance $\mathrm{A}_2 \,..\, p_2$ from the same vertex to the second principal point. The latter we have already got an expression for; the former, or the back focal length, we must now proceed to formulate. After the first refraction, in the case of a collective lens, the axial pencil of parallel rays is converged to f_1; let $\mathrm{A}_1 \,..\, f_1 = \grave{u}$. Then

$$\frac{\mu}{\grave{u}} = \frac{\mu - 1}{r} - \frac{1}{u},$$

of which

$$\frac{1}{u} = o,$$

therefore

$$\frac{\mu}{\grave{u}} = \frac{\mu - 1}{r}, \quad \frac{1}{\grave{u}} = \frac{\mu - 1}{\mu r}, \text{ and } \grave{u} = \frac{\mu r}{\mu - 1}.$$

Next

$$\mathrm{A}_2 \,..\, f_1 = \grave{u} - t = \frac{\mu r}{\mu - 1} - t = \frac{\mu r - (\mu - 1)t}{\mu - 1}.$$

At the second refraction we have

$$\frac{\mu}{\mathrm{A}_2 .. f_1} = \frac{\mu - 1}{s} - \frac{1}{\mathrm{A}_2 .. \mathrm{F}},$$

therefore

$$\frac{1}{\mathrm{A}_2 .. \mathrm{F}} = \frac{\mu - 1}{s} - \frac{\mu}{\mathrm{A}_2 .. f},$$

but in this case $\mathrm{A}_2 \,..\, f$ is by convention intrinsically a negative quantity with respect to the second surface, which we have seen to be equal to

$$-\frac{\mu r - (\mu - 1)t}{\mu - 1},$$

so that

$$\frac{\mu}{A_2..f_1} = -\frac{\mu(\mu - 1)}{\mu r - (\mu - 1)t},$$

therefore

$$\frac{1}{A_2..F} = \frac{\mu - 1}{s} + \frac{\mu(\mu - 1)}{\mu r - (\mu - 1)t} = \frac{(\mu - 1)\{\mu r - (\mu - 1)t\} + \mu(\mu - 1)s}{s\{\mu r - (\mu - 1)t\}},$$

therefore

$$\frac{1}{A_2..F} = \frac{\mu(\mu - 1)(r + s) - (\mu - 1)^2 t}{s\{\mu r - (\mu - 1)t\}},$$

and

$$A_2..F = \frac{s\{\mu r - (\mu - 1)t\}}{(\mu - 1)\{\mu(r + s) - (\mu - 1)t\}}, \qquad \text{V A.}$$

Formula for the back focal length.

Add $A_2 .. p_2$ to this from V., and we get

$$A_2 .. F + A_2 .. p_2 = \frac{s\{\mu r - (\mu - 1)t\}}{(\mu - 1)\{\mu(r + s) - (\mu - 1)t\}} + \frac{ts}{\mu(r + s) - t(\mu - 1)},$$

$$= \frac{s\{\mu r - (\mu - 1)t\} + (\mu - 1)ts}{(\mu - 1)\{\mu(r + s) - (\mu - 1)t\}}$$

$$= \frac{\mu rs}{(\mu - 1)\{\mu(r + s) - (\mu - 1)t\}} \qquad \text{VI.}$$

Formula for the equivalent focal length of a thick positive lens.

This, then, is the formula for the equivalent principal focal length E of a thick positive lens. If a small infinitely thin positive lens of principal focal length $p_2 .. F$ were placed at p_2, it would form an image at F of distant objects of the same dimensions as that formed by the thick lens, of which latter it is the equivalent.

In the case of the double concave lens, Fig. 10*b*, we have $A_2 .. F + A_2 .. p_2$, also

$$= \frac{s\{\mu r - (\mu - 1)t\}}{(\mu - 1)\{\mu(r + s) - (\mu - 1)t\}} + \frac{ts}{\mu(r + s) - t(\mu - 1)},$$

in which case, of course, the latter expression for $A_2 .. p_2$ comes out minus and as a deduction from the back focal length A_2F, since t in this case must be entered as a negative quantity; so that, just as in the case of the collective lens, we get the same expression for the equivalent principal focal length, viz.—

$$\frac{\mu rs}{(\mu - 1)\{\mu(r + s) - (\mu - 1)t\}} = E. \qquad \text{VII.}$$

Formula for the equivalent focal length of thick dispersive lens.

Now if the lens were infinitely thin, the reciprocal of its principal focal length would be simply $(\mu-1)\left(\frac{1}{r}+\frac{1}{s}\right)$. Calling this $\frac{1}{\mathrm{F}}$ and subtracting it from $\frac{1}{\mathrm{E}}$ we get

$$\frac{(\mu-1)\{\mu(r+s)-(\mu-1)t\}}{\mu rs}-\frac{(\mu-1)(r+s)}{rs}=\frac{1}{\mathrm{E}}-\frac{1}{\mathrm{F}};$$

therefore

$$\frac{1}{\mathrm{E}}-\frac{1}{\mathrm{F}}=-\frac{t(\mu-1)^2}{\mu rs},$$

so that

$$\frac{1}{\mathrm{E}}=(\mu-1)\left(\frac{1}{r}+\frac{1}{s}\right)-\frac{t(\mu-1)^2}{\mu rs}\quad\text{or}\quad\frac{1}{\mathrm{F}}-\frac{t(\mu-1)^2}{\mu rs};$$

therefore

$$\frac{1}{\mathrm{E}}=\frac{1}{\mathrm{F}}\left\{1-\mathrm{F}\frac{t(\mu-1)^2}{\mu rs}\right\}$$

$$=\frac{1}{\mathrm{F}}\left\{1-\frac{rs}{(\mu-1)(r+s)}\cdot\frac{t(\mu-1)^2}{\mu rs}\right\};$$

therefore

Formula giving modification due to thickness, as a percentage.

$$\frac{1}{\mathrm{E}}=\frac{1}{\mathrm{F}}\left\{1-\frac{t(\mu-1)}{\mu(r+s)}\right\}\quad\text{or}\quad\frac{1}{\mathrm{F}}+\triangle\frac{1}{\mathrm{F}}. \qquad \text{VIII.}$$

This is perhaps the most convenient and significant mode of expressing the modification of the power of any lens whatsoever which is due to thickness; it expresses it in the form of a percentage of gain or loss as compared with the power which the lens would have if it were infinitely thin. It shows a loss of power in the case of double convex lenses, a gain in power in the case of double concave lenses, no alteration in power in the case of plano-convex, plano-concave, convexo-plane, concavo-plane lenses, for in all four cases $r+s$ becomes infinity; while in the case of a collective meniscus, when $r+s$ becomes negative, a greater and greater *relative* gain in power, consequent in thickness, is attained as the radius of the concave surface approaches to equality with the radius of curvature of the convex surface; while, lastly, in the case of the dispersive meniscus a loss of power ensues on an increase of thickness, since both numerator and denominator of the function of t become negative.

Effect of thickness upon power in various cases.

We have now arrived at the formula for the equivalent principal focal length E of any lens whatsoever, and also have located the geometric centre C and the two principal points p_1 and p_2, from the latter of which the equivalent principal focal length is measured.

We have next to inquire whether, in cases wherein the entering rays are more or less divergent or convergent—that is, when the entering rays are either diverging from a near object on the left of the lens or converging towards a real image to the right of the lens—the thick lens still maintains the same principal focal length, or departs from it. Fig. 14*a* or 14*b* illustrates such a case. It is of the highest practical importance to know whether the law of conjugate foci for a thin lens $\frac{1}{v}=\frac{1}{F}-\frac{1}{u}$ or $\frac{1}{F}=\frac{1}{u}+\frac{1}{v}$ still holds good. In short, does $\frac{1}{p_2 \ldots F}=\frac{1}{E}-\frac{1}{Q \ldots p_1}$; that is, is $\frac{1}{p_2 \ldots F}+\frac{1}{Q \ldots p_1}$ a constant. We shall be in a better position to answer this question when we have dealt with the problem by means of a device or theorem which is more general in its applications than any method which has been hitherto devised. This we will deal with in the next section.

Inquiry. Is the power of thick lenses a constant quantity?

SECTION II

THE THEOREM OF ELEMENTS

Power of a single surface is a highly inconstant entity.

We have seen in the last section that the Formula II. relating to refraction of an axial pencil of rays at a single surface is by no means such a simple formula as the Formula III., which applies to the corresponding case of refraction of an axial pencil of rays by a lens bounded by two surfaces. In the case of the single surface, Fig. 4*a*, for instance, if the rays are strongly divergent, then a large amount of positive refraction takes place; but supposing the entering rays are *converging* to the centre of curvature C, then no refraction takes place; while if the rays are converging still more to any point between C and A, then there ensues refraction of a negative character. Thus, from the practical point of view of refractive effect, we may disregard the so-called "optical invariant" of the late Professor Abbe as applied to a single refracting surface. Clearly a single surface is a somewhat puzzling and inconstant entity, which varies in its effects enormously according to circumstances. But not so the lens bounded by two refracting surfaces; for whatever conditions of divergence or convergence may characterise the entering pencil of rays, the lens always adds or subtracts a constant refractive effect of its own which is expressed by $(\mu - 1)\left(\frac{1}{r} + \frac{1}{s}\right)$ or $\frac{1}{F}$.

The power of a thin lens is a constant quantity.

Thick lenses compounded of infinitely thin elements and a parallel plane plate.

Let us see, then, whether we cannot express any thick lens in terms of two complete lenses. Let Figs. 15*a*, *b*, *c*, and *d* be four various thick lenses. Each one of these may be considered to be built up of plano-convex, plano-concave, convexo-plane, or concavo-plane lenses of infinite thinness, each lens consisting of any two of the above and containing between them a piece of plane parallel glass of a thickness equal to the axial thickness of the whole lens. For instance, the collective meniscus, Fig. 15*a*, may be considered to be built up of a convexo-plane infinitely thin lens e_1 at the left-hand vertex of the whole lens, and a plano-concave lens e_2 of infinite thinness at the right-hand vertex, the two enclosing between them a plate of parallel plane glass of thickness $= t$. These two infinitely thin lenses we will call elements. They are indicated in black in Figs. 15. In Fig. 15*b* we have a convexo-plane element at e_1, and a plano-convex element at e_2. In Fig. 15*c* we have a concavo-plane element at e_1, and a plano-concave element at e_2, both dispersive, while in Fig. 15*d* we have a concavo-plane lens at e_1, and a plano-convex lens at e_2, the latter being negative with respect to the more powerful first element, and the whole lens a dispersive meniscus. Now the reciprocal value of the principal focal length of any element or the power is $(\mu - 1)\left(\frac{1}{r} + \frac{1}{s}\right)$; but as one surface is always plane, therefore either $\frac{1}{r}$ or $\frac{1}{s}$ becomes zero, and the power then resolves itself into either $\frac{\mu - 1}{r}$ or $\frac{\mu - 1}{s}$. The principal focal length of e_1 being called f_1, then $\frac{1}{f_1} = \frac{\mu - 1}{r_1}$, and for the second element e_2, $\frac{1}{f_2} = \frac{\mu - 1}{s}$ or $\frac{\mu - 1}{r_2}$ if we call all radii r_1, r_2, r_3, etc.

Elements defined and explained.

Power of an element defined.

But before proceeding further, we must ascertain what is the effect of the plate of plane parallel glass upon the pencils of rays traversing it in passing from one element to the other.

Effect of the plane parallel plate.

Fig. 16*a* represents a parallel plane plate of glass of thickness $A_1 \,..\, A_2$, and Q_1 is a point from which a pencil of rays diverges and passes perpendicularly through the plate; that is, the central or principal ray $Q_1 \,..\, P$ of the pencil is normal to the plate. Let $Q_1 \,..\, A_1 = u$ and $A_1 \,..\, A_2 = t$. After refraction at the first surface the rays diverge from the point q, such that $q \,..\, A_1 = \mu u$ (μ being the refractive index). Therefore when striking the second surface they are diverging from a point q at a distance from A_2 equal to $\mu u + t$. Then after refraction from the second surface they diverge again from a new point Q_2, such that $Q_2 \,..\, A_2 = \frac{\mu u + t}{\mu} = u + \frac{t}{\mu}$. That is, on emerging at A_2, after passage through the plate, the rays are diverging just as if they had passed without any refraction through an air space equal to $\frac{t}{\mu}$.

Transference of radiant point formulated.

Let Fig. 16*b* represent a corresponding pencil of rays converging into the parallel plate. In both cases any small oblique pencil may be regarded as part of a larger pencil whose central ray $P \,..\, Q_1$ is perpendicular to the plane surfaces, so that any displacements are along this perpendicular central ray as before. The entering rays are converging to Q_1. Let

Case of slightly oblique pencils.

$A_1 \ldots Q_1$ be u. These rays, after refraction, converge in lesser degree to q, such that $A_1 \ldots q = \mu(A_1 \ldots Q_1)$ or μu. Then when striking the second surface they are obviously converging to a point at a distance to the right of A_2 equal to $\mu(A_1 \ldots Q_1) - t$ or $\mu u - t$, and after refraction converge more strongly to a point at a distance to the right of A_2 equal to $\frac{\mu u - t}{\mu} = u - \frac{t}{\mu} = A_2 \ldots Q_2$. Here again the rays on emerging at A_2 are converging, just as if they had passed without any refraction through an air-space equal to $\frac{t}{\mu}$.

In Fig. 16*a* the distance

$$Q_1 \ldots Q_2 = u + t - (Q_2 \ldots A_2) = u + t - \left(u + \frac{t}{\mu}\right) = t - \frac{t}{\mu} = t\left(1 - \frac{1}{\mu}\right) = t\left(\frac{\mu - 1}{\mu}\right).$$

In Fig. 16*b* the distance

$$Q_1 \ldots Q_2 = A_2 \ldots Q_2 + t - (A_1 \ldots Q_1) = \left(u - \frac{t}{\mu}\right) + t - u = t\left(1 - \frac{1}{\mu}\right) = t\left(\frac{\mu - 1}{\mu}\right).$$

Displacement of Q a constant function of the thickness of parallel plate.

Hence by passage through the plate the origin or apex Q_1 of the pencil is simply displaced a distance equal to $t\left(\frac{\mu - 1}{\mu}\right)$ along a perpendicular from Q_1 to the plate, and in the same direction as the light is travelling.

If the point Q_1 is anywhere in the interior of the plate, we still arrive at the same result. Therefore, so far as our present purposes are concerned, we may consider the elements e_1 and e_2 in Figs. 15 to be separated by an air-space equal to $\frac{t}{\mu}$ instead of glass of thickness t.

Hence if Fig. 17 represents any lens whatsoever (except convexo-plane and the reverse), then we may consider it, for our present purposes, to consist of two small infinitely thin convexo-plane and plano-convex elements e_1 and e_2 separated by an air-space equal to $\frac{t}{\mu}$; that is, Fig. 18 is the equivalent of Fig. 17. Thus we consider the two elements to be brought nearer together by an amount equal to $t - \frac{t}{\mu}$ or $t\left(\frac{\mu - 1}{\mu}\right)$, while all conjugate distances, such as that from e_1 to an object Q on the left, or that from e_2 to its image q on the right, remain exactly as before. Also the distances from e_1 to the first principal point p_1, and e_2 to the second principal point p_2, remain undisturbed, as we will see later. Therefore the *total* distance $Q \ldots q$ between conjugate focal planes is altered by $+$ or $- t\left(\frac{\mu - 1}{\mu}\right)$ according to circumstances, as shown on comparing

PLATE. IV.

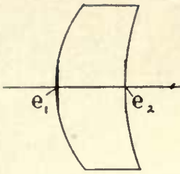


Fig. 15.a.

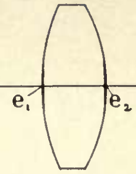


Fig. 15.b.

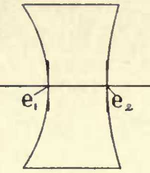


Fig. 15.c.

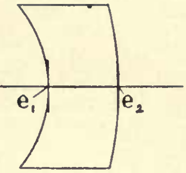


Fig. 15.d.

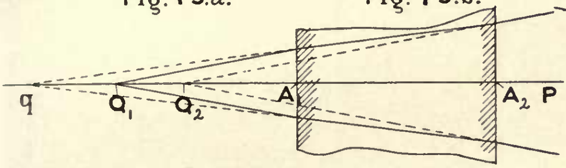


Fig. 16.a.

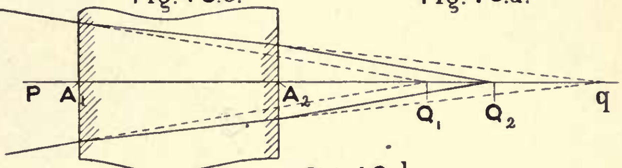


Fig. 16. b.

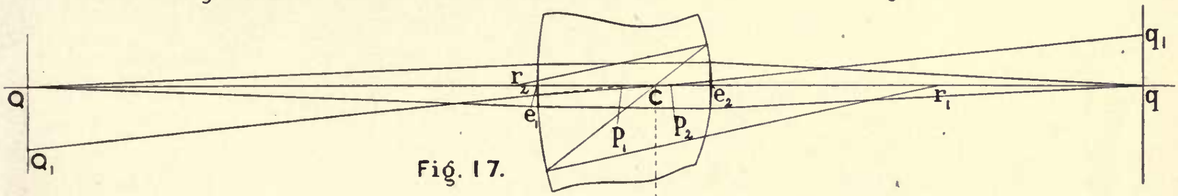


Fig. 17.

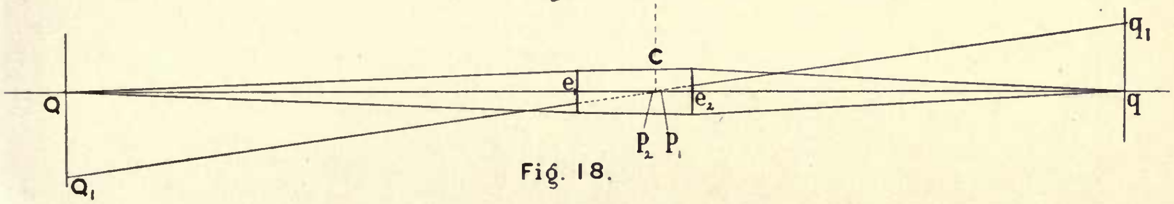


Fig. 18.

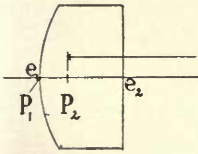


Fig. 18.a.

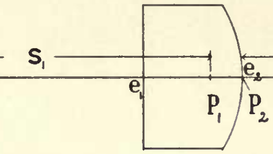


Fig. 18.b.

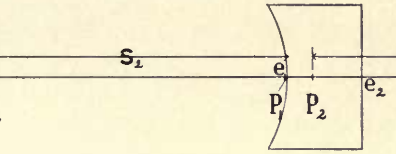


Fig. 18.c.

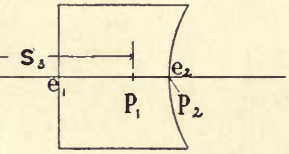


Fig. 18.d.

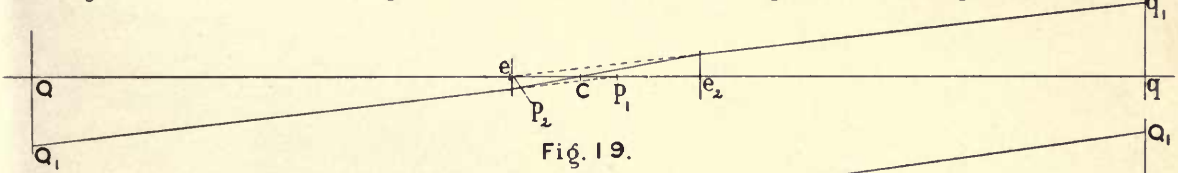


Fig. 19.

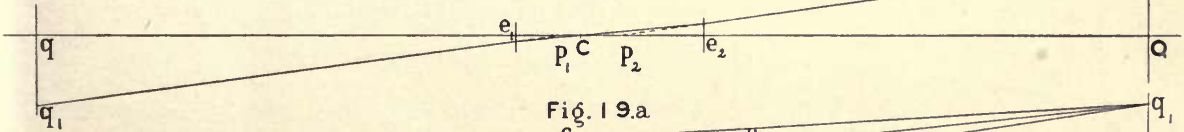


Fig. 19.a

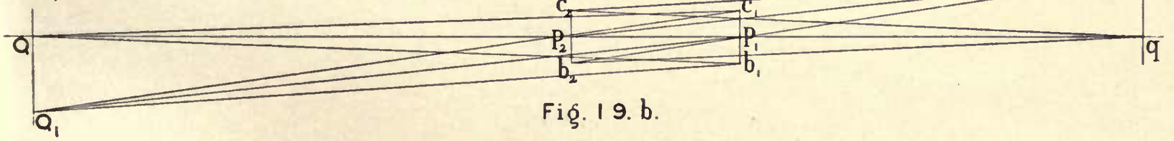


Fig. 19. b.

PLATE. IV.

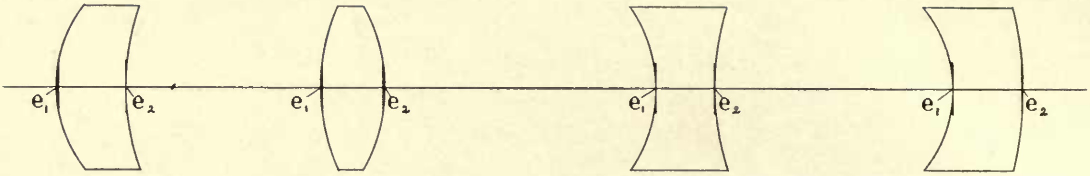


Fig. 15.a. Fig. 15.b. Fig. 15.c. Fig. 15.d.

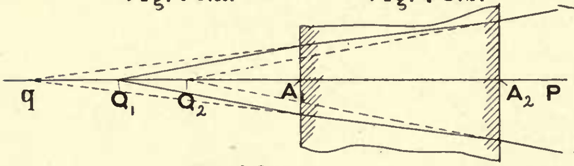


Fig. 16.a.

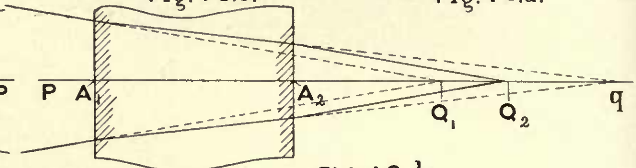


Fig. 16. b.

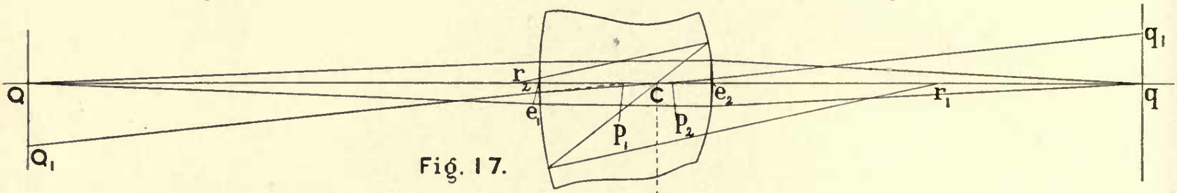


Fig. 17.

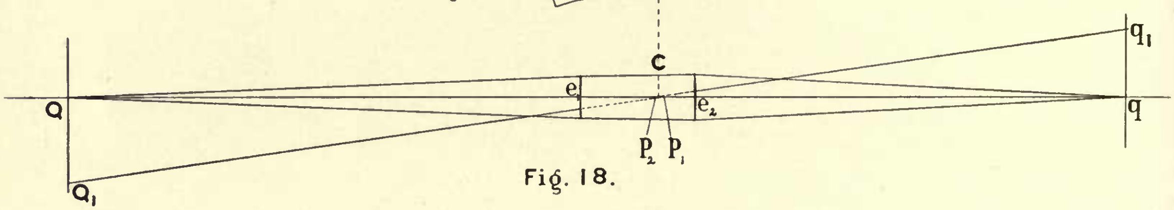


Fig. 18.

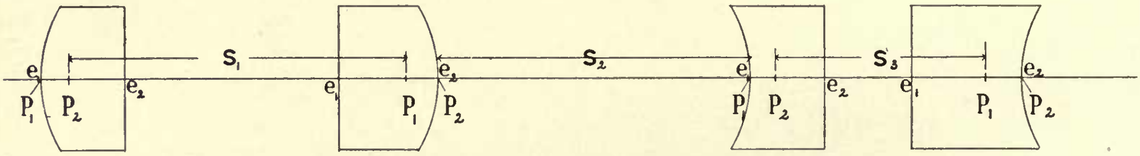


Fig. 18.a. Fig. 18.b. Fig. 18.c. Fig. 18.d.

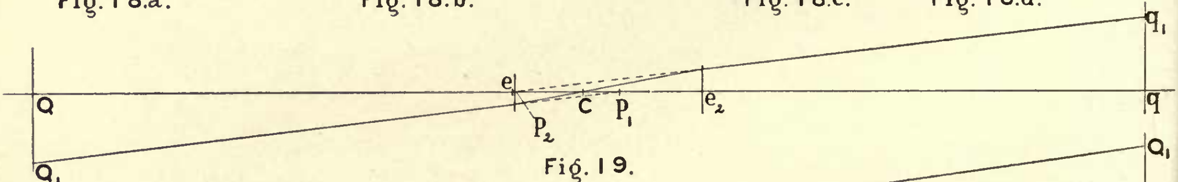


Fig. 19.

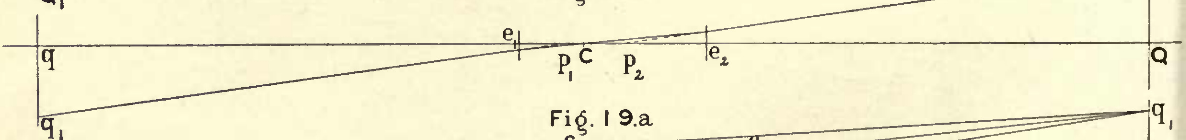


Fig. 19.a

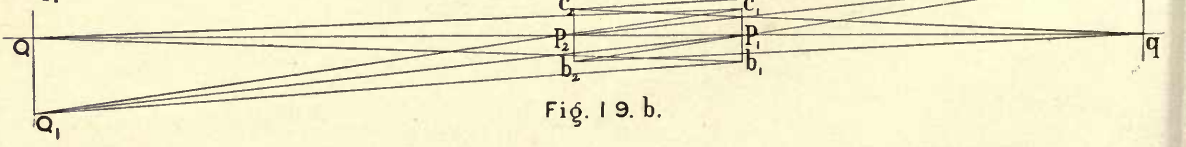


Fig. 19. b.

Fig. 18 with Fig. 17. But so long as all the distances, whether of conjugate focal planes or of principal points, measured from e_1 and e_2 respectively, remain exactly as when treating the lens as a solid entity, we need then have nothing to do with the fact that the distance from the first principal point p_1 to the second element e_2 and the distance from the second principal point p_2 to the first element e_1 are altered by $t\left(\frac{\mu-1}{\mu}\right)$; we can ignore it altogether, for those distances never come into account in any formulæ whatever that are of practical importance.

Conjugate focal distances and positions of principal points undisturbed by theorem of elements.

The cases of convexo-plane, plano-convex, concavo-plane, and plano-concave lenses, as in Figs. 18*a*, *b*, *c*, and *d*, call for special remark.

Cases of thick lenses having one surface plane.

We must bear in mind that, in all such cases, the geometric centre of the lens is at the vertex or point where the curved surface cuts the axis, and therefore that point (e_1 in 18*a*, e_2 in 18*b*, e_1 in 18*c*, and e_2 in 18*d*) is an element as well as the first principal point in 18*a*, the second principal point and element in 18*b*, the first principal point and element in 18*c*, and the second principal point and element in 18*d*, while the other principal point, whether it be the first or the second, is always at an apparent distance from the other one (at the vertex of curvature) equal to $t\frac{\mu-1}{\mu}$ and $\frac{t}{\mu}$ from the plane surface. For $e_1..p_2$ in 18*a*, $p_1..e_2$ in 18*b*, $e_1..p_2$ in 18*c*, and $p_1..e_2$ in 18*d*, each $=t\left(\frac{\mu-1}{\mu}\right)$, their distances from the plane surfaces being $\frac{t}{\mu}$. And we have already seen that the principal equivalent focal length of all lenses having one surface plane is in no way altered by thickness, however great.

Therefore in treating such lenses we may take any focal distances u or v that may be measured from the central or axial point of the *plane* surface, add $+\frac{t}{\mu}$ in the case of collective lenses, and add $-\frac{t}{\mu}$ in the case of dispersive lenses. Then u or v, as the case may be, will be referred to the principal point.

Focal distances measured from vertex of plane surface to be corrected.

Of course the addition of $\frac{t}{\mu}$ is algebraical, for if the rays of a pencil emerging from the plane surface of Fig. 18*a* are converging, then v is positive, and $+\frac{t}{\mu}$ is an extension of that distance; but if the rays of the pencil after emerging from the plane surface are diverging, then v is negative, and $\frac{t}{\mu}$ becomes a deduction from its numerical value.

Case where collective and dispersive lenses are ranged on a common axis.

But the same rule of adding $\pm\frac{t}{\mu}$ will not quite apply to the measurements of axial distances between neighbouring lenses of collective and dispersive types mixed. For instance, the distance between the two lenses 18*a* and 18*b*, which have their plane sides towards one another, is indicated by the line s_1; that is, to the original air-space $e_2 \,..\, e_1$ the distances $\frac{t_1}{\mu}$ and $\frac{t_2}{\mu}$ have to be added at each end.

The distance between 18*b* and 18*c* is simply $e_2 \,..\, e_1$ as the two lenses are presenting their curved sides and two elements to one another, while the distance between 18*c* and 18*d* is $e_2 \,..\, e_1$ with $\frac{t_3}{\mu}$ and $\frac{t_4}{\mu}$ added on at each end. It might here be urged that these latter are two dispersive lenses, and therefore $\frac{t_3}{\mu}$ and $\frac{t_4}{\mu}$ are negative quantities, so that they become deductions from the numerical value of the distance $e_2 \,..\, e_1$ if the latter is positive. Here is a seeming inconsistency. But we must remember that if we are dealing with the two dispersive lenses 18*c* and 18*d* *alone*, then we treat them as positive entities, in which case both their thicknesses and any separation between them would be treated as negative quantities, so that s_3 would be the sum of $-(p_2 \,..\, e_2)$, $-(e_2 \,..\, e_1)$, and $-(e_1 \,..\, p_1)$.

An apparent inconsistency explained.

Conventions to be observed in case of mixed lenses having plane surfaces.

But if we are tracing pencils through a series of collective and dispersive lenses ranged on a common axis, we can then treat *all* axial distances between such lenses as positive, provided the principal focal lengths of all dispersive lenses are considered negative relatively to the said distances and to the principal focal lengths of the collective lenses. Therefore in Figs. 18*a*, *b*, *c*, and *d*, if, as usual, we consider the distances s_1 and s_2 positive, then s_3 would also be positive besides $\frac{t_3}{\mu}$ and $\frac{t_4}{\mu}$, while the powers of 18*c* and 18*d* would be negative, and the powers of 18*a* and 18*b* positive. And this is the most reasonable convention to follow in the case of a series of mixed lenses. Such matters constantly demand the exercise of careful discrimination.

SECTION III

THEORY OF EQUIVALENT FOCAL LENGTHS AND PRINCIPAL POINTS OF LENS COMBINATIONS

IN Section I. we have already worked out formulæ for the equivalent principal focal lengths of thick lenses and for the distances of the two principal points from the two vertices. We will now prove the identity of the formulæ obtained from the theory of elements with the above formulæ already worked out for single thick lenses, and then prove that the sum of the reciprocals of the conjugate foci, as measured from the principal points, is invariably constant and equal to the equivalent principal focal length.

Previous formulæ to be confirmed by the theorem of elements.

Constancy of equivalent focal length to be proved.

Fig. 19 represents two elements, exaggerated in diameter for clearness, separated by an axial air-space or distance s_1. Let $\frac{1}{f_1}$ be the reciprocal of the principal focal length of e_1, and $\frac{1}{f_2}$ that of e_2, or the respective powers of the two elements. Let C be the geometric centre, such that

$$e_1 \,..\, C : e_2 \,..\, C :: f_1 : f_2.$$

It is then plain that any slightly oblique ray passing through C will impinge upon e_1 and e_2 under exactly similar conditions, and will meet with exactly equal deviations when refracted through the elements, and therefore the rays after refraction both ways, $Q_1 \,..\, p_1$ and $p_2 \,..\, q_1$, will be parallel to one another, and if produced backwards will cut the axis at p_1 and p_2, which two points are the principal points of the combination.

Let the distance e_1 to p_1 be P_1, and e_2 to p_2 be P_2.

We then have

$$C \,..\, e_1 = s\frac{f_1}{f_1 + f_2},$$

and if we are to make P_1 positive we have

$$\frac{1}{P_1} = \frac{1}{C \ldots e_1} - \frac{1}{f_1} = \frac{f_1 + f_2}{sf_1} - \frac{1}{f_1} = \frac{f_1 + f_2 - s}{sf_1},$$

and

Distance of first principal point from first element.

$$P_1 = \frac{sf_1}{f_1 + f_2 - s}. \qquad \text{IX}_{\text{A}}.$$

Also we have

$$C \ldots e_2 = s\frac{f_2}{f_1 + f_2};$$

and if we are to make P_2 positive we have

$$\frac{1}{P_2} = \frac{1}{C \ldots e_2} - \frac{1}{f_2} = \frac{f_1 + f_2}{sf_2} - \frac{1}{f_2} = \frac{f_1 + f_2 - s}{sf_2},$$

and

Distance of second principal point from second element.

$$P_2 = \frac{sf_2}{f_1 + f_2 - s}. \qquad \text{IX}_{\text{B}}.$$

Now if r be the radius of the curved surface of e_1, and s that of e_2, then $\frac{1}{f_1} = \frac{\mu - 1}{r}$ and $\frac{1}{f_2} = \frac{\mu - 1}{s}$. Also, if e_1 and e_2 were the elements of a solid thick lens of thickness t, we have by our theorem $s = \frac{t}{\mu}$ and $t = \mu s$.

Substituting these values in Formula IXA. we get

Identity of Formulæ IV. and IXA.

$$P_1 = \frac{\frac{t}{\mu}\frac{r}{\mu - 1}}{\frac{r}{\mu - 1} + \frac{s}{\mu - 1} - \frac{t}{\mu}} = \frac{\frac{t}{\mu}\frac{r}{\mu - 1}}{\frac{\mu(r + s) - t(\mu - 1)}{\mu(\mu - 1)}} = \frac{tr}{\mu(r + s) - t(\mu - 1)},$$

which is identical with Formula IV. arrived at in Section I.

Next we will work out the back focal length $e_2 \ldots q$, supposing the rays entering e_1 to be parallel, in which case q is on the principal focal plane. The image formed by e_1 in this case is at a distance f_1 from e_1 and a negative distance equal to $f_1 - s$ behind e_2, therefore we have

$$\frac{1}{e_2 \ldots q} = \frac{1}{f_2} - \left(\frac{1}{-(f_1 - s)}\right) = \frac{1}{f_2} + \frac{1}{f_1 - s}.$$

Calling $e_2 \ldots q = \text{B}$, we have

$$\frac{1}{\text{B}} = \frac{f_1 + f_2 - s}{f_2(f_1 - s)}$$

and

Formula for back focal length for two separated elements.

$$\text{B} = \frac{f_2(f_1 - s)}{f_1 + f_2 - s}. \qquad \text{IX}_{\text{C}}.$$

Then to get the equivalent principal focal length, or E, we have

$$E = B + P_2 = \frac{f_2(f_1 - s)}{f_1 + f_2 - s} + \frac{sf_2}{f_1 + f_2 - s},$$

and

$$E = \frac{f_1 f_2}{f_1 + f_2 - s}. \qquad \text{X.}$$

Formula for the equivalent focal length of two separated elements.

Putting $\frac{r}{\mu - 1}$ for f_1, $\frac{s}{\mu - 1}$ for f_2, and $\frac{t}{\mu}$ for s, as before, we then have

$$E = \frac{\frac{r}{\mu - 1} \cdot \frac{s}{\mu - 1}}{\frac{r}{\mu - 1} + \frac{s}{\mu - 1} - \frac{t}{\mu}} = \frac{\frac{rs}{(\mu - 1)^2}}{\frac{\mu(r + s) - t(\mu - 1)}{\mu(\mu - 1)}} = \frac{\mu rs}{(\mu - 1)\{\mu(r + s) - t(\mu - 1)\}},$$

which is identical with the Formula VI. for the equivalent focal length of a solid lens of thickness μs or t which we obtained in Section I. It will be found that Formula X. is universally true for couples of elements, provided our former conventions as to lenses are adhered to. If one of the elements is collective and the other dispersive, the stronger element should give the character to the lens, while the f for the weaker element should be entered in the formula as a negative quantity. For instance, if e_2 is the stronger, having $f_2 = 9$ and dispersive, while f_1 is 10 and collective, then, as in the case of the lens, we should consider the character of the combination, *by first intention* as it were, to be dispersive, and the separation s, which say $= 2$, to be relatively a minus quantity, just as t was in the case of a dispersive lens; so that E becomes, since f_1 is negative,

Concrete example of the use of signs and of the highly important influence of separation on the character of a pair of elements or lenses.

$$\frac{(-10)(+9)}{-10 + 9 + 2} = -90.$$

Now as we assume the lens, by first intention, to be a dispersive lens, but with a positive sign, it is clear that E, being minus relatively, indicates that a real image is formed, and the combination, owing to the separation, acts as a collective lens, although by first intention it was dispersive. Thus the separation has reversed the character of the couple.

If, on the other hand, we insert f_1 as a positive quantity in the Formula X., and f_2 as a negative quantity, and therefore s as a positive quantity, we shall then in the same case get $E = +90$, which comes to the same thing, E being $+$ or of the same sign as f_1, which is collective, and so indicating a collective resultant lens or combination.

This variance between the character of a lens or combination of elements by first intention and in actual result should be clearly borne in mind.

Inquiry. Is the equivalent focal length a constant?

Having now got the equivalent principal focal length E of the combination on the supposition that the entering pencils consist of parallel rays, we may now profitably return to our inquiry, whether, supposing the entering pencils to consist of either divergent or convergent rays, the sum of the reciprocals of the conjugate focal distances $Q \ldots p_1$ and $p_2 \ldots q$ will always be equal to the reciprocal of the equivalent principal focal length, and therefore constant. It is of the highest importance to know this.

The first thing we want is the formula for the back focal length $e_2 \ldots q$, supposing the rays entering e_1 are *not* parallel. Let $Q \ldots e_1 = u_1$, and the conjugate focal distance after refraction through e_1 be v_1, and the amended distance from the focus so formed by e_1 to the element e_2 be u_2, and the conjugate focal distance after refraction through e_2 be v_2. Then v_2 is the back focal distance required, and we have

$$\frac{1}{v_1} = \frac{1}{f_1} - \frac{1}{u_1} = \frac{u_1 - f_1}{f_1 u_1},$$

and

Back conjugate focal distance for first element.

$$v_1 = \frac{f_1 u_1}{u_1 - f_1}.$$

Then

Front conjugate focal distance for second element.

$$u_2 = v_1 - s = \frac{f_1 u_1}{u_1 - f_1} - s = \frac{f_1 u_1 - s(u_1 - f_1)}{u_1 - f_1}.$$

Then, since the rays are converging into e_2, u_2 becomes a minus quantity, therefore

$$\frac{1}{v_2} = \frac{1}{f_2} - \frac{1}{-u_2} = \frac{1}{f_2} + \frac{u_1 - f_1}{f_1 u_1 - s(u_1 - f_1)}$$

$$= \frac{f_1 u_1 - s(u_1 - f_1) + f_2(u_1 - f_1)}{f_2\{f_1 u_1 - s(u_1 - f_1)\}} = \frac{f_1 u_1 - (s - f_2)(u_1 - f_1)}{f_2\{f_1 u_1 - s(u_1 - f_1)\}},$$

Back conjugate focal distance for second element.

and

$$v_2 = \frac{f_2\{f_1 u_1 - s(u_1 - f_1)\}}{f_1 u_1 - (s - f_2)(u_1 - f_1)}.$$

Add $e_2 \ldots p_2$ or P_2 to this in order to obtain the distance of the focus q from the second principal point p_2, and we have

$$P_2 + v_2 = \frac{s f_2}{f_1 + f_2 - s} + \frac{f_2\{f_1 u_1 - s(u_1 - f_1)\}}{f_1 u_1 - (s - f_2)(u_1 - f_1)},$$

and

$$\frac{1}{P_2+v_2}=\frac{(f_1+f_2-s)\{f_1u_1-(s-f_2)(u_1-f_1)\}}{sf_2\{f_1u_1-(s-f_2)(u_1-f_1)\}+(f_1+f_2-s)f_2\{f_1u_1-s(u_1-f_1)\}}.$$

After multiplying out denominator and cancelling we get

$$\frac{1}{P_2+v_2}=\frac{(f_1u_1-su_1+sf_1+f_2u_1-f_1f_2)(f_1+f_2-s)}{f_1f_2(f_1u_1+sf_1+f_2u_1-su_1)}.$$

Reciprocal of back conjugate focal length measured from second principal point.

Then the other conjugate focal distance $= Q\,.\,.\,p_1$ or u_1+P_1

$$=u_1+\frac{sf_1}{f_1+f_2-s}=\frac{u_1(f_1+f_2-s)+sf_1}{f_1+f_2-s}.$$

Therefore

$$\frac{1}{u_1+P_1}=\frac{f_1+f_2-s}{u_1f_1+u_1f_2-u_1s+sf_1},$$

Reciprocal of front conjugate focal length measured from first principal point.

therefore

$$\frac{1}{u_1+P_1}+\frac{1}{v_2+P_2}=\frac{(u_1f_1+u_1f_2-u_1s+f_1s-f_1f_2)(f_1+f_2-s)+f_1f_2(f_1+f_2-s)}{f_1f_2(u_1f_1+u_1f_2-u_1s+f_1s)},$$

which

$$=\frac{(u_1f_1+u_1f_2-u_1s+f_1s)(f_1+f_2-s)+(f_1f_2-f_1f_2)(f_1+f_2-s)}{f_1f_2(u_1f_1+u_1f_2-u_1s+sf_1)}$$

$$=\frac{f_1+f_2-s}{f_1f_2}\text{ or }\frac{1}{E}=\text{constant}.$$

Sum of above two reciprocals $=\frac{1}{E}$.

Thus the mutually dependent variables u_1 and v_2, the front and back focal distances respectively, have eliminated themselves, and we find that the sum of the reciprocals of the conjugate focal distances as measured from their respective principal points p_1 and p_2 is constantly equal to $\frac{1}{E}$. If the reader will apply the same processes to a combination of three separated elements, he will arrive at just the same result, although the process is much more lengthy. Therefore the combination of two thin lenses or elements, however widely separated they may be, behaves like a simple thin lens of principal focal length E, such that $\frac{1}{V}=\frac{1}{E}-\frac{1}{U}$ if we put U for u_1+P_1 and V for P_2+v_2. It only differs from a simple thin lens in that the two principal points are widely separated instead of both merging in the lens centre. Fig. 19*a* presents the case of two dispersive elements.

It is commonly remarked that a thing cannot be in two places at once, but here we have an optical combination of equivalent focal

A compound lens exists practically in two positions at once.

length E (that is, it forms an image of infinitely distant objects on exactly the same scale as would be formed by a simple thin lens of principal focal length E); but from the point of view of Q, or the left hand, this equivalent simple lens is supposed to be placed at p_1, while from the point of view of q, or the right hand, it is supposed to be placed at p_2. It thus presents a dual aspect.

Above curious feature illustrated.

Fig. 19b illustrates this curious feature. It represents the essentials of Fig. 19, p_1 and p_2 being the first and second principal points, and $Q \,..\, Q_1$ and $q \,..\, q_1$ the two conjugate focal planes in which lie either an object or its image. The planes drawn through the two principal points perpendicular to the optic axis $Q \,..\, q$ are generally known as the principal planes, and can be shown to have the curious property that if any direct or oblique pencils of rays, such as $Q \,..\, p_1$ and $Q_1 \,..\, p_1$, strike centrally upon the first principal plane p_1 at certain points at certain distances from the optic axis, then the same rays will start from the second principal plane at similar points at the same distances from the axis (and on the same side of it). For instance, the principal ray $Q \,..\, p_1$, together with two outer rays $Q \,..\, c_1$ and $Q \,..\, b_1$, constitute the axial pencil striking the first principal plane at c_1, p_1, and b_1. Also let the principal ray $Q_1 \,..\, p_1$, together with $Q_1 \,..\, c_1$ and $Q_1 \,..\, b_1$, constitute an oblique pencil also striking the first principal plane at c_1, p_1, and b_1. Draw straight lines from these points parallel to the optic axis to intersect the second principal plane at c_2, p_2, and b_2; then these become the starting-points for the rays of the corresponding conjugate pencils $p_2 \,..\, q$ and $p_2 \,..\, q_1$ in such manner that the principal emergent ray $p_2 \,..\, q_1$ is parallel to the principal entering ray $Q_1 \,..\, p_1$.

A real pupil at the geometric centre implies two equal virtual pupils at the two principal planes.

The proof of this theorem is really a simple one, for we have already seen that if we take two infinitely thin lenses L_1 and L_2 of focal lengths f_1 and f_2 separated by a distance s, then the first principal point is the image of the geometric centre C as formed by L_1, and the second principal point is the image of the geometric centre as formed by L_2. But the geometric centre is symmetrically disposed to the two lenses, and if $f_1 = 3f_2$, then the geometric centre is three times as far from L_1 as from L_2. Therefore the image of C formed by L_1 is magnified or diminished in exactly the same degree as the image of C formed by L_2. Consequently, if we imagine a circular aperture or pupil to be placed at C, then the image of it formed in the first principal plane by L_1 will be exactly equal to the other image of it formed in the second principal plane by L_2. The two principal planes are in this way shown to be planes of unit magnification relatively to one another. Therefore if the bounding rays of any pencil whatever strike the first

The principal planes are planes of unit magnification.

principal plane at distances d and d' from the axis, they will start from the second principal plane also at points distant by d and d' from the axis, although when actually passing through the plane of the geometric centre the distances d and d' may be more or less reduced or increased. Moreover, these distances d and d' invariably keep to the same side of the axis; for since the images of the imaginary aperture placed at the geometric centre are formed by L_1 and L_2 in the two principal planes under similar conditions, therefore if the image of our pupil at C formed by L_1 at p_1 is the same way up as the original, then the other image of C formed by L_2 at p_2 is also the same way up, or if one image at p_1 is reversed, then so is the other image at p_2.

The two principal pupils the same way up.

It is interesting to note how the pencils of rays are set back in their course, as it were, by the distance $p_1 - p_2$ between the focal centres, which therefore constitutes in this case an overlapping of the conjugate focal distances, and corresponding shortening of the distance $Q \mathinner{..} q$. This theorem, that any two separated lenses on a common axis act as a simple thin lens of equivalent principal focal length E, is highly significant, and the important corollary follows from it, that all optical systems, however complex, exhibit two final principal points, and that the sum of the reciprocals of the conjugate focal distances measured from those points is constant.

All lens systems have two final principal points.

Proof of above theorem.

For, supposing we have three thin lenses e_1, e_2, e_3, of principal focal lengths f_1, f_2, f_3, arranged on a common axis, as in Fig. 20, Plate V. Then the couple e_1 and e_2 have their geometric centre at c_1, and their two principal points at p_1 and p_2, and have an equivalent principal focal length = E. Then, from the point of view of e_3, the combination $e_1 + e_2$ is tantamount to a simple lens of principal focal length = E placed at p_2. It therefore follows that we have a new geometric centre C such that $(p_2 \mathinner{..} \mathrm{C}) : (\mathrm{C} \mathinner{..} e_3) :: \mathrm{E} : f_3$. Then the point C refracted by the equivalent lens at p_2 will be apparently transferred to P_1, $p_1 \mathinner{..} P_1$ being in this case conjugate to $p_2 \mathinner{..} \mathrm{C}$, and C is also transferred to P_2 by the refraction of e_3, and we have two new principal points P_1 and P_2 for the whole combination of three lenses, which latter possesses a new equivalent principal focal length which we may call E_3, *which is also a constant with respect to the three lenses* (so long as the separations are constant). It will be seen that

$$\frac{1}{p_1 \mathinner{..} \mathrm{P}_1} = \frac{1}{p_2 \mathinner{..} \mathrm{C}} - \frac{1}{\mathrm{E}}$$

and

$$\frac{1}{\mathrm{P}_2 \mathinner{..} e_3} = \frac{1}{\mathrm{C} \mathinner{..} e_3} - \frac{1}{f_3}.$$

Therefore, from the left hand, or from the point of view of rays entering the combination, P_1 is the first principal point; while from the right hand, or from the point of view of rays leaving the combination, P_2 is the second principal point.

Principal points of a three-lens combination investigated.

We now require formulæ giving the distances $e_1 \ldots P_1$ and $P_2 \ldots e_3$, or P_3 and P_4 respectively.

Let $e_1 \ldots e_2 = s_1$, and $e_2 \ldots e_3 = s_2$.

First of all, from Formula IXA. we have by analogy

$$p_1 \ldots P_1 = \frac{SE}{E + f_3 - S},$$

in which

$$S = s_2 + (p_2 \ldots e_2) = s_2 + \frac{s_1 f_2}{f_1 + f_2 - s_1}$$

and

$$E = \frac{f_1 f_2}{f_1 + f_2 - s_1}.$$

Therefore after substituting these values we get

$$p_1 \ldots P_1 = \frac{\left(s_2 + \dfrac{s_1 f_2}{f_1 + f_2 - s_1}\right)\dfrac{f_1 f_2}{f_1 + f_2 - s_1}}{\dfrac{f_1 f_2}{f_1 + f_2 - s_1} + f_3 - \left(s_2 + \dfrac{s_1 f_2}{f_1 + f_2 - s_1}\right)},$$

$$= \frac{\left\{\dfrac{s_2(f_1 + f_2 - s_1) + s_1 f_2}{f_1 + f_2 - s_1}\right\}\dfrac{f_1 f_2}{f_1 + f_2 - s_1}}{\dfrac{f_1 f_2 + f_3(f_1 + f_2 - s_1) - s_2(f_1 + f_2 - s_1) - s_1 f_2}{f_1 + f_2 - s_1}},$$

therefore

$$p_1 \ldots P_1 = \frac{\{s_2(f_1 + f_2 - s_1) + s_1 f_2\} f_1 f_2}{(f_1 + f_2 - s_1)\{f_1 f_2 + f_3(f_1 + f_2 - s_1) - s_2(f_1 + f_2 - s_1) - s_1 f_2\}}.$$

But we must add the distance $e_1 \ldots p_1$ to this in order to obtain the required distance $e_1 \ldots P_1$; and

$$e_1 \ldots p_1 = \frac{s_1 f_1}{f_1 + f_2 - s_1}$$

(see Formula IXA.); and on adding this to the above formula for $p_1 \ldots P_1$ we get

$$\frac{\{s_2(f_1 + f_2 - s_1) + s_1 f_2\} f_1 f_2 + s_1 f_1 \{f_1 f_2 + f_3(f_1 + f_2 - s_1) - s_2(f_1 + f_2 - s_1) - s_1 f_2\}}{(f_1 + f_2 - s_1)\{f_1 f_2 + f_3(f_1 + f_2 - s_1) - s_2(f_1 + f_2 - s_1) - s_1 f_2\}},$$

which reduces down to

$$\frac{f_1(f_2 s_1 + f_2 s_2 + f_3 s_1 - s_1 s_2)}{f_2(f_1 - s_1) + (f_3 - s_2)(f_1 + f_2 - s_1)} = e_1 \ldots \mathrm{P}_1 = \mathrm{P}_3. \qquad \text{XIA.}$$

Formula locating first principal point for three elements.

Next we require a formula for the distance $\mathrm{P}_2 \ldots e_3$, measured from the second principal point P_2 of the triple combination to the element e_3.

From Formula IXB. we have by analogy

$$\mathrm{P}_2 \ldots e_3 = \frac{\mathrm{S}f_3}{\mathrm{E} + f_3 - \mathrm{S}}, \text{ in which S, as above, } = s_2 + \frac{s_1 f_2}{f_1 + f_2 - s_1},$$

and

$$\mathrm{P}_2 \ldots e_3 = \frac{\left(s_2 + \dfrac{s_1 f_2}{f_1 + f_2 - s_1}\right) f_3}{\dfrac{f_1 f_2}{f_1 + f_2 - s_1} + f_3 - \left(s_2 + \dfrac{s_1 f_2}{f_1 + f_2 - s_1}\right)},$$

which reduces down to

$$\frac{f_3(f_2 s_2 + f_2 s_1 + f_1 s_2 - s_1 s_2)}{f_2(f_1 - s_1) + (f_3 - s_2)(f_1 + f_2 - s_1)} = \mathrm{P}_2 \ldots e_3 = \mathrm{P}_4. \qquad \text{XIB.}$$

Formula locating second principal point for three elements.

It is plainly evident that the Formula XIB. is the symmetrical complement of Formula XIA. For, if we trace the light backwards through the combination, then f_3 becomes f_1, s_2 becomes s_1, while f_2 remains f_2, and thus the one formula may be turned into the other. Next, we require a formula for the equivalent principal focal length E_3 of such a combination of three lenses or elements.

By analogy from Formula X. we derive

$$\mathrm{E}_3 = \frac{\mathrm{E}f_3}{\mathrm{E} + f_3 - \mathrm{S}},$$

in which, as in last two cases,

$$\mathrm{S} = s_2 + p_2 \ldots e_2, \text{ or } s_2 + \frac{s_1 f_2}{f_1 + f_2 - s_1},$$

and

$$\mathrm{E} = \frac{f_1 f_2}{f_1 + f_2 - s_1},$$

therefore

$$\mathrm{E}_3 = \frac{\dfrac{f_1 f_2 f_3}{f_1 + f_2 - s_1}}{\dfrac{f_1 f_2}{f_1 + f_2 - s_1} + f_3 - \left(s_2 + \dfrac{s_1 f_1}{f_1 + f_2 - s_1}\right)},$$

which reduces down to

Formula for equivalent focal length for three elements.

$$\frac{f_1f_2f_3}{f_2(f_1-s_1)+(f_3-s_2)(f_1+f_2-s_1)}=E_3. \qquad \text{XII.}$$

We may now pass on to the consideration of a combination of four separated lenses or elements. Let Fig. 21 represent four elements e_1, e_2, e_3, and e_4, separated by the three distances s_1, s_2, and s_3. Our line of procedure is to consider this as a combination of two couples, viz. e_1 and e_2, having their two principal points at p_1 and p_2; and e_3 and e_4, having their principal points at p_3 and p_4. Then the distance $p_2 \ldots p_3$ or S is obviously the real separation between these two couples, whose respective equivalent principal focal lengths we will denote by E_1 and E_2. Then the separation between them is obviously equal to $p_2 \ldots e_2 + s_2 + e_3 \ldots p_3 = S$, so that generally the equivalent principal focal length E_4 of the whole combination

$$=\frac{E_1E_2}{E_1+E_2-S},$$

which

$$=\frac{\dfrac{f_1f_2}{f_1+f_2-s_1}\cdot\dfrac{f_3f_4}{f_3+f_4-s_3}}{\dfrac{f_1f_2}{f_1+f_2-s_1}+\dfrac{f_3f_4}{f_3+f_4-s_3}-\left\{\dfrac{s_1f_2}{f_1+f_2-s_1}+s_2+\dfrac{s_3f_3}{f_3+f_4-s_3}\right\}}$$

$$=\frac{f_1f_2f_3f_4}{f_1f_2(f_3+f_4-s_3)+f_3f_4(f_1+f_2-s_1)-s_1f_2(f_3+f_4-s_3)-s_2(f_1+f_2-s_1)(f_3+f_4-s_3)-s_3f_3(f_1+f_2-s_1)},$$

and finally

Formula for equivalent focal length for four elements.

$$E_4=\frac{f_1f_2f_3f_4}{(f_1+f_2-s_1)(f_3f_4-f_3s_3)+(f_3+f_4-s_3)\{f_2(f_1-s_1)-s_2(f_1+f_2-s_1)\}}. \qquad \text{XIII.}$$

It is obvious that the two couples have between them a new centre of symmetry, C, or the optical centre of the whole combination, so located that it divides the distance $p_2 \ldots p_3$ between the second and third principal points into two parts such that $p_2 \ldots C : C \ldots p_3 :: E_1 : E_2$. Then the point C, refracted by E_1, is transferred to P_1 such that $p_1 \ldots P_1$ is conjugate to $p_2 \ldots C$. In the same way the point C, refracted by E_2, is transferred to P_2 such that $p_4 \ldots P_2$ is conjugate to $C \ldots p_3$. We now want formulæ for the distances of the two principal points P_1 and P_2 from the outside elements e_1 and e_4. In working out the principal points for two collective elements we found that if the first principal point fell to the right hand of e_1 then the distance $e_1 \ldots p_1$ was positive, and if it fell to the left hand, it was negative; while if the second principal point fell to the left of e_2 then the distance $p_2 \ldots e_2$ was positive, and if it fell to the right hand it was negative. Bearing this in mind, it will be seen that in this case the required distance $e_1 \ldots P_1 = (e_1 \ldots p_1) + (p_1 \ldots P_1)$, the latter being negative in Fig. 21, and also that $e_4 \ldots P_2 = (p_4 \ldots e_4) + (p_4 \ldots P_2)$, the latter also being negative in Fig. 21. So we have $e_1 \ldots P_1 = (e_1 \ldots p_1) + (p_1 \ldots P_1)$ (algebraically)

$$= \frac{s_1 f_1}{f_1 + f_2 - s_1} + \frac{SE_1}{E_1 + E_2 - S},$$

in which S or $p_2 \ldots e_3 = (p_2 \ldots e_2) + s_2 + (e_3 \ldots p_3)$, which

$$= \frac{s_1 f_2}{f_1 + f_2 - s_1} + s_2 + \frac{s_3 f_3}{f_3 + f_4 - s_3}.$$

On substituting this value of S in the above, we get

$$e_1 \ldots P_1 \text{ or } \overline{P}_1 = \frac{s_1 f_1}{f_1 + f_2 - s_1} + \frac{\left\{\dfrac{s_1 f_2}{f_1 + f_2 - s_1} + s_2 + \dfrac{s_3 f_3}{f_3 + f_4 - s_3}\right\}\dfrac{f_1 f_2}{f_1 + f_2 - s_1}}{\dfrac{f_1 f_2}{f_1 + f_2 - s_1} + \dfrac{f_3 f_4}{f_3 + f_4 - s_3} - \left\{\dfrac{s_1 f_2}{f_1 + f_2 - s_1} + s_2 + \dfrac{s_3 f_3}{f_3 + f_4 - s_3}\right\}},$$

which reduces down to

$$e_1 \ldots P_1 \text{ or } \overline{P}_1$$

$$= \frac{f_1\{(f_2 s_1 + f_2 s_2 - s_1 s_2)(f_3 + f_4 - s_3) + f_3(f_2 s_3 + f_4 s_1 - s_1 s_3)\}}{(f_1 + f_2 - s_1)(f_3 f_4 - f_3 s_3) + (f_3 + f_4 - s_3)\{f_2(f_1 - s_1) - s_2(f_1 + f_2 - s_1)\}} \quad \text{XIVA.}$$

Four elements. Formula for position of first principal point.

The distance of the second principal point P_2 from e_4 is obviously $e_4 \ldots p_4 + p_4 \ldots P_2$, analogously to the last case, and is expressed by

$$\frac{s_3 f_4}{f_3 + f_4 - s_3} + \frac{SE_2}{E_1 + E_2 - S},$$

in which S', as before, is the distance $p_2 \ldots p_3$, and E_2 is

$$\frac{f_3 f_4}{f_3 + f_4 - s_3}.$$

Therefore

$$e_4 \ldots P_2 \text{ or } \overline{P}_2 = \frac{s_3 f_4}{f_3 + f_4 - s_3} + \frac{\left\{\dfrac{s_1 f_1}{f_1 + f_2 - s_1} + s_2 + \dfrac{s_3 f_3}{f_3 + f_4 - s_3}\right\}\dfrac{f_3 f_4}{f_3 + f_4 - s_3}}{\dfrac{f_1 f_2}{f_1 + f_2 - s_1} + \dfrac{f_3 f_4}{f_3 + f_4 - s_3} - \left\{\dfrac{s_1 f_1}{f_1 + f_2 - s_1} + s_2 + \dfrac{s_3 f_3}{f_3 + f_4 - s_3}\right\}},$$

which reduces down to

$$e_4 \ldots P_2 \text{ or } \overline{P}_2$$

$$= \frac{f_4\{(f_3 s_3 + f_3 s_2 - s_2 s_3)(f_1 + f_2 - s_1) + f_2(f_3 s_1 + f_1 s_3 - s_1 s_3)\}}{(f_1 + f_2 - s_1)(f_3 f_4 - f_3 s_3) + (f_3 + f_4 - s_3)\{f_2(f_1 - s_1) - s_2(f_1 + f_2 - s_1)\}} \quad \text{XIVB.}$$

Four elements. Formula for position of second principal point.

Symmetry of Formulæ XIVA. and XIVB.

On comparing Formulæ XIVA. and XIVB. it will again be noticed that they are symmetrical to one another: f_1 in the first corresponds to f_4 in the second, f_2 corresponds to f_3, f_3 to f_2, and f_4 to f_1, while s_1 corresponds to s_3 and s_2 to s_2. Hence one formula may be converted into the other by supposing the light to traverse the system in the reverse direction.

Formulæ relating to more than four elements undesirably complex.

We have now got general formulæ for the equivalent focal lengths, and the positions of the two principal points for any combinations of two to four separated elements or thin lenses, stated in terms of the principal focal lengths of the several elements or lenses concerned and the separations between them; and we have found these formulæ relating to a four-lens system to be sufficiently complex to deter us from proceeding any further on the same lines; that is, were we to work out formulæ for a five-lens, six-lens, and eight-lens systems, all likewise expressed in terms of the principal focal lengths of the several elements or lenses involved and their respective separations, we should arrive at undesirably bulky formulæ. In such cases the results are perhaps best arrived at by the building up or cumulative process, yielding formulæ in which equivalent principal focal lengths of two or four lenses together constitute the terms. In this way we may deal with the case of five lenses as follows:—

Case of five elements.

Let e_1, e_2, e_3, e_4, and e_5, Fig. 22, be the five elements involved. Let E_4 be the equivalent principal focal length of the first four elements, $\overline{P}_1$ be the distance from first element e_1 to first principal point P_1 of the same, and $\overline{P}_2$ be the distance from second principal point P_2 to the fourth element e_4. Then we may treat the whole as a combination of a simple lens of E.F.L. $= E_4$ placed at P_2 with another simple lens of E.F.L. $= f_5$ placed at e_5, the distance between them being $P_2 \,..\, e_5$ or $\overline{P}_2 + s_4$. Therefore the equivalent principal focal length of the whole combination will be (see Formula X.)

Equivalent focal length for five elements.

$$E_5 = \frac{E_4 f_5}{E_4 + f_5 - (\overline{P}_2 + s_4)}. \qquad \text{XV.}$$

The distance of the *new* first principal point $\mathbf{P}_1'$ of the five-lens combination from e_1 will then be $(P_1 \,..\, \mathbf{P}_1') + (P_1 \,..\, e_1)$, or $(P_1 \,..\, \mathbf{P}_1') + \overline{P}_1 =$ say $\overline{\mathbf{P}}_1'$, for which the formula will be (see IXA.)

Five elements. Position of first principal point.

$$\overline{\mathbf{P}}_1' = \frac{(s_4 + \overline{P}_2)E_4}{E_4 + f_5 - (s_4 + \overline{P}_2)} + \overline{P}_1, \qquad \text{XVA.}$$

and the formula for the distance of the second new principal point $\mathbf{P}_2'$ from e_5 will be (see IXB.)

PLATE. V.

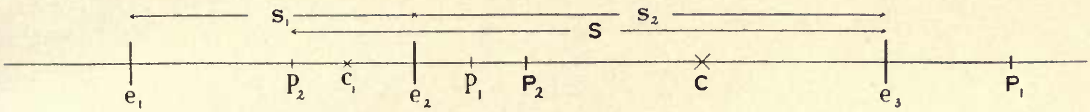


Fig. 20.

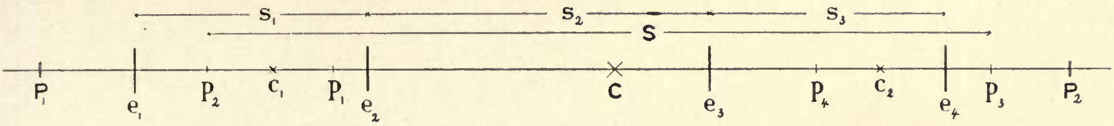


Fig. 21.

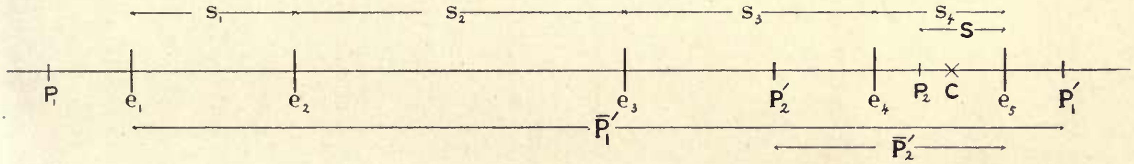


Fig. 22.

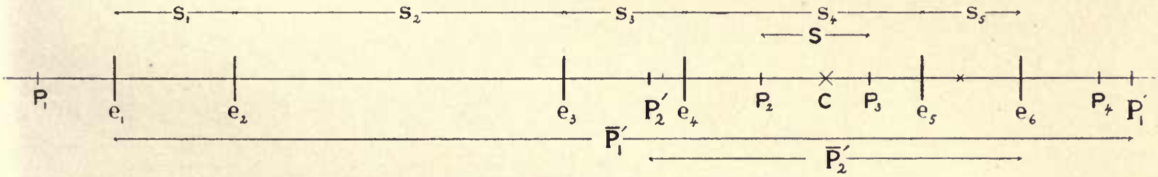


Fig. 23.

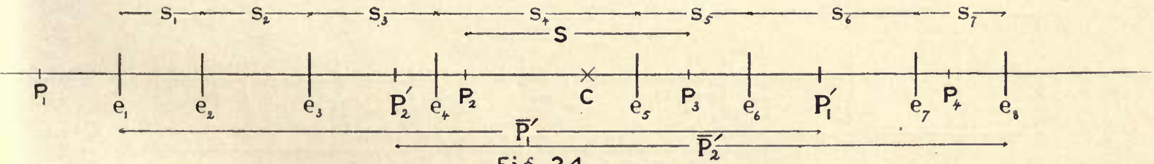


Fig. 24.

PLATE. V.

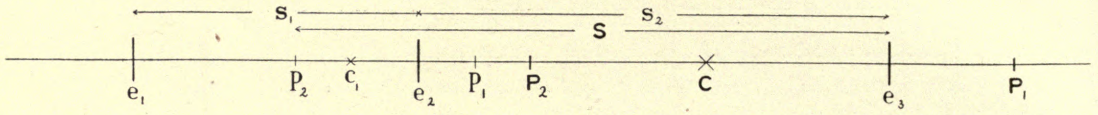


Fig. 20.

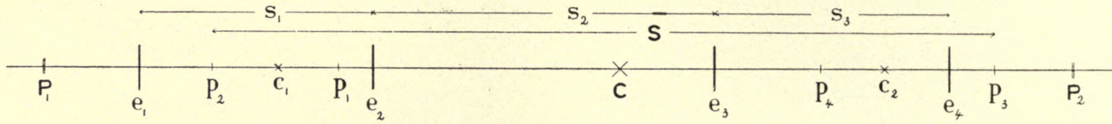


Fig. 21.

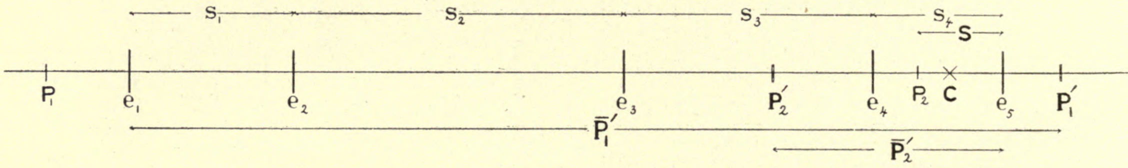


Fig. 22.

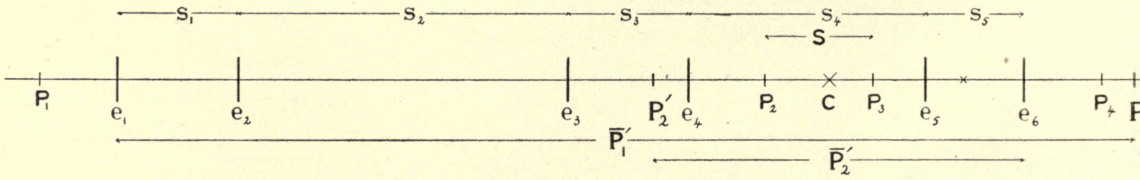


Fig. 23.

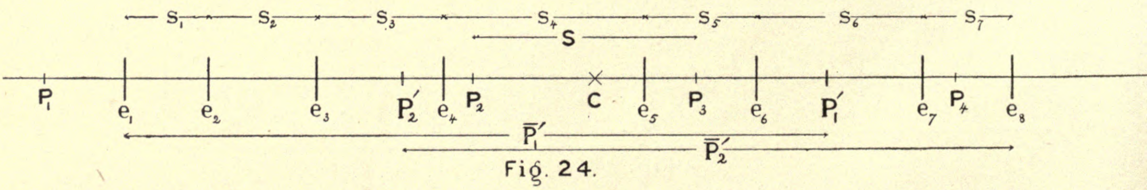


Fig. 24.

$$\bar{\mathbf{P}}_2' = \frac{(s_4 + \bar{P}_2)f_5}{E_4 + f_5 - (s_4 + \bar{P}_2)}. \qquad \text{XVB.}$$

Five elements. Position of second principal point.

Formulæ for Six Thin Lenses or Elements

We may treat this as a combination of four lenses of E.F.L. $= E_4$ with another combination of two lenses of E.F.L. $= E_2$. (See Fig. 23.)
Then if P_1 = first principal point of the four-lens combination, and $\bar{P}_1$ its distance $e_1 \ldots P_1$ from e_1,
and P_2 = the second principal point of the four-lens combination, and $\bar{P}_2$ = distance $e_4 \ldots P_2$ from e_4,
P_3 = the first principal point of the two-lens combination,
and $\bar{P}_3$ = distance $P_3 \ldots e_5$, then we have
P_4 = the second principal point of the two-lens combination, and $\bar{P}_4$ its distance from e_6.

$$E_6 = \frac{E_4 E_2}{E_4 + E_2 - (s_4 + \bar{P}_2 + \bar{P}_3)}. \qquad \text{XVI.}$$

Equivalent focal length for six elements.

Then if $\mathbf{P}_1'$ is the new first principal point and $\mathbf{P}_2'$ the second one for the whole combination, then putting $\bar{\mathbf{P}}_1'$ for the distance $e_1 \ldots \mathbf{P}_1'$, and $\bar{\mathbf{P}}_2'$ for $\mathbf{P}_2' \ldots e_6$, we have

$$\bar{\mathbf{P}}_1' = \frac{(s_4 + \bar{P}_2 + \bar{P}_3)E_4}{E_4 + E_2 - (s_4 + \bar{P}_2 + \bar{P}_3)} + \bar{P}_1 \qquad \text{XVIA.}$$

Six elements. Position of first principal point.

and

$$\bar{\mathbf{P}}_2' = \frac{(s_4 + \bar{P}_2 + \bar{P}_3)E_2}{E_4 + E_2 - (s_4 + \bar{P}_2 + \bar{P}_3)} + \bar{P}_4. \qquad \text{XVIB.}$$

Six elements. Position of second principal point.

Another way is to treat a six-lens combination as a combination of three couples of E.F.L.s respectively $= E_1$, E_2, and E_3, then apply Formulæ XIA., XIB., and XII.

Formulæ for Eight Thin Lenses or Elements

This consists of two four-lens combinations, whose respective E.F.L.s we may call E_4' and E_4''. (See Fig. 24.)
Let P_1 be the first principal point of first four-lens combination, and $\bar{P}_1$ its distance from e_1.
Let P_2 be the second principal point of first four-lens combination, and $\bar{P}_2$ its distance from e_4.

Let P_3 be the first principal point of second four-lens combination, and $\bar{P}_3$ its distance from e_5.

Let P_4 be the second principal point of the second four-lens combination, and $\bar{P}_4$ its distance from e_8.

Let $\mathbf{P}_1' =$ the first principal point of the eight-lens combination, and $\bar{\mathbf{P}}_1' =$ the distance $e_1 \ldots \mathbf{P}_1'$.

Let $\mathbf{P}_2' =$ the second principal point of the eight-lens combination, and $\bar{\mathbf{P}}_2' =$ the distance $\mathbf{P}_2' \ldots e_8$.

Equivalent focal length for eight elements.

Then
$$E_8 = \frac{E_4' E_4''}{E_4' + E_4'' - (s_4 + \bar{P}_2 + \bar{P}_3)}, \qquad \text{XVII.}$$

Eight elements. Position of first principal point.

$$\bar{\mathbf{P}}_1' = \frac{(s_4 + \bar{P}_2 + \bar{P}_3)E_4'}{E_4' + E_4'' - (s_4 + \bar{P}_2 + \bar{P}_3)} + \bar{P}_1, \qquad \text{XVIIA.}$$

Eight elements. Position of second principal point.

and
$$\bar{\mathbf{P}}_2' = \frac{(s_4 + \bar{P}_2 + \bar{P}_3)E''_4}{E_4' + E_4'' - (s_4 + \bar{P}_2 + \bar{P}_3)} + \bar{P}_4, \qquad \text{XVIIB.}$$

Various methods of treatment.

While combinations of eight separate lenses may seldom occur, yet combinations of four thick lenses are frequently employed, and we have seen that such cases may be treated as cases of eight elements, the elements appertaining to each solid lens being considered to be separated by a distance s equal to $\frac{t}{\mu}$; while if any lenses are cemented together or in contact, then, in the above formulæ, the separation s_2, s_4, or s_6 (or whichever it may be, in its natural order) should be entered as equal to O, while the principal focal lengths of the elements in contact may be entered as usual, even when of equal refractive indices, in which case they exactly neutralise one another and may be treated as non-existent. Or if of different refractive indices and in contact or cemented, then, as one is necessarily a collective element and the other a dispersive element, the difference of their powers may be taken as the power of one resultant element, and thus the calculations, which are inevitably tedious in complicated cases, be considerably simplified. Or the E.F.L. and principal points of each thick lens may be worked out separately, resulting in a combination of four equivalent lenses, whose effective separations of course depend upon the relative positions of their principal points; then any of the above formulæ suitable to the case may be employed.

Cemented lenses, etc.

Having once obtained the principal equivalent focal length of any one more or less complicated combination of lenses, and the position of the two principal points (sometimes called nodal points) with reference to the first element or first vertex of the combination, and the last element or last vertex of the combination respectively, then, as all conjugate focal lengths are to be measured from those principal points, the positions of all images or original plane objects and their images can always be correctly assigned with reference to the first and last vertices of the combination, if so desired, *provided that the optical corrections of the system are at least approximately well carried out.*

Good optical corrections assumed.

For it must be borne in mind that the above lines of reasoning and the consequent formulæ are based upon the theorems of Gauss, which are abstractions in the sense that they would be of no practical value whatever if applied to lens combinations thrown haphazard together in such manner that no approach to flat, distinct, and rectilinear images were made at all. The more perfect the images formed by complex lens systems, so much the nearer to absolute accuracy become the deductions from the Gauss theory as embodied in the formulæ which we have arrived at in this section.

The above theorems acknowledge no optical aberrations.

A few illustrative examples of the application of the formulæ to known combinations may now be given.

Examples.

Let a sphere of glass of refractive index $= 1{\cdot}5$ and of radius r be treated by the method of elements. Then

Case of refracting sphere.

$$\frac{1}{f_1} = \frac{1}{f_2} = \frac{\cdot 5}{r} = \frac{1}{2r}$$

and

$$f_1 = f_2 = 2r$$

and

$$s = \frac{t}{\mu} = \frac{2r}{1{\cdot}5} = \frac{4}{3}r,$$

and the E.F.L. by Formula X.

$$= \frac{(2r)(2r)}{2r + 2r - \dfrac{4r}{3}} = \frac{4r^2}{\dfrac{12r - 4r}{3}} = 4r^2 \times \frac{3}{8r} = \frac{3}{2}r,$$

E.F.L. of sphere.

while either the first or second positions of principal points are given by Formula IXA. or IXB., so that

$$\overline{\mathrm{P}}_1 = \mathrm{P}_2 = \frac{2r \cdot \dfrac{4}{3}r}{2r + 2r - \dfrac{4}{3}r} = \frac{\dfrac{8}{3}r^2}{\dfrac{8}{3}r} = r.$$

Centre and principal points coincide.

Thus if the lens is a solid sphere, then the distances of the two principal points are both + and coincide with the centre of the sphere, but if the combination is of two elements separated by an air-space equal to $\frac{t}{\mu}$, then the two principal points would overlap by a distance equal to $t\frac{\mu-1}{\mu}=\frac{t}{3}$; but the performance of the spherical solid lens and its equivalent combination of elements are exactly identical from the exterior point of view, so far as the E.F.L. and conjugate focal distances and relative scale of images are concerned.

Huygenian Eye-pieces

The usual and older form of the Huygenian eye-piece (Fig. 25, Plate VI.) consisted of two lenses of principal focal lengths 3 and 1, placed at a distance apart equal to half the sum of their focal lengths, that being the necessary condition for the variously coloured images being of equal size. In many cases, however, the ratio of 2 to 1 for the principal focal lengths is adopted, the same rule for separation of course prevailing. Treating the case generally we have E.F.L., or

Two cases of Huygenian eye-pieces.

$$\mathrm{E}=\frac{f_1 f_2}{f_1+f_2-\frac{f_1+f_2}{2}}=\frac{f_1 f_2}{\frac{f_1+f_2}{2}}=2\left(\frac{f_1 f_2}{f_1+f_2}\right),$$

and

$$\frac{1}{\mathrm{E}}=\frac{1}{2}\left(\frac{1}{f_1}+\frac{1}{f_2}\right),$$

so that the power of the combination is the mean of the powers of the two lenses. For instance

$$\text{if } f_1=3 \text{ and } f_2=1, \text{ we get E.F.L.} = 1\tfrac{1}{2}, \text{ and } \frac{1}{\mathrm{E}}=\tfrac{1}{2}(\tfrac{1}{3}+1)=\tfrac{2}{3},$$

$$\text{and if } f_1=2 \text{ and } f_2=1, \text{ we get E.F.L.} = 1\tfrac{1}{3}, \text{ and } \frac{1}{\mathrm{E}}=\tfrac{1}{2}(\tfrac{1}{2}+1)=\tfrac{3}{4}.$$

The position of the first principal point p_1 is given by

Principal points of Huygenian eye-pieces.

$$\overline{\mathrm{P}}_1=\frac{f_1\cdot\frac{f_1+f_2}{2}}{f_1+f_2-\frac{f_1+f_2}{2}}=+f_1, \text{ and } \overline{\mathrm{P}}_2=\frac{f_2\frac{f_1+f_2}{2}}{f_1+f_2-\frac{f_1+f_2}{2}}=+f_2.$$

Thus Figs. 25 and 26 represent the essential features of any such combination having $s=\frac{f_1+f_2}{2}$.

It is clear that since, as in all previous cases, the distance from the first vertex to the principal focal plane $f = E - \overline{P}_1$, and $\overline{P}_1$ is in this case the larger, therefore $E - \overline{P}_1$ is a minus quantity and indicates that the image formed by the eye-piece at f of distant objects on the right is a virtual one. On the other hand, $E - \overline{P}_2$ is a positive quantity, as $\overline{P}_2$ is the lesser, and indicates that a real image is formed at F_2 of a distant object on the left. It is well known that a real image of the relatively distant object glass to the left is formed at F_2. Thus either of the distances $p_1 \ldots F_1$ or $p_2 \ldots F_2$ represents the principal equivalent focal length of the combination.

Image in first principal focal plane is a virtual one.

Image formed in second principal focal plane is a real one.

It is clear also that when used with a telescope whose objective is to the left hand, the eye-piece must be so placed that the primary image formed by the objective must be made to fall upon the first principal focal plane f, in order that the rays emerging from the eye-piece may be parallel and fit for normal vision.

Condition of use with a telescope.

For it is clear that the rays converging to the image in the first principal focal plane f will, after refraction by the first lens, be converged to a real image in $F_1 \ldots F_1$, in which plane also is the second principal point p_2, where it is also in the principal focal plane of the second lens. This coincidence of the position of the real image formed between the lenses with the position of the second principal point is characteristic of combinations wherein $s = \frac{f_1 + f_2}{2}$, but it is a matter concerning the internal economy of the combination as it were; and we must remember that the formulæ we have worked out for equivalent focal lengths and positions of the principal points, in themselves deal with resultants and take no explicit notice of what goes on between the lenses, but only deal with the positions of objects or images from or to which the rays are proceeding before they enter the system and after they emerge from it. Thus in Fig. 26, with regard to the rays entering the combination, a simple thin lens (having a principal focal length equal to the E.F.L. of the combination) may be imagined to be placed at the first principal point p_1, so that the entering rays converging to a real image at f and $f \ldots p_1$ are about equal to the E.F.L. of the system; while, after emergence from the eye-piece, the rays of pencils are either parallel, as if coming from a distant virtual image on the left hand, slightly divergent from a nearer virtual image, or else slightly convergent to a real image on the right hand; but in all cases the focal distance of such image, which is conjugate to the distance $f \ldots p_1$, is

Formulæ of this section deal with resultant effects only.

An elementary lens equivalent to the eye-piece.

measured from the second principal point p_2. Therefore, supposing I is a particular point in the first image at f, and we join I by a straight line $I..p_1$ to the first principal point p_1, then if we draw another straight line through p_2 parallel to $I..p_1$, it will cut the plane of the second conjugate image at the point where the image of the point I is formed therein (assuming distortion to be eliminated).

Another aspect of the question.

While we are dwelling on the case of the Huygenian eye-piece, Fig. 26, we may, with much advantage, discuss an aspect of this question of equivalent focal lengths of lens combination which may well appear puzzling to those studying the question for the first time.

Principal rays do not pass through geometric centre.

In our treatment of thick lenses and combinations of two thin lenses or elements we have assumed the centre or principal rays of oblique pencils of rays to pass through the geometric centre of the said lens or pair of elements, but in Figs. 25 and 26 this does not take place at all, and, in fact, the principal rays of oblique pencils are shown to cross the optic axis, not at the geometric centre C, but at or near F_2, the second principal focal plane. Now it is the size of the distant object glass to the left that defines the sizes of the pencils of light entering the eye-piece, and we have seen that an image of the object glass is formed very near to F_2, through which image pass all the more or less oblique pencils of light emerging from the eye-piece. This image is the exit pupil of the eye-piece, and its centre or the point on the axis where it occurs is the exit pupil point of the eye-piece.

The exit pupil of an eye-piece.

Definition of pupil; in the case of an image of a real stop.

The pupil point or points of an optical combination may then be defined as the point or points where the principal rays traversing the combination, or their projections, cross the axis. In this case the pupil point is where an image of the object glass would be formed by L_1. If the object glass on the left is brought nearer to the eye-piece, then the pupil point will, of course, move towards the right. The aperture of the object glass may then be regarded as the entrance pupil of the eye-piece, the pupil being an image of it formed by L_1, and the exit pupil is an image of that image formed by L_2.

The entrance pupil.

Case where stop and pupil are one.

But cases of other optical combinations may be imagined, such as photographic lenses, wherein the stop or diaphragm may be somewhere in the middle of the combination, and be an actual stop and not merely an image of another stop. In some cases the diaphragm or stop forming the pupil may be placed exactly at the geometric centre of the combination, as for simplicity has been assumed in working out the formulæ in this section, but in very many cases it is not so placed.

Pupil not necessarily placed at geometric centre of a combination.

In fact, the position of the pupil point of any combination is totally independent of the position of the geometric centre, and therefore of

PLATE. VI.

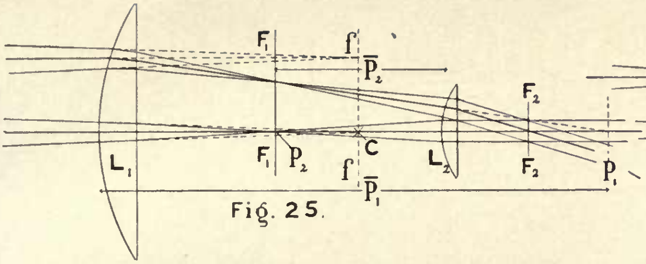


Fig. 25.

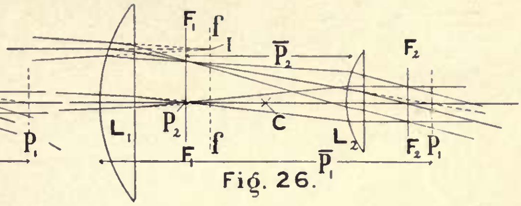


Fig. 26.

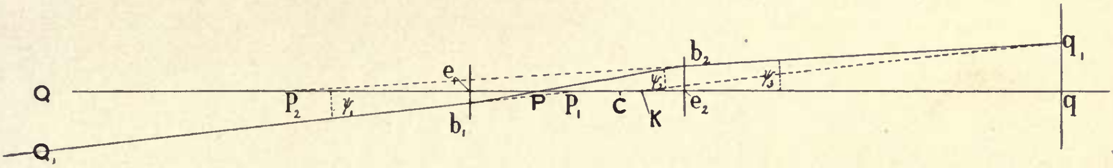


Fig. 27.

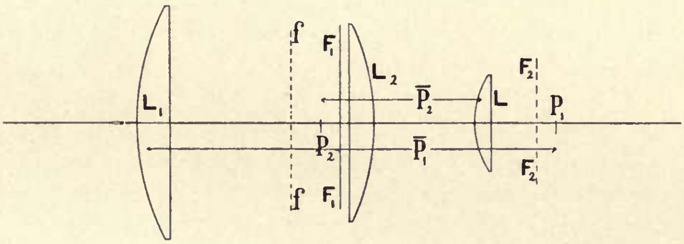


Fig. 28.

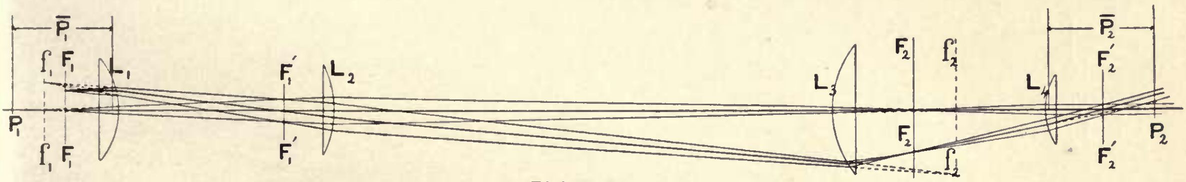


Fig. 29.

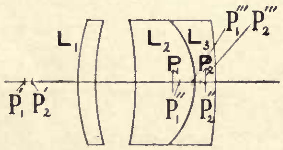


Fig. 30.

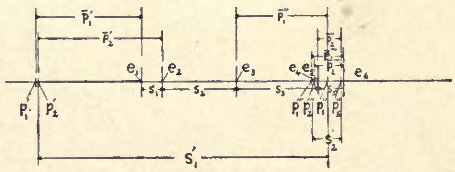


Fig. 30.a.

PLATE. VI.

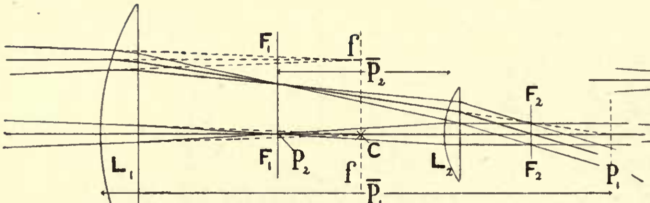


Fig. 25.

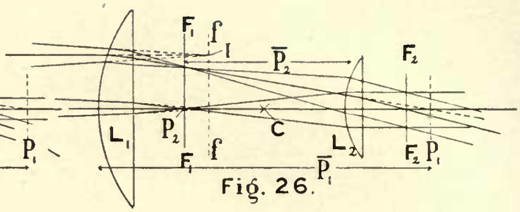


Fig. 26.

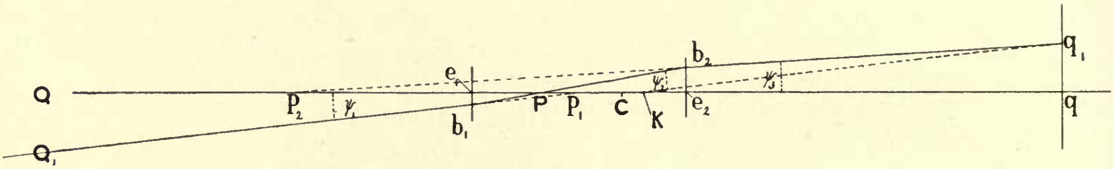


Fig. 27.

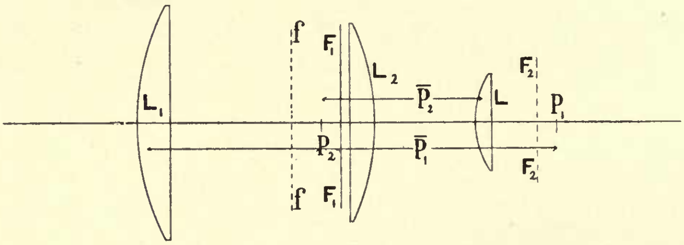


Fig. 28.

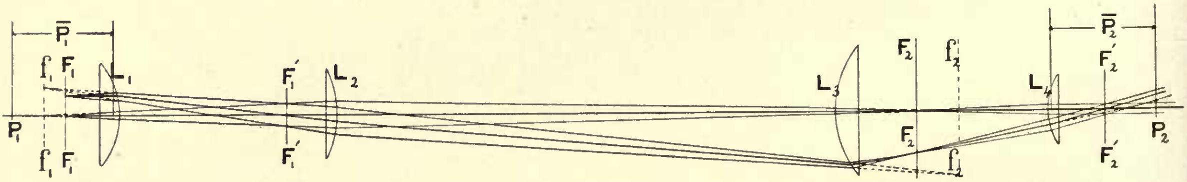


Fig. 29.

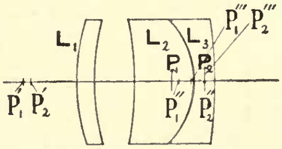


Fig. 30.

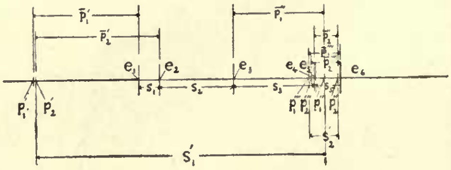


Fig. 30.a.

the two principal points. But it might be thought that if the pupil point is widely removed from the geometric centre, as is the case in the Huygenian eye-piece, then the equivalent focal length of the combination might be quite different, and that our formula for the same would no longer hold good. This matter is certainly worth inquiring into. In the first place, the theorem of homogeneous pencils as explained on page 11, may be applied here. For although we are considering the oblique pencils traversing the Huygenian eye-piece as avoiding the geometric centre (and therefore the principal points) of the combination, yet if we imagine such pencils to be homogeneous, but very much enlarged in angular aperture, then we arrive at a state of things in which, although the principal or central rays of all such pencils still avoid the geometric centre, yet there is sure to be some *one* ray in each pencil which does actually pass through the geometric centre and the two principal points, and since such centre-traversing rays are proceeding to or from the same image points as the principal rays of the same pencils (*ex hypothesi*), we therefore clearly see that the relative sizes and positions of the conjugate images should not be disturbed by the fact that the real pupil point in an optical system does not coincide with the geometric centre, or that the apparent pupil points do not coincide with the principal points, *if we assume that the final image approximates to perfection in all respects.*

The geometric centre traversed by *one* ray of each oblique pencil.

Assuming that the theorem of the homogeneous pencil holds good we may prove the case algebraically thus:—

Proof that the E.F.L. is independent of position of the pupil.

Let e_1 and e_2, Fig. 27, be two thin lenses or elements, and let P be the position of the stop where the principal rays of oblique pencils are constrained to cross the optic axis $Q \,..\, q$, and let P be placed anywhere not necessarily coincident with the geometric centre, which may be at C for instance. Let $Q_1 \,..\, b_1 \,..\, P \,..\, b_2 \,..\, q_1$ be one of the oblique principal rays proceeding from an infinitely distant point Q_1 on the left hand to the image point of it at q_1 in the principal focal plane $q \,..\, q_1$. Before entering e_1 this principal ray is proceeding to p_1, the first pupil point, which is the apparent position of P as refracted by e_1, and on emerging from e_2 it proceeds apparently from p_2, the second pupil point, which is the apparent position of P as refracted by e_2. Let the separation $e_1 \,..\, e_2 = S$, and let $e_1 \,..\, p_1 = C_1$, $e_1 \,..\, P = D_1$, $P \,..\, e_2 = S - D_1 = C_2$, and $p_2 \,..\, e_2 = D_2$; so that we have C_1 and D_1 conjugates as well as C_2 and D_2.

Then we have

$$\frac{1}{C_1} = \frac{1}{D_1} - \frac{1}{f_1} = \frac{f_1 - D_1}{D_1 f_1}$$

and

$$C_1 = \frac{D_1 f_1}{f_1 - D_1},$$

$$\frac{1}{D_2} = \frac{1}{S - D_1} - \frac{1}{f_2} = \frac{f_2 - (S - D_1)}{f_2(S - D_1)}$$

and

$$D_2 = \frac{f_2(S - D_1)}{f_2 - (S - D_1)}.$$

Let angle $Q_1 \ldots p_1 \ldots Q = \psi_1$, angle $b_2 \ldots P \ldots e_2 = \psi_2$, and angle $q_1 \ldots p_2 \ldots q = \psi_3$; then

$$\tan \psi_2 = \tan \psi_1 \frac{C_1}{D_1}$$

and

$$\tan \psi_3 = \tan \psi_2 \frac{C_2}{D_2};$$

therefore

$$\tan \psi_3 = \tan \psi_1 \frac{C_1 C_2}{D_1 D_2} = \tan \psi_1 \frac{D_1 f_1}{f_1 - D_1} \cdot \frac{1}{D_1} \frac{S - D_1}{1} \frac{f_2 - (S - D_1)}{f_2(S - D_1)},$$

and

$$\frac{\tan \psi_3}{\tan \psi_1} = \frac{f_1(f_2 + D_1 - S)}{f_2(f_1 - D_1)}.$$

Now the back focal distance $e_2 \ldots q$ or B (from Formula IXc.)

$$= \frac{f_2(f_1 - S)}{f_1 + f_2 - S};$$

therefore in order to get the distance $p_2 \ldots q$ we must add D_2 and B together, when we have

$$p_2 \ldots q = \frac{f_2(S - D_1)}{f_2 + D_1 - S} + \frac{f_2(f_1 - S)}{f_1 + f_2 - S},$$

which, after multiplying out and cancelling, reduces to

$$p_2 \ldots q = \frac{f_2^2(f_1 - D_1)}{(f_1 + f_2 - S)(f_2 + D_1 - S)}.$$

Now, it is evident that if we draw a straight line from q_1 parallel to the incident principal ray $Q_1 \ldots p_1$, and therefore making the same angle ψ_1 with the axis, it will cut the axis at the point K, where a simple thin lens of equivalent focal length E of the combination would have to be placed in order to project on $q \ldots q_1$ an image of identical size, and therefore the distance $K \ldots q$ will be the equivalent focal length of the combination; but obviously

$$\text{E or K} \ldots q = (p_2 \ldots q)\frac{\tan \psi_3}{\tan \psi_1} \text{ or } (\text{B} + \text{D}_2)\frac{\tan \psi_3}{\tan \psi_1}$$

$$= \frac{f_2^2(f_1 - \text{D}_1)}{(f_1 + f_2 - \text{S})(f_2 + \text{D}_1 - \text{S})} \cdot \frac{f_1(f_2 + \text{D}_1 - \text{S})}{f_2(f_1 - \text{D}_1)} = \frac{f_1 f_2}{f_1 + f_2 - \text{S}} = \text{E}.$$

Thus the question of the position of the pupil point as measured by D_1 has eliminated itself, and the equivalent focal length is shown to be a function of the principal focal lengths of the lenses and their separations, and quite independent of the position of the pupil point within or without the system.

Another method of deriving the E.F.L.

There is still another method of working out the equivalent focal lengths of any combinations, which treats all images by projection from the several lens centres or the points on the axis where the elements occur, by which the back focal length is arrived at. The back focal length is then multiplied by $\frac{\tan \psi_{n+1}}{\tan \psi_1}$, wherein n is the number of elements, ψ_{n+1} the angle made with the optic axis by a straight line joining the last lens centre or element to the particular image point q_1, and ψ_1 being the angle made with the optic axis by a ray from the infinitely distant object point Q_1 striking the first element or lens centre. It is thus based upon the theorem of central projection, and leads directly to precisely the same formulæ for equivalent focal lengths and indirectly to the same formulæ for principal points.

The Ramsden Eye-piece

This well-known form of eye-piece is supposed to consist of two lenses of equal focal length separated by the focal length of either. Under these conditions it is clear that the geometric centre is half way between them, and therefore the first principal point coincides with the second lens and first principal focal plane, while the second principal point coincides with the first lens and the second principal focal plane. In practice, however, the two lenses are fixed rather closer together than this, even at the sacrifice of perfect oblique achromatism.

Three-Lens Huygenian Eye-piece

A few more concrete examples may now be examined. For instance, Fig. 28 represents a form of three-lens Huygenian eye-piece which is often used by Continental opticians.

$$f_1 = 6{\cdot}2 \qquad f_2 = 5{\cdot}7 \qquad f_3 = 2{\cdot}2$$
$$s_1 = 2{\cdot}6 \qquad s_2 = 1{\cdot}2$$

From these figures Formula XII. gives $E_3 = +2{\cdot}6$, Formula XIA. gives $\overline{P}_1 = +5{\cdot}04$, and Formula XIB. gives $\overline{P}_2 = +1{\cdot}92$. Thus P_1, the first principal point, is a long way back, even behind the last lens; and if pencils of parallel rays enter the lens from the left, then a real image is formed in the principal focal plane F_2–F_2; and if pencils of parallel rays enter from the right hand, then a virtual image is formed in the other principal focal plane F_1–F_1 from the point of view of an observer to the left hand. Therefore, if an object glass away to the left forms a real image at $F_1 \ldots F_1$, in such a manner that it would be actually formed at $F_1 \ldots F_1$ were L_1 not there, then the pencils emerging from L_3 will consist of parallel rays in proper condition to be received by a normal eye with its pupil placed in or near $F_2 \ldots F_2$, near which an image of the distant object glass will be formed. In this case a real image will be formed between L_2 and L_3 in the plane $f \ldots f$.

The Three-Lens Erecting Eye-piece

This is an old and discarded device which may be compared to a Huygenian eye-piece with a supplementary collective lens placed a long way in front of it, whose office it is to throw into the Huygenian eye-piece an inverted image of the primary telescopic image. Although this combination can be made into an achromatic eye-piece, yet the impossibility of obtaining a well-corrected large field of view has led to its disuse. Such a three-lens eye-piece may also be regarded as practically a four-lens eye-piece in which the power of the second lens has become zero.

The Four-Lens Erecting Eye-piece

Let us now turn our attention to the well-known four-lens or erecting eye-piece. This is a construction subject to much variety consistently with good performance, but Fig. 29 may be taken as a fair sample of the construction. Here

$$f_1 = 1 \qquad f_2 = 1{\cdot}25 \qquad f_3 = 1{\cdot}25 \qquad f_4 = {\cdot}80$$
$$s_1 = 1{\cdot}3 \qquad s_2 = 4{\cdot}0 \qquad s_3 = 1{\cdot}2$$

Here from Formula XIII. we get $E_4 = -{\cdot}31$, from Formula XIVA. we get $\overline{P}_1 = -{\cdot}605$, and from Formula XIVB. we get $\overline{P}_2 = -{\cdot}635$. We thus find that the E.F.L. of such a combination is negative, while the negative values for $\overline{P}_1$ and $\overline{P}_2$ indicate that the two principal points are both outside the system, as shown.

One of the most striking points about the four-lens eye-piece is its clumsiness. In the present case it is seen that the length over all the lenses is about thirteen times the E.F.L., and it is almost impossible to compress such an eye-piece into less than seven times the E.F.L. without sacrificing flatness of field and other good qualities.

Positions of the two images of distant objects on the left and with distant object on the right.

If pencils of parallel rays enter from the left, then a real image (upside down) is first formed at $F_1' \ldots F_1'$ at a distance f_1 behind L_1, and then another real and upright image is formed at $F_2' \ldots F_2'$, the second principal focal plane of the system, and at a distance $P = E_4$ to the left of the second principal point P_2. If, on the other hand, we suppose pencils of parallel rays to enter the system from the right, then a real image (upside down) is formed at $F_2 \ldots F_2$ at a distance $= f_4$ to left of L_4, and another, upright, image is formed in $F_1 \ldots F_1$, the first principal focal plane, situated at a distance $= E_4$ to the right of the first principal point P_1.

Use with a telescope.

Conversely, if an object glass to the left forms an upside-down real image at $F_1 \ldots F_1$, then after passage through the first three lenses an erect real image is formed at $F_2 \ldots F_2$ in the principal focus of L_4, and the rays emerge from L_4 parallel and in condition to be received by a normal eye with its pupil placed somewhere near $F_2' \ldots F_2'$ (where an image of the distant object glass is formed).

To all intents and purposes, and regarded from the left hand, the combination is equal to a thin dispersive lens of principal focal length $= E_4$ placed at P_1 at its principal focal length *inside* of the primary image $F_1 \ldots F_1$, while from the point of view of the right hand the combination is equivalent to a thin dispersive lens of principal focal length $= E_4$ placed at P_2, with the rays emerging from it in parallel condition, but with the principal rays of the pencils diverging from an exit pupil point in or slightly to the right of $F_2 \ldots F_2$, where an image of the object glass is formed. But since such equivalent dispersive lens placed at P_2 is an abstraction, there is nothing to prevent the pupil of the observer's eye being advanced to the plane F_2'–F_2', where it is obviously in a position to take in the whole field of view, instead of a small portion of it, which it would be restricted to were a real equivalent dispersive lens placed at P_2.

The Cooke Process Lens

We will now take, as a further example of the application of these formulæ, a form of photographic lens designed for copying diagrams, of which Fig. 30 gives a section.

L_1 and L_2 are of the same glass having $\mu_D = 1{\cdot}6103$.
L_3 is made of glass having $\mu_D = 1{\cdot}524$ $(= M_D)$.
The radii counting from left to right are as follows:—

Curves of process lens.

$$r_1 = +1{\cdot}264 \quad r_2 = -1{\cdot}48 \quad r_3 = -2{\cdot}09 \quad r_4 = +{\cdot}553$$
$$r_5 = -{\cdot}5325 \quad r_6 = +2{\cdot}8$$

The thicknesses of the lenses L_1, L_2, L_3, are respectively—

$$t_1 = +{\cdot}105 \quad t_2 = {\cdot}358 \quad t_3 = {\cdot}110$$

Air-space $A_1 = {\cdot}232$. Air-space $A_2 = {\cdot}0053$.

Three pairs of elements.

We will now treat this combination as one of six elements arranged in three pairs. Fig. 30*a* shows it rendered into six elements separated by five air-spaces s_1, s_2, s_3, s_4, and s_5, of which

$$s_1 = \frac{t_1}{\mu_D} \quad s_2 = A_1 \quad s_3 = \frac{t_2}{\mu_D} \quad s_4 = A_2 \quad s_5 = \frac{t_3}{M_D}.$$

The first step is to take the elements in three consecutive pairs corresponding to the three lenses, and find their equivalent focal lengths by Formula X., and the positions of their principal points by Formulæ IXA and IXB.

E.F.L. of each lens and position of its principal points required

The second step is to obtain the equivalent focal length of the combination of three lenses (or sets of two elements) and the positions of their respective principal points, by Formulæ XII. and XIA. and XIB.

Calling the principal focal lengths of the several elements f_1 and f_2, etc., we find

$$f_1 = +2{\cdot}0711 \mid f_2 = -2{\cdot}425 \parallel f_3 = -3{\cdot}4246 \mid f_4 = +{\cdot}90611 \parallel$$
$$f_5 = -1{\cdot}01622 \mid f_6 = +5{\cdot}3435,$$

and

$$s_1 = \frac{t_1}{\mu_D} = {\cdot}0652 \mid s_2 = A_1 = {\cdot}232 \mid s_3 = \frac{t_2}{\mu_D} = {\cdot}2223 \quad s_4 = A_2 = {\cdot}0053 \quad s_5 = \frac{t_3}{M_D} = {\cdot}0722.$$

We then get

L_1

L_1. Equivalent focal length.

$$E_1 = \frac{(2{\cdot}0711)(-2{\cdot}425)}{2{\cdot}0711 - 2{\cdot}425 - {\cdot}0652} = +11{\cdot}984,$$

L_1. Position of first principal point.

$$\bar{p}_1{}' = \frac{({\cdot}0652)(2{\cdot}0711)}{2{\cdot}0711 - 2{\cdot}425 - {\cdot}0652} = -{\cdot}3222$$

(to left of and outside of lens),

$$\bar{p}_2{}' = \frac{(\cdot0652)(-2\cdot425)}{2\cdot0711 - 2\cdot425 - \cdot0652} = +\cdot3772$$

(to left of and within second vertex).

L_1. Position of second principal point.

L_2

$$E_2 = \frac{(-3\cdot4246)(\cdot90611)}{-3\cdot4246 + \cdot90611 - \cdot2223} = +1\cdot1322,$$

L_2. Equivalent focal length.

$$\bar{p}_1{}'' = \frac{(\cdot2223)(-3\cdot4246)}{-3\cdot4246 + \cdot90611 - \cdot2223} = +\cdot2777$$

(to right of and within first vertex),

L_2. Position of first principal point.

$$\bar{p}_2{}'' = \frac{(\cdot2223)(\cdot90611)}{-3\cdot4246 + \cdot90611 - \cdot2223} = -\cdot07349$$

(to right of and outside second vertex).

L_2. Position of second principal point.

L_3

(treated as a positive entity)

$$E_3 = \frac{(+1\cdot01622)(-5\cdot3435)}{+1\cdot01622 - 5\cdot3435 + \cdot0722} = 1\cdot27617,$$

L_3. Equivalent focal length.

$$\bar{p}_1{}''' = \frac{(-\cdot0722)(1\cdot01622)}{1\cdot01622 - 5\cdot3435 + \cdot0722} = +\cdot01724$$

(to left of and outside first vertex),

L_3. Position of first principal point.

$$\bar{p}_2{}''' = \frac{(-\cdot0722)(-5\cdot3435)}{1\cdot01622 - 5\cdot3435 + \cdot0722} = -\cdot09064$$

(to left of and within second vertex).

L_3. Position of second principal point.

Fig. 30*a* shows on an enlarged scale the positions of the six elements with their virtual separations and the principal points for the three combinations of two elements representing the three lenses. Thus $p_1{}'$ and $p_2{}'$ are the principal points for the first lens, consisting of $e_1 + e_2$; $p_1{}''$ and $p_2{}''$ are the principal points for the second lens, consisting of $e_3 + e_4$; and $p_1{}'''$ and $p_2{}'''$ are the principal points for the third lens, consisting of $e_5 + e_6$. We now want the separations between these equivalent lenses. First we want $s_1{}'$ which obviously

$$= \bar{p}_2{}' + s_2 + \bar{p}_1{}'' = +\cdot3772 + \cdot232 + \cdot2777 = \cdot887.$$

Then we want $s_2{}'$, which obviously

$$= s_4 + \bar{p}_2{}'' + \bar{p}_1{}''' = +\cdot0053 - \cdot07349 - \cdot01724 = -\cdot08543.$$

We must bear in mind that we have, according to our usual procedure, treated the dispersive lens in itself as a positive entity, but that in adding up a series of collective and dispersive lenses, we must then prefix the minus sign before the E.F.L. of a dispersive lens or any of its functions yet dealt with. Hence $\bar{p}_1'''$, which is a plus quantity relatively to the dispersive lens, becomes a minus quantity in the above expression for s_2'.

Having now got the values of the three E.F.L.'s and the two separations, we may then work out the E.F.L. of the whole combination from Formula XII., thus stated

E.F.L. of the Process lens.

$$\text{E.F.L.} = \frac{E_1E_2E_3}{E_2(E_1 - s_1') + (E_3 - s_2')(E_1 + E_2 - s_1')},$$

$$= \frac{(+11{\cdot}984)(+1{\cdot}1322)(-1{\cdot}27617)}{(1{\cdot}1322)(11{\cdot}984 - {\cdot}887) + (-1{\cdot}27617 + {\cdot}08543)(11{\cdot}984 + 1{\cdot}1322 - {\cdot}887)}$$

$$= \frac{(11{\cdot}984)(1{\cdot}1322)(-1{\cdot}27617)}{-1{\cdot}9975} = +8{\cdot}6685.$$

The next important matter is the determination of the two principal points of the combination. By analogy with Formula XIA. we have

$$\bar{P}_1 = \frac{E_1(E_2s_1' + E_2s_2' + E_3s_1' - s_1's_2')}{E_2(E_1 - s_1') + (E_3 - s_2')(E_1 + E_2 - s_1')}$$

$$= \frac{(11{\cdot}984)(1{\cdot}00426 - {\cdot}096724 - 1{\cdot}13196 + {\cdot}075776)}{-1{\cdot}9975}$$

$$= \frac{E_1(-{\cdot}14865)}{-1{\cdot}9975} = +{\cdot}8918 = \bar{P}_1.$$

Also by analogy with Formula XIB. we have

$$\bar{P}_2 = \frac{E_3(E_2s_2' + E_2s_1' + E_1s_2' - s_1's_2'}{E_2(E_1 - s_1') + (E_3 - s_2')(E_1 + E_2 - s_1')}$$

$$= \frac{(-1{\cdot}27617)(-{\cdot}09672 + 1{\cdot}00426 - 1{\cdot}0238 + {\cdot}07577)}{-1{\cdot}9975}$$

$$= \frac{(-1{\cdot}27617)(-{\cdot}0405)}{-1{\cdot}9975} = -{\cdot}0259 = \bar{P}_2.$$

Further factors to be allowed for.

It should here be pointed out that our Formulæ XIA. and XIB. gave the distances $\bar{P}_1$ and $\bar{P}_2$ of the two principal points from the two outer lenses or elements, on the supposition that the three members of the system were simple or infinitely thin lenses, in which case their two principal points would be merged together in the centre of each such lens or element. But in the case before us each of the three lenses is a compounded lens, having its two principal points more or less widely separated; and it is obvious that the distance $\overline{\mathrm{P}}_1$, which we have just worked out, is measured, not from e_1, but from p_1', the first principal point of the first lens L_1. Hence $\overline{p}_1'$ has to be algebraically added to it in order to obtain the corrected distance $e_1 \cdot\cdot \mathrm{P}_1$ or $\overline{\mathbf{P}}_1$.

Likewise the distance $\overline{\mathrm{P}}_2$, which we have worked out, is measured from p_2''', the second principal point of the third lens L_3, so that $\overline{p}_2'''$ must be algebraically added to $\overline{\mathrm{P}}_2$ in order to obtain the corrected distance $e_6 \,..\, \mathrm{P}_2$ or $\overline{\mathbf{P}}_2$, so that

Final principal points of the process lens.

$$\overline{\mathbf{P}}_1 = \overline{\mathrm{P}}_1 + \overline{p}_1' = +\cdot 8918 - \cdot 3222 = +\cdot 57$$

and

$$\overline{\mathbf{P}}_2 = \overline{\mathrm{P}}_2 + \overline{p}_2''' = -\cdot 0259 - (-\cdot 0906) = +\cdot 0647.$$

(Here the sign of $\overline{p}_2'''$ for the dispersive lens has to be reversed.) This particular combination will be seen to afford a capital illustration of the application of our formulæ, as it embodies certain features characteristic of meniscus lenses, which may easily lead astray a student taking up investigations of this sort for the first time. There cannot be too much care bestowed upon the matter of signs; for in prolonged and intricate optical calculations errors in signs are more likely to occur, and are often more difficult to detect, than errors in mere arithmetic.

Pitfalls as to signs.

Back focal length.

There is a very common term used in connection with the focal lengths of lens combinations, and that is the Back Focal Length, or the distance from the outer vertex of the last lens to the image formed by the lens of infinitely distant objects.

It is obvious that the back focal length is simply the algebraic difference between the equivalent focal length and the distance of the second principal point from the outer apex of the last lens, or

$$\text{B.F.L.} = \mathrm{E} - \overline{\mathbf{P}}_2.$$

Nodal points and focal centres.

The principal points of lens combinations are also often termed nodal points and focal centres. These terms more fully emphasise the fact that a straight line drawn from a certain point Q_1 in the first conjugate image or object to the first nodal point is always parallel to a straight line drawn from the second nodal point to the point q_1 in the final conjugate image where the image of the aforesaid point Q_1 is formed.

Certain defects in lens systems which may more or less disguise this normal law of projection, will be dealt with in subsequent Sections.

What constitutes a telescope.

Judged by the formulæ we have been dealing with in the present inquiry, the combination of lenses forming a telescope is of peculiar theoretical interest. For the condition of clear vision through a telescope for normal eyesight demands that the primary image of distant objects formed in the principal focal plane of the object glass shall also be in the principal focal plane of the eye-piece. Therefore the separation s between the object glass and the eye-piece $= F + f$ or the sum of their principal equivalent focal lengths. Then in the formula for the E.F.L. of the combination

E.F.L. of telescope = infinity.

$$E = \frac{Ff}{F + f - s}$$

we have $F + f - s = 0$ and $E =$ infinity. Also our formulæ for $\overline{P}_1$ and $\overline{P}_2$, or

A telescope has no principal points.

$$\frac{sF}{F + f - s} \text{ and } \frac{sf}{F + f - s},$$

severally = infinity. Thus the image is formed at the geometric centre of the combination forming the telescope, but it has no focal power and no principal points, although it may possess immense magnifying power. The subject of magnifying power will be best dealt with in a subsequent Section (IX.) relating to distortion.

SECTION IV

SPHERICAL ABERRATION OF SIMPLE AND COMBINED LENSES AND CONDITIONS OF ITS ELIMINATION—VON SEIDEL'S FIRST CONDITION

Spherical Aberration of Direct or Axial Pencils

So far we have assumed that, in all cases of refraction of axial pencils of rays by a spherical surface or their reflection from any spherical surface, the rays so refracted or reflected will still diverge from or converge to definite points situated in the conjugate focal planes.

Aberration not hitherto considered.

It requires, however, a very slight practical or theoretical acquaintance with optics to convince one of the existence of what is known as Spherical Aberration, or the aberration or wandering of the outer rays of direct pencil from the theoretical conjugate focal point which we have hitherto assumed. In our investigation of this phenomenon we shall find it most convenient to deal with the case of spherical refracting surfaces and lenses first, and with the case of spherical reflecting surfaces afterwards. We will first follow the method pursued by Henry Coddington in his *Treatise on the Reflection and Refraction of Light*, Part I., pp. 56 *et seq.*, also 90 *et seq.*

Method pursued by Coddington.

Let Fig. 31, Plate VII., be a typical case of a convex refracting surface RAR' of radius r, on which is impinging a cone or pencil of rays diverging from the point Q_1, the axis of the pencil or the principal ray passing through the centre of curvature O. After refraction the rays converge again, the rays ultimately near the axis focusing at Q_1' and the marginal rays Q . . R and Q . . R' at the point Q_2'.

Diagrams explained

From R drop R . . P perpendicular to $Q_1 \,.\,.\, Q_1'$. It must of course be understood that in the diagrams the distance R . . P or y, which measures the semi-aperture of the pencil, is much exaggerated relatively to the radius of curvature, in order to make it easier to follow the diagram. Let A be the vertex of the surface and let O be the centre of its curvature. Then it is evident that $\angle Q_1RO$ is the supplement to the angle

Construction.

of incidence, while $\angle Q_2'RO$ is the angle of refraction. Hence sin $Q_1RO = \mu$ sin $Q_2'RO$, also we have

$$\left.\begin{aligned}\frac{Q_1..O}{Q_1..R}=\frac{\sin Q_1RO}{\sin Q_1OR}=\frac{\sin\angle\text{ Incidence}}{\sin Q_1OR}\\ \frac{Q_2'..O}{Q_2'..R}=\frac{\sin Q_2'RO}{\sin Q_2'OR}=\frac{\sin\angle\text{ Refraction}}{\sin Q_1OR}\end{aligned}\right\}=\frac{\mu}{1};$$

in which $\mu =$ the refractive index; therefore

The fundamental equation.

$$\mu\frac{Q_2'..O}{Q_2'..R}=\frac{Q_1..O}{Q_1..R}. \qquad (1)$$

Let $Q_1..A = u$, $O..R = r$, $A..Q_1' = u_1$, and $A..Q_2' = u_2$; and let $R..P = y$, $Q_2'..O = u_2 - r$, and $Q_1..O = u + r$. Then we have

$$Q_2'..R = Q_2'..A - \text{vers}\,(A..P) + \text{vers}\,(a..P)$$

$$= u_2 - \frac{y^2}{2r} + \frac{y^2}{2u_2} = u_2 - y^2\left(\frac{1}{2r} - \frac{1}{2u_2}\right);$$

therefore

$$\frac{1}{Q_2'..R} = \frac{1}{u_2} + \frac{y^2}{2u_2^2}\left(\frac{1}{r} - \frac{1}{u_2}\right) = \frac{1}{u_2}\left\{1 + \frac{1}{u_2}\left(\frac{1}{r} - \frac{1}{u_2}\right)\frac{y^2}{2}\right\}.$$

We have also

$$Q_1..R = Q_1..A + \text{vers}\,(A..P) + \text{vers}\,(P..b)$$

$$= u + \frac{y^2}{2r} + \frac{y^2}{2u} = u + y^2\left(\frac{1}{2r} + \frac{1}{2u}\right);$$

therefore

$$\frac{1}{Q_1..R} = \frac{1}{u} - \frac{y^2}{2u^2}\left(\frac{1}{r} + \frac{1}{u}\right) = \frac{1}{u}\left\{1 - \frac{1}{u}\left(\frac{1}{r} + \frac{1}{u}\right)\frac{y^2}{2}\right\}.$$

Therefore Equation (1) expands to

$$\mu\frac{u_2 - r}{u_2}\left\{1 + \frac{1}{u_2}\left(\frac{1}{r} - \frac{1}{u_2}\right)\frac{y^2}{2}\right\} = \frac{u + r}{u}\left\{1 - \frac{1}{u}\left(\frac{1}{r} + \frac{1}{u}\right)\frac{y^2}{2}\right\}.$$

By dividing both sides by r and reducing we get

$$\mu\left(\frac{1}{r} - \frac{1}{u_2}\right) + \frac{\mu}{u_2}\left(\frac{1}{r} - \frac{1}{u_2}\right)^2\frac{y^2}{2} = \left(\frac{1}{r} + \frac{1}{u}\right) - \frac{1}{u}\left(\frac{1}{r} + \frac{1}{u}\right)^2\frac{y^2}{2};$$

therefore

$$-\frac{\mu}{u_2} = \frac{1}{r} + \frac{1}{u} - \frac{\mu}{r} - \frac{1}{u}\left(\frac{1}{r} + \frac{1}{u}\right)^2\frac{y^2}{2} - \frac{\mu}{u_2}\left(\frac{1}{r} - \frac{1}{u_2}\right)^2\frac{y^2}{2},$$

or

$$\frac{\mu}{u_2} = \frac{\mu - 1}{r} - \frac{1}{u} + \left\{\frac{1}{u}\left(\frac{1}{r} + \frac{1}{u}\right)^2 + \frac{\mu}{u_2}\left(\frac{1}{r} - \frac{1}{u_2}\right)^2\right\}\frac{y^2}{2}.$$

PLATE. VII.

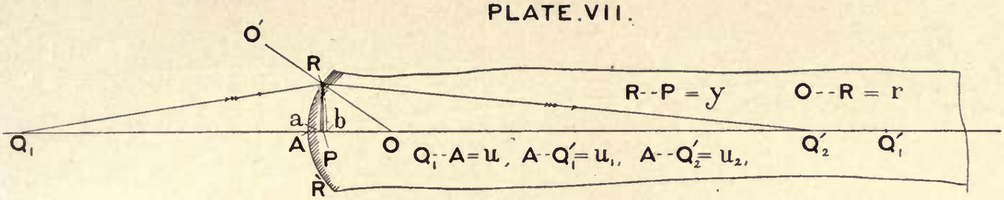


Fig. 31.

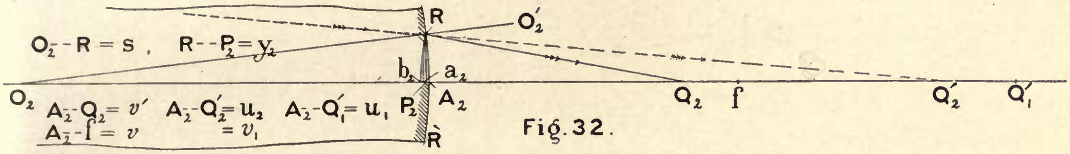


Fig. 32.

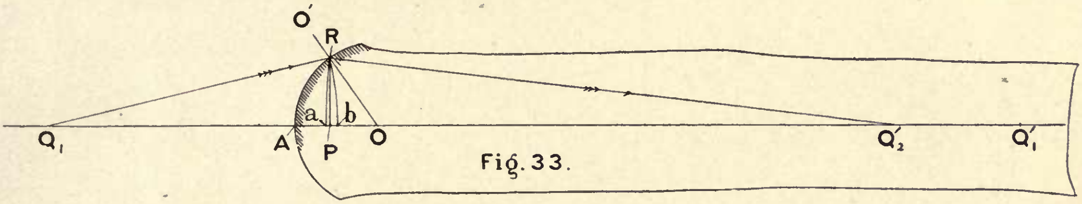


Fig. 33.

Figs. 34.

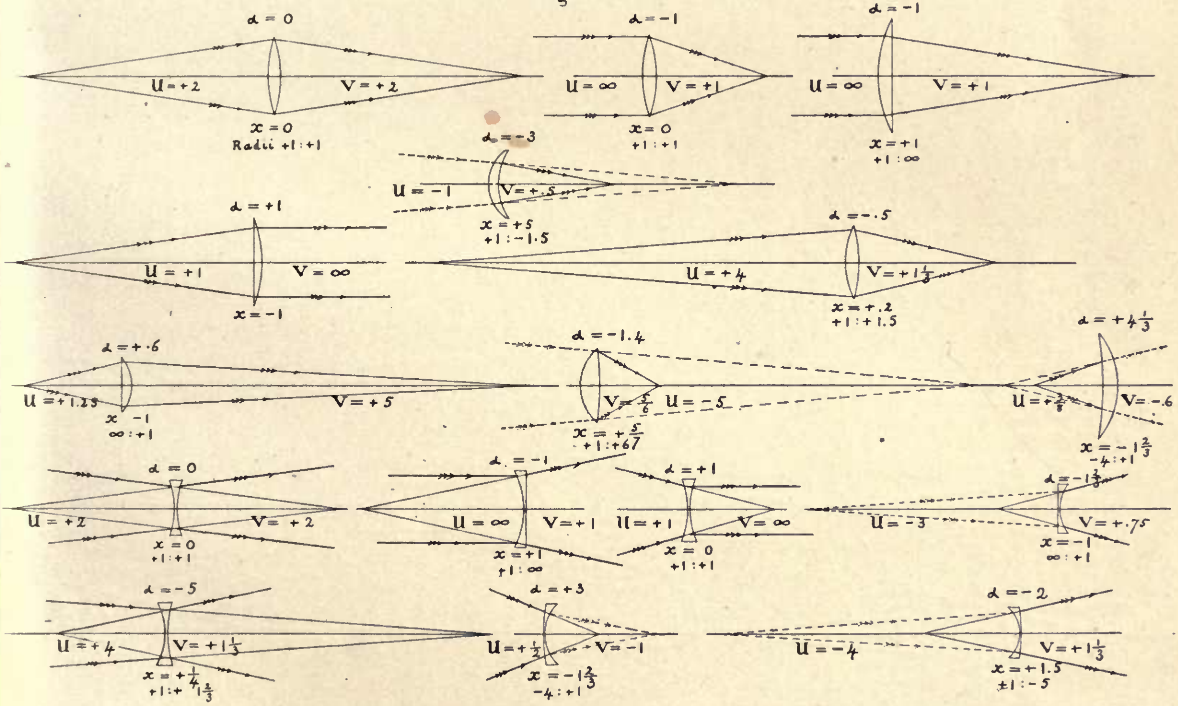


PLATE.VII.

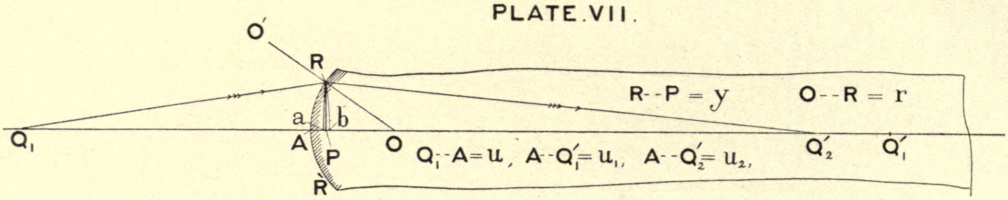


Fig. 31.

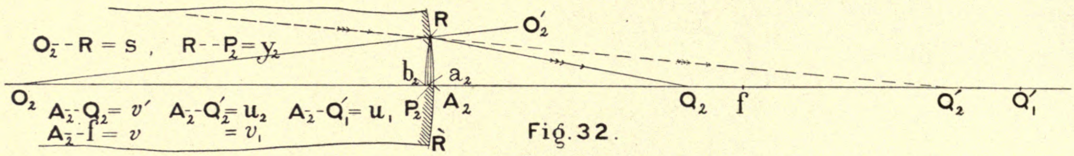


Fig. 32.

