A

SYSTEM

OF

MECHANICS,

FOR THE USE OF

THE STUDENTS

IN THE

UNIVERSITY OF DUBLIN.

BY THE

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TO

THE REV. DR. LLOYD,

THIS TREATISE,

UNDERTAKEN AT HIS REQUEST,

IS DEDICATED

BY

HIS FRIEND, AND FORMER PUPIL,

THE AUTHOR.

PREFACE.

IT has long been acknowledged that the course of Physics appointed for the Undergraduates is by no means commensurate with the present enlarged state of that department of science: of late years, however, Stack's Treatise on Optics, and Brinkley's System of Astronomy, have in part obviated this complaint. The first of these works, though the theoretic parts are not sufficiently diffuse, may serve for the purpose of instruction for several years to come; by which time the science of Optics will have assumed a new face. The latter contains a great mass of information, conveyed in a pleasing form, and has already contributed much to the improvement of the students; in particular it has familiarised them with certain parts of trigonometry which had been little attended to. The department of Mechanics is alone neglected, and left in the same state as it was a hundred years ago, though it is of more practical utility than any other part of the course. The treatise of Helsham was, when published, an excellent book; but it is manifestly obsolete, and in several places erroneous; it only considers the equilibrium of machines, which is deduced in an unsatisfactory manner from the principle of Virtual Velocities, and the theory of motion on the inclined plane, and of the circular pendulum, is almost disgraceful.

Hamilton's four Lectures are little better than Helsham: the composition of motion is placed at the end of them instead of the beginning: the third Lecture is a group of unconnected topics thrown together without order; and the second deceives the student by the easy manner in which it disposes of the subject of Capillary attraction. Under these circumstances, the better class of students had recourse to the treatises of Wood and Vince; but these are imperfect, being deficient in practical information; it has therefore been long desired that a treatise should be prepared, adapted to the method of instruction pursued in the University.

It may require some explanation why one so little known, and so imperfectly qualified for the task, as the Author of this treatise, should have ventured on this undertaking, to the exclusion of many, his superiors in academical rank and in talents. Under the system pursued at present in

Trinity College, its Fellows can scarcely be expected to devote themselves to any work of research, or even of compilation; constantly employed in the duties of tuition, which harass the mind more than the most abstract studies, they can have but little inclination, at the close of the day, to commence a new career of labour. How different is this from the state of the English Universities, where the tutors constitute a very small part of the body, and the remainder have both leisure and incitement to pursue their peculiar studies, and increase the literary fame of their Alma Mater by their publications. In the present case the author happened to be less occupied than most of his brethren, yet he was engaged from seven to eight hours daily in academical duties for the year during which he composed this work.

This may in part account for its defects; but a considerable difficulty was presented by its being necessary to provide for the instruction of three different classes of students, and he is not certain that he has succeeded in the attempt. The first of these classes is composed of those who, from defective intellect, or from idleness, want the capacity or inclination to acquire abstract knowledge: such persons can make no use of principles, but they may learn results; algebraic, or even geometric proofs are beyond their acquire-ment, but they possess memory, and it may be stored with useful information. They are therefore to be loaded with as little theory as possible: but the composition of Motion and Forces, its simpler applications, the construction of machines, and many similar topics, should be required of them at examinations, altogether making a considerable portion of the text.

The second class, which fortunately predominates in point of number, have faculties of a higher order, and require a wider field; of them may be required a knowledge of Elementary Algebra, Trigonometry, and the simple analytic properties of the Conic Sections. These will read the entire of the text, except perhaps the properties of the Logarithmic, and the entire of it may be expected from the candidates for honors at the quarterly examinations.

Still this is not sufficient, and there remains a third class which is entitled to some notice; it consists of those who wish to pursue the course of Mathematical study which has lately been opened here, and are disposed to acquire the Transcendental Analysis. For them notes are added to each chapter, containing such propositions as seem too complicated to be admitted into the text, or by including applications of the Differential and Integral calculus, become inadmissible into the general course of instruction. This part the Author fears will be found most defective; in particular unity of plan is wanting; but he hopes for some indulgence. When he began the work, which was necessarily printed as it was written, he did not contemplate any extensive introduction of the Calculus: sensible of the absurdity of treating Dynamical enquiries without some Fluxional process, he intended to lay down the elementary principles of the Differential method, (which are in fact as easy as common Algebra) and refer to them as occasion offered, rather than use Prime and Ultimate Ratios, Indivisibles, or any other disguised Fluxions. This is partially effected in the Notes to the third chapter, which, with a little developement, and the addition of Taylor's Theorem, would be sufficient; but he then feared to go so far. Since that chapter was printed, his views have been extended, partly by the extension given to the courses of the Gold Medal, and Bishop Law's Premium, and partly by his own experience: he has found, with surprise and pleasure, that several Undergraduates in the Senior Classes, possess a knowledge of the Calculus much surpassing that which he had hesitated to require, and he determined to use it more freely in the subsequent parts, without confining himself to the differential theorems above mentioned; more particularly as they must be superseded by the elementary work on this subject, which Mr. Lardner is publishing for the use of Undergraduates.

In one respect, its free introduction is im-peratively demanded; it is necessary for the character of the nation, that the higher branches of Mathematics should be more widely cultivated. The erroneous notions on this head, which prevail on the Continent, can scarcely be credited: as an example it may be mentioned, that a foreign adventurer lately ventured to assert of an eminent English Mathematician, that he could not integrate an easy differential equation of the second order! It is certainly true, that for a long time the progress of this science was slow in these countries; this arose from the example of Newton, though it might have been expected to produce a contrary effect. That great man, while he felt all the advantages of the method of Fluxions, was led by his unfortunate preference of the ancient geometry to publish under a geometric form, the results which he obtained by his newly invented analysis. This gave the bias to his followers; and while the inventions of Taylor, Maclaurin, and others, were caught and used on the Continent, it may be said that none of the scholars of Newton were capable of going beyond him; a sure evidence of the impropriety of the methods adopted by them. Since the beginning of this century, the prejudice in favour of geometry has declined; many both here and in England have devoted themselves to analysis, and with success more than proportionate to the duration of their plan is wanting; but he hopes for some indulgence. When he began the work, which was necessarily printed as it was written, he did not contemplate any extensive introduction of the Calculus: sensible of the absurdity of treating Dynamical enquiries without some Fluxional process, he intended to lay down the elementary principles of the Differential method, (which are in fact as easy as common Algebra) and refer to them as occasion offered, rather than use Prime and Ultimate Ratios, Indivisibles, or any other disguised Fluxions. This is partially effected in the Notes to the third chapter, which, with a little developement, and the addition of Taylor's Theorem, would be sufficient; but he then feared to go so far. Since that chapter was printed, his views have been extended, partly by the extension given to the courses of the Gold Medal, and Bishop Law's Premium, and partly by his own experience: he has found, with surprise and pleasure, that several Undergraduates in the Senior Classes, possess a knowledge of the Calculus much surpassing that which he had hesitated to require, and he determined to use it more freely in the subsequent parts, without confining himself to the differential theorems above mentioned; more particularly as they must be superseded by the elementary work on this subject, which Mr. Lardner is publishing for the use of Undergraduates.

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It is evidently desirable that an early impulse should be given to the Students: the Lectures and Examinations of Dr. Lloyd effectually provide for the diffusion of very high mathematical knowledge among the candidates for Fellowships; but it seems advisable to begin lower on the scale, and with this view, the Author has composed the Notes on the second part, so that they contain a sketch, though very elementary, of the application of analysis to Dynamics. It is not intended that the solutions there given should be used to compare the new methods with the old; for the problems solved are of the simplest kind, and evidently beneath the power of the means employed: in particular, they are sometimes deficient in brevity, but they are always attainable by an uniform route, and to a certain degree independent of any peculiar talent in the investigator. Among the examples may be mentioned the attraction of a sphere, Parabolic projectiles, the orbit described by a force in the inverse duplicate ratio of the distance, which Newton solved only indirectly and the circular pendulum: in particular the doctrine of Moments of Inertia, and the theory of the Centre of Oscillation, will be found worthy of notice, and the principle of D'Alembert, which has scarcely found its way into the English treatises on this subject, notwithstanding its immense power in Dynamical enquiries. By the perusal of these the Student will acquire the habits of analysis, and be prepared for more difficult problems. This triple division of the treatise is indicated by the Table of Contents placed at the end of the volume: those Articles before which no mark appears are to be read by all the Students; those to which an Asterisk * is prefixed may be omitted by those who are satisfied with inferior judgments. Of the Notes, those which have no mark prefixed involve the Calculus, and are not to be required of Undergraduates under a penalty in case of failure; those marked ‡ may be classed with the Asterisks of the Chapters.

It remains to apologize for the Typographical errors, which it is feared are numerous; the employment of the Author prevented him from attending to the correction of the Press, and the printing of formulæ requires a certain familiarity with symbols, which is not commonly possessed by Compositors. It will however be found, that the latter parts are much less incorrect than the commencement.

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A SYSTEM

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MECHANICS, &c.

THE Science of Mechanics includes the notions of Matter, Motion and Force; the first of these possesses the attributes of Solidity and Inertia; it is divisible as far as our senses or even our imagination can go, but we are certain that its ultimate particles must be indivisible, or at least, that they are never divided in the operations of nature. In the strict sense of the term no mass of Matter is solid, for no mass is destitute of pores which are capable of containing other substances, thus wood contains air between its fibres; and air water, diffused in it in the state of vapour; whether the atoms of which sensible masses are composed be solid or not it is impossible to determine, for the arguments commonly used are, some doubtful, and some false. This much is certain, that these particles act on each other without being in actual contact (*a*), by means of powers connected with them by the Author of all things, and that hardness, strength and toughness, arise from the balance of two antagonist forces, one resisting the approach of the molecules composing a body, the other their separation: the first of these may be named the Repulsive, the other the Cohesive forces of the atoms. Where the forces act with great intensity, the body presents the quality of Hardness, resisting any attempt to separate the particles; where their sphere of action is of some extent, it is termed Elastic, and possesses the power of resuming its original dimension, when the power which compressed or extended it ceases to act. If the repulsive force be diminished, or the cohesive increased, the bulk of the body must diminish, as when it is compressed by an external force, or when its temperature is reduced. Heat being always an antagonist to cohesion, by its action solids become fluid, and liquids are changed into vapours, the intermediate steps being marked and measured by their expansion.

Inertia is a term invented to express that quality of matter by which it is indifferent as to rest or motion, that passiveness to every impulsewhich is so decidedly its attribute. Were there no other being in the universe it must be for ever unmoved and dead, were it once put in motion it must move for ever; and they who dreamed that the universe was caused by the fortuitous concourse of atoms, shewed their absolute want of observation. That matter has no power to put itself in motion every one will readily admit, but it is thought by some hard to conceive how it can be indifferent to rest; at the first view it appears that all motions decay, and that as some cause is required for their beginning, so it is necessary to maintain them: but if we examine more minutely, we find that there exist powers, capable of producing this loss of motion, and to which therefore it must be attributed, such as the resistance of the air, friction, stiffness of cordage: if these be diminished, for they can never be removed, the motion is prolonged, and to such a degree, as decidedly shews, that if they were removed, the motion would be perpetual. The quality of absolute

Inertia belongs only to matter in the abstract, for every atom of it with which we are acquainted acts on others, being the vehicle of the energies, by which the Governor of the Universe has ordered his work to be swayed: Gravitation resides in every particle of the solar system, Electricity and Magnetism and Heat, are in this globe almost omnipresent, and the actions of bodies on light, and the play of Chemical affinities, indicate the existence of countless forces resident in matter. But the effects of these are obviously distinct from matter, and cannot ultimately be explained by any material agency: besides we see that they cannot affect our conclusions, for in our inquiries we are aware of their influence, and allow for it, considering them as unconnected with matter; as instances, we reason as if rods were inflexible, cords pliable, and machinery void of weight, but merely conveyers of forces, and we obtain conclusions, true only in the abstract, but capable of being corrected for particular circumstances.

Motion is hard to be defined, but the mathematical conception of it is abundantly simple, if the distance of any point in space from another be supposed to change, or more accurately, if its distances from 3 perpendicular planes suffer any change, it must have moved, and the motion is translated into analytical language, by expressing the 3 perpendiculars as functions of the time, and supposing them to vary (*b*). The direction of a point's motion is found by drawing a right line through two successive places of it, and its quantity is expressed by comparing the space described with the time, the ratio of these being the *velocity*: for example, if a body move over 2 feet in one second, and another over 6 feet in two seconds, their velocities are said to be 2 and 3. If the velocity continue the same for the whole time of the motion, this is called Uniform: if it increase, the motion is said to be Accelerated, if it decrease, Retarded. In Uniform motion the Space described is as the product of the Velocity and Time: for $V \propto \frac{S}{T}$ therefore $S \propto V \times T$: and if the time of motion be the same, $S \propto V$, or if the velocity be given, $S \propto T$. Numerical examples of these facts can be easily supplied by the reader. (*c*)

As a consequence of Inertia, it follows, that no motion can begin without a cause, and we are therefore led to investigate its origin when it occurs; but if the conception of motion be not without its difficulty, this more metaphysical research is enveloped in obscurity, so that Locke declared himself unable to conceive any clear idea of active power, but from the consideration of immaterial agents. The nature of the causes of motion which we call Forces, is of no consequence in Mechanics, as they are measured by the motions which they produce, or are capable of producing in a given time, if not counteracted. If they act uniformly, the Force is measured by the ratio of the Velocity to the Time in which it has been generated, or in symbols $F \propto \frac{V}{T}$, and even if the Force and Velocity vary with the Space gone over, this proposition is still true, taking an indefinitely small moment of time, for during it the variation of force may be considered as insensible. The sources of motion with which we are acquainted are, the Energy of animated beings, the forces to which we have already alluded as implanted in matter, and the impulsion of a body which communicates its motion to another, this last being scarcely entitled to the name of force. Where one body communicates motion to another, the Quantities of motion lost and gained are equal, and they are measured by the Quantities of matter multiplied into the Velocities; thus if the striking body be doubled, its quantity of motion must be doubled, and if its velocity also be doubled, its motion is fourfold, and in the same way when a force generates motion its energy is as the product of the mass moved into the velocity produced. The strength of animals is more manageable than most other movers, but the employment of it is narrowed by the limited velocity which they can produce, and by the variable nature of their exertions; and it is too often attended with circumstances revolting to humanity. The most powerful forces which man has subjected to his industry are those of gravitation and expansion; a mass of solid matter descending from a height, a stream of water or a current of air afford potent movers, which are made useful by means of Machinery. Still more energetic are the forces causing Expansion, the Elastic force of Steam and the yet more formidable agency of Gunpowder give the means of exciting almost unlimited velocity. To devise the means of applying these to use in the most advantageous manner is the object of Practical Mechanics, and for the perfection of this art, both theory and experiment must lend their aid, as it is equally absurd to despise the investigations of the analyst without understanding them, and to found elaborate researches on false data.

The Science may be divided into two branches, the first and simplest treats of forces, not as producing motion, but as causing *Equilibrium* by their mutual opposition; in this part of the subject time is not an element of the calculus, and little more than the first principles of mathematical science are required to develope its results. But when we proceed to investigate the motions of a system of bodies acting on each other, we find ourselves beset with difficulties, some insurmountable by all the resources of Analysis, and many scarcely mastered by the exertions of the most sublime Geometry, while to reward our labour we meet at every step results the most striking and useful. It is our wish to give in this treatise a brief sketch of these two departments, sufficient to prepare the way for a complete course of study, and if any one approach *non invita Minerva*, to shew a glimpse of the acquirements which will crown his exertion.

Notes on Chap. I. (*a*) That no particles are in contact is evident from the fact that every body is compressible into less bulk: it is made manifest to the senses, by sending an electric discharge through a chain even when stretched by weights, a spark being seen at every link; and also by the rings of colours seen between two convex lenses, shewing that there is always space between them.

(*b*) According to this method the distance from the origin of the co-ordinates is $\sqrt{x^2 + y^2 + z^2}$, and from another point marked by the co-ordinates a, b and c, $\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$. If z and c are $= o$ the motion is performed in the same plane, and if y and b, the motion is rectilineal in the axis of x.

(*c*) If the velocity be not uniform, yet as it varies indefinitely little in an instant of time, we may still say that $V = dS \div dT$, denoting by d prefixed to a quantity, what Newton calls the *ultimate value* of its increment, or in other words taking the increment indefinitely small.

CHAP. II.

1. The object in all enquiries of Statics being the conditions of equilibrium, we proceed to examine its simplest case, namely, where two forces acting in opposition keep a particle at rest: in this case it may be taken as an axiom that they must be equal, and act in the same line, so that if they acted separately they would produce equal and opposite velocities.

2. When three forces in the same plane, acting on a particle, keep it at rest, one of them must be equal and opposite to the united action of the other two; therefore to determine the condition of equilibrium, we must know how to find the quantity and direction of the force resulting from the action of two or more. This problem is known by the name of the Composition of Forces, and is analogous to, but not identical with, the Composition of motion or impulse, with which, as simplest, we begin. If a moving body receive an impulse in the direction of its motion, its velocity will be increased; if in the opposite direction it will be retarded, and move with less velocity, but in either case it will continue in the same line; if the impulse make an angle with its direction, the velocity and direction are both changed, and it deviates from the line of its original motion. Let a body move in the line AE, fig. 1, perpendicular to AB, if it receive an impulse in the direction CE parallel to AB, it is indeed driven side-

ways, but *the rate at which its distance from* AB *varies,* or its velocity *in that direction* remains the same. If this be denied, let the impulse change this velocity, and it cannot be denied that the application of an equal impulse in the opposite direction must double the change. the impulse DE makes the same angle with AE that CD does, and therefore has the same effect in accelerating the particle E. But this is absurd, for CE and DE are equal and opposite, and therefore their united effect is nothing. Let now a body move so that in a time T it would describe the line AC fig. 2, at A let it receive an impulse such that if previously at rest it would describe the perpendicular AB in the time T, then it will move in the diagonal of the parallelogram CB so as to arrive at D in the time T. For the impulse AB does not change the velocity of departure from AB; therefore after the time T is expired the body's distance from that line must be = AC, as if the impulse had not acted; or it must be somewhere in CD. By similar reasoning it appears that at the end of T the body must be in BD, and therefore it must be in their intersection. If the motions AC, AB were proportionally lessened, T would be lessened in thesame proportion,and as the body would be still found at the extremity of the diagonal of the new parallelogram, which must be some where in AD, the space described by the body with this compound motion $\propto$ the time, and therefore its velocity is uniform. From the composition of motions at right angles, the general case may be derived, for, fig. 5, the motion AD is the result of AF and AE; AE = AH+HE, HE=BD=AC; now AF, AH compound AB and the result of AE and AH together with AC= Result of AB, and AC.

In general, therefore, the result of two impulses is the diagonal of the parallelogram, whose sides represent them in quantity and direction.

Instances of the composition of Velocities or Impulses are frequent, and it is easily comprehended, but in respect of pressures the demonstration is not so satisfactory. In fact the preceding proof supposes the body actually to move, whereas in all cases of equilibrio, the body remains at rest. We may perhaps fairly measure a pressure by the velocity which it would produce, if not opposed by another force; but if this should not be deemed satisfactory, we have given another in our notes, whose principles are stated thus: If three equal forces act on a point, making angles of 120° with each other, they keep it at rest, for no cause can be assigned why one should prevail over another. Here one of them EA fig. 3, must be equal to the result of the other two; the result of equal forces must bisect the angle under them, for no cause of its inclination to one rather than the other can be assigned, and hence AD is their result, bearing to one of them a ratio of $2 \cos \frac{1}{2} 120° : 1$. Secondly, if two equal forces act at an angle a, and if the result equal one $\times 2 \cos \frac{1}{2} a$, then if the same forces act at half that angle, the law holds, or the result equals one $\times 2 \cos \frac{1}{4} a$. ($a$). If the law be true of forces acting at the angles a and b, and at their difference, then it is still true when the forces act at an angle equal to their sum (b). Hence being true of 120° it holds with respect to all the angles obtained by a binary division of this angle, and of all sums of these; which ultimately extends to all angles whatsoever. The law thus proved of equal forces can be extended to all others, fig. 4, BA is equivalent to GA, AE; AC is equivalent to AE, AF, but GA, AF are equal and opposite, therefore 2 AE or AD is equivalent to AB, AC.

(c)

3. The result of two forces makes, with their directions, angles whose sines are inversely as them, for fig. 6, AB : AD : : sin BCA or CAD : sin BAC.

The result of two forces is the base of a triangle whose sides are the forces, and vertical angle the supplement of that at which they act, and therefore calling the three $A\ B$ and C, $\frac{A}{\sin (a)} = \frac{B}{\sin (b)} = \frac{C}{\sin (a+b.)}$

In all composition of forces there must be a loss of power, for one side of a triangle must always be less than the other two.

If a force act on a body in motion at right angles to its course, its velocity estimated in that direction remains unchanged; and if it be required to estimate its action in any given angle with its direction, it is to be multiplied into the cosine of the angle, thus, fig. 7, if it be required to know the effect of the force AC in a direction parallel to BC, let fall the $\perp$ AB, and the forces AB, BC are equivalent to AC, but BC : AC : : cosine **C** : 1, therefore BC = AC × cos C; and if a body at C be prevented from leaving the line BC, it will be urged along it by a force BC. (*d*).

4. We can now determine the equilibrium of our three forces, for we have seen that two of them compound a result expressed by the side of the triangle ABC fig. 6; the third must be equal and opposite to this; if therefore the three be as the sides of the triangle under their directions, a body on which they act will remain at rest.

If several forces act on a point in the same plane, it will be at rest if they be as the sides of the Polygon under their directions, for PA and AB fig. 8, compound PB, this with BC compounds PC, which with CD gives

PD opposite in direction to the last force, and equal to it. (*e.*)

NOTES TO CHAP. II.

(*a*) Let the angle EAG fig. $2 = a$ and the force EA, F, from this and an equal force in the direction of AD results AB, and bears to it a relation depending on the angle BAG $= \frac{1}{2}\ a$; now the result of AB and AC must = result of EA and AH together with 2 AI = 2 AE; but result of EA and AH by hypothesis $= 2\ F \times \cos (a)$ therefore result of AB and AC $= 2\ F\ (\cos (a) + 1)$, but it has to AB the same relation that AB has to F, let AB $= F \times f\ a$, $f\,a$ denoting a function of a expressing the relation between F and AB, then $F \times f\ a \times f\ a = AB \times f\ a = 2\ F (\cos\ (a) + 1) = F \times 4 \cos.^2 (\frac{1}{2}\ a)$ or $f\,a = 2 \cos. (\frac{1}{2}\ a.)$

(*b*) Call the angles DAB and DAF, a and b, the result of EA and AE′, is $F \times f(a + b)$ and AD $= 2\,F \times \cos\,(b)$, the result of AD and AD′ is therefore $4\ F \times \cos\ (b) \times \cos\ (a)$ = result of EA and AE′ + result of AB and AB′ $= F \times f (a + b) + 2\ F \times \cos\ (a - b)$, therefore $f (a + b) + 2 \cos (a - b) = 4 \cos\ (a) \cos\ (b) = 2 \cos (a + b) + 2 \cos\ (a - b)$ or, $f (\ a + b) = 2 \cos\ (a + b)$.

(*c*) The direct solution of the composition of forces is given by Lagrange, in his Mecanique, but on a principle which is by no means self-evident, Laplace's requires too much knowledge of the integral calculus for a treatise so elementary as this: where the forces are at right angles perhaps the following may be satisfactory, in fig. 4 call AB x, AC y, and their result $2\,z$, and angle BADθ Let the action of x in the direction of $z = x f\,\theta$, then considering the forces GA, AF $= z$; $x = 2zf(\theta)$ and $y = 2zf(90^\circ - \theta)$ therefore $2z = xf(\theta) + yf(90^\circ - \theta) = 2zf^2(\theta) + 2zf^2(90^\circ - \theta)$ or $= f^2(\theta) + f^2(90^\circ - \theta)$ hence $f\theta$ is the sine or cosine of θ, to determine which it is, suppose $y = 0$ in which case $\theta = 0$, and $2z = x$ therefore $f(\theta) = \cos(\theta)$ and $x = 2z\cos.(\theta)$.

(*d*) Let the force be referred to 3 rectangular axes, let γ be the angle which its direction makes with the axis of z, and ε that of its projection with the axis of x, then F is composed of a force parallel to z and another in the direction AC fig 9, and this latter again is resolvable into two in the directions of y and x; denoting them by Z, Y and X, $Z = F \times \cos\gamma$, $X = F, \cos\varepsilon \sin\gamma$ and $Y = F \sin\gamma, \cos\varepsilon$. It is often more convenient to refer them to the angles, which the direction of F makes with the three axes; if a sphere were described round the centre A, $a\,b\,c$ its intersection with the planes of the solid angle is a right-angled spherical triangle, and therefore $\cos\alpha = BAD = \cos\varepsilon \sin\gamma$, and $\cos\beta = \sin\gamma \sin\varepsilon$ therefore $Z = F\cos\gamma$, $X = F\cos\alpha$, and $Y = \cos\beta$; If the values of $\cos\alpha$ and $\cos\beta$ be squared and added to $\cos^2\gamma$, we obtain $1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$ and therefore $F^2 = Z^2 + Y^2 + X^2$ or the result of 3 perpendicular forces is equal to the square root of the sum of their squares.

2. If many forces act on the same point, each may be resolved in the direction of the three axes, and as the component forces are all in the same line, their sums or differences are the forces which compound the total result, thus let R be the result of the forces F′, F″ &c. $R\cos\alpha = F'\cos\alpha' + F''\cos\alpha'' +$ &c. $R\cos\beta = F'\cos\beta' + F''\cos\beta'' +$ &c. considering those forces positive which tend to increase the distance of the point from the origin of the coordinates. Where the forces and their direc-

tions are given, we as before call their components in the direction of $x\ y\ z$, X, Y, Z and the direction of their total result R can be found as above.

(*e*) In order that any number of forces meeting at the same point may be in equilibrio, we must have separately $X = 0, Y = 0, Z = 0$, for otherwise their result cannot vanish. Hence if 4 forces be in Equilibrio, they must be as the diagonal and sides of the parallelopiped under their directions.

2. If the forces act in the same plane, Z necessarily vanishes and the equations $X = 0$, $Y = 0$, are sufficient, we will apply them to determine the equilibrium of 3 forces. For simplicity we will take the direction of F'' for the axis of x, and $X = F \cos a + F' \cos a' - F'' = 0$, and $Y = F \sin a - F' \sin a' = 0$, this last equation gives $\frac{F}{\sin a'} = \frac{F'}{\sin a}$ and multiplying the terms of each equation by the coefficient of F' in the other $F (\cos a \sin a' + \sin a \cos a') = F'' \sin a'$ hence $\frac{F''}{\sin (a + a')} = \frac{F'}{\sin a} = \frac{F}{\sin a'}$, or each force must be as the sine of the angle made by the other two.

3. If any number of forces act in the same plane, we have seen that $R \cos a = F' \cos a' +$ &c. and $R \sin a = F' \sin a'$ &c. let there be taken a point in the plane and connected with the point of application of the forces, call the angle which it makes with the axis of x, m, then $R \cos a \sin m = F' \cos a' \sin m + F'' \cos a'' \sin m +$ &c. and $R \sin a \cos m = F' \sin a' \cos m + F'' \sin a'' \cos m$ &c. or $R \sin a - m = F' \sin (a' - m) + F'' \sin (a'' - m) +$ &c. but these sines are as the perpendiculars on the directions of the forces from the point assumed, and if these be denoted by the symbol p, $R p = F' p', + F'' p'', +$ &c. and in case of equilibrium $R p = o$. These

products are called *moments*, and as is easily seen express the effort of the forces to produce motion round that point, their consideration will be resumed in the next chapter.

CHAP. III.

5. We have hitherto supposed the forces to be applied at the same point, but this is seldom the case nor is it necessary: if a force act on a point, connected with another by a line incapable of extension, that other must be influenced as by a direct application of it; if the connecting line be inflexible and the action transverse, it not merely transmits the force, but also sometimes modifies it in a remarkable manner. When several forces act on a cord, which may be supposed in theory incapable of transmitting lateral force from its flexibility, two must act in the direction of its axis, and the rest will strain it into a polygon. Fig. 10 represents such a system, we will not determine its conditions of equilibrium at present, but merely remark, that there must be equilibrium at every angle; at B the forces A′ and B′ are opposed by the tension of BC, this is the result of C′ and the tension of CD, and so on. It follows from this, that the action on the cord is the same as if all the forces were applied at each point, and from these data the angles of the Polygon can be determined.

6. If two parallel forces act at the extremities of an inflexible line, it may be kept in equilibrio by a third

applied to some intermediate point. Fig 11 Let AH be taken to represent the force A′ on its direction, and BI for B′, we introduce no change in the system, by adding two equal and opposite forces in the direction of AB, let them be represented by FH and IE, then the result of the compound forces AF and BE, must be the same as that of A′ and B′, it must also be equal to their sum and must be parallel to them as there exists no cause to make it incline to either of them. Let it be DC′ then the triangles FAH and ADC, are similar, as also EBI and BDC, therefore FH : AH :: AC : DC and $AH \times AC = FH \times DC$, and $BI \times BC = DC \times IE$, therefore the intercepts of AB between the forces and their result, are inversely as the forces. If then a force opposite and equal to their sum act at the point found by this proportion, the line AB will remain at rest.

7. If two forces F and F′ be applied at the points A and C, and their result R′ at B, draw through these points any parallel lines; F: F′ :: AB: BC, or calling the intercepts between the parallels and P, a' r' and a' $F : F' :: r' - a' : a - r'$ and $Fa - Fr' = F'r' - F'a'$ or $Fa + F'a' = \overline{F + F'} \times r' = R' \times r'$ or the sum of the *Moments* of the two forces is equal to the *Moment* of their result, calling F a the moment of that force with respect to P. If there be a third force F'', substituting for the two first their result R', $F'' \times a'' + R'r' = (R' + F'') \times r$, and substituting for R′ and r', we have $Fa + F'a' + F''a'' = (F + F' + F'')\ r$, and the proposition is true of any number of forces. The proposition is true even though the a s are not drawn from the same point: let A and C be as before the points of application, and let a denote the ordinate CF, then $F : F' ::$ BH : CI or as $r' - a' : a - r'$ and $F \times a + F' \times a' = R'\ r'$ and as before $Fa + Fa' + F''a'$ &c. $= r\,(F + F' + F''$ &c.). If the point P

should lie between the lines of direction of the forces, or if the line F' F should have forces on each side of it, those which fall below it should be affected with different signs from the rest.

8. What has been delivered respecting parallel forces is capable of extensive practical application, as every body with which we are acquainted is a system of such forces acting on the particles of which it is composed; the descent of heavy bodies and their pressure on any thing which prevents it are matter of constant observation, and lead us to infer that there exists in the earth a force capable of producing these effects, which is therefore termed the force of *Gravity.* As the path of falling bodies is every where perpendicular to the earth's surface, it must act in the direction of its radii, and therefore over a small portion of space its lines of direction must be nearly parallel. But it does not emanate from the earth's centre only, for the vicinity of a mountain draws the Astronomer's plummet from the vertical line, and the attraction of a metallic mass is sufficient to affect a balance of Torsion; (*a*) we may therefore infer with great probability, that every particle of matter in this planet gravitates, and it will afterwards appear that this conclusion may be extended to the solar system. At any height to which we can ascend above the surface of the earth, it may be considered unvaried, for though it decreases, its decrement is inappreciable in our experiments. Its measure is the weight of a body, and this is as the quantity of matter which it contains, for the effect of a force is as the matter moved, multiplied by the velocity excited, but all bodies fall with equal velocities, and therefore, every particle is urged equally. (*b*).

From this it is evident, that the gravitation of each particle of a body is one of a system of parallel forces, acting on points connected by their cohesion, as we have supposed the points of application to be by lines, the result of the whole is the weight, and it passes through a point which can be determined, at which, if an opposite force be applied, the body must be supported. This point is often called the centre of Gravity, but it should in strictness be called by another name, as its properties have no peculiar connexion with gravity, but refer to any parallel forces. It possesses some remarkable properties; in considering the action of a bodies weight, we may suppose it without magnitude, and all its mass condensed in that centre; if it be the point of support, the body will remain at rest in any position, for however the direction of parallel forces be changed, the result passes through the same point; thirdly a body suspended freely cannot remain at rest, unless this point be in the vertical passing through the point of suspension. Fig 13, let S be the suspension; if G be not in the vertical let GI be the action of its weight applied at G, this may be resolved into two forces at right angles, one drawing G from S which is resisted by the cohesion of the body = GI × cosine of S, as SL is parallel to GI, the other GI × sin S, which being unopposed, will produce a rotatory motion round S. If the sine of S be = 0 that is if S be = 0 or = 180°, it is in equilibrium but with this difference between the two cases, that in the former if removed a little from the position of rest it will return to it, in the latter the equilibrium will be still more disturbed and the body will upset. From this property we derive a practical method of finding the centre, let the body be suspended from two points in succession, and it must be in the intersection of the vertical planes drawn through them.

9. From the theory of parallel forces it follows that if perpendiculars be let fall from every particle of a body

on any plane, the sum of all their moments = the weight × the perpendicular of the centre of gravity; or denoting the sum by S (pa), G the ordinate of the centre= $\frac{S.(p\,a)}{S\,(p)}$ a being the perpendicular from each particle, and p its mass, and by computing three of these ordinates for 3 axes at right angles we determine its place, but this process can seldom be performed without the aid of the integral calculus: we will determine its position in a few of the mest obvious cases. If the body were a plane surface its *CG* must be in the plane, for its moment with respect to that must be = 0; if the particles of the body taken by pairs be equidistant on each side from an axis, in which case the body is called symmetrical with respect to that axis, the CG must be in it; for since there are as many positive as negative ordinates, S $(p\,a)$ — S $(p\,a)$ = moment of the body, and = 0, therefore G vanishes. (c)

10. The *CG* of two equal particles is midway between them; that of two unequal divides their distance inversely as their weights, these follow from the equations of Art. 7. That of a right line is its point of bisection, for let AB be divided into a number of eqnal and indefinitely small parts, there are an equal number of them on each side of G, and the moment of A a, or A $a \times a$ G, = B $a' \times a$. G, but as they are on different sides of G one must be positive and the othe other negative, their sum is therefore =0; similarly the sum of the moments of $a\ b$ and $a'\ b'$ are = 0 and therefore that of the whole line; and hence the distance of the *CG* from G = 0. The *CG* of a parallelogram is the intersection of its diagonals, for fig 15, bisecting the opposite sides, BC bisects, every line parallel to EF, and AD those parallel to EH, the figure is therefore symmetrical with respect to both BC and AD therefore the *CG* must be found

in each of them and therefore it is their intersection. The Circle or Ellipse is symmetrical to any diameter, its centre is therefore its *CG*. A Cylinder or Rectangular beam is symmetrical to any two planes passing through the axis and also to a plane ⊥ to it passing through its middle point, that therefore is the required centre. The Triangle is symmetrical to the bisector of its base, for this bisects every line parallel to it, the *CG* therefore is the intersection of two bisectors which is $\frac{2}{3}$ of the whole distant from the vertex. This result may be also obtained by computing the moment of the triangle and dividing it by the area; draw any two parallels to the base at the indefinitely small distance $d\,x$, then $y \times d\,x$ is the area of the elementary trapezium (fig 16) $a\,b\,c\,d$ and $y \times x\,d\,x$ its moment, the sum of all these moments is that of the triangle. Now $y\,x\,d\,x$ is obviously the increment of a pyramid whose height is EB, fig 17, and base the rectangle under and BE, at the distance x from the vertex, the moment of the triangle is therefore equal to the pyramid $= \frac{1}{3}\,AC \times BE^2$ but the area $= BE \times \frac{1}{2}\,AC$ therefore $G = \frac{2}{3}\,BE$. (*d*)

11. The *CG* of any system may be found where those of the bodies which compose it are known: connect any two, and divide the distance between them in the inverse proportion of their weights; if we suppose their weights concentrated in the point thus found, by comparing them with a third we can find the centre of three and so of all, or still better by multiplying each weight into its distance from a given plane and dividing by their sum. Thus let it be required to find the centre of six equal weights placed at the angles and middle points of the sides of a triangle, fig 18 find their moments with respect to AE, those of AF, and E, are nothing, those of B and D are $W \times CF$ and that of C the same, therefore $G\,F = \frac{2\,W \times CF}{6\,W} = \frac{1}{3}\,CF$.

NOTES TO CHAP. III.

(*a*) The balance of Torsion is an instrument used to measure very minute attractive or repulsive forces, it consists of a horizontal rod, suspended at its centre by a wire which is twisted by its revolution; the elastic force of the wire tends to bring it back to its first position, and as this is found to be proportional to the angle of torsion, the angle measures the force which twists it. In the experiments alluded to, leaden balls two inches in diameter were fixed at the extremities of the rods, and on bringing near them leaden masses weighing 100 pounds they were attracted through an arch of 15 degrees, the force required to produce this torsion was ascertained by a method which shall be afterwards explained, and the result was that the attraction of the earth is about $\frac{1}{2}$ of what would be observed were its mean density equal to that of lead.

(*b*) When a body presents much surface, the resistance of the air which is proportional to this diminishes the velocity of descent, but when this source of error is removed all bodies fall with the same rapidity; thus if a slip of paper and a coin fall separately, they descend at a very unequal rate, but if the paper be laid on the upper surface of the coin they will not separate in their fall: the true proof however is derived from the Pendulum and shall be given when we treat of that instrument.

(*c*) Before we proceed to find the sums of moments it may be useful to explain the principle of the method of finding them. Let X be any function of x, then if x be augmented by a quantity D x, the function must receive a corresponding change DX, D being the characteristic of the successive increments of a quantity.

DX will often be a series ascending by the powers of Dx but it may be reduced to its first term if Dx be taken less than any assignable quantity, which is denoted by the characteristic d, or differential, then $\frac{dX}{dx} = X'$ this being another function of x different from the other but depending on it by a certain relation. From that relation therefore if X$'$ be given we can determine X and we must therefore investigate this relation in some of the ordinary cases. The differential of ax is adx. for if two successive values of x differ by dx, those of the product must differ by adx.

The differential of xy where both x and y vary is found in the same manner, for $(x + dx)(y + dy) - xy = ydx + xdy + dydx$, this latter term may be neglected and $d(xy) = ydx + xdy$.

The differential of $\frac{x}{y}$ is $\frac{ydx - xdy}{y^2}$ for $\frac{x}{y} \times y = x$; by the preceding paragraph,

$$d\left(\frac{x}{y}\right) \times y + \frac{x}{y} \times dy = dx,$$

$$d\left(\frac{x}{y}\right) = \frac{dx}{y} - \frac{xdy}{y^2} = \frac{ydx - xyd.}{y^2}$$

The differential of $x^n = nx^{n-1} dx$, for $\frac{x^n - x'^n}{x - x'} = x^{n-1} + x' x^{n-2} + x'^2 x^{n-3} .. + x'^{n-1}$ now if we take the ultimate value of $x - x'$ which is when they are equal, the second member of the equation which consists of n terms becomes $nx^{n-1} = \frac{d\left(x^n\right)}{dx}$ or $d(x^n) = nx^{n-1} \times dx$.

Thirdly if an arch vary, the differential of the sine of $z = dz \times \cos. z$ and that of its cosine $= - dz \times \sin. z$. To find these we must observe that the first is the ultimate difference between sin $(z + Dz)$, and sin z, but sin $(z + Dz) =$ sin z. cos $Dz +$ cos z sin Dz but if the *ultimate value* of Dz be taken, cos $dz = 1$ and sin $dz = dz$, for a very small arch coincides with its sine, hence d (sin z) $= dz$. cos z, and the differential of the cosine may be found in the same manner; the logarithmic differential is explained further on, and these are sufficient for our use.

2. If a differential be given which corresponds to any of those described we can find its primitive function or Integral; this process is denoted by the sign $\int$; thus $\int adx$ must be ax, $\int nx^{n-1} dx$ must be x^n, and to integrate we must undo what is done in taking the differential, thus in the last case we must increase the exponent of the variable by unity and divide by the exponent thus increased and by the differential of the variable.

3. The integral must be taken within certain limits for it may not begin with x, and its true value is the difference of two values corresponding to the extreme values of x.

This may be more easily understood by considering that a quantity may consist of a constant and variable part; in taking the differential the constant quantity disappears, and in integrating does not reappear: thus the differential of $\int X\,dx + C$ is $X\,dx$, and C may be thus removed from an equation. The correction may be determined where any value of the integral is known, with the corresponding value of x; for instance if it be known that both vanish together, let W be the value of $\int X\,dx$ when $x = 0$ then $C + W = 0$ and $C = -W$.

The method of integration by parts may be mentioned here as we shall use it often, it is this, $d(x \times X) = X\,dx + x\,dX$, therefore integrating, $xX = \int X\,dx + \int x\,dX - C$ or $\int X\,dx = xX - \int x\,dX + C$, the advantage of this is that $\int x\,dX$ is often much simpler than $\int X\,dx$.

4. Let it be required to find the Moment of the surface ADC with respect to PD, fig 19 this while x increases by dx increases by $2\,yx\,dx$ therefore $M = \int 2\,yx\,dx$ and if this be divided by A the area of the curve or by $\int 2\,y\,dx$ we obtain G the distance of the *CG* from D. From what has been said these integrals must be corrected by subtracting from their values when $x = BD$ those when it is $= 0$. If the body be a solid of Revolution, $G = \frac{\int py^2\,x\,dx}{\int py^2\,dx}$ p being the ratio of the circumference of a circleto the diameter. (*d*) Let it be required to find the G of a triangle, here $y = \frac{bx}{a}$ and $\frac{M}{A}$

$$= \frac{\frac{b}{a}\int x^2\,dx}{\frac{b}{a}\int x.\,dx} = \frac{\frac{1}{3}x^3}{\frac{1}{2}x^2}$$ these when $x = a$ are

$\frac{2\,a^3}{3\,a^2}$ and as M begins with x the other values are $= 0$ therefore $G = \frac{2}{3}\,a$. In the cone

$$G = \frac{\int x^3\,dx}{\int x^2\,dx} = \frac{3\,x^4}{4\,x^3}$$

or when $x = a$, $G = \frac{3}{4}\,a$. If we did not wish for the *CG* of the whole triangle but for that of a trapezium whose height is h we should take the integrals from $x = a - h$ to $x = a$ and the distance of that point from the occurse of the sides would be

$$\tfrac{3}{4}\left(\frac{3a^2 - 3ah + h^2}{2a - h}\right)$$

Its distance from the base is $\frac{h}{3}\left(\frac{3a - 2h}{a - h}\right)$

The distance of the CG of a conical frustum from its base is if its diameters D and d be nearly equal, $\frac{h}{6} \times \frac{D + 2d}{2D - d}$

To find that of a circular arc let AB $= z$ and CB unity, then $2\,dz \times \cos z$ is the differential of the moment, and G $= \frac{\int 2 \cos z\, dz}{2z} = \frac{2 \sin z}{2z}$ or G : radius as BD : BAD.

To find that of the segment BAD, M $= \int -$ $2 \sin z . \cos z . d \cos z = \int 2 \sin^2 z \cos z . dz = \int 2 \sin^2 z \times d \sin z = \tfrac{2}{3} \sin^3 . z$ while the area $= \int -2 \sin z \times d \cos z = \int 2 \sin^2 z . dz = \int (1 - \cos 2z)\, dz = z + \tfrac{1}{2} \sin 2z$ therefore

$$G = \frac{\tfrac{2}{3} \sin^3 z}{z - \tfrac{1}{2} \sin 2z}$$

The distance of the CG of a spherical surface from its centre $= \frac{\int 2p . \sin z . \cos z . dz}{\int 2p . \sin z . dz} \quad \frac{p . \sin^2 z}{2p(1 - \cos z)}$ $= \tfrac{1}{2}(1 + \cos z)$ and of course it is at the middle point of the versesine.

That of a spherical segment is

$$\frac{\int p \sin^2 z \cos z \times - d . \cos z}{\int p . \sin^2 z \times - d . \cos . z} = \frac{\int p \sin^3 z\, dz \sin z}{\int p (\cos z - 1)\, d \cos . z}$$

$= \frac{\tfrac{1}{4} \sin^4 z}{\tfrac{1}{3}(\cos^3 z - 3 \cos z + 2}$ If the segment be a hemisphere it is $\tfrac{3}{8}$ of radius.

In the Conical Paraboloid it is $\frac{\int pax^2\,dx}{\int paxdx} = \frac{\frac{1}{3}x^3}{\frac{1}{2}x^2} = \frac{2}{3}x.$

These examples are sufficient to explain this method, and we shall have occasion to refer to many of their results again; I have therefore introduced them though they might have been found in the ordinary works on Fluxions, to exercise the student in the method of integration.

CHAP. IV.

12. We have already enumerated the various moving powers which are at our command, and have stated that they are applied to use by the intervention of Machinery; This, besides applying them to the resistances to be overcome by them, serves also to change their direction, energy, and velocity. Without such instruments it would rarely be possible to use any force except human strength, and the employment even of this would be very limited: in the least complicated of all efforts, that of raising a weight, as soon as it surpassed the strength of one labourer great difficulty would be experienced, for it would scarcely be possible to unite that of many, and they would produce an effect proportionably less as their number increased; but if it be made possible for one man to act against a resistance tenfold his unaided strength, and at the same time it be easy to combine the efforts of many with undiminished effect, there can be no limits to the tasks which they may perform. But the work to be done is often much more complicated than the mere raising a weight, and often requires a degree of velocity

much greater than that of the mover ; for example in grinding corn it is found that this is done best when the upper millstone has a velocity of 20 feet in a second, and this could not be given by the exertion of men, for it would require them to move at the rate of 12 miles in the hour, which much surpasses their power even if unloaded. In general there is a certain velocity at which an animal's strength is entirely expended in moving its body, and some is always expended thus, but this loss is found to increase as the square of the velocity, and therefore it is useful to give the speed by machinery and make the animal move slowly. Thus if the work done by a man walking at 2½ miles an hour for a day be 1, if he double his rate of walking it will be 0, for 5 miles is his maximum velocity, but if the speed be doubled by machinery the energy is ½. The more minute detail of these principles belongs to Dynamics. For the other movers which have a velocity not variable at our pleasure, and which move only in right lines, it is evident that machinery must be used both to collect and to regulate their power. It has been shewn that motive forces are as their energies, measured by the masses they can move, multiplied into the velocities: if then we excite motion at the beginning of a train of machinery, the force which we apply must be communicated to its working point or termination, except what may be lost by friction in its passage, therefore the energy of the working point must be, to that of the power applied, inversely as their velocities. Therefore no force is generated in Machines, all that is effected by them, is to move a light body with speed, or a great weight slowly; and therefore they who have sought for a perpetual Motion have displayed their ignorance of the Elements of Mechanics. In the ordinary employment of Machines

Time is not considered, and the question to be resolved is what mechanical advantage is gained, or what weight will make equilibrium to a given power. This we will now examine, the consideration of Machines in motion must be deferred till we treat of Dynamics.

13. The Cord is the simplest possible of Machines not merely as a conveyer of force but as affording by particular application great mechanical advantage. fig 21, Let a cord BCA be attached at B to a firm support and a force P act at C, it is required to find the energy W excited in the direction CA. Supposing W in equilibrio with P, and the tension produced on CB, these three forces must be as the sides of the triangle under their directions, therefore W : P :: AC : CD, but when the angle BCA is very obtuse, the advantage is great, for CD is small in comparison of AC. In this arrangement the effect must be a maximum when CD is perpendicular to CB. If the point C be permitted to move along the Cord the angle C must be bisected by the line CD, for in that case the tensions of AC and CB must be equal and therefore the triangle CAD Isosceles, hence ACD = DCB. In this case W : P :: sin $\frac{1}{2}$ C : sin C, or as sin $\frac{1}{2}$ C : 2 sin $\frac{1}{2}$ C $\times$ cos $\frac{1}{2}$ C, or as 1 : 2 cosine of $\frac{1}{2}$ C. (*a*)

14. The next in point of simplicity is the Lever, without which man would be unable to execute any work of art and which enters into almost every mechanical combination: this Machine is a bar, supposed inflexible, and devoid of weight, which is used to produce equilibrium between two powers. Let P and W acting in the directions AD and BD (fig 22 and 23) be applied to two points of the Lever AB, their result must clearly pass through it, and if a support be placed when the result intersects it, the Lever must be kept at rest. There is no difficulty in finding this point, for by Art. 3. P : W :: sin CDB : sin CDA, or if we take on AD and

DB lines in the proportion of P : W and complete the parallelogram, its diagonal intersects the Lever at the point where the prop or fulcrum should be placed. If the result be not perpendicular to the Lever, it may make it slide on the fulcrum. The ratio of the forces is most easily represented by the inverse ratio of the perpendiculars on their directions from the fulcrum thus determined, for they are as the sines of the angles made by the result DC with directions of the forces, and the condition of equilibrium is expressed by saying that the forces must be inversely as the perpendiculars on their directions from the fulcrum If the forces be represented by the inverse perpendiculars, the line EF is the pressure on the fulcrum. The equilibrium does not depend on the figure of the Lever, and P and W will keep at rest any one whose fulcrum is in the line DC, and points of application of the forces in AD and BD : if ECF were taken, its arms would be the least possible and the forces acting perpendicularly to them would tend solely to produce rotation round C. In this case the products of the forces and arms are equal, and express what have been already described as Moments of forces, being clearly their powers of producing rotation, and we might have derived the equilibrium of the Lever at once from what has been demonstrated in note(*d*)chap. 2, that the sum of the Moments must be = 0 ; for the sake of clearness we have preferred a separate demonstration.

If the forces act in parallel lines the point D is at an infinite distance, and if the Lever be straight since EC : CF :: AC : CB fig 24, the power and weight are inversely as the arms, a conclusion which might have been drawn from the consideration that the result of parallel forces divides the distance between them in their inverse proportion ; and we see that if they be weights at the extremities of the arms, the fulcrum must be at their *CG*. If the distance of the power from the fulcrum be greater than that of the weight to be raised, it ma echani-less in the same proportion, and if it be increased as is neces-librium must be broken. But though it be than coul raise a great weight with a small power, y giously a raised through a less space than the power d , for the perpendicular ascent and descent are EA and FB, which are as the arms, or inversely as the power and weight.

There are reckoned 3 kinds of Lever, distinguished by the place of the fulcrum ; in those of the first order, the fulcrum is between the power and weight, in the second, the weight is between the power and fulcrum, and in the third, the power between the fulcrum and weight. In the three the mechanical advantage depends on superior distance from the fulcrum, and therefore in the third kind is always on the side of the weight : this kind therefore is never used except from necessity or where it is desired to augment velocity. The second is most useful, as in it the pressure on the fulcrum is only the difference of the weights instead of the sum, and its length is that of the longer arm of an equivalent Lever of the first order.

Examples of a Lever of the first kind are afforded by shears, pincers, and instruments of the same kind, brakes of pumps, the crow used in quarries whose point is introduced under a fragment of rock, the edge of that below it being the fulcrum. A Pyrometer or instrument adapted to measure the minute expansions of bodies by heat, is sometimes constructed on this principle, fig 25. It consists of a light Lever, turning on a centre and provided with a graduated arch ; a rod of the substance to be examined is placed with one end resisting on a firm support, and the other bearing by a point on A which is pressed against it by a spring : heat of known intensity is then applied, and the motion of its extremity is pointed out on the arch, being magnified in the proporti[illegible]f AC : CD. Sometimes a succession of point w[illegible]he long arm of each bearing on the short result be n[illegible] and it is obvious that the motion is in-cide on the [illegible]portion of the product of the short arm[illegible]duct of the long. The Gunlock is a combination of Levers of this kind and is mentioned for the ingenious method by which a variable mover is made to produce a uniform power: the flint is carried by a bent Lever, urged by a strong spring which must act more powerfully the more it is bent, but the arm of the Lever on which it acts is curved, so that as it is bent it approaches to the centre of motion, greater energy being compensated by shorter Leverage.

As specimens of Levers of the second kind we may present the Oar, where the water by its resistance against the blade is the fulcrum, and the weight is applied at the contact of the oar with the side of the vessel; here from the advantageous method in which a man's strength is applied, great effect is produced, and when machinery is moved by human strength the motion of Rowing should be used. The treddle of the turning lathe and the chipping knife, one of whose extremities is fixed to a bench, while the other is a long lever producing powerful pressure on any body placed under its edge, even doors are of this order; the hinges being the fulcrum, the strength of the bolt the power, and a force attempting to break it open the weight. In general, the first kind is used where the power and resistance act in the same direction, the second where they are opposite.

The most remarkable example of the third kind occurs in the structure of animals, where the bones are levers of this order, the joints being fulcra, and the attachment of the muscles the application of the power, this is mostly very near the joint, and as the direct pull of muscular conraction is immensely great, notwithstanding the mechanical disadvantage we possess as much strength as is necessary, with much greater rapidity of motion than could otherwise have been obtained without prodigiously augmenting the bulk of our bodies.

It is sometimes required to produce an immense pressure at a given point, and at the same time to bring the Lever rapidly into that position, as in the Stanhope Press, where the part which presses on the Types must descend through a considerable height and exert a severe pressure only at its contact with the paper; for this the arrangement of fig 26 is used, the Lever ACB is connected with ED by a piece jointed at B and D; in the position shewn by the figure a power acting at right angles to the lever ACB has almost irresistible energy to turn ED round E; at a distance from this position it loses its mechanical advantage, but turns the other with much greater rapidity.

The Lever is sometimes used to distribute a load proportionally to the strength of the bearers; and from the preceding demonstrations there is no difficulty in conceiving that if a load be put in the place of a fulcrum, it will produce pressures inversely as the perpendiculars from it on the directions of the sustaining forces, or if these be parallel, as the arms. Horses are commonly harnessed in this manner, and it has the advantage of equalizing their draught, as if one presses forward he acts at a disadvantage; the dotted lines in fig 27 shew this, as the distances from B the point to which the load is attached are AB and BC, while in the ordinary position of the bar they are equal, being both EI. (*b*)

We have hitherto supposed the Lever void of weight, but it always is heavy, and it is necessary to allow for its weight; if its *CG* be in the fulcrum or in the vertical passing through it at the position of equilibrium, it requires no correction as its moment = 0. In any other case, let the weight of the instrument be W′ and GI the distance of its *CG* from the vertical passing through the fulcrum, then in equilibrio $P \times AE + W' \times GI = W \times BF$. If the Lever be straight GI is as half the difference of the arms.

15. The Lever is used to ascertain the Weights of bodies by comparing them with others that are known, it then is called a Balance, some of the most useful of which we will describe. The simplest and best is a Lever of the first order with its arms equal in length and weight, its fulcrum is an axis of hard steel whose extremities are formed into knife edges on which it turns, the arms have at their ends similar knife edges from which dishes are suspended to hold the weights, and an index or tongue rises perpendicularly over the axis. It is evident that if equal Weights be placed in the dishes the Balance will be in equilibrio, as its arms are equal; and by the position of the *CG* of the Beam or Lever below the axis, it must be horizontal when in equilibrio and the index perpendicular. If the weights be unequal, the Beam will incline and then indicate a difference.

Simple as this machine is, it requires many adjustments, and it is very rare to meet one perfect. In the first place the arms must be of equal length, this is ascertained by counterpoising any body by Weights and then transposing them, if the arms be equal there will still be equilibrium, but if not, that which inclines is the longer. Balances designedly unequal are sometimes used for the purposes of fraud, as by putting the weight on the shorter arm it will be balanced by less than an equal weight of the commodity sold; however, even with snch an instrument the true weight can be ascertained, for let W and w be the apparent weights of x, A and a the lengths of the arms, then $W : x :: A : a$ and $x : w :: A : a$ therefore $W : x :: x : w$ and $x = \sqrt{Ww}$ A buyer would suffer no loss if he made the seller weigh half in one dish and half in the other, as if the balance be deceitful he would get more than his due. Even where the most rigorous attention has been paid to the construction of the beam, this requisite may not be attained, and therefore in nice weighing, the method given above should be used, or still better, the body to be weighed should be counterpoised with fine sand or some similar substance, it should then be removed and weights put in its place to balance the counterpoise, they must be its true weight.

In the second place, the CG must be in a line passing through the axis perpendicular to the beam; this is examined by placing a mark against one of the ends of the beam and reversing it, if the other arm point to the mark, the beam must be horizontal. 3dly, the index must be in the line passing through the axis and CG, this is known by reversing the beam. 4thly, the points of suspension ought to be in the plane passing through the edges of the axis as otherwise the ratio of the arms varies with the inclination of the beam, fig 29, let AB be the line joining the points of suspension and C the axis, then the moments of the weight are as AF and FL whose difference is 2 CD × sine of inclination, 5thly, the line joining them, should be horizontal when the balance is unloaded, for otherwise the arms may become unequal when weights are put into it, by the flexibility of the beam. Fig 30, let ACB be the line connecting the points of suspension and the axis, let it become a curve by the flexure of the beam, and B will recede from C while A approaches it; the first of these may be examined by trying whether a given difference of weights on either side produces the same inclination, and the second by trying the equality of the arms with different loads. A balance thus adjusted will have no tendency to upset whatever be its load, which if the line joining the points of suspension passed above the axis, might happen; for by the third property of the CG, a body cannot be in a state of stable equilibrium unless the CG be below the support, and conceiving the mass of the dishes and weights collected in the points of suspension, their CG would be in the line joining them, and that of the whole, between that line and the CG of the beam unloaded; but it is obvious that as the load is increased the common CG must approach nearer to the line joining the points of suspension, till it rises above the axis. In that case the beam might remain at rest with equal weights in the dishes but on the least oscillation it would descend, that arm which was inclined preponderating; balances are sometimes made thus intentionally, that they may be unable to weigh any body above a certain weight, but this is a bad practice as we have already proved.

The balances used in commerce are seldom so carefully constructed as to be perfectly adjusted in all or any of these points, but the instruments which are used in Philosophical researches, particularly Chemical, must be of the utmost truth and sensibility, and we will briefly mention their peculiarities of construction. The beam is in these made inflexible by means which shall be afterwards explained; one of its points of suspension is carried by a screw towards and from the axis, by moving which the arms can be made equal; the other has a similar motion up and down by which the three points of support are brought into the same right line; in the vertical passing through the axis is a weight adjustable by a screw to bring the CG into any required distance from it, as on this depends the sensibility of the balance or the ratio of the least sensible difference to the load. In fact the balance may be considered as a lever bent at right angles, the difference of weights being applied at the extremity of the arm, and their sum together with the weight of the beam and dishes acting at the centre of gravity, the less therefore this latter distance is, the less will be the difference of weights required to counterpoise the energy acting at it. Opposite to one of the extremities of the beam is a graduated arch shewing its inclination to the horizon, the index being liable to interfere with the adjustments; (*c*) the knife edges bear on planes of agate or hard steel which are set level by a plumb-line, and there is a contrivance to bear the beam when not in use and relieve the edges of its weight; the whole is included in a glass case to secure it from the agitations of the air. A good balance will easily turn with the ten thousandth part of the load, but some are mentioned which possessed a sensibility ten times as great, one made by Troughton weighed six pounds and turned with $\frac{1}{100}$ of a grain, that belonging to the Royal Society of London shews the $\frac{1}{10000000}$ of the weight.

16. The Steelyard or Statera, is a lever of the first order but its arms are unequal; the shorter is made thick and massive, so that with a scale dish attached to it as in the balance to hold the body weigh'd, the instrument may be in equilibrio; the axis resembles that already described, and its edge, that of the suspension, and the upper surface of the longer arm must be in the same right line. The upper surface is divided into equal parts, each division being a certain submultiple of the distance between the axis and point of suspension, and a sliding weight is suspended on it by a ring of hard steel formed on its inner circumference to a sharp edge,

the horizontal position of the beam is marked by an index at right angles In weighing with this instrument the body is placed in the dish and the weight is slided along the divided arm till it is in equilibrio, and the division at which it rests shews the weight; the theory of it is evident, for the moment of the weight is as its distance, therefore at any other distance it will be proportionally increased or diminished, let it weigh a pound and let the shorter arm be four inches, then if the counterpeise be placed four inches from the axis it balances a pound, if its distance be 8, the body in the scale is 2 pounds, on this supposition every inch of distance would be four ounces, and the machine might determine any weight under 10 pounds: if now the counterpoise were removed, and another substituted weighing 4 pounds, then every inch correspouds to a pound and the indication of the divisions must be multiplied by 4, and thus by a very few weights we possess an extensive scale.

This instrument cannot determine weight with the same accuracy as the balance, yet if carefully made it would be of great use to the Philosopher, from the facility of its use where the utmost precision is not required; such have been made and found extremely convenient, they carry several weights at once, the largest is placed near the position of equilibrium and it is perfected by moving one much smaller, which of course indicates a minute fraction of the other; where great delicacy is necessary it is carried by a micrometer screw: with one counterpoise, the instrument gives a scale of specific gravities of fluids, with another of gases and with a third it corresponds to a series of weights in decimal progression. As commonly made it is but a rude instrument, and from the preponderance of the longer arm there is no ready method of verifying it; it has in general two axes one nearer to the point of suspension than the other, the counterpoise therefore answers to two different scales which are marked on two sides of the beam: in these the divisions ought not to commence from the axis, but from the point where the counterpoise should be placed to keep the beam horizontal without any load.

The steelyard on a large scale is used to weigh waggons and other great weights, it is commonly called a *weighing machine.* Its mechanism is exhibited in fig 31, it consists of two levers ABA, each of these bears at A, on steel points, and the lower surfaces of B are formed into edges which rest on a pin projecting from the lever CD whose fulcrum is at C. Four steel points shewn at E in the upper surface of the levers ABA support a platform on wheih the waggon is driven, and the counterpoise is placed on a dish at D. A weight placed on the platform is to its energy at B as AB: EB, and this again to the energy which it produces at D as CD : CG, the weight therefore is to its counterpoise as CD × AB : CG × EB. The levers ABA should be so bent that the bearings A and B on their lower surface and E on their upper may be in the same horizontal plane, they should also be precisely equal and similar.

17 A variety of the Steelyard in which the weight is fixed and the point of suspension moveable deserves notice, it is called by some the *Danish balanee* ; it is a rod, terminated at one end by a heavy knob and at the other by a hook to which any body may be appended; for an axis it has a sharp edged ring or even a loop of cord, which is shifted till equilibrium is produced; a scale of divisions is marked on the rod and that at which

it stands is the weight of the body examined; if the series of weights be in arithmetical progression, that of divisions is in harmonical *(d.)* The instrument is not adapted to give very accurate results but it is extremely portable and simple.

18. A balance whose arms are not in directum is sometimes used as it gives the weight without any manipulation, fig 32, ACB is such a beam with a fixed weight P at A and an axis at C; if a body be appended at B, supposing the beam void of weight, $W \times CB \times \sin FCB = P + AC \times \sin ACF$, but the angle ACB is usually right, therefore $W = P \times \frac{AC}{CB} \times \text{tangent } ACF$: the weight is therefore shewn by an index CI on a graduated quadrant divided so that the tangents of the angles are in an arithmetical progression, and placed so that the index points to zero when the balance is unloaded. The sensibility of the instrument being inversely as the weight required to move it through a small angle is inversely as the square of the secant or directly as the square of the cosine of ACF. *(e)*.

NOTES TO CHAP. IV.

(a) If the point C move along the cord, it describes an ellipse whose transverse axis is the length of the cord and foci are the points A and B, hence if the direction of C′ be given we can find the point at which C will rest supposing the cord firmly attached at A and B, for let the ellipse be described, draw a tangent to it at right angles to the direction of the force and suppose it represented by a weight sustained by the cord, this will rest

at the point of contact; the point of contact is easily determined, for CD is parallel to the perpendicular from the focus, therefore draw through A a line parallel to CD and with B as centre and the length of the cord as radius describe an arch cutting it, connect the intersection with B and from the middle point of AF draw the perpendicular EC which gives tho point C.

(*b.*) If a load be carried by a lever of this kind and firmly attached to it so that it cannot swing, the ratio of the arms varies with the inclination; for in fig 35 let G be the centre of gravity of the load, it must be considered as acting at V, and the effective arms are AD — GD × tangent of inclination and BD + GD × tangent of inclination, therefore A's share of the burthen is increased by the inclination; if GD had been negative, that is if G fell below AB the reverse would have been the case.

(*c.*) Let W and *w* be two weights nearly equal and D their difference, G the CG of the beam and them supposed concentrated in the points of suspension, then considering the balance as a bent lever ACG, the difference acting at A tends to incline it, and the weights W + *w* + *m* (the weight of the balance and its appendages) applied at G, tend to make it horizontal, therefore $D \times AC = (2W - D + m) \times CG \times$ tang inclination or neglecting D in the second member, $D = \frac{CG}{CA} \times$ tang $I \times (2W + m)$, D therefore is as the tangent of inclination and if the arch be divided to an arithmetical scale of tangents, it will serve to measure small differences where minute fractional weights are not at hand.

(*d*) Let AB be a Danish balance whose CG is C and weight W, the loop is at R it is required to determine the weight applied at B, let RB be x and CB a then y, the weight at B, $\times\ x = W \times a - x$, if then there be erected perpendiculars proportional to the weights at the extremities of the xs the curve obtained will shew the relation between the weights and distances, now $(y + W)\,x = W\,a$ is the equation of an equilateral hyperbola one of whose asymptotes is CD and the other a parallel to GR drawn at the distance BC = W, take CI : CB as the largest weight required is to W, let CI be divided according to the scale required, and through the points thus found drawing parallels and dropping perpendiculars from their intersections with the curve, we obtain the requisite graduation.

(e) Let the weight of the bent Lever be V the distance of its CG from the axis D the angle ACF θ, T its value when the balance is unloaded, and τ when the weight P is detached then $W \times CB\ (= b) \times \cos\theta = P \times AC\ (= a) \times \sin\theta + V\,D \times \sin(\theta - \tau)$ or $W\,b = (Pa + VD \times \cos\tau)$ $\tan\theta - VD \times \sin\tau$, but if $W = o$, $VD \sin\tau = (Pa + VD \cos\tau)$ tangent T therefore $W = \frac{1}{b} \times (Pa + VD \cos\tau)(\tan\theta - \tan T.)$ If P be very heavy in comparison of V, T may be neglected, and $W = \frac{a}{b} \times P \times$ tangent θ for $D = \frac{P}{V} \times \frac{\sin T}{\sin(\tau - T)} \times a$ or T may be made to vanish by a sliding weight on the index.

CHAPTER V.

19. If a body be placed on a horizontal plane it remains at rest, for the force of gravity acts perpendicularly on the plane and is opposed by its reaction, but if the

plane be inclined,as its surface is oblique to the direction of gravity, this may be resolved into two, one parallel and one perpendicular to it; the latter of these is counteracted by its reaction, being its pressure on the plane, the former produces motion. If a second force be applied to the body and be resolved in directions parallel and perpendicular to the plane, the sum of the perpendicular forces is the pressure, and that of the parallel, the tendency to move along the plane. It may happen that the action of the second force parallel to the plane is equal and opposite to that of gravity, and in that case the body must be supported. (*a*) It is perhaps simpler to consider the two forces as producing a result perpendicular to the plane, as in that case they cannot generate motion. Let, fig 39, the body B be acted on by gravity in the direction BV and a force P in that of BF, their result must be perpendicular to IH, therefore in BE, draw EV and EL parallel to the directions of the forces and they must be as the sides of the parallelogram, therefore P : W :: sin LEB or angle I: sin LBE, but the sine of LBE is the cosine of LBH, therefore if any power sustain a body on an inclined plane it is to the weight as the sine of inclination of the plane: cosine of the angle made by the power and plane.

If the power act in the direction of the plane, the cosine of LBH is unity, and P = W × sin I, or is to W :: HD : HI; if it act parallel to the base of the plane LBH = I, and P = W × tangent I, or is to the weight as the height to the base.

20. The result of P and W is the pressure on the plane, call it R, it is to W :: BE : LE or as sin L : cos LBH; the pressure then is to the weight as cosine of the angle which the power makes with the horizon to cosine of the angle which it makes with the plane. If

LBH is = 0, R : W :: cos I : 1, and if it be = — I, or the force be parallel to the base, R : W :: 1 : cos I. From these principles it follows that the inclined plane is a mechanical power as the Power may be less than the Weight raised in any ratio ; but here as in all other machines we lose in time what we gain in power, for, to raise the Weight through the height the Power must traverse the length of it, these spaces being inversely as their energies.

21. The relation between the power and weight does not depend on the magnitude of the plane but merely on its inclination ; but a curve surface and its tangent plane have the same inclination, therefore the formulas of Art. 19 and 20 determine the force required to sustain a body on a curved surface. We will give an instance or two of the method. Let AF, fig 39, be the section of a cylindrical surface, draw FE a tangent, the force required to sustain a body at F is to its weight as sine of E or cosine of C : radius, and the pressure on the cylinder is as the sine of the same angle. (*b*)

Another example may be given in the Cycloid, a curve remarkable in the annals of Mechanics, but it is necessary first to describe its nature and properties. If a circle AB, fig 40 called the Generating Circle carrying a pencil in its circumference, were rolled along the line AL, the pencil would trace the curve LBM, in which BA is the axis and LM the base. Its first property is that the intercept of any ordinate PV between the curve and the circle BVA is equal to the arch BV, for E is the place where the generating circle touches LM when the pencil is at P, and LE = PE the arch which has rolled off it, and PE = VA, for their verse sines are equal being the distance between the parallels PV, LA, therefore LEP = VAE being equal to the angles in the alternate segments, which stand on the equal arches PE, VA, or PE is parallel to VA, and PA is a parallelogram, so that PV = EA, but LA AVB and LE = VA, therefore EA = PV = arch VB. It follows from this that the Cycloid is described by a compound of two equal motions, one parallel, to the base and the other the rotation of the circle, for MN = PV and Mn = PV — nN = MP, but the compound of equal motions bisects the angle made by their direction, and if MP be extremely small so as to be coincident with the tangent, MP and Mn are the sides of a parallelo gram whose diagonal is the tangent of the Cycloid; but the angle NVX is bisected by the chord BV, and it is equal to MPX, therefore n PX = BVX and n P is parallel to BV, the tangent of the curve to the chord of the generating circle. Thirdy, let the Cycloid LFK be described by the same generating circle, describe EFI and draw PE, EF, the arch PE = LE = DI = IFE — IF = EF, therefore arch LH = arch VA, and their chords are equal and parallel, but they are parallel to PE and EF respectively, and PE, EF are in directum and are equal, therefore if in the tangent of the Cycloid there be taken a portion equal to twice the chord, its extremity is in a curve equal and similar to that given; If a flexible thread were unwound off the semicycloid LFK, it would evidently trace the semicycloid LPB, for PF the straight part of it must be equal to the cycloidal arch LF, as it has been wound off it, but this is double the chord for P n : V s : : PF : VA : : 2 : 1 but P n and V s are the simultaneous increments of the cycloidal arch PB and the chord BN, these therefore are as their increments. (c)

Having proved these properties which we shall find useful hereafter, we can find the power required to support a body on a cycloidal surface; its inclination is the same as that of the tangent or chord, therefore P : W : : BX : BV, or as BV : BA by similar triangles, or as PB : 2 BA, the power is therefore as the distance from the vertex, the pressure on the cycloid being as PF.

22. If two bodies connected by a cord be supported on two inclined planes disposed as in fig 41, their weights must be inversely as the sines of the inclinations, or directly as the lengths of the planes, for the power required to sustain a body on AB = W × sine A is equal to the power on BC = w × sin C, but sin A : sin C : : BC : BA. The equilibrium is manifestly independent of the position of the bodies on the planes: this proposition may also be proved from the consideration that the common CG of the two weights must neither ascend nor descend. (*d*

23. We have hitherto supposed bodies to touch the planes on which they rested at a single point, but this is seldom the case, and the pressure on the plane must be divided among the points of contact; if the body touch only in two points as when a beam is sustained on two props, its under surface being the plane, the pressure on each of them is if they be equidistant from the vertical passing through the CG $\frac{1}{2}$ W × cos inclination of the beam, the remainder of the weight tending to make it slide on the props. (*e*) The magnitude of the surface of contact with the plane, makes a considerable change in the motion of the body: it slides, if the vertical through its CG, commonly called the Line of Direction, falls within its base, and otherwise it rolls. Thus a cube will roll if the inclination exceed 45° and a hexagon at 30° while a cylinder cannot be made to slide, but rolls with the least inclination.

24. If a beam or other body be laid on two inclined planes, in order that it may be in equilibrio the pres-

sures must compound a force perpendicular to the horizon and passing through the CG of the beam. That the first of these conditions may be fulfilled, the pressures must be reciprocally as the sines of the inclinations of the planes, and that the second may have place, those sines must be as the cosines of the angles which the beam makes with the planes. *(f)*

NOTES TO CHAP. V.

(*a*) Call the angle LBH θ, then as the body is acted on by three forces P, W and R the reaction of the plane, we have, making IH the axis of the abscissæ,— $W. \cos I + P \sin \theta + R = 0$, and $- W \sin I + P \cos \theta = 0$, hence $P = \frac{W \times \sin I}{\cos \theta}$ and $R = W \left(\cos I - \frac{\sin I \sin \theta}{\cos \theta}\right)$

$= W. \frac{\cos (I + \theta)}{\cos \theta}$.

If the weight be sustained by two forces, the equations are

$- W. \cos I + P \sin \theta + p \sin \vartheta + R = 0$, and $- W \sin I + P \cos \theta + p \cos \vartheta = 0$, hence

$$p = \frac{W \sin I - P \cos \theta}{\cos. \vartheta}$$

$$\text{and } R = \frac{W. \cos (I + \vartheta) - P \sin (\theta - \vartheta.)}{\cos \vartheta}$$

(*b*) This formula is easily extended to the other conic sections, let DE fig 42 be an Ellipse or Hyperbola, CD is parallel to the tangent, therefore $P = W \times \frac{DV}{DC}$

but $DV^2 = b^2 - y^2 = \frac{b^2 x^2}{a^2}$ and $DC^2 = \frac{a^4 \pm e^2 x^2}{a^2}$,

whence $P = \frac{W \times b\, x}{\sqrt{a^4 \pm e^2 x^2}}$ If e vanish as in the circle

it is $= \frac{W \times x}{a}$

And if we substitute for b^2, $\frac{a\,p}{2}$ and for x, $a - \xi$, and suppose a infinite, we obtain for the parabola.

$$P = W \times \frac{\sqrt{p}}{\sqrt{p + 4\,\xi}}$$

(*c*) The inclination of the tangent and length of the arch may be obtained from the general principle, that if the points of section coalesce, the secant coincides with the tangent, or that the curve is traced by two motions parallel to the axes of coordinates, see fig 42, where the motions may be considered uniform through the differentials $d\,x$ and $d\,y$, therefore the diagonal is the corresponding differential motion of the tracing point, and $d\,s = \sqrt{d\,x^2 + d\,y^2}$ and $\frac{d\,y}{d\,x} =$ tang A.

In the Cycloid calling y the ordinate PX, and BX fig 40, x, and z the arch VNB, $y = z +$ sin z, and $x = 1 -$ cos z, AB $= 2$, therefore $\frac{d\,x}{d\,y}$ the tangent of n PV $= \frac{d\,z \sin z}{d\,z\,(1 + \cos z)} = \frac{\sin z}{1 + \cos z} =$ tang $\frac{1}{2}\,z$, but the angle BVX $=$ A $= \frac{1}{2}\,z$, therefore n P is parallel to BV.