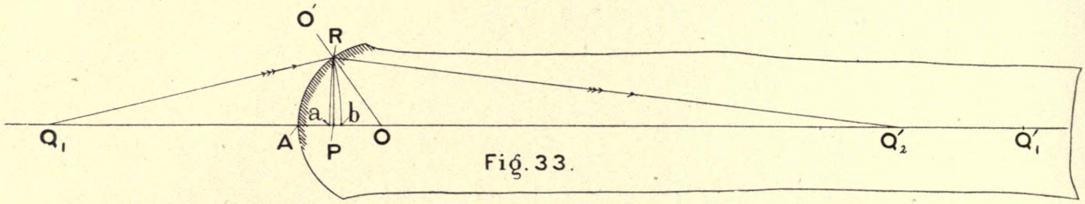
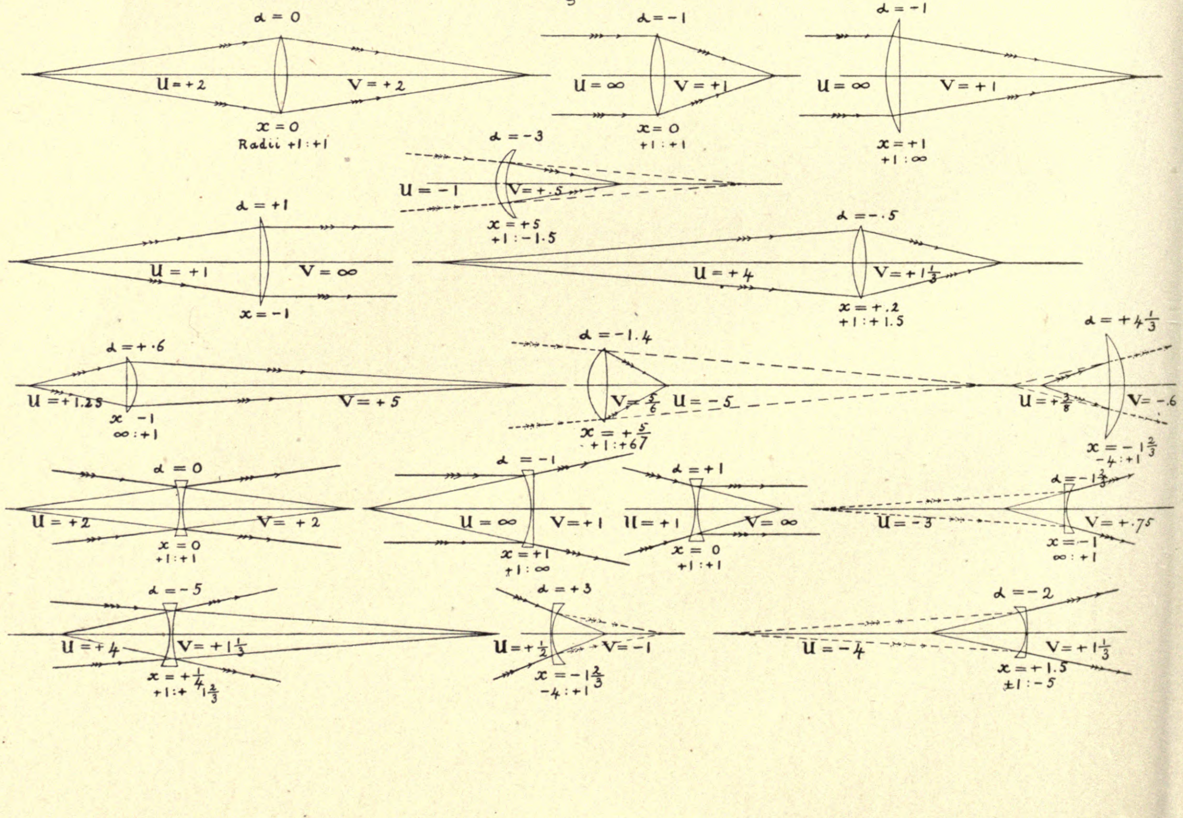


Fig. 33.

Figs. 34.



We may now insert approximate values of u_2 in the above coefficient of y^2, treating it as equal to u_1, so that in the corrections we may assume that

First approximate values to be inserted in formulæ.

$$\frac{\mu}{u_2} = \frac{\mu}{u_1} = \frac{\mu - 1}{r} - \frac{1}{u} \text{ (see Formula II., Section I.),}$$

and

$$\frac{1}{u_2} = \frac{\mu - 1}{\mu r} - \frac{1}{\mu u},$$

and therefore $\left(\frac{1}{r} - \frac{1}{u_2}\right)^2$ becomes $\frac{1}{\mu^2}\left(\frac{1}{r} + \frac{1}{u}\right)^2$, and the above equation becomes

$$\frac{\mu}{u_2} = \frac{\mu - 1}{r} - \frac{1}{u} + \left\{\frac{1}{u}\left(\frac{1}{r} + \frac{1}{u}\right)^2 + \left(\frac{\mu - 1}{r} - \frac{1}{u}\right)\left(\frac{1}{r} + \frac{1}{u}\right)^2 \frac{1}{\mu^2}\right\}\frac{y^2}{2},$$

which further reduces to the more convenient form

$$\frac{\mu}{u_2} = \frac{\mu - 1}{r} - \frac{1}{u} + \frac{\mu - 1}{2\mu^2}\left(\frac{1}{r} + \frac{1}{u}\right)^2\left(\frac{1}{r} + \frac{\mu + 1}{u}\right)y^2. \quad \text{XVIII. (R.)}$$

First refraction. Reciprocal of corrected second focal distance.

As before, we will number all important formulæ, such as the above, with Roman numerals, and all of minor importance, but useful as steps in the investigation, with ordinary numerals.

The function of y^2 in XVIII. is the correction to be applied to the reciprocal value $\frac{\mu}{u_1}$ or $\frac{\mu}{A..Q_1'}$, expressing the reciprocal of the length of the ultimate or paraxial rays, in order to convert it into $\frac{\mu}{A..Q_2'}$; and the distance $Q_1'..Q_2'$, or the longitudinal aberration *within the glass*, is therefore

$$-\frac{\mu - 1}{2\mu^2}\left(\frac{1}{r} + \frac{1}{u}\right)^2\left(\frac{1}{r} + \frac{\mu + 1}{u}\right)\frac{y^2}{\mu}u_1^2. \quad \text{XVIII. (L.)}$$

Linear value of above aberration.

It is desirable to call all corrections to the reciprocal values of distances R corrections, and all corrections to the linear values of such distances L corrections.

Formula XVIII. will be found to interpret itself in all cases if due regard is paid to the conventions which we laid down on page 10. If the entering rays are converging, and u therefore minus, there is obviously no aberration if either $r = -u$ or $= -\frac{u}{\mu + 1}$.

Let it now be supposed that the pencil of light is refracted a second time by a second spherical surface closely following the first, as shown in Fig. 32, wherein Q_2' is the point on the axis to which the

The second refraction.

ray R . . Q_2' is converging in Fig. 31. Supposing the collective lens which is formed by these two spherical surfaces to be very thin, and of a sharp edge at R or R', then we have R . . P in Fig. 31 = R . . P_2 in Fig. 32, or $y_1 = y_2$. Supposing in Fig. 32 that the ray Q_2 . . R is travelling right to left, originating from Q_2, and entering the convex surface R . . A_2, then putting $Q_2 \ldots A_2 = v$, and $A_2 \ldots Q_2' = v_1$, and radius $O_2 \ldots R = s$, we have by application of Formula XVIII. (R.)

$$\frac{\mu}{v_1} = \frac{\mu - 1}{s} - \frac{1}{v} + \frac{\mu - 1}{2\mu^2}\left(\frac{1}{s} + \frac{1}{v}\right)^2\left(\frac{1}{s} + \frac{\mu + 1}{v}\right)y^2;$$

therefore

Second refraction. Reciprocal of last focal distance.

$$\frac{1}{v} = \frac{\mu - 1}{s} - \frac{\mu}{v_1} + \frac{\mu - 1}{2\mu^2}\left(\frac{1}{s} + \frac{1}{v}\right)^2\left(\frac{1}{s} + \frac{\mu + 1}{v}\right)y^2. \qquad (2)$$

But v_1 in Fig. 32 is identical with u_2 in Fig. 31, if the axial thickness of the lens is zero, only we must remember that the ray R . . Q_2' is converging into the second surface; and while the distance A . . Q_2' or $A_2 \ldots Q_2'$ is positive relatively to the first surface, it is negative relatively to the second surface, by convention. So that in the last Formula (2) we have

$$\frac{\mu}{v_1} = -\frac{\mu}{u_2}$$

and

$$-\frac{\mu}{v_1} = +\frac{\mu}{u_2} \text{ of Formula XVIII. (R.).}$$

We may now insert the full expression for $\frac{\mu}{u_2}$ from Formula XVIII. (R.) in Formula (2), and thus obtain

$$\frac{1}{v} = \frac{\mu - 1}{s} + \left\{\frac{\mu - 1}{r} - \frac{1}{u} + \frac{\mu - 1}{2\mu^2}\left(\frac{1}{r} + \frac{1}{u}\right)^2\left(\frac{1}{r} + \frac{\mu + 1}{u}\right)y^2\right\}$$
$$+ \frac{\mu - 1}{2\mu^2}\left(\frac{1}{s} + \frac{1}{v}\right)^2\left(\frac{1}{s} + \frac{\mu + 1}{v}\right)y^2.$$

In the last function of $\frac{1}{s}$ and $\frac{1}{v}$ we must of course assume v to be its first approximate value as the focal distance conjugate to u. On adding together we then get

Spherical aberration of complete lens.

$$\left.\begin{aligned}\frac{1}{v} = (\mu - 1)\left(\frac{1}{r} + \frac{1}{s}\right) - \frac{1}{u} + \frac{\mu - 1}{2\mu^2}\Bigg\{&\left(\frac{1}{r} + \frac{1}{u}\right)^2\left(\frac{1}{r} + \frac{\mu - 1}{u}\right)\\ &+ \left(\frac{1}{s} + \frac{1}{v}\right)^2\left(\frac{1}{s} + \frac{\mu + 1}{v}\right)\Bigg\}y^2.\end{aligned}\right\} \text{XIX. (R.)}$$

Thus we arrive at the formula of the second approximation, which contains also the old formula of the first approximation, viz.—

$$\frac{1}{v} = (\mu - 1)\left(\frac{1}{r} + \frac{1}{s}\right) - \frac{1}{u} \text{ or } \frac{1}{v} = \frac{1}{F} - \frac{1}{u},$$

which states the relationship between the conjugate focal distances u and v, which we previously obtained as Formula III., but we have gone further than in that case and arrived at a formula for the deviation from the strict conjugate relationship, a correction which has to be applied to the value of v obtained from Formula III. This correction is the spherical aberration, and is seen to vary as y^2 or the square of the distance from the axis of the point in the lens where the particular ray dealt with traverses the lens.

If, in Fig. 32, f is the point where rays ultimately close to the axis come to focus after refraction at both surfaces, such that

$$\frac{1}{A..f} = \frac{1}{F} - \frac{1}{u} = \frac{1}{v},$$

then the distance $f - Q_2$ will be the longitudinal value of the spherical aberration, which will be expressed by the formula

Linear value of above spherical aberration.

$$\Delta v = -\frac{\mu - 1}{2\mu^2}\left\{\left(\frac{1}{r} + \frac{1}{u}\right)^2\left(\frac{1}{r} + \frac{\mu + 1}{u}\right) + \left(\frac{1}{s} + \frac{1}{v}\right)^2\left(\frac{1}{s} + \frac{\mu + 1}{v}\right)\right\}y^2v^2, \quad \text{XIX. (L.)}$$

provided that the longitudinal aberration is small compared with the distance v, not exceeding 10 per cent or so. Should the aberration from Formula XIX. (R.) exceed 10 per cent of $\frac{1}{v}$, then its longitudinal value is best obtained by the formula $v - \frac{1}{\frac{1}{v} + ay^2}$, wherein ay_2 is the aberration as given in Formula XIX. (R.).

Later on we will put the Formula XIX. into a much more convenient and general shape. It will be seen that owing to the essentially approximate nature of the statement of such quantities as versines of the curves, which necessarily form the foundation on which this formula is built, no very great accuracy can be expected from it when y becomes large compared with the radii of curvature of the lens in question, and it is strongly advisable to pursue the investigation further and arrive at some idea of the modifications to the formula rendered necessary, if we are to approach still more closely to accuracy. But as the working out of the formula of the third approximation is very long and much more difficult, the reader is quite at liberty to omit it during the first perusal of this book, especially as the formulæ of the second approximation will be found to form a complete system quite independently of the formulæ of the third approximation. He may then resume his perusal at page 64.

The Investigation pursued to the Third Approximation

The diagram, Fig. 33, represents a case in which R..P or y is considerably increased relatively to the radius of curvature O..R or r. About Q_1 as a centre and Q_1..R as radius, draw the arc R..b, b being its intersection with the axis, and about Q_2' as a centre and with Q_2'..R as radius draw the arc R..a, a being the intersection with the axis. Then A..a is the difference in length between Q_2'..R and Q_2'..A, and is the difference between the versines A..P and a..P. For the purpose of a more accurate third approximation it is not sufficiently exact to write

Versines according to second approximation.

$$\text{vers. A..P} = \frac{y^2}{2r}, \quad \text{vers. } a\text{..P} = \frac{y^2}{2(\text{A..Q}_2')}, \text{ and vers. P..}b = \frac{y^2}{2(\text{A..Q}_1)}.$$

It is evident that as a second step in accuracy, though not a final one, we may write

Versines according to third approximation.

$$\text{vers. A..P} = \frac{y^2}{2r - (\text{A..P})}, \quad \text{vers. } a\text{..P} = \frac{y^2}{2\{(\text{P} - \text{Q}_2') + (a\text{..P})\} - (a\text{..P})}, \text{ and}$$

$$\text{vers. P..}b = \frac{y^2}{2\{(\text{Q}_1\text{..A}) + (\text{A..P}) + (\text{P..}b)\} - (\text{P..}b)},$$

in which expressions we may enter approximate values of the terms in the denominators.

In the statement of vers. a..P, the distance P..Q_2' occurs, which differs from P..Q_1' by the longitudinal aberration Q_1'..Q_2', which is a function of the quantity x which we want to arrive at. In stating a value for the versine a..P we cannot afford to neglect this aberration Q_1'..Q_2' as a deduction from the radius of curvature of the arc R..a.

Let Q_1..A $= u$ and A..$Q_1' = u_{/}$ (the first approximate value for paraxial rays) and A..$Q_2' = u_2$ as before. Then let

$$\frac{1}{\text{A..Q}_2'} = \frac{1}{\text{A..Q}_1'} + x = \frac{1}{u_{/}} + x,$$

so that the longitudinal aberration Q_1'..$Q_2' = -xu_{/}^2$. As the basis of our inquiry we still have the strictly true relationship

$$\mu\frac{Q_2'\,..\,O}{Q_2'\,..\,R}=\frac{Q_1\,..\,O}{Q_1\,..\,R}.$$

The fundamental equation.

Then

$$(Q_2'\,..\,O)=(Q_2'\,..\,A)-(A\,..\,O)=(Q_1'\,..\,A)-(Q_1'\,..\,Q_2')-(A\,..\,O)\,;$$

therefore

$$Q_2'\,..\,O=u_{/}-xu_{/}^2-r\,; \tag{3}$$

Formula for $Q_2'\,..\,O$.

also we have

$$Q_1\,..\,O=u+r. \tag{4}$$

Formula for $Q_1\,..\,O$.

Then

$$Q_2'\,..\,R=(A\,..\,Q_2')-(A\,..\,P)+(a\,..\,P)$$

$$=(u_{/}-xu_{/}^2)-\frac{y^2}{2r-\dfrac{y^2}{2r}}+\frac{y^2}{\begin{array}{c}2\{(P\,..\,Q_2')+(a\,..\,P)\}-(a\,..\,P)\\ \text{or } 2(P\,..\,Q_2')+(a\,..\,P)\end{array}}$$

$$=(u_{/}-xu_{/}^2)-\frac{y^2}{2r-\dfrac{y^2}{2r}}+\frac{y^2}{2\left((A..Q_2')-\dfrac{y^2}{2r}\right)+\dfrac{y^2}{2(A\,..\,Q_2')}}$$

$$=(u_{/}-xu_{/}^2)-\frac{y^2}{2r-\dfrac{y^2}{2r}}+\frac{y^2}{2\left(u_{/}-xu_{/}^2-\dfrac{y^2}{2r}\right)+\dfrac{y^2}{2(u_{/}-xu_{/}^2)}}$$

$$=(u_{/}-xu_{/}^2)-\frac{y^2}{2}\left(\frac{1}{r-\dfrac{y^2}{4r}}\right)+\frac{y^2}{2}\left(\frac{1}{u_{/}-xu_{/}^2-\dfrac{y^2}{2r}+\dfrac{y^2}{4}\left(\dfrac{1}{u_{/}}+x\right)}\right)$$

$$=u_{/}-xu_{/}^2-\frac{y^2}{2}\left(\frac{1}{r}+\frac{y^2}{4r^3}\right)+\frac{y^2}{2}\left(\frac{1}{u_{/}}+x+\frac{y^2}{2ru_{/}^2}-\frac{y^2}{4u_{/}^3}-\frac{y^2}{4u_{/}^2}x\right);$$

therefore

$$Q_2'\,..\,R=u_{/}-xu_{/}^2+\frac{y^2}{2}x-\frac{y^4}{8u_{/}^2}x-\overbrace{\frac{y^2}{2}\left(\frac{1}{r}-\frac{1}{u_{/}}\right)}^{a}-\frac{y^4}{4}\left(\frac{1}{2r^3}-\frac{1}{ru_{/}^2}+\frac{1}{2u_{/}^3}\right).$$

We now want the reciprocal value of $Q_2'\,..\,R$, and as we wish to preserve all functions of y^4, the term (a) must be developed to two terms in the sense that $\dfrac{1}{u-a}=\dfrac{1}{u}+\dfrac{a}{u^2}+\dfrac{a^2}{u^3}$.

Therefore we get

$$\frac{1}{Q_2'\,..\,R}=\frac{1}{u_{/}}+x-\frac{y^2}{2u_{/}^2}x+\frac{y^4}{8u_{/}^4}x+\left\{\frac{y^2}{2u_{/}^2}\left(\frac{1}{r}-\frac{1}{u_{/}}\right)+\frac{y^4}{4u_{/}^3}\left(\frac{1}{r}-\frac{1}{u_{/}}\right)^2\right\}$$
$$+\frac{y^4}{4u_{/}^2}\left(\frac{1}{2r^3}-\frac{1}{ru_{/}^2}+\frac{1}{2u_{/}^3}\right),$$

and

Formula for $\frac{1}{Q_2'..R}$.

$$\left.\begin{aligned}\frac{1}{Q_2'..R} &= \frac{1}{u_{\prime}} + x - \left(\frac{y^2}{2u_{\prime}^2} - \frac{y^4}{8u_{\prime}^4}\right)x + \frac{y^2}{2u_{\prime}^2}\left(\frac{1}{r} - \frac{1}{u_{\prime}}\right)\\ &\qquad + \frac{y^4}{4u_{\prime}^2}\left(\frac{1}{2r^3} + \frac{1}{r^2u_{\prime}} - \frac{3}{ru_{\prime}^2} + \frac{3}{2u_{\prime}^3}\right).\end{aligned}\right\} \quad (5)$$

Next we have

$$Q_1..R = (Q_1..P) + (P..b) = (Q_1..A) + (A..P) + (P..b)$$

$$= u + \frac{y^2}{2r - \frac{y^2}{2r}} + \frac{y^2}{2\left(u + \frac{y^2}{2r} + \frac{y^2}{2u}\right) - \frac{y^2}{2u}}$$

$$= u + \frac{y^2}{2}\left(\frac{1}{r - \frac{y^2}{4r}}\right) + \frac{y^2}{2}\left(\frac{1}{u + \frac{y^2}{2r} + \frac{y^2}{4u}}\right)$$

$$= u + \frac{y^2}{2}\left(\frac{1}{r} + \frac{y^2}{4r^3}\right) + \frac{y^2}{2}\left(\frac{1}{u} - \frac{y^2}{2ru^2} - \frac{y^2}{4u^3}\right);$$

therefore

$$Q_1..R = u + \frac{y^2}{2}\left(\frac{1}{r} + \frac{1}{u}\right) + \frac{y^4}{4}\left(\frac{1}{2r^3} - \frac{1}{ru^2} - \frac{1}{2u^3}\right).$$

Here again we want the reciprocal value of $Q_1..R$, which, analogously to our procedure in arriving at Formulæ (5), may be stated thus—

$$\frac{1}{Q_1..R} = \frac{1}{u} - \left\{\frac{y^2}{2u^2}\left(\frac{1}{r} + \frac{1}{u}\right) - \frac{y^4}{4u^3}\left(\frac{1}{r} + \frac{1}{u}\right)^2\right\} - \frac{y^4}{4u^2}\left(\frac{1}{2r^3} - \frac{1}{ru^2} - \frac{1}{2u^3}\right)$$

$$= \frac{1}{u} - \frac{y^2}{2u^2}\left(\frac{1}{r} + \frac{1}{u}\right) + \frac{y^4}{4u^2}\left(\frac{1}{r^2} + \frac{2}{ru} + \frac{1}{u^2}\right)\frac{1}{u} - \frac{y^4}{4u^2}\left(\frac{1}{2r^3} - \frac{1}{ru^2} - \frac{1}{2u^3}\right),$$

and

Formula for $\frac{1}{Q_1..R}$.

$$\frac{1}{Q_1..R} = \frac{1}{u} - \frac{y^2}{2u^2}\left(\frac{1}{r} + \frac{1}{u}\right) - \frac{y^4}{4u^2}\left(\frac{1}{2r^3} - \frac{1}{r^2u} - \frac{3}{ru^2} - \frac{3}{2u^3}\right). \quad (6)$$

The insertion of Formulæ (3), (4), (5), and (6) in the fundamental equation.

On substituting Formulæ (3), (4), (5), and (6) in our basis equation

$$\mu(Q_2'..O)\frac{1}{Q_2'..R} = (Q_1..O)\frac{1}{Q_1..R},$$

we then get

$$\begin{aligned}\mu(u_{\prime} - xu_{\prime}^2 - r)\Big[\frac{1}{u_{\prime}} + x - \left(\frac{y^2}{2u_{\prime}^2} - \frac{y^4}{8u_{\prime}^4}\right)x + \frac{y^2}{2u_{\prime}^2}\left(\frac{1}{r} - \frac{1}{u_{\prime}}\right)\\ + \frac{y^4}{4u_{\prime}^2}\left(\frac{1}{2r^3} + \frac{1}{r^2u_{\prime}} - \frac{3}{ru_{\prime}^2} + \frac{3}{2u_{\prime}^3}\right)\Big]\end{aligned}$$

$$= (u + r)\left[\frac{1}{u} - \frac{y^2}{2u^2}\left(\frac{1}{r} + \frac{1}{u}\right) - \frac{y^4}{4u^2}\left(\frac{1}{2r^3} - \frac{1}{r^2u} - \frac{3}{ru^2} - \frac{3}{2u^3}\right)\right].$$

In expanding this equation we may legitimately omit functions of x^2 or x^3, as x is a relatively small quantity, and also omit functions of xy^4. Then we get, after cancelling out a few terms,

$$\left\{\mu - \mu\frac{y^2}{2u_{\prime}}x + \mu\frac{y^2}{2u_{\prime}}\left(\frac{1}{r}-\frac{1}{u_{\prime}}\right) + \mu\frac{y^4}{4u_{\prime}}\left(\frac{1}{2r^3}+\frac{1}{r^2u_{\prime}}-\frac{3}{ru_{\prime}^2}+\frac{3}{2u_{\prime}^3}\right) - \mu\frac{r}{u_{\prime}} - \mu rx\right.$$

$$\left. + \mu r\frac{y^2}{2u_{\prime}^2}x - \mu r\frac{y^2}{2u_{\prime}^2}\left(\frac{1}{r}-\frac{1}{u_{\prime}}\right) - \mu r\frac{y^4}{4u_{\prime}^2}\left(\frac{1}{2r^3}+\frac{1}{r^2u_{\prime}}-\frac{3}{ru_{\prime}^2}+\frac{3}{2u_{\prime}^3}\right) - \mu\frac{y^2}{2}\left(\frac{1}{r}-\frac{1}{u_{\prime}}\right)x\right\}$$

for the first side of the equation, which then becomes

$$\left\{\mu - \mu rx + \mu r\frac{y^2}{2u_{\prime}^2}x - \mu\frac{y^2}{2r}x + \mu\frac{y^2}{2u_{\prime}}\left(\frac{1}{r}-\frac{1}{u_{\prime}}\right) - \mu\frac{r}{u_{\prime}} - \mu r\frac{y^2}{2u_{\prime}^2}\left(\frac{1}{r}-\frac{1}{u_{\prime}}\right)\right.$$

$$\left. + \mu\frac{y^4}{4u_{\prime}}\left(\frac{1}{2r^3}+\frac{1}{r^2u_{\prime}}-\frac{3}{ru_{\prime}^2}+\frac{3}{2u_{\prime}^3}\right) - \mu r\frac{y^4}{4u_{\prime}^2}\left(\frac{1}{2r^3}+\frac{1}{r^2u_{\prime}}-\frac{3}{ru_{\prime}^2}+\frac{3}{2u_{\prime}^3}\right)\right\},$$

which

$$=\left\{-\mu rx + \mu r\frac{y^2}{2u_{\prime}^2}x - \mu\frac{y^2}{2r}x + \mu - \mu\frac{r}{u_{\prime}} + \left(\mu\frac{y^2}{2u_{\prime}} - \mu r\frac{y^2}{2u_{\prime}^2}\right)\left(\frac{1}{r}-\frac{1}{u_{\prime}}\right)\right.$$

$$\left. + \left(\mu\frac{y^4}{4u_{\prime}} - \mu r\frac{y^4}{4u_{\prime}^2}\right)\left(\frac{1}{2r^3}+\frac{1}{r^2u_{\prime}}-\frac{3}{ru_{\prime}^2}+\frac{3}{2u_{\prime}^3}\right)\right\}.$$

So that the whole equation now takes the form

$$\left\{-\mu rx + \frac{y^2}{2}\left(\frac{\mu r}{u_{\prime}^2}-\frac{\mu}{r}\right)x + \mu - \mu\frac{r}{u_{\prime}} + \frac{y^2}{2u_{\prime}}\left(\mu-\mu\frac{r}{u_{\prime}}\right)\left(\frac{1}{r}-\frac{1}{u_{\prime}}\right)\right.$$

$$\left. + \frac{y^4}{4u_{\prime}}\left(\mu-\mu\frac{r}{u_{\prime}}\right)\left(\frac{1}{2r^3}+\frac{1}{r^2u_{\prime}}-\frac{3}{ru_{\prime}^2}+\frac{3}{2u_{\prime}^3}\right)\right\}$$

$$=(u+r)\left[\frac{1}{u}-\frac{y^2}{2u^2}\left(\frac{1}{r}+\frac{1}{u}\right)-\frac{y^4}{4u^2}\left(\frac{1}{2r^3}-\frac{1}{r^2u}-\frac{3}{ru^2}-\frac{3}{2u^3}\right)\right].$$

By dividing both sides by μr, and keeping functions of x on the left-hand side we then get

Both sides divided by μr.

$$-x+\frac{y^2}{2}\left(\frac{1}{u_{\prime}^2}-\frac{1}{r^2}\right)x$$

$$=-\frac{1}{r}+\frac{1}{u_{\prime}}-\frac{y^2}{2u_{\prime}}\left(\frac{1}{r}-\frac{1}{u_{\prime}}\right)^2-\frac{y^4}{4u_{\prime}}\left(\frac{1}{r}-\frac{1}{u_{\prime}}\right)\left(\frac{1}{2r^3}+\frac{1}{r^2u_{\prime}}-\frac{3}{ru_{\prime}^2}+\frac{3}{2u_{\prime}^3}\right)$$

$$+\frac{u+r}{\mu r}\left\{\frac{1}{u}-\frac{y^2}{2u^2}\left(\frac{1}{r}+\frac{1}{u}\right)-\frac{y^4}{4u^2}\left(\frac{1}{2r^3}-\frac{1}{r^2u}-\frac{3}{ru^2}-\frac{3}{2u^3}\right)\right\},$$

from which

$$x\left\{1-\frac{y^2}{2}\left(\frac{1}{u_/^2}-\frac{1}{r^2}\right)\right\}$$

$$=\frac{1}{r}-\frac{1}{u_/}+\frac{y^2}{2u_/}\left(\frac{1}{r}-\frac{1}{u_/}\right)^2+\frac{y^4}{4u_/}\left(\frac{1}{r}-\frac{1}{u_/}\right)\left(\frac{1}{2r^3}+\frac{1}{r^2u_/}-\frac{3}{ru_/^2}+\frac{3}{2u_/^3}\right)$$

$$-\frac{u+r}{\mu ru}+\frac{u+r}{\mu ru}\left\{\left(\frac{1}{r}+\frac{1}{u}\right)\frac{y^2}{2u}+\frac{y^4}{4u}\left(\frac{1}{2r^3}-\frac{1}{r^2u}-\frac{3}{ru^2}-\frac{3}{2u^3}\right)\right\}.$$

$u_/$ to be expressed in terms of u and r.

It is now desirable to express $u_/$ or A . . Q_1' in terms of u and r, for

$$\frac{1}{u_/}=\frac{1}{\mu}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)=\frac{(\mu-1)u-r}{\mu ru},$$

also

$$\frac{1}{r}-\frac{1}{u_/}=\frac{\mu u-(\mu-1)u+r}{\mu ru}=\frac{u+r}{\mu ru}.$$

After substituting these values in the equation and cancelling we then get

$$x\left\{1-\frac{y^2}{2}\left(\frac{u^2(1-2\mu)-2ur(\mu-1)+r^2}{\mu^2r^2u^2}\right)\right\}$$

$$=\frac{y^2}{2}\left(\frac{(\mu-1)u-r}{\mu ru}\right)\left(\frac{u+r}{\mu ru}\right)^2+\frac{y^4}{4}\left(\frac{(\mu-1)u-r}{\mu ru}\right)\left(\frac{u+r}{\mu ru}\right)\left\{\frac{1}{2r^3}+\frac{(\mu-1)u-r}{\mu r^3u}\right.$$

$$\left.-\frac{3}{r}\left(\frac{(\mu-1)u-r}{\mu ru}\right)^2+\frac{3}{2}\left(\frac{(\mu-1)u-r}{\mu ru}\right)^3\right\}+\left(\frac{u+r}{\mu ru}\right)\left\{\left(\frac{1}{r}+\frac{1}{u}\right)\frac{y^2}{2u}\right.$$

$$\left.+\frac{y^4}{4u}\left(\frac{1}{2r^3}-\frac{1}{r^2u}-\frac{3}{ru^2}-\frac{3}{2u^3}\right)\right\};$$

from which

$$x\left\{1-\frac{y^2}{2}\left(\frac{u^2(1-2\mu)-2ur(\mu-1)+r^2}{\mu^2r^2u^2}\right)\right\}$$

$$=\frac{y^2}{2\mu^3}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)\left(\frac{1}{r}+\frac{1}{u}\right)^2+\frac{y^2}{2\mu u}\left(\frac{1}{r}+\frac{1}{u}\right)^2$$

$$+\frac{y^4}{4\mu^2}\left(\frac{\mu-1}{r}-\frac{1}{u}\right)\left(\frac{1}{r}+\frac{1}{u}\right)\left\{\frac{1}{2r^3}+\frac{\mu-1}{\mu r^3}-\frac{1}{\mu r^2u}-\frac{3}{r}\left(\frac{(\mu-1)^2u^2-2ur(\mu-1)+r^2}{\mu^2r^2u^2}\right)\right.$$

$$\left.+\frac{3}{2}\left(\frac{(\mu-1)^3u^3-3(\mu-1)^2u^2r+3(\mu-1)ur^2-r^3}{\mu^3r^3u^3}\right)\right\}$$

$$+\frac{y^4}{4\mu u}\left(\frac{1}{u}+\frac{1}{r}\right)\left(\frac{1}{2r^3}-\frac{1}{r^2u}-\frac{3}{ru^2}-\frac{3}{2u^3}\right);$$

from which we derive

$$x\left\{1-\frac{y^2}{2}\left(\frac{u^2(1-2\mu)-2ur(\mu-1)+r^2}{\mu^2r^2u^2}\right)\right\}=\frac{\mu-1}{2\mu^3}\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{u}\right)y^2$$

$$+\frac{y^4}{4\mu}\left(\frac{1}{r}+\frac{1}{u}\right)\overbrace{\left\{\frac{\begin{array}{c}3r^4-3\mu^4r^4+12r^3u-6\mu^4r^3u-6\mu r^3u+18r^2u^2\\ -2\mu^4r^2u^2+2\mu^2r^2u^2-18\mu r^2u^2+12ru^3+\mu^3ru^3\\ +\mu^4ru^3+4\mu^2ru^3-18\mu ru^3+3u^4-6\mu u^4+\mu^3u^4+2\mu^2u^4\end{array}}{2\mu^4r^4u^4}\right\}}^{(a)}.$$

On multiplying both sides of the equation by

$$\left\{1+\frac{y^2}{2}\left(\frac{u^2(1-2\mu)-2ur(\mu-1)+r^2}{\mu^2r^2u^2}\right)\right\}$$

(the function of y^2 being supposed to amount to less than $\frac{1}{10}$th), we then get, if we neglect functions of y^6,

Elimination of function of x.

$$x=\frac{\mu-1}{2\mu^3}\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{u}\right)y^2$$

$$+y^4\frac{\mu-1}{4\mu^3}\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{u}\right)\left(\frac{u^2(1-2\mu)-2ur(\mu-1)+r^2}{\mu^2r^2u^2}\right) \qquad (b)$$

$$+\frac{y^4}{4\mu}\left(\frac{1}{r}+\frac{1}{u}\right)\overbrace{\left\{\frac{3r^4,\text{ etc., etc.}}{2\mu^4r^4u^4}\right\}}^{(a)}. \qquad (c)$$

We may now add together all the functions of y^4 contained in (b) and (c), (b) being expressed in the form

Functions of y^4 sorted out.

$$\frac{y^4}{4\mu}\left(\frac{1}{r}+\frac{1}{u}\right)\left\{\left(\frac{1}{r}+\frac{1}{u}\right)\left(\frac{1}{r}+\frac{\mu+1}{u}\right)\left(\mu-1\right)\left(\frac{u^2(1-2\mu)-2ur(\mu-1)+r^2}{\mu^4r^2u^2}\right)\right\}.$$

After multiplying out the factors contained in the large brackets and adding them to the terms in (a), we get, after much reducing and cancelling out,

$$x=\frac{\mu-1}{2\mu^3}\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{u}\right)y^2=\left(\begin{array}{c}\text{aberration by second approximation}\\ \text{as per Formula XVIII. (R.)}\div\mu\end{array}\right)$$

$$\left.\begin{array}{l}+\frac{y^4}{8\mu}\left(\frac{1}{r}+\frac{1}{u}\right)\left(\frac{1}{\mu^4r^4}+\frac{1}{\mu^4u^4}+\frac{4}{\mu^4r^3u}+\frac{6}{\mu^4r^2u^2}+\frac{4}{\mu^4ru^3}\right.\\ \quad+\frac{4}{\mu^2r^2u^2}-\frac{2}{\mu^2r^4}-\frac{2}{\mu^2r^3u}+\frac{6}{\mu^2ru^3}+\frac{2}{\mu^2u^4}-\frac{3}{\mu r^3u}-\frac{8}{\mu r^2u^2}\\ \quad\left.-\frac{4}{\mu ru^3}+\frac{1}{\mu r^4}+\frac{1}{r^3u}-\frac{2}{r^2u^2}-\frac{6}{ru^3}-\frac{3}{u^4}\right).\end{array}\right\}\text{XX. (R.)}$$

First refraction. Formula for the spherical aberration of the third approximation.

Thus we again arrive at Formula XVIII. (R.) of the second approximation (but divided by μ), while in Formula XX. (R.) we have the corrective formula of the third approximation. This generally is a correction of very much smaller value than the correction of the second approximation. If we were to pursue the investigation still further, that is, were we to develop the fundamental equation given on page 59 to higher and higher degrees of accuracy, then we should obtain a series of formulæ for the spherical aberration, first to the second and third approximations, being the above functions of y^2 and y^4, and the following approximations, being functions of y^6, y^8, etc., or rising even powers of y, and also increasing in complexity.

Aberrations of ascending orders theoretically interminable.

Formulæ of the fourth approximation generally undesirable.

The Formula XX. is not too complex, especially after it has been transformed into a more convenient and general form, to be sometimes useful in the higher problems which have frequently to be dealt with; but approximations of still higher orders are for practical purposes undesirable.

We have now got in Formulæ XVIII. (R.) $\div \mu$ and XX. (R.) taken together a fairly exact corrective x to the reciprocal value of the distance A . . Q_1' or $u_{/}$, such that $\frac{1}{u_{/}} + x = \frac{1}{\text{A} \,..\, \text{Q}_1'}$, while $\frac{1}{u_{/}} = \frac{\mu - 1}{\mu r} - \frac{1}{\mu u}$. It must be borne in mind that we are dealing with the distance $u_{/}$ *as measured within the substance of the glass or other refracting medium.* It is easily seen, therefore, that if the pencil of rays we are dealing with is refracted into air again at a second surface closely following the first, then, quite apart from any further spherical aberration imparted at the second surface, the spherical aberration imparted at the first surface may be looked upon as an angular deviation from the true direction, which will be multiplied by the refractive index on being refracted through the second surface. An aberration correction of value a inside of the glass becomes μa on being refracted out of the glass. Therefore our value of x, the aberration correction, must be multiplied by μ to bring it outside the lens, when we may add the formula to the analogous formula appertaining to the refraction at the second surface, just as we did before when we took Formula XVIII. of the second approximation for the first surface, and then added to it the corresponding formula for the second surface, thus obtaining the Formula XIX. for the complete lens. Adapting that method to our present case, our formula for the spherical aberration to the third approximation for the whole lens is expressed thus—

Distinction between air value and glass value of the aberration.

Lens. Formula of the second approximation again emerges.

$$X = \frac{\mu - 1}{2\mu^2} \left\{ \left(\frac{1}{r} + \frac{1}{u}\right)^2 \left(\frac{1}{r} + \frac{\mu + 1}{u}\right) + \left(\frac{1}{s} + \frac{1}{v}\right)^2 \left(\frac{1}{s} + \frac{\mu + 1}{v}\right) \right\} y^2 \qquad \text{XXI. (R.)}$$

$$\left.\begin{aligned}
&+\frac{y^4}{8}\left\{\left(\frac{1}{r}+\frac{1}{u}\right)\left(\frac{1}{\mu^4r^4}+\frac{1}{\mu^4u^4}+\frac{4}{\mu^4r^3u}+\frac{6}{\mu^4r^2u^2}+\frac{4}{\mu^4ru^3}+\frac{4}{\mu^2r^2u^2}-\frac{2}{\mu^2r^4}-\frac{2}{\mu^2r^3u}\right.\right.\\
&\left.+\frac{6}{\mu^2ru^3}+\frac{2}{\mu^2u^4}-\frac{3}{\mu r^3u}-\frac{8}{\mu r^2u^2}-\frac{4}{\mu ru^3}-\frac{1}{\mu r^4}+\frac{1}{r^3u}-\frac{2}{r^2u^2}-\frac{6}{ru^3}-\frac{3}{u^4}\right)\\
&+\left(\frac{1}{s}+\frac{1}{v}\right)\left(\frac{1}{\mu^4s^4}+\frac{1}{\mu^4v^4}+\frac{4}{\mu^4s^3v}+\frac{6}{\mu^4s^2v^2}+\frac{4}{\mu^4sv^3}+\frac{4}{\mu^2s^2v^2}-\frac{2}{\mu^2s^4}-\frac{2}{\mu^2s^3v}\right.\\
&\left.\left.+\frac{6}{\mu^2sv^3}+\frac{2}{\mu^2v^4}-\frac{3}{\mu s^3v}-\frac{8}{\mu s^2v^2}-\frac{4}{\mu sv^3}-\frac{1}{\mu s^4}+\frac{1}{s^3v}-\frac{2}{s^2v^2}-\frac{6}{sv^3}-\frac{3}{v^4}\right)\right\}.
\end{aligned}\right\}\ \text{XXII. (R.)}$$

Formula of the third approximation complete.

These two corrections are to be added to the value of $\frac{1}{v}$, when $\frac{1}{v}=\frac{1}{\text{F}}-\frac{1}{u}$ simply. So that if in any given case we work out the value of $\frac{1}{v}+\text{X}$, then we may take its reciprocal for the longitudinal value of the corrected conjugate focal distance of the two rays which are refracted through the lens at the height y from the optic axis. Or if the aberration is small relatively to $\frac{1}{v}$, then we may take the linear or longitudinal value of the aberration as $-v^2\text{X}$, so that, since for a collective lens X is nearly always positive, the longitudinal aberration is a deduction from v when v is positive, and an increase to v when v is negative or the emergent rays diverging.

Present formulæ clumsy and inconvenient in form.

It is clear that the formulæ we have now arrived at for the spherical aberration of a thin lens do not easily lend themselves to analytical problems, such as finding the form of a lens requisite to give or to counteract a certain known amount of spherical aberration, and the next desirable step is to put the formulæ into a shape that is better adapted to manipulation, as well as more elegant and simple.

Introduction of a more Scientific Notation

Coddington's device explained.

Here we cannot conceivably do better than adopt the beautiful device apparently invented by Coddington and explained on page 110 of his work before referred to. He shows how the reciprocal values of the radii r and s, and of the conjugate focal distances u and v, may be expressed by the use of two terms x and a. It may shortly be explained thus. Since $\frac{1}{v}+\frac{1}{u}$ for the ultimate axial pencils $=\frac{1}{\text{F}}$ and

$$\frac{1}{r}+\frac{1}{s}=\frac{1}{(\mu-1)}\frac{1}{\text{F}}=\frac{1}{\rho}.$$

Let

$$\frac{1}{u} = \frac{1+a}{2F} \tag{7}$$

and let

$$\frac{1}{v} = \frac{1-a}{2F}, \tag{8}$$

so that

$$\frac{1+a}{2F} + \frac{1-a}{2F} = \frac{1}{F};$$

let

$$\frac{1}{r} = \frac{1+x}{2(\mu-1)F} = \frac{1+x}{2\rho}, \tag{9}$$

and let

$$\frac{1}{s} = \frac{1-x}{2(\mu-1)F} = \frac{1+x}{2\rho}, \tag{10}$$

so that

$$\frac{1+x}{2(\mu-1)F} + \frac{1-x}{2(\mu-1)F} = \frac{1}{(\mu-1)F} = \frac{1}{\rho} = \frac{1}{r} + \frac{1}{s}.$$

a. **The characteristic of the conditions of vergency.**

Therefore a becomes the characteristic of the state of convergence or divergence of the rays constituting the axial pencil traversing the lens, in relation to the power of the lens. For instance, if the rays of the entering axial pencil are parallel, or $\frac{1}{u} = 0$, then $a = -1$; while if the conjugate foci are equal, or $\frac{1}{u} = \frac{1}{v}$, then $a = 0$; while if the rays of the emergent pencil are parallel, and $\frac{1}{u} = \frac{1}{F}$ and $\frac{1}{v} = 0$, then $a = +1$. In short, we may style the term a the characteristic of the vergency of the pencil traversing the lens.

x. **The characteristic of the shape of a lens.**

Also x becomes the characteristic of the *shape* of the lens. If the lens is equiconvex and $\frac{1}{r} = \frac{1}{s}$, then $x = 0$; if convexo-plane, then $x = +1$; if plano-convex, $x = -1$. If meniscus, such that $r = 1$ and $s = -3$, then x is $+2$; and if the same meniscus is reversed, then x is -2. Fig. 34, Plate VII., gives numerous self-explanatory illustrations of the application of the two terms x and a to different cases. This device of numerical characteristics represented by a and x is invaluable in practical analytical calculations.

After substituting the above expressions for $\frac{1}{r}$, $\frac{1}{s}$, $\frac{1}{u}$, and $\frac{1}{v}$ in the Formulæ XXI. and XXII., and arranging the terms in descending powers of x and ascending powers of a, we get

$$\frac{y^2}{8f^3} \cdot \frac{1}{\mu(\mu-1)}\left\{\frac{\mu+2}{\mu-1}x^2 + 4(\mu+1)ax + (3\mu+2)(\mu-1)a^2 + \frac{\mu^3}{\mu-1}\right\} \text{XXIII. (R.)}$$

Lens. Coddington's formula for spherical aberration.

$$\left.\begin{aligned} + \frac{y^4}{128\mu^3(\mu-1)^4 f^5}\Big\{ &(\mu^3 - 2\mu^2 + \mu - 5)x^4 \\ &+ 2(\mu-1)(-\mu^3 - 9\mu^2 - 10)ax^3 \\ &+ 3(\mu-1)^2(-8\mu^3 - 13\mu^2 - 2\mu - 10)a^2x^2 \\ &+ 2(\mu-1)^3(-18\mu^3 - 16\mu^2 - 4\mu - 10)a^3x \\ &+ (\mu-1)^4(-15\mu^3 - 9\mu^2 - 3\mu - 5)a^4 \\ &+ (-8\mu^5 + \mu^4 + 2\mu^3 - 5\mu^2)x^2 \\ &+ 2(\mu-1)(-18\mu^5 + 4\mu^4 - \mu^3 - 5\mu^2)ax \\ &+ (\mu-1)^2(-30\mu^5 + 6\mu^4 - 4\mu^3 - 5\mu^2)a^2 \\ &+ (-3\mu^7 + 3\mu^6 - \mu^5)\Big\}. \end{aligned}\right\} \text{XXIV. (R.)}$$

Formula for the spherical aberration by third approximation, in terms of a and x.

In Formula XXIII. we again have in a more convenient form Coddington's Formula XIX. for the spherical aberration to the second degree of approximation, while XXIV. is a further correction to it worked out on the same lines to the third degree of approximation; both of them being corrections to $\frac{1}{v}$, the latter being ascertained by the simple law of conjugate focal lengths, $\frac{1}{v} = \frac{1}{F} - \frac{1}{u}$. The first is a function of $\frac{y^2}{f^3}$, the second is a function of $\frac{y^4}{f^5}$. We shall find, on further investigation of cases of axial pencils traversing combinations of lenses, and especially separated lenses, that many other corrections arise which are also functions of y^4, and which it will be desirable to work out, where possible, and add to the same category of corrections as XXIV.

Other corrections of the order y^4.

It is easily seen that these formulæ will interpret themselves correctly in all conceivable cases.

It will be as well to call the Formula XXIV. the *Intrinsic* Aberration Function of the order y^4. For we shall find that although certain other aberration functions of the same order y^4 will have to be considered, yet they will turn out to be functions of Formula XXIII.—that is, they will be products of the latter formula into another function of y^2, and are therefore functions of y^4 in that sense only.

It will be found that corrections involving higher powers of y than y^4 involve degrees of cumbrousness and complexity which are out of all proportion to their importance or utility.

If the reader will apply the reasoning of this Section to the corresponding case of a dispersive lens, in which preferably u, $u_{\prime}$, and v, as well as r and s, are all positive for convenience in reasoning, he will arrive at precisely the same formulæ. That is, it is best to assume the rays to be converging into the first concave surface of the dispersive lens, to be diverging after first refraction, and more strongly diverging after the refraction at the second surface, which is also concave.

Conditions under which the aberration may be 0 or negative.

It is clear that there is only one term in Coddington's formula (which may be conveniently referred to as $\frac{y^2}{8f^3}A'$, while the Formula XXIV. may be termed $\frac{y^4}{128f^5}A''$) which can ever be negative, and that is the second term, involving ax, so that the only possible way of approaching to freedom from aberration in a simple lens is to make a and x of opposite signs; therefore if rays are strongly diverging into a positive lens and a is positive, then x must be negative, and *vice versâ*. For instance, if $\mu = 1{\cdot}5$, then we have

$$\frac{\mu+2}{\mu-1}x^2 + 4(\mu+1)ax + (3\mu+2)(\mu-1)a^2 + \frac{\mu^3}{\mu-1} = 7x^2 + 10ax + 3{\cdot}25a^2 + 6{\cdot}75,$$

and this will equate to 0 if $a =$ at least $+4{\cdot}45$, when x will be about $-3{\cdot}15$, implying a strong meniscus form with its hollow side turned to receive the divergent rays. With a still higher plus value for a, a value for x may be found to give a certain amount of negative aberration. This fact is utilised in many systems of condenser lenses whereof the member nearest the source of light is made of a pronounced meniscus type.

On differentiating the Formula XXIII. with respect to x we have

Formula XXIII. differentiated with respect to x.

$$\Delta_x\left(\frac{y^2}{8f^3}A'\right) = \frac{y^2}{8f^3}\left\{\frac{2(\mu+2)x + 4(\mu-1)(\mu+1)a}{\mu(\mu-1)^2}\right\}\Delta x, \qquad (11)$$

which will equate to 0 when

Condition of minimum aberration.

$$x = -\frac{2(\mu-1)(\mu+1)}{(\mu+2)}a; \qquad (12)$$

so that if $\mu = 1{\cdot}5$, then x, for minimum possible spherical aberration, must be $-\frac{5}{7}a$.

If the entering rays are parallel and $a = -1$, then $x = +\frac{5}{7}$, so that the radii of curvature will be as $2:12$ or $1:6$; while if μ is about $1{\cdot}67$, then $x = +1$, or the lens of minimum aberration is convexo-plane. It will be meniscus if the refractive index is still higher.

If we suppose $\mu = 1{\cdot}5$, then the Formulæ XXIII. and XXIV. work out to

Values of the two orders of aberration when $\mu = 1{\cdot}5$.

$$+\frac{y^2}{6f^3}\left\{7x^2 + 10ax + 3{\cdot}25a^2 + 6{\cdot}75\right\}$$

$$+\frac{y^4}{27f^5}\left\{\begin{array}{l} -4{\cdot}625x^4 - 33{\cdot}625ax^3 - 60{\cdot}1875x^2 - 51{\cdot}94a^2x^2 - 55{\cdot}55a^2 \\ \qquad\qquad - 28{\cdot}19a^3x - 131{\cdot}06ax - 5a^4 - 24{\cdot}7 \end{array}\right\}.$$

From this it appears that the corrections of the order y^4 must be always of a negative character when a and x are of the same sign, as when parallel rays fall upon a plano-convex lens, *i.e.* when $a = -1$ and $x = -1$; but it will be found that if parallel rays fall upon a convexo-plane lens, in which case $a = -1$ and $x = +1$, then the functions of ax^3, a^3x, and ax come out positive and nearly neutralise the negative terms.

For instance, if $f = 1$, $y = {\cdot}25$, and $\mu = 1{\cdot}5$, $a = -1$, $x = +1$, then $\frac{y^2}{f^3}A'$ gives

Convexo-plane lens refracting parallel rays.

$$\frac{1}{96}\frac{1}{f^3}\left\{7 - 10 + 3{\cdot}25 + 6{\cdot}75\right\} = +\frac{7}{96}\frac{1}{f^3},$$

and $\frac{y^4}{f^5}A''$ gives

$$\frac{1}{(16)(16)(27)}\frac{1}{f^5}\left\{\begin{array}{l} -4{\cdot}625 + 33{\cdot}625 - 60{\cdot}1875 - 51{\cdot}94 - 55{\cdot}55 + 28{\cdot}19 \\ \qquad\qquad + 131{\cdot}06 - 5 - 24{\cdot}7 \end{array}\right\}$$

$$= \frac{1}{(16)(16)(27)}\frac{1}{f^5}\left(-9{\cdot}1\right) = -\frac{1}{768}\frac{1}{f^5},$$

or only $\frac{1}{56}$th part of the correction to the second approximation. But if x also $= -1$, then the first formula gives

Plano-convex lens refracting parallel rays.

$$\frac{1}{96}\frac{1}{f^3}\left(27\right) \text{ or } +\frac{27}{96}\frac{1}{f^3},$$

and the second formula gives

$$\frac{1}{(16)(16)(27)}\frac{1}{f^5}\left\{-395\right\} = -\frac{14{\cdot}6}{256}\frac{1}{f^5} = -\frac{6{\cdot}2}{96}\frac{1}{f^5},$$

or nearly a quarter of the aberration of the order y^2. But if f is doubled while y keeps constant, then the aberration $\frac{y^2}{f^3}A'$ is reduced to $\frac{1}{8}$th part, while the aberration $\frac{y^4}{f^5}A''$ is reduced to $\frac{1}{32}$nd part.

These conclusions apply with equal truth to the corresponding concavo-plane and plano-concave dispersive lenses when refracting parallel rays.

However, we must not treat this formula as if it represented the only aberration correction of the order y^4 which has to be dealt with. For in the case of thick lenses of large relative apertures, or a system of separated lenses, the formulæ, before alluded to, which are functions of y^4 into the aberrations of the second approximation, may often exceed in importance the intrinsic formulæ of the third approximation.

We have hitherto assumed that the thickness of the lens to which this Formula XXIII. refers is too small to sensibly affect its accuracy, but in general practice cases very often occur in which the thicknesses of the lenses concerned are so considerable that no approach to accuracy could be made without making proper allowance for it. Here we shall again find that the Theorem of Elements will enable us to effectually get over the difficulty.

A thick lens requires special treatment.

Application of the Theorem of Elements to Thick Lenses

Let Figs. 35*a* and 35*b*, Plate VIII., represent two thick lenses, one a collective lens and one a dispersive lens, the conjugate focal distances $Q_1 \ldots A_1$ and $A_2 \ldots Q_2$ being also positive in each of the two cases. Let tangents to the two vertices A_1 and A_2 of the lenses be drawn. These then represent planes perpendicular to the optic axis, and as we imagine two elements to be located at the two vertices, these planes may appropriately be called Element Planes.

Element planes.

Moreover, if we are treating these thick lenses in accordance with the Theorem of Elements, it is obvious that the two element planes are also the bounding planes or surfaces of the imaginary plate of parallel glass which is supposed to lie between the two elements.

Let $b_1 \ldots A_1$ and $b_2 \ldots A_2 = Y_1$ and Y_2 respectively, and let $b_1' \ldots c_1$ and $b_2' \ldots c_2 = y_1$ and y_2 respectively.

Now so far, in working out the formula for spherical aberration for a curved surface like $A_1 \ldots b_1'$, we have assumed y (or $b_1' \ldots c_1$) to express the perpendicular distance of b_1' (the point *on the curved surface* where the ray in question is refracted) from the optic axis.

But we might have assumed y to mean *not* $b_1' \ldots c_1$, but $A_1 \ldots b_1$, that is the height Y_1 of the point where the same ray cuts the element plane, instead of the height where the ray cuts the curved surface; and it is obvious that the plan of measuring our y's along the two element planes of any lens presents the advantage of great simplicity, and renders it perfectly easy to assign the values of the successive Y's for a ray traversing a series of thick or widely separated lenses.

Simplicity gained by assuming the y's to lie in the element planes.

PLATE. VIII.

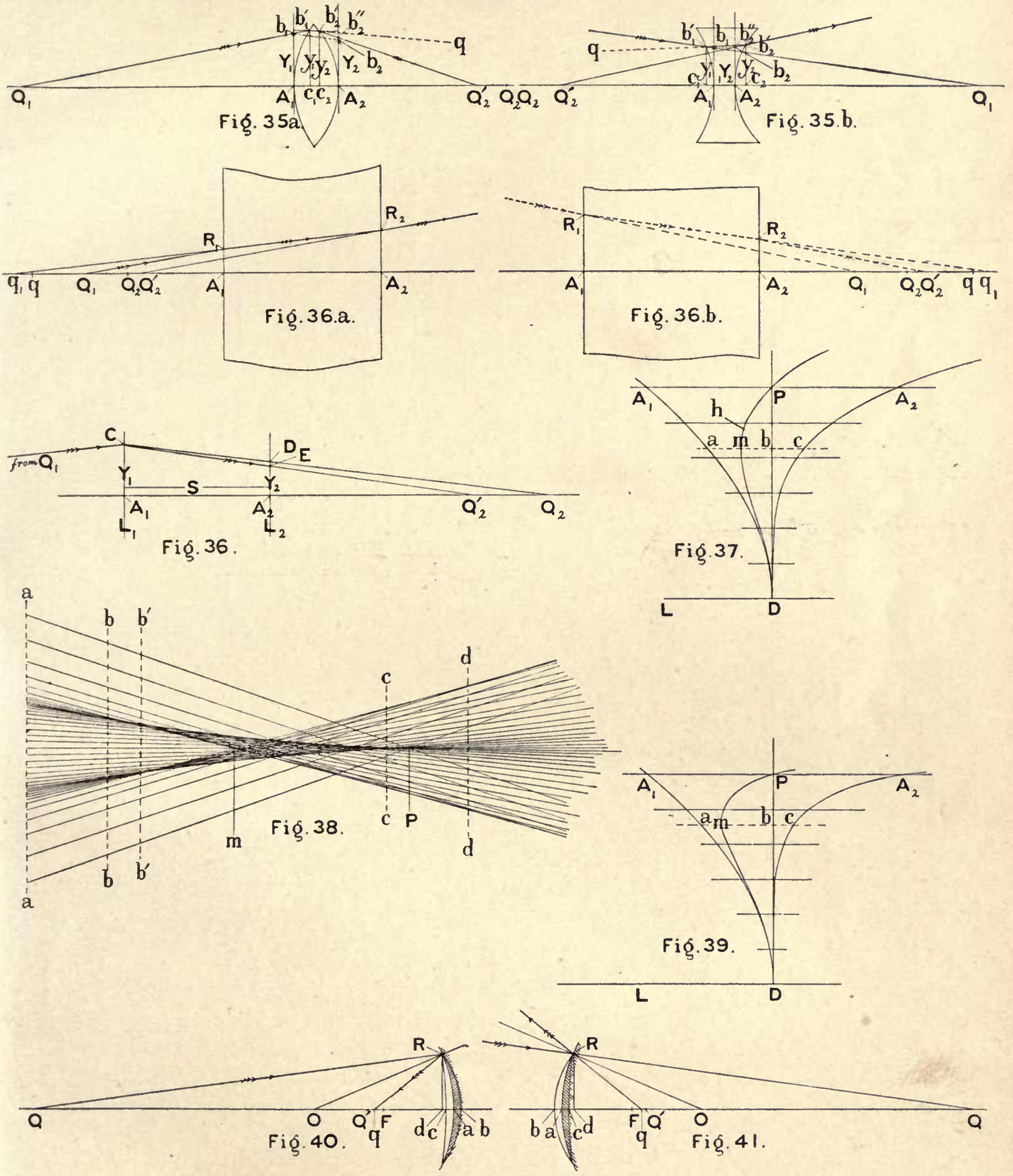


Fig. 35a.

Fig. 35.b.

Fig. 36.a.

Fig. 36.b.

Fig. 36.

Fig. 37.

Fig. 38.

Fig. 39.

Fig. 40.

Fig. 41.

PLATE. VIII.

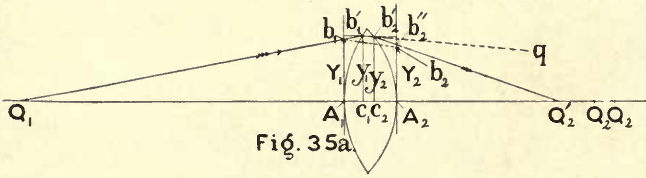


Fig. 35a.

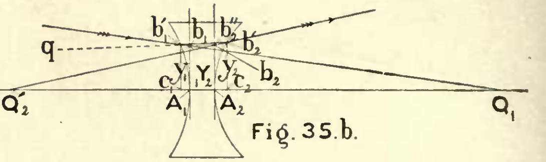


Fig. 35.b.

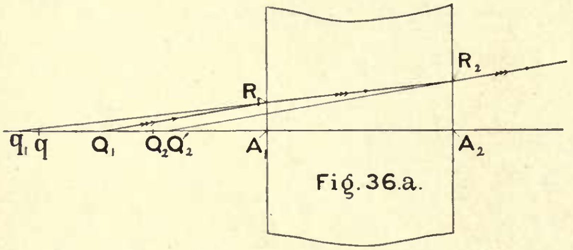


Fig. 36.a.

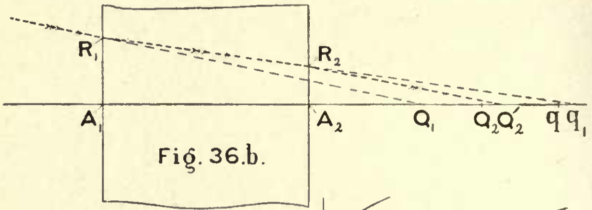


Fig. 36.b.

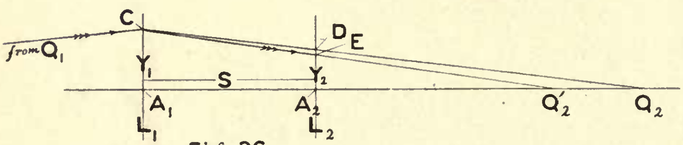


Fig. 36.

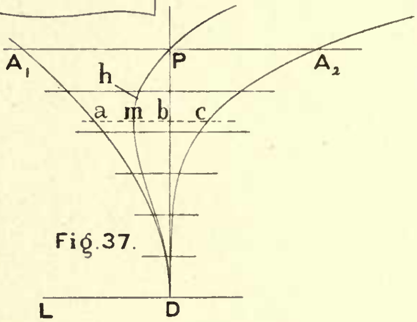


Fig. 37.

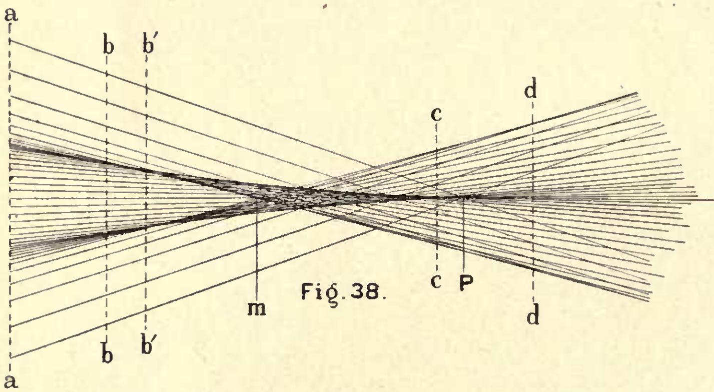


Fig. 38.

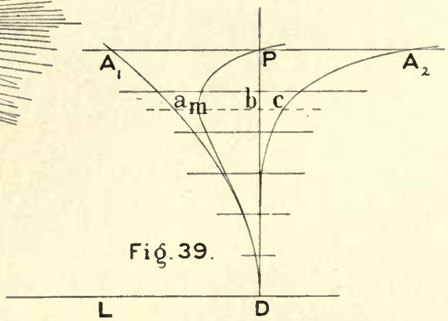


Fig. 39.

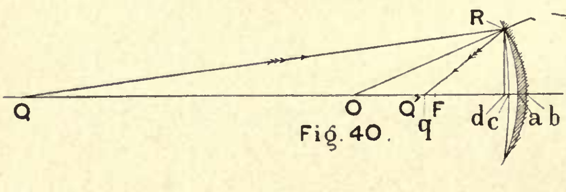


Fig. 40.

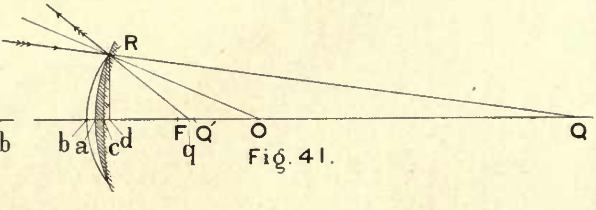


Fig. 41.

That once granted, then of what nature will be the corrections to the spherical aberration following upon the nonconformity between the Y's measured in the element planes and the y's measured up to the points where the same rays strike the curved surfaces? We shall soon see that these corrections are comprised under the order of functions of Y^4, and of higher even powers of Y. Of course, if the entering rays are parallel, there is then no disparity between Y_1 and y_1, and no disparity between Y_2 and y_2, if the emergent rays are parallel. If we treat the whole lens as a self-contained entity, then if $Q_1 \ldots A_1 = u$, and $A_2 \ldots Q_2 = v$, as before, and $\frac{1}{F} = (\mu - 1)\left(\frac{1}{r} + \frac{1}{s}\right)$, we find that

$$y_1 = Y_1 + (b_1 \ldots b_1')\frac{Y_1}{u} = Y_1 + \left(\frac{Y_1^2}{2r}\right)\frac{Y_1}{u} = Y_1 + \frac{Y_1^3}{2ru} = Y_1\left(1 + \frac{Y_1^2}{2ru}\right),$$

so that

$$y_1^2 = Y_1^2\left(1 + 2\frac{Y_1^2}{2ru}\right). \tag{13}$$

Similarly

$$y_2^2 = Y_2^2\left(1 + 2\frac{Y_2^2}{2sv}\right). \tag{14}$$

The above two formulæ serve to indicate the general nature of the corrections involved, and we will return to a more exact investigation of this matter at a later stage.

After this we will assume our y's to lie in the element planes except where otherwise stated; therefore we will retain the symbol y in place of the symbol Y which we employed in the above inquiry.

Thick lens. Form of the aberrations of the two surfaces.

We will first consider the thick lenses in Figs. 35a and 35b, in terms of the aberrations of the two surfaces. The rays radiating from Q_1 are supposed, after refraction at the first surface, to converge to a point q situated at a distance $\acute{u}$ $(= A_1 \ldots q)$ from A_1, that distance being an intra-glass measurement. In the case of Fig. 35b they are supposed to be diverging from q after first refraction, a condition analogous to that of Fig. 35a. Now, the spherical aberration of the first surface as yielded by Formula XVIII. (R.) is a correction to the first approximate value of $\frac{1}{A_1 \ldots q}$ or $\frac{1}{\acute{u}}$, and the longitudinal aberration is obtained by multiplying XVIII. by $-\acute{u}^2$, as in Formula XVIII. (L.).

How aberration of first surface is transferred to second vertex.

We will call the longitudinal aberration so obtained $-y_1^2 a_1 \acute{u}^2$. Now, we wish to transfer the value of the aberration of the first surface to a new reference point A_2, so that we can add it to the aberration of the second surface. Therefore we have an aberration from the first surface denoted by y^2a_1, implying a linear aberration equal to $-y^2a_1\acute{u}^2$, which, regarded from the new point A_2, or the vertex of the second surface, will be equal to $-\left(\frac{-y^2a_1\acute{u}^2}{(\acute{u}-t)^2}\right) = y_1^2a_1\frac{\acute{u}^2}{(\acute{u}-t)^2}$ in the case of the collective lens, and $y_1^2a_1\frac{\acute{u}^2}{(\acute{u}+t)^2}$ in the case of the dispersive lens, as an R correction. If we now add in the aberration of the second surface we have the joint aberration, referred to the point A_2, expressed by

$$y_1^2a_1\frac{\acute{u}^2}{(\acute{u}-t)^2} + y_2^2a_2$$

Sum of the aberration of the two surfaces.

for the collective lens, and

$$y_1^2a_1\frac{\acute{u}^2}{(\acute{u}+t)^2} + y_2^2a_2$$

for the dispersive lens, as R corrections. Furthermore, if the y's are so small that the versines of the curves are small and negligible quantities, we then have

$$y_2 = y_1\left(\frac{\acute{u}-t}{\acute{u}}\right) \text{ and } y_2^2 = y_1^2\left(\frac{\acute{u}-t}{\acute{u}}\right)^2$$

Relationship between the two y's.

for collective lens, and

$$y_2 = y_1\left(\frac{\acute{u}+t}{\acute{u}}\right) \text{ and } y_2^2 = y_1^2\left(\frac{\acute{u}+t}{\acute{u}}\right)^2$$

for dispersive lens, which is a very simple relationship.

Let us now treat the same lens by the method of elements.

We may then denote the conjugate focal distances for the first element by u_1 and v_1, and those for the second element by u_2 and v_2, so that $\frac{1}{u_1}+\frac{1}{v_1}=\frac{1}{f_1}$, and $\frac{1}{u_2}+\frac{1}{v_2}=\frac{1}{f_2}$.

Same thick lens treated by method of elements.

Then at A_1 we must imagine a convexo-plane element, and at A_2 a plano-convex element in the case of the collective lens; and a concavo-plane and plano-concave element at A_1 and A_2 respectively in the case of the dispersive lens. The rays which converge to or diverge from q after refraction by the first surface of the first element will, after refraction at the second or plane surface of the element, converge to or diverge from a new point distant from A_1 by $\frac{\acute{u}}{\mu}$ $(=v_1)$. Then, since the separation between the two elements is $\frac{t}{\mu}$, we shall have a spherical aberration for the first element, which may be called $y_1^2(a_1+p_1)$, p_1 being the aberrative function for the second or plane surface. This aberration becomes

$$y_1^2(a_1+p_1)\left(\frac{\frac{\acute{u}}{\mu}}{\frac{\acute{u}}{\mu}-\frac{t}{\mu}}\right)^2 = y_1^2(a_1+p_1)\left(\frac{v_1}{v_1-\frac{t}{\mu}}\right)^2$$

when referred to the second element at A_2 for the collective lens, and

$$y_1^2(a_1+p_1)\left(\frac{\frac{\acute{u}}{\mu}}{\frac{\acute{u}}{\mu}+\frac{t}{\mu}}\right)^2 = y_1^2(a_1+p_1)\left(\frac{v_1}{v_1+\frac{t}{\mu}}\right)^2$$

for the dispersive lens; while y_2^2 will be

$$y_1^2\left(\frac{\frac{\acute{u}}{\mu}-\frac{t}{\mu}}{\frac{\acute{u}}{\mu}}\right)^2 = y_1^2\left(\frac{v_1-\frac{t}{\mu}}{v_1}\right)^2$$

for the collective lens, and y_2^2 will be

$$y_1^2\left(\frac{\frac{\acute{u}}{\mu}+\frac{t}{\mu}}{\frac{\acute{u}}{\mu}}\right)^2 = y_1^2\left(\frac{v_1+\frac{t}{\mu}}{v_1}\right)^2$$

for the dispersive lens; but

$$\frac{\frac{\acute{u}}{\mu}}{\frac{\acute{u}}{\mu}\mp\frac{t}{\mu}} \quad \text{or} \quad \frac{v_1}{v_1\mp\frac{t}{\mu}}$$

is obviously equal to $\frac{\acute{u}}{\acute{u}\mp t}$, which we got before for the solid lens, and the same applies to their reciprocals; only, in the case of the imaginary elements separated by $\frac{t}{\mu}$ we have supposed to exist the inner plane surfaces of the said elements, which do not exist in the solid lens.

Interpretation of a thick lens by theorem of elements.

But it is clear that we can legitimately imagine the two inner plane surfaces of the two elements to exist in the solid lens, provided that we also imagine to exist a solid parallel plate of glass of thickness t lying between and touching the said two elements.

There would then be four plane surfaces to be imagined, two bounding the elements and two bounding the parallel plate. At each one of such plane surfaces, provided that the rays traversing the interior of the lens are not parallel, a certain amount of aberration takes place. We may call the spherical aberration for the first element, as before, $y_1^2(a_1+p_1)$, a_1 being the spherical aberration of the curved surface and p_1 the aberration of the second or plane surface of the element. Then the spherical aberration of the first surface of the parallel plate may be written $y_1^2(p_1')$; the spherical aberration of the second surface of the parallel plate written $y_2^2(p_2')$; and the spherical aberration of the second element may be $y_2^2(p_2+a_2)$, in which p_2 is the spherical aberration of the first or plane surface of the second element, and a_2 that of the curved surface. So that the whole series of aberrations, referred to the point A_2, may be expressed by

The six constituents of the whole aberration.

$$\left\{y_1^2(a_1+p_1)+y_1^2p_1'\right\}\left(\frac{\acute{u}}{\acute{u}-t}\right)^2+y_2^2p_2'+y_2^2(p_2+a_2).$$

Now it is plain that if a pencil of rays passes, however obliquely, from one piece of glass bounded by a plane surface into another piece of glass of the same refractive index and bounded by another plane surface in close contact with the plane surface of the first piece of glass, then no refraction and therefore no aberration whatsoever can take place. In other words, the refraction or aberration which takes place when the pencil of rays emerges from the first piece of glass into air is exactly neutralised by the opposite refraction or aberration ensuing on the same pencil being refracted again immediately into the second piece of glass, so that the two plane surfaces might be absent and the glass be solid and homogeneous so far as any optical effect upon the pencil of rays is concerned.

Therefore in our series of aberrations it is clear that $y_1^2p_1+y_1^2p_1'=0$ and $y_2^2p_2'+y_2^2p_2=0$, and therefore the whole series is equivalent to $y_1^2a_1\left(\frac{\acute{u}}{\acute{u}-t}\right)^2+y_2^2a_2$ (for a collective lens), which is what we arrived at when treating the lens by surfaces.

But we can put another interpretation upon the above series of aberrations. We wish to retain the elements as actual entities, and they necessarily imply two surfaces. The aberration of the first element necessarily includes the aberration of its plane second surface, likewise the aberration of the second element necessarily includes the aberration of its plane first surface. Hence we may group the series of aberrations in the following manner *consistently with the same total result*—

Another interpretation of the sum of the six aberrations.

$$y_1^2(a_1+p_1)\left(\frac{\acute{u}}{\acute{u}-t}\right)^2+\left\{y_1^2p_1'\left(\frac{\acute{u}}{\acute{u}-t}\right)^2+y_2^2p_2'\right\}+y_2^2(p_2+a_2).$$

As we are making a point of retaining the aberrations of the two plane surfaces of the elements, we must therefore retain, in order to balance the former, the aberrations of the two plane surfaces of the parallel glass plate separating the elements. The latter aberrations are gathered together within the centre brackets, and represent the aberration (of the same nature as spherical aberration) produced by the parallel glass plate of thickness t. Also we have seen that the term

$$\frac{\frac{\acute{u}}{\mu}}{\frac{\acute{u}}{\mu}+\frac{t}{\mu}} \left(\text{or } \frac{v_1}{v_1+\frac{t}{\mu}}\right)$$

used in case of the two elements separated by a distance $=\frac{t}{\mu}$ comes to exactly the same thing as the $\frac{\acute{u}}{\acute{u}+t}$ in the formulæ strictly applying to the solid lens.

Therefore our general conclusion is (1st) that the spherical aberration of a solid thick lens, when referred to its second vertex A_2, is equal to the sum of the spherical aberration of its two elements, separated by $\frac{t}{\mu}$, referred to the position of the second element, plus the aberration of a parallel glass plate of the same thickness as the solid lens, also referred to its second surface; and (2nd) that y_2 for the second element

Aberrations of the two elements added to that of the parallel plane plate.

$$=y_1\frac{\frac{\acute{u}}{\mu}+\frac{t}{\mu}}{\frac{\acute{u}}{\mu}}=y_1\frac{v_1+\frac{t}{\mu}}{v_1}=y_1\frac{\acute{u}+t}{\acute{u}}$$

if we measure the two y's in the two element planes respectively, while $\frac{\acute{u}}{\mu}$ is obviously equal to v_1 for the first element or the focal distance conjugate to u_1, such that $\frac{1}{v_1}=\frac{1}{f_1}-\frac{1}{u_1}$, wherein $\frac{1}{f_1}$ is the power of the first element or $\frac{\mu-1}{r}$, and $\frac{1}{u_1}=\frac{1}{Q_1 \, . \, . \, A_1}$.

The y's to be measured in the element planes.

Aberration of a Parallel Plane Plate

Our next step, therefore, is to find an expression for the aberration of a parallel glass plate of any thickness.

Let Fig. 36a represent a case of a divergent pencil traversing a

Refraction of a normal pencil through parallel plane plate.

thick parallel plane plate of thickness $A_1 \,..\, A_2$, and Fig. 36*b* a case of a convergent pencil of rays traversing a similar plate. The principal ray in each case, $q \,..\, A_2$ and $A_1 \,..\, q$, passes perpendicularly through both surfaces and therefore suffers no refraction. Q_1 is the origin or apex of the pencil.

Let $Q_1 \,..\, A_1 = u_1$, and be considered positive in the case of Fig. 36*a* and negative in the case of Fig. 36*b*. Let the semi-diameter $R_1 \,..\, A_1$ of the pencil be called a_1. Let q be the conjugate focus to Q_1 by first approximation—that is, let $q \,..\, A_1 = \mu u_1 = v_1$, and let $q_1 \,..\, A_1 = x_1$. For the ray $Q_1 \,..\, R$ after refraction at R proceeds in a direction which (if it has to be produced backwards) cuts the principal ray at q_1, further from A_1 than q, so that $q \,..\, q_1$ is the longitudinal aberration to which the ray $Q_1 \,..\, R$ is subject.

Let the angle $R_1Q_1A_1 = \phi$ and the angle $R_1q_1A_1 = \phi'$. These are obviously the angles of incidence and refraction respectively. Then we have, as on page 49 of Coddington's work,

$$\begin{aligned} R_1 \,..\, q_1 : R_1 \,..\, Q_1 &:: \sin R_1Q_1q_1 : \sin R_1q_1Q_1 \\ &:: \sin R_1Q_1A_1 : \sin R_1q_1A_1 \\ &:: \quad \mu \quad : \quad 1 \end{aligned}$$

that is,

$$R_1 \,..\, q_1 = \mu(R_1 \,..\, Q_1).$$

But

$$R_1 \,..\, q_1 = \frac{x_1}{\cos\phi'} \text{ and } R_1 \,..\, Q_1 = \frac{u_1}{\cos\phi},$$

therefore

The exact formula.

$$\frac{x_1}{\cos\phi'} = \mu\frac{u_1}{\cos\phi} \text{ and } x_1 = \mu\frac{\cos\phi'}{\cos\phi}u_1$$

exactly.

This can be reduced into an approximately accurate algebraic form, thus—

Since
$$\cos\phi' = \frac{q_1 \,..\, A_1}{q_1 \,..\, R_1} \text{ and } \frac{1}{\cos\phi} = \frac{Q_1 \,..\, R_1}{Q_1 \,..\, A_1},$$

therefore above equation becomes

$$x_1 = \mu\frac{q_1 \,..\, A_1}{q_1 \,..\, R_1} \cdot \frac{Q_1 \,..\, R_1}{Q_1 \,..\, A_1}u_1,$$

which

$$= \mu u_1 \frac{(q_1 \,..\, A_1)\left((Q_1 \,..\, A_1) + \dfrac{a_1^2}{2(Q_1 \,..\, A_1)}\right)}{\left((q_1 \,..\, A_1) + \dfrac{a_1^2}{2(q_1 \,..\, A_1)}\right)(Q_1 \,..\, A_1)},$$

in which we may insert the approximate values of $q_1 \ldots A_1$, $= \mu u_1$, and for $Q_1 \ldots A_1$ write u_1, making

$$x_1 = \mu u_1 \left\{ \frac{\mu u_1 \left(u_1 + \frac{a_1^2}{2u_1} \right)}{\left(\mu u_1 + \frac{a_1^2}{2\mu u_1} \right) u_1} \right\} = \left\{ \frac{\mu u_1 + \mu \frac{a_1^2}{2u_1}}{\mu u_1 + \frac{a_1^2}{2\mu u_1}} \right\} \mu u_1$$

$$= \left(\mu^2 u^2 + \frac{\mu^2 a_1^2}{2} \right) \frac{2\mu u_1}{2\mu^2 u_1^2 + a_1^2} = \frac{2\mu^3 u_1^3 + 2\mu^3 u_1 \frac{a_1^2}{2}}{2\mu^2 u_1^2 + a_1^2}$$

$$= (2\mu^3 u_1^3 + \mu^3 u_1 a^2) \left(\frac{1}{2\mu^2 u_1^2} - \frac{a_1^2}{4\mu^4 u_1^4} \right) = \mu u_1 + \mu \frac{a_1^2}{2u_1} - \frac{a_1^2}{2\mu u_1}$$

$$= \mu u_1 + \frac{\mu^2 a_1^2 - a_1^2}{2\mu u_1} = \mu u_1 + \frac{\mu^2 - 1}{\mu} \cdot \frac{a_1^2}{2u_1};$$

therefore we get

$$x_1 = \mu u_1 + \frac{\mu^2 - 1}{\mu} \cdot \frac{a_1^2}{2u_1}$$

and therefore

$$\frac{1}{x_1} = \frac{1}{\mu u_1} - \left(\frac{\mu^2 - 1}{\mu} \frac{a_1^2}{2u_1} \right) \frac{1}{\mu^2 u_1^2}$$

and

$$\frac{\mu}{x_1} = \frac{1}{u_1} - \frac{\mu^2 - 1}{\mu^2} \cdot \frac{a_1^2}{2u_1^3}. \qquad (15)$$

First plane surface. Formula of second approximation.

It is clear that this formula applies to both cases, 36*a* and 36*b*, and that the aberration is of a minus character, implying an extension of the first approximate distance $A_1 \ldots q$. We can also derive Formula (15) from the Formula XVIII. expressing the spherical aberration of a single *spherical* surface. For the plane surface is but a spherical surface of infinite radius, so that $\frac{1}{r}$ in XVIII. becomes zero, and the result is Formula (15) (with a conventional difference of sign), which confirms our result. Further, it will be readily seen that the case of the convergent rays entering left to right into the plane surface is but the reversal, as it were, of the case of divergent rays passing out of the glass from right to left, and the same formula can be applied. Therefore the same formula which applies to the converging rays entering in Fig. 36*b* will apply also to the diverging rays leaving the glass in Fig. 36*a*.

Course of rays considered reversed.

Turning our attention to this case, then let

$$A_2 \ldots Q_2 = u_1 + \frac{t}{\mu} = v_2 \text{ and } A_2 \ldots Q_2' = x_2,$$

$$A_2 \ldots q = u_2 \text{ and } A_2 \ldots R_2 = a_2.$$

Then we also have the relations

$$\frac{a_1}{u_1} = \frac{a_2}{v_2},$$

and therefore the following identities hold good—

$$a_1 = a_2\frac{u_1}{v_2} = a_2\frac{v_1}{u_2} = a_2\frac{\mu v_2 - t}{\mu v_2},$$

$$u_2 = \mu v_2 = v_1 + t = \mu u_1 + t,$$

$$u_1 = v_2 - \frac{t}{\mu} \text{ and } v_2 = u_1 + \frac{t}{\mu}.$$

Then we have at the second surface

$$\frac{\mu}{u_2} = \frac{1}{v_2} - \frac{\mu^2 - 1}{\mu^2} \cdot \frac{a_2^2}{2v_2^3};$$

therefore

$$\frac{1}{v_2} = \frac{\mu}{u_2} + \frac{\mu^2 - 1}{2\mu^2} \cdot \frac{a_2^2}{v_2^3}. \tag{16}$$

This expresses the aberration of the pencil of divergent rays emerging from the second surface, on the condition, of course, that the rays are diverging from a fixed point at a distance $= u_2$ within the substance of the glass. After being refracted outwards they are subject to the aberration given in above Formula (16); and this aberration is of the opposite tendency to that which the rays met with on entering the glass, and implies a shortening of the first approximate value $\frac{u_2}{\mu}$.

But we have now to add the aberration produced at the first surface to that produced at the second.

Aberration of first surface transferred to second surface.

In order to transfer the aberration produced at the first surface to the new reference point A_2, we must multiply (15) by $\left(\frac{\mu u_1}{\mu u_1 + t}\right)^2$, thus getting

$$\frac{\mu}{x_1 + t} = \frac{\mu}{u_2}(\text{of Formula (16)}) = \frac{\mu}{\mu u_1 + t} - \frac{\mu^2 - 1}{2\mu^2} \cdot \frac{a_1^2}{u_1^3}\left(\frac{\mu u_1}{\mu u_1 + t}\right)^2$$

independently of the second refraction. On adding the aberration of the second surface from (16) to the above, we then get

$$\frac{1}{x_2} = \frac{\mu}{\mu u_1 + t} + \frac{\mu^2 - 1}{2\mu^2}\frac{a_2^2}{v_2^3} - \frac{\mu^2 - 1}{2\mu^2}\frac{a_1^2}{u_1^3}\left(\frac{\mu u_1}{\mu u_1 + t}\right)^2.$$

This should, for the sake of practical convenience, be expressed in terms of v_2 and a_2, so that

$$\frac{1}{x_2}=\frac{1}{u_1+\frac{t}{\mu}}+\frac{\mu^2-1}{2\mu^2}\left\{\frac{a_2^{\ 2}}{v_2^{\ 3}}-\frac{\left(a_2\frac{\mu v_2-t}{\mu v_2}\right)^2}{u_1^{\ 3}}\frac{\mu^2u_1^{\ 2}}{(\mu u_1+t)^2}\right\}$$

$$=\frac{1}{u_1+\frac{t}{\mu}}+\frac{\mu^2-1}{2\mu^2}\left\{\frac{a_2^{\ 2}}{v_2^{\ 3}}-\frac{a_2^{\ 2}}{u_1}\left(\frac{\mu v_2-t}{\mu v_2}\right)^2\frac{\mu^2}{(\mu v_2)^2}\right\}$$

$$=\frac{1}{v_2}+\frac{\mu^2-1}{2\mu^2}\left\{\frac{a_2^{\ 2}}{v_2^{\ 3}}-\frac{a_2^{\ 2}}{v_2-\frac{t}{\mu}}\left(\frac{v_2-\frac{t}{\mu}}{v_2}\right)^2\frac{1}{v_2^{\ 2}}\right\}$$

$$=\frac{1}{v_2}+\frac{\mu^2-1}{2\mu^2}\left\{\frac{a_2^{\ 2}}{v_2^{\ 3}}-\frac{a_2^{\ 2}}{v_2^{\ 3}}\left(\frac{v_2-\frac{t}{\mu}}{v_2}\right)\right\}=\frac{1}{v_2}+\frac{\mu^2-1}{2\mu^2}\left\{\frac{a_2^{\ 2}}{v_2^{\ 3}}\left(1-1+\frac{t}{v_2\mu}\right)\right\};$$

therefore

$$\frac{1}{x_2}=\frac{1}{v_2}+\frac{\mu^2-1}{2\mu^3}\frac{a_2^{\ 2}}{v_2^{\ 4}}t. \qquad \text{XXV. (R.)}$$

Aberration of a parallel plane plate.

If the same line of reasoning is applied to Fig. 36*b* the same result will be obtained, provided that u_1 and v_2 are considered negative; but if they are also considered positive then the spherical aberration will work out with a minus sign before it. In fact, we find that the aberration given by a parallel plate of glass is always of a negative character, if we compare its influence with that of a collective lens under normal conditions. If a pencil of divergent rays traverses a parallel plate, then the outer rays of the pencil on emergence are diverging from a point nearer to the second surface than the point indicated by the first approximation; while in the case of a convergent pencil of rays the outer rays after emergence are converging to a point farther from the second surface than the point indicated by the first approximation. In short, the aberration is of the character of that yielded by a dispersive lens, and we shall afterwards find that this analogy holds good in other respects also.

We also find from XXV. that the amount of the aberration increases inversely as the fourth power of the distance of that point from the second surface from which or to which the emergent rays are diverging or converging, and therefore there is no aberration in the case of u_1 or v_2 being infinite or the rays parallel.

We also find from our formula that

Linear value of the above aberration.

$$x_2 = v_2 - \frac{\mu^2 - 1}{2\mu^2} \cdot \frac{a_2^2}{v_2^2} \cdot \frac{t}{\mu}, \tag{17}$$

and therefore the latter term is the linear aberration, which thus varies inversely as v_2^2, directly as a_2^2, and directly as t.

Therefore it is plain that when the pencils of rays traversing the interior of thick lenses are strongly convergent or divergent, and the pencils are of wide aperture, the parallel plate aberration may be very considerable.

A Detailed Confirmation of the Theorem of Elements

Having worked out the Formula XXV. for the aberration produced by a parallel plate, we are now in a position to give the general confirmation of the theorem of elements as applied to thick lenses. This proof can best be presented in the form of a balance-sheet (see p. 81), on one side of which we insert the successive aberrations of the six surfaces in their order, two belonging to the first element, two to the parallel plate, and two to the second element; while on the other side we gather together the aberrations of the first pair of surfaces and express them as the aberration for the first element, the aberrations of the third and fourth plane surfaces and express them as the aberration of the parallel plate, and the aberrations of the two last surfaces and express them as the aberration of the second element. Then in comparing the one side with the other the identity of the two sums is clearly established, while at the same time it is also clearly seen on looking down the left-hand side that the whole sum for the six surfaces is identical with the sum of the aberrations of the first and sixth surfaces only, the intervening aberrations neutralising one another.

The notation.

The notation is as follows:—y_1 is the height of the ray where it cuts the first element plane, y_2 is the height of the ray where it cuts the second element plane, u_1 and v_1 are the conjugate focal distances for the first element, $\acute{u}_1$ is the distance from first vertex to the point to which the rays are converging after refraction by the first *surface*, and u_2 and v_2 are the first and second conjugate focal distances for the second element, so that $\acute{u}_1$ and $\acute{u}_2$ are within glass measurements, so that $\acute{u}_2 = \acute{u}_1 - t$ (t being the thickness), and therefore

$$\frac{\acute{u}_1 - t}{\acute{u}_1} = \frac{\acute{u}_2}{\acute{u}_2 + t} = \frac{u_2}{u_2 + \dfrac{t}{\mu}}.$$

Aberrations expressed for the Six Consecutive Surfaces

$$\text{1st Surface } + \frac{\mu-1}{2\mu^2}\left(\frac{1}{r_1}+\frac{1}{u_1}\right)^2\left(\frac{1}{r_1}+\frac{\mu+1}{\mu_1}\right)y_1^2\left\{\frac{u_2+\frac{t}{\mu}}{u_2}\right\}^2 = \text{Formula XVIII.}$$

$$\text{2nd Surface } + \frac{\mu-1}{2\mu^2}\left(\frac{1}{r_2}+\frac{1}{v_1}\right)^2\left(\frac{1}{r_2}+\frac{\mu+1}{v_1}\right)y_1^2\left\{\frac{u_2+\frac{t}{\mu}}{u_2}\right\}^2 = \text{Do.}$$

which, since

$$\frac{1}{r_2}=0, = \frac{\mu^2-1}{2\mu_2}\frac{1}{v_1^3}y_1^2\left\{\frac{u_2+\frac{t}{\mu}}{u_2}\right\}^2.$$

$$\text{3rd Surface } - \underbrace{\frac{\mu^2-1}{2\mu^2}\frac{1}{v_1^3}y_1^2}_{=\text{Formula (15).}}\left\{\frac{u_2+\frac{t}{\mu}}{u_2}\right\}^2 = -\frac{\mu^2-1}{2\mu^2}y_1^2\frac{1}{\left(u_2+\frac{t}{\mu}\right)^3}\left\{\frac{u_2+\frac{t}{\mu}}{u_2}\right\}^2$$

$$\left.\begin{matrix}\text{2nd Surface} \\ \text{3rd Surface}\end{matrix}\right\} = 0.$$

$$\text{4th Surface} + \frac{\mu^2-1}{2\mu^2}\frac{1}{u_2^3}\left\{y_1^2\left(\frac{u_2}{u_2+\frac{t}{\mu}}\right)^2\right\}.$$

$$\text{5th Surface} - \frac{\mu^2-1}{2\mu^2}\left(\frac{1}{s_1}+\frac{1}{u_2}\right)^2\left(\frac{1}{s_1}+\frac{\mu+1}{u_2}\right)\left\{y_1^2\left(\frac{u_2}{u_2+\frac{t}{\mu}}\right)^2\right\}.$$

which, since

$$\frac{1}{s_1}=0, = -\frac{\mu^2-1}{2\mu^2}\frac{1}{u_2^3}\left\{y_1^2\left(\frac{u_2}{u_2+\frac{t}{\mu}}\right)^2\right\}.$$

$$\left.\begin{matrix}\text{4th Surface} \\ \text{5th Surface}\end{matrix}\right\} = 0.$$

$$\text{6th Surface} + \frac{\mu-1}{2\mu^2}\left(\frac{1}{s_2}+\frac{1}{v_2}\right)^2\left(\frac{1}{s_2}+\frac{\mu+1}{v_2}\right)\left\{y_1^2\left(\frac{u_2}{u_2+\frac{t}{\mu}}\right)^2\right\} = \text{Formula XVIII.}$$

Aberrations expressed for Two Elements and a Parallel Plate

Aberration of First Element

$$= \frac{\mu-1}{2\mu^2}\left\{\left(\frac{1}{r_1}+\frac{1}{u_1}\right)^2\left(\frac{1}{r_1}+\frac{\mu+1}{u_1}\right)+\left(\frac{1}{r_2}+\frac{1}{v_1}\right)^2\left(\frac{1}{r_2}+\frac{\mu+1}{v_1}\right)\right\}y_1^2\left(\frac{u_2+\frac{t}{\mu}}{u_2}\right)^2$$

$$= \text{Formula XIX.}$$

Aberration of Parallel Plate

$$-\frac{\mu^2-1}{2\mu^2}\frac{1}{\left(u_2+\frac{t}{\mu}\right)^3}y_1^2\left\{\frac{u_2+\frac{t}{\mu}}{u_2}\right\}^2+\frac{\mu^2-1}{2\mu^2}\frac{1}{u_2^3}\left\{y_1^2\left(\frac{u_2}{u_2+\frac{t}{\mu}}\right)^2\right\}$$

$$= \frac{\mu^2-1}{2\mu^2}\frac{1}{u_2^3}\left\{\frac{u_2^2}{\left(u_2+\frac{t}{\mu}\right)^2}-\frac{u_2}{u_2+\frac{t}{\mu}}\right\}y_1^2 = \frac{\mu^2-1}{2\mu^2}\frac{1}{u_2^3}\left\{\frac{u_2^2-\left(u_2^2+u_2\frac{t}{\mu}\right)}{\left(u_2+\frac{t}{\mu}\right)^2}\right\}y_1^2$$

$$= +\frac{\mu^2-1}{2\mu^2}\frac{1}{u_2^3}\left(\frac{-u_2\frac{t}{\mu}}{\left(u_2+\frac{t}{\mu}\right)^2}\right)y_1^2$$

$$= -\frac{\mu^2-1}{2\mu^2}\cdot\frac{1}{u_2^4}\cdot\frac{t}{\mu}y_2^2 = \text{Formula XXV.}$$

Aberration of Second Element

$$+\frac{\mu-1}{2\mu^2}\left\{\left(\frac{1}{s_1}+\frac{1}{u_2}\right)^2\left(\frac{1}{s_1}+\frac{\mu+1}{u_2}\right)+\left(\frac{1}{s_2}+\frac{1}{v_2}\right)^2\left(\frac{1}{s_2}+\frac{\mu+1}{v_2}\right)\right\}\left\{y_1^2\left(\frac{u_2}{u_2+\frac{t}{\mu}}\right)^2\right\}$$

$$= \text{Formula XIX.}$$

On both sides of this balance-sheet all the aberrations are referred to the second vertex of the lens or to the second element, and y_2 is expressed in terms of y_1, so that $y_2 = y_1\left(\frac{u_2}{u_2 + \frac{t}{\mu}}\right)$; also the aberrations of the parallel plate are similarly treated, so that y_2 becomes $y_1\left(\frac{u_2}{u_2 + \frac{t}{\mu}}\right)$, while r_1 and r_2 are radii of the first element, and s_1 and s_2 those of the second element.

In the above formulæ it has been the more convenient for our purpose to consider u_2 as a positive quantity, and the sign prefixed to the formula for each surface shows whether the aberration is + or — with respect to the final results. But after gathering together the two last formulæ into one formula for the second element, the convention of u_2 being — is resumed.

A Practical Illustration

As a further confirmation of the above theorem, and as an arithmetical illustration of the practical application of Coddington's Formula XXIII. and the above Formula XXV. to a thick lens, treated by the method of elements, we will take the case of a lens of principal focal length $= 1{\cdot}5$, such that $\left(\frac{1}{r} + \frac{1}{s}\right)(\mu - 1) = \frac{1}{1{\cdot}5}$, $\mu = 1{\cdot}50$, $r_1 = 1$, and r_2 or $s = 3$, while the central thickness $t = {\cdot}75$.

Treatment by elements.

We will suppose u_1 to be infinite and the entering rays parallel.

Powers of the two elements.

The power of the first element $= \frac{\mu - 1}{r_1} = \frac{1}{2}$, therefore $f_1 = 2$.

The power of the second element $= \frac{\mu - 1}{r_2} = \frac{{\cdot}5}{3}$, therefore $f_2 = 6$.

Let us suppose y_1 to be ${\cdot}40$; then since $u_2 = v_1 - \frac{t}{\mu}$ and

Relation between the two y's.

$$y_2 = y_1\frac{u_2}{v_1} - y_1\frac{v_1 - \frac{t}{\mu}}{v_1} = y_1\frac{2 - {\cdot}50}{2} = y_1\frac{3}{4},$$

therefore

$$y_2 = {\cdot}30.$$

Values of the two a's.

Then we have $a_1 = -1$ and $x_1 = +1$, while at the second element we have

$$\frac{1 + a_2}{2f_2} = -\frac{1}{1{\cdot}5};\ \therefore\ 1 + a_2 = -\frac{12}{1{\cdot}5} = -8$$

and

$$a_2 = -9, \text{ while } x_2 = -1.$$

For the first element Formula XXIII. gives for the spherical aberration

$$\frac{(\cdot 4)^2}{6f^3}\{7 - 10 + 3\cdot 25 + 6\cdot 75\} = \frac{\cdot 16}{8}\{1\tfrac{1}{6} - 1\tfrac{2}{3} + \cdot 5416 + 1\cdot 125\}$$
$$= \frac{1}{50}\{1\cdot 1666\} = \cdot 02333,$$

Aberration of the first element.

which quantity we must transfer to the second element by multiplying it by $\left(\frac{v_1}{u_2}\right)^2 = \left(\frac{2}{1\cdot 5}\right)^2 = \frac{16}{9}$, which makes it ·04148.

Above transferred to second vertex.

Then the aberration of the second element

$$= \frac{(\cdot 30)^2}{6(6)^3}\{7 + 10(9) + 3\cdot 25(9)^2 + 6\cdot 75\}$$
$$= \frac{\cdot 09}{216}\{1\tfrac{1}{6} + 1\tfrac{2}{3}(9) + \cdot 5416(81) + 1\cdot 125\}$$
$$= \frac{\cdot 09}{216}(61\cdot 16) = \quad \cdot 0255$$

Aberration of the second element.

Add ·04148 brought forward from first element.

Total (2 elements) = + ·06698

From this must be deducted the parallel plate aberration given by

Aberration of the parallel plate.

$$\frac{\mu^2 - 1}{2\mu^3}\,\frac{a_2^{\,2}}{v_2^{\,4}}t.$$

Here a_2 is the same as y_2, which in this case = ·30, and v_2 is the same as u_2, which in this case = 1·5, so that we have

$$\frac{2\cdot 25 - 1}{2 \times 3\cdot 375}\left(\frac{(\cdot 3)}{(1\cdot 5)^4}\right)^2(\cdot 75) = \frac{1\cdot 25}{6\cdot 75}\,\frac{\cdot 09}{5\cdot 0625}(\cdot 75) = \frac{1\cdot 25}{9}\cdot\frac{\cdot 09}{5\cdot 0625}$$
$$\doteq \frac{\cdot 0125}{5\cdot 0625} = \cdot 00247.$$

So that we have

Aberration of the two elements =	+ ·06698
Aberration of the parallel plate =	− ·00247
Corrected aberration of lens =	+ ·06451

Total of the three aberrations.

Alternative Treatment of the same Case

We will now treat the aberration of this lens as simply the sum of the spherical aberrations of the two surfaces, for which purpose we must employ Formula XVIII. (R.), which is

Alternative treatment by two surfaces.

$$\frac{\mu-1}{2\mu^2}\left\{\left(\frac{1}{r_1}+\frac{1}{u_1}\right)^2\left(\frac{1}{r_1}+\frac{\mu+1}{u_1}\right)\right\}y_1^2 \text{ for the first surface,}$$

and

$$\frac{\mu-1}{2\mu^2}\left\{\left(\frac{1}{r_2}+\frac{1}{v_2}\right)^2\left(\frac{1}{r_2}+\frac{\mu+1}{v_2}\right)\right\}y_2^2 \text{ for the second surface.}$$

In this case, after the parallel entering rays have been refracted by the first surface, they will converge to a point (by first approximation) behind the first vertex by a distance $= \acute{u}_1 = r_1\dfrac{\mu}{\mu-1} = 3r_1$, and will then be converging into the second surface to a point $= 3r_1 - t = 3 - \cdot 75 = 2\cdot 25$ behind the second vertex (which is a negative distance), and then by the formula

Value of v_2 ascertained.

$$\frac{1}{v_2} = \frac{\mu-1}{r_2} - \frac{\mu}{v_2'} = \frac{\cdot 5}{3} - \left(-\frac{1\cdot 5}{2\cdot 25}\right) = \frac{1}{6} + \frac{1}{1\cdot 5} = \frac{5}{6} = \frac{1}{1\cdot 2}$$

we get $v_2 = 1\cdot 2$.

Then we also have $y_2 = y_1\dfrac{2\cdot 25}{3} = y_1\dfrac{3}{4}$, just as when we treated the lens by the method of elements. So we again have $y_1 = \cdot 40$ and $y_2 = \cdot 30$.

Relation between the two y's.

Then the aberration at the first surface

Aberration of the first surface.

$$= \frac{\cdot 5}{2(2\cdot 25)}\{(1+0)^2(1+0)\}(\cdot 40)^2 = \frac{1}{9}(1)(\cdot 16) = \cdot 01777.$$

This aberration has now to be transferred to the vertex of the second surface by multiplying it by

Above transferred to second vertex.

$$\left(\frac{\acute{u}_1}{v_2'}\right)^2 \text{ or } \left(\frac{3}{2\cdot 25}\right)^2 = \left(\frac{4}{3}\right)^2 = \frac{16}{9},$$

just as when we treated the lens by the method of elements, so that we have $\cdot 01777 \times \dfrac{16}{9} = \cdot 0316$ to add in to the aberration of the second surface, which is

Aberration of the second surface.

$$\frac{\cdot 5}{2(2\cdot 25)}\left\{\left(\frac{1}{3}+\frac{1}{1\cdot 2}\right)^2\left(\frac{1}{3}+\frac{2\cdot 5}{1\cdot 2}\right)\right\}(\cdot 30)^2$$

$$= \frac{1}{9}\left\{\left(\frac{4+10}{12}\right)^2\left(\frac{4+25}{12}\right)\right\}(\cdot 09) = (\cdot 01)\left\{\left(\frac{7}{6}\right)^2\left(\frac{29}{12}\right)\right\}$$

$$= (\cdot 01)\left(\frac{49}{36}\right)\left(\frac{29}{12}\right) = (\cdot 01)\frac{1421}{432} = (\cdot 01)(3\cdot 29) = \cdot 0329$$

Add aberration from first surface $= \cdot 0316$

$\cdot 0645$

Total of the two aberrations identical with the last result.

Thus the aberration for the whole lens referred to the second vertex, obtained by treating the lens as a solid entity of thickness $= t$ bounded by two spherical surfaces, gives exactly the same result as we got by supposing the lens to consist of two infinitely thin elements with a parallel plate of glass of thickness t lying between them. If so, then why should we not always compute the spherical aberration of such thick lenses by the formulæ applying to surfaces, and not trouble ourselves with the method of elements? To which question it may be replied that while the student is perfectly at liberty to apply the formulæ for surfaces when computing spherical aberrations, yet when it comes to working out various other corrections of great importance, to be dealt with in subsequent Sections, it will be found that the method of elements simplifies and renders quite feasible problems which mere surface formulæ would be quite inadequate to deal with, at any rate without risk of hopeless confusion arising. Moreover, we have already seen at the beginning of Section II. that a refracting surface is not a constant entity. That being so, it may be conceded that it is as well, for many obvious reasons, to adopt the same general method throughout all optical computations.

The principal advantages of the theorem of elements yet to be explained.

Investigation of certain other Aberrations of the Third Order

We have yet to apply Formula XXIV. or $\frac{y^4}{f^5}A''$ to this lens, but before doing so it will be as well to work out the other aberrations of the order y^4 to which the lens is subject. We will return to Figs. 35*a* and 35*b*, representing a biconvex and a biconcave lens touched at each vertex by the element plane $A_1 \ldots b_1$ and $A_2 \ldots b_2$ respectively.

First the versine corrections.

Let Q_1 be the origin of the pencil, and $Q_1 \ldots b_1$ a ray impinging on the lens surface at b_1' but cutting the first element plane at b_1, while $Q_1 \ldots b_1' \ldots b_2' \ldots Q_2'$ is the actual course of the ray dealt with, which finally cuts the optic axis at Q_2' considerably short of Q_2, where it would cut the axis were there no aberration. Now we have assumed so far that the first refraction takes place in the first element plane, so that the straight line $b_1 \ldots b_2$ represents the course of the ray within the glass, if it were refracted by a small portion of glass surface really placed at b_1. It is obvious enough that this is practically the case for any ray from Q_1 passing through the lens much nearer the axis. Now supposing the ray after the first refraction at the curved surface (supposed to be placed at b_1) converges to a point q within the glass, then it is obvious that the refracted ray $b_1 \ldots q$ will cut the second element plane at a point b_2 such that $A_2' \ldots b_2$ or Y_2 will be equal to

$$(A_1 \,..\, b_1 \text{ or } Y_1)\frac{A_2 \,..\, q}{A_1 \,..\, q}; \text{ that is, } Y_2 \text{ will be } Y_1\frac{(A_1 \,..\, q) - T}{A_1 \,..\, q},$$

or, what is the same thing,

$$Y_2 = Y_1 \frac{\dfrac{A_1 \,..\, q}{\mu} - \dfrac{T}{\mu}}{\dfrac{A_1 \,..\, q}{\mu}}.$$

But $\dfrac{A_1 \,..\, q}{\mu}$ is the distance v_1 from the first element to the point on the axis to which the ray $Q_1 \,..\, b_1$ would be refracted by passage through the second or plane surface of the first element in addition to the first or curved surface; so that our equation

$$Y_2 = Y_1\left(\frac{\dfrac{A_1 \,..\, q}{\mu} - \dfrac{T}{\mu}}{\dfrac{A_1 \,..\, q}{\mu}}\right) \text{ is the equivalent of } Y_2 = Y_1\left(\frac{v_1 - \dfrac{T}{\mu}}{v_1}\right).$$

That point being settled, we may return to the determination of the actual or corrected heights $b_1' \,..\, c_1$ and $b_2' \,..\, c_2$, at which the ray is refracted by the two surfaces of the lens. Let these two heights be called y_1 and y_2 respectively. We have to find a formula expressing y_1 in terms of Y_1, and y_2 in terms of Y_2, when

$$Y_2 = Y_1 \frac{v_1 - \dfrac{T}{\mu}}{v_1}, \text{ or, what is the same thing, when } Y_2 = Y_1 - \frac{T}{\mu}\frac{Y_1}{v_1}.$$

Lens considered to be divided into three portions.

We may now consider the lens to be composed of three portions—a convexo-plane lens of thickness $A_1 \,..\, c_1$, $b_1' \,..\, c_1$ being its plane surface; a parallel plate of glass of thickness $c_1 \,..\, c_2$ $(= t)$; and another plano-convex lens of thickness $c_2 \,..\, A_2$, of which $b_2' \,..\, c_2$ is the plane surface. Thus the ray is refracted at the two sharp edges b_1' and b_2' of these two lenses. It may then be assumed that the distance t becomes an air-space of thickness $\dfrac{t}{\mu}$, so far as our present purposes are concerned.

It is clear that the vertical difference between y_1 and Y_1 is the horizontal distance $b_1 \,..\, b_1'$ multiplied by $\dfrac{Y_1}{Q_1 \,..\, A_1}$ or $\dfrac{Y_1}{u_1}$.

But $b_1 - b_1'$ is the versine of the curve of radius r_1 for the semi-chord $b_1' \,..\, c_1$. It is sufficiently accurate to suppose that

$$b_1 \,..\, b_1' = \frac{(Y_1)^2}{2r_1}.$$

Then we have

$$y_1 = Y_1 + \frac{Y_1^2}{2r_1} \times \frac{Y_1}{u_1} = Y_1 + \frac{Y_1^3}{2r_1 u_1}. \qquad (18)$$

Expression for y_1 in terms of Y_1, etc.

We next proceed to find the value of y_2 in terms of y_1. It is plain that

$$y_2 = y_1 - \frac{t}{\mu}\frac{Y_1}{v_1}; \text{ in which } \frac{t}{\mu} = \frac{1}{\mu}\left(T - \frac{Y_1^2}{2r_1} - \frac{Y_2^2}{2r_2}\right),$$

so that

$$y_2 = y_1 - \frac{1}{\mu}\left(T - \frac{Y_1^2}{2r_1} - \frac{Y_2^2}{2r_2}\right)\frac{Y_1}{v_1};$$

Expression for y_2 in terms of y_1, etc.

in which we may insert for y_1 the value given above in (18), so that

$$y_2 = Y_1 + \frac{Y_1^3}{2r_1 u_1} - \frac{1}{\mu}\left(T - \frac{Y_1^2}{2r_1} - \frac{Y_2^2}{2r_2}\right)\frac{Y_1}{v_1}$$

or

$$y_2 = Y_1 - \frac{T}{\mu}\frac{Y_1}{v_1} + \frac{Y_1^3}{2r_1 u_1} + \frac{Y_1^3}{2\mu r_1 v_1} + \frac{Y_2^2 Y_1}{2\mu r_2 v_1};$$

Expression for y_2 in terms of Y_1 and Y_2, etc.

in which, as we have already seen,

$$Y_1 - \frac{T}{\mu}\frac{Y_1}{v_1} = Y_2, \text{ so that } y_2 = Y_2 + \frac{Y_1^3}{2r_1 u_1} + \frac{Y_1^3}{2\mu r_1 v_1} + \frac{Y_2^2 Y_1}{2\mu r_2 v_1}.$$

As the last three terms are small quantities compared to Y_2 we may say that

$$y_2^2 = Y_2^2 + \frac{Y_1^3 Y_2}{r_1 u_1} + \frac{Y_1^3 Y_2}{\mu r_1 v_1} + \frac{Y_2^3 Y_1}{\mu r_2 v_1};$$

therefore

$$y_2^2 = Y_2^2\left\{1 + \left(\frac{1}{r_1 u_1} + \frac{1}{\mu r_1 v_1}\right)\frac{Y_1^3}{Y_2} + \frac{Y_1 Y_2}{\mu r_2 v_1}\right\}.$$

In this formula we can express Y_1 in terms of Y_2, so that

$$Y_1^3 = Y_2^3\left(\frac{v_1}{-u_2}\right)^3 \text{ and } Y_1 = Y_2\left(\frac{v_1}{-u_2}\right),$$

remembering that if v_1 is positive (the rays converging) relatively to the first element, then the reduced distance u_2 $\left(= -\left(v_1 - \frac{T}{\mu}\right)\right)$ is negative relatively to the second element. Therefore we get

$$y_2^2 = Y_2^2\left\{1 + \left(\frac{1}{r_1 u_1} + \frac{1}{\mu r_1 v_1}\right)Y_2^2\frac{v_1^3}{(-u_2)^3} + \frac{Y_2^2}{\mu r_2 v_1}\cdot\frac{v_1}{-u_2}\right\}$$

or

$$y_2^2 = Y_2^2\left\{1 + Y_2^2\left(-\frac{v_1^3}{r_1 u_1 u_2^3} - \frac{v_1^2}{\mu r_1 u_2^3} - \frac{1}{\mu r_2 u_2}\right)\right\}. \qquad (19)$$

Expression for y_2^2 in terms of Y_2^2, etc.

If the rays are converging into the second element, as in the diagram, then, as u_2 in this case would be negative, all the above terms would arithmetically work out positive. We saw from Formula (18) that

$$y_1 = \mathrm{Y}_1 + \frac{\mathrm{Y}_1^3}{2r_1u_1},$$

therefore

Expression for y_1^2 in terms of Y_1, etc.

$$y_1^2 = \mathrm{Y}_1^2 + \frac{\mathrm{Y}_1^4}{r_1u_1} = \mathrm{Y}_1^2\left(1 + \frac{\mathrm{Y}_1^2}{r_1u_1}\right).$$

So that, having now obtained expressions for y_1^2 and y_2^2 in terms of Y_1^2 and Y_2^2, we may state the aberration of the first element to be

Aberration of first element corrected for versine.

$$\frac{\mathrm{Y}_1^2}{8f_1^3}(\mathrm{A}_1')\left(1 + \frac{\mathrm{Y}_1^2}{r_1u_1}\right), \tag{20}$$

and the aberration of the second element to be

Aberration of second element corrected for versine.

$$\frac{\mathrm{Y}_2^2}{8f_2^3}(\mathrm{A}_2')\left\{1 + \mathrm{Y}_2^2\left(-\frac{v_1^3}{r_1u_1u_2^3} - \frac{v_1^2}{\mu r_1u_2^3} - \frac{1}{\mu r_2u_2}\right)\right\} \tag{21}$$

These formulæ, however, are open to objection in their present form. In the application of (20), for instance, to the first element of a thick positive lens in which the first surface is concave and therefore r_1 is negative, and still supposing that the entering rays are diverging into the first element, as in Fig. 35a, it is plain that y_1 will be less than Y_1, instead of greater, so that $\frac{\mathrm{Y}_1^2}{r_1u_1}$ should turn out negative if the formula is quite self-interpreting. But obviously r_1 should be entered as a negative quantity; moreover, by our conventions previously laid down, u_1 should also be entered as a negative quantity, and therefore $\frac{\mathrm{Y}_1^2}{r_1u_1}$ would remain positive, which is obviously wrong.

In order to render Formulæ (20) and (21) quite self-interpreting, we may leave u_2^2 and v_1^2 intact, while putting

$$\frac{1}{(\mu-1)f_1} \text{ for } \frac{1}{r_1},\quad \frac{1}{(\mu-1)f_2} \text{ for } \frac{1}{r_2},\quad \frac{1+a_1}{2f_1} \text{ for } \frac{1}{u_1}, \text{ etc.}$$

Then $\frac{1}{r_1u_1}$ becomes $\frac{1}{(\mu-1)f_1}\cdot\frac{1+a_1}{2f_1}$, and therefore Formula (20) becomes

Formula (20) in self-interpreting form.

$$\frac{\mathrm{Y}_1^2}{8f_1^3}(\mathrm{A}_1')\left\{1 + \frac{\mathrm{Y}_1^2}{2(\mu-1)f_1^2}(1+a_1)\right\} \tag{XXVI.}$$

Obviously if r_1 and f_1 become negative, then by convention $\frac{1}{u_1}$ becomes negative with respect to f_1, and $(1+a_1)$ is therefore negative. In like manner Formula (21) becomes

$$\left.\begin{aligned}\frac{Y_2^2}{8f_2^3}(A_2')\left\{1+Y_2^2\left(-\frac{(1+a_1)(1+a_2)}{2(\mu-1)(1-a_1)f_1f_2}\cdot\frac{v_1^2}{u_2^2}\right.\right.\\ \left.\left.-\frac{(1+a_2)}{2\mu(\mu-1)f_1f_2}\cdot\frac{v_1^2}{u_2^2}-\frac{1+a_2}{2\mu(\mu-1)f_2^2}\right)\right\}.\end{aligned}\right\}\ \text{XXVII.}$$

Formula (21) in self-interpreting form.

Since f_1 in the denominators of the first two functions in the inside brackets may be expressed as nf_2, it is evident that the corrections in the inside brackets in both Formulæ XXVI. and XXVII. are aberrations of the order $\frac{Y^4}{f^5}$ similarly to the intrinsic aberration functions of the third approximation. It is clear that these formulæ may be applied to any pair of elements constituting a thick lens.

Thus the corrections that have to be added to the first values of the aberration to the order Y^2, as ascertained from Y_1 and Y_2 in the element planes, are functions of Y^4 and of the aberration of the second approximation as expressed in Formula XXIII. Precisely the same formula will be obtained by the same course of reasoning in the case of the negative lens, Fig. 35*b*, although in the intermediate processes the signs of T and t are different.

As these corrections are consequent upon the curved surfaces retreating from the element planes, we may fitly call them the *versine corrections of the order* Y^4, in distinction from the intrinsic aberrative corrections of the order Y^4 as expressed in Formula XXIV.

Above versine corrections distinguished from intrinsic corrections of the same order.

Practical Application of the Intrinsic Aberration of the Order Y^4 to the same Lens as before

As an instance of the arithmetical application of these aberration formulæ of the order Y^4 we will take the same lens of radii 1 and 3, thickness ·75, $Y_1=·40$, and $Y_2=·30$, with entering rays parallel, for which we worked out an aberration of the order Y^2 equal to $+·0645$.

Applying the Intrinsic Aberration Formula XXIV. we get for the first element, since $x_1=+1$, and $a_1=-1$,

$$\frac{(·40)^4}{27(2)^5}\left\{\begin{aligned}-4·625+33·625-60·1875-51·94-55·55+28·19+131·06\\-5-24·7\end{aligned}\right\}$$

$$=\frac{·0256}{(27)(32)}\{-9·125\}=-\frac{·0086}{32}=-·00027,$$

Intrinsic aberration of the third order for first element.

or about $\frac{1}{90}$th of the aberration of the order Y^2, which was $+·02333$.

Above aberration transferred to second vertex.

In order to transfer this to the second element, we must, as before, multiply by $(\frac{4}{3})^2 = \frac{16}{9}$, thus getting $-\cdot 00048$.

For the second element, with $x_2 = -1$, and $a_2 = -9$, as before, we get

$$\frac{(\cdot 30)^4}{27(6)^5}\left\{\begin{matrix} -4\cdot 625 - 302\cdot 625 - 60\cdot 1875 - 4206\cdot 9 - 4499\cdot 3 - 20548\cdot 7 \\ -1179\cdot 6 - 32958\cdot 7 - 24\cdot 7 \end{matrix}\right\}$$

Intrinsic aberration of the third order for second element.

$$= \frac{\cdot 0081}{(27)(6)^5}\{-63785\} = -\cdot 00246,$$

or about $\frac{1}{10}$th of the aberration of the order Y^2, which was $+\cdot 0255$.

So that we have

	$-\cdot 00048$ for first element
and	$-\cdot 00246$ for second element.
Total . .	$-\cdot 00294$

Total of above.

for the intrinsic aberration corrections of the order Y^4.

Aberration of the third order for parallel plate not important.

To work out a formula for the aberration of the parallel glass plate also to the order Y^4 would scarcely be of any importance, for, as a rule, even the parallel plate aberrations of the order Y^2 are small compared to the aberrations of the elements.

Application of the Versine Corrections to the same Lens

We will now turn to the versine corrections of the order Y^4 for the above lens. At the first element we have

$$\frac{Y_1^2}{8f_1^3}(A_1')\left(\frac{Y_1^2}{r_1u_1}\right),$$

Versine correction for first element = 0.

which $= 0$, since $u_1 =$ infinity.

At the second element we have, as applying to this case,

$$\frac{Y_2^2}{8f_2^3}(A_2')\left(-\frac{v_1^3}{r_1u_1u_2^3} - \frac{v_1^2}{\mu r_1u_2^3} - \frac{1}{\mu r_2u_2}\right)Y_2^2,$$

in which, since $u_1 =$ infinity, the first term vanishes. In the remaining two terms $v_1 = 2$, $\mu = 1\cdot 5$, $r_1 = 1$, $r_2 = 3$, and $u_2 = -\left(v_1 - \frac{T}{\mu}\right) = -\left(2 - \frac{\cdot 75}{1\cdot 5}\right) = -1\cdot 5$, so that the formula becomes

$$(\cdot 0255)\left(-\frac{4}{(1\cdot 5)(1)(-3\cdot 375)} - \frac{1}{(1\cdot 5)(3)(-1\cdot 5)}\right)(\cdot 30)^2$$

Versine corrections for second element.

$$= \cdot 0255\left(\frac{4}{5\cdot 0625} + \frac{1}{6\cdot 75}\right)(\cdot 09) = (\cdot 0255)\left(\frac{1}{1\cdot 07}\right)(\cdot 09) = +\cdot 0021,$$

an amount which goes a long way towards neutralising the intrinsic aberration of the order Y^4, which was $-\cdot 0029$. We could here have employed Formula XXVII. for the second element with a like result.

The possibility of the intrinsic functions being neutralised completely by the versine corrections in the case of thick lenses at once suggests itself, but space does not permit of a full inquiry into the conditions under which this may take place, although it is a question of much interest.

Further Aberration Corrections of the Third Order, due to Aberrations of preceding Lenses

Our next task is to consider the nature of further aberration corrections of the order Y^4 which arise in a system of two or more lenses separated by substantial intervals.

Let Fig. 36 represent two collective lenses or elements L_1 and L_2 separated by an interval S_1, and $Q_1 \ldots C \ldots Q_2'$ be a ray refracted by L_1 at C. Let Q_2 be the point by first approximation to which the ray would be refracted by L_1 were there no aberration, but Q_2' the point to which it is actually refracted. Thus $Q_2 \ldots Q_2'$ is the longitudinal aberration. It is plain that at L_2, Y_2, or the height up to the point $D = Y_1 \frac{v_1 - S_1}{v_1}$ simply; but the height y_2 up to the point E, where the ray actually cuts the plane of L_2, is less than Y_1 by an amount that is a function of the aberration of L_1. Let

$$L_1 \ldots Q_2' = v_1', \quad L_1 \ldots Q_2 = v_1,$$

and let

$$L_2 \ldots Q_2' = u_2', \quad L_2 \ldots Q_2 = u_2.$$

Then we have

$$y_2 = Y_1 \frac{u_2'}{v_1'} = Y_1 \frac{v_1 - S_1 - \frac{Y_1^2}{8f_1^3}A_1'v_1^2}{v_1 - \frac{Y_1^2}{8f_1^3}A_1'v_1^2}$$

$$= Y_1\left(v_1 - S_1 - \frac{Y_1^2}{8f_1^3}A_1'v_1^2\right)\left(\frac{1}{v_1} + \frac{Y_1^2}{8f_1^3}A_1\right)$$

$$= Y_1\left\{\frac{v_1 - S_1}{v_1} - \frac{Y_1^2}{8f_1^3}A_1'v_1 + (v_1 - S_1)\frac{Y_1^2}{8f_1^3}A_1'\right\}$$

$$= Y_1\left\{\frac{v_1 - S_1}{v_1} - S_1\frac{Y_1^2}{8f_1^3}A_1'\right\} = Y_1\frac{v_1 - S_1}{v_1} - S_1\frac{Y_1^3}{8f_1^3}A_1 ;$$

in which $v_1 - S_1$ obviously $= -u_2$, so that

$$y_2 = Y_1\left(\frac{-u_2}{v_1}\right) - S_1\frac{Y_1^3}{8f_1^3}A'_1 = -Y_1\frac{u_2}{v_1}\left(1 + S_1\frac{v_1}{u_2}\frac{Y_1^2}{8f_1^3}A'_1\right);$$

and since the correction is generally small compared to 1, then we may assume that

Formula for y_2^2 as modified by aberration of first lens or element.

$$y_2^2 = Y_1^2\left(\frac{u_2}{v_1}\right)^2\left(1 + 2S_1\frac{v_1}{u_2}\frac{Y_1^2}{8f_1^3}A'_1\right). \qquad (22)$$

This formula is open to the objection that if L_2 were dispersive, then $\frac{v_1}{u_2}$ would be positive instead of negative, and the correction to Y_2 would come out as an increment instead of the decrement, which it so obviously is. But we can make the formula universally self-interpreting by adopting the same device as in the case of the versine corrections, thus arriving at

Above formula in self-interpreting form.

$$y_2^2 = Y_1^2\left(\frac{u_2}{v_1}\right)^2\left\{1 + \frac{Y_1^2}{4f_1^3}A'_1\left(\frac{1+a_2}{1-a_1}\right)\frac{f_1}{f_2}S_1\right\} \qquad \text{XXVIII.}$$

Now, if f_2 is dispersive, it is negative relatively to f_1, so that $\frac{f_1}{f_2}$ is negative, while $1 + a_2$ and $1 - a_1$ are both positive, therefore the correction to Y_1 comes out negative.

The spherical aberration of L_2 may now be written in the form

$$\frac{Y_2^2}{8f_2^3}(A'_2)\left\{1 + \frac{Y_1^2}{4f_1^3}A'_1\left(\frac{1+a_2}{1-a_1}\right)\frac{f_1}{f_2}S_1\right\},$$

or, if we express Y_2 in terms of Y_1, in the form

Whole expression for the aberration of L_2, including that of the third order.

$$Y_1^2\left(\frac{u_2}{v_1}\right)^2\frac{1}{8f_2^3}(A'_2)\left\{1 + \frac{Y_1^2}{4f_1^3}A'_1\left(\frac{1+a_2}{1-a_1}\right)\frac{f_1}{f_2}S_1\right\}; \qquad \text{XXVIIIA.}$$

so that the aberration of the order Y_1^4, when separated out, is

Aberration of the third order for L_2 isolated.

$$Y_1^4\left(\frac{u_2}{v_1}\right)^2\frac{1}{8f_2^3}(A'_2)\frac{1}{4f_1^3}(A'_1)\left(\frac{1+a_2}{1-a_1}\right)\frac{f_1}{f_2}S_1. \qquad (23)$$

The Y's modified by aberration of preceding lenses.

In this case we may say that the modification of Y_2 at the second lens and the consequent modification of its aberration is due to borrowed aberration. Let it now be supposed that another lens is added to the right hand of L_2 and at a distance $= s_2$ from it. Then it is evident that the aberration of L_1 will not only affect Y_2, but will generally affect Y_3 in still greater degree, since L_3 is further removed from L_1. The aberration of L_1 will be transferred right through L_2 on to L_3. Not only so, but L_2 will add (if it is a collective lens) its own aberration to the aberration of L_1 passing through it, and therefore Y_3 will be affected by the two aberrations borrowed from L_1 and L_2.

We will here refer forward to Fig. 96, Plate XX., which represents a case of four collective lenses or elements in succession, so arranged that all the u's and v's are equal and positive. The first lens *only* is supposed to give an aberration whose linear amount is $Q_1 \, . \, . \, q_1$, while the other three lenses are supposed to be free from aberration and to simply copy through from focus to focus the aberration given by L_1; yet the cumulative effect upon the successive Y's is most marked, and they grow larger and larger as we proceed from left to right.

The cumulative effect of aberration upon the succeeding Y's.

Of course, if L_2, for instance, is a dispersive lens, then the effect of its aberration on Y_3 will more or less neutralise the effect of the aberration of L_1.