If we square and add the differentials, $ds = dz \sqrt{(1 + 2\cos z + \cos^2 z + \sin^2 z)} = dz \sqrt{2 \times \overline{1 + \cos z}} = 2\,dz \times \cos \tfrac{1}{2} z = 4\,d\,(\sin \tfrac{1}{2} z)$ but $2 \sin \tfrac{1}{2} z =$ BV, and $\int 4\,d\,(\sin \tfrac{1}{2} z) = 4 \sin \tfrac{1}{2} z = 2$ BV.

(*d*) Let G be the *CG* of the two weights, draw parallels to the planes, then $W : w :: x : WE : FW : y$, hence

$FW = \frac{W \times y}{w}$ and $WE = \frac{w\,x}{W}$ but these with y and x are s the length of the cord, therefore

$$s = \frac{\overline{W + w} \times y}{w} + \frac{\overline{W + w} \times x}{W}.$$

Hence y and x are the coordinates of a right line, and to find where it meets the planes put $y = 0$, and

$$x' = \frac{W\,s}{W + w} \quad \text{again put } x = 0 \text{ and}$$

$$y' = \frac{w\,s}{W \times w}$$ therefore $x' : y' :: W : w$, but that the line connecting them may be horizontal $x' : y' :: L : l$, the lengths therefore are as the weights: this principle is sometimes useful in determining equilibrium.

(*e*) A body is supported on an inclined plane by n points to determine which we have three equations

$$R + R' + R'' \text{ \&c.} = W \times \cos I,$$

and as the result must pass through the *CG*, their moment with respect to x and y must be null, let the parallel to the intersection of the plane with the horizon at the projection of the *CG* be the axis of x, and the perpendicular to it passing along the plane be that of y, then

$$R\,x + R'\,x' + R''\,x'' \text{ \&c.} = 0$$

and

$$R\,y + R'\,y' + R''\,y'' \text{ \&c.} = 0.$$

If the body be a beam resting on two props, the x s vanish and the x' s are equal and opposite, therefore the pressures are equal and each $= \tfrac{1}{2}\,W \times \cos I$.

(*f*) Let a body whose *CG* is G rest on two planes at A and B, then sin m: sin a :: DG : GA, and sin b : sin n :: BG : DG, or sin m, sin b: sin n sin a :: BG : AG, let $\frac{AG}{BG} = r$ and sin n sin $a = r$ sin m sin b, but $m + n = G - (a + b) = s$, therefore sin $a \times$ sin $(s - m) =$ sin a (sin s cos m — cos s sin m) $= r$ sin m sin b, and tang $m = \frac{\sin a \sin s}{r \sin b + \sin a \cos s}$ If AG, G B were in directum sin $s =$ sin $(a + b)$ and we s hould have tang $m = r$ cotang $a - (r + 1)$ cotang$(a + b)$, and if r were $= 1$, and $a + b = 90°$, tang $m =$ cotang a.

CHAPTER VI.

25. We have provcd that a force, applied to a cord by any means which permit lateral motion, must bisect the angle under the parts of it and produce a tension $= \frac{F}{2 \cos a}$ a being half their mutual inclination. If a cord passing through a ring attached to some fixed point be connected with a weight, the reaction of the ring may be considered as such a force, it therefore must be to the weight as 2 cos a : 1 and as the tension; at the extremities of the cord are equal, the power which supports a weight must be equal to it : in general the angle $a = 0$, and the pressure on the ring is twice the weight; a fixed ring therefore serves only to change the direction of a force without changing its energy. But if the ring were attached to the weight the case would be different;

as before the tension is uniform throughout the cord, one end of which is attached to a fixed point and the other held by the hand or by any other power, this tension is $\frac{W}{2 \cos a}$ and is equal to the power exerted in sustaining it, therefore P : W : : 1 : 2 cos a, or when the cords are vertical, as 1 : 2.

We have supposed that the cord is inflected by means of a ring, but friction and the rigidity of cordage would in that case interfere very materially with the results, and it is usual to bend them on the circumference of a wheel grooved at its circumference to receive them and turning on an axis or pin: it will subsequently be shewn that the impediments of the cord's motion are thus much diminished, and the introduction of the wheel does not change in any respect the theory; for fig 46 the cord TG being a tangent to the circle acts as if continued to I, and the result of its tensions passes through C so that in theory it is the same whether a weight be hung on at I or at C; The Wheel or Sheave is placed in a cavity mortised in a Block, which prevents the rope from slipping out of the groove, and affords a ready mode of attaching the machine to any point; the Sheave and Block are named a Pulley, and any assemblage of them is a Tackle.

What has been said of the fixed ring applies to the fixed Pulley, fig 47 ; for equilibrium P and W are equal, and this even when the parts of the cord are not parallel. Fig 48 represents a tackle consisting of a single moveable pulley B and a fixed one A to change the direction of the power, the end of the cord is attached to a fixed point C ; here the cords are parallel and cos $\frac{1}{2}$ $a = 1$ therefore W : tension of either part of the cord as 2 : 1, but the tension of BC is resisted by the fixed point C, and that of BA is communicated by A to P, if P then be $\frac{1}{2}$ W it will support it. Had the cords not been parallel the power must have been to the weight :: $1 : 2 \cos \frac{1}{2} a$. Fig 49 shews another containing a single moveable Pulley, here the cord instead of passing from C to a fixed point is directed by a second fixed pulley B back to C and is fastened to its block at E, the tension must be equal through the system, call it P; that of CB is employed to diminish the weight which is to supply the place of the fixed point C in fig 48, therefore the weight which remains to produce the tension is W — P, which must = 2 P if the cords are parallel, or $2 P \times \cos \frac{1}{2} a$ in any other case, 3 P is therefore in common = W.

26. These combinations are the elements of all other Tackles, fig 50 is one more complex; as before the tensions are all equal to P, two of these act against the weight at F, two at E and two at D, the sum of the whole being equal to W, therefore W = 6 P. If the cord instead of being attached to the fixed part of the system had passed over a fourth fixed pulley and been fastened at D, it will be easily understood that W = 7 P: in general, where there are n pullies there are $2n$ or $(2n + 1)$ tensions according as the termination of the cord is at the fixed or moveable part of the system, and the power is to the weight :: $1 : 2n$ or $2n + 1$. In these systems to raise the weight 1 foot, 1 foot must pass the pulley furthest from the power which in this figure is D, 2 feet must pass C, 3 E and in general the quantities which pass the pullies of any tackle are in arithmetical progression, the first term and common difference being the height to which the weight is raised, the coefficient of those belonging to that part to which the weight is fastened being the even numbers, and of the others the odd: if then the diameters of the sheaves be in a similar

arithmetical series, they will all make the same number of revolutions, thereby wearing equally and if they be ranged side by side diminishing friction. In this arrangement, however, the actual diameters of the sheaves must not be in this proportion, for as the power of a rope is applied at its axis, ½ its thickness must be added to the radius of the sheave to obtain the virtual diameter, in fact the first term of the series of diameters is d and the common difference $d + 2\,t$ the thickness of the rope; if the grooves fit the rope the diameters of the outer edges should be as 1 2 3, &c.

27. Where a tackle is required to exert a great power through a small space, the arrangements fig 51 or 52 may be used; in 51 half the weight is sustained by a, and half by the axis of B, half of this again is communicated to C and balanced by the power P; P therefore : W : : 1 : $2 \times 2 \times 2$ or 8; in general if the power be unity its energy at the axis of the n^{th} pulley is 2^n. The Power produces an energy at d 2 P and at $c =$ P, that at $b = 2$ P, and that at $a = 4$ P, the sum of these is 9 P, therefore if the tackle were inverted and $a\ b\ c\ d$ were a weight it would be more powerful, fig 52 is on this principle.

The tension P is communicated to W by the pulley A, B communicates 2 P, and C 4 P, and D 8 P; the energy acting against W is therefore $P \times (1 + 2 + 4 + 8)$ $= P \times 15$. In general $W = P \times \left(2^{n+1} - 1\right)$ n being the number of moveable pullies in the tackle, for this coefficient is the sum of a geometrical series whose first term is unity, its ratio 2 and the number of its terms $n + 1$. Analogous to this are the systems called Spanish Burtons shewn in fig 53 and 54, in the first of which the Power : to the Weight as 1 : 4, and in the second as 1 : 5.

In the ordinary tackles there is a loss of power from the obliquity of the ropes, which when the fixed and moveable blocks approach becomes considerable: Mr. Smeaton has proposed to diminish this by the following arrangement, the blocks contain two sets of sheaves of unequal size, the rope to which the power is applied enters in the middle of the larger set and proceeds to its extremity, it is then conducted through the whole of the smaller and at the other extremity returns to the larger and is finally attached in its middle, in this way the application of the power causes no obliquity in the blocks.

It has been proposed to unite the sheaves and let them turn on pivots or axles, proportioning them to turn in the same time, which would certainly diminish the friction; but in general $\frac{1}{3}$ of the power is thus lost. Very great mechanical advantage cannot therefore be gained by this machine, but from its portable nature and cheapness it is highly useful in ordinary cases, particularly in ships where its application is most extensive.

CHAPTER VII.

28. It is the defect of the Lever that the space through which it can raise a weight is very limited, but the Wheel-and-Axle is free from this defect. Let, fig 55, BE be the section of the axle projected on the plane of the wheel, the power if applied by a cord coil'd on the wheel acts perpendicularly, and its moment is P × AC,

and this must be equal to the moment of W = W × BC, the Power and weight are therefore inversely as the diameters of the cylinders at which they act.

If the direction of the power be not a tangent to the wheel its moment is = P′ × HC. The axis of the wheel and the axle, or the line drawn through C perpendicular to the plane of the wheel, must pass through the centre of gravity of the Power and weight, as also of the machine itself; if therefore we compound the weight, power, and weight of the machine, the one applied at the *CG*, and the others at the intersection of a line joining A and B with the axis, this result decomposed into two parallel to it will give the pressures on the pivots or Gudgeons of the machine. If the actions of the power and weight be not parallel, their directions must be produced till they meet, and the line drawn from their occurse to the axis is in the direction of their pressure. The power is applied to the circumference in various ways, but the weight is generally attached by a cord to the axle, which coils on it as the machine is turned; the thickness of the rope is therefore to be added to the diameter of the axle, and even more if the rope make two turns, a correction is also required for the obliquity of its turns, and may be made if t is inconsiderable in respect of d by diminishing D by that part of it whose coefficient is t_2 divided by twice the square of the circumference of the axle *(a)*. The Power is sometimes applied by pins projecting from the circumference as in the wheel used in steering ships, sometimes the exterior or interior are formed into steps up which men or other animals walk, and turn it by their weight.

In water mills the weight or impulse of water acts at the circumference, and in wind-mills the oblique action of the wind is diffused over several radii; on the whole the entire circumference is used but rarely. Most frequently one radius only is used carrying at its extremity a handle perpendicular to the plane which it describes in a revolution; it is then called a winch and as is evident supplies the place of the complete wheel with the inconvenience of producing a greater variation of the moving force, for a man exerts a very variable strength in different parts of the revolution, in two parts nearly opposite he exerts an action of 70 pounds, and at the intermediate points only of 25; if, however, two workmen are employed by disposing these winches at right angles to each other, the two can work for a day against 70. The Capstan and Windlass are other modifications of the wheel and axle, here the axle is pierced with holes into which levers are introduced, which are shifted from one to another as the machine is turned; in the capstan the axle is vertical and the levers are pushed horizontally, the other is horizontal and is wrought by the men throwing their weight on the bars; both are provided with catches which drop into notches and prevent the weight from receding on any intermission of the power. The wheel-and-axle is used to raise ore &c. out of mines; the rope makes a few turns round an axle of considerable diameter, and as one end with its basket is drawn up the shaft, an empty one is lowered by the other. Considerable mechanical advantage may be gained by making the axle of unequal diameters and attaching the rope so that as it coils on one it unwinds from the other; fig 56, neglecting the inclination of the cords each of them bears $\frac{1}{2}$ W, and the moments of their tension with respect to C are $\frac{1}{2}$ W $\times$ AC and $\frac{1}{2}$ W $\times$ DC, but as the cord is coiled in opposite directions on the two cylinders, the difference of the moments $\frac{W}{2}(AC - DC)$ is the energy of the weight to turn the axle, but this is the same as if it acted on a cylinder whose diameter was the difference of the radii.

If the diameters be twelve and ten inches moved by a winch of 18, the condition of equilibrium will be the same as in a wheel of 36 and an axle of 1, this would enable two men to raise about 3000 pounds which would require a rope three inches in circumference, and it would be impossible to wind it on an axle of an inch diameter, or to find materials sufficiently strong to make the axle of such small dimensions; but by this contrivance the axle is virtually diminished at pleasure, the necessary strength and diameter being retained.

29. Where the resistance is variable its energy may be made uniform by varying the diameter of the axle so that it may act by a less Lever as it is more powerful, and by this means the power is not overcome by the resistance when great, and when the latter diminishes it works with greater velocity; contrivances of this kind are used in cranes which vary the diameter of the axle at pleasure; the same principle has been used in an instrument to measure the intensity of the wind by windmill sails raising a weight on a conical axle which turn until the cord arrives at a place where the moment of the weight balances the impulse. Were the resistance a weight which increased or decreased uniformly in its ascent, it might be counterpoised by another whose cord coils on a spiral of such a nature that its radii are as the angles which they make with a given line, this is used in Gasometers (*b.*) But the most ingenious mode of equalizing a varying action is that used in the fusee of a watch; the mover is a spiral spring, coiled in a box to which its outer extremity is fastened, the other is secured to an axle on which the box turns, a chain is fastened to the box and coiled round it so that by pulling it the spring is coiled into smaller spires: as it is unwound from the box it is wound on the Fusee or axle of the first wheel of the watch,

and the force which it exerts to return to its first position moves the wheel work; now this force is nearly as the number of turns which the spring has made, or the length of chain unwound, and by making the fusee decrease as the quantity of chain wound on it increases the motive force is nearly constant, the outline of the fusee should be the Hyperbola. (*c*,)

NOTES TO CHAPTER VII.

(*a*.) The rope forms a helix on the axle making with the plane perpendicular to its axis an angle whose cosine : sine :: circumference of a cylinder whose diameter is $d + t : t$, hence cosine of it

$$= \frac{p\,(d+t)}{\sqrt{p^2\,(d+t)^2 + t^2}} = \frac{1}{\sqrt{1 + \frac{t^2}{p^2\,(d+t)^2}}}$$

now the power acts in the plane of rotation and must be reduced to the direction of the rope, or

$$W \times \frac{d+t}{\cos \text{ obliquity}} = PD = W \times \sqrt{1 + \frac{t^2}{p^2\,(d+t)^2}}$$

$\times\, d + t$, develope the radical and we have

$$PD = W\,(d+t) \times \left(1 + \tfrac{1}{2}\,\frac{t^2}{p^2\,(d+t)^2} - \&c.\right)$$ and

as t is small in comparison of d we may stop at the second turn. If d be 10 inches and t 1, the effect of this obliquity is about $\frac{1}{200}$ of W.

(*b*.) This spiral which is called by the name of Archimedes is generated by the motion of a point moving uniformly along a line which revolves round one extremity with a uniform angular velocity and can easily be traced by points; its application to counterpoise a resistance which varies uniformly is easily understood, and it is generally used, to change circular into uniform reciprocating motion. For the purpose mentioned in the text it is merely an approximation: and the proper spiral is the involute of the circle, continued through several turns. See note *(a)* of the next chapter, from which it can be easily understood that as the involute turns uniformly round its centre, the vertical tangent recedes uniformly from it: the diameter of the generating circle is known by comparing the weight of the rope with the load; this gives the ratio of the greatest distance to the radius of the shaft, and as it is diminished in each revolution by a circumference of the circle, the latter can be determined when the number of revolutions is known. A similar correction might be made by properly shaping the drum on which the rope is wound, as in the Fusee; but the curve is known by a differential equation, whose integral cannot be given in a finite shape.

(c) In fig 57, B is the box containing the spring, F the fusee; the chain when the spring is unbent is coiled regularly on the box, and as it is wound on the fusee it ascends up with a uniform motion; its distance therefore from the bottom of the box is as the number of turns made, or as the restitutive force of the spring: but the diameter of the fusee where the force is applied must be inversely as it, or as the distance from the base of the fusee, supposing it continued downwards so far that the spring would be inactive when the chain is wound off it; its shape is therefore a portion of the solid described by the revolution of an Hyperbola round its asymptote.

CHAP. VIII.

30. In the wheel and axle the weight is raised by a cord fastened to the axle, but this is not the sole application of the machine, by far the most extensive use of it is on a different principle. If the circumference of the axle revolved in contact with that of a second wheel, so connected that they could not slide, on turning the first wheel the second must turn with it: in fig. 58 AB is the radius of the first wheel, CB of its axle, CD of the second wheel, ED of the second axle; let the machine be turned by any power applied at A, its moment to turn the wheel is $P \times AB$, and the energy which it exerts at $C = P \times \frac{AB}{BC}$, but as the axle is connected with the second wheel, any force impressed on the one is communicated to the other; therefore $P \times \frac{AB}{BC} \times \frac{CD}{DE}$ is the energy exerted at E, and if there be any number of wheels and axles, it may be shewn in the same way that in equilibrium $P \times D \times D' \times D''$, &c. $= W \times d \times d' \times d''$, &c.

The velocity of the point A : that of C :: AB : BC, and velocity of C to that of E : : CD : DE, therefore vel. A : vel. E :: $AB \times CD : BC \times DE$, but this is the inverse ratio of the power and weight; therefore in wheel-work, as in other machinery, what is gained in energy is lost in time. The Angular Velocities of any two working together are inversely as their diameters, and they turn in opposite directions. The connection which we have supposed to exist between the circumferences in contact, may be produced in various ways; if the touching surfaces be rough, as if they are faced with leather or wood cut across the grain, the action of friction is found sufficient when the resistance is not considerable. Wheels are often driven by a band, strap, or chain, as in the Turning Lathe, in which motion is communicated to a large heavy wheel which turns one much smaller by a band with a speed of from 1 to 10 revolutions in a second or even more; the axle of this latter is provided with means of holding the substance to be worked, and as it revolves, by the application of cutting instruments it is fashioned into any shape whose cross section is a circle. Machinery is often driven by straps descending from horizontal shafts running along the buildings of great manufactories and turned by water wheels or steam engines, the wheels round which they pass are loose on their axles, with which they can be connected at pleasure, enabling each workman to stop or move the machine which he superintends, independent of the rest.

31. But the most usual form of wheel-work is where the circumferences are indented into teeth locking into each other, and it ensures their action unless the strain be so great as to fracture the teeth; but minute attention is required to their figure, that they may communicate the motion uniformly and steadily. Supposing that the teeth have plane surfaces as shewn in fig 59, when the tooth D comes into contact with C it acts on it in a direction oblique to the radius, and as it moves, on the angle of C slides on the plane surface of D producing much friction and grinding itself away, till they come to the position marked A and B, when the surfaces separate again and the point of B slides on the face of A for the rest of the time of their contact; when they pass the line of the centres the point of application of the force which an instant before was at *b*, suddenly changes to *a*, producing a jolt in the action. Without going more mi-

nutely into the investigation, it may suffice to mention that the teeth should be curved so as to roll on each other, as much as possible without sliding, and to bear on each other perpendicularly. T and S in the same figure represent teeth possessing these properties, formed by evolving a thread off a circle equal to the diameter of the wheel, and as MN the perpendicular passing through their contact is a tangent to both wheels, therefore the force is communicated in the direction perpendicular to the radius of each wheel, and at a distance from the centre equal to it. (*a.*)

32. In the estimation of the power of machinery acting by teeth, it is not usual to introduce the diameters of the wheels and axles, and their place is supplied by the number of teeth; if t be the thickness of each tooth and n their number, nt is the periphery of the wheel, but as wheels which work together must have their teeth of the same size, n is as D, and we may say that $\frac{W}{P} = \frac{N.N'}{n.n'}$ &c. or that the power is to the weight as the product of the numbers of the pinions : product of the numbers of the wheels; a pinion being an axle cut into teeth. Where it is desired to communicate a given angular motion with the least possible number of teeth, the numbers of the pinions should be about $\frac{1}{4}$ the numbers of the wheels; where the weight and friction are required a minimum they should be about $\frac{1}{3}$ (*b*) : 1 to 5 is probably a good proportion. The numbers of a wheel and pinion should be prime to each other, as in this case every tooth of one comes into contact with every one of the other, and the wear is distributed uniformly, and as the pinion revolves more frequently than the wheel its teeth should be a little stronger. The wheel is not always in the same plane with its pinion, when they are at right angles it is named a Crown

wheel, when in any intermediate direction, the wheel-work is named Bevil Gear. To comprehend its action we must suppose two right cones, of equal length, and the diameters of their bases inversely as the angular velocities required, applied so that their vertices may coincide, and that they may touch in a side of the cones, if then one be turned on its axis, it will drive the other; fig 60, at any distance from the vertex draw parallel planes to the bases, their intersections with the conical surfaces are circles, and the velocities of their circumferences are as the velocities of the bases, therefore the angular velocity of each surface is the same throughout, and any frusta equidistant from the vertices will supply the place of entire cones, the teeth of these wheels are cut not parallel as in the common kind, but converging to the vertex of the cone; the angles of the cones may easily be found, for the motion is communicated at the angle BAC, and this is to be divided so that sin BAD: sin DAC, inversely as the angular velocities of the wheels. *(c.)*

38. As examples of wheel-work we may mention the Crane and the Watch, the one augmenting the power, the other increasing the velocity. The first of these machines is essentially a combination of wheels and pinions fixed in a frame which can be turned to any part of the horizon, with a projecting arm over which a rope or chain passes from the last axle; it is furnished with a Ratchet Wheel and Brake to sustain the weight at any height or regulate its descent, the latter is an arch pressed by a Lever on the circumference of a wheel and retarding its motion by friction, the other is a wheel whose teeth are inclined in one direction, so when it turns in the opposite a catch is raised by each tooth and passes over it, but drops into the intervals and pre-

vents any return, see fig 61. The proportion of the wheel-work is easily determined by the power required; for example, if it be required that two men shall be able to raise ten tons, this may be done by various combinations so that the problem is indeterminate except as far as it is limited by other considerations: it has been shewn that $\frac{W}{P} = \frac{R}{r} \times \frac{N}{n} \times \frac{N'}{n'}$ &c. R and r being the radii of the winch by which the machine is turned and the axle on which the rope is wound, and as two men can work against 70 pounds $\frac{W}{P}$, $= 320$. $\frac{R}{r}$ is in practice $\frac{18}{6}$ therefore $106 = \frac{N}{n} \times \frac{N'}{n'} \times$ &c. the first member is to be resolved into factors which are the fractions $\frac{N}{n}$ &c.but these should be about 5 or 6, and if we take the three factors 6, 6 and 4 we shall obtain a machine which shall have the power required, and an overplus of $\frac{1}{4}$ for friction, it will therefore consist of 3 pinions of 6 leaves and three wheels of 36, 86 and 24 teeth.

The Watch is moved by a spring connected with a fusee as already described; the fusee is on the axle of a wheel which drives a pinion whose axle carries the minnte hand, its wheel drives a second pinion and wheel which moves the last of the series called the Crown Wheel. In general the numbers of the wheel-work are $\frac{50}{10}$, $\frac{60}{8}$, $\frac{64}{8}$, therefore while the fusee wheel makes one turn, the crown wheel makes 300; but by a contrivance which shall be afterwards described this is constrained to revolve once in a minute, an index carried by it points seconds on a circle divided into 60 parts, the axle of the second wheel drives by a pinion of 6 a wheel of 72 which moves another turning on the first axle with the same speed carrying the hour index which makes $\frac{1}{12}$ of a revolution while the minute hand turns once ; the spring is wound up by a ratchet on the fusee which permits motion in one direction but not in the other.

34. The combinations of Wheel-work are endless, there is scarcely a motion even of the hand which cannot be performed by it. The study of this part of Machinery is highly interesting, but our limits do not permit us to go further into the subject, we will therefore conclude with describing some contrivances appertaining to this species of machines which are very useful. Where an engine is driven by a power acting constantly in the same direction, it is often necessary to reverse the motion; this may be effected by a shaft having two pinions at such a distance that they cannot both act on the teeth of the crown wheel A, fig 62, the shaft moves in its Bushes so that one or other may be brought to act, and as the opposite points of the circumference move in opposite directions the wheel must turn different ways as the motion is applied at B or C.

35. It is often required in machines to convert a rotatory motion into a reciprocating one, and vice versâ; this is most easily done by a crank or axle twice bent at right angles, fig 63, as it makes a revolution, the rod B which is connected with it at A so that it can turn freely, is raised and lowered through twice AD. If on the other hand a reciprocating force act by B, it will make the crank revolve if the machinery connected with it have inertia sufficient to continue the motion past the points above and below the centre of the circle described by it, as the force pulls directly to and from that point in those two places and therefore does not produce rotation. The resistance acts very variably against a crank, sup-

posing it to act in vertical lines the energy which turns it is P × AC, while the moment of the resistance is W × BA, its energy is therefore as the sine of the angular distance from the vertical, and is nothing at E and F, increasing to a maximum at G and H. This irregularity of action is often beneficial; where a great mass of matter is put in motion and suddenly urged in a contrary direction, the shock on the machinery is prodigious, but the crank changes the motion gradually and acts with a mechanical advantage which at the commencement of the action as in the Lever of the Stanhope Press is almost infinite; as an example we may refer to machines which raise water from a great depth with pumps and are wrought by cranks. In the descending stroke the engine acquires a considerable velocity, till the crank passes the vertical and is opposed by the weight of a column of water many hundred feet in length, if this acted by the whole radius of the crank, it would stop the engine or tear it to pieces, but at that instant the moment of the power exceeds its moment beyond all calculation, and it is slowly put in motion; as it acquires velocity the energy of the power decreases, and its velocity augments until it has attained its maximum. Where a reciprocating power acts, this is a very defective mode of producing rotation, as much force is expended in pressing on the axle of the crank. *(d)*

The change of reciprocating into rotatory motion is also effected by an invention of Mr. Watt, commonly called the Sun and Planet Wheel, fig 65; the rod which conveys the power is fixed to a toothed wheel of the radius AB so that it cannot turn, the centre of this is connected with the centre of the wheel to be driven by the brace AC, and on moving the rod the wheel BC revolves; every point of the area of BD describes an equal circle, and therefore moves with the velocity of A, therefore the periphery of BE, which is connected with that of BD by its teeth, must also have the same velocity as A, therefore the angular velocities of a point in BE and of A must be inversely as their radii or as AE : BE. In general the wheels are equal and the axle makes two revolutions for one reciprocation.

Circular motion may be changed into rectilineal uniform motion by Rack-work; a straight bar is cut on one of its surfaces into teeth which work in those of a wheel, and as this is turned on its axis the rack moves through a space equal to its circumference; by this means the adjustments of Philosophical instruments are in general moved, though its motion is by no means so smooth as that of the screw; it is also used in the machine called a Jack to raise great weights, being urged by a powerful train of wheel-work.

If a circle roll on the concave circumference of another twice its diameter, a point in its periphery describes a right line; fig 66, the arches PE PA are equal, for the parts of one have been successively applied in the rolling to those of the other, therefore since angles on equal arches are inversely as the radii, ADE $= \frac{1}{2}$ PCE $=$ PDE, the point P is therefore always in the line DA. Hence if the circles act by teeth and are kept in contact, a rod attached at P will reciprocate through AR, if a circular motion be applied at D.

In one important machine a rectilineal motion is used to move the extremity of a Lever, which as it describes its arch, approaches and recedes from the vertical through its fulcrum, this lateral motion is obviated by a system of Levers known by the name of Parallel Motion; fig 67, as the Lever AC turns round C a perpendicular from A continually approaches C, but in the same way a perpendicular from F recedes from C and the motion of F is communicated by the jointed frame AEBE, all whose angles move on centres, to E; this point therefore is at once moved by two circular motions curved opposite ways, and it is possible by a proper proportion of the radii, to make the curvatures compensate each other through a certain space. In reality the path of L is not a right line but a curve with a point of contrary flexure.

In some operations pestles or stampers are raised by pins or Wipers projecting from a revolving shaft which elevate them to a certain height, and passing them allow them to descend on the materials subjected to their action: they are often straight pins, but the objections which apply to straight teeth of wheels are equally powerful here and are obviated by forming the wipers into curves of such a nature that they may act uniformly; one easy method is shewn in fig 68, where the curve is a spiral of Archimedes and the arm **A** is raised through equal spaces with equal angles of rotation. If the wiper be straight the arm on which it acts must be curved into a cycloidal arch, and if the body to be raised move not in a vertical line but in a circle, it is still possible to produce uniform motion by a uniform power. (*e.*)

NOTES TO CHAP. VIII.

(a.) There are two modes of shaping the teeth of wheels, in one they are formed into Epicycloids or curves formed by a point in the circumference of one circle

while it rolls on another, in the other they are Involutes of circles; the latter of these seems the best as the curves are more easily described and it admits of more teeth acting at once. It is unnecessary to enter minutely into the details of this investigation, we will shew that an equable motion can be procured by either of them, referring those desirous of further information to Brewster's Notes on Ferguson, or Hachette Traitè des Machines.

In fig 69, EPI is part of an epicycloid whose base is the circle EAB and generating circle IVB; the arch EA = arch PA = arch VB, but EAB = BVI therefore AB = VI, but AB or VI : OX :: AC : OC; or VI : PV :: radius of base to distance of P from its centre.

The Epicycloid is described by a motion compounded of two, one round the centre V the other round C, which are in the above proportion of AC to PC, for they are represented by the differential increments of BV and PV; their directions make, with the tangent of the curve, angles whose sines are in their inverse proportion, but their directions are perpendicular to NP, PC, if therefore we divide the angle NPC so that the sines of its parts may be inversely as the motions, we obtain a line perpendicular to the tangent; draw PA, sin NPA or PAN : sin APC :: PC : PN : motion in arch PA : motion in PO, therefore PA is perpendicular to the tangent.

Let B and C be the centres of two wheels, fig 70, the teeth of the one are formed by planes in the direction of its radii, and those of the other are Epicycloids generated by a circle $\frac{1}{2}$ the diameter of the wheel B, the motion will be communicated uniformly from C to B; for let DB be the face of a tooth, it touches the

Epicycloid at D, therefore DA is perpendicular to DB, or the line in which motion is communicated from the tooth ED to the tooth DB passes through A the contact of the two wheels, and their angular velocities must be inversely as CH : DB, or CA : AB. This is perhaps more distinctly seen by considering EB as an interior Epicycloid (we have already shewn that when the generating circle is half the base, the Epicycloid is a right line); for as the teeth begin to act at the line of the centres, FA, EA are described by their wheels in the same time and are equal, but the angles standing on equal arches are inversely as the radii. The right line is not the only interior Epicycloid which can be used, but this is obviously more simple than where both sets of teeth are to be adjusted to a curve. It is evident that the tooth ED cannot act on the other before they come to the line CB; if this be required the teeth of both wheels must be in part plane and in part epicycloidal as in fig 71. Before they arrive at CB, the plane surfaces of the teeth of C press on the curved parts of those belonging to B, afterwards the curved parts of the former act on the planes of the latter.

The Involute of a circle is the limit of the Epicycloid, being described by a right line rolling on a circle, and therefore if the teeth be made according to it they will act uniformly; fig 72, AC and DB are parts of the effective circumferences of the wheels, suppose the thread AB wound off A and on B, it will always be perpendicular to the involutes EC and ED, and a tangent to both circles; but the teeth communicate motion in the perpendicular to their surfaces, therefore force is communicated in the tangent to the two circles, and their peripheries being connected by the thread or by the action of the involutes which acts similarly, move with equal velocity, their angular velocities are inversely as their radii.

The teeth of Racks may be constructed on the same principles, for as a right line is a circle of infinite radius, if the teeth of the wheel be epicycloids, its teeth must be Cycloidal; if involutes, right lines: fig 73, EC and CD are generated by the same circle, AC is perpendicular to the epicycloid and as they have a common tangent, to the cycloid, but as they must always be in contact the arches of the generating circles intercepted between the contact and the points A and H must be equal, but these are as the motions of the wheel and rack. In fig 74, BE is an involute, therefore always a tangent to CB the tooth of the rack, and BA the distance from the line of the centre = the arch evolved, therefore the motions of the wheel and rack are equal.

(*b*.) (1.) If there be two numbers so related that $a^x = y$, x is the Logarithm of y, Logarithms measure the ratio of their numbers for

$$\frac{y}{u} = \frac{a^{ly}}{a^{lu}} = a^{ly-lu}.$$

The logarithm of any power of y is its logarithm × exponent of the power.

The differential of a logarithm is as the differential of the number divided by the number, for

$$a^{x+dx} - a^{x} = y + dy - y, \text{ and } a^{dx} = 1 + \frac{dy}{y}$$

let $a = 1 + b$, and raise this by the binomial theorem to the power dx and

$$1 + \frac{dy}{y} = 1 + b \times dx + \frac{b^2}{2} \times dx.\,dx - 1 + \frac{b^3}{2.3} \times$$

$dx.dx - 1.\ dx - 2$ &c. but all the powers of dx but the first vanish, and the equation becomes

$$\frac{dy}{y} = dx \times \left(b - \frac{b^2}{2} + \frac{b^3}{3} - \text{\&c.}\right)$$

or if the sum of the series be called $\frac{1}{m}$

$$\frac{dy}{y} = \frac{dx}{m}.$$

There are many kinds of logarithms depending on the magnitude of a which is called the base, for on this depends the quantity m called the Modulus of the system. If $m = 1$, the logarithms are called Napier's from their discoverer or Hyperbolic, and the integral of $\frac{dy}{y} = hl.y$ If these logarithms be multiplied by the modulus of a system they give its logarithms for

$$hl.y = hl.a \times x$$

but $hl.a = \frac{1}{m}$ and $x = hl.y \times m$. These propositions relative to logarithms are indispensable, and they are introduced here to save the trouble of a reference to other works.

(2.) If in a machine the number of teeth in the wheels be to that in their pinions as $y : 1$, if there be x wheels the angular velocity at the end of the train is as y^x, for it is as

$$\frac{N}{n} \times \frac{N'}{n} \times \text{\&c.}$$

but these factors are all $= y$ and the product is that power of y whose index is x; hence log. $y \times x =$ log a calling a the ratio of the angular velocities of the first and last axles: if it be required that the number of teeth should be a minimum, this is as $(y + 1)\,x$, or substituting for x, as $(y + 1)\ \frac{la}{ly}$.

When any function is at a maximum or minimum, its differential $= 0$, in this instance

$$\frac{dy \times ly - (y + 1)\ d.\,ly}{l^2 y} = 0$$

or $ly = \frac{y + 1}{y}, = 1 + \frac{1}{y}$ and y is 3.59 nearly.

But there are reasons for doubting whether this is the best ratio of the numbers, it might be preferable to use that which would give the least number of teeth and of wheels or $(y + 1) x^2$ a minimum which gives $ly = 2 + \frac{2}{y}$ and y is 9.2 nearly. If the weight of wheel-work and number of wheels be a minimum $l.y = 1 + \frac{1}{y^2}$ and $y =$ 3.02.

(*c*) The semiangles of the cones may be determined by the common formulas of trigonometry, or may be derived from a very simple construction; for take, fig 75, the angle A = to that required for the communication of the motion, and AB to AC as the angular velocities bisect BC, and the sines of the angles are inversely as the adjacent sides. For the proper shape of the teeth of Bevelled wheels, see Hachette.

(*d*) Though a Crank is a useful mode of changing rotatory motion into reciprocating, yet it does not perform the converse of the problem so well, for much of a reciprocating power is lost in pressing on the axis; it has been shewn that the energy of any force to turn the crank is as the sine of the angle which its direction makes with the radius, and its total effect in a revolution must be as sum of all the sines or the integral of the product sine of angle $z \times dz$. In the simplest case where the force acts in parallel lines, E the effect $= \int \sin z.dz = -\cos z$, but this integral must be taken from $z = 0$ to $z = 180°$ and is therefore $= 2$, but if the same force had acted perpendicularly at the extremity of the radius, its action would have been $1 \times p = 3.14$, the effect is therefore to the force applied in half a revolution :: 2 : 3.14; more than $\frac{1}{3}$ is therefore lost.

Let us now suppose the force to act in the line AB and be connected with the crank by a rod CB $= l$, AC $= 1$ and angle B $= u$, then the force acting at C $\propto$ $\cos u \times \sin C = u + z$, or as $\sin z. \cos^2 u + \sin u \cos n$ $\cos z$, but $\sin u = \frac{\sin z}{l}$ and $\cos u = \frac{\sqrt{l^2 - \sin^2 z}}{l}$

hence

$$d\,E = \frac{dz}{l^2}\left\{\sin z(l^2 - \sin^2 z) + \cos.\, z \sin z \times \sqrt{l^2 \sin_2 z)}\right\}$$

or putting for l^2, $\lambda^2 + 1$,

$$dE = \frac{1}{l^2}\left\{d.\cos z \times \left(\lambda^2 + \cos^2 z\right) + \cos z \times d.\cos z \times \sqrt{\lambda^2 + \cos^2 z}\right\}$$

or integrating

$$E = -\frac{1}{3l^2}\left\{3\,\lambda^2 \cos z + \cos^3 z + \left(\lambda^2 + \cos^2\right)^{\frac{3}{2}}\right\}$$

when $z = 180°$ this is $\frac{1}{3l^2}\left(3\,\lambda + 1\right) - \left(\lambda^2 + 1\right)^{\frac{3}{2}}$

and when $z = o$ it is

$$\frac{1}{3l^2}\left\{-3\,\lambda^2 - 1 - \left(\lambda^2 + 1\right)^{\frac{3}{2}}\right\}$$

and the difference of these is the complete integral

$$= \frac{1}{3l^2}\left(6\,\lambda^2 + 2\right) = \frac{6l^2 - 4}{3\,l^2}$$

If l were 10 times AC, $E = 1.9$ nearly, and the loss would be greater if it were shorter.

(*e*) It is evident that the spiral described in the text will raise the stamper with a uniform motion since the increment of the angle $\propto$ increment of radius, and the same thing may be effected by attaching a curved arm to the stamper which is acted on by a radius CA. in fig 76, while the radius revolves from A to B through the arch dz, the arm is raised through DB, $= dy + dz \times \cos z$, but the height encreases uniformly or as the angle $C = z$ and we may suppose $dh = dz$, hence $dy + dz \cos z = dz$, and integrating $y = z - \sin\ z$ but when $z = 0$ let $y = 180°$ therefore the integral from 0 to 180 is $y = 180° - z + \sin z$ or $z' + \sin z'$ which is the equation of the cycloid whose generating circle has a radius $= 1 = AC$.

By a similar process the figure of a Cam or Heart wheel to depress or raise a lever with a uniform motion in an arch of a circle may be investigated, and we obtain the differentials of the radius of the required spiral, and the angle which it makes with a given line in functions of the inclination of the lever to the horizon which on integration give their contemporaneous values; as the results are complicated we should not give them, but in their place assign a method of tracing the proper curve by points. Let it be proposed to raise the lever AD through the arch A while the shaft BC revolves through AB; if the lever is required to move with a given angular velocity, divide AZ into any number of equal parts, divide AB into the same number, take from AH an arch e H = that subtending the angle ECA and describes an arch with the radius EC, its intersection with the radius through e is in the spiral. In the same way the angle IC f = ACF, and CP = CF, as also angle KCg = ACG, and NC to CG. In this way any required number of points may be found, and connecting them we obtain, if their number be considerable, a close approximation to the proper curve. If it were required to raise the lever with a motion according to any other law, it would be necessary to vary the divisions of the arch according to that law, but the rest of the process would be the same.

CHAP. IX.

37. If a right angled triangle whose base = the circumference of a cylinder be wrapped on its surface so that its altitude is parallel to the axis, the hypothenuse becomes a line named a Helix; and if other similar triangles

be applied in the same way, the curve may be continued at pleasure. Let another Helix be traced between the spires of the first and parallel to it, and let the materials of the cylinder be cut away in the space above the first and below the second, we obtain a convex screw, which consists of a Helical thread or ridge running on a cylinder. In general the generating Helices bisect the distance between each others spires, and therefore the projecting threads are of equal thickness with the hollows between them. The Helices may be traced on the surface of a Concave Cylinder, and we obtain a Concave Screw. If several Helices be traced between the spires of the first, the result is a many threaded screw.

From the equation of a right line, it may be traced by two uniform motions at right angles, and the helix may be traced by carrying a point uniformly along a cylinder while it revolves; we are thus enabled to cut screws with rapidity, the cylinder is made to revolve in the Lathe, and a tool applied whose edge is composed of a number of teeth, either squares or equilateral triangles; this is moved steadily from right to left with such a motion that in one revolution it moves through the thickness of a tooth, and it cuts a single screw; if it be moved with double speed, a double screw.

38. From the Genesis of the screw it is obvious that every part of the surface of the thread is equally inclined to the base of the cylinder, it may therefore be reduced to the inclined plane, fig 78. Let AB be a small portion of it, considering this as an inclined plane, the force parallel to AC which will support a weight laid on it is $= W \times$ tang A, but tang A is the quotient of the perpendicular by the base of the right angled triangle which traced the helix, and its base is the circumference of the cylinder, its perpendicular the interval between

two spires or threads; the power therefore applied at the surface of the cylinder : to the weight which it can sustain on the thread :: I the interval : *pd* the circumference. The weight is made to rest on the thread by a concave screw exactly fitting the convex, for as the threads of the latter are lodged in the hollow thread of the former, they bear any pressure which may be applied to it, and if the convex screw be turned on its axis while the concave or nut as it is commonly termed, is prevented from revolving, this will rise through the interval of two threads in every revolution of the screw. This instrument is commonly turned by a lever or wheel, to which if any power P be applied its moment at the surface of the cylinder $= \frac{P \times D}{d}$, but this must equal the energy of the weight in producing motion round the axis $= \frac{W \times I}{pd}$, and $P \times p\,D = W \times I$ or

P : W :: I : circumference described by the power.

The screw is sometimes used to raise a weight through a small space though other machines perform this more effectually, but its chief mechanical use is to produce pressure, which it performs effectually as the friction of the screw and nut keeps it in its place and maintains the pressure when produced; it is therefore the agent in most Presses. The thread should be square in large screws, as this form is stronger than the equilateral under less surface and of course friction, and the frame of the press should be of cast iron as otherwise it will yield and diminish the pressure; but if the materials pressed be yielding so that they contract under it, then the elasticity of the frame is useful, or the screw may be made to continue to press through the medium of a strong spring.

39. The screw may be used to measure minute distances, for its point advances through I while the radius by which it is turned describes 360°. Instruments for this purpose are named Micrometers, their principal part is a fine and accurate screw which carries a frame, across which a fine wire is stretched at right angles to the axis of a telescope or microscope and in the plane of its principal image; by turning the screw the wire moves parallel to itself and its distance from a parallel fixed wire is the magnitude of the portion of the image included between them, and from this by the principles of Optics we draw the magnitude of the object. This distance is indicated by a circular head on the screw which is graduated; thus if the interval be $\frac{1}{100}$ of an inch, and the circumference of the head 3 inches divided into 60 parts, for every one of these which passes an index, the wire moves through $\frac{1}{6000}$ of an inch. Smeaton asserts that he had used a screw which agreed with itself to $\frac{1}{60000}$.

40. The power of the screw in producing pressure depends on the fineness of its thread, and this is limited by the strength of materials as if too fine it would be broken from the cylinder, and on the other hand Micrometer screws cannot be made beyond a certain interval; the screw of Mr. Hunter is therefore worthy of attention. Its convex screw is hollow and contains a second screw of a finer thread which can rise or descend but is prevented from turning, and it is evident that in a revolution the second screw advances through the difference of the intervals, the instrument being equivalent to a screw of that fineness of thread.

41. The Endless Screw is supposed to be the engine with which the Geometer of Syracuse launched by his single strength one of Hiero's gallies when challenged by that sovereign to give a specimen of the use of Me-

chanics. It is a screw whose nut is supplied by a wheel cut with teeth equal to the threads and with the same obliquity; every revolution of the screw moves on the wheel one tooth, and $P \times \frac{pD}{I}$ = energy of the power at the circumference of the wheel, and this × diameter of wheel (= V) = W × d the diameter of the axle, hence $P \times D \times \frac{pV}{I} = W \times d$, but I × number of teeth $= p$ V, and $P \times D \times n = Wd$, or P : W :: diameter of axle : number of teeth × twice the radius of the winch which turns the screw.

CHAPTER X.

42. The Wedge is sufficiently simple in its operation, but there is some difficulty about its theory, for the power by which it is urged is commonly an impulse, while the resistance to be overcome is a pressure whose quantity cannot easily be appreciated, and whose direction is not certainly known. If the pressure to be overcome act perpendicularly on the sides of the wedge, and the power be a pressure perpendicular to the back, the power must be to the resistance as the back of the wedge to its side, for it is kept in equilibrio by the pressures on its three sides, they therefore must be as the sides of a triangle parallel to their directions, or as the sides of a triangle perpendicular to them, that is, as the sides of the Wedge; this is in general isosceles, therefore the pressures on AC and BC are equal, and that on AB : that on AC :: AB : AC. If the pressure on AC act in lines parallel to AB, then AC may be considered as

an inclined plane whose base is CD to which the direction of the power is parallel, but it has been demonstrated, that a power parallel to the base : W the pressure which it can counteract : : AD : DC; an equal power is required to counteract the pressure on CB, the power is therefore to the resistance as AB : CD. The direction of the pressure is in practice included between these limits, and therefore the ratio of the power and weight is intermediate between these ratios.

The Wedge is almost always urged by percussion, and this makes it extremely powerful, insomuch that some paradoxes have been maintained as to the incomparable nature of percussion and pressure ; this is not the proper place for entering into the subject, and at present we will observe that a blow can be compared to a pressure acting for the instant of time during which the motion of the striking body is destroyed ; this must be many times its weight and much exceeds the resistance opposed to it during that instant.

43. The Wedge is sometimes employed to raise weights which no other machine could move, by its means large vessels of war weighing at least 3000 tons are lifted from their supports by the strength of a few men driving them with a Battering-ram. It is the agent in the press of the Oil-mill ; the crushed seeds of certain plants are introduced into hair bags and placed in a row separated by partitions of hard wood, wedges are inserted between them and driven by heavy beams raised by Wipers on a revolving shaft ; the oil is thus so completely expressed that the cake which remains is almost of the consistance of wood. But the chief use of the wedge is in penetrating and dividing bodies, all cutting instruments being varieties of it, and its power is greater the sharper its angle, with this restriction that if it be too acute the edge may be de-

ficient in strength and, instead of dividing the body on which it acts, may itself be crushed. The edge of tools for wood is formed by planes at an angle of about 30°, that of iron is about 50° or 60°, and that for brass about 80° or 90°; and all tools which act by pressure are sharper than those driven by a blow. They are made of steel, which has the property of becoming hard by immersion in cold water when ignited, but the hardness thus acquired must be reduced by tempering or exposure to a certain degree of heat different according to the use for which the tool is intended. The wedge is also used as well as the screw, to unite the parts of machines; pins, bolts and nails being wedges retained in their place by the action of friction.

CHAPTER XI.

44. We have hitherto supposed the parts of machinery capable of relative motion without any loss of power, but this is true only in theory; if the surfaces in contact were perfectly smooth and had no adhesion they might slide on each other with perfect freedom, but even after the most careful polishing they remain studded with asperities which must lock into each other and be broken bent or compressed in the motion; any of these requires force to effect it, and the expenditure of power in overcoming this resistance, is considered as counteracting a force opposing the motion, which is denominated Friction. The action of this modifies extremely the theoretical conclusions which have been deduced concerning the

mechanical powers, insomuch that machines often require many times the calculated power to perform their work: it is therefore of the utmost importance to be able to make the requisite allowance for its action, and diminish its influence as much as possible; with this view we shall state the leading facts known about it.

Friction may for the most part be considered a uniform force independent of the velocity; when the rubbing surfaces are rough it increases with the rapidity of motion, but by a continuance of the motion they become smooth and this irregularity disappears. Where the surfaces are given in magnitude and nature, friction is as the pressure, being to it in a ratio which is constant for the same body, thus the forces required to draw equal weights on a horizontal plane are with dry wood from 3 to 4 tenths of the weight, with stone one half, and with metals about one fifth. These ratios are nearly independant of the magnitude of the surfaces; strictly speaking, the friction is the sum of two forces, one varying as the pressure alone, the other proportional to the surface and independant of pressure, which being constant, is less sensible when the latter is increased than in experiments on a small scale, and unless the surfaces are soft is never considerable. When the surfaces remain in contact for some time, the force required to begin motion is much greater than that which is sufficient to maintain it, sometimes 4 times as much; this increase of friction is caused by their adhesion, and the augmentation of it is much greater when they are highly polished, (this enabling them to come into closer contact) particularly in metals where if oil, which prevents this effect, be not used, they adhere to such a degree that they are *cut up* and destroyed. Oil, grease or plumbago should be supplied in every case, as they reduce the friction to about $\frac{1}{10}$ of the pressure.

45. The simplest case of friction is when a load is drawn along a horizontal plane; and here it is to be observed that since the friction is as the pressure, less force will be required for the draught if it act obliquely so as to diminish the pressure. Let the force act at the angle A, fig 80, the part of it which draws the body along the plane is $= \frac{F \times AC}{AB} =$ F. cos A, and that which acts perpendicularly to the plane and diminishes the pressure is F $\times$ sin A, and denoting by f the ratio of friction to pressure, $f \times$ F sin **A** is the diminution of friction, which is the same as an addition to the moving force, this therefore $=$ F $\times$ (cos A $+ f$. sin A), and supposing it just equal to the friction, F is inversely as cos A $+ f$ sin A, and when this latter is a maximum, the force required to draw a given weight is a minimum. To determine the value of A, fig 81, on BC describe a segment of a circle capable of containing the angle whose cotangent is f, draw from B a diameter, and B is the angle required, for letting fall the perpendicular CE, BE is the cosine of B, and ED $=$ EC $\times$ cotang D $=$ sin B $\times f$, BD is therefore $=$ cos B $+ f$ sin B, and it is a maximum, for the diameter is the greatest chord; the angle B is the complement of D and therefore its tangent $= f$. If the road be wood this angle is 18 degrees, if sand stone 40°, if iron 11°.

On the inclined plane friction assists the power in sustaining the weight, but opposes it if the weight is to be raised; its effect is found by multiplying the pressure on the plane into f, but it is seldom that bodies slide on planes, as if of a form approaching to the cylinder they move by rolling, which changes the nature of the friction. (a.)

46. The friction of the teeth of wheel-work is not considerable if they be properly shaped, though with either epicycloidal teeth or those which are involutes some always remains; that of the gudgeons on which they turn is more considerable, it varies as the pressure, and as the diameter of the gudgeon, for it opposes motion acting at the arm of a lever equal to its radius. The *bushes* in which the axle turns should be of a different material, as friction is always less when the surfaces are heterogeneous; iron turning in bell metal is commonly used, which gives $f = \frac{1}{12}$, but where the strain is slight, iron axles turning in Lignum vitæ may be employed as these give $f = \frac{1}{20}$.

Where the wheel can be placed horizontally it may turn on pivots, which are conical points bearing on a surface hollowed to receive them; this permits us to diminish the diameter of the bearing surface, and concentrates the friction at $\frac{2}{3}$ of its base from the axis: the semiangle of a pivot bearing a considerable strain should not be less than 45; some mechanics have made their pivots rest on oil, but the stuffing required to prevent its escape round the axle is perhaps equivalent to the bearing surface in producing friction. (b.)

47. The friction of the screw is very great, being about $\frac{1}{3}$ of the power; it may be investigated by supposing the thread to be an inclined plane, on which a body is sustained by a force parallel to its base. (c.)

48. If a cylinder roll along a horizontal plane, supposing both perfectly hard there would be no resistance to its motion, for the line of contact of its surface with the plane does not slide on it but is lifted off it by the rotation, but in practice it always compresses the plane,

and by sinking in the hollow occasions a resistance analogous to friction; if the road be tolerably firm this is very trifling, and heavy weights are often removed in this manner, which however has the inconvenience of requiring frequent alteration of the position of the rollers on which the load rests. The resistance to the motion of a wooden roller on a wooden road is according to Coulomb $\frac{1}{28} \times \frac{P}{D}$ in pounds, P being in pounds and D in inches; were they iron it would be about $\frac{1}{14}$ of this.

49. The inconvenience of shifting the rollers is obviated by making the load rest on axles passing through their centres, for as it is drawn along the rollers revolve, and the friction is transferred from their circumferences to those of their axles; but it must be much less sensible than if a load were made to slide, for its resistance to the force of draught is the same as would act against a power applied at the circumference of the roller to turn it, and therefore is equal to its movement there, or $= f\,W \times \frac{d}{D}$, or it is lessened in the ratio of the diameter of the roller to that of the axle, even supposing the load to slide on a road of polished iron. The friction of the axle diminishes very little the power of the roller, and this latter is accordingly but sseldom used in its simple state, being much less convenient than the ordinary form of wheels. These are narrow cylinders of considerable diameter, formed of a circular rim connected with a Nave by the spokes; this contains a box which turns on its axle. The axle is made slightly conical that it may not wear loose, and as if the axis of the cone were horizontal the obliquity of the pressure would force the box along it, it is bent downward so that the plane touching its lower surface is horizontal; hence the bearing surface of the wheel must also be conical, but this deviation from the cylin-

ders hould be as minute as possible, for such wheels have a tendency to move in circles, which requires additional power to draw them in right lines, and augments the friction of the axles and the destruction of the roads.

In order that the wheels may wear equally they should be equidistant from the centre of gravity, and as the resistance is increased by their sinking in the road their diameter should be as large as possible; for we have seen that this retardation is inversely as the diameter. The large wheel has also an advantage in passing over any obstacle; let O, fig 81, be an irregularity in the road, CB the line of draught, making with the horizon an angle of 14° degrees, which is found by experiment to be the most advantageous for the exertion of a horse; the wheel is a lever whose fulcrum is at O, and CB and CA are the directions of the weight and power; these are to each other inversely as the perpendiculars on their directions, but it is obvious that if the height of O be given, the ratio of EO to OD, or of the power to the weight is greater the less the radius, for $\frac{EA}{CA}$ is the versine of AO, and therefore sin $\frac{1}{2}$ AO $\propto \sqrt{CA}$ inversely, therefore it is increased by diminishing AC; but AO is less than a quadrant, therefore its sine increases with the sine of its half, therefore ACO increases and OCB decreases, and therefore sin OCB decreases or increases less than sin ACO. *(d.)*

The weight which can be drawn on a road by a given power depends on its hardness and smoothness, and these requisites are obtained in the highest degree by Rail-roads of Iron on which the wheels run. On one of these a good horse can draw as much as 15 tons without injury on a level.

50. The application of wheels to carriages suggest the means of diminishing the friction of axles, for if the bush in which one of them turns be larger than it, so as to admit a number of rollers arranged round it, the motion will be changed from sliding to rolling; this method is, however, much inferior to the use of friction wheels, fig 83, the Gudgeon D rests on the circumferences of the wheels CE and BF, and as it revolves turns them, the friction of their axles acts in retarding D with an energy $= W f \times \frac{CH}{CA}$, while if they were fixed it would be $W f$, and as CH may be much less than CA we may reduce the friction in almost any degree, limited only by the increase of friction which their weight produces. *(c.)*

Half of the load is borne by each axle and is equal to $\frac{W}{2 \sin C}$ and therefore where the radii of the wheels and axles are given, the friction is inversely as sin C, this angle should be as great as possible or the distance CB as little, it is therefore probable that if the gudgeon rest on a single friction wheel and be kept in its place by pivots, the friction will be more diminished than by the use of two.

51. Friction, besides decreasing the work to be done by a machine, is injurious by destroying the surfaces which move in contact; this effect takes place with extreme rapidity unless they be lubricated with some fluid or unctuous matter, for they adhere and are torn away; if the velocity be considerable much heat is evolved, and this still further increases the friction; the pivots and axles expand, and by their pressure still further augment the friction, till they sometimes become actually ignited. In some great works a small pump supplies a stream of

water to the bearings which are exposed to great pressure, and there is often contrived in the box or bush of an axle, a reservoir of oil which obviates these dangers. These and all other means of lessening friction should be employed, and it must be kept in mind that the great beauty of a machine consists in its simplicity, as the less complicated it is the nearer its effects approach to theory.

NOTES TO CHAP. IX.

(a.) Retaining the notation of note (*a*) Chap. 5. the pressure on the plane is $= W \cos I - P \sin \theta$, this produces a friction $= f\,(W \cos I - P \sin \theta)$ and the forces parallel to the plane, which must be equal in equilibrium, are $W \times \sin I = P \cos \theta \pm f\, W \cos I \mp f\, P \sin \theta$; or $W (\sin I \mp f \cos I) = P (\cos \theta \mp f \sin \theta)$, and

$$P = \frac{W (\sin I \mp f \cos I}{\cos \theta \mp f \sin \theta}$$

the upper sign being taken where the weight is barely sustained, the lower when it is on the point of being raised. If $\theta = - I$ or the power act parallel to the base

$$P = W \frac{\text{tang } I \mp f}{1 \pm f \text{ tang } I}$$

If the power be parallel to the plane

$$P = W (\sin I \mp f \cos I,$$

and if the friction be sufficient to support the body on the plane, $\sin I = f \cos I$, or $\text{tang } I = f$, its inclinanation being the angle which was found for the best direction of draught.

(*b.*) If the power and weight in the lever be not inversely as the perpendiculars on their lines of action, their result cannot pass through the fulcrum, fig 84, let ED be its direction; it is equivalent to the two forces CF and FE, the first of which presses on the fulcrum, and the second tends to make the lever revolve round C; but the force DF produces friction on the gudgeon or axle C, and if the angle FDE have its tangent $= f$, this friction which $= f \times$ DF $=$ FE being equal and opposite to the force which tends to move the lever, it will remain at rest. The result of P and W is obviously a given quantity, for it is the base of a triangle of which they are the sides and whose vertical angle is the supplement of D, therefore $=$ to $\sqrt{(P^2 + W_2 + 2\,PW \times \cos D)}$ and this multiplied by sin FDE gives the friction which opposes the power with the moment $F \times r$ the radius of the axle, and therefore the effect of it is as was stated in the text as the diameter. If $P = \pi + p$, π being a force which would equilibrate with W if there were no friction, pa must equal the moment of the friction, but $\pi = \frac{Wb}{a}$ therefore $\sqrt{P^2 + W^2 + 2\,PW \cos D} =$

$$\sqrt{b^2 + a^2 + 2\,ab \cos D) \times \frac{W^2}{a^2} + 2\frac{pW}{a} \times (b + a \cos D) + p^2}$$

but as r is very small in comparison of a or b, p is small in comparison of W, and as sine FDE $= \frac{f}{\sqrt{1+f^2}}$,

$$\frac{W}{a} \times \sqrt{a^2 + b^2 + 2\,ab \cos D} \times \frac{f\,r}{\sqrt{1+f^2}} = pa;$$

and adding to both sides of the equation $\frac{Wb}{a}$,

$$P = W \times \left\{ \frac{b}{a} + \left(\frac{r}{a^2} \times \frac{f}{\sqrt{1+f^2}} \times \sqrt{b^2 + a^2 + 2\,ab \cos o} \right\}\right.$$

If the directions of the forces be parallel $D = o$ and

$$P = W \times \left\{ \frac{b}{a} + \left(\frac{r}{a^2} \times (a + b) \right) \times \frac{f}{\sqrt{1+f^2}} \right\}$$

(c.) The force required to raise a weight by a screw is the same as that required to raise it on an inclined plane of the same inclination as the thread when acting parallel to its base: therefore

$$P \times \frac{D}{d} = W \times \frac{\text{tang I} + f}{1 - f \times \text{tang I}}$$

but $\text{tang I} = \frac{\text{Interval of thread}}{pd}$ and therefore

$$P \times D = Wd \times \left(\frac{I + pdf}{pd - f\,I} \right).$$

(d.) In fig 82, the angle ACO $= \varphi$ and ACB $= \tau + 90°$, therefore $P \times (\cos \tau . \cos \varphi + \sin \tau . \sin \varphi)$ $= W \sin \varphi$ and $P \times (\sin \tau + \frac{\cos \tau}{\text{tang } \varphi}) = W$; but $1 + \text{tang}^2 \varphi = \frac{1}{\cos^2 \varphi} = \frac{1}{(1 - \frac{h}{r})^2}$ h being the height of the obstacle and r the radius of the wheel, and developing, $\text{tang}^2 \varphi = \frac{2h}{r} + \frac{3h^2}{r^2}$ but in practice we may stop at the second term and $P (\sin \tau + \frac{\cos \tau \times \sqrt{r}}{\sqrt{2h}}) = W$: but it is obvious that P is less as its coefficient is greater, and this increases with r. This expression shews also that while r is below a certain magnitude an advantage is obtained by obliquity of draught, for let τ increase, its sine increases and its cosine decreases, and the coefficient is a maximum when $\text{tang } \tau = \sqrt{2h} = \text{tang } \varphi$. Large wheels are also useful in rapid motion because they cause the CG of the load to describe a path of less curvature in surmounting an obstacle, but there is a limit to their use, for as has been stated a horse draws best at an angle of about 14 degrees with the horizon, and therefore on an average the height of the axle cannot exceed four feet.

(e.) Let W be the load which a friction wheel bears, and let its weight be as the square of its diameter $= sD^2$, this must be added to the load, and the friction at its axle is $f\,(W + sD^2)$, but the diameter of its axle $\delta = d \times \frac{(W + sD^2)^{\frac{1}{3}}}{W^{\frac{1}{3}}}$, d being the diameter of that which would barely support the load, and the strength of cylinders being as the cubes of their diameters; the moment of the friction is therefore as $\frac{(W + s\,D^2)^{\frac{4}{3}}}{D}$, and this is a minimum when sD^2 the weight of this friction wheel $= \frac{3}{5}$ of the pressure produced by the load on the axle of it.

CHAPTER XII.

52. The strength of cordage is a subject which is by no means to be omitted in a treatise of Mechanics, for in almost every instance it is the instrument by which the action of a machine is applied to raise heavy bodies. They are composed of the fibres which constitute the rind of certain vegetables; these are obtained separately from the other parts, by exposing the plants to putrefaction, which acts less rapidly on them than on the ligneous matter; this becomes brittle, and is removed from the fibres, which still retain much of their original strength, by percussion with a proper instrument. This process, which is very hazardous to the strength of the material, is not necessary except for the finer quality of work; as where the matter is sufficiently resisting it may be prepared without previous maceration.

A number of these fibres united form a cord, but as their length is not great it is necessary to be able to connect many lengths of them, and this is effected by the same means which makes them act with united strength. This is extremely simple, a number of them are twisted together, and when the spinner comes near the end of any parcel he twists in among them the beginning of a new one; thus every fibre acts, for the twisting presses them together, and produces a friction which makes every one partake of the strain applied to the cord, and prevents them from slipping out of their place. A cord may thus be extended to any length; in this state it has no permanence, as if its extremitles were released it would untwist by its elasticity, and return to the state of unwrought fibres. It is however far stronger than any other kind of cord, and is sometimes, where great strength is required, in a small bulk, made permanent by smearing its surfacc with glue or tar.

Yarns are united by a process, extremely simple and effective. If two yarns twisted in the same direction be connected at their extremities, each of them tends to untwist, therefore the parts of their peripheries in contact would move in opposite directions; those points therefore will be a centre round which the rest of the yarns will be carried by the remaining forces, and the two must be twisted together: see fig 85, where the two circles are sections of the yarns; the point of contact must remain at rest, the point A is urged in the direction AB, and C in the direction CD. The cord thus formed will not lose its twist, and each fibre is less twisted than in the original yarn. Many yarns thus united are called a Strand; three or four strands, twisted so as to untwist their fibres, will unite in the same way, and form a Hawser or Shroud; and these, by similar treatment, afford a Cable.

All these ropes derive their strength from their twist, which must be such that the friction on each fibre is equal to its strength, and therefore they are shorter than the yarns of which they are composed, in the proportion of 2 : 3 : but the torsion of the fibres, though necessary, weakens them extremely, it is obvious that a fibre may be twisted till it breaks, and though it be not carried so far, yet the strain on it must employ part of its strength; and it also is made to act obliquely against the load. Hawsers are half as strong as the yarns of which they are made; cables $\frac{2}{3}$, being ultimately less twisted; shrouds are nearly as strong, but cannot be bent without injury. Ropes are lately invented, in which the lengths of the strands are proportioned to their place in it, and these are $\frac{7}{8}$ of the original strength. In round numbers a good Hawser will bear $\frac{1}{5}$ as many tons as there are inches in the square of its circumference.

Chains are sometimes used instead of cords; they are more flexible and stronger, but their weight is much greater; they are about 10 times stronger, and 7 heavier than ropes of the same section as their links.

53. We have already mentioned what allowance is to be made for the thickness of cordage; it remains to indicate the effect of its inflexibility when bent round a pulley or axle: the theory which we shall give is collected from experiments by induction, and its application in any particular instance should be regulated by actual trial. According to the observations of its inventor, when a cord not very rigid is drawn over a fixed pulley to raise a weight, that part of it which is on the side of the power is nearly vertical, while the other forms a curve, so that a vertical drawn through the weight passes at a

distance from the centre greater than radius; the weight therefore acts by a longer lever than the power, and the difference is the effect of the rigidity. Now this must be inversely as the radius of the pulley, or directly as the curvature given to it, and as some determinate though unknown power of the thickness of the cord; it may be represented by $\frac{ct^x}{r}$, but it consists of two parts, one the force required to bend the cord without any load on it, the other arising from its tension, the expression therefore assumes the form of $t^x \times (y + u\mathrm{W})$, therefore

$$\mathrm{P} = \mathrm{W} + \frac{t^x}{r} \times (y + u\mathrm{W}.)$$

This equation contains three unknown quantities, which are thus determined: Take any specimen of cord, pass it over a pulley whose friction on its axle has been previously ascertained, and annex to its ends equal weights; add weights to one of these till it begins to preponderate; this addition is the amount of the friction: repeat the experiment with different weights and pullies, take any 3 of these results and equal them to the quantity $\frac{t^x}{r} \times (y + u\mathrm{W})$ putting for r and W their values in that experiment. The three equations thus obtained determine the three quantities, and the other results serve to verify the formula. This method of constructing a formula from observations is often used, and gives results of high practical utility in cases where analysis is utterly at fault. It is thus shewn that within certain limits the value of x is nearly 1.5, or that the stiffness varies in the sesquiplicate ratio of the thickness, (*u.*) The quantity $\frac{t^x}{r}(y)$ is in cords of 1, 1 $\frac{2}{3}$, and 2 $\frac{1}{3}$ inches circumference, respectively 0.2, 1.2 and 4.2, r being 2 inches and $\frac{t^x}{r} \times u$ W = 2.2, 5, and 9, W being 100 in pounds.

54. The friction of ropes is used in lowering weights; a cord is coiled once or twice round a cylinder, and very little force is required to suitain a weight. If f be $\frac{1}{10}$ for rope on metal, any power will sustain 2.7 times itself if the rope make $1\frac{1}{2}$ turn, and if the number of turns increase in arithmetical, the weight which can be sustained increases in geometrical progression. (*b.*)

NOTES TO CHAP. XII.

(*a.*) Since ropes bear weights as the squares of their diameters, and the rigidity is as the diameter raised to the power $\frac{3}{2}$, the axles on which ropes are wound should be as $\frac{3}{4}$th power of the greatest weight to be raised.

(*b.*) In the circumference while an arch z increases by dz, the pressure on the cylinder increases by the pressure on dz; this by art 25. is the tension of the rope t, multiplied into twice the cosine of half the angle made by the tangents drawn through the extremities of dz, or into twice the sine of $\frac{dz.}{2r}$ By the series for the sine of an arch,

$$\sin \frac{dz}{2r} = \frac{dz}{2r} - \frac{dz^3}{8.23.r^3} \text{ \&c.}$$

and stopping at the first term of it, we have calling p the pressure

$$dp = t \times \frac{dz}{r}$$

The tension is equal to the power applied to sustain the weight + the friction of the rope on z, the portion of the circumference between dz and the beginning of its contact, or

$$t = P + f\,p \text{ and } dt = fdp = \frac{ftdz,}{r}$$

and $\frac{dt}{t} = \frac{fdz}{r}$. The integral of the first member is $h.\log(t)$, and therefore $h.l\,(t) - C = \frac{fz}{r}$ but when $z = 0$, $t = P$, hence $h.l\,(t) = \frac{fz}{r}, + hl.\,(P)$, and passing from Logarithms to numbers,

$$t = P \times e^{\frac{fz}{r}}$$

but $t = W$, and if the rope make n turns $\frac{z}{r} = 2np$ and therefore

$$W = P \times e^{fn \times 2p}$$

CHAPTER XIII.

55. The strength of the materials used in Mechanical combinations is exerted in five different ways; they are exposed to a direct pull, to compression, transverse strain or flexure, torsion, and percussion. The strength which resists extension is the result of a quality of matter named cohesion, to which we have referred as in conjunction with a force of repulsion, causing the appearances which are usually considered results of solidity. To conceive properly the action of these forces, we must keep in mind that each particle of a body is at a distance from the rest; if they be forced to approach within this limit, repulsion is exerted; if they be withdrawn beyond it, cohesion opposes the extension; both act according to the same law, being as the extension or compression while they are small, and if the body be homogeneous, probably in every case;

but they differ in this, that there is a certain limit beyond which cohesion does not act, and if it be exceeded a total separation takes place; while there seems to be no limit to the compressibility of a body.

Cohesive strength is proportional to the surface of fracture in bodies of uniform texture, and is measured by the weight required to tear them asunder: in those substances, however, which possess ductility, the surface of fracture is not the true surface whose cohesion is overcome by the weight; these stretch considerably on its first application, and their diameter is gradually contracted until they yield; the force therefore required to tear them suddenly, is much greater than that which they can bear for a length of time, even iron which suspends 27 tons for every square inch of its section, cannot be trusted in any structure with more than 15. Hard steel which cannot be thus stretched is far stronger, bearing nearly 80. Experiments on wood are much more irregular than those on metals, from the irregularity of its fibres; there is also a considerable variety in specimens from the different parts of the same tree. Oak bears about four tons per inch, and fir about two and a half. Where a rod used to suspend a body is of considerable length, its weight must be added to the load, and therefore its diameter should be greater above than below; in theory its outline should be a logarithmic curve. (*a*.)

56. The strength of bodies to resist compression is far more difficult of investigation, and the greatest geometers have made mistakes in their analysis of its action; if a force be applied to a rectangular beam in the direction of and at its axis, it will compress it, diminishing its length by a quantity proportional to the compressing force: if the force be increased, the beam, if

of friable materials, splits off in two pieces, leaving a wedge whose angle depends on the ratio between the pressure and that mode of cohesion which presents the particles of the pillar from sliding off on each side, called by some the lateral adhesion; in cast iron the angle is nearly 90°. If it be like wood of a fibrous texture, its fibres bulge out in the middle, being forced asunder by the repulsion excited in them, and thus give warning of their weakness , if they be hooped so as to prevent this, their strength is much increased. While exposed to a longitudinal stress, the slightest transverse strain is sufficient to break a pillar, which would have carried its load with perfect safety. There is no relation between the cohesive and repulsive forces; fir, which suspends little more than half as much as oak, will carry twice as much; and it is said that cast iron resists compression with a force six times greater than its cohesion.

If the compressing force be not applied exactly in the axis, it will bend the column; for the repulsive forces acting against it on each side of a line drawn through its point of application must be equal, but the number of particles between it and the surface nearest to it, is less than that of the rest of the section; and if they were equally compressed throughout, their action could not be equal to that of the others; they must therefore be more compressed, which augments their repulsion, and compensates for inferiority of number; the column therefore must bend, as one of its surfaces becomes shorter than the other. If the point of application be removed still further from the axis, the particles between it and the near surface are still more compressed, until when its distance is $\frac{1}{6}$ of the depth of the column (supposed rectangular); the remote surface is in its natural state, if it exceed $\frac{1}{6}$ the state of compression ceases before we arrive at the remote surface, and there is a longitudinal section of the column neither compressed nor extended; this is called the *Neutral* section or line. The portion of the column between this and the remote surface is in a state of extension, being lengthened by the general flexure *(b)*. It is obvious that a similar flexure would be produced by a force applied obliquely to the axis, or by a transverse strain even when it acted directly, and that the strength is much increased if it be prevented from bending by lateral braces applied at its middle.

57. The transverse strength is derived from the cohesion, but their relation is not easily ascertained. If the particles resisted equally, so that they all gave way at once, the *Transverse strength* of a rectangular beam would be half its *Absolute* strength, for let AC be its section, fig 86, if this be torn asunder by a strain, the two parts turn on the line DC as a fulcrum, and every particle resists the separation by its cohesion multiplied into its perpendicular distance from DC; but the sum of the particles multiplied into their distance from any line is the moment of the area with respect to that line = area AC × GE, this in a rectangle is $\frac{1}{2}$ AD, therefore calling f the cohesion of a square inch, the transverse strength is $\frac{1}{2}$ AD × AB × f, or the absolute strength acting by $\frac{1}{2}$ leverage.

This hypothesis concerning the manner in which a transverse strain is resisted, is due to Galileo, and it may easily be applied to every body; according to it the cohesion of the section acts in the same manner as if it were concentrated in the CG, the fulcrum of the lever by which a force is supposed to act against it is the line touching its lower surface, and the arm by which the force acts is the distance of its application from the place of fracture: calling this l, W the weight which produces fracture : the absolute strength :: d' : l, d' being the distance of the CG from the line of fracture. According to this a triangular bar is twice as strong when its base is uppermost as when its vertex. The principle on which Galileo's theorem rests is quite gratuitous, for the particles do not yield at once, but are exposed to very different degrees of strain; those at the top are previous to the fracture separated to the limit of their cohesion, and resisting with their utmost energy, while those nearer the fulcrum are not acting with their whole force; at length the uppermost stratum yields, and that below it must withstand the same strain with less leverage, it also is torn, and the whole beam is broken without the entire of its strength being called into action; Galileo's hypothesis therefore overrates the transverse strength.

58. Mariotte's hypothesis is more near the truth; according to it the fulcrum is in the lower surface, and the resistance of each particle is as its extension; this latter supposition is strictly true, and its results are more conformable to facts. To examine its application in the case of the rectangular beam, conceive parallel lines drawn as near as possible, the area of one of these elementary parallelograms is AB × Bf, the cohesion of the particles in it is as their extension, or as their distance from the centre of motion = FE, and this cohesion is also acting by the leverage FE, the energy of the parallelogram AB × Ff in resisting fracture is AB × FE² × Ff, but the sum of all the FE² × Ff is (see art 10,) a pyramid whose base is FE² and altitude FE, but this is $\frac{FE^3}{3}$ which when FE is the entire depth $= \frac{d^3}{3}$. The cohesion of a particle at AB is exerted to the utmost,

call it f. and, $\frac{f}{d} \times FE$ is the cohesion of the parallelogram $b \times Ff$, therefore the total transverse strength is $f \times \frac{bd^2}{3}$, or the tranverse strength : absolute strength :: $\frac{d}{3} : l$. For some other applications of this method see the note (*c.*)

59. Even Mariotte's hypothesis overrates the transverse strength, for the fulcrum is not in the lower surface, but much nearer the upper, and when beams are broken a great part of their substance contributes nothing to the strength. If a beam of willow be sawed across, and the cut filled by a wedge of hard wood, it will bear a strain which would have broken it without this preparation ; here it is obvious that only $\frac{1}{3}$ of the wood was acting, and that the rest injured its strength. In fact the upper surface and the parts adjoining it are in a state of extension, but below them is a *neutral line*, and all the rest is in a state of compression. Suppose the total effect of the repulsive forces collected in a point, that will be the fulcrum of the lever of fracture, and the extremity of its short arm is at the centre of effort of the cohesion ; this arm is therefore less than in Mariotte's hypothesis, but we know too little of the interior mechanism of bodies to assign its magnitude.