The formulæ giving the modifications of the aberrations of the third and fourth lenses due to aberrations borrowed from the preceding lenses are naturally more complex and unwieldy than XXVIII., and it will suffice to give the complete expressions for the spherical aberrations of the third and fourth lenses of a series of four widely separated elements or thin lenses, without detailing their working out. The student may easily verify the formulæ for himself. We have already obtained the expression for the second lens or element in Formula XXVIIIA., and we will adhere to the highly convenient expedient of expressing all the Y's of the succeeding lenses in terms of Y_1.

All the Y's to be expressed in terms of Y_1.

Then the formula for the corrected spherical aberration of the third lens is, in self-interpreting form,

$$\left.\frac{1}{8f_1{}^3}A'_3Y_1{}^2\left(\frac{u_2u_3}{v_1v_2}\right)^2\left\{1+S_1\frac{Y_1{}^2}{4f_1{}^3}A'_1\frac{1+\alpha_2}{1-\alpha_1}\frac{f_1}{f_2}+S_2\frac{1+\alpha_3}{1-\alpha_2}\frac{f_2}{f_3}\left(\frac{Y_1{}^2}{4f_2{}^3}A'_2\frac{u_2{}^2}{v_1{}^2}+\frac{Y_1{}^2}{4f_1{}^3}A'_3\frac{v_1{}^2}{u_2{}^2}\right)\right\}\right\};\ \text{XXVIIIB.}$$

Whole expression for the aberration of L_3, including that of the third order.

and the formula for the fourth lens is

$$\left.\begin{aligned}\frac{1}{8f_4{}^3}A'_4Y_1{}^2\left(\frac{u_2u_3u_4}{v_1v_2v_3}\right)^2\Bigg[1&+S_1\frac{Y_1{}^2}{4f_1{}^3}A'_1\frac{1+\alpha_2}{1-\alpha_1}\frac{f_1}{f_2}\\&+S_2\frac{1+\alpha_3}{1-\alpha_2}\frac{f_2}{f_3}\left(\frac{Y_1{}^2}{4f_2{}^3}A'_2\frac{u_2{}^2}{v_1{}^2}+\frac{Y_1{}^2}{4f_1{}^3}A'_1\frac{v_1{}^2}{u_2{}^2}\right)\\&+S_3\frac{1+\alpha_4}{1-\alpha_3}\frac{f_3}{f_4}\left\{\frac{Y_1{}^2}{4f_3{}^3}A'_3\left(\frac{u_2u_3}{v_1v_2}\right)^2+\frac{Y_1{}^2}{4f_2{}^3}A'_2\left(\frac{u_2v_3}{v_1u_3}\right)^2\right.\\&\left.+\frac{Y_1{}^2}{4f_1{}^3}A'_1\left(\frac{v_1v_2}{u_1u_2}\right)^2\right\}\Bigg]\end{aligned}\right\}\ \text{XXVIIIC.}$$

Whole expression for the aberration of L_4, including that of the third order.

The formula for the fifth lens would evidently contain ten terms, and that for the sixth lens fifteen terms. In the case of large apertures and separations, the corrections of the order Y^4 may form a large percentage of the spherical aberrations of the order Y^2.

Vergency Variations consequent upon the Aberrations of one or more preceding Lenses

There is now a further modification of the aberration of L_2 in Fig. 36 to be considered, which, strictly speaking, applies even when L_2 is in contact with L_1, but applies with much greater force if S is large compared with v_1.

Hitherto in assessing the value of the vergency characteristic a for any lens or element, we have assumed that there is a fixed point Q from which or to which the rays are diverging or converging before entering. But in Fig. 36 it is clear that in the case of L_2 the entering rays are converging to a varying point Q_2', which recedes farther and farther from Q_2 in proportion to Y_1^2, the recession being a function of the spherical aberration of L_1.

We may regard $Q_2 \ldots Q_2'$ as a variation of either v_1 or u_2, and since in Fig. 36 u_2 is minus, we have

$$\Delta u_2 = -\left(-\frac{Y_1^2}{8f_1^3}A_1'v_1^2\right) = \frac{Y_1^2}{8f_1^3}A_1'v_1^2,$$

therefore

$$\Delta\frac{1}{u_2} = -\frac{Y_1^2}{8f_1^3}A_1'\frac{v_1^2}{u_2^2};$$

so that we have

$$\frac{1+a_2+\Delta a_2}{2f_2} = \frac{1}{u_2} - \frac{Y_1^2}{8f_1^3}A_1'\frac{v_1^2}{u_2^2},$$

in which

$$\frac{1+a_2}{2f_2} = \frac{1}{u_2},$$

therefore

$$\frac{\Delta a_2}{2f_2} = -\frac{Y_1^2}{8f_1^3}A_1'\frac{v_1^2}{u_2^2}$$

and

Vergency variation for L_2 due to aberration of L_1.

$$\Delta a_2 = -f_2\left(\frac{Y_1^2}{4f_1^3}A'_1\frac{v_1^2}{u_2^2}\right). \qquad (24)$$

In the same way we find that

Vergency variation for L_3 due to aberrations of $L_1 + L_2$.

$$\Delta a_3 = -f_3\left\{\frac{Y_1^2}{4f_1^3}A'_1\left(\frac{v_1v_2}{u_2u_3}\right)^2 + \frac{Y_1^2}{4f_2^3}A'_2\left(\frac{u_2v_2}{v_1u_3}\right)^2\right\} \qquad (25)$$

and

Vergency variation for L_4 due to aberrations of L_1, L_2, and L_3.

$$\Delta a_4 = -f_4\left\{\frac{Y_1^2}{4f_1^3}A_1\left(\frac{v_1v_2v_3}{u_2u_3u_4}\right)^2 + \frac{Y_1^2}{4f_2^3}A'_2\left(\frac{u_2v_2v_3}{v_1u_3u_4}\right)^2 + \frac{Y_1^2}{4f_3^3}A'_3\left(\frac{u_2u_3v_3}{v_1v_2u_4}\right)^2\right\} \qquad (26)$$

Now if we differentiate the formula for spherical aberration of the second lens with respect to a_2 we get

Differentiation of the spherical aberration formula with respect to a.

$$\frac{Y_2^2}{8f_2^3\mu_2(\mu_2-1)}\left\{4(\mu_2+1)x_2+2(3\mu_2+2)(\mu_2-1)a_2\right\}da_2,$$

in which we may substitute Formula (26) for da_2, and $Y_1^2\left(\frac{u_2}{v_1}\right)^2$ for Y_2^2, and then get

Complete formula for variation in a_2 consequent from aberration of L_1.

$$-f_2\frac{Y_1^4}{4f_1^3}A'_1\frac{1}{8f_2^3}\frac{1}{\mu_2(\mu_2-1)}\left\{4(\mu_2+1)x_2+2(3\mu_2+2)(\mu_2-1)a_2\right\}. \quad \text{XXIX.}$$

In this formula Y_2^2 has been expressed as $Y_1^2\left(\frac{u_2^2}{v_1^2}\right)^2$, which has cancelled out the $\frac{v_1^2}{u_2^2}$ of (24), and as f_1 can be expressed as nf_2, we see that the correction is of the order $\frac{Y_1^4}{f_2^5}$ and is the expression for the variation in the spherical aberration of L_2 consequent upon the variation in a_2 due to the aberration of L_1. In the same way the complete expressions for the functions of da_3 and da_4 can be worked out.

In these two cases of the effects of the aberration of one lens upon another we have assumed that the rays entering the first or left-hand lens are either diverging from or converging to a definite point on the axis.

But if we have to look upon these rays as principal rays, each such ray being the central ray of a pencil, then it often happens that such principal rays are constrained to pass through a definite point on the axis *after* passage through one, two, or perhaps all of the lenses of a series, owing to a diaphragm with a circular aperture being placed at the desired crossing point.

In such a case, of course, it is the more simple and convenient to regard the rays as travelling from right to left, and the formulæ expressing the corrections to the aberrations consequent on borrowed aberrations may then be worked in inverse order.

However, these considerations do not strictly apply in the present section, but only when we come to deal with the optical characteristics of lenses other than spherical aberration, and especially distortion.

Summary of the Spherical Aberrations of the Order Y^4

On summing up these spherical aberrations of the order Y_4, we have for each element or thin lens—

First, as applying to all single lenses, and in the case of all

First, the intrinsic aberration functions.

elements, the intrinsic aberration function of the order $\frac{Y^4}{f^5}$ as expressed by Formula XXIV.

Second, the versine corrections to the aberration.

Second, as applying to all single lenses, and in all cases, the versine corrections to the aberration of the order $\frac{Y^4}{f^5}$ as expressed in Formula XXVI. for the first element of a thick lens, and also of the order $\frac{Y^4}{f^5}$ as in Formula XXVII. for the second lens element. Thus in a series of lenses, Formula XXVI. applies to the first, third, fifth, seventh elements, etc., and Formula XXVII. to the second, fourth, sixth elements, etc.

Third, the corrections to the Y's due to aberrations of preceding lenses.

Third, but only where separations exist between lenses or elements, the corrections to the aberration of a lens or element due to the variation in its Y caused by borrowed aberration of the order $\frac{Y^4}{f^5}$ as expressed in Formulæ XXVIIIA., B, and C.

Fourth, the corrections to the α's due to aberrations of preceding lenses.

Fourth, but only in the case of one lens being preceded by others, and especially if widely separated, the corrections to the aberration of a lens or element due to the variation of its vergency characteristic, and caused by borrowed aberration of the order $\frac{Y^4}{f^5}$ as expressed in Formula XXIX.

Hybrid Spherical Aberrations

Let it now be supposed that in a system of lenses the above aberrations of the order Y^4 do not neutralise one another, but that there is a perceptible balance left over; then the question arises, can they be neutralised by a contrary overplus of aberration of the order Y^2? We shall soon see that they cannot.

An aberration of the second order cannot be properly neutralised by a contrary aberration of the third order.

Let it be supposed that Y represents the extreme semi-aperture of a system of lenses in which we are seeking to eradicate all the spherical aberration, and that there is a residue of minus aberration of the order Y^4. Then, of course, it is quite possible and practicable to counteract this residue by leaving in the system a residue of plus aberration of the order Y^2, so that we have

$$f_{I}Y^2 + f_{II}Y^4 = 0, \qquad (27)$$

in which f_I represents a certain coefficient of Y^2, and f_{II} represents a certain coefficient of Y^4. Then it is obvious that the relationship of these two coefficients is given by

$$f_{II} = -f_{I}\frac{1}{Y^2}. \qquad (28)$$

Let us now take another measure of the semi-aperture, smaller than Y, and call it y. Then since the coefficients and their relationship are constant, the only variable being y, then we have $f_{/}y^2+f_{//}y^4$ to express the aberration for the smaller semi-aperture y, and if we differentiate this expression with respect to y we get

$$(2f_{/}y+4f_{//}y^3)dy. \tag{29}$$

Then it is plain that we can equate this differential coefficient to 0, thus: $2f_{/}y+4f_{//}y^3=0$, in which (from 28) $f_{//}=-f_{/}\frac{1}{Y^2}$; so that we then have

$$2f_{/}y-4f_{/}\frac{y^3}{Y^2}=0 \text{ or } 1-2\frac{y^2}{Y^2}=0,$$

and

$$y^2=\frac{Y^2}{2}. \tag{30}$$

Evidently, then, at a distance from the axis such that $y=\frac{Y}{\sqrt{2}}$, there is a maximum deviation from a true balance of the two orders of aberration, and the amount of this maximum deviation may be easily determined as follows:—

The point of maximum hybrid aberration.

Since

$$y^2=\frac{Y^2}{2}, \therefore\ y^4=\frac{Y^4}{4},$$

therefore at the height $\frac{Y}{\sqrt{2}}$ from the axis the state of the aberration is given by an expression exactly analogous to (27), viz. $f_{/}y^2+f_{//}y^4$ becomes $f_{/}\frac{Y^2}{2}+f_{//}\frac{Y^4}{4}$, in which $-f_{/}\frac{1}{Y^2}$ may be substituted for $f_{//}$ (from (28), so that we then have

$$f_{/}\frac{Y^2}{2}-f_{/}\frac{Y^2}{4}, \text{ which } = +\frac{f_{/}Y^2}{4}, \tag{31}$$

Maximum hybrid aberration is one-fourth of the aberration of the second order and of the same sign.

or exactly one-fourth part of the + aberration of the order Y^2 to which the ray passing through at the extreme semi-aperture Y is subject.

This theorem is illustrated in a striking and convincing manner by the diagram, Fig. 37.

Let L..D be the optic axis of a system of lenses of semi-aperture =D..P, placed somewhere towards the left hand, and let A_2..P represent the longitudinal value of a residual amount of negative spherical aberration of the order Y^4 to which the edge ray is subject. Then let there be introduced such an amount of positive spherical aberration of the order Y^2 as will neutralise the negative aberration of the order Y^4.

That is, $A_1 .. P = P .. A_2$, and represents the longitudinal value of the positive spherical aberration of the order Y^2. Then, as these two aberrations for the edge ray are equal and opposite, the said ray will, of course, focus at P in the same plane as D, the focus for ultimate centre rays as given by formulæ of first approximation.

But if the abscissæ of the curve $D–A_1$ are made to vary, as y^2 or the square of the height from L..D of any point in the curve, and the abscissæ of the curve $D .. A_2$ are made to vary, as y^4 or the fourth power of the height from L..D, then it is easy to see that the resultant curve joining loci of actual focal points for rays traversing the system at different heights from the axis will be the curve $D..m..P$, having its maximum abscissa at m, where $y^2 = \frac{Y^2}{2}$, and that $m..b$ will be exactly a quarter of $P .. A_1$ or $P .. A_2$.

Zone of aberration explained.

Here we have the explanation of a phenomenon familiar to many opticians who have attempted optical systems of large relative aperture, and found it impossible to obtain a well-defined axial image of a point owing to the presence of what we may fitly call "a zone of aberration," which exhibits itself in the form of a bright diffuse zone or annulus within the cone of rays, which is visible through an eye-piece placed either inside of the focus or beyond it.

While the edge rays at the height Y from the axis and ultimate centre rays may be brought to the same focus, yet the rays traversing the system at a height equal to $\frac{Y}{\sqrt{2}}$ intersect the optic axis at perhaps a considerable distance either short of or beyond the focal point for axial and edge rays. The reason why, when the eye-piece is placed well within or beyond the focus, the phenomenon gives rise to a bright zone, is rendered plain by means of the diagram, Fig. 38, which accurately represents the rays coming to focus in a case where there is hybrid aberration, brought about as in Fig 37. If the eye-piece is made to focus upon a plane somewhere about $a..a$, it is evident that a condensation of rays occurs about half-way between centre and periphery of the circular penumbra or section of the cone of rays. On approaching the focus, as at position $b..b$, the condensation of rays is still more marked, but it occurs now relatively nearer to the centre, while at $b'..b'$ the zone of aberration is at its most distinct phase and has a radius of about one-fourth of the radius of the whole penumbra. The extreme edge ray focuses or cuts the optic axis at P,

Phenomena at the focus.

which is supposed to be the focal point also for the rays ultimately close to the axis, as given by the formulæ of first approximation. The whole distance $m \,..\, P$ along the axis over which the hybrid aberration spreads itself of course corresponds to the maximum distance $m \,..\, b$ in Fig. 37.

Opposite effects at the two sides of the focus.

If the eye-piece is made to focus upon planes beyond the focus in this case, then a ring of rarefaction or a comparatively dark ring will show itself, corresponding to the bright ring visible inside focus. In the plane $c \,..\, c$ the central bright nucleus is very marked.

It is clear from Fig. 37 that the bright zone of aberration will always show itself on the same side of the focus as the aberration of the order Y^2, while a corresponding dark zone will show itself on the same side of the focus as the opposing aberration of the order Y^4.

It is the existence of outstanding aberration of the third approximation or of the order Y^4, as represented by $P \,..\, A_2$ in Fig. 37, which is supposed to have necessitated our having in the system an equal and opposite aberration of the second approximation or of the order Y^2, as represented by $A_1 \,..\, P$; and we have seen that the incongruity between the two orders of aberration gives rise to a maximum amount of hybrid aberration whose amount $m \,..\, b$ is always one-fourth of the amount of the aberration $A_1 \,..\, P$ of the order y^2 to which this extreme ray is subject.

Favourable effect of dividing up powers of lenses upon a zone of aberration.

We have also seen that all the aberrations of the order Y^4 which arise in a lens or system of lenses are functions of $\frac{Y^4}{f^5}$. From this it follows that if in place of each lens of a combination we substitute two lenses, each being of half the power or double the focal length of the original, then, instead of an aberration represented by $\frac{Y^4}{f^5}$, we have an aberration represented by $2\left(\frac{Y^4}{(2f)^5}\right)$ or $\frac{1}{16}\frac{Y^4}{f^5}$.

Thus, supposing we are troubled with a zone of aberration at the focus of any given system, and it cannot be eliminated by opposing plus aberrations of the order Y^4 against minus aberrations of the same order, then we can at once reduce the zone to one-sixteenth part (as a general proposition) by the expedient of splitting up the lenses, or at any rate the most violently curved one, into two lenses each of half the power of the original.

It is also evident that the linear amount of hybrid aberration in any given case and the consequent intensity of the zone will be multiplied 16 times on doubling the aperture.

The next higher order of a zone of aberration.

It is also worth while to glance at the case of the hybrid aberration which arises when we correct a certain amount of aberration of the fourth approximation, or of the order Y^6 for the extreme ray by an equal and opposite amount of aberration of the second approximation, or of the order Y^2. Fig. 39 illustrates this case.

We then have $f_{I}Y^2 + f_{III}Y^6 = 0$, from which

$$f_{III} = -f_I\frac{Y^2}{Y^6} = -f_I\frac{1}{Y^4};$$

therefore, substituting, we have $f_I y^2 - f_I\frac{1}{Y^4}y^6$ to represent the hybrid aberration for any other height of ray $= y$.

On differentiating this we have

$$\left(2f_I y - 6y^5 f_I\frac{1}{Y^4}\right)dy = 0, \therefore \left(1 - 3y^4\frac{1}{Y^4}\right)dy = 0;$$

and on equating this expression to 0 we get

$$3\frac{y^4}{Y^4} = 1 \text{ and } y^4 = \frac{Y^4}{3}, \therefore y = \frac{Y}{\sqrt[4]{3}} = \cdot 7598Y.$$

Then it is for this height of ray y that the maximum amount of hybrid aberration occurs, and its amount will be given by

$$f_1\left(\frac{Y}{\sqrt[4]{3}}\right)^2 - f_1\frac{1}{Y^4}\left(\frac{Y}{\sqrt[4]{3}}\right)^6$$

$$= f_1Y^2\left(\frac{1}{\sqrt[4]{3}}\right)^2 - f_1Y^2\left(\frac{1}{\sqrt[4]{3}}\right)^6 = f_1Y^2\left\{\left(\frac{1}{\sqrt[4]{3}}\right)^2 - \left(\frac{1}{\sqrt[4]{3}}\right)^6\right\}$$

$$= f_1Y^2(\cdot 577 - \cdot 192) = f_1Y^2(\cdot 385).$$

Where the hybrid aberration is at its maximum.

Hence the maximum amount of the hybrid aberration occurs for a ray which traverses the system at a distance from the axis equal to about three-fourths of the extreme semi-aperture, and the amount of it is about three-eighths of the outstanding aberration of the order Y^2 to which the extreme ray is subject.

Aberration of the order Y^6 generally small compared to that of the order Y^4.

But of course the amount of aberrations of the order Y^6 will, generally speaking, be but a small fraction of the aberrations of the order Y^4. Hence we may regard the hybrid aberration curve as a combination of the curve of Fig. 37 with a much flatter curve of the character of Fig. 39. The latter will have the effect of raising an elevation or wave on the curve of Fig. 37 at about h.

An Important Corollary

One very obvious corollary from all the preceding investigation is—That if for any optical system the aberrations of the two higher

Conditions under which formulæ of the second approximation are accurate

orders Y^4 and Y^6 are eliminated or of an imperceptible and negligible amount, then our formulæ of the order Y^2, as applied to elements, etc., will be *strictly accurate.*

A suitable test case.

The best possible test case for this proposition is provided by an optical system whose curves are strictly spherical, which is known not to show any perceptible zone of aberration at the focus, and whose focal distance for the ray traversing the extreme edge of the aperture has been proved by the most rigorous possible trigonometrical calculation to be exactly equal to the focal distance for rays ultimately close to the axis, as determined by the formulæ of the first approximation.

Application of the Method of Elements to a large Telescope Object Glass

The following astronomical objective of 12-inches aperture and focal length of 176·13 inches measured from the vertex of the fourth surface serves as a capital example of the application of the formulæ for spherical aberration of the order Y^2 which we have worked out.

Radii of Curves, etc.

Specification of 12-inches aperture objective.

Collective Lens	Dispersive Lens
$r_1 = +59{\cdot}8''$ $r_2 = +90{\cdot}15''$.	$r_3 = -84{\cdot}7''$ $r_4 = -410''$.
Centre thickness = 1″.	Centre thickness = 1″.
Refractive index of the crown glass for C ray = 1·5146 $= \mu$.	Refractive index of the flint glass for the C ray = 1·6121 = M.

The focal length for parallel rays measured from the vertex of the fourth surface, as trigonometrically calculated for the C rays, is—

$$\begin{aligned} \text{for the ultimate centre rays} &= 176{\cdot}1306'' \\ \text{and for the ray 6 inches from the axis} &= 176{\cdot}1272 \\ \hline \text{Aberration undercorrected by} &\quad -{\cdot}0034'' \end{aligned}$$

We will now apply the algebraic formulæ of the second approximation to this objective, by the method of elements. We have

$$\frac{1}{f_1} = \frac{{\cdot}5146}{59{\cdot}8}, \ \therefore f_1 = 116{\cdot}2068 = v_1, \text{ from which subtract } \frac{t_1}{\mu}, \text{ which} = {\cdot}66024$$

$$\begin{aligned} &\quad {\cdot}66024 \\ \hline u_2 &= -115{\cdot}54656; \end{aligned}$$

$$\frac{1}{f_2} = \frac{\cdot 5146}{90 \cdot 15}, \therefore f_2 = 175 \cdot 1846;$$

$$\frac{1}{v_2} = \frac{1}{f_2} - \frac{1}{u_2} = \frac{1}{175 \cdot 1846} + \frac{1}{115 \cdot 54656}, \therefore v_2 = +69 \cdot 6244''.$$

The axial separation between vertices of second and third surfaces is $\cdot 013''$; and subtracting this from v_2 we get

$$u_3 = +69 \cdot 6114'', \qquad \frac{1}{f_3} = \frac{\cdot 6121}{84 \cdot 7}, \qquad \text{and } f_3 = 138 \cdot 3761;$$

$$\frac{1}{v_3} = \frac{1}{f_3} - \frac{1}{u_3} = \frac{1}{138 \cdot 3761} - \frac{1}{69 \cdot 6114}, \therefore v_3 = -140 \cdot 08,$$

the rays being convergent.

From v_3 subtract

$$\frac{t_2}{\mathrm{M}} = \cdot 6203,$$

and we get

$$u_4 = +139 \cdot 4597''.$$

$$\frac{1}{f_4} = \frac{\cdot 6121}{410}, \therefore f_4 = 669 \cdot 825,$$

then

$$\frac{1}{v_4} = \frac{1}{f_4} - \frac{1}{u_4} = \frac{1}{669 \cdot 825} - \frac{1}{139 \cdot 4597} = -\frac{1}{176 \cdot 1306}.$$

Therefore $v_4 = -176 \cdot 1306$, as stated above, and the distance is minus only with respect to the dispersive lens, since the rays are convergent. So we now have

$u_1 = \infty$	$f_1 = 116 \cdot 2068\ (+)$	$v_1 = +116 \cdot 2068$
$u_2 = -115 \cdot 54656$	$f_2 = 175 \cdot 1846\ (+)$	$v_2 = +69 \cdot 6244$
$u_3 = +69 \cdot 6114$	$f_3 = 138 \cdot 3761\ (-)$	$v_3 = -140 \cdot 08$
$u_4 = +139 \cdot 4597$	$f_4 = 669 \cdot 825\ (-)$	$v_4 = -176 \cdot 1306$

We may now assess the values of the characteristics a and x.

First element.

$$\frac{1 + a_1}{2f_1} = 0, \therefore a_1 = -1; \ x_1 = +1.$$

$$\frac{1 + a_2}{2f_2} = \frac{1}{u_2}, \therefore \frac{1 + a_2}{350 \cdot 3692} = -\frac{1}{115 \cdot 54656}, \text{ from which } 1 + a_2 = -3 \cdot 03228,$$

so that

Second element.

$$a_2 = -4 \cdot 03228; \ x_2 = -1.$$

$$\frac{1+a_3}{2f_3}=\frac{1}{u_3},\ \therefore\ \frac{1+a_3}{276\cdot7522}=\frac{1}{69\cdot6114},\ \text{from which } 1+a_3=+3\cdot97567,$$

so that

$$a_3=+2\cdot97567\ ;\ x_3=+1.$$ Third element.

$$\frac{1+a_4}{2f_4}=\frac{1}{u_4},\ \therefore\ \frac{1+a_4}{1339\cdot650}=\frac{1}{139\cdot4597},\ \text{from which } 1+a_4=+9\cdot60605,$$

so that

$$a_4=+8\cdot606\ ;\ x_4=-1.$$ Fourth element.

We have next to express the y's or heights of the ray from the axis where it cuts each element plane in terms of the corresponding y_1 at the first element plane.

We have

The y's expressed in terms of y_1.

$$y_2=y_1\frac{u_2}{v_1},\qquad \therefore\ y_2^2=y_1^2\left(\frac{u_2}{v_1}\right)^2;$$

$$y_3=y_2\frac{u_3}{v_2}=y_1\frac{u_2u_3}{v_1v_2},\qquad \therefore\ y_3^2=y_1^2\left(\frac{u_2u_3}{v_1v_2}\right)^2;$$

$$y_4=y_3\frac{u_4}{v_3}=y_1\frac{u_2u_3u_4}{v_1v_2v_3},\qquad \therefore\ y_4^2=y_1^2\left(\frac{u_2u_3u_4}{v_1v_2v_3}\right)^2.$$

Next we must transfer the spherical aberrations of all four elements to one common reference point, which is, of course, the vertex of the fourth surface or the locus of the fourth element.

Calling the aberration function

$$\frac{1}{8f^3}\,\frac{1}{\mu(\mu-1)}\left\{\frac{\mu+2}{\mu-1}x^2+4(\mu+1)ax+(3\mu+2)(\mu-1)a^2+\frac{\mu^3}{\mu-1}\right\}y^2$$

by the symbol $\frac{1}{8f_2^3}A'_1y_1^2$ for the first element, $\frac{1}{8f_1^3}A'_2y_2^2$ for the second element, etc., then the aberration of the first element transferred to the fourth will be expressed by

Aberration of first element transferred to fourth.

$$\frac{1}{8f_1^3}A'_1\left(\frac{v_1}{u_2}\right)^2\left(\frac{v_2}{u_3}\right)^2\left(\frac{v_3}{u_4}\right)^2y_1^2=\frac{1}{8f_1^3}A'_1\left(\frac{v_1v_2v_3}{u_2u_3u_4}\right)^2y_1^2.$$

The aberration of the second element transferred to the fourth is

Aberration of second element transferred to fourth.

$$\frac{1}{8f_2^3}A'_2\left(\frac{v_2}{u_3}\right)^2\left(\frac{v_3}{u_4}\right)^2y_2^2=\frac{1}{8f_2^3}A'_2\left(\frac{v_2v_3}{u_3u_4}\right)^2\left(\frac{u_2}{v_1}\right)^2y_1^2.$$

The aberration of the third element transferred to the fourth is

Aberration of third element transferred to fourth.

$$\frac{1}{8f_3^3}A'_3\left(\frac{v_3}{u_4}\right)^2y_3^2=\frac{1}{8f_3^3}A'_3\left(\frac{v_3}{u_4}\right)^2\left(\frac{u_2u_3}{v_1v_2}\right)^2y_1^2;$$

and the aberration of the fourth element is

Aberration of fourth element.

$$\frac{1}{8f_4^{\,3}}A'_4 y_4^{\,2} = \frac{1}{8f_4^{\,3}}A'_4\left(\frac{u_2u_3u_4}{v_1v_2v_3}\right)^2 y_1^{\,2}.$$

It is interesting to note in the above functions of the u's and the v's that, after we have corrected the aberrations to one reference point, and also expressed the y's in terms of y_1, we then always get a function containing $n-1$ terms in both numerator and denominator when the number of elements $= n$, and that as we pass from one element to the next the first term in the numerator disappears, and appears again as the last term in the new denominator; and the first term of the denominator disappears, and appears again as the last term of the new numerator.

The full statement of the aberration of the first element is

Aberration of first element fully stated.

$$\frac{1}{8f_1^{\,3}}\left(\frac{v_1v_2v_3}{u_2u_3u_4}\right)\frac{1}{\mu(\mu-1)}\left\{\frac{\mu+2}{\mu-1}x_1^{\,2} + 4(\mu+1)a_1x_1 + (3\mu+2)(\mu-1)a_1^{\,2} + \frac{\mu^3}{\mu-1}\right\}y_1^{\,2}$$

which

$$= \frac{1}{8}\left(+\cdot 0000057005 - \cdot 0000083952 + \cdot 00000281063 + \cdot 00000563544\right)y_1^{\,2}$$

$$= +\frac{1}{8}(\cdot 00000575137 y_1^{\,2}) \text{ altogether.}$$

The full statement of the aberration of the second element is

Aberration of second element fully stated.

$$\frac{1}{8f_2^{\,3}}\left(\frac{v_2v_3u_2}{u_3u_4v_1}\right)^2\frac{1}{\mu(\mu-1)}\left\{\frac{\mu+2}{\mu-1}x_2^{\,2} + 4(\mu+1)a_2x_2 + (3\mu+2)(\mu-1)a_2^{\,2} + \frac{\mu^3}{\mu-1}\right\}y_1^{\,2},$$

which

$$= \frac{1}{8}(+\cdot 00000162637 + \cdot 00000965812 + \cdot 0000130381 + \cdot 00000160782)y_1^{\,2}$$

or

$$+\frac{1}{8}(\cdot 0000259304 y_1^{\,2}) \text{ altogether.}$$

The full statement of the aberration of the third element is

Aberration of third element fully stated.

$$\frac{1}{8f_3^{\,3}}\left(\frac{v_3u_2u_3}{u_4v_1v_2}\right)^2\frac{1}{M(M-1)}\left\{\frac{M+2}{M-1}x_3^{\,2} + 4(M+1)a_3x_3 + (3M+2)(M-1)a_3^{\,2} + \frac{M^3}{M-1}\right\}y_1^{\,2},$$

which

$$= \frac{1}{8}(+\cdot 00000225053 + \cdot 0000118572 + \cdot 00000261036 + \cdot 0000141306)y_1^{\,2}$$

or

$$-\frac{1}{8}(\cdot 0000308487 y_1^{\,2}) \text{ altogether ;}$$

but as f_3 is minus, the element being dispersive, therefore f_3^3 gives a minus sign to above total.

The full statement of the aberration of the fourth element is

Aberration of fourth element fully stated.

$$\frac{1}{8f_4^3}\left(\frac{u_2u_3u_4}{v_1v_2v_3}\right)^2\frac{1}{\mathrm{M}(\mathrm{M}-1)}\left\{\frac{\mathrm{M}+2}{\mathrm{M}-1}x_4^2+4(\mathrm{M}+1)a_4x_4 + (3\mathrm{M}+2)(\mathrm{M}-1)a_4^2+\frac{\mathrm{M}^3}{\mathrm{M}-1}\right\}y_1^2,$$

which

$$=\frac{1}{8}(+\cdot00000001949-\cdot00000029702+\cdot00000102371+\cdot0000000226)y_1^2$$

or

$$-\frac{1}{8}(\cdot00000076881y_1^2)\text{ altogether};$$

and again, as this is a dispersive element, and f_4^3 is minus, the above is minus aberration.

Summing up, we have

Aberrations of collective and dispersive elements respectively.

for e_1	$+\cdot00000575137y_1^2$		for e_3	$-\cdot0000308487y_1^2$	
for e_2	$+\cdot0000259304y_1^2$		for e_4	$-\cdot0000007688y_1^2$	
	$\frac{1}{8}(+\cdot00003168177y_1^2)$			$\frac{1}{8}(-\cdot0000316175y_1^2)$	
	for collective lens			for dispersive lens	

So the total aberration for the four elements or two lenses is

Sum of the aberrations of collective and dispersive lenses.

$$\frac{1}{8}\left\{\begin{array}{l}+\cdot0000316818y_1^2\\-\cdot0000316175y_1^2\end{array}\right.$$
$$\frac{1}{8}(+\cdot0000000643y_1^2)$$

If now we take y at its full value of 6 inches, then $\frac{y^2}{8}=4\cdot5$, so the full correction to $\frac{1}{v_4}$ for the edge ray is $+\cdot0000002894$, and this $\times -v_4^2$ or $-(176\cdot13)^2=-\cdot00896''$, which is the longitudinal value of the spherical aberration at the focus. But there are the parallel plate corrections to be added in yet, and although in this particular case their amount is small and does not seriously affect the result, yet the case serves as an example of their application.

It is obvious that in applying the Formula XXV. to the case of the first parallel plate of thickness 1″, the a_2 for its second surface is the same as y_2, which $=y_1\frac{u_2}{v_1}$, and the v_2 of the plate is the same thing as the u_2 in the present case.

Therefore, the first parallel plate correction is, in the first place,

Aberration of first parallel plate.

$$\frac{\mu^2-1}{2\mu^3}\frac{y_2^2}{u_2^4}t_1\text{ or }\frac{\mu^2-1}{2\mu^3}y_1^2\left(\frac{u_2}{v_1}\right)^2\frac{t}{u_2^4}.$$

But this has to be referred to the fourth element. It is now a correction to $\frac{1}{u_2}$; so that we must multiply by $\left(\frac{v_2}{u_3}\right)\left(\frac{v_3}{u_4}\right)^2$, and then our formula becomes

Value of above.

$$t_1\frac{\mu^2-1}{2\mu^3}y_1^{\,2}\left(\frac{v_2v_3}{v_1u_2u_3u_4}\right)^2 = -(\cdot 00000000415)y_1^{\,2},$$

which is a correction to $\frac{1}{u_4}$ or $\frac{1}{v_4}$.

The second parallel plate correction is already a correction to $\frac{1}{u_4}$ or $\frac{1}{v_4}$, viz.—

$$\frac{M^2-1}{2M^3}\cdot\frac{y_4^{\,2}}{u_4^{\,4}}t_2, \text{ in which } y_4^{\,2} = y_1^{\,2}\left(\frac{u_2u_3u_4}{v_1v_2v_3}\right)^2$$

so that we have

Aberration of second parallel plate.

$$t_2\frac{M^2-1}{2M^3}\frac{y_1^{\,2}}{u_4^{\,2}}\left(\frac{u_2u_3}{v_1v_2v_3}\right)^2,$$

which works out to

Value of above.

$$-(\cdot 00000000049)y_1^{\,2},$$

which added to the previous amount gives

Total for two parallel plates.

$$-(\cdot 00000000464)y_1^{\,2},$$

and since

$$y_1 = 6, \text{ this } = -\cdot 000000167,$$

which must be deducted from the total we found for the spherical aberration

$$\begin{array}{r} = +\cdot 000000289 \\ -\cdot 000000167 \\ \hline +\cdot 000000122 \end{array}$$

Final total aberrations of objective.

which is the final correction to $\frac{1}{v_4}$, so that the final longitudinal error at the focus is obtained by multiplying the above final result by $-v_4^{\,2}$ or $-(176\cdot 13)^2$, giving a final spherical aberration at the focus of $-\cdot 0038$ inches, which scarcely perceptibly differs from the $-\cdot 0034$ which was arrived at by a rigorous trigonometrical calculation of the course of the same edge ray through the objective.

Aberration of the third order quite imperceptible.

It is theoretically true that for this objective there exists an aberration of the order Y^4, but it is an imperceptibly small amount of about $+\cdot 0004$ longitudinally, resulting in a zone of rays focusing $-\cdot 0001$ short of the focus for edge and centre rays. The aperture of such an objective would have to be at least 24 inches, giving a zone of aberration of $-{\cdot}0001 \times (2)^4 = -{\cdot}0016$ in., before it would become perceptible at the focus by the most refined optical tests.

The chief value of the above example is illustrative, and there is no necessity in practice for being accurate to so many decimal places or for adopting the device of elements in a case of an ordinary double objective whose aperture is only $\frac{1}{15}$th of its focal length; for were it the case that there existed at the focus a longitudinal aberration of $+$ or $-\frac{1}{10}$th of an inch, it would be possible to correct it by departing from true spherical curves, either by parabolising the figures of the surfaces or the reverse, thus bringing about a slight deviation for the rays which increases as y^2 approximately. Therefore it by no means follows that, because a given optical combination yields an axial image of a point of light which shows no trace of outstanding spherical aberration, therefore a calculation of the course of an edge ray, either algebraic or trigonometric, will also show no aberration. Hence the desirability of comparing the results of an algebraic calculation with the results of a rigid trigonometric calculation if we wish to thoroughly test the accuracy of the former.

How a small aberration may be neutralised by departing from spherical curves.

Many optical designers would prefer to employ trigonometric calculations of spherical aberration rather than any other, even in the case we have just dealt with. Indeed, it is doubtful whether in the case of some of the highly complex constructions of five or more thick lenses forming modern microscope objectives, any method can be as easily applied as the trigonometrical one, provided that not only the focus for the extreme edge rays relatively to the ultimate centre rays is calculated, but also the focus for the rays passing the aperture at a height y equal to about $\frac{3}{4}$ths of the full semi-aperture. Thus any discrepancy between the focus for the intermediate zone of rays and the joint focus for the central and edge rays would at once indicate the presence of an aberration of the order y^4 and perhaps y^6. Or, supposing the focus for the edge rays not to coincide with the focus for ultimate centre rays as calculated by the formulæ of the first approximation, then the calculated relative position of the focus for the zone of radius $\frac{3}{4}$ths would at once show any departure from the law of the aberration varying as y^2 simply, and thus reveal the presence of an aberration of the next higher order.

Trigonometrical methods often preferred for axial pencils.

It is certainly true that the trigonometrical method is very much more applicable to broad axial pencils than to any other case of refraction that can arise.

Although trigonometrical calculations of the course of a ray through an optical system are often highly desirable, yet these are merely mechanical processes which, more especially when applied to oblique and eccentric pencils, do not lend themselves at all to analysis. They are empirical and uninstructive, or at any rate barren of enlightenment unless a large number of calculations are carried out in which certain factors, such as radii or separations, are varied, and the results of such variations carefully noted. All this involves much empirical work; whereas by the aid of algebraic formulæ, although they may be not quite so exact, leading principles can be established, and the *tendencies* of the corrections consequent upon the variation of any one term can always be worked out with very little trouble, and it is by the intelligent grasp of the general tendencies that an optical construction may be varied in its parts until the utmost possible perfection is realised.

Empirical nature of trigonometrical methods.

An Example of the Practical Analytical Application of Formula XXIII.

Before dealing with the spherical reflector, we will give another useful example of the practical analytical application of Formula XXIII., or Coddington's formula for spherical aberration.

While we have seen that if we wish to arrive at a correct estimate of the total aberrations of the second approximation for thick lenses, we must treat them by the method of elements, still we must not lose sight of the fact that for analytical purposes, when planning out new combinations of lenses whose thicknesses are not great compared with their focal lengths, we may with approximate accuracy treat such lenses as wholes, and then, if we desire greater accuracy, check the aberrations by the application of the method of elements.

The designing of a cemented objective of two lenses.

For instance, we may wish to design an object glass for telescopes with the interior surfaces of the two lenses of equal but opposite radii of curvatures, so that the two lenses will touch all over, and can be cemented together by Canada balsam. Let the crown glass lens be outermost and have a refractive index $= \mu_1 = 1·5$, and the flint glass have a refractive index $= \mu_2 = 1·6$, and let the ratio of focal lengths for crown and flint be 3 : 5, so that $F_1 = +3$, and $F_2 = -5$.

Then, since the rays entering the first or crown glass lens are parallel, we have $a_1 = -1$; then u_2 for the second lens $= F_1 = 3$; and we have

$$\frac{1+a_2}{10} = +\frac{1}{3}, \therefore\ 1 + a_2 = 3\tfrac{1}{3} \text{ and } a_2 = +2\tfrac{1}{3}.$$

Now we can express x_2 for the second or negative lens in terms of

x_1; for, as the two contiguous radii of curvature have to be equal and will be of the same sign (as the lenses are of opposite sign), we have

How x_2 may be expressed in terms of x_1.

$$\frac{1+x_2}{2(\mu_2-1)5}=\frac{1-x_1}{2(\mu_1-1)3},$$

$$\therefore\ 1+x_2=(1-x_1)\frac{5}{3}\left(\frac{\cdot 6}{\cdot 5}\right)=(1-x_1)2=2-2x_1, \text{ and } x_2=1-2x_1,$$

so that the spherical aberration for the combination is

$$\frac{y^2}{8(3)^3\cdot 75}\left\{7x_1^2-10x_1+10\right\}$$
$$-\frac{y^2}{8(5)^3\cdot 96}\left\{6(1-2x_1)^2+10\cdot 4(2\tfrac{1}{3})(1-2x_1)+4\cdot 08(2\tfrac{1}{3})^2+6\cdot 83\right\},$$

which we must then equate to 0, getting

$$\frac{1}{(27)(\cdot 75)}\left\{7x_1^2-10x_1+10\right\}-\frac{1}{(125)(\cdot 96)}\left\{6(1-4x_1+4x_1^2)\right.$$
$$\left.+(10\cdot 4)(2\tfrac{1}{3})(1-2x_1)+(5\tfrac{4}{9})(4\cdot 08)+6\cdot 83\right\}=0,$$

$$\frac{1}{20\cdot 25}\left\{7x_1^2-10x_1+10\right\}-\frac{1}{120}\left\{(6-24x_1+24x_1^2)+(24\cdot 266-48\cdot 532x_1)\right.$$
$$\left.+22\cdot 21+6\cdot 83\right\}=0,$$

$$\cdot 345x_1^2-\cdot 493x_1+\cdot 493-(\cdot 05-\cdot 20x_1+\cdot 20x_1^2+\cdot 2022-\cdot 4044x_1$$
$$+\cdot 185+\cdot 057)=0,$$

$$\cdot 145x_1^2+\cdot 111x_1-\cdot 001=0,$$
$$x_1^2+\cdot 765x_1=\cdot 007,$$
$$x_1^2+\cdot 76x_1+(\cdot 38)^2=\cdot 007+\cdot 145=\cdot 152\,;$$
$$\therefore\ x_1+\cdot 38=\mp\sqrt{\cdot 152}=\mp\cdot 39,$$
$$x_1=-\cdot 38\mp\cdot 39=+\cdot 01, \text{ or } -\cdot 77.$$

Hence the crown lens, if placed outermost, must be practically equiconvex, or else have its radii in the ratio 177 to 23, or nearly 8 : 1.

Two solutions of the equation.

It can be shown that if we have the two lenses with principal focal lengths in the ratio $1:-1\cdot 875$, and the refractive indices $1\cdot 5$ and $1\cdot 62$ respectively, then in the same manner we get the equation in final form

$$x_1^2+\cdot 486x_1=-\cdot 025,$$
$$x_1^2+\cdot 486x_1+(\cdot 243)^2=-\cdot 025+\cdot 059,$$
$$x_1+\cdot 243=\mp\sqrt{\cdot 034}=\mp\cdot 18,$$

and

$$x_1=-\cdot 243\mp\cdot 18\,;$$

therefore finally

$$x_1 = -\cdot 063 \text{ or } -\cdot 423.$$

A very slight increase in the focal ratio over the above 1·875 : 1 will render the equation insoluble, the nearest approach to freedom from spherical aberration being made when $x =$ about $-\cdot 2$.

Limits of focal ratio for two lenses to be cemented.

The ratio 1·9 : 1 for the principal focal lengths with the refractive indices 1·52 and 1·62 is just about the limit, a higher ratio of focal lengths producing undercorrected spherical aberration.

Two often useful formulæ are the differentials of the spherical aberration with respect to the two characteristics a and x, which we will here give.

First, the differential with respect to a:—

Differential of $\frac{1}{8f^3}A'y^2$ with respect to a.

$$d_a\left(\frac{1}{8f^3}A'y^2\right) = \frac{y^2}{8f^3}\left\{\frac{4(\mu+1)}{\mu(\mu-1)}x + 2a\left(\frac{3\mu+2}{\mu}\right)\right\}da. \qquad \text{XXX.}$$

Second, the differential with respect to x is

Differential of $\frac{1}{8f^3}A'y^2$ with respect to x.

$$d_x\left(\frac{1}{8f^3}A'y^2\right) = \frac{y^2}{8f^3}\left\{\frac{2(\mu+2)}{\mu(\mu-1)^2}x + \frac{4(\mu+1)}{\mu(\mu-1)}\right\}dx. \qquad \text{XXXI.}$$

By means of these formulæ the effect of any contemplated change in a or x for any lens is easily ascertained; or, on the other hand, the value of dx or da required to effect a given small change in the spherical aberration is soon arrived at.

It will be as well to repeat here the formula for the least circle of confusion—that is, the smallest section or circular aperture through which the rays of a pencil subject to spherical aberration will pass. It is practically the best possible approach to a focus that the pencil is capable of, and its linear diameter is worked out by Coddington on page 12 of his work.

Thus the linear diameter of the least circle of confusion is

Linear diameter of least circle of confusion.

$$\frac{av}{2}\left(\cdot\frac{a^2}{8f^3}A'\right), \qquad \text{XXXII.}$$

and its angular diameter subtended at the lens centre is therefore

Angular diameter of least circle of confusion.

$$\frac{a}{2}\left(\frac{a^2}{8f^3}A'\right), \qquad \text{XXXIII.}$$

wherein a is the semi-aperture of the pencil at the lens, v is the second conjugate focal distance, and $\left(\frac{a^2}{8f^3}A'\right)$ represents the spherical aberration, as a correction to $\frac{1}{v}$, as usual. Thus the angular value of the least circle of confusion varies inversely as the cube of the focal length when a is constant, and as the cube of the aperture when v is constant. For simple lenses of relatively small aperture, however, the circle of confusion consequent upon the differently coloured rays being refracted to different foci far exceeds the least circle of confusion consequent upon the spherical aberration, a matter which we may have occasion to refer to again in Section X., on Achromatism.

The Aberration of a Spherical Reflector

We will conclude this Section by working out the formula for the spherical aberration for an axial pencil of rays directly reflected from a spherical reflector, either of concave or convex form. In this case we cannot do better than follow Coddington's method as explained on page 18 of his work.

Coddington's procedure followed.

Let Fig. 40 represent a divergent pencil impinging on a concave mirror, and Fig. 41 a convergent pencil impinging on a convex mirror. Let the radius r in both cases be considered intrinsically positive, in which case the distance Q . . a or u will be positive by the conventions laid down on page 7.

Let Q′ be the focal point by first approximation.

Let the circular curve a–R have its centre at O, so that $r = \mathrm{O}\,..\,a = \mathrm{O}\,..\,\mathrm{R}$.

Then it is clear that the ray Q . . R or R . . Q makes an angle QRO with the radius or perpendicular O . . R, which is equal to the angle ORq made with it by the reflected ray; therefore sin QRO = sin ORq, and we also have sin ROq = sin ROQ, so that we have the strict relationship

The fundamental equation.

$$\frac{\mathrm{O}\,..\,q}{q\,..\,\mathrm{R}} = \frac{\mathrm{O}\,..\,\mathrm{Q}}{\mathrm{Q}\,..\,\mathrm{R}}. \qquad (32)$$

About q as a centre draw through R the arc R . . b cutting the axis at b; about Q as a centre draw through R the arc R . . c cutting the axis at c, and from R drop R . . d perpendicular to the axis; and let R . . $d = y$, let $a\,..\,q_1$ the required corrected focal distance $= v'$, and let a . . Q′ the focal distance by first approximation $= v$ as usual.

Now in the above equation (32) the distance O . . q evidently $= r - (v - xv^2)$ if we denote the linear aberration Q′ . . q by xv^2; also, if the angle RQ′a is not large, we may say that $q\,..\,\mathrm{R} = \mathrm{Q}'\,..\,\mathrm{R} - xv^2$. But it will be found that the introduction of xv^2 into both the numerator and denominator of the ratio $\dfrac{\mathrm{O}\,..\,q}{q\,..\,\mathrm{R}}$ will not affect the result as regards the formula of the second order of approximation, which we are proceeding to work out, and therefore its introduction is only required if a formula of the third order, involving y^4, is wanted.

The introduction of the aberration itself into the fraction $\frac{O..q}{q..R}$ unnecessary.

This was clearly shown in the course of working out the aberration of the third order for a spherical refracting surface on page 54, wherein the introduction of the required aberration x into the more exact statement of the fundamental equation did not lead to any modification of the formula of the second approximation itself, but only to modifications of the formula of the third approximation. Since, however, the aberration of a spherical reflector is already much smaller than in the case of a lens of the same relative aperture, even in the most favourable case, it is scarcely worth while working out a formula of the third order of approximation.

Formula of the third approximation undesirable.

Therefore we may assume that

$$(O..q)=(O..a)-(a..Q')=r-v,$$

$$(q..R)=(a..Q')+(a..b)=(a..Q')+\{(b..d)-(a..d)\};$$

$$\therefore\ q..R=(a..Q')+\left(\frac{y^2}{2v}-\frac{y^2}{2r}\right)=v+\frac{y^2}{2}\left(\frac{1}{v}-\frac{1}{r}\right),\text{ and }\frac{1}{q..R}=\frac{1}{v}-\frac{y^2}{2v^2}\left(\frac{1}{v}-\frac{1}{r}\right).$$

Then we have

$$O..Q=u-r;$$

then

$$Q..R=(Q..a)-(a..c)=u-\{(a..d)-(c..d)\}$$

$$=u-\left(\frac{y^2}{2r}-\frac{y^2}{2u}\right)=u-\frac{y^2}{2}\left(\frac{1}{r}-\frac{1}{u}\right),\text{ and }\frac{1}{Q..R}=\frac{1}{u}+\frac{y^2}{2u^2}\left(\frac{1}{r}-\frac{1}{u}\right).$$

Therefore, on putting the whole equation together, we get

$$(r-v)\left\{\frac{1}{v}-\frac{y^2}{2v^2}\left(\frac{1}{v}-\frac{1}{r}\right)\right\}=(u-r)\left\{\frac{1}{u}+\frac{y^2}{2u^2}\left(\frac{1}{r}-\frac{1}{u}\right)\right\}$$

or

$$\frac{r-v}{v}\left\{1-\frac{y^2}{2v}\left(\frac{1}{v}-\frac{1}{r}\right)\right\}=\frac{u-r}{u}\left\{1+\frac{y^2}{2u}\left(\frac{1}{r}-\frac{1}{u}\right)\right\}.$$

On dividing both sides by r we then get

$$\left(\frac{1}{v}-\frac{1}{r}\right)\left\{1-\frac{y^2}{2v}\left(\frac{1}{v}-\frac{1}{r}\right)\right\}=\left(\frac{1}{r}-\frac{1}{u}\right)\left\{1+\frac{y^2}{2u}\left(\frac{1}{r}-\frac{1}{u}\right)\right\};$$

$$\therefore\left(\frac{1}{v}-\frac{1}{r}\right)-\frac{y^2}{2v}\left(\frac{1}{v}-\frac{1}{r}\right)^2=\frac{1}{r}-\frac{1}{u}+\frac{y^2}{2u}\left(\frac{1}{r}-\frac{1}{u}\right)^2.$$

Now by first approximation

$$\frac{1}{v}=\frac{1}{f}-\frac{1}{u}\text{ or }=\frac{2}{r}-\frac{1}{u},\ \therefore\ \frac{1}{v}-\frac{1}{r}=\frac{1}{r}-\frac{1}{u};$$

therefore the equation becomes

$$\frac{1}{v'}-\frac{1}{r}=\frac{1}{r}-\frac{1}{u}+\frac{y^2}{2}\left(\frac{1}{u}+\frac{1}{v}\right)\left(\frac{1}{r}-\frac{1}{u}\right)^2, \text{ in which } \frac{1}{u}+\frac{1}{v}=\frac{2}{r} \text{ or } \frac{1}{f};$$

therefore finally we get

$$\frac{1}{v'}=\frac{1}{f}-\frac{1}{u}+\frac{1}{r}\left(\frac{1}{r}-\frac{1}{u}\right)^2 y^2. \qquad \text{XXXIV. (R.)}$$

Spherical aberration of reflector.

Hence if u is infinite and the impinging rays are parallel, the aberration becomes $\frac{y^2}{r^3}$ simply or $\frac{y^2}{8f^3}$; whereas in the case of a lens of principal focus f, of glass of refractive index $=1{\cdot}5$, and of the shape to give the minimum possible aberration for parallel rays (when x would be $+\frac{5}{7}$ and a be -1), the aberration would be $\frac{y^2}{8f^3}\left(8\frac{4}{7}\right)$. So that the reflector shows to very great advantage compared to a lens of the same aperture and focal length, even when most favourably shaped.

Aberration of spherical reflector much smaller than that of corresponding lens.

It will be remembered that the Formula XVIII. that we arrived at for the aberration in the case of a single spherical surface of radius r was

$$\frac{\mu-1}{2\mu^3}\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{u}\right)y^2.$$

Now in the case of reflection it is legitimate to consider the refractive or reflective index to be -1; that is, the sine of the angle of incidence $=-1$ (the sine of the angle of reflection).

Reflection assumes the refractive index to be -1.

If, then, we put $\mu=-1$ in the above formula for the refracting surface, we then get

$$\frac{-2}{2(-1)}\left(\frac{1}{r}-\frac{1}{u}\right)^2\left(\left(\frac{1}{r}\right)+\frac{-1+1}{u}\right)y^2$$
$$=\frac{1}{r}\left(\frac{1}{r}-\frac{1}{u}\right)^2 y^2,$$

which is identical with Formula XXXIV. This analogy will be found in later Sections to apply in all corresponding cases between a reflecting and a refracting surface of the same radius, so that we have only to stipulate $\mu=-1$ in order to convert the refraction formula into the corresponding reflection formula.

SECTION V

CENTRAL OBLIQUE REFRACTION OF PENCILS THROUGH THIN LENSES OR ELEMENTS

We have now investigated the spherical aberrations to which a direct pencil of rays is subject whose central or principal ray coincides with the optic axis of the lens or lens system, and our next task is to trace out what happens to those pencils of rays which are refracted centrally but more or less obliquely through a thin lens or element—that is, in such manner that the principal ray of each pencil traverses the centre of the lens or element.

Angle of obliquity $=\phi$.

It is obvious that we here have to do with a new variable in the shape of the angle ϕ formed by the principal ray of each pencil with the optic axis. The extended images which it is sought to obtain by means of optical systems such as the telescope, microscope, and the photographic or lantern projection lens, are always flat images of plane objects. In the case of the telescope or the photographic lens when used on distant objects, the oblique pencils of rays entering them consist of practically parallel rays, which may be considered as originating from points in an infinitely distant plane. The image in the case of the telescope has to be presented to the eye in that state best adapted to simultaneously distinct vision over a considerable angular extent of field; that is, the image presented to the eye must be approximately flat. This condition of flatness of image applies with still greater force to the camera and lantern projection lens; and as often as not they have to form flat and well-defined images of strictly plane objects.

Flat images required of optical systems.

Conjugate focal planes assumed.

Therefore, throughout our investigations of oblique pencils we shall treat all such pencils of rays as diverging from points which lie in a plane normal to the optic axis, or else as converging to points in a plane normal to the optic axis, and all such planes that pass through points on the optic axis which are conjugate to one another, we will call conjugate focal planes. We will also assume the existence of planes tangent to the vertices of curvature of any lens, or, in other words, the same element planes which we assumed in the last Section, reserving the consideration of any corrections to our formulæ depending upon the versines or departure of the spherical surfaces from such element planes for Section XI.

The element planes again assumed.

We shall then find that the position of the focus or mutual crossing point for the two extreme rays of an oblique pencil, as defined by its distance from the lens centre, measured parallel to the optic axis, is essentially a matter of the spherical aberrations which take place at each surface of the lens as well as of other corrections of a somewhat different character. Let Figs. 42 and 42a represent the case of oblique refraction of a pencil through the first surface of a double convex and double concave lens whose optic axis is $P\,..\,p$.

Notation, etc., explained.

Let r' be the centre of curvature, a_1 the vertex of the surface, and $r'\,..\,a_1$ the radius of curvature, or shortly r, and let $P\,..\,Q$ be the original plane object, and Q a radiant point in it. Let the angle of obliquity $P\,..\,a_1\,..\,Q$ be called ϕ, and the angle $P\,..\,r'\,..\,Q$ be called θ.

Let $P\,..\,a = U$, $Q\,..\,d_1 = u$, and $d_1\,..\,q = \grave{u}$.

Let points e_1 and h_1 mark the limits of the aperture with which we are dealing, reckoned in the element plane. Then the two extreme rays of our pencil lying in the plane of the diagram, or in what we term *the primary plane*, will be the two rays from Q which strike the element plane at e_1 and h_1; but it is clear at the outset that besides these extreme rays in the primary plane there are also the two extreme rays to be considered which radiate from Q and strike the top and the bottom of the aperture, perpendicularly above and below the plane of the diagram, such that the perpendicular joining their points of incidence on the element plane passes through the point a_1. Now we shall always call the plane of the diagram, or the plane containing the optic axis and the oblique principal ray $Q\,..\,a_1$, the Primary Plane, and the plane perpendicular to the primary plane, but containing the oblique principal ray $Q\,..\,a_1$, the Secondary Plane. These terms correspond respectively to what German optical writers generally term the Meridional Plane and the Sagittal Plane.

Primary and secondary planes defined.

Thus our two extreme rays $Q\,..\,e_1$ and $Q\,..\,h_1$ lying in the plane of the diagram are the primary or meridional rays of the oblique pencil, while the two extreme rays in the secondary plane are the secondary or sagittal rays.

Investigation of the Focal Point for the Two Extreme Rays contained in the Secondary Plane

Rays in secondary plane dealt with first.

Now, as the focus for the two secondary rays is much more easily investigated and located than the focus for the primary rays, we will deal with the former first.

The height from the normal ray at which refraction takes place.

It is clear that the distance from a_1 to *either of* the points where the two secondary rays impinge on the element plane is equal to $a_1 \ldots e_1$ or $a_1 \ldots h_1$, that is, to the radius of the circular aperture, which we will call A. Then the distance from c_1, where the oblique normal ray $Q \ldots r'$ passing through the centre of curvature cuts the element plane, to the point where either of the two secondary rays cuts it, is obviously equal to $\sqrt[2]{\{(a_1 \ldots c_1)^2 + A^2\}}$, and this expression then gives us the value of y_1 or the height of the secondary ray, where refracted, from the normal ray $Q \ldots r'$ passing through the centre of curvature, which latter is clearly the axial ray with reference to the pencil under consideration.

Normal ray defined.

Here it may be objected that $a_1 \ldots c_1$ as measured in the element plane is incorrect, inasmuch as it should be measured perpendicular to $Q \ldots r'$. This is quite true, but it will be shown in Section XI. that the corrections which have to be added in order to make up for this and other analogous departures from strict truth are corrections of a higher order. While the formulæ which we shall arrive at in this Section are functions of $\tan^2 \phi$, the formulæ of higher orders are functions of $\tan^4 \phi$ or of $A^2 \tan^2 \phi$, and generally not nearly so important in a quantitative sense. We have, then, at the first surface,

$$y_1^2 = (a_1 \ldots c_1)^2 + A^2,$$

or, shortly,

$$y_1^2 = B_1^2 + A^2 \text{ (if we put } a_1 \ldots c_1 = B_1) \qquad (1a)$$

We may then make the dotted line $Q \ldots g_1 \ldots q'$ represent one of these secondary rays, so that $c_1 \ldots g_1$ is equal to y_1.

Turning now to the refraction at the second surface as shown in Figs. 43 and 43*a*, let q and q' be the same points as in Figs. 42 and 42*a*, q' being the point to which the rays in the secondary plane are converging after the *first* refraction. Let $q \ldots s'$ be drawn from q to the second centre of curvature s', cutting the second surface at d_2 and the element plane at c_2. Then with reference to the second surface and the emergent pencil $s' \ldots q$ is the axial or normal ray. Then our two secondary rays cutting the element plane above and below a_2 will be refracted through the surface at a height from $s' \ldots q$ equal to $\sqrt{(a_2 \ldots c_2)^2 + A^2}$; that is, PLATE. IX.

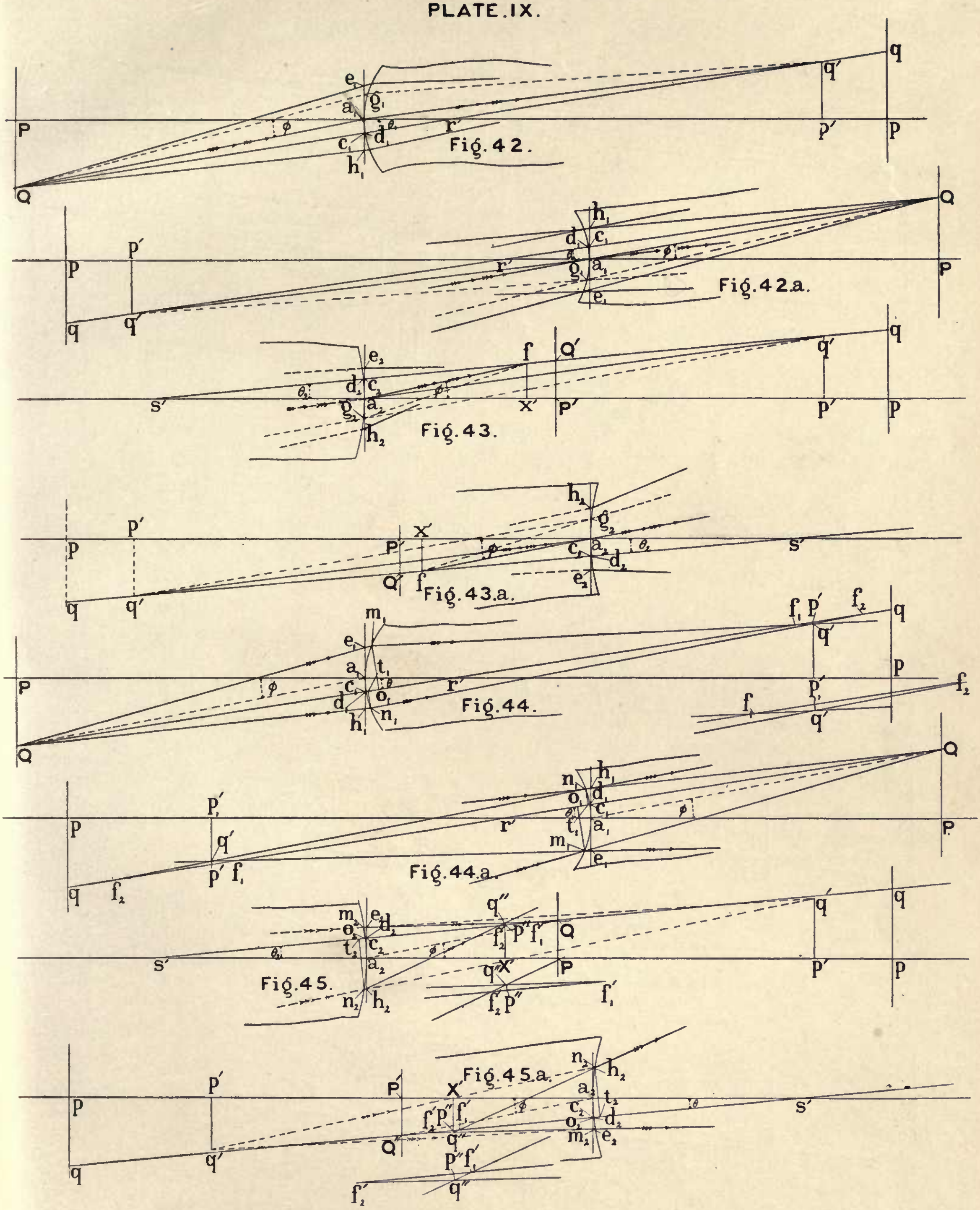


Fig. 42.

Fig. 42.a.

Fig. 43.

Fig. 43.a.

Fig. 44.

Fig. 44.a.

Fig. 45.

Fig. 45.a.

PLATE. IX.

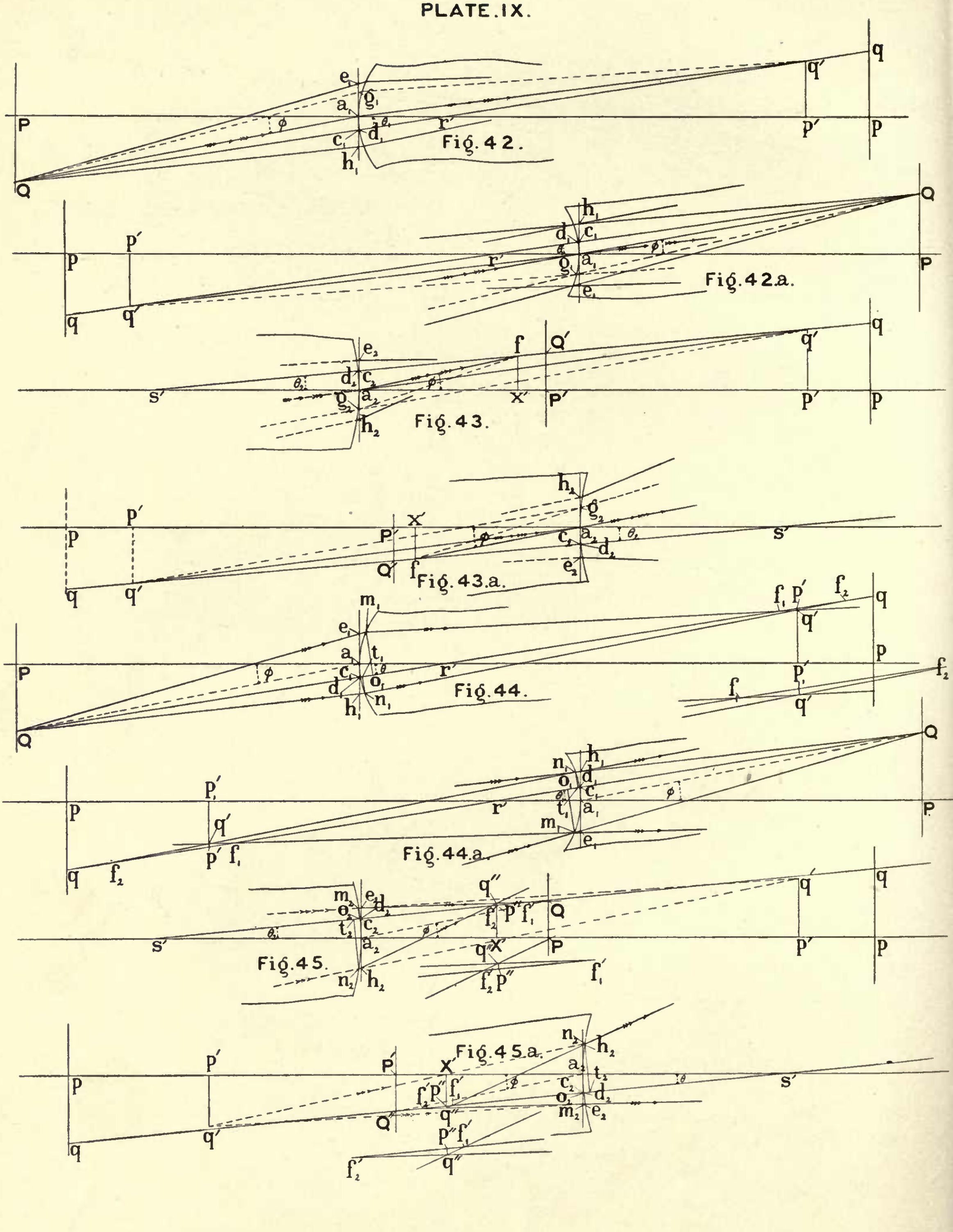


Fig. 42. Fig. 42.a. Fig. 43. Fig. 43.a. Fig. 44. Fig. 44.a. Fig. 45. Fig. 45.a.

$$y_2^2 = (a_2 \,..\, c_2)^2 + A^2,$$

or, shortly,

$$y_2^2 = B_2^2 + A_2^2. \tag{1b}$$

Secondary plane. Value of y_2^2.

Let $c_2 \,..\, g_2$ represent y_2^2, and $g_2 \,..\, f$ one of the two secondary rays. Let the radius $s \,..\, a_2 = s$, and the second conjugate focal distance $a_2 \,..\, \mathrm{P}'$ as measured along the axis be V, and let $d_2 \,..\, f$ be v and $d_2 \,..\, q'$ be $\grave{v}$.

We may then state the values of y_1^2 and y_2^2 as follows:—

$$y_1^2 = B_1^2 + A^2 = \left(\mathrm{U} \tan\phi \frac{r}{\mathrm{U}+r}\right)^2 + A^2 ; \tag{2}$$

Detailed value of y_1^2.

$$y_2^2 = B_2^2 + A^2 = \left(\mathrm{V} \tan\phi \frac{s}{\mathrm{V}+s}\right)^2 + A^2. \tag{3}$$

Detailed value of y_2^2.

Also

$$\mathrm{Q} \,..\, d_1 \text{ or } u = \mathrm{U} + (\mathrm{U}\tan\phi)^2 \frac{1}{2(\mathrm{U}+r)} \text{ approx.} ; \qquad (4)\ \mathrm{L}.$$

Value of u in terms of U and r.

$$\therefore \quad \frac{1}{u} = \frac{1}{\mathrm{U}} - \frac{1}{\mathrm{U}^2}(\mathrm{U}\tan\phi)^2 \frac{1}{2(\mathrm{U}+r)} ;$$

$$\therefore \quad \frac{1}{u} = \frac{1}{\mathrm{U}} - \tan^2\phi \frac{1}{2(\mathrm{U}+r)} . \qquad (5)\ \mathrm{R}.$$

Value of $\frac{1}{u}$ in terms of U and r.

Neglecting aberration $\frac{\mu}{\grave{u}} = \frac{\mu-1}{r} - \frac{1}{u}$, and substituting from (5), we get

$$\frac{\mu}{\grave{u}} = \frac{\mu-1}{r} - \frac{1}{\mathrm{U}} + \tan^2\phi \frac{1}{2(\mathrm{U}+r)}. \qquad (6)\ \mathrm{R}.$$

Value of $\frac{\mu}{\grave{u}}$ in terms of U and r without aberration.

Next, as a basis for converting $\grave{u}$ $(= d_1 \,..\, q')$ for the first surface into $\grave{v}$ $(= c_2 \,..\, q')$ for the second surface we have the equation, putting t for the axial thickness,

$$\grave{u} - \frac{(q' \,..\, p')^2}{2(\grave{u}-r)} - t = (p \,..\, a_2) = \grave{v} - \frac{(q' \,..\, p')^2}{2(\grave{v}+s)},$$

Equation connecting $\grave{u}$ and $\grave{v}$.

in which we have supposed a thickness t to exist, which afterwards eliminates itself so far as our purposes are concerned. Therefore

$$\grave{v} = \grave{u} - t - (q' \,..\, p')^2 \left\{ \frac{1}{2(\grave{u}-r)} - \frac{1}{2(\grave{v}+s)} \right\},$$

wherein

$$(q' \,..\, p')^2 = (\mathrm{P} \,..\, \mathrm{Q})^2 \left(\frac{\grave{u}-r}{\mathrm{U}+r}\right)^2 = \left(\mathrm{U} \tan\phi \frac{\grave{u}-r}{\mathrm{U}+r}\right)^2 ;$$

therefore

$$\frac{1}{\grave{v}} = \frac{1}{\grave{u}} + \frac{t}{\grave{u}^2} + \frac{1}{\grave{u}^2}\left\{\mathrm{U} \tan\phi \frac{\grave{u}-r}{\mathrm{U}+r}\right\}^2 \left\{\frac{1}{2(\grave{u}-r)} - \frac{1}{2(\grave{v}+s)}\right\}$$

and
$$\frac{\mu}{\grave{v}} = \frac{\mu}{\grave{u}} + t\frac{\mu}{\grave{u}^2} + \frac{\mu}{\grave{u}^2}\left\{\mathrm{U}\tan\phi\,\frac{\grave{u}-r}{\mathrm{U}+r}\right\}^2\left\{\frac{1}{(\grave{u}-r)} - \frac{1}{(\grave{v}+s)}\right\}\frac{1}{2}.$$

Substituting in above the value of $\frac{\mu}{\grave{u}}$ from (6) we get

Value of $\frac{\mu}{\grave{v}}$ excluding the aberration.

$$\frac{\mu}{\grave{v}} = \frac{\mu-1}{r} - \frac{1}{\mathrm{U}} + \tan^2\phi\,\frac{1}{2(\mathrm{U}+r)} + t\frac{\mu}{\grave{u}^2} + \frac{\mu}{\grave{u}^2}\left\{\mathrm{U}\tan\phi\,\frac{\grave{u}-r}{\mathrm{U}+r}\right\}^2\left\{\frac{1}{\grave{u}-r} - \frac{1}{\grave{v}+s}\right\}\frac{1}{2}.$$

Now to above we must add the spherical aberration due to the first surface, taking y_1^2 from (2), so that we then get the complete value of $\frac{\mu}{\grave{v}}$ as follows :—

Value of $\frac{\mu}{\grave{v}}$ including the aberration of first surface.

$$\left.\begin{aligned}\frac{\mu}{\grave{v}} = \frac{\mu-1}{r} - \frac{1}{\mathrm{U}} + \tan^2\phi\,\frac{1}{2(\mathrm{U}+r)} + t\frac{\mu}{\grave{u}^2} + \frac{\mu}{\grave{u}^2}\left\{\mathrm{U}\tan\phi\,\frac{\grave{u}-r}{\mathrm{U}+r}\right\}^2\left\{\frac{1}{\grave{u}-r}\right.\\ \left. - \frac{1}{\grave{v}+s}\right\}\frac{1}{2} + \frac{\mu-1}{2\mu^2}\left\{\frac{1}{r}+\frac{1}{\mathrm{U}}\right\}^2\left\{\frac{1}{r}+\frac{\mu+1}{\mathrm{U}}\right\}\overbrace{\left\{\left(\mathrm{U}\tan\phi\,\frac{r}{\mathrm{U}+r}\right)^2 + A^2\right\}}^{=y_1^2}\end{aligned}\right\} \text{(7) R.}$$

Turning now to the refraction at the second surface we have $\grave{v}$ negative as the rays are converging; therefore, including its spherical aberration, we have

Value of $\frac{1}{v}$ including aberration of second surface.

$$\frac{1}{v} = \frac{\mu-1}{s} - \underset{\text{(7) R.}}{\underset{\|}{\left(\frac{\mu}{-\grave{v}}\right)}} + \frac{\mu-1}{2\mu^2}\left\{\frac{1}{s}+\frac{1}{\mathrm{V}}\right\}^2\left\{\frac{1}{s}+\frac{\mu+1}{\mathrm{V}}\right\}\left\{\left(\mathrm{V}\tan\phi\,\frac{s}{\mathrm{V}+s}\right)^2 + A^2\right\} \text{(8) R.}$$

Length v to be reduced to the axis.

Then, after having got the value of v $(=d_2\,.\,.f)$, we have to reduce that distance to the axis. Drop the perpendicular $f\,.\,.x'$ to the lens axis, then evidently

$$x'\,.\,.a_2 = \overbrace{d_2\,.\,.f}^{\overset{v}{\|}} - \frac{(x\tan\phi)^2}{2(x+s)},\ \text{wherein } x = \text{corrected distance } x'\,.\,.a_2;$$

$$\therefore \frac{1}{x} = \frac{1}{v} + \frac{1}{v^2}\,\frac{(x\tan\phi)^2}{2(x+s)},$$

in which small correction we can put V for x, and say

Reciprocal value $\frac{1}{x}$ when x is measured parallel to axis.

$$\frac{1}{x} = \frac{1}{v} + \frac{1}{\mathrm{V}^2}\,\frac{\mathrm{V}^2\tan^2\phi}{2(\mathrm{V}+s)} \quad\text{or}\quad \frac{1}{x} = \frac{1}{v} + \tan^2\phi\,\frac{1}{2(\mathrm{V}+s)}, \qquad \text{(9) R.}$$

which last expression is symmetrical to the other end correction in Formula (6). After adding (9) to Formula (8), while substituting Formula (7) for $\frac{\mu}{\grave{v}}$ therein, we then get the complete formula—

$$
\left.\begin{aligned}
\frac{1}{x} &= (\mu-1)\left(\frac{1}{r}+\frac{1}{s}\right)-\frac{1}{U}+t\frac{\mu}{\grave{u}^2}+\tan^2\phi\,\frac{1}{2(U+r)}\\
&\qquad = \text{first end correction},\\
&+\frac{\mu-1}{2\mu^2}\left\{\frac{1}{r}+\frac{1}{U}\right\}^2\left\{\frac{1}{r}+\frac{\mu+1}{U}\right\}\underbrace{\left\{\overbrace{\left(U\tan\phi\,\frac{r}{U+r}\right)^2}^{=B_1^2}+A^2\right\}}_{=y_1^2}\\
&\qquad = \text{first surface spherical aberration},\\
&+\frac{\mu}{\grave{u}^2}\left\{U\tan\phi\,\frac{\grave{u}-r}{U+r}\right\}^2\left\{\frac{1}{\grave{u}-r}-\frac{1}{\grave{v}+s}\right\}\frac{1}{2}\\
&\qquad = \text{correction for converting } \grave{u} \text{ into } \grave{v},\\
&+\frac{\mu-1}{2\mu^2}\left\{\frac{1}{s}+\frac{1}{V}\right\}^2\left\{\frac{1}{s}+\frac{\mu+1}{V}\right\}\underbrace{\left\{\overbrace{\left(V\tan\phi\,\frac{s}{V+s}\right)^2}^{=B_2^2}+A^2\right\}}_{=y_2^2}\\
&\qquad = \text{second surface spherical aberration},\\
&+\tan^2\phi\,\frac{1}{2(V+s)}\\
&\qquad = \text{second end correction}.
\end{aligned}\right\}(10)\,\text{R.}
$$

Complete formula for $\frac{1}{x}$.

As in general the middle correction of the above is small relatively to $\frac{1}{U}$, $\frac{1}{V}$, $\frac{1}{\grave{u}}$, and $\frac{1}{\grave{v}}$, we may again insert approximate values of $\grave{u}$ and $\grave{v}$; and since

Approximate values to be inserted in the corrections.

$$\frac{\mu}{\grave{u}} = \frac{\mu-1}{r} - \frac{1}{U} \text{ by first approximation,}$$

$$\frac{\mu}{\grave{u}^2} \text{ reduces to } \frac{1}{\mu}\left(\frac{U(\mu-1)-r}{rU}\right)^2,$$

$$\grave{u}-r \text{ reduces to } \frac{r(U+r)}{U(\mu-1)-r}, \text{ and } \frac{1}{\grave{u}-r} \text{ to } \frac{U(\mu-1)-r}{r(U+r)};$$

also since

$$-\frac{\mu}{-\grave{v}} = \frac{1}{V} - \frac{\mu-1}{s} \text{ or } \frac{\mu}{\grave{v}} = \frac{1}{V} - \frac{\mu-1}{s}, \therefore \frac{1}{\grave{v}+s} \text{ reduces to } \frac{s-V(\mu-1)}{s(V+s)}.$$

After separating out from Formula (10) the products of the two spherical aberrations into A^2 and also substituting the above values of $\frac{\mu}{\grave{u}^2}$, $\grave{u}-r$, $\frac{1}{\grave{u}-r}$, and $\frac{1}{\grave{v}+s}$, we then get

$$\frac{1}{x} = \frac{1}{F} - \frac{1}{U} + t\frac{\mu}{\grave{u}^2} + \underbrace{\frac{\mu-1}{2\mu^2}\left\{\left(\frac{1}{r}+\frac{1}{U}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{U}\right)+\left(\frac{1}{s}+\frac{1}{V}\right)^2\left(\frac{1}{s}+\frac{\mu+1}{V}\right)\right\}A^2}$$

Includes the aberration of all pencils of semi-aperture A.

$$= \text{spherical aberration of all pencils of semi-aperature } A,$$

Aberration of first surface.

Corrections converting $\grave{u}$ into $\grave{v}$.

Aberration of second surface.

The two end corrections.

$$\left.\begin{aligned}
&\underbrace{+\frac{\mu-1}{2\mu^2}\left(\frac{1}{r}+\frac{1}{\mathrm{U}}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{\mathrm{U}}\right)\left(\tan\phi\frac{\mathrm{U}r}{\mathrm{U}+r}\right)^2}_{=B_1^2} && (11)\ \mathrm{R.}\\
&+\frac{1}{\mu}\left(\frac{\mathrm{U}(\mu-1)-r}{r\mathrm{U}}\right)^2\left\{\mathrm{U}\tan\phi\frac{r(\mathrm{U}+r)}{\mathrm{U}(\mu-1)-r}\cdot\frac{1}{\mathrm{U}+r}\right\}^2\left\{\frac{\mathrm{U}(\mu-1)-r}{r(\mathrm{U}+r)}+\frac{\mathrm{V}(\mu-1)-s}{s(\mathrm{V}+s)}\right\}\frac{1}{2} && (12)\ \mathrm{R.}\\
&\underbrace{+\frac{\mu-1}{2\mu^2}\left(\frac{1}{s}+\frac{1}{\mathrm{V}}\right)^2\left(\frac{1}{s}+\frac{\mu+1}{\mathrm{V}}\right)\left(\tan\phi\frac{\mathrm{V}s}{\mathrm{V}+s}\right)^2}_{=B_2^2} && (13)\ \mathrm{R.}\\
&\underbrace{+\tan\phi\left(\frac{1}{\mathrm{U}+r}+\frac{1}{\mathrm{V}+s}\right)\frac{1}{2}}_{=\text{the two end corrections from (10).}} && (14)\ \mathrm{R.}
\end{aligned}\right\}\ \mathrm{I.\ (R.)}$$

The expressions (11), (12), (13), and (14) together constitute what we will call the normal curvature errors, as corrections to the reciprocal of the conjugate focal distance of the axial pencil of rays of semi-aperture A.

Complex as these expressions are, they nevertheless simplify down, without any further compromise, to the simple expression

$$\frac{\tan^2\phi}{2\mathrm{F}}\cdot\frac{\mu+1}{\mu},$$

so that our complete formula becomes

Includes the aberration common to all pencils of semi-aperture $\grave{A}$.

$$\frac{1}{x}=\frac{1}{\mathrm{F}}-\frac{1}{\mathrm{U}}+t\frac{\mu}{\grave{u}^2}+\underbrace{\frac{\mu-1}{2\mu^2}\left\{\left(\frac{1}{r}+\frac{1}{\mathrm{U}}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{\mathrm{U}}\right)+\left(\frac{1}{s}+\frac{1}{\mathrm{V}}\right)^2\left(\frac{1}{s}+\frac{\mu+1}{\mathrm{V}}\right)\right\}A^2}_{\mathrm{II.\ (R.)}}$$

The normal curvature error in secondary plane.

$$+\frac{\tan^2\phi}{2\mathrm{F}}\cdot\frac{\mu+1}{\mu}. \qquad \mathrm{III.\ (R.)}$$

The reader is strongly recommended to verify these reductions for himself.

Thus in III. we arrive at the same result as did Coddington by a considerably different method, in which he neglected the spherical aberration of the pencil, as expressed in II.

The Rays in the Primary Plane

We will now trace through the lens the two rays which are refracted at the extreme ends of that diameter of the lens lying in the plane of the paper—in other words, symmetrical pairs of rays in the primary plane.

Let the two rays Q . . e_1 and Q . . h_1 impinge upon the element plane at e_1 and h_1 at equal perpendicular distances $= A$ (the semi-aperture) from the lens axis P . . r' (Figs. 44 and 44a).

Then, correctly, the distances or y's of these two rays from the normal ray Q . . r' . . q are respectively m_1 . . o_1 and n_1 . . t_1; but for our present purposes we will assume y_1 to be e_1 . . c_1 in the element plane, and y_2 to be h_1 . . c_1, also in the element plane. Then, approximately, if a_1 . . $c_1 = B_1$ as before,

The two y's to be estimated in the element plane.

$$y_1^2 = \left(A + r \tan \phi \frac{\mathrm{U}}{\mathrm{U} + r}\right)^2 = (A + B_1)^2,$$

Expression for y_1^2.

$$y_2^2 = \left(A - r \tan \phi \frac{\mathrm{U}}{\mathrm{U} + r}\right)^2 = (A - B_1)^2.$$

Expression for y_2^2.

It is evident that the ray Q . . e_1 meets with more spherical aberration than the ray Q . . h_1, so that while the former is refracted to f_1 the latter is refracted to f_2 on the normal or axial ray Q . . r' . . f_2, and therefore the point q' where they intersect will be slightly to one side of the oblique axial ray Q . . r' . . f_2.

Let x_1 denote the required distance d_1 . . q'. Let f_1 denote the distance d_1 . . f_1, and let f_2 denote the distance d_1 . . f_2.

Draw q' . . p' perpendicular to the oblique axis Q . . r' . . f_2. Then we have the equation

$$y_1 \frac{x_1 - f_1}{f_1} = (q' . . p') = y_2 \frac{f_2 - x_2}{f_2}$$

The fundamental equation.

or

$$y_1(x_1 - f_1)f_2 = y_2(f_2 - x_1)f_1,$$

from which

$$\frac{1}{x_1} = \left(\frac{y_1}{f_1} + \frac{y_2}{f_2}\right)\left(\frac{1}{y_1 + y_2}\right).$$

But f_1 and f_2 involve y_1^2 and y_2^2 respectively, since they are affected by the spherical aberration.

Now that part of the expressions for $\frac{\mu}{f_1}$ and $\frac{\mu}{f_2}$ which is common to both of them is the term $\frac{\mu}{u}$, and denoting the spherical aberration by the term $\omega_1 y^2$, we then have

$$\frac{\mu}{f_1} = \frac{\mu}{u} + \omega_1 y_1^2 \text{ and } \frac{\mu}{f_2} = \frac{\mu}{u} + \omega_1 y_2^2,$$

$$\frac{1}{f_1} = \frac{1}{u} + \frac{\omega_1}{\mu} y_2^2 \text{ and } \frac{1}{f_2} = \frac{1}{u} + \frac{\omega_1}{\mu} y_2^2;$$

then
$$\frac{1}{x_1} = \left(\frac{y_1}{f_1} + \frac{y_2}{f_2}\right)\left(\frac{1}{y_1 + y_2}\right)$$
becomes
$$\frac{1}{x_1} = \left\{y_1\left(\frac{1}{\grave{u}} + \frac{\omega_1}{\mu}y_1^2\right) + y_2\left(\frac{1}{\grave{u}} + \frac{\omega_1}{\mu}y_2\right)^2\right\}\frac{1}{y_1 + y_2} = \frac{\frac{1}{\grave{u}}(y_1 + y_2) + \frac{\omega_1}{\mu}(y_1^3 + y_2^3)}{y_1 + y_2};$$
$$\therefore \quad \frac{1}{x_1} = \frac{1}{\grave{u}} + (y_1^2 + y_2^2 - y_1y_2)\frac{\omega_1}{\mu},$$
and

$\frac{\mu}{x_1}$ corrected for the compounded aberration of first surface.

$$\therefore \quad \frac{\mu}{x_1} = \frac{\mu}{\grave{u}} + (y_1^2 + y_2^2 - y_1 y_2)\omega_1, \tag{15) R.$$

$$\therefore y_1^2 + y_2^2 - y_1y_2 = \left\{\begin{array}{ll} \overbrace{A^2 + 2Ar\tan\phi\frac{u}{\mathrm{U}+r}}^{=A^2+2AB_1} & + \overbrace{\left(r\tan\phi\frac{\mathrm{U}}{\mathrm{U}+r}\right)^2}^{=+B_1^2} \\ + A^2 - 2Ar\tan\phi\frac{\mathrm{U}}{\mathrm{U}+r} & + \left(r\tan\phi\frac{\mathrm{U}}{\mathrm{U}+r}\right)^2 \\ - A^2 \quad \ldots\ldots & + \left(r\tan\phi\frac{\mathrm{U}}{\mathrm{U}+r}\right)^2, \end{array}\right.$$
which in skeleton form is equivalent to
$$y_1^2 + y_2^2 - y_1y_2 = \left\{\begin{array}{l} +(A^2 + 2AB + B^2) \\ +(A^2 - 2AB + B^2) \\ -(A_2 \qquad\quad - B^2) \end{array}\right\} = A^2 + 3B^2,$$
and

Value of the function of y_1 and y_2.

$$y_1^2 + y_2^2 - y_1y_2 = A^2 + 3\tan^2\phi\left(\frac{r\mathrm{U}}{\mathrm{U}+r}\right)^2;$$
therefore in full

Value of the compounded aberration of first surface.

$$\left.\begin{array}{l} \frac{\mu}{x_1} = \underbrace{\frac{\mu-1}{r} - \frac{1}{\mathrm{U}} + \tan^2\phi\frac{1}{2(\mathrm{U}+r)}}_{=\mathrm{C}} \\ \qquad + \underbrace{\frac{\mu-1}{2\mu^2}\left(\frac{1}{r}+\frac{1}{\mathrm{U}}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{\mathrm{U}}\right)}_{=\omega_1}\underbrace{\left\{A^2 + 3\tan^2\phi\left(\frac{r\mathrm{U}}{\mathrm{U}+r}\right)^2\right\}}_{=(y_1^2+y_2^2-y_1y_2)} \end{array}\right\} \tag{16) R.$$

Turning now to the refraction of the same two rays at the second surface, Figs. 45 and 45*a*, we have the upper ray Q . . e_1 . . q after both refractions cutting the normal or axial ray s' . . q' at the point f_1', while the lower ray Q . . h_1 . . q meets with more spherical aberration and cuts the oblique axis s' . . q' at f_2'.

Therefore the two rays intersect or come to a focus at q'' a little to one side of $s'..q_1'$. From q'' draw $q''..p''$ perpendicular to $s'..q_1'$. Let $\frac{1}{f_1'} = \frac{1}{d_2..f_1'}$ and $\frac{1}{f_2'} = \frac{1}{d_2..f_2'}$ and $\frac{1}{x_2} = \frac{1}{d_2..q_2}$. Then as in the previous case, supposing $e_2..c_2 = Y_1$ and $c_2..h_2 = Y_2$, we have

$$Y_1 \frac{f_1' - x_2}{f_1'} = q''..p'' = Y_2 \frac{x_2 - f_2'}{f_2'} \text{ and } \frac{1}{x_2} = \left(\frac{Y_1}{f_1'} + \frac{Y_2}{f_2'}\right)\frac{1}{Y_1 + Y_2}$$

and, as before,

$$\frac{1}{x_2} = \frac{1}{v} + (Y_1^2 + Y_2^2 - Y_1 Y_2)\omega_2$$

$\frac{1}{x_2}$ corrected for compounded aberration of second surface.

and

$$Y_1^2 + Y_2^2 - Y_1Y_2 = \left\{ \begin{array}{l} \overbrace{A^2 - 2As\tan\phi\frac{V}{V+s}}^{=A^2 - 2AB_2} + \overbrace{\tan^2\phi\left(\frac{sV}{V+s}\right)^2}^{=+B_2^2} \\ + A^2 + 2As\tan\phi\frac{V}{V+s} + \tan^2\phi\left(\frac{sV}{V+s}\right)^2 \\ - A^2 \qquad\qquad\qquad\quad + \tan^2\phi\left(\frac{sV}{V+s}\right)^2; \end{array}\right.$$

$$\therefore Y_1^2 + Y_2^2 - Y_1Y_2 = A^2 + 3\tan^2\phi\left(\frac{sV}{V+s}\right)^2;$$

$$\therefore \frac{1}{x_2} = \frac{1}{v} + \omega_2\left\{A^2 + 3\tan^2\phi\left(\frac{sV}{V+s}\right)^2\right\},$$

or, more fully,

$$\frac{1}{x} = \frac{\mu - 1}{s} - \frac{\mu}{-v} + \frac{\mu - 1}{2\mu^2}\left(\frac{1}{s} + \frac{1}{V}\right)^2\left(\frac{1}{s} + \frac{\mu - 1}{V}\right)\left\{A^2 + 3\tan^2\phi\left(\frac{sV}{V+s}\right)^2\right\}. \quad (17)\ R.$$

Value of the compounded aberration of second surface.

Drop $q''..X$ perpendicular from q'' to the axis $s..q$, then, as in the previous case, $a_2..X$ or $X = v - \frac{(q''..X)^2}{2(X+s)}$, and, approximately,

$$\frac{1}{X} = \frac{1}{v} + \frac{1}{V^2}\left(\frac{V^2\tan^2\phi}{2(V+s)}\right) = \frac{1}{v} + \tan^2\phi\frac{1}{2(V+s)}.$$

On summing up all corrections in their order we then get

$$\frac{1}{X} = \underbrace{\frac{1}{F} - \frac{1}{U} + t\frac{\mu}{u^2} \qquad + \tan^2\phi\frac{1}{2(U+r)}}_{=\text{ first end correction,}}$$

Involves $\frac{1}{u}$ expressed in terms of U and r.

$$+ \underbrace{\frac{\mu - 1}{2\mu^2}\left(\frac{1}{r} + \frac{1}{U}\right)^2\left(\frac{1}{r} + \frac{\mu + 1}{U}\right)\left\{A^2 + \overbrace{3\tan^2\phi\left(\frac{rU}{U+r}\right)^2}^{=A^2 + 3B_1^2}\right\}}_{=\text{ spherical aberration of first surface,}}$$

Compounded spherical aberration of first surface.

Corrections converting $\grave{u}$ into $\grave{v}$.

$$+\frac{\mu}{\grave{u}^2}\left(\mathrm{U}\tan\phi\frac{\grave{u}-r}{\mathrm{U}+r}\right)^2\left(\frac{1}{\grave{u}-r}-\frac{1}{\grave{v}+s}\right)\frac{1}{2}\ \text{(conversion of } \grave{u} \text{ into } \grave{v})$$

Compounded spherical aberration of second surface.

$$+\frac{\mu-1}{2\mu^2}\left(\frac{1}{s}+\frac{1}{\mathrm{V}}\right)^2\left(\frac{1}{s}+\frac{\mu+1}{\mathrm{V}}\right)\overbrace{\left\{A^2+3\tan^2\phi\left(\frac{\mathrm{V}s}{\mathrm{V}+s}\right)^2\right\}}^{=A^2+3B_2{}^2}$$

= spherical aberration of second surface,

Correction converting $\frac{1}{v}$ into $\frac{1}{\mathrm{V}}$.

$$+\tan^2\phi\frac{1}{2(\mathrm{V}+s)}$$

= second end correction.

Then after selecting out the product of A^2 into the sum of the two aberrations and substituting approximate values of $\frac{\mu}{\grave{u}^2}$, $\grave{u}-r$, $\frac{1}{\grave{u}-r}$, and $\frac{1}{\grave{v}-s}$, as we did in the case of the analogous formulæ for rays in the secondary plane, we then get

Includes the aberration of all pencils of semi-aperture A.

$$\frac{1}{\mathrm{X}}=\frac{1}{\mathrm{F}}-\frac{1}{\mathrm{U}}+t\frac{\mu}{\grave{u}^2}+\frac{\mu+1}{2\mu^2}\left\{\left(\frac{1}{r}+\frac{1}{\mathrm{U}}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{\mathrm{U}}\right)+\left(\frac{1}{s}+\frac{1}{\mathrm{V}}\right)^2\left(\frac{1}{s}+\frac{\mu+1}{\mathrm{V}}\right)\right\}A^2$$

Aberration of first surface.

$$+\frac{\mu-1}{2\mu^2}\left(\frac{1}{r}+\frac{1}{\mathrm{U}}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{\mathrm{U}}\right)\left\{3\tan^2\phi\left(\frac{\mathrm{U}r}{\mathrm{U}+r}\right)^2\right\} \qquad (18)\,\mathrm{R}.$$

Aberration of second surface.

$$+\frac{\mu-1}{2\mu^2}\left(\frac{1}{s}+\frac{1}{\mathrm{V}}\right)^2\left(\frac{1}{s}+\frac{\mu+1}{\mathrm{V}}\right)\left\{3\tan^2\phi\left(\frac{\mathrm{V}s}{\mathrm{V}+s}\right)^2\right\} \qquad (19)\,\mathrm{R}.$$

Corrections converting $\grave{u}$ into $\grave{v}$.

$$\left.\begin{aligned}&+\frac{1}{\mu}\left(\frac{\mathrm{U}(\mu-1)-r}{r\mathrm{U}}\right)^2\left\{\mathrm{U}\tan\phi\frac{r(\mathrm{U}+r)}{\mathrm{U}(\mu-1)-r}\cdot\frac{1}{\mathrm{U}+r}\right\}^2\left\{\frac{\mathrm{U}(\mu-1)-r}{r(\mathrm{U}+r)}\right.\\&\qquad\qquad\left.+\frac{\mathrm{V}(\mu-1)-s}{s(\mathrm{V}+s)}\right\}\frac{1}{2}\end{aligned}\right\}(20)\,\mathrm{R}.$$

The two end corrections.

$$+\tan^2\phi\left(\frac{1}{\mathrm{U}+r}+\frac{1}{\mathrm{V}+s}\right)\frac{1}{2}. \qquad (21)\,\mathrm{R}.$$

The above expressions (18), (19), (20), and (21) therefore together constitute the normal curvature errors to which the rays in the primary plane are subjected when refracted centrally as well as obliquely by the lens. Analogously to the last case, all these expressions simplify down to the simple expression $\frac{\tan^2\phi}{2\mathrm{F}}\frac{3\mu+1}{\mu}$, so that the complete formula becomes

Includes the aberration of all pencils of semi-aperture A.

$$\underbrace{\frac{1}{\mathrm{X}}=\frac{1}{\mathrm{F}}-\frac{1}{\mathrm{U}}+t\frac{\mu}{\grave{u}^2}+\frac{\mu-1}{2\mu^2}\left\{\left(\frac{1}{r}+\frac{1}{\mathrm{U}}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{\mathrm{U}}\right)+\left(\frac{1}{s}+\frac{1}{\mathrm{V}}\right)^2\left(\frac{1}{s}+\frac{\mu+1}{\mathrm{V}}\right)\right\}A^2}_{\mathrm{IV.\ (R.)}}$$

The normal curvature error in primary plane.

$$+\frac{\tan^2\phi}{2\mathrm{F}}\cdot\frac{3\mu+1}{\mu}. \qquad \mathrm{V.\ (R.)}$$

As regards the correction for obliquity we have again arrived at the same result as did Coddington, only we have in Formula IV. added the spherical aberration which is common to *all* the pencils, whether direct or oblique. We have recapitulated these processes chiefly in order to form an introduction to more important results yet to be arrived at, also bearing in mind the principle that complex investigations of this sort are understood in less time and with less effort when all processes (except perhaps reductions) are given in full.

Result is the same as for infinitely thin pencil.

The differential process as applied to infinitely narrow oblique pencils by Coddington and other writers, resulting in Formulæ VI. and VII., also leads to Formulæ III. and V. with less trouble, it is true; but the developments dealt with in subsequent Sections of this work and the corrections of the third order of Section XI. could not be derived from them.

If the reader takes the trouble to pursue the same lines of reasoning in the case of a negative lens with the entering rays converging and the emergent rays diverging, or the cases of

The formulæ universally true.

Entering rays converging into a positive lens

or

Entering rays diverging into a negative lens,

he will again arrive at the same formulæ, if due regard is paid to the conventions already laid down.

The further convention with regard to meniscus lenses must be also observed, viz. that the radius of the deeper curve shall be considered positive and characteristic of the lens and the radius of the shallower curve negative relatively, so that the spherical aberration corrections and curvature errors for the shallower surface will come out negative with respect to the same corrections for the deeper surface, and the result for the whole lens be the algebraic difference. Then the final formulæ emerge just as before.

As to the expression $t\frac{\mu}{u^2}$, it will be found to be but another way of expressing the correction, due to thickness, to be applied to the reciprocal value of $\frac{1}{v}$ (by first approximation), and it has no further significance in the present investigations.

The term $t\frac{\mu}{u^2}$ does not affect the present formulæ.

Having now got the corrections for curvature of image formed by pencils traversing the lens obliquely but centrally,

$$\frac{\tan^2\phi}{2F}\cdot\frac{\mu+1}{\mu} \text{ and } \frac{\tan^2\phi}{2F}\cdot\frac{3\mu+1}{\mu}$$

in secondary planes and primary planes respectively, and these being small corrections relatively to the values of $\frac{1}{F}$ or $\frac{1}{V}$ if the angle of obliquity ϕ is not more than a few degrees, therefore the linear or longitudinal (L.) corrections are expressed by

Secondary plane. Linear value.

$$-F^2\frac{\tan^2\phi}{2F}\cdot\frac{\mu+1}{\mu} \text{ or } -V^2\frac{\tan^2\phi}{2F}\cdot\frac{\mu+1}{\mu} \text{ in secondary planes,}$$

and

Primary plane. Linear value.

$$-F^2\frac{\tan^2\phi}{2F}\cdot\frac{3\mu+1}{\mu} \text{ or } -V^2\frac{\tan^2\phi}{2F}\cdot\frac{3\mu+1}{\mu} \text{ in primary planes,}$$

and we may therefore treat these quantities as the versines of the curved images formed by rays in the two planes, and calling the required radii of curvature of the two images R and R′ we have

$$2R=\frac{(F\tan\phi)^2}{\frac{F\tan^2\phi}{2}\cdot\frac{\mu+1}{\mu}} \text{ or } =\frac{(V\tan\phi)^2}{V^2\frac{\tan^2\phi}{2F}\cdot\frac{\mu+1}{\mu}}=2F\frac{\mu}{\mu+1},$$

and

$$2R'=\frac{(F\tan\phi)^2}{\frac{F\tan^2\phi}{2}\cdot\frac{3\mu+1}{\mu}} \text{ or } =\frac{(V\tan\phi)^2}{V^2\frac{\tan^2\phi}{2F}\frac{3\mu+1}{\mu}}=2F\frac{\mu}{3\mu+1};$$

therefore

Radius of curvature of image, secondary plane.

$$R=F\frac{\mu}{\mu+1} \tag{22}$$

and

Radius of curvature of image, primary plane.

$$R=F\frac{\mu}{3\mu+1} \tag{23}$$

Curvature of image is approximately constant.

whether $V=F$ or whatever its value may be. Thus the curvature of image for some distance from the optic axis is independent of the distance V of the image from the lens, and depends solely upon F and upon the refractive index μ of the glass, and is independent of the shape of the lens. Supposing $\mu=1{\cdot}5$, then the radii of curvatures are respectively $\frac{3}{5}F$ and $\frac{3}{11}F$.

If we take the difference between the R corrections

$$\frac{\tan^2\phi}{2F}\cdot\frac{3\mu+1}{\mu} \text{ and } \frac{\tan^2\phi}{2F}\cdot\frac{\mu+1}{\mu},$$

Expression for the astigmatism of a central oblique pencil.

we then get

$$\frac{\tan^2\phi}{2F}\left(\frac{3\mu+1}{u}-\frac{\mu+1}{\mu}\right) \text{ or } \frac{\tan^2\phi}{F}$$

as the R correction expressing the astigmatism at the oblique focus for any degree of obliquity ϕ.

The same simple expression also applies in the case of a spherical reflecting surface. Clearly no variations in the refractive index can affect the astigmatism, nor do they in any substantial sense affect the curvature errors. For, supposing the refractive index is 1·6 instead of 1·5, we then get radii of curvatures of $F\frac{1\cdot6}{2\cdot6} = F(\cdot6154)$ instead of $F(\cdot6)$, when $\mu = 1\cdot5$; and $F\frac{1\cdot6}{5\cdot8} = F(\cdot276)$ instead of $F(\cdot\dot{2}72\dot{7})$, when $\mu = 1\cdot5$. So that it would require a refractive index of a very abnormal character to much affect the results; for even if μ were ∞, then F and $\frac{F}{3}$ would become the radii of curvatures. But when we come to deal with combinations of collective and dispersive lenses, we shall find variations in refractive indices of one unit of the first decimal place of the highest importance.

Small effect of $\Delta\mu$ upon normal curvature errors.

We may here with advantage compare our results with the exact formulæ for oblique central pencils worked out by Coddington, and given on page 120 of his work. He adopted the course of supposing the pencil of rays to be an infinitely narrow one, and therefore the effective aperture and thickness of the lens to be vanishing quantities; he then worked out the oblique focal distances by a strictly differential method, arriving at the formula

$$\frac{1}{v} = \left(\mu\frac{\cos\phi'}{\cos\phi} - 1\right)\left(\frac{1}{r} + \frac{1}{s}\right)\cos\phi - \frac{1}{u} \qquad \text{VI.}$$

Secondary plane. Exact formula for thin pencil.

in the secondary plane, and

$$\frac{\cos\phi}{v} = \left(\mu\frac{\cos\phi'}{\cos\phi} - 1\right)\left(\frac{1}{r} + \frac{1}{s}\right) - \frac{\cos\phi}{u} \qquad \text{VII.}$$

Primary plane. Exact formula for thin pencil.

in the primary plane, in which

u is the oblique distance from the radiant point Q to the lens centre.

v is the oblique distance from the lens centre to the corresponding conjugate focal point.

ϕ is the angle of obliquity as before.

ϕ' is the angle of obliquity of the principal ray after refraction, such that $\sin\phi = \mu\sin\phi'$.

r is the radius of the first surface, and

s is the radius of the second surface.

This formula is by its nature accurate for all angles of obliquity, and Fig. 46, Plate X., represents the primary and secondary curves

Comparison of the exact curves of image with those of the second approximation III. and V.

deduced from it when the incident rays are parallel or u is infinite, while the two curves indicated by dots are those obtained by the application of Formulæ III. and V. as herein worked out.

The lens is supposed to be located at L in each case. The curve for rays in secondary planes is drawn as a full line, and that for rays in the primary plane as a closely dotted line.

Fig. 46a shows the primary and secondary curves obtained when u (axial value) $= -1$, and the two widely dotted curves are obtained from Formulæ III. and V.

Fig. 47 represents the case when $u = v = 2f$, when the focal distance is double what it is in the case of Fig. 46.

Thus it will be seen that our Formulæ III. and V. fall off in accuracy when the angle of obliquity becomes large; but they are exceedingly useful formulæ, lending themselves easily to analytical processes, while the accurate Formulæ VI. and VII. involve the use of trigonometric tables in their application.

Normal curvature corrections involving the aperture of the oblique pencil.

It will be shown algebraically in Section XI. that the differences between the approximate dotted curves and the accurate solid curves are made up of corrections of the higher orders, involving functions of $\tan^4 \phi$, $\tan^6 \phi$, etc. We shall also find that when the aperture of the oblique pencil becomes large enough to show perceptible spherical aberration, then among the corrections of such higher orders we find corrections involving the square and higher powers of the aperture, so that the curve traced out by the foci of the two extreme rays of a pencil of large aperture will not be exactly of the same character as the curve traced out by the foci of two rays infinitely close to the principal ray. This means that the amount of the spherical aberration of a very oblique pencil of semi-aperture A will not be the same as the spherical aberration of the axial pencil of semi-aperture A.

It is, however, obvious that while in any system of separated lenses or elements the principal rays of the pencils may cross the axis just where one lens or element occurs, and thus be refracted obliquely but centrally through the same, yet such principal rays must traverse most of the lenses eccentrically as well as obliquely. In the next Section we will deal with such cases of eccentric oblique refraction; but before proceeding to that it will be as well to deal with a few very useful formulæ in connection with the curvature errors which we have arrived at in the shape of Formulæ III. and V., or

$$+\frac{\tan^2 \phi}{2f} \cdot \frac{\mu + 1}{\mu} \text{ in secondary planes,}$$

and $$\frac{+\tan^2\phi}{2f}\cdot\frac{3\mu+1}{\mu} \text{ in primary planes.}$$

It is often very desirable to know the effect of a change in the refractive index upon these curvature corrections.

We will first deal with the case of the curvature being constant; that is, $\frac{1}{r}+\frac{1}{s}$ or $\frac{1}{\rho}$ is constant, so that $\frac{1}{f}$ or $\frac{\mu-1}{\rho}$ is variable as μ varies.

In secondary planes we have

$$d_\mu\left\{\frac{\mu-1}{2\rho}\cdot\frac{\mu+1}{\mu}\right\}\tan^2\phi$$

$$=\frac{\tan^2\phi}{2\rho}\cdot\frac{\mu\{(\mu-1)+(\mu+1)\}-(\mu^2-1)}{\mu^2}d\mu$$

$$=\frac{\tan^2\phi}{2\rho}\left(\frac{\mu^2+1}{\mu^2}\right)d\mu$$

$$=\frac{\tan^2\phi}{2\rho}\left(1+\frac{1}{\mu^2}\right)d\mu; \qquad \text{VIII.}$$

Secondary plane. Variation in curvature error due to $d\mu$ when $\frac{1}{\rho}$ is constant.

so that if the curvature of the lens is constant, then the curvature of image increases with μ.

In primary planes we have

$$d_\mu\left\{\frac{\mu-1}{2\rho}\cdot\frac{3\mu+1}{\mu}\right\}\tan^2\phi$$

$$=\frac{\tan^2\phi}{2\rho}\cdot\frac{\mu\{(\mu-1)+(3\mu+1)\}-(\mu-1)(3\mu+1)}{\mu^2}d\mu$$

$$=\frac{\tan^2\phi}{2\rho}\cdot\frac{4\mu^2-(3\mu^2-2\mu-1)}{\mu^2}d\mu$$

$$=\frac{\tan^2\phi}{2\rho}\left(\frac{\mu+1}{\mu}\right)^2 d\mu; \qquad \text{IX.}$$

Primary plane. Variation in curvature error due to $d\mu$ when $\frac{1}{\rho}$ is constant.

and again the curvature of image increases with μ.

But if f is kept a constant, then we find in secondary planes that

$$d_\mu\frac{\tan^2\phi}{2f}\cdot\frac{\mu+1}{\mu}=\frac{\tan^2\phi}{2f}\,\frac{\mu-(\mu+1)}{\mu^2}d\mu$$

$$=\frac{\tan^2\phi}{2f}\left(-\frac{1}{\mu^2}\right)d\mu; \qquad \text{X.}$$

Secondary plane. Variation in curvature error due to $d\mu$ when $\frac{1}{f}$ is constant.

so that for a constant focal length the higher refractive index, implying shallower curves for the lens, yields a flatter image.

In primary planes we have

$$d_\mu \frac{\tan^2\phi}{2f} \cdot \frac{3\mu+1}{\mu} = \frac{\tan^2\phi}{2f} \cdot \frac{\mu 3d\mu - (3\mu+1)d\mu}{\mu^2}$$

Primary plane. Variation in curvature error due to $d\mu$ when $\frac{1}{f}$ is constant.

$$= \frac{\tan^2\phi}{2f}\left(-\frac{1}{\mu^2}\right)d\mu. \qquad \text{XI.}$$

So we get the same differential as in the case of the secondary plane.

This we should, of course, expect, since the astigmatism as measured by

$$\frac{\tan^2\phi}{2f} \cdot \frac{3\mu+1}{\mu} - \frac{\tan^2\phi}{2f} \cdot \frac{\mu+1}{\mu} = \frac{\tan^2\phi}{f} = \text{constant},$$

whatever may be the value of μ, and therefore the changes in curvature consequent upon $d\mu$ must be identical in the two planes.

The Spherical Reflector

We have yet to consider the case of a spherical reflecting surface and its effect upon pencils of rays reflected obliquely but centrally. Let Fig. 47*a*, Plate X., represent a spherical reflector of semi-aperture C. . E_1 or C . . $E_2 = A$. Let Q . . Q′ be a finitely distant flat object perpendicular to the axis C . . Q. Let O be the centre of curvature, the radius being O . . C $= r$.

Primary Plane

We will deal with rays in the primary plane first.

Draw a straight line Q′ . . O . . S from Q′ through the centre of curvature; this then becomes the theoretical axis of the oblique pencil, so that S . . E_1 and S . . E_2 are the two heights for the two extreme rays, which heights we will call y_1 and y_2. It is clear that if f* is the ultimate focal point for rays close to the oblique axis Q′ . . S, then the ray Q′ . . E_2, after reflection, will cut Q′ . . S at a point f_2, the ray Q′ . . E_1 from the upper edge will, after reflection, cut Q′ . . S at f_1, and f . . f_2 and f . . f_1 will be the linear spherical aberrations proportional to y_2^2 and y_1^2, and these two rays reflected from the extreme edges of the mirror will cut one another at a point q slightly outside of the oblique normal ray Q′ . . S. Draw q . . p perpendicular to C . . Q. Then, as in the case of oblique refraction at a spherical surface, we may put Q′ . . S $= u$, f . . S $= v'$, and q . . S $= x$, and let the angle of obliquity Q′CQ $= \phi$. Then we have the fundamental equation—

* The ultimate focal point f has been omitted, but should be shown a little to the right hand of f_2.

PLATE. X.

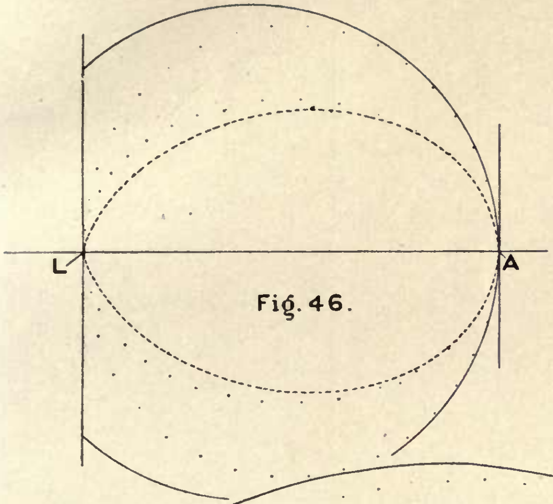


Fig. 46.

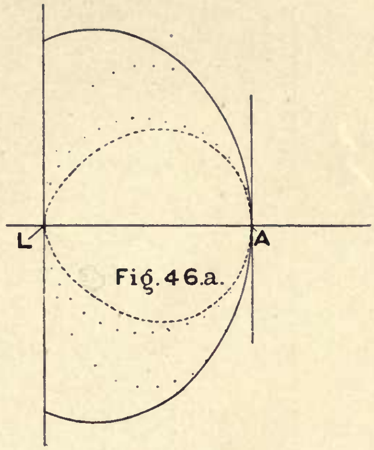


Fig. 46.a.

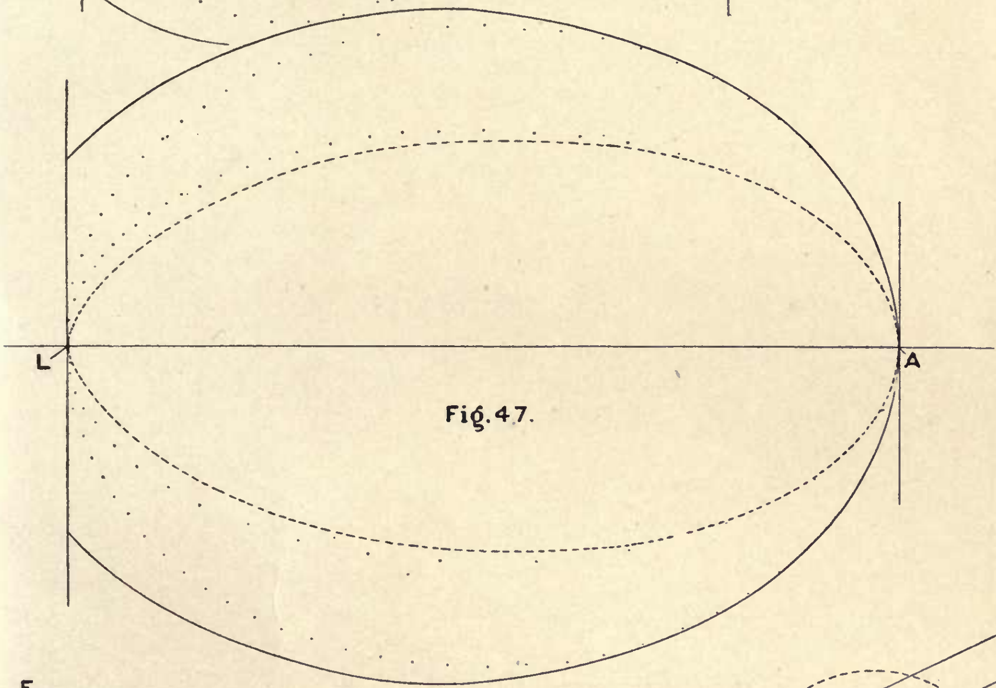


Fig. 47.

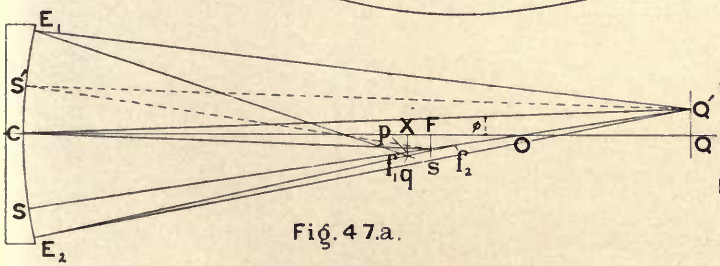


Fig. 47.a.

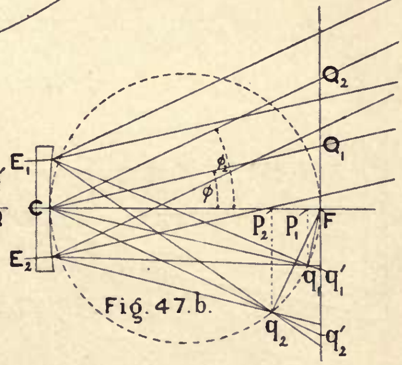


Fig. 47.b.

PLATE. X.

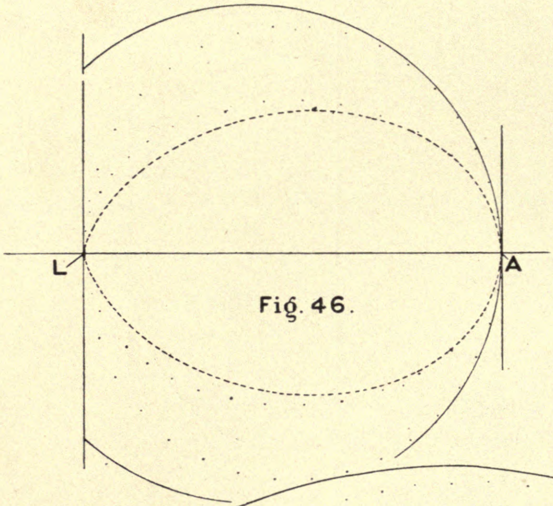


Fig. 46.

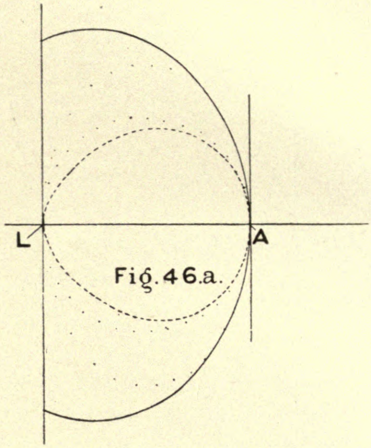


Fig. 46.a.

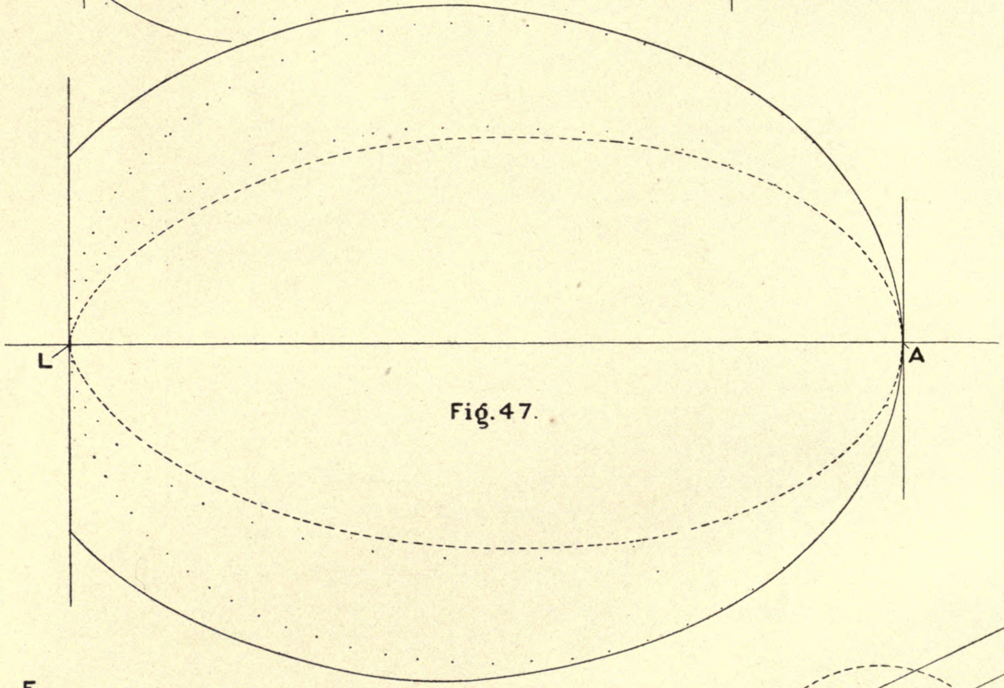


Fig. 47.

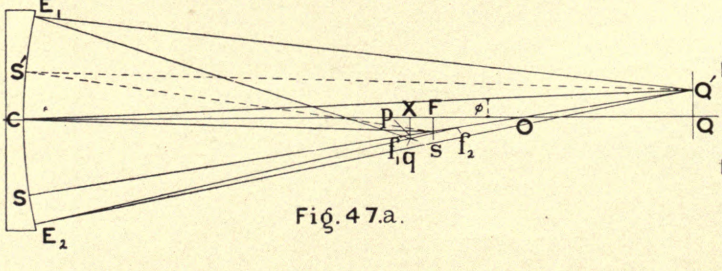


Fig. 47.a.

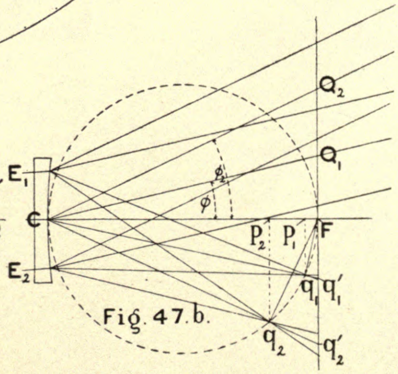


Fig. 47.b.

$$(f_1\,..\,p)\frac{y_1}{f_1\,..\,\mathrm{S}}=(p..q)=(f_2\,..\,p)\frac{y_2}{f_2\,..\,\mathrm{S}},$$

The fundamental equation.

in which, if we put f_1 for $\mathrm{S}\,..\,f_1$, and f_2 for $\mathrm{S}\,..\,f_2$, we have $f_1\,..\,p=x-f_1$, and $f_2\,..\,p=f_2-x$; therefore

$$(x-f_1)\frac{y_1}{f_1}=(f_2-x)\frac{y_2}{f_2},$$

from which

$$\frac{1}{x}=\left(\frac{y_1}{f_1}+\frac{y_2}{f_2}\right)\frac{1}{y_1+y_2} \qquad (24\text{A})$$

Value of $\frac{1}{x}$ deduced from above.

But f_1 and f_2 involve spherical aberration corrections which are functions of y_1^2 and y_2^2 respectively. That part of the expressions for $\frac{1}{f_1}$ and $\frac{1}{f_2}$ which are common to both of them is of course $\frac{1}{v}$, or $\frac{1}{f\,..\,\mathrm{S}}$; then if we put A′ for the aberration function, which, as we have seen in Section IV., is $\frac{1}{r}\left(\frac{1}{r}-\frac{1}{u}\right)^2$, then we have

$$\frac{1}{f_1}=\frac{1}{v}+\mathrm{A}'y_1^2 \text{ and } \frac{1}{f_2}=\frac{1}{v}+\mathrm{A}'y_2^2,$$

and Equation (24A) becomes

$$\frac{1}{x}=\left\{y_1\left(\frac{1}{v}+\mathrm{A}'y_1^2\right)+y_2\left(\frac{1}{v}+\mathrm{A}'y_2^2\right)\right\}\frac{1}{y_1+y_2}$$

$$=\frac{\frac{1}{v}(y_1+y_2)+\mathrm{A}'(y_1^3+y_2^3)}{y_1+y_2}=\frac{1}{v}+\mathrm{A}'(y_1^2+y_2^2-y_1y_2),$$

so that we get finally

$$\frac{1}{x}=\frac{1}{v}+\mathrm{A}'(y_1^2+y_2^2-y_1y_2). \qquad (24\text{B})$$

Value of $\frac{1}{x}$ when corrected for compounded aberration.