If the body be perfectly elastic, that is if it resist compression and extension equally, we can determine the strength of the beam. Let AB, fig 87, be the plane of fracture, since the material is equally compressible and extensible, the same number of particles will be compressed and extended, the neutral line is therefore at the middle of the depth ; take AC as the extension of a particle at the surface which is as its cohesion, that of any other particle is as its distance from NE, the sum of all the cohesions is therefore as the triangle AHC, and their centre of effort is at the same distance from NE as its CG, or its distance is $\frac{2}{3}$ AH. In the same way it may be shewn that the repulsive forces are as the triangle IHB, and their centre of efforts at the same distance below NE, the distance between these centres $= \frac{2}{3}$d the depth of the beam is the leverage by which the cohesion acts; but were the beam torn asunder by a direct pull, its cohesion would be represented by AC $\times$ AB which is four times the triangle ACH, but the direct cohesion is fbd, therefore $\frac{1}{4}f \times bd \times \frac{2}{3}d$ or $\frac{1}{6}fbd^2$ is the energy which opposes fracture, hence the transverse : absolute strength :: $\frac{d}{6} : l$. (d.)

60. These three hypotheses differ in the quantity of the strength which they assign, but all give the result that the strength of a beam is as its breadth $\times$ the square of its depth and inversely as its length, and we can thus draw some important practical conclusions.

The strength of the same beam according as its different sides are uppermost, is as the vertical sides, and therefore where strength is required, the beams should be narrow and deep.

Similar beams are strong as the square of their length, if we do not consider their weight; but if we take it into account, their power to bear their own weight is inversely as their length, for their weight is as l^3, and their strength $\propto \frac{bd^2}{Wl} \propto \frac{1}{l}$. Hence it is evident that the proper dimensions of a machine cannot be deduced from trials on a small scale, without an increase of bulk more than proportional; a beam of oak 1 foot square would break by its own weight if it projected beyond its support 70 feet.

If it be required to cut out of a horizontal plank, a beam of uniform strength to bear a weight at its extremity, bd^2 must be as l. and b as l, its outline will therefore be an Isosceles triangle (*e.*)

61. If a beam be supported at its extremities, it is twice as strong as a beam of half the length fixed at one end and bearing a weight at the other, call the distances of the weight from the extremities D and d, then its power to break the beam is the same as that of a weight equal to the pressure on one of the props acting at the distance of it from the point of application: but the pressure on one of the props $= \frac{W \times d}{D + d}$, multiply this by D its leverage, and we obtain the stress $= \frac{W \times D\,d}{L}$. The stress produced by the weight at its point of application is therefore as the rectangle under its distances from the props, and a maximum when it is applied in the middle: in this case it is $\frac{W \times L}{4}$, but the stress on a beam of $\frac{1}{2}$ L fixed at one extremity, the point of application being the other, is $\frac{WL}{2}$.

If a beam be not merely supported, but secured at its extremities so that they cannot rise, as if they are built in a wall, the strength of it is doubled; for if it were cut through in the middle, each of its halves would bear half the weight, and the stress on their extremities would be $\frac{1}{2}$ W $\times$ $\frac{1}{4}$ L equal to the stress on the beam in the preceding case; if then it bear a given stress when loosely supported, and an equal stress when cut across, if its cohesion be added to this latter strength, it will bear twice as much. It is seldom, however, that we can secure the extremities sufficiently to derive the full increase which might be obtained.

The stress on any point of a beam loosely supported, produced by a weight applied at another, is as the rectangle under the distance of the point from one prop, and the weight from the other; for, fig 88, supposing the beam fixed in a wall at W, and the part WC projecting from it, the stress produced by the weight at W is the same as that occasioned by a weight equal to the pressure at C applied there, or it is $W \times \frac{AWC}{AC}$, but the stress at W : stress at X :: WC : CX, therefore stress at $X = W \times \frac{AW \times CX}{AC}$. Hence if the weight be applied in the middle, the stress which it produces at any point is as its distance from the prop, and a beam of uniform depth should be a Rhombus, to be equally strong throughout when thus loaded. *(f)*

62. A square beam of given length resists tranverse strain in proportion to the cube of its diameter, while the quantity of materials in it is as the square of the diameter. If therefore the central part were cut out, and it were thus made a square pipe, its weight would be more diminished than its strength; for example, if its external diameter were double its internal, its strength would be $\frac{7}{8}$ of a solid beam of the diameter, and its weight $\frac{6}{8}$. By this means we can build hollow masts and beams much stronger than the timber of which they are composed, for instance, a beam 8 inches square may be sawn into planks 1 inch thick, they may be formed into two tubes of 10 by 8, whose united strength is to that of the original beam as 832 : 512. This contrivance is widely used in the structure of organized bodies, where strength is not more necessary than lightness: the

quills of birds, which are astonishingly stiff in proportion to their weight, the bones of animals, and the stems of grasses, and canes, are familiar examples of the wisdom of Him who looked on creation, and behold it was very good ; to develope part of the skill with which He formed the meanest of his creatures, overtasks the mightiest intellect; but that which we can fathom is enough to reveal to any reflecting mind, the unspeakable glory of its Author. (*g.*)

63. Stiffness or the power of resisting flexure is analogous to transverse strength, but follows different laws, it is inversely as the flexure produced by a given force. If the beam be fixed at one end, and bear a weight at the other, the stiffness is as $\frac{bd^3}{l^3}$. (*h.*)

64. In Torsion as in other strains the cohesion appears proportional to the space through which the particles are separated. In bodies which are homogeneous, it is as the cube of the diameter supposing them cylinders, as in the case of axles, &c. The force thus exerted, as measured by the weight which when applied at the circumference is able to twist it asunder, is about $\frac{1}{3}$ the absolute strength, and it is evident that in this case, as well as the two last, a tube is much stronger than a solid cylinder of equal weight and length. When a wire is twisted, it returns to its original position with a force proportional to the angle through which it has been twisted, this principle is used in the balance of torsion already described ; in different wires, this force is as the 4th power of the diameter, and inversely as the length. (*i.*)

NOTES TO CHAP. XIII.

(*a.*) A weight is attached at B, fig 89, to a rod formed by the revolution of the curve DC round its axis, and it is required that it should be equally strong throughout; the section whose radius is y, bears along with the weight W, the weight of that portion of the rod intercepted between y and BC $= a$; while x increases by dx, the area of the section which is as the cohesive strength must increase, as the portion of the rod between y and a increases. The area of the section is py^2, and its cohesion fpy^2, its differential is $2\,fpy\,dy =$ differential of the rod's weight $= s \times fpy^2\,dx$, s being the weight of a cubic unit of the material of which it is composed, hence $\frac{2f}{s} \times \frac{dy}{y} = dx$, and integrating $\frac{2f}{s} \times h.l\,y = x$, but when $x = 0$, $y = a$, the complete integral is therefore $\frac{2f}{s} \times hl.\left(\frac{y}{a}\right) = x$ and passing from logarithms to numbers $y = A^x$ and the curve is a logarithmic whose subtangent is $\frac{2f.}{s}$

(*b.*) AB, fig 90, is the summit of a rectangular pillar whose depth AC is $=$ D, NL is the neutral line, a force is applied at F which compresses the rectangle NB, let EB $= u$, and ac the compression produced at CB, then drawing two parallels to CB indefinitely near $b.dx$ is the magnitude of the differential rectangle, and $bdx = \frac{2cx}{u}$ its repulsive force, for this is as the space through which it has been compressed, or as x; to find the effect of all these forces we must take their moment with respect to NE, and divide it by their sum, which will give the distance of the centre of their effort from the neutral line. The moment $= \int \frac{b \times 2\, cx^2 dx}{u} = \frac{2}{3} \frac{bcx^3}{u}$, and the sum is $c \times bx$, for it is as the repulsive force of NEBC, supposing the compression equal throughout, but the compression decreases uniformly from BC to NE, and in a mean quantity it is $\frac{1}{2}\, 2c$; the distance of the centre of repulsive force is therefore $\frac{2}{3} \frac{cbx^3}{u \times cbx}$ and when $x = u$, it is $\frac{2}{3} u$: but the centre of repulsion must be at the point of application of the force which produced it. If NE be in a surface of the pillar, calling $\frac{1}{2}$ the depth a, $u = 2a$, and the distance of F from NE $= \frac{8}{6} a$ and deducting a its distance from the axis is $\frac{a}{3}$. In this demonstration we suppose the part AE to have no action. If the neutral lines fell outside of AD, we should integrate from $x = u - 2a$ to $x = u$, and we would find that the distance of F from the axis must be less than $\frac{a}{6}$. But in general part of the column is compressed, and part extended, and the results depend on the ratio between the cohesive and repulsive forces; if they be equal, that is if the column be perfectly elastic, we can determine the position of the neutral line.

In this case the repulsive and cohesive forces are as the squares of their distance from the neutral line, fig 91, let AB be a line drawn parallel to the depth of the pillar N, its intersection with the neutral line, call NC x, and CF y, then the compression is as $(a + x)^2$, and the extension as $(a - x)^2$; but the force applied tends to tear asunder the extended part by a lever whose fulcrum is at the centre of compression, and whose extremities are one at F and the other at the centre of cohesion, but one force acting on a lever must be the difference between the other and the pressure on the fulcrum, therefore the force at F $\propto (a+x)^2 - (a-x)^2 = 4ax$, and its moment is $4ax \times (y + \frac{1}{3}x - \frac{2}{3}a)$ because its leverage is NF — $\frac{2}{3}$ NB, and this is equal to the cohesion $\times \frac{4}{3}a$ the distance between the centres $= \frac{4}{3}a(a-x)^2$, and we obtain

$$3\,yx = a^2$$

As before if $x = a$, $y = \frac{1}{3}a$, if $x = o$ y is infinite.

3. Let CL, fig 92, be a portion of a column bent by an external force, producing its top and bottom till they meet, RL is ultimately the radius of curvature, but RL : LC :: $ac : an$, and LC : an as the compressing force to the space through which the column is compressed, or where the material of the pillar is given in a given ratio $m : 1$, RL : $an = \frac{a^2}{3y} :: m : 1$, the radius of curvature is therefore at any part $\frac{a^2m}{3y}$. m is called the Modulus of Elasticity, a term which will be explained when we treat of the elasticity of air.

(c.) The differential of the section of the beam is $b \times dx$, and its cohesion $\frac{f}{D} \times xdx$, but as it acts by the leverage x, its moment with respect to DC expresses the opposition to fracture arising from it; the integral of $\frac{f}{D} \times bx^2dx$ is therefore the transverse strength, equal when x is D to $fb\,D \times \frac{D}{3}$, or to the absolute strength, acting at the distance of $\frac{1}{3}$ the depth.

2 To find the strength of a triangle with its vertex downwards, call b its base, a its altitude, x the distance from the vertex, and y the parallel to the base. If the beam be loaded till it yield, a particle in b exerts a co-

hesion f, and therefore the cohesion of a particle in y is $\frac{fx}{a}$, the element of the triangle is ydx, and the moment of cohesion of ydx is the differential of the transverse strength, but $y = \frac{b}{a} \times x$ therefore

$$ds = \frac{fb}{a^2} \times x^3 dx$$

(the differential of the cohesion being multiplied by x to obtain its moment). Integrating

$$s = \frac{fb}{a^2} \times \tfrac{1}{4} x^4$$

which is complete, for s is o when x is o, if $x = a$, $s = \frac{1}{4} fba^2$, but the absolute strength is $f \times \frac{ba}{2}$, therefore the transverse = absolute acting by the leverage $\frac{1}{2}a$. The strength of a triangular beam is therefore $\frac{3}{4}$ of a circumscribing rectangular, and its weight only one half.

3. If it be required to obtain the strength of the trapezium between b and B, it may be done by integrating from $x = a - h$ to $x = a$, observing that the cohesion of a particle is no longer $\frac{fx}{a}$ but $f\frac{(x - \mathrm{H})}{h}$ and its moment is taken with respect to B; this gives

$$s = \frac{fb}{12a} \times (3h + 4\mathrm{H}) \times h^2$$

for the trapezium with its broader side uppermost. The section of the beam is

$$\frac{bh}{2a} \times (h + 2\mathrm{H})$$

and dividing s by its absolute cohesion, we find the distance of the centre of effort

$$\frac{h}{6a} \times \left(\frac{3h + 4\mathrm{H}}{h + 2\mathrm{H}}\right).$$

4. When the vertex of the triangle is uppermost, the origin of x is in b and $y = \frac{b}{a} \times a - x$, hence

$$d s = \frac{fb}{a^2} \times (a - x)\, x^2\, dx$$

and

$$s = \frac{fb}{a^2} \times \frac{ax^3}{3} - \frac{x_4}{4}$$

which when $x = a$ is $\frac{fba^2}{12}$, and dividing this by the cohesion we obtain $\frac{a}{6}$ for the leverage by which the absolute strength acts; in this position the beam is only $\frac{1}{3}$ as strong as when its base is up. To determine the strength of a trapezium with its smaller side up, we put the cohesion of a particle $\frac{fx}{h}$, and integrate from $x = o$ to $x = h$, this gives

$$s = \frac{fb}{12a} \times h^2 (h + 4\text{H})$$

and we obtain for the distance of the centre of effort from b,

$$\frac{h}{6} \times \frac{h + 4\,\text{H}.}{h + 2\,\text{H}.}$$

5. To find the strength of a cylinder, call the distance from its bottom $r - x$, the cohesion of the differential of the area $- 2\, ydx$ is

$$\frac{f}{2r} \times 2y \times r - x \times - dx$$

and its moment or

$$ds = - \frac{f}{r} \times (r - x)^2\, ydx$$

in which y is to be replaced by its value derived from the equation of the circle. To integrate this easily we may express these quantities as circular functions and

$$ds = fr^3 \times (1 - \cos z)^2 \times \sin^2 z \times dz$$

$$= fr^3 (dz \times (\sin^2 z - 2 \sin^2 \cos z + \sin^2 \cos^2 z)$$

$$= fr^3 \times dz \frac{(1 - \cos 2 z)}{2}$$

$$+ \tfrac{1}{8} (1 - \cos 4 z) - 2 \sin^2 \cos z.$$

$\frac{s}{fr^3}$ is $= \frac{z}{2} + \frac{z}{8}$ and other quantities of the form $m \sin nz$. these are $= o$ when z is o and 180°, and therefore may be neglected, s is therefore $fr^3 \times \frac{5}{8} \times p$, p being the circumference whose diameter is unity. or $fr^3 p \times \frac{5}{8} r$, but $fr^2 p$ is the absolute cohesion, which therefore acts by a leverage of $\frac{5}{16}$ of the diameter.

6. If the beam be a semicylinder with the plane of section downwards,

$$ds = - fr^3 \times 2 \sin z \cos^2 z \times d \cos z$$

$$= - fr^3 \times \frac{\sin^2 2 z \times 4 dz}{8}$$

$$= - \frac{fr^3}{4} \times \frac{(1 - \cos 4 z)}{4} \times 4 dz$$

and

$$s = - \frac{fr^3}{4} \times z - \tfrac{1}{4} \sin 4 z.$$

this integral is to be taken from $z = 90$ to $z = 0$, and therefore

$$s = \frac{fr^3}{4} \times \frac{p}{2},$$

the absolute strength is $\frac{fr^2 p}{2}$ and therefore it acts in resisting a transvere strain by the lever $\frac{1}{4} r$, these applications of this method may suffice.

(*d.*) 1. In the third hypothesis where the lower part of the beam is compressed, and the upper extended, the moment of the cohesive forces with respect to the neutral line must be equal to that of the repulsive, as if either of these were greater than the other, it would bend the beam till they balanced each other, hence if the cohesion and repulsion of a particle be equal at equal distances, the neutral line of a rectangular beam bisects its depth: the action of the beam under these circumstances is the same as if the cohesion of NABE, fig 95, were applied at a point in h, and the repulsive force of CNED at another in H; this latter being the fulcrum of the lever, by which a weight acting at the distance l, breaks the beam. By the preceding note it appears that the energy of the cohesion of the rectangle AE, to prevent motion round NE, is the same as if its absolute strength were collected in a point distant from NE $\frac{1}{3}h$, the centre of repulsion is also distant from it $\frac{1}{3}$ H; these are equal, and their sum is $\frac{1}{3}a = \frac{1}{3}(H+h)$. The moment of the cohesion of AE with respect to the centre of repulsion, is therefore $fbh \times \frac{a}{3} = \frac{fba^2}{6}$, for it is double the moment with respect to NE.

2. Let the section be a triangle, and let the trapezium cut off by the NL be extended, the moment of its cohesion with respect to NL = that of the repulsion of NAL, fig 96, but (note *c*, No. 3,) the first of these quantities is $\frac{fb}{12a} \times h^2\ (3h + 4H)$ and the second is $\frac{fb}{12a} \times H^3$ and as $h + H = a$, we obtain by substituting

$$h \;+ 3\,ha = a^2$$

which gives

$$h = a \times \frac{\sqrt{13} - 3}{2}$$

and

$$H = a. \times \frac{5 - \sqrt{13}}{2}$$

shewing that NL quam proxime bisects the area of the section.

The moment of cohesion of the trapezium with respect to NL × distance between the centres of cohesion and repulsion, and divided by that of the centre of cohesion from NL, or what is simpler, the distance of the centres × the absolute strength of the trapezium is the transverse strength: the absolute strength is $\frac{fbh}{2a} \times$ $(2\,a - h)$; and the distance between the centres is

$$\frac{a - h}{6} + \frac{h}{6} \frac{(4\,a - h}{2a - h} \text{ therefore}$$

$$s = \frac{fba}{2} \times \frac{h}{6a} \times 2a + h$$

but $\frac{fba}{2}$ is the absolute strength of the whole triangle, and the lever by which it acts is, substituting for h its value, $\frac{a}{12} \times 5 - \sqrt{13}$.

If the distance between the centres be multiplied by the absolute strengh of the triangular part of the area, we obtain the strength of the beam with its vertex up

$$= \frac{fba}{2} \times \frac{a}{6} \times \frac{7\sqrt{13} - 23.}{7 - \sqrt{13}}$$

In the first of these positions, the absolute strength of the beam acts by a leverage $= a \times 0.\,1162$, in the second by $a \times 0.1099$. The strength of the beam is therefore nearly equal in either position, and is only of what it should have been were its materials incompressible.

By considering the beam as composed of two trapezia, whose height $= a - v$, if we investigate its strength on the supposition that v is small, the term into which

v enters is positive, and therefore the strength of a triangle is increased by cutting off its edge.

3. If the beam be cylindric, its neutral line will pass through its centre; by note (c) No. 6. the distance of the centre of cohesion is $\frac{r}{4}$, the distance between the centres of cohesion and repulsion is therefore $\frac{r}{2}$, but the absolute strength of a semicircle is $\frac{fpr^2}{2}$, therefore the strength of the cylinder is $fpr^2 \times \frac{r}{4}$ or it acts by the leverage of $\frac{1}{8}$ the diameter.

If we knew the ratio between the compressibility and extensibility, we could determine the position of the NL, and ascertain the ratio of the transverse to the absolute strength a priori, but at present we are ignorant of this in almost every instance.

(e.) On any of these hypotheses, the strength of a beam $\propto ba^2$, if therefore it be required to cut the strongest beam out of a given cylinder, we must find the right angled triangle on its diameter, the square of one of whose sides $\times$ into the other is a maximum. When fig 97, AB^2, $\times$ BC is a maximum, the square of this product must also be a maximum; but the squares of the sides are as AE and EC, call AC a, and EA, x; $(a - x)\ x^2$ is a maximum, therefore its differential

$$(2\ xa - 3\ x^2)\ dx = 0$$

hence $x = \frac{2}{3}\ a$: from this it is obvious that $AB^2 = 2BC^2 = \frac{2}{3}\ AC_2$. This maximum is of considerable use, it may be presented under the form. $\sin^2 \times \cos.\ C =$ maximum, and the value of C is then 54°. 44′ an angle as we shall subsequently find, possessing many remarkable properties.

2. If the strongest rectangular beam of a given girt be required, $p = a + b$, and $a^2p - a^3 =$ maximum, and as before $a = \frac{2p}{3}$ and $b = \frac{1}{2} a$, the depth therefore should be $\frac{1}{2}$ the depth.

3. If a beam project horizontally from a wall bearing a weight at its extremity, it has been shewn that the stress at any part is as $\frac{ba^2}{l}$, and therefore it may be made of uniform strength by properly varying these quantities.

If b be constant, as if the beam be cut out of a vertical slab or plank, $a^2 \propto l$, and the outline of the beam is a common parabola, whose parameter p is such that the absolute strength of $pb = Wn$, W being the weight supported, and n the ratio of the leverage by which the cohesion acts to a.

If the section be always similar, $ba^2 \propto a^3 \propto l$, and the outline is a cubical parabola.

If the beam be loaded by a weight equally distributed along it, the stress at any section is as the weight which it bears, and the leverage by which it acts; but the weight is as l, and the leverage as $\frac{l}{2}$, hence the strength which is as $ba^2 \propto l^2$, therefore if b be constant, $a \propto l$, and the outline is straight, if a is unvaried $b \propto l^2$, the curve being a parabola referred to the tangent at its vertex, if $b \propto a$ the curve is called a Semicubic Parabola.

The strength of a beam to bear its own weight is of more complicated investigation, for the weight depends on the shape which we wish to find. The stress at a, fig 99, is the moment of the portion of the beam la, or it is $G \times \int badl$, G being the distance of the CG from

a: if l increase by $d.l$, the moment is increased by the moment of $abd.l \times d.l + \int bad.l \times d.l$, for dG may be taken equal to dl. The first of these terms is indefinitely less than the second, for they are as $abdl : \int abdl$, or in a ratio less than any assignable one, and we may say that while l increases by $d.l$, the stress increases by $d.l \times \int bad.l$, but the strength must increase as the stress, and therefore calling m a coefficient depending on the transverse strength and weight of a square unit of the material, we have

$$m\,d\,(ba^2) = d.l \times \int bad.l,$$

from this we can deduce the relation between b or a, and l, the section being as above supposed rectangular, and one of its sides constant. Let b be constant, then taking the differentials, and assuming $d.l$ constant

$$md^2\,(a^2) = dl^2 \times a,$$

multiply both sides of the equation by $d\,(a^2)$,

$$\frac{md\,(a^2) \times d^2\,(a^2)}{dl^2} = a\,d\,(a^2) = 2\,a^2 da,$$

$$\frac{2}{3}a^3 + \text{C} = \frac{m\,(d\,(a^2))^2}{2\,dl^2} = m \times \frac{2\,da^2}{dl^2} \times a^2, \text{ or}$$

$$\frac{\sqrt{ra^3 + \text{C}}}{\sqrt{2\,m} \times a} = \frac{da}{dl}$$

we have therefore

$$\frac{dl}{\sqrt{2\,m}} = \frac{ada}{\sqrt{ra^3 + \text{C}}}$$

this transcendental cannot be integrated by common means, and we must develope the radical, and integrate the terms of the series separately, which gives

$$\frac{l}{\sqrt{2\,m}} = \text{C}' +$$

$$\frac{1}{\sqrt{r}} \times \left(\frac{\sqrt{a}}{2} - \frac{\text{C}}{r \times a^{\frac{5}{2}}} \times \left(\frac{1}{5} - \frac{3.1}{4.11} \times \frac{\text{C}}{ra^3} + \&\text{c.}\right)\right.$$

but l must vanish with a, and this cannot be unless C and C′ are both $= o$, in which case the equation is

$$l = \frac{\sqrt{m}}{2r} \times \sqrt{a}$$

the equation of the parabola referred to its tangent at the vertex.

Where a is constant we have

$$ma.d_2 b = dl_2 \times b,$$

$$\frac{ma.dbd^2b}{dl^2} = bdb, \text{ and } ma \times \frac{db^2}{dl^2} = b^2 + C,$$

$$\text{hence } \frac{dl}{\sqrt{ma}} = \frac{db}{\sqrt{b^2 + C}}$$

and by Lacroix. Cal. Int. Art 162.

$$\frac{l}{\sqrt{ma}} = h.log. \left(b + \sqrt{b^2 + C}\right) + C'$$

but as l and b vanish together,

$$C' = - h.log \sqrt{C} \text{ so that}$$

$$\frac{l}{\sqrt{ma}} = h.log \left(\frac{b + \sqrt{b^2 + C}}{\sqrt{C}}\right)$$

C may be determined from the condition, that the beam is of a given length and section.

(f.) If a beam be loosely supported at its ends, in order that it be equally strong to bear a given weight at any part, a^2b must be as $D \times (L - D,)$ hence if b be constant the outline is an Ellipsis, if a be given, it is a Parabola whose axis is a perpendicular at the middle of the beam. If a weight be uniformly distributed along the beam, fig 88, call CX, x, W the total weight along the beam, and AW D : the differential of the stress at W is $\frac{W}{L^2} \times D \times xdx$, integrating, the stress arising from $WC = \frac{W}{2L^2} \times Dx^2$ or making $x = d = \frac{W}{2L^2} \times$

$Dd \times d$, to this we must add for the stress arising from AW $\frac{W}{2L^2} \times Dd \times D$ and the sum is $\frac{W}{2} \times \frac{Dd}{L}$ being exactly half the stress which the weight would have produced if concentrated at that point.

(g.) The strength of a cylinder is as r^3, that of the tube is therefore as $R^3 - r^2R$, or as t the thickness $\times$ $R \times (r + R)$, which when t is small and constant is nearly as R^2. The quantity of matter in the tube is as $R^2 - r^2$, or as $2\,Rt$, t is therefore as $\frac{1}{R}$ and the strength of the tube where the quantity of matter is given as the diameter.

This is calculated on the first hypothesis, supposing all the parts of a cylinder to resist fracture with equal energies, but on the other suppositions which have been noticed, the portion which we imagine removed, is not exerting its full cohesive force, and therefore the tube is stronger than in the above proportion of $R^2 - r^2 : R^2$. On the hypothesis of equal compression and extension, the NL passes through the centre, and the extreme extension of each cylindric portion which has been denoted by the symbol f is as its radius: the f' of the inner cylinder is therefore $\frac{r}{R} \times f$, and its strength as $f \times \frac{r}{R} \times r^3$ the strength of the tube is therefore to that of the solid cylinder :: $R^3 - \frac{r^4}{R} : R^3$, or as $R^4 - r^4 : R^4$. If the quantity of matter = that of a cylinder of equal length, and of the radius r', as $R^2 - r^2 = r'^2$, the strength of the tube is to that of this cylinder when t is evanescent as $2\,R : r'$.

Tubes, though stronger and stiffer than solid beams, are much less capable of resisting a blow, and therefore cannot be used indiscriminately; they are much stengthened by introducing transverse partitions, which prevent their shapes from being changed, as is exemplified in the joints of reeds.

(h.) Let NL, fig 100, be the neutral line of a beam bent by a transverse strain, let the part A be extended, and B be compressed, take EG a small portion of the length of the beam, draw the perpendiculars to NL, IK and MR, let IM be lengthened in the flexure to MH, draw HE, meeting MR at R; MR is ultimately the radius of the circle equicurve to the beam, call KI a, and EI $\frac{a}{n}$, n being a quantity depending on the ratio of the extensibility, to the compressibility of the body, then R or RE : EI :: MI : IH, but MI : IH in a given ratio $1 : f'$, for MI the natural length of that portion is increased to MH by the stress s, and the extension is as the extending force and the length, f' is therefore its extensibility by a stress s, but the extensibility of a given length is as the stress, therefore $f' : f$ the extension at the instant of fracture :: s : S the stress which breaks the beam; this on any ratio of the cohesive and repulsive strength is as $fvba^2$, therefore $f' = \frac{fs}{fvba^2}$ and $R = \frac{a}{nf'} = \frac{a}{ns} \times vba^3$. If the stress arise from the action of a weight hung at the extremity of the beam, $s = wl$ and $R = \frac{v}{n} \times \frac{ba^3}{wl}$ but the deflection of a curve from its tangent is as the square of the differential arch divided by the RC. If the flexure be minute, so that R is very great, the length is a differential arch of the curve, and the deflection $\propto \frac{l^2}{R} \propto \frac{l^3}{ba^3}$. The stiffness is inversely

as the deflection produced by a given weight, and therefore is as the reciprocal of this fraction.

(i.) Suppose the section of the axle or wire divided into a number of concentric rings of evanescent breadth; let one of the parts into which it is divided by this section, receive a motion of torsion while the other is held fast, there must ensue a separation of the particles of the two axles in contact, which is to a certain extent resisted by their cohesion. The effect of the cohesion of any ring is as the number of particles in it, as the leverage by which they act, and the distance to which they are separated; the effect of these three causes is at the distance x ftom the centre $\frac{f}{r} x \times x \times 2\ pxdx$, and integrating, T the resistance to torsion $= \frac{1}{2} \frac{fx^4p}{r}$ which when $x = r$ becomes $\frac{1}{2} r \times fpr^2$, or equal to the absolute strength acting at the distance $\frac{1}{2} r$. and therefore it is double the transverse ssrength on the hypothesis of equal cohesive and repulsive forces. Conceive now the axle divided by an indefinite number of such sections, and let a torsion be applied to one of the differential cylinders into which it is thus resolved, this would turn through a given angle were those above and below it fixed immoveably till its cohesion yielded or was in equilibrium with the torsion: but if those contiguous to it be permitted to move, since they experience the same force, they will also move through the same angle, therefore each section on to the extremities is equally twisted relative to that contiguous: hence the angle of torsion required to excite a given resistance is as the length of the axle or wire, and where the angle is given the force of torsion is inversely as the length. Where the length is given the force is as shewn above, proportional to f,

the separation of the particles of two contiguous sections at the circumference of the cylinder, but this is as the angle of torsion. Lastly, if wires of equal length be equally twisted, the forces are as the squares of their sections, for they are as fr^3, and f is as r. This proportionality of the force to the angle has been already noticed, and it obtains even in those bodies where it might least be expected, such as lead and clay. These within certain limits obey these laws, and when the twisting force is removed return to their original position, but if twisted too far, they take a *Set*, or in other words their particles assume a new arrangement; and resist any displacement from it as they did at first, if still further strained, fracture is of course the result.

CHAPTER XIV.

65. From the principles delivered in the last Chapter, it may be inferred that the transverse strength of any material is much inferior to the resistance which it offers to a direct thrust or pull, and experience shews that the difference is even greater than that indicated by theory; it is therefore essential, that the practical mechanic should know how to dispose the bodies of which his machinery are constructed, so that their strength may be exerted with the least possible exposure to this dangerous strain, and to contrive that every piece may resist in the direction of its length. In this consists the science of Carpentry, which in its most important department, teaches the mode of constructing trussed framing of every kind,

as levers, roofs, centres, and wooden or iron bridges; in its minor details it shews the methods of uniting firmly the compound parts of any of the above mentioned structures. These we cannot touch on, though they are replete with curious information and of the utmost practical importance, and we can only glance on the leading principles of its theory.

One of the simplest cases of the change of transverse into direct strain is shewn in fig 101, the beam AB projecting from a wall, bears a weight at its extremity; the strain of this tends to break the beam across at B, but if another beam AD be added, its extremity at A mortised into the other, so that it cannot slip, and D abutting firmly against the wall, the action is quite different, and absolutely independent of the transverse strength, we might even suppose the beams connected with the wall by hinges; in that case if AD were removed, the weight would descend, describing an arch of a circle round B, in doing this it must approach to D, therefore the strain on AD tends to make A approach D, or to compress the beam, on the other hand if the cohesion of AB were destroyed, the point A describing a circle round D, would recede from B, AB is therefore pulled. The supposition that the joints at B and D are flexible, is in all great works scarcely different from the truth, for it is absolutely impossible that they could resist for an instant, if the load acted transversely on them, and the strength must be entirely derived from the framing. The stresses on AB and AD are easily ascertained, for the two are equivalent to the weight; we are given the directions of the three forces, and can thence construct the parallelogram of forces, making the vertical AW the diagonal. The triangle ABD is similar to the triangle *abc*, therefore the stress on AB : stress on AD :: AB : AD, and this again : W :: AB : BD. These stresses are each greater than W, from the acuteness of the angle A, the figure is therefore defective, but it is not offered as an advantageous arrangement. Since AB is pulled, its place may be supplied by a cord or an iron rod, and this substitute is often used in roofs for those parts which are pulled, technically named Ties, as they can thus be made far lighter with the same degree of strength. Instead of AD, a tie AE may be employed, and in that case AB is compressed or is a Strut.

For a second example, let us take the beam AB, fig 102, bearing a weight at its middle point; if it be sufficiently great the beam must bend, and at last yield, but let there be fixed at its centre an upright piece KP, and let its extremities be connected with A and B by Ties, then till KP be crushed, or AK torn, AB cannot fail. To compute the stress on each part, suppose the beam inverted, a prop at P, the beam divided there, and a weight = ½ W hung at each extremity; their stress on AK : ½ W :: AK : KP, and the stress on KP = W or is as 2 KP; AB is compressed. Such a truss, but in an inverted position with respect to the figure, is often contained in the beams or girders which support floors, when it is necessary to make them of considerable length, and no support can be obtained for their middle points; it is made of iron, and adds greatly to the stiffness, as well as strength. A still stronger lever is shewn in fig 103, where the stress on each part is obviously proportional to its length, the weight acting at A or B being represented by CP; its superiority in strength to fig 102 is manifest.

66. It has been stated, art 15, that the beam of a balance should be nearly inflexible, and we now see how this requisite may be obtained; one very simple mode of stiffening it, is the connecting the extremities of the arms by steel wires, with the top of the tongue or index; by this it becomes a framed lever which cannot bend as long as the index resists, but as it must be made very slender, this contrivance does not completely remove the imperfection. A more effectual method is to make the beam a light parallelogram, ABCD, fig 104, the axis is fixed in BD, and this piece is tubular below it to hold a sliding weight for the purpose of raising or lowering the C.G; to prevent the sides of the parallelogram from bending, light braces as BE, EF are sometimes added, and sometimes the space is filled by a number of rings touching each other and the sides, but these are very weak. Some of the finest balances ever made were formed of two hollow cones united at their bases; one of them whose sensibility is mentioned under that head, sustained 6 pounds in each scale, though the brass of which it was made was little more than $\frac{1}{100}$ of an inch in thickness, and it gave no signs of bending.

Wheels are in general sufficiently strong by their figure without any framing; the circular rim throws the strain on the radii, its tendency is to twist them off the axle, and it unavoidably acts transversely. Where the circumference is large, as in water wheels, its weight requires some support to keep it in shape, and the same remark applies to the large astronomical circles now used in observatories, where stiffness is absolutely essential; but it would lead us too far to dwell more on this subject.

67. Roofs are systems of frames used to support the lead, slates, or other materials employed to cover houses;

these are ranged in parallel rows, and are crossed by bearers on which the covering immediately rests, and which transfer its weight to the frames. Each of these is similar to the rest, and it is therefore sufficient to consider one of them. In warm climates, Roofs are nearly flat, but in these countries they must have a certain pitch or inclination to throw off rain or snow, and this has the additional advantage of increasing the strength by a judicious distribution of the materials. Suppose the rafters AB, BC, fig 105, bearing on the walls W W′, and resting against each other at B, such a frame could scarcely bear any weight, for when loaded B would sink, and A and C receding horizontally, would thrust against the walls with a force which their cohesion could not resist. Supposing the covering of uniform weight, the load on each of the rafters may be conceived to be collected in its CG, call it W, and its vertical pressure at A and B are ½ W, the pressure at A is supported by the wall, and therefore that at B is alone to be considered, add to it an equal pressure from the other rafter, and the pressure acting at B = W; but this is resolvable into pressures in the direction of AB and BC, draw AE and EC parallel to them, and BE the diagonal representing W, AB is the thrust on that rafter, and this being resolved in the direction AC, gives the horizontal thrust which tends to push away the walls. The relation between these quantities is easily expressed in terms of the inclination of the roof, for W the load on one rafter : to thrust on that rafter :: BE : BA :: 2 sin inclination to 1. It is therefore $= \frac{W}{2 \sin I}$, and the horizontal thrust : W :: ½ AC : BE :: 1 : tang I, and $= \frac{W}{2 \text{ tang } I}$. Hence as we dimininish the inclination we increase both thrusts, and as we increase it we increase the size and weight of the roof; the most advantageous pitch is probably an inclination of 45°. *(a.)*

As the horizontal thrust endangers the stability of the edifice, its action must be withstood, which is effectually done by connecting A and C by the Tie beam AC; for this prevents those points from receding, and no pressure is exerted on the walls but in a vertical direction. The frame is now a triangle, and is the simplest of all, being the element of which more complicated framing is composed, for it is the only figure which keeps its shape if its angles are flexible, as they cannot be altered without lengthening or shortening one of its sides.

In small roofs the triangle is sufficiently strong, but the Tie beam has often to support a floor and ceiling, it is also liable to bend by its own weight; its middle is therefore suspended from the summit of the roof by a King-post fig 106, KP. A further addition is sometimes made, as the rafters are in a state of compression, while at the same time the weight of the covering presses them transversely, a strain which has already been referred to as very dangerous; the Struts KE, KF are therefore used, which throw the weight on the King-post.

Every part of this roof is compressed or stretched, and we can easily determine the strain which each beam suffers, and proportion our dimensions accordingly. The weight of the covering being uniformly diffused, produces a stress at E $= \frac{1}{2}$ W, (see note *f* preceding chap.) the thrust on EK is therefore $\frac{1}{4} \frac{(W + \text{weight of AR})}{\sin I}$ $= W'$, the stress on KP $= \frac{1}{2}$ weight of AC $+ W' \times 2 \sin I = W''$, that on AP $= \frac{\frac{1}{2} \text{weight of KP} + W''}{2 \sin I}$ nearly. If we neglect the weight of the beams, KP is pulled by a strain = W, and that on EK is half that on AP.