Now if $\frac{1}{\mathrm{F}}$ = the reciprocal of the principal focal length or $\frac{2}{r}$, we have

$$\frac{1}{v}=\frac{1}{\mathrm{F}}-\frac{1}{\grave{u}},$$

and

$$\grave{u} \text{ or } \mathrm{Q}'\,..\,\mathrm{S} \text{ obviously} = \mathrm{Q}\,..\,\mathrm{C} \text{ or } u+\frac{(\mathrm{Q}\,..\,\mathrm{Q}_1)^2}{2(\mathrm{O}\,..\,\mathrm{Q})}=u+\frac{(u\tan\phi)^2}{2(u-r)};$$

$$\therefore\ \frac{1}{\grave{u}}=\frac{1}{u}-\frac{\tan^2\phi}{2(u-r)} \text{ and } \frac{1}{v}=\frac{1}{\mathrm{F}}-\frac{1}{u}+\tan^2\phi\,\frac{1}{2(u-r)},$$

so that Equation (24B) becomes

Abbreviated value of $\frac{1}{x}$.

$$\frac{1}{x}=\frac{1}{\mathrm{F}}-\frac{1}{u}+\tan^2\phi\frac{1}{2(u-r)}+\mathrm{A}'(y_1^2+y_2^2-y_1y_2).$$

Now

$$y_1=A+(\mathrm{S}\,..\,\mathrm{C})=A+(\mathrm{Q}\,..\,\mathrm{Q}')\frac{r}{u-r}:$$

Expression for y_1.

$$\therefore\ y_1=A+(u\tan\phi)\frac{r}{u-r};$$

and similarly

$$y_2=A-(\mathrm{S}\,..\,\mathrm{C})$$

Expression for y_2.

$$\therefore\ y_2=A-(u\tan\phi)\frac{r}{u-r}.$$

Therefore we have

$$y_1^2+y_2^2-y_1y_2=\left\{\begin{array}{l} A^2+2A(u\tan\phi)\frac{r}{u-r}+(u^2\tan^2\phi)\frac{r^2}{(u-r)^2}\\ +A^2-2A(u\tan\phi)\frac{r}{u-r}+(u^2\tan^2\phi)\frac{r^2}{(u-r)^2}\\ -A^2\qquad\qquad\qquad\qquad +(u^2\tan\phi)\frac{r^2}{(u-r)^2};\end{array}\right.$$

Value of the function of y_1 and y_2.

$$\therefore\ y_1^2+y_2^2-y_1y_2=A^2+3\tan^2\phi\left(\frac{ur}{u-r}\right)^2;$$

Full value of $\frac{1}{x}$.

$$\therefore\ \frac{1}{x}=\frac{1}{\mathrm{F}}-\frac{1}{u}+\tan^2\phi\frac{1}{2(u-r)}+\mathrm{A}'\left\{A^2+3\tan^2\phi\left(\frac{ur}{u-r}\right)^2\right\}.\quad(24\text{C})$$

Distance x to be reduced to the axis.

We must next reduce the oblique distance $q\,..\,\mathrm{S}$ or x to the axis C . . Q of the mirror. From q draw $q\,..\,\mathrm{X}$ perpendicular to the mirror axis, so that C . . X, or X for short, becomes the required corrected distance. It is clear that $\mathrm{X}=x+\frac{(q\,..\,\mathrm{X})^2}{2(r-v)}$, in which v may be put as its first approximate value such that $\frac{1}{v}=\frac{1}{\mathrm{F}}-\frac{1}{u}$. Then we have

$$(q\,..\,\mathrm{X})^2=\left\{(\mathrm{Q}\,..\,\mathrm{Q}')\frac{r-v}{u-r}\right\}^2=\left\{u\tan\phi\frac{r-v}{u-r}\right\}^2;$$

$$\therefore\ \mathrm{X}=x+\left\{u\tan\phi\frac{r-v}{u-r}\right\}^2\frac{1}{2(r-v)}$$

and

Value of $\frac{1}{\mathrm{X}}$.

$$\frac{1}{\mathrm{X}}=\frac{1}{x}-\left\{u\tan\phi\frac{r-v}{u-r}\right\}^2\frac{1}{2(r-v)}\cdot\frac{1}{x^2},$$

in which we may put

$$\frac{1}{x^2}=\left(\frac{1}{\mathrm{F}}-\frac{1}{u}\right)^2,$$

so that

$$\frac{1}{\mathrm{X}}=\frac{1}{x}-\left(u\tan\phi\frac{r-v}{u-r}\right)^2\frac{1}{2(r-v)}\left(\frac{1}{\mathrm{F}}-\frac{1}{u}\right)^2. \qquad (24\mathrm{D})$$

Value of $\frac{1}{\mathrm{X}}$ amplified.

On inserting the previously worked out value of $\frac{1}{x}$ from 24C we then get

$$\frac{1}{\mathrm{X}}=\frac{1}{\mathrm{F}}-\frac{1}{u}+\tan^2\phi\frac{1}{2(u-r)}+\mathrm{A}'\left\{A^2+3\tan^2\phi\left(\frac{ur}{u-r}\right)^2\right\}$$
$$-\left(u\tan\phi\frac{r-v}{u-r}\right)^2\frac{1}{2(r-v)}\left(\frac{1}{\mathrm{F}}-\frac{1}{u}\right)^2. \qquad (24\mathrm{E})$$

Value of $\frac{1}{\mathrm{X}}$ with all corrections.

Now $\mathrm{A}'A^2$ is obviously the spherical aberration for the direct or axial pencil originating from Q on the axis and of semi-aperture a (which was y in our investigation of the direct spherical aberration in Section IV.), and should be kept separate; so that after inserting $\frac{1}{r}\left(\frac{1}{r}-\frac{1}{u}\right)^2$ for A$'$, we then get $\left(\text{since } \frac{1}{r}-\frac{1}{u}=\frac{u-r}{ru} \text{ and } v=\frac{ur}{2u-r}\right)$

$$\frac{1}{\mathrm{X}}=\frac{1}{\mathrm{F}}-\frac{1}{u}+\frac{1}{r}\left(\frac{1}{r}-\frac{1}{u}\right)^2A^2$$
$$+3\tan^2\phi\frac{1}{r}+\tan^2\phi\frac{1}{2(u-r)}-u^2\tan^2\phi\left(\frac{r}{2u-r}\right)\frac{1}{2(u-r)}\left(\frac{2u-r}{ur}\right)^2. \qquad (24\mathrm{F})$$

Full value of $\frac{1}{\mathrm{X}}$ after separating out the common aberration.

Then the last line of the above simplifies down thus—

$$=3\tan^2\phi\frac{1}{r}+\tan^2\phi\frac{1}{2(u-r)}-\tan^2\phi\frac{2u-r}{2r(u-r)}$$
$$=\tan^2\phi\left\{\frac{1}{2(u-r)}+\frac{3}{r}-\frac{2u-r}{2r(u-r)}\right\}=\tan^2\phi\left\{\frac{r+6(u-r)-(2u-r)}{2r(u-r)}\right\}$$
$$=\tan^2\phi\frac{4(u-r)}{2r(u-r)}=\tan^2\phi\frac{2}{r}=\tan^2\phi\frac{1}{\mathrm{F}};$$

so that finally we get

$$\frac{1}{\mathrm{X}}=\frac{1}{\mathrm{F}}-\frac{1}{u}+\frac{1}{r}\left(\frac{1}{r}-\frac{1}{u}\right)^2A^2$$

Includes the aberration common to all pencils.

$$+\tan^2\phi\frac{1}{\mathrm{F}}. \qquad \mathrm{XII.}$$

The normal curvature error in primary plane.

From this it appears that the radius of curvature of the image formed by rays in primary planes is

The radius of curvature of the primary image.

$$\frac{1}{2}\left\{\frac{(v\tan\phi)^2}{\tan^2\phi\dfrac{v^2}{F}}\right\}=\frac{F}{2}.$$

Secondary Plane

Here it is clear that the y is equal to the distance from the point S, where the ray through the centre of curvature strikes the plane of the mirror, obliquely up to the top edge of the aperture, perpendicularly above C, so that

Expression for y^2.

$$y^2=(S..C)^2+A^2 \text{ or }=\left\{(Q..Q')\left(\frac{r}{u-r}\right)\right\}^2+A^2=\left\{u\tan\phi\frac{r}{u-r}\right\}^2+A^2,$$

so that the spherical aberration to which the two extreme rays in the secondary plane are subject is expressed by

The spherical aberration of the secondary rays.

$$A'\left\{A^2+\tan^2\phi\left(\frac{ur}{u-r}\right)^2\right\},$$

while the other corrections for reducing the distances concerned to the axis are the same as before, so that following the analogy of Formula (24E) we have

$$\frac{1}{X}=\frac{1}{F}-\frac{1}{u}+\tan^2\phi\frac{1}{2(u-r)}+A'\left\{A^2+\tan^2\phi\left(\frac{ur}{u-r}\right)^2\right\}$$
$$-\left(u\tan\phi\frac{r-v}{u-r}\right)^2\frac{1}{2(r-v)}\left(\frac{1}{F}-\frac{1}{u}\right)^2; \quad (24\text{G})$$

so that after insertion of the term A′ in full we get

Full value of $\frac{1}{X}$ after separating out the common aberration.

$$\frac{1}{X}=\frac{1}{F}-\frac{1}{u}+\frac{1}{r}\left(\frac{1}{r}-\frac{1}{u}\right)^2A^2$$
$$+\frac{1}{r}\tan^2\phi+\tan^2\phi\frac{1}{2(u-r)}-u^2\tan^2\phi\left(\frac{r}{2u-r}\right)\frac{1}{2(u-r)}\left(\frac{2u-r}{ur}\right)^2, \quad (24\text{H})$$

the last line of which

$$=\tan^2\phi\frac{1}{r}+\tan^2\phi\frac{1}{2(u-r)}-\tan^2\phi\frac{2u-r}{2r(u-r)}$$
$$=\tan^2\phi\left\{\frac{1}{r}+\frac{1}{2(u-r)}-\frac{2u-r}{2r(u-r)}\right\}$$
$$=\tan^2\phi\left\{\frac{2(u-r)+r-(2u-r)}{2r(u-r)}\right\}=0.$$

Secondary plane. The normal curvature error = 0.

So that there is no curvature error for rays in secondary planes and the image is flat.

From these results it follows that the curvature error $\tan^2\phi\frac{1}{F}$ in primary planes represents also the astigmatism of oblique pencils, and it is thus seen that it is exactly the same as for a lens of the same principal focal length. For a lens we have the formulæ for curvature of oblique pencils—

Curvature error in primary plane and astigmatism identical.

$$\frac{\tan^2\phi}{2F}\cdot\frac{\mu+1}{\mu} \text{ in secondary planes,}$$

Normal curvature errors for lenses.

and

$$\frac{\tan^2\phi}{2F}\cdot\frac{3\mu+1}{\mu} \text{ in primary planes,}$$

their difference, or the astigmatism, being $\tan^2\phi\frac{1}{F}$.

If in the above two formulæ we insert $\mu = -1$, we then get

Result of assuming refractive index in above $= -1$.

$$\frac{\tan^2\phi}{2F}\ \frac{-1+1}{-1} = 0 \text{ in secondary planes,}$$

and

$$\frac{\tan^2\phi}{2F}\ \frac{-3+1}{-1} = \tan^2\phi\frac{1}{F} \text{ in primary planes,}$$

which agree with the curvature errors which we have already worked out.

This last formula, however, can be shown to be inexact, for there are corrections of higher orders, functions of $\tan^4\phi$, $\tan^6\phi$, etc., but of little practical importance in this case, wherein the spherical aberrations involved are generally very small.

The curvature corrections for a spherical mirror as worked out by Coddington by the application of the differential process to infinitely narrow oblique pencils are given on pages 22 to 24 of his work in the form

$$\frac{\cos\phi}{v} = \frac{1}{F} - \frac{\cos\phi}{u} \text{ in primary planes,} \qquad (24\text{I})$$

Exact formulæ for normal curvature errors for spherical mirror.

and

$$\frac{1}{v} = \frac{\cos\phi}{F} - \frac{1}{u} \text{ in secondary planes,} \qquad (24\text{J})$$

in which formulæ u is the oblique distance $Q'..C$ of our Fig. 47a, and v is the oblique distance $(= C..q)$ of q, the focus, from C, the centre of the mirror surface. These formulæ are exact for infinitely narrow pencils, and practically accurate for cases in which the aperture of the mirror does not amount to one-tenth part of the principal focal length.

If in the above two formulæ we suppose $\frac{1}{u}$ to vanish, we then get

Impinging rays parallel.

in primary planes $v = F\cos\phi$. Fig. 47*b* shows a spherical mirror of principal focal length $= C..F$, and of radius of curvature $= 2(C..F)$.

A pencil of parallel rays whose principal ray is $Q_1..C$ is incident upon the mirror at an angle $Q_1CF = \phi$, and of course reflected off at the same angle. Let a circle of radius $\frac{F}{2}$ be drawn touching the mirror centre at C, and the principal focal point or plane at F. Here, then, $C..q_1$ is Coddington's v. But in order to compare his formulæ with those we have worked out we must first reduce his oblique distance v to the axis by drawing $q_1..p_1$ perpendicular to the axis $C..F$. Let $C..p_1 = V$. Then it is clear that

$$V = (C..q_1)\cos\phi = v\cos\phi,$$

and we have seen above that

$$v = F\cos\phi,$$

therefore

$$V = F\cos^2\phi.$$

Primary image is then formed on a spherical surface.

It is clear that the formula worked out by Coddington differentially implies that the locus of curvature for the oblique foci is a circle of radius $\frac{F}{2}$, for the triangle CFq_1 is always a right-angled triangle having its right angle at q_1, so that v or $C..q_1$ invariably $= F\cos\phi$.

Secondary image is always flat.

In secondary planes we have by Coddington's formula

$$v = \frac{F}{\cos\phi} = F\sec\phi,$$

which, of course, requires a plane image to satisfy that condition, the focus for secondary rays falling at q_1' when the focus for primary rays falls at q_1.

SECTION VI

ECCENTRIC OBLIQUE REFRACTION OF PENCILS THROUGH THIN LENSES OR ELEMENTS

In the last Section we have assumed the central or principal ray of every oblique pencil to pass through the centre A_1 of the lens or element. We have now to consider the case wherein the point where the principal rays cross the optic axis is removed from A_1 or the lens centre to another point on the optic axis, under which condition the principal rays of oblique pencils will strike the element plane at distances from the lens centre A_1 varying in proportion to the tangent of the angle of obliquity. It is clear, then, that the distance C from A_1 to the point O_1, where a principal ray of an eccentric oblique pencil cuts the element plane, is the new factor which has to be introduced into the investigation. It will be best to deal with the rays in secondary planes first.

The new factor introduced into the case.

Secondary Plane

In Fig. 48, Plate XI., D'.. D' is a stop or diaphragm having a circular aperture of diameter $= 2S$, placed axially in front of a spherical lens surface, compelling the principal rays, such as Q .. O_1, to cross the lens axis at G. As before, P .. r' is the axis of the lens, and Q is the point in plane P .. Q from which the oblique and eccentric pencil of rays radiates. Let $U =$ P .. a_1, $u' = d_1 .. q'$, and $u =$ Q .. d_1, c_1 being where Q .. r' cuts the element plane; $r =$ radius of curvature, r' being the centre of same, and q' the point where the two extreme rays in the secondary plane come to focus. It is evident that q' is strictly upon the normal ray Q .. r' projected. Let $\phi =$ angle of obliquity Pa_1Q, $\theta =$ angle Pr'Q, and D = distance of diaphragm from a_1, the vertex and centre of the lens, or from the element plane. Let the two extreme rays Q .. n_1 and Q .. w_1 passing the diaphragm in the primary plane cut

Notation.

the element plane at points n_1 and w_1. Let the central ray or principal ray of the eccentric oblique pencil, which goes through the centre G of the diaphragm, cut the element plane at O_1. Then $a_1 \ldots O_1$ is the linear eccentricity of the pencil, and, as we have seen, is the new factor in the case. As before, we will reserve the consideration of the higher corrections arising from the departure of the curve from the element plane for a subsequent Section, XI. Now the two rays in the secondary plane, or the plane perpendicular to the paper (and containing the oblique principal ray $Q \ldots O_1$), whose focus q' we wish to locate, are evidently the two rays just grazing the upper and lower limits of the aperture in D'..D', and striking the element plane at two points, say n'_1 and w'_1, immediately above and below the point O_1; and it is obvious that the square of the distance from c_1 to either of the said points n'_1 or w'_1 is equal to

The two rays in the secondary plane defined.

Value of the two y's.

$$(c_1 \ldots O_1)^2 + (O_1 \ldots n'_1)^2 = y^2. \tag{25A}$$

Now, calling the semi-diameter of the aperture in the diaphragm S we have

$$(O_1 \ldots n'_1)^2 = \left(S \frac{U}{U - D}\right)^2,$$

which is the semi-aperture of the pencil where it cuts the element plane. Also we have

$$c_1 \ldots O_1 = (O_1 \ldots a_1) + (a_1 \ldots c_1),$$

of which

The eccentricity C defined.

$$O_1 \ldots a_1 = (P \ldots Q)\frac{D}{U - D} = U \tan\phi \frac{D}{U - D}, \tag{25B}$$

which is our new factor C; and

$$a_1 \ldots c_1 = r \tan\theta = r \tan\phi \frac{U}{U + r} = B_1,$$

as before; therefore

$$(c_1 \ldots O_1)^2 = \left(U \tan\phi \frac{D}{U - D} + r \tan\phi \frac{U}{U + r}\right)^2 = (C + B_1)^2;$$

and since

$$y^2 = (c_1 \ldots O_1)^2 + (O_1 \ldots n'_1)^2,$$

therefore

Value of the y's in detail.

$$y^2 = \left(\tan\phi \frac{UD}{U - D} + \tan\phi \frac{Ur}{U + r}\right)^2 + \left(S \frac{U}{U - D}\right)^2, \tag{25C}$$

and this value for y^2 must be entered as a coefficient in the formula for the spherical aberration at the first refraction.

PLATE. XI.

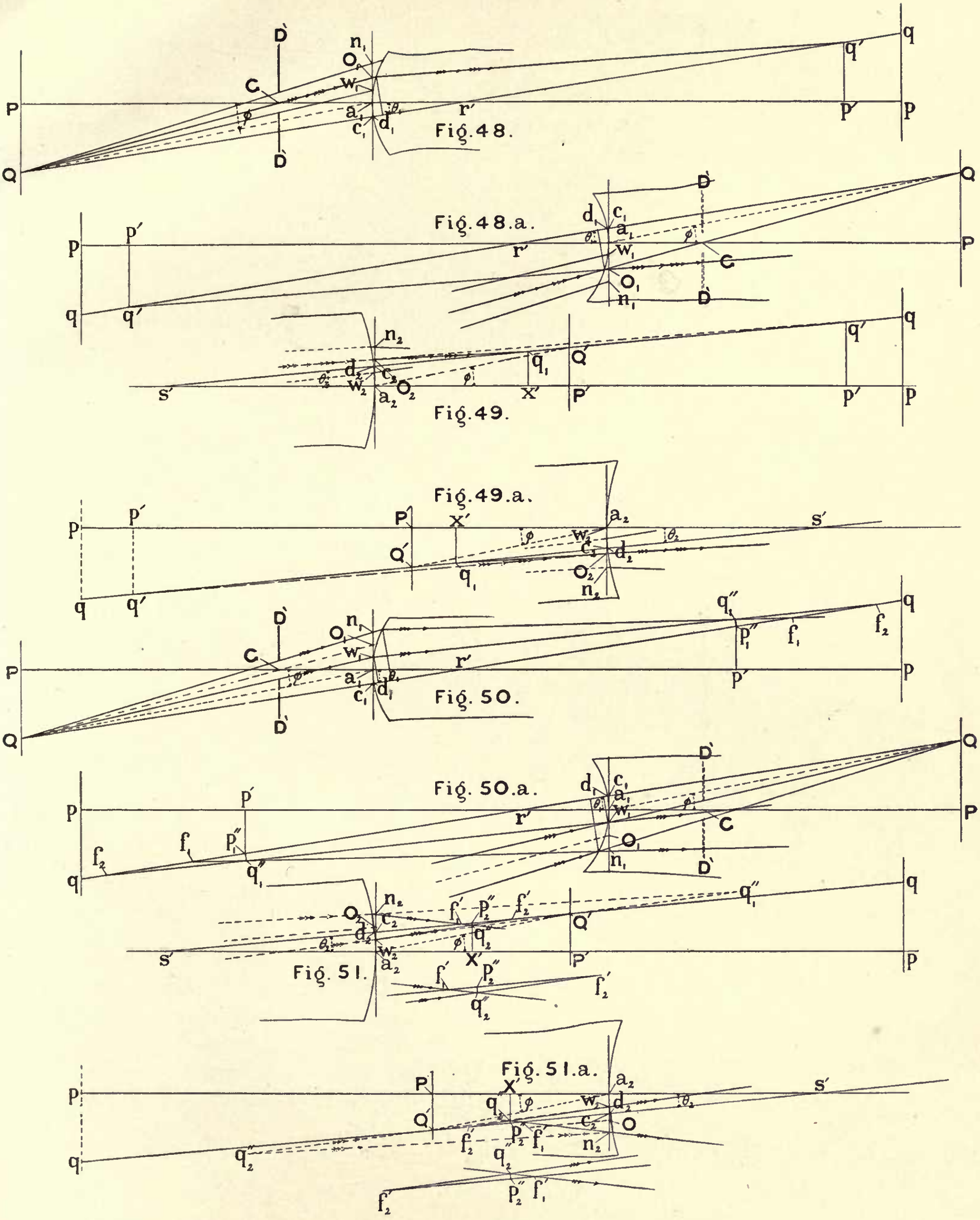


Fig. 48.

Fig. 48.a.

Fig. 49.

Fig. 49.a.

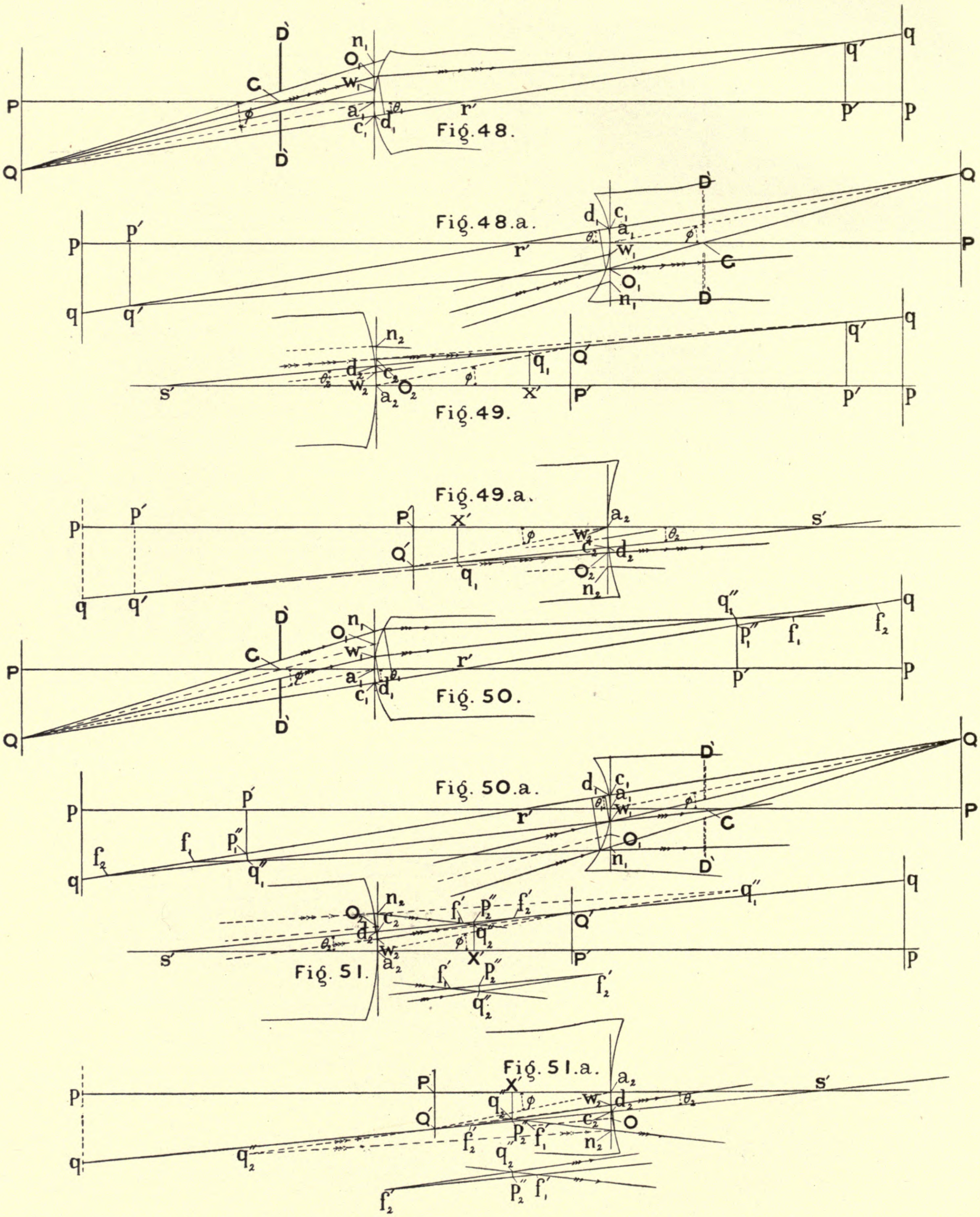
Fig. 50.

Fig. 50.a.

Fig. 51.

Fig. 51.a.

PLATE. XI.



The Refraction at the Second Surface

Turning now to the refraction at the second surface (see Fig. 49) we have the same two rays, $n'_1 \ldots q'$ and $w'_1 \ldots q'$, converging towards the point q' before entering the second surface.

Join q' of our last Fig. 48 to s', the centre of curvature of the second surface, cutting the second element plane at c_2. Then $s' \ldots c_2 \ldots q'$ is the second oblique axis. Adopting the same construction as in Fig. 48, we have the points n_2 and w_2 where the two extreme rays in the primary plane cut the element plane, and the point O_2 where the centre or principal ray cuts the element plane. Then supposing the upper ray in the secondary plane to strike the element plane at n'_2, we have, as before, since the lens is thin,

$$(O_2 \ldots n'_2)^2 = (O_1 \ldots n'_1)^2 = S^2\left(\frac{U}{U-D}\right)^2,$$

and since

$$(C_2 \ldots n'_2)^2 \text{ or } Y^2 = (C_2 \ldots O_2)^2 + (O_2 \ldots n'_2)^2,$$

$$\therefore\ (C_2 \ldots n'_2)^2 = \left\{(a_2 \ldots O_2) - (a_2 \ldots c_2)\right\}^2 + (O_2 \ldots n'_2)^2.$$

Now we may take the eccentricity $a_2 \ldots O_2$ to be the same as $a_1 \ldots O_1$ for the first surface, for it is the distance from the lens axis of the point where the principal ray of the pencil cuts the lens, and we are supposing the lens so thin as to admit of no variation in $a_1 \ldots O_1$ as the pencil traverses the lens. Therefore we may assume that

$$C = a_2 \ldots O_2 = \tan\phi\frac{UD}{U-D}, \qquad (26\text{A})$$

The eccentricity C.

as in the case of the first surface, while $a_2 \ldots c_2$ or B_2 (analogous to $a_1 \ldots c_1$ of the first surface) is approximately equal to

$$V\tan\phi\frac{s}{V+s};$$

$$\therefore\ (c_2 \ldots n'_2)^2 = \left(\tan\phi\frac{UD}{U-D} - \tan\phi\frac{Vs}{V+s}\right)^2 + \left(S\frac{U}{U-D}\right)^2 = Y^2, \qquad (26\text{B})$$

Value of the Y's in detail.

and this is the coefficient of the aberration at the second refraction.

Reverting to Formula I., page 120, Section V., giving the complete statement of corrections applicable to the secondary rays of the central oblique pencil, it will easily be seen that the R correction (see Formula (14)) expressing the differences between the oblique $\frac{1}{u}$ and $\frac{1}{v}$ and the axial $\frac{1}{U}$ and $\frac{1}{V}$ respectively will apply just the same in our present

All the corrections worked out in Section V. also apply to present case.

case of the eccentric oblique pencil. Also the expression (12), which gives the R correction necessary for converting the $\frac{1}{u}$, of the first refraction into the $\frac{1}{v}$, of the second refraction, will equally be required, while expressions (11) and (13) for the two spherical aberrations will be replaced by corresponding expressions with those values of y and Y given above in (25C) and (26B) substituted therein. Hence we get the following formula, after selecting out the joint spherical aberrations for the semi-aperture $S\frac{\mathrm{U}}{\mathrm{U-D}}$ which constitute a correction common to all the pencils, whether axial or otherwise—

$$\left.\begin{aligned}\frac{1}{\mathrm{X}}=\frac{1}{\mathrm{F}}-\frac{1}{\mathrm{U}}+t\frac{\mu}{u^2}+\frac{\mu-1}{2\mu^2}\Big\{&\left(\frac{1}{r}+\frac{1}{\mathrm{U}}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{\mathrm{U}}\right)\\&+\left(\frac{1}{s}+\frac{1}{\mathrm{V}}\right)^2\left(\frac{1}{s}+\frac{\mu+1}{\mathrm{V}}\right)\Big\}\left(S\frac{\mathrm{U}}{\mathrm{U-D}}\right)^2\end{aligned}\right\}\qquad(27)$$

from (25C)

$$+\frac{\mu-1}{2\mu^2}\left(\frac{1}{r}+\frac{1}{\mathrm{U}}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{\mathrm{U}}\right)\left(\tan\phi\frac{\mathrm{UD}}{\mathrm{U-D}}+\tan\phi\frac{\mathrm{U}r}{\mathrm{U}+r}\right)^2\qquad(28)$$

$$\left.\begin{aligned}+\frac{1}{\mu}\left(\frac{\mathrm{U}(\mu-1)-r}{r\mathrm{U}}\right)^2\left\{\mathrm{U}\tan\phi\frac{r(\mathrm{U}+r)}{\mathrm{U}(\mu-1)-r}\cdot\frac{1}{\mathrm{U}+r}\right\}^2\Big\{&\frac{\mathrm{U}(\mu-1)-r}{r(\mathrm{U}+r)}\\&+\frac{\mathrm{V}(\mu-1)-s}{s(\mathrm{V}+s)}\Big\}\frac{1}{2}\end{aligned}\right\}\qquad(29)$$

from (26B)

$$+\frac{\mu-1}{2\mu^2}\left(\frac{1}{s}+\frac{1}{\mathrm{V}}\right)^2\left(\frac{1}{s}+\frac{\mu+1}{\mathrm{V}}\right)\left(\tan\phi\frac{\mathrm{UD}}{\mathrm{U-D}}-\tan\phi\frac{\mathrm{V}s}{\mathrm{V}+s}\right)^2\qquad(30)$$

$$+\tan^2\phi\left(\frac{1}{\mathrm{U}+r}+\frac{1}{\mathrm{V}+s}\right)\frac{1}{2},\qquad(31)$$

in which formula X = the horizontal distance $a_2 \ldots \mathrm{X}'$.

Then (28) becomes

$$\left.\begin{aligned}\frac{\mu-1}{2\mu}\left(\frac{1}{r}+\frac{1}{\mathrm{U}}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{\mathrm{U}}\right)\Big\{&\left(\frac{\mathrm{UD}}{\mathrm{U-D}}\right)^2+2\frac{\mathrm{UD}}{\mathrm{U-D}}\cdot\frac{\mathrm{U}r}{\mathrm{U}+r}\\&+\left(\frac{\mathrm{U}r}{\mathrm{U}+r}\right)^2\Big\}\tan^2\phi,\end{aligned}\right\}\qquad(28\mathrm{A})$$

and (30) becomes

$$\left.\begin{aligned}\frac{\mu-1}{2\mu^2}\left(\frac{1}{s}+\frac{1}{\mathrm{V}}\right)^2\left(\frac{1}{s}+\frac{\mu+1}{\mathrm{V}}\right)\Big\{&\left(\frac{\mathrm{UD}}{\mathrm{U-D}}\right)^2-2\frac{\mathrm{UD}}{\mathrm{U-D}}\cdot\frac{\mathrm{V}s}{\mathrm{V}+s}\\&+\left(\frac{\mathrm{V}s}{\mathrm{V}+s}\right)^2\Big\}\tan^2\phi,\end{aligned}\right\}\qquad(30\mathrm{A})$$

from which we can again select out from (28A) and (30A) the function of the two aberrations

$$\frac{\mu-1}{2\mu^2}\left\{\left(\frac{1}{r}+\frac{1}{\mathrm{U}}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{\mathrm{U}}\right)+\left(\frac{1}{s}+\frac{1}{\mathrm{V}}\right)^2\left(\frac{1}{s}+\frac{\mu+1}{\mathrm{V}}\right)\right\}\left(\frac{\mathrm{UD}}{\mathrm{U}-\mathrm{D}}\right)^2\tan^2\phi, \qquad (32)$$

so we then get for the whole formula, after somewhat simplifying down Formula (29),

$$\left.\begin{aligned}\frac{1}{\mathrm{X}}=\frac{1}{\mathrm{F}}-\frac{1}{\mathrm{U}}+t\frac{\mu}{u'^2}+\frac{\mu-1}{2\mu^2}\left\{\left(\frac{1}{r}+\frac{1}{\mathrm{U}}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{\mathrm{U}}\right)\right.\\ \left.+\left(\frac{1}{s}+\frac{1}{\mathrm{V}}\right)^2\left(\frac{1}{s}+\frac{\mu+1}{\mathrm{V}}\right)\right\}\left(S\frac{\mathrm{U}}{\mathrm{U}-\mathrm{D}}\right)^2\end{aligned}\right\} \qquad (27)$$

$$+\frac{\mu-1}{2\mu^2}\left\{\left(\frac{1}{r}+\frac{1}{\mathrm{U}}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{\mathrm{U}}\right)+\left(\frac{1}{s}+\frac{1}{\mathrm{V}}\right)^2\left(\frac{1}{s}+\frac{\mu+1}{\mathrm{V}}\right)\right\}\left(\frac{\mathrm{UD}}{\mathrm{U}-\mathrm{D}}\right)^2\tan^2\phi \qquad (32)$$

$$\left.\begin{aligned}&+\frac{\mu-1}{2\mu^2}\left\{\left(\frac{1}{r}+\frac{1}{\mathrm{U}}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{\mathrm{U}}\right)\right\}\left\{2\frac{\mathrm{UD}}{\mathrm{U}-\mathrm{D}}\cdot\frac{\mathrm{U}r}{\mathrm{U}+r}+\left(\frac{\mathrm{U}r}{\mathrm{U}+r}\right)^2\right\}\tan^2\phi \qquad (33)\\ &+\frac{\mu-1}{2\mu^2}\left\{\left(\frac{1}{s}+\frac{1}{\mathrm{V}}\right)^2\left(\frac{1}{s}+\frac{\mu+1}{\mathrm{V}}\right)\right\}\left\{-2\frac{\mathrm{UD}}{\mathrm{U}-\mathrm{D}}\cdot\frac{\mathrm{V}s}{\mathrm{V}+s}+\left(\frac{\mathrm{V}s}{\mathrm{V}+s}\right)^2\right\}\tan^2\phi \qquad (34)\end{aligned}\right\}\left(\begin{matrix}\text{from (28)}\\ \text{and (30)}\end{matrix}\right)$$

$$+\frac{1}{\mu}\left\{\frac{\mathrm{U}(\mu-1)-r}{r(\mathrm{U}+r)}+\frac{\mathrm{V}(\mu-1)-s}{s(\mathrm{V}+s)}\right\}\frac{1}{2}\tan^2\phi \qquad (29\mathrm{A})$$

$$+\left(\frac{1}{\mathrm{U}+r}+\frac{1}{\mathrm{V}+s}\right)\frac{1}{2}\tan^2\phi. \qquad (31)$$

Adoption of the shape and vergency characteristics x and a.

Before simplifying down the above complex formula it is expedient to adopt Coddington's device which was explained on pages 65 and 66. Recapitulating, we have—since $\frac{1}{\mathrm{V}}+\frac{1}{\mathrm{U}}$ for the ultimate axial pencils $=\frac{1}{\mathrm{F}}$ and $\frac{1}{r}+\frac{1}{s}=\frac{1}{(\mu-1)}\frac{1}{\mathrm{F}}$, and

$$\frac{1}{\mathrm{U}}=\frac{1+a}{2\mathrm{F}} \qquad (35)$$

and

$$\frac{1}{\mathrm{V}}=\frac{1-a}{2\mathrm{F}}, \qquad (36)$$

so that

$$\frac{1+a}{2\mathrm{F}}+\frac{1-a}{2\mathrm{F}}=\frac{1}{\mathrm{F}},$$

then

$$\frac{1}{r}=\frac{1+x}{2(\mu-1)\mathrm{F}} \qquad (37)$$

and
$$\frac{1}{s}=\frac{1-x}{2(\mu-1)\mathrm{F}}, \tag{38}$$
so that
$$\frac{1+x}{2(\mu-1)\mathrm{F}}+\frac{1-x}{2(\mu-1)\mathrm{F}}=\frac{1}{(\mu-1)\mathrm{F}} \text{ or } \frac{1}{\rho}.$$

After substituting the above values of $\frac{1}{r}$, $\frac{1}{s}$, $\frac{1}{\mathrm{U}}$, and $\frac{1}{\mathrm{V}}$ in the above Formulæ (27), (32), (33), (34), (29A), and (31), excepting in those expressions involving D, which for the present it is desirable to keep intact, the above formulæ simplify down to the following:—

Secondary Plane.

Reciprocal of back focal distance corrected for thickness.

$$\frac{1}{\mathrm{X}}=\frac{1}{\mathrm{F}}-\frac{1}{\mathrm{U}}-\frac{1}{\mu}(x-a)^2\frac{t}{4\mathrm{F}^2} \tag{39}$$

Spherical aberration for all pencils.

$$\left.+\frac{1}{8\mathrm{F}^3\mu(\mu-1)}\left\{\frac{\mu+2}{\mu-1}x^2+4(\mu+1)ax+(3\mu+2)(\mu-1)a^2+\frac{\mu^3}{\mu-1}\right\}\Big(\underbrace{S\frac{\mathrm{U}}{\mathrm{U}-\mathrm{D}}}_{A^2}\Big)^2\right\} \quad \text{I. from (27)}$$

Normal curvature error.

$$+\frac{\tan^2\phi}{2\mathrm{F}}\cdot\frac{\mu+1}{\mu}\text{ ; from (29A) and (31)} \tag{40}$$

Eccentricity correction dependent on spherical aberration.

$$\left.+\frac{\tan^2\phi}{8\mathrm{F}^3\mu(\mu-1)}\left\{\frac{\mu+2}{\mu-1}x^2+4(\mu+1)ax+(3\mu+2)(\mu-1)a^2+\frac{\mu^3}{\mu-1}\right\}\left(\frac{\mathrm{DU}}{\mathrm{U}-\mathrm{D}}\right)^2\right\} \quad \text{II. from (32)}$$

Eccentricity correction dependent on coma.

$$+\frac{\tan^2\phi}{4\mathrm{F}^2(\mu-1)}\left\{4\mu a+\frac{2(\mu+1)}{\mu}(x-a)\right\}\frac{\mathrm{DU}}{\mathrm{U}-\mathrm{D}}\text{ ; from (33) and (34).} \quad \text{III.}$$

Formulæ (39) and I. apply to *all* pencils.

In (39) and I. we have the complete formulæ for the reciprocal of the distance from the back apex a_2 of the focus of the axial pencil whose semi-aperture at the element plane is $S\frac{\mathrm{U}}{\mathrm{U}-\mathrm{D}}$ or A, corresponding to Formula XXIII., page 67, for a pencil of semi-aperture $= y$. These Formulæ (39) and I. apply to all pencils, whether axial, oblique, or eccentric.

Formula (40) applies to *all* oblique pencils, whether central or not.

Formula (40) is the same again as Formula III., page 120, which we before worked out for the full lens aperture and for oblique but central pencils. It is now seen that it applies to all oblique pencils whether central or eccentric.

Formula II. applies only to eccentric pencils.

Formula II. is a further function of the spherical aberration of the lens applying *only* to eccentric pencils. Since the spherical aberration is almost invariably positive or of the same sign as the power of the lens, this correction is also almost invariably positive.

We may call II. the diaphragm correction or *stop correction* dependent upon the spherical aberration of the lens and the degree of eccentricity.

In III. we have a further correction applying *only* to eccentric pencils. It is a second stop correction due to the presence of *coma* in the lens or eccentric oblique refraction. It is as well, before entering more closely into the nature of these stop corrections II. and III. and their causes, to first investigate the case of the rays of the same eccentric oblique pencils contained in the primary plane.

Formula III. applies only to eccentric pencils.

Rays of Eccentric Oblique Pencils contained in the Primary Plane

In this case we may follow much the same lines of construction as we did in tracing rays in the primary planes of central oblique pencils, Figs. 44 and 45. In Figs. 50 and 50*a* let n_1 and w_1 be the two points where the two extreme rays in the primary plane passed by the stop D'..D' strike the first element plane of the lens. Join the radiant point Q to r', the centre of curvature, and produce it to the ultimate focus of the pencil at q. Obviously ray Q..n_1 meets with more spherical aberration than does ray Q..w_1, and therefore intersects the normal oblique ray Q..r'..q at f_1 nearer to the lens than the point f_2 where ray Q..w_1 intersects Q..r'..q.

Let $c_1..n_1 = y_1$ and $c_1..w_1 = y_2$. Let O_1 be the point where the principal or central ray of the pencil cuts the element plane. Let aperture of stop $= 2S$ as before. Let q_1'' be the point to be found where rays Q..n_1..f_1 and Q..w_1..f_2 intersect one another. It evidently lies somewhat to one side of the oblique axis Q..r'..q by the small distance $q_1''..p_1''$ measured perpendicular to the lens axis P..r'. Let x_1 stand for the desired distance of q_1'' from the vertex d_1; that is, $x_1 = d_1..q_1''$. Let $d_1..f_1 = f_1$, and $d_1..f_2 = f_2$. Then pursuing a process analogous to that pursued in the case of Fig. 44, page 121, with the difference that in this case the two y's are on the same side of the normal ray Q..r'..q, we have

$$y_1\frac{f_1 - x_1}{f_1} = (q_1''..p_1'') = y^2\frac{f_2 - x_1}{f_2},$$

The fundamental equation.

from which

$$\frac{1}{x_1} = \frac{y_2 f_1 - y_1 f_2}{f_1 f_2 (y_2 - y_1)} = \left(\frac{y_2}{f_2} - \frac{y_1}{f_1}\right)\frac{1}{y_2 - y_1}.$$

Then adopting the same device as before we get

$$\frac{1}{x_1} = \left\{y_2\left(\frac{1}{\grave{u}} + \frac{\omega_1}{\mu}y_2^2\right) - y_1\left(\frac{1}{\grave{u}} + \frac{\omega_1}{\mu}y_1^2\right)\right\}\frac{1}{y_2 - y_1}$$

$$=\frac{\frac{1}{u}(y_2-y_1)+\frac{\omega_1}{\mu}(y_2^3-y_1^3)}{y_2-y_1}=\frac{1}{u}+\frac{\omega_1}{\mu}(y_1^2+y_2^2+y_1y_2);$$

$$\therefore\ \frac{\mu}{x_1}=\underbrace{\frac{\mu-1}{r}-\frac{1}{\mathrm{U}}+\tan^2\phi\frac{1}{2(\mathrm{U}+r)}}_{=\frac{\mu}{u}}+\omega_1(y_1^2+y_2^2+y_1y_2). \qquad (41)$$

Here it may be remarked that we now get $\omega_1(y_1^2+y_2^2+y_1y_2)$ instead of the $\omega_1(y_1^2+y_2^2-y_1y_2)$ which we arrived at in Section V., dealing with central oblique refraction (Fig. 44). But this difference is simply due to the fact that in Fig. 44 we had the two extreme primary rays refracted on opposite sides of the normal oblique ray Q .. r', so that the two y's were also on opposite sides; whereas in this case of Fig. 50 we have both the extreme primary rays refracted on the same side of the normal oblique ray Q .. r', so that the two y's are now on the same side. This leads to a difference in the statement of our fundamental equation, for in the earlier case of Fig. 44 it was

When the y's are on opposite sides of the normal oblique ray.

$$y_1\frac{(x_1-f_1)}{f_1}=(q'\,..\,p')=y_2\frac{f_2-x}{f_2},$$

but in this case of Fig. 50 it is

When the y's are on the same side.

$$y_1\frac{f_1-x}{f_1}=(q_1''\,..\,p_1'')=y_2\frac{f_2-x}{f_2}.$$

But if we put

$$C=a_1\,..\,\mathrm{O}_1=\tan\phi\frac{\mathrm{UD}}{\mathrm{U-D}},\ B_1=a_1\,..\,c_1=r\tan\theta \text{ or } \tan\phi\frac{\mathrm{U}r}{\mathrm{U}+r}, \text{ and}$$

$$A=(\mathrm{O}_1\,..\,n_1)=(\mathrm{O}_1\,..\,w)_1=S\frac{\mathrm{U}}{\mathrm{U-D}},$$

we shall then find that

$$y_1^2+y_2^2+y_1y_2\text{ (of Fig. 50)}=A^2+3(B_1+C)^2=y_1^2+y_2^2-y_1y_2\text{ (of Fig. 44)},$$

since in Fig. 50

$$y_1=A+(B_1+C)\text{ and }y_2=-A+(B_1+C),$$

while in Fig. 44 we may consider that

$$y_1=A+(B+C)\text{ and }y_2=A-(B_1+C)$$

Identity of the final results.

wherein $C=0$ as the refraction is central, so that in all cases we arrive at the same result when $y_1^2+y_2^2+y_1y_2$ or $y_1^2+y_2^2-y_1y_2$ are expressed in terms of A (the semi-aperture of the pencil in the element plane) and B_1 and C.

Now approximately

$$y_1^2 = \left\{ \mathrm{U} \tan \phi \frac{\mathrm{D}}{\mathrm{U-D}} + r \tan \theta_1 + S \frac{\mathrm{U}}{\mathrm{U-D}} \right\}^2 = (B_1 + C + A)^2$$

and $\left(\begin{array}{l} \text{and } r \tan \theta \\ = \tan \phi \dfrac{\mathrm{U}r}{\mathrm{U}+r} \end{array} \right)$

$$y_2^2 = \left\{ \mathrm{U} \tan \phi \frac{\mathrm{D}}{\mathrm{U-D}} + r \tan \theta - S \frac{\mathrm{U}}{\mathrm{U-D}} \right\}^2 = (B_1 + C - A)^2 ;$$

$$\therefore \; y_1^2 = \left\{ \tan \phi \left(\frac{\mathrm{UD}}{\mathrm{U-D}} + \frac{\mathrm{U}r}{\mathrm{U}+r} \right) + S \frac{\mathrm{U}}{\mathrm{U-D}} \right\}^2,$$

and

$$y_2^2 = \left\{ \tan \phi \left(\frac{\mathrm{UD}}{\mathrm{U-D}} + \frac{\mathrm{U}r}{\mathrm{U}+r} \right) - S \frac{\mathrm{U}}{\mathrm{U-D}} \right\}^2,$$

and

$$y_1 y_2 = \left\{ \tan^2 \phi \left(\frac{\mathrm{UD}}{\mathrm{U-D}} + \frac{\mathrm{U}r}{\mathrm{U}+r} \right)^2 - \left(S \frac{\mathrm{U}}{\mathrm{U-D}} \right)^2 \right\};$$

$$\therefore \; y_1^2 + y_2^2 + y_1 y_2 =$$

$$\left\{ \begin{array}{l} \tan^2 \phi \left(\dfrac{\mathrm{UD}}{\mathrm{U-D}} + \dfrac{\mathrm{U}r}{\mathrm{U}+r} \right)^2 + 2 \tan \phi \left(\dfrac{\mathrm{UD}}{\mathrm{U-D}} + \dfrac{\mathrm{U}r}{\mathrm{U}+r} \right) \left(S \dfrac{\mathrm{U}}{\mathrm{U-D}} \right) + \left(S \dfrac{\mathrm{U}}{\mathrm{U-D}} \right)^2 \\ + \tan^2 \phi \left(\dfrac{\mathrm{UD}}{\mathrm{U-D}} + \dfrac{\mathrm{U}r}{\mathrm{U}+r} \right)^2 - 2 \tan \phi \left(\dfrac{\mathrm{UD}}{\mathrm{U-D}} + \dfrac{\mathrm{U}r}{\mathrm{U}+r} \right) \left(S \dfrac{\mathrm{U}}{\mathrm{U-D}} \right) + \left(S \dfrac{\mathrm{U}}{\mathrm{U-D}} \right)^2 \\ + \tan^2 \phi \left(\dfrac{\mathrm{UD}}{\mathrm{U-D}} + \dfrac{\mathrm{U}r}{\mathrm{U}+r} \right)^2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad - \left(S \dfrac{\mathrm{U}}{\mathrm{U-D}} \right)^2 \end{array} \right\}$$

$$\left. \begin{array}{l} \therefore \; y_1^2 + y_2^2 + y_1 y_2 = \left(S \dfrac{\mathrm{U}}{\mathrm{U-D}} \right)^2 + 3 \tan^2 \phi \left\{ \left(\dfrac{\mathrm{UD}}{\mathrm{U-D}} \right)^2 \right. \\ \qquad\qquad\qquad \left. + 2 \dfrac{\mathrm{UD}}{\mathrm{U-D}} \cdot \dfrac{\mathrm{U}r}{\mathrm{U}+r} + \left(\dfrac{\mathrm{U}r}{\mathrm{U}+r} \right)^2 \right\}. \end{array} \right\} \quad (42)$$

Full value of the function of y_1 and y_2.

The Refraction at the Second Surface

At the second refraction, illustrated in Figs. 51 and 51a, after adopting the same construction and putting x_2 for the required distance of the focus q_2'' from the second vertex d_2, f_1 for the distance from d_2 of the intersection of ray $\mathrm{Q} \ldots n_2 \ldots f_1'$ with the normal oblique ray $s' \ldots q$, f_2 for the distance from d_2 of the intersection of ray $\mathrm{Q} \ldots w_2 \ldots f_2'$ with the same normal oblique ray, Y_1 for $c_2 \ldots n_2$, and Y_2 for $c_2 \ldots w_2$, we then have, as in the cases of Figs. 44 and 45,

$$y_1 \frac{x_2 - f_1}{f_1} = (q_2'' \ldots p_2'') = \mathrm{Y}_2 \frac{f_2 - x_2}{f_2}, \qquad (43)$$

The fundamental equation.

from which
$$\frac{1}{x_2} = \frac{1}{v} + \omega_2(Y_1^2 + Y_2^2 - Y_1Y_2), \tag{44}$$

as in Fig. 45, wherein the two Y's were, as in Fig. 51, on opposite sides of the oblique axis $s'..q$. It is clear that the eccentricity C or $a_2..O_2$ of Fig. 51 is equal to the $a_1..O_1$ of Fig. 50. Also A is the same at both surfaces; only B_1 and B_2 are different. In this case

$$Y_1^2 = \left\{\left(U\tan\phi\frac{D}{U-D} - s\tan\phi\frac{V}{V+s}\right) + S\frac{U}{U-D}\right\}^2$$
$$= \left\{(C - B_2) + A\right\}^2 = c_2..n_2,$$

$$Y_2^2 = \left\{-\left(U\tan\phi\frac{D}{U-D} - s\tan\phi\frac{V}{V+s}\right) + S\frac{U}{U-D}\right\}^2$$
$$= \left\{-(C - B_2) + A\right\}^2 = c_2..w_2,$$

$$\therefore\ Y_1^2 + Y_2^2 - Y_1Y_2 \begin{cases} \tan^2\phi\left(\dfrac{UD}{U-D} - \dfrac{Vs}{V+s}\right)^2 + 2\tan^2\phi\left(\dfrac{UD}{U-D} - \dfrac{Vs}{V+s}\right) \\ \qquad \times\left(S\dfrac{U}{U-D}\right) + \left(S\dfrac{U}{U-D}\right)^2 \\ +\tan^2\phi\left(\dfrac{UD}{U-D} - \dfrac{Vs}{V+s}\right)^2 - 2\tan^2\phi\left(\dfrac{UD}{U-D} - \dfrac{Vs}{V+s}\right) \\ \qquad \times\left(S\dfrac{U}{U-D}\right) + \left(S\dfrac{U}{U-D}\right)^2 \\ -\left\{-\tan^2\phi\left(\dfrac{UD}{U-D} - \dfrac{Vs}{V+s}\right)^2 + \left(S\dfrac{U}{U-D}\right)^2\right\} \end{cases}$$

$$\left.\begin{aligned} \therefore\ Y_1^2 + Y_2^2 - Y_1Y_2 = \left(S\frac{U}{U-D}\right)^2 + 3\tan^2\phi\Bigg\{\left(\frac{UD}{U-D}\right)^2 \\ - 2\frac{UD}{U-D}\cdot\frac{Vs}{V+s} + \left(\frac{Vs}{V+s}\right)^2\Bigg\}; \end{aligned}\right\} \tag{45}$$

therefore the sum of the compounded aberrations at the two surfaces is

$$\omega_1(y_1 + y_2 + y_1y_2) + \omega_2(Y_1^2 + Y_2^2 - Y_1Y_2)$$
$$= (\omega_1 + \omega_2)\left\{S^2\frac{U^2}{(U-D)^2} + 3\tan^2\phi\left(\frac{UD}{U-D}\right)^2\right\} + \omega_1\left\{6\tan^2\phi\frac{UD}{U-D}\cdot\frac{Ur}{U+r}\right\}$$
$$- \omega_2\left\{6\tan^2\phi\frac{UD}{U-D}\cdot\frac{Vs}{V+s}\right\} + \omega_1\left\{3\tan^2\phi\left(\frac{Ur}{U+r}\right)^2\right\}$$
$$+ \omega_2\left\{3\tan^2\phi\left(\frac{Vs}{V+s}\right)^2\right\}.$$

We have then to add to the above the two end corrections for obliquity (31), and also the correction (29) or (29A) for converting $\frac{1}{u}$ into $\frac{1}{v}$; and then after gathering together all corrections and putting $\frac{1}{X}$ for the corrected reciprocal of the final axial or horizontal distance $a_2 \ldots X'$ of the final focus q_2'' from the back vertex a_2 of the lens, we get, after cancelling out in (49) and (50), the following complete formulæ:—

$$\frac{1}{X}=\frac{1}{F}-\frac{1}{U}+t\frac{\mu}{u^2} \tag{46}$$

$$+\frac{\mu-1}{2\mu^2}\left\{\left(\frac{1}{r}+\frac{1}{U}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{U}\right)+\left(\frac{1}{s}+\frac{1}{V}\right)\left(\frac{1}{s}+\frac{\mu+1}{V}\right)\right\}\overbrace{\left(S\frac{U}{U-D}\right)^2}^{A^2} \tag{47}$$

$$+\frac{\mu-1}{2\mu^2}\left\{\left(\frac{1}{r}+\frac{1}{U}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{U}\right)+\left(\frac{1}{s}+\frac{1}{V}\right)^2\left(\frac{1}{s}+\frac{\mu+1}{V}\right)\right\}\left(\frac{UD}{U-D}\right)^2 3\tan^2\phi \tag{48}$$

$$\left.\begin{aligned}&+\left[\frac{\mu-1}{2\mu^2}\left\{\left(\frac{1}{r}+\frac{1}{U}\right)\left(\frac{1}{r}+\frac{\mu+1}{U}\right)\right\}\right.\\&\qquad\left.-\frac{\mu-1}{2\mu^2}\left\{\left(\frac{1}{s}+\frac{1}{V}\right)\left(\frac{1}{s}+\frac{\mu+1}{V}\right)\right\}\right]\left(\frac{UD}{U-D}\right)6\tan^2\phi\end{aligned}\right\} \tag{49}$$

$$+\frac{\mu-1}{2\mu^2}\left\{\left(\frac{1}{r}+\frac{\mu+1}{U}\right)+\left(\frac{1}{s}+\frac{\mu+1}{V}\right)\right\}3\tan^2\phi \tag{50}$$

$$+\frac{1}{2\mu}\left\{\frac{U(\mu-1)-r}{r(U+r)}+\frac{V(\mu-1)-s}{s(V+s)}\right\}\tan^2\phi \text{ (from (29A))} \tag{51}$$

$$+\frac{1}{2}\left(\frac{1}{U+r}+\frac{1}{V+s}\right)\tan^2\phi \text{ (from (31))} \tag{52}$$

After again adopting the same device as in the last corresponding case of rays in the secondary plane, the above complex formula reduces down to

Primary Plane.

Reciprocal of back focal distance, corrected for thickness.

$$\frac{1}{X}=\frac{1}{F}-\frac{1}{U}-\frac{1}{\mu}(x-a)^2\frac{t}{4F^2} \tag{53}$$

Spherical aberration for all pencils.

$$\left.\begin{aligned}&+\frac{1}{8F^3\mu(\mu-1)}\left\{\frac{\mu+2}{\mu-1}x^2+4(\mu+1)ax+(3\mu+2)(\mu-1)a^2\right.\\&\qquad\qquad\left.+\frac{\mu^3}{\mu-1}\right\}\underbrace{\left(S\frac{U}{U-D}\right)^2}_{A^2}\end{aligned}\right\} \text{(54) or I.}$$

Normal curvature error.

$$+\frac{\tan^2\phi}{2F}\cdot\frac{3\mu+1}{\mu} \tag{55}$$

Eccentricity correction dependent on spherical aberration.

$$\left.\begin{aligned}&+\frac{3\tan^2\phi}{8F^3\mu(\mu-1)}\left\{\frac{\mu+2}{\mu-1}x^2+4(\mu+1)ax+(3\mu+2)(\mu-1)a^2\right.\\&\qquad\qquad\left.+\frac{\mu^3}{\mu-1}\right\}\left(\frac{DU}{U-D}\right)^2\end{aligned}\right\} \text{IV.}$$

Eccentricity correction dependent on coma.

$$+\frac{3\tan^2\phi}{4F^2(\mu-1)}\left\{4\mu a+\frac{2(\mu+1)}{\mu}(x-a)\right\}\frac{DU}{U-D}. \qquad \text{V.}$$

The Formula (54) is the spherical aberration common to *all* pencils of light passing through the stop. Formula (55) is the normal curvature error for all oblique pencils, central or eccentric. Formula IV. gives the stop correction for all eccentric oblique pencils due to the spherical aberration of the lens; while Formula V. gives the stop correction for the same pencils due to coma in the lens. All these are R corrections to be applied to the first approximate value of $\frac{1}{V}$, as obtained from $\frac{1}{V}=\mu-1\left(\frac{1}{r}+\frac{1}{s}\right)-\frac{1}{U}$, or $\frac{1}{F}-\frac{1}{U}$.

Ratio between the Eccentricity Corrections in the two planes.

Thus the R corrections due to the presence of the stop, viz. IV. and V., for rays in primary planes come out just three times the corresponding stop corrections for rays in secondary planes, viz. II. and III.

Conventions under which the formulæ are universally true.

The student may with advantage pursue the same processes in the case of positive and negative lenses and meniscus lenses with the entering rays both divergent and convergent, the stop being real, and either in front of or behind the lens, or else virtual only, adhering always to the following conventions, consistently with those already laid down on page 10.

Collective Lenses.

Rays constituting the pencils.

Collective Lenses or Menisci

Entering rays diverging,	U is +	intrinsically.
" " converging,	U is −	"
Emergent rays converging,	V is +	"
" " diverging,	V is −	"

Principal rays.

Stop in front of lens and real, or entering principal rays diverging	D′ is +	intrinsically.
Stop behind lens and virtual, or entering principal rays converging	D′ is −	"
Stop behind lens and real, or emergent principal rays converging	D″ is +	"
Stop in front of lens and virtual, or emergent principal rays diverging	D″ is −	"

Thus we may write D′ for the distance from lens to where the principal rays cross the optic axis *before* entering the lens, and D″ for the refracted distance, conjugate to the former, between the lens and the point where the principal rays cross the optic axis *after* refraction.

Dispersive Lenses and Menisci

Dispersive Lenses. Rays constituting the pencils.

Entering rays converging, U is + intrinsically
" " diverging, U is − "
Emergent rays diverging, V is + "
" " converging, V is − "

Principal rays.

Stop behind lens and virtual, or entering principal rays converging } D′ is + intrinsically.
Stop in front of lens and real, or entering principal rays diverging } D′ is − "
Stop in front of lens and virtual, or emergent principal rays diverging } D″ is + "
Stop behind lens and real, or emergent principal rays converging } D″ is − "

Seeing that such principal rays are compelled to cross the axis of the lens at the centre of the stop, or at any image of such stop, therefore that centre has to be regarded as an axial point from which such principal rays are diverging or to which they are converging, and since these principal rays are refracted by the lens in precisely the same manner as any other rays, therefore it is universally true that D′ and D″, in relation to any one lens in any particular case, are conjugate focal distances, such that

$$\frac{1}{D''}=\frac{1}{F}-\frac{1}{D'}. \tag{56}$$

Introduction of the new vergency characteristic β for the principal rays.

Therefore we can carry Coddington's device one step further and let β stand as the characteristic of the state of divergence or convergence of the principal rays with respect to the lens, so that

$$\frac{1+\beta}{2F}=\frac{1}{D'} \text{ and } \frac{1-\beta}{2F}=\frac{1}{D''}. \tag{57}$$

β is thus closely analogous to a, and may be called *the vergency characteristic for the principal rays.* Then $\left(\frac{DU}{U-D}\right)^2$ converts into $\frac{4F^2}{(\beta-a)^2}$ and $\frac{DU}{U-D}$ into $\frac{2F}{\beta-a}$, since the D we have so far been dealing with was the front conjugate distance D′, relating to the *entering* principal rays.

Therefore Formula IV. becomes

The spherical aberration Eccentricity Correction.

$$\frac{3\tan^2\phi}{2F\mu(\mu-1)}\,\frac{1}{(a-\beta)^2}\left\{\frac{\mu+2}{\mu-1}x^2+4(\mu+1)ax+(3\mu+2)(\mu-1)a^2+\frac{\mu^3}{\mu-1}\right\}, \quad \text{VI.}$$

and Formula V. becomes, after multiplying by $\frac{1}{\mu}(\mu)$,

$$\frac{3\tan^2\phi}{2F\mu(\mu-1)}\left\{4\mu^2\alpha + 2(\mu+1)(x-\alpha)\right\}\frac{1}{\beta-\alpha},$$

or more conveniently

The comatic Eccentricity Correction.

$$-\frac{3\tan^2\phi}{F\mu(\mu-1)}\frac{1}{(\alpha-\beta)}\left\{(2\mu+1)(\mu-1)\alpha + (\mu+1)x\right\}. \qquad \text{VII.}$$

So that these two stop corrections may be bracketed together thus—

The above two corrections combined.

$$\left.\frac{3\tan^2\phi}{2F\mu(\mu-1)}\frac{1}{(\alpha-\beta)^2}\left[\frac{\mu+2}{\mu-1}x^2 + 4(\mu+1)\alpha x + (3\mu+2)(\mu-1)\alpha^2 + \frac{\mu^3}{\mu-1} - 2\left\{(2\mu+1)(\mu-1)\alpha + (\mu+1)x\right\}(\alpha-\beta)\right]\right\} \text{VIII.}$$

We may often have occasion to write this formula in the abbreviated form

Abbreviated form for the Eccentricity Corrections.

$$\frac{3\tan^2\phi}{2F}\frac{1}{(\alpha-\beta)^2}\left\{A' - 2(\alpha-\beta)C'\right\}. \qquad (58)$$

For rays in secondary planes the $3\tan^2\phi$ is replaced by $\tan^2\phi$.

Subject to the conventions as to the intrinsic signs of U, V, D′, and D″, these formulæ are universally true of all lenses, provided their axial thicknesses are very small. Calling the inevitable curvature errors $\frac{\tan^2\phi}{2F}\cdot\frac{\mu+1}{\mu}$ and $\frac{\tan^2\phi}{2F}\frac{3\mu+1}{\mu}$, which are incidental to central oblique pencils, the *normal curvature corrections*, then Formula VI. expresses what is nearly always a plus stop correction due to the joint effect of the spherical aberration and the selective action of the stop upon eccentric pencils, while Formula VII. expresses what is a very variable stop correction, sometimes plus and sometimes minus, due to the joint effect of coma, or eccentric oblique refraction, and the selective action of the stop.

Thus diaphragm or stop corrections may be defined broadly as corrections applicable to oblique pencils refracted *eccentrically* through a lens, causing *more or less serious departures from its normal curvature corrections.* It is more convenient to call these diaphragm corrections *eccentricity corrections*, or E.C.s for brevity.

Comparison of above results with Coddington's formulæ.

Turning now to the comparison of these results with those worked out by Coddington, more especially in his Prop. 123, p. 132, it might be thought on first inspection that they are quite at variance.

In secondary planes he arrived at the formula for an infinitely thin pencil refracted eccentrically through a lens—

$$\frac{1}{k} = \frac{1}{F} - \frac{1}{h} + \left(V + \frac{1}{\mu}\right)\frac{1}{k^2}\frac{z^2}{2F}; \qquad (59)$$

wherein his

$$\frac{1}{k} = \text{our } \frac{1}{X}, \; \frac{1}{h} = \text{our } \frac{1}{U}, \text{ and } \frac{z^2}{k^2} = \text{our } \tan^2\phi\,;$$

while his term V:

$$= \frac{1}{\mu(\mu-1)}\frac{1}{(a-\beta)^2}\left\{\frac{\mu+2}{\mu-1}x^2 + 2(\mu+1)(a-\beta)x + 2(\mu+1)(\mu-1)a\beta + \mu(\mu-1)\beta^2 + \frac{\mu^3}{\mu-1}\right\} \text{VIII}_{\text{A}}.$$

Coddington's Formula.

It follows from his method that his $\left(V+\frac{1}{\mu}\right)\frac{1}{k^2}\cdot\frac{z^2}{2F}$ in secondary planes and $\left(3V+\frac{1}{\mu}\right)\frac{1}{k^2}\cdot\frac{z^2}{2F}$ in primary planes are inclusive formulæ, embracing not only the corrections due to eccentric refraction of oblique pencils, but also the corrections due to their central refraction.

If, however, we take the normal curvature corrections

$$\frac{\tan^2\phi}{2F}\left(\frac{\mu+1}{\mu} \text{ or } \frac{3\mu+1}{\mu}\right)$$

in the form

$$\frac{\tan^2\phi}{2F}\left\{\left(1+\frac{1}{\mu}\right) \text{ or } \left(3+\frac{1}{\mu}\right)\right\}$$

and add them to our corresponding Formula VIII. we get

$$\frac{\tan^2\phi}{2F}\left[\frac{1 \text{ or } 3}{\mu(\mu-1)(a-\beta)^2}\left\{\left(\frac{\mu+2}{\mu-1}x^2 + 4(\mu+1)ax + (3\mu+2)(\mu-1)a^2 + \frac{\mu^3}{\mu-1}\right) - 2\Big((2\mu+1)(\mu-1)a(a-\beta) + (\mu+1)x\Big)(a-\beta) + \mu(\mu-1)(a-\beta)^2\right\} + \frac{1}{\mu}\right] \text{VIII}_{\text{B}}.$$

This will be found to reduce exactly to Coddington's

$$\frac{\tan^2\phi}{2F}\left\{\left(V+\frac{1}{\mu}\right) \text{ or } \left(3V+\frac{1}{\mu}\right)\right\}.$$

Formula VIII. confirmed by Coddington's results.

Hence it is evident that in his formula he got the normal curvature corrections, the E.C.s due to spherical aberration and the E.C.s due to coma all mixed up together in a manner unfortunately most inconvenient for practical purposes.

Mixed-up nature of Coddington's formulæ.

It is a most curious fact that throughout Coddington's work there is no allusion to such a well-recognised thing as "coma"; indeed it is doubtful whether he could have been aware of its existence without at least attempting to work out a formula for it and its effects. On

Coddington apparently unaware of coma.

page 159, in the course of discussing aplanatic combinations of lenses in contact, he says: "The next question that offers itself is the advantage to be derived from a combination of lenses when a pencil passes through it centrically but obliquely. It will, however, easily be seen that as the effects of obliquity in this case are totally independent of the form of a single lens, so they cannot be removed or diminished by any combination." While this statement is quite true in regard to the normal curvature of image, yet the possibility of coma being either present or absent is entirely overlooked. Every practical optician is aware that some objectives for telescopes are extremely sensitive to being thrown out of square, while others are not; the former show strong coma at the foci of even slightly oblique pencils, while the latter show little or none, but only pure astigmatism, while simple lenses show the same differences, only there is spherical aberration superadded. Such objectives without coma give better definition for a considerable angular distance from the axis than do those whose oblique images are marred by coma or eccentric oblique refraction; although the normal curvature of image and astigmatism can be shown to vary only slightly in different cases. We will revert to this subject again with greater advantage at the end of Section VIII. The phenomenon of coma is not only deeply interesting, but of great practical importance, and we will reserve a more thorough investigation into its properties for Section VIII.

Coma met with in every-day optical practice.

Before concluding this Section, we may with advantage consider a question that may already have occurred to the reader with regard to Formula VIII. for the Eccentricity Corrections.

Incongruous nature of the two Eccentricity Corrections.

Since the E.C.s consequent upon the spherical aberration of the lens vary as $\frac{1}{(\alpha-\beta)^2}$, and the E.C.s consequent upon coma in the lens vary as $\frac{1}{\alpha-\beta}$, and since the value of $\left(\frac{1}{\alpha-\beta}\right)^2$ increases more rapidly than does $\frac{1}{\alpha-\beta}$ when the stop is removed farther from the lens, therefore the plus E.C.s consequent upon the spherical aberration must rapidly overtake in value the comatic E.C.s, therefore we should expect that there should be a limit to the distance of the stop, beyond which it will be impossible to obtain an excess of minus comatic E.C.s, or even a neutral balance of minus comatic E.C.s against plus aberration E.C.s.

Limits to the useful position of the stop.

In other words, if we want to modify the normal curvature of images in the direction of *flattening* them, we must take care that our stop is not placed too far from the lens, or else the plus aberration

E.C.s will inevitably prevail and the images be more curved than before.

Now, if we have eccentric refraction of oblique pencils through a simple thin lens, and we wish to preserve the normal curvature of the images, then we must equate the E.C.s to 0; that is, we must have

$$\left.\begin{aligned} &\frac{\mu+2}{\mu-1}x^2 + 4(\mu+1)ax + (3\mu+2)(\mu-1)a^2 + \frac{\mu^3}{\mu-1} \\ &\qquad - 2(a-\beta)\{(2\mu+1)(\mu-1)a + (\mu+1)x\} = 0 \end{aligned}\right\} \quad \text{VIIIc.}$$

Condition for equating E.C.s to 0.

This formula yields the following quadratic equation:—

$$\left.\begin{aligned} &x^2 + 2\frac{(\mu^2-1)}{(\mu+2)}(a+\beta)x + \frac{(\mu^2-1)^2}{(\mu+2)^2}(a+\beta)^2 \\ &= \left(\frac{\mu-1}{\mu+2}\right)^2(2\mu^2+4\mu+1)a^2 - 2\left(\frac{\mu-1}{\mu+2}\right)^2(\mu^2+3\mu+1)a\beta \\ &\qquad + \left(\frac{\mu^2-1}{\mu+2}\right)^2\beta^2 - \frac{\mu^3}{\mu+2}. \end{aligned}\right\} \quad \text{VIIId.}$$

Quadratic equation derived from above.

In order that E.C.s may be *just possibly* eliminated, it is obvious that we must have the right-hand side of the equation equal to 0, from which, since a is a known quantity, we may derive the limiting value of β, and then obtain the necessary correlative value of x from the left-hand side.

In this way we may derive the following limiting values for β and x:—

If $\mu = 1{\cdot}5$, and $a = -1$, then

$$\beta = \begin{cases} +1{\cdot}45 \\ \text{or } -3{\cdot}93 \end{cases} \quad \text{and } x = \begin{cases} -{\cdot}15 \\ \text{or } +1{\cdot}76 \end{cases}$$

When $\mu = 1{\cdot}5$.

If $\beta = +1{\cdot}45$, then the stop is ·817F in front of the lens.

If $\beta = -3{\cdot}93$, then the stop is at a distance = ·405F behind the lens. In either case x indicates the meniscus form of collective lens with the concave side facing the stop.

If $\mu = 1{\cdot}6$ and a again $= -1$, then

$$\beta = \begin{cases} +1{\cdot}523 \\ \text{or } -2{\cdot}757 \end{cases} \quad \text{and } x = \begin{cases} -{\cdot}225 \\ \text{or } +1{\cdot}62 \end{cases}$$

When $\mu = 1{\cdot}6$.

If $\beta = +1{\cdot}523$, then the stop is ·79F in front of the lens.

If $\beta = -2{\cdot}757$, then the stop is ·53F behind the lens.

So that the above stop distances are the maximum permissible if we wish to get our images flatter than the normal by means of E.C.s.

Hence we cannot expect to obtain a flat final image from such a combination as a Cooke portrait or astro-photographic lens if the separations between the simple lenses composing it exceed the limits implied in the above Formula VIIID.

It is often useful to know the effect upon the Eccentricity Corrections of a lens (as expressed in Formula VIII. of this Section) of slight alterations in the value of a, β, or x, and we will here give the differentials of the E.C. formula with respect to these three characteristics for rays in primary planes.

1st, with respect to a—

Differential of the Eccentricity Corrections when α varies.

$$d_\alpha \frac{3\tan^2\phi}{2f}\frac{1}{(\alpha-\beta)^2}\left\{A' - 2(\alpha-\beta)C'\right\}$$

$$= \frac{3\tan^2\phi}{f}\left[-\frac{1}{(\alpha-\beta)^3}\left\{A'\right\} + \frac{1}{(\alpha-\beta)^2}\left\{\frac{3(\mu+1)}{\mu(\mu-1)}x + \frac{5\mu+3}{\mu}\alpha\right\} - \frac{1}{\alpha-\beta}\left\{\frac{2\mu+1}{\mu}\right\}\right]d\alpha, \qquad \text{IX.}$$

from which we see that the effect of a change in the divergency of the entering rays is somewhat complex.

2nd, with respect to β—

Differential of the E.C.s when β varies.

$$d_\beta \frac{3\tan^2\phi}{2f}\frac{1}{(\alpha-\beta)^2}\left\{A' - 2(\alpha-\beta)C'\right\}$$

$$= +\frac{3\tan^2\phi}{f}\left[\frac{1}{(\alpha-\beta)^3}\left\{A'\right\} - \frac{1}{(\alpha-\beta)^2}\left\{C'\right\}\right]d\beta, \qquad \text{X.}$$

which is necessarily an expression of a much simpler nature than the last.

3rd, with respect to x—

Differential of the E.C.s when x varies.

$$d_x \frac{3\tan^2\phi}{2f}\frac{1}{(\alpha-\beta)^2}\left\{A' - 2(\alpha-\beta)C'\right\}$$

$$= \frac{3\tan^2\phi}{f}\frac{1}{(\alpha-\beta)^2}\left[\left\{\frac{(\mu+2)}{\mu(\mu-1)^2}x + \frac{2(\mu+1)}{\mu(\mu-1)}\alpha\right\} - (\alpha-\beta)\frac{\mu+1}{\mu(\mu-1)}\right]dx, \qquad \text{XI.}$$

which is perhaps the most useful of the above three differentials.

SECTION VII

ON SYSTEMS OF LENSES AND THE APPLICATION OF THE THEOREM OF ELEMENTS TO THICK LENSES

SOME consequences of the greatest practical importance follow from the various formulæ arrived at in the last Section.

Single Lenses.

Why normal curvature errors cannot be neutralised by E.C.s in both planes at once.

First of all, since $\frac{\mu+1}{\mu}$ and $\frac{3\mu+1}{\mu}$ represent the relative normal curvature corrections of any simple lens, and as these functions stand generally in the ratio of 1 to 2·2, while the Eccentricity Corrections in primary planes are always three times the corresponding E.C.s in secondary planes, it follows that the two normal curvature errors of a simple lens cannot possibly be simultaneously neutralised by E.C.s, due to the presence of a stop placed anywhere on the optic axis. If the normal curvature errors in primary planes are neutralised by E.C.s, so that the image formed by rays in primary planes is got quite flat, in which case

$$\text{E.C.s (in pr. plane)} = \frac{\tan^2\phi}{2F}\cdot\frac{3\mu+1}{\mu},$$

then $\frac{\tan^2\phi}{2F}\left(\frac{\mu+1}{\mu}-\frac{1}{3}\cdot\frac{3\mu+1}{\mu}\right)$ will represent the remaining curvature error for rays in secondary planes. This is equivalent to $\frac{\tan^2\phi}{2F}\left(\frac{2}{3}\cdot\frac{1}{\mu}\right)$, so that the radius of curvature of the image formed by rays in secondary planes will be

Curvature of secondary image when primary image is flat.

$$F\frac{3\mu}{2} \text{ when the primary image is flat.} \qquad \text{XII.}$$

If $\mu = 1{\cdot}5$, then $F\frac{3\mu}{2} = 2\frac{1}{4}F$.

Conditions of the formation of an anastigmatic image.

Or the E.C.s due to an axial stop may be of such value that the curvature of image in primary and secondary planes is equalised, and there is therefore no oblique astigmatism.

If we put x for the curvature error of such anastigmatic image, then the conditions are such that

$$\frac{\tan^2\phi}{2F}\left(\frac{3\mu+1}{\mu}\right)-x=3\left\{\frac{\tan^2\phi}{2F}\left(\frac{\mu+1}{\mu}\right)-x\right\}, \qquad \text{XIIa.}$$

from which it is evident that

$$x=\frac{\tan^2\phi}{2F}\,\frac{1}{\mu}, \qquad (60)$$

Curvature of the anastigmatic image.

so that the versine of the curve $=\dfrac{F\tan^2\phi}{2\mu}$ and the radius of curvature of the anastigmatic image $=\mu F$, or the principal focal length $\times$ the refractive index.

This condition of the anastigmatic image is also attained when, in Coddington's formulæ $\dfrac{\tan^2\phi}{2F}\left(V+\dfrac{1}{\mu}\right)$ in secondary planes and $\dfrac{\tan^2\phi}{2F}\left(3V+\dfrac{1}{\mu}\right)$ in primary planes, the value of V is 0. Obviously these results also apply to two or more collective lenses or two or more dispersive lenses on the same axis.

Combined Lenses in Contact

But by far the most important practical corollaries follow from the applications of these formulæ to combinations of collective with dispersive lenses, and we will first suppose that such lenses have no appreciable axial thicknesses and are in actual contact.

An important inquiry.

Problem.—Is it possible, by any combination of collective and dispersive lenses, to get the joint normal curvature errors in primary planes just three times the corresponding errors in secondary planes, and thus be in the right relation for being simultaneously neutralised by E.C.s ?

Let P = principal focal length of the collective lens, and
N = " " " of dispersive lens,
μ_p = refractive index of the glass of the collective lens.
μ_n = refractive index, for the same ray, of the glass of the dispersive lens.

Then, if we write N negative, we must stipulate that

$$\frac{\tan^2\phi}{2}\left\{\frac{1}{P}\,\frac{3\mu_p+1}{\mu_p}-\frac{1}{N}\,\frac{3\mu_n+1}{\mu_n}\right\}=3\left[\frac{\tan^2\phi}{2}\left\{\frac{1}{P}\,\frac{\mu_p+1}{\mu_p}-\frac{1}{N}\,\frac{\mu_n+1}{\mu_n}\right\}\right],$$

from which

Condition which renders a flat and anastigmatic image possible.

$$\frac{\tan^2\phi}{2}\left\{\frac{1}{P}\,\frac{2}{\mu_p}-\frac{1}{N}\,\frac{2}{\mu_n}\right\}=0 \qquad \text{or} \qquad \frac{1}{P\mu_p}-\frac{1}{N\mu_n}=0, \qquad \text{XIII.}$$

or the powers of the lenses must be in direct ratio to the respective refractive indices of the glasses of which they are composed, or their principal focal lengths be in inverse ratio to the same.

Thus we arrive at a result which is one form of what of late years has been known as the Petzval condition. Fifty years ago or more it was laid down by Joseph Petzval that the radius of curvature of an anastigmatic image close to the optic axis, formed by two or more collective or dispersive lenses, was given by the following formula—

The Petzval Theorem.

$$\frac{1}{r} = \Sigma\frac{1}{f_n\mu_n} \text{ or } \frac{1}{f_1\mu_1} + \frac{1}{f_2\mu_2} + \text{etc.,} \qquad \text{XIIIA.}$$

in which r is the radius of the anastigmatic image; and that if one lens of a double combination is collective and the other dispersive, and the powers such that

$$\frac{1}{f_1\mu_1} - \frac{1}{f_2\mu_2} = 0 \qquad (61)$$

(which is the same as the above Formula XIII.), then the radius of curvature of the anastigmatic image becomes infinity and the image flat. It is strange that no optical writers seem to have come across Petzval's proof of this theorem, which up to very recent years has been regarded as of merely academic interest, not capable of practical realisation. It is easy to prove that Petzval was quite justified in giving the former formula for the reciprocal of the radius of curvature of the anastigmatic image.

Confirmation of the Petzval Theorem.

For let $x \tan^2 \phi$ be the R correction to the reciprocal value of the combined focal length F of two lenses in contact; then, if the final image is free from astigmatism, $F^2(x \tan^2 \phi)$ is the versine of such anastigmatic curved image. Therefore we have the equation

$$\tan^2\phi\left[\left\{\frac{1}{P}\frac{3\mu_p+1}{\mu_p} - \frac{1}{N}\frac{3\mu_n+1}{\mu_n}\right\} - x\right] = 3\tan^2\phi\left[\left\{\frac{1}{P}\frac{\mu_p+1}{\mu_p} - \frac{1}{N}\frac{\mu_n+1}{\mu_n}\right\} - x\right],$$

which condition follows from the fact that the primary E.C.s (due to the presence of an axial stop) required for throwing back the curved image formed by central oblique rays in primary planes on to the curve of the anastigmatic image are always three times the secondary E.C.s required for throwing the image formed by central oblique rays in secondary planes on to the same anastigmatic image. From this equation we get

Value of the curvature correction for anastigmatic image.

$$2x = \frac{1}{P\mu_p} - \frac{1}{N\mu_n} \text{ or } x = \frac{1}{2}\left(\frac{1}{P\mu_p} - \frac{1}{N\mu_n}\right),$$

and the versine of the curve of the anastigmatic image

$$=\frac{1}{2}\left(\frac{1}{P\mu_p}-\frac{1}{N\mu_n}\right)F^2\tan^2\phi. \tag{62}$$

Then if $r=$ the required radius of curvature, then

$$\frac{(F\tan\phi)^2}{2r}=\frac{1}{2}\left(\frac{1}{P\mu_p}-\frac{1}{N\mu_n}\right)F^2\tan^2\phi$$

and

Reciprocal of the radius of the anastigmatic image.

$$\frac{1}{r}=\frac{1}{P\mu_p}-\frac{1}{N\mu_n}; \tag{XIV.}$$

and in this equation, which represents the Petzval theorem, the meaning of XIII. is much extended.

Impossibility of obtaining a real anastigmatic image without E.C.s.

Although it can be proved to be absolutely impossible to get a real image free from astigmatism from a contact combination of thin lenses without having a stop placed somewhere on the axis to compel the oblique pencils to traverse the lenses eccentrically, and thus become subject to E.C.s of the proper amount, yet Petzval made no mention of such a condition.

For if a pair of lenses fulfils the condition XIII., and consequently $\frac{1}{N}=\frac{\mu_n}{\mu_p}\frac{1}{P}$, then the simple sum of their normal curvature errors, quite apart from E.C.s,

$$=\tan^2\phi\left\{\frac{1}{2P}\,\frac{\mu_p+1}{\mu_p}-\frac{1}{2N}\,\frac{\mu_n+1}{\mu_n}\right\}\text{in secondary planes,} \tag{63}$$

and

$$\tan^2\phi\left\{\frac{1}{2P}\,\frac{3\mu_p+1}{\mu_p}-\frac{1}{2N}\,\frac{3\mu_p+1}{\mu_n}\right\}\text{in primary planes,} \tag{64}$$

and, after writing $\frac{1}{P}\frac{\mu_n}{\mu_p}$ for $\frac{1}{N}$, the above formulæ

$$=\frac{\tan^2\phi}{2P}\cdot\frac{\mu_p-\mu_n}{\mu_p}\text{ and }\frac{\tan^2\phi}{2P}\cdot\frac{3(\mu_p-\mu_n)}{\mu_p}$$

respectively. Hence the radius of curvature of the secondary image $=P\frac{\mu_p}{\mu_p-\mu_n}$, and of the primary image $=P\frac{\mu_p}{3(\mu_p-\mu_n)}$. Then, if $\frac{1}{F}$ is the power of the combination when there is no separation between the lenses, it follows that

$$\frac{1}{F}=\frac{1}{P}-\frac{\mu_n}{\mu_p}\frac{1}{P}=\frac{1}{P}\cdot\frac{\mu_p-\mu_n}{\mu_p},$$

from which

$$\frac{1}{P}=\frac{1}{F}\,\frac{\mu_p}{\mu_p-\mu_n}\text{ and }P=F\frac{\mu_p-\mu_n}{\mu_p}. \tag{65}$$

On substituting this value of P in the previous two formulæ for the radii of curvatures of the two images we get

$$\text{Radius of secondary image} = \left(\mathrm{F}\frac{\mu_p - \mu_n}{\mu_p}\right)\frac{\mu_p}{\mu_p - \mu_n} = \mathrm{F} \text{ simply.} \qquad (66)$$

$$\text{Radius of primary image} = \left(\mathrm{F}\frac{\mu_p - \mu_n}{\mu_p}\right)\frac{\mu_p}{3(\mu_p - \mu_n)} = \frac{\mathrm{F}}{3} \text{ simply.} \qquad (67)$$

Radii of the two normal images when Formula XIII. is fulfilled.

It is interesting to observe, then, that the normal curvatures of the two images yielded by a compound lens fulfilling the condition XIII. are the same as if the lens were a simple lens of the same focal length, but made of glass having an infinitely high refractive index.

Above combination equal to a simple lens of infinite refractive index.

So that we may regard the particular case of the Petzval Formula XIIIA. being equated to 0, as in Formula XIII., as a device for making a lens whose refractive index is virtually infinity, with regard to its influence on the compound normal curvature errors.

Further necessity for E.C.s to get a flat image.

Therefore it is quite clear, from what has preceded, that E.C.s must perform a part in this compound lens, if the two images are to be simultaneously thrown back into a plane image. That is, eccentric oblique refraction is absolutely necessary to the attainment of the desired flat and anastigmatic image, in the case of contact combinations fulfilling Formula XIII.

It is plain that the Petzval condition XIII. demands that if the combination is to have a positive focus, the collective lens must, in order to possess the preponderating power, be made of a glass of higher refractive index than that of which the dispersive lens is made (so that $\mathrm{P}\mu_p = \mathrm{N}\mu_n$, or the principal focal lengths are in inverse ratio to their refractive indices), a condition which was impossible to fulfil consistently with achromatism until the era of the new optical glasses was inaugurated at Jena.

The new Jena glasses.

The new dense barium crown glasses combining a refractive index as high as 1·61 with a dispersive power as low as $\frac{1}{58}$ for rays C to F, and the new crown or very light flint glasses having a refractive index of 1·52 to 1·54 with a dispersive power as high as $\frac{1}{47}$, were the creations of the celebrated firm of Herren Schott & Gen., of Jena, who thus rendered it possible to embody the Petzval condition in combinations of two or more lenses in contact. Dr. Hugo Schroeder's concentric lens was apparently the first photographic lens in which Petzval's sum $\Sigma\frac{1}{f_n\mu_n}$ was equated to 0 with any degree of success; but not only does the far too small difference of refractive indices yet available render it impossible to get much focal power from such combinations,

The Concentric Lens.

The first contact combination equating the Petzval Formula to 0.

The small balance of power available.

but the fact that Schroeder made it a condition in his concentric lens that the plano-convex collective lens of high refractive index should be cemented to the plano-concave dispersive lens of low refractive index, precluded him from the advantage of freedom from spherical aberration. A reference to Fig. 52 renders it evident that any ray entering the dispersive lens parallel to the axis is refracted away from the axis, so that its distance from the axis when traversing the collective lens is greater than its distance from the axis when traversing the dispersive lens. This variation in y_2 would have little significance if the glass of the collective lens were of lower refractive index than that of the dispersive lens, but in the case of this abnormal pair of glasses the variation in y_2 introduces an aberration of the third order which is fatal to the elimination of spherical aberration, so that, as a matter of fact, sharp definition, even on the axis, could not be secured with any larger aperture than about $\frac{F}{22}$, or $\frac{F}{32}$ in larger-sized lenses. Afterwards Dr. Rudolph of Jena, in Germany, got over this difficulty with considerable success by adopting the expedient of opposing two cemented combinations A and B, A comprising an abnormal pair of a collective and a dispersive lens, of which the collective lens had the higher refractive index, while B was a normal pair in which the collective lens had the lower refractive index.

Imperfect correction against spherical aberration.

Dr. Rudolph's anastigmat.

Combination A was undercorrected for spherical aberration, but this defect was counteracted by the opposite fault in B; also a rough approximation to the Petzval condition was secured by a suitable division of the powers of the lenses relatively to their refractive indices. In this way much larger relative apertures were obtained. Later Dr. Rudolph, closely followed by Emile von Hoegh, devised a still better symmetrical construction for each half of the lens, which was made to consist of a double concave dispersive lens cemented between an inner meniscus collective lens and an outer double convex collective lens, the refractive index of the dispersive lens being approximately a mean between the high refractive index of the double convex collective lens on the one side and the low refractive index of the meniscus collective lens on the other side. Dr. von Hoegh's lens is generally known as the Goerz lens.

Dr. Rudolph's and Dr. Emile von Hoegh's improved anastigmat.

In each half lens the so-called Petzval condition,

Petzval condition not quite fulfilled.

$$\frac{1}{P_1\mu_p} - \frac{1}{N\mu_n} + \frac{1}{P_2\mu_p'}, = 0, \tag{68}$$

was almost but not quite fulfilled. In order to fulfil it exactly, either the power of the dispersive lens would have to be increased, or its refractive index decreased, but the exigencies of cemented combinations preclude the simultaneous fulfilment of other conditions, consistently with sufficient power being obtained. As the extreme differences of refractive indices between the new abnormal pairs of glasses are as 1·6 to 1·5, it is evident that any contact combination of thin lenses fulfilling the Petzval condition must have the power of the collective lens or lenses equal to 16, as against 15 for the power of the dispersive lens or lenses, the resulting power of the combination being 1. or only $\frac{1}{16}$th of the power of the collective lens or lenses. This is a limitation implying the use of very powerful or strongly curved lenses in order to gain a comparatively long focused combination, whose normal curvature errors in primary planes are three times the normal curvature errors in secondary planes, and therefore in the proper relation for being simultaneously neutralised by E.C.s left in the system for that purpose.

Powerful constituent lenses result in relatively small power.

The Case of Separated Lenses or Elements

So far, then, we have considered the application of the formulæ arrived at to combinations of very thin lenses in contact. We have yet to consider their application to either thin lenses more or less widely separated, or to thick lenses considered either singly or in combination. Some twelve years ago, in the course of thinking over the general results arrived at in the last two Sections, especially in relation to the normal curvatures of image characteristic of simple or achromatic lenses, it suddenly occurred to the author that since the normal curvatures of image due to any lens, whether simple or compound, are fixed by its refractive indices and power alone, and are independent of the state of the rays entering the lens, whether convergent, divergent, or parallel, then it should follow that the normal curvature errors of an achromatic and aberration-free collective lens should be neutralised by the normal curvature errors of an achromatic and aberration-free dispersive lens of the same power (and made of the same glasses) placed at a considerable distance behind the collective lens; while the combination would, *as a result of the separation*, have considerable power or yield a positive focus, so long as the rays from the collective lens are convergent to a distance behind the dispersive lens less than the principal focal length of the latter, or more especially when the rays entering the first or collective lens are parallel. But such complete neutralisation of normal curvature errors could obviously

How collective and dispersive lenses of equal powers may neutralise each other's normal curvature errors even when separated.

not ensue if any E.C.s were allowed to interfere, therefore both these achromatic and aberration-free lenses must be free from coma or give symmetrical oblique refraction; otherwise pencils of rays traversing one of the lenses centrally, but the other necessarily eccentrically, would be subject to E.C.s, and their final foci be either shortened or extended, and thus the desired result be prevented. This idea led to further experiments and calculations, which we will now deal with.

The above two lenses must be free from coma.

Effect of separations on the formulæ.

We must first ascertain how the formulæ which have been arrived at, are to be applied to combinations of thin lenses on a common axis, but having considerable separations between them. In Fig. 53 let L_1 represent such a compound collective lens free from coma and aberration, of principal focal length $=f_1$, and L_2 a compound dispersive lens also free from coma and aberration, and made of the same glasses, and having the same principal focal length f_2 $(= -f_1)$. Let the rays entering L_1 be parallel. Then at the distance f_1 behind L_1 is formed the curved image $s\,..\,s$ due to rays in secondary sections of oblique pencils, and the still more curved image $p\,..\,p$ due to rays in primary sections of the same oblique pencils. The dispersive lens L_2 will project an enlarged image of these to a distance b behind it, such that $\frac{1}{b} = \frac{1}{f_1 - s} - \frac{1}{f_2}$, where a plane anastigmatic image will be formed. Or treating the said plane as an origin for the pencils in the reverse direction, it is evident that after such direct and oblique divergent pencils (such as that from q) have been refracted by L_2, they will then virtually radiate from points in the curved surfaces, $s\,..\,s$ in secondary planes and $p\,..\,p$ in primary planes, which are exactly the same curved images as are yielded by the positive lens L_1, so that all the pencils will emerge strictly parallel leftwards from L_1. The theorem that the normal curvature errors of two equal collective and dispersive lenses will neutralise one another, even when the lenses are widely separated, is thus almost self-evident when once pointed out; but the more general theorem that the curvature errors and E.C.s of a system of separated lenses are the simple sum of the curvature errors and E.C.s of the individual lenses, and that the power gained by separation is a net gain and carries with it no curvature corrections whatever, requires further demonstration. It might at first be thought that the fact that the centre of each lens of a separated system views the same point of the original object or its image under different angles of obliquity, and views the same curvature error from different distances, would lead to unavoidable complications, but this is not so.

The power gained by separation between collective and dispersive lenses is an unqualified net gain.

In Fig. 53 let ϕ = the original angle of obliquity of a central or eccentric pencil impinging on L_1. As throughout the foregoing processes, the angle ϕ is always the angle contained between the optic axis and that ray to or from the real or virtual radiant or focal point Q which passes through the centre of the lens. The corresponding oblique focal point about Q, to which the rays converge after refraction by L_1, subtends a new angle θ at the centre of L_2. Let us assume that the linear aberrations of Q′ from the focal plane P..P..P do not exceed $\frac{1}{10}$th part of f_1, as is the case if the angle ϕ does not exceed 14 degrees. Let δ_1 = any R corrections, including normal curvature errors and E.C.s, for the first lens; let δ_2 = the similar R corrections for L_2—in neither case amounting to more than 10 per cent of $\frac{1}{f_1}$ or $\frac{1}{f_2}$ respectively.

Demonstration of the above theorem.

Then $\frac{1}{f_1}+\frac{\tan^2\phi}{2f_1}\delta_1$ is the reciprocal value of the corrected focal length of the oblique pencil we are dealing with, and if the same pencil traversed L_2 under the same angle of obliquity ϕ, then the corrected reciprocal value of the back focus would be

$$\frac{1}{B}=\frac{1}{f_1-s}-\left\{\frac{1}{f_2}+\frac{\tan^2\phi}{2f_2}\delta_2\right\}, \qquad (69)$$

The two angles ϕ and θ assumed to be equal.

supposing $\frac{\tan^2\phi}{2f_1}\delta_1$ for the first lens is for the moment neglected. But the second lens L_2 views Q under the angle θ, and it is evident that

$$\tan\theta=\frac{f_1}{f_1-s}\tan\phi.$$

Tan θ in terms of tan ϕ.

Also the R corrections for the oblique pencil traversing L_1, expressed by $\frac{\tan^2\phi}{2f_1}\delta_1$ will from the point of view of the second lens become $\left(\frac{f_1}{f_1-s}\right)^2\frac{\tan^2\phi}{2f_1}\delta_1$, or increased in inverse proportion to the square of the distance; for generally if v = the *linear* amount of the curvature error in question (referred to the axis) and is a small quantity compared to f_1 or f_1-s, then

The same R correction as viewed from L_1 and L_2 respectively.

General argument.

$$\frac{1}{f_1-v}=\frac{1}{f_1}+\frac{v}{f_1^2};$$

and then if f_1 becomes f_1-s, then

$$\frac{1}{(f_1-s)-v}=\frac{1}{f_1-s}+\frac{v}{(f_1-s)^2},$$

so that the R correction from the point of view of L_2 is $\frac{v}{(f_1-s)^2}$, as against $\frac{v}{f_1^2}$ for the same R correction from the point of view of L_1; but

$$\frac{v}{(f_1-s)^2}=\frac{v}{f_1^2}\left(\frac{f_1}{f_1-s}\right)^2,$$

and moreover $\frac{v}{f_1^2}$ is only another way of expressing $\frac{\tan^2\phi}{2f_1}\delta_1$, therefore with reference to L_2 the R correction from L_1 is $\left(\frac{f_1}{f_1-s}\right)^2\frac{\tan^2\phi}{2f_1}\delta_1$, as above.

Next, the R correction to which the same pencil is subjected on traversing L_2 under the new angle of obliquity θ is evidently

$$\tan^2\theta\frac{1}{2f_2}\delta_2,\text{ which }=\left\{\left(\frac{f_1}{f_1-s}\right)^2\tan^2\phi\right\}\frac{1}{2f_2}\delta_2.$$

Therefore the sum of the R corrections for both lenses from the point of view of L_2 becomes

$$\left(\frac{f_1}{f_1-s}\right)^2\frac{\tan^2\phi}{2f_1}\delta_1+\left(\frac{f_1}{f_1-s}\right)^2\frac{\tan^2\phi}{2f_2}\delta_2$$

or

Sum of the R corrections for the two lenses.

$$\left(\frac{f_1}{f_1-s}\right)^2\frac{\tan^2\phi}{2}\left(\frac{1}{f_1}\delta_1+\frac{1}{f_2}\delta_2\right). \qquad (70)$$

And if this last expression is multiplied by B^2, or the back focal length squared, we shall then get the linear value, reduced to the axis, of the sum of the R corrections of the two lenses. As we have seen before, $\frac{1}{B}=\frac{1}{f_1-s}+\frac{1}{f_2}$, and $B=\frac{f_2(f_1-s)}{f_2+(f_1-s)}$, so that (70) × B^2 becomes

Linear value of the above.

$$\left\{\frac{f_2(f_1-s)}{f_2+(f_1-s)}\right\}^2\left(\frac{f_1}{f_1-s}\right)^2\frac{\tan^2\phi}{2}\left(\frac{1}{f_1}\delta_1+\frac{1}{f_2}\delta_2\right). \qquad (71)$$

Next, in order to reduce this to an R correction of the reciprocal of the equivalent focal length of the whole combination, we must divide (71) by $(\text{E.F.L.})^2$ or the square of the equivalent focal length of the whole combination. Now the E.F.L. is the axial distance of the back principal point from the final image plane, at which point a pin-hole would have to be placed in order to throw an image of the same dimensions as that yielded by the combined lenses; on which supposition the E.F.L. is equal to $B\frac{f_1}{f_1-s}$, which

$$=\frac{f_1f_2}{f_2+(f_1-s)}, \qquad (72)$$

from Formula X., Section III. Therefore (71) $\div$ (E.F.L.)^2

$$=\left\{\frac{f_2(f_1-s)}{f_2+(f_1-s)}\right\}^2\left(\frac{f_1}{f_1-s}\right)^2\frac{\tan^2\phi}{2}\left(\frac{1}{f_1}\delta_1+\frac{1}{f_2}\delta_2\right)\left\{\frac{f_2+(f_1-s)}{f_1f_2}\right\}^2,$$

which

$$=\frac{\tan^2\phi}{2}\left(\frac{1}{f_1}\delta_1+\frac{1}{f_2}\delta_2\right) \qquad (73)$$

simply.

Sum of the R corrections assessed upon the E.F.L.

The same line of reasoning pursued in the case of separated combinations of three or more lenses leads to the same important result. That is, the R corrections of a series of separated lenses sum up as a correction to the reciprocal value of the E.F.L., and no notice need be taken of the successive modifications of tan ϕ at each lens. We need only take the sum of the R corrections appertaining to the several lenses and multiply them by (E.F.L.)^2 $\tan^2\phi$ in order to convert them into their linear value at the final image, taking care to insert for tan ϕ the tangent of the angle contained between the optic axis and a principal ray proceeding from any selected point in the original object or image to the first principal point of the combination. That is, the two principal points are the points to which the angles of obliquity ϕ should be referred. Then it is clear that, if the original object is infinitely distant and the rays of pencils parallel, it becomes quite a matter of indifference whether the angle ϕ is referred to the outer vertex of the first lens or to the first principal point. Clearly there is no difference in such a case. With regard to the second conjugate focal distance, it is obvious that tan ϕ must also be measured from the second principal point.

Same important theorem applies to three or more separate lenses.

Tan ϕ should always be referred to the two principal points.

The Gain in Power due to Separation

Now the reciprocal of the E.F.L., or $\frac{1}{\mathrm{F}}$ for brevity, or the equivalent *power* of the combination (73), $=\frac{1}{\text{E.F.L.}}=\frac{f_2+f_1-s}{f_1f_2}$, as we have seen above, and this is made up of two parts, viz. $\frac{1}{f_1}+\frac{1}{f_2}$, or the simple difference of the powers of the two lenses, and $\frac{-s}{f_1f_2}$, which is the gain of power due entirely to separation, so that while $\frac{1}{f_1}+\frac{1}{f_2}$ may be zero if the powers of the two lenses are equal, one collective and the other dispersive, yet there remains a considerable surplus power, represented by $\frac{-s}{f_1f_2}$, in the case of the same two lenses separated.

But as we have seen from Formula (73) the curvature errors or E.C.s appertain solely to $\frac{1}{f_1}+\frac{1}{f_2}$, therefore the great gain in power represented by $\frac{-s}{f_1 f_2}$ is an unqualified net gain and carries with it no normal curvature aberrations whatsoever.

The gain in power due to separation is a net gain.

A practical illustration.

Supposing we have $f_1 = 1, f_2 = -1$, and $s = \cdot 25$, then $\frac{-s}{f_1 f_2} = +\cdot 25 = \frac{1}{4}$, or the equivalent power of the combination, entirely due to separation, is one-quarter of the power of the collective lens—a very considerable amount, especially if we compare it with the case of two lenses in contact fulfilling the so-called Petzval condition; the collective lens being of power 16 and the dispersive lens of power 15, and the resulting power of the combination being only 1, or $\frac{1}{16}$th part of the power of the collective lens. Now it obviously remains perfectly true, that even in the case of a separated pair of a collective and dispersive lens such as we have been dealing with, the condition XIII. must be fulfilled if a flat final image, free from astigmatism, is to be secured; and it still remains true that $\frac{1}{P\mu_p} - \frac{1}{N\mu_n} = \frac{1}{r}$ (see XIV.) if that condition is *not* fulfilled; but since the radius of curvature r of the anastigmatic image is the same whether the two lenses be in contact or separated, it is obvious that the shortening of the E.F.L. due to separation means virtually a flattening of the anastigmatic image, for $\frac{r}{\text{E.F.L.}}$ becomes much greater than if there were no separation. Therefore it follows that a departure from the so-called Petzval condition, which would lead to serious astigmatism in the final image of mean flatness thrown by a contact combination, would lead to a much less serious astigmatism in the final image of mean flatness thrown by the same two lenses when separated. For instance, let us take two lenses, one collective, of focal length 15 and refractive index 1·5, and the other dispersive, of focal length 16 and refractive index also 1·5, thus not fulfilling the Petzval condition at all. The radius of curvature of the anastigmatic image thrown by these two lenses in contact is given by

The Petzval condition also applies to separated lenses.

The radius of the anastigmatic image independent of the separation.

When the Petzval condition may be largely ignored.

An instance.

Reciprocal of the radius of the anastigmatic image.

$$\frac{1}{r} = \frac{1}{15(1\cdot 5)} - \frac{1}{16(1\cdot 5)} = \frac{1}{22\cdot 5} - \frac{1}{24} = \frac{1}{360},$$

while

Power when in contact.

$$\frac{1}{F} = \frac{1}{15} - \frac{1}{16} = \frac{1}{240},$$

so that $r=(1\cdot5)F$. Then if the two lenses be separated by a distance $=4$, we find

$$\frac{1}{\text{E.F.L.}}=\frac{15-16-4}{-16\times15}=\frac{1}{240}+\frac{1}{60}=\frac{5}{240}=\frac{1}{48}.$$

Power when separated.

Thus we have $r=360$ as before, while the E.F.L. is reduced to 48, so that r is now $(7\cdot5)F$ instead of $(1\cdot5)F$. Thus the effect of a departure from the Petzval condition is reduced to a vanishing quantity, so that if we construct photographic lenses on the principle of gaining a considerable proportion or all of their power by separation, then we need no longer be restricted to carrying out the Petzval condition; we can ignore it to some extent in favour of a more general and elastic rule, viz. that the power of the dispersive lens must be approximately equal to the power of the collective lens, or the sum of the powers of the collective lenses if there are more than one.

This is one of the two supplementary principles which underlie the Cooke photographic lenses, and many others which have been introduced since they were first made public.

Anastigmats could have been produced by the aid of the old crown and flint glasses only.

A practical instance.

And now it will be easily seen that a true anastigmat might have been made long before the advent of the new Jena glasses. For instance, we will take a crown glass collective lens of refractive index $=1\cdot5$, and whose $\frac{1}{f_1}=\frac{1}{16}$, and a dense flint glass dispersive lens of refractive index $=1\cdot6$ whose $\frac{1}{f_2}=\frac{1}{15}$, the two being separated by $s=7$. Here the Petzval condition is fulfilled, but if the lenses are in contact the power is $-\frac{1}{240}$ and the system is dispersive, but the power when separated by 7 is $\frac{16-15-7}{(-15)(16)}=\frac{-6}{-240}=+\frac{1}{40}$. When put into the triplet form, like a Cooke lens, a very fair rectilinear anastigmat lens could be and has been produced, but not so good as when the newer Jena glasses are employed. The Cooke lens of aperture $\frac{F}{4\cdot5}$, known as Series $1a$, for astronomical photography, is practically an anastigmat in which the refractive index of the dispersive lens is considerably higher than that of the two collective lenses, and the Petzval condition is very considerably departed from, yet the final image is quite flat and shows only a trace of astigmatism within an angle of 20 degrees.

The Cooke lens for astronomical photography.

Before proceeding to the question of thick lenses it is desirable to arrive at two more very useful formulæ relating to contact or separated combinations. If the final image yielded by a photographic lens has a little residual astigmatism away from the axis, it yet remains desirable to attain an approximately flat image, and two useful compromises suggest themselves.

When the primary image is made flat.

1. The image formed by rays in primary planes may be got flat, leaving the image formed by rays in secondary planes still somewhat curved concave to the lens. In such case what will be the radius of curvature (r) of such secondary image?

It is evident that the primary E.C.s which throw the primary image back on to the focal plane must be equal to

$$-\tan^2\phi\left(\frac{1}{2P}\frac{3\mu_p+1}{\mu_p}-\frac{1}{2N}\frac{3\mu_n+1}{\mu_n}\right).$$

If $\frac{1}{P}$ = the power of the collective and $\frac{1}{N}$ that of the dispersive lens, the simultaneous secondary E C.s will then be

$$-\frac{\tan^2\phi}{3}\left(\frac{1}{2P}\frac{3\mu_p+1}{\mu_p}-\frac{1}{2N}\frac{3\mu_n+1}{\mu_n}\right),$$

which latter must then be subtracted from the normal curvature errors in secondary planes, so that we have

$$\tan^2\phi\left(\frac{1}{2P}\frac{\mu_p+1}{\mu_p}-\frac{1}{2N}\frac{\mu_n+1}{\mu_n}\right)-\frac{\tan^2\phi}{3}\left(\frac{1}{2P}\frac{3\mu_p+1}{\mu_p}-\frac{1}{2N}\frac{3\mu_n+1}{\mu_n}\right) \quad (74)$$

to express the R curvature correction for the final image, which reduces to

$$\tan^2\phi\left(\frac{1}{P}\frac{1}{3\mu_p}-\frac{1}{N}\frac{1}{3\mu_n}\right),$$

so that

$$\frac{(F\tan\phi)^2}{2r}=\tan^2\phi\left(\frac{1}{P}\frac{1}{3\mu_p}-\frac{1}{N}\frac{1}{3\mu_n}\right)F^2,$$

Reciprocal of the radius of secondary image when the primary image is flat.

and finally

$$\frac{1}{r}=\frac{2}{3}\left(\frac{1}{P\mu_p}-\frac{1}{N\mu_n}\right). \qquad \text{XV.}$$

When the mean image is flat.

2. Perhaps the best possible compromise is attained when the primary image is as much overcorrected as the secondary image is undercorrected, the focal plane lying midway between the two curves, the primary curve convex to the lens and the secondary curve concave to the lens. Thus the circles of least confusion are made to fall upon the focal plane. The formula for the normal curvature errors of the combination, with respect to circles of least confusion, obviously

Mean normal curvature errors.

$$=\tan^2\phi\left\{\frac{1}{2P}\frac{2\mu_p+1}{\mu_p}-\frac{1}{2N}\frac{2\mu_n+1}{\mu_n}\right\},$$

and therefore the E.C.s for circles of least confusion, or the mean E.C.s, must be supposed equal to the above; therefore it follows that the E.C.s in secondary planes will be equal to one-half of the mean E.C.s and equal to

$$-\tan^2\phi\,\frac{1}{2}\left\{\frac{1}{2\mathrm{P}}\,\frac{2\mu_p+1}{\mu_p}-\frac{1}{2\mathrm{N}}\,\frac{2\mu_n+1}{\mu_n}\right\},$$

Value of the E.C.s in secondary planes.

and in primary planes the E.C.s will be equal to

$$-\tan^2\phi\,\frac{3}{2}\left\{\frac{1}{2\mathrm{P}}\,\frac{2\mu_p+1}{\mu_p}-\frac{1}{2\mathrm{N}}\,\frac{2\mu_n+1}{\mu_n}\right\}.$$

Value of the E.C.s in primary planes.

Therefore the final curvature R correction in secondary planes will be

$$\tan^2\phi\left[\left\{\frac{1}{2\mathrm{P}}\,\frac{\mu_p+1}{\mu_p}-\frac{1}{2\mathrm{N}}\,\frac{\mu_n+1}{\mu_n}\right\}-\frac{1}{2}\left\{\frac{1}{2\mathrm{P}}\,\frac{2\mu_p+1}{\mu_p}-\frac{1}{2\mathrm{N}}\,\frac{2\mu_n+1}{\mu_n}\right\}\right], \quad (75)$$

Curvature errors minus E.C.s in secondary planes.

which reduces to

$$\tan^2\phi\left\{\frac{1}{2\mathrm{P}}\,\frac{1}{2\mu_p}-\frac{1}{2\mathrm{N}}\,\frac{1}{2\mu_n}\right\},$$

so that the versine of the image curve

$$=\frac{(\mathrm{F}\tan\phi)^2}{2r}=\tan^2\phi\left\{\frac{1}{4\mathrm{P}\mu_p}-\frac{1}{4\mathrm{N}\mu_n}\right\}\mathrm{F}^2$$

and

$$\frac{1}{r}=\frac{1}{2}\left\{\frac{1}{\mathrm{P}\mu_p}-\frac{1}{\mathrm{N}\mu_n}\right\}. \qquad \text{XVI.}$$

Reciprocal of the radius of either secondary or primary image.

The three Formulæ XIV., XV., and XVI. give at a glance, as it were, the degree of approximation to an anastigmatic focal plane attainable in any suggested combination of lenses of known powers and refractive indices, whose combined equivalent focal length, if separations exist, can also be derived from Formula (72) if only two lenses are employed, or from the more complex formulæ given in Section II. if there are more than two. No photographic lens of separated lenses can be made to give achromatic and rectilinear images with less than three constituent lenses, and if P_1 is the P.F.L. of the first collective lens L_1, N the P.F.L. of the dispersive middle lens L_2, and P_2 the P.F.L. of the back collective lens L_3, s_1 the separation between L_1 and L_2, and s_2 the separation between L_2 and L_3, then the E.F.L. of the combination for parallel rays is given by the formula

The minimum number of lenses required for a photographic lens.

$$\left(\frac{1}{\mathrm{P}_1}-\frac{1}{\mathrm{N}}+\frac{1}{\mathrm{P}_2}\right)+\left\{\frac{s_1(\mathrm{P}_2-\mathrm{N})+s_2(\mathrm{P}_1-\mathrm{N})-s_1s_2}{\mathrm{P}_1\mathrm{N}\mathrm{P}_2}\right\}. \qquad \text{XVII.}$$

Triplet lens. The increment to power due to separations.

The first part of the above formula is the simple sum of the powers, or the E.F.L. of the three lenses if thin and placed in contact, while the latter part of the formula gives the further increment to power due entirely to the separations.

A Significant Corollary relating to Eye-pieces

We have just alluded to the increment to power accruing to a combination of two collective and one dispersive lens, consequent upon separation. Referring back to Section III., p. 46, we found that a certain four-lens erecting eye-piece whose lenses were of focal lengths 1, 1·25, 1·25, and ·80, only gave an equivalent focal length of $-\cdot 31$.

Cases where separations lead to loss of power.

Here is a case wherein the separations have led to a noticeable decrement to power; for while the sum of the powers of the four lenses is $\frac{1}{\cdot 26}$ we have $\frac{1}{\text{E.F.L.}} = \frac{1}{\cdot 31}$. Therefore while the normal curvature errors will be proportional to $\frac{1}{\cdot 26}$, and consequently the radius of curvature of the anastigmatic image be proportional to ·26, yet the E.F.L. is $-\cdot 31$ only. That is, the radius of curvature of the anastigmatic image, if formed, will be smaller (in the ratio $\frac{26}{31}$) than the radius of the same image if a simple equivalent lens of E.F.L. = ·31 were used.

The above eye-piece is a comparatively favourable case, having s_2 or the second separation greater than usual, which leads to increment to power. In most cases shortness is aimed at, when the $\frac{1}{\text{E.F.L.}}$ of course grows smaller compared to the sum of the powers of the four lenses, and therefore the curvature errors of the final image are bound to increase. A flat or nearly flat image for rays in primary planes is generally aimed at, and therefore it will be seen that the less is the power realised in the combination, the more relatively violent will be the curvature of the same final image as formed by rays in secondary planes. The shorter is such an eye-piece, the more difficult it becomes to attain a satisfactorily flat field of view.

Huygenian eye-pieces.

We also saw that in the case of the Huygenian eye-piece with lenses of focal lengths 3 and 1, we got

$$\frac{1}{\text{E.F.L.}} = \frac{2}{3}, \text{ while } \frac{1}{f_1} + \frac{1}{f_2} = \frac{4}{3}.$$

Hence the curvature of image will be twice as strong as that for the equivalent lens.

In the case where the two lenses were of the focal lengths 2 and 1, then

$$\frac{1}{\text{E.F.L.}} = \frac{3}{4}, \text{ while } \frac{1}{f_1} + \frac{1}{f_2} = \frac{3}{2};$$

and here again we have the same disadvantage.

In the case of the Ramsden eye-piece of lenses of focal lengths 1 and 1 and separation 1, the same argument again applies, but as the separation is generally about $\frac{7}{8}$ or $\frac{3}{4}$, leading to a gain in power, the construction comes out about on a par with an ordinary four-lens erecting eye-piece.

Ramsden eye-piece.

The practical conclusion of these arguments is that the ideal eye-piece is one which consists of a single lens, self corrected for spherical and chromatic aberration by being built up of a dispersive lens, and one, or better still, two collective lenses. If it consists of a dispersive lens between two collective lenses, then any effective separation between the components (in the form of thickness perhaps) counts for gain in power and not loss as in the eye-pieces just considered. Then if the image has to be erected, crossed doubly reflecting prisms can be resorted to.

The ideal eye-piece.

Erection of the inverted image by reflecting prisms.

The modified form of Kellner eye-piece now so commonly employed in prismatic telescopes does not fall far short of this ideal, and it must be conceded that the images that it yields are not only superior to those yielded by four-lens erecting eye-pieces in regard to angular extent and flatness of field and freedom from astigmatism, but also as regards freedom from certain other curvature errors and E.C.s of a higher order which we shall glance at in Section XI.

The eye-piece used in prismatic telescopes.

It will also be seen that the use of a pair of double total reflecting prisms between the eye-piece and the objective rather helps to flatten the image formed by the latter. For they are equivalent to placing a pair of thick plane parallel plates in the path of the pencils of converging rays whose principal rays radiate from the centre of the objective, so that the oblique foci are subject to parallel plate corrections tending to throw them back relatively to the axial focus. This relieves the eye-piece of a certain amount of eccentricity corrections. It will, however, be seen that the position of the prisms relatively to the primary image will make no difference to their flattening effect upon the same.

The favourable effect of the reflecting prisms.

Application of the Theorem of Elements

So far as we have yet proceeded, it has been assumed that the axial thicknesses of the lenses we have been dealing with have been

Thicknesses cannot always be neglected.

quite negligible quantities, very small compared with the radii or focal lengths of the lenses in question. While excessive axial thicknesses in the lenses building up optical systems are objectionable for obvious reasons, and as much as possible to be avoided, yet thicknesses far too great to be neglected in our computations arise in most cases. Now the formulæ of the order $\tan^2 \phi$ arrived at are in their very nature and origin more and more exact in their results in inverse proportion to the fourth power of the angles of obliquity (ϕ) dealt with; and, if a pencil of rays crossing the axis of a lens system at a given diaphragm point is traced through all the other lenses at a small enough degree of obliquity, it may obviously traverse all the lenses very closely to their centres, even if the lens system is of considerable length. In Sections II. and III., etc., we have already dealt with the theorem of elements as applied to thick lenses, and we will now see how the same theorem may be applied in the computation of normal curvature errors and E.C.s. Let Fig. 54 be a double convex lens and Fig. 55 a meniscus collective lens, Fig. 54*a* a double concave lens and Fig. 55*a* a meniscus dispersive lens.

How the theorem of elements is to be applied.

Recapitulating, it is obvious that close to the axis the double convex lens may be considered to be built up of two infinitely thin elementary lenses e_1 and e_2, e_1 being convexo-plane, and e_2 plano-convex, the two enclosing between them a parallel plate of glass of a thickness equal to t, the axial thickness of the lens.

It is not quite so obvious, but nevertheless is demonstrable, that any departures from exactness in the formulæ of this Section, due to the refraction of the pencils through outer parts of the lenses where the thicknesses are widely different to the central thicknesses, take the form of corrections of the higher orders $\tan^4 \phi$ and $\tan^6 \phi$, etc. These higher developments will be dealt with in Section XI.

The corrections of the third order, etc.

In the same way the collective meniscus lens may be considered built up of a convexo-plane elementary lens e_1, and a plano-concave elementary lens e_2, enclosing between them a parallel plate of glass of a thickness equal to t, the axial thickness of the lens. If r and s are, as before, the two radii of curvatures, then the power of e_1 is simply $+\frac{\mu - 1}{r}$, and the power of e_2 simply + or $-\frac{\mu - 1}{s}$, while x, the characteristic of the shape of each elementary lens, is $+1$ simply for e_1, and -1 simply for e_2. Then in assessing the consecutive values u_1 and a_1 with respect to e_1, and u_2 and a_2 with respect to e_2, or the respective axial distances from which or to which the axial pencils diverge before refraction, we must look upon e_1 and e_2 as two distinct lenses separated by an air-space equal to $\frac{t}{\mu}$.

Also in the case of slightly oblique and eccentric pencils, the principal rays of which cross the optic axis at any known diaphragm point at a known distance D_1' or D_1'' in front of or behind e_1 (according to which β_1 is assessed), we can always assess the value of D_2' and β_2 with respect to e_2 consistently with the same supposition, viz. that e_1 and e_2 are two separate lenses separated by an air-space equal to $\frac{t}{\mu}$. In this way the values of a and β for each element may be arrived at in a very simple way.

The Effects of a Parallel Plane Plate upon Obliquely Refracted Pencils

We have next to consider whether, besides the influence exerted by the parallel plate on the spherical aberration of the axial pencil, it has any influence upon the corrections of the oblique pencils that should be taken into account. It is obvious enough that if the rays constituting pencils emerge in a condition of parallelism from e_1, and consequently traverse the parallel glass plate in a condition of parallelism, then the plate cannot possibly exert any influence upon them, and they will emerge from the plate and enter e_2 still in a parallel condition. But if the rays of pencils are converging to or diverging from points at a distance from the plate, not very large compared with t, then the plate exerts an influence on oblique pencils which it is necessary to investigate before we are in a position to properly bring the theorem of elements into practical application. We already have the complete Formula XXV., Section IV., for the spherical aberration (to use an expression which is here rather a misnomer) of a direct pencil refracted through a flat parallel plate, but for our present purpose we shall first require Formula (15), Section IV., which gives the spherical aberration occurring at the first flat surface, which formula was of the form

No effect upon pencils of parallel rays.

$$\frac{\mu}{\dot{u}} = \frac{1}{u} - \frac{\mu^2 - 1}{\mu^2}\frac{a^2}{2u^3} \text{ or } \frac{1}{u} - \omega_1 a^2, \qquad (76)$$

Aberration at first plane surface.

in which a is the semi-aperture of the pencil at the first surface.

We can now bring this formula into requisition when investigating the case of oblique pencils.

Notation.

Let Fig. 56 represent the case of a divergent oblique pencil $n_1 \ldots Q \ldots w_1$. Let $Q \ldots A_1 = u$, and let $\mu u = \dot{u}$. Then let Fig. 56a be the corresponding case of a convergent oblique pencil, both entering into a plane glass surface. In the first case u_1 and $\grave{u}_1$ should be considered positive, and in the second case negative.

Let semi-aperture of pencil $B_1 \,..\, n_1$ or $B_1 \,..\, w_1 = a$, as before. Let $A_1 \,..\, w_1 = y_1$ and $A_1 \,..\, n_1 = y_2$, and $A_1 \,..\, B_1 = H_1$; and let the angle between the principal ray $B_1 \,..\, Q$ and the perpendicular $A_1 \,..\, Q$ be called χ. Since ray $Q \,..\, n_1$ is the most oblique, it therefore meets with more aberration than ray $Q \,..\, w_1$, and after refraction cuts the perpendicular $Q \,..\, A_1$ at f_2 farther away from the surface than f_1 for the refracted ray $q_1 \,..\, w_1$. Let q_1 be the desired point where the two extreme rays $Q \,..\, n_1$ and $Q \,..\, w_1$ intersect in the primary plane after refraction. Draw $q_1 \,..\, p_1$ at right angles to the axis or perpendicular $Q \,..\, A_1$.

Then if we put x_1 for $q_1 \,..\, A_1$ or the corrected value of $\grave{u}$, f_1 for $f_1 \,..\, A_1$, and f_2 for $f_2 \,..\, A_1$, then

The fundamental equation.

$$(A_1 \,..\, n_1)\frac{x_1 - f_2}{f_2} = p_1 \,..\, q_1 = (A_1 \,..\, w_1)\frac{x_1 - f_1}{f_1},$$

from which

$$x_1 = (y_2 - y_1)\frac{f_1 f_2}{f_1 y_2 - f_2 y_1} \text{ and } \frac{1}{x_1} = \frac{f_1 y_2 - f_2 y_1}{f_1 f_2 (y_2 - y_1)}.$$

Adopting our device used on former occasions, let

$$\frac{1}{f_1} = \frac{1}{\mu u} - \frac{\omega_1}{\mu} y_1^2 \text{ and } \frac{1}{f_2} = \frac{1}{\mu u} - \frac{\omega_1}{\mu} y_2^2,$$

then

$$\frac{1}{x_1} = \frac{1}{\grave{u}} - \frac{\omega_1}{\mu} \cdot \frac{y_2^3 - y_1^3}{y_2 - y_1} = \frac{1}{\grave{u}} - \frac{\omega_1}{\mu}(y_1^2 + y_2^2 + y_1 y_2);$$

$$\therefore \frac{1}{x_1} = \frac{1}{\mu u} - \frac{\mu^2 - 1}{2\mu^3 u^3}(y_1^2 + y_2^2 + y_1 y_2).$$

Now

$$\begin{aligned} y_1^2 &= (H_1 - a_1)^2 = (u \tan\chi - a_1)^2 = u^2 \tan^2\chi - 2au \tan\chi + a_1^2, \\ y_2^2 &= (H_1 + a_1)^2 = (u \tan\chi + a_1)^2 = u^2 \tan^2\chi + 2au \tan\chi + a_1^2, \\ y_1 y_2 &= (H_1^2 - a_1^2) = (u^2 \tan^2\chi - a_1^2) = u^2 \tan^2\chi \qquad\qquad - a_1^2; \end{aligned}$$

$$\therefore \frac{\omega_1}{\mu}(y_1^2 + y_2^2 + y_1 y_2) = \frac{\omega_1}{\mu}(3u^2 \tan^2\chi + a_1^2);$$

Value of the compounded aberration. Primary plane.

$$\therefore \frac{1}{x_1} = \frac{1}{\mu u} - \frac{\mu^2 - 1}{2\mu^3 u^3}(3u^2 \tan^2\chi) - \frac{\mu^2 - 1}{2\mu^3 u^3} a_1^2,$$

and

Primary plane. Obliquity correction + aberration.

$$\frac{\mu}{x_1} = \frac{1}{u} - \frac{\mu^2 - 1}{2\mu^2 u} 3 \tan^2\chi - \frac{\mu^2 - 1}{2\mu^2 u^3} a_1^2. \tag{77}$$

PLATE. XII.

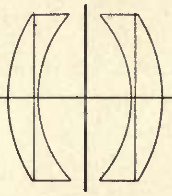


Fig. 52.

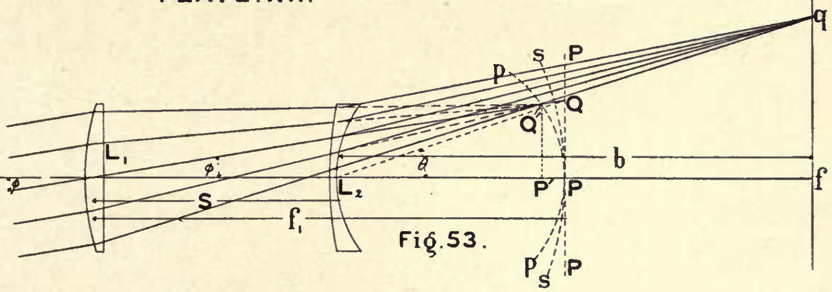


Fig. 53.

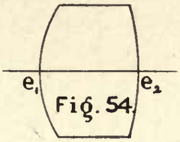


Fig. 54.

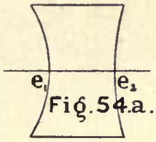


Fig. 54.a.

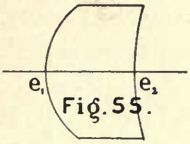


Fig. 55.

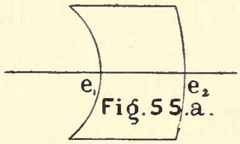


Fig. 55.a.

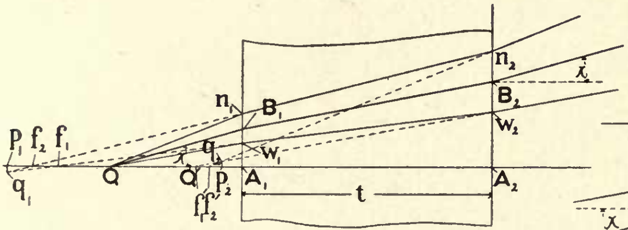


Fig. 56.