68. Roofs cannot always consist of two rafters, and their outline is frequently a polygon; in this case the beams of which it is composed should be in equilibrio with each other, as thus they are disposed in the form most advantageous for the complete exertion of their strength; the method of determining the form which possess this requisite, depends on a principle developed in the note (*b*): it may be easily ascertained mechanically in any given case by this method; compute the weight which must be borne on each rafter, construct a model of the required number of rafters, and load their extremities with weights equal to half the load which they must carry, let the extremities of it be fixed in a horizontal line, and it will arrange itself in the inverted form required for the roof. If the frame be constructed according to this figure, it will sustain the load laid on it although in a tottering equilibrium, but if it be trussed so that it must preserve its shape, we obtain the strongest possible roof with rafters in that proportion. Fig 107, shews a roof of four rafters, called a Kirb or Mansard, from the architect who invented it; the rafters AB,CD bear against the ends of a Straining piece BD; BH, DI, are called Queen-posts, they serve to prevent the Tie beam from sinking by its own weight, and stiffen the trapezium ABCD; suppose them taken away, and a force applied at B, that angle would sink down, and D would rise, but we see that it is tied down by DI; the King-post supports BD, and the braces FH, EI prevent AB and DC from bending: the chief advantage of this roof is the quantity of room afforded by the parallelogram BI, but it is not so strong as it might be made with a very small increase of timber. *(c.)*

69. One of the most difficult problems in this department of mechanics, is the construction of the *Centre*, or scaffolding which supports an arch while building. The load to be sustained is enormous, and in the course of the work it is distributed in the most unequal manner, so that any thing like equilibration is out of the question. Centres differ from roofs in being polygons of a far greater number of sides, and in the ultimate points of resistance: in roofs the thrust is withstood by the Tie beam, but centres rest on piers sufficiently strong to bear it without injury: a polygon of beams is therefore inscribed in the arch, and one interior to it is used to keep it in shape, the sides of the former being struts, those of the latter ties. See fig 108, where a very simple one is exhibited; fig 109 shews one more complicated: in both the spaces between the Polygon and the curve must be supposed filled with blocks or framing, so that the arch may rest on them.

70. Frames of wood or iron resembling Centres, are often used as bridges, some of the latter indeed on a most stupendous scale, but the principle continues the same, being the combination of systems of trusses; an iron bridge in general consists of several ribs of cast iron springing at unequal heights, the lowest are most curved, and like the various polygons of a centre, the strength of the whole is united by transverse framing. Fig 110 is a simple construction of a wooden bridge, AB is one of the beams of the road way, its middle hangs from C by CP, which is supported by CD, CE, the smaller truss AEP supports AP, and forms a railing at the same time. Sometimes they are made as in fig 111.

NOTES TO CHAP. XIV.

(a.) Call half the span of a roof b, then the length of a rafter is $\frac{b}{\cos I}$ which is proportional to the quantity of materials in the roof; $\frac{b}{\cos \sin I}$ or $\frac{b}{\sin 2 I}$ is as the thrust in the direction of AB, and the horizontal thrust is as $\frac{b}{\sin I}$; when the thrust in the direction of AB is a minimum, the sine of 2 I is a maximum, or $2 I = 90°$, and I 45°, the same gives the expence of the roof and its horizontal thrust, a minimum.

(b.) Let the Polygonal frame *abc*, fig 112, bear weights at its angles, which we suppose perfectly flexible; there must be equilibrium among the forces which act at each angle, for otherwise the angle where the forces are not equivalent to each other must move. The forces at the angle where W is applied, are W, the thrust along P*a*, and that along *bb′* which we call t; but the reaction of *bb′* against *cc′* = its pressure on P*a*; and by Art 3,

$$\frac{W}{\sin (a - b)} = \frac{t}{\sin a}$$

hence $\frac{W. \sin a}{\sin (a - b)} = t$; by the same argument $\frac{W'' \sin d}{\sin (c - d)} = t'$, but $\frac{t}{\sin c} = \frac{t'}{\sin b}$ therefore

$$\frac{W \sin a, \sin b}{\sin (a - b)} = \frac{W'' \sin c \sin d,}{\sin (c - d)}$$

or the weights sustained at the angles of a polygon in equilibrium, are as the sines of the angles directly and inversely as the products of the cosines of the inclinations of the rafters meeting at each angle. In this equation if we substitute for W and W″ their values in t and t', we obtain

$$t \times \sin b = t' \times \sin d$$

or the thrusts are reciprocally as the cosines of inclination, and hence it follows that the horizontal thrust of each rafter is the same.

(*c.*) Let it be proposed to determine the form of an equilibrated roof of four equal rafters, bisect AI, fig 113, and erect a perpendicular; the middle angle of the roof will be in it, and from the equality of AC, CI, that of angle A and angle I may be inferred, as also of F and B; and $\frac{\sin \text{ABE}}{\sin \text{ABC}} = \frac{\sin \text{HCB}}{\sin \text{FCB}}$, or retaining the notation of the last note

$$\frac{\sin a}{\sin(b-a)} = \frac{\sin b}{\sin 2b} = \frac{1}{2 \cos b}$$

hence $\sin b . \cos a = 3 \cos b \sin a$, or $\text{tang } b = 3 \text{ tang } a$. Call the angle CAH which is known from the given span and height of the roof m, then $\text{cotang. } m + x = \text{tang } a$, x being the angle BAD, and $\text{cotang. } m - x = \text{tang } b$, therefore

$$3 \text{ tang. } m - x = \text{tang. } m + x$$

and since

$$\frac{\text{tang } (m + x)}{\tan (m - x)} = \frac{\sin. 2m + \sin 2x}{\sin. 2m - \sin 2x}$$

it follows that

$$2 \sin 2x = \sin 2m$$

from which the length of AB can easily be found.

CHAPTER XV.

71. The principles on which the stability of an arch depends, are different from those of the combinations which we have hitherto considered: in these the load is resisted by cohesive or repulsive strength of the materials, and the frame is made to keep its shape by bracing the angles; but in those arches which are constructed of stone, weight is resisted by weight, and the stability depends on this, that no part can sink down without raising some other which may be made sufficiently heavy to withstand its action.

72. An Arch consists of Piers on which it rests, of Voussoirs or Arch-stones, which are truncated wedges, and of a Roadway which is in theory supposed to be a prolongation of the Voussoirs; the interior curve of the arch is called the Intrados, the exterior the Extrados. The most obvious mode of investigating the theory of arches is to suppose the Voussoirs in equilibrio, and from this to determine the distribution of pressure; we can thus determine either the Intrados or Extrados where the other is given. The arch thus constructed will stand by the mutual action of its parts, and if an additional load be laid on it, its figure will be slightly changed, but will still, unless the disturbance exceed certain limits, stand firmly. This is by no means a complete solution of the problem, but it is all which can be attained in the present state of Mechanical analysis.

73. In fig 114, P, P′ are the Piers, the Voussoirs V, are supposed of equal length, and the inequality of pressure required for their equilibrium may be obtained by

laying earth, &c., on their upper surface; their faces are also supposed polished, and capable of sliding on each other without friction; this is not actually the case, but we cannot introduce into our calculations such a complicated element as the friction of two surfaces separated by a layer of mortar, and it shall be shewn that this omission cannot lead to any dangerous error. Were the structure of the arch attempted by piling the Voussoirs at P successively on each other, we should find that at a certain inclination they would slide by their own weight on those below them, the angle depending on the friction of the surfaces; call its complement a, the joint which makes this angle with the vertical is called a Joint of Fracture, and this is properly the origin of the arch, all below it being a part of the Pier. Produce these joints Ii and I′i' till they meet, then the portion ICI′ acts as a wedge to thrust aside the piers, and by sect 42, in the wedge the impelling force is to the parallel pressure on the side as the back to the length, or as 2 sin IEG or of a : 1; but the impelling force is the weight of ICI′, call this A, and the pressure on a pier $= \frac{\frac{1}{2} A}{\sin a}$. From the same section it follows that $\frac{A}{2}$: horizontal thrust :: tang a : 1.

Let us now suppose the arch ICI′ to be divided at M by a joint making with the vertical an angle ε; the surfaces of that joint are pressed together by a force compounded of the horizontal thrust, and the tendency of the parts IM and MCI′ to fall in, turning round on the piers as fulcra: let the mass of IM be named m, its tendency to slide on the inclined plane I i produces a thrust at the point M$m = \frac{m}{\text{tang } a}$ which, as the arch is equilibrated = thrust of MCI′, or $\frac{A - m}{\text{tang } a}$, half the sum of these is therefore = one of them, or the horizontal pressure at Mm = $\frac{A}{2} \times \frac{1}{\text{tang } a}$. This force is perpendicular to the other force which is compounded with it, and therefore is to their result the perpendicular pressure on the surface of the joint :: cos ε: 1, for the perpendicular to the joint and horizontal line make the same angle as the joint and vertical; it is therefore

$$= \frac{A}{2} \times \frac{1}{\text{tang } a \times \cos \varepsilon}$$

Conceive now that there is inserted at this joint an indefinitely thin Voussoir whose weight is m', this is a wedge impelled by a power m', and resisted by $\frac{A}{2 \text{ tang } a} \times \frac{1}{\cos \varepsilon}$ but as the vertical in whose direction m' acts makes with the axis of the wedge (or with the joint, which coincides with it) an angle = ε, resolving it in the direction of this latter we obtain by the property of the wedge $m' \times \cos \varepsilon \times MP = \frac{A}{2 \text{ tang } a} \times \frac{1}{\cos \varepsilon} \times Mn$, but P being the intersection of two perpendiculars indefinitely near is the centre of the circle which has the same curvature as the Intrados at that point, and therefore MP = Radius of Curvature; hence in an equilibrated arch m' the load at any point $\times \cos^2 \varepsilon \times R = \frac{A \times Mn}{2 \text{ tang } a}$ and therefore is inversely as $R \times \cos^2 \varepsilon$. ($a$.)

74. From this proposition we can easily deduce the construction of an equilibrated arch, for the shape of the Intrados along with the Specific Gravity of the materials used in filling up the roadway, afford the data which are required to determine the Extrados. The practice however requires the use of the differential calculus: and we

can only exhibit its application in the case of a circular Intrados. The Voussoirs being supposed of a given depth, the load which they carry is the weight of the column of earth which stands on them, and this is as $MH \times Mn \times \cos \varepsilon$. fig 115. Hence taking PL for the axis of the abscissa $MH \times \cos^3 \varepsilon =$ constant quantity $= CV$, for at C $\cos \varepsilon = 1$, if therefore we take $MH : CV :: CP^3 : MN^3$, the point H is in the Extrados, and thus the entire curve may be traced. It is of a very unpromising aspect, for it has a point of contrary flexure, and if the arch spring at right angles to the horizon a vertical asymptote, requiring an infinite load at its origin, but for other curves the Extrados does not deviate so much from a practicable shape, see note (*b*.)

75. The inverse of this Problem, namely, to find the Intrados when the Extrados is given is yet more difficult, it requires in general two successive integrations, and cannot be attempted here, (*c*.) but in its stead the mechanical method proposed by Hooke and afterwards by Robison may be given. It depends on the fact, that a flexible cord loaded according to any law will dispose itself in equilibrio, and that if it were inverted keeping the same shape, it would stand as an arch, if therefore pieces of chain be appended along it, and trimmed till their lower extremities are in a line which is the required Extrados, then the polygon formed by the chains from which they hang, is an approximation to the Intrados required for the state of equilibrium.

76. In the construction of an arch, two considerations are of the utmost importance to the stability of the arches, the first of these is the resistance of the piers on which the entire strain ultimately bears, and the second the strength of the Voussoirs. A pier may yield either by

being upset or by sliding horizontally, the thrust separating the strata of its masonry; the first of these can never happen in large arches, for the direction of the tangential pressure falls within the base of the pier, and therefore tends rather to keep it firm than overthrow it. The thrust cannot be withstood by any adhesion of the cement used in the construction, for this is utterly insignificant when compared to the immense force exerted against the piers; but as the friction of stone is very considerable, it is possible to load the pier so that the strata against which the thrust is exerted cannot be displaced. It varies as we have shewn inversely as the tangent of the inclination of the arch at its spring, low pitched arches have therefore much more of it than such as rise at a greater angle, those which spring perpendicularly would have none, but as such cannot be equilibrated from the infinite height of the Extrados at that point, we must be content to diminish it as much as possible. Some of the large iron bridges are made very flat, but only where the abutments are rock; and where a pier is intermediate between two arches, the thrust is a matter of indifference as the opposite pressures counterbalance each other.

77. The requisite magnitude of the Voussoirs may be derived from the known strength of the stone used in building, the dimensions of the arch are given, and therefore its weight, but the pressure on the Voussoirs at the crown, is $\frac{A}{2} \times \frac{1}{\text{tang } a}$ and that at the spring $\frac{A}{2} \times \frac{1}{\sin a}$ the depths of the Voussoirs at these two places must therefore be in the proportion of $\cos a : 1$, in order that they may be equally strong.

78. The theory of equilibrated arches is very beautiful, and therefore entitled to a place in an elementary

work, but it must be confessed that it is not of great practical utility; this the reader must have surmised from the notorious fact that semicircular arches have been often constructed, which have stood well though their Extrados is nearly horizontal: but the hypothesis on which our calculation is founded, is widely different from the actual conditions of the problem, the Voussoirs instead of being polished and frictionless, as we suppose, are rough, and no pressure which they can bear wiihout being crushed, is able to force them back; therefore they cannot be considered as wedges, and the arch must be supposed continuous and flexible. If it be overloaded at the crown, this sinks, and the haunches rise, the curvature increases at some points and decreases at others; at the former the Voussoirs continue in contact at their upper angles separating below, at the latter the points open above, and the arch resolves itself into a polygon, whose angles are at these separations. This brings the action of an overloaded arch to the operation of a frame of carpentry; but as we cannot secure the angles by Ties, we must depend on the weight and friction of the superincumbent materials, and fortunately these are in general sufficient. In iron bridges where there are no joints, the stress of a load applied at any part tends to depress that part below the curve of equilibration, and raise others above it; and if the piers be firm, the arch cannot fail but by a transverse fracture.

It is unnecessary to go further into this subject, and we will conclude with this one remark, that when a weight is laid on the crown of an arch, its action is not confined to the part immediately below it, but is propagated laterally, and produces the same effect as if the curve of the Extrados were raised at its vertex, this therefore requires a new Intrados; if this fall within the

Voussoirs, the arch will be firm, but if they be so small that it passes outside, the stucture will probably fall, and thus the magnitude of the arch-stones may be determined.

79. Domes differ from arches, in that the thrust which so much endangers the latter, augments the stability of the former by forcing inwards their joints: they are much stronger, and may even be open at the summit, but the investigation of their properties is too difficult to find a place here; it is enough to mention that the Equilibrated dome is the weakest of all. *(d)*.

NOTES TO CHAP. XV.

(*a*). Conceive an arch divided by a joint into the portion *m* and A — *m*, and suppose them urged by vertical forces proportional to their mass, they may be considered as wedges; drawing the vertical *ab* and raising perpendiculars at A and B, fig 116, the weight of the wedge *m* : pressure at B : : *ab* : *ac* : sin *acb* : sin *abc*, but angle *acb* = angle D and angle *abc* = comp *bd*A = comp *a*, and therefore pressure at B : *m* : : cos *a* : sin D = sin $(a - \varepsilon)$: it is therefore $= \dfrac{m. \cos a.}{\sin (a - \varepsilon)}$ The pressure at B produced by the remainder of the arch BF is in the same way proved $= (A - m) \dfrac{\times \cos a}{\sin (a + \varepsilon)}$ and as these are equal in case of equilibrium,

$$\frac{A - m}{m} = \frac{\sin (a + \varepsilon)}{\sin (a - \varepsilon)} = \frac{\text{tang } a + \text{tang } \varepsilon}{\text{tang } a - \text{tang } \varepsilon}$$

hence

$$\frac{A - 2m}{A} = \frac{\text{tang}\,\varepsilon}{\text{tang}\,a} \qquad (1).$$

An equation which must be satisfied in order that the pressures may be equal and opposite at each joint.

For the weight of the Voussoirs we may substitute any other vertical force, and for tang ε, its value (as ε is the inclination of the tangent of the Intrados if the joints be perpendicular to the curve) $\frac{dy,}{dx,}$ and taking the differentials

$$-dm = \frac{A}{2\,\text{tang}\,a} \times \frac{d^2y}{dx} \qquad (2)$$

an expression of the force acting on ds, the differential of the Intrados. This may be otherwise expressed for

$$-\frac{ds^3}{dxd^2y} = \text{Radius of Curvature.}$$

and therefore

$$dm = \frac{A}{2\,\text{tang}\,a} \times \frac{ds^3}{dx^2 \times R} \qquad (3.)$$

but $\frac{dx}{ds} = \cos\varepsilon$, and therefore the second member becomes

$$\frac{A}{2\,\text{tang}\,a} \times \frac{ds}{R \times \cos^2\varepsilon} \qquad (4)$$

a formula which may be useful where the radius of curvature is known.

This latter formula may also be derived from the theorem of note (b), Chap. 14: for considering the Intrados as a Polygon of an indefinite number of sides, the load at any point $\propto$

$$\frac{\text{sin of angle of contact}}{\text{sin}^2\text{ of angle made by curve with vertical}}$$

but if ds be taken constant, the sine of the angle of contact is as the Sagitta, or inversely as R, hence

$\propto \frac{1}{R \times \cos^2 \iota}$. In general the vertical force m' is the weight of materials resting on the Intrados, and where they press only in a vertical direction, it is as h their height at that point $\times dx \times \cos \iota$, and hence

$$h = \frac{A}{2 \tan g. a} \times \frac{d^2 y}{dx^2} = \frac{C}{R \times \cos^3 \iota} \qquad (5.)$$

If the force be as h, and act perpendicularly to the Intrados, a formula may be deduced from the theory of the wedge to express the law of its variation; but the process would be rather difficult, and we may use another principle, namely, that the pressures of two contiguous Voussoirs on each other must be equal and opposite: call f the force at any point, f : the pressure which it produces perpendicularly to the side : : the back : the length, but when a Voussoir is indefinitely thin, its back is ds, and its side R ; hence $\frac{f \times R}{ds}$ is the pressure produced by such a Voussoir, and this is constant, therefore = C, but $f \propto h.ds$, therefore

$$h = \frac{C.}{R} \qquad (6)$$

This formula is probably more conformable to the natural state of the case than the former, as earth or sand exercise a considerable lateral pressure, and when mixed with water act nearly as actual fluids.

(*b.*) We shall give one or two instances of the application of these formulas. In the first place let the proposed Intrados be a conic section, take its axis, supposed to pass through the crown, for the axis of y, and the tangent through its extremity for that of x, and the equation of the curve is

$$\pm x^2 = ny^2 \pm my,$$

the upper signs relating to the Hyperbola, the lower to the Ellipse; resolving,

$$y \pm \frac{m}{2n} = \pm \frac{\sqrt{m^2 \pm 4nx^2}}{2n}$$

hence

$$dy = \pm \frac{2x.dx}{\sqrt{m^2 \pm 4nx^2}}$$

and

$$d^2y = \pm \frac{2m^2 \times dx^2}{\overline{m^2 \pm 4nx^2}^{\frac{3}{2}}}$$

Using the first formula

$$h = \frac{d^2y}{dx^2} \times C,$$

we obtain

$$h = 2C \times \pm \frac{m^2}{(m^2 \pm 4nx^2)^{\frac{3}{2}}}$$

to determine C we must put $x = 0$, and calling the thickness of the arch at the crown H, $H = \pm \frac{2C}{m}$ and $C = \pm H \times \frac{m}{2}$ so that

$$h = \frac{m^3 \times H}{(m^2 \pm 4nx^2)^{\frac{3}{2}}}$$

If the proposed curve be a parabola, $n = 0$, and the second member becomes H, shewing that the Extrados is a parabola similar to the intrados. In the other sections we have $m = \pm \frac{2a^2}{b}$ and $n = \frac{a^2}{b^2}$ which give

$$h = \frac{H \times a^3}{(a^2 \pm x^2)^{\frac{3}{2}}}$$

In the Hyperbola, the second member decreases continually as x increases, and therefore the thickness of the arch constantly decreases, vanishing when x is infinite but in the Ellipsis, it increases as x increases, and when

$x = a$, h is infinite. At the crown, the Extrados is concave towards the curve, but it has a point of contrary flexure unless H be too great, whose place is determined by referring the Extrados to the transverse axis of the Ellipse, in which case its equation is

$$y = \frac{a^3 H}{\overline{a^2 - x^2})^{\frac{3}{2}}} + \frac{b}{a}\sqrt{a^2 - x^2}$$

taking $d^2 y$, and putting it $= 0$ we obtain the equation

$$x^2 = a^2 + m - \sqrt{m^2 + \frac{10\, a^2 m}{3}}$$

m being $= \frac{9}{8}\frac{a2}{b} \times$ H, for the abscissa of the point of contrary flexure. This expression is impossible, if H be greater than $\frac{1}{6}\, b$, the Extrados being then totally convex towards the Intrados; and as the roadway cannot readily be extended beyond the point of contrary flexure, we can thus determine the thickness of the crown where the span is given. But as has been already observed, this hypothesis respecting the pressure of the materials, does not strictly agree with the actual state of things, the second formula is nearest the truth, and they are both limits between which the actual curve of the Extrados lies.

Cycloidal arches are sometimes used, and we may give the investigation for them. R as has been shewn $= 2$ CO, and $\varepsilon =$ angle COA, and its cosine is $\frac{\text{CO}}{\text{AO}}$

h therefore $= \frac{\text{C} \times \text{AO}^3}{2\ \text{CO}^4}$: call AO $2\,r$, and $\text{CO}^2 = 2\,ry$,

therefore $h = \frac{\text{C} \times r}{y^2}$ but at the vertex $\text{H} = \frac{\text{C}r}{4r^2}$ and

henee $h = \frac{4r^2 \text{H}}{y^2}$

In the second formula $h = \frac{C}{R}$ to apply this to the Conic Sections we must know the expression for the Radius of Curvature. The general value for this is $\frac{ds^3}{d^2y\ dx}$ when dx is constant; and using the values found above for dy and d^2y, $ds^2 = dx^2 + dy^2$

$$= dx^2 \frac{(m^2 + 4x^2 (1 \pm n)}{m^2 \pm 4nx^2}$$

$$\text{R therefore} = \frac{dx^3 \times \left(\frac{m^2 + 4x^2 (1 \pm n)}{m^2 \pm 4nx^2}\right)^{\frac{3}{2}}}{dx^3 \times \frac{\pm 2m^2}{\overline{m^2 \pm 4nx^2})^{\frac{3}{2}}}}$$

$$= \frac{\overline{m^2 + 4x^2 \times (1 \pm n)})^3}{\pm 2m^2},$$

but at the vertex of the curve $x = 0$, and $R' = \frac{\pm m}{2}$, therefore at the vertex $H = \frac{C \times 2}{m}$ and $C = \frac{mH}{2}$ therefore

$$h = \frac{m^3 H}{\overline{m^2 + 4x^2 (1 \pm n)})^{\frac{3}{2}}}$$

this in the parabola becomes

$$\frac{m^3 H}{\overline{m^2 + 4x^2}|^{\frac{3}{2}}}$$

in the other sections

$$\frac{a^6\ H}{\overline{a^4 \pm c^2 x^2})^{\frac{3}{2}}}$$

an expression which does not become infinite in the Ellipse, when $x = a$ being then $= \frac{a^3}{b^3} \times H$. In the

Cycloid $h = \frac{C}{\sqrt{2ry}}$ and we obtain, finding for C its value,

$$h = \frac{\sqrt{2r}\,H.}{\sqrt{y}}$$

(c.) Where the law according to which the pressure varies, is given in terms of the co-ordinates of the Intrados, or where the equation of the Extrados is known, the Intrados may be determined, for the differential equation which is obtained can in general be integrated, although the results are not of a very simple form; we select the simplest cases as examples. Let it be given that the arch is of uniform thickness, and dm is as ds, therefore $ds = \frac{C\,d^2y}{dx}$ and multiplying both members by dy, we obtain

$$dy = \frac{C\,dy\,d^2y}{dx\,ds} = \frac{C\,dy\,d^2y}{dx \times \sqrt{dy^2 + dx^2}}$$

and integrating

$$y + C' = C \times \frac{\sqrt{dy^2 + dx^2}}{dx}$$

squaring and transposing

$$\frac{dy}{dx} = \frac{\sqrt{(y + C')^2 - C^2}}{C}$$

and to determine C′, if we observe that $\frac{dy}{dx}$ is the tangent of the angle which the tangent makes with the horizon, and that it $= 0$, when $y = 0$ we have $\sqrt{C'^2 - C^2} = 0$, and the equation becomes

$$dx = \frac{C.dy}{\sqrt{y^2 + 2\,Cy}}$$

the differential equation of the Catenary, as might have been inferred from the consideration that the equilibrium of a cord fixed at its extremities would not be disturbed if the force of Gravity became negative; the integration of this equation gives x in terms of y, but it is unnecessary to give it. The most useful case of this problem is to determine the Intrados being given the Extrados a right line; we may take it for the axis of x, and the quantity which we have denominated h, is the ordinate of the Intrados, and therefore

$$\frac{C\,d^2y}{dx^2} = y.$$

this Equation is integrable if multiplied by dy, and we obtain

$$\frac{C\,dy^2}{dx^2} = y^2 + C',$$

and

$$\sqrt{C}\,\frac{dy}{dx} = \sqrt{y^2 + C'}$$

but when

$$y = H, \frac{dy}{dx} = o,$$

therefore

$$C' = - H^2 \text{ and}$$

$$\frac{dx}{\sqrt{C}} = \frac{dy}{\sqrt{y^2 - H^2}}$$

whose integral is

$$\frac{x}{\sqrt{C}} = h.log.\left(y + \sqrt{y^2 - H^2}\right) + C''$$

and as $x = 0$ when $y = H$, $C'' = -\log H$, and the equation gives

$$\frac{x}{\sqrt{C}} = h.log\left(\frac{y + \sqrt{y^2 - H^2}}{H}\right)$$

the equation of a curve resembling the Catenary, but not identical with it except for a particular value of H. The constant C is determined by the given span and height of the arch.

(*d*.) The principles of Dome Vaulting are derived from the theorems given above, by considering an Element of the Dome, bounded by two planes, inclined at a very small angle, and intersecting in its axis; this is an arch whose transverse breath is not as in ordinary arches, uniform, but it decreases from the spring to the vertex. Let PRSTV, fig 118, be such a portion; supposing the Dome a surface of revolution, *rvxst* (the differential which was called *dm*) is as *tz* the thickness at that part $\times$ *zx* or *ds* $\times$ *rz* which is as *x*, therefore by formula (3) $-dm$ or $t \times x \times ds = \frac{C \times ds^3}{R \times dx^2}$ or $t = -\frac{C\,ds^2}{R\,x\,dx^2}$ an expression which determines the thickness of the dome where the Intrados is given. In a Spherical surface $ds = -\frac{dx}{y}$ and therefore $t = \frac{C}{R\,y^2x}$ At the vertex *x* is $= 0$, and therefore $t =$ infinite as also when $x = R$, and it might thence be inferred that such a dome could not be constructed: but it must be observed that this proposition supposes the parts of the vaulting perfectly equilibrated, so that they have no tendency to fall in, and there is no pressure in the vertical joints; suppose the crown of such a dome to be lightened, or even removed entirely, the lower parts immediately tend to fall in, but this is impossible, for each horizontal course is an arch, and its Voussoirs being equally pressed inwards, must uphold each other. It is therefore evident that the only requisite for the stability of a dome is that no part should have any tendency to yield outward, and therefore any curve of the same height and span which lies within the curve of equilibration, will stand firmly, provided that its support be able to withstand its horizontal thrust, and this may be always ensured by surrounding the lowest course with an iron hoop. To return to the Sphere, *t* must be susceptible of a minimum when y^2x is a maximum, or when $y^2 = \frac{2}{3}$, this point is therefore 36° from the vertex; and the thickness of the dome must be increased below it, for as we have shewn all above it may be reduced to the minimum thickness.

A spherical dome may stand although of uniform thickness, provided too large a segment be not taken, the limit is where the pressure of the incumbent mass is in equilibrio with the weight of a course supposed to act as a wedge, to find it take the quotient of Equation (4) by Equation (1) or

$$\frac{-dm}{\frac{A}{2}-m} = \frac{ds}{R \times \text{tang}\,\varepsilon \times \cos\varepsilon^2} = \frac{ds}{R \times \sin\varepsilon \times \cos\varepsilon}$$

dm we know $= txds$, and we can easily express $\frac{A}{2} - m$, it being a spherical surface whose sine is x, and cosine y. The differential of this is as xds = in the circle to rdy, but this is the differential of a cylindric surface of the radius r, and height y, therefore the surfaces of a sphere and circumscribed cylinder cut off by planes parallel to their contact, are equal, $\frac{A}{2} - m$, is therefore as $r\,(r-y)$

therefore $$\frac{x}{r^2 - ry} = \frac{1}{r \sin\varepsilon \cos\varepsilon} = \frac{r}{xy}$$

$$\text{and } x^2y = r^2\,(r-y),$$

$$y\,(r^2 - y^2) = r^2\,(r-y),$$

$$y\,(r+y) = r^2,$$

$$\text{and hence } y = \frac{-r \pm \sqrt{5r^2}}{2}$$

$= \frac{1}{2}\, r\,(\sqrt{5} - 1)$, and $x = \sqrt{y}$; y is nearly 618, r being unity, and the arch of which it is the cosine is about 52°; a larger segment cannot be used without hooping.

They who wish to pursue this subject further, may consult Robison's papers in the Lncyclopedia Britannica, Bossut on the Equilibrium of arches, and Nimmo in Brewster's Encyclopedia Art. Bridge, where the corrections to be made in the application of the theory of equilibration, are investigated; this last writer may be studied with great advantage by those who are anxious to know this department of architecture practically.

NOTES TO CHAP. XVI.

80. The propositions which have been delivered hitherto, apply chiefly to coherent masses of matter which are denominated Solids, but there exists a numerous class possessed of peculiar properties, which therefore require a separate consideration; these are the forms of matter in which the ratio of the cohesive to the repulsive force is very much diminished, so that the particles of which they are composed are perfectly moveable in any direction with respect to each other. Of these again there are two kinds: the first are denominated Liquids, and though their particles are thus moveable, yet they cannot easily be made to approach nearer than their original position, and therefore these bodies are scarcely compressible, but the other class of Fluids have no cohesion, or at least it is exceeded by the repulsive force, which is alone obvious to our research, and tends to disperse the atoms, if not counteracted by external force. If that pressure be increased or diminished, the bulk of the body varies with it, and the repulsive force follows the same law. These are called Elastic Fluids, and their Equilibrium is a particular case of that of Fluids in general, we shall therefore give the theory of liquids first, and afterwards shew how far it is modified by supposing the Fluid of variable density.

81. Before we proceed to this investigation, it may be expedient to remark, that these three states of matter do not arise from any ultimate difference in the nature of the atoms of which they are composed; for any of them may be rednced to either of the others by varying the action of heat. For example, a piece of Copper is hard and cohesive; let its bulk be measured, and its temperature be increased, it is found to expand in all its dimensions: at a certain heat, its cohesion is perceptibly diminished, and at length it melts, becoming liquid, and as long as the required temperature is maintained, it continues so; but the effect of heat is not limited here, for if it be very much increased the melted copper boils, and is changed into vapour which assumes the appearance of flame. Mercury is another example of this truth, if the temperature be augmented it boils and rises in vapour; if reduced, it congeals, becoming a malleable metal. This may seem to belong to Chemistry rather than Mechanics, but we shall hereafter find it useful, and we will proceed to investigate the conditions of the equilibrium of fluids.

82. The principle commonly used is that any pressure is uniformly propagated through a fluid; this expresses its peculiar nature with sufficient distinctness, and is easily applicable to the problem. To use it let us conceive a vessel filled with a fluid, which for the present we suppose void of weight; let the orifice of the vessel be closed by a piston capable of sliding in it freely, a force F applied to this will press on the fluid, and this pressure will be diffused through its whole mass equally: the result is a pressure on the sides of the vessel uniform throughout, and every portion of them equal in area to the piston, sustains a force $= F$, call the bulk of the piston unity, and Fs is the pressure on any surface s.

From this the Equation of Equilibrium may be deduced, but it is unnecessary to investigate it as we shall obtain all that we require by a more easy though indirect method. (a)

83. If we now suppose the fluid to gravitate, each portion of it is pressed by the weight of those incumbent on it, and the pressure is unequal at different depths, but still that arising from the action of the piston is uniformly diffused. If the thickness of the stratum of fluid be inconsiderable its weight may be neglected, and S being the surface of the piston AB, fig 119, the pressure on it is FS; let there be imagined a circular aperture in CD, also closed by a piston *ab*, and the pressure on it is F*s*, it will therefore be balanced by a weight equal to F*s* laid on *ab*, this weight may be that of a column of the fluid which is represented by H*s*, H being its height F therefore = H, and if the vessel be continued up to E and filled with fluid to that height, the piston *ab* which serves only to transmit external force may be suppressed, and AB is pressed by a force H × S. If another aperture *s'* were made in CD the pressure against it would be H*s'*, and for equilibrium it must be covered by a column of the depth H, but it is obvious that the upper surfaces of these columns are equidistant from CD. The entire of CD may thus be removed, and if it be replaced by a column H × CD, the pressure on AB is the same as before, and we thence infer that the pressure on a horizontal surface covered by a fluid, is the weight of a column of which it is the base, and whose height is its depth below the surface, without reference to the quantity of fluid. For AB we may substitute the reaction of an inferior stratum of a fluid, and it follows that the pressure on any portion of it is as its depth below the surface.

As the pressure is uniform over a horizontal surface, the height of the incumbent fluid must be equal in every part, and therefore the upper surface is horizontal. (b.)

Before we proceed to develope further, the consequences of this remarkable proposition which is the basis of Hydrostatics, it may not be amiss to describe the experimental mode of exhibiting it to view; for some are staggered by the position that the Hydrostatic pressure is not as the quantity of fluid, and a particular case of this law is honoured by the appellation of the Hydrostatic Paradox. Therefore let figures 120.1,2,3 be three vessels in which the apertures AB are of the same diameter, these are closed by similar brass plates which are at first held against them by cords. The three vessels are then immersed to the line CD in a cistern, and on loosening the cords, the plates do not fall from the vessels being pressed against them by the surrounding water; let water be cautiously poured into them, and when it attains a certain height over the brass, which is the same in the three, the plates fall off; here it is visible that the pressure on the upper surfaces of them must be equal, and that the quantity of water employed is very unequal. Again let the vessels, fig 121, be set in a bason of Mercury so that their edge is one inch below the surface, the metal rises in them to the external level, now pour water into them, the Mercury sinks in them below that outside, and when the water is 13 inches deep the mercury is completely expelled from each vessel, and as the part of each immersed in the metal is equal and similar, it contained the same weight of Mercury, but this is the measure of the pressure of the water.

84. As a consequence of the proportionality of the pressure to the height, it follows that a fluid stands at the same level in communicating vessels however curved

or winding the communication may be; and therefore tubes connected with a reservoir of water, deliver it at any height not exceeding that of the supply, a fact, the importance of which can scarcely be estimated when we consider what stupendous works the ancients executed to supply their cities with this necessary fluid, and apparently without knowing that it would find its level.

85. The same property enables us to determine a horizontal plane at any time with facility by means of a Spirit Level; this instrument is a glass tube closed at the ends, and nearly full of alcohol; it is secured in a metallic box which is in most instances connected with a telescope adjusted so that its line of collimation is parallel to the surface of the alcohol. It is obvious that any point visible at the intersection of the wires of the telescope must be in the horizontal plane passing through the observers eye, and by raising perpendiculars from the ground at various points, and knowing the intercepts of them between it and the plane of collimation, we know the vertical ordinates of its surface. Thns in fig 122, let L be the instrument, LH is horizontal, a graduated rod is held upright at *a*, and a moveable disk slided along it till it appears at the cross wires, the graduation gives *ca*, the rod is then removed to *f*, and *fh* ascertained in the same way, the difference of these is the difference of level of *a* and *f*. Thus the perpendiculars of all points whose distance from LH does not exceed *ba*, the length of the levelling staff is ascertained, the level is then transported to a higher station, and the observations pursued. *(c.)*

In levelling it must be remembered that LH is parallel to the tangent of the earth at L, but not at H, and that H is therefore above the level of L, let LB, fig 123, represent a portion of the meridian, EB is this difference of level, and it is nesrly equal to the versine FB $= \frac{LB^2}{2R}$, but 2 R is nearly 8,000 miles, and therefore $\frac{1}{8000}$ of the square of the distance in miles gives the difference of true and apparent level.

86. The principle that pressure is uniformly diffused has lately been applied in the Hydrostatic Press, an instrument which has nearly superseded the Screw Press, where rapid action is not necessary. Without entering into detail, the machine consists of two hollow cylinders fitted with pistons, and the smaller is provided with the valves of a forcing pump; call the diameters of the pistons D and d, and if the space between them be filled with water, the pistons will be in equilibrio if loaded with weights in the proportion of their areas. If therefore the small pump be worked with a force P, as it forces water into the large cylinder AB, it will lift the piston of it with an energy $P \times \frac{D^2}{d^2}$, and without any friction except that occasioned by the two pistons, which is very small in comparison of the common cases of friction. The action of the press is relaxed by a stopcock in the pipe of communication which permits the water to escape, and in favourable situations the pressure of a head of water of sufficient altitude may be substituted for the little pump, the press being then merely a modification of the Hydrostatic Bellows. To compute the power of the Press, suppose the great Cylinder 12 inches wide, and the Pump ½ the areas are as 576 : 1, but a man can easily apply downwards, a force of 56 pounds which may be augmented tenfold by a lever, the energy of this on the body to be pressed is therefore 144 Tons, and nearly all this is actually exerted.