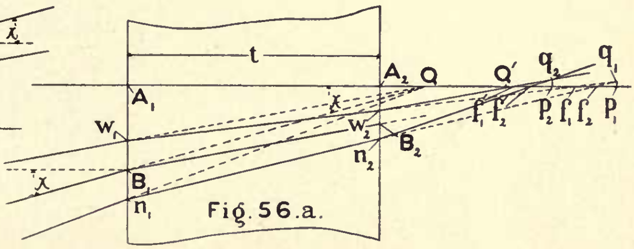


Fig. 56.a.

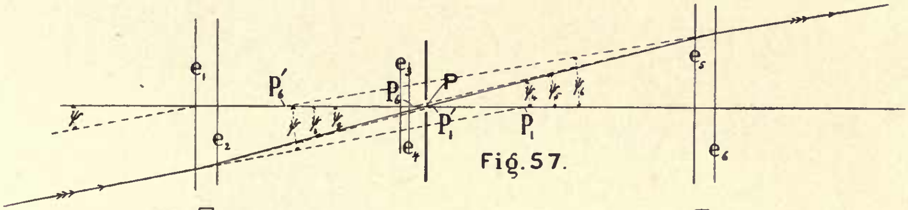


Fig. 57.

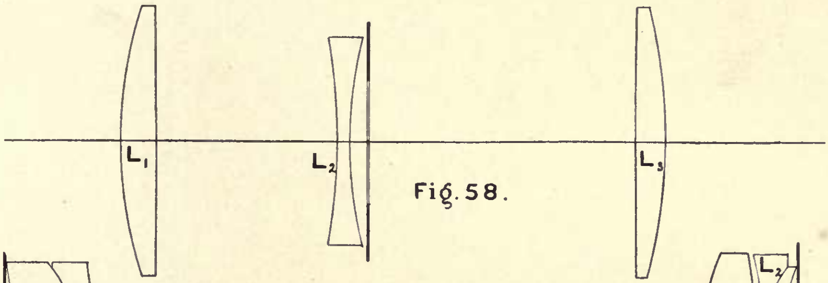


Fig. 58.

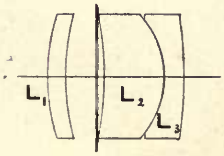


Fig. 59.

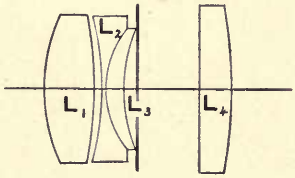


Fig. 60.

PLATE. XII.

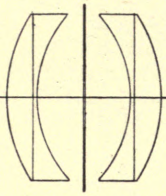


Fig. 52.

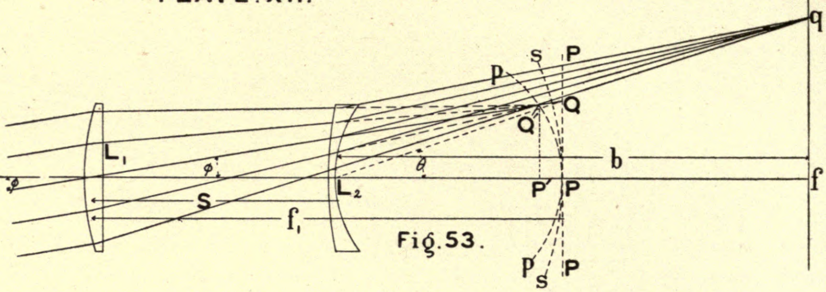


Fig. 53.

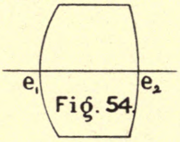


Fig. 54.

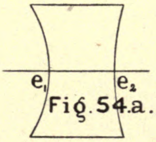


Fig. 54.a.

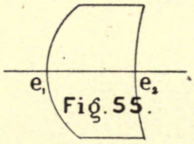


Fig. 55.

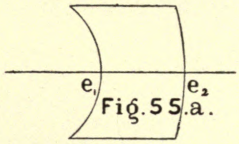


Fig. 55.a.

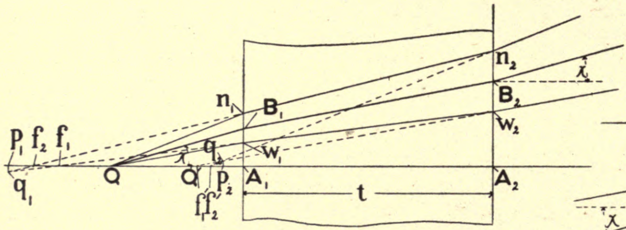


Fig. 56.

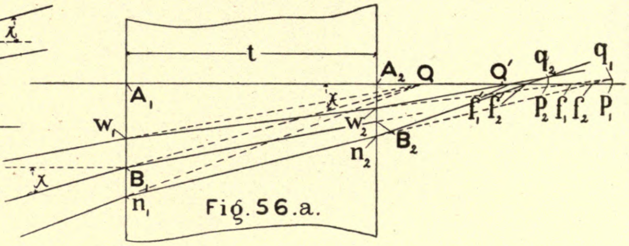


Fig. 56.a.

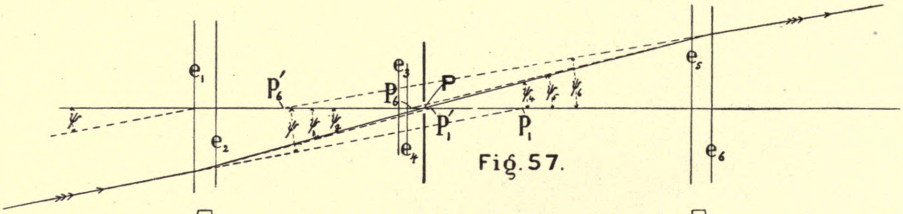


Fig. 57.

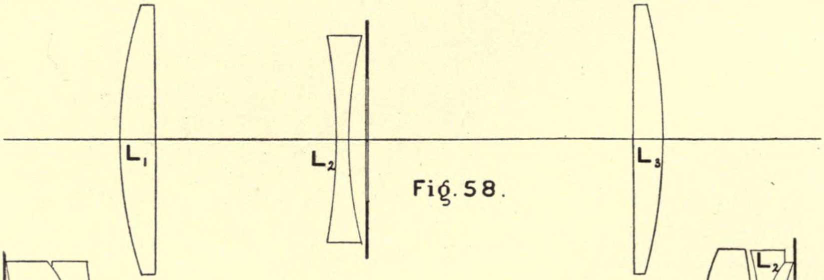


Fig. 58.

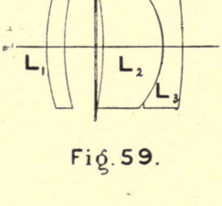


Fig. 59.

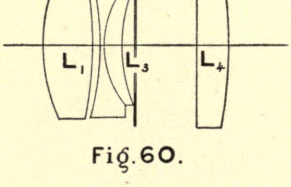


Fig. 60.

In the secondary plane we have

$$y^2 = H^2 + a_1^2 = u^2 \tan^2 \chi + a_1^2 ;$$

$$\therefore \frac{1}{x_1} = \frac{1}{\mu u} - \frac{\mu^2 - 1}{2\mu^3 u^3}(u^2 \tan^2 \chi + a_1^2),$$

and

$$\frac{\mu}{x_1} = \frac{1}{u} - \frac{\mu^2 - 1}{2\mu^2 u} \tan^2 \chi - \frac{\mu^2 - 1}{2\mu^2 u^3} a_1^2. \qquad (78)$$

Secondary plane. Obliquity correction plus aberration.

Hence, as in the case of eccentricity corrections, the correction for obliquity or the function of $\tan^2 \chi$ is three times as much in the primary plane as in the secondary plane.

Ratio between the corrections in the two planes.

Second Surface

We will pursue the investigation in the primary plane. At second surface of Fig. 56 we have the same state of things as is represented in Fig. 56*a* at the first surface, only that in the latter figure we must imagine the light to be passing from right to left, instead of from left to right, and under either supposition the Formula (77) equally applies, so that we have

$$\frac{\mu}{v'} = \frac{1}{v} - \frac{\mu^2 - 1}{2\mu^2 v} 3 \tan^2 \chi - \frac{\mu^2 - 1}{2\mu^2 v^3} a_2^2,$$

and therefore

$$\frac{1}{v} \text{ corrected or } \frac{1}{x_2} = \frac{\mu}{v'} + \frac{\mu^2 - 1}{2\mu^2 v} 3 \tan^2 \chi + \frac{\mu^2 - 1}{2\mu^2 v^3} a_2^2 ; \qquad (79)$$

Second surface. Oblique correction and aberration.

wherein v = corrected value of $Q' \ldots A_2$ of Fig. 56 (corresponding to $Q \ldots A_1$ of Fig. 56*a*),

and v' = first approximate value of $q_1 \ldots A_2$ of Fig. 56 (corresponding to $q_1 \ldots A_1$ of Fig. 56*a*).

But in order to express v and v' for the second refraction in terms of u and $\grave{u}$ at the first refraction, we must put $\grave{u} + t$ for v', and $\dfrac{\grave{u} + t}{\mu}$ or $u + \dfrac{t}{\mu}$ for v; also if a_2, the semi-aperture of the pencil at the second surface, is to be expressed in terms of a_1, we then have

$$a_2 = a_1 \frac{\grave{u} + t}{\grave{u}} = a_1 \frac{\mu u + t}{\mu u} = a_1 \frac{u + \dfrac{t}{\mu}}{u} = a_1 \frac{v}{u}.$$

On inserting the above values of v', v, and a_2 in (79) we then have

$$\frac{1}{x_2}=\frac{\mu}{\dot{u}+t}+\frac{\mu^2-1}{2\mu^2\left(u+\frac{t}{\mu}\right)}3\tan^2\chi+\frac{\mu^2-1}{2\mu^2\left(u+\frac{t}{\mu}\right)^3}\left(a_1\frac{u+\frac{t}{\mu}}{u}\right)^2$$

Formula (79) in terms of u and a.

$$=\frac{1}{u+\frac{t}{\mu}}+\frac{\mu^2-1}{2\mu^2\left(u+\frac{t}{\mu}\right)}3\tan^2\chi+\frac{\mu_2^2-1}{2\mu^2\left(u+\frac{t}{\mu}\right)}\frac{a_1^2}{u^2}. \tag{80}$$

Addition of the formulæ for the two surfaces.

To these aberrations at the second refraction we have yet to add the corresponding aberrations due to the first refraction. First, in order to refer the R corrections to $\frac{\mu}{\dot{u}}$ to the new reference point A_2, we must multiply them by $\left(\frac{\dot{u}}{\dot{u}+t}\right)^2$ or by $\left(\frac{u}{u+\frac{t}{\mu}}\right)^2$ before adding them in to Equation (80).

Thus summing up the aberrations at both surfaces we get

$$\frac{1}{x_2}=\frac{1}{u+\frac{t}{\mu}}-\frac{\mu^2-1}{2\mu^2u}\left(\frac{u}{u+\frac{t}{\mu}}\right)^2 3\tan^2\chi-\frac{\mu^2-1}{2\mu^2u^3}\left(\frac{u}{u+\frac{t}{\mu}}\right)^2 a_1^2$$

(from first refraction, Formula (77)),

$$+\frac{\mu^2-1}{2\mu^2\left(u+\frac{t}{\mu}\right)}3\tan^2\chi+\frac{\mu^2-1}{2\mu^2\left(u+\frac{t}{\mu}\right)}\frac{a_1^2}{u^2}$$

(from second refraction, Formula (80)),

and the sum of these aberrations

$$=\left\{\frac{\mu^2-1}{2\mu^2\left(u+\frac{t}{\mu}\right)}-\frac{\mu^2-1}{2\mu^2\left(u+\frac{t}{\mu}\right)}\cdot\frac{u}{\left(u+\frac{t}{\mu}\right)}\right\}3\tan^2\chi+\left\{\frac{\mu^2-1}{2\mu^2u}\cdot\frac{1}{\left(u+\frac{t}{\mu}\right)}\cdot\frac{1}{u}-\frac{\mu^2-1}{2\mu^2u\left(u+\frac{t}{\mu}\right)^2}\right\}a_1^2$$

$$=\frac{\mu^2-1}{2\mu^2\left(u+\frac{t}{\mu}\right)}\left(1-\frac{u}{u+\frac{t}{\mu}}\right)3\tan^2\chi+\left\{\frac{\mu^2-1}{2\mu^2u\left(u+\frac{t}{\mu}\right)}\left(\frac{1}{u}-\frac{1}{u+\frac{t}{\mu}}\right)\right\}a_1^2$$

$$=\frac{\mu^2-1}{2\mu^2\left(u+\frac{t}{\mu}\right)}\frac{\frac{t}{\mu}}{\left(u+\frac{t}{\mu}\right)}3\tan^2\chi+\frac{\left(\mu^2-1\right)\frac{t}{\mu}}{2\mu^2u^2\left(u+\frac{t}{\mu}\right)^2}a_1^2\,;$$

therefore finally

$$\frac{1}{x_2}=\frac{1}{u+\frac{t}{\mu}}+\frac{(\mu^2-1)\frac{t}{\mu}}{2\mu^2\left(u+\frac{t}{\mu}\right)^2}3\tan^2\chi+\frac{(\mu^2-1)\frac{t}{\mu}}{2\mu^2\left(u+\frac{t}{\mu}\right)^2}\frac{a_1{}^2}{u^2}, \quad \text{XVIII A.}$$

The oblique correction and aberration in terms of u and a_1.

or, as we shall find it more convenient to deal with the pencil as an emergent one, we may therefore express these corrections in terms of a_2 and v. Then, since $\frac{a_2{}^2}{v^2}$ obviously $=\frac{a_1{}^2}{u^2}$, we arrive at the formula

Parallel plate. The oblique correction and aberration in terms of v and a_2.

$$\frac{1}{x_2}=\frac{1}{v}+\frac{(\mu-1)(\mu+1)\frac{t}{\mu}}{2\mu^2v^2}3\tan^2\chi+\frac{(\mu-1)(\mu+1)\frac{t}{\mu}}{2\mu^2v^2}\cdot\frac{a_2{}^2}{v^2}. \quad \text{XVIII. (R.)}$$

If the rays are convergent and v negative, these corrections become negative relatively to v.

In the secondary plane $\tan^2\chi$ replaces $3\tan^2\chi$. This formula can be applied, as regards the correction for obliquity, to any thicknesses of lenses with which we have to deal, the axial part of the lens being supposed to be occupied by a parallel glass plate of the same thickness as that of the lens, only with this difference. We have seen that we need take no notice of the modifications in $\tan\phi$ in a system of separated lenses when computing E.C.s, because the effects of such variations are neutralised by corresponding inverse variations in the distances; but in the case of our parallel plates the nature of the case is in one sense different, the angle χ being the angle made between the optic axis and the principal ray of the oblique pencil entering or leaving the plate, whereas the angle ϕ is the angle included between the optic axis and a ray drawn from the oblique image point Q to the principal point of any lens system.

How the angle χ is to be derived.

Therefore in computing our parallel plate corrections we must always insert the actual angle of obliquity under which the principal ray of the pencil enters or leaves the plate, and this angle χ may be considerably different from the original angle ϕ, which is always assessed in relation to the first principal point of the system; but χ is easily calculated from ϕ.

Let Fig. 57 represent the essentials of Fig. 58—that is, a collective lens L_1, a dispersive lens L_2, and a collective lens L_3 in succession; and let P be the point on the axis where the principal rays of all pencils traversing the system are made to cross—that is, P is the pupil point, where a stop of variable aperture is placed.

Let it be carefully noted that the axial glass thicknesses in this diagram (57) are supposed to be drawn equal to $\frac{1}{\mu}$th of their real amounts, as shown in Fig. 58, and also that the refractions, shown apparently as surface refractions, are really the refractions due to the passage of the principal ray through the successive infinitely thin elements e_1, e_2, e_3, etc., e_1 being convexo-plane, e_2 plano-convex, e_3 concavo-plane, etc. The principal ray is traced through the system as a solid line. Every refraction of the principal ray at an element plane leads to an apparent shifting of the diaphragm point P. For rays first entering the system the apparent diaphragm point is at p_1, and that is what is known as the entrance pupil point of the system; while the axial point p_6', from which the principal rays apparently diverge on emerging from the system, is the exit pupil point of the lens.

Successive pupil points.

For e_1 the front pupil or diaphragm point, or the point to which the principal rays are converging *before* entering, is p_1, and the corresponding diaphragm point to which the principal ray converges *after* refraction by e_1 is p_1'; that is, p_1 and p_1' are conjugate foci with respect to the element e_1. We will denote the distance $e_1 .. p_1$ by D_1', and the distance $e_1 .. p_1'$ by D_1'', so that D_1' and D_1'' are conjugate focal distances. Either of these distances D_1' or D_1'' determines the characteristic quantity β_1 for the element e_1, and we can either write

$$\frac{1+\beta_1}{2f_1} = \frac{1}{D_1'} \text{ or } \frac{1-\beta_1}{2f_1} = \frac{1}{D_1''}$$

(wherein f_1 stands for the principal focal length of e_1), and so determine β_1. For e_2 the front diaphragm point is p_1' or p_2, and the back diaphragm point is p_2' or p_3. Distance $e_2 .. p_2 = D_2'$, and $e_2 .. p_2' = D_2''$, either giving β_2, and so on.

As it is scarcely possible to exhibit clearly all the various refractions to which the principal ray is subject in Fig. 57, unless it were on a much larger scale, therefore the minor refractions exerted by e_3 and e_4 are not represented therein.

Now it is evident that if ψ is the first angle between the incident principal ray and the axis before refraction by e_1, then ψ_1, representing the angle between the principal ray and the axis *after* refraction by e_1, will also be the required angle of either incidence or emergence under which the principal ray enters or leaves the parallel glass plate of thickness t_1, and it will be greater than ψ, since e_1 is collective. If the lens L_1 had been drawn in its actual thickness, it is evident that the principal ray would have had to be shown traversing it at a smaller angle $= \frac{\psi_1}{\mu}$, or the angle of obliquity in the *interior* of the glass plate. But by considering e_1 and e_2 not as mere surfaces, but as complete though infinitely thin lenses, and also substituting an air-space equal to $\frac{t_1}{\mu_1}$ in place of t_1, then ψ_1 becomes the angle of incidence or emergence into and out of the first glass plate, which is what we really want. Moreover, all the diaphragm distances D_1', D_2', and D_1'', D_2'', etc., etc., for the principal rays, and the image distances u_1, u_2, and v_1 and v_2, etc., etc., for rays constituting the pencils, all come out to their proper values by means of this simple device.

The expression of $\tan\psi_1$, $\tan\psi_2$, etc., in terms of $\tan\psi$.

Now it is evident that

$$\tan\psi_1 = \tan\psi\,\frac{D_1'}{D_1''},\ \tan\psi_2 = \tan\psi\,\frac{D_1'D_2'}{D_1''D_2''},\ \tan\psi_3 = \tan\psi\,\frac{D_1'D_2'D_3'}{D_1''D_2''D_3''},$$

$$\tan\psi_4 = \tan\psi\,\frac{D_1'D_2'D_3'D_4'}{D_1''D_2''D_3''D_4''},\ \tan\psi_5 = \tan\psi\,\frac{D_1'D_2'D_3'D_4'D_5'}{D_1''D_2''D_3''D_4''D_5''},$$

and finally

$$\tan\psi_6 = \frac{D_1'D_2'D_3'D_4'D_5'D_6'}{D_1''D_2''D_3''D_4''D_5''D_6''}\tan\psi.$$

Hence in applying the oblique correction of Formula XVIII. (R.) the original $\tan\psi$ must be multiplied by the corresponding factor $\frac{D_1'}{D_1''}$ or $\frac{D_1'D_2'D_3'}{D_1''D_2''D_3''}$, as the case may be. It is clear that if the rays of pencils entering L_1 are parallel as if coming from an infinitely distant object, then angle ψ is the same as ϕ.

All these diaphragm stop or pupil distances have, in the first place, to be worked out in any given lens system, as a necessary step to deriving the characteristics β_1, β_2, etc., for each element.

The Transference of the Parallel Plate Corrections to the Final Focus

But we have yet to carry these parallel plate corrections through to the final focus and convert them into corrections to $\frac{1}{\text{E.F.L.}}$ of the system. Referring to Formula XVIII. we have two corrections to the reciprocal value of the perpendicular distance v from the second plate surface of the point from which or to which the pencil diverges or converges. The second formula is a function of the aperture of the pencil, and is of the same nature as spherical aberration, and we have already dealt with it in Section IV. It applies to all pencils, whether axial or oblique, and may for our present purposes be left out of consideration, leaving us only the oblique correction to $\frac{1}{v}$ expressed as

$$\frac{(\mu^2-1)\frac{t}{\mu}}{2\mu^2 v^2} 3 \tan^2 \chi. \tag{81}$$

For our purposes we must now convert this into a correction to the linear value of v by multiplying it by $-v^2$, and then we get

Parallel plate. Linear value of oblique correction.

$$-\frac{\mu^2-1}{2\mu^2} \cdot \frac{t_1}{\mu} 3 \tan^2 \chi. \qquad \text{XIX. (L.)}$$

This is the absolute linear value of the oblique correction due to a parallel plate of thickness t_1. It is thus seen to be independent of the amount of u or of v, and is merely a function of the thickness, angle of obliquity ϕ, and refractive index μ. Referring to Fig. 57, it will be readily seen that after we have got the linear oblique correction due to passage through the parallel plate t_1 from Formula XIX. (L.), we can then express it as a correction to $\frac{1}{u_2}$ by multiplying it by $\left(\frac{1}{u_2}\right)^2$; we transform it back again to its linear value at the conjugate focal distance v_2 by multiplying by v_2^2, so that the linear correction to v_2 after refraction through e_2 is expressed by

$$\frac{\mu^2-1}{2\mu^2} \frac{t}{\mu} 3 \tan^2 \chi_1\left(\frac{v_2^2}{u_2^2}\right). \tag{82}$$

It must be borne in mind that *all* parallel plate corrections, reduced to linear value, are essentially of positive value with respect to the final focal distance of a collective system; there is, therefore, no question of signs to trouble us. They all take the form of linear transferences of oblique foci from left to right, or in the direction in which the light travels through the system. For the same reasons these corrections considered as reciprocal corrections, as in Formula (81), are all of negative import with respect to the final focal power, if the latter is positive; and since their value in the primary plane is three times their value in the secondary plane, they amount for all practical purposes to the same thing as minus eccentricity corrections.

The oblique plate corrections all of same sign ultimately.

Having now got Formula (82) expressing the linear correction to v_2, we then express it as a correction to $\frac{1}{u_3}$ by multiplying by $\frac{1}{u_3^2}$, and then reduce to its value as a linear correction to v_3 by multiplying by v_3^2, when we get

$$\frac{\mu^2-1}{2\mu^2}\frac{t}{\mu}3\tan^2\phi_1\frac{v_2^2v_3^2}{u_2^2u_3^2}, \qquad (83)$$

and so on, until after refraction through e_6 we get, as the linear correction to the oblique final conjugate focal distance v_6, the amount

$$\frac{\mu^2-1}{2\mu^2}\cdot\frac{t}{\mu}\left(\frac{v_2v_3v_4v_5v_6}{u_2u_3u_4u_5u_6}\right)^2 3\tan^2\phi_1. \qquad (84)$$

Then to convert this into a correction to the reciprocal of the equivalent focal length of the combination we must multiply (84) by $\left(\frac{1}{\text{E.F.L.}}\right)^2$. Also $\tan^2\chi_1=\tan^2\psi_1=\tan^2\psi\left(\frac{D_1'}{D_1''}\right)^2$, as we have seen before. After inserting these values we therefore get, for the case before us,

$$-\frac{\mu_1^2-1}{2\mu_1^2}\cdot\frac{t_1}{\mu_1}\left(\frac{v_2v_3v_4v_5v_6}{u_2u_3u_4u_5u_6}\right)^2 3\tan^2\psi\left(\frac{D_1'}{D_1''}\right)^2\left(\frac{1}{\text{E.F.L.}}\right)^2. \qquad (85)\,\text{R.}$$

First oblique plate correction transferred to final focus.

In the same way the final oblique plate correction due to the second parallel plate of thickness t_2 is expressed as

$$-\frac{\mu_2^2-1}{2\mu_2^2}\cdot\frac{t_2}{\mu_2}\left(\frac{v_4v_5v_6}{u_4u_5u_6}\right)^2\left(\frac{D_1'D_2'D_3'}{D_1''D_2''D_3''}\right)^2 3\tan^2\psi\left(\frac{1}{\text{E.F.L.}}\right)^2; \qquad (86)\,\text{R.}$$

Second oblique plate correction transferred to final focus.

and, finally, the third glass plate of thickness t_3 gives us

$$-\frac{\mu_3^2-1}{2\mu_3^2}\cdot\frac{t_3}{\mu_3}\left(\frac{v_6}{u_6}\right)^2\left(\frac{D_1'D_2'D_3'D_4'D_5'}{D_1''D_2''D_3''D_4''D_5''}\right)^2 3\tan^2\psi\left(\frac{1}{\text{E.F.L.}}\right)^2. \qquad (87)\,\text{R.}$$

Third oblique plate correction transferred to final focus.

As the quantities D′ and D″ and u and v have always to be worked out for each element at the outset for the purpose of arriving at the characteristics a and β for each element, the application of the above formulæ entails very little extra work. There is another way of working in these parallel plate corrections, but the above method is the simplest and most straightforward.

Having now explained the nature of the method of calculating the normal curvature errors and eccentricity corrections, etc., of any optical system, so as to define the state of the final image with regard to flatness, curvature, or astigmatism, we will conclude with three series of carefully checked calculations as applied to three different optical constructions of which the curves, thicknesses, separations, and refractive indices were all known with reasonable accuracy, and whose final images were also carefully observed and accurately measured.

Instances of the Practical Application of the Formulæ of this Section to actual Lens Constructions

1st. A Series 1*c* Cooke Lens for Stellar Photography of 6·5 inches aperture and 43·05 inches measured equivalent focal length (Fig. 58). As the foci for the D ray lend themselves best to visual measurement, we will take the heads of the calculations for that ray—

	L_1	L_2	L_3
Refractive indices.	$\mu_D = 1{\cdot}5180$	$\mu_D = 1{\cdot}6035$	$\mu_D = 1{\cdot}5180$
	Radii	Radii	Radii
Radii of surfaces.	$r_1 = +10{\cdot}64$ $r_2 = +72{\cdot}45$	$r_3 = -14{\cdot}54$ $r_4 = -10{\cdot}35$	$r_5 = +67{\cdot}35$ $r_6 = +13$
Thicknesses and equivalent air-spaces.	$t_1 = {\cdot}83 \quad \frac{t_1}{\mu_1} = {\cdot}547$	$t_2 = {\cdot}325 \quad \frac{t_2}{\mu_2} = {\cdot}203$	$t_3 = {\cdot}75 \quad \frac{t_3}{\mu_3} = {\cdot}494$

Separations. Axial air-space $A_1 = 4{\cdot}39$, and $A_2 = 6{\cdot}85$.

Diaphragm or pupil point where principal rays cross the axis is taken as being ·40 inches behind vertex of fourth surface.

The powers of the six elements are therefore

Powers of the six elements.

$$\frac{1}{f_1} = \frac{{\cdot}518}{10{\cdot}64} = \frac{1}{20{\cdot}54} \qquad \frac{1}{f_2} = \frac{{\cdot}518}{72{\cdot}45} = \frac{1}{139{\cdot}865} \qquad \frac{1}{f_3} = \frac{{\cdot}6035}{14{\cdot}54} = \frac{1}{24{\cdot}092}$$

$$\frac{1}{f_4} = \frac{{\cdot}6035}{10{\cdot}35} = \frac{1}{17{\cdot}15} \qquad \frac{1}{f_5} = \frac{{\cdot}518}{67{\cdot}35} = \frac{1}{130{\cdot}019} \qquad \frac{1}{f_6} = \frac{{\cdot}518}{13} = \frac{1}{25{\cdot}096}$$

The first entering rays are supposed to be parallel and $u_1 = \infty$, starting from which we get the following data for the six successive elements (each element being styled by E_x)—

Values of u, v, a, and x for the successive elements.

E_1

$$\frac{1}{f_1} = \frac{1}{20{\cdot}54}$$

$u_1 = \infty$ from which $a_1 = -1$

$v_1 = +20{\cdot}54$ (convergent and plus) $x_1 = +1$

E_2

$$\frac{1}{f_2} = \frac{1}{139{\cdot}865}$$

$u_2 = 20{\cdot}54 - {\cdot}547 = -19{\cdot}993$ (convergent and minus)

$v_2 = +17{\cdot}492$ (convergent and plus) from which $a_2 = -14{\cdot}992$

$x_2 = -1$

E_3

$$\frac{1}{f_3} = \frac{1}{24{\cdot}092}$$

$u_3 = 17{\cdot}492 - 4{\cdot}39 = +13{\cdot}102$ (convergent and plus)

$v_3 = -28{\cdot}723$ (convergent and minus) $a_3 = +2{\cdot}677$

$x_3 = +1$

E$_4$

$\frac{1}{f_4} = \frac{1}{17 \cdot 15}$ $u_4 = 28{\cdot}723 - {\cdot}203 = +28{\cdot}52$ (convergent and plus)

$v_4 = 43{\cdot}018$ (divergent and plus) $a_4 = +{\cdot}203$

$x_4 = -1$

E$_5$

$\frac{1}{f_5} = \frac{1}{130 \cdot 019}$ $u_5 = 43{\cdot}018 + 6{\cdot}85 = 49{\cdot}868$ (divergent and plus)

$v_5 = 80{\cdot}895$ (divergent and minus) $a_5 = +4{\cdot}214$

$x_5 = +1$

E$_6$

$\frac{1}{f_6} = \frac{1}{25 \cdot 096}$ $u_6 = 80{\cdot}895 + {\cdot}494 = 81{\cdot}389$ (divergent and plus)

$v_6 = +36{\cdot}285$ (convergent and plus and = back focal length)

$a_6 = -{\cdot}383$

$x_6 = -1$

We have now to assess the value of β for each element. Starting from the pupil point or the intercrossing point of the principal rays placed at ·40 inch behind the fourth element, we have

Values of D′, D″, and β for the successive elements.

for E$_4$ $D_4'' = -{\cdot}40$ behind, and $D_4' = +{\cdot}391$ (conjugate to D_4'') $\therefore \beta_4 = +86{\cdot}75$

for E$_3$ $D_3'' = -({\cdot}391 + {\cdot}203) = -{\cdot}594$, and $D_3' = +{\cdot}58$ $\therefore \beta_3 = +82{\cdot}12$

for E$_2$ $D_2'' = {\cdot}58 + 4{\cdot}39 = +4{\cdot}97$, and $D_2' = -5{\cdot}153$ $\therefore \beta_2 = -55{\cdot}284$

for E$_1$ $D_1'' = 5{\cdot}153 + {\cdot}547 = +5{\cdot}70$, and $D_1' = -7{\cdot}89$ $\therefore \beta_1 = -6{\cdot}207$

for E$_5$ $D_5' = 6{\cdot}85 - {\cdot}40 = +6{\cdot}45$, and $D_5'' = -6{\cdot}786$ $\therefore \beta_5 = +39{\cdot}316$

for E$_6$ $D_6' = 6{\cdot}786 + {\cdot}494 = +7{\cdot}28$, and $D_6'' = -10{\cdot}25$ $\therefore \beta_6 = +5{\cdot}894$

Then the E.C.s in secondary planes, as ascertained from Formula VIII., Section VI. (substituting $\tan^2\phi$ for $3\tan^2\phi$), may be expressed shortly as

$$\frac{\tan^2\phi}{2F}\frac{1}{(a-\beta)^2}\left\{A' - 2(a-\beta)C'\right\},$$

and come out as follows:—

Eccentricity Corrections.

for E$_1$ E.C.s = + ·00276 $\tan^2\phi$
for E$_2$ „ = + ·0102389 „
for E$_3$ „ = + ·0053134 „
for E$_4$ „ = − ·0014101 „
for E$_5$ „ = + ·0036181 „
for E$_6$ „ = − ·0152812 „

{These two being dispersive elements, the signs of the E.C.s have to be reversed before summing up. [refers to E$_3$ and E$_4$]

E.C.s for E$_3$ and E$_6$ =	− ·020594 $\tan^2\phi$
E.C.s for E$_1$, E$_2$, E$_4$, and E$_5$ =	+ ·018027 „
Total for system	− ·002567 $\tan^2\phi$

Total of above, secondary plane.

The normal curvature errors in secondary planes of the four collective elements as ascertained by

$$\frac{\tan^2\phi}{2}\cdot\frac{\mu+1}{\mu}\left(\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_5}+\frac{1}{f_6}\right) = +\cdot085734\tan^2\phi$$

and the same for the two dispersive elements $= -\cdot081032$ „

Total normal curvature errors, secondary plane.

therefore the total normal curvature errors in secondary planes $= +\cdot004702\tan^2\phi$

The normal curvature errors in primary planes of the four collective elements as ascertained by

$$\frac{\tan^2\phi}{2}\,\frac{3\mu+1}{\mu}\left(\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_5}+\frac{1}{f_6}\right) = +\cdot189105\tan^2\phi$$

and the same for the two dispersive elements $= -\cdot180847$ „

Total normal curvature errors, primary plane.

therefore the total normal curvature errors in primary planes $= +\cdot008258\tan^2\phi$

Parallel plate corrections in secondary planes for L_1 as ascertained by

First plate oblique corrections, secondary plane.

$$\tan^2\phi\,\frac{(\mu_1{}^2-1)\frac{t_1}{\mu_1}}{2\mu_1{}^2}\left(\frac{D_1'}{D_1''}\right)^2\left(\frac{v_2v_3v_4v_5v_6}{u_2u_3u_4u_5u_6}\right)^2\left(\frac{1}{\text{E.F.L.}}\right)^2 = -\cdot00070028\tan^2\phi$$

and for L_2 as ascertained by

Second plate oblique corrections, secondary plane.

$$\tan^2\phi\,\frac{(\mu_2{}^2-1)\frac{t_2}{\mu_2}}{2\mu_2{}^2}\left(\frac{D_1'D_2'D_3'}{D_1''D_2''D_3''}\right)^2\left(\frac{v_4v_5v_6}{u_4u_5u_6}\right)^2\left(\frac{1}{\text{E.F.L.}}\right)^2 = -\cdot000078 \text{ „}$$

and for L_3 as ascertained by

Third plate oblique corrections, secondary plane.

$$\tan^2\phi\,\frac{(\mu_3{}^2-1)\frac{t_3}{\mu_3}}{2\mu_3{}^2}\left(\frac{D_1'D_2'D_3'D_4'D_5'}{D_1''D_2''D_3''D_4''D_5''}\right)^2\left(\frac{v_6}{u_6}\right)^2\left(\frac{1}{\text{E.F.L.}}\right)^2 = -\cdot00002537 \text{ „}$$

Total of same.

Total $= -\cdot00080366\tan^2\phi$

and three times that quantity in primary planes.

Summary.

On summing up in secondary planes we have

$+\cdot004702\tan^2\phi$ for normal curvature errors,
$-\cdot002567\tan^2\phi$ for eccentricity corrections (E.C.s),
$-\cdot000804\tan^2\phi$ for parallel plate corrections,

Final total, secondary plane.

$+\cdot001331\tan^2\phi$ being the final error, which it is now desirable to express as a linear deviation from the focal plane. To that end it must be multiplied by $-(\text{E.F.L.})^2$.

Let ϕ be $7\frac{1}{2}$ degrees, for which the tangent $=\cdot132$.

Then the linear deviation, in the secondary plane, from the

Final linear error.

focal plane at that angle is $+\cdot00133\times-(\cdot132\times43\cdot05)^2 = -\cdot043$ inch,

Observed error.

while the actually measured deviation was $-\cdot040$ inch.

Primary Plane

On summing up in the primary plane we have

$+\cdot008258 \tan^2 \phi$ for normal curvature errors,
$-\cdot007701 \tan^2 \phi$ for eccentricity corrections (E.C.s),
$-\cdot002412 \tan^2 \phi$ for parallel plate corrections,

$-\cdot001855 \tan^2 \phi$ being the final error, from which the linear error at $7\frac{1}{2}$ degrees from the axis **Final total, primary plane.**

$$=(-\cdot001858) \times -(\cdot132 \times 43\cdot05)^2 = +\cdot059 \text{ inch,}$$ **Final linear error.**

while the actually measured deviation was $+\cdot030$ inch. **Observed error.**

Thus the measured deviation in the secondary plane agrees more exactly with the calculated result than the deviation in the primary plane. The whole field of this lens did not extend to much more than 10 degrees from the axis. We shall have occasion to refer to these residual discrepancies in Section XI.

Process Lens

The next example is shown in section in Fig. 59. It is a lens specially designed for copying or process work, also composed of only three lenses. The following curves, etc., are for an E.F.L. of 8·55 inches.

L_1	L_2	L_3	
$\mu_{1D}=1\cdot6103$	$\mu_{2D}=1\cdot6103$	$\mu_{3D}=1\cdot5240$	**Refractive indices.**
$r_1=+1\cdot264$	$r_3=-2\cdot09$	$r_5=-\cdot5325$	**Radii.**
$r_2=-1\cdot48$	$r_4=+\cdot553$	$r_6=+2\cdot8$	
$t_1=\cdot105 \quad \frac{t_1}{\mu_1}=\cdot0652$	$t_2=\cdot358 \quad \frac{t_2}{\mu_2}=\cdot222$	$t_3=\cdot110 \quad \frac{t_3}{\mu_3}=\cdot0722$	**Thicknesses.**
$A_1=\cdot232$		$A_2=\cdot0053$	**Separations.**

Focal lengths and characteristics.

	f	α	β	x
E_1	$f_1=+2\cdot0711$	$\alpha_1=-1$	$\beta_1=-16\cdot796$	$x_1=+1$
E_2	$f_2=-2\cdot425$	$\alpha_2=+1\cdot4179$	$\beta_2=+27\cdot945$	$x_2=-1$
E_3	$f_3=-3\cdot4246$	$\alpha_3=-\cdot3978$	$\beta_3=-132\cdot711$	$x_3=+1$
E_4	$f_4=+\cdot90611$	$\alpha_4=-\cdot6462$	$\beta_4=+5\cdot626$	$x_4=-1$
E_5	$f_5=-1\cdot01622$	$\alpha_5=+\cdot8552$	$\beta_5=-6\cdot1186$	$x_5=+1$
E_6	$f_6=+5\cdot3432$	$\alpha_6=-\cdot2423$	$\beta_6=+28\cdot877$	$x_6=-1$

E.C.s, Secondary Plane

Eccentricity Corrections, secondary plane.

for $E_1 =$	$+ \cdot 0053128 \tan^2\phi$	for $E_2 =$	$- \cdot 0183698 \tan^2\phi$
for $E_3 =$	$+ \cdot 0034821$,,	for $E_4 =$	$- \cdot 4621330$,,
for $E_5 =$	$+ \cdot 4544450$,,	for $E_6 =$	$- \cdot 0222200$,,
	$+ \cdot 4632399 \tan^2\phi$		$- \cdot 5027228 \tan^2\phi$
			$+ \cdot 4632399$,,
		Total E.C.s $=$	$- \cdot 0394829 \tan^2\phi$

Total.

Normal Curvature Errors

Total normal curvature errors, secondary plane.

$$\frac{\tan^2\phi}{2}\frac{\mu+1}{\mu}\left(\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}+\frac{1}{f_4}\right) = +\cdot 714920 \tan^2\phi$$

$$\frac{\tan^2\phi}{2}\frac{\mu_3+1}{\mu_3}\left(\frac{1}{f_5}+\frac{1}{f_6}\right) = -\cdot 659896 \text{ ,,}$$

$$+\cdot 055024 \tan^2\phi$$

Normal curvature errors in primary plane $+\cdot 14020 \tan^2\phi$

Oblique plate corrections, secondary plane.

Parallel plate corrections for	$L_1 =$	$- \cdot 0060907 \tan^2\phi$
,, ,,	$L_2 =$	$- \cdot 0029855$,,
,, ,,	$L_3 =$	$- \cdot 0001118$,,
Total		$- \cdot 0091880 \tan^2\phi$

Summary

Final totals.

	Secondary Plane.	Primary Plane.
Total normal curvature errors	$+\cdot 055024 \tan^2\phi$	$+\cdot 140200 \tan^2\phi$
Total E.C.s	$-\cdot 039483$,,	$-\cdot 118449$,,
Total parallel plate corrections	$-\cdot 009188$,,	$-\cdot 027564$,,
Final error	$+\cdot 006353 \tan^2\phi$	$-\cdot 005813 \tan^2\phi$

Taking the angle of obliquity ϕ to be 14° 2′, whose tangent is ·25, and multiplying above results by $-(\text{E.F.L.})^2$, we get

Calculated error. $-(+\cdot 006353)(\cdot 25)^2(8\cdot 55)^2 = -\cdot 029$ inch in secondary plane,

Observed error. while actual measurement gave $-\cdot 005$ inch in secondary plane,

Calculated error. $-(-\cdot 005813)(\cdot 25)^2(8\cdot 55)^2 = +\cdot 0265$ inch in primary plane,

Observed error. while actual measurement gave $+\cdot 05$ inch in primary plane.

Discrepancies.

Owing to the difficulty in accurately measuring the radii in such deep curved combinations, such discrepancies as the above may be partly due to statements of radii being inexact.

But the principal cause of the discrepancy is due to the unmistakable presence of minus corrections of the order $\tan^4\phi$, which will be better understood after reading Section XI.

Series III*a*. Cooke Lens

This is composed of four lenses, the dispersive lens being compound; see Fig. 60.

E.F.L. = 10 inches.

L_1	L_2	L_3	L_4	
$\mu_D = 1{\cdot}5101$	$\mu_D = 1{\cdot}5365$	$\mu_D = 1{\cdot}6110$	$\mu_D = 1{\cdot}5101$	**Refractive indices.**
$r_1 = +2{\cdot}158$	$r_3 = -3{\cdot}472$	$r_5 = +1{\cdot}150$	$r_7 = +12{\cdot}65$	**Radii.**
$r_2 = +4{\cdot}655$	$r_4 = -1{\cdot}150$	$r_6 = -1{\cdot}910$	$r_8 = +5{\cdot}843$	
$t_1 = {\cdot}603$	$t_2 = {\cdot}044$	$t_3 = {\cdot}218$	$t_4 = {\cdot}393$	**Thicknesses.**

$A_1 = {\cdot}08$ $A_2 = 0$ $A_3 = {\cdot}90$ **Separations.**

Diaphragm or pupil point ·25 behind vertex of sixth surface.

Focal lengths and characteristics.

E_1	$f_1 = +4{\cdot}2305$	$a_1 = -1$	$\beta_1 = -8{\cdot}566$	$x_1 = +1$
E_2	$f_2 = +9{\cdot}1256$	$a_2 = -5{\cdot}764$	$\beta_2 = -38{\cdot}634$	$x_2 = -1$
E_3	$f_3 = -6{\cdot}4716$	$a_3 = +3{\cdot}943$	$\beta_3 = +33{\cdot}012$	$x_3 = +1$
E_4	$f_4 = -2{\cdot}1435$	$a_4 = -{\cdot}019$	$\beta_4 = +10{\cdot}410$	$x_4 = -1$
E_5	$f_5 = +1{\cdot}8822$	$a_5 = -{\cdot}105$	$\beta_5 = -9{\cdot}262$	$x_5 = +1$
E_6	$f_6 = +3{\cdot}126$	$a_6 = +{\cdot}912$	$\beta_6 = +26{\cdot}008$	$x_6 = -1$
E_7	$f_7 = +24{\cdot}8$	$a_7 = -{\cdot}31$	$\beta_7 = +75{\cdot}307$	$x_7 = +1$
E_8	$f_8 = +11{\cdot}45$	$a_8 = -1{\cdot}607$	$\beta_8 = +23{\cdot}624$	$x_8 = -1$

E.C.s, Secondary Plane

Eccentricity Corrections, secondary plane.

E_2	$+{\cdot}074013 \tan^2\phi$	E_1	$-{\cdot}0000772 \tan^2\phi$
E_4	$+{\cdot}104650$,,	E_3	$-{\cdot}0839500$,,
E_6	$+{\cdot}001617$,,	E_5	$-{\cdot}1002700$,,
E_7	$+{\cdot}001347$,,	E_8	$-{\cdot}0226700$,,
	$+{\cdot}181627 \tan^2\phi$		$-{\cdot}2069672 \tan^2\phi$
			$+{\cdot}181627$,,
		Total =	$-{\cdot}02534 \tan^2\phi$

Total.

Normal Curvature Errors

Normal curvature errors, secondary plane.

$$\frac{\tan^2\phi}{2}\frac{\mu_1+1}{\mu_1}\left\{\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_7}+\frac{1}{f_8}\right\} = +\cdot393624\tan^2\phi$$

$$\frac{\tan^2\phi}{2}\frac{\mu_2+1}{\mu_2}\left\{\frac{1}{f_3}+\frac{1}{f_4}\right\} = -\cdot512616 \quad ,,$$

$$\frac{\tan^2\phi}{2}\frac{\mu_3+1}{\mu_3}\left\{\frac{1}{f_5}-\frac{1}{f_6}\right\} = +\cdot171320$$

$$\text{Total} = +\cdot052320\tan^2\phi$$

Normal curvature errors in primary plane $= +\cdot11631\tan^2\phi$.

Total oblique plate corrections, secondary plane.

Parallel plate corrections for $L_1 = -\cdot0108010\tan^2\phi$
,, ,, $L_2 = -\cdot0005574$,,
,, ,, $L_3 = -\cdot0048564$,,
,, ,, $L_4 = -\cdot0000536$,,

Total $-\cdot0162684\tan^2\phi$ in secondary plane.

Summary

	Secondary Plane.	Primary Plane.
Nor. curv. errors	$+\cdot05232\tan^2\phi$	$+\cdot11631\tan^2\phi$
E.C.s . .	$-\cdot02534$,,	$-\cdot07602$,,
Par. plate corr. .	$-\cdot01627$,,	$-\cdot04881$,,
Final totals.	$+\cdot01071\tan^2\phi$	$-\cdot00852\tan^2\phi$

Supposing the angle of obliquity to be 14° 2′ as before, then after multiplying above final errors by $-(\tan^2\phi)(\text{E.F.L.})^2$ or by $-(\cdot25)^2(10)^2$ we get linear deviations from the plane image of $-\cdot067$ in secondary planes and $+\cdot053$ in primary planes. The actually observed errors were $-\cdot04$ in secondary planes and no perceptible error in primary planes, with the lens stopped down to $\frac{F}{10}$.

In the three concrete instances given it will be observed that the thickness of a lens exerts influence in two ways upon the oblique pencils refracted through it: first and most important, the separation between the two elements very largely alters the relationship between the several D's, and consequently the β's, for the two elements; and secondly, by introducing a parallel glass plate. This last generally gives rise to much smaller effects than the first, and yet in the three instances given it is too large to be neglected. There is no manageable formula whereby a thick lens can be treated as a whole.

Eccentric Oblique Reflection from a Spherical Reflector

It would scarcely be worth the necessary space to work out fully and independently the formulæ applying to eccentric oblique reflection for a spherical mirror, as their practical applications do not compare in importance with the corresponding formulæ relating to lenses. However, there is a short cut to the formulæ relating to a spherical reflector which may be followed with advantage. We have already noted, in connection with the formulæ for spherical aberration and central oblique refraction, that the refraction formulæ may be transformed into the corresponding reflection formulæ by the simple device of substituting the value -1 for μ. Let us take the formula for E.C.s in the secondary plane, which is

$$\frac{\tan^2\phi}{2f}\frac{1}{(\alpha-\beta)^2}\frac{1}{\mu(\mu-1)}\left[\left\{\frac{\mu+2}{\mu-1}x^2+4(\mu+1)\alpha x+(3\mu+2)(\mu-1)\alpha^2+\frac{\mu^3}{\mu-1}\right\}\right.$$
$$\left.-2(\alpha-\beta)\{(2\mu+1)(\mu-1)\alpha+(\mu+1)x\}\right],$$

and make the substitution therein of -1 for μ, and we then get

$$\frac{\tan^2\phi}{2f}\frac{1}{(\alpha-\beta)^2}\frac{1}{2}\left[\left\{-\frac{1}{2}x^2+0+2\alpha^2+\frac{1}{2}\right\}-2(\alpha-\beta)\{2\alpha+0\}\right].$$

If the power of a lens is concentrated into one surface only, then the other surface is plane and x is $+$ or -1. In the case of a spherical reflecting surface the power is also concentrated into one surface, and $x = +$ or -1; it does not matter which. Therefore the term containing x^2 cancels out and there remains simply

$$\frac{\tan^2\phi}{2f}\frac{1}{(\alpha-\beta)^2}\{\alpha^2-(\alpha-\beta)(2\alpha)\}, \qquad \text{XX.}$$

while in the primary plane $\tan^2\phi$ becomes $3\tan^2\phi$, and the correction is of course extra to the normal curvature error $\frac{\tan^2\phi}{F}$.

Here, just as in the case of the lens,

$$\frac{1+\alpha}{2f}=\frac{1}{u} \text{ and } \frac{1+\beta}{2f}=\frac{1}{D'},$$

D′ being the distance of the stop from the mirror vertex. Thus α is the vergency characteristic for the rays constituting pencils, and β the vergency characteristic for the principal rays. If the reader will pursue the investigation in detail and *ab initio* for a mirror with a stop placed in front, he will arrive at precisely the same formula as that which we have just derived by substituting -1 for μ and 1 for x.

We have in the Gregorian and Cassegrain forms of reflecting telescope two cases to which the above formula applies, for it is clear that while there is central oblique reflection from the large concave mirror, yet there is eccentric oblique reflection from the small concave or convex mirror as the case may be. But, as the angular extent of field taken in by even the lowest power eye-piece rarely exceeds a degree, the question as to which form of reflecting telescope gives the flattest final image is of little practical consequence. Such telescopes are essentially very ill adapted, owing to their construction, for taking photographic views covering an angle of view at all comparable to what can be embraced by refracting instruments.

SECTION VIII

COMA AND THE SINE CONDITION—VON SEIDEL'S SECOND CONDITION—CENTRAL OBLIQUE REFRACTION

It is now our object to investigate much more closely than we have yet done the nature of that phenomenon known to practical opticians as coma, and sometimes as side-flare. We shall find that many of its manifestations are of an exceedingly interesting nature, of great theoretical interest as well as of great practical importance. For a small amount of coma at the oblique focus of a point in a distant object formed by a lens system may cause much more mischief to the definition than either astigmatism or spherical aberration, or both combined, so that it is eminently desirable to arrive at reliable formulæ of the second approximation by the employment of which it shall be possible to eliminate coma from any desired lens system.

Great importance of coma.

In Section VI. we arrived at Formulæ VI. and VII., which together give the Eccentricity Correction or modification to the normal curvature of image due to the presence of an axial stop or diaphragm causing the pencils to traverse the lens eccentrically instead of centrally. Formulæ VI. will be seen at once to be a function of the spherical aberration of the lens.

Now it is obvious that if we have two thin lenses in contact so arranged as to give equal and opposite spherical aberrations, as is the case in the object glass of a telescope, then as the compound lens gives no axial spherical aberration, and Formula I., Section VI., proves that the spherical aberration for the oblique eccentric pencil is the same as for the axial pencil of the same aperture, therefore there should not ensue any eccentricity correction due to pencils traversing the compound lens eccentrically. This is certainly the case, and Formula VI., if applied to the two lenses, will be found to be zero. For the formula for the spherical aberration for the axial pencil is, written shortly,

Spherical aberration for a pair of lenses in contact.

$$\frac{1}{8f_1^3}(A'_1)y_1^2 + \frac{1}{8f_2^3}(A'_2)y_1^2 = 0, \qquad (1)$$

and the formula for E.C.s in the primary plane, also abbreviated, is as follows—

Spherical aberration E.C.s for same pair of lenses.

$$\frac{3\tan^2\phi}{2f_1}\frac{1}{(\alpha_1-\beta_1)^2}A'_1 + \frac{3\tan^2\phi}{2f_2}\frac{1}{(\alpha_2-\beta_2)^2}A'_2 \qquad (2)$$

which should also be expected to $= 0$. That this is really the case is evident from the following relations, which obviously exist in the case of two thin lenses in contact. For supposing both to be collective we have

Relations between the characteristics for a pair of lenses in contact.

$$\frac{1}{u_2} = -\frac{1}{v_1} \text{ and } \frac{1}{D_2'} = -\frac{1}{D_1''};$$

that is,

$$\frac{1+\alpha_2}{2f_2} = -\frac{1-\alpha_1}{2f_1} \text{ and } \frac{1+\beta_2}{2f_2} = -\frac{1-\beta_1}{2f_1},$$

$$1+\alpha_2 = -(1-\alpha_1)\frac{f_2}{f_1} \text{ and } 1+\beta_2 = -(1-\beta_1)\frac{f_2}{f_1},$$

$$\therefore\ \alpha_2 = -(1-\alpha_1)\frac{f_2}{f_1} - 1 \text{ and } \beta_2 = -(1-\beta_1)\frac{f_2}{f_1} - 1;$$

$$\therefore\ \alpha_2 - \beta_2 = -(1-\alpha_1)\frac{f_2}{f_1} - 1 + (1-\beta_1)\frac{f_2}{f_1} + 1$$

Relations between $\alpha_2 - \beta_2$ and $\alpha_1 - \beta_1$.

$$\therefore\ (\alpha_2 - \beta_2) = (\alpha_1 - \beta_1)\frac{f_2}{f_1}. \qquad (3)$$

From the above Equation (1) obviously $A'_2 = -A'_1\left(\frac{f_2}{f_1}\right)^3$, so that if we take Equation (2) and substitute therein this value for A'_2 and the value of $(\alpha_2 - \beta_2)$ from Formula (3) we then get

$$\frac{3\tan^2\phi}{2}\left\{\frac{1}{f_1}\frac{1}{(\alpha_1-\beta_1)^2}A'_1 + \frac{1}{f_2}\frac{1}{(\alpha_1-\beta_1)^2\left(\frac{f_2}{f_1}\right)^2}A'_1\left(\frac{f_2}{f_1}\right)^3\right\},$$

No axial spherical aberration implies no spherical aberration E.C.s.

$$\text{which} = \frac{3\tan^2\phi}{2}\left\{\frac{1}{f_1}\frac{1}{(\alpha_1-\beta_1)^2} - \frac{1}{f_1}\frac{1}{(\alpha_1-\beta_1)^2}\right\}A'_1 = 0.$$

Hence in the case of a combination of thin lenses in contact from which the spherical aberration is eliminated for an axial pencil, there are therefore no E.C.s consequent on spherical aberration. But it by no means follows that the combination is free from coma or side-flare for pencils refracted through it obliquely. That is, if we imagine a diaphragm to be placed in front of or behind such a compound lens, then the application of Formula VII. to the two lenses will not necessarily give a zero result; in other words, coma may be strongly in evidence.

For this formula gives us the modification to the normal curvature of image consequent upon the selective action of the stop upon the rays of oblique pencils which are characterised by coma, so that we may call VII. the formula for comatic E.C.s, just as we may conveniently call VI. the formula for aberration E.C.s.

The Formulation of Coma

The question now arises, whether from the comatic E.C., Formula VII., we can derive other formulæ which will give us not only the actual size of the comatic flare at the focus when the whole aperture is in use and the refraction oblique and central, but also the size of the comatic flare when the pencils are not only oblique but eccentric, owing to the presence of a stop. These formulæ are of such vital importance as to justify a thorough investigation for central oblique pencils, while we may leave the case of the coma at the foci of eccentric pencils to the next Section. In the course of working out such formulæ we are also helped to a much clearer understanding of the phenomenon, and the course of the rays which produce it.

Let $L..L_1$, Fig. 61, represent a lens, Q the oblique radiant point in the plane P .. Q, p the conjugate focal point or image of P, and q the conjugate focal point or image of Q as formed by the ultimate oblique centre rays close to Q .. C; and let it be supposed that the lens is free from every defect excepting coma, which in this case is *inward* coma, that is, having the flare eccentric towards the optic axis P .. C .. p, the brightest and most condensed end being at q on the oblique axis Q .. C .. q, and the most diffused end at e. Then as our Formula VII. **Line of argument.** for comatic E.C.s is absolutely independent of the aperture of the lens, and obviously equates to 0 when oblique pencils are centrally refracted (since in that case β becomes infinity), and as we have seen that the normal curvature errors are also independent of aperture, therefore, since spherical aberration is supposed absent, the conclusion is that any pairs of rays refracted through the lens at equal distances from and on opposite sides of the oblique axis Q .. C .. q come to a focus in the same plane as q, the focus for the ultimate rays close to Q .. C .. q. But if such pairs of oblique rays focussed at the same point as the ultimate central oblique rays, that is, if the oblique pencils were homocentric, then evidently there could be no comatic E.C.s under Formula VII. Therefore, since they focus or intersect in the same plane as do the ultimate central oblique rays, but not at the same point as the latter, the only other possible explanation is that they focus in the same plane, but at a different distance from the lens axis. For instance, in the case of Fig. 61, if the ultimate central oblique rays focus at q, then the extreme pair of rays Q . . L and Q . . L_1 focus at e, and other pairs of rays refracted by the lens at points nearer to its centre will focus at intermediate points in the line q . . e. We have now to find how these focal points are distributed along the line q . . e in the focal plane. It is obvious from the foregoing that the primary section of the cone of rays is at a minimum at q . . e, in the plane wherein symmetrical pairs of rays such as Q . . L and Q . . L_1 intersect after refraction. If now we can find the point f where the ray Q . . L after refraction crosses the centre ray Q . . C . . q, then clearly the distance $(f\,..\,q)\dfrac{\text{C}\,..\,\text{L}}{\text{C}\,..\,f}$ will give q . . e, the length of the comatic flare.

A device for obtaining length of comatic flare.

In order to get at this we must imagine a stop S_1 . . S_1 to be so placed centrally on the lens axis as to just let pass simultaneously the centre ray Q . . C . . q and the extreme ray Q . . L . . f; then it is obvious that f . . e will be the longitudinal value of the stop correction or E.C. as a variation of V or C . . p, the back conjugate focal distance, which is due to that particular position of the stop and degree of obliquity ϕ.

Let S = semi-aperture of stop, and A semi-aperture of lens. Let d . . C as usual = D, P . . C = U, and C . . p = V. Then

$$\text{C}\,..\,h = \text{V} - \text{V}^2\frac{3\tan^2\phi}{4\text{F}^2(\mu-1)}\left\{4\mu a + \frac{2(\mu+1)}{\mu}(x-a)\right\}\frac{\text{DU}}{\text{U}-\text{D}},$$

by comatic E.C. Formula V. (This form of the formula is the most convenient for our present purpose.) We then have the relations

$$\text{D}\tan\phi = \text{S} = A\frac{r\,..\,d}{r\,..\,\text{C}} = A\frac{(r\,..\,\text{C}) - \text{D}}{r\,..\,\text{C}} = A\frac{\left(\dfrac{A}{A+(\text{P}\,..\,\text{Q})}\right)\text{U} - \text{D}}{\left(\dfrac{A}{A+(\text{P}\,..\,\text{Q})}\right)\text{U}};$$

$$\therefore\ \text{D}\tan\phi = \frac{A\text{U} - \text{D}\{A + (\text{P}\,..\,\text{Q})\}}{\text{U}},\ \text{wherein P}\,..\,\text{Q} = \text{U}\tan\phi;$$

$$\therefore\ \text{DU}\tan\phi = A\text{U} - \text{D}A - \text{DU}\tan\phi,$$

so that our condition that the stop just allows the extreme ray Q . . L and centre ray Q . . C to pass demands that

PLATE. XIII.

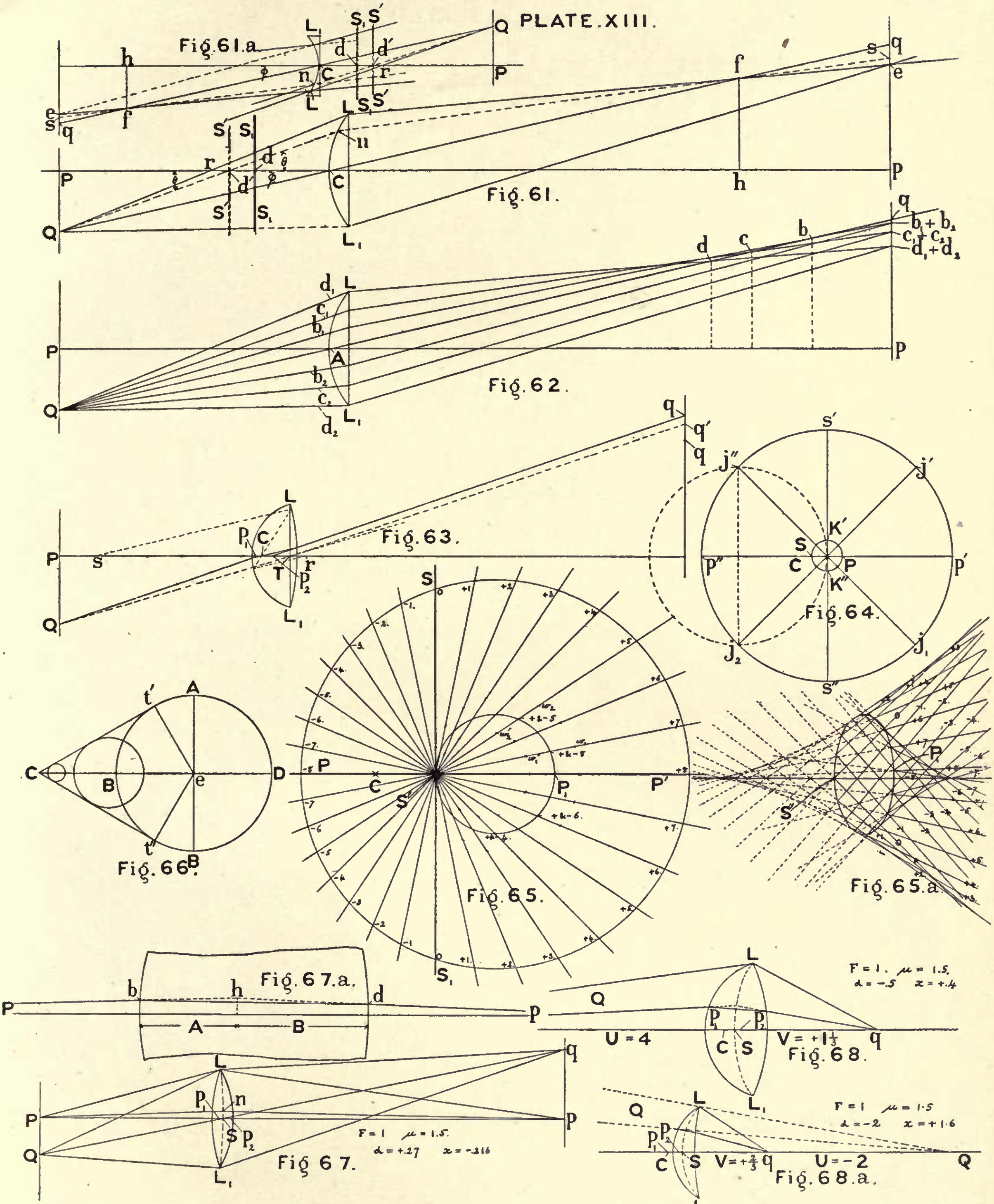


Fig. 61. a.
Fig. 61.
Fig. 62.
Fig. 63.
Fig. 64.
Fig. 65.
Fig. 65. a.
Fig. 66.
Fig. 67. a.
Fig. 67.
F = 1. μ = 1.5.
α = +.27 x = -.216
Fig. 68.
F = 1. μ = 1.5.
α = -.5 x = +.4
U = 4
V = +1⅓
Fig. 68. a.
F = 1 μ = 1.5
α = -2 x = +1.6
V = +⅔
U = -2

PLATE. XIII.

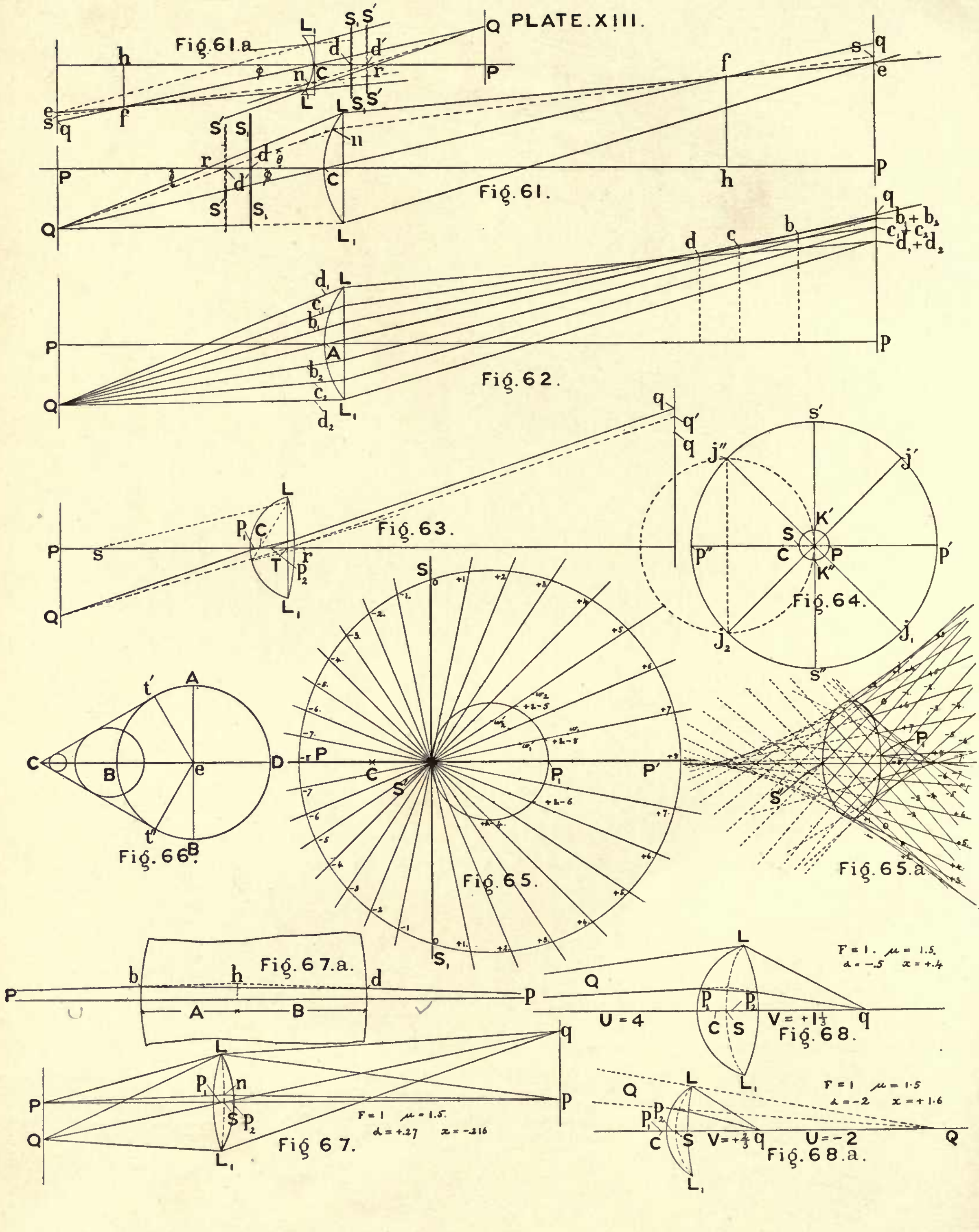


Fig. 61.a. Fig. 61. Fig. 62. Fig. 63. Fig. 64. Fig. 65. Fig. 65.a. Fig. 66. Fig. 67.a. Fig. 67. Fig. 68. Fig. 68.a.

$$D = \frac{AU}{2\,U \tan\phi + A}.$$

Substituting this value of D in the factor $\frac{DU}{U-D}$ in Formula V., expressing the E.C.s due to the coma, we then get

$$\frac{DU}{U-D} = \frac{\dfrac{AU^2}{2U\tan\phi + A}}{\dfrac{2U^2\tan\phi + AU - AU}{2U\tan\phi + A}} = \frac{A}{2\tan\phi};$$

so that Formula V. becomes

$$\frac{3\tan^2\phi}{4F^2(\mu-1)}\left\{4\mu a + \frac{2(\mu+1)}{\mu}(x-a)\right\}\frac{A}{2\tan\phi},$$

or, more conveniently,

$$\frac{3\tan\phi}{4F^2\mu(\mu-1)}\left\{(2\mu+1)(\mu-1)a + (\mu+1)x\right\}A, \qquad (4)$$ **The correction to $\frac{1}{V}$.**

which is the correction to $\frac{1}{V}$ required to convert it into $\frac{1}{C\,..\,h}$. Therefore the required *linear* stop correction $h\,..\,p$ or $f\,..\,e$ is obtained by multiplying (4) by $-V^2$, unless V is very large compared to F, and then $q\,..\,e$ or the length of the comatic flare will be obtained by multiplying $f\,..\,e$ or $h\,..\,p$ by $\frac{A}{C\,..\,h}$, or approximately by $\frac{A}{V}$; so that

$$\begin{aligned} q\,..\,e &= -\frac{3\tan\phi}{4F^2\mu(\mu-1)}\left\{(2\mu+1)(\mu-1)a + (\mu+1)x\right\}V^2\frac{A^2}{V} \\ &= -\frac{A^2\,3\tan\phi}{4F^2\mu(\mu-1)}\left\{(2\mu+1)(\mu-1)a + (\mu+1)x\right\}V, \end{aligned}$$

in which, resorting to our former device, we may substitute $\frac{2F}{1-a}$ for V, thus arriving at

$$q\,..\,e = -\frac{A^2\,3\tan\phi}{2F\mu(\mu-1)}\left\{(2\mu+1)(\mu-1)a + (\mu+1)x\right\}\frac{1}{1-a}, \qquad \text{I.}$$ **Formula for the length of the comatic flare.**

This, then, is the formula for the length of the comatic flare, supposing that the other aberrations are absent. It is evident that it is not affected by the stop $S_1\,..\,S_1$, which we have used as a stepping-stone in the line of reasoning, being taken away, thus bringing the full aperture into use. The formula therefore applies to the full aperture $2A$ of the lens. It is now seen that the length of the coma increases as the square of the aperture, other factors being constant, and therefore the lateral displacement (like $q..e$) of the foci for symmetrical pairs of rays increases as the square of the distance from the oblique axis or ray $Q..C..q$ of the points where they impinge on the lens.

Distribution of the rays in the primary plane.

We are now in a position to construct a diagram of the course of the rays in the primary plane, which gives rise to coma, in more detail. Fig. 62 illustrates the same case as Fig. 61, only with the coma farther exaggerated for the sake of clearness, and with more rays filled in.

Here the pair of rays $Q..b_1$ and $Q..b_2$ refracted at distance $= 1$ from the oblique axis $Q..A$ come to focus at point $b_1 + b_2$ at 1 unit from q, the focal point for central rays; the rays $Q..c_1$ and $Q..c_2$ refracted at distance $= 2$ from the oblique axis $Q..A$ come to focus at point $c_1 + c_2$ at 4 units from q; while the pair of rays $Q..d_1$ and $Q..d_2$ refracted at distance $= 3$ from the oblique axis $Q..A$ come to focus at point $d_1 + d_2$ situated 9 units from q, and so on as the square of the aperture.

A corollary.

It follows, as an obvious corollary from the law of the length of the coma increasing as the square of the aperture, that, provided the length of the coma is very small compared to its distance from the lens, as is usually the case, then the distances $q..b$, $q..c$, and $q..d$ from the focus to the points where the rays $Q..b_1$, $Q_1..c_1$, and $Q..d_1$ intersect the central oblique ray $Q..A..q$ must vary as the aperture, or as the respective distances $A..b_1$, $A..c_1$, and $A..d_1$. The coma in Fig. 62 is too much exaggerated to permit of this relationship being properly shown.

The distribution of the brightness.

In the primary plane it is clear that the rays are most crowded together at the end q of the coma, and most diffused at the other end e where $d_1 + d_2$ intersect. Hence the former is the bright end, and the latter the diffused end of the flare.

Supposing that the lens were divided into concentric rings or zones, and each zone in turn allowed to throw an image of the point Q on to the plane $p..q$, it is very evident that as the image of Q formed by the two extreme rays in primary planes falls at $d_1 + d_2$ nearer to the optic axis than the foci for smaller zones of the lens, therefore the

Variation in focal lengths for different lens zones.

equivalent focal length may be said to vary for different zones; the larger the zones of the lens the smaller the equivalent focus of such zones. In other words, the equivalent focal length differs from that of the ultimate central portion by amounts varying as $\tan\phi$ and as the square of the aperture. This property of a lens subject to coma has been well emphasised by Professor Silvanus Thompson, who has

"Zonal aberration."

applied the term "zonal aberration" to the phenomenon.

It is clearly of the greatest practical importance, when estimating or eliminating coma in a combination of lenses, to have an expression for the *angular* value of the comatic flare, that is $\frac{q\,..\,e}{V}$. Of course this is obtained by multiplying the Formula I. by $\frac{1}{V}$ or $\frac{1-a}{2F}$, by which we get

The angular value of the coma, as subtended at lens centre.

$$\frac{q\,..\,e}{V} = -\frac{3\tan\phi}{4F^2\mu(\mu-1)}\left\{(2\mu+1)(\mu-1)a+(\mu+1)x\right\}A^2. \qquad \text{II.}$$

It is clear that in the case of Fig. 61 we have both a and x positive, while at the same time the coma $q\,..\,e$ is inwards or towards the optic axis $P\,..\,p$. It is very important to adopt a convention with regard to the sign of coma. In Formula II. the angular coma comes out negative. We will consider any such comatic flare to be negative which is inward, or whose diffused end lies towards the optic axis (or whose bright end C (Fig. 66) lies away from the optic axis); and this rule must apply whether the coma is real or whether it is merely virtual, and irrespective of whether the lens in question is collective or dispersive. For instance, Fig. 61*a* gives, on a smaller scale, the case of a dispersive lens corresponding exactly to the case of the collective lens in Fig. 61. Here, also, it will be easily seen that the coma $e\,..\,q$ is likewise inward or towards the optic axis. Also both a and x are positive. Therefore it is clear that the minus sign must still prefix Formulæ I. and II. with respect to the dispersive lens; and then, as we shall see farther on, the comatic functions of a series of lenses can all be simply added together, and there will be no need for reversing the signs of the functions for dispersive lenses before summing up. The case is intrinsically quite different to that of the eccentricity corrections.

Conventions as to signs of coma.

In short, the fact that the formula for coma is a function of $\frac{1}{f^2}$ shows that the sign of f may be ignored. Moreover, the sign of the lens is implied in the sign of a.

The Part Played by the Secondary Rays in Coma Formation

We may now turn our attention to the consideration of symmetrical pairs of rays contained in the secondary plane, any two rays refracted through the lens at equal distances above and below A. Since we are assuming the existence of coma without astigmatism (a condition which is hypothetical in the case of a simple lens except under very special cases of *eccentric* refraction, but quite possible and quite common in the case of certain compound lenses), we have, of course, to assume that a pair of rays in the secondary plane intercross or focus in the same focal plane $q..p$ as do the pair of rays in the primary plane, and it is obvious that they will focus somewhere in the straight line $p..q$ lying in the primary plane and passing through the optic axis.

Difficult nature of the inquiry.

The line of reasoning whereby the position of this focal point for two rays refracted at the distance A from the lens centre in the secondary plane is determined is long and difficult, and perhaps it is unnecessary for our purpose to do more than give a brief sketch of it by the help of Fig. 63.

This method consists in assuming the two rays Q..T′ and Q..T″ in the secondary plane to be refracted through the sharp edge of the lens immediately above and below the point T, and finding by spherical trigonometry how much the vertical plane containing these two rays after refraction is angularly deviated (in the primary plane) from the plane containing the same two rays before refraction; for it can be shown that such a deviation always takes place. In Fig. 63 the two incident rays Q..T′ and Q..T″ respectively have to be represented by one straight line Q..T, and the two emergent rays T′..q' and T″..q' by another straight line T..q'; but those two straight lines are not one; they form a small angle with one another at T, and the angular displacement of T..q' with respect to Q..T is outwards or away from the optic axis.

Having got a general expression for this deviation (which depends upon the shape of the lens, etc.), we next compare it with the lateral parallel displacement which occurs to the central ray which passes through the two principal points, p_1 and p_2, of the lens and its geometric centre, as shown by the solid lines. We then arrive at the formula, III., for the angular displacement of the focus q_1 for two rays in the secondary plane from the focus q for the ray passing through the geometric centre of the lens—that is, the angular value of $q..q'$ subtended at T—

Angular value of interval between oblique central ray and secondary focus.

$$\frac{q..q'}{\mathrm{V}} = -\frac{\tan\phi}{4\mathrm{F}^2\mu(\mu-1)}\left\{(2\mu+1)(\mu-1)a+(\mu+1)x\right\}A^2. \qquad \text{III.}$$

Thus we obtain a value which is just one-third of Formula II. So that if, in Fig. 63, q is the point where the central ray strikes the focal plane, and q' is the point where the two rays Q..T′ and Q..T″ in the secondary plane come to focus, then if we make $q..q'' = 3(q..q')$, q'' will be the point where the two rays Q..E′ and Q..E″ in the primary plane come to focus, the two sets of rays belonging to the same zone or circle of the lens, which we have assumed to coincide with its sharp edge.

The Diameter of the Coma in the Secondary Plane

The following line of reasoning for obtaining the diameter of the comatic flare in the secondary plane may be pursued consistently with the theorem of coma which we have just explained.

We have supposed that the four rays which, two by two, impinge upon the two extremities of the secondary axis of the comatic circle and define its size in the secondary plane, are refracted through the lens zone at points 45 and 135 degrees in both directions from the neutral point p' (Fig. 64), that is, rays from j', j'', j_1, and j_2. Confining our attention to the pair j'' and j_2 immediately above and below the point n, as shown in dotted lines in Fig. 61, we have $\mathrm{C}\,..\,n = (\mathrm{C}\,..\,\mathrm{L}) \cos 45^\circ$ $= A\frac{1}{\sqrt{2}}$ (A being the semi-aperture of the lens). The dotted circle in Fig. 64 then represents the eccentric zone limited by the stop $\mathrm{S}'\,..\,\mathrm{S}'$, and its radius is obviously $\frac{A}{\sqrt{2}}$. We have already found the crossing point f for the two rays $\mathrm{C}\,..\,f$ and $\mathrm{L}\,..\,f$, which gave us the linear E.C. in primary plane ($=f\,..\,e$), from which we got $q\,..\,e$. We now want the corresponding E.C. for the two rays $n\,..\,s$ in the secondary plane passing above and below n; and in order to find it we must imagine the diaphragm moved back from d to d', such that $\mathrm{Q}\,..\,d'$ produced passes through n; then, calling the diaphragm distance ($d'\,..\,\mathrm{C}$) D', for short, we have, if angle $\mathrm{P}\,..\,d'\,..\,\mathrm{Q} = \theta$,

$$\mathrm{D}' = \frac{A\frac{1}{\sqrt{2}}}{\tan\theta} = A\frac{1}{\sqrt{2}}\,\frac{\mathrm{U}-\mathrm{D}'}{\mathrm{U}}\,\frac{1}{\tan\phi}$$

and dividing by D' we get

$$A\frac{1}{\sqrt{2}}\,\frac{1}{\tan\phi}\,\frac{\mathrm{U}-\mathrm{D}'}{\mathrm{U}\mathrm{D}'} = 1, \therefore \frac{\mathrm{D}'\mathrm{U}}{\mathrm{U}-\mathrm{D}'} = \frac{A}{\sqrt{2}}\,\frac{1}{\tan\phi}.$$

Hence the required E.C. is expressed by Formula III., Section VI., with the above value of $\frac{\mathrm{D}'\mathrm{U}}{\mathrm{U}-\mathrm{D}'}$ inserted; that is,

$$\frac{\tan^2\phi}{4\mathrm{F}^2(\mu-1)}\left\{4\mu a + \frac{2(\mu+1)}{\mu}(x-a)\right\}A\frac{1}{\sqrt{2}}\,\frac{1}{\tan\phi},$$

which is more conveniently expressed as

$$\frac{A \tan\phi}{2\sqrt{2}F^2\mu(\mu-1)}\left\{(2\mu+1)(\mu-1)a+(\mu+1)x\right\}. \qquad (5)$$

Then the linear E.C. obtained by multiplying by $-V^2$ is

$$\frac{2A \tan\phi}{\sqrt{2}\mu(\mu-1)}\left\{(2\mu+1)(\mu-1)a+(\mu+1)x\right\}\frac{1}{(1-a)^2}. \qquad (6)$$

Then the secondary diameter of the comatic flare is obtained by multiplying (6) by $\frac{\text{aperture in secondary plane}}{V}$, that is, by $\dfrac{\frac{2A}{\sqrt{2}}}{V}$ or $\frac{2A}{\sqrt{2}}\,\frac{1-a}{2F}$. So that we get

Diameter of the coma in the secondary plane.

$$\frac{A^2 \tan\phi}{F\mu(\mu-1)}\left\{(2\mu+1)(\mu-1)a+(\mu+1)x\right\}\frac{1}{1-a} \qquad \text{IV.}$$

for the secondary axis of the comatic flare, which is just two-thirds of the value given by our previous Formula I. for the primary axis of the flare.

To trace out mathematically what happens to the rays from Q other than those we have dealt with, and which are refracted through the sharp edge or belong to the same lens zone, is a much more difficult task. It has, however, been undertaken by Professor Finsterwalder and others, and the results may be shortly explained by Fig. 64.

Structure of Pure Coma

We will now give a brief explanation of the comatic flare, while reserving until later the general proof that this theorem of coma necessarily implies the ratio of 3 to 1 for the E.C.s in primary and secondary planes respectively.

Let the circle $s'..p'..s''..p''$ of Fig. 64 represent one of the concentric zones of a lens, the optic axis of such lens being perpendicular to the paper. Let C be the point in the distant focal plane where the ray passing through the geometric centre of the lens strikes; let P be the point where the two rays in the primary plane, $p'..$P and $p''..$P, come to focus; and S be the point where the two rays in the secondary plane, $s'..$S and $s''..$S, come to focus, C..S being $\frac{1}{3}$ of C..P. About a point half-way between S and P draw the circle S..K′..P of diameter = S..P. This circle we will call a comatic circle, on which strike all the rays refracted through the zone $s'..p''..s''..p'$ of the lens, only the way in which the striking points are distributed around the comatic circle is

The comatic circle.

a peculiar one. Starting from p' in the primary plane, the point towards which P (the point of the comatic circle most remote from the centre ray C) lies, we may reckon our rays by their angular distance from p' measured along the zone. The ray from j', a point 45° from p', will strike the comatic circle at K′ at a point 90° from P; the ray from s', 90° from p', strikes the comatic circle at S, 180° from P; the ray j'', 135° from p', strikes the comatic circle at K″, 270° from P, and so on. That is, every ray passing through the lens zone at an angle θ from the neutral point p' strikes the comatic circle at a point situated by 2θ from the corresponding neutral point P. Thus all the striking points of rays are subjected to what may be termed a degree of torsional displacement equal to θ.

Distribution of the rays round the comatic circle.

The torsion imparted to the rays.

In Fig. 64 the comatic circle, for clearness, is shown too large in proportion to the size of the lens zone. Fig. 65 shows the structure of the comatic circle far more truly, for it is constructed on the supposition that the lens zone is infinitely large compared to the comatic circle, so that the inclination of all the rays shown therein to the primary plane P . . P′ is the true measure of their angular distribution round the lens zone. Also it is supposed that the diagram 65 represents a view of the comatic circle as if looking along the oblique central ray, so that the lens zone would, strictly speaking, appear as an ellipse. But the angle of obliquity is assumed to be small enough to allow us to treat the lens zone as a circle, of immense size compared to the diagram. As a corollary from this torsional effect on all rays (except the neutral pair striking the comatic circle at P_1), it follows that every point in the comatic circle is the mutual striking point of two rays originating from two points in the corresponding lens zone which are 180° apart or diametrically opposite So that each straight line drawn across Fig. 65 represents two rays, one from one point in the lens zone, and the other from the opposite point. A marked feature of the case is that all the rays cut the straight line drawn from the lens centre to the intersection S′ of the two rays S . . S′ and S_1 . . S′ in the secondary plane; but let it be noted that these intersections are at different distances from the plane of the diagram or comatic circle, so that the seeming intersection of all the rays at S′ is apparent only.

Every point in the comatic circle receives two opposite rays.

A common intersection axis for all rays from each lens zone.

Fig. 65*a* is designed to elucidate these points further. It is a perspective view of the comatic circle and the same rays coming from the lens zone as those shown in Fig. 65, wherein the rays are numbered -1, -2, $+1$, $+2$, etc. The $+$ sign means that the ray in question, after intersecting the comatic circle, proceeds to cut the ray projected through S′ from the centre of the lens, at a point beyond the plane of the comatic circle; while the − sign means that the ray in question cuts the projected central line before it intersects the comatic circle. Thus rays of the same sign and number cut the axis of intersection at the same point, and those of equal numerical values, but opposite signs, cut the axis of intersection at points equidistant from, but on opposite sides of, the plane of the comatic circle. The two rays marked s and s_1 in the secondary plane cut the comatic circle at one point S′, also shown in Fig. 65a. For the sake of clearness, each ray is drawn as a solid line up to its intersection with the comatic circle, and as a dotted line after its intersection. Also each ray is marked with the same numbers and signs as in Fig. 65, so that each ray may be identified in both diagrams. The relative aperture of the lens zone is assumed to be very large.

Distribution of the rays along the common intersection axis.

The Distribution of the Comatic Circles formed by Different Lens Zones

The next Figure, 66, shows a series of comatic circles and their relative distribution for a series of lens zones of semi-apertures = 1, 2, and 3, from which it will be easily seen that the two tangents to the series of comatic circles embrace an angle of 60°, and intersect at the point C where the central ray cuts the focal plane. For we have seen from Formula III. that the distance C . . B, from the central ray C to the point B where the two rays in the secondary plane intersect, is $\frac{1}{3}$ of C . . D. Therefore, assuming the comatic circle t' . . B . . t'' . . D, with its centre at e, to exist, we have $\frac{t' \,..\, e}{C \,..\, e} = \frac{1}{2}$ $= \sin \angle(t' \,..\, C \,..\, e) = \sin\ 30°$, therefore the angle between the two tangents is 60°. Such an expanding series of comatic circles makes the well-known balloon-shaped side-flare or coma instead of a point of light at C. Then C is the end of the coma at which the greatest intensity of light concentration occurs, while D, the opposite extremity, is marked by the greatest diffusion of light. We will call C the root of the coma, and D its extremity. If the extremity of a comatic flare lies towards the optic axis of a lens, then the coma is negative or −; if it lies away from the optic axis, then the coma is positive or +. The signs preceding Formulæ I., II., and III. are arranged to always give results in accordance with the above convention, bearing in mind that no difference of sign is required to be made in applying these formulæ to dispersive lenses, of which instances will be given later.

Outline of the comatic flare defined.

The student wishing to study the formation of coma corresponding to any particular lens zone cannot do better than take one-half of a Goerz Double Anastigmat, with the stop to the front to receive nearly parallel rays from a distant bright point of light. The lens may be rendered opaque except for a narrow zone near the edge of its aperture, and then, on examining the focus with an eye-piece, while tilting the lens to a certain degree of obliquity, a very fine example of pure coma without much admixture of astigmatism may be obtained, and the duplex circle of Fig. 70 may be watched as it closes up to focus. It is particularly instructive to cover up half the zone, when, at the focus, a complete ring of light will still be obtained.

How pure coma may be exhibited.

A half lens zone shows an apparently complete comatic circle.

The Sine Condition

By many optical authorities, especially on the Continent, it has been asserted that if a lens fulfils what is called "the Sine Condition," it will then show no coma. The late and much lamented Professor Abbe, of Jena, was the first to prove that if a lens $L..L_I$ (see Fig. 67) is so shaped relatively to the conjugate axial foci P and p that $\frac{\sin LPS}{\sin LpS} =$ constant for all values of $L..S$ or y, then pencils refracted obliquely but centrally through the lens, such as pencils LQL_I and LqL_1, will be free from coma. It can be proved that if the lens fulfils the sine condition, then, if we take a new point of origin Q to one side of the axis, but in the same focal plane as P, the length of path $Q..L + L..q =$ the length of path $Q..L_1 + L_1..q$, and therefore two elements of a wave of light starting together from Q meet again at q simultaneously upon a common point situated on the central oblique ray, there being, therefore, no lateral displacement. But to plan a lens that will fulfil the sine condition in any particular case by trigonometric methods is far more laborious than arriving at a direct result by a simple algebraic formula, and it may easily be proved that our formula for eliminating coma, $(2\mu + 1)(\mu - 1)a + (\mu + 1)x = 0$, can be deduced directly from Professor Abbe's sine condition, and is the algebraic expression of that condition. Let us consider any pair of conjugate rays such as $P..n$ and $n..p$ (Fig. 67), and suppose they are each produced into the lens until they meet at n, then the perpendicular $n..S$ is common to the two triangles nPS and npS, and $\frac{\text{sine } npS}{\text{sine } nPS} = \frac{n..P}{n..p}$ simply.

The sine condition implies equal "optical lengths" for extreme rays of an oblique pencil.

The sine condition made the basis for our formula for no coma.

Then let us consider a pair of conjugate rays refracted extremely closely to the lens axis (see enlarged diagram of the centre of the lens, Fig. 67a). If the two conjugate rays $P..b$ and $p..d$ are produced inwards and meet at h, then in the extremely narrow triangle hbd the base $b..d$ is the course of the ray within the lens, the angle hbd is the angle of deviation at the first surface, and the angle hdb is the angle of deviation at the second surface, but at such extremely small angles, the angles of incidence or emergence and angles of deviation bear the constant relation $\mu : \mu - 1$, and we may say that the angle of incidence of the ray $P..b$ is to the angle of emergence of the ray $d..p$ as $h..d$ is to $h..b$; so that ultimately when h is brought down to the axis it will be so placed as to divide the thickness t of the lens into two parts—A, corresponding to $b..h$, and B corresponding to $h..d$. Then

$$\frac{A}{B} = \frac{\text{angle of emergence of } d..p}{\text{angle of incidence of } P..b}. \tag{7}$$

Let $P..L$ and $L..p$ in Fig. 67 be another pair of conjugate rays refracted by the extreme thin edge of the lens; then it is obvious that the sine condition demands that

$$\frac{P..L}{L..p} = \frac{P..n}{n..p} \text{ or } \frac{P..h}{h..p}, \text{ but } \frac{P..h}{h..p} = \frac{(P..b)+(b..h)}{(d..p)+(d..h)} = \frac{U+A}{V+B};$$

therefore

$$\frac{P..L}{L..p} = \frac{U+A}{V+B}. \tag{8}$$

Now let perpendicular $L..S = y$, then

$$P..L = U + \frac{y^2}{2U} + \frac{y^2}{2r} = U + \frac{y^2}{2}\left(\frac{1}{U} + \frac{1}{r}\right), \tag{9}$$

$$L..p = V + \frac{y^2}{2V} + \frac{y^2}{2s} = V + \frac{y^2}{2}\left(\frac{1}{V} + \frac{1}{s}\right). \tag{10}$$

Reverting to Formula (7), giving the ratio between A and B, it is obvious that the ultimate angle of emergence of ray $d..p$ is expressed by $\left(\frac{1}{s} + \frac{1}{V}\right)$, and the ultimate angle of incidence of ray $P..b$ is similarly expressed by $\left(\frac{1}{r} + \frac{1}{U}\right)$. Therefore putting t for the central thickness of the lens we have

$$A = t\,\frac{\frac{1}{s} + \frac{1}{V}}{\left(\frac{1}{r} + \frac{1}{U}\right) + \left(\frac{1}{s} + \frac{1}{V}\right)} \text{ and } B = t\,\frac{\frac{1}{r} + \frac{1}{U}}{\left(\frac{1}{r} + \frac{1}{U}\right) + \left(\frac{1}{s} + \frac{1}{V}\right)};$$

therefore Formula (8) becomes

$$\frac{U + \frac{y^2}{2}\left(\frac{1}{U} + \frac{1}{r}\right)}{V + \frac{y^2}{2}\left(\frac{1}{V} + \frac{1}{s}\right)} = \frac{U + t\dfrac{\frac{1}{s} + \frac{1}{V}}{\left(\frac{1}{r} + \frac{1}{U}\right) + \left(\frac{1}{s} + \frac{1}{V}\right)}}{V + t\dfrac{\frac{1}{r} + \frac{1}{U}}{\left(\frac{1}{r} + \frac{1}{U}\right) + \left(\frac{1}{s} + \frac{1}{V}\right)}}, \tag{11}$$

in which we may put

$$t = \frac{y^2}{2r} + \frac{y^2}{2s} = \frac{y^2}{2}\left(\frac{1}{r} + \frac{1}{s}\right),$$

so that (11) becomes

$$\frac{U + \frac{y^2}{2}\left(\frac{1}{U} + \frac{1}{r}\right)}{V + \frac{y^2}{2}\left(\frac{1}{V} + \frac{1}{s}\right)} = \frac{U + \frac{y^2}{2}\left(\frac{1}{r} + \frac{1}{s}\right)\dfrac{\frac{1}{s} + \frac{1}{V}}{\left(\frac{1}{r} + \frac{1}{U}\right) + \left(\frac{1}{s} + \frac{1}{V}\right)}}{V + \frac{y^2}{2}\left(\frac{1}{r} + \frac{1}{s}\right)\dfrac{\frac{1}{r} + \frac{1}{U}}{\left(\frac{1}{r} + \frac{1}{U}\right) + \left(\frac{1}{s} + \frac{1}{V}\right)}}. \tag{12}$$

On resorting to the former device of making $\frac{1+x}{2f(\mu-1)} = \frac{1}{r}$, $\frac{1-x}{2f(\mu-1)} = \frac{1}{s}$, $\frac{1+a}{2f} = \frac{1}{U}$, and $\frac{1-a}{2f} = \frac{1}{V}$, and substituting these in the smaller terms, then (12) becomes **Reductions.**

$$\frac{U + \frac{y^2}{2}\left\{\frac{(1+x) + (\mu-1)(1+a)}{2(\mu-1)f}\right\}}{V + \frac{y^2}{2}\left\{\frac{(1-x) + (\mu-1)(1-a)}{2(\mu-1)f}\right\}} = \frac{U + \frac{y^2}{4f\mu(\mu-1)}\left\{(1-x) + (\mu-1)(1-a)\right\}}{V + \frac{y^2}{4f\mu(\mu-1)}\left\{(1+x) + (\mu-1)(1+a)\right\}}. \tag{13}$$

From (13) we derive

$$\frac{4Uf(\mu-1) + y^2\{(1+x) + (\mu-1)(1+a)\}}{4Vf(\mu-1) + y^2\{(1-x) + (\mu-1)(1-a)\}} = \frac{4Uf\mu(\mu-1) + y^2\{(1-x) + (\mu-1)(1-a)\}}{4Vf\mu(\mu-1) + y^2\{(1+x) + (\mu-1)(1+a)\}},$$

from which we get, on reducing to a common denominator and leaving out the latter,

$$\begin{aligned}
&16UVf^2\mu(\mu-1)^2 + 4y^2Vf(\mu-1)\{(1-x) + (\mu-1)(1-a)\} \\
&\qquad + 4y^2Uf\mu(\mu-1)\{(1-x) + (\mu-1)(1-a)\} + y^4\{(1-x) + (\mu-1)(1-a)\}^2 \\
&- 16UVf^2\mu(\mu-1)^2 - 4y^2Vf\mu(\mu-1)\{(1+x) + (\mu-1)(1+a)\} \\
&\qquad - 4y^2Uf(\mu-1)\{(1+x) + (\mu-1)(1+a)\} - y^4\{(1+x) + (\mu-1)(1+a)\}^2 = 0.
\end{aligned}$$

Neglecting functions of y^4, which belong to a higher order of approximation and are small compared to the other terms, we get

$$\left.\begin{array}{l} V\{(1-x)+(\mu-1)(1-a)\}+U\mu\{(1-x)+(\mu-1)(1-a)\} \\ -V\mu\{(1+x)+(\mu-1)(1+a)\}-U\{(1+x)+(\mu-1)(1+a)\} \end{array}\right\}=0.$$

Then on writing $\frac{2f}{1+a}$ for U, and $\frac{2f}{1-a}$ for V, and multiplying all terms by $(1-a)(1+a)$, we get,

$$\left.\begin{array}{l} (1+a)\{(1-x)+(\mu-1)(1-a)\}+\mu(1-a)\{(1-x)+(\mu-1)(1-a)\} \\ -\mu(1+a)\{(1+x)+(\mu-1)(1+a)\}-(1-a)\{(1+x)+(\mu-1)(1+a)\} \end{array}\right\}=0,$$

and this simplifies down to

Conclusion from fulfilment of the sine condition.

$$(2\mu+1)(\mu-1)a+(\mu+1)x=0, \qquad (14)$$

which, as we have already seen in Formulæ I., II., and III., etc., is the condition of no coma, which we previously worked out from quite different premises.

It can also be shown that if, when the sine condition is fulfilled, the incident and emergent rays are produced to intersect within the lens, then the radius R of the circular curve $L\,..\,S\,..\,L_1$ along which the pairs of conjugate rays thus intersect is given by the formula—

Reciprocal of the radius of the sine surface.

$$\frac{1}{R}=-\frac{1}{U}+\frac{1}{V}.$$

Thus, when U is infinite $R=V$; when $U=V$, R is infinite, and the surface $L\,..\,S\,..\,L_1$ is flat; but when $V>U$, then the curve of radius R is reversed in sign and faces convex to the longer conjugate focus.

We may call this spherical surface of radius R the sine surface. When a lens is free from coma, or fulfils the sine condition, two important corollaries can be deduced from the conditions prevailing—and these are, firstly, that the point S, Fig. 67, where the sine surface cuts the optic axis, is always exactly in a straight line between any original radiant point Q and its image q; and secondly, this point S is so situated with respect to the two principal points, p_1 and p_2, of the lens as to divide $p_1\,..\,p_2$ into two parts, such that $(p_1\,..\,S):(S\,..\,p_2)::U:V$.

Two corollaries.

Therefore S falls between the two principal points if both U and V are positive, as in Fig. 68; but if U and V are of different signs and the conjugate foci on the same side of the lens, as in Fig. 68a, then the point S falls outside the principal points, and in this case behind them.

Some Manifestations of Coma

Returning now to the consideration of the structure of coma, we have seen that, in the absence of other aberrations, a lens manifesting coma forms for each zone of the objective or lens a duplex circle in the focal plane, whose actual diameter is given by Formula IV., and its angular diameter, as viewed from the lens centre, by two-thirds of Formula II. Thus for any given lens zone the diameter of the comatic circle varies as the tangent of the angle of obliquity of the incident pencil; and for any given angle of obliquity the diameters of the comatic circles and the distances of their centres from the oblique central or principal ray alike vary as the square of the diameters of the corresponding lens zones.

When the focal plane is departed from.

It now becomes interesting to inquire what sort of figures will be traced out by the rays going to form such comatic circles—first, when the focal plane is departed from either towards or away from the lens; and, second, when that usual accompaniment of coma, viz. astigmatism, is also present.

In the case of pure coma.

We will first of all deal with pure coma as projected upon planes nearer to or farther from the lens than the focal plane in which the duplex comatic circle is formed. Here Fig. 65 will at once help us to form an idea of the figure traced out by the rays on a plane somewhat nearer to the lens. This figure represents what would be seen by the eye placed in and looking in a direction parallel to the straight line joining the centre of the lens to the centre of the comatic circle. Therefore, since the inclinations of all the converging rays to the plane of the diagram are equal, if we mark off on each ray a point such as w_1, w_2, etc., such that the distances from all such points to the points where the same rays cut the comatic circle are equal, then the curve $w_1 \,..\, w_2$ and $w_1' \,..\, w_2'$, etc., through all these points will be one of the out-of-focus comatic curves. The resemblance to a hypocycloid is at once apparent. In fact, it has been proved by Finsterwalder (what is in entire conformity with the formulæ we have worked out) that the comatic curve traced out by the rays from any one lens zone is such a curve as would be traced out by a point in a uniformly rotating circle whose centre is simultaneously travelling at half the rate and in the same direction around another fixed circle. Fig. 69, Plate XIV., illustrates this.

Hypocycloidal nature of the curves.

In all the figures $r \,..\, r$ is the rotating circle, and $f \,..\, f$ the fixed circle that the centre of the former travels round. While the centre of $r \,..\, r$ travels once uniformly round $f \,..\, f$ the circle $r \,..\, r$ has rotated uniformly on itself twice. Now $r\,..\,r$ is the same size as the comatic circle in the focal plane, and thus represents the amount of torsion to which the rays are subjected; while the fixed circle $f\,..\,f$ may be zero or of any size, for it simply represents the circle traced by the hollow coned surface of rays upon the selected plane of projection (supposing that the rays were all refracted accurately to a point at the centre of the comatic circle). Thus the size of $f\,..\,f$ simply depends upon the distance of our plane of projection from the focal plane. If the plane of projection coincides with the focal plane, then $f\,..\,f$ vanishes to a point, and in that case we have to imagine the rotating circle $r\,..\,r$ rotating on itself twice while its centre remains stationary, which hypothetical case explains the duplex comatic ring. It is really a double loop in its ultimate closed-up form. Fig. 70 *a* and *á* show two phases of the comatic curve at equal distances on each side of the focal plane in which the comatic circle O is formed, followed by three more out-of-focus phases *b*, *c*, *d*. All these and the following figures have been traced out by the employment of a geometric machine in accordance with the above law of coma formation.

Out-of-focus coma for five concentric lens zones.

Next, let us take a lens giving pure coma, and consider the tracings made near the focal plane by each of five concentric zones of the lens of radii, 1, 2, 3, 4, and 5. Then at the focus we shall have a figure like Fig. 66, a series of duplex comatic circles, but at a little distance on either side of the focus we shall get Fig. 71.

The effect of adding astigmatism.

Next we may consider the effect of the usual astigmatism being added to the coma. The effect of astigmatism is, at the focus for rays in the primary plane, to substitute a short and nearly straight focal line for the point, and at the focus for rays in the secondary plane to substitute another straight focal line of the same length as the former for the point, these two focal lines being at right angles to one another. Consequently, the figure to be expected in the plane of each focal line is the figure that will be traced by a point in the comatic circle rotating on itself twice, while its centre travels with a harmonic motion up and down the whole length of the focal line. Fig. 69*a* illustrates this action, at O within the primary focus, at P the primary focus, at L the least circle, at S the secondary focus, and at O′ beyond the latter; while in Fig. 72, P is the figure thus traced at the focus for the two rays in the primary plane which mutually intersect at the point *p*. Then, if the plane of projection is transferred to a position half-way between the two focal lines or at the circle of least confusion, we get the tracing L; and then, on transferring the plane of projection to the secondary focal line where the two rays in the secondary plane

PLATE. XIV.

Fig. 69. a.

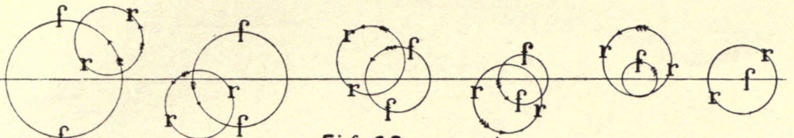
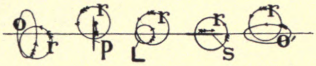


Fig. 69.



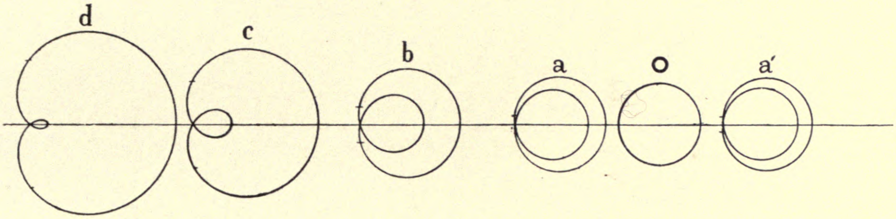


Fig 70.

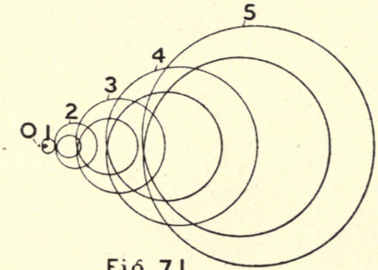


Fig. 71.

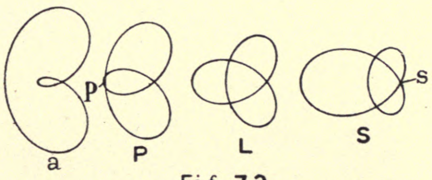


Fig. 72.

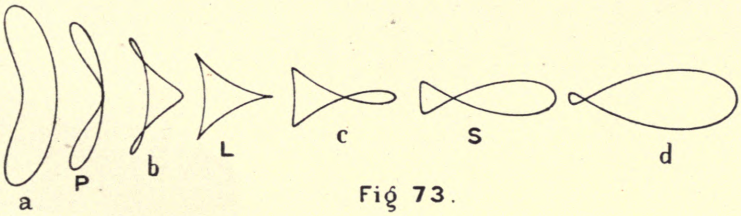


Fig 73.

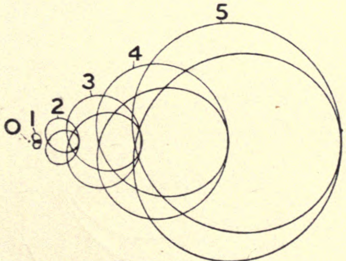
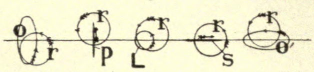


Fig. 74.

PLATE. XIV.

Fig. 69 a.



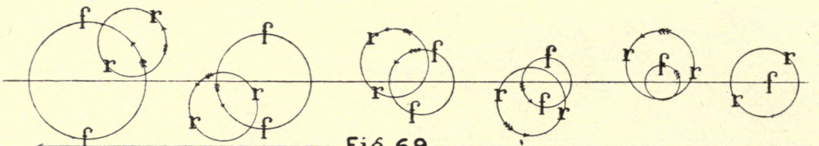


Fig. 69.

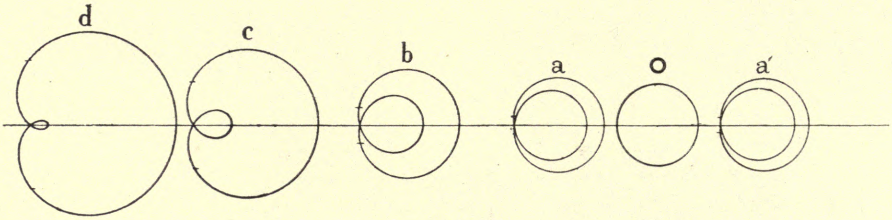


Fig 70.

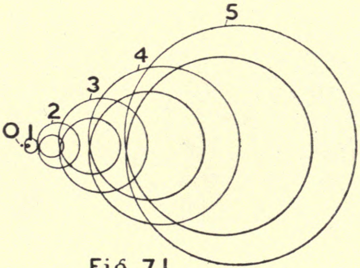


Fig. 71.

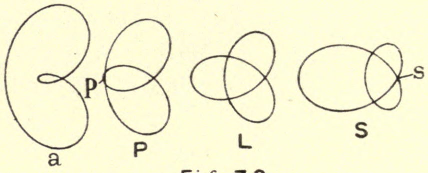


Fig. 72

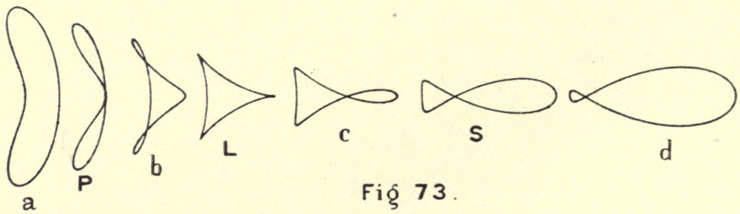


Fig 73.

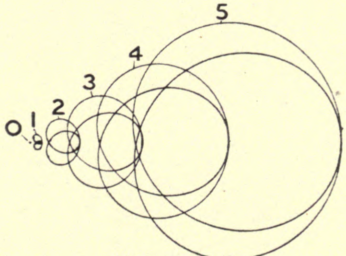


Fig. 74.

intersect, we get the tracing S, *s* being the intersection point for the two rays from the zone which lie in the secondary plane. Tracing *a* is taken within the primary focus.

Phases of astigmatic coma from the same lens zone.

Fig. 73 is a series of phases of astigmatic coma, all for the same lens zone, in a case where the degree of astigmatism bears a still greater proportion to the comatic circle. *a* is within the primary focus, P is at the primary focus, *b* half-way between the primary focus and the least circle, L is at the least circle, *c* is half-way between the latter and the secondary focus, *s* is at the secondary focus, and *d* beyond it.

Astigmatic coma for five concentric lens zones, primary focus.

Fig. 74 is the complete series of tracings for five-lens zones in a case of coma combined with very moderate astigmatism, taken in the focus for primary rays for all zones, as the lens is supposed to be free from spherical aberration.

Astigmatism more pronounced, primary focus.

Fig. 75 P, Plate XV., is the complete comatic formation for five-lens zones at the primary focus, in a case where the astigmatism is more pronounced than in Fig. 74.

Same at least circle, and at secondary focus.

Fig. 75 L is the phase of the same which occurs at the least circle, and Fig. 75 S the phase of the same which occurs at the secondary focus.

Same with astigmatism still stronger.

Figs. 76 P, L, and S show the phases, corresponding to the last, of astigmatic coma in a case where the astigmatism is relatively still more violent.

Form of the astigmatic loop varies for each lens zone.

Throughout all cases of astigmatic coma it will be noticed that the form of the loop is different for each lens zone. For it is obvious that the length of the focal line increases as the diameter of the corresponding lens zone, whereas the comatic circle, whose rotation and travel produce the loop, increases as the square of the corresponding lens zone. Hence for the smaller lens zones the straight line formation predominates, and for the larger lens zones the circular element or loop-like effect predominates. Figs. 75 P and 76 P both show this feature.

Beautiful nature of the effects.

The phase of coma indicated in Fig. 76 S, when all the infinite series of zones are filled in, as in the actual case of real coma formed by an aberration-free object glass, is perhaps the most beautiful, being a shell-like formation which at first sight looks complicated and puzzling.

Spherical aberration adds a further complication.

The comatic formations yielded at the oblique foci produced by uncorrected lenses are still further complicated by the fact that the foci for each lens zone vary by spherical aberration, but by the kind permission of Professor Silvanus Thompson * we are enabled to here reproduce some actual sketches taken by him at the oblique foci of a

Prof. S. Thompson's experiments.

* And also by permission of the Royal Photographic Society.

simple plano-convex lens whose face was divided up by annuli of black varnish into a series of concentric transparent zones of finite width. Of course a good deal of colour fringe which was actually present does not show in these reproductions, which will be seen to exhibit practically the same character as the curves we have just dealt with. A full account of his experiments was given in a most interesting and instructive paper printed in the *Photographic Journal* for December 1901; which should be carefully studied by all interested in this branch of optics. Some of the paradoxical consequences of coma therein described are exceedingly interesting.

Varying phases for different lens zones.

If Fig. 78 E be carefully observed, it will be noticed that the tracing of light for the outermost zone is at the focus for the rays in the primary plane, and the curve is in the same phase as any one of the curves in Fig. 76 P. But the curves in Fig. 78 E for the smaller lens zones are more open loops, for, owing to the spherical aberration, the two primary rays of such zones focus beyond the plane in which the comatic curves were taken. In short, the effect of spherical aberration upon the comatic curves is to cause the latter to assume more or less different phases for the different lens zones.

The great broadening out of the outermost zone tracing so marked in Fig. 77 F is of course due to the outer lens zone having a finite and appreciable width, the loops for the outer edge and inner edge of the zone being widely different, owing in large part to the spherical aberration, while the zones between these two all contribute their light to intermediate loops.

Our comatic loops verified by Dr. Steinheil's trigonometrical calculations.

Fig. 79*a* illustrates the figures obtained by Dr. Adolph Steinheil by elaborate trigonometrical calculations applied to the case of the 6-inch refracting telescope at Königsberg made by the celebrated Frauenhofer. He selected four zones of the objective, as in Fig. K, and calculated the oblique foci for eight rays equally distributed round each of the said zones, and found where they impinged on the plane passing through the axial focus (see G and H) on a second plane ·35 of a millimetre nearer the objective (see I and J), and on a third plane ·70 of a millimetre nearer the objective (see K and L). He thus arrived at the comatic formations H, J, and L, whose identity with our previous results is plainly evident. He then, after a few alterations in the curves of the objective, got it to give symmetrical oblique refraction, the sine condition being fulfilled, and the resulting oblique foci shown in Fig. 79*b*, N, P, and R, then showed pure astigmatism only.

PLATE.XV.

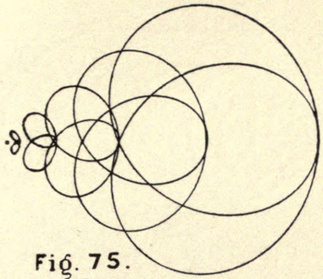


Fig. 75.
P

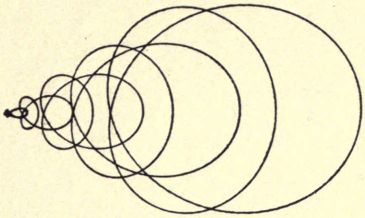


Fig. 75.S.

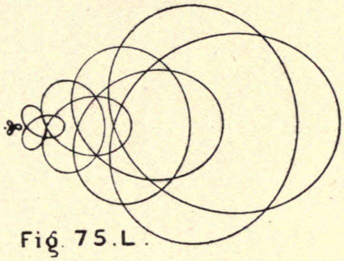


Fig. 75.L.

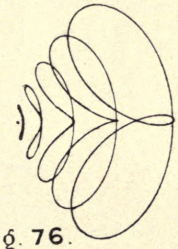


Fig. 76.
P

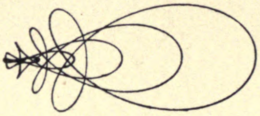


Fig. 76.S.

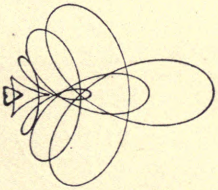


Fig. 76.L.

PLATE. XV.

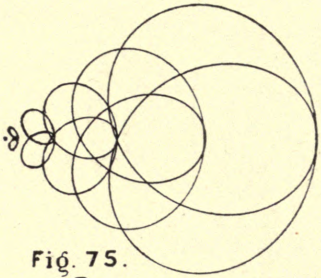


Fig. 75.
P

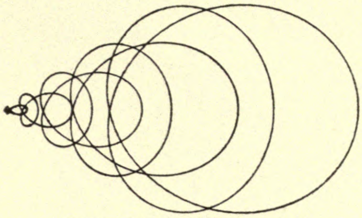


Fig. 75. S.

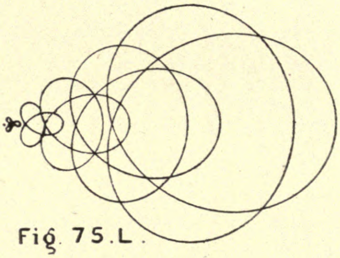


Fig. 75. L.

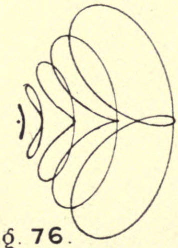


Fig. 76.
P

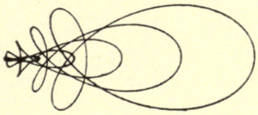


Fig. 76. S.

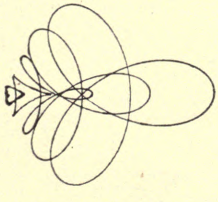


Fig. 76. L.

General Proof of the Theorem of Coma

Having now given a certain explanation of the formation of coma and shown many figures synthetically formed by way of illustration, and others either drawn from actual experiment or trigonometrical calculation, all of which confirm one another, it will now be as well to give a general proof that our theorem of coma will necessarily lead to all comatic eccentricity corrections in the primary plane being three times as much as the simultaneous eccentricity corrections in the secondary plane.

In Fig. 79c let C be the centre of an aberration-free objective yielding coma, and let the eccentric circle c_1 .. G .. c_2 .. H represent the outline of a pencil of rays where it impinges upon the plane of the lens. Then C .. f is the eccentricity. Let the radius or semi-aperture of the eccentric pencil f .. c_1 or f .. c_2 be r. About C describe the circle R_3 .. R_3, touching circle c_1 .. G .. H at G, another circle R_2 .. R_2 passing through c_1 and c_2 at the upper and lower extremities of the secondary diameter of the pencil, and another circle R_1 .. R_1 touching the circle c_1 .. G .. c_2 .. H at H. Then G and H are the points where the two extreme rays in the primary plane are refracted through the lens, while c_1 and c_2 are the points where the two extreme rays in the secondary plane are refracted. Turning our attention to the oblique focus (Fig. 79d) formed by light filling the whole aperture R_1 .. R_1, we have the lens zone R_1 .. R_1 forming the duplex ring R_1', the lens zone R_2 .. R_2 forming the duplex ring R_2', and the lens zone R_3 .. R_3 forming the duplex ring R_3'. Here let it be borne in mind that Fig. 79d is really very small compared with the lens aperture R_1 .. R_1.

We will assume that the distance, such as C .. h, between the central ray C and the outermost point of any duplex ring is N times the radius of the duplex ring. We have so far assumed this ratio to be 3 : 1, but as it is desirable to make this proof quite general in its bearing and be applicable also to comatic formations of a higher order, we will assume the outermost point of each comatic circle to be displaced from the central ray by a distance equal to N times the radius of each comatic circle.

Secondary Plane

Here we may proceed as follows:—

First we may express the radii R_2 and R_3 of the two-lens zones

$R_2 \ldots R_2$ and $R_3 \ldots R_3$ in terms of R_1, the radius of the outermost zone, and of r, the radius $f \ldots c_1$ of the eccentric pencil; thus

$$R_2^2 = (C \ldots c_1)^2 = (c_1 \ldots f)^2 + (C \ldots f)^2 = r^2 + (R_1 - r)^2 = R_1^2 - 2R_1 r + 2r^2,$$

Radius of second lens zone.

$$\therefore\ R_2 = \sqrt{R_1^2 - 2Rr + 2r^2}\,; \tag{15}$$

then we have

Radius of third lens zone.

$$R_3 = R_1 - 2r. \tag{16}$$

Along the lens zone $R_2 \ldots R_2$ mark off the arc $c_1 \ldots b$ equal to $d \ldots c_1$, and join d to b by the chord $d \ldots b$. Also join a to b by straight line $a \ldots b$, and then from the centre c draw $c \ldots e$ perpendicular to $a \ldots b$, and bisecting the latter at e.

Then for the moment we will assume the circle $R_2 \ldots R_2$ to represent the comatic circle formed by lens zone $R_2 \ldots R_2$; in which case we have the ray refracted through the lens zone at c_1 striking the comatic circle at b, $c_1 \ldots b$ being the torsion imparted to the ray in the comatic circle. Then since $c_1 \ldots b = c_1 \ldots d$, therefore the chord $b \ldots d$ is bisected at n, and angle $bCc_1 = c_1Cd$. But angle bad = one-half of angle bcd, therefore angle $bad =$ angle bCc_1. But angle bad is also equal to Cba. Therefore angle $Cba =$ angle bCc_1. Therefore $a \ldots b$ is parallel to $C \ldots c_1$ and $e \ldots b$ is equal to $C \ldots n$, which latter obviously $= C \ldots f$, so that we have

$$a \ldots b = 2(e \ldots b) = 2(C \ldots n) = 2(C \ldots f) = 2(R_1 - r),$$

from which we then derive

Ratio between radius of second comatic circle and path of secondary rays projected on it.

$$\frac{a \ldots b}{R_2} = \frac{2(R_1 - r)}{\sqrt{R_1^2 - 2R_1 r + 2r^2}}. \tag{17}$$

Turning now to the real comatic circle R_2' in Fig. 79*d*, which is formed by lens zone $R_2 \ldots R_2$, we have k_1 as the point where the ray from c_1 strikes the comatic circle, and $A_2 \ldots k_1$ is obviously parallel to $a \ldots b$ of Fig. 79*c*. Now we have already seen, from Figs. 65 and 65*a*, that the perpendicular to the diagram drawn through A_2 is a sort of axis through which pass *all* rays from $R_2 \ldots R_2$ which intersect the comatic circle $k_1 \ldots A_2 \ldots k_2$. Therefore the two rays in the secondary plane from c_1 and c_2 which strike the comatic circle at k_1 and k_2 respectively, will intersect one another at a point somewhere on the perpendicular through A_2, whose distance from the plane of the diagram can be expressed in terms of $A_2 \ldots k_1$ or $A_2 \ldots k_2$.

Now clearly

$$A_2 \ldots k_1 = R_2' \times \frac{a \ldots b}{R_2} = R_2' \frac{2(R_1 - r)}{\sqrt{R_1^2 - 2R_1 r + 2r^2}}$$

PLATE. XVI.

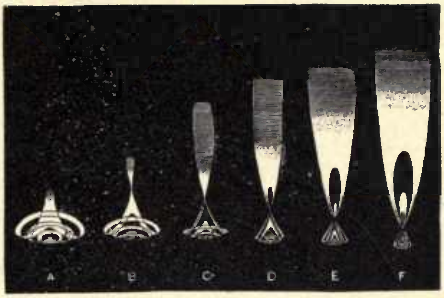


Fig. 77

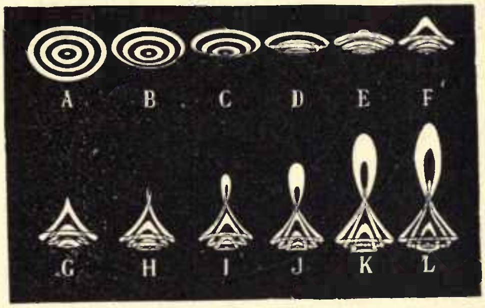


Fig. 78.

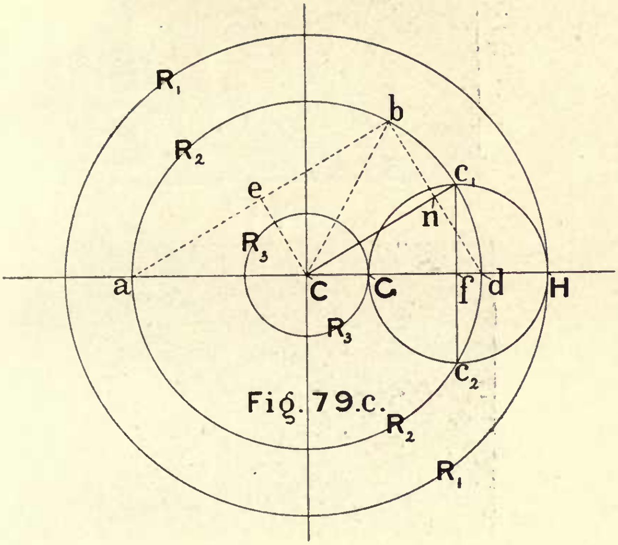


Fig. 79.c.

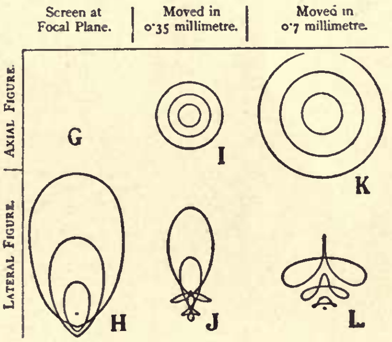


Fig 79.a.

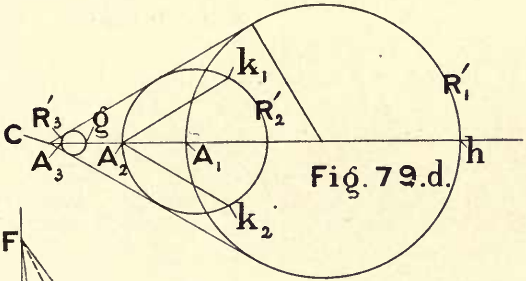


Fig. 79.d.

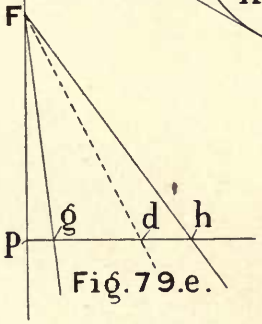


Fig. 79.e.

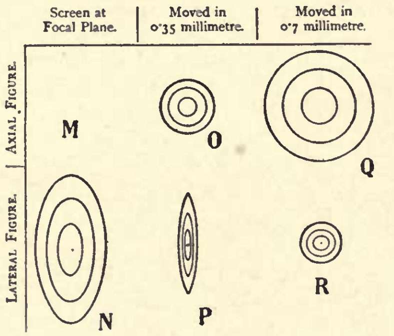


Fig. 79.b

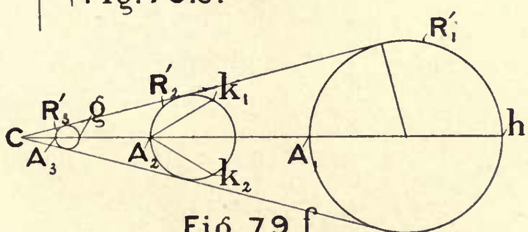


Fig. 79.f.

PLATE. XVI.

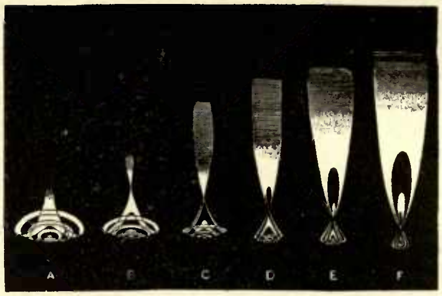


Fig. 77

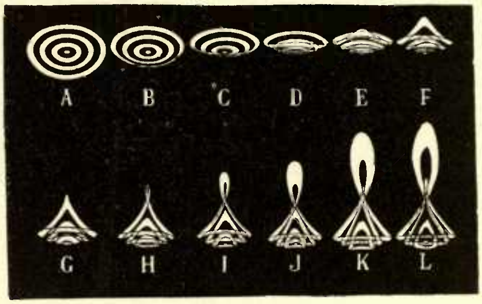


Fig. 78.

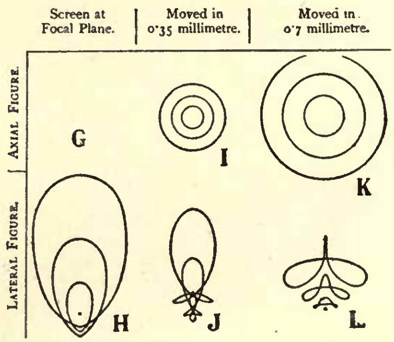


Fig 79.a.

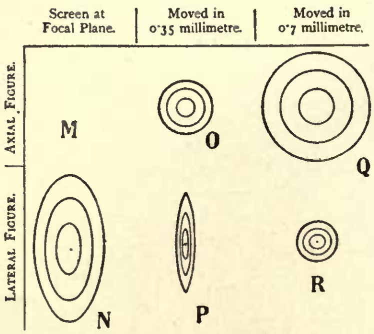


Fig. 79.b

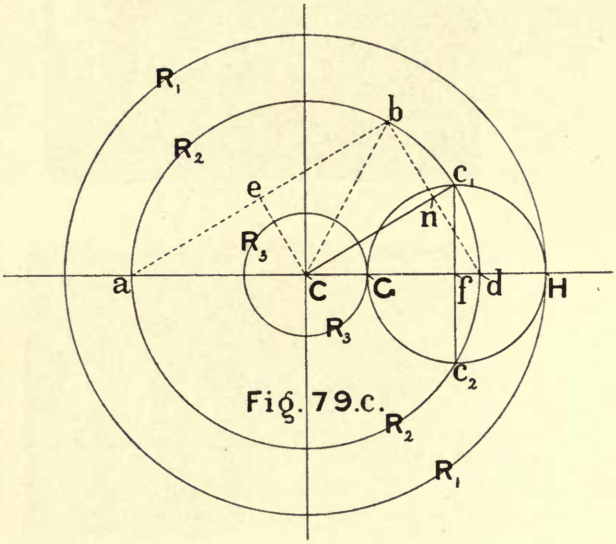


Fig. 79.c.

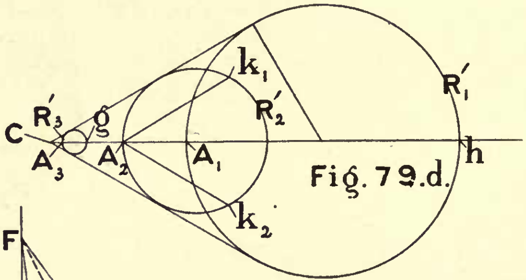


Fig. 79.d.

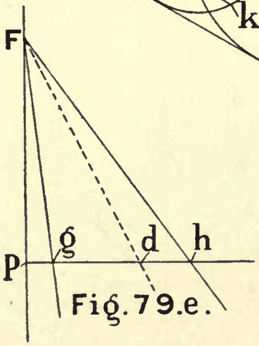


Fig. 79.e.

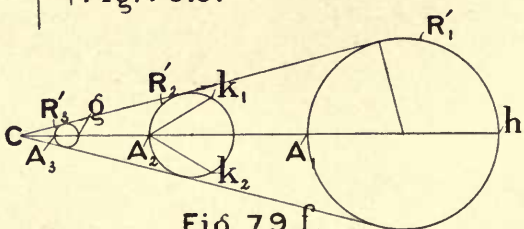


Fig. 79.f.

$$=R_1'\frac{R_2^2}{R_1^2}\cdot\frac{2(R_1-r)}{\sqrt{R_1^2-2R_1r+2r^2}} \qquad (18)$$

Length of secondary ray as projected on second comatic circle.

If now $A_2 \ldots k_2$ is multiplied by $\frac{1}{\tan\theta_2}$, θ_2 being the angle made with the central ray by any of the rays refracted through the lens zone $R_2 \ldots R_2$, we shall then arrive at the distance within or beyond the plane of the diagram at which the two rays $k_1 \ldots A_2$ and $k_2 \ldots A_2$ intersect, and this is the required linear E.C. in the secondary plane.

Now if we write θ_1 for the angle made with the central ray by the rays refracted through the outer lens zone $R_1 \ldots R_1$, then we have, if $F =$ the focal length,

$$\tan\theta_1 = \frac{C \ldots H}{F};$$

Semi-angle of cone of rays from outer lens zone.

and if we take $\tan\theta_1$ as the unit we have

$$\tan\theta_2 = \frac{C \ldots d}{F} = \tan\theta_1\frac{C \ldots d}{C \ldots H} = \tan\theta_1\frac{R_2}{R_1}, \qquad (19)$$

Semi-angle of cone of rays from second lens zone.

and

$$\tan\theta_3 = \frac{C \ldots G}{F} = \tan\theta_1\frac{C \ldots G}{C \ldots H} = \tan\theta_1\frac{R_3}{R_1}. \qquad (20)$$

Semi-angle of cone of rays from third lens zone.

Therefore the linear E.C. in the secondary plane (from 18)

$$=R_1'\cdot\frac{R_2^2}{R_1^2}\cdot\frac{2(R_1-r)}{\sqrt{R_1^2-2R_1r+2r^2}}\cdot\frac{1}{\tan\theta_1\frac{R_2}{R_1}}$$

$$=R_1'\cdot\frac{R_2^2}{R_1^2}\cdot\frac{2(R_1-r)}{\sqrt{R_1^2-2R_1r+2r^2}}\cdot\frac{R_1}{R_2}\cdot\frac{1}{\tan\theta_1},$$

in which, as we have seen,

$$R_2 = \sqrt{R_1^2-2R_1r+2r^2},$$

so that finally our linear E.C.

$$=R_1'\cdot\frac{2(R_1-r)}{R_1}\cdot\frac{1}{\tan\theta_1}, \qquad \text{V.}$$

Linear E.C. in the secondary plane.

with which we have yet to compare the linear E.C. in the primary plane, which we will now proceed to formulate.

Primary Plane

Here we have to deal with the two rays refracted through the lens at G and H. The ray from G strikes the comatic circle R_3' at the point g, while the ray from H strikes the comatic circle R_1 at h, and we first require the linear distance $g\,..\,h$.

Now $g\,..\,h = NR_1' - NR_3' = N(R_1' - R_3')$

$$= N\left\{R_1' - R_1'\frac{R_3^2}{R_1^2}\right\} = N\left\{R_1' - R_1' \cdot \frac{(R_1 - 2r)^2}{R_1^2}\right\}$$

$$= N\left\{R_1' - R_1'\left(\frac{R_1^2 - 4R_1r + 4r^2}{R_1^2}\right)\right\};$$

Distance between points where the two primary rays strike the plane of the coma.

$$\therefore\ g\,..\,h = N\left\{R_1'\left(1 - \frac{R_1^2 - 4R_1r + 4r^2}{R_1^2}\right)\right\}. \qquad (21)$$

In Fig. 79*e* let $p\,..\,h$ be the plane of the diagram Fig. 79*d*, and $g\,..\,F$ and $h\,..\,F$ the two rays we are dealing with which intersect at F beyond the plane of the coma $p\,..\,h$.

We have just obtained a formula for the distance $g\,..\,h$, and now what we want is the linear E.C. correction $F\,..\,p$ measured parallel to the ray through the centre c of the lens and perpendicular to the plane $p\,..\,h$ of the comatic rings. Let x represent this required distance $F\,..\,p$.

First we have the ray $H\,..\,h\,..\,F$ making the angle θ_1 with the centre ray or with $F\,..\,p$; the other ray $G\,..\,g\,..\,F$ makes the angle θ_3 with $F\,..\,p$ (while each secondary ray $c_1\,..\,d\,..\,F$ makes the angle θ_2 with $F\,..\,p$). Thus we have

$$x\tan\theta_1 - x\tan\theta_3 = (g\,..\,h);$$

$$\therefore\ x(\tan\theta_1 - \tan\theta_3) = (g\,..\,h);$$

Distance behind comatic plane where the two primary rays intersect.

$$\therefore\ x = (g\,..\,h)\frac{1}{\tan\theta_1 - \tan\theta_3}. \qquad (22)$$

On substituting in this the values of $g\,..\,h$ and $\tan\theta_3$ already worked out in Formulæ (21) and (20) we have

$$x = N\left\{R'_1\left(1 - \frac{R_1^2 - 4R_1r + 4r^2}{R_1^2}\right)\right\}\frac{1}{\tan\theta_1\left(1 - \frac{R_3}{R_1}\right)}$$

$$= N\left\{R'_1\left(\frac{4R_1r - 4r^2}{R_1^2}\right)\right\}\frac{1}{\tan\theta_1}\cdot\frac{R_1}{R_1 - R_3}$$

$$= N\left\{R'_1\left(\frac{4r(R_1 - r)}{R_1^2}\right)\right\}\frac{R_1}{R_1 - (R_1 - 2r)}\cdot\frac{1}{\tan\theta_1} = N\left\{R'_1\left(\frac{4r(R_1 - r)}{R_1^2}\right)\frac{R_1}{2r}\cdot\frac{1}{\tan\theta_1};\right.$$

therefore, finally,

Linear E.C. in the primary plane.

$$x\ (\text{or } p\,..\,F) = N\left\{R'_1\cdot\frac{2(R_1 - r)}{R_1}\cdot\frac{1}{\tan\theta_1}\right\}. \qquad \text{VI.}$$

Thus we get a linear E.C. which is N times the corresponding E.C. in the secondary plane, a result quite independent of the value of N, which, in the comatic formations of the second order that we have been dealing with, is 3 to 1.

Let it be supposed that N = 5 ; then the sort of coma that would be formed at the focus, supposing coma of the second order and other aberrations to be absent, would partake of the character of Fig. 79*f*, wherein the length C . . *h* = five times the radius of the outermost comatic circle which touches at *h*, and so on.

Form of coma that will give ratio of 5:1 between primary and secondary E.C.s.

When we come to deal with the curvature errors and E.C.s of the third order in Section XI. we shall have occasion to revert to this Fig. 79*f*.

The Elimination of Coma from Combinations of Thin Lenses in Contact

Before leaving the subject of coma it is desirable to deal with a problem relating to telescope objectives which often calls for solution. In the first place, it is clear that since the lenses composing such objectives are in contact, and generally thin compared to their focal lengths, therefore it may be said that points in the image away from the axis are formed by pencils of rays which are refracted obliquely but centrally through the lenses, any diaphragm corrections due to eccentric oblique refraction being so small compared to the normal curvature errors as to be negligible ; so that it cannot be supposed that any one form of telescope objective presents any substantial advantage over another form, as regards the flatness of its image, or the amount of its astigmatism for oblique foci. It may be said that the radius of curvature for the image formed by rays in primary planes is somewhat less than $\frac{3}{11}$ths of the principal focal length, and that for the image formed by rays in secondary planes somewhat less than $\frac{3}{5}$ths of the principal focal length. But since the extent of image utilised in such cases seldom amounts to more than two degrees from the axis, these curvature errors do not seriously matter, so we have the fact that the principal factor which determines the superiority of one form of objective over another as regards its definition away from the optic axis is simply the presence or absence of coma. For instance, a double achromatic objective with the collective lens placed first and of a meniscus or convexo-plane form will yield a very considerable amount of inward coma at its oblique foci which, at even five minutes of arc from the axis, is considerable enough to spoil definition; while if

Curvature of image scarcely varies in telescope objectives.

But coma at oblique foci is very variable.

the collective lens is plano-convex and still placed first, the opposite sort of coma will prevail, although it will not be quite so bad as in the former case.

Coma causes sensitiveness to squaring on.

It is also obvious that forms of objectives characterised by strong coma will be very sensitive to being slightly thrown out of square, a highly undesirable condition, for the mischief caused to definition by such coma may far exceed the mischief caused by the inevitable astigmatism.

Coma quite avoidable.

We cannot get rid of the normal curvature of the images nor the astigmatism in thin contact combinations, but we can get rid of the coma, and therefore it is of the highest importance in the case of telescope objectives, especially when designed for photographic purposes, that they should be designed free from coma, and to that end we may proceed as follows:—

Formula II. of this Section gives us the angular value of the coma yielded by any lens, so that in the case of the two lenses constituting a telescope objective that is to be free from coma, we have

Condition for elimination of coma from a two-lens combination.

$$\left.\begin{aligned} &-\frac{A^2\ 3\tan\phi}{4F_1^2\mu_1(\mu_1-1)}\left\{(2\mu_1+1)(\mu_1-1)\alpha_1+(\mu_1+1)x_1\right\} \\ &-\frac{A^2\ 3\tan\phi}{4F_2^2\mu_2(\mu_2-1)}\left\{(2\mu_2+1)(\mu_2-1)\alpha_2+(\mu_2+1)x_2\right\} \end{aligned}\right\}=0. \qquad \text{VII.}$$

Let $F_1 = +1$ and $F_2 = -\frac{5}{3}$, $\alpha_1 = -1$, the collective lens being placed first. Then

$$\frac{1+\alpha_2}{2(\frac{5}{3})} = 1, \text{ so that } \alpha_2 = +2\tfrac{1}{3}.$$

Let $\mu_1 = 1{\cdot}5$ and $\mu_2 = 1{\cdot}6$. Then, leaving out common factors, we have

$$-\left\{\frac{4}{1{\cdot}5}(-1)+\frac{2{\cdot}5}{(1{\cdot}5)({\cdot}5)}x_1\right\}-\left(\frac{3}{5}\right)^2\left\{\frac{4{\cdot}2}{1{\cdot}6}(2\tfrac{1}{3})+\frac{2{\cdot}6}{(1{\cdot}6)({\cdot}6)}x_2\right\}=0,$$

from which finally we derive

Relation between the x's for no coma.

$$x_2 = -3{\cdot}43x_1 + {\cdot}474.$$

We may now insert this value of x_2 in our formulæ for spherical aberration for the two lenses and equate them to 0, thus—

$$\frac{1}{8({\cdot}75)}\left\{7x_1^2 - 10x_1 + 3{\cdot}25 + 6{\cdot}75\right\}$$

$$-\left(\frac{3}{5}\right)^3\frac{1}{8(\cdot 96)}\left\{6(\underbrace{-3\cdot 43x_1+\cdot 474}_{=x_2})^2+10\cdot 4(\underbrace{-3\cdot 43x_1+\cdot 474}_{=x_2})(2\tfrac{1}{3}) + 4\cdot 08(2\tfrac{1}{3})^2+6\cdot 83\right\}=0,$$

from which we get

$$1\tfrac{1}{6}x_1^2-1\tfrac{2}{3}x_1+1\tfrac{2}{3}-\cdot 028\left\{\begin{matrix}(70\cdot 59x_1^2-19\cdot 5x_1+1\cdot 35)\\(-83\cdot 23x_1+11\cdot 502)\\+22\cdot 21+6\cdot 83\end{matrix}\right\}=0,$$

which reduces to

$$-\cdot 81x_1^2+1\cdot 21x_1+\cdot 493=0,$$
$$+x_1^2-1\cdot 5x=+\cdot 608,$$
$$x_1^2-1\cdot 5x+(\cdot 75)^2=\cdot 608+\cdot 5625,$$
$$x_1-\cdot 75=\pm\sqrt{1\cdot 1705}=\pm 1\cdot 082;$$
$$\therefore\ x_1=\cdot 75\pm 1\cdot 082$$
$$=-\cdot 332 \text{ or } +1\cdot 832.$$

The first result is the most convenient, as it implies radii in the ratio $\frac{4}{3}$ to $\frac{2}{3}$ or 2 to 1, in which case we have

$$x_2=-3\cdot 43\left(-\frac{1}{3}\right)+\cdot 474=+1\cdot 617,$$

or radii in about the ratio of $\frac{1}{2\cdot 6}:\frac{1}{-\cdot 6}$ or $+1:-4\frac{1}{3}$, which implies a concavo-convex dispersive lens.

Among useful formulæ is one for the spherical aberration of a single lens free from coma.

In order to be free from coma we must have

$$-\frac{3A^2\tan\phi}{4F^2\mu(\mu-1)}\left\{(2\mu+1)(\mu-1)a+(\mu+1)x\right\}=0,$$

from which

$$x=-\frac{(2\mu+1)(\mu-1)}{(\mu+1)}a. \qquad \text{VIII.}$$

Relation between x and a in single lens free from coma.

Then, on substituting this value of x in the formula for spherical aberration, we get

$$\frac{y^2}{8F^3}\frac{1}{\mu(\mu-1)}\left\{\frac{\mu+2}{\mu-1}\frac{(2\mu+1)^2(\mu-1)^2}{(\mu+1)^2}a^2-4(\mu+1)\frac{(2\mu+1)(\mu-1)}{(\mu+1)}a^2 + (3\mu+2)(\mu-1)a^2+\frac{\mu^3}{\mu-1}\right\}.$$

After adding together the three functions of a^2 and reducing, we get

Spherical aberration of simple lens free from coma.

$$\frac{y^2}{8F^3}\left\{\frac{\mu^2}{(\mu-1)^2}(1-a^2)\right\}, \qquad \text{IX.}$$

which is a simple expression for the spherical aberration of a lens free from coma. We have seen before that a simple lens gives the least possible spherical aberration when

Condition of least spherical aberration.

$$x'' = -2\frac{(\mu+1)(\mu-1)}{\mu+2}a. \qquad \text{X.}$$

Then

$$\text{X.} - \text{VIII.} = x'' - x' = -\frac{2(\mu^2-1)(\mu+1)+(2\mu+1)(\mu-1)(\mu+2)}{(\mu+1)(\mu+2)}a;$$

Difference between conditions of least spherical aberration and no coma.

$$\therefore\ x'' - x' = \frac{\mu(\mu-1)}{(\mu+1)(\mu+2)}a. \qquad \text{XI.}$$

If $a = -1$ and $\mu = 1{\cdot}5$, then the above

$$= \frac{{\cdot}75}{(2{\cdot}5)(3{\cdot}5)}(-1) = -\frac{{\cdot}75}{8{\cdot}75} = -\frac{1}{11\frac{2}{3}} \text{ or } -{\cdot}086,$$

so that the difference between the two values of x required to fulfil the conditions of freedom from coma and minimum aberration is only a small one.

Let us now consider the lens from another point of view. Suppose we wish the lens to satisfy the condition that if a varies or the vergency of the entering rays alters, then the spherical aberration shall remain constant, or, at any rate, vary in the least possible degree. We must then differentiate the spherical aberration formula with respect to a, and we have

Differential of spherical aberration with respect to a.

$$d_a\frac{1}{8f^3}(A')y_1^2 = \frac{y^2}{8f^3}\frac{1}{\mu(\mu-1)}\left\{4(\mu+1)x + 2(3\mu+2)(\mu-1)a\right\}da, \qquad \text{XII.}$$

which equates to 0 when

Condition of constancy of aberration when vergency varies.

$$x''' = -\frac{(3\mu+2)(\mu-1)}{2(\mu+1)}a. \qquad \text{XIII.}$$

Here again it is instructive to compare this formula with VIII. and X. For instance, we find that

$$\text{XIII.} - \text{VIII.} = \frac{-(3\mu+2)(\mu-1)+2(2\mu+1)(\mu-1)}{2(\mu+1)} = \frac{\mu(\mu+1)}{2(\mu+1)}a;$$

Difference between condition of vergency insensitiveness and of no coma.

$$\therefore\ x''' - x' = \frac{\mu(\mu-1)}{2(\mu+1)}a. \qquad \text{XIV.}$$

If $a = -1$ and $\mu = 1{\cdot}5$, the above $= \frac{{\cdot}75}{5} = -{\cdot}15$; and again we find there is not a very great difference between the values of x for fulfilling the two conditions of constancy of aberration when a varies, and freedom from coma. Of course, the same methods may be extended to compound lenses such as telescope objectives, and it will be found that the form of objective which we worked out as free from coma with $x_1 = -\cdot 332$ will also not differ very seriously from the form of objective necessary to give the least possible change in the spherical aberration when a varies, as, for instance, when the entering rays become slightly divergent instead of parallel. To fulfil this condition x_1 would have to be about $-\cdot 40$. Thus there is not such a large discrepancy between the two conditions as has been asserted by some writers.

Discrepancy between above conditions not very great.

Spherical and Parabolic Reflectors at Open Aperture

We have already had several instances before us of the conversion of any formula relating to refraction into the corresponding one relating to reflection by simply inserting the value -1 for μ. In this case, also, it will be found that the formula for coma at the oblique focus of a spherical reflector at open aperture may be obtained from the Formula II. for the angular value of the coma for a lens of open aperture. The latter formula was

$$-\frac{3\tan\phi}{4\mathrm{F}^2}\frac{1}{\mu(\mu-1)}\left\{(2\mu+1)(\mu-1)a+(\mu+1)x\right\}A^2. \qquad (23)$$

Here there need be no ambiguity about the meaning of x in the case of the above formula, since $(\mu+1)$ becomes $=0$, while a is -1, as in the case of the lens when the entering rays are parallel, while it is 0 if the rays are diverging from the centre of curvature, and $+1$ if they are diverging from the principal focus. Our formula therefore becomes

$$-\frac{3\tan\phi}{4\mathrm{F}^2}\frac{1}{2}\left\{(-1)(-2)a+0\right\}A^2$$
$$=-\frac{3\tan\phi}{4\mathrm{F}^2}(a)A^2. \qquad \text{XV.}$$

Angular coma in case of central oblique reflection.

Let it be supposed that the semi-aperture A is 1 foot, and the principal focal length 20 feet, and entering rays parallel as usual. so that $a=-1$; then the coma will be $+$ and outward. and its angular amount $\tan\phi\,\frac{3}{1600}$. If $\tan\phi=\frac{1}{100}$, then at 2·4 inches from the axis we shall have coma whose angular value at the mirror centre will be $\frac{3}{160{,}000}$, and its linear value will be $\frac{60}{160{,}000}=\frac{1}{2666}$th of a foot or $\frac{1}{222}$nd part of an inch, a very small quantity.

SECTION VIIIa

COMA AT THE FOCI OF ECCENTRIC OBLIQUE PENCILS

So far we have got the universal Formula II., giving the angular diameter of the longer axis of the comatic flare (as subtended at the centre of the lens) on the assumption that the principal ray of the oblique pencil passes through the centre of the lens.

Central oblique refraction exceptional.

But in the numerous cases of systems of more or less separated lenses it is the exception rather than the rule for central oblique refraction to take place; in most cases the principal rays of such pencils are refracted through the lenses at considerable distances from their centres, and as it is highly important to be in a position to eliminate coma at the oblique foci of such lens systems, we must therefore work out the formulæ appropriate to the eccentric oblique pencils refracted through them.

Two sorts of coma.

In the first place, a very little consideration will show that there are two sorts of coma, or rather coma caused in two different ways, to be dealt with in the case under consideration. First, there is coma which is simply part of the general coma already dealt with, which may be present in the lens and show at full symmetrical aperture. Second, there is coma resulting from the presence of spherical aberration in the *central* oblique pencil. Indeed, this sort of coma may manifest itself in the case of a direct axial pencil limited by an eccentrically placed stop.

Direct axial pencil limited by eccentric stop.

For instance, let Figs. 80 and 80*a* represent an uncorrected lens with an axial pencil, refracted eccentrically through it, owing to the presence of the circular but eccentrically placed stop. Then let Figs. 81 and 81*a* represent cases in which the pencil is obliquely refracted by the lens, but the stop is central and of an aperture allowing of the same aperture of the pencil where traversing the lens, as in Figs. 80 and 80*a*. Then such oblique pencil is subject to the same spherical aberration as the axial pencil of the same aperture; but we will suppose that there is no coma of the sort that we have yet dealt with; in other words, we will assume that the lens gives symmetrical oblique refraction. Of course, it will also give considerable astigmatism, but for the sake of simplicity we will assume the astigmatism to be absent and the focus to be exactly the same as for the axial pencil.

Symmetrical oblique refraction assumed.

It is at once obvious from the Diagrams 80 and 81 that there will ensue an eccentric formation at the focus whose structure in the primary plane is perhaps more clearly shown in Figs. 80*b* and 81*b*.

Suppose we arrange our stop *s* . . *s* so as to pass the central ray at one extreme of its aperture, and the outer ray at the other extreme of its aperture, as shown in Fig. 81, and that we place a ground glass screen perpendicular to the optic axis at the point *f* where the extreme outer ray passed by the stop intersects the centre ray. Let Fig. 82 represent a view of this screen when looking towards the centre of the lens, *a* . . *b* . . *c* the periphery of the lens, and *d* . . *e* . . *f* the outline of the eccentric pencil where it traverses the lens. We can then plot out the figure thrown on the screen or plane of the diagram by the rays which are refracted through the lens at points in the zone *d* . . *e* . . *f* of the eccentric pencil, in the following manner. From *f*, which is the point where both the centre ray Q . . *f* and the ray from *g* (the other extremity of the eccentric pencil) strike the screen, radial lines may be drawn to as many points in the circumference or zone *d* . . *e* . . *f* as may be desired, say points every ten degrees apart as measured from *f*. Then the lengths of these lines from *f* to the points where they cut the eccentric zone *d* . . *e* . . *f* . . *g* will give the values of the *y*'s or the distances from the lens centre of the points in the lens where each ray is refracted, from which the relative longitudinal spherical aberrations of such rays may be calculated, and from those the distances from the central ray *f* to the points where each ray cuts the screen or the plane of the diagram. It is obvious that all such displacements on the screen must take place along the radial lines drawn from *f*; all rays, except the extreme one, cut the central ray through *f* at points on the latter situated farther from the lens in calculable degrees, that is, at points nearer to the observer. Having worked out the point on each radial line where the corresponding ray from the zone *d* . . *e* . . *f* . . *g* cuts the plane of the diagram, and joining all such points together, we obtain the curve shown, which is exactly the same sort of curve as in Fig. 76 P, resulting from coma combined with astigmatism. For it is evident that while we are at the focus for the two extreme rays from the zone contained in the primary plane, yet we should have to retreat farther from the lens before we arrived at the focus for the two rays from w_1 and w_2 on the

How the comatic loop is derived.

The result is an astigmatic comatic loop.

zone which are contained in the secondary plane and strike the comatic loop at w_1' and w_2'. Hence there is astigmatism introduced by the selective action of the stop. We have already seen from Formula VI., Section VI., for E.C.s, that if we place a diaphragm in front of a collective lens having positive spherical aberration, so as to cause a pencil to traverse the lens eccentrically, then the E.C. consequent on spherical aberration will always be positive; that is, the intersection point for rays both in primary and secondary planes will be brought much nearer to the lens, and by three times as much in primary planes as in secondary planes, which last condition implies the existence of the astigmatism which we have independently arrived at in Fig. 82.

The same loop derivable from the axial pencil with eccentric stop.

It is obvious, also, that the comatic curve obtained in Fig. 82 may be derived also from the case of Figs. 80 and 80*a*; but of course the combination of an axial pencil limited by an eccentric stop does not occur in practice. Now let O be the point on the screen where the principal ray Q . . *h*, or the ray through the centre of the stop or of the eccentric zone or circle *d* . . *e* . . *f*, cuts the plane of the diagram; then the line O . . *f* will be the length of the whole comatic formation in the primary plane, for any comatic curves traced out by rays from smaller zones than *d* . . *e* . . *f* . . *g* will all be found to lie between O and *f*, as in Fig. 76 P.

Investigation of the Coma due to General Spherical Aberration

Construction.

We may now proceed to work out a formula for the length of such a comatic formation in the primary plane in the following manner. Let Figs. 83 and 83*a* represent a case of an oblique and eccentric pencil, limited by the stop *s* . . *s*, refracted through a lens at *a*. The origin or focus of the oblique pencil is Q, and its focus for rays ultimately close to the oblique axis Q . . *a* . . a_1 is at a_1. Let the ray Q . . *k* grazing the lower edge of the stop focus at *b* on the oblique axis, the principal ray Q . . *c* passing through the centre of the stop focus at c_1, and the other extreme ray Q . . *t* focus at *d*, so that a_1 . . *b*, a_1 . . c_1, and a_1 . . *d* are the longitudinal spherical aberrations, being therefore proportional to $(a\,..\,k)^2$, $(a\,..\,c)^2$, and $(a\,..\,t)^2$ respectively. Let the angle of obliquity P . . *a* . . Q or ϕ be measured at the lens or element centre as usual.

Let *f* be the point where the two extreme rays Q . . *t* and Q . . *k* passing the stop intersect, and through *f* draw *e* . . *f* . . *g* perpendicular to the optic axis P . . *a*. Then the size of the comatic formation is evidently at a minimum in *e* . . *f* . . *g*, where the two extreme rays in

PLATE. XVII.

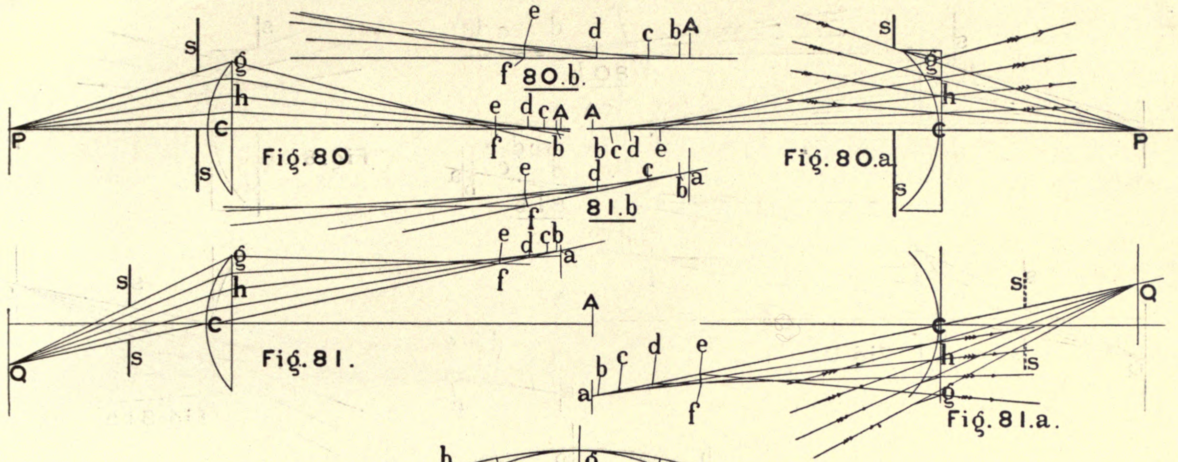


Fig. 80 Fig. 80.a. Fig. 81. Fig. 81.a.

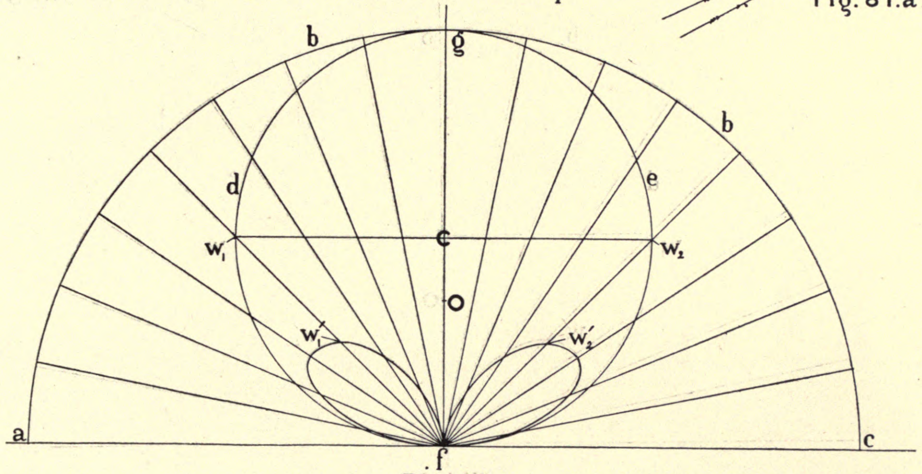


Fig. 82.

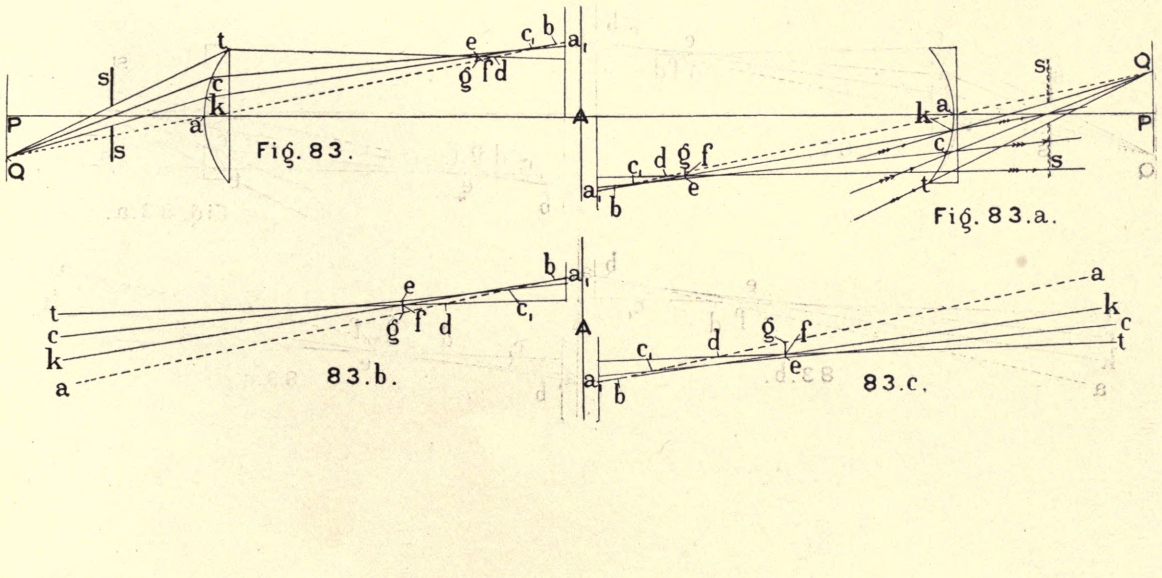


Fig. 83. Fig. 83.a.

PLATE. XVII.

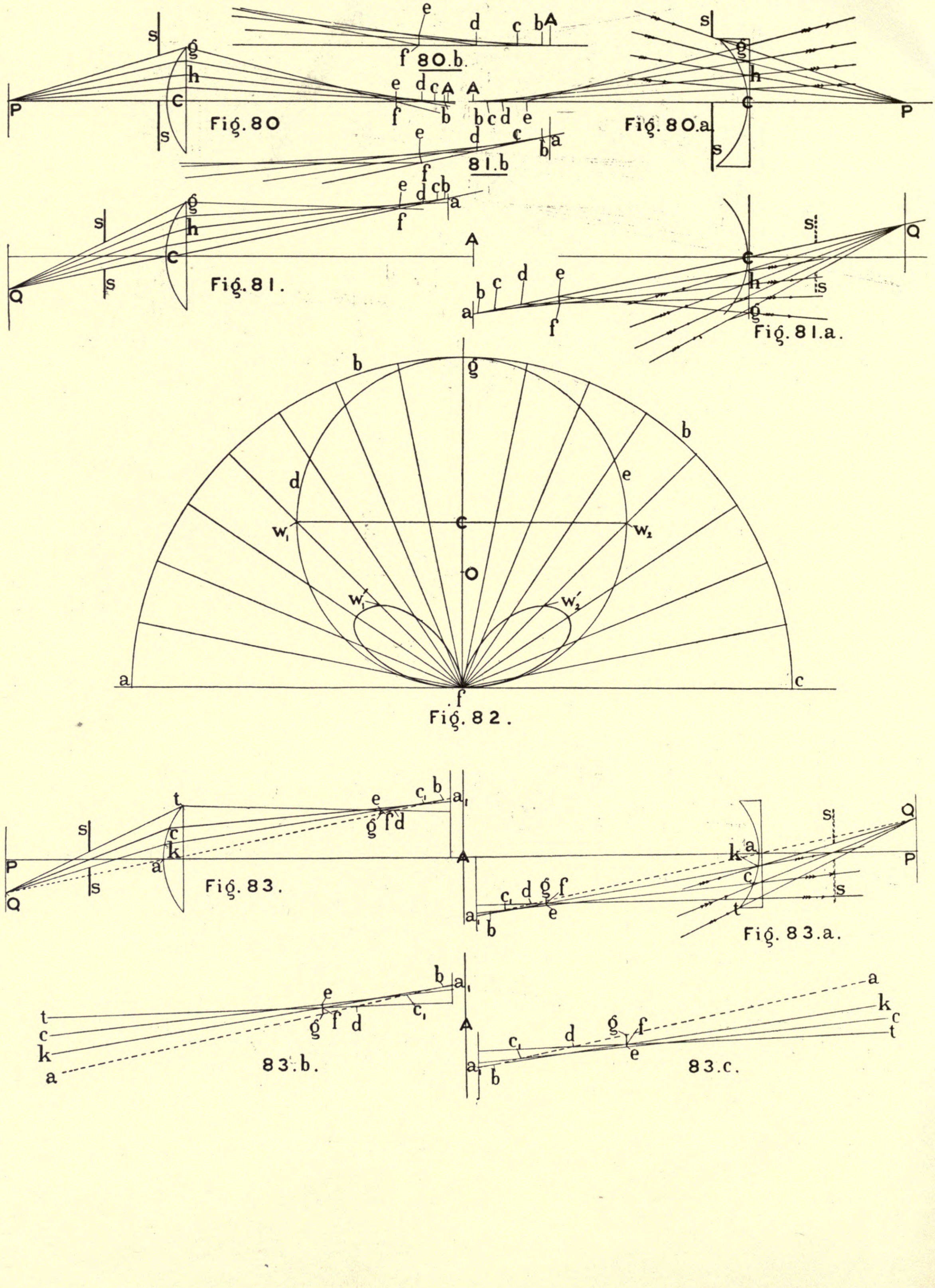


Fig. 80
80.b.
Fig. 80.a.
81.b
Fig. 81.
Fig. 81.a.
Fig. 82.
Fig. 83.
Fig. 83.a.
83.b.
83.c.

the primary plane focus, and the total length of the coma is obviously given by $e\,..\,f$.

Let $\mathrm{P}\,..\,a = \mathrm{U}$, and $a\,..\,a_1$ referred to the optic axis be V as usual.

Let the semi-aperture of the stop be S and the semi-aperture of the pencil where it traverses the lens be A. Let the vertical distance from a to c, where the principal ray cuts the lens, be L,* and let the distance of the stop from the lens = D. Let the formula for spherical aberration be stated shortly as $\frac{y^2}{8f^3}(\mathrm{A}')$, in which f is the principal focal length of the lens. For y we shall have in turn to substitute various other values. Then we have the following expressions for the longitudinal spherical aberrations :—

$$(a_1\,..\,b) = \frac{(a\,..\,k)^2}{8f^3}(\mathrm{A}')\mathrm{V}^2 = \frac{(\mathrm{L}-A)^2}{8f^3}(\mathrm{A}')\mathrm{V}^2\,;$$

$$(a_1\,..\,c_1) = \frac{(a\,..\,c)^2}{8f^3}(\mathrm{A}')\mathrm{V}^2 = \frac{(\mathrm{L})^2}{8f^3}(\mathrm{A}')\mathrm{V}^2\,;$$

$$(a_1\,..\,d) = \frac{(a\,..\,t)^2}{8f^3}(\mathrm{A}')\mathrm{V}^2 = \frac{(\mathrm{L}+A)^2}{8f^3}(\mathrm{A}')\mathrm{V}^2.$$

Then we have the following relations :—

$$(b\,..\,e)\frac{\mathrm{L}-A}{\mathrm{V}} = (f\,..\,g) = (d\,..\,g)\frac{\mathrm{L}+A}{\mathrm{V}}, \qquad (24)$$

in which $(d\,..\,g) = (b\,..\,e) - (b\,..\,d) = (b\,..\,e) - \{(a_1\,..\,d) - (a_1\,..\,b)\}$.

$$\therefore\ (d\,..\,g) = (b\,..\,e) - \left\{\frac{\mathrm{V}^2}{8f^3}(\mathrm{A}')(\mathrm{L}+A)^2 - \frac{\mathrm{V}^2}{8f^3}(\mathrm{A}')(\mathrm{L}-A)^2\right\};$$

$$\therefore\ (b\,..\,e)\frac{\mathrm{L}-A}{\mathrm{V}} = \left[(b\,..\,e) - \frac{\mathrm{V}^2}{8f^3}(\mathrm{A}')\left\{(\mathrm{L}+A)^2 - (\mathrm{L}-A)^2\right\}\right]\frac{\mathrm{L}+A}{\mathrm{V}}, \text{ (from (24))}$$

$$\therefore\ (b\,..\,e)\frac{\mathrm{L}-A}{\mathrm{V}} = (b\,..\,e)\frac{\mathrm{L}+A}{\mathrm{V}} - \frac{\mathrm{V}^2}{8f^3}(\mathrm{A}')\frac{\mathrm{L}+A}{\mathrm{V}}(\mathrm{L}^2 + 2\mathrm{L}A + A^2 - \mathrm{L}^2 + 2\mathrm{L}A - A^2);$$

$$\therefore\ (b\,..\,e)\left(\frac{\mathrm{L}-A}{\mathrm{V}} - \frac{\mathrm{L}+A}{\mathrm{V}}\right) = -\frac{\mathrm{V}^2}{8f^3}(\mathrm{A}')\frac{\mathrm{L}+A}{\mathrm{V}}(4\mathrm{L}A) = -\frac{\mathrm{V}^2}{2f^3}(\mathrm{A}')\frac{\mathrm{L}+A}{\mathrm{V}}\mathrm{L}A\,;$$

$$\therefore\ (b\,..\,e)\frac{-2A}{\mathrm{V}} = -\frac{\mathrm{V}^2}{2f^3}(\mathrm{A}')\frac{\mathrm{L}+A}{\mathrm{V}}\mathrm{L}A\,;$$

$$\therefore\ (b\,..\,e) = \frac{\mathrm{V}^2}{4f^3}(\mathrm{A}')(\mathrm{L}+A)\mathrm{L}. \qquad (25)$$

Also from (24)

$$(f\,..\,g) = (b\,..\,e)\frac{\mathrm{L}-A}{\mathrm{V}} = \frac{\mathrm{V}^2}{4f^3}(\mathrm{A}')(\mathrm{L}+A)\mathrm{L}\frac{\mathrm{L}-A}{\mathrm{V}} \text{ (from (25))};$$

* It is clear that L is the same thing as the eccentricity C of Section VI.

$$\therefore \ (f\,.\,.\,g) = \frac{\mathrm{V}}{4f^3}(\mathrm{A}')(\mathrm{L}^2 - A^2)\mathrm{L}\,; \qquad (26)$$

Also

$$(e\,.\,.\,g) = (c_1\,.\,.\,e)\frac{\mathrm{L}}{\mathrm{V}} = \left\{(b\,.\,.\,e) - (b\,.\,.\,c_1)\right\}\frac{\mathrm{L}}{\mathrm{V}} = \left\{(b\,.\,.\,e) - (a_1\,.\,.\,c_1) + (a_1\,.\,.\,b)\right\}\frac{\mathrm{L}}{\mathrm{V}}\,;$$

$$\therefore \ (e\,.\,.\,g) = \left\{\frac{\mathrm{V}^2}{4f^3}(\mathrm{A}')(\mathrm{L}+A)\mathrm{L} - \frac{\mathrm{L}^2}{8f^3}(\mathrm{A}')\mathrm{V}^2 + \frac{(\mathrm{L}-A)^2}{8f^3}(\mathrm{A}')\mathrm{V}^2\right\}\frac{\mathrm{L}}{\mathrm{V}},$$

$$= \frac{1}{8f^3}(\mathrm{A}')\mathrm{V}^2\left\{2(\mathrm{L}^2 + \mathrm{L}A) - \mathrm{L}^2 + \mathrm{L}^2 - 2\mathrm{L}A + A^2\right\}\frac{\mathrm{L}}{\mathrm{V}}$$

$$= \frac{1}{8f^3}(\mathrm{A}')\mathrm{V}^2\left\{2\mathrm{L}^2 + A^2\right\}\frac{\mathrm{L}}{\mathrm{V}}\,;$$

$$\therefore \ (e\,.\,.\,g) = \frac{1}{8f^3}(\mathrm{A}')(2\mathrm{L}^2 + A^2)\mathrm{LV}. \qquad (27)$$

Now $(e\,.\,.\,f) = (e\,.\,.\,g) - (f\,.\,.\,g)$;

$$\therefore \ (e\,.\,.\,f) = \frac{1}{8f^3}(\mathrm{A}')(2\mathrm{L}^2 + A^2)\mathrm{LV} - \frac{2\mathrm{V}}{8f^3}(\mathrm{A}')(\mathrm{L}^2 - A^2)\mathrm{L}\,;$$

Formula for the length of the aberration coma.

$$\therefore \ (e\,.\,.\,f) = \frac{1}{8f^3}(\mathrm{A}')(3A^2)\mathrm{LV}. \qquad (28)$$

Its length varies as the eccentricity.

Hence $e\,.\,.\,f$, or the length of the coma, varies directly as the eccentricity L. Formula (28) may be put into more general and convenient form by substituting $\frac{2\mathrm{F}}{1-\alpha}$ for V, and U tan $\phi\,\frac{\mathrm{D}}{\mathrm{U}-\mathrm{D}}$ or tan $\phi\,\frac{2\mathrm{F}}{\beta-\alpha}$ for L, and then we get

$$(e\,.\,.\,f) = \frac{1}{8f^3}(\mathrm{A}')(3A^2)\tan\phi\,\frac{2\mathrm{F}}{\beta-\alpha}\cdot\frac{2\mathrm{F}}{1-\alpha}\,;$$

$$\therefore \ (e\,.\,.\,f) = \frac{3A^2}{2f}(\mathrm{A}')\tan\phi\,\frac{1}{\beta-\alpha}\cdot\frac{1}{1-\alpha}. \qquad (29)$$

Generalisation of the formula.

Now since the diaphragm is nearer the lens than Q, then β in the above Formula (29) will be of positive value and numerically greater than α; therefore $\beta - \alpha$ will be positive. Also, since V is positive and real, therefore $1 - \alpha$ will also be positive. Also A$'$ is positive, therefore $e\,.\,.\,f$ will be positive also. But we shall find it convenient to treat $e\,.\,.\,f$ as a negative quantity, for the coma is obviously inward coma, a flare lying towards the optic axis; e is the position of the centre or principal ray of the eccentric pencil, and therefore $e\,.\,.\,f$ is a diminution of the distance from the optic axis. We must therefore reverse the sign of $e\,.\,.\,f$ by writing

$$(e\,..f) = \frac{3A^2}{2f}(\mathrm{A}') \tan\phi \frac{1}{\alpha - \beta} \cdot \frac{1}{1-\alpha},$$

or, in full,

$$\left.\begin{aligned}(e\,..f) = \tan\phi \frac{3A^2}{2f} \frac{1}{\mu(\mu-1)} \left\{\frac{\mu+2}{\mu-1}x^2 + 4(\mu+1)\alpha x + (3\mu+2)(\mu-1)\alpha^2 + \right. \\ \left. \frac{\mu^3}{\mu-1}\right\} \times \frac{1}{\alpha-\beta} \cdot \frac{1}{1-\alpha}.\end{aligned}\right\} \text{XVI.}$$

Full formula for length of the aberration coma.

But the most convenient formula of all is one expressing the angular value of $e\,..f$ as viewed from the lens centre, which is of course obtained by multiplying the above formula by $\frac{1}{\mathrm{V}}$ or by $\frac{1-\alpha}{2f}$, by which we then get

$$\left.\begin{aligned}\frac{e\,..f}{\mathrm{V}} = \tan\phi \frac{3A^2}{4f^2} \frac{1}{\mu(\mu-1)} \left\{\frac{\mu+2}{\mu-1}x^2 + 4(\mu+1)\alpha x + (3\mu+2)(\mu-1)\alpha^2 \right. \\ \left. + \frac{\mu^3}{\mu-1}\right\} \frac{1}{\alpha-\beta}.\end{aligned}\right\} \text{XVII.}$$

Universal formula for the angular value of the aberration coma.

Fig. 83a and c illustrates the analogous case of a dispersive lens in which also β is + and numerically greater than α, so that $\alpha - \beta$ is again negative and therefore gives a minus value to Formula XVII. This is as it should be, for it is plain from the diagram that the coma produced is again inward or towards the optic axis. Since the formula is a function of $\frac{1}{f^2}$, it is evident that the sign of f has no influence on the sign of the result; in fact, the sign of the lens is really implied in the value of $\alpha - \beta$. Thus it will be found that Formula XVII. is universally true of all cases. We may now turn our attention to the case of the coma of eccentric and oblique pencils consequent upon coma proper.

Investigation of the Coma Proper at the Foci of Eccentric Oblique Pencils

Fig. 84 represents a case of a collective lens giving pure inward coma at the focus of a central oblique pencil, spherical aberration and astigmatism being supposed to be absent, while Fig. 84a represents the corresponding case of a dispersive lens. Fig. 84b shows on a larger scale the structure of the focus for the collective lens. As in the last case, A is the semi-aperture of the eccentric pencil where it strikes the lens. f = the principal focal length of the lens; L = the eccentricity or the height A..C from the lens axis at which the

Construction.

principal ray strikes the lens. Q..A is the central oblique ray passing through the lens centre at an angle of obliquity $= \phi$; b is the point where the extreme ray Q..k..b passing the stop s, and nearest the lens centre, intersects or focuses on the central oblique ray; c is the point where the principal ray Q..C..c focuses on the central oblique ray; and d is the point where the extreme ray Q..t, passed by the stop s..s and most remote from the lens centre, intersects the central oblique ray. Then the two extreme rays passed by the stop, Q..t and Q..k, intersect one another at the point f. Through f draw e..f..g perpendicular to the optic axis; then e..f is the length of the coma at the focus of the eccentric oblique pencil as limited by the stop s..s.

Referring back to our method of finding the length of the coma yielded by the open lens (not shielded by any stop), we obtained a formula (4) having its application to Fig. 61. This formula expressed the eccentricity correction to be applied to $\frac{1}{V}$ in order to convert it into $\frac{1}{c..h}$ for any given semi-aperture A of the lens, on the supposition that the hypothetical stop was always so placed as to just pass the central oblique ray and the other ray cutting the lens at the semi-aperture A from the lens centre. We may apply that formula again in the present case of Fig. 83 or 84. It was

$$\frac{3 \tan \phi}{4F^2\mu(\mu - 1)}\left\{(2\mu + 1)(\mu - 1)a + (\mu + 1)x\right\}A,$$

which we may write shortly as $\frac{3 \tan \phi}{4F^2}(C')A$.

In the present case it is obvious that the linear distance $a..b$ is the above eccentricity correction $\frac{3 \tan \phi}{4F^2}(C')A$, with the semi-aperture A..k or L–A substituted for the former A, and the whole multiplied by V^2, so that

$$(a..b) = \frac{3 \tan \phi}{4F^2}(C')(L - A)V^2. \tag{30}$$

Likewise

$$(a..c) = \frac{3 \tan \phi}{4F^2}(C')(L)V^2 \tag{31}$$

and

$$(a..d) = \frac{3 \tan \phi}{4F^2}(C')(L + A)V^2. \tag{32}$$

We may now proceed in a manner analogous to the last case. We have

PLATE. XVIII.

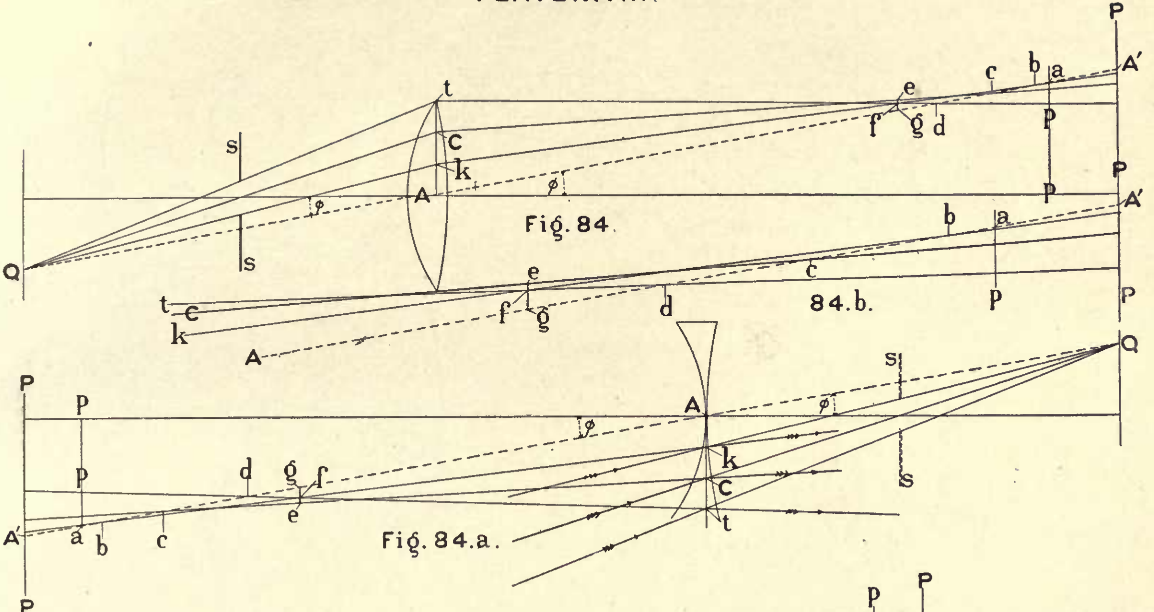


Fig. 84.

84. b.

Fig. 84. a.

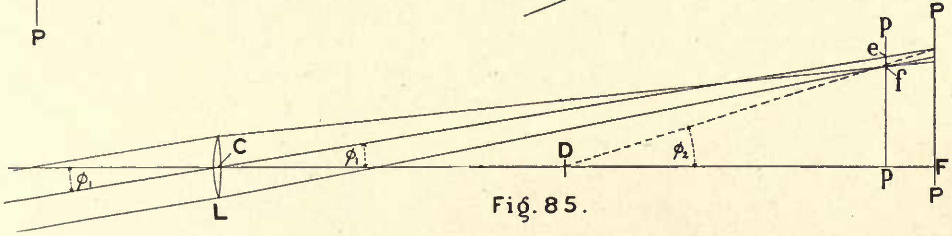


Fig. 85.

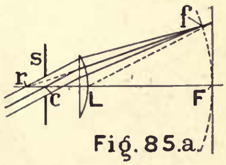


Fig. 85. a.

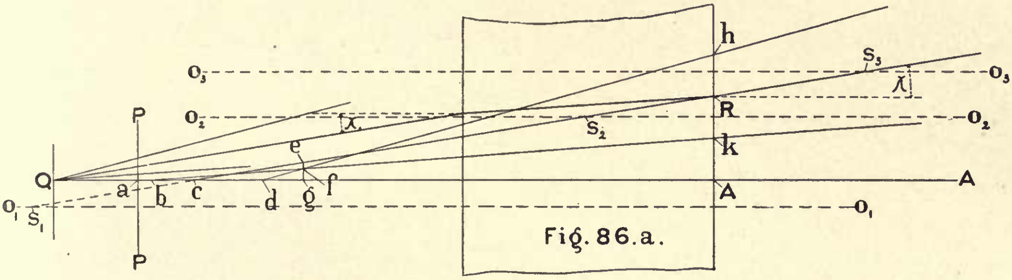


Fig. 86. a.

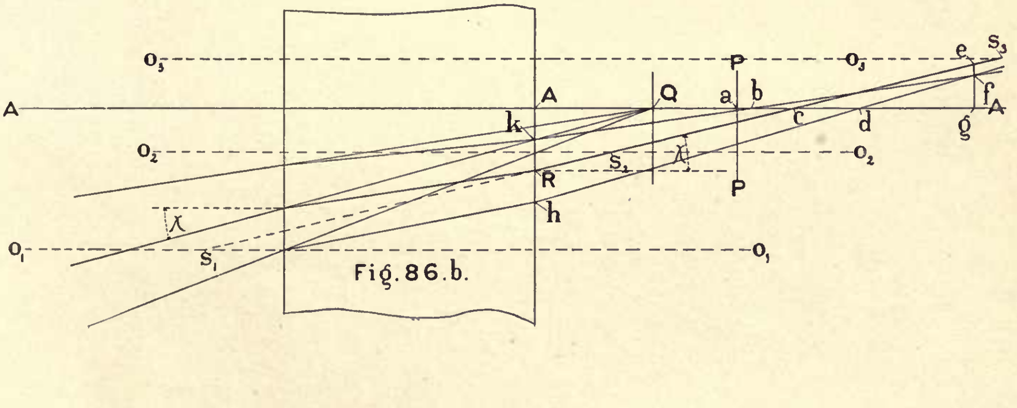


Fig. 86. b.

PLATE. XVIII

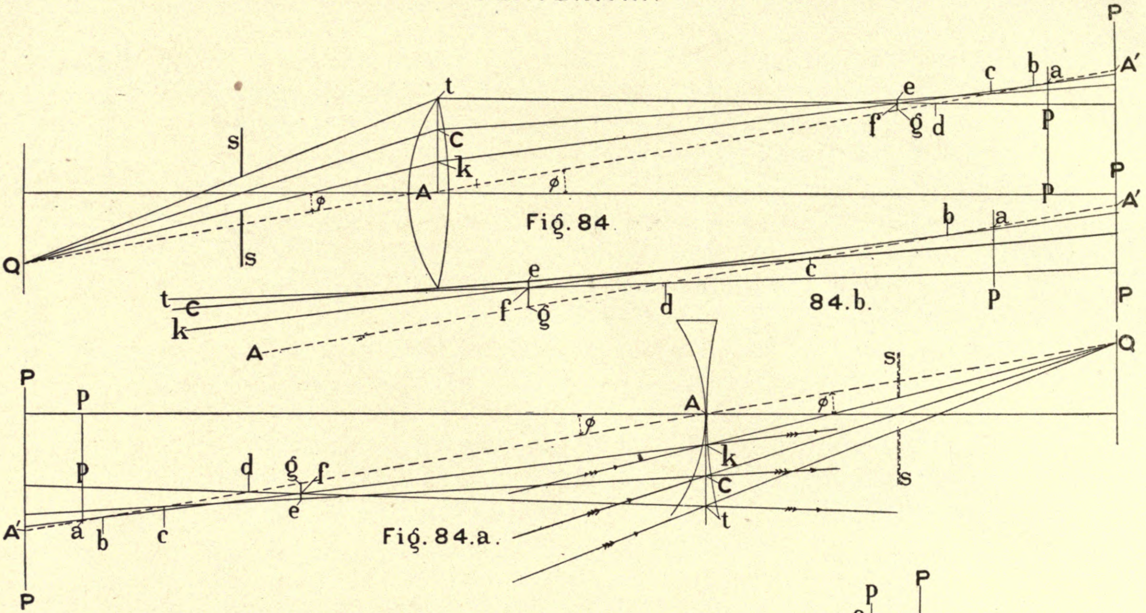


Fig. 84

84. b.

Fig. 84. a.

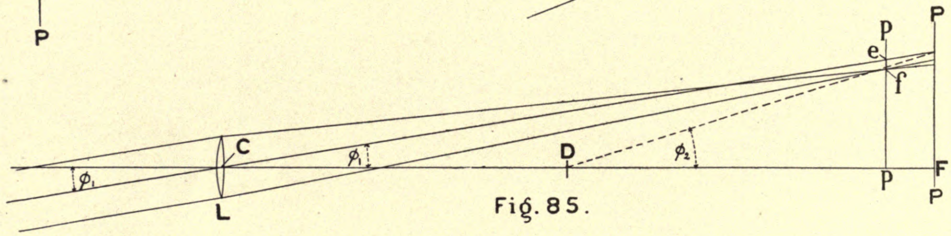


Fig. 85.

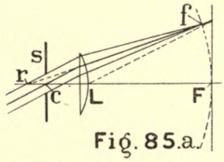


Fig. 85. a.

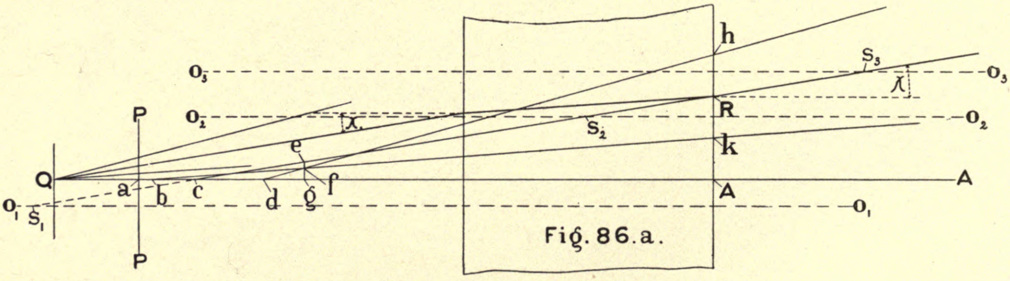


Fig. 86. a.

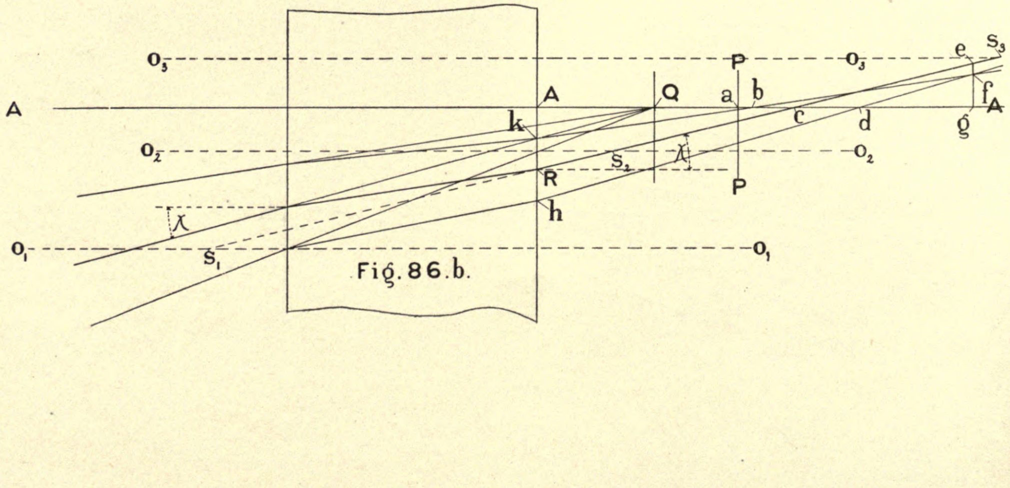


Fig. 86. b.

$$(b..e)\frac{\text{L}-A}{\text{V}}=(f..g)=(d..g)\frac{\text{L}+A}{\text{V}}, \tag{33}$$

in which $(d..g)=(b..e)-(b..d)=(b..e)-\{(a..d)-(a..b)\}$;

$$\therefore\ (d..g)=(b..e)-\Big\{(\text{L}+A)-(\text{L}-A)\Big\}\frac{3\tan\phi}{4\text{F}^2}(\text{C}')\text{V}^2; \tag{33a}$$

$$\therefore \text{ from (33) and (33}a\text{)}, (b..e)\frac{\text{L}-A}{\text{V}}=\left[(b..e)-(2A)\frac{3\tan\phi}{4\text{F}^2}(\text{C}')\text{V}^2\right]\frac{\text{L}+A}{\text{V}};$$

$$\therefore\ (b..e)\left\{\frac{\text{L}-A}{\text{V}}-\frac{\text{L}+A}{\text{V}}\right\}=-(2A)\frac{3\tan\phi}{4\text{F}^2}(\text{C}')\text{V}^2\frac{\text{L}+A}{\text{V}};$$

$$\therefore\ (b..e)\ \frac{-2A}{\text{V}}=-2A\,(\text{L}+A)\frac{3\tan\phi}{4\text{F}^2}(\text{C}')\text{V},$$

and

$$(b..e)=(\text{L}+A)\frac{3\tan\phi}{4\text{F}^2}(\text{C}')\text{V}^2. \tag{34}$$

Also, from (33)

$$(f..g)=(b..e)\frac{\text{L}-A}{\text{V}};$$

$$\therefore\ (f..g)=\frac{3\tan\phi}{4\text{F}^2}(\text{C}')(\text{L}^2-A^2)\text{V}. \tag{35}$$

Also

$$(e..g)=(c..e)\frac{\text{L}}{\text{V}}=\Big\{(b..e)-(b..c)\Big\}\frac{\text{L}}{\text{V}}=\Big\{(b..e)-(a..c)+(a..b\Big\}\frac{\text{L}}{\text{V}};$$

$$\therefore\ (e..g)=\Big\{(\text{L}+A)-\text{L}+(\text{L}-A)\Big\}\frac{3\tan\phi}{4\text{F}^2}(\text{C}')\text{V}^2\left(\frac{\text{L}}{\text{V}}\right);$$

$$\therefore\ (e..g)=(\text{L})\frac{3\tan\phi}{4\text{F}^2}(\text{C}')\text{VL}=\text{L}^2\frac{3\tan\phi}{4\text{F}^2}(\text{C}')\text{V}. \tag{36}$$

Therefore $e..f$, the required quantity, may now be arrived at from (35) and (36), thus

$$(e..f)=(e..g)-(f..g)=\text{L}^2\frac{3\tan\phi}{4\text{F}^2}(\text{C}')\text{V}-\frac{3\tan\phi}{4\text{F}^2}(\text{C}')(\text{L}^2-A^2)\text{V};$$

$$\therefore\ (e..f)=A^2\,\frac{3\tan\phi}{4\text{F}^2}(\text{C}')\text{V},$$

or, in full,

Formula for the length of the coma proper.

$$(e..f)=-A^2\,\frac{3\tan\phi}{4\text{F}^2\mu(\mu-1)}\Big\{(2\mu+1)(\mu-1)a+(\mu+1)x\Big\}\text{V}. \qquad \text{XVIII.}$$

Now we have assumed the coma in our diagram to be inward or towards the axis, the E.C.s being positive or an addition to the value of $\frac{1}{V}$. This would certainly be the case if, for instance, $x = +1$ and $a = 0$ or $-\cdot 5$; but as we have laid down the rule that inward coma is to be considered negative and outward coma positive, we must prefix the negative sign to the above formula as shown. Next, if we divide by V we shall then obtain the angular value of the coma as viewed from the lens centre, getting finally

Universal formula for the angular value of the coma proper.

$$\frac{e..f}{V} = -A^2 \frac{3\tan\phi}{4F^2\mu(\mu-1)}\left\{(2\mu+1)(\mu-1)\alpha + (\mu+1)x\right\}. \qquad \text{XIX.}$$

On comparing this result with Formula II., formerly arrived at for the angular value of the coma for the lens at open aperture, we find that the two formulæ are identical, although A is now eccentric; that is, for a given pair of conjugate focal planes and a given degree of obliquity the angular value of the coma is simply a function of the square of the semi-aperture of the pencil where it is refracted, and is quite independent of the degree of eccentricity of the pencil where it traverses the lens, and therefore of the distance of the stop from the latter. In this respect it differs from the aberration coma. Thus the amount and character of the coma will not be affected if the stop is moved across the optic axis in its own plane. Given a fixed aperture of the stop, then the only way in which the distance of the stop from the lens can affect the coma is by modifying the semi-aperture of the pencil where it cuts the lens, since the latter is equal to the semi-aperture of the stop multiplied by $\frac{U}{U-D'}$ or $\frac{V}{V-D''}$ as the case may be. We may now combine Formulæ XVII. and XIX. for the spherical aberration coma and the coma proper respectively for an eccentric oblique pencil into one, thus—

Formula for angular value of both sorts of coma.

$$\left.\begin{aligned}\frac{e..f}{V} = A^2\cdot\frac{3\tan\phi}{4F^2\mu(\mu-1)}\cdot\frac{1}{\alpha-\beta}\Bigg[&\left\{\frac{\mu+2}{\mu-1}x^2 + 4(\mu+1)\alpha x + (3\mu+2)(\mu-1)\alpha^2\right.\\ &\left.+\frac{\mu^3}{\mu-1}\right\} - (\alpha-\beta)\left\{(2\mu+1)(\mu-1)\alpha + (\mu+1)x\right\}\Bigg].\end{aligned}\right\}\text{XX.}$$

Thus the interior functions in the formula are closely analogous to those in the formula for E.C.s, VIII., Section VI., only $4F^2$ replaces 2F, and $\frac{1}{\alpha-\beta}$ replaces $\frac{1}{(\alpha-\beta)^2}$, while the comatic function is reduced to a half.

If the same processes are followed in the similar case of the dispersive lens, exactly the same formula will be arrived at, provided that our convention is adhered to which makes inward coma or flare towards the optic axis negative, and outward coma positive, irrespective of whether the lens in question be collective or dispersive, for, as we have seen, that matter really tells in the sign of $\alpha - \beta$ for the lens in question.

A good test case for the correctness of signs in Formula XX. in their application to collective and dispersive lenses is one in which a plano-convex collective lens is placed in contact with a concavo-plane dispersive lens of the same radius of curvature and of the same index of refraction. Thus it is clear that, especially if cemented together, the two lenses will merely form a parallel plate of glass, and act as such. Then the Formulæ XX. for the two lenses will in this case be found to equate to 0 in all circumstances, since $\alpha_2 = -\alpha_1$, $\beta_2 = -\beta_1$, and $x_2 = -x_1$, and therefore $(\alpha_2 - \beta_2) = -(\alpha_1 - \beta_1)$.

Coma in Relation to E.C.s and Normal Curvature Errors, etc. Some Interesting Corollaries.

Many important deductions may be drawn from the formulæ arrived at in this and previous Sections.

1. Supposing that in the case of eccentric oblique refraction through a simple lens the E.C.s are eliminated, leaving the normal curvature errors of the lens intact, then what will be the result as to the presence or absence of coma at the foci of oblique pencils? Such a condition has often to be fulfilled or closely approached in Cooke lenses.

First of all we have for the elimination of E.C.s from a lens the condition

$$\frac{\tan^2\phi}{2f}\left\{\frac{1}{(\alpha-\beta)^2}\mathrm{A}' - 2\frac{1}{(\alpha-\beta)}\mathrm{C}'\right\} = 0,$$

from which we derive

$$\mathrm{C}' = \frac{\mathrm{A}'}{2(\alpha-\beta)}. \qquad \text{XXA.}$$

Condition of elimination of E.C.s.

On the other hand we have for the elimination of coma from a lens under the same circumstances the condition

$$3A^2\frac{\tan\phi}{4f^2}\left\{\frac{1}{(\alpha-\beta)}\mathrm{A}' - \mathrm{C}'\right\} = 0, \qquad \text{XXA.}$$

from which

$$\mathrm{C}' = \frac{\mathrm{A}'}{(\alpha-\beta)}. \qquad \text{XXB.}$$

Condition of elimination of coma.

Hence it is clear that when E.C.s are eliminated there will be a preponderance of spherical aberration coma at the foci of eccentric oblique pencils, for which the formula will be

Formula for coma when E.C.s are eliminated.

$$3A^2\frac{\tan\phi}{4f^2}\left\{\frac{1}{(\alpha-\beta)}A' - \frac{1}{2(\alpha-\beta)}A'\right\}$$

$$= 3A^2\frac{\tan\phi}{4f^2}\cdot\frac{1}{2(\alpha-\beta)}A'. \qquad \text{XXc.}$$

2. Let it be supposed that the E.C.s are so arranged as to neutralise the normal oblique astigmatism of the lens, then what will be the condition of the oblique foci as to coma?

For the elimination of astigmatism we have

$$\frac{2\tan^2\phi}{2f}\left\{\frac{1}{(\alpha-\beta)^2}A' - 2\cdot\frac{1}{(\alpha-\beta)}C'\right\} = -\frac{\tan^2\phi}{f}, \qquad (37)$$

from which we derive

$$A' - 2(\alpha-\beta)C' = -(\alpha-\beta)^2$$

and

Condition of elimination of astigmatism.

$$C' = \frac{A' + (\alpha-\beta)^2}{2(\alpha-\beta)} \qquad \text{XXd.}$$

as the condition of no astigmatism.

If now we insert this value of C' into the above formula for coma, XXa., we get

Formula for angular coma when there is no astigmatism.

$$3A^2\frac{\tan\phi}{4f^2}\left\{\frac{1}{(\alpha-\beta)}A' - \frac{A' + (\alpha-\beta)^2}{2(\alpha-\beta)}\right\}$$

$$= 3A^2\frac{\tan\phi}{4f^2}\left\{\frac{A' - (\alpha-\beta)^2}{2(\alpha-\beta)}\right\}, \qquad \text{XXe.}$$

which expresses the angular value of the coma when there is no astigmatism. Then it is clear that if $A' = (\alpha - \beta)^2$ there will be no coma at the foci of eccentric oblique pencils.

Case of a lens giving an astigmatic image free from coma.

Fig. 85*a* illustrates an example of this case which will be already familiar to many readers. It is the case of a plano-convex lens of crown glass receiving parallel rays passed through a stop fixed at a distance $D' = \frac{F}{3}$ in front of it. Here, the refractive index being 1·5, it is clear that after refraction by the first plane surface the centre point c of the stop s will be transferred to $r\left(=\frac{F}{2}\right)$, which is then the centre of curvature of the second surface, and therefore the principal rays will all impinge upon the second surface as if diverging from the centre of curvature, and will consequently meet with perfectly symmetrical refraction, and there will be neither astigmatism nor coma at the oblique focus f.

Here we have $x = -1$, $a = -1$, $\beta = +5$, and $(a-\beta) = -6$. A′ works out to

$$\frac{1}{\cdot 75}\left\{7 + 10 + 3\cdot 25 + 6\cdot 75\right\} = \frac{1}{\cdot 75}(27) = 36.$$

C′ works out to

$$\frac{1}{\cdot 75}\left\{2(-1) + (2\cdot 5)(-1)\right\} = -6.$$

Therefore if we insert these values of A′ and C′ in above Formula XXE., we then have

$$3A^2\frac{\tan\phi}{4F^2}\left\{\frac{36-(-6)^2}{2(-6)}\right\} = 0.$$

Formula for coma =0.

But it is evident that other conditions may be found, leading to no astigmatism, which will yet permit of the presence of coma, especially when u is less than F and v negative, and therefore a greater than $+1$.

We may now inquire what will be the formula for E.C.s when coma is eliminated. The formula for E.C.s in the primary plane is

$$\frac{3\tan^2\phi}{2f}\left\{\frac{1}{(a-\beta)^2}A' - 2\frac{1}{a-\beta}C'\right\},$$

and if for C′ we substitute its value from XXB., which holds good when coma is eliminated, we then have

$$\frac{3\tan^2\phi}{2f}\left\{\frac{1}{(a-\beta)^2}A' - 2\frac{1}{(a-\beta)^2}A'\right\};$$

and therefore the E.C.s

$$= \frac{3\tan^2\phi}{2f}\left\{-\frac{1}{(a-\beta)^2}A'\right\}. \qquad \text{XXF.}$$

Formula for E.C.s when coma is eliminated.

Lastly, we have the formula for astigmatism,

$$\frac{\tan^2\phi}{f}\left\{\frac{1}{(a-\beta)^2}A' - 2\frac{1}{(a-\beta)}C' + 1\right\} \text{ (from (37))},$$

in which we may substitute the value of C′ which holds good when there is no coma, and we then have

$$\frac{\tan^2\phi}{f}\left\{\frac{1}{(a-\beta)^2}A' - 2\frac{1}{(a-\beta)^2}A' + 1\right\},$$

Formula for astigmatism when there is no coma.

which finally

$$=\frac{\tan^2\phi}{f}\left\{\frac{(\alpha-\beta)^2-\mathrm{A}'}{(\alpha-\beta)^2}\right\}. \qquad \text{XXG.}$$

In the course of the preliminary planning out of optical systems, such generalisations as the above are often useful.

Application of the Formulæ to a Series of Separated Lenses

We saw that the formulæ for eccentricity corrections were functions of $\tan^2\phi$, and had to be multiplied by V^2 or F^2 in order to reduce them to their longitudinal value as corrections to the focal length, and that in adding together the functions for a series of separated lenses no notice need be taken of the successive modifications of the angle ϕ for the different lenses, all that was required being the simple algebraic sum of the corrections for all the lenses; so, in the case of a series of separated lenses we may in the same way apply the Formula XIX. for coma directly to each lens in turn, for the formula is a function of $\tan\phi$ simply, and the linear amount of coma yielded by each lens is obtained by multiplying by V. Fig. 85 shows a lens L giving a certain length of coma $e\,..\,f$. It obviously makes no difference to the linear value of $e\,..\,f$ whether we assume it to be referred to the point C at the centre of the lens and in terms of $\tan\phi_1$, or to the point D and in terms of $\tan\phi_2$. For supposing Formula XIX. gives us a certain value $\mathrm{M}\tan\phi_1$ for the angular value of the coma as viewed from C; then, supposing $\mathrm{C}\,..\,\mathrm{F}=\mathrm{V}$, the linear value of the coma is simply $\mathrm{MV}\tan\phi_1$. If, on the other hand, we assume that D is the position of the back lens of the combination and that V or $\mathrm{C}\,..\,\mathrm{F}=n(\mathrm{D}\,..\,\mathrm{F})$, or $\mathrm{D}\,..\,\mathrm{F}=\frac{\mathrm{V}}{n}$, then obviously $\tan\phi_2=n\tan\phi_1$, and therefore the length L of the coma referred to the point D is given by

$$\mathrm{L}=(\mathrm{M}\tan\phi_1)\mathrm{V}=\mathrm{M}(n\tan\phi_1)\frac{\mathrm{V}}{n}=(\mathrm{M}\tan\phi_2)(\mathrm{D}\,..\,\mathrm{F}),$$

which is the same result. But it is clear that the semi-aperture A of the oblique eccentric pencil where it traverses each lens in turn must be carefully inserted.

For brevity let us write Formula XX. as simply

$$\frac{3\tan\phi}{4\mathrm{F}^2}\cdot\frac{1}{\alpha-\beta}\left\{\mathrm{A}'-(\alpha-\beta)\mathrm{C}'\right\}A^2;$$

then for two lenses or elements in succession, whether separated or not, the formula will take the form

$$\left.\begin{aligned}&\frac{3\tan\phi}{4F_1^2}\frac{1}{\alpha_1-\beta_1}\left\{A'_1-(\alpha_1-\beta_1)C_1'\right\}A_1^2\\&\qquad+\frac{3\tan\phi}{4F_2^2}\frac{1}{\alpha_2-\beta_2}\left\{A'_2-(\alpha_2-\beta_2)C_2'\right\}A_1^2\left(\frac{u_2}{v_1}\right)^2\end{aligned}\right\}\text{XXI.}$$

Formulæ for coma for two lenses in succession.

and for three elements or lenses in succession, whether separated or not,

$$\left.\begin{aligned}&\frac{3\tan\phi}{4F_1^2}\frac{1}{\alpha_1-\beta_1}\left\{A'_1-(\alpha_1-\beta_1)C_1'\right\}A_1^2\\&\quad+\frac{3\tan\phi}{4F_2^2}\frac{1}{\alpha_2-\beta_2}\left\{A'_2-(\alpha_2-\beta_2)C_2'\right\}A_1^2\left(\frac{u_2}{v_1}\right)^2\\&\qquad+\frac{3\tan\phi}{4F_3^2}\frac{1}{\alpha_3-\beta_3}\left\{A'_3-(\alpha_3-\beta_3)C_3'\right\}A_1^2\left(\frac{u_2u_3}{v_1v_2}\right)^2\end{aligned}\right\}\text{XXII.}$$

Formulæ for coma for three lenses in succession.

and so on up to any number of lenses or elements in succession; the semi-aperture of the pencil where it traverses each lens or element plane being expressed in terms of the semi-aperture of the pencil at the first lens or element plane of the series.

Coma produced by Oblique Refraction through a Parallel Plane Plate

However, our formula for coma is not yet quite complete, for in the case of thick lenses we have to deal with two elements and a parallel plate, and we must now work out a formula for the coma produced when a pencil of converging or diverging rays is refracted obliquely through a parallel plane plate. That spherical aberration coma is produced in such a case is evident from the inspection of Figs. 86*a* and 86*b*, and also from experiment.

Construction.

Let A . . h be the second surface of a piece of parallel plane glass of thickness $=t$ and refractive index $=\mu$. Let b . . K and d . . H be the two extreme rays of the oblique pencil, and c . . R the middle or principal ray of the same. Let a be the focal point for the rays ultimately close to the normal Q . . A, which, if the pencil were indefinitely extended, would be a ray perpendicular to the plane surfaces. Then we must imagine that the origin of the pencil or the point from which all the rays originally start is at a point Q on A . . a produced backwards and at a distance to the left of a equal to $t\frac{\mu-1}{\mu}$, and the diagrams chiefly represent the course of the rays *after* emergence from the second surface. Then, as we have seen in Section IV., page 79, the rays are subjected to a negative aberration which, as a correction to $\frac{1}{v}$ or $\frac{1}{A\,..\,a}$, was found to be $\frac{(\mu^2-1)t}{2\mu^3v^4}a_2^{\,2}$, in which a_2 was the distance of each ray, where it cut the second surface, from the normal ray A .. Q.

On multiplying the above formula by v^2 we then get the longitudinal aberration for any ray, so that we have

$$(\mathrm{A}\,..\,b) = \frac{(\mu^2-1)t}{2\mu^3v^2}(\mathrm{A}\,..\,k)^2, \tag{38}$$

$$(\mathrm{A}\,..\,c) = \frac{(\mu^2-1)t}{2\mu^3v^2}(\mathrm{A}\,..\,\mathrm{R})^2, \tag{39}$$

$$(\mathrm{A}\,..\,d) = \frac{(\mu^2-1)t}{2\mu^3v^2}(\mathrm{A}\,..\,h)^2. \tag{40}$$

Let the angle of obliquity enclosed between the principal ray c .. R and the normal ray A .. Q be χ, and let A .. R = L and R .. h = R .. $k = A$ (the semi-aperture of the pencil).

It is evident that the length of the coma is e .. f, f being the point at the extremity of the coma where the two extreme rays of the pencil intersect, which, as is always the case where there is coma, lies to one side of the principal ray.

We may now follow a line of reasoning analogous to that we pursued in the case of working out the spherical aberration coma produced by a lens on an eccentric pencil; as follows:—

$$(b\,..\,g)\frac{\mathrm{L}-A}{v} = (f\,..\,g) = (d\,..\,g)\frac{\mathrm{L}+A}{v}, \tag{41}$$

in which

$$(d\,..\,g) = (b\,..\,g) - (b\,..\,d) = (b\,..\,g) - \{(a\,..\,d) - (a\,..\,b)\};$$

$$\therefore\ (d\,..\,g) = (b\,..\,g) - \frac{(\mu^2-1)t}{2\mu^3v^2}\left\{(\mathrm{L}+A)^2 - (\mathrm{L}-A)^2\right\} = (b\,..\,g) - \frac{(\mu^2-1)t}{2\mu^3v^2}(4\mathrm{L}A);$$

$$\therefore \text{ from (41)}\quad (b\,..\,g)\frac{\mathrm{L}-A}{v} = \left\{(b\,..\,g) - \frac{(\mu^2-1)t}{2\mu^3v^2}(4\mathrm{L}A)\right\}\frac{\mathrm{L}+A}{v};$$

$$\therefore\ (b\,..\,g)\left\{\frac{\mathrm{L}-A}{v} - \frac{\mathrm{L}+A}{v}\right\} = -4\frac{(\mu^2-1)t}{2\mu^3v^2}\,\frac{\mathrm{L}A(\mathrm{L}+A)}{v};$$

$$\therefore\ (b\,..\,g) = \frac{(\mu^2-1)t}{\mu^3v^2}\mathrm{L}(\mathrm{L}+A). \tag{42}$$

Also

$$(f\,..\,g) = (b\,..\,g)\frac{\mathrm{L}-A}{\mathrm{V}} = \frac{(\mu^2-1)t}{\mu^3v^2}\mathrm{L}(\mathrm{L}+A)\frac{\mathrm{L}-A}{v};$$

$$\therefore\ (f\,..\,g) = \frac{(\mu^2-1)t}{\mu^3 v^3}\mathrm{L}(\mathrm{L}^2 - A^2). \tag{43}$$

Also

$$(e\,..\,g) = (c\,..\,g)\frac{\mathrm{L}}{v} = \Big\{(b\,..\,g) - (b\,..\,c)\Big\}\frac{\mathrm{L}}{v} = \Big\{(b\,..\,g) - (a\,..\,c) + (a\,..\,b)\Big\}\frac{\mathrm{L}}{v};$$

$$\therefore\ (e\,..\,g) = \frac{(\mu^2-1)t}{2\mu^3 v^2}\Big\{2\mathrm{L}(\mathrm{L}+A) - \mathrm{L}^2 + (\mathrm{L}-A)^2\Big\}\frac{\mathrm{L}}{v};$$

$$\therefore\ (e\,..\,g) = \frac{(\mu^2-1)t}{2\mu^3 v^3}(2\mathrm{L}^2 + A^2)\mathrm{L}. \tag{44}$$

Then

$$(e\,..f) = (e\,..\,g) - (f\,..\,g) = \frac{(\mu^2-1)t}{2\mu^3 v^3}(2\mathrm{L}^2 + A^2)\mathrm{L} - \frac{(\mu^2-1)t}{2\mu^3 v^3}2\mathrm{L}(\mathrm{L}^2 - A^2);$$

$$\therefore\ (e\,..f) = \frac{3(\mu^2-1)}{2\mu^3 v^3}\mathrm{L}A^2, \tag{45}$$

in which formula $\mathrm{L} = v \tan\chi$, so that

$$(e\,..f) = 3\tan\chi\frac{(\mu^2-1)t}{2\mu^3 v^2}A^2, \tag{46}$$

and then the angular value of the coma subtended at A is given by

$$\frac{e\,..f}{v} = 3\tan\chi\frac{(\mu^2-1)t}{2\mu^3 v^3}A^2. \tag{47}$$

How the sign of the coma is to be determined.

We have now got the numerical value of the coma; but its sign demands very special consideration, chiefly for the reason that the optic axis of the glass plate is indeterminate, or may be any straight line perpendicular to the surfaces. But the optic axis of the lens system, of which the plate is a part, is always definable.

A parallel plane plate has no axis.

In Fig. 86*a* let it be supposed that the optic axis of the system is O_1–O_1, then obviously the coma $e\,..f$ is inwards or towards the optic axis; but if the optic axis is at $O_2\,..\,O_2$ or $O_3\,..\,O_3$ the same coma becomes outward or from the optic axis. In the same way if, in Fig. 86*b*, the optic axis is at $O_1\,..\,O_1$ the coma is inward, and if at O_3–O_3, then it is outward. We therefore require a sign determinant, and the following convention will answer our purpose in all cases in which no element occurs at the second surface of the plate. Let the distance $\mathrm{A}\,..\,a$ or v be considered a positive quantity when the rays emerging from the glass plate are diverging, as in Fig. 86*a*, and a negative quantity when the emergent rays are converging, as in Fig. 86*b*. Also, if the principal ray of the oblique pencil is diverging from the point where it crosses the optic axis, then let the distance D'' from such point on the left to

A sign determinant required.

the second surface be also considered a positive quantity. But if such point, when the principal ray cuts the optic axis, is to the right hand of the second surface, so that the principal ray emerges converging to the optic axis, then let the distance D'' in question be considered negative.

On referring back to Formula (47) it will be seen that we have $\frac{1}{v}$ on the left-hand side of the equation and $\frac{1}{v^3}$ on the other, so that if v is negative, then both sides become negative. Therefore we must regard the Formula (47) for the angular value of the coma as in itself always a positive quantity, as is the case with Formula (46), and the sign must be settled by a sign determinant in the form of $(v - \mathrm{D}'')$. We will now show how this device works out. In Fig. 86*a* let the optic axis be $\mathrm{O}_1 \ldots \mathrm{O}_1$; then the point where the principal ray $a \ldots \mathrm{R}$ cuts the axis $\mathrm{O}_1 \ldots \mathrm{O}_1$ is away to the left at s_1 at a + distance D'' from the second surface, which is greater than $\mathrm{A} \ldots a$ or v; therefore v–D'' is negative, and gives a negative sign to the angular coma, which is inward. Then let $\mathrm{O}_2 \ldots \mathrm{O}_2$ be the optic axis; then s_2 becomes the crossing point for the principal rays, while v remains as before, and v–D'' is now positive, while the coma is outward.

Next let the optic axis be considered to be at O_3–O_3; then s_3 becomes the crossing point for principal rays, and D'' is now minus, so that v–D'' is still positive, as is the coma, which is clearly outward.

Turning to Fig. 86*b*, if the optic axis is at $\mathrm{O}_1 \ldots \mathrm{O}_1$, then s_1 is the crossing point for principal rays, and D'' is positive, while v is negative, so that v–D'' is negative and the coma is inward. But if the optic axis is at $\mathrm{O}_3 \ldots \mathrm{O}_3$, then both v and D'' are negative; but D'' is greater than v, so that v–D'' is positive, and the coma has become positive.

This device covers the case of the parallel glass plate, supposing it is either a detached and independent unit in a lens system with an air-space on either side of it, or if it forms part of a convexo-plane or concavo-plane lens, in which case no element occurs at the second surface.

Case wherein an element occurs at the second surface.

But if, as is usual, an element does occur at the second surface, then we have only to refer to those data which have had to be worked out for the various elements in order to find a simple sign determinant in the form of $(u–\mathrm{D}')f$ for that element which occurs at the second surface. If the element is a collective one, then f is entered as positive, but if a dispersive one, then f must be entered negative, while the u and the D′ must be entered with those signs prefixed which have been already assigned in accordance with the conventions laid down on pages 148 and 149, Section VI.

Thus, then, when no element occurs at the second plane surface we have the formula—

Parallel Plane Plate.

Formula for angular coma with no element at second surface.

$$\frac{e \,.\,.\, f}{v} = 3 \tan \chi \frac{(\mu^2 - 1)t}{2\mu^3 v^3} A^2, \text{ with } (v\text{–D}'') \text{ as sign determinant; } \quad \text{XXIIIA.}$$

or if there is an element at the second plane surface, then

Same when element occurs at second surface.

$$\frac{e \,.\,.\, f}{v} = 3 \tan \chi \frac{(\mu^2 - 1)t}{2\mu^3 v^3} A^2, \text{ with } (u - \text{D}')f \text{ as sign determinant.} \quad \text{XXIIIB.}$$

The aberrations in the diagram are much exaggerated, for clearness, and the crossing points for principal rays are of course determined by formulæ of the first approximation only, all aberrations being ignored. In this way the point where the principal rays of pencils entering a lens system cross the axis (generally the stop centre or its image) is determined in the first instance; and supposing, as usual, that the angle made by the principal ray with the optic axis at the first element is ψ, then the angle χ which the same principal ray makes with any particular parallel plane plate may be obtained in the way described on page 179, Section VII., where it was shown that if n elements precede any given parallel glass plate, then

$$\tan \chi = \tan \psi \frac{\text{D}_1'\text{D}_2' \,.\,.\, \text{D}_n'}{\text{D}_1''\text{D}_2'' \,.\,.\, \text{D}_n''}, \text{ etc.;}$$

while the semi-aperture A of the pencil where it cuts the second surface of such parallel glass plate may be obtained in the manner described on page 103, for it is the same thing as the semi-aperture y for the axial pencil. Supposing there is an element at the second surface of any given parallel glass plate, and it is the nth element of the series, then

$$(A_n)^2 = \left(\frac{u_2 u_3 \,.\,.\, u_n}{v_1 v_2 \,.\,.\, v_{n-1}}\right)^2 A_1^2. \qquad (48)$$

If there is no element at the second surface, then the focal distance v of the emergent pencil may be specially assessed with respect to the second surface of the parallel plate, as also the focal distance (or D'') for the principal rays in accordance with Formula XXIIIA.

Application of the Formulæ for Coma to two Actual Lens Systems

We will now conclude this Section with two examples of the actual application of the formulæ for coma to two of the photographic lenses that we dealt with in Section VII.

Stellar Cooke lens of 43-in. focus.

First we will take the Series 1*c* Stellar Cooke Lens, whose curves and all other data were given on page 182, Section VII. Recapitulating, we have the formula for E.C.s in the abridged form (in secondary planes)—

$$\frac{\tan^2\phi}{2f}\frac{1}{(a-\beta)^2}\Big\{\mathrm{A}'-2(a-\beta)(\mathrm{C}')\Big\}, \qquad (49)$$

while we have seen that the formula for coma is

$$\frac{3\tan\phi}{4f^2}\frac{1}{a-\beta}\Big\{\mathrm{A}'-(a-\beta)\mathrm{C}'\Big\}A^2; \qquad (50)$$

from which it is seen that if we take the function 2C′ already worked out for the E.C.s, and divide by two, and multiply the whole formula by $\frac{3(a-\beta)}{2f}A^2$, and substitute $\tan\phi$ for $\tan^2\phi$, then we shall arrive at the angular comatic corrections for each lens or element in turn. Proceeding in this way we get, taking each element in succession (the actual sign of $a-\beta$ being indicated over each)—

First element.

$$\overset{\mathrm{E}_1}{+}\qquad (+\cdot00784-\cdot00254)\frac{3(a_1-\beta_1)}{2f_1}A_1^2\tan\phi=+\cdot02129\tan\phi.$$

Second element.

$$\overset{\mathrm{E}_2}{+}\qquad (+\cdot002597+\cdot003820)\frac{3(a_2-\beta_2)}{2f_2}A_1^2\left(\frac{u_2}{v_1}\right)^2=+\cdot02775\tan\phi.$$

Third element.

$$\overset{\mathrm{E}_3}{-}\qquad (+\cdot0002383+\cdot0025376)\frac{3(a_3-\beta_3)}{2f_3}A_1^2\left(\frac{u_2u_3}{v_1v_2}\right)^2=-\cdot07709\tan\phi.$$

Fourth element.

$$\overset{\mathrm{E}_4}{-}\qquad (+\cdot0000437-\cdot0007269)\frac{3(a_4-\beta_4)}{2f_4}A_1^2\left(\frac{u_2u_3u_4}{v_1v_2v_3}\right)^2=+\cdot02863\tan\phi.$$

Fifth element.

$$\overset{\mathrm{E}_5}{-}\qquad (+\cdot00046152+\cdot00157827)\frac{3(a_5-\beta_5)}{2f_5}A_1^2\left(\frac{u_2u_3u_4u_5}{v_1v_2v_3v_4}\right)^2=-\cdot006145\tan\phi.$$

Sixth element.

$$\overset{\mathrm{E}_6}{-}\qquad (+\cdot011511-\cdot013396)\frac{3(a_6-\beta_6)}{2f_6}A_1^2\left(\frac{u_2u_3u_4u_5u_6}{v_1v_2v_3v_4v_5}\right)^2=+\cdot00533\tan\phi.$$

E_1	$+\cdot02129 \tan\phi$	E_3	$-\cdot07709 \tan\phi$
E_2	$+\cdot02775$,,	E_5	$-\cdot00614$,,
E_4	$+\cdot02863$,,		$-\cdot08323 \tan\phi$
E_6	$+\cdot00533$,,		$+\cdot08300$,,
	$+\cdot08300 \tan\phi$		$-\cdot00023 \tan\phi$ = total for all elements.

Total angular coma for six elements.

This result implies a minute amount of inward coma at the oblique focus; but we have yet to work out and add in the parallel plate comatic corrections.

First Plate

For the first parallel plate we have the angular coma

$$= 3 \tan \chi_1 \frac{(\mu^2-1)t_1}{2\mu^3 u^3} A_2{}^2$$

with $(u_2 - D_2{}')f_2$ as sign determinant, which

$$= 3 \tan \psi \frac{D_1{}'}{D_1{}''} \frac{(\mu^2-1)t}{2\mu^3 u_2{}^3} A_1{}^2 \left(\frac{u_2}{v_1}\right)^2, \qquad (51)$$

wherein $\tan \psi = \tan \phi$, as the original object plane is infinitely distant. This formula gives $-\cdot000076175 A_1{}^2 \tan \phi$, $(u_2 - D_2{}')f_2$ being $(-)(+)$.

Second Plate

For the second parallel plate we have the angular coma

$$= 3 \tan \chi_2 \frac{(\mu^2-1)t_2}{2\mu^3 u^3} A_4{}^2$$

with $(u_4 - D_4{}')f_4$ as sign determinant, which

$$= 3 \tan \phi \frac{D_1{}'D_2{}'D_3{}'}{D_1{}''D_2{}''D_3{}''} \frac{(\mu^2-1)t}{2\mu^3 u_4{}^3} A_1{}^2 \left(\frac{u_2 u_3 u_4}{v_1 v_2 v_3}\right)^2, \qquad (52)$$

(which gives the result $-\cdot000005878 A_1{}^2 \tan \phi$, $(u_4 - D_4{}')f_4$ being $(+)(-)$.

Third Plate

For the third parallel plate we have the angular coma

$$= 3 \tan \chi_3 \frac{(\mu^2-1)t_3}{2\mu^3 v^3} A_6{}^2,$$

with $(u_6 - D_6{}')f_6$ as sign determinant, which

$$= 3 \tan \phi \frac{D_1{}'D_2{}'D_3{}'D_4{}'D_5{}'}{D_1{}''D_2{}''D_3{}''D_4{}''D_5{}''} \frac{(\mu^2-1)t_3}{2\mu^3 u_6{}^3} A_1{}^2 \left(\frac{u_2 u_3 u_4 u_5 u_6}{v_1 v_2 v_3 v_4 v_5}\right)^2, \qquad (53)$$

which gives the result $+\cdot 0000007215 A_1^2$, $(u_6 - D_6')f_6$ being $(+)(+)$. So we have

$$\begin{array}{ll}
-\cdot 000076175 A_1^2 & \text{for } L_1 \\
-\cdot 000005878 A_1^2 & \text{for } L_2 \\
\hline
-\cdot 000082053 A_1^2 & \text{for } L_1 + L_2 \\
+\cdot 000000722 A_1^2 & \text{for } L_3 \\
\hline
\end{array}$$

Total parallel plate corrections for three lenses.

$$\text{Total} = -\cdot 000081331 A_1^2 \tan\phi$$

$$\left.\begin{array}{r}\text{Add previous total} \\ \text{from the six elements}\end{array}\right\} = \underline{-\cdot 00023 A_1^2 \tan\phi}$$

Final total.

$-\cdot 000311 A_1^2 \tan\phi$ Total angular value of the coma at final focus.

On multiplying this result by (E.F.L.) $\tan\phi\, A_1^2$ we shall get the linear value of the inward coma at any angle ϕ from the axis.

Let $\tan\phi = \frac{1}{12}$ for about 5 degrees, $A_1 = 3$ inches (the full aperture was $6\frac{1}{2}$ inches), while the E.F.L. is 43 inches, then our multiplier is $(43)\left(\frac{1}{12}\right)(9) = 32\frac{1}{4}$, and $(-\cdot 00031)(32\frac{1}{4}) = -\cdot 01$ inch. This is more than the coma which was sensibly inward actually measured; indeed, at about 7 degrees from the axis there was no coma at all. The existence of just perceptible inward coma at from 1 to 6 degrees from the axis, its absence at about 7 degrees, to be superseded by more and more outward coma as 10 to 12 degrees was approached, was a characteristic which manifests itself in the final image of many such combinations, and is explained in exactly the same way as we explained the existence of zones of aberration. For besides the comatic corrections of the order $\tan\phi$, for which we have worked out the formulæ, there exist comatic corrections of higher orders whose formulæ will be more complex in inverse ratio to their relative numerical importance. Hence, if we refer back to Fig. 39 and let the curves represent two orders of comatic corrections which are left over at the final focal plane and are equal and opposite at any given distance from the axis, so as to bring about absence of coma at that point, then at a point somewhere between that neutral point and the axis there will occur a maximum of coma of the same character as the lower and most important order of coma for which we have worked out the formulæ, while outside of the neutral point the coma of the higher order will more and more prevail. In this case we have slight residual negative coma of the order $\tan\phi$ pitted against residual positive coma of the higher order $\tan^3\phi$, so that inside the neutral point slight inward coma prevails, and

Length of the coma at 5 degrees from the axis.

Positive coma of a higher order present.

outside of it outward coma prevails, and would show up much more strongly were not the effective aperture of the combination for oblique pencils largely reduced by the obliquity.

The Cooke Process Lens.

For our second illustration we will fall back upon the process lens, Fig. 59, whose radii, etc., and E.C.s are all given on pp. 185 and 186. Here again, in order to convert the eccentricity corrections for each element into comatic corrections, we must first halve the inside comatic E.C.s, and then multiply the whole aberration E.C.s plus half the comatic E.C.s by $\frac{3A^2(\alpha-\beta)}{2f}$, and substitute tan ϕ for $\tan^2\phi$; we then obtain the following comatic corrections for each element in turn:—

First element.

$$\overset{+\quad E_1}{(+\cdot0063848-\cdot0005360)}\frac{3(\alpha_1-\beta_1)}{2f_1}A_1^2\tan\phi=+\cdot051125A_1^2\tan\phi.$$

Second element.

$$\overset{-\quad E_2}{(+\cdot0018885+\cdot0082405)}\frac{3(\alpha_2-\beta_2)}{2f_2}A_1^2\left(\frac{u_2}{v_1}\right)^2\tan\phi=-\cdot15590A_1^2\tan\phi.$$

Third element.

$$\overset{+\quad E_3}{(+\cdot0000786-\cdot0017803)}\frac{3(\alpha_3-\beta_3)}{2f_3}A_1^2\left(\frac{u_2u_3}{v_1v_2}\right)^2\tan\phi=-\cdot088849A_1^2\tan\phi.$$

Fourth element.

$$\overset{-\quad E_4}{(+\cdot3032244-\cdot3826785)}\frac{3(\alpha_4-\beta_4)}{2f_4}A_1^2\left(\frac{u_2u_3u_4}{v_1v_2v_3}\right)^2\tan\phi=+\cdot81219A_1^2\tan\phi.$$

Fifth element.

$$\overset{+\quad E_5}{(+\cdot3120633-\cdot383254)}\frac{3(\alpha_5-\beta_5)}{2f_5}A_1^2\left(\frac{u_2u_3u_4u_5}{v_1v_2v_3v_4}\right)^2\tan\phi=-\cdot71453A_1^2\tan\phi.$$

Sixth element.

$$\overset{-\quad E_6}{(+\cdot0022287-\cdot0122244)}\frac{3(\alpha_6-\beta_6)}{2f_6}A_1^2\left(\frac{u_2u_3u_4u_5u_6}{v_1v_2v_3v_4v_5}\right)^2\tan\phi=+\cdot080489A_1^2\tan\phi.$$

E_1	$+\cdot051125$	E_2	$-\cdot15590$	
E_4	$+\cdot81219$	E_3	$-\cdot088849$	$-\cdot95928$
E_6	$+\cdot080489$	E_5	$-\cdot71453$	$+\cdot94380$
	$+\cdot943804$		$-\cdot95928$	Total $=-\cdot01548A_1^2\tan\phi$.

Total angular coma for six elements.

We have yet to add the three parallel plate corrections. In this case we supposed the rays constituting the pencils entering the first lens to be parallel; therefore $\tan\psi = \tan\phi$.

First Plate

The formula for the first parallel plate is therefore

$$3\tan\phi\,\frac{D_1'}{D_1''}\,\frac{(\mu^2-1)t_1}{2\mu^3 u_2^3}A_1^{\,2}\left(\frac{u_2}{v_1}\right)^2 \tag{54}$$

with $(u_2 - D_2')f_2$ as sign determinant, which gives us $-\cdot007877A_1^{\,2}\tan\phi$.

Second Plate

$$3\tan\phi\,\frac{D_1'D_2'D_3'}{D_1''D_2''D_3''}\,\frac{(\mu^2-1)t_2}{2\mu^3 u_4^2}A_1^{\,2}\left(\frac{u_2u_3u_4}{v_1v_2v_3}\right)^2 \tag{55}$$

with $(u_4 - D_4')f_4$ as sign determinant, which gives us $+\cdot002329A_1^{\,2}\tan\phi$.

Third Plate

$$3\tan\phi\,\frac{D_1'D_2'D_3'D_4'D_5'}{D_1''D_2''D_3''D_4''D_5''}\cdot\frac{(\mu^2-1)t_3}{2\mu^3 u_6^2}A_1^{\,2}\left(\frac{u_2u_3u_4u_5u_6}{v_1v_2v_3v_4v_5}\right)^2 \tag{56}$$

(with $u_6 - D_6')f_6$ as sign determinant $= +\cdot0000327A_1^{\,2}\tan\phi$.

Summing up we have—

for second plate	$+\cdot002329$
for third plate	$+\cdot000033$
	$+\cdot002362$

Total parallel plate corrections for three lenses.

Final total.

for first plate . .	$-\cdot007877$
for second and third plate	$+\cdot002362$
Total plate corrections	$-\cdot005515A_1^{\,2}\tan\phi$
Total from elements .	$-\cdot01548$,,
Final total .	$-\cdot02100A_1^{\,2}\tan\phi$

Supposing A_1 the semi-aperture $= \frac{1}{5}$th inch, and $\tan\phi = \cdot25$ (for about 14 degrees), and the E.F.L. $= 8\cdot5$ inches, then the linear amount of the inward coma at 14 degrees from the optic axis will be given by

Length of the coma at 14 degrees from the axis.

$(-\cdot021)\left(\frac{1}{4}\right)\left(\frac{1}{25}\right)(8\cdot5) = -\cdot00178$, a very minute amount of inward coma. As a matter of fact, there was a just perceptible inward coma at that angle of obliquity visible with a high-power eye-piece, while the lens showed unusually free from comatic aberration of higher orders.

It will be noticed that the parallel plate in the first lens gives a comatic effect about 3·5 times as strong as the much thicker second plate, owing to the fact that the rays are converging more strongly through the first plate than they are through the second plate.

These particular instances do not show relatively very strong comatic corrections for the parallel plates, and these might legitimately be neglected; but cases of much thicker lenses may sometimes occur, or cases in which the divergence or convergence of the rays through the plates is relatively very much stronger, leading to very serious comatic corrections which cannot be neglected. Such cases are, perhaps, the most likely to happen in microscope objectives; so that our formulæ for coma of the order tan ϕ would be incomplete without those applying to parallel plates.

Coma for parallel plates may often be ignored.

Coma at the Foci of Eccentric Oblique Pencils Reflected from a Spherical Mirror

In the case of the spherical reflector, we may occasionally have to deal with the eccentric oblique refraction of pencils, as occurs off the small concave or convex spherical mirror of the Gregorian or Cassegrain reflecting telescopes. Here again if we take Formula XX. and substitute -1 for μ, we then arrive at the formula—

$$\frac{e\,.\,.\,f}{\mathrm{V}} = A^2 \cdot \frac{3\tan\phi}{4\mathrm{F}^2} \cdot \frac{1}{(a-\beta)}\left\{a^2 - (a-\beta)a\right\}, \qquad \text{XXIV.}$$

Angular coma at foci of eccentric oblique reflected pencils.

which is the universal expression for the angular value of the comatic flare subtended at the vertex of the mirror; the vergency characteristics a and β being assessed in accordance with the usual conventions, and A being, as usual, the semi-aperture of the pencil where it impinges upon the mirror.

SECTION IX

DISTORTION AND RECTILINEARITY OF IMAGES—VON SEIDEL'S FIFTH CONDITION

WE now have to consider another very important condition which has to be fulfilled by any optical combinations that are designed to project on to a plane surface images of exterior objects which extend to many degrees from the optic axis, and are at the same time required to resemble the original in the sense that the linear distances of image points from the axis point shall be strictly proportional to the tangents of the angles that the corresponding points in the original subtend at the front apex of the lens or at any other point on the axis of the same. Fig. 87, Plate XIX., illustrates the ideally simple case of the projection of an image of the original flat object B..C on to a flat screen *b*..*c* by means of a pinhole P.

The simplest case of image projection by a pinhole.

Let A..*a* be the straight line drawn through the pinhole P perpendicular to both planes B..C and *b*..*c*, and we may regard it as the optic axis; let C, D, and B be three points in the original whose images are projected, in straight lines, to *c*, *d*, and *b*, their respective image points; then we have $\frac{a..b}{\mathrm{A..B}} = \frac{a..d}{\mathrm{A..D}} = \frac{a..c}{\mathrm{A..C}}$, and also the tangents $\frac{a..b}{a..\mathrm{P}} = \frac{\mathrm{A..B}}{\mathrm{A..P}}$, $\frac{a..d}{a..\mathrm{P}} = \frac{\mathrm{A..D}}{\mathrm{A..P}}$, and $\frac{a..c}{a..\mathrm{P}} = \frac{\mathrm{A..C}}{\mathrm{A..P}}$, so that we not only have a constant ratio between all radial distances in the image and all radial distances in the original, but also a constant equality between the tangents of angles subtended by points in the original and the tangents of angles subtended by the corresponding image points. And it is clear that these relations will continue to hold good whatever may be the ratio between what we may term the focal distances A..P and P..*a*.

The pinhole replaced by a thin lens.

Next we may suppose the pinhole to be enlarged, and a small and very thin collective lens to be inserted in it, after which we shall have the relationship $\frac{1}{P..a}=\frac{1}{F}-\frac{1}{A..P}$ if the most distinct image is to be projected on to $b..c$. But we are passing our narrow pencils of rays through the centre of the lens in this case, and, as we have seen in Section I., rectilinear projection ensues with reasonable accuracy throughout a very large angle of view. But let us go further and suppose that we have two separated lenses, as in Fig. 88, which we will suppose to be plano-convex with their convexities turned towards one another, and of equal powers.

Case of separated lenses or elements.

Let there be a screen or stop placed half-way between them to compel the effective pencils to cross the axis at S, the geometric centre of the system, and let the two conjugate focal distances $A..L_1$ and $L_2..a$ be equal, so that the image is equal to the original. Let B..C..S..D..E be the course of an oblique principal ray from B. Let p_1 be the first principal point, being the image of the stop centre S as formed by the lens L_1, and presented to outward view; and let p_2 be the second principal point or the image of the stop centre, similarly formed by the lens L_2. We are now going further than we did in Section I., and must therefore take notice of the spherical aberration of the two lenses, for we are supposing the angle of obliquity $\frac{A..B}{A..p_1}$ to be considerable, so that the ray B..C traverses L_1 and L_2 at a substantial distance from their centres. Under these circumstances it is clear that the image of S formed at p_1 or p_2 is subject to spherical aberration; the ray S..C (tracing it backwards) after refraction at C seems to proceed from q_1, and not from p_1; similarly, a ray S..D after refraction at D proceeds from q_2, and not from p_2. Now, under the circumstances of perfect symmetry prevailing in Fig. 88, this aberration obviously does not interfere in the least with the perfect similarity and equality existing between the image and the original; we have the ray B..C entering L_1 as if proceeding to q_1; after refraction at C it then proceeds through the stop centre S and cuts L_2 at D at a height from the axis equal to that of C in L_1; and after refraction there proceeds, as if from the point q_2, and strikes the screen at E, and E..A is exactly equal to B..A, since the two triangles Aq_1B and Eq_2a are equal and similar.

Stop placed at geometric centre.

How the problem is affected by spherical aberration.

The condition of symmetry.

But it is clear that the principal rays entering L_1 and the principal rays leaving L_2 are neither converging to nor diverging from the two definite and fixed principal points p_1 and p_2, although that may be practically true for principal rays very little inclined to the axis.

Hence our first important inference is that the radiation of principal rays from a definite principal point after passage, or their convergence to a definite principal point before passage, is not always a necessary condition for rectilinear projection. But we shall soon see that definite principal points are absolutely essential if we are to have the condition of rectilinear projection for *all* ratios of conjugate focal distances, and not merely for the one ratio which involves symmetry, and which in Fig. 88 is also one of equality.

Condition of universally correct projection.

Let Fig. 89 reproduce in exaggerated degree the case of Fig. 88. We have the original AB and its equal image a..E. p_1 and p_2 are the two principal points as fixed by formulæ of the first approximation, q_1 and q_2 the same as they appear by spherical aberration. B..C..S..D..E is the actual course of the principal ray, but B..b..S..d..E is the course which the ray would take were there no spherical aberration affecting the principal points, for before entering L_1 it would, if produced, pass through p_1, and after leaving L_2 would, if produced backwards, pass through p_2. This ideal course for the principal ray is shown as a dotted line. Let the actual course of the ray and the ideal course be produced away from the lenses beyond the object and image planes. Then we have the two courses intercrossing at B and at E, and there is no distortion with the conjugate focal planes in that position. But let the original plane object be removed farther back to F..Q, when the image will be formed in a new and nearer plane q..f, and we have not only unequal conjugate focal distances, but it is plain that we shall also have distortion. For, supposing that G, a point in the original, and its image point g, were both upon the dotted line of the ideal ray, then we should have no distortion, for G..b and d..g are by hypothesis parallel, they make equal angles with the axis with equal tangents, and radiate to and from fixed principal points. But the actual ray cuts the object plane at Q, inside of G, while the actual ray after passage cuts the image plane at q, outside of g. To a smaller original F..Q there corresponds a larger image q..f. Therefore if we suppose our original point Q to be coincident with G instead of inside it, then its image point will be transferred from q to r, still farther outside of g, and g..r will be the linear distortion or the deviation from the position of correct projection.

Discrepancy between ideal and real course of principal rays.

Linear amount of the distortion.

If on the plane F..G we have a series of true squares, like Fig. 90, then the image will be distorted into the form shown in Fig. 90*a*.

So we clearly see that if any lens is to be universally free from distortion, and not merely so under one condition of a certain ratio of conjugate focal distances, then not merely must there be a constant ratio (not necessarily equality) between the tangent of the angle made with the axis by the incoming principal ray and the tangent of the

The tangent condition.

PLATE. XIX.

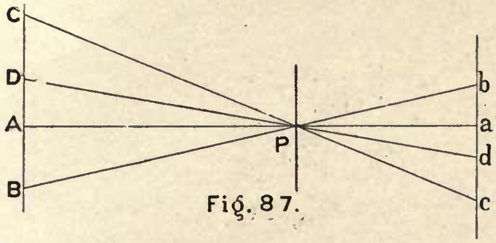


Fig. 87.

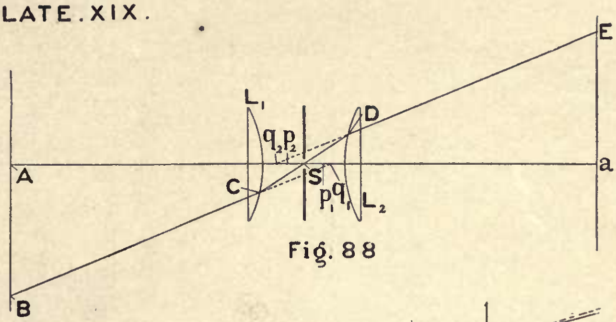


Fig. 88

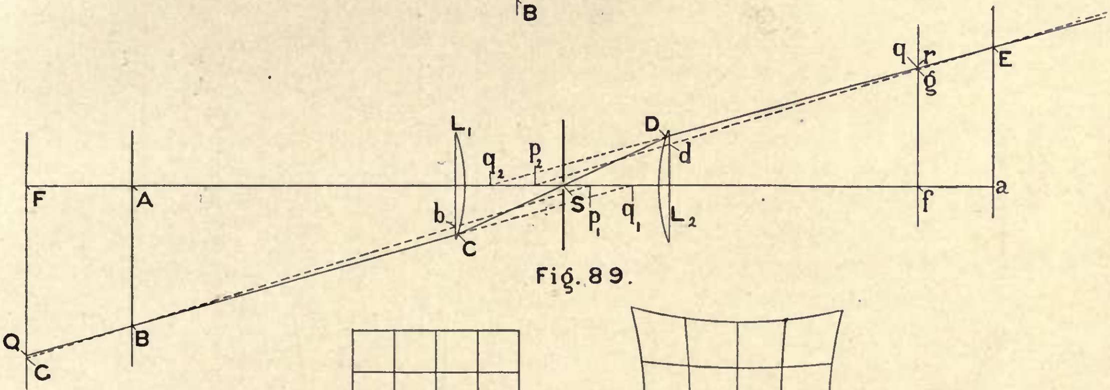


Fig. 89.

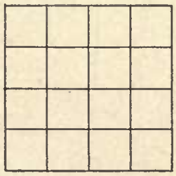


Fig. 90.

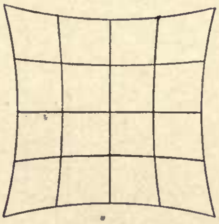


Fig. 90.a.

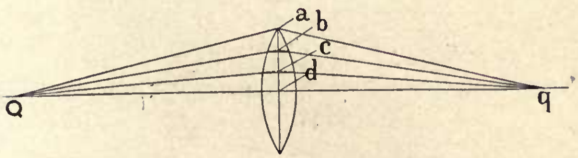


Fig. 91

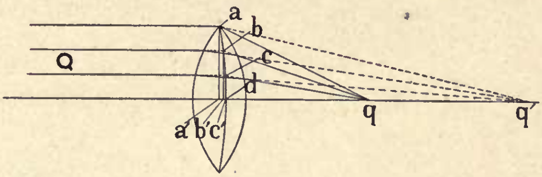


Fig. 92.

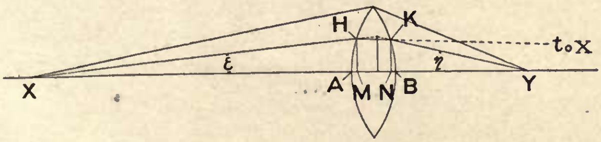


Fig. 93.

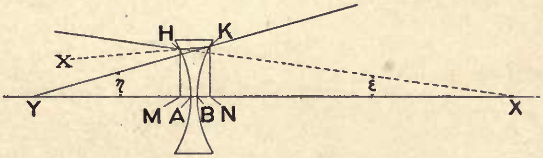


Fig. 93.a.

PLATE. XIX.

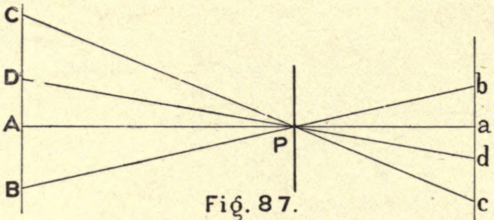


Fig. 87.

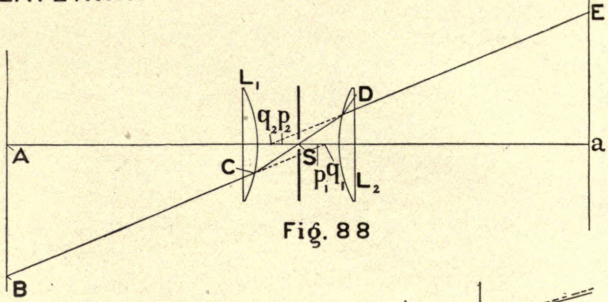


Fig. 88

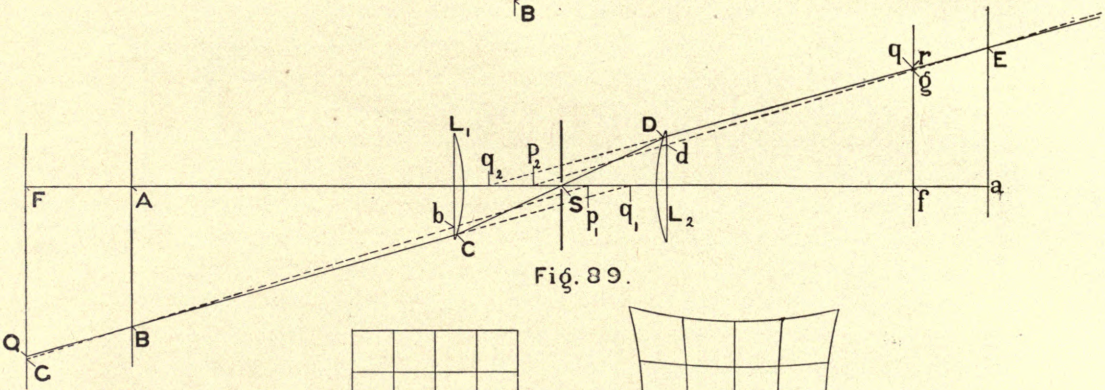


Fig. 89.

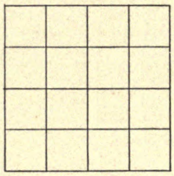


Fig. 90.

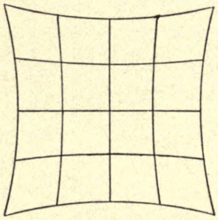


Fig. 90.a.

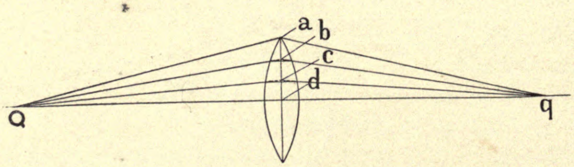


Fig. 91

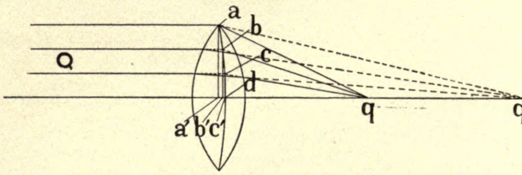


Fig. 92.

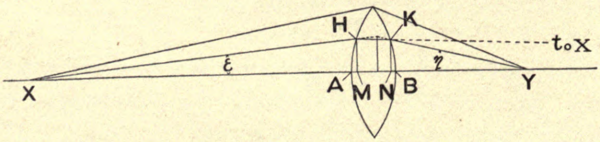


Fig. 93.

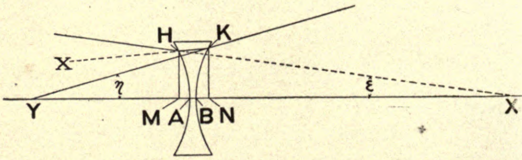


Fig. 93.a.

angle made with the axis by the same outgoing principal ray, but the incoming and the outgoing principal rays must alike be converging to or radiating from fixed points on the axis. And as such fixed points are always either the centres of stops themselves or else images of stop centres, as in Figs. 88 and 89, therefore we must have the images of such stop centres formed free from spherical aberration. In Figs. 88 and 89 the stop actually coincides with the geometric centre of the combination, and its two images p_1 and p_2 are therefore principal points; but as often as not the stop in a combination is not placed at the geometric centre, and therefore its images are not principal points, but are usually by Continental optical writers spoken of as pupil points, for they are points at the centres of apertures or their images to which or from which the principal rays of the pencils converge or diverge. But our above condition of freedom from distortion applies just as truly to such pupil points as to principal points; we must have aberration-free images of the stop, or pupil points, combined with a constant ratio of tangents of the angles made with the axis by the entering principal rays and the same principal rays when emergent. If the stop happens to coincide with the geometric centre, as in Fig. 89, then we have not merely a constant ratio of tangents, but equality of tangents and parallelism between the incoming and outgoing principal rays, so long as the two lenses, as in Fig. 89, are symmetrically shaped with respect to the point S.

Condition that the pupil points are aberration free.

Pupil points not necessarily principal points.

The ratio of the tangents of the angles made with the axis by the entering principal rays to the tangents of the angles made with the axis by the same outgoing principal rays, is a matter which can be legitimately considered on the supposition that there is no spherical aberration or that the formulæ of first approximation only strictly apply throughout the lens aperture.

Then the further effects of the spherical aberration may be investigated afterwards and the formulæ accordingly modified.

The Tangent Condition

Coddington's methods first employed.

Up to a certain stage we cannot here do better than follow the method and the notation employed by Coddington in his before-mentioned work, pages 121 to 131, although we shall find that it is possible to carry the processes further than he did, thereby arriving at results of greater simplicity and convenience in application. His methods were really founded upon or suggested by a certain paper on "The Spherical Aberration of Eye-pieces," published in the Cambridge

Philosophical Transactions, by Sir George Airy, the leading pioneer of British optical science. Let Fig. 91 represent an equiconvex lens under the condition of equal conjugate foci, spherical aberration being supposed absent. It is clear that, under these circumstances the rays enter and leave the lens under precisely the same conditions, the angles of incidence and emergence are equal, as are the angles of refraction within the glass, so that the course of the rays within the glass is parallel to the axis. Therefore it follows that if the entering and emergent rays are produced inwards, they must intersect one another exactly half-way between the two surfaces; that is, every incident ray will cut the corresponding emergent ray on a straight line passing through the sharp edge of the lens and perpendicular to the axis, cutting the latter at d, the centre of the lens. Clearly, then, tan $a\mathrm{Q}d = \tan$ aqd, tan $b\mathrm{Q}d = \tan$ bqd, and tan $c\mathrm{Q}d = \tan$ cqd, and a constant ratio, here equal to unity, prevails between the tangent of the angle made with the axis by the incoming ray and the angle made with the axis by the corresponding outgoing ray. That the locus of the intersection points of entering and emergent rays produced is a straight line passing through the sharp edge of the lens and perpendicular to the optic axis is clearly the necessary condition for this constancy of tangent ratios. But it is by no means always fulfilled. For instance, let it be supposed that the point Q is moved a very great distance away along the axis to the left, so that the entering rays become practically parallel, then we have the condition of things shown in Fig. 92. The parallel entering rays after refraction at the first surface converge within the glass to a point q' distant from the first vertex by three times the radius (if $\mu = 1{\cdot}5$), and then after refraction at the second surface converge to q, the final focus. If now we produce these exterior rays to intersect, we shall find they no longer intersect on a straight line, but on a circular curve $a \,.\,.\, b \,.\,.\, c \,.\,.\, d$, convex towards the focus q. Supposing the three entering rays strike the lens at heights 1, 2, and 3 from the axis, then we may regard the minute angles they make with the axis to have their tangents in the proportions 1, 2, and 3; but not so for the emergent rays, for we still have heights 1, 2, and 3 as the numerators in our tangents for angles cqd, bqd, and aqd, but the denominators are respectively $c' \,.\,.\, q$, $b' \,.\,.\, q$, and $a' \,.\,.\, q$, which vary considerably, so that tan cqd is $\frac{c \,.\,.\, c'}{c' \,.\,.\, q}$, and considerably in excess of one-third of tan aqd, which is $\frac{a \,.\,.\, a'}{a' \,.\,.\, q}$.

Condition of equal symmetry.

Consequence of altering vergency.

The tangent surface.

We will now investigate the formulæ expressing the relationship between the tangents of the angles of the rays entering and the same rays leaving a lens. Fig. 93 illustrates the case of a collective lens and Fig. 93*a* the corresponding case of a dispersive lens, both curves and thicknesses being exaggerated for clearness. The same notation and the same line of reasoning apply to both cases.

Notation.