87. The pressure of an immersed surface is easily computed where every part of it is equally distant from the upper surface of the fluid; but if it be oblique to the horizon or curved, it is necessary to find the sum of all the unequal columns incumbent on its parts. To this end suppose the surface divided into a number of indefinitely small portions, let *ab* be the profile of one of them, fig 125, and SH the upper surface, the pressure on *ab* is various in its different parts, for the pressure at *a* is as *ae*, and that at *b* as *bf*, but the difference of these is, if *ab* be very small, evanescent in comparison of *ae*; therefore denoting the element *ab* by *ds*, $ds \times pc$ is the pressure on that small portion, and the sum of all these is the pressure on the surface; but from what is delivered in art 9, the sum of these is the moment of the surface with respect to SH, and this is $= S \times GI$, G being the Centre of Gravity of the surface S; so that the pressure on any surface is the weight of a column of the fluid whose base is the surface and altitude the depth of the Centre of Gravity.

88. Let us apply this to a rectangular surface whose profile is AB, fig 126, its centre of Gravity is at the point of bisection of AB; GI is therefore $\frac{1}{2}$ BD, and the pressure on the rectangle AB is $AB \times \frac{DB.}{2}$ This theorem enables us to determine a limit to the requisite thickness of floodgates and dykes, and to proportion their strength according to the stress on them; thus the stress on AG : stress on AB :: $AG^2 : AB^2$, and therefore the thickness should be as the square of the depth, and should decrease from the bottom to the surface. *(d.)* The locus of the pressure is a right line, for the pressure at G is as GI, erect a perpendicular $Gi = GI$, and draw *Aid*, any perpendicular B*d* represents the pressure

at B, and the sum of the pressures ∝ triangle ABD; such a triangle would be kept in equilibrio by a force Mn passing through its CG, or the surface AB will be in equilibrio if supported against the pressure by a force applied at $\frac{2}{3}$ of its depth. This point is called the centre of pressure. (e.)

89. The structure of Dykes depends on these same principles, but they must be combined with the resistance of earth to determine rules for this department of building. The pressure of water produces three different effects on an embankment; it tends to overturn it ronnd the outer edge of its base, if resolved in the horizontal and vertical directions the horizontal result tends to make it slide horizontally in strata, and the vertical presses it down and opposes the other by increasing the friction and adhesion.

The first of these effects is nullified by making the inclination of the slopes of the embankment not less than 90°, for if AD′, fig 127, be the exterior face, and the angle D′AB be acute draw D′E perpendicular to AB, this is the direction of the pressure on it, and it is evident that the portion D′AE will be torn off, but when DA is the face making an obtuse angle, the direction of the pressure falls within the base, and therefore augments its stability.

The horizontal thrust is constant where the depth is given, being as its square, but the friction is as the weight of MLA, ∝ area MLA, gg 128, and this must be as AL^2, iherefore AMD is a right line, and the bank should be terminated by planes. In works of this kind it must be remembered that the stability of it depends in a great measure on the closeness of its workmanship, for if there be any fissure into which the water can penetrate, it produces a Hydrostatic pressure which tends to float up the parts of the bank over it.

NOTES TO CHAP. XVI.

(*a*). To find the differential equation which determines the equilibrium of a fluid acted on by any forces, let us consider an elementary parallellopid in it, and taking the axes of the co-ordinates parallel to its sides, these will be dx, dy, and dz, fig 129. The powers acting on the particles must be resolved in the directions of the axis, call the results f, f' and f'', let the density of the Parallellopiped be s, then fs, $dx\ dy\ dz$ is the energy by which it is urged in the direction of x; $f's.\ dx\ dy\ dz$, that in the direction of y, But the result of the forces which act on the particles of the fluid must be a pressure P, which varies in different parts of the fluid, and therefore is a function of the co-ordinates. In order to Equilibrium, the effect of this pressure on two opposite surfaces must be unequal, and the difference must be equivalent and opposite to 'the effect of the forces on $dx\ dy\ dz$. Since P is a function of x, y, z, take its differential on the hypothesis of x variable, and it is $\frac{dP}{dx} \times dx$ according to the common notation (Lacroix. Cal. Diff. Abr. Art 126); and this multiplied into the surface $dy \times dz$ on which it acts, gives the difference of pressure on the surfaces of the parallellopiped which pass through the extremities of dx. This is equal to $\frac{dP}{dx} \times dx\ dy\ dz = sfdxdydz$, therefore $\frac{dP}{d} = sf$, and in the same way $\frac{dP}{dy} = sf'$ and $\frac{dP}{dz} = sf''$. but (see Lacroix art 123)

$$dP = \frac{dP}{dx} \times dx + \frac{dP}{dy} \times dy + \frac{dP}{dz} \times dz,$$

and substituting for the partial differential co-efficients $\frac{dP}{dx}$, $\frac{dP}{dy}$, $\frac{dP}{dz}$, their values sf, sf', and sf'', we obtain

$$dP = s\,(fdx + f'dy + f''dz)$$

for the equation determining the pressure. If the second member be not an exact differential, as no value of P can be assigned, the equilibrium is impossible, which is ascertained by the values of $\frac{d\,(sf)}{dy}$, $\frac{d\,(sf')}{dx}$, &c. for these should be equal (Lacroix Cal. Int. Abr. art. 126).

Where the fluid is acted on by a single constant force as that of gravity, f and f' vanish and $dP = sF\,dz$, and integrating $P = sFz + C$, and if we suppose the origin at the surface of the fluid

$$P = s\,Fz,$$

or the pressure is as the density of the fluid and its depth.

(*b*.) If the force acting on the fluid emanate from the origin of the co-ordinates, and be as a function of the distance, the strata of equal pressure are spherical surfaces; for calling the force at a distance r $F(r)$, this multiplied by the cosine of the angle which r makes with x or by $\frac{x}{r} = f$, $f' = F(r) \times \frac{y}{r}$ and $f'' = F(r) \times \frac{z}{r}$, but as the pressure is to be constant in the stratum, $dP = 0$, and therefore

$$0 = sF(r) \times \frac{xdx + ydy + zdz}{r},$$

$$0 = xdx + ydy + zdz,$$

and integrating

$$C^2 = x^2 + y^2 + z^2,$$

the equation of a Sphere. At the surface the pressure $= 0$, and therefore this surface is also Spherical.

(c.) The tube of the level is ground on the interior so as to be a portion of a circle, and it is evident that the action of the level is the same as that of a plumb-line whose length is the radius of its curvature; but in practice it is found that from the cohesion of the fluid, and other causes, its accuracy is only equal to a plumb-line $\frac{1}{25}$ of its Radius, which is easily found from the motion of the bubble corresponding to a given inclination. In levelling, besides the correction for the curvature of the earth, a correction must be made for the refraction of the air; as the point where the line of collimation cuts the distant vertical is higher than the observer, the air there is rarer, and therefore the light proceeding from that point does not come in a right line. Without giving the investigation here, as it properly belongs to optics, the effect of this source of error is to make the correction for the earths curvature $\frac{1}{7}$ too great in a mean state of the barometer and thermometer. in this climate.

(d.) Let SA fig 130, be the projection of any surface on the plane of z and y, and let udz be the element of the surface corresponding to dz, an expression sufficiently general for our present purpose, then the perpendicular pressure on udz is, calling the density of the fluid unity, equal to $uzdz$, and its integral eorrected to vanish with z, is the total pressure on the whole surface; u being a function of z is given by the equation of the curve. (1).

This equation is not very useful, for the use of knowing the pressure is that we may apportion the resistance to it, but the whole of the perpendicular pressure is not efficacious in bursting the vessel containing the fluid; and it is therefore necessary to know, 1st, the horizontal, 2dly. the vertical pressure, and 3dly the direction of their result, and the distance of it from the surface of the fluid.

If the element of the surface be vds, ds being that of the curve SA $\int vrds$ is the perpendicular pressure, and resolving it in the horizontal direction, the horizontal pressure is

$$\text{Hor. P.} = \int vzds \times \frac{dz}{ds} \quad \text{or} \int vzdz \qquad (2)$$

and the vertical is

$$\text{Vert. P.} = \int vzdy \qquad (3)$$

and therefore the tang. of the angle made by the result with the horizon is

$$= \frac{\int vzdy}{\int vzdz} \qquad (4)$$

and the value of the result is

$$\sqrt{(\int vzdy)^2 + (\int vzdz)^2}$$

Let us now suppose the horizontal forces to tend to turn SA round S then $\int vz^2dz$ is their total effort, and if Z be the z of their result, $Z \times \int vzdz = \int vz^2dz$

$$\text{and } Z = \frac{\int vz^2dz}{\int vzdz} \qquad (5)$$

By a similar process the place of the result of the vertical pressures may be found, and if required, that of their compound may be determined.

Let the projection SA, (fig. 131) be a right line, and let v be constant, the horizontal action $= \int vzdz = \frac{1}{2}vz^2$, and the vertical action $= \int v$ tang. $azdz = v$ tang. $a \times \frac{1}{2}z^2$, and their resnlt makes with the horizon an angle whose tangent

$$= \frac{\frac{1}{2}vz^2 \text{ tang. } a}{\frac{1}{2}vz^2}$$

or the result is perpendicular to SA, a conclusion which is obvious a priori. The integral of vz^2dz is $\frac{1}{3}vz^3$, and therefore $Z = \frac{2}{3}z$. It is evident that this value of Z and that of the horizontal pressure are the same, whatever be the nature of the curve SA.

The pressure on a rectangular surface as a Dock Gate, is uniform in any horizontal section, and increases uniformly as we descend in the vertical section; it must, therefore, be made stronger at the bottom than at the top. Suppose it constructed of materials to which the reasoning of chap. 13, is applicable, we can ascertain its construction by a similar mode of reasoning to that used in note *(f)* chap. 13t Call a the total depth of the gate, the pressure at x is, supposing $v = 1$, xdx. and the stress which it produces at (z) is by art. 61. $\frac{\overline{a-z} \times x^2 dx}{a}$ whose integral gives $\overline{a-z} \times \frac{1}{3}z^3$. for the stress at (z) produced by the fluid above it. For that below it, the pressure is xdx and the stress it produces $\frac{z}{a} \times \overline{a-x} \times xdx$ whose integral is

$$= \frac{zx^2}{2} - \frac{zx^3}{3a}$$

which taken from $x = z$ to $x = a$ gives

$$\frac{z^4}{3a} + \frac{za^2}{6} - \frac{z^3}{2}$$

The sum of this and the preceding integral is the total stress at $(z) = \frac{z}{6}(a^2 - z^2)$ but the strength of the timber of which the gate is constructed is as the square of its thickness therefore

$$t^2 \times C = z \times \overline{a^2 - z^2};$$

t^2 is a maximum when $z = \frac{a}{\sqrt{3}}$ or $.578a$.

The law according to which its thicknes should vary in a horizontal section has been already determined; but the strength of it is much increased by making it in two parts abutting on each other like the rafters of a roof, a construction which is necessary to give a passage to shipping. The angle of inclination is susceptible of a minimum; for, on the one hand the stress on AC, fig. 133, being as AC^2 is excessive, if angle C be not sufficiently obtuse, and on the other, the thrust on AC in the direction of its length, augments with that angle. The pressures on the gates are perpendicular to them, let them be represented by CD and CF; these compound a result = 2CE, but this is also the result of the thrusts, and therefore the thrust is to the pressure :: CB : CD : 1 ; tang angle B. If it be required that the thrust × stress be a minimum $\frac{AC^2 \times AC}{\text{tang. B}} = \text{min},$ but $AC = \frac{AE}{\cos B}$ and this product will be a min. when $\cos^3 B \times \text{tang } B$ is a maximum, or where $\cos^2 B \times \sin B$ is a max. B is then 36.16 and the angle C is 190° nearly, but this is only a limit, for it must be remembered that transverse strains diminish the longitudinal strength.

We will conclude this note with an application of the theory to a surface of revolution, and take the simplest, namely, the cone. Call AB, fig. 134, l, and CAB a, $y = (l - z)$ tang a, and $v = 2\,p$ tang $a \times (l - z)$, Z is therefore

$$= \frac{\int 2\,p \text{ tang } a\,(lz^2 - z^3)\,dz}{\int 2\,p \text{ tang } a \times (lz - z^2)\,dz}$$

$$= \frac{\frac{lz^3}{3} - \frac{z^4}{4}}{\frac{lz^2}{2} - \frac{z^3}{3}} = \frac{\frac{1}{2}\,(4\,lz - 3\,z^2)}{3\,l - 2\,z}$$

If the integrals be taken between o and l, or if we consider the whole surface, $Z = \frac{1}{2}\,l$, or if we suppose z small in comparison of l, as is the case in large vats which are mostly frusta of cones, we may develope the fraction, which

becomes

$$\frac{\frac{1}{2}(4\,lz-3\,z^2)}{3\,l}\times\left\{1+\frac{2z}{3l}+\frac{4z^2}{9l^2}+\&c.\right\}$$

or

$$\tfrac{2}{3}\,z\left\{1-\frac{1}{18}\,\frac{z}{l}-\frac{1}{27}\,\frac{z^2}{l^2}\;\&c.\right\}$$

shewing the placc at which the strongest hooping must be placed: other examples will readily occur to the Student, and are so easily managed, that we conceive it unnecessary to go into more detail.

CHAP. XVII.

90. It has been stated that the pressure on a horizontal surface, is the weight of a column of the incumbent fluid, whose base is the surface, and height the depth to which it is immersed. Is the pressure on a given surface the same with different fluids? This is easily tried, let a quantity of water be poured into the Siphon AB, fig. 135, it will stand at the same level in each leg; pour into B, olive, oil which not being miscible with water is separated from it by a visible surface at O, as it is poured, the surface O sinks, and the other surface of the water rises in the other leg: draw a horizontal line HO, and as the water in the space HO of the tube would be at rest if the legs were empty above it, we may neglect it, and consider the surface O as pressed upwards by the column

of water HW, and downwards by one of oil OL, these are equal, and therefore the weight of any bulk of oil is as much less than that of the same bulk of water, as OL is greater than WH.

This is confirmed by weighing a bottle filled with water, and afterwards with oil, when the weights are found to differ in this ratio.

This fact is expressed by saying, that the Specific Gravity of oil is less than that of water, meaning by this term the proportional weights of equal bulks, which are sa called, as being peculiar to each different body, and the result of the above experiment is thus expressed, wheu two immiscible fluids have a common surface, their altitudes above it are reciprocally as their Specific Gravity. The theorem for the pressure on a surface, is therefore, $P = G \times Z \times S$, G being the weight of a unit of bulk, and Z the depth of the surface.

91. Specific Gravities are of such importance in every branch of experimental enquiry, that it is necessary to enter into some detail of the methods by which they are found. In the first place it is obvious that the numbers denoting them are not restricted to any magnitude, expressing merely proportionals; some one must therefore be assumed as an origin of the series which will determine that of the rest. All Philosophers have agreed in chusing water as the standard to which other bodies are referred, it is every where procurable in abundance, and when purified by distillation from the saline materials which it contains, or even in the state of rain water, it is at a given temperature identical in its properties. It is next to be determined what number shall represent *its* specific gravity; unity is the most obvious, but there is a particular motive which causes many to assume 1000; a cubic foot of rain water weighs 1000 avoirdupois ounces,

the Specific Gravity of any other substance will therefore on this scale express in ounces the weight of a cubic foot, and if its dimensions be given in feet, multiplying its bulk by its Specific Gravity we obtain its weight, a process highly useful to the Architect and Engineer.

Denoting the bulk of a body by the symbol M, its weight by W and its Specific Gravity by S, it is evident that SM $= W$, for S is the weight of the unit of bulk, and repeated as often as M contains units, it is the weight of the body. From this expression it follows that $S = \frac{W}{M}$, W being in ounces and M in feet. This expression is apparently not homogeneous, for it requires us to divide a weight by bulk, but we may suppose M to be the weight of a mass of water equal to the bulk of the body, and the process for finding Specific Gravities comes to this; find the weight of a body, find that of an equal bulk of water, their quotient is the Specific Gravity on the hypothesis of water $= 1$, or if it be 1000, the decimal point is to be moved three places to the right.

92. This process gives much accuracy to our operations, for we can weigh with much greater exactness than we can measure; and it only remains to shew how M is determined. In the case of fluids the most obvious method is the best; provide a bottle furnished with a stopper accurately fitted to it, for the sake of diminishing calculation, its capacity may be 1000 grains of rain water, counterpoise the empty bottle, and then fill it with the fluid to be examined; its increase of weight, is that of the contained fluid, and this obviously is its Specific Gravity, water being 1000. To perform the experiment accurately, the stopper should be tubular or have a notch cut along it

that the fluid may not be compressed on introducing it; or a stopper may be dispensed with by having a cylindric neck to the bottle with a mark at a certain height, up to which it is to be filled.

93. If the specific gravity of a fluid be taken at different times in this manner, it will scarcely ever be found the same in two successive trials. This discrepancy arises from the effects of the variation of temperature on the bulk of the glass bottle and its contents. If the bottle is adjusted to hold 1000 grs. of water, at 40° of Fahrenheit's thermometer, and the air is at 70, by this increase of heat the bottle is enlarged, so as to be capable of holding 1000.4 grs. of water as dense as it is at 40°, but water expands much more than glass; and if the bottle preserved its original capacity it would only hold 998 grains of water at 70. These two errors counteract each other in some degree, but it is necessary to use a correction, for which see the note (*a*).

94. The expansion of liquids by heat affords a ready means of estimating the temperature; for this purpose a bulb is blown on a glass tube of small diameter, into which any liquid is introduced mercury being that commonly employed, till the bulb and a portion of the tube are filled. If the fluid be heated it expands more than the bulb, and must therefore rise in the tube; but from the minute capacity of this, in comparison of the quantity of liquid in the bulb, a minute fraction of the latter occupies a considerable length of the former. But that it may perfectly satisfy the wants of the philosopher its indications must be definite and referred to a certain standard. Two determinate temperatures are sufficient to groduate the instrument, and these are attained by

plunging it in melting ice, marking the place of the liquid in the tube on a scale attached to it, and performing the same experiment with boiling water. The first of these points is invariable, and the other nearly so; and thus limits of the graduation are obtained, the interval between which is divided into any number of equal parts. In this country, the point at which ice melts is marked 32°, and that of boiling water 212, the interval being 180, and the 0 or Zero is below the most intense cold of this climate: this graduation is named after its inventor Fahrenheit. On the Continent the Centesimal thermometer is used, in which the freezing point is Zero, and the boiling point 100. To make the instrument complete the top of the stem must be closed, and as the air contained in the part of it which is not occupied by the liquid would be condensed by the expansion of the latter, before it is closed the bulb is heated till the liquid rises to the top of the tube; at that moment the point of the tube is melted and sealed, and on cooling the liquid sinks, leaving a vacuum above it.

95. The importance of the thermometer may apologize for this digression; and to return to our subject, the specific gravity of a solid may, like that of a liquid, be taken by the weighing bottle. Weigh it, then introduce it into the bottle, and fill this with water, its weight $W' = W + 1000 - M$; for the weight of water is $1000 - M$, the solid occupying the place of M grains of water. M therefore $= 1000 + W - W'$ and hence S can be found. This method, where the solid is in small fragments, is the best which can be practised, and in the case of fine powders, the only one

admissible; but it cannot be applied to large masses of matter, and we must determine M by other means.

96. We have seen that a fluid remains at rest if its surface be horizontal; in this state every part of it must be pressed upwards by a force equal to the action of gravity on it. This upward pressure must be equally exerted on any body immersed in the fluid as on that portion of the fluid which it displaces, and therefore we can ascertain the quantity displaced by weighing the solid in air and weighing it in water; the difference of these weights is the weight of a quantity of water equal in bulk to the solid, as is shewn by a pretty experiment. Let a cylinder of brass be turned so as exactly to fill a small bucket, let the cylinder and bucket be suspended from the scale of a balance and counterpoised; if then the cylinder be immersed in water, the other scale will preponderate; but on filling the bucket with water, the equilibrium is restored. This principle is the source of a multitude of facts which we cannot afford time to develope; among them we may however remark, that if the weight of the immersed solid exceed that of an equal bulk of water, it will be urged downwards by a force which is equal to its weight multiplied into the difference between its Specific Gravity and unity: if its Specific Gravity be the same as that of water, it will remain at rest in any part of the fluid, and if less it will rise to the surface and remain at rest when the part immersed is to the whole body as specific gravity of the body to that of water; for in that case $MS = ms$ but these products are equal, the first to the weight of the body, the latter to that of the quantity of water displaced by it; these conclusions are, as is obvious, not confined to water only, but apply to every fluid. (*b*)

97. By the assistance of the theorem just mentioned,

the Specific Gravity of a solid may easily be found ; suspend it from a balance by fine wire, or, still better, by a horse hair, which being nearly of the Specific Gravity of water, cannot affect the result: weigh it exactly, and placing under it a vessel of water, raise this till it is immersed, it will float on the surface till weights are put into its scale equal to the weight of the displaced fluid; this addition is therefore the Loss of Weight which it experiences when weighed in water, so called although the weight is not lost, but only transferred to the fluid ; the weight divided by the loss is therefore the specific gravity at that temperature, which must be determined by a thermometer immersed in the fluid at the moment of observation; or the result may be reduced to any fixed temperature by the principles delivered in note (*a*.)

98. If the body be lighter than water, as it will always float on it, we must have a cage of such a weight, that it may sink even when it contains the body in question, immerse the empty cage in water and counterpoise it, introduce the body into the scale from which it hangs and determine its weight in air, then introduce it into the cage, and the weight which must be put into the empty scale to restore the equilibrium is its loss. Bodies of this description owe their small Specific Gravity for the most part to the presence of air in their cavities or pores and therefore we may err in determining it, as some of this air always escapes on immersion. All woods are heavier than water, even Cork if long exposed under the air pump sinks in it, and in this way we can determine its real density ; but if we desire to obtain the minimum Specific Gravity of any of these substances, their pores must be closed by a coat of varnish.

99. Saline substances and many vegetable and animal products are soluble in water, so that this fluid cannot

be used, but many of them are not affected by alcohol, and none by mercury; this latter however can only be used in the weighing bottle, as nothing except gold or platina can sink in it. The equation $S = \frac{W}{M}$ shews that where the bulk is given $S \propto W$, therefore $S : s$ the SG of alcohol :: $W : M'$ the loss of weight in it, therefore $S = \frac{s\,W}{M'}$ or take the Specific Gravity, as if the fluid were water and multiply the result by the Specific Gravity of the fluid.

100. If a solid be weighed in different fluids, the losses are as the Specific Gravities, and thus we can determine the Specific Gravity of any fluid; for $1 : s :: m : m'$, and therefore, $S = \frac{m'}{m}$ or $m' \times \frac{1}{m}$, and $\frac{1}{m}$ may be inscribed on the solid which is generally a ball of glass. The use of a balance is made unnecessary, by having a series of glass bubbles, whose Specific Gravity increases in arithmetical progression; several of these are thrown into the fluid to be tried, and that which neither sinks nor floats is of its Specific Gravity *(c)*

This method is liable to some sources of inaccuracy from which the weighing bottle is free. The want of perfect fluidity, or the viscidity of the liquid impedes the accurate determination of the loss of weight: and the temperature of the solid produces an error if it differ from that of the water or other fluid used; not by its expansion, for that of solids is inappreciable in such experiments, but if it be hotter than the water, the particles of this in contact with it become heated, and therefore specifically lighter than the rest of the fluid; they therefore ascend and their place is taken by others, which in their turn ascend producing an upward current as long as the solid is hotter than the fluid; were it colder there would be a descending current, and in either case its influence would be a source of error in the determination of the loss. It is also said, that the Specific Gravity of a solid is diminished by reducing it to powder when it is tried by this method, and the effect is attributed to the adhesion of the liquid to the surface of the solid, which is supposed capable of augmenting its density at the surface of contact.

101. We have seen that the part of a floating body immersed is to its whole bulk :: Specific Gravity of the fluid on which it floats : its own Specific Gravity: it will therefore sink to different depths in different liquids, for calling the part immersed I, $I = \frac{M \times s}{S}$ but as $M s$ is constant, $I \propto \frac{1}{S}$ and therefore $S \propto \frac{1}{I}$ This principle is employed to determine the Specicc Gravity of fluids by means of the Hydrometer, which in its simplest form consists of a ball A, fig 136, to which is attached beneath a bulb B filled with mercury to balance it and keepit upright, and a slender cylindrical stem graduated according to a scale of equal parts. The mode of using it is obvious, for if made to float on any liquor the degree of its stem intersected by the surface of the fluid shews its Specific Gravity after the necessary corrections for temperature have been made.

We have said that the stem should be cylindrical, because it is the only shape of whose accuracy we can be certain, at least with common workmen, and for the same reason have divided it into equal parts; but this gives a series of values of I in Arithmetical Progression, and therefore the corresponding values of S are in Harmonical, Harmonicals being the reciprocals of Arithmeticals.

If then the stem be cylindrical it should be divided in Harmonical progression, or if the Arithmetical be preferred, the diameter of the stem must increase upwards for $S-S' \propto \frac{1}{I} - \frac{1}{I'}$ or as $\frac{I'-I}{II'}$ or when $I'-I$ is very small as $\frac{I'-I}{I}$, if therefore $S-S'$ is constant which is the case when the graduation is equal, $I'-I \propto I^2$ but $I'-I$ being the portion of the stem included between two successive divisions is nearly a cylinder, whose height is given and whose base is the thickness of the stem at that place, the thickness is therefore as I. An approximation to the proper shape is made by making the stem a frustum of a cone with its base uppermost, but it is perhaps better to use the cylindrical stem and equal division, ascertaining the Specifie Gravity by reference to a table. (*d*).

102. An instrument constructed on these principles cannot unite delicacy and extent of scale, and is much inferior as a philosophical instrument to that which we proceed to describe. In this the stem carries a cup to receive additional weights, by means of which it is sunk to the same depth in every experiment, and as $SI = s\,M$ or W, $S \propto W$, but $W = W'$ the weight of the instrument $+\ w$ the additional weight, and therefore by a suitable series of weights we can ascertain S. If the weight required to bring the instrument to the mark on its stem in any fluid be w, and that of the instrument when floating in the same manner on water be W, then $S = \frac{W \mp w}{I}$ but when $S = 1, w = 0$ therefore $I = W$ and $S = 1 \mp \frac{w}{W}$; the upper sign being used when the fluid is lighter than water.

This instrument may be used to weigh any body lighter than the additional weight necessary to sink it in water, for if this be 1000 grs. it is obvious that the weight is 1000—the weight added to the cup; and its Specific Gravity can be easily determined, for the weight which ballasts the Hydrometer is also in the form of a cup, and if the body be placed in this it is equally evident, that we obtain its weight in water. The Hydrometer requires a correction for the action of the air on that part which is not immersed, and also for the adhesion of the fluid, which is variable at different temperatures; *(e)* This force gives rise to the phænomena of capillary attraction, some of which are detailed in the note; but they are far too difficult to be discussed minutely in a work like this. *(f)*.

103. To complete the theory of the Hydrometer, we should give the conditions of stable equilibrium for bodies floating on a fluid, but here also we must content ourselves with very general notions referring those anxious for more extensive information to the treatises of Prony, Bouguer, and Atwood. In the first place, S must be less than *s*, and this may be effected even with the heaviest materials by making the body hollow, thus buoys and even ships are constructed of Iron. 2dly, the centres of Gravity of the immersed part, and of the whole must be in the same vertical, for the action of the fluid, on the part immersed passes vertically through its CG, and that it may counteract the weight of the whole body, it must pass through its CG also.

104. The equilibrium thus obtained may be either stable or unstable; supposing the body moved from the position of equilibrium, the action of the fluid may tend to bring it back or to remove it further from that state, and this takes place according as the point where

the result of the pressure of the fluid meets the line passing through the centres in the state of equilibrium, which is called the Metacentre, is situated above or below the centre of gravity of the body. An example of these two states is afforded by an elliptic cylinder, whose profile is ACB, fig 137, floating on a fluid of twice its density in which case the surface of the fluid FL (called the plane of flotation) passes through its centre. If either AB or CD be vertical the cylinder is obviously in equilibrio, but in any intermediate position, the action of the fluid tends to bring it to the position in which CD is vertical, for E is its CG, and that of the part immersed is in EF the line which bisects all parallels to FL, a vertical drawn through it must therefore meet AB below E, and therefore AB will ascend till it is horizontal; the equilibrium is therefore stable only when AB is horizontal. If the metacentre coincided with E, the equilibrium would be neutral as in the case of a sphere, for whatever its position may be, the two centres of gravity are always in the same vertical. The stability of ships depends on these principles, and much curious matter is connected with this branch of Hydrostatics which we are constrained to pass by for want of room. *(g)*.

NOTES TO CHAP. XVII.

(a) Let the capacity of the bottle be = *mno* and let *e* be the expansion of the unit of length for 1° of Fahren-

heit, then each of those factors being increased by a heat of $t°$ the capacity becomes $m + met \times n + net \times o + oet$ or $mno\,(1+et)^3$, subtract from this mno and the increase of capacity is obtained $= mno\,(3\,et + 2e^2t^2 + e^3t^3)$ but as et is always a very small fraction, the correction is sufficiently accurate if the capacity of the bottle be multiplied by three times the linear expansion of glass multiplied into the temperature, e for flint glass is about 0.0000045.

The expansion of solids is sensibly proportional to the temperature; that of fluids follows a continually increasing progression whose law is not known a priori, but it is possible to construct from observation a formula which shall represent the expansion within certain limits. The quantity e is still a function of t, and therefore may be represented by a series of the form $E + at + at^2 + ct^3$ &c. The coefficients $a\,b\,c$ &c. being constant quantities and all very small fractions. As the expansion is nothing when $t = 0$, $E = 0$, and the formula is sufficiently accurate, if we put $e = at + bt^2 + ct^3$; to determine the coefficients, a thermometer is filled with water and exposed along with a mercurial one to various temperatures, thus the expansion of it is known by correcting for that of glass; any three of these values of e give three equations from which $a\,b$ and c may be found. Biot gives formulas for the true and apparent expansions of water expressing t in degrees of Reaumur's thermometer, which begins at the freezing point and indicates boiling water by 80. See his Physique. The first of these altered to Fahrenheit's graduation is this,

$$e = -0.0000\,2439\,T + 0.0000020029\,T^2 - 0.00000000236\,T^3$$

where T is the temperature according to Fahrenheit

— 32. In ordinary cases the term affected with T^3 is obviously insignificant.

In this formula it is remarkable that the coefficient of T is negative, and as the sign of e depends on this while T is small, it is evident that for some degrees above the freezing point, water contracts instead of expanding; there must then be a temperature at which the density of water is a maximum: when this is the case $\frac{de}{dt} = 0$ or

$$0 = -0.00002439 + 0.0000020029 \times 2T$$
$$-0.00000000236 \times 3T^2$$

of which the root that gives e a minimum is T = 6. 15. the maximum density of water is therefore at 38°. 15 of F.

This determination agrees sufficiently well with observation, if we consider that a quantity near a maximum is little changed by a considerable alteration in its variable.

It may seem irrelevant to my subject, but I cannot refrain from remarking on this property which is found in water only of all the fluids with which we are acquainted, that it affords a striking proof of the wisdom displaced in the arrangement of our globe, for had water contracted down to its freezing point and below it, (for fluids can be cooled many degrees below it, the temperature suddenly rising at the instant of congelation) then the temperature of a lake or sea would have continually decreased, for as the surface cooled that portion becoming heavier than the rest would sink, and thus the whole mass would at last be cooled to the freezing point; after that it is possible that it might be frozen solid, and would probably never melt. But as water becomes of less Specific Gravity, by being cooled below 38 or 40, when the whole mass has cooled down to this temperature, any subsequent cooling can affect only its surface, and therefore that alone is congealed.

In the adjustment of the weighing bottle, it must be observed that when first weighed it is full of air: at a mean pressure and temperature the Specific Gravity of air is 0.0013, and therefore if the capacity of the bottle be 1000 grs. of water, as the water displaces the air, the apparent weight of the water should be 1000—1.3 or 998.7 grs.; it is needless to give a formula for the various circumstances of pressure and temperature. A correction might also be applied for the varying pressure of the atmosphere, which diminishes unequally the weight of the bottle and of the brass weights which counterpoise it. Suppose two bodies whose weights are W, and w, their Specific Gravities S and s to be balanced in a fluid whose Specific Gravity is Σ, the weight of each is diminished by the weight of an equal bulk of the fluid, and therefore $W - \frac{W}{S} \times \Sigma = w - \frac{w}{s} \times \Sigma$, or $W - w$, the difference of weight $= \frac{(W s - w S)}{S s} \Sigma$ or as $W = w$ very nearly $= W \frac{(S - s)}{S s} \times \Sigma$. If therefore the weights be equal at any value of Σ, they will be unequal at all others. This correction is however in the present instance scarcely appreciable.

(b) It may happen that a body of intermediate Specific Gravity floats between two immiscible fluids. Let its Specific Gravity be Σ, those of the fluids S and s, M and m the portions of the body immersed in them, then its weight $= (M + m)\,\Sigma = MS + ms$, the sum of the weights of the fluids displaced or $M\,(S - \Sigma) = m\,(\Sigma - s.)$

(c) It is a problem of frequent occurrence to determine the Specific Gravity of a mixture of known composition, let the weights of the components and

compound be W, w, υ their bulks M, m, μ, their Specific Gravities S, s, σ, we have the equations,

$MS + ms = (M+m)\,\sigma = \mu\sigma$, and $\frac{W}{S} + \frac{w}{s} = \frac{W+w}{\sigma} = \frac{\upsilon}{\sigma}$

Hence,

$$\sigma = \frac{MS + ms}{M + m} = \frac{(W+w)\,S\,s}{W\,s + w\,S}.$$

If equal weights be mixed, $\sigma = \frac{2\,S\,s}{S+s}$, if equal weights,

$$\sigma = \frac{S + s}{2}$$

being in the first case a Harmonic, in the second an Arithmetic mean between S and s.

These formulas are of no use in the case of mixtures of liquids, or of solids united by fusion, as alloys of metals; for in such compounds the equation $\mu = M + m$ does not hold, there being what is called a Penetration of dimensions. Thus if two measures of water be mixed with 1 of sulphuric acid, the compound instead of being of the bulk 3 will occupy only the bulk 2.8, at the same time giving out much heat; this alteration of dimensions takes place in every case of chemical union, and we may safely conclude that where it is not observed, the compound is merely a mechanical mixture.

(d). In the common Hydrometer $S = \frac{W}{I}$; if it be required that $S \propto x$ the part of the stem above the surface of the fluid, let a be the value of x, when $S = 1$, then $S = \frac{x}{a} = \frac{W}{I}$ and $\frac{dx}{a} = \frac{W dI}{I^2}$, but $I = M$ the whole bulk of the hydrometer minus the part of the stem which is not immersed, suppose the stem a solid of revolution, and this $= \int py^2 dx$

or $\frac{dx}{a} = \frac{Wpy^2 dx}{(M - \int py^2 dx)^2}$ $M - \int py^2 dx = \sqrt{apW}\, y$, taking the differentials

$$-py^2dx = \sqrt{apW}\, dy,$$

$$dx = \sqrt{\frac{aW}{p}} \times -\frac{dy}{y^2},$$

$$x+C = \sqrt{\frac{aW}{p}} \times \frac{1}{y}$$

which is of this form, $yx + dy = b^2$ the equation of the hyperbola referred to its asymptote.

(e) By note *(b)* when a body floats between two fluids with a part in each, $MS+ms = (M+m)\ \Sigma$, this is the case with the hydrometer; suppose that it sinks to the same degree in a fluid of specific gravity $S+S'$, independent of the air, then S' is the quantity to be deducted from the indication of the instrument to attain the true result; and $M(S+S') = M+m)\ \Sigma = MS+ms$, $MS' = ms$, $S' = \frac{m\ s}{M}$; now this instrument is chiefly used to ascertain the strength of spirituous liquors, suppose, as an extreme case, that it is immersed in alcohol specific gravity 0.8, then $m : M :: 1 : 4$, and $S' = \frac{0.0013}{4} = 0.0003$, making the result inaccurate in the fourth place of decimals: Nicholson's hydrometer is even less affected by this correction,

(f) It is not our intention to give a general theory of capillary phenomena, for this would lead us rather too far; it is sufficient to give a slight notice of their existence, and explain some of the simplest of the appearances. When a glass tube of small bore is dipped in water or alcohol, this fluid rises in it above its level, and to a height greater as the tube is narrower; what is the cause of this? obviously the reciprocal attraction of the glass and fluid. These forces being of the nature of cohesion act only at insensible distances, and are probably similar functions of the distance, differing only in

the coefficient. To investigate their nature, suppose HT a cylindrical tube (fig. 138.) plunged in the liquor HV, suppose a portion of the fluid solidified in continuation with TV, so as to form a similar tube; this imaginary tube will attract the fluid film V*v* *downwards* with a force equal to the attraction of the fluid towards itself, which call —A′ as it is opposite to the ascent of the fluid, but the glass tube attracts the same film V*v* upwards with a force A; that portion of it at F exerts an equal action, and the intermediate parts of the tube produce no effect, as each raises the fluid below it, and depresses that above it. The whole force is therefore 2A—A′ and as that is positive, null, or negative, the fluid will rise, keep its level, or be depressed. The forces A and A′ are in all probability as the surfaces exerting them, and therefore the forces of ascent, as $2\,pr\,(2a - a')$ a and a' denoting the coefficients which give the forces A and A′ when multiplied into the surface. This ascensional force is opposed by hydrostatic pressure of the elevated column FH, = to $S \times pr^2\left(h + \frac{r}{3}\right)$ h being the height and r the radius of the tube, for it is observed that the upper surface of the column is, if the tube be capillary, a concave hemisphere; the bulk of the hemisphere is $\frac{2}{3}\,pr^3$, and therefore that of the Meniscus contained between its surface and a tangent plane perpendicular to the tube is $\frac{1}{3}\,pr^3$, which must be added to pr^2h in order to obtain the total weight of the elevated column, hence

$$2 \times \left(\frac{2a - a'}{s}\right) = \frac{r}{3} + h \times r = V$$

putting

$$V = 2 \times \frac{(2\,a - a')}{S}$$

V is constant where the nature of the fluid and the tube is unchanged, and this formula is found to represent very precisely the elevations; where $\frac{r}{3}$ may be neglected in respect of h we have $h \propto \frac{1}{r}$ the common theorem.