Let $X..A = b$ $\quad B..Y = c$ $\quad H..M = y(= K..N$ approximately)
$A..B = t$ $\quad A..x = b'$ $\quad$ Angle $HXA = \epsilon$ $\quad$ Angle $KYB = \eta$
Radius of first surface $= r$, $\quad$ Radius of second surface $= s$.

Then we have $\tan\eta = \dfrac{K..N}{N..Y}$, and $\tan\epsilon = \dfrac{H..M}{M..X}$;

The tangent ratio.

$$\therefore \frac{\tan\eta}{\tan\epsilon} = \frac{K..N}{N..Y}\cdot\frac{M..X}{H..M} = \frac{K..N}{H..M}\cdot\frac{M..X}{N..Y},$$

Also

$$\text{vers}\,(A..M) = \frac{y^2}{2r} \qquad \text{vers}\,N..B = \frac{y^2}{2s}$$

$$\frac{K..N}{H..M} = \frac{x..N}{x..M} = \frac{b' - t + \dfrac{y^2}{2s}}{b' - \dfrac{y^2}{2r}} = \left(b' - t + \frac{y^2}{2s}\right)\left(\frac{1}{b'} + \frac{y^2}{2rb'^2}\right)$$

$$= 1 - \frac{t}{b'} + \frac{y^2}{2s}\frac{1}{b'} + \frac{y^2}{2r}\frac{1}{b'} - t\frac{y^2}{2r}\left(\frac{1}{b'}\right)^2,$$

The thickness may be neglected.

in which we may neglect functions of the thickness t (which is independent of y), especially as we shall eventually apply our formulæ to elements of no thickness and parallel plates in the case of having to deal with very thick lenses. Therefore we may write—

$$\frac{K..N}{H..M} = 1 + \frac{y^2}{2}\left(\frac{1}{r} + \frac{1}{s}\right)\frac{1}{b'}; \tag{1}$$

next

$$M..X = b + \frac{y^2}{2r} = b\left(1 + \frac{y^2}{2br}\right); \tag{2}$$

$$N..Y = c + \frac{y^2}{2s} = c\left(1 + \frac{y^2}{2cs}\right);$$

$$\therefore \frac{1}{N..Y} = \frac{1}{c}\left(1 - \frac{y^2}{2cs}\right); \tag{3}$$

$$\therefore \frac{M..X}{N..Y} = b\left(1 + \frac{y^2}{2br}\right)\frac{1}{c}\left(1 - \frac{y^2}{2cs}\right) = \frac{b}{c}\left\{1 + \frac{1}{2}\left(\frac{1}{br} - \frac{1}{cs}\right)y^2\right\}; \tag{4}$$

$$\therefore \frac{\text{K..N}}{\text{H..M}} \cdot \frac{\text{M..X}}{\text{N..Y}} = \left\{1 + \frac{y^2}{2b'}\left(\frac{1}{r} + \frac{1}{s}\right)\right\} \frac{b}{c}\left\{1 + \frac{y^2}{2}\left(\frac{1}{br} - \frac{1}{cs}\right)\right\};$$

$$\therefore \frac{\tan \eta}{\tan \epsilon} = \frac{b}{c}\left\{1 + \frac{y^2}{2}\frac{1}{b'}\left(\frac{1}{r} + \frac{1}{s}\right) + \frac{y^2}{2}\left(\frac{1}{br} - \frac{1}{cs}\right)\right\}$$

The tangent ratio.

$$\therefore \frac{\text{K..N}}{\text{H..M}} \cdot \frac{\text{M..X}}{\text{N..Y}} = \frac{b}{c}\left[1 + \frac{y^2}{2}\left\{\frac{1}{r}\left(\frac{1}{b} + \frac{1}{b'}\right) - \frac{1}{s}\left(\frac{1}{c} - \frac{1}{b'}\right)\right\}\right]. \qquad (5)$$

Here it is desirable to express b' in terms of μ, r, and b. We have

$$\frac{\mu}{b'} = \frac{\mu - 1}{r} - \frac{1}{b}; \therefore \frac{1}{b'} = \frac{\mu - 1}{\mu r} - \frac{1}{\mu b} = \frac{b(\mu - 1) - r}{\mu r b};$$

$$\therefore \frac{1}{b} + \frac{1}{b'} = \frac{\mu r + b(\mu - 1) - r}{\mu r b} = \frac{(\mu - 1)r + (\mu - 1)b}{\mu r b} = \frac{\mu - 1}{\mu}\left(\frac{1}{r} + \frac{1}{b}\right). \qquad (6)$$

Relatively to c and the second surface we also have

$$\frac{\mu}{b'} = \frac{1}{c} - \frac{\mu - 1}{s} = \frac{s - (\mu - 1)c}{cs} \text{ and } \frac{1}{b'} = \frac{s - c(\mu - 1)}{\mu c s};$$

$$\therefore \frac{1}{c} - \frac{1}{b'} = \frac{1}{c} - \frac{s - c(\mu - 1)}{\mu c s} = \frac{\mu s - s + c(\mu - 1)}{\mu c s} = \frac{\mu - 1}{\mu}\left(\frac{1}{c} + \frac{1}{s}\right); \qquad (7)$$

therefore on substituting we get

The tangent ratio in terms of μ, r, s, b, and c.

$$\frac{\tan \eta}{\tan \epsilon} = \frac{b}{c}\left[1 + \frac{\mu - 1}{2\mu}\left\{\frac{1}{r}\left(\frac{1}{r} + \frac{1}{b}\right) - \frac{1}{s}\left(\frac{1}{c} + \frac{1}{s}\right)\right\}y^2\right]. \qquad (8)$$

Remembering that the so-called rays that we are dealing with are principal rays, each of which is supposed to be the central ray of a pencil or cone of rays which is limited by an aperture of which X is the centre, we may now adopt the device described on page 149, Section VI., and use the characteristic β, substituting $\frac{1 + \beta}{2f}$ for $\frac{1}{b}$, and $\frac{1 - \beta}{2f}$ for $\frac{1}{c}$, assessing the signs of b and c according to the conventions there laid down, and we will also adopt the characteristic x for the shape of the lens, so that

The characteristics β and x introduced.

$$\frac{1 + x}{2(\mu - 1)f} = \frac{1}{r} \text{ and } \frac{1 - x}{2(\mu - 1)f} = \frac{1}{s}.$$

On substituting these values we find that Formula (8) works out to

The tangent ratio in final form.

$$\frac{\tan \eta}{\tan \epsilon} = \frac{b}{c}\left[1 + \frac{y^2}{4f^2}\frac{1}{\mu(\mu - 1)}\left\{(\mu + 1)x + (\mu - 1)\beta\right\}\right], \qquad \text{I.}$$

which we may briefly write $\frac{b}{c}\left\{1 + \frac{y^2}{4f^2}\text{T}'\right\}$.

When $(\mu+1)x+(\mu-1)a=0$, then the intersection points of rays entering the lens and the same rays leaving the lens all lie on a plane passing through the sharp edge of the lens, and the tangent condition is fulfilled.

The Effect of Spherical Aberration upon the Distortion

We may now consider the further addition to our formulæ consequent upon the introduction of the spherical aberration of the lens.

Figs. 94 and 94*a*, Plate XX., illustrate the case. The principal rays from or to X, instead of converging to or diverging from Z, as supposed before, really converge to or diverge from z owing to spherical aberration. Thus $z\,..\,\mathrm{Z}$ is the linear spherical aberration whose value is expressed shortly as $\frac{y^2}{8f^3}(\mathrm{A}')c^2$, if $y=\mathrm{H}\,..\,\mathrm{M}$ as before, and if, in the full value of A', β is substituted for a. Therefore the true value of $\mathrm{B}\,..\,z$ or c' is $c-\frac{y^2}{8f^3}(\mathrm{B}')c^2$, writing B' instead of A', because we are dealing with the question of the spherical aberration of *principal* rays; and so the true value of $\frac{1}{c'}$ or $\frac{1}{\mathrm{B}\,..\,z}$ is $\frac{1}{c}+\frac{y^2}{8f^3}\mathrm{B}'$, which may be written in the form $\frac{1}{c}\left\{1+\frac{c}{2f}\frac{y^2}{4f^2}\mathrm{B}'\right\}$, in which $\frac{c}{2f}$ may be expressed as $\frac{2f}{1-\beta}\cdot\frac{1}{2f}=\frac{1}{1-\beta}$;

$$\therefore\ \frac{1}{c'}=\frac{1}{c}\left\{1+\frac{1}{1-\beta}\frac{y^2}{4f^2}\mathrm{B}'\right\}.$$

On substituting this value of $\frac{1}{c}$, corrected in accordance with the spherical aberration in Formula I., we then get

$$\frac{\tan\eta}{\tan\epsilon}=\frac{b}{c}\left[1+\frac{y^2}{4f^2}\left\{\mathrm{T}'+\frac{1}{1-\beta}\mathrm{B}'\right\}\right], \tag{9}$$

which in full is

$$\left.\begin{aligned}\frac{\tan\eta}{\tan\epsilon}=\frac{b}{c}\Bigg[1+\frac{y^2}{4f^2}\frac{1}{\mu(\mu-1)}\Bigg(\left\{(\mu+1)x+(\mu-1)\beta\right\}+\frac{1}{1-\beta}\bigg\{\frac{\mu+2}{\mu-1}x^2\\ +4(\mu+1)\beta x+(3\mu+2)(\mu-1)\beta^2+\frac{\mu^3}{\mu-1}\bigg\}\Bigg)\Bigg],\end{aligned}\right\}\mathrm{II.}$$

Coddington's formula expressing ratio between tangents of emerging and entering principal rays.

in which b and c are the conjugate focal distances by first approximation, so that $\frac{1}{b}+\frac{1}{c}=\frac{1}{f}$ simply.

Coddington's formula not universal.

This is Coddington's formula for the relationships of $\tan\eta$ and $\tan\epsilon$ for one lens. We shall, however, soon see that it is not an universal formula, and will not interpret itself in all circumstances. In the case of Figs. 94 and 94a we have supposed X to be the point where the principal rays cross the optic axis, and the spherical aberration only affects the value of c by reducing it, therefore $\frac{\tan \eta}{\tan \epsilon}$ is increased in value by the aberration.

When crossing point of principal rays is defined after passage.

But let us suppose that the point where the principal rays cross the optic axis is defined *after* passage through the lens; let there be a stop at Z in the case of the collective lens instead of at X; then it will be A..X or b that will be reduced by spherical aberration, and $\frac{\tan \eta}{\tan \epsilon}$ should obviously suffer a decrease from the normal $\frac{b}{c}$. But it is clear that the value of β, if the stop were at Z, might be anything between -1 and $+1$, so that $\frac{1}{1-\beta}$ would still be of positive value, while we want a negative value in order to make $\frac{\tan \eta}{\tan \epsilon}$ less by the spherical aberration.

Formula varies according to position of the stop.

Coddington showed that in any case in which the crossing point of the principal rays is defined after passage, then $-\frac{1}{1+\beta}$ must be substituted for $\frac{1}{1-\beta}$ in Formula II., and this works out quite correctly.

Two or more lenses in succession.

He then proceeded to adapt the above Formula II. to the cases of two or more lenses in succession. In Fig. 95 let L_1 the first lens be receiving principal rays diverging from a point X_1 on the axis to the left, then after refraction they are subject to spherical aberration, and the ray figured above crosses the axis at z_1 instead of at Z_1 the ultimate focus, and passes on to the second lens L_2. It is clear that while $Z_1 \ldots z_1$ is a decrement to c_1 it is an increment to $A_2 \ldots Z_1$ or b_2. Therefore the statement of $\frac{\tan \eta_2}{\tan \epsilon_2}$ for the second lens needs modification in order to cover the variation of b_2 consequent on the variation of c_1. Coddington made the necessary correction, and thereby obtained the Formula IIA. which is applicable to two lenses in succession, such as a Huygenian or Ramsden eye-piece; but in extending the application to the case of a four-lens or erecting eye-piece, which was one of the main objects in view throughout his investigation of distortion, he made a strange omission.

Spherical aberration must be carried forward to following lenses.

For in his series of formulæ, while carrying the spherical aberration of L_1 through to L_2, that of L_2 through to L_3, and that of L_3 through to L_4, he omitted to carry the aberration of L_1 through L_2 on to L_3 and L_4; nor did he carry the aberration of L_2 through L_3 on to L_4.

PLATE. XX.

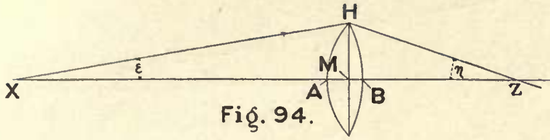


Fig. 94.

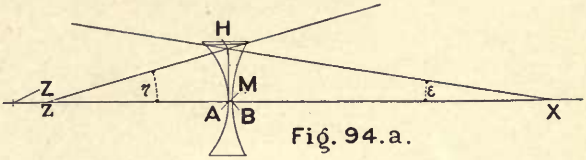


Fig. 94.a.

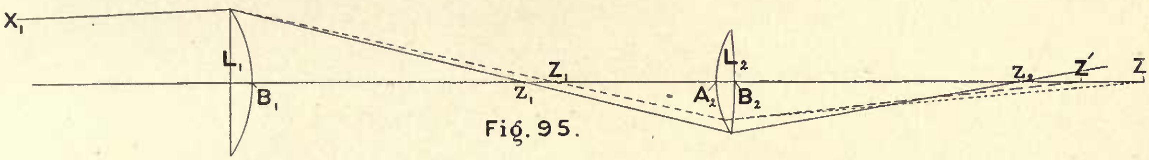


Fig. 95.

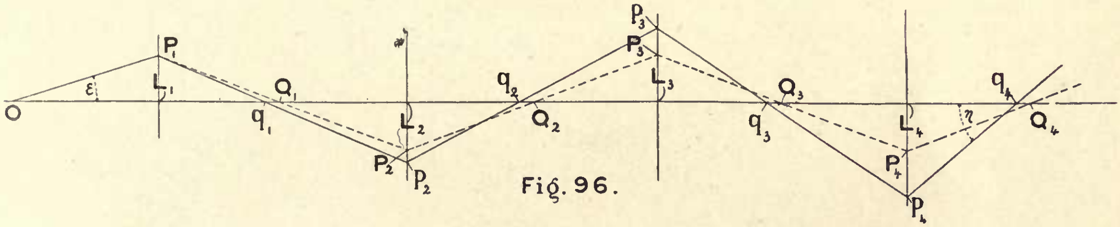


Fig. 96.

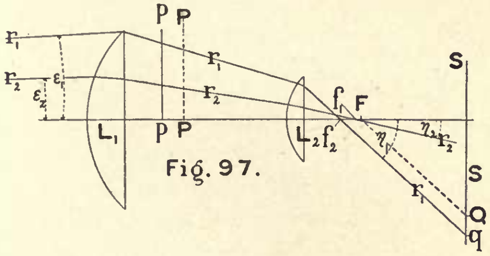


Fig. 97.

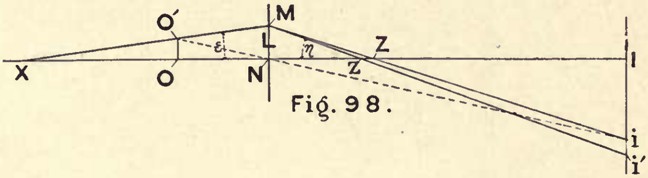


Fig. 98.

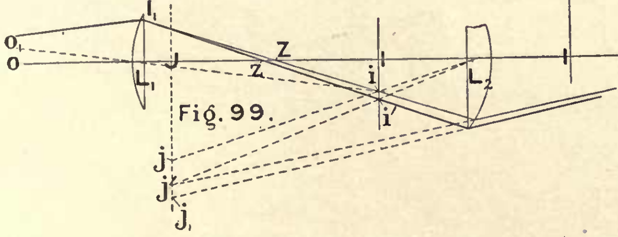


Fig. 99.

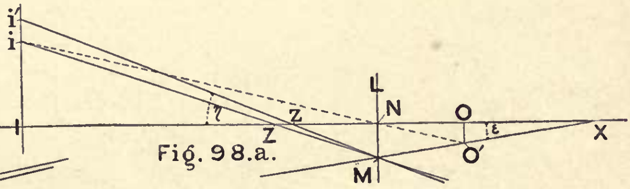


Fig. 98.a.

PLATE. XX.

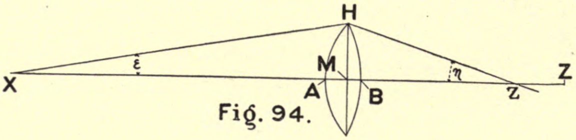


Fig. 94.

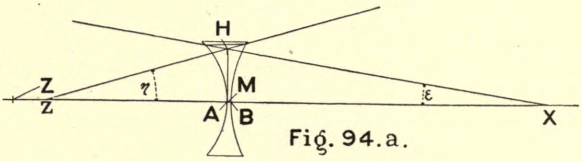


Fig. 94.a.

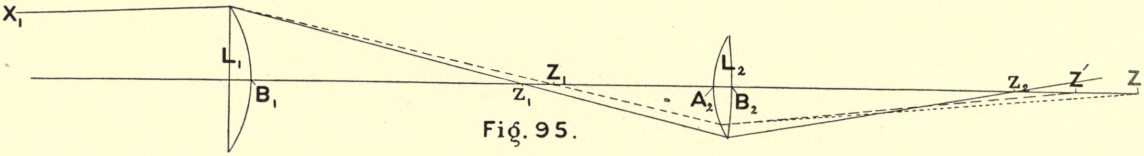


Fig. 95.

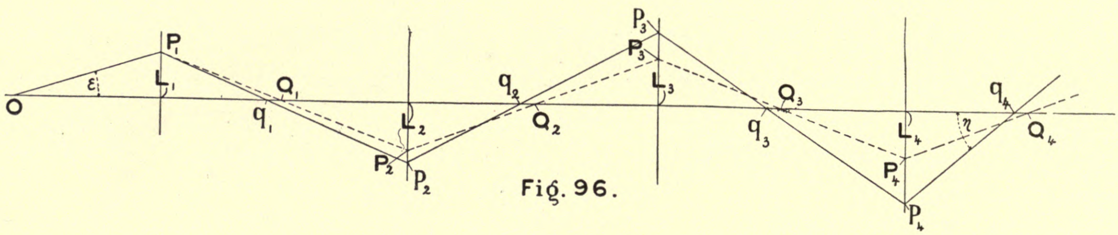


Fig. 96.

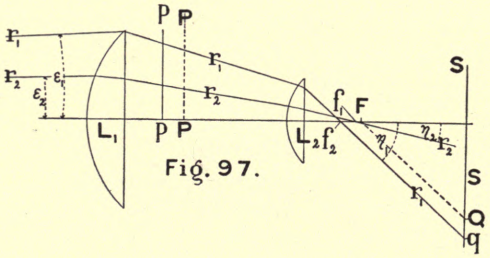


Fig. 97.

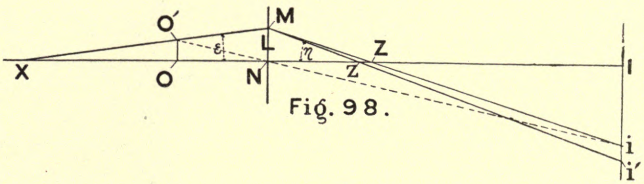


Fig. 98.

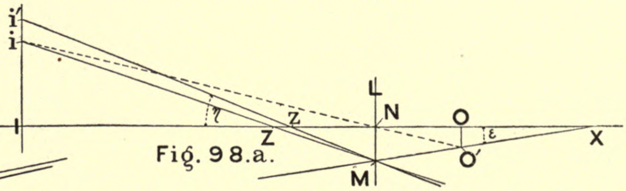


Fig. 98.a.

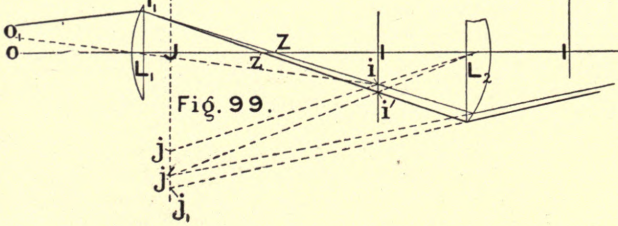


Fig. 99.

But the omitted operations can be shown to be as important and sometimes much more important than the processes which he retained. Fig. 96 represents a case which furnishes a capital illustration of the necessity for carrying the aberration of any one lens right through to the following lenses. Let there be four lenses, L_1, L_2, L_3, and L_4, all of equal focal lengths and equal separations, the latter being four times the focal length of any one of the lenses. Let it be supposed that a set of principal rays is radiating from a fixed point O at a distance in front of L_1 equal to twice its focal length. Let $O..P_1$ be one of these principal rays forming an angle ϵ_1 with the axis. Let it be supposed that all four lenses are quite free from aberration, and also that the tangent condition is fulfilled, so that the refractions all take place in one plane perpendicular to the optic axis and passing through the lens centres (equiconvex lenses are here implied). Then it is obvious that the course of the principal ray through the series is $O..P_1..Q_1..P_2..Q_2..P_3..Q_3..P_4..Q_4$, and what takes place at one lens is a repetition of what takes place at any other, and the emergent ray makes an angle η_4 with the axis equal to ϵ_1. Next, let it be supposed that a very slight spherical aberration is introduced in L_1, so that the principal ray $O..P_1$, instead of being refracted accurately to Q_1, is refracted to q_1, so that $Q_1..q_1$ is the linear aberration. Supposing this to be a small quantity, say 1 per cent of $L_1..Q_1$, then we have the ray striking the second lens plane at a height $L_2..p_2$ which will be 2 per cent greater than $L_2..P_2$. Then the image point of q_1 thrown by L_2 will obviously be q_2, and $Q_2..q_2$ will be very nearly equal to $q_1..Q_1$, as the conjugate focal distances are equal and the variation very small. Let $O..L_1 = u_1$, $L_1..Q_1 = v_1$, $L_1..q_1 = v'_1$, $q_1..L_2 = u'_2$, $L_2..q_2 = v'_2$, $L_2..P_2 = y_2$, and $L_2..p_2 = y'_2$.

Four lenses in succession.

All four lenses first supposed free from aberration.

Effect of introducing spherical aberration in first lens.

Then the increment to $L_3..P_3$ will be 4 per cent, and that of $L_4..P_4$ will be 6 per cent. But it is not our purpose to take notice of the variations in the y's in our functions of T' and B', because they involve corrections of a higher order, namely, of the order y_1^4. What we are chiefly concerned with are the *new* functions of B' and y^2 which have to be introduced in order to express the cumulative increment to $\tan\eta$, for evidently

Cumulative effect of the aberration of first lens.

$$\frac{\tan\eta_4}{\tan\epsilon_1} = \frac{u_1}{v'_1}\cdot\frac{u'_2}{v'_2}\cdot\frac{u'_3}{v'_3}\cdot\frac{u'_4}{v'_4} = \left(\frac{u_1}{v_1 - \cdot 01 v_1}\right)\left(\frac{u_2 + \cdot 01 u_2}{v_2 - \cdot 01 v_2}\right)\left(\frac{u_3 + \cdot 01 u_2}{v_3 - \cdot 01 v_3}\right)\left(\frac{u_4 + \cdot 01 u_4}{v_4 - \cdot 01 v_4}\right),$$

and on writing $u = v = 1$, the above becomes

$$(1 + \cdot 01)\{(1 + \cdot 01)(1 + \cdot 01)\}\{(1 + \cdot 01)(1 + \cdot 01)\}\{(1 + \cdot 01)(1 + \cdot 01)\},$$

$$\therefore \frac{\tan \eta_4}{\tan \epsilon_1} = (1 + \cdot 07),$$

or we may say that

$$\tan \eta_4 = \frac{y'_6}{v'_6} = \frac{y_6(1 + \cdot 06)}{v_6(1 - \cdot 01)} = \frac{y_6}{v_6}(1 + \cdot 07).$$

Hence Coddington's omission to transfer all the aberrations through the series is fatal to the accuracy of his formulæ for more than two lenses in succession. It will be as well, however, to repeat here his formula for two separated lenses in succession, which is quite correct although very unwieldy—

Coddington's distortion formula for two lenses in succession.

$$\left.\begin{aligned}\frac{\tan \eta}{\tan \epsilon} = \frac{b_1 b_2}{c_1 c_2}\Bigg[1 + \left\{ T_1' + \frac{1}{1-\beta_1}\left(1 + 2\frac{f_1}{f_2}\frac{1+\beta_2}{(1-\beta_1)(1-\beta_2)}\right)B_1' \right\}\frac{y_1^2}{4f_1^2} \\ + \left(T_2' + \frac{1}{1-\beta_2}B_2'\right)\frac{y_2^2}{4f_2^2}\Bigg].\end{aligned}\right\} \text{IIA.}$$

The student will find his formulæ for lenses in series dealt with on pages 162 to 172 of his work, and, after perusing the same, will be obliged to concede that, even as they stand, they are very complex and ill adapted for practical purposes, especially when any variations in the position of the limiting stop always render certain modifications necessary. If, however, the omitted functions for the transferred aberrations were also taken into account, then Coddington's formulæ for three or four lenses, when completed, would become unmanageably complex, or at any rate full of pitfalls for the unwary. This is essentially the case in a method which seeks to interpret distortion only in terms of the relationship between the tangents for finally emergent principal rays and the tangents for the same rays before entering.

Formulæ for three or more lenses highly complicated.

Application of Formula IIA. to a Huygenian eye-piece.

Let Fig. 97 represent a Huygenian eye-piece, for which Coddington's two-lens formula is quite correct. Let it be supposed that an objective away to the left is projecting a truly rectilinear image on to the plane P..P (if L_1 were not interposed). Let two principal rays from the centre of the objective be considered, one $r_1 \ldots r_1$ aiming for a point in the outskirts of the image, and one $r_2 \ldots r_2$ aiming for a point in the image very near the optic axis. After these two rays are refracted by L_1 they proceed, through a new and imperfect image formed at $p \ldots p$ in the principal focal plane of L_2, on to L_2, by which they are again refracted, $r_1 \ldots r_1$ to cross the axis at f_2, and $r_2 \ldots r_2$ at f_1; $F \ldots f_2$ being the linear spherical aberration. But the rays constituting the emergent pencils represented

Emergent rays parallel.

by these principal rays emerge in a very nearly parallel state, as if coming from an infinitely distant object, that being the state of the rays best adapted for distinct vision by the normal human eye. Therefore so long as tan η_1 bears the same ratio to tan ϵ_1 as tangent η_2 bears to tan ϵ_2, the eye will notice no distortion, and straight lines in the distant object will appear to the eye through the telescope as straight lines wherever they may occur in the field of view. That is what takes place when the functions of y^2 in Coddington's Formula IIA. equate to 0. But let us consider what will happen, supposing we no longer confine ourselves to receiving the emergent rays into the eye, but draw out the eye-piece with a view to throwing a real image of the object (the sun for instance) onto a white screen S..S at a little distance behind the eye-piece. It is clear that such an image will no longer be free from distortion. For the principal rays, although emerging in the right direction, as implied in the constancy of $\frac{\tan \eta}{\tan \epsilon}$, will be subject to a lateral displacement consequent on the aberration F..f_2. If they all radiated from F there would be no distortion on the screen S..S, and the ray r_1..r_1 would strike the screen at Q; but instead of that it strikes the screen at q, and Q..q is the linear distortion or displacement of the image point q from the correct position Q. The linear amount of this distortion Q..q varies as the cube of the distance from the axis. On an infinitely big image, either virtual or real, the absolute displacement Q..q is relatively a vanishing quantity; but relatively to the image formed on S..S it may be a very large quantity.

Lateral displacement of emergent principal rays.

Now the amount of linear spherical aberration of principal rays taking place in the case of a four-lens eye-piece is very much greater than in the case before us, and the student will find, what is well known to many opticians, that if he takes an erecting telescope free from distortion and directs it to an object containing straight lines, and then pulls out the eye-piece until it throws an image onto a ground glass screen a few inches behind the eye lens, he will then see that the positive distortion of the straight lines, or pincushion distortion as it is often called, is very marked.

An experiment with a four-lens eye-piece.

On the other hand, let an extremely short-sighted person use the same telescope on the same object. He requires a virtual image a few inches from his eye to be formed, and therefore pushes the eye-piece nearer to the objective than its normal position; when he will see all the straight lines distorted in the opposite sense, for there will be strongly marked negative or barrel-shaped distortion.

Formulæ of greater scope required.

It is quite plain, then, that Coddington's formulæ are quite inadequate to deal with cases in which real or virtual images are formed at finite distances, instead of at infinite distances. We therefore require formulæ of perfectly general application, and the following lines of reasoning will guide us to what we want, as well as lead to much greater simplicity. So far, all that has been taken into account is, first, that the rays constituting pencils finally emerging shall be parallel as though emanating from an infinitely distant image, and, second, the constancy or otherwise of $\frac{\tan \eta}{\tan \epsilon}$ for principal rays traversing the system at varying heights from the optic axis, and therefore traversing the several lenses at varying degrees of obliquity.

Extension of the Inquiry

As yet the positions of the planes where the various real or virtual images are formed have not been properly considered. Let Fig. 98 represent a collective lens L, placed behind a real image O . . O′, such real image being projected without any distortion from X, which point may perhaps mark the centre of a telescope objective, and is thus the point on the optic axis from which the principal rays of the pencils going to form the image O . . O′ radiate. Let the distance from O to the lens be greater than the P.F.L. of the lens, so that it projects another real image of O . . O′ at I . . i'. Then as X . . N is greater than O . . N, therefore the focal point Z conjugate to X will be nearer to the lens than I . . i'. Supposing Z is the ultimate point by first approximation, then z is the real point where the principal ray XMz crosses the axis before proceeding to i', and Z . . z is the linear aberration.

Let Fig. 98a represent the corresponding case of a dispersive lens, exactly the same notation applying. It is best always to choose for our typical examples cases in which all the quantities are conventionally positive. What we now want is a formula expressing the relationship between the size of the image I . . i' and the size of the original image O . . O′ presented to the lens. That is, we want to find out by how much the ratio between the radial dimensions of the two images as painted by the eccentric principal ray X . . M . . i' departs from constancy or from the ideal or normal relationship expressed by $\frac{v}{u}$.

Relationship between the sizes of the images.

Here we are assuming that the conjugate focal distances b and c for principal rays are measured from the point N, the axial point of the tangent surface M..N. We must next inquire, from what point must the conjugate focal distances u and v be measured, if aberration-free refraction of the principal rays at M in the tangent surface is to lead to rectilinear projection or an image of O..O′ that is free from distortion? Is N the required centre of projection?

Must u and v be measured from N?

The theorem that a lens through which are refracted a system of eccentric pencils, which fulfils the tangent condition and is free from spherical aberration, also fulfils the condition of central projection through the point N, may be proved algebraically thus—

If Formulæ I. and II. equate to 0, then is central projection implied?

In Fig. 99*b* let N..M be a lens fulfilling the tangent condition for a system of principal rays radiating from Q. That being the case, then all refractions of such principal rays will virtually take place in the plane M..N. Let the lens also be aberration free for all distances, so that the law of conjugate focal distances by first approximation will strictly hold good.

Let F = the principal focal length of the lens L. Let Q..N $= b$, and let q be the focus conjugate to Q, so that

$$\frac{1}{\mathrm{N}..q} \text{ or } \frac{1}{c} = \frac{1}{\mathrm{F}} - \frac{1}{b}.$$

Construction.

Let $p..d$ be a plane image or object placed anywhere between Q and L, and perpendicular to the axis. Then let p be a point in such plane image which also lies upon the principal ray Q..M. From p draw the straight line p..N through the centre N of the tangent surface, and produce it onwards until it intersects the refracted principal ray M..q..g at g. From g draw $g..f$ perpendicular to the axis.

Let the distance N..f be v, and d..N be u. Assuming N to be the centre of projection, then the question is, what must be the relationship between v and u?

Since the point g is on the line of projection from the original p through the centre N of the tangent surface,

$$\therefore\ f..g = (p..d)\frac{v}{u} \text{ or } \mathrm{O}\frac{v}{u}, \text{ if } p..d = \mathrm{O};$$

$$f..g \text{ also} = (v - c)\frac{\mathrm{Y}}{c}, \text{ if } \mathrm{Y} = \mathrm{M}..\mathrm{N}.$$

Therefore we get

$$\mathrm{O}\frac{v}{u} = (v - c)\frac{\mathrm{Y}}{c}, \text{ in which } \mathrm{Y} = \mathrm{O}\frac{b}{b - u};$$

$$\therefore\ \mathrm{O}\frac{v}{u} = (v - c)\frac{\mathrm{O}\dfrac{b}{b-u}}{c}; \qquad (10)$$

$$\therefore \frac{v}{u} = \frac{v-c}{b-u} \cdot \frac{b}{c}, \text{ and } v = \frac{v-c}{b-u} \cdot \frac{b}{c} \cdot u;$$

$$\therefore v = \frac{vbu}{c(b-u)} - \frac{cbu}{c(b-u)}; \tag{11}$$

$$\therefore v\{c(b-u) - bu\} = -cbu,$$

$$v = -\frac{cbu}{c(b-u)-bu}, \text{ and } \frac{1}{v} = -\frac{c(b-u)-bu}{cbu},$$

in which expression we may put

$$c = \frac{1}{\frac{1}{F} - \frac{1}{b}} = \frac{bF}{b-F},$$

and then we have

$$\frac{1}{v} = \frac{-\frac{bF}{b-F}(b-u) + bu}{\frac{bF}{b-F}bu}$$

$$= \frac{-b^2F + b^2u}{b^2uF} = \frac{b^2(u-F)}{b^2uF};$$

N is proved to be the common reference point for conjugate distances u and v as well as b and c.

$$\therefore \frac{1}{v} = \frac{1}{F} - \frac{1}{u}. \tag{12}$$

So that the simple law of conjugate focal distances, connecting Q . . N and N . . q (or b and c) for the principal rays, co-operating with rectilinear central projection through N for the corresponding image points p and g, also satisfies the same simple law of conjugate focal distances for the two image distances u and v; that is, a distortion-free lens forms its image in strict conformity with the condition of rectilinear projection through the centre of the tangent surface.

Effect of the separation between the principal points.

We have now to inquire whether the above line of reasoning will apply to a lens having appreciable central thickness, that is, will the above theorem apply when the lens thickness is such as to lead to very appreciable separation between the two principal points? In order to answer this question we must know how the point N is situated with respect to the principal points.

Fig. 99c represents a thick collective lens. The tangent surface is of course the plane containing the sharp edge M of the lens, and N is the axial point of the same. C is the geometric centre of the lens, and p_1 and p_2 are the two principal points, while a_1 and a_2 are the two vertices, the radii being r and s as usual.

Formulæ IV. and V., page 14, fix the positions p_1 and p_2, or the distances $a_1 \,.\,.\, p_1$ and $a_2 \,.\,.\, p_2$, as respectively

$$\frac{tr}{\mu(r+s) - t(\mu-1)} \text{ and } \frac{ts}{\mu(r+s) - t(\mu-1)}.$$

Now as $t(\mu - 1)$ is generally a very small quantity compared to $\mu(r+s)$, representing as it does the very small effect upon the positions of p_1 and p_2 exercised by the refraction of the two curved surfaces as compared to two plane surfaces, we may legitimately omit it and write

$$a_1 \,.\,.\, p_1 = \frac{t}{\mu}\frac{r}{r+s} \text{ and } a_2 \,.\,.\, p_2 = \frac{t}{\mu}\frac{s}{r+s};$$

$$\therefore\ p_1 \,.\,.\, p_2 = t - \{(a_1 \,.\,.\, p_1) + (a_2 \,.\,.\, p_2)\} = t\frac{\mu - 1}{\mu}.$$

Then we have $a_2 \,.\,.\,$N obviously $= t\dfrac{r}{r+s}$, therefore

$$(a_2 \,.\,.\, \mathrm{N}) - (a_2 \,.\,.\, p_2) \text{ or } \mathrm{N} \,.\,.\, p_2 = t\frac{r}{r+s} - \frac{t}{\mu}\cdot\frac{s}{r+s};$$

$$\therefore\ \mathrm{N} \,.\,.\, p_2 = \frac{t}{\mu}\cdot\frac{\mu r - s}{r+s}, \tag{13}$$

and

$$\frac{\mathrm{N} \,.\,.\, p_2}{p_1 \,.\,.\, p_2} = \frac{t}{\mu}\cdot\frac{\mu r - s}{r+s} \div \frac{t(\mu-1)}{\mu} = \frac{\mu r - s}{(\mu-1)(r+s)}, \tag{14}$$

which expresses the proportion borne by $\mathrm{N} \,.\,.\, p_2$ to the separation $p_1 \,.\,.\, p_2$ between the principal points.

This formula can be written in a more convenient form, in terms of x,

Position of N with reference to the principal points.

$$\frac{\mathrm{N} \,.\,.\, p_2}{p_1 \,.\,.\, p_2} = \frac{(\mu-1) - (\mu+1)x}{2(\mu-1)}. \tag{15}$$

Now let it be supposed that the two conjugate focal distances for principal rays b and c bear the same ratio to one another as $p_1 \,.\,.\,$N to $\mathrm{N} \,.\,.\, p_2$, and therefore that

$$\frac{c}{b+c} = \frac{\mathrm{N} \,.\,.\, p_2}{p_1 \,.\,.\, p_2},$$

so that

$$\frac{1+\beta}{2} = \frac{(\mu-1) - (\mu+1)x}{2(\mu-1)};$$

from this we get

$$\beta = \frac{(\mu-1) - (\mu+1)x}{\mu - 1} - 1,$$

and therefore

$$\beta = -\frac{\mu+1}{\mu-1}x. \tag{16}$$

But we have seen that the tangent condition is fulfilled when $(\mu-1)\beta+(\mu+1)x=0$, which is the same thing.

Point N distant from the principal points in proportion to b and c.

The conclusion is, then, that when the tangent condition is fulfilled the tangent surface cuts the optic axis so as to divide the distance between the principal points into two portions $p_1..\mathrm{N}$ and $\mathrm{N}..p_2$ respectively, proportional to b and c. Therefore if two principal planes are drawn through the two principal points (Fig. 99*b*) parallel to M..N they will obviously be cut by Q..M and M..g at equal heights. Also, by the law of principal points, the ray $p_2'..g'$ through the second principal point is parallel to the ray $p'..p_1$ through the first principal point. Therefore the conjugate distances b and u on the one hand and c and v on the other hand will be measured from the principal planes. So that if we suppose the gap between the two principal planes to be closed up by sliding the two halves of the diagram into one another, as it were, we then arrive at the state of things first assumed in our inquiry, for p_1' and p_2' will become merged in N, while k_1' and k_2' will be simultaneously merged in a common point M. The only difference made by the thickness, if not excessive, is in the value of $\frac{1}{\mathrm{F}}$, but the equation $\frac{1}{b}+\frac{1}{c}=\frac{1}{u}+\frac{1}{v}$ of course always holds good, and we still have the equivalent of central projection of the image through the point N. Thus in Fig. 99*b* the dotted lines and accented letters indicate the state of things when the separation between the two principal points is allowed for, and the full lines and unaccented letters the state of things when the gap between the principal planes is closed up.

Above theorems therefore apply to the two principal planes.

The thickness only alters value of F.

Formula for tangent condition fairly accurate for thick lenses.

It will now be seen that, with regard to the fulfilment of the tangent condition or any departures from it, it is scarcely necessary to the attainment of accuracy to treat a thick lens by elements, although it becomes desirable to do so when the thickness becomes excessive, for the refractive effect of the curved surfaces (as compared with flat surfaces) upon the linear positions of the principal points grows as the square of the thickness, and leads to the above theorems becoming inapplicable.

Let us now revert to Figs. 98 and 98*a*; and, as usual, let

$$\mathrm{X..N}=b \text{ and } \mathrm{N..Z}=c,$$
$$\mathrm{M..N}=y,\ \mathrm{O..N}=u, \text{ and } \mathrm{N..I}=v,$$
$$\angle \mathrm{MXN}=\epsilon \text{ and } \angle \mathrm{M}z\mathrm{N}=\eta,$$

and let

$$X\,..\,O = d_1, \text{ and } z\,..\,I = d_2.$$

Then we have

$$I\,..\,i' = d_2 \tan \eta, \text{ and } O\,..\,O' = d_1 \tan \epsilon\,;$$

$$\therefore \quad \frac{I\,..\,i'}{O\,..\,O'} = \frac{d_2 \tan \eta}{d_1 \tan \epsilon}, \tag{17}$$

wherein

$$d_2 = v - c + \frac{y^2}{8f^3}B'c^2, \text{ and } d_1 = b - u\,;$$

$$\therefore \quad \frac{I\,..\,i'}{O\,..\,O'} = \frac{v - c\left(1 - \frac{y^2}{8f^3}B'c\right)}{b-u} \cdot \frac{\tan \eta}{\tan \epsilon}$$

$$= \frac{v}{b-u}\left\{1 - \frac{c}{v}\left(1 - \frac{1}{1-\beta}B'\frac{y^2}{4f^2}\right)\right\}\frac{\tan \eta}{\tan \epsilon}, \tag{18}$$

in which we may next insert the value of $\frac{\tan \eta}{\tan \epsilon}$ already worked out, and which was expressed shortly in Formula (9) as

$$\frac{\tan \eta}{\tan \epsilon} = \frac{b}{c}\left\{1 + \frac{y^2}{4f^2}\left(T' + \frac{1}{1-\beta}B'\right)\right\}\,;$$

so that Formula (18) amplifies to

$$\frac{I\,..\,i'}{O\,..\,O'} = \frac{bv}{c(b-u)}\left\{1 - \frac{c}{v}\left(1 - \frac{1}{1-\beta}B'\frac{y^2}{4f^2}\right)\right\}\left\{1 + \frac{y^2}{4f^2}\left(T' + \frac{1}{1-\beta}B'\right)\right\}, \tag{19}$$

in which we may now, following Coddington's useful device, substitute $\frac{2f}{1+\alpha}$ for u, $\frac{2f}{1-\alpha}$ for v, $\frac{2f}{1+\beta}$ for b, and $\frac{2f}{1-\beta}$ for c, on which $\frac{bv}{c(b-u)}$ becomes $\frac{(1+\alpha)(1-\beta)}{(1-\alpha)(\alpha-\beta)}$, and $\frac{c}{v}$ becomes $\frac{1-\alpha}{1-\beta}$.

On substituting these values in Equation (19) we then get

$$\frac{I\,..\,i'}{O\,..\,O'} = \frac{(1+\alpha)(1-\beta)}{(1-\alpha)(\alpha-\beta)}\left\{\left(1 - \frac{1-\alpha}{1-\beta}\right) + \frac{1-\alpha}{(1-\beta)^2}B'\frac{y^2}{4f^2}\right\}\left\{1 + \frac{y^2}{4f^2}T' + \frac{y^2}{4f^2}\frac{1}{1-\beta}B'\right\}$$

$$= \frac{(1+\alpha)(1-\beta)}{(1-\alpha)(\alpha-\beta)}\left\{\frac{\alpha-\beta}{1-\beta} + \frac{1-\alpha}{(1-\beta)^2}B'\frac{y^2}{4f^2} + \frac{\alpha-\beta}{1-\beta}\frac{y^2}{4f^2}T' + \frac{\alpha-\beta}{1-\beta}\frac{y^2}{4f^2}\frac{1}{1-\beta}B'\right\},$$

which, if we neglect functions of $\frac{y^4}{f^4}$,

$$= \frac{(1+\alpha)}{(1-\alpha)}\left(\frac{1-\beta}{\alpha-\beta}\right)\left\{\frac{\alpha-\beta}{1-\beta} + \frac{\alpha-\beta}{1-\beta}\frac{y^2}{4f^2}T' + \frac{(1-\beta)}{(1-\beta)^2}B'\frac{y^2}{4f^2}\right\}$$

$$= \left(\frac{1+\alpha}{1-\alpha}\right)\left\{1 + \frac{y^2}{4f^2}T' + \frac{1}{\alpha-\beta}B'\frac{y^2}{4f^2}\right\},$$

in which
$$\frac{1+a}{1-a}=\left(\frac{1+a}{2f}\right)\left(\frac{2f}{1-a}\right)=\frac{v}{u};$$

so that the formula finally becomes
$$\frac{\mathrm{I}\,..\,i'}{\mathrm{O}\,..\,\mathrm{O}'}=\frac{v}{u}\left\{1+\left(\mathrm{T}'+\frac{1}{a-\beta}\mathrm{B}'\right)\frac{y^2}{4f^2}\right\}, \tag{20}$$

which in full is

Universal formula for distortion of image.

$$\left.\frac{v}{u}\left[1+\frac{1}{\mu(\mu-1)}\left(\left\{(\mu+1)x+(\mu-1)\beta\right\}+\frac{1}{a-\beta}\left\{\frac{\mu+2}{\mu-1}x^2 + 4(\mu+1)\beta x+(3\mu+2)(\mu-1)\beta^2+\frac{\mu^3}{\mu-1}\right\}\right)\frac{y^2}{4f^2}\right]\right\}\text{III.}$$

Thus we find that the change required in the formula for $\frac{\tan\eta}{\tan\epsilon}$ in order to convert it into a statement of the ratio between the radial dimensions of the two conjugate images is an unexpectedly simple one, involving the simple substitution of $\frac{1}{a-\beta}$ for $\frac{1}{1-\beta}$ in the spherical aberration function, and $\frac{v}{u}$ for $\frac{b}{c}$. If the reader will pursue the same process in the case of X being nearer the lens than O .. O′, the case of the stop being placed behind the lens, or any other case he likes to choose, he will arrive at the same formula; in fact, it is quite universal and interprets itself in all cases.

Applications of Formula III. to Combinations of Lenses

We will now show how this formula simplifies the problem of arriving at the distortion produced by a series of separated or non-separated lenses in succession, even when employed for projecting real images on to plane surfaces at finite distances.

Two separated lenses.

Let Fig. 99 represent two lenses in succession placed in alignment behind either a real plane object $\mathrm{O}\,..\,\mathrm{O}_1$ or an image projected by another lens. Let it be supposed that the lenses are very thin, and that the principal rays cross the axis somewhere about z, and then proceed to intersect the conjugate focal plane $\mathrm{I}\,..\,i'$ where an image (in this case inverted) of $\mathrm{O}\,..\,\mathrm{O}_1$ is formed. From O_1 draw $\mathrm{O}_1\,..\,\mathrm{L}_1\,..\,i$ straight through the lens centre, then i in the plane $\mathrm{I}\,..\,i'$ will be the correct place for the image of the point O_1 to be formed if there is no distortion; but owing to the operation of distortion the image of the point O_1 is really formed at i', and $i\,..\,i'$ is the linear distortion; which, for example, may be 10 per cent of the correct radial dimension I . . i, which latter, of course, $=(\mathrm{O}\,..\,\mathrm{O}_1)\frac{v_1}{u_1}$. This exaggerated radial dimension I . . i' is then presented as an image to the lens L_2. It is clear that if L_2 is so circumstanced as to form an image of I . . i' without in itself exercising any distorting effect, then if we draw a straight line from the centre of L_2 through i' to cut the conjugate focal plane J . . j' at j', then j' becomes the image point of the point i', whereas the image of the true point i would be thrown to j; therefore $j\,..\,j'$ is the correct projection or image of the linear distortion $i\,..\,i'$; that is, the lens L_2 will simply form a correct image of what is presented to it if it is free from distortion, while if it does exercise any distortion itself, it is obvious that it will add its own distortion, $j'..j_1$, for instance, to that which is already presented to it. If the two distortions are of opposite signs and equal, then the final image will, of course, be a true image of the original.

Our Formula III. simply represents an increment or decrement to the ideal radial distance from the optic axis of any image point located or defined by a principal ray passing through the lens at a given height y from the axis, and is therefore quite independent of the sign of the lens; in fact, $\frac{1}{f^2}$ is always positive, and the sign of the lens is really always implied in the term $\frac{1}{a-\beta}$ in the spherical aberration function, and in β in the function of the tangent condition. Therefore the distortion functions involving y^2 for a series of lenses will be the simple sum of the distortion functions for the individual lenses. In the case of two lenses, we have the image to object ratio for the first lens

Simple summation of the distortion formulæ for a series of lenses.

$$\frac{\mathrm{I}\,..\,i'}{\mathrm{O}\,..\,\mathrm{O}'}=\frac{v_1}{u_1}\left\{1+\left(\mathrm{T}_1'+\frac{1}{a_1-\beta_1}\mathrm{B}_1'\right)\frac{y_1^2}{4f_1^2}\right\},$$

and the image to object ratio for the second lens is given by

$$\frac{v_2}{u_2}\left\{1+\left(\mathrm{T}_2'+\frac{1}{a_2-\beta_2}\mathrm{B}_2'\right)\frac{y_2^2}{4f_2^2}\right\}.$$

On multiplying these two formulæ together we get

$$\frac{\mathrm{I}\,..\,i'}{\mathrm{O}\,..\,\mathrm{O}'}=\frac{v_1v_2}{u_1u_2}\left\{1+\left(\mathrm{T}_1'+\frac{1}{a_1-\beta_1}\mathrm{B}_1'\right)\frac{y_1^2}{4f_1^2}+\left(\mathrm{T}_2'+\frac{1}{a_2-\beta_2}\mathrm{B}_2'\right)\frac{y_2^2}{4f_2^2}+\int\frac{y_1^2y_2^2}{16f_1^2f_2^2}\right\},\quad(21)$$

Distortion formulæ for two lenses in succession.

from which the function of $\frac{y_1^2y_2^2}{16f_1^2f_2^2}$ may be left out, as it is a correction of the order y^4. Therefore the total distortion of the series is the sum of the distortions of the individual lenses. But it is obvious that y will have to be inserted at its proper value for each lens; and all the y's may be expressed in terms of y_1, for

$$y_2 = y_1\frac{b_2}{c_1},\ y_3 = y_2\frac{b_3}{c_2} = y_1\frac{b_2b_3}{c_1c_2},\ \text{etc.}$$

So that the formulæ for a series of n lenses or elements must be written in abbreviated form,

Distortion formulæ for three or more lenses in succession.

$$\left.\begin{aligned}\frac{v_1v_2\ldots v_n}{u_1u_2\ldots u_n}\Bigg\{1 &+ \left(\mathrm{T}_1{}' + \frac{1}{\alpha_1-\beta_1}\mathrm{B}_1{}'\right)\frac{y_1^2}{4f_1^2} \\ &+ \left(\mathrm{T}_2{}' + \frac{1}{\alpha_2-\beta_2}\mathrm{B}_2{}'\right)\frac{y_1^2}{4f_2^2}\left(\frac{b_2}{c_1}\right)^2 \\ &\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ &+ \left(\mathrm{T}_n{}' + \frac{1}{\alpha_n-\beta_n}\mathrm{B}_n{}'\right)\frac{y_1^2}{4f_n^2}\left(\frac{b_2\ldots b_n}{c_1\ldots c_{n-1}}\right)^2\Bigg\}.\end{aligned}\right\}\ \text{IV.}$$

In such cases y_1 for the first lens may be taken to be $b_1 \tan\phi$, which connects the functions with the angle of obliquity of the pencil of rays in question.

An objection to the validity of the above formulæ.

It will be as well to now consider an objection that may be raised to this series of formulæ, and at first sight a very plausible objection. It may be urged against it that it does not allow for curvature of image.

Let L_1, Fig. 99a, be a collective lens which by central oblique pencils forms an image $q_1..F_1$ which for rays in primary planes is curved as usual to a radius equal to about $f_1\frac{\mu}{3\mu+1}$ or $\frac{3}{11}$ths f_1. If so, then will not the primary focal point at q_1, and not its projection O_1 on the focal plane, form an object, as it were, from the point of view of a second lens placed at L_2? Let L_2 be a dispersive lens of the same power and material as L_1, and let it project an enlarged image of $O_1..F_1$ or $q_1..F_1$ on to another plane $O_2..F_2$, which image, if L_2 is free from E.C.s, will be a flat one.

First case. E.C.s of L_2 assumed to neutralise the normal curvature errors.

Now the primary focal line q_1 is formed on the oblique principal ray $L_1..O_1$ (unless there is coma, but that is dealt with by separate formulæ), and assuming L_2 free from distortion and coma, and at the same time to have no curvature of image, in the sense that the E.C.s balance the normal curvature errors and therefore the lens projects a flat primary image of a flat object, then the image of the point q_1 will be projected to q_2 on the refracted principal ray and on $L_2..q_1$ produced, so that the versine or curvature error $q_2..s_2$ will be $\left(\frac{v_2}{u_2}\right)^2$

times $q_1 \ldots s_1$. Also the focal plane $q_2 \ldots f_2$ will in this case be conjugate to $q_1 \ldots f_1$ in exactly the same sense that $O_2 \ldots F_2$ is conjugate to $O_1 \ldots F_1$. The curvature of image would in this case be copied through from one image to the other, without the point O_2 being disturbed. For O_2 would then be the centre of an out-of-focus oval, being a section of the eccentric pencil of rays whose axis is $l' \ldots q_2$.

But now it may be urged that supposing the E.C.s of L_2 are eliminated so that its normal curvature errors become equal and opposite to those of L_1, then it will throw upon $F_2 \ldots O_2$ a flat image, and if we still assume the line of central projection $L_2 \ldots q_1$ to be produced to cut the focal plane $F_2 \ldots O_2$ at q_3, then should not we expect a focused image to be formed at q_3 instead of the previous out-of-focus image at O_2, so that we now have a distortion of linear value $O_2 \ldots q_3$, where before we had none, due to a change in the curvature corrections of L_2?

Second case. E.C.s of L_2 eliminated, leaving the normal curvature errors free play.

Assuming that to be the case, yet there is nothing essentially inconsistent with our distortion formulæ, for we must remember that the formulæ for E.C.s and those for distortion have some functions of x in common, and it cannot therefore be expected that changes can be made in the curvature corrections of L_2 without changes also taking place in the distortion corrections, unless perhaps L_2 is a compound lens.

Formulæ for E.C.s and for distortion interconnected.

First, we have assumed L_2 to have its curvature errors neutralised by E.C.s and to form an image q_2 of the original q_1, the image q_2 being projected to O_2 in an out-of-focus condition; and, secondly, we have assumed E.C.s to be eliminated and the normal curvature errors to have free play in L_2, counteracting those in L_1, so that it must be assumed to project an image of q_1 at q_3 or thereabouts. But the change in the x or x's in the formulæ for E.C.s for L_2, if it is a simple lens, necessary to do this will also bring about plus increments in the distortion corrections, which will now indicate a new path $l' \ldots q_3$ for the refracted principal ray, shown dotted in Fig. 99*a*; and this new path will result, not only from a variation in the tangent condition in L_2, but also from the increase in its spherical aberration.

A plus increment to the E.C.s in L_2 implies a plus increment to the distortion.

But supposing we could assume variations in the curvature errors of the different lenses to occur without at all affecting their distortion corrections, then it is clear that such variations in the curvature errors would simply cause the foci for rays in primary planes to slide to and fro along the path of the principal ray, as, for instance, q_2 might be supposed to slide to and fro along $q_2 \ldots O_2$. Thus $q_2 \ldots O_2$ may be regarded as the image in two dimensions of $q_1 \ldots O_1$.

If distortion is constant, changes in image curvature cause image points to slide along the principal rays.

Thus our formula need not concern itself with anything but the conjugate focal planes, and it is the point O_1 on the first focal plane $O_1 \ldots F_1$ which it is the business of the lens L_2 to project correctly, for although O_1 may be somewhere inside an out-of-focus patch of light, yet it is where the principal ray strikes the focal plane, and as long as O_1 is correctly projected it cannot be said that there exists any distortion, however bad the image may be in other respects.

Thus a system of formulæ which only takes note of the paths of the principal rays and of the points where they intersect the successive conjugate image planes and formulates the deviation of those points from their true and proper positions in such image planes, is none the less accurate because some or all of the images may be more or less curved. The interconnection between the distortion formula in such a case as this and the formula for E.C.s, together with the formulæ for coma and spherical aberration, is highly interesting, but exceedingly involved; and it can be shown that the last three formulæ all have an indirect bearing upon the course of the principal ray as prescribed by the distortion formula.

Distortion corrections of a higher order.

In the course of a previous discussion in Section IV. of the influence upon spherical aberration of large separations between the lenses, we found that their tendency was to set up relatively strong aberrations of the higher orders y^4 and y^6, etc., and it is clear that the spherical aberration functions in our distortion formulæ are liable to precisely the same modifications, a matter to which we shall refer again when we come to consider the case of the well-known four-lens erecting eye-piece.

The Distortion produced by a Parallel Plane Plate

But before we are exactly in a position to apply our formulæ to very thick lenses by the method of elements, we must first work out the formula for the distortion produced by a parallel plane plate of glass, or other transparent substance.

That distortion is produced in such a case is rendered evident by inspection of Fig. 100, representing an oblique converging pencil whose principal ray is $R \ldots B \ldots c$ emerging from the second surface of a parallel glass plate, and Fig. $100a$, a divergent pencil emerging in the same manner. As we are studying the effect of the plate only, we must assume that before entering the plate the rays of the pencil are converging to or diverging from a true point—for instance, the point Q_1. Let straight lines $Q \ldots P$ be drawn through Q_1 perpendicular to the plane surfaces. Such perpendiculars will, of course, pass through

PLATE. XXI.

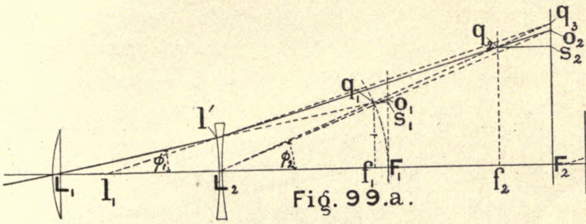


Fig. 99. a.

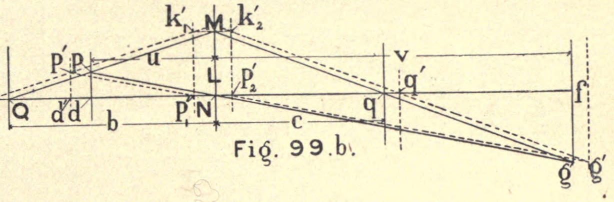


Fig. 99. b.

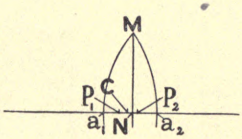


Fig. 99. c.

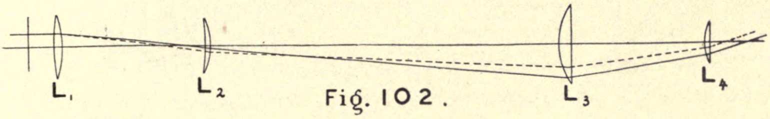


Fig. 102.

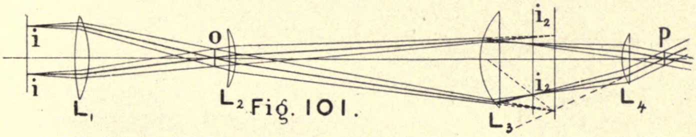


Fig. 101.

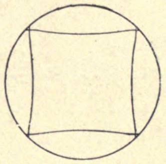


Fig. 103.

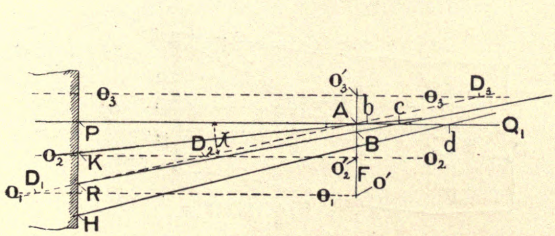


Fig. 100.

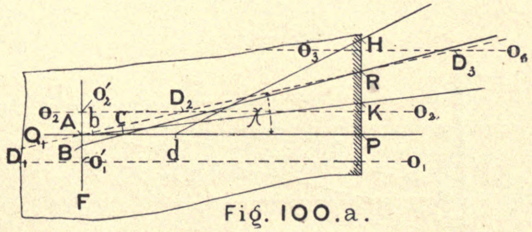


Fig. 100. a.

PLATE. XXI.

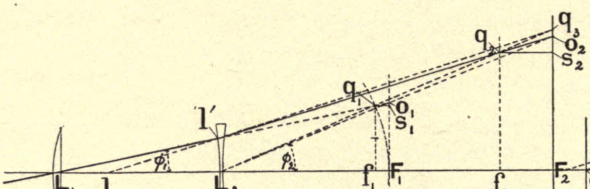


Fig. 99.a.

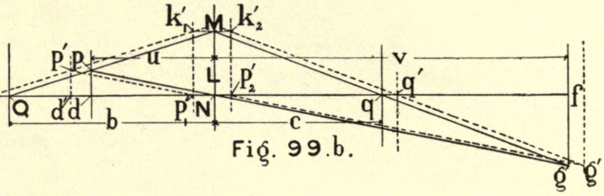


Fig. 99.b.

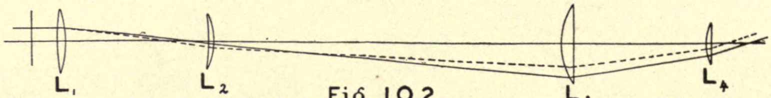


Fig. 102.

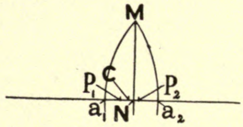


Fig. 99.c.

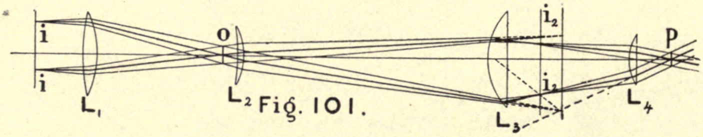


Fig. 101.

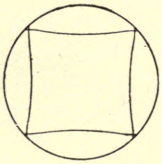


Fig. 103.

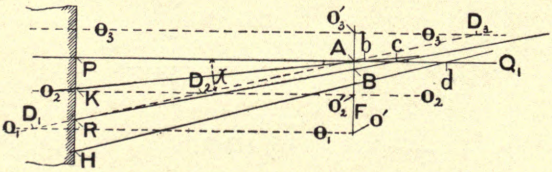


Fig. 100.

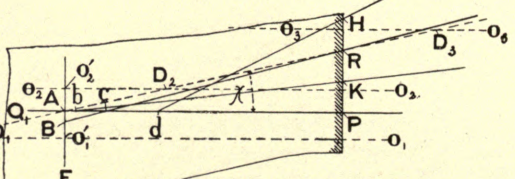


Fig. 100.a.

the ultimate focus A after refraction, according to the first approximation. Through A draw the focal plane A..F parallel to the surfaces. It is obvious that if the focus were formed at A, as it would be by a thin perpendicular pencil, there would be no distortion; but the oblique rays are subject to aberration, the ray K..b intersects the normal ray P..A at b, the principal ray R..c intersects it at c, and the ray H..d at d, and the longitudinal aberrations A..b, A..c, and A..d are proportional respectively to $(P..K)^2$, $(P..R)^2$, and $(P..H)^2$. But the principal ray R..c, when produced, cuts the focal plane A..F at B to one side of the true point A. A..B is then the linear or absolute value of the distortion, and our problem is to express it in terms of the radial dimensions of the image, which, of course, necessitates our knowing the whereabouts of the optic axis of the system, of which the parallel plate forms a part.

In the first place, we are supposed to know the angle of obliquity PAR or χ; we required and ascertained it before for other parallel plate corrections.

Then we also have the formula for the linear aberration c..A from page 80, Section IV., which was

$$\frac{(\mu^2-1)t}{2\mu^3 v^4}a^2(v^2),$$

wherein in this case a, the semi-aperture of the larger direct pencil, is P..R, which we will call h, while $v=$ P..A. It is clear that $h=v\tan\chi$;

$$\therefore\ c\,..\,A=\frac{(\mu^2-1)t}{2\mu^3 v^2}(h)^2=\frac{(\mu^2-1)t}{2\mu^3 v^2}(v\tan\chi)^2, \qquad (22)$$

also

$$A\,..\,B=(c\,..\,A)\tan\chi=\frac{(\mu^2-1)t}{2\mu^3 v^2}(v\tan\chi)^2\tan\chi. \qquad (23)$$

Formula for the linear distortion yielded by parallel plane plate.

But so far there is nothing to determine the sign of the distortion.

Let $O_1..O_1$, $O_2..O_2$, and $O_3..O_3$ represent three possible and different positions of the optic axis. Then A..O_1', A..O_2', and A..O_3' are the respective radial dimensions of the image, in terms of which we want to express the displacement A..B. Let D_1, D_2, and D_3 be the points where the principal ray cuts the optic axes $O_1..O_1$, $O_2..O_2$, and $O_3..O_3$.

Then, in pursuance of the conventions previously adopted, the distance from D_1 to the second surface is + in both cases, for the principal ray is diverging from D_1 on emergence. The distance from D_2 to the second surface is − in Fig. 100, as the principal ray is converging to D_2, but is shown to be + in Fig. 100*a*, as the principal ray is diverging from D_2 after emergence.

The distance from D_3 to the second surface is shown − in both cases, as the principal ray is converging to D_3 after emergence. Let these distances be c_1, c_2, and c_3 respectively.

In Fig. 100 the distance A . . P or v is −, and in Fig. 100*a* is +. Then, if the above conventions are adhered to, we have

$A \, . \, . \, O_1' = (v - c_1) \tan \chi$ and is − in both cases;
$A \, . \, . \, O_2' = (v - c_2) \tan \chi$ and is − in Fig. 100 and + in Fig. 100*a*;
$A \, . \, . \, O_3' = (v - c_3) \tan \chi$ and is + in both cases.

Evidently, then, $\dfrac{A \, . \, . \, B}{(v-c)\tan \chi}$ gives the distortion as a fraction of the radial dimension of the image. Then A . . B in the numerator, having no sign, may always be considered +, but $(v - c)$ in the denominator acts as a sign determinant.

In full, then, the fractional distortion is

$$\frac{(\mu^2 - 1)t}{2\mu^3(v-c)} \tan^2 \chi. \tag{24}$$

Normally the ratio between the sizes of the two conjugate images in the case of a parallel plate is simply unity, therefore we find the corrected ratio to be

Formula for the fractional distortion yielded by parallel plane plate.

$$\left\{1 + \frac{(\mu^2 - 1)t}{2\mu^3(v-c)} \tan^2 \chi\right\}. \qquad \text{V.}$$

In the case of an optical combination containing thick lenses the quantities from which we can pick out v and c have to be assessed at the outset, as we have seen before. But we must remember in this case that while v and c may be known quantitatively, yet their signs must not necessarily be taken in connection with or with respect to the element following the parallel plate, but must be assessed with respect to the parallel plate itself in strict conformity with the above convention. Should any parallel plate not be followed by an element, still the quantities v and c are easily inferred from the values v and c or v and D'' of the preceding element. Under these circumstances Formula V. will be found to interpret itself in all cases, and give a positive result when the displacement A . . B is from the optic axis, and a minus result when it is towards the optic axis.

Some Concrete Examples of the Application of the Distortion Formulæ

Process lens.

We will now take the process lens whose curves and other data were given on page 185, Section VII., and work out its total distortion by the Formula IV. we have arrived at, taking the quantities α and β, x and f, etc., as before arrived at. Then we get the following quantities for each element, the values of the function of T′ and those of B′ being stated separately, or shortly as fT′ and fB′, and assuming y_1 to be ·05 inch:—

	e_1	
$fT_1' = -\cdot 00113275$		$fB_1' = +\cdot 0095122$
		Total $- +\cdot 0083795$

	e_2	
$fT_2' = +\cdot 0008095$		$fB_2' = -\cdot 0062888$
		Total $-\cdot 0054792$

	e_3	
$fT_3' = -\cdot 00018385$		$fB_3' = +\cdot 00127723$
		Total $+\cdot 00109338$

	e_4	
$fT_4' = +\cdot 0007870$		$fB_4' = -\cdot 0130917$
		Total $-\cdot 0123047$

	e_5	
$fT_5' = -\cdot 0006549$		$fB_5' = +\cdot 0111008$
		Total $+\cdot 0104459$

	e_6	
$fT_6' = +\cdot 0006872$		$fB_6' = -\cdot 0048547$
		Total $-\cdot 0041675$

Total for e_1	$+\cdot 0083795$	Total for e_2	$-\cdot 0054792$
" e_3	$+\cdot 0010934$	" e_4	$-\cdot 0123047$
" e_5	$+\cdot 0104459$	" e_6	$-\cdot 0041675$
	$+\cdot 0199188$		$-\cdot 0219514$
			$+\cdot 0199188$
		Final total for six elements	$= -\cdot 0020326$

Distortion for six elements.

indicating a slight negative or barrel-shaped distortion. But we have yet to add the parallel plate corrections.

For the first plate P_1 we have $\tan \chi_1 = \frac{b_1}{c_1} \tan \phi$, in which $\tan \phi = \frac{y_1}{b_1} =$ in this case $\frac{\cdot 05}{\cdot 2622} = \tan 10° \, 47'$, and the formula for P_1 is

$$\frac{(\mu^2-1)t_1}{2\mu^3(v_1-c_1)}\tan^2\phi\left(\frac{b_1}{c_1}\right)^2;$$

while in the case before us v_1 and c_1 are the same quantitatively as u_2 and b_2 of the second elements; and as the rays of the pencils emerging from the plate are converging, v_1 must be put $-2\cdot0059$; also the principal rays are convergent, so that $-c_1$ becomes $-(-\cdot1676) = +\cdot1676$; so that v_1-c_1 becomes $-1\cdot8383$, and the distortion is $-$. It works out to $-\cdot01383\tan^2\phi$.

P_2

In this case $\tan\chi_2 = \tan\phi\,\frac{b_1b_2b_3}{c_1c_2c_3}$, and v_2 and c_2 are quantitatively the same as u_4 and b_4 of the fourth element. The rays of the emergent pencils are divergent, and

$$v_2 = +\ 5\cdot122 = u_4,$$

and the principal rays also are divergent,

$$\therefore\quad c_2 = (+\ \cdot2735) = b_4,$$

$$\therefore\ v_2 - c_2 = +4\cdot8485,$$

Distortion of second parallel plane plate.

and the distortion is therefore positive, and it works out to

$$+\cdot01596\tan^2\phi.$$

P_3

In this case $\tan\chi_3 = \tan\phi\,\frac{b_1b_2b_3b_4b_5}{c_1c_2c_3c_4c_5}$, and v_3 and c_3 are quantitatively the same as u_6 and b_6 of the sixth element. The rays of the emergent pencils are divergent, and

$$v_3 = +14\cdot1046 = u_6,$$

and the principal rays are divergent,

$$\therefore\ c_3 = (+)\cdot3577 = b_6,$$

$$\therefore\ v_3 - c_3 = +13\cdot7469,$$

Distortion of third parallel plane plate.

and the distortion is therefore positive, and works out to

$$+\cdot015976\tan^2\phi.$$

We then get a distortion for

$$\begin{array}{lr} P_2 = & +\cdot01596\tan^2\phi \\ P_3 & +\cdot01598\tan^2\phi \\ \hline & +\cdot03194\tan^2\phi \\ P_1 & -\cdot01383\tan^2\phi \\ \hline \text{Total for parallel plates} = & +\cdot01811\tan^2\phi \end{array}$$

Total distortion of plates.

On multiplying by $\tan^2\phi$, which we saw was $\left(\frac{\cdot05}{\cdot26223}\right)^2$, we then get

a total distortion for the three plates $= +\cdot00498$
to which we have to add the distortion
for six elements $= -\cdot00203$

Final total for whole lens.

and our grand total is $+\cdot00295$

The E.F.L. was $8{\cdot}55''$ and (E.F.L.) $\tan\phi = 1{\cdot}63$ inches, so that at a distance from the optic axis $= 1{\cdot}63$ inches, corresponding to an angle of $10^\circ{\cdot}47'$, the linear distortion is $(+{\cdot}00295)(1{\cdot}63) =$ about $+{\cdot}005$ inches, an amount barely perceptible by any but very delicate tests. As a matter of fact, this lens was very carefully corrected for rectilinearity, and at much greater angles from the axis very slight negative distortion was just perceptible. Having now dealt with a case in which the relative separations are not large, it will be as well to apply the same formulæ to the well-known cases of the Huygenian eye-piece, and the four-lens erecting eye-piece, in which the separations are very considerable.

Huygenian Eye-piece

Let this be the usual combination of two convexo-plane lenses of focal lengths 3 inches and 1 inch separated by a distance $s = 2$ inches.

Then as the image is formed in the principal focal plane of L_2, or 1 inch in front of it, it falls therefore half-way between the two lenses. The E.F.L. of the eye-piece $= 1{\cdot}5$ inches.

The characteristics and other data.

We may assume the principal rays entering L_1 to be parallel if the focal length of the object glass forming the image is relatively very long, so that $\beta_1 = -1.$

Also the rays are converging into L_1 as if to form an image $1{\cdot}5$ inch behind L_1, therefore $\alpha_1 = -5.$

The principal rays are converging into L_2 to a point 1 inch behind it, therefore $\beta_2 = -3,$

also $\alpha_2 = +1,$

$\therefore\ \alpha_1 - \beta_1 = -4,$ and $\alpha_2 - \beta_2 = +4.$

Also $y_2 = y_1 \dfrac{1}{3}.$ Let $\mu = 1{\cdot}5.$

The distortion for L_1 works out to $+\dfrac{1}{108}y_1^2,$

and for L_2 works out to $+\dfrac{17}{108}y_1^2,$

Total $\dfrac{18}{108}$ or $\dfrac{1}{6}y_1^2,$

Final result.

which, if $y_1 = {\cdot}2$, gives a distortion of $+\dfrac{1}{150}$. This is at an angular distance from the centre of the apparent field of view, such that $\tan\phi = \dfrac{{\cdot}2}{1{\cdot}5} = \dfrac{1}{7{\cdot}5}$.

The distortion yielded by an equivalent convexo-plane lens.

Supposing we substituted a single convexo-plane lens of the same power for this eye-piece, it would have to be $1{\cdot}5$ in focal length, while y would be the same as the y_1 of the eye-piece $= \cdot 2$, and β would then become -1. In that case the distortion would work out to $+\frac{88}{108}y_1^2$ or nearly five times as much as the eye-piece. The difference is partly due to the fact that in the eye-piece the principal rays are strongly convergent into the eye lens instead of parallel, which causes a much closer approach to the fulfilment of the tangent condition (which requires β_2 to be -5) than in the case of the simple equivalent lens, but principally because of the relative reduction in y_2. For supposing an equivalent simple lens is substituted for the eye-piece, then its y would necessarily be equal to the y_1 of the above eye-piece, and if $y_1 = \frac{1}{n}$th of f_1 (and y_2 then $= \frac{1}{n}$th of f_2), it is clear that y_1 would be $\frac{2}{n}$ths of f the focal length of the equivalent lens. Thus the principal rays are caused to be refracted through the eye lens of a Huygenian eye-piece three times as close to the axis as in the case of the equivalent lens, while the power of the eye lens is $\frac{3}{2}$ of the equivalent lens, so that the relative distortion of the eye lens, other things being equal, may be expected on that account alone to be reduced to $\left(\frac{1}{3}\right)^2\left(\frac{3}{2}\right)^2 = \frac{1}{4}$th.

Causes of the inferiority of the single lens.

The formula for distortion for the Huygenian eye-piece will be found to work out to about a minimum, when $x_1 = 0$ and $x_2 = +1$, in which case the field lens is equiconvex, and the eye lens convexo-plane, when the total distortion is $+\frac{5}{108}y_1^2$. But such a combination has certain other disadvantages.

When the distortion of Huygenian eye-piece is at a minimum.

Sometimes Huygenian eye-pieces are constructed with a ratio of focal lengths between the field lens and eye lens of 2 to 1, which enables a flatter field of view to be obtained than with the ratio 3 to 1; but with the ratio 2 to 1 the approach to freedom from distortion is not quite so good.

The Four-Lens Erecting Eye-piece

Its inventor.

This well-known and useful optical device seems to have been arrived at quite empirically by the monk De Rheita, who evidently had been experimenting with various combinations of lenses in series in conjunction with a telescopic objective. But the theory of it was not worked out until very many years later, by Sir George Airy and Henry

The course of the rays through four-lens eye-piece.

Coddington, and even then not in one sense completely. Fig. 101 shows the course of a couple of pencils of rays through such an eye-piece, from their points of origin in the first object or aerial image $i \,..\, i$ to their again concentrating into a second aerial inverted image $i_2 \,..\, i_2$ in the principal focal plane of the eye lens L_4, so that after emergence from the latter the rays constituting the pencils are parallel and fit for vision by the normal eye placed behind it at P.

Position and function of first stop.

Since the objective of the telescope is supposed to be placed at a considerable distance to the left hand, and the principal rays of the various pencils or cones of rays are supposed to radiate from the centre of the objective, therefore such principal rays are brought to a focus at O at a distance behind L_1 equal to or a little more than its principal focal length; not only so, but an image of the aperture of the objective is formed at that position, where it is usual to place a stop with a circular aperture a little larger than such image of the objective, whose office it is to screen off stray light reflected from the interior of the tubes.

Then a second image of the objective or an inverted image of O is again formed behind the eye lens at P; that is, the principal rays again come to a focus or cross the axis at P, where the pupil of the eye is placed to receive them and the pencils of rays which they represent. But, as we shall see later, this second image of the objective, or exit pupil, is an exceedingly rough and imperfect one.

Fig. 101 is a correct drawing to scale of a four-lens eye-piece which was specially adjusted with great care to show an apparently rectilinear image when used as a magnifier on a set of straight lines ruled on a flat surface placed at $i \,..\, i$, the eyesight of the observer being normal. The object was to see whether the sum of the formulæ for distortion for the four lenses would in that case work out to zero. The stop at O was at a distance $= f_1$ behind L_1. The data for this combination were as follows, the refractive index being 1·53 for all four lenses:—

$f_1 = 1{\cdot}9''$ $\quad x_1 = -\frac{1}{3}$ $\quad b_1 = \infty$ $\quad c_1 = +1{\cdot}9''$ $\quad \therefore\ \beta_1 = -1$

Separation $s_1 = 2{\cdot}24''$ $\quad u_1 = +\ {\cdot}76$ $\quad v_1 = -1{\cdot}27$ $\quad \therefore\ \alpha_1 = +4$

and $\alpha_1 - \beta_1 = +5$

$f_2 = 2{\cdot}26''$ $\quad x_2 = -2$ $\quad b_2 = +\ {\cdot}34$ $\quad c_2 = -\ {\cdot}40$ $\quad \therefore\ \beta_2 = +12{\cdot}3$

$s_2 = 5{\cdot}24''$ $\quad u_2 = +3{\cdot}51$ $\quad v_2 = +6{\cdot}35$ $\quad \therefore\ \alpha_2 = +\ {\cdot}29$

and $\alpha_2 - \beta_2 = -12$

$f_3 = 2{\cdot}03''$ $\quad x_3 = +1$ $\quad b_3 = +5{\cdot}64$ $\quad c_3 = +3{\cdot}17$ $\quad \therefore \beta_3 = -{\cdot}28$

$s_3 = 2{\cdot}13''$ $\quad u_3 = -1{\cdot}115$ $\quad v_3 = +{\cdot}72$ $\quad \therefore \alpha_3 = -4{\cdot}64$

and $\alpha_3 - \beta_3 = -4{\cdot}36$

$f_4 = 1{\cdot}41''$ $\quad x_3 = +1$ $\quad b_4 = -1{\cdot}04$ $\quad c_4 = +{\cdot}60$ $\quad \therefore \beta_4 = -3{\cdot}7$

$u_4 = +1{\cdot}41$ $\quad v_4 = \infty$ $\quad \therefore \alpha_4 = +1$

and $\alpha_4 - \beta_4 = +4{\cdot}7$

From which we get the following values of the distortion when $y_1 = {\cdot}20''$:—

$$\left.\begin{array}{lll}
L_1 & \dfrac{1}{4f_1^2}\left\{T_1' + \dfrac{1}{\alpha_1 - \beta_1}B_1'\right\}y_1^2 & = +{\cdot}00512 \\
L_2 & \dfrac{1}{4f_2^2}\left\{T_2' + \dfrac{1}{\alpha_2 - \beta_2}B_2'\right\}y_1^2\left(\dfrac{b_2}{c_1}\right)^2 & = -{\cdot}00190 \\
L_3 & \dfrac{1}{4f_3^2}\left\{T_3' + \dfrac{1}{\alpha_3 - \beta_3}B_3'\right\}y_1^2\left(\dfrac{b_2 b_3}{c_1 c_2}\right)^2 & = -{\cdot}00203 \\
L_4 & \dfrac{1}{4f_4^2}\left\{T_4' + \dfrac{1}{\alpha_4 - \beta_4}B_4'\right\}y_1^2\left(\dfrac{b_2 b_3 b_4}{c_1 c_2 c_3}\right)^2 & = +{\cdot}0223
\end{array}\right\} \begin{array}{l}\text{Total} \\ = +{\cdot}0235\end{array}$$

The total result is a positive distortion of about $2\frac{1}{3}$ per cent, which, although small in itself, is in excess of the distortion yielded by any one of the four lenses. But $2\frac{1}{3}$ per cent of distortion could scarcely go unperceived under a searching test. How is it that this apparent discrepancy between theory and practice arises? It is partly due to the fact that a good deal of the personal equation arises in the case of a series of straight lines or chords viewed through a circular aperture. The real image formed in the principal focal plane of the eye lens is bounded or limited by the field diaphragm within the circular aperture of which it is formed.

The personal equation.

Parallel straight lines viewed through a circular aperture may appear distorted.

Now, it can be shown that a series of parallel straight lines viewed, *without any lenses whatever*, through a circular aperture do not appear to be straight to all observers; to some, including the author, they invariably appear somewhat barrel-shaped, as if by the presence of negative distortion, while a square drawn with sides so curved inwards as to represent a case of 2 per cent of positive distortion at the corners (and therefore 1 per cent at the middle of the sides) appears to be perfectly rectilinear when viewed through a circular aperture just well clearing the corners. The reader should try this experiment for himself, and will then become convinced of the difficulty there is in saying whether an eye-piece is really free from distortion or not.

Distortion of the third order.

Furthermore, in the four-lens eye-piece, consisting as it does of four widely separated lenses, the distortion corrections of the higher order

y^4 in some cases may form a very appreciable fraction of those which we have formulated of the order $\frac{y^2}{f^2}$, and this is chiefly true of the corrections affecting the eye lens. To be sure Coddington, on pages 168 to 170 of his work, in dealing with the four-lens eye-piece, makes it appear that the distortion formulæ of the order $\frac{y^2}{f^2}$ for the four lenses may be reduced to zero; but we have seen that he neglected in working out his formulæ to allow for the spherical aberration of the first lens being carried through to the third and fourth lenses, and that of the second to the fourth, operations which, as we have already seen, are really as vitally important in his scheme as carrying forward the aberrations of each lens to the next following lens, which he did allow for. Hence his conclusions on page 170 were erroneous.

We have seen that the formula for distortion which we have worked out is quite independent of such accumulated variations of b and c in each lens, that is, so far as the formulæ of the order $\frac{y^2}{f^2}$ are concerned. But Fig. 102 will help us to see that the aberrations exerted by each lens upon the principal rays must necessarily have an effect upon the distortions of the following lenses which we cannot altogether neglect. In Fig. 102 the deviation of the principal ray from its theoretical course is a little exaggerated for the sake of clearness. The solid lines indicate the theoretical course of two principal rays through the lenses according to the formulæ of the first approximation, by which the values of b and c, and therefore β for each lens are assessed. But the dotted lines indicate the actual course of the same principal ray, which deviates largely from the theoretical course, especially at the eye lens. It is clear that our method of expressing the y for each lens in turn in terms of y_1 deviates more and more from the truth as we work towards the eye lens, and this fact is just as important whether we work out our distortion completely by Coddington's scheme or by our own. After allowing for these modifications of y_2, y_3, and y_4, and β_2, β_3, and β_4, it can be shown that our principal ray, striking L_1 at a height $y_1 = \cdot 20$ from the axis, is subject to a distortion of the order y_2^4, y_3^4, and y_4^4, equal to $\frac{1}{5}$th part of the distortion of $2\frac{1}{3}$ per cent previously arrived at and of the opposite sign.

Departure of the actual path of a principal ray from the ideal path.

On page 91, Section IV. (Fig. 36), we showed how, when two lenses are separated from one another on a common axis, the spherical aberration of the first lens gave rise to a spherical aberration in the second lens of the order y_2^4, and similarly for any subsequent lenses; and the same influences operate in the case of the four-lens eye-piece. Moreover, there exists for each lens the intrinsic aberrations of the order y^4, not only as regards the spherical aberration, but also the aberrations from the tangent condition. So that the distortion formulæ for a four-lens erecting eye-piece, supposing we take all of the order y^4 into account, as well as those of the order y^2, are of a highly complex nature.

Hybrid distortion.

The fact that the corrections against distortion are generally of a hybrid nature, involving the opposition of these two orders of corrections, is made apparent by rigidly testing the rectilinearity of an eye-piece which has an extra large field of view. It will then be found that there exists a small amount of positive or pincushion distortion of straight lines in the inner zones of the field of view, while in the outermost zone there is quickly increasing negative or barrel-shaped distortion of straight lines. This is illustrated in exaggerated form in Fig. 103.

Above explained.

The case is exactly illustrated by means of Fig. 37, in which the left-hand curve may be taken to represent $+$distortion of the order y^2 and the right-hand curve $-$distortion of the order y^4. These neutralise each other at a certain distance D from the axis or centre of the field of view; but at a distance equal to $\frac{D}{\sqrt{2}}$ from the axis there occurs a maximum of $+$distortion equal to $\frac{1}{4}$th of the distortion that occurs at D, and outside that a rapidly increasing $-$distortion.

In the case of certain forms of four-lens erecting eye-pieces largely favoured by Continental opticians, and consisting of four compound and achromatic lenses, this compound curvature of straight lines, consequent upon a still greater degree of distortion of the order y^4 opposed by distortion of the opposite sign of the order y^2, is still more noticeable.

Hybrid distortion increases as the fourth power of the angular field of view.

It is clear that since the distortion of the order y^4 increases as $\tan^4 \eta$ or the fourth power of the semi-diameter of the apparent field of view, therefore the size of the latter cannot be very much increased without the hybrid distortion showing itself in an aggressive manner. Doubling the size of the field of view will multiply the defect sixteen times.

Cooke Photographic Lenses

These lenses, which are composed of two simple collective lenses containing between them a simple dispersive lens, form good practical examples of the embodiment of the formula—

$$\frac{y_1^2}{4f_1^2}\left\{T_1' + \frac{1}{\alpha_1 - \beta_1}B'\right\} + \frac{y_2^2}{4f_2^2}\left\{T_2' + \frac{1}{\alpha_2 - \beta_2}B'\right\} + \frac{y_3^2}{4f_3^2}\left\{T_1' + \frac{1}{\alpha_3 - \beta_3}B'\right\} = 0,$$

for the two collective lenses of focal lengths f_1 and f_3 are separated from the dispersive lens by separations s_1 and s_2, which are proportional to f_1 and f_3; and when the distances from the object to L_1 and from L_3 to the image are also proportional to f_1 and f_3, and L_1 and L_3 are symmetrically shaped with respect to one another, then clearly the conditions of vergency as well as of shape of the lenses L_1 and L_3 are all symmetrical if the principal rays are supposed to cross the optic axis at the centre of the lens L_2; so that $\frac{1}{\alpha_2 - \beta_2} = \frac{1}{\infty} = 0$, also $\alpha_1 - \beta_1 = -(\alpha_3 - \beta_3)$, and $\alpha_1 = -\alpha_3$, $\beta_1 = -\beta_3$, $x_1 = -x_3$, etc. Therefore the system is free from distortion, and practically remains so under all normal conditions.

Magnification

We have yet to consider the important question of the magnifying powers of lens systems.

It is quite obvious that if the eye views a distant flat object and fixes itself upon some central point C, then various other points in the object will seem to be distant from C by certain angles ϕ_1, ϕ_2, etc.; and their apparent distances from C as measured in the plane of the object will be proportional to $\tan\phi_1$, $\tan\phi_2$, etc.

On approaching to a distance equal to $\frac{1}{n}$th of the first distance, the apparent distances of the same points from C will be proportional to $n\tan\phi_1$, $n\tan\phi_2$, etc.

If, instead of approaching n times nearer, an optical contrivance causes principal rays to make angles equal to $n\tan\phi_1$, $n\tan\phi_2$, etc., with the axial line through C, in place of $\tan\phi_1$ and $\tan\phi_2$, etc., then clearly the magnifying power $= n$.

So that if, in the case of the telescope, we write $\tan\phi$ for the tangent of the angle included between the optic axis and the principal ray from any point in the distant object, and $\tan\phi'$ for the angle made with the optic axis by the same principal ray after emerging from the instrument, then clearly $\frac{\tan\phi'}{\tan\phi}$ will express the magnifying power.

This is of course equivalent to the ratio $\frac{\tan\eta}{\tan\epsilon}$ in Airy's and Coddington's Formulæ II. for the distortion of eye-pieces; in which

$\tan \epsilon = \tan \phi$, or the original visual angle subtended at the object glass, and $\tan \eta = \tan \phi'$, the angle for the same principal ray on emergence.

Formula for the magnification of a telescope.

The simplest way, however, of expressing $\frac{\tan \phi'}{\tan \phi}$ is in its equivalent form $\frac{F}{f}$, in which F = the equivalent focal length of the object glass, and f the E.F.L. of the eye-piece.

Use of the dynamometer.

Supposing neither F nor f are exactly known, then the familiar device of measuring the diameter of the image of the aperture of the object glass formed just beyond the eye lens with a dynamometer, when the telescope is focused for distant objects, and dividing the same into the aperture of the object glass, may always be relied upon to give fairly exact results. Theoretically the method is quite exact, as the following reasoning will show.

Proof of the accuracy of the dynamometer.

When set for normal eyesight the first principal point of the eye-piece is distant from the second principal point of the objective by a distance equal to $F + f$. Now let $F = mf$, so that m is the magnifying power. Then the two conjugate focal distances, with respect to the eye-piece, of the object glass and its image will clearly be

$$(m+1)f \text{ and } \frac{1}{\frac{1}{f} - \frac{1}{(m+1)f}}$$

$$= (m+1)f \text{ and } \frac{1}{\frac{m}{(m+1)f}} \text{ or } (m+1)f \text{ and } \frac{m+1}{m}f;$$

and consequently the image of the objective will be $\frac{1}{m}$th of the original size; and therefore the ratio m expresses the magnifying power of the telescope.

The only thing which militates against the accuracy of this method is the violent spherical aberration to which the image of the object glass is subject in many cases.

Also many cases arise in the case of three- or four-lens eye-pieces in which the image formed behind the eye-piece is not really an image of the objective at all, but is an image of the stop between the first and second lens of the eye-piece, which is, either intentionally or not, made too small to pass the full image of the object glass thrown into it by the first lens.

In such cases the best plan is to place an artificial circular aperture of smaller size over the object glass and divide its aperture by the diameter of the image of the same formed by the eye-piece.

The Simple Microscope

Here we have to deal with a somewhat different state of things, for the apparent size of the original objects, which are close at hand in the first instance, is evidently quite arbitrary; a short-sighted person may view an object with his naked eye 6 inches away, and see it magnified three times relatively to a person who can only see it clearly with the naked eye at 18 inches away. Therefore the convention has been adopted of accepting 10 inches as the standard distance at which the normal naked eye can comfortably view small objects, and therefore all microscope magnifying powers are estimated relatively to that conventional standard.

The conventional standard of distance.

First, it is clear that in the case of using lenses of low magnifying power the short-sighted person will clearly have an advantage, as he can place his magnifier nearer to the object and deal with more divergent rays than the long-sighted; and, again, the question is further complicated by the variation occurring in the distance of the eye behind the lens.

Advantage of being short-sighted.

Let f be the E.F.L. of the lens, u its distance from the object, and D the distance of the eye from the lens, all in inches. Then the conjugate focal distance v will be $\dfrac{1}{\frac{1}{f}-\frac{1}{u}}=\dfrac{fu}{u-f}$, and the distance of the image from the eye will be

$$v-\mathrm{D}=\frac{fu}{u-f}-\mathrm{D}=\frac{fu-\mathrm{D}(u-f)}{u-f}.$$

If the eye were at the lens centre, then clearly the conventional magnifying power would be $\frac{10}{u}$ quite independently of the position of the second conjugate image, but the eye is at a distance from the image which is reduced by D, therefore the magnifying power becomes

$$\frac{10}{u}\cdot\frac{v}{v-\mathrm{D}}=\frac{10}{u}\,\frac{\frac{fu}{u-f}}{\frac{fu-\mathrm{D}(u-f)}{u-f}}=\frac{10}{u}\cdot\frac{fu}{fu-\mathrm{D}(u-f)}=\frac{10f}{fu-\mathrm{D}(u-f)}. \qquad \text{VI.}$$

Formula for the magnification of a simple microscope.

As a general rule v is a minus quantity, since the emergent rays constituting the pencils are diverging. If they are converging, then of course D gives a gain in magnifying power instead of a loss.

The Compound Microscope

Here there is a real image of the original formed behind the objective, and this image is viewed through an eye-piece, which yields a further magnifying power.

Let $F =$ the E.F.L. of the objective, and f that of the eye-piece, and U and V the conjugate focal distances of the object and image respectively, and let it be assumed that the rays emerge parallel from the eye-piece.

If the eye were placed at the first principal point of the objective it would see the object under a magnification equal to $\frac{10}{U}$; and if it could turn to the second principal point and look the other way it would see the conjugate image under exactly the same visual angle, and the magnifying power would still be $\frac{10}{U}$.

If the eye then views the conjugate image through the eye-piece, the magnifying power will be obviously increased in the ratio $\frac{V}{f}$; therefore the whole magnifying power will be

$$\frac{10}{U} \cdot \frac{V}{f}. \qquad (25)$$

Now we may call V, or the distance from the second principal point of the objective to the enlarged image, the effective length of tube, which may also be written as nF, so that we have

$$\frac{1}{U} = \frac{1}{F} - \frac{1}{nF} = \frac{n-1}{nF}, \qquad (26)$$

so that our formula becomes

Formula for the magnification of a compound microscope.

$$10\left(\frac{n-1}{nF}\right)\frac{nF}{f} = 10\left(\frac{n-1}{f}\right). \qquad \text{VII.}$$

As in the compound microscope an image of the objective is formed just behind the eye-piece, therefore the eye cannot be far removed from the latter if the whole field of vision is to be seen; nor, in the case of high-power eye-pieces at any rate, will the state of divergence of the emergent rays very appreciably affect the truth of the above simple formula.

SECTION X

ACHROMATISM

So far as we have yet proceeded, we have generally treated the rays refracted by any particular lens, element, or parallel plate as if the refractive index μ were a fixed quantity.

Our next task is to consider what follows from the refractive index, varying, as it does, for the differently coloured rays usually constituting the pencils of light refracted through lenses.

It may fairly be assumed that the reader will be quite familiar with the simpler formulæ relating to achromatism, yet for the sake of completeness it is desirable to recapitulate the usual formulæ, and then pass on to the new theorems and formulæ contained in this Section. **Elementary formulæ.**

First of all from our familiar formula for a thin lens—

$$\frac{1}{\mathrm{F}} = (\mu - 1)\left(\frac{1}{r} + \frac{1}{s}\right),$$

we deduce

$$\Delta_\mu\left(\frac{1}{\mathrm{F}}\right) = (\Delta\mu)\left(\frac{1}{r} + \frac{1}{s}\right);$$

and since

$$\frac{1}{r} + \frac{1}{s}\left(\text{or } \frac{1}{\rho} \text{ for brevity}\right) = \frac{1}{(\mu - 1)\mathrm{F}},$$

$$\therefore \quad \Delta_\mu\left(\frac{1}{\mathrm{F}}\right) = \frac{\Delta\mu}{\mu - 1} \cdot \frac{1}{\mathrm{F}}. \qquad \text{I.}$$

Variation of the power of a lens due to colour variation.

So that the variation of the power of a lens consequent upon a variation $\Delta\mu$ in the refractive index is equal to the power of the lens multiplied by $\frac{\Delta\mu}{\mu - 1}$, which is the well-known expression for the dispersive power of the glass for the range of rays dealt with.

Then from the formula for conjugate foci $\frac{1}{V}=\frac{1}{F}-\frac{1}{U}$ we derive, if U is constant and F varies, as in Formula I.,

$$\Delta_{\mu}\frac{1}{V}=\frac{\Delta\mu}{\mu-1}\frac{1}{F},$$

so that

Linear value of chromatic aberration.

$$\Delta V = -\frac{\Delta\mu}{\mu-1}\frac{1}{F}V^2. \qquad (1)$$

Thus the linear chromatic aberration, as measured along the optic axis, varies directly as the square of V, the distance to which it is projected by the lens, just as in the case of spherical aberration, only with this difference, that the linear chromatic aberration is quite independent (except in the higher orders) of the aperture or form of the lens and of the state of divergence or otherwise of the entering rays. Thus the characteristics a and x do not as yet enter into the case at all, and the chromatic aberration depends only upon the power of the lens and the dispersive power of its material.

But it is quite clear that the aperture of the lens must exert a proportional effect upon the size of the least circle of chromatic aberration through which the range of coloured rays will pass. This least circle is obviously situated half-way between the focal points for the two extreme colours concerned, and its diameter is equal to half the linear chromatic aberration multiplied by the ratio of aperture to the conjugate focal distance V, or $\frac{2a}{V}$, wherein a is the semi-aperture of the pencil or lens. So that the diameter of the least circle of chromatic aberration is expressed by

Diameter of least circle of chromatic confusion.

$$\frac{1}{2}\left(\frac{\Delta\mu}{\mu-1}\frac{1}{F}V^2\right)\frac{2a}{V}$$

$$=a\left(\frac{\Delta\mu}{\mu-1}\frac{1}{F}V\right), \qquad \text{IA.}$$

and its angular diameter as subtended at the lens centre is

Angular value of above subtended at objective.

$$a\left(\frac{\Delta\mu}{\mu-1}\frac{1}{F}\right), \qquad \text{IB.}$$

which shows that, supposing the aperture is constant, the angular diameter of the least circle of chromatic aberration varies inversely as F, a fact which was realised in a very practical manner by astronomers and opticians such as Huygens and Hevelius in the early days of the simple objective, for they made a great point of having the focal lengths of their telescopes as long as possible, 120 feet being nothing unusual.

We have also seen in Section IV., page 110, that the least circle of confusion consequent upon spherical aberration has an angular diameter which varies inversely as the cube of the focal length when the aperture is constant.

If we put two thin lenses in contact, with a view to producing an achromatic image in the conjugate focal plane of the compound lens, then we must fulfil the equation

Two lenses in contact. Condition of axial achromatism.

$$\frac{\Delta\mu_1}{\mu_1 - 1}\frac{1}{F_1} + \frac{\Delta\mu_2}{\mu_2 - 1}\frac{1}{F_2} = 0, \qquad \text{II.}$$

in which $\Delta\mu_1$ or $\Delta\mu_2$ refer to the respective differences in refractive indices for any two coloured rays of the spectrum that may be fixed upon. These are generally the orange-red ray known as the C ray, and the blue-green ray known as the F ray.

Since in all known glasses the refractive index increases as we ascend the spectrum from red to violet, and $\Delta\mu$ is always of the same sign for different lenses when it refers to the same spectrum interval, therefore it is clear that $\frac{1}{F_1}$ and $\frac{1}{F_2}$ must be of opposite signs, and that

Dispersive powers in proportion to focal lengths.

$$\frac{\Delta\mu_2}{\mu_2 - 1} = -\frac{\Delta\mu_1}{\mu_1 - 1}\frac{F_2}{F_1}; \qquad (2)$$

that is, the dispersive powers of the glasses forming the lenses must be in inverse proportion to their powers or in direct proportion to their focal lengths.

Also, since the resultant power of the contact combination is simply $\frac{1}{F_1} + \frac{1}{F_2}$, it is clear that the fulfilment of Equation II. demands that the lens of the greater power shall be made out of glass of the least dispersive power, and then its power will prevail over the other. So that if the combination is to have positive power, then the collective lens must be made of the glass of the lower dispersive power; and if the combination is to have negative power, then the dispersive lens must be made out of the glass of the lower dispersive power.

Thus if

$$\frac{\Delta\mu_2}{\mu_2 - 1} = \frac{5}{3}\frac{\Delta\mu_1}{\mu_1 - 1},$$

then $\frac{1}{F_2}$ will be $\frac{3}{5}$ths $\frac{1}{F_1}$, and the power of the combination will be

$$\frac{1}{F_1}-\frac{3}{5}\frac{1}{F_1}=\frac{2}{5}\frac{1}{F_1}.$$

The glasses usually used for achromatic objectives.

This is the ratio of dispersive powers generally prevailing in the glasses used for ordinary telescope objectives, the collective lens being generally made of a crown glass having a dispersive power of $\frac{1}{60}$ for the spectrum interval C to F, and the dispersive lens out of a dense flint glass having a dispersive power for the same spectrum interval equal to $\frac{1}{36}$. It is clear that any contact combinations of a collective with a dispersive lens may be achromatic for all degrees of divergence or convergence of the entering rays.

Thin Lenses Separated by an Interval

Should an interval s exist between the two lenses, Formula II. will no longer apply. Since $\frac{\Delta\mu_1}{\mu_1-1}\frac{1}{F_1}$ is the chromatic aberration of the first lens, and $\frac{1}{f_1}\cdot\frac{\Delta\mu_1}{\mu_1-1}v_1^2$ is the longitudinal chromatic aberration as measured along the axis, or the chromatic variation of v_1, therefore from the centre of the second lens as a reference point the chromatic aberration of the first lens

$$=\frac{1}{f_1}\frac{\Delta\mu_1}{\mu_1-1}\frac{v_1^2}{u_2^2}\text{ or }\frac{1}{f_1}\frac{\Delta\mu_1}{\mu_1-1}\frac{v_1^2}{(s-v_1)^2},$$

which must be neutralised by the chromatic aberration of the second lens. Therefore the formula for achromatism is

Two separated lenses. Condition of axial achromatism.

$$\frac{1}{f_1}\frac{\Delta\mu_1}{\mu_1-1}\frac{v_1^2}{u_2^2}+\frac{1}{f_2}\frac{\Delta\mu_2}{\mu_2-1}=0. \qquad \text{III.}$$

So that the greater is the separation multiplier $\left(\frac{v_1}{u_2}\right)^2$ the greater is the chromatic aberration which the second lens has to counteract.

But, since $\left(\frac{v_1}{u_2}\right)^2$ can be made practically equal to unity by assuming v_1 to be a very large quantity compared to s, as when the rays leaving L_1 are about parallel, the formula in such circumstances becomes practically the same as Formula II.

Axial achromatism of two separated lenses not constant.

Hence it is clear that while Formula III. may be equated to 0 for any given value of u_1, yet if u_1 varies considerably and thus causes

$\left(\frac{v_1}{u_2}\right)^2$ to vary, the condition of achromatism will no longer hold good. Hence no separated combination of a collective with a dispersive lens can possibly be achromatic for all degrees of divergence or convergence of the entering rays. While the equivalent focal length of the combination is constant, as we saw in Section III., yet the chromatic aberration varies according to the radiant distance u_1. But it can be shown that under certain circumstances a combination of three or more separated thin lenses may yield a practically constant chromatic aberration under all circumstances likely to occur in practice.

Axial achromatism of three separated lenses may be practically constant.

We may now extend Formula III. to a larger number of separated lenses.

Supposing we have three lenses, then from the centre of the last lens as a reference point the chromatic aberration of the first lens as a variation of $\frac{1}{u_3}$ is

$$\frac{1}{f_1}\frac{\Delta\mu_1}{\mu_1 - 1}\left(\frac{v_1}{u_2}\frac{v_2}{u_3}\right)^2,$$

and that of the second lens is

$$\frac{1}{f_2}\frac{\Delta\mu_2}{\mu_2 - 1}\left(\frac{v_2}{u_3}\right)^2,$$

and that of the third lens is

$$\frac{1}{f_3}\frac{\Delta\mu_3}{\mu_3 - 1}\text{ simply.}$$

Proceeding in the same way for n number of lenses we get the general formula $\Delta\frac{1}{v_n}$

$$=\frac{1}{f_1}\frac{\Delta\mu_1}{\mu_1 - 1}\left(\frac{v_1 v_2 \ldots v_{n-1}}{u_2 u_3 \ldots u_n}\right)^2+\frac{1}{f_2}\frac{\Delta\mu_2}{\mu_2 - 1}\left(\frac{v_2 \ldots v_{n-1}}{u_3 \ldots u_n}\right)^2 \ldots + \frac{1}{f_n}\frac{\Delta\mu_n}{\mu_n - 1}, \qquad \text{IV.}$$

Condition of axial achromatism for a series of separated lenses.

which is strictly applicable also to a series of elements.

The Linear Chromatic Aberration of a Parallel Glass Plate

But in order to apply the formulæ to thick lenses by the method of elements, we must next work out the formulæ for the chromatic aberration of a parallel plate of glass.

Let Fig. 104 represent a case of a divergent pencil of white light originating from Q and passing perpendicularly through the parallel plate of thickness $A_1 \ldots A_2$ or t, and Fig. 104a the corresponding case of a perpendicular pencil of rays converging to Q. After refraction at the first surface the less refrangible rays, such as the red, will be divergent from or convergent to r_1, such that $r_1 \,..\, A_1 = \mu_r(Q \,..\, A_1)$, while the more refrangible rays, such as the blue, will be divergent from or convergent to b_1, such that $b_1 \,..\, A_1 = \mu_b(Q \,..\, A_1)$; so that the distance between b_1 and r_1 will be $Q \,..\, A_1(\mu_b - \mu_r)$, or, shortly, $u(\Delta\mu)$. Then as a correction to the reciprocal value of the distance $r_1 \,..\, A_2$ in the case of Fig. 104, the quantity $u(\Delta\mu)$ becomes $\frac{u\Delta\mu}{(r_1 \,..\, A_2)^2}$; that is,

$$\frac{1}{b_1 \,..\, A_2} = \frac{1}{r_1 \,..\, A_2} - \frac{u\Delta\mu}{(\mu_r u + t)^2} = \frac{1}{\mu_r u + t} - \frac{u\Delta\mu}{(\mu_r u + t)^2}.$$

After refraction at the second surface $\frac{1}{\mu_r u + t}$ becomes $\frac{\mu_r}{\mu_r u + t}$ or $\frac{1}{u + \frac{t}{\mu_r}}$ or $\frac{1}{v}$, which $= \frac{1}{r_2 \,..\, A_2}$, and $\frac{u\Delta\mu}{(\mu_r u + t)^2}$ becomes $\frac{\mu_r u\Delta\mu}{(\mu_r u + t)^2}$.

Now, supposing the other ray, or the blue ray, were also radiating from the same point r_1 as the red ray before refraction at the second surface, then after refraction we should have the blue rays apparently radiating from b_2', such that the distance $b_2' \,..\, A_2$ would be equal to $\frac{r_1 \,..\, A_2}{\mu_b}$, which $= \frac{r_1 \,..\, A_2}{\mu_r + \Delta\mu}$ or $\frac{\mu_r u + t}{\mu_r + \Delta\mu}$, so that

$$\frac{1}{b_2' \,..\, A_2} = \frac{\mu_r + \Delta\mu}{\mu_r u + t} = \frac{\mu_r}{t\mu_r u + t} + \frac{\Delta\mu}{\mu_r u + t}$$

$$= \frac{1}{v} + \frac{\Delta\mu}{\mu_r u + t}, \qquad (3)$$

so that $\frac{\Delta\mu}{\mu_r u + t}$ is the increment to $\frac{1}{v}$ due to colour consequent upon the second refraction only. But we have seen that the chromatic aberration brought over from the first surface and referred to the point A_2 was $-\frac{\mu_r u\Delta\mu}{(\mu_r u + t)^2}$, so that the chromatic aberrations of both surfaces are

$$\frac{\Delta\mu}{\mu_r u + t} - \frac{\mu_r u\Delta\mu}{(\mu_r u + t)^2} = \Delta\mu\frac{\mu_r u + t - \mu_r u}{(\mu_r u + t)^2} = \frac{t}{(\mu_r u + t)^2}\Delta\mu.$$

Parallel plane plate. The chromatic variation of $\frac{1}{v}$.

But $\mu_r u + t = \mu_r v$, so that the chromatic aberration becomes

$$+\frac{t\Delta\mu}{\mu_r^2 v^2} \text{ as a correction to } \frac{1}{v}, \qquad \text{V.}$$

PLATE. XXII.

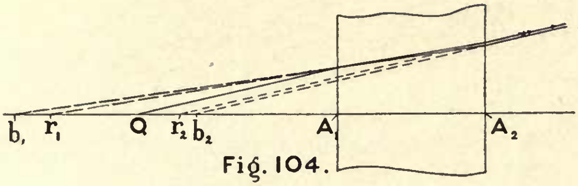


Fig. 104.

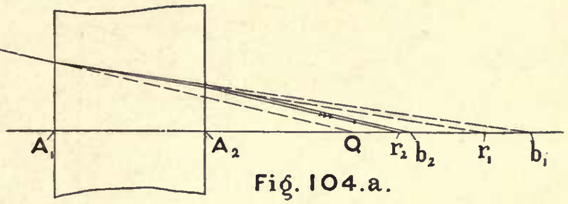


Fig. 104.a.

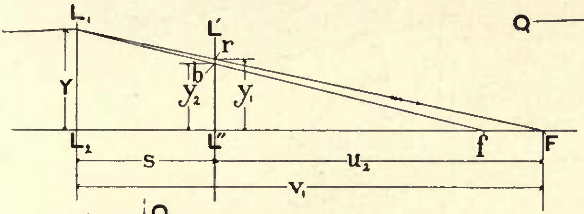


Fig. 105

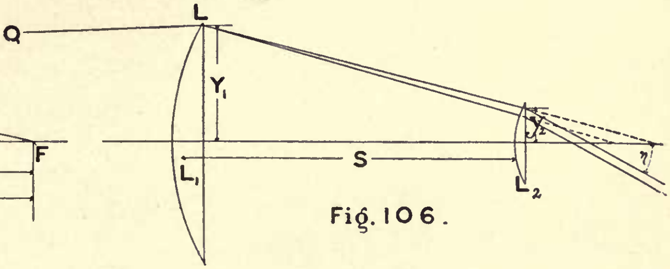


Fig. 106.

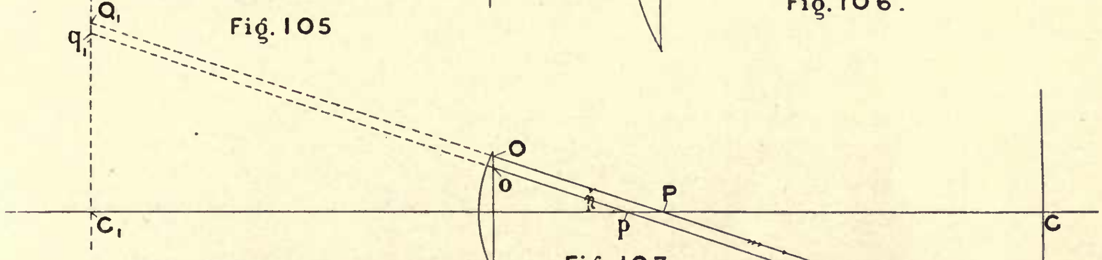


Fig. 110.a.

Fig. 107.

Fig. 110.b.

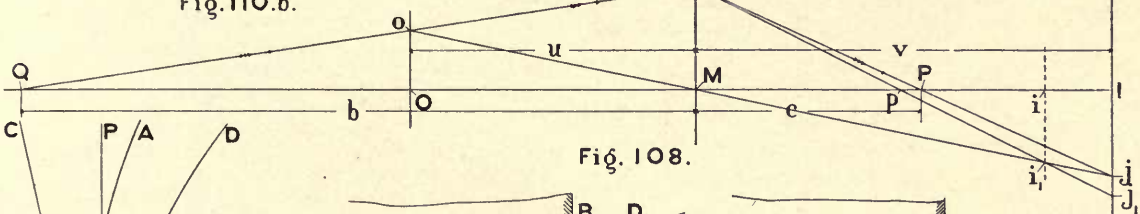


Fig. 108.

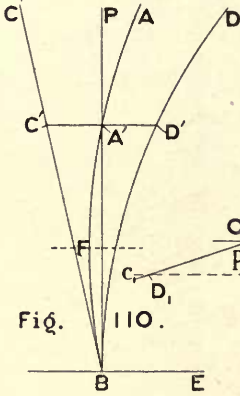


Fig. 110.

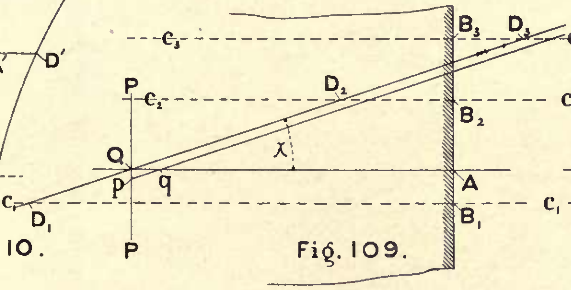


Fig. 109.

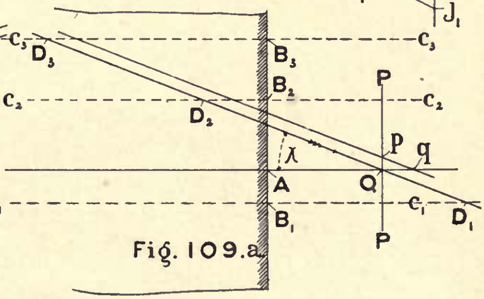


Fig. 109.a.

PLATE. XXII.

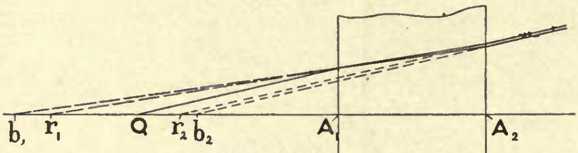


Fig. 104.

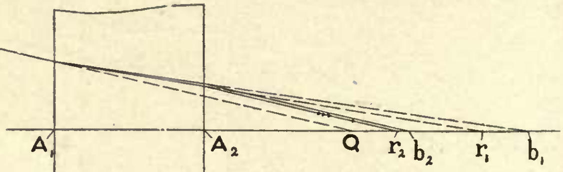


Fig. 104.a.

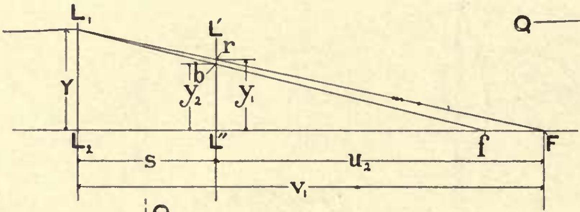


Fig. 105

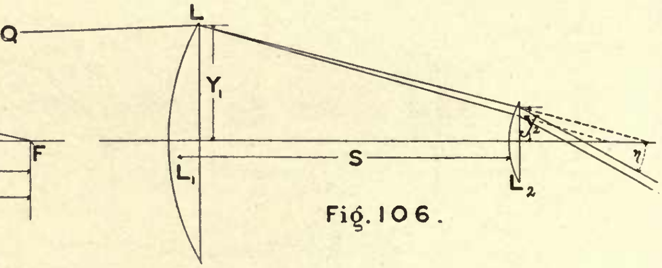


Fig. 106.

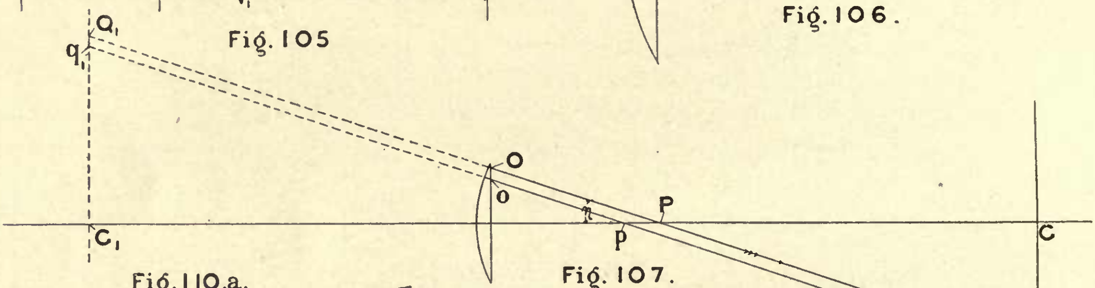


Fig. 110.a.

Fig. 107.


Fig. 110.b.

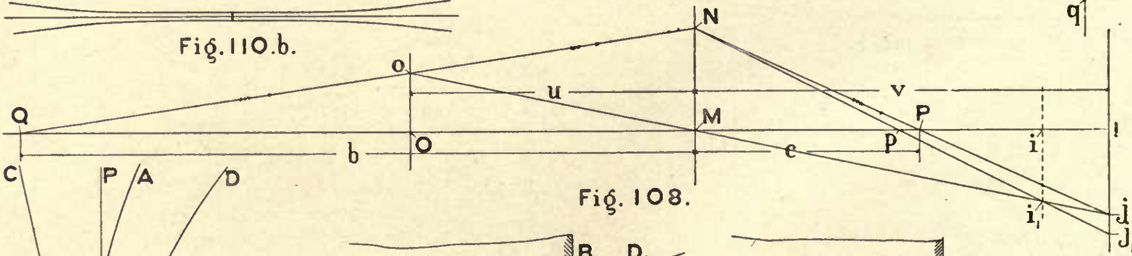


Fig. 108.

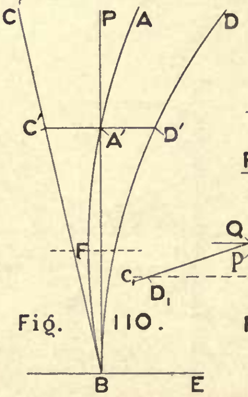


Fig. 110.

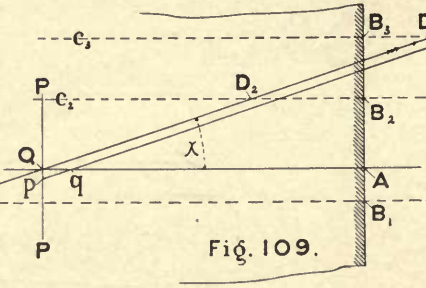


Fig. 109.

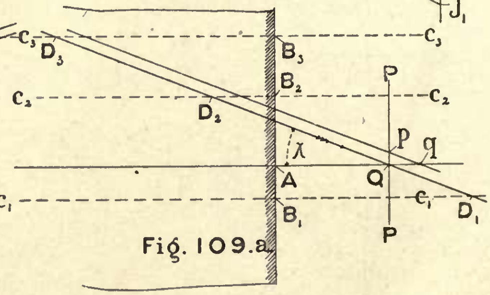


Fig. 109.a.

and the linear value $r_2 \,.\,.\, b_2$ of the aberration is simply

$$t\frac{\Delta\mu}{\mu_r^{\,2}}. \qquad \text{VI.}$$

The linear chromatic variation of v.

If $\Delta\mu$ refers to a large interval of spectrum and the glass is highly dispersive, it is more correct to write

$$t\frac{\Delta\mu}{\mu_r\mu_b} \text{ or } t\frac{\mu_b - \mu_r}{\mu_r\mu_b}. \qquad \text{VIa.}$$

The same line of reasoning applied to Fig. 104*a* leads to the same result, provided we consider v negative, so that in the case of Fig. 104 we have

$$\frac{1}{b_2 \,.\,.\, A_2} = \frac{1}{v} + \frac{\Delta\mu}{\mu_r^{\,2}v^2}t,$$

and in the case of Fig. 104*a* we have

$$\frac{1}{b_2 \,.\,.\, A_2} = -\frac{1}{v} + \frac{\Delta\mu}{\mu_r^{\,2}v^2}t.$$

In both cases we find the linear chromatic aberration $r_2 \,.\,.\, b_2$ ranges left to right; a plus increment to μ implies a transference of the focal point in the same direction as the light is travelling, and in this sense the effect of a parallel plate is similar to that of a dispersive lens—only with this difference, that while the chromatic aberration of a dispersive lens is $-\frac{1}{f}\frac{\Delta\mu}{\mu - 1}$ and thus independent of u or v, in the case of the parallel plate the chromatic aberration $\frac{\Delta\mu}{\mu_r^{\,2}v^2}t$ varies inversely as v^2, and of course vanishes when v becomes infinite and the rays parallel.

The dispersion always in one direction.

We might have arrived at the same result more shortly in this way. Since the linear transference of the focal point due to passage through a parallel plate is, as we have seen in Section I., $t\frac{\mu - 1}{\mu}$, then in differentiating with respect to μ we get $t\frac{\mu\Delta\mu - (\mu - 1)\Delta\mu}{\mu^2} = t\frac{\Delta\mu}{\mu^2}$, only we should have missed noting the effects taking place at each surface.

Chromatic Variation of the Spherical Aberration

So far we have studied the effects of μ being a variable upon the formulæ of the first approximation, and it is now desirable to investigate the effect of μ being a variable upon the spherical aberration of a lens. It is a subject of considerable importance in the theoretical designing of achromatic object glasses of larger aperture, and especially when of relatively short focal length. For it is somewhat futile to take elaborate precautions to ensure any two colours being refracted exactly to the same focal point by formulæ applying to ultimate central rays, and also get the spherical aberration of the rays of the one colour perfectly corrected and then allow the rays of the other colour to be subject to strong spherical aberration, thus to a large extent discounting the advantages of achromatism.

Importance of two colours being free from spherical aberration.

If we take the formula for spherical aberration,

$$\frac{1}{8f^3\mu(\mu-1)}\left\{\frac{\mu+2}{\mu-1}x^2+4(\mu+1)ax+(3\mu+2)(\mu-1)a^2+\frac{\mu^3}{\mu-1}\right\}y^2, \quad (4)$$

and for $\frac{1}{f^3}$ put $\left(\frac{1}{r}+\frac{1}{s}\right)^3(\mu-1)^3$, or simply $\left(\frac{\mu-1}{\rho}\right)^3$, since $\frac{1}{f}$ is a variable depending on μ, we may then write it in the form

$$\left.\begin{array}{r}\frac{1}{8\rho^3}\left\{\frac{(\mu+2)(\mu-1)}{\mu}x^2+\frac{4(\mu+1)(\mu-1)^2}{\mu}ax+\frac{(3\mu+2)(\mu-1)^3}{\mu}a^2\right. \\ \left.+\frac{\mu^3(\mu-1)}{\mu}\right\}y^2.\end{array}\right\} \quad (5)$$

On differentiating with respect to μ we shall then find that

$$d_\mu\left\{\frac{1}{8f^3}(\text{A}')y^2\right\}$$

$$\left.\begin{array}{r}=\frac{1}{8\rho^3}\left\{\left(1-\frac{2}{\mu^2}\right)x^2+4\left(2\mu-1-\frac{1}{\mu^2}\right)ax+\left(9\mu^2-14\mu+3+\frac{2}{\mu^2}\right)a^2\right. \\ \left.+(3\mu^2-2\mu)\right\}y^2d\mu.\end{array}\right\}\text{VII.}$$

Differential of the spherical aberration with respect to μ.

Supposing $\mu = 1{\cdot}5$ this works out to

$$\frac{1}{8\rho^3}\left\{\frac{1}{9}x^2+6\frac{2}{9}ax+3{\cdot}13a^2+3{\cdot}75\right\}y^2d\mu.$$

If $\frac{1}{\rho}=1$ (for a focal length of 2), $x = +1$, and $a = +1$, and $y=\frac{1}{4}$, then the formula works out to $+\frac{13}{128}d\mu$; and since $\cdot01$ is a very liberal allowance for $d\mu$ for the brighter part of the spectrum in the case of glasses of low refractive index, we then get

$$d\mu\left\{\frac{1}{8f^3}\text{A}'y^2\right\}=\frac{\cdot13}{128}=\text{ about }\cdot001.$$

But the spherical aberration in such a case would be

$$\frac{1}{8}\frac{1}{(2)^3\cdot 75}\left\{7+10+3\cdot 25+6\cdot 75\right\}\frac{1}{16}=\frac{1}{48}(27)\frac{1}{16}=\frac{27}{768}=\frac{1}{28},$$

or 36 times the above variation due to $d\mu$.

Such a small quantity as this might almost be neutralised by parabolising the curves of an object glass or the reverse if there were only rays of one colour to be dealt with; but it is clear that if we have perfect correction for spherical aberration for one colour, whether it be by a perfect balance of curves or by figuring, then a very minute amount of spherical aberration for another colour will be perceptible under high magnifying powers, so that the correct balancing of the spherical aberration for all colours as far as possible assumes a great importance. This means that in the case of a double achromatic object glass it is desirable to fulfil the condition

$$d_{\mu 1}\left\{\frac{1}{8f^3}A'_1 y_1\right\}-d_{\mu 2}\left\{\frac{1}{8f_2^{\,3}}A'_2 y_1\right\}=0\,; \qquad (6)$$

or if it does not or cannot equate to 0, then we must introduce another influence to effect it. In the case of an ordinary achromatic objective with the collective lens at the front and double convex, and the dispersive lens double concave or concavo-convex, but in close contact with the collective lens, it will be found that the chromatic variation of the spherical aberration as expressed shortly in (6), and in detail for the collective lens in Formula VII., is negative; that is, the dispersive lens exerts the greater influence, so that the more refrangible rays are over-corrected for spherical aberration when the less refrangible rays are accurately corrected.

The separation device adopted by Gauss.

Apparently Gauss was the first to point out that a separation between the two lenses could be made to neutralise this defect.

Let Fig. 105 represent two lenses separated, $L_1 \ldots L_2$ being half the collective lens and $L' \ldots L''$ half the dispersive lens, and $L_2 \ldots F$ the optic axis.

Let F be the focal point by first approximation for the red ray (ray C) and f the focal point for the blue ray (ray F) for the collective lens, so that $F \ldots f$, or shortly δ, is the linear chromatic aberration which

$$=\frac{1}{f_1}\cdot\frac{\Delta\mu_1}{\mu_1-1}v_1^{\,2}.$$

Let the semi-aperture of L_1 be Y and the semi-aperture of L_2 or the height $L'' \ldots r$ be y_1. We will assume the red ray $L_1 \ldots F$ to be the standard ray which gives the values $\frac{1}{f_1}$ and $\frac{1}{f_2}$.

Let $L_2 \ldots F$ be v_1 and $L'' \ldots F$ be u_2, and the separation $L_2 \ldots L''$ be s.

Let the height $L'' \ldots b$ to where the blue ray $L_1 \ldots f$ cuts the second lens be called y_2. Then our object is to express y_2 in terms of y_1, allowing for the dispersive effect of the first lens.

First we have

$$y_1 = Y\frac{u_2}{v_1}, \qquad (7)$$

$$y_2 = Y\frac{u_2 - \delta}{v_1 - \delta} = Y\left(u_2 - \frac{1}{f_1}\frac{\Delta\mu_1}{\mu_1 - 1}v_1^2\right)\left(\frac{1}{v_1} + \frac{1}{f_1}\frac{\Delta\mu_1}{\mu_1 - 1}\right)$$

$$\therefore \; y_2 = Y\left(\frac{u_2}{v_1} - \frac{1}{f_1}\frac{\Delta\mu_1}{\mu_1 - 1}v_1 + \frac{1}{f_1}\frac{\Delta\mu_1}{\mu_1 - 1}u_2\right)$$

$$= Y\left\{\frac{u_2}{v_1} - \frac{1}{f_1}\frac{\Delta\mu_1}{\mu_1 - 1}(v_1 - u_2)\right\} = Y\left\{\frac{u_2}{v_1} - \frac{1}{f_1}\frac{\Delta\mu_1}{\mu_1 - 1}s\right\}$$

$$= Y\frac{u_2}{v_1} - Y\frac{u_2}{v_1}\left(\frac{v_1}{u_2}\frac{\Delta\mu_1}{\mu_1 - 1}s\right), \text{ in which } Y\frac{u_2}{v_1} = y_1; \qquad (8)$$

so that

$$y_2 = y_1 - y_1\left(\frac{v_1}{u_2}\frac{\Delta\mu_1}{\mu_1 - 1}s\right) \text{ and } y_2 = y_1\left\{1 - \frac{v_1}{u_2}\frac{\Delta\mu_1}{\mu_1 - 1}s\right\};$$

so that finally

Two separated lenses. Value of y_2 in terms of y_1.

$$y_2^2 = y_1^2\left\{1 - 2\frac{v_1}{u_2}\cdot\frac{\Delta\mu_1}{\mu_1 - 1}s\right\} \qquad \text{VIII.}$$

So that for the second or dispersive lens the spherical aberration may be written shortly

$$y_1^2\left\{1 - 2\frac{v_1}{u_2}\frac{\Delta\mu_1}{\mu_1 - 1}s\right\}\frac{1}{8f_2^3}A'_2,$$

and since f_2 is negative, therefore the aberration is negative, and the variation

$$\left(-2\frac{v_1}{u_2}\frac{\Delta\mu_1}{\mu_1 - 1}s\right)y_1^2\frac{1}{8f_2^3}A'_2$$

comes out a positive one, and may be made to neutralise the negative result of Formula (6) as applied to the same combination.

Oblique images formed by a separated double objective cannot be achromatic.

Unfortunately, however, the separation between the two lenses of a double objective exercises a most prejudicial effect on the images a little removed from the axis; it is impossible to have the different coloured oblique images of any given star depicted at the same point on the focal plane, the blue images falling farther from the axis than the red (if the collective lens is at the front), while there is also a large amount of coma, so that the available field of good definition is very much restricted.

OBLIQUE ACHROMATISM AND CHROMATIC MAGNIFICATION

The foregoing remarks about the double separated objective brings us to the question of the conditions which determine whether an optical system forming an image of a real object, distant or otherwise, shall paint the said image on a dimensional scale which shall be independent of the colour or refrangibility of the various rays making up the pencils of white or mixed light diverging from the original object.

We have just noticed that an achromatic objective consisting of two separated lenses with the collective lens to the front is only achromatic for the axial image, and that the oblique image of a star is not a true image, that it is drawn out into a minute spectrum, the red end of which lies towards the optic axis. If the dispersive lens were at the front, then the opposite state of things would result, and the blue end of the spectrum would lie nearest to the optic axis.

It will be as well in the first instance to recapitulate the inquiry made by Sir George Airy and Henry Coddington into the conditions for securing oblique achromatism or equal magnification for the different colours that have to be fulfilled in the case of two-lens Huygenian or Ramsden eye-pieces or three- or four-lens erecting eye-pieces.

It is assumed in all such cases that the oblique pencils of rays emerging from such eye-pieces are made up of parallel rays, that is, that they are proceeding from an apparently very distant or infinitely distant virtual and magnified image.

Coloured constituents of principal rays emerging separated but parallel.

Such being the case, then it is clear that if the oblique image of any point of white light, such as a star, is to appear to the eye as one white image, then the variously coloured rays constituting the mixed oblique pencil must be emerging parallel to one another, and whether or not there happens to be any lateral separation of such variously coloured pencils of rays does not matter, provided that the virtual image is infinitely distant.

We saw in Section IX., pages 247 to 254, that freedom from distortion in such a case depended upon the ratio of the tangent of the angle of emergence of the principal rays to the tangent of the angle of incidence being a constant throughout the field of view, and that Formula IIA. was, for two lenses in succession—

$$\frac{\tan\eta}{\tan\epsilon}=\frac{b_1}{c_1}\frac{b_2}{c_2}\left[f+\frac{y_1^2}{4f_1^2}\left\{T_1'+\frac{1}{1-\beta}B_1'\right\}+\text{ etc.}\right].$$

Now if we differentiate the functions T′ and B′ for each lens with respect to μ, we shall find that the variation in the functions corresponding to $d\mu$ comes out very small compared to the functions themselves; we worked out a case on pages 288 and 289, where the chromatic variation in $\frac{1}{8f^3}A'y^2$ was only $\frac{1}{36}$th part of the latter, and in most cases likely to occur in practice it would amount to still less. Now the function A′ is almost exactly similar to B′. And since the distortion functions in eye-pieces rarely amount to more than 5 per cent of the radial dimensions of the image, it is not to be expected that $\frac{1}{36}$th part of that, or less, would be at all noticeable.

Chromatic variation of the distortion very small.

So that we need not in ordinary practice trouble ourselves about the chromatic variation of the distortion functions.

It is in the exterior magnification function $\frac{b_1 b_2 \ldots b_n}{c_1 c_2 \ldots c_n}$ (for n number of lenses) that we must look for the vastly more important chromatic variation; for it is plain enough that all the terms with the exception of b_1 are variables; they depend upon focal lengths, and the focal lengths are different for the different colours.

Conditions of oblique achromatism of eye-pieces for normal vision.

Let Fig. 106 represent two thin lenses L_1 and L_2 in succession, of focal lengths f_1 and f_2, of the same glass, and separated by an interval s (less than f_1). Let principal rays be diverging from an axial point Q to the left, so that $Q \ldots L_1 = b_1$. If these two lenses are used as an eye-piece for a telescope or microscope then Q will represent the centre of the objective. Also, in order to suit normal vision, the rays constituting the pencils of any one colour emerging from L_2 must be considered parallel, so that $v_2 = \infty$. In such case it is clear that if the variously coloured images are to appear all of the same size, then a multi-coloured principal ray, which enters the eye-piece all as one, must, after being split up by the first lens into a fan of diversely coloured rays, emerge from the second lens with such variously coloured constituent rays parallel to one another, when they will all appear to originate from one and the same point in the infinitely distant image.

Tan η not to vary with μ.

Therefore for all eye-pieces the condition for achromatism for oblique pencils is that tan η = constant for different values of μ; that is, that $d_\mu \tan\eta = 0$.

Therefore we first want to express tan η in terms of b_1, y_1, f_1, f_2, s, and y_2.

We have

$$\tan\eta = \frac{y_2}{c_2};\quad y_2 = y_1\frac{c_1 - s}{c_1};\quad \frac{1}{c_1} = \frac{1}{f_1} - \frac{1}{b_1} = \frac{b_1 - f_1}{f_1 b_1};$$

$$\therefore\ y_2 = y_1\left(\frac{b_1 f_1}{b_1 - f_1} - s\right)\frac{b_1 - f_1}{f_1 b_1}; \tag{9}$$

$$\frac{1}{c_2} = \frac{1}{b_2} + \frac{1}{f_2} = \frac{1}{c_1 - s} + \frac{1}{f_2} = \frac{f_2 + (c_1 - s)}{f_2(c_1 - s)}$$

$$= \frac{f_2 + \dfrac{b_1 f_1}{b_1 - f_1} - s}{f_2\left(\dfrac{b_1 f_1}{b_1 - f_1} - s\right)}; \tag{10}$$

$$\therefore\ \tan\eta = \frac{y_2}{c_2} = y_1\left(\frac{b_1 - f_1}{b_1 f_1} - s\right)\frac{b_1 - f_1}{f_1 b_1} \cdot \frac{f_2 + \dfrac{b_1 f_1}{b_1 - f_1} - s}{f_2\left(\dfrac{b_1 f_1}{b_1 - f_1} - s\right)}$$

$$= y_1 \frac{b_1 - f_1}{f_1 b_1}\left(1 + \frac{b_1 f_1}{b_1 - f_1}\frac{1}{f_2} - \frac{s}{f_2}\right);$$

$$\therefore\ \tan\eta = y_1\left\{\frac{1}{f_1} - \frac{1}{b_1} + \frac{1}{f_2} - \left(\frac{1}{f_1} - \frac{1}{b_1}\right)\frac{s}{f_2}\right\}. \tag{11}$$

Two-lens eye-piece. Value of $\tan\eta$.

We now have to differentiate this expression with respect to μ. Leaving out $\frac{1}{b_1}$, which is a constant, we have

$$d_\mu y_1\left\{\frac{1}{f_1} + \frac{1}{f_2} - \frac{s}{f_1 f_2} + \frac{s}{b_1 f_2}\right\} = 0. \tag{12}$$

The differential of $\frac{1}{f_1}$ is $\frac{1}{f_1}\frac{\Delta\mu}{\mu - 1}$, of $\frac{1}{f_2}$ is $\frac{1}{f_2}\frac{\Delta\mu}{\mu - 1}$, and of $\frac{1}{f_1 f_2}$ is $\frac{2}{f_1 f_2}\frac{\Delta\mu}{\mu - 1}$; therefore we have

$$\left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{2s}{f_1 f_2} + \frac{s}{b_1 f_2}\right)\frac{\Delta\mu}{\mu - 1} = 0;$$

$$\therefore\ \frac{2s}{f_1 f_2} - \frac{s}{b_1 f_2} = \frac{1}{f_1} + \frac{1}{f_2},\ \therefore\ 2s - s\frac{f_1}{b_1} = f_1 + f_2;$$

$$\therefore\ s\left(2 - \frac{f_1}{b_1}\right) = f_1 + f_2,$$

and

$$s = \frac{f_1 + f_2}{2 - \dfrac{f_1}{b_1}}. \qquad \text{IXA.}$$

Two-lens eye-piece. Separation necessary to oblique achromatism.

If b_1 is large relatively to f_1, then we arrive at the well-known rule of the separation being half the sum of the focal lengths. If b is relatively small, then the lenses must be more widely separated.

In order to secure better corrections for astigmatism, distortion, or coma, it is sometimes desirable that the two lenses of such an eye-piece shall be made of glasses of different dispersive power.

Let $\frac{\Delta M}{M-1}$ = the dispersive power of the first lens or field lens, and $\frac{\Delta\mu}{\mu-1}$ = the dispersive power of the second or eye lens, and let the dispersive ratio

$$\frac{\frac{\Delta M}{M-1}}{\frac{\Delta\mu}{\mu-1}} = r;$$

then it can be shown that

Condition of oblique achromatism for a two-lens eye-piece.

$$s = \frac{f_1 + rf_2}{(1+r) - \frac{f_1}{b_1}}; \qquad \text{IXB.}$$

from which it appears that a stronger dispersive power in the field lens leads to a smaller separation, and a stronger dispersive power in the eye lens to a greater separation.

It will scarcely be necessary here to recapitulate the much more complex and lengthy processes of the same nature which have to be gone through in order to arrive at the condition for oblique achromatism for eye-pieces consisting of three and four separated lenses. Let it suffice to simply state the results. The reader will find the investigation in full in Coddington's work, Part I., pages 259 to 268.

Condition of Oblique Achromatism for a Three-Lens Eye-piece

Let f_1, f_2, and f_3 be the principal focal lengths of the three lenses, all being made of the same sort of glass, and s_1 and s_2 the first and second separations, and b_1 for the first lens being assumed infinite or relatively large. Then the achromatic condition is

Condition of oblique achromatism for a three-lens eye-piece.

$$f_1f_2 + f_1f_3 + f_2f_3 - 2f_1s_2 - 2f_2s_1 - 2f_2s_2 - 2f_3s_1 + 3s_1s_2 = 0. \qquad \text{IXC.}$$

Condition of Oblique Achromatism for a Four-Lens Eye-piece

Let f_1, f_2, f_3, and f_4 be the principal focal lengths of the four lenses, all of the same sort of glass, and s_1, s_2, and s_3 the three separations in order, and let b_1 be considered infinite or relatively very large. Then the achromatic condition is

$$\left.\begin{aligned} & f_1f_2f_3+f_1f_2f_4+f_1f_3f_4+f_2f_3f_4 \\ & -2f_2f_3s_1-2f_2f_4s_1-2f_3f_4s_1 \\ & -2f_1f_3s_2-2f_1f_4s_2-2f_2f_3s_2-2f_2f_4s_2 \\ & -2f_1f_2s_3-2f_1f_3s_3-2f_2f_3s_3 \\ & +3f_3s_1s_2+3f_4s_1s_2+3f_2s_1s_3+3f_3s_1s_3+3f_1s_2s_3+3f_2s_2s_3 \\ & -4s_1s_2s_3 \end{aligned}\right\}=0. \qquad \text{X.}$$

Condition of oblique achromatism for a four-lens eye-piece.

The condition for a five-lens eye-piece works out to a very much more cumbersome formula.

Fig. 107 will help us to realise the very restricted usefulness of all these formulæ. It represents the last or eye lens of one of these eye-pieces, preferably that of a four-lens eye-piece.

Since the lenses are all simple, therefore the chromatic aberrations all sum up together, so that at the position about P, where the principal rays cross the optic axis and where a rough image of the object glass is formed, the crossing point p for the blue rays is very much nearer the lens than the crossing point P for the red rays. We have two oblique principal rays—one red, Q_1 . . O . . P . . Q, and one blue, $q_1 \ldots o \ldots p \ldots q$—which entered the eye-piece as *one* ray, finally emerging separately but parallel to one another, and to the normal eye with its pupil placed at P or p the two rays seem as one.

Lateral displacement of coloured constituents of the principal ray.

But supposing we wish to use the eye-piece for projecting a real image of what is seen in the telescope or microscope on to a screen G . . Q at a short distance to the right, and for that purpose draw out the eye-piece. It is perfectly clear that such an image cannot be achromatic, for the red ray will strike the screen at Q and the blue ray at q; so that the blue image of any extended object will be painted on a larger scale than the red image.

Real image larger for the more refrangible rays.

On the other hand, let it be supposed that a very short-sighted person uses the eye-piece. He will have to push the eye-piece farther in towards the objective, in order that the emergent rays of pencils may be divergent as though proceeding from a virtual image Q_1 . . G_1 8 or 10 inches to the left hand. It is again clear that such an image cannot appear achromatic, for the blue principal ray appears to be coming from a point q_1 nearer to the axis than the point Q_1 for the corresponding red ray; the red image is now painted on a larger scale than the blue image.

Virtual image larger for the less refrangible rays.

Supposing we want to project real or virtual images to or from finite distances, then what help or enlightenment can we possibly obtain from formulæ of the nature

Constancy of tangent ratios useless where real images are formed.

$$d_\mu \tan \eta = 0\,?$$

Such formulæ, however useful they may be for eye-pieces, are absolutely useless for working out the oblique achromatism of combinations, such as photographic lenses, which are expected to form real images of real objects at finite distances.

Formulæ of Perfectly General Application

We must therefore seek for a formula of perfectly general application, and in so doing may with advantage pursue the same method or line of reasoning that we followed in arriving at our general formula for distortion in the last Section.

In Fig. 108 let a principal ray radiate from Q and take the eccentric course Q . . N . . P . . j through the lens M . . N. We are supposing the lens free from spherical aberration and the tangent condition fulfilled, since we are discussing solely the effects of variations in the refractive index. Let there be an image formed at o . . O whose radial dimension is o. From o draw o . . M through the centre of the lens, and produce it to cut the conjugate image plane I . . j at j.

Let it be assumed that the principal ray Q . . N . . j is of the standard colour, for which the refractive index μ applies, and that the conjugate images o . . O and I . . j also apply to rays of the same standard colour. It is clear that another more refrangible coloured ray coincident with Q . . N before refraction will take a different course N . . p . . j_1 after refraction, and j . . j_1 will be the linear dispersion between the two colours. Now what we want is an expression for j . . j_1 in terms of I . . j, or the radial dimension of the blue image in terms of the radial dimension of the corresponding red image, supposing we fix upon those two colours. Let I . . j be i, and I . . j_1 be i_1, and o . . O be o; Q . . M be b, O . . M be u, M . . P be c, M . . I be v, and M . . p be c_1.

Then we have $i = o\frac{v}{u}$,

$$\text{also } o\frac{b}{b-u} = \text{M . . N} = i\frac{c}{v-c};$$

$$\therefore i = o\frac{b}{b-u}\cdot\frac{v-c}{c} = o\frac{b}{c}\cdot\frac{v-c}{b-u} = o\frac{v}{u}, \tag{13}$$

and

$$i_1 = o\frac{b}{b-u}\cdot\frac{v-c_1}{c_1}, \text{ wherein } c_1 = c - \frac{1}{f}\frac{\Delta\mu}{\mu-1}c^2;$$

so that

$$i_1 = o\frac{b}{b-u}\times\frac{v-\left(c-\frac{1}{f}\frac{\Delta\mu}{\mu-1}c^2\right)}{c-\frac{1}{f}\frac{\Delta\mu}{\mu-1}c^2};$$

$$\therefore\ i_1 = o\frac{b}{b-u}\left(v - c + \frac{1}{f}\frac{\Delta\mu}{\mu-1}c^2\right)\left(\frac{1}{c} + \frac{1}{f}\frac{\Delta\mu}{\mu-1}\right)$$

$$= o\frac{b}{b-u}\left\{\frac{v-c}{c} + \frac{1}{f}\frac{\Delta\mu}{\mu-1}c + (v-c)\frac{1}{f}\frac{\Delta\mu}{\mu-1}\right\}$$

$$= o\frac{b}{b-u}\left\{\frac{v-c}{c} + \frac{v}{f}\frac{\Delta\mu}{\mu-1}\right\}$$

$$= o\frac{b}{b-u}\cdot\frac{v-c}{c}\left\{1 + \frac{cv}{v-c}\frac{\Delta\mu}{\mu-1}\frac{1}{f}\right\},$$

in which the outside function $= O\frac{v}{u}$, from Formula (13);

$$\therefore\ i_1 = o\frac{v}{u}\left\{1 + \frac{cv}{v-c}\frac{1}{f}\frac{\Delta\mu}{\mu-1}\right\}. \tag{14}$$

On adopting Coddington's device we find that

$$\frac{cv}{v-c} = \frac{\frac{2f}{1-\beta}\cdot\frac{2f}{1-\alpha}}{\frac{2f}{1-\alpha} - \frac{2f}{1-\beta}} = \frac{2f}{\alpha-\beta};$$

so that finally we get

$$\frac{i_1}{o} = \frac{v}{u}\left\{1 + \frac{2}{\alpha-\beta}\cdot\frac{\Delta\mu}{\mu-1}\right\}, \qquad \text{XI.}$$

Single lens. Universal formula for ratio between object and coloured image of same.

a very simple and convenient formula which can be applied with the greatest ease to any number of lenses or elements in series. The term f has disappeared, but its value is really implied in $\alpha - \beta$, which terms are, of course, assessed with regard to the ray of standard colour. On applying the same line of reasoning to the corresponding case of a dispersive lens, or any other cases whatever, exactly the same formula will be arrived at.

An objection.

An objection may be raised to the above formula on the ground that $\alpha - \beta$ is in itself a variable, for it varies as f, which varies inversely as $\mu - 1$; but if we insert the variation in $\alpha - \beta$, we then get for our formula—

$$\frac{i_1}{o} = \frac{v}{u}\left\{1 + \frac{2}{(\alpha-\beta)\left(1 - \frac{\Delta\mu}{\mu-1}\right)}\frac{\Delta\mu}{\mu-1}\right\}$$

$$= \frac{v}{u}\left\{1 + \frac{2}{\alpha-\beta}\left(\frac{\Delta\mu}{\mu-1} + \frac{(\Delta\mu)^2}{(\mu-1)^2}\right)\right\}.$$

The correction involved is thus seen to be of the order $\left(\frac{\Delta\mu}{\mu-1}\right)^2$ or the square of what is already a very small quantity. Hence it may be legitimately neglected.

In applying Formula XI. to a series of lenses in succession, it is clear that a lens will copy or transfer forward any want of chromatic conformity in the radial dimensions of any image presented to it by the preceding lens or lenses, and at the same time add its own chromatic error, and so on. Therefore the expression for a series of n lenses is

Universal formula for ratio between object and final coloured image for a series of lenses.

$$\frac{i_n}{o} = \frac{v_1 v_2 \ldots v_n}{u_1 u_2 \ldots u_n}\left\{1 + \frac{2}{a_1 - \beta_1}\cdot\frac{\Delta\mu_1}{\mu_1 - 1} + \frac{2}{a_2 - \beta_2}\cdot\frac{\Delta\mu_2}{\mu_2 - 1}\ldots + \frac{2}{a_n - \beta_n}\cdot\frac{\Delta\mu_n}{\mu_n - 1}\right\}. \quad \text{XII.}$$

Then, if all the lenses are made of the same sort of glass, the condition of oblique achromatism is simply

Series of lenses. Condition of oblique achromatism when all of same dispersive power.

$$\frac{1}{a_1 - \beta_1} + \frac{1}{a_2 - \beta_2} + \ldots + \frac{1}{a_n - \beta_n} = 0. \qquad \text{XIII.}$$

Let us apply this formula to the ordinary Huygenian eye-piece wherein $f_1 = 3$, $f_2 = 1$, and which we have seen is achromatic when $s = 2$, provided that $b_1 = \infty$. Then we have

$$b_1 = \infty \quad \text{and } \beta_1 = -1 \qquad b_2 = -1 \quad \text{and } \beta_2 = -3$$

$$v_1 = +1 \qquad \therefore\ u_1 = -1{\cdot}5 \qquad \therefore\ a_1 = -5 \qquad u_2 = f_2 \quad \therefore\ a_2 = +1$$

$$a_1 - \beta_1 = -4 \qquad \text{and } a_2 - \beta_2 = +4$$

$$\therefore\ \frac{1}{a_1 - \beta_1} + \frac{1}{a_2 - \beta_2} = 0.$$

Axially, however, the Huygenian eye-piece is perceptibly under-corrected for colour, for although the variously coloured images are of the same size on an infinitely distant plane for the standard colour, yet they are formed in greatest distinctness in different planes.

Four-lens erecting eye-piece.

Next let us take the case of a four-lens erecting eye-piece given on p. 266, Part I., of Coddington's work, which fulfilled the condition

$$\Delta\mu\frac{\tan\eta}{\tan\epsilon} = 0.$$

The focal lengths of the lenses were

$$f_1 = 3 \qquad s_1 = 4 \qquad f_2 = 4 \qquad s_2 = 6 \qquad f_3 = 4 \qquad s_3 = 5{\cdot}13 \qquad f_4 = 3$$

From these data we get

$$\begin{array}{lll} u_1 = +1{\cdot}35 & v_1 = -\ 2{\cdot}44 & \therefore\ a_1 = +3{\cdot}46 \\ u_2 = +6{\cdot}44 & v_2 = +10{\cdot}55 & \therefore\ a_2 = +\ {\cdot}24 \\ u_3 = -4{\cdot}55 & v_3 = +\ 2{\cdot}13 & \therefore\ a_3 = -2{\cdot}75 \\ u_4 = +3 & v_4 = \infty & \therefore\ a_4 = +1 \end{array}$$

$$\begin{array}{lll} b_1 = \infty & c_1 = +3 & \therefore\ \beta_1 = -1 \\ b_2 = +1 & c_2 = -1\tfrac{1}{3} & \therefore\ \beta_2 = +7 \\ b_3 = +7\tfrac{1}{3} & c_3 = +8{\cdot}8 & \therefore\ \beta_3 = +\ {\cdot}09 \\ b_4 = -3{\cdot}67 & c_4 = +1{\cdot}65 & \therefore\ \beta_4 = -2{\cdot}63 \end{array}$$

$$(\alpha_1 - \beta_1) = +4{\cdot}46 ; (\alpha_2 - \beta_2) = -6{\cdot}76 ; (\alpha_3 - \beta_3) = -2{\cdot}84 ; (\alpha_4 - \beta_4) = +3{\cdot}63 ;$$

$$\therefore \frac{1}{\alpha_1 - \beta_1} + \frac{1}{\alpha_2 - \beta_2} + \frac{1}{\alpha_3 - \beta_3} + \frac{1}{\alpha_4 - \beta_4}$$

$$= \frac{1}{4{\cdot}46} + \frac{1}{3{\cdot}63} - \left(\frac{1}{6{\cdot}76} + \frac{1}{2{\cdot}84}\right) = \frac{8{\cdot}09}{16{\cdot}18} - \frac{9{\cdot}6}{19{\cdot}20} = 0 .$$

So that the final images formed by rays in different colours are all of the same size as thrown on the infinitely distant plane, although formed in different planes, for parallel to the axis the eye-piece is very far from being achromatic. But this imperfection is generally neutralised by giving to the object glass with which such an eye-piece is used an equal amount of over-corrected colour aberration.

Axial colour aberration strong.

Of course, the axial colour aberration of a four-lens eye-piece is much more serious than that of a Huygenian or Ramsden eye-piece. Such eye-pieces are thus only achromatic in the sense that the variously coloured images appear to be of the same size.

Oblique Chromatic Aberration of a Parallel Plane Plate

As very thick lenses must be treated by the method of elements and parallel plane plates before we can accurately apply these formulæ, we must next work out the expression for the chromatic variation in the size of an image viewed through or thrown through a parallel plane plate.

Fig. 109 is a case of principal rays diverging through a parallel plate and emanating from a real flat object or image P . . P, and Fig. 109*a* the case of rays converging through the plate towards a flat image P . . P on the right hand.

Let Q be the point from which the rays are diverging or to which they are converging, *after* passage through the plate. From Q draw Q . . A perpendicular to the surfaces.

Then we have seen from Formula VI. that the linear dispersion Q . . q is $t\frac{\Delta\mu}{\mu^2}$.

Now if χ is the angle (as usual) made by the ray in question with the perpendicular to the plate, then it is clear that the lateral chromatic displacement in the plane of the image is simply

$$t\frac{\Delta\mu}{\mu^2}\tan\chi = Q..p,$$

which must then be expressed in terms of the radial dimensions of the image. In order to express the radial dimension of the image we must know where the optic axis of the system lies.

Location of the optic axis.

In Fig. 109, if the optic axis is $B_1..D_1..c_1$, then we have $B_1..P = v$, and $B_1..D_1 = C$, and the distance from Q to $B_1..D_1$ or $A..B_1$ is the radial dimension of the image, which obviously equals $(C - v)\tan\chi$, and is $+$.

If $B_2..D_2..c_2$ is the optic axis, then $B_2..P = v$, and $B_2..D_2 = C$, and $B_2..A$ is the radial dimension of the image, which $= (C - v)\tan\chi$, and is $-$.

If $D_3..B_3..c_3$ is the optic axis, then $B_3..P = v$, and $B_3..D_3 = C$, and $A..B_3$ is the radial dimension of the image, which $= (C - v)\tan\chi$, and is $-$.

Three successive positions for the optic axis are likewise shown in Fig. 109*a*.

Conventions.

By our convention for parallel plates we have

$B_1..P$ or v is a plus quantity in Fig. 109,
and a minus quantity in Fig. 109*a*.
$B_1..D_1$ or C is plus in Fig. 109, and minus in Fig. 109*a*.
$B_2..D_2$ or C is plus in Fig. 109, and minus in Fig. 109*a*.
$B_3..D_3$ or C is minus in Fig. 109, and plus in Fig. 109*a*.

With reference specially to the lowest optic axis $B_1..D_1..c_1$, all terms are of the same sign, and we have

$$(A..B_1) - (Q..p) = (C - v)\tan\chi - t\frac{\Delta\mu}{\mu^2}\tan\chi = \left((C - v) - t\frac{\Delta\mu}{\mu^2}\right)\tan\chi$$
$$= p..c_1, \quad (15)$$

or the reduced radial dimension of image, due to the increment to μ. On dividing this expression by $A..B_1$ we have

Parallel plane plate. Ratio between differently coloured images.

$$\frac{p..c_1}{A..B_1} = \left\{(C - v) - t\frac{\Delta\mu}{\mu^2}\right\}\tan\chi \cdot \frac{1}{(C - v)\tan\chi};$$

$$\therefore \frac{p..c_1}{A..B_1} = 1 - t\frac{\Delta\mu}{\mu^2}\left(\frac{1}{C - v}\right). \qquad \text{XIV.}$$

This formula will be found to interpret itself correctly in all cases if the signs of C and v are entered in strict accordance with our conventions.

Illustrations of the conventions.

In Fig. 109, Case 1, C is $+$ and largest, and v is $+$;
$\therefore$ $C - v$ is plus, and $Q..p$, the radial dispersion, is relatively minus.
In Case 2, C is $+$ and smallest, and v is plus;
$\therefore$ $C - v$ is minus, and $Q..p$ is relatively plus.

In Case 3, C is minus, and v is plus;
$\therefore$ C − v is minus, and Q . . p is relatively plus.
In Fig. 109a, Case 1, C is minus, and v is minus, but numerically smaller;
$\therefore$ C − v is minus, and Q . . p plus.
Case 2, C is plus, and v is minus, but greater;
$\therefore$ C − v is plus, and Q . . p is relatively minus.
Case 3, C is plus, and v is minus, but smaller;
$\therefore$ C − v is plus, and Q . . p is relatively minus.

Practical application of the formulæ to the process lens.

As an actual instance of the practical application of the formulæ which we have arrived at for both axial and oblique achromatism, we cannot do better than take the case of the process lens of $8\frac{1}{2}$ in. E.F.L., Fig. 59, whose curves and other data were given on pages 185 and 186.

First we will deal with the axial achromatism by Formulæ III. to VI. of this Section.

The spectrum interval is C to F, and the data are

$$\mu_1 = 1{\cdot}6103 \text{ for the D ray, and } \Delta\mu_1 = {\cdot}01080, \text{ C to F}$$
$$\mu_2 = 1{\cdot}5240 \quad ,, \quad ,, \quad \text{and } \Delta\mu_2 = {\cdot}01028, \quad ,,$$

so that

$$\frac{\Delta\mu_1}{\mu_1 - 1} = \frac{1}{56{\cdot}5} \quad \text{and} \quad \frac{\Delta\mu_2}{\mu_2 - 1} = \frac{1}{51}.$$

The v's and u's are

$$\begin{array}{lll} v_1 = +2{\cdot}071 & v_2 = -11{\cdot}6067 & v_3 = +4{\cdot}90 \\ v_4 = +1{\cdot}1008 & v_5 = +14{\cdot}032 & v_6 = +8{\cdot}603 \\ u_1 = \infty & u_2 = +2{\cdot}006 & u_3 = +11{\cdot}375 \\ u_4 = +5{\cdot}122 & u_5 = +1{\cdot}095 & u_6 = +14{\cdot}105 \end{array}$$

Axial chromatic errors.

By Formula IV. we have for the sum of the chromatic aberrations of all the elements, all referred to the last element,

First element.

$$\frac{1}{f_1}\frac{\Delta\mu_1}{\mu_1 - 1}\left(\frac{v_1v_2v_3v_4v_5}{u_2u_3u_4u_5u_6}\right)^2 \text{ for the first element } = +{\cdot}008675,$$

Second element.

$$\frac{1}{f_2}\frac{\Delta\mu_1}{\mu_1 - 1}\left(\frac{v_2v_3v_4v_5}{u_3u_4u_5u_6}\right)^2 \text{ for the second element } = -{\cdot}045520,$$

Third element.

$$\frac{1}{f_3}\frac{\Delta\mu_1}{\mu_1 - 1}\left(\frac{v_3v_4v_5}{u_4u_5u_6}\right)^2 \text{ for the third element } = -{\cdot}004726,$$

Fourth element.

$$\frac{1}{f_4}\frac{\Delta\mu_1}{\mu_1 - 1}\left(\frac{v_4v_5}{u_5u_6}\right)^2 \text{ for the fourth element } = +{\cdot}01952,$$

Fifth element.

$$\frac{1}{f_5}\frac{\Delta\mu_2}{\mu_2 - 1}\left(\frac{v_5}{u_6}\right)^2 \text{ for the fifth element } = -{\cdot}019098,$$

Sixth element.

$$\text{and } \frac{1}{f_6}\frac{\Delta\mu_2}{\mu_2 - 1} \text{ for the sixth element } = +{\cdot}003670.$$

On adding together the six colour aberrations we get

Totals.

E_1	$+ \cdot 008675$	$E_2 =$	$- \cdot 0069497$
E_4	$+ \cdot 019521$	$E_3 =$	$- \cdot 0047265$
E_6	$+ \cdot 003670$	$E_5 =$	$- \cdot 019098$
	$+ \cdot 031866$		$- \cdot 030774$
	$- \cdot 030774$		
Total $=$	$+ \cdot 001092$		

Chromatic errors of the three parallel plates.

We have now to add the chromatic aberrations of the three parallel plates. Formula V. gives us $t_1 \frac{\Delta\mu_1}{\mu^2} \frac{1}{v^2}$ for the first plate, in which v is the same quantity as u_2 of the second element. In order to transfer this chromatic correction to the sixth element we must obviously multiply by $\left(\frac{v_2 v_3 v_4 v_5}{u_3 u_4 u_5 u_6}\right)^2$ just as we did for the second element; so that the chromatic aberrations for the three parallel plates must be stated as

First plate.

$$\left(t_1 \frac{\Delta\mu_1}{\mu_1{}^2} \frac{1}{u_2{}^2}\right)\left(\frac{v_2 v_3 v_4 v_5}{u_3 u_4 u_5 u_6}\right)^2 = - \cdot 0010349. \qquad (a)$$

Second plate.

$$\left(t_2 \frac{\Delta\mu_1}{\mu_1{}^2} \frac{1}{u_4{}^2}\right)\left(\frac{v_4 v_5}{u_5 u_6}\right)^2 = - \cdot 0000568. \qquad (b)$$

Third plate.

$$\left(t_3 \frac{\Delta\mu_2}{\mu_2{}^2} \cdot \frac{1}{u_6{}^2}\right) = - \cdot 0000024. \qquad (c)$$

So, finally, we have

Chromatic aberrations of the three plates $=$	$- \cdot 001094$
Chromatic aberrations of the six elements $=$	$+ \cdot 001091$
Total . .	$- \cdot 000003$

Final total.

On multiplying this final result by $-(v_6)^2$ or the square of the back focal length, we then get a small residue of over-corrected chromatic aberration equal to about ·00022, which is a negligible quantity.

Taking next the oblique chromatic corrections, we have for the six elements, by Formula XII., also for the spectrum interval C to F,

Oblique colour errors for the two collective lenses.

$$\left.\begin{array}{l} \frac{\Delta\mu_1}{\mu_1 - 1} \cdot \frac{2}{\alpha_1 - \beta_1} = \\ \frac{\Delta\mu_1}{\mu_1 - 1} \frac{2}{\alpha_2 - \beta_2} = \\ \frac{\Delta\mu_1}{\mu_1 - 1} \frac{2}{\alpha_3 - \beta_3} = \\ \frac{\Delta\mu_1}{\mu_1 - 1} \frac{2}{\alpha_4 - \beta_4} = \end{array}\right\} \left\{\frac{1}{15 \cdot 796} - \frac{1}{26 \cdot 527} + \frac{1}{132 \cdot 313} - \frac{1}{6 \cdot 2722}\right\} \frac{2}{56 \cdot 5} = - \cdot 00447$$

$$\left.\begin{array}{l}\dfrac{\Delta\mu_2}{\mu_2-1}\dfrac{2}{a_5-\beta_5}= \\ \dfrac{\Delta\mu_2}{\mu_2-1}\dfrac{2}{a_6-\beta_6}=\end{array}\right\}=\left\{\frac{1}{6{\cdot}974}-\frac{1}{29{\cdot}119}\right\}\frac{2}{51} \qquad = +{\cdot}004277.$$

Oblique colour error for the dispersive lens.

To these we must add the three parallel plate corrections by Formula XIV.

For the first plate we have

$C = b_2$ and is therefore convergent and minus,
$v = u_2$ and is therefore convergent and minus;

so that

$$C - v = -{\cdot}16756 + 2{\cdot}006 = 1{\cdot}838;$$

$$\therefore\ -t_1\frac{\Delta\mu_1}{\mu_1^2}\frac{1}{C-v} = -{\cdot}0002379,$$

Oblique colour error of first plate.

t_1 being $\cdot 105$, $\Delta\mu$ being $\cdot 0108$, and μ being $1{\cdot}6103$.

For the second plate we have

$C = b_4$ and is therefore divergent and plus,
$v = u_4$ and is therefore divergent and plus;

so that

$$C - v = +{\cdot}2735 - 5{\cdot}122 = -4{\cdot}8485,$$

and

$$-t_2\frac{\Delta\mu_1}{\mu_1^2}\frac{1}{C-v} = +{\cdot}0003075,$$

Oblique colour error of second plate.

t_2 being $\cdot 358$.

For the third plate we have

$C = b_6$ and is therefore divergent and plus,
$v = u_6$ and is therefore divergent and plus;

so that

$$C - v = +{\cdot}3577 - 14{\cdot}1046 = -13{\cdot}747,$$

and

$$-t_3\frac{\Delta\mu_2}{\mu_2^2}\frac{1}{C-v} = +{\cdot}000035417,$$

Oblique colour error of third plate.

t_3 being $\cdot 110$, $\Delta\mu_2$ being $\cdot 01028$, and μ_2 being $1{\cdot}524$.

So that, finally, we have

The chromatic errors for six elements	$= -{\cdot}000193$
The chromatic errors for the three plates	$= +{\cdot}000105$
Final total . .	$= -{\cdot}000088$

Total oblique colour error for whole system.

If we take a point 4 inches from the axis we have a chromatic difference in the radial dimension of the image equal to

Linear value of above, four inches from axis.

$$4(-\cdot000088) = -\cdot000352 \text{ inch,}$$

which is an imperceptible amount, and as a matter of fact no oblique colour aberration was noticeable in the image under the most careful tests.

Cooke Photographic Lenses

Any of the wider-angled Cooke lenses of three simple lenses afford capital illustrations of the practical embodiment of the condition

$$\frac{2}{\alpha_1 - \beta_1} \cdot \frac{\Delta\mu}{\mu - 1} + \frac{2}{\alpha_2 - \beta_2} \cdot \frac{\Delta M}{M - 1} + \frac{2}{\alpha_3 - \beta_3} \cdot \frac{\Delta\mu}{\mu - 1} = 0,$$

for the normal arrangement of the combination implies two collective lenses of the same glass and of focal lengths f_1 and f_3, enclosing between them a dispersive lens of focal length f_2, the two separations s_1 and s_2 being proportional to f_1 and f_3 respectively, and also the distances from the object to L_1 and from L_3 to the image are proportional to f_1 and f_3 respectively; therefore everything is symmetrical with respect to the centre of L_2, where the principal rays are supposed to cross the optic axis. Thus $\frac{1}{\alpha_2 - \beta_2} = \frac{1}{\infty} = 0$, and obviously $\frac{1}{\alpha_1 - \beta_1} = -\frac{1}{\alpha_3 - \beta_3}$; so that above equation is fulfilled, and the oblique image is achromatic, and remains practically so under all conditions.

Oblique Chromatic Corrections of a Higher Order

On reverting to the effect of separation between two lenses upon the spherical aberrations of the second lens for different colours, which on page 289 we worked out with special reference to an object glass, arriving at Formula VIII., we can easily see that if the separation becomes large compared with the focal length of the first lens, then the variation in the second y, consequent upon $d\mu$, may become very serious, possibly reducing it by a quarter or a third; so that y_2 for the blue rays may be, for instance, $\frac{7}{10}$ths of the y_2 for the red rays, which would mean that y_2^2 for blue would be but a half of y_2^2 for red; and therefore, roughly speaking, the spherical aberration of the second lens for the blue (principal) rays falling upon it would be only half of the spherical aberration for the red rays.

Distortion of each lens affected by the chromatic errors of preceding lenses.

This means that that part of the distortion formula for the second lens depending on its spherical aberration will be seriously modified in accordance with the colour variation of the preceding lens; that is, y_2 will be modified in accordance with Formula VIII., and

β_2 in accordance with another, which we scarcely need work out, for the main point is that these colour aberrations affecting the spherical aberration distortions for each of several lenses in succession, excepting the first, are corrections of the order y^2, and it is clear that they must come into force in the familiar case of our four-lens eye-piece, and especially when the second separation is largely increased for the purpose of gaining magnifying power. But we have already seen that the oblique chromatic errors of the second order of approximation are of the form

$$\frac{v_1 \ldots v_n}{u_1 \ldots u_n}\left(1 + \frac{2}{a_1 - \beta_1}\,\frac{\Delta\mu_1}{\mu_1 - 1} \cdots + \frac{2}{a_n - \beta_n}\,\frac{\Delta\mu_n}{\mu_n - 1}\right),$$

so that the absolute radial colour aberrations, if any, are thus a constant percentage of the radial dimensions of the final image.

But the variations in the distortions due to spherical aberration of any lens in a separated series, caused by the colour aberrations of the preceding lens or lenses, are of the order y^2, as shown in Formula VIII.

Hybrid Oblique Colour Aberrations

It is then of importance to inquire what will happen if in a four-lens eye-piece we have a residue of oblique colour aberration of the second order, or of the order y, as we may conveniently term it, which is either accidentally or intentionally corrected by aberrations of the third order y^2, but of the opposite sign. Fig. 110 illustrates what we should expect to be the result. Let B . . P be an axis of measurement so that the horizontal distances from B . . P to the oblique straight line B . . C shall represent the oblique colour aberrations of the second order y, which thus increase directly as the vertical distances from B, which latter represent y as well as the radial dimensions of the image. At the other side of B . . P we have the curve B . . D, its abscissæ increasing as the square of the heights above the optic axis B . . E. It is thus seen to be approximately a circular curve, and represents the oblique chromatic errors of the third order y^2.

Effect of correcting a chromatic error of the order y by another of the order y^2.

At the height B . . A′ we have the abscissæ A′ . . C′ and A′ . . D′ equal and opposite, so that the curve B . . A′ . . A, which is the resultant of the two, will then cross B . . P at A′. It will easily be seen that the resultant curve B . . A′ . . A is also a circular one. While at A′ we have no colour aberration, yet at F, half-way between B and A′, we get a maximum of colour aberration of the same sign as the original aberration of the second order; while at points in the image outside of A′ we get a colour aberration of the opposite sign to that of the original aberration of the second order, and increasing as the square of the distance from A′. Thus we may get a final image which in a middle zone of the field of view is achromatic, but half-way between that zone and the centre shows slight colour aberration, the blue image being, for instance, the largest, while round the margin of the field of view the red image is largest.

Zones of oblique chromatic error.

Such irrationalities between corrections of two different orders are very liable to show themselves in very long eye-piece combinations, presenting a large field of view, not only with respect to colour aberrations and distortion, but also with respect to the coma and corrections for curvature of image.

It will now be seen that the optical theory of a four-lens eye-piece is very much more complex than it appears to be at first sight.

The Secondary Spectrum

So far we have dealt with the different effects of lenses and systems of lenses upon rays of only two colours whose refractive indices differ from one another by $\Delta\mu_1$ for one glass, and by $\Delta\mu_2$ for another glass, and so on; and if we have considered any rays intermediate between such two selected rays, it has been on the tacit understanding that if μ_1 = the refractive index for one ray, and $\mu_1 + \Delta\mu_1$ that for the other, and again, if $\mu_1 + \Delta'\mu_1$ = the refractive index for an intermediate ray for one glass, and $\mu_2 + \Delta'\mu_2$ the refractive index for the same intermediate ray for the other glass, then we have assumed that

$$\frac{\Delta'\mu_1}{\Delta'\mu_2} = \frac{\Delta\mu_1}{\Delta\mu_2},$$

or that the dispersive ratio between the two glasses for one part of the spectrum interval chosen is equal to the dispersive ratio for the other part of the spectrum interval.

A constant ratio of dispersions for different parts of the spectrum between two glasses hitherto assumed.

Unfortunately, however, there are no two glasses differing in dispersive power sufficiently to be combined into an achromatic object glass which have a constant ratio of dispersive power for different regions of the spectrum, and it is this *irrationality of dispersion*, as it is called, which gives rise to that residual colour aberration at the axial focus which is well known as the "Secondary Spectrum."

Irrationality of dispersion.

The following table gives the difference of refractive indices $\Delta_1\mu$, $\Delta_2\mu$, $\Delta_3\mu$, $\Delta_4\mu$, etc., etc., for ordinary crown glass and ordinary dense flint glass respectively for the spectrum intervals D to A′, F to D, C to F, and F to G′.

	D to A′. Red and Orange.		F to D. Yellow and Green.		C to F. Red to Green.		F to G′. Green to Blue.	
Crown	− ·00553	·643	− ·00605	·703	+ ·00860	1·000	+ ·00487	·566
	$\Delta\mu_1$	$\vee$	$\Delta\mu_2$	$\wedge$	$\Delta\mu_3$	$\|$	$\Delta\mu_4$	$\wedge$
Flint .	− ·01034	·605	− ·01220	·714	+ ·01709	1·000	+ ·01041	·609

Proportional sectional dispersions for crown and flint glasses.

As experience has shown that about the best working achromatism is secured when the two rays C and F are brought to one focus, therefore a contact combination of the above two glasses is so arranged that

$$\frac{\cdot 00860}{\rho_1} - \frac{\cdot 01709}{\rho_2} = 0, \tag{16}$$

Condition for bringing C and F rays to one focus.

where $\dfrac{1}{\rho_1} = \left(\dfrac{1}{r_1} + \dfrac{1}{s_1}\right)$ for the crown glass lens, and $\dfrac{1}{\rho_2} = \left(\dfrac{1}{r_2} + \dfrac{1}{s_2}\right)$ for the flint glass lens.

The dispersive interval C to F is generally taken as unity for each glass; then clearly any other dispersive interval may be expressed in terms of the former. Accordingly, the figures in the second column for each dispersive interval express the latter in terms of the dispersive interval C to F. In this way it is clearly shown that for the interval D to A′ the crown glass exercises a relatively higher dispersion than the flint glass, for the region F to D the flint has the relatively higher dispersion, while for F to G′ the flint has very decidedly the higher dispersion.

The interval C to F, or $\Delta\mu$(C to F), usually taken as unity.

It is clear that if Formula (16) is fulfilled, and the two rays C and F are refracted to the same focus, then the linear secondary spectrum at the principal focus yielded by the objective will be, as a variation of F,

$$-F^2\left(-\frac{\cdot 00553}{\rho_1} + \frac{\cdot 01034}{\rho_2}\right) \text{ for the interval D to A}',$$

Formula for the chromatic error D to A′.

$$-F^2\left(-\frac{\cdot 00605}{\rho_1} + \frac{\cdot 01220}{\rho_2}\right) \text{ for the interval F to D},$$

Formula for the chromatic error F to D.

and

$$-F^2\left(\frac{\cdot 00487}{\rho_1} - \frac{\cdot 01041}{\rho_2}\right) \text{ for the interval F to G}';$$

Formula for the chromatic error F to G′.

and it is clear that there will be prevailing dispersion of the crown lens along the axis from D to A′, the A′ ray focusing beyond the D ray; from F to D the dispersion of the flint lens will predominate, and the D ray will focus inside of C and F; while for the region

F to G′ the flint glass dispersion will again prevail, and the G′ ray will focus considerably beyond the C and F rays.

As an example we will take the case of a double objective of 30 feet focal length composed of the crown and flint glasses whose main characteristics have been given above, only the values of $\frac{1}{\rho_1}$ and $\frac{1}{\rho_2}$ are so calculated as to cause a ray half-way between B and C of the spectrum to focus to the same axial point as the ray F, which arrangement is likely to give the best colour correction for an objective of that size (upwards of 2 feet aperture).

VARIATION OF F FOR THE DIFFERENT COLOURS (IN INCHES) FOR A TELESCOPE OBJECTIVE 30 FEET E.F.L.

Table of chromatic errors of a 30-foot double objective.

Ray . .	A′	B	C	D_2	Minimum	E	F	G′	*h*	H_1
Variation	+ ·33	+ ·05	− ·04	− ·20	− ·24	− ·19	0	+ ·80	+ 1·40	+ 1·88

It will be noticed that the total is 2·02 inches, and that the largest minus variation occurs about half-way between D_2 and E, where it is − ·24. This is about the brightest part of the spectrum from a visual point of view, and since the maximum light concentration obviously occurs at the minimum focus where a high value of $\Delta\mu$ or range of spectrum may coincide with a very small variation from the minimum focal point, it is highly important that this light concentration should coincide with the position in the spectrum of the greatest visual intensity, unless the objective is specially designed for photographic purposes, when the greatest effectiveness and best definition is obtained by arranging for the minimum focal length and maximum light concentration to occur for a ray a little on the less refrangible side of the G′ ray (the hydrogen blue ray), at which position in the spectrum the usual photographic plate is most sensitive.

The minimum focus.

Chromatic correction for astro-photographic purposes.

We will here give the variations in F for such a telescopic objective for photographic purposes of the same focal length of 30 feet.

VARIATION OF F FOR THE DIFFERENT COLOURS (IN INCHES) FOR A PHOTOGRAPHIC TELESCOPE OBJECTIVE 30 FEET FOCAL LENGTH

Table of chromatic errors of a 30-foot astro-photographic objective.

Ray . . .	A′	B	C	D_2	E	F	G′	*h*	H_1
Variation	+ 2·16	+ 2·20	+ 1·38	+ ·88	+ ·40	+ ·16	0	+ ·05	+ ·18

Here it will be seen that the brightest visual rays are scattered along the axis for two inches or so beyond the photographic focus, and the image of a star formed by the G′ ray is therefore surrounded by a large halo of wasted light, which imprints itself more and more on the photographic plate as the exposure is extended; and that is why the photographs of the brighter stars come out so abnormally large when those of small magnitude have just imprinted themselves.

Triple Telescope Objectives

The only way known of getting rid of the secondary spectrum is by resorting, if possible, to a combination of one dispersive lens enclosed between two collective lenses, the two latter being made of two different sorts of glass, so chosen that the mean of their partial relative dispersion $\frac{\Delta'\mu_1}{\rho_1} + \frac{\Delta'\mu_3}{\rho_3}$, etc., for various regions of the spectrum shall correspond as closely as possible with the corresponding relative partial dispersions $\frac{\Delta'\mu_2}{\rho_2}$, etc., etc., for the same spectrum regions for the glass used for the dispersive lens.

In this way the glasses employed in the Cooke Photo-Visual Objective were chosen; with the result that the linear secondary colour aberrations for such an objective of 30-foot focus are reduced to less than one-tenth part of an inch for the whole range of spectrum A′ to H_1, which is only one-twentieth part of the 2·02 inches, the total axial chromatic error given above for the ordinary double objective of the same focal length.

Secondary spectrum reduced to one-twentieth.

Why the Secondary Spectrum of Large Double Objectives does not render Clear Vision impossible

Returning to the case of the visually corrected objective, it can be shown that if the usually accepted theory of the formation of the image by rays of any one colour is correct, then anything like distinct vision through a 30-foot objective of 18-inch to 24-inch aperture would be impossible.

Fig. 110*a*, Plate XXII., shows a section of the usual conception of the cone of rays converging to form the well-known spurious disc or star image at the focus, and then diverging again, so that the beam of rays takes the form of two straight-sided cones with both their points cut away to the diameter of the spurious disc. If this really represented the case, then only a very small fraction (about 15 per cent) of the light refracted through a 30-foot objective would be utilised for defining purposes, all the rest being wasted. Happily, however, the real section near the focus of the converging and diverging beam of rays is as in Fig. 110*b*; the angle between the two sides of each cone decreases as the spurious disc is approached, or tails off into the cylindrical shape. This can be proved by experiment, and it is a strange fact that while mathematicians have spent a good deal of work upon the conformation of the spurious disc and its surrounding diffraction rings as they are formed in the focal plane, yet none have entered upon an investigation of the conformation of the cone of rays along the axis as it approaches the spurious disc. Such an investigation, based upon the wave theory of light, should be most instructive and of the highest importance.

The tapering-off of the cone of rays near the focus.

The tapering-off most marked with large relative apertures.

It can also be proved by experiment that the tailing off into the cylindrical shape takes place in a more marked degree in the case of cones of rays of large angular aperture than in the case of cones of small angular aperture, which fact tells in favour of objectives of relatively large aperture, and discounts their other disadvantages in a substantial degree. However, we are here trenching on the borderland between geometrical and physical optics, with the latter of which this work does not profess to deal. For further information on this subject the reader is referred to a paper entitled "The Secondary Colour Aberrations of the Refracting Telescope in relation to Vision," in the *Monthly Notices of the Royal Astronomical Society*, vol. liv. No. 2, also to "Description of a Perfectly Achromatic Refractor," in the same publication, vol. liv. No. 5; both by the author.

SECTION XI

A BRIEF SKETCH OF THE NORMAL AND OTHER CURVATURE ABERRATIONS OF THE THIRD ORDER TAN$^4\phi$, ETC.

Importance of a plane image.

PERHAPS the most important corrections that the optical designer has to take into consideration in the course of working out photographic lenses are those relating to the curvature of image or the deviations of the image from an ideal flatness.

We found that the deviations from a plane image as calculated by the formulæ of Sections V. and VI. applied to the three lenses given as examples in Section VII. differed appreciably from the actually measured results.

These discrepancies are indeed scarcely too large to be accounted for by inexactness in the measurement of the curvatures, especially in the cases of the deep curves employed in the process lens and the four-lens Cooke lens.

It can be shown, for instance, that an increment of plus value in the convex curvature of a lens of low refractive index, together with a rather smaller minus increment in the convex curvature of a lens of high refractive index, may have the effect of quite reversing the character of a small residual oblique astigmatism without affecting the principal focal length of the combination; while the increments in question may easily escape all but the most exact methods of measurement.

But in the cases worked out the character of the image curvatures at still greater distances from the optic axis proves that the discrepancies are chiefly due to the presence of curvature aberrations of a higher order than those we have yet dealt with.

CENTRAL OBLIQUE REFRACTION

The Three Corrections to the y's

First of all, for the purpose of calculating the y's, we have assumed the refractions to take place in a plane tangent to the vertex of each surface or element, and we may first consider the nature of the corrections which would have to be applied in order to allow for the y's being reduced to perpendicularity to the normal oblique ray passing through the centre of curvature, since all the formulæ for spherical aberration assume the y's to be measured at right angles to the aforesaid ray.

The element plane to be departed from.

Primary Planes

Reverting to Section V., page 121, dealing with the question of oblique rays passing centrally through a lens, we had at the first surface, Fig. 44a, the equation

$$\frac{1}{x_1}=\left(\frac{y_1}{f_1}+\frac{y_2}{f_2}\right)\frac{1}{y_1+y_2}, \text{ leading to } \frac{\mu}{x_1}=\frac{\mu}{u}+\omega_1(y_1^2+y_2^2-y_1y_2), \qquad (1)$$

in which x_1 denoted the required oblique distance from d, the oblique vertex, to q, the crossing point of the two extreme rays in the primary plane.

These y's were the distances $c\,..\,e$ and $c\,..\,e_1$ reckoned *in the element plane,* as shown in Figs. 111 and 112. Now it is clear from these figures that the y's, so reckoned, become more and more incorrect as the aperture and the angle of obliquity ϕ increase. The y's are subject to three corrections: (1) the correction for obliquity; (2) the versine correction, and (3) the correction for the positions of the y's or the lateral separation generally existing between them.

The corrections for obliquity defined.

(1) The corrections for obliquity consist in converting the distances $c\,..\,e$ and $c\,..\,e_1$ in the element plane into the distances $e\,..\,g$ and $e_1\,..\,g_1$ measured perpendicularly to the normal oblique ray $Q\,..\,r$.

The versine corrections defined.

(2) The versine correction is due to the retreat of the spherical surface from the element plane. Let the extreme ray $Q\,..\,e$ be produced to cut the spherical surface at k; from k draw $k\,..\,l$ perpendicular to the normal ray $Q\,..\,r$; then through e draw $e\,..\,h$ parallel to $Q\,..\,r$, and cutting $k\,..\,l$ at h; then the distance $k\,..\,h$ is the versine correction applicable to $e\,..\,g$ in order to convert it into $k\,..\,l$, which latter is the real y upon which the spherical aberration should correctly be based.

The corrections for positions defined.

(3) Still another species of correction has yet to be applied—a correction *not* of the values of the y's, but a correction for their *positions.* Fig. 111 explains this. We must bear in mind that the Formula (1) gives the value of $\frac{1}{d\,..\,q}$ or $\frac{1}{x}$, that is, the reciprocal value of the focal distance $d\,..\,q$ measured along the normal oblique ray $Q\,..\,c\,..\,p$, with absolute correctness, provided that—

PLATE. XXIII.

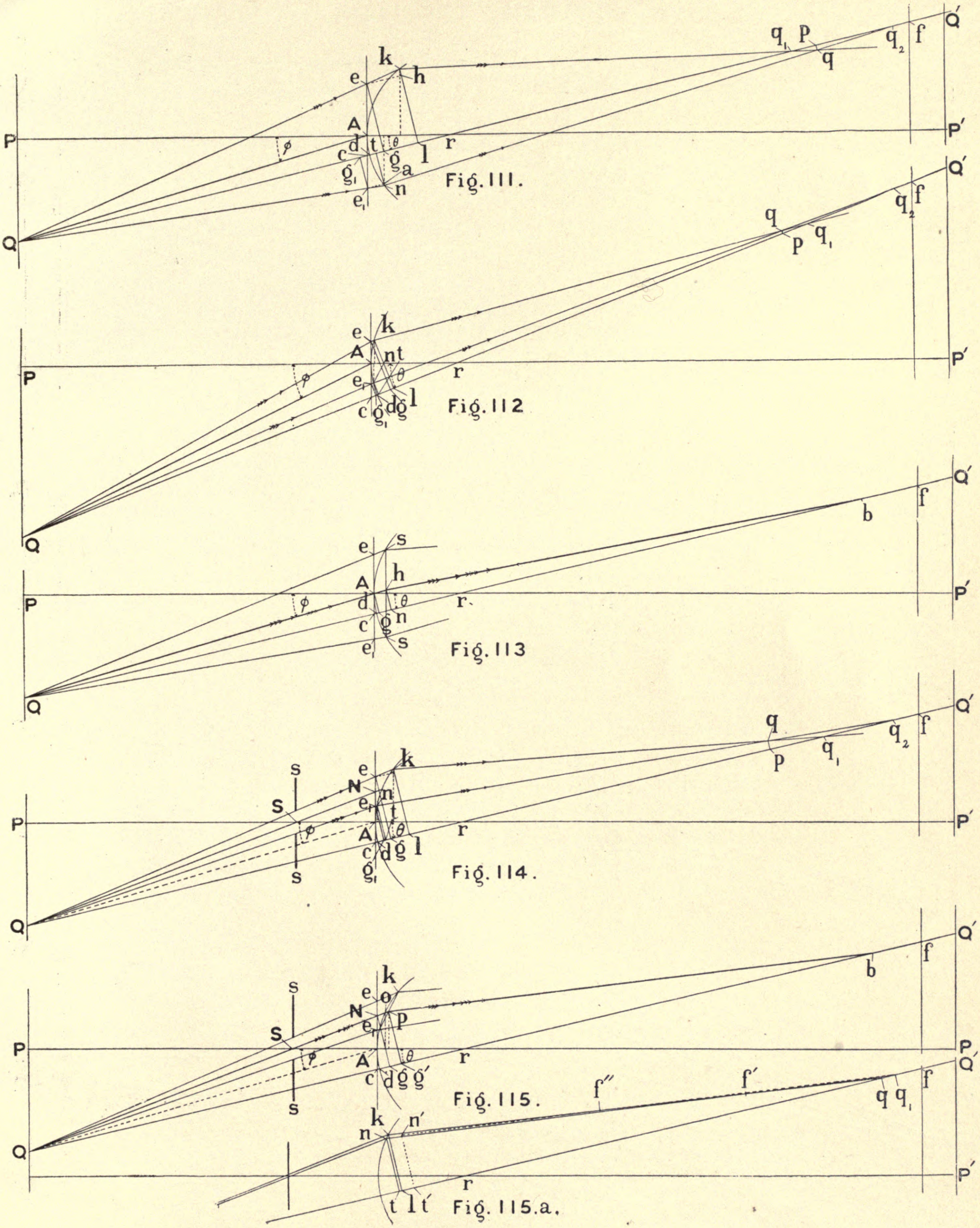


Fig. 111.

Fig. 112

Fig. 113

Fig. 114.

Fig. 115.

Fig. 115.a.

PLATE. XXIII.

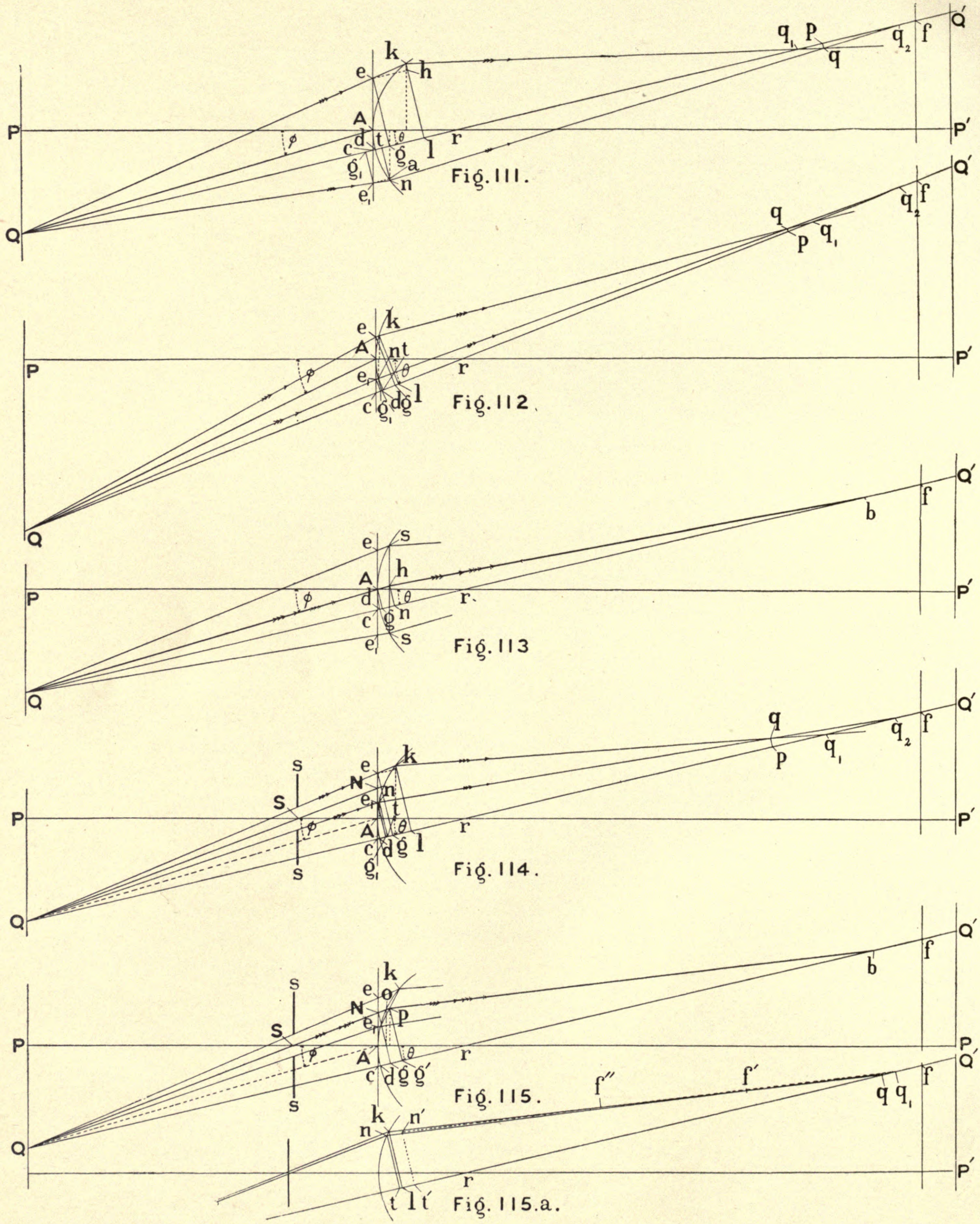


Fig. 111.

Fig. 112

Fig. 113

Fig. 114.

Fig. 115.

Fig. 115.a.

1st. The spherical aberration function ω is correctly formulated.

2nd. The values of the two y'''s, $k\,.\,.\,l$ and $n\,.\,.\,t$, are correctly given ; and

3rd. The two y'''s are at equal distances from the focus q, that is, that $l\,.\,.\,q = t\,.\,.\,q$, in which case the two y'''s would be in one straight line. But it is plain that this can only happen when either the radius r is infinite and the refracting surface is plane, or when $\grave{u}$ or $d\,.\,.\,q$ is infinite, when, of course, the separation $t\,.\,.\,l$ becomes a relatively vanishing quantity.

It is clear in Fig. 111 that the lateral translation of $k\,.\,.\,l$ or y_1'' towards the right hand, while assuming its length to remain constant, must cause the crossing point q to move to the left hand, nearer the lens; that is, the correction due to the separation of the y'''s is in this case of a plus nature, since it adds to the value of $\frac{1}{x}$.

Correction for positions only applies in primary planes.

It is also clear that this separation of the two y'''s gives rise to a correction to $\frac{1}{x}$ which operates in the primary plane only. We shall also find that it works out as a function of $\tan^4\phi$ and $a^2\tan^2\phi$, and therefore comes under the head of the formulæ of the third approximation. We will now treat these corrections more explicitly.

The Correction for Obliquity

Primary Plane

It will be better to deal with the question in general terms, first taking the obliquity correction.

Notation.

Let a = the semi-aperture $A\,.\,.\,e$ or $A\,.\,.\,e_1$ of the pencil where it crosses the element plane. Let b = the distance $A\,.\,.\,c$ from the lens vertex to the point in the element plane where the normal oblique ray $Q\,.\,.\,r$ cuts it.* Then if the angle $PAQ = \phi$, and $QrA = \theta$, as before, and $P\,.\,.\,A = u$, as usual, then b or $A\,.\,.\,c = r\tan\theta$ $= r\tan\phi\,\frac{u}{u+r}$. Let $e\,.\,.\,g = y_1'$ and $e_1\,.\,.\,g_1 = y_2'$.

Then, as in our earlier inquiry, in Fig. 111,

$$y_1^2 = (a+b)^2 \text{ and } y_2^2 = (a-b)^2 \text{ and } y_1y_2 = (a^2-b^2).$$

Then in the right-angled triangles $c\,.\,.\,e\,.\,.\,g$ and $c\,.\,.\,e_1\,.\,.\,g_1$ it is clear that

$$(e\,.\,.\,g)^2 \text{ or } y_1'^2 = y_1^2 - y_1^2\sin^2 ceg \text{ and } (e_1\,.\,.\,g_1)^2 \text{ or } y_2'^2 = y_2^2 - y_2^2\sin^2 ce_1g\,;$$

* In this Section the terms a, b, and c will supersede the corresponding terms A, B, and C of Sections V. to VIIIA., as they are more convenient for manipulation.

similarly

$$(y_2'')^2 = y_2'^2\left(1+\frac{a^2}{ru}\right) = y_2^2(1-e^2)\left(1+\frac{a^2}{ru}\right)$$

and

$$-(y_1''y_2'') = -y_1y_2(1-e^2)\left(1+\frac{a^2}{ru}\right);$$

Primary plane. Value of the functions of y_1'', y_2'', and ω_1.

$$\therefore\ \{(y_1'')^2+(y_2'')^2-y_1''y_2''\}\omega_1 = (a^2+3b^2)(1-e^2)\left(1+\frac{a^2}{ru}\right)\omega_1$$

$$= \left\{(a^2+3b^2) - a^2e^2 - 3b^2e^2 + (a^4+3a^2b^2)\frac{1}{ru}\right\}\omega_1, \qquad (4)$$

the two last terms being the new terms consequent on the versine corrections.

Secondary Plane

Here we see from Fig. 113 that if the chord $s\,..\,s$ represents the circular aperture of the lens surface seen edgeways of semi-aperture $=a$, plus a correction shortly to be dealt with; then the right-angled triangle, whose hypothenuse is the y required, consists of the side $h\,..\,n$, and a vertical side above h, perpendicular to the diagram, so that $y'^2 = (h\,..\,n)^2 +$ (the vertical from $h)^2$, and clearly

$$(h\,..\,n)^2 = \left((\mathrm{A}\,..\,g) + \frac{a^2}{2r}\frac{\mathrm{A}\,..\,g}{u}\right)^2 = (\mathrm{A}\,..\,g)^2\left(1+\frac{a^2}{ru}\right)$$

$$= (\mathrm{A}\,..\,c)^2(1-e^2)\left(1+\frac{a^2}{ru}\right) \doteq b^2(1-e^2)\left(1+\frac{a^2}{ru}\right),$$

and the vertical side over h

$$= \text{vertical side over A } (=a) + \frac{a^2}{2r}\frac{a}{u} = a\left(1+\frac{a^2}{2ru}\right);$$

$$\therefore\ (\text{vertical side over } h)^2 = a^2\left(1+\frac{a^2}{ru}\right),$$

so that

$$\omega_1 y'^2 = \left\{b^2(1-e^2)\left(1+\frac{a^2}{ru}\right) + a^2\left(1+\frac{a^2}{ru}\right)\right\}\omega_1$$

Secondary plane. Value of $y'^2\omega_1$.

$$= \left\{(a^2+b^2) - b^2e^2 + (a^4+a^2b^2)\frac{1}{ru}\right\}\omega_1, \qquad (5)$$

the two last terms being consequent upon the versine corrections. The term $a^4\frac{1}{ru}$, which is independent of the angle of obliquity ϕ, is thus seen to be common to primary and secondary planes, and is, in fact, a function of the spherical aberration consequent upon the axial or oblique pencil expanding in diameter as it traverses the distance between the element plane and the spherical surface.

We also see that the functions of a^2b^2 or $a^2 \tan^2 \phi$ and of $\tan^4 \phi$ or b^2e^2 are three times as great in primary planes as in secondary planes.

ECCENTRIC OBLIQUE REFRACTION

We may now deal with the more complex y's involved in the case of eccentric pencils on the same lines, taking as our basis equation

$$\frac{1}{x_1} = \left(\frac{y_2}{f_2} - \frac{y_1}{f_1}\right)\frac{1}{y_2 - y_1}, \text{ leading to } \frac{\mu}{x_1} = \frac{\mu}{u} + \omega_1(y_1{}^2 + y_2{}^2 + y_1y_2)$$

The basis equation.

(see page 143, Section VI.).

Let $Q..e$ and $Q..e_1$ (Fig. 114) be the extreme rays in primary planes of an eccentric pencil limited by a stop s, as in our earlier Fig. 50. Let N be the point where the principal ray through the centre of the stop strikes the element plane, and let A be the vertex where the curved surface cuts the optic axis $P..r$ and touches the element plane. Let $c..e = y_1$ and $c..e_1 = y_2$ as before. Then c, the new constituent in both y's due to the eccentricity of the pencil, is the distance A..N which $= (P..Q)\frac{S..A}{P..S} = u \tan \phi \frac{D}{u - D}$, when, as usual, ϕ is the angle PAQ and $D = S..A$. So that b and c are both functions of $\tan \phi$. Let us then denote A..N or $\tan \phi \frac{Du}{u - D}$ by the symbol c, $A..c$ or $r \tan \phi \frac{u}{u + r}$ being b, and the semi-aperture of the pencil $N..e$ or $N..e_1$ where it cuts the element plane being a, as before.

Obliquity Corrections to the y's

Primary Plane

Here let $e..g = y_1'$ and $e_1..g_1 = y_2'$.

In the right-angled triangle $e..c..g$ we have as before

$$(e..g)^2 \text{ or } y_1'^2 = (e..c)^2 - (c..g)^2\,;$$

that is,

$$y_1'^2 = y_1{}^2 - y_1{}^2 \tan^2 \theta\,;$$

$$\therefore\ y_1'^2 = y_1{}^2\left(1 - \tan^2 \phi \frac{u}{u + r}\right) = y_1{}^2(1 - e^2),$$

and similarly

$$e_1 - g_1 \text{ or } y_2'^2 = y_2{}^2(1 - e^2)$$

$$y_1^2 = (b + c + a)^2 \qquad \therefore\ y_1'^2 = (b + c + a)^2(1 - e^2)$$
$$y_2^2 = (b + c - a)^2 \qquad \therefore\ y_2'^2 = (b + c - a)^2(1 - e^2)$$
$$y_1 y_2 = (b + c)^2 - a^2 \qquad \therefore\ y_1' y_2' = \{(b + c)^2 - a^2\}(1 - e^2)$$

and

$$(y_1'^2 + y_2'^2 + y_1' y_2')\omega_1 = \{(a^2 + 3b^2) + 3c^2 + 6bc)\}(1 - e^2)\omega_1$$
$$= \{(a^2 + 3b^2) + 6bc + 3c^2\}\omega_1 \text{ plus}$$

Primary plane. Value of obliquity functions in terms of y_1', y_2', and ω_1.

the new terms in the shape of functions of e^2, which are

$$(- a^2e^2 - 3b^2e^2 - 3c^2e^2 - 6bce^2)\omega_1, \tag{6}$$

which are clearly functions of $a^2 \tan^2 \phi$, 3 $\tan^4 \phi$, 3 $\tan^4 \phi$, and 6 $\tan^4 \phi$ respectively.

Secondary Plane

Turning to Fig. 115, it is clear that c . . N or $b + c$ only is subject to the obliquity correction, so that $(b + c)^2$ modified for the obliquity $= (b + c)^2(1 - e^2)$ and

$$y'^2\omega_1 = \{(b + c)^2(1 - e^2) + a^2\}\omega_1$$
$$= \{(b^2 + 2bc + c^2)(1 - e^2) + a^2\}\omega_1$$
$$= (b^2 + 2bc + c^2 + a^2)\omega_1$$

Secondary plane. Value of obliquity functions in terms of $y'^2\omega_1$.

$+$ the new terms in the shape of functions of e^2, which are

$$(- b^2e^2 - c^2e^2 - 2bce^2)\omega_1, \tag{7}$$

all minus functions of $\tan^4 \phi$, and, as usual, one-third of the corresponding corrections in primary planes; but the function of a^2e^2 or $a^2 \tan^2 \phi$ is again absent.

Versine Corrections to the y's

Primary Plane

Reverting to Fig. 114, let k . . $l = y_1''$ and n . . $t = y_2''$.

Here the versines of the curved surface with respect to the element plane measured parallel to P . . A are obviously proportional to $(c + a)^2$ and $(c - a)^2$, and the increment to y_1' or e . . $g = (c + a)^2 \dfrac{1}{2r} \cdot \dfrac{y_1}{u}$ approximately,

$$\therefore\ y_1''^2 = y_1'^2\left\{1 + (c + a)^2\frac{1}{ru}\right\},$$

and similarly

$$y_2''^2 = y_2'^2\left\{1 + (c - a)^2\frac{1}{ru}\right\}$$

and

$$y_1''y_2'' = y_1'y_2'\left\{1 + (c^2 + a^2)\frac{1}{ru}\right\};$$

$$\therefore y_1''^2 + y_2''^2 + y_1''y_2''$$

$$= (b+c+a)^2(1-e^2)\left\{1 + (c^2 + 2ac + a^2)\frac{1}{ru}\right\}$$

$$+ (b+c-a)^2(1-e^2)\left\{1 + (c^2 - 2ac + a^2)\frac{1}{ru}\right\}$$

$$+ \{(b+c)^2 - a^2\}(1-e^2)\left\{1 + (c^2 + a^2)\frac{1}{ru}\right\},$$

and the new terms consequent upon the versine corrections are

$$(b^2 + c^2 + a^2 + 2ab + 2ac + 2bc)(c^2 + 2ac + a^2)\frac{1}{ru}$$

$$+ (b^2 + c^2 + a^2 - 2ab - 2ac + 2bc)(c^2 - 2ac + a^2)\frac{1}{ru}$$

$$+ (b^2 + 2bc + c^2 - a^2)(c^2 + a^2)\frac{1}{ru},$$

which, after multiplying out and cancelling, gives us

$$\left\{(3b^2c^2 + 6bc^3 + 3c^4) + 12a^2c^2 + 14a^2bc + (3a^2b^2 + a^4)\right\}\frac{1}{ru}\omega_1, \qquad (8)$$

Primary plane. Value of functions of y_1'', y_2'', and ω_1.

in which the terms $3a^2b^2 + a^4$ appertain to the central oblique pencil also. So we have

$$(3b^2c^2 + 6bc^3 + 3c^4) \text{ all functions of } \tan^4\phi,$$

$$\left.\begin{matrix} +\,12a^2c^2 \\ +\,14a^2bc \end{matrix}\right\}\text{both functions of } a^2\tan^2\phi,$$

and the functions of a^4 and $3a^2b^2$ before worked out for central oblique pencils.

Secondary Plane

Turning to Fig. 115, it will be seen that in the right-angled triangle, whose two sides including the right angle are $p\,..\,g'$ and a'', the latter being perpendicular to the plane of the diagram and over the point p; evidently $p\,..\,g' = \mathrm{N}\,..\,g$ subject to a double versine correction approximately equal to

$$\left(\frac{c^2}{2r} + \frac{a^2}{2r}\right)\frac{(b+c)\left(1 - \frac{e^2}{2}\right)}{u};$$

a is also subject to a double versine correction approximately equal to

$$\left(\frac{c^2}{2r} + \frac{a^2}{2r}\right)\frac{a}{u};$$

so that we have

$$p\,..\,g' = \mathrm{N}\,..\,g + \left(\frac{a^2 + c^2}{2ru}\right)(b + c)\left(1 - \frac{e^2}{2}\right)$$

$$= (b + c)\left(1 - \frac{e^2}{2}\right) + (b + c)\left(1 - \frac{e^2}{2}\right)\left(\frac{a^2 + c^2}{2ru}\right)$$

$$= (b + c)\left(1 - \frac{e^2}{2}\right)\left(1 + \frac{a^2 + c^2}{2ru}\right),$$

and

$$(p\,..\,g')^2 = (b + c)^2(1 - e^2)\left(1 + \frac{a^2 + c^2}{ru}\right),$$

also

$$a'' = a + \frac{a^2 + c^2}{2ru}a = a\left(1 + \frac{a^2 + c^2}{2ru}\right)$$

and

$$a''^2 = a^2\left(1 + \frac{a^2 + c^2}{ru}\right);$$

so that the hypothenuse squared, after correction, or

$$y''^2 = (p\,..\,g')^2 + a''^2 = \left\{(b + c)^2(1 - e^2) + a^2\right\}\left(1 + \frac{a^2 + c^2}{ru}\right)$$

and

$$y''^2\omega_1 = \left\{(b^2 + 2bc + c^2)(1 - e^2) + a^2\right\}\left(1 + \frac{a^2 + c^2}{ru}\right)\omega_1;$$

so that the new terms consequent upon the versine corrections are

$$(b^2 + 2bc + c^2 + a^2)\left(\frac{a^2 + c^2}{ru}\right)\omega_1$$

$$= (a^2b^2 + 2a^2bc + a^2c^2 + a^4 + c^2b^2 + 2bc^3 + c^4 + a^2c^2)\frac{1}{ru}\omega_1;$$

Secondary plane. Value of $y''^2\omega_1$.

$$\therefore\; y''^2\omega_1 = (\underline{a^2b^2} + 2a^2bc + 2a^2c^2 + b^2c^2 + 2bc^3 + c^4 + \underline{a^4})\frac{1}{ru}\omega_1, \qquad (9)$$

in which $a^2b^2 + a^4$ appertain to the central oblique pencil.

Here it is instructive to notice that while the terms $a^2b^2 + b^2c^2 + 2bc^3 + c^4$ are one-third of the corresponding terms in the primary plane, the term $2a^2c^2$ is only one-sixth part and $2a^2bc$ only one-seventh part of the corresponding term in the primary plane, while the term a^4 is common to both planes.

The Corrections for the Separation between the two y's

Central Oblique Refraction

So far we have been considering, in a qualitative sense, the nature of the small corrections which have to be applied to the two y's in order to convert them into the y'''s.

We will now deal with those corrections which are due to the separation between the two y'''s to which we have previously alluded.

We may again legitimately express the necessary corrections in terms of the uncorrected y's, since to express them in terms of the corrected y'''s would lead to functions of the order $\tan^6 \phi$ which are beyond the scope of this inquiry.

First we have our fundamental equation, with reference to Fig. 111,

$$y_1 \frac{x - f_1}{f_1} = p \,..\, q = y_2 \frac{f_2 - x}{f_2},$$

The basis equation.

wherein f_1 so far has been held to mean the distance $d \,..\, q_1$, whereas it should be the distance $l \,..\, q$ (Fig. 111), the versine $d \,..\, l$ being deducted. Similarly f_2 has been held, so far, to mean the distance $d \,..\, q_2$, whereas it should be the distance $t \,..\, q_2$, the versine $d \,..\, t$ being deducted. But as regards the numerator $x - f_1$, it is clear that since $x - f_1$ is simply the distance $q_1 \,..\, p$, therefore if we deducted the versine $c \,..\, l$ from f_1 we should also have to deduct it from x; that is, the terms $(x - f_1)$ and $(f_2 - x)$ are not affected by our corrections; but obviously f_1 and f_2 in the denominators must be corrected for the versines, so that the above equation becomes

$$y_1 \frac{x - f_1}{f_1 - \dfrac{y_1^2}{2r}} = y_2 \frac{f_2 - x}{f_2 - \dfrac{y_2^2}{2r}};$$

$$\therefore \; y_1(x - f_1)\left(\frac{1}{f_1} + \frac{y_1^2}{2r}\frac{1}{f_1^2}\right) = y_2(f_2 - x)\left(\frac{1}{f_2} + \frac{y_2^2}{2r}\frac{1}{f_2^2}\right),$$

$$y_1\left\{\frac{x}{f_1} + x\frac{y_1^2}{2rf_1^2} - 1 - \frac{y_1^2}{2r}\frac{1}{f_1}\right\} = y_2\left\{1 + \frac{y_2^2}{2r}\frac{1}{f_2} - \frac{x}{f_2} - x\frac{y_2^2}{2r}\frac{1}{f_2^2}\right\},$$

$$x\left(\frac{y_1}{f_1} + \frac{y_2}{f_2}\right) + x\frac{y_1^3}{2rf_1^2} + x\frac{y_2^3}{2r}\frac{1}{f_2^2} = y_1 + y_2 + \frac{y_1^3}{2rf_1} + \frac{y_2^3}{2rf_2},$$

$$x\left\{\left(\frac{y_1}{f_1} + \frac{y_2}{f_2}\right) + \frac{1}{2r}\left(\frac{y_1^3}{f_1^2} + \frac{y_2^3}{f_2^2}\right)\right\} = y_1 + y_2 + \frac{1}{2r}\left(\frac{y_1^3}{f_1} + \frac{y_2^3}{f_2}\right). \qquad (10)$$

We may now express f_1 and f_2 in terms of $d \,..\, f$ or $\dot{u}$ and the spherical aberration, so that we may put

$$\frac{1}{f_1} = \frac{1}{\dot{u}} + \frac{\omega}{\mu} y_1^2 \text{ and } \frac{1}{f_2} = \frac{1}{\dot{u}} + \frac{\omega}{\mu} y_2^2;$$

so we then get

$$x\left\{y_1\left(\frac{1}{\grave{u}}+\frac{\omega}{\mu}y_1^2\right)+y_2\left(\frac{1}{\grave{u}}+\frac{\omega}{\mu}y_2^2\right)+\frac{y_1^3}{2r}\left(\frac{1}{\grave{u}^2}+2\frac{\omega}{\mu}\cdot\frac{y_1^2}{\grave{u}}\right)+\frac{y_2^3}{2r}\left(\frac{1}{\grave{u}^2}+2\frac{\omega}{\mu}\frac{y_2^2}{\grave{u}}\right)\right\}$$

$$=y_1+y_2+\frac{y_1^3}{2r}\left(\frac{1}{\grave{u}}+\frac{\omega}{\mu}y_1^2\right)+\frac{y_2^3}{2r}\left(\frac{1}{\grave{u}}+\frac{\omega}{\mu}y_2^2\right);$$

$$\therefore\ x\left\{\frac{y_1+y_2}{\grave{u}}+\frac{\omega}{\mu}(y_1^3+y_2^3)+(y_1^3+y_2^3)\frac{1}{2r\grave{u}^2}+\frac{\omega}{\mu}(y_1^5+y_2^5)\frac{1}{r\grave{u}}\right\}$$

$$=y_1+y_2+(y_1^3+y_2^3)\frac{1}{2r\grave{u}}+\frac{\omega}{\mu}(y_1^5+y_2^5)\frac{1}{2r};$$

$$\therefore\ \frac{1}{x}=\frac{\dfrac{y_1+y_2}{\grave{u}}+(y_1^3+y_2^3)\dfrac{1}{2r\grave{u}^2}+\dfrac{\omega}{\mu}(y_1^5+y_2^5)\dfrac{1}{r\grave{u}}+\dfrac{\omega}{\mu}(y_1^3+y_2^3)}{(y_1+y_2)+(y_1^3+y_2^3)\dfrac{1}{2r\grave{u}}+\dfrac{\omega}{\mu}(y_1^5+y_2^5)\dfrac{1}{2r}}.$$

Since the y's are generally much smaller quantities than r and $\grave{u}$, we may treat the second and third terms of the denominator as variants of (y_1+y_2), so that $\frac{1}{x}$ becomes

$$\left\{\frac{y_1+y_2}{\grave{u}}+(y_1^3+y_2^3)\frac{1}{2r\grave{u}^2}+\frac{\omega}{\mu}(y_1^5+y_2^5)\frac{1}{r\grave{u}}+\frac{\omega}{\mu}(y_1^3+y_2^3)\right\}$$

multiplied by

$$\left\{\frac{1}{y_1+y_2}-\frac{y_1^3+y_2^3}{(y_1+y_2)^2}\frac{1}{2r\grave{u}}-\frac{\omega}{\mu}\frac{y_1^5+y_2^5}{(y_1+y_2)^2}\frac{1}{2r}\right\},$$

and after multiplying out we get

$$\frac{1}{x}=\left\{\frac{1}{\grave{u}}+\frac{y_1^3+y_2^3}{y_1+y_2}\cdot\frac{1}{2r\grave{u}^2}+\frac{\omega}{\mu}\cdot\frac{y_1^5+y_2^5}{y_1+y_2}\cdot\frac{1}{r\grave{u}}+\frac{\omega}{\mu}\cdot\frac{y_1^3+y_2^3}{y_1+y_2}\right.$$

$$-\frac{y_1^3+y_2^3}{y_1+y_2}\cdot\frac{1}{2r\grave{u}^2}-\frac{(y_1^3+y_2^3)^2}{(y_1+y_2)^2}\frac{1}{4r^2\grave{u}^3}-\frac{\omega}{\mu}\cdot\frac{(y_1^3+y_2^3)(y_1^5+y_2^5)}{(y_1+y_2)^2}\cdot\frac{1}{2r^2\grave{u}^2}$$

$$-\frac{\omega}{\mu}\cdot\frac{(y_1^3+y_2^3)^2}{(y_1+y_2)^2}\cdot\frac{1}{2r\grave{u}}$$

$$-\frac{\omega}{\mu}\frac{y_1^5+y_2^5}{y_1+y_2}\frac{1}{2r\grave{u}}-\frac{\omega}{\mu}\frac{(y_1^3+y_2^3)(y_1^5+y_2^5)}{(y_1+y_2)^2}\frac{1}{4r^2\grave{u}^2}-\left(\frac{\omega}{\mu}\right)^2\text{etc.},-\left(\frac{\omega}{\mu}\right)^2\text{etc.}$$

Here we may neglect the last two terms, and also the two terms (the seventh and tenth) involving $\frac{1}{r^2\grave{u}^2}$, since they involve functions of the order y_1^6 and y_2^6, which we are not dealing with. The second and fifth terms cancel one another, while the third and ninth add together, so that we get finally

$$\left.\begin{aligned}\frac{1}{x}=\frac{1}{\grave{u}}+\frac{\omega}{\mu}\cdot\frac{y_1^3+y_2^3}{y_1+y_2}+\frac{\omega}{\mu}\frac{y_1^5+y_2^5}{y_1+y_2}\frac{1}{2r\grave{u}}-\frac{\omega}{\mu}\frac{(y_1^3+y_2^3)^2}{(y_1+y_2)^2}\frac{1}{2r\grave{u}}\\-\frac{(y_1^3+y_2^3)^2}{(y_1+y_2)^2}\frac{1}{4r^2\grave{u}^3}.\end{aligned}\right\}\quad(11)$$

Primary plane. Value of $\frac{1}{x}$, corrected for separation between the y's.

The last term in this formula need not be heeded, as it does not involve the spherical aberration at all; for if, in our original equation,

$$y_1\frac{x-f_1}{f_1}=p\,.\,.\,q=y_2\frac{f_2-x}{f_2},$$

we suppose that there is no aberration whatsoever, and therefore that $f_1=\grave{u}=f_2$, and yet suppose that the two y's are at unequal distances from q', and then correct $\grave{u}$ for the versines as before, we then get

$$y_1\frac{x-\grave{u}}{\grave{u}-\dfrac{y_1^2}{2r}}=y_2\frac{\grave{u}-x}{\grave{u}-\dfrac{y_2^2}{2r}},$$

which finally works out to

$$\frac{1}{x}=\frac{1}{\grave{u}}-\frac{(y_1^3+y_2^3)^2}{(y_1+y_2)^2}\frac{1}{4r^2\grave{u}^3},$$

which means that the distance x is to be measured from a point very slightly to the left of the vertex d_1 by a minute amount varying inversely as $\grave{u}$.

This curious result doubtless follows upon our assuming the versines to vary exactly as y^2, which is not strictly true. Anyway this term has nothing to do with our present purposes and may be ignored, so that we have, after dividing out the functions of y_1 and y_2,

$$\begin{aligned}\frac{1}{x}=\frac{1}{\grave{u}}&+\frac{\omega}{\mu}(y_1^2+y_2^2-y_1y_2)\\&+\frac{\omega}{\mu}(y_1^4+y_2^4-y_1^3y_2+y_1^2y_2^2-y_1y_2^3)\frac{1}{2r\grave{u}}\\&-\frac{\omega}{\mu}(y_1^4+y_2^4-2y_1^3y_2+3y_1^2y_2^2-2y_1y_2^3)\frac{1}{2r\grave{u}}\end{aligned}$$

Here the first line is the result of the second approximation, which we have had to deal with before in Sections V. and VI. After adding together the second and third lines we get finally

Primary plane. Reduced value of $\frac{1}{x}$ corrected for separation between the y's.

$$\frac{1}{x}=\frac{1}{\grave{u}}+\frac{\omega}{\mu}(y_1^2+y_2^2-y_1y_2)+\frac{\omega}{\mu}(y_1^3y_2-2y_1^2y_2^2+y_1y_2^3)\frac{1}{2r\grave{u}}.\quad(12)$$

We have in the process leading to this result dealt with Fig. 111, in which the two y's are at opposite sides of the normal oblique ray, and we have treated both y's as positive quantities.

Under this assumption, then, y_2 in the next Figure (112) would have to be considered negative, so that the above $y_1^3 y_2 - 2y_1^2 y_2^2 + y_1 y_2^3$ would become three negative quantities. Now it is clear that if a, the aperture of the pencil, vanishes, then y_1 and y_2 become numerically equal, and the above correction of the third order will not therefore vanish, and a little consideration and a reference to Fig. 115a will show that this correction to the oblique focal length x, due to a lateral separation between the two y's, should not vanish when the aperture of the pencil vanishes, for the smaller is the aperture the smaller is the angle between the two rays bounding the pencil in the primary plane; and assuming their two focal points q and q_1 on the oblique normal ray $r \,..\, Q'$ to remain fixed, it is clear that the position of their intersection point q becomes highly sensitive to even a most minute lateral separation between the y's. For instance, if the two rays through k and n focus at fixed points q and q_1, while $n \,..\, t$ is transferred laterally to $n' \,..\, t'$ without changing its length, then the point of intersection of the two rays will be transferred from f' to f''. But the formula of course vanishes when either one of the y's $= 0$, since then one ray becomes the normal oblique ray $Q \,..\, f$.

Sensitiveness of x to the separation between the y's in the case of narrow pencils.

If the reader will carry out a similar investigation in the case of Fig. 112, treating both y's as positive quantities, he will arrive at the correction $-\frac{\omega}{\mu}(y_1^3 y_2 + 2y_1^2 y_2^2 + y_1 y_2^3)\frac{1}{2r\grave{u}}$, which when worked out and expressed in terms of a and b for central oblique refraction, or of a, b, and c for eccentric oblique refraction, will lead to exactly the same formulæ as Nos. (13) and (14) below.

We may proceed to convert (12) of the third order as follows:—First, in the case of Fig. 111 for central oblique refraction we have $y_1 = a + b$ and $y_2 = a - b$, so that $\frac{1}{x} = \frac{1}{\grave{u}} + \frac{\omega}{\mu}(a^2 + 3b^2)$ (which has been dealt with before) plus the following new terms—

$$+\frac{\omega}{\mu}\left\{\begin{array}{l} a^4 + 2a^3b - 2ab^3 - b^4 \\ -(2a^4 - 4a^2b^2 + 2b^4) \\ +a^4 - 2a^3b + 2ab^3 - b^4 \end{array}\right\}\frac{1}{2r\grave{u}},$$

and the function of the third order is finally

Primary plane. Corrections to $\frac{1}{x}$ for the separation between the y's.

$$+\frac{\omega}{\mu}\frac{1}{2r\grave{u}}(4a^2b^2 - 4b^4), \tag{13}$$

for central oblique refraction.

Eccentric Oblique Refraction

Turning now to the case of eccentric oblique refraction, Fig. 114, wherein the two y's are again on the same side of the central oblique ray Q . . r, we have

$$y_1 = (b + c) + a \text{ and } y_2 = (b + c) - a.$$

This being the case, we shall find that the value of the consequent function

$$\frac{1}{2r\grave{u}}(-y_1^3y_2 - 2y_1^2y_2^2 - y_1y_2^3)\frac{\omega}{\mu},$$

expressed in terms of a, b, and c, works out to

Primary plane. Corrections to $\frac{1}{x}$ for the separations between the y's.

$$\frac{\omega}{\mu}\left\{(4a^2b^2 - 4b^4) + 8a^2bc + 4a^2c^2 - 24b^2c^2 - 16b^3c - 16bc^3 - 4c^4\right\}\frac{1}{2r\grave{u}}. \qquad (14)$$

The first two terms apply to the central oblique refraction which we have just worked out, while the last six terms, all involving c, follow from the eccentricity of the pencil. It is interesting to note how the terms of Formulæ (13) and (14) equate to 0, when in (13) $a = b$, or in (14) $a = b + c$, for, of course, when this is the case, y_2 becomes zero, or, in other words, the lower ray coincides with the normal oblique ray, so that the case is fully met by the usual spherical aberration formula $\frac{\omega}{\mu}y_1^2$. It might at first be thought that Formulæ (13) and (14) should equate to 0 when a vanishes, but this is not so, for we have seen that the narrower is the pencil the greater is the sensitiveness of the position of the focus to a minute lateral separation between the two y's.

The separation correction only valid in the primary plane.

These corrections of the third order consequent upon the relative lateral displacement of the two y's, obviously come into force in the primary plane only, and there is nothing corresponding to them in the secondary plane.

It is clear that such corrections as (13) and (14) could not apply to a parallel glass plate or a plane surface, since r would become infinite and the value of the formulæ vanish.

It is also clear that when we come to add to the functions of the third order for the first surface the corresponding functions for the second surface, then $\frac{\omega_1}{\mu}$, which is the " inside glass " value of the spherical aberration, will become ω_1, or the outside glass value for the same aberration.

We have now considered all the corrections of the third order which have to be applied in order to convert the y's of the second approximation into the y's of the third approximation, the correcting formulæ being functions of ω_1 and ω_2, or the spherical aberration formulæ of the second approximation for the two surfaces which resulted in the formulæ for curvature errors previously worked out in Sections V. and VI., which, as applied to the single surface that we have been considering, was

$$\left\{\frac{\mu-1}{2\mu^2}\left(\frac{1}{r}+\frac{1}{u}\right)^2\left(\frac{1}{r}+\frac{\mu+1}{u}\right)\right\}y^2$$

(see Formula XVIII. (R.), Section IV.), and the corrections that we have been dealing with in this Section are of course all products of the corrections to y_1 or y_2 into the part of the above formula included in the large brackets.

THE INTRINSIC SPHERICAL ABERRATION OF THE THIRD ORDER

But we have yet to consider the intrinsic spherical aberration of the third order in its application to oblique rays; that is, we have to find what are the modifications to the curvature corrections consequent upon our taking into account Formula XX. (R.) of Section IV. (page 63), which is a function of y^4, and therefore, in its present application, of $\tan^4\phi$. We will first deal with the case of

Central Oblique Refraction

Primary Plane

Here we must revert again to the fundamental equation dealt with on page 121, Section V., applying also to Fig. 111—

$$\frac{1}{x_1}=\left(\frac{y_1}{f_1}+\frac{y_2}{f_2}\right)\frac{1}{y_1+y_2},$$

in which we must now stipulate that

$$\frac{\mu}{f_1}=\frac{\mu}{u}+\omega_1 y_1{}^2+\chi_1 y_1{}^4 \text{ and } \frac{\mu}{f_2}=\frac{\mu}{u}+\omega_1 y_2{}^2+\chi_1 y_2{}^4,$$

the last terms expressing the intrinsic spherical aberration of the order y^4 as given in Formula XX. (R.). We are not now to consider any corrections to the y's involved in χy^4, since such procedure would only result in corrections of the order y^6, etc., but have to find what is the result of the introduction of this new term on the curvature corrections in the primary plane. We have then

$$\frac{1}{f_1}=\frac{1}{u}+\frac{\omega}{\mu}y_1^2+\frac{\chi}{\mu}y_1^4,$$

$$\frac{1}{f_2}=\frac{1}{u}+\frac{\omega}{\mu}y_2^2+\frac{\chi}{\mu}y_2^4,$$

so that the equation

$$\frac{1}{x}=\left(\frac{y_1}{f_1}+\frac{y_2}{f_2}\right)\frac{1}{y_1+y_2}$$

now becomes

$$\frac{1}{x}=\left\{y_1\left(\frac{1}{u}+\frac{\omega}{\mu}y_1^2+\frac{\chi}{\mu}y_1^4\right)+y_2\left(\frac{1}{u}+\frac{\omega}{\mu}y_2^2+\frac{\chi}{\mu}y_2^4\right)\right\}\frac{1}{y_1+y_2}$$

$$=\frac{\frac{1}{u}(y_1+y_2)+\frac{\omega}{\mu}(y_1^3+y_2^3)+\frac{\chi}{\mu}(y_1^5+y_2^5)}{y_1+y_2};$$

$$\therefore\ \frac{\mu}{x}=\frac{\mu}{u}+\omega(y_1^2+y_2^2-y_1y_2)+\chi(y_1^4+y_2^4-y_1^3y_2-y_2^3y_1+y_1^2y_2^2). \quad (15)$$

Primary plane. Value of $\frac{\mu}{x}$ including functions of χ.

The functions of ω have been already worked out, and we may confine our attention to the functions of χ. We have, in Fig. 111,

$$y_1=(a+b)$$
$$y_2=(a-b);$$

$$\begin{array}{rl} \therefore\ y_1^4 = & a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\ y_2^4 = & a^4-4a^3b+6a^2b^2-4ab^3+b^4 \\ -y_1^3y_2 = & -(a^4+2a^3b \qquad\qquad -2ab^3-b^4) \\ -y_2^3y_1 = & -(a^4-2a^3b \qquad\qquad +2ab^3-b^4) \\ +y_1^2y_2^2 = & a^4 \qquad\qquad -2a^2b^2 \qquad\qquad +b^4 \\ \hline & a^4 \qquad\qquad +10a^2b^2 \qquad\qquad +5b^4 \end{array}$$

therefore

$$\frac{\mu}{x}=\frac{\mu}{u}+\omega(a^2+3b^2)+\chi(a^4+10a^2b^2+5b^4). \quad (16)$$

Primary plane. Value of $\frac{\mu}{x}$ from (15) after reduction.

The functions ωa^2 and χa^4 are, of course, the spherical aberrations of the two orders to which all pencils of semi-aperture a are subject, whether axial or oblique.

Secondary Plane

Here $y^2=$ simply a^2+b^2, and

$$\therefore\ y^4=a^4+2a^2b^2+b^4,$$

and we have therefore

$$\frac{\mu}{x'}=\frac{\mu}{u}+\omega(a^2+b^2)+\chi(a^4+2a^2b^2+b^4). \quad (17)$$

Secondary plane. Value of $\frac{\mu}{x'}$ after reduction.

Here again ωa^2 and χa^4 are the spherical aberrations of the two orders common to all pencils, axial or oblique; while it will be seen that the correction $\chi(2a^2b^2+b^4)$ is only *one-fifth part* of the corresponding correction in primary planes.

Eccentric Oblique Refraction

Primary Plane

Here we have, in Fig. 114,

$$y_1^2=(b+c+a)^2$$
$$y_2^2=(b+c-a)^2.$$

The conditions are here the same as dealt with on page 143, Section VI., Fig. 50, and we have

$$\frac{1}{x}=\left(\frac{y_2}{f_2}-\frac{y_1}{f_1}\right)\frac{1}{y_2-y_1}$$
$$=\left\{y_2\left(\frac{1}{u}+\frac{\omega}{\mu}y_2^2+\frac{\chi}{\mu}y_2^4\right)-y_1\left(\frac{1}{u}+\frac{\omega}{\mu}y_1^2+\frac{\chi}{\mu}y_1^4\right)\right\}\frac{1}{y_2-y_1}$$
$$=\left\{\left(y_2-y_1\right)\frac{1}{u}+\frac{\omega}{\mu}(y_2^3-y_1^3)+\frac{\chi}{\mu}(y_2^5-y_1^5)\right\}\frac{1}{y_2-y_1};$$

Value of $\frac{\mu}{x}$ including functions of χ.

$$\therefore\ \frac{\mu}{x}=\frac{\mu}{u}+\omega(y_1^2+y_2^2+y_1y_2)+\chi(y_2^4+y_1y_2^3+y_1^2y_2^2+y_1^3y_2+y_1^4).\quad(18)$$

The functions of ω have already been dealt with.

On working out the functions of χ in terms of a, b, and c, we get

Primary plane. Functions of χ, y_1, and y_2 after reduction.

$$\chi(\underline{a^4}+\underline{10a^2b^2}+10a^2c^2+30b^2c^2+20a^2bc+20b^3c+20bc^3+\underline{5b^4}+5c^4$$
$$=\chi[a^4+\{\underbrace{10a^2(b^2+2bc+c^2)}_{\text{order } a^2\tan^2\phi}+\underbrace{5(b^4+4b^3c+6b^2c^2+4bc^3+c^4)}_{\text{order }\tan^4\phi}\}],\quad(19)$$

in which the underlined terms $a^4+10a^2b^2+5b^4$ relate, as we have seen, to central oblique pencils also.

Secondary Plane

Here $y^2=$ simply $(b+c)^2+a^2$, and

$$y^4=(b+c)^4+2a^2(b+c)^2+a^4,$$

and

$$\frac{\mu}{x_1}=\frac{\mu}{u}+\omega\left\{(b^2+2bc+c^2)+a^2+\text{etc.}\right\}$$

Secondary plane. Function of χ after reduction.

$$+\chi\left\{a^4+2a^2(b^2+2bc+c^2)+(b^4+4b^3c+6b^2c^2+4bc^3+c^4)\right\},\quad(20)$$

in which the terms $a^4 + 2a^2b^2 + b^4$ apply to central oblique pencils. The functions of ω have already been dealt with.

Thus we find that the corrections involving the intrinsic spherical aberration of the order y^4 are five times as great in primary planes as in secondary planes, and that all the terms are represented in both planes.

It is advisable to now gather together our results in the form of a table as follows :—

Functions of ω_1

Primary Plane

2nd Approximation		Terms to be added for 3rd Approximation		
Central Oblique Pencils.	Add for Eccentric Pencils.	Central Oblique Pencils.	Add for Eccentric Pencils.	
a^2+3b^2	$+6bc+3c^2$	$-a^2e^2-3b^2e^2$	$-3c^2e^2-6bce^2$	Obliquity corrections.
		$+(a^4+3a^2b^2)\dfrac{1}{ru}$	$(3b^2c^2+6bc^3+3c^4+14a^2bc+12a^2c^2)\dfrac{1}{ru}$	Versine corrections.
		$(4a^2b^2-4b^4)\dfrac{1}{2r\grave{u}}$	$+(-24b^2c^2-16bc^3-16b^3c-4c^4+8a^2bc+4a^2c^2)\dfrac{1}{2r\grave{u}}$	Separation corrections.

Secondary Plane

a^2+b^2	$+2bc+c^2$	$-0-b^2e^2$	$-c^2e^2-2bce^2$	Obliquity corrections.
		$+(a^4+a^2b^2)\dfrac{1}{ru}$	$(b^2c^2+2bc^3+c^4+2a^2bc+2a^2c^2)\dfrac{1}{ru}$	Versine corrections.

Functions of χ_1

Primary Plane

Central Oblique Pencils	Add for Eccentric Pencils
$a^4+(10a^2b^2+5b^4)$	$(30b^2c^2+20bc^3+20b^3c+5c^4+20a^2bc+10a^2c^2)$.

Secondary Plane

$a^4+(2a^2b^2+b^4)$	$(6b^2c^2+4bc^3+4b^3c+c^4+4a^2bc+2a^2c^2)$.

The Functions of ω_1

Leaving the functions of $\dfrac{1}{2r\grave{u}}$ out of present consideration it will be seen that the functions of ω_1 under the head of second approximation have already been fully worked out for both surfaces of a lens in Sections V. and VI., wherein we found that the term $6bc$ or $2bc$

resulted in the formulæ for comatic stop corrections or E.C.s, and $3c^2$ and c^2 in the formulæ for spherical aberration E.C.s.

Therefore the terms $(14a^2bc$ and $2a^2bc)\frac{1}{ru}$ for primary and secondary planes in the third approximation are comatic functions involving the semi-aperture squared, and the high ratio of 7 : 1 between primary and secondary planes instead of the 3 : 1 for the second approximation is significant of much that requires working out.

The two terms $(12a^2c^2$ and $2a^2c^2)\frac{1}{ru}$ imply spherical aberration stop corrections dependent upon the aperture of the pencil, whose influence is six times as powerful in primary planes as in secondary planes.

All the other terms with one exception imply the usual ratio of 3 : 1 in primary and secondary planes.

The exception alluded to is the term $-a^2e^2$ in the obliquity corrections which does not appear at all in the secondary plane. This is also a highly significant term, and explains a phenomenon commonly observable at the foci of oblique pencils passing through certain photographic lenses, and that is a sort of double coma. For instance, when a little way inside of the focus the section of the oblique cone of rays shows over-correction for spherical aberration in the primary plane, and the primary plane only, while in the secondary plane the spherical aberration may be about correct. Thus there appears to be a side flare both towards the optic axis and away from it.

Double side flare.

The terms $12a^2c^2$ and $2a^2c^2$ may also tend either to aggravate or to mitigate the above effect.

As regards the corrections, functions of $\frac{1}{2r\dot{u}}$, which follow from the lateral separation between the two y's, although they apply only in the primary plane, yet their quantitative value may usually be regarded as by no means unimportant compared to the obliquity and versine corrections.

The Functions of χ_1

Turning to the functions of χ_1, or the intrinsic spherical aberration of the third order, it is interesting to see that the corrections in the primary plane are exactly five times as much as in the secondary plane.

The significance of this discrepancy between the ratios 3 : 1 and 5 : 1 for the functions of ω_1 and χ_1 respectively, together with the presence of the separation corrections in the primary plane only (supposing we leave all corrections involving a, or the aperture of the pencil, out of consideration), will shortly become apparent in studying the actually measured or calculated curvature of image corrections for certain photographic lenses, figured on Plate XXIV.

The peculiar comatic formation which will satisfy the ratio of 5 : 1 between the E.C.s of the third order was shown on Plate XVI., Fig. 79*f*, as being formed of a series of duplex comatic circles distributed over a length equal to five times the radius of the largest one; while the size of the formation will vary as $\tan^3 \phi$ instead of as $\tan \phi$.

We have so far dealt with all the oblique curvature aberrations of the second and third orders which are functions of the spherical aberrations at the surface or surfaces; but the series of terms would not be complete without also taking into account the end corrections, and corrections for converting $\dot{u}$ into v', carried to the third approximation. These corrections are those marked first end correction for converting $\dot{u}$ into v', and second end correction respectively, in the group of Formulæ (10) on page 119, Section V. It will be found that a third approximation will lead to corrections of the order $\tan^4 \phi$; which will apply equally to both primary and secondary planes.

The complete reduction of the formulæ of the third order highly laborious.

But the complete working out and reduction of all these aberrations of the third order, and their expression in terms of a, β, and x, as far as may be, implying the addition of the terms for both surfaces of the lens or element, would involve very much more space than we have at our disposal; and their complete discussion would require a volume to itself, although we should expect a much greater simplification in the final results for one lens or element.

Not only would the aberrations of the third order which intrinsically appertain to each lens or element require discussion, but also those which we may conveniently call the borrowed aberrations of the third order which arise in the case of several lenses in succession.

Corrections of one lens affected by preceding lenses.

For instance, a highly curved image thrown by a first lens will, from the point of view of a second lens placed at some distance behind it, lead to variations in $\tan \phi_2$ and a_2, dependent upon the first angle of obliquity ϕ; which may often be too considerable to be ignored.

Also the image of the stop centre thrown by the first lens may be subject to a considerable spherical aberration leading to variations in b_2, again dependent upon the first angle of obliquity ϕ.

These aberrations of image curvature of the third approximation present an ample field for the exercise of a higher order of mathematical skill than has generally been called for in the present work.

Some Practical Examples of Hybrid Curvature Errors

Before concluding this Section it will be instructive to reproduce in Plate XXIV., by the kind permission of Dr. Moritz von Rohr, a few diagrams from his most valuable and painstaking work, *Theorie und Geschichte des Photographisches Objectivs*, which furnish illustrations of certain of these curvature aberrations of the third order which we have been dealing with. These graphic curves show the deviations from true flatness, in primary planes by the dotted line, and in secondary planes by a solid line, of the images of distant objects thrown by various types of photographic lenses. They were worked out by careful calculation, on the supposition that the stop of the lens was in its usual working position, but reduced almost to a point ; that is, the curves traverse the foci of infinitely narrow oblique and eccentric pencils. Thus all corrections of the third order involving a (the aperture), such as we have lately been dealing with, are eliminated. Therefore, if the stop of any of the lenses were opened out to considerable working aperture, as in practical use, it would by no means follow that the curves of aberrations from the flat image would remain like these diagrams; indeed, in many cases the curves would become very substantially modified, in some cases favourably and in other cases unfavourably, a fact which somewhat discounts the value of these diagrams from the practical photographer's point of view.

Dr. von Rohr's graphs of curvature errors for narrow pencils.

Errors involving the aperture not considered.

Each of the lenses here dealt with is supposed to be placed on the left hand, and to be 3·5 inches equivalent focal length on the scale of the plate; the ordinates represent angular distances from the optic axis ; the abscissæ represent the aberrations from the plane image, but for the sake of clearness these are *four times exaggerated.*

Every 5 degrees are marked off along the vertical, and every millimetre of horizontal aberration along the horizontal base line, which represents the optic axis.

Dr. Steinheil's lenses.

Fig. 116 is the curve for Steinheil's Orthostigmat Lens, Fig. 118 for his Antiplanat, and Fig. 120 for his Rapid Antiplanat.

These three curves are substantially of the same character. The broad features are the under-corrected field and over-corrected astigmatism within 20 degrees of the axis. The image formed by rays in primary planes (dotted) is more nearly flat than the image formed by rays in the secondary plane (solid). This failure to come up to a plane image simultaneously is due to the imperfect approach to the fulfilment of the Petzval condition.

PLATE. XXIV.

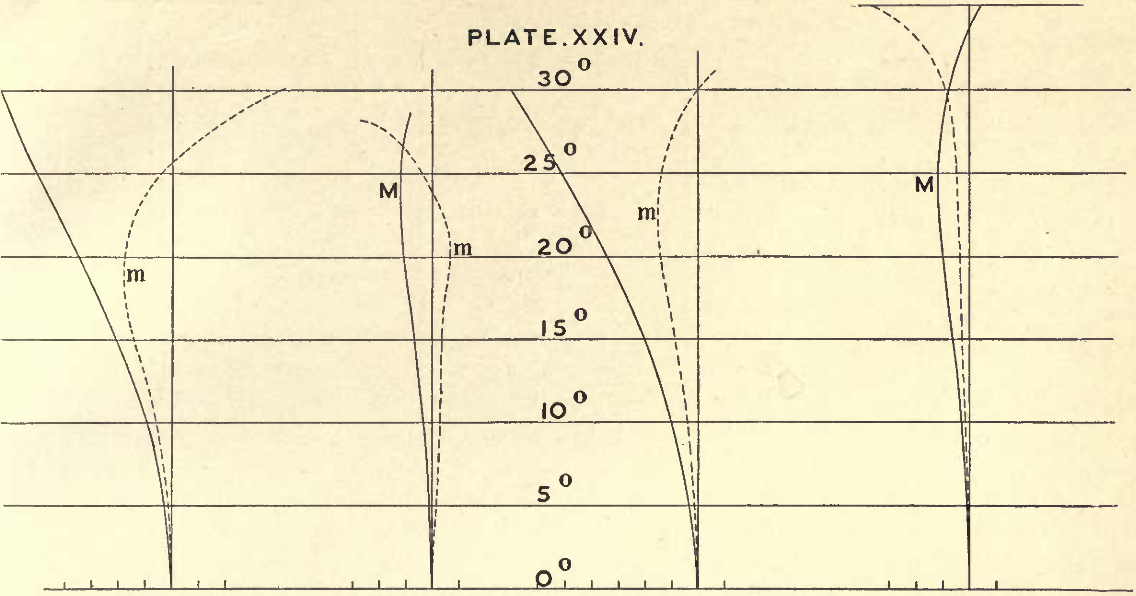


Fig. 116. Fig. 117. Fig. 118. Fig. 119.

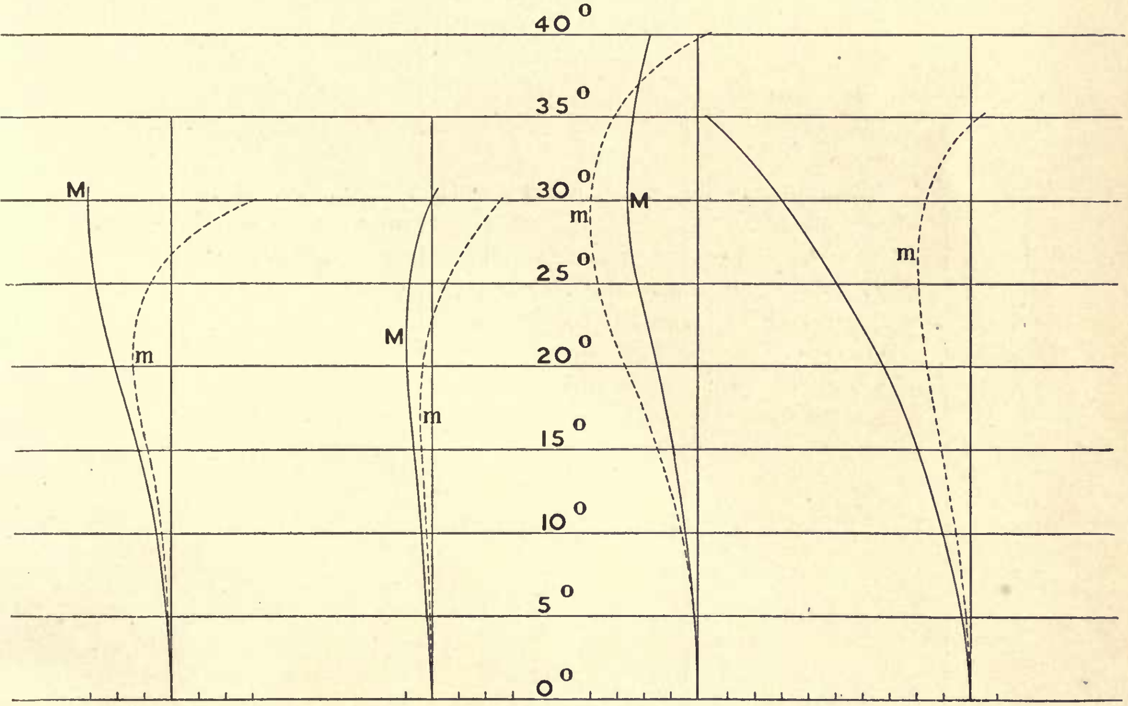


Fig. 120. Fig. 121. Fig. 122 Fig. 123.

PLATE. XXIV.

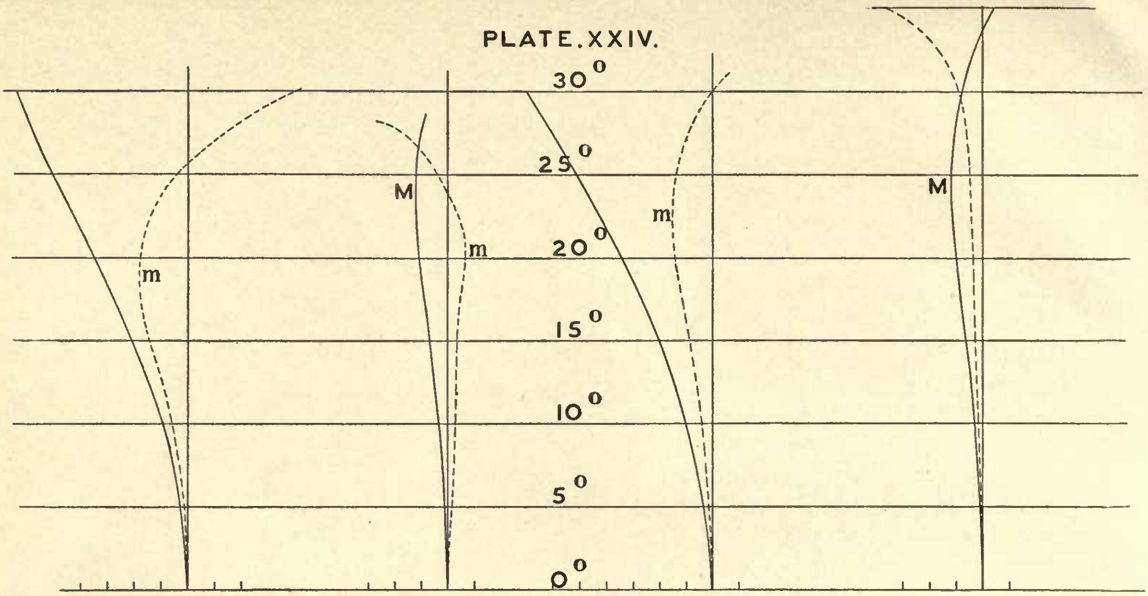


Fig. 116. Fig. 117. Fig. 118. Fig. 119.

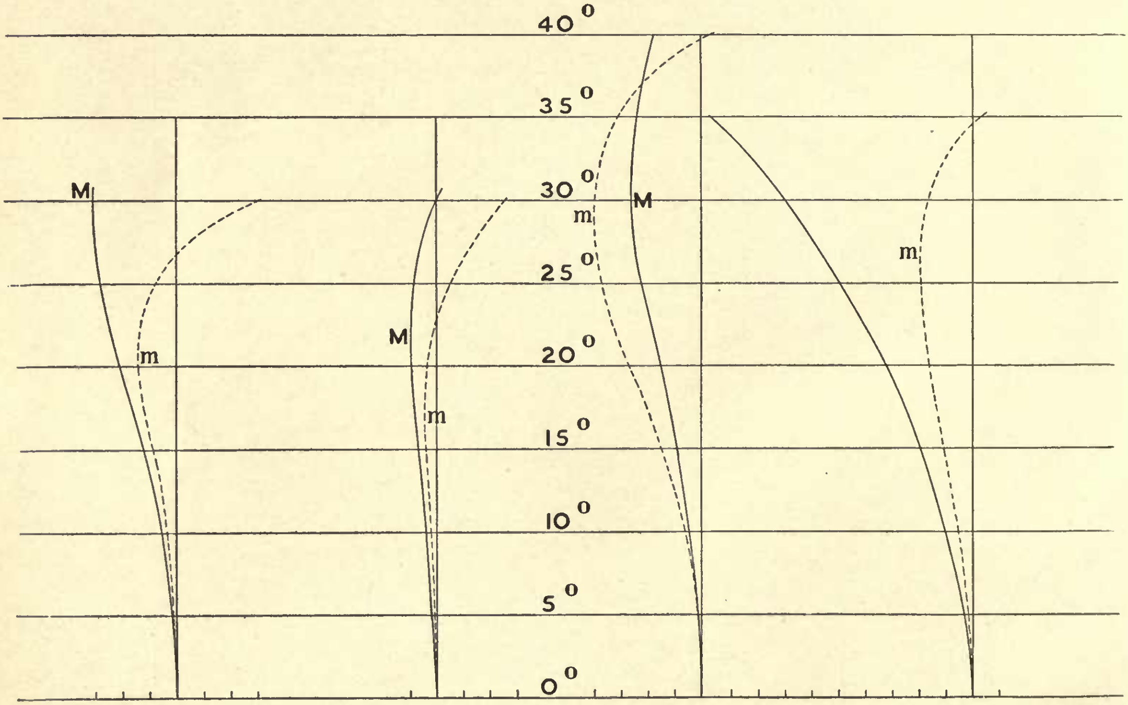


Fig. 120. Fig. 121. Fig. 122 Fig. 123.

Here we have strongly marked plus curvature errors tending to round images (concave to the lens) of the second order or varying as $\tan^2 \phi$, against which are working minus curvature errors of the third order varying as $\tan^4 \phi$. Hence the latter rapidly overtake and more than neutralise the former as we get away beyond 25 degrees, but the $+$ curvature error for rays in primary planes is at a maximum at 20 to 22 degrees, but apparently beyond 30 degrees for rays in secondary planes. But the curve of errors is of the same general character in the two planes, although the maxima and points of crossing back over the focal plane do not coincide.

Dr. Rudolph's Wide-Angle Anastigmat.

Fig. 122 for Dr. Rudolph's Wide-Angle Anastigmat furnishes a capital example of the same general features as the last three, excepting that the maxima much more nearly coincide, and the astigmatism is reversed.

Old Ross Doublet Lens.

Fig. 123 for an old type Ross Doublet Lens is a case similar to the preceding for rays in primary planes, but it is doubtful whether the curve for rays in secondary planes shows any decided tendency to a maximum followed by a curve back again; indeed, aberrations of the order $\tan^4 \phi$ appear to be only slight, while yet strong in the primary plane. These curves may be taken as fairly typical of the curvature errors exhibited by the old-fashioned Rapid Symmetrical and Rectilinear Lenses, excepting that the curve for rays in primary planes does not always retreat from the lens at the outskirts of the field.

Cooke Lens, Series V.

Fig. 121 for a Cooke Lens, Series V., indicates a very much closer approximation to an anastigmatic flat field; not only is the Petzval condition more nearly fulfilled, but a good deal of anastigmatic flatness is also gained by the separation between the lenses.

Here we have a residuum of $+$ curvature errors of the order $\tan^2 \phi$ in both primary and secondary planes counteracted by $-$ curvature errors of the order $\tan^4 \phi$; the latter at the outskirts of the field asserting themselves so much as to throw the images back behind the focal plane. The maximum for secondary rays is at about 22 degrees, and that for primary rays at about 18 degrees from the optic axis.

Old Cooke Lens, Series 111*a*.

Fig. 117 gives the curves of errors for an old form of Cooke Lens, Series 111*a* (the lens figured in Fig. 60, Plate XII.), and Fig. 119 gives the curves of errors for the well-known Goerz Double Anastigmat (the older cemented doublet). These two cases are of the same general character, excepting that in Fig. 117 the primary image is by first intention curved back convex to the lens, and is slightly concave to the lens in Fig. 119. But the most remarkable characteristic lies in

Independence of the curvature errors of the third order in the two planes.

the fact that the curvature errors of the order $\tan^4 \phi$ are decidedly negative for rays in secondary planes, but positive for rays in primary planes. Hence the manner in which the two curves cross one another at 27 and 30 degrees respectively, after which there follows a rapid mutual separation.

Now it is clear that were the ratio between the aberrations of the third order invariably 3 : 1 or any other fixed ratio between the primary and secondary planes, then such graphs as these could never arise. But since (leaving all terms containing a out of consideration, as we are dealing here with pencils of infinitely small aperture) the third order functions of $\frac{\omega}{\mu}$ are in the ratio 3 : 1 in the primary and secondary planes, while the third order functions of χ are in the ratio 5 : 1 in the primary and secondary rays, then we can clearly see that in the case of the functions of $\frac{\omega}{\mu}$ being of the opposite sign to the functions of χ we may easily have the total aberrations of the third order plus in one plane while they are minus in the other.

The separation corrections to the y's, existing, as we have seen, only in the primary plane, cause a still further degree of independence between the curvature errors in the two planes.

The disturbing effect of aperture upon the curvature errors of the third order.

And the scope for vagaries of this sort is still more enlarged when we come to deal with the images thrown by pencils of relatively large aperture, for we have seen that in the primary plane there are functions of a that are seven and six times the corresponding functions in the secondary plane.

Therefore it is that, if we take the lenses we have dealt with and open out their apertures and locate their oblique foci (by obtaining the best possible distinctness of image), we may find the curvature errors come out substantially different to those shown on Plate XXIV.

Future progress depends upon elimination of curvature errors of third and fourth orders.

It is clear, then, that it is not always practicable to determine the working character of a lens by calculating its curvature errors for infinitely narrow pencils only. It will easily be seen that the future progress of photographic lenses towards perfection depends chiefly upon the successful elimination of the curvature errors of the order $\tan^4 \phi$, and the doing of it with the simplest possible lens construction.

Printed by R. & R. CLARK, LIMITED, *Edinburgh*.