If two parallel planes be dipped in a fluid it will rise between them, but the formula is not exactly the same; here the upper surface is curved only in one direction' and instead of a hemisphere it is a semicylinder; let their distance be r the half diameter of the tube, and their length l, then the attractive force is as $2\,l \times (a - a')$; the weight of the prismatic column raised is $S \times lrh$, and that of the fluid included between its summit and the cylindric surface is $Sl\left(\frac{r^2}{2} - \frac{pr^2}{8}\right)$ therefore

$$V = r\left(h + \frac{r}{2}\left(1 - \frac{p}{4}\right)\right)$$

here, as in the preceding instance, r is nearly $\propto \frac{1}{p}$, and the equation shews that the ascent of a fluid between planes is $\frac{1}{2}$ that in tubes whose diameter is equal to their interval. If the planes meet at one of their perpendicular edges, the distance between them is as the distance from the point of occurse, and the ascent being inversely as this distance, the surface of the elevated fluid is an hyperbola, of which the occurse is one asymptote, and the level of the fluid the other.

These attractions are the agents in many curious phenomena, which it is impossible for us to pursue; indeed any thing beyond the notice which we have given would involve us in the most complicated geometry: and we are compelled to omit the most curious part of the enquiry, namely, the effects of the curvature of the upper surface ; this differs with different fluids, depending ultimately on the angle of contact with the tube, which itself is a function of the forces A and A′: and the ascent varies with it, where it is concave the fluid ascends, where it is convex the fluid is depressed, as in the case of mercury. It may also be remarked that it follows from the minute analysis of the question, that the capillary attractions are incomparably greater than that of gravity, insomuch that the film of fluid, in contact with a solid body, may be so compressed as to exceed many times its ordinary density, a fact which has been already referred to in this chapter. See on this subject Dr. T. Young, Laplace Sup. Mec. Cel. and Count Rumford on the cohesion of the surface of fluids.

(*g*). The finding the positions of equilibrium of a floating body is reducible to this problem, to cut from a body a portion in a given ratio to the whole by a plane perpendicular to the line joining the CGs of the whole and the part cut off; for the part immersed is always in a given ratio to the whole, and the action of the fluid to support it may be reduced to a force passing through the CG of the immersed part, and at right angles to the surface of the flnid. This problem is not always very simple, we will give two of the simplest examples of it.

Let there be a beam, whose section is the Isosceles ABC 139, whose CG is at G, it will obviously be in equilibrio when BD is perpendicular to the fluid; to ascertain whether it have any other points, suppose it inclined till FL represent the surface of the fluid, the triangle FBL is of a given magnitude, and therefore FL is always a tangent to the hyperbola HCT, BT bisects FL, and therefore the CG of the triangle FBL is at

g and $Bg = \frac{2}{3}$ BT, draw the normal TV, and parallel to it g S, this is the direction of the pressure of the fluid; if s be at G the body is in equillibrium, if above it, it tends to diminish the inclination of the body, if below it, to augment it.

As BT : Bg :: BV : BS : 3 : 2 ∵ BS $= \frac{2}{3}$ BV. Draw the ordinate TI and BV = BI + IV = abscissa + subnornal = (see Anal. Geon. Art. 58)

$$x + \frac{b^2}{a^2} x,$$

therefore

$$BS = \frac{2}{3} \left\{ \frac{b^2 + a^2}{a^2} x. \right\}$$

When x = BC or a the expression is

$$\frac{\frac{2}{3} b^2 + a^2}{a}$$

or

$$\frac{\frac{2}{3} \quad a}{\cos^2 (\text{angle ABD}).}$$

and if a be less than

$$BD \times \cos^2 (\text{angle ABD})$$

the vertical position of equilibrium is unstable. The value of

$$BS \text{ or of } \frac{x}{\cos^2 m}$$

obviously increases with x, and this with the inclination so, that if s were originally below G, it will coincide with it when

$$x = BD \times \cos^2 m.$$

If then the vertical position be unstable, there are two others stable; if it be stable there are no others.

Let the section be rectangular with two angles immersed, its centre of gravity is in the perpendicular raised at the point of bisection of AD, fig. 140, and the plane of flotation must cut off a trapezium of a given magnitude ABL, therefore the plane of flotation must always pass through E; call ED h, BD b, and tang angle F. m, bisect LD, draw MF, it bisects h at I, and the central gravity of the trapezium is in it at g.

$$BS = BI + IH + HS$$

which we must calculate. By the process used in the note *(d)* of Chap. 3, we find the distance of the centre of gravity of a trapezium from its base

$$= \frac{m\left(FD \times \frac{4b^2}{2} - \frac{8}{3} b^3\right)}{2\,bh}$$

and FD $= b \pm \frac{h}{m}$ hence

$$gN = \frac{m\left(2b^3 + \frac{2b^2h}{m} - \frac{8}{3} b^3\right)}{2bh}$$

$$= \frac{6\,b^2\,h - 2b^3\,m}{6b\,h}$$

$$= \frac{3\,bh - mb^2}{3\,h}$$

Hence we obtain

$$g\,H = b - g\,N$$

or

$$= \frac{3\,bh - 3\,bh + mb^2}{3\,h}$$

$$= \frac{mb^2}{3h}$$

but

$$HS = \frac{g\,H}{m} = \frac{b^2}{3\,h}$$

and

$$IH = \frac{m^2\,b^2}{3\,h};\ IB = \tfrac{1}{2}\,h;$$

hence

$$BS \quad \frac{\frac{1}{2} h^2 + \frac{1}{3} m^2 b^2 + \frac{1}{3} b^2}{h}$$

$$= \frac{1}{2} h + \frac{b^2}{3 h \times \cos^2 F}$$

which must be greater than BG in the case of stability. If angle F be evanescent, its cosine = 1 and the height of the Metacentre is

$$\frac{1}{2} h + \frac{b^2}{3h}$$

If we take the differential of this, and put it = o, we obtain for its minimum height

$$h = b \times \sqrt{\frac{2}{3}} ;$$

and if it be required to find the depth of immersion at which the equilibrium is neutral, calling BG, H, we have

$$H = \frac{h}{2} + \frac{b^2}{3h}$$

which gives

$$h^2 - 2 Hh = - \frac{2 b^2}{3},$$

$$h = H \pm \sqrt{H^2 - \frac{2 b^2}{3}}$$

which is impossible if G^2 is less than $\frac{2}{3} b^2$, shewing that a beam whose breadth is to its depth in a greater ratio than 5 : 4 can have no neutral equilibrium; and it is also obvious that there are two values of h, at which the stability is equal, one greater and the other less than G. Those who wish for more information on this subject may consult Bossut's Hydroynamics.

CHAPTER XVIII.

Of the Mechanical properties of Gases.

105. The class of fluids which now claims our consideration is widely different from liquids in its physical properties; water is many hundred times denser than the heaviest of them, they are nearly all invisible, all perfectly elastic, all expansible by heat; and what could scarcely have been expected, all equally dilated by a given augmentation of temperature. They differ however in their elasticity, at least in appearance, for some of them, if their particles be approximated beyond a certain limit, condense into fluids or solids, evolving much heat; and this, whether the condensation is occasioned by reducing their temperature, or by mechanical pressure; while other elastic fluids continue unchanged by any cold which we can produce, or any pressure we can apply. From this difference arises a distinction: those which preserve their elasticity are named Gases, the others are termed Vapours, and we will treat of them in this order.

106. As the mechanical properties of Gases are the same, we may confine ourselves to that one which is of most importance to mankind the Air in which we breathe, as our conclusions respecting it may be extended if necessary to others. This fluid is not elementary, being composed of several ingredients, differing most remarkably in their chemical properties, four-fifths of its gaseous part is a gas irrespirable and extinguishing flame; the remaining fifth, on the other hand, if

breathed powerfully excites the vital energies, and if applied to a burning body augments combustion to a degree far beyond its usual intensity. With these are mixed a variable proportion of the vapour of water, and, at times, certain subtle agents arising from the putrefaction of organized matter, which are detected only by their effects on animals, determining the peculiar insalubrity of certain seasons and situations. But though thus compound its mechanical habitudes are the same as if it were a Homogeneous gas.

107. The invisibility of air may at first seem to make the investigation of its properties diffieult; but this has been obviated by a contrivance afterwards to be described, and we can measure air and pour it from one vessel to another with the utmost facility. That it possesses the property of filling space is shewn by a very simple experiment; in the cistern AB (fig. 141,) immerse a bell glass provided with a stop-cock at its summit, and the water does not rise within the jar to a greater height than CE, but the surface of the water C is forced upwards by the pressure of a column = CD, and there must be a body in the jar which resists this: this can be nothing but air, and as it is elastic, it is compressed by the column of water; and hence the water rises a little in the jar. Let a bladder, previously moistened to make it flexible, be attached to the stop-cock, and let it be turned; the water immediately rises in the jar to the external level, and the bladder is distended by the air which had previously occupied the jar; if pressure be made on it, the air is forced hack, and the water in the jar descends; and if the jar be raised up in the cisrern, the water in it keeps its level, and the bladder collapses, the air returning into the jar.

108. If the jar be immersed down to the stop cock and the air permitted to escape, closing the stop cock, and raising it again, we find that the water does not sink as it would if the stop cock were open, but remains suspended. See fig. 142. This cannot arise from an adhesion of the water to the glass; such an adhesion does indeed exist, but cannot sustain a column more than $\frac{1}{10}$ of an inch high, whereas DX may be many feet, and the sustaining force is obviously destroyed by admitting air at the top of the jar. The appearance is completely explained by supposing that air has weight; and as the earth is encompassed by an ocean of it, this produces a pressure on the bodies which support it according to the laws of other fluids, which have been explained in the two preceding chapters. Let the pressure of the Atmosphere (by which term is understood that total mass of air attached to our globe) on the unit of surface be P, then the surface CE of the fluid, supposing the plane AB continued through the jar, is urged upwards by a force = surface CE $\times$ P while on the other hand it is urged downwards by the pressure of the elevated water, equal to surface CE$\times$ED: this latter force is in general less than the other, and therefore the water is supported. If the preceding sections have been well understood, this is easily apprehended, and experiment shews that the reasoning is correct; for if the cistern and jar in this state be inclosed in a vessel connected with an air-pump, an instrument which shall be described hereafter, at first no change is produced, for the sides of the vessel maintain on the included air, the pressure which the atmosphere had previously exerted; and even on exhausting, the water does not immediately sink, until the elasticity of the air

in the vessel is diminished beyond a certain degree. At length the water sinks, and if the air pump be good it descends almost to the external level, and on readmitting the air, it ascends to its original station.

109. This simple experiment is fraught with remarkable results, and was sufficient when rightly understood to change the whole state of physical science: a modification of it is familiar to every one; if a tube open at both ends be dipped in a fluid, and the air partially exhausted, the pressure on its surface in the tube being diminished, the fluid rises in it. This is vulgarly ascribed to a power of suction, but such a power is not very clear, and they who followed the guidance of Aristotle in Physics were in no small degree embarrassed about it; their conclusion was that nature abhorred a vacuum, which would be inevitably produced if the water did not rise. This passed current for some time, but new observations were made, and Galileo, who had learned prudence under the discipline of the Inquisition taught that nature's abhorrence of a Vacuum was limited, not exceeding 34 feet of water; for in fact if the jar in our experiment were more than 34 feet high, when it is raised above the water in the cistern, the water would be supported in it till it exceeded that height, at which it would stand, leaving above it a Vacuum, or at least a space void of air.

110. From this fact we can determine the value of P in an approximate manner, and can devise a much more accurate mode of observing it: a jar 34 feet high is rather unmanageable, but if we use a fluid heavier than water it is evident that it will be sustained at a height as much less than 34 feet, as its Specific Gravity is greater than that of water. Mercury is such a fluid, and the experiment made with it is called, from him who first made it, the

Torricellian experiment. To perform it, choose a glass tube of about half an inch bore, seal it hermetically at one end, or in other words close it by melting the glass; fill it with mercury previously heated and close it with the finger, invert the tube and plunge its orifice, still closed by the finger, in a vessel of mercury; on withdrawing the finger, the metal sinks in the tube and stands at the average height of 29.5 inches, leaving above it a space void of air, which is spoken of by the name of the Torricellian vacuum. The height of the mercury is not always the same, but varies within certain limits, from causes at present unknown; it sometimes rises in this country as high as 30.5 and very rarely descends to 28, and from the connection of these changes with meteorological phænomena, the Torricellian tube is vulgarly called the weather-glass; its scientific appellation, *Barometer*, shews the real nature of its indications.

111. That the observations made with this instrument may be valuable to the Philosopher, several precautions must be used in its construction. A scale of inches and tenths is always annexed to the tube, that the altitude may be observed, and it is evident that the division marked 0 should be level with the mercury in the cistern in which the tube is plunged, but this is often otherwise, for supposing it accurate at one time, yet when the state of the atmosphere changes, and the mercury e. g. sinks in the tube, it must rise in the cistern above the Zero or 0 of the scale, and in that case the observer registers an altitude greater than the true by the rise in the cistern. This is obviated by having the cistern moveable, so that its surface may be brought to the Zero. In the second place the tube should be not less than half an inch internal diameter, for in narrower

tubes, an action of the nature of those called Capillary takes place, by which the mercury is held below the height at which it would be in equilibrio, and this error is inversely as the diameter of the tube. Thirdly, the tube as well as the mercury must be freed from adhering air and moisture, which as will be seen immediately would materially interfere with the accuracy of the instrument; this can only be effected by boiling the metal in the tube.

Many contrivances for enlarging the scale of the Barometer are described by various authors, as the Rectangular, the Diagonal, the Wheel Barometer the invention of Dr. Hooke, and the only one which still keeps its ground, and the Conical, which for its ingenuity is described in the note (*a*): but the friction in them is so great that it more than compensates the encreased extent of the scale, and as 0.01 or even 0.001 of an inch is perfectly visible on the vernier of the common barometer, this seems fully adequate to meteorological purposes. One of the best modes of extending its scale is given in Nicholson's Journal by Dr. Wilson.

112. For certain purposes the barometer must be made portable, a quality which it is far from possessing in its common construction, as independent of the hazard of introducing air into it, the tube is often fractured by the percussion of the mercury against its top. It is rendered secure in two ways: the bottom of the cistern is in some made of leather, which is pressed upwards till the mercury is forced to the top of the tube, when the instrument may be inverted and carried in that position: when an observation is to be made, the instrument is hung on Jimmals, which permit it to assume a perpendicular position; these are two rings, the instrument rests on the inner, this is supported on the

outer by pivots at the extremities of its diameter, and the outer rests on a support by other pivots at right angles to the former, see fig 143, for a section of them. When the barometer is thus suspended, the bottom of the cistern is lowered till the surface of the mercury in it is at a given height, and the instrument is adjusted. A much simpler instrument was invented by Dr. Hamilton, and described in the Irish Transactions; the cistern is a cylinder of Ivory or box, closed at top by a cork through which the tube passes, so that its orifice is at the centre of the cylinder, and it is filled $\frac{2}{3}$ with mercury fig. 144. It is obvious that we can invert such an instrument without the orifice ever rising above the surface, and that it is therefore portable; but it could scarcely have been expected that the pressure of the atmosphere, or even its minute variations could act so rapidly through the pores of cork as it is found to do. Sir H. Englefield afterwards discovered that the air acted even through the pores of box, and therefore omitted the cork. As the mercury in the cistern of these instruments rises and falls, a correction to the observed altitude is necessary.

113. From considering the weight of air we proceed to its elasticity, which quality it possesses in the highest degree, being capable of indefinite compression and expansion; it is said that some gases have been compressed to twice the density of water, and in the receiver of an excellent air pump it expanels into 3000 times its volume under the pressure of the atmosphere. From its elasticity, we may always infer the existence of a force which confines it and prevents its particles from receding; this must be in equilibrium with the elasticity, and if it be augmented the elasticity must increase equally. Under such an augmentation of com-

pression, the volume of the air diminishes, and it becomes an important object of enquiry to determine the relation subsisting between the density and the compressing force. For this purpose, a portion of mercury is introduced into the bend of the recurved tube AB fig 145, whose shorter branch contains a portion of air, freed from aqueous vapour by means to be described when we treat of the air pump: the short branch must be provided with a graduation expressing equal parts of its capacity. Observe the bulk of the included air, when the surfaces of the mercury are on a level; in this state the air sustains the pressure of the atmosphere, or that of a column of mercury equal to the height of the Barometer at the time of experiment, which for the future we will call the standard altitude, and denote by the symbol A. Call its volume M, then if mercury be poured into the branch A it will be found that it also rises in B though less remarkably. Now the confined air is pressed by A, and also by a column equal to the difference of level between C and D, or calling this latter a, the pressure on it is $A+a$, and if we observe its bulk M′ by means of the graduation, we find that $M' \times (A+a) = M\,A$ or as this latter is constant that $M' \propto \frac{1}{A+a}$. The density of a given quantity of air is inversely as the bulk which it occupies, and therefore is directly as $A+a$. This law is therefore when expressed in words, that the density of air at a given temperature is to the force compressing it in a constant ratio.

For pressures less than A the same law is proved by an apparatus even more simple, the tube AB fig 146, contains a portion of mercury CD, which unless the tube be too wide, cannot fall out even when the open end B is downwards, as the air and mercury are unable to change places; if this be laid in a horizontal position, the air AC supports no pressure except A; calling DC a, in the position shewn in the figure, the pressure is $A-a$ as the weight of mercury acts against the pressure of the atmosphere; on inverting the tube, the two pressures act in the same direction and it is found that MA, M′ ($A-a$) and M″ ($A+a$) are all equal; more mercury may then be introduced and we thus have a new value of a.

The same may be otherwise shewn: in filling the Torricellian tube with mercury, leave a part unfilled, we thus introduce on inverting the tube, a bulk M of air under the pressure A above the mercury, which by its elasticity depresses it below the standard altitude; that depression must therefore measure the compressing force, as is evident if we consider that the column which is supported acts in opposition to the atmosphere pressure and therefore the included air is pressed only by their difference or A—a, If the tube be graduated, we can observe M′ and as in the preceding experiments $M' \times (A-a) = MA$. Hence we may infer that in every instance the density is as the compressing force, and from this law it has been shewn by Newton, that the particles of air must repel each other with forces inversely as the distance between them, and that the repulsion is in liquids probably inversely as some very high negative power of it. *(b)*.

114. Since the density of air is as its compressing force, D : C : : 1 : H, (calling the density of the air at its ordinary state 1 and the pressure of the atmosphere H) therefore $\frac{C}{D} = H$, H is therefore the Ratio of D to C, or it is the force in equilibrio with the elasticity of air

at the density represented by unity. This force we have hitherto measured by A the weight of the column of mercury sustained by it, but it is desirable to have another measure of it derived from the air itself; in fact, if the atmosphere were of uniform density, its height above any place would measure the pressure exerted by it there, and though this is not the case, yet we may compute its height if it were uniform and the number thus obtained is H. *(c)*.

115. Since a Homogeneous Atmosphere of the height H is in equilibrio with a column of mercury A, $H \times 1 = A \times S'$, S′ being the Specific Gravity of mercury, taken on the supposition of air unity, $= \frac{S}{s}$; when A is 29.9 the Specific Gravity of air is 0.001299, and that of mercury 13.588, the temperature being at the freezing point; therefore H = 26000 feet nearly.

It has been said that H is the ratio of D to C, and it remains the same however they vary; this is evident from the formula $H = \frac{A \times S}{s}$ for if A becomes A' s becomes s', and the Specific Gravities being as the densities, and these as the compressing forces, $A : A' :: s : s'$, and $\frac{A}{s} = \frac{A'}{s'}$ or it is always of the same value, and therefore H is an invariable quantity.

116. Though H does not depend on the density of the air, it does on its temperature, for heat increases its elasticity and therefore augments the ratio of D to C. To ascertain the amount of this change let the tube described in sect 113 be placed horizontally in a vessel filled with water which may be heated to any required degree between 32 and 212: thus the air contained in AC is heated and expands, and its expansion E is observed as

well as the elevation of temperature. The air is of the bulk M at the beginning of the experiment and is in equilibrio with the pressure of the atmosphere; after it is heated its volume becomes M + E, and the compressing force is the same as before, therefore since the bulks are inversely as the densities, and $H : H' :: \frac{C}{D} : \frac{C}{D}$,

$$H : H' :: C \times M : C \times (M + E) :: 1 : 1 + \frac{E}{M}$$

therefore $H' = H \times \left(1 + \frac{E}{M}\right)$ This formula shews that H is increased by heat in the same proportion as M; and as it is found that $\frac{E}{M}$ is proportional to the increase of temperature, if we set out from 32°, and call the expansion produced by one degree e, and the excess of temperature above 32 t, $\frac{E}{M} = et$ and $H' = H \times (1 + et)$. The coefficient e is $\frac{1}{480}$ nearly or as a decimal, 0,0021, where the centesimal Thermometer is used it is $\frac{1}{250}$ nearly.

117. The density of the air depending on the pressure must diminish as we ascend in the atmosphere, for the density at any height is as the force there, that is as the weight of the superincumbent portion of the Atmosphere. It is not possible to express this pressure by any very simple function of the height, for the density varies through the incumbent column, and we must have recourse to aid a little beyond the limits of common geometry.

Let fig. 147 be the profile of a column of air resting on the earth, and supposed to extend upwards to the boundary of the atmosphere: conceive it divided by parallel planes into strata of evanescent thickness, as at CD: as CE the thickness of one of them is very small, the density may be supposed not to vary from C to E. Let AB be assumed to represent the density at A, and take RD : CD or AB:: density at C : density at A; do the same for every other stratum, and the line passing through all these points is a curve of such a nature, that the density at any height BI is denoted by the perpendicular to it SI. From the evanescent magnitude of CE the figure RF is quam proxime a rectangle, and therefore is to CF:: RD : CD; but if the air which fills the space CF were condensed into the space RF, its density would be increased in the reciprocal proportion of the spaces, or as RD : CD:: density at C : density at A, and therefore the air when condensed into RF would ba of the same density as at A. In the same way it may be shown, that if the other strata were compressed horizontally till they were included between the curve and its axis, that the curvelinear column thus formed would be throughout of uniform density; and as it is evident that the weight of the air diffused through any space CI is equal to that of the condensed column RI, which is proportional to the area RI, it follows that the pressure on any section CD is represented by the area above it, supposing the curve indefinitely extended. This will no doubt appear difficult to those who are not familiar with the Infinitesimal Calculus, but an attentive reader will easily comprehend it, and it is worth some labour; but to proceed, since the compressing force is as the area, and the density as the ordinate RD, it is evident that the Area is as the ordinate, which dedetermines the nature of the curve to be that which is named the Logarithmic.

118. The nature of this curve is, that if in its axis AI there be taken segments BP BR, &c. in arithmetical progression, the corresponding ordinates or perpendiculars BA, PV and RD are in geometrical progression. From this follow two others: since equidistant ordinates are in geometrical progression, their differences must be in like manner continually proportional, and in the same ratio; and it is easily inferred, that if PO be equal to CE the thickness of a stratum, VX : VP in a constant ratio. The chord VN coincides with the evanescent arch and the line VT is ultimately the tangent of the curve, and VX : VP :: XN or PO : PT; VX : VP is a constant ratio, therefore PO : PT and as PO is always supposed of the same magnitude, PT is the same in every part of the curve. Snch a line is called a subtangent. The space VO is quam prox. = VP × PO, the space RF = RDDF= × RD × PO, and therefore the sum of all the elementary spaces, of which the Area above PV is made up = PO × snm of all the equidistant ordinates. Their sum may be found by the note ou the 11th Prop. of the 6th Book of Euclid; for they are in a decreasing geometrical series, and their sum is therefore a 3d proportional to

$$\text{VX and VP; VX} = \frac{VP \times PO}{PT}$$

therefore the sum of ordinates

$$= \frac{VP^2 \times PT}{VP \times PO}$$

and sum of ordinates × PO = VP × PT. The *indefinite* Area above an ordinate is therefore eqnal to the rectangle under it and the subtangent.

110. If a Logarithmic were constructed, in which the distance between ordinates in the proportion of 10 : 1 is = 1, and the subtangent is 0.43429, &c. it is possible, if we know any ordinate in numbers, to compute its distance from the point B, which is said to *Measure its*

ratio to AB; thus if BP were = 2, PV would be $\frac{1}{100}$ of AB, for bisecting BP, this would be a mean proportional between AB and PV, but as its distance from B = 1 it is $\frac{1}{10}$ AB, therefore PV = $\frac{1}{100}$ AB. The natural numbers from 1 to 10,000 have been assumed as ordinates, and the corresponding distances from B have been computed; tbey are called Logarithms, and are of the utmost use in arithmetical operations, for in the curve if BP = DI, AB : PV :: RD : SI, therefore AB×SI = RD×PV. AB is in the case of numbers numbers always = 1, and therefore SI = RD × PV, but BI the distance of SI from B = ID+DB = BP+BD, or if we take a number whose Logarithm is the sum of the Logarithms of two otber numbers, that number is equal to their product.

110. This may appear digression, but it will, I hope, be admitted, that the importance of Logarithms merits some notice, more particularly as some idea of their nature is indispensible in what follows, and we may now proceed in our investigation of the density of the atmosphere. From the nature of the Logarithmic, it is evident, that if the heights above the eartb be taken in Arithmetical, the densities are in Geometrical progression, or that the heights are as the logarithms of the densities, considering the density at the surface = unity. The height may therefore be obtained by multiplying the Logarithm of the density into a given quantity, and this co-efficient is easily found: for since the ordinate of the Atmospheric curve is the density, and it multiplied into the subtangent gives the area, or compressing force, and since C = H×D, it follows that the subtangent of the atmospheric curve is the height of the atmosphere supposed homogeneous. If then the Common

Logarithms of the densities be multiplied by H, and divided by their own subtangent, we obtain others which are the heights corresponding to the densities: for in different curves the intercepts between two ordinates in a given ratio are as the subtangents. *(d)*

120. The variation of density is obviously proportioned to the fall of the mercury in the barometer, and hence we have a ready method of measuring the difference of elevation of any two places to which a barometer can be transported, for it must be equal to the difference of the Logarithms of the altitudes of the barometer at the places multiplied by a coefficient which is in this climate 60148, in feet or nearly 10000 in fathom. This conclusion is however true, only where the temperature at both stations is near the freezing point, if it be different, the coefficient whose numerator is H, changes, H becoming $H \times (1 + et')$. We may assume the temperature $t' =$ to the arithmetical mean of the observed temperatures T and t, and as H becomes on this hypothesis

$$H\left(1 + e \times \frac{(T+t)}{2}\right)$$

we obtain the difference of elevation in latitude 53

$$= 60076 \left\{1 + \frac{T+t}{2 \times 480}\right\} \left(\log(A) - \log(a)\right) \quad (e)$$

With good instruments this method of levelling gives a degree of precision which can scarcely be exceeded by Trigonometrical observations; and its readiness, together with the short time which it requires, have enabled philosophic travellers to add much to our stock of geographical knowledge. For this reason we have dwelt at some length on the developement of those principles on which it rests, and hope that it has been sufficiently explained to make it intelligible to all but those of the meanest capacity.

NOTES.

(a). The Conical Barometer is a tube slightly conical, sealed at its narrowest end, and filled with mercury; on inverting it, as the tube is of small diameter, the metal does not fall out, though its orifice is not plunged in mercury, but it is suspended at a certain height by the pressure of the atmosphere. Suppose this to vary, and become less, the column must descend; but in descending it comes to a wider part of the tube, and as its bulk is given, it must become shorter, at last it becomes equal to the pressure, and is again supported, but it has descended through a much greater space than the mere change of *A*. Let a be the tangent of the half angle of the cone, M the quantity of mercury, and x the distance of the upper surface from the vertex of the cone, then

$$M = \tfrac{1}{3} p \left(\overline{A+x}\,^3 - x^3\right) \times a^2$$

by the ordinary value of a conical frustum, hence,

$$\frac{3\,M}{p\,a^2} = A^3 + 3\,A^2\,x + 3\,A\,x^2$$

or taking the differentials as we are considering only minute variations,

$$0 = (3\,A^2 + 6Ax)\,dx + (3\,A^2 + 6Ax + 3\,x^2)\,dA$$

or

$$-dx = dA \times \frac{\overline{A+x}^2}{A^2+2Ax} = dA \times \frac{\overline{A+x}^2}{2Ax \times \left(1+\frac{A}{2x}\right)}$$

As $\frac{A}{2x}$ is small, developing the denominator by the Binomial theorem and rejecting all terms affected with negative powers of x we obtain the motion of the upper surface or

$$-dx = dA\left(\frac{3A+2x}{4A}\right).$$

(b) Let the repulsion of two particles vary as $\frac{1}{d^n}$ d being the distance between them; M the volume of any bulk of air, is the product of three linear factors and is therefore $= d^3 \times V$, V being the product of the numbers of times that d is contained in those factors: when a given quantity of air occupies different bulks, $D \propto \frac{1}{M}$ and therefore in this case $\propto \frac{1}{d^3}$, V being constant while the quantity of air remains the same, $D^{\frac{1}{3}} \propto \frac{1}{d}$ and $D^{\frac{n}{3}} \propto \frac{1}{d^n}$ or as the repulsive force of two particles. The pressure on any surface $\propto$ numbers of particles pressing it and $\propto$ the force of each, but the surface $= d^2 \times$ number of particles, if the surface be given the number of particles $\propto \frac{1}{d^2} \propto D^{\frac{2}{3}}$ and therefore the pressure on it is as $D^{\frac{n+2}{3}}$ but this pressure being the elasticity of the air is obviously equal to the compressing force. If C be as D, $\frac{n+2}{3} = 1$ and therefore

$n = 1$ or the repulsive is as $\frac{1}{d}$.

(c) The quantity H is the height of a column of air capable of compressing a given bulk of it into half its volume; it is also the Modulus of Atmospheric Logarithms; from analogy, a similar symbol is used to represent the elasticity of other substances and is called the Modulus of Elasticity. It is the height of a column of the substance whose weight would compress a portion of it into half its bulk, supposing that its elasticity were in every state as its density. The modulus is to the length of the body, if it be a solid as any experimental pressure is to the condensation produced by it; bnt it is more easily deduced from the theory of sonorous bodies.

(d) From the equation $HD = C$ we derive $H \times d D = d C$ and $\frac{dD}{D} = \frac{dC}{C}$, but dC is obviously the weight of a column of air whose height is the differential of the distance from the earths centre x, call the force of gravity at that distance g,

$dC = -gD\,dx$; and $\frac{dC}{C} = -\frac{gD\,dx}{HD} = -\frac{gdx}{H}$

and supposing gravity to be as the n^{th} power of the distance from the centre, $g = \frac{G \times x^n}{R^n}$ hence,

$$-H\frac{dD}{D} = \frac{G}{R^n} \times x^n dx$$

whose integral is

$$-H \times h.\log(D) = \frac{G}{R^n} \times \frac{x^{n+1}}{n+1} + C$$

and correcting on the supposition that $D = 1$ when $x = R$ we have

$$-\,H \times h.\log(D) = \frac{G}{R^n} \times \frac{x^{n+1} \quad R^{n+1}}{n+1}.$$

If the Hyperbolic Logarithms of the tables be multiplied by H, we obtain a table of Atmospherical Logarithms which may be used for finding the densities; or as the common, or Briggs's Logarithms are equal to the Hyperbolic Logarithms of the same number × 0.43429 &c. we may put $N = \frac{H}{0.43428\ \&c.}$ and write the equation

$$-\,N \times L(D) = \frac{G}{R_n} \times \frac{x^{n+1}\,R^{n+1}}{n+1}$$

denoting by L the tabular logarithm.

In nature $n = -2$ and the second number of the equation is =

$$GR^2 \times \frac{1}{x} - \frac{1}{R} = GR \times \frac{x - R}{x} \quad (1.)$$

If we call $h = x - R$ the elevation above the surface, and G unity, it becomes $R \times \frac{h}{R+h}$ and developing the denominator, it is $\frac{Rh}{R} \times 1 - \frac{h}{R} + \&c.$ or $h - \frac{h^2}{R} \times$ &c., shewing that the logarithm of the density is nearly as the elevation, the error being $\frac{1}{4}$ of a yard in the first mile.

If the densities be taken in geometrical progression the distances from the centre are in harmonic: for in this case, log (D) — log (D′) is constant, and therefore so is $\overline{1 - \frac{R}{x}} - \overline{1 - \frac{R}{x'}}$ or $\frac{1}{x'} - \frac{1}{x}$ The reciprocals of the values of x are therefore in Arithmetic progression.

If the force be as the distance, $n = 1$ and the general equation becomes

$$-N \times L(D) = \frac{G}{R} \times \frac{x^2 - R^2}{2} \quad (2.)$$

This law of force obtains below the surface of the earth and we may enquire what density air would have at the earth's centre, if a column of it reached there; put $x = 0$ and $L(D) = \frac{R}{2N} = 180$, so that the number expressing its density with respect to common air would consist of 180 places of figures, that of Platina the heaviest solid being under 5. Even water though so far less compressible than air, would if a column of it reached to the centre, be more than a million of times denser than it is at present, for its modulus of elasticity is about 28 times H, and therefore L (D) = 6 . 5 nearly; and it is easily computed that at the depth of 0.165 of the earth's radius it would be denser than lead. This may perhaps account for the high Specific Gravity of the earth.

Lastly, if we suppose gravity constant, $n = 0$ and—$N \times L(D) = G \times (x - R)$ and this is sufficient for practice as we have already shewn the amount of its error. (3).

(*d*). If the temperature were constant, the formula (3) would suffice for barometrical measurement, but as this is not the case we must introduce its variation into our formula. It has been shown by Laplace that the hypothesis of a decrease of temperature in arithmetical progression, while the height increases in arithmetical progression, represents observations with sufficient accuracy. If then we observe the temperature at any height, and ascend through an elevation z, the temperature is lower, and by a certain submultiple of z; hence $dt = ndz$. If now we resume the theorem—$\frac{dD}{D} = \frac{gdx}{H}$ and put g constant, and dz for dx, it becomes

$$\frac{-dD}{D} = \frac{G}{H} \times dz,$$

Substitute for dz, $\frac{-dt}{n}$ and for H, $H \times (1+et)$ and we obtain

$$\frac{-dD}{D} = \frac{-G}{nH} \times \frac{dt}{1+et}$$

and expanding

$$\frac{1}{1+et}$$

the second member

$$= \frac{-G}{nH} \times dt \left\{1-et+e^2 t^2 \ \&c.\right\}$$

If now we integrate the equation $-dt = ndz$ we find $T-t = nz$, T being the temperature at the lower station; and integrating the second equation we have

$$-h \log (D) = \frac{-G}{nH}\left\{t - \frac{et}{2}^2 + \&c.\right\} + C$$

in which we may neglect all terms affected with powers of e above the first. We may obtain the correction by putting D, $= \Delta$ when $t = T$ or $z = 0$ and obtain for the complete integral

$$h. \log(\Delta) - h. \log(D) = \frac{G}{nH}\left\{T-t-e \times \frac{T^2-t^2}{2}\right\}$$

putting for n its value $\frac{T-t}{z}$ this becomes

$$h \log\left(\frac{\Delta}{D}\right) = \frac{Gz}{H}\left\{1 - e + \frac{T \times t}{2}\right\}$$

$$h \log\left(\frac{\Delta}{D}\right) \frac{\times H}{1-e \times \frac{T+t}{2}} = G z$$

and developing

$$N \times L\left(\frac{\Delta}{D}\right) \times \left\{1+e \times \frac{T+t}{2}\right\} = Gz.$$

This formula is not yet sufficiently precise, for it is necessary to allow for the effect of the difference of temperature on the Specific Gravity of the mercury itself, this becoming more dense by the cold; its contraction is uniform, and, denoting its expansion and temperatures at the two stations by the Greek letters, a the height observed at the upper : $\acute{a}$ the height if it were at the temperature Θ:: 1: $1+\epsilon(\Theta-\theta)$, $a' = a \times (1+\epsilon(\Theta-\theta)$ the formula becomes then, putting A and a' for Δ and D

$$N \times L\left(\frac{A}{a\,(1+\epsilon\,\Theta-\theta)}\right) \times \left\{1+e \times \frac{(T+t)}{2}\right\} = z.$$

This formula admits of two corrections on account of the variation of gravity; from equation (1) it is evident that our value of z is in reality the value of $z\,(1 - \frac{z}{r})$ at least if we omit the powers of $\frac{z}{r}$ higher than the first, and therefore we must subtract from the value given by our formula, its square divided by 21 million, which is in round numbers the value of r in feet. The mercury stands higher than it ought to do in the upper barometer for the same reason, and it may be corrected in the logarithmic formulas by adding to log (A) 2 Log. $(1+z)$, but neither of these are of much importance. The value of N as determined by Ramond is 60148, which in latitude 53 becomes 60076, that of $\epsilon = \frac{1}{9741}$, that of $e = \frac{1}{480}$: the formula becomes with these values

$$z = 60076\left\{1+\frac{T+t}{960}\right\} \times \left\{L\,(A) - L\,(a, - L\,(1+\frac{\Theta-\theta}{9741}\right\}$$

As $\Theta-\theta$ is very small, its *h. l.* is quam prox. the first term of its developement, and its $L = \frac{0.43429}{9741}$, &c. $\times\, \Theta-\theta$, or

$(\Theta - \theta) \times 0.00004458$. As the computation must be made by Logarithms, we give a logarithmic formula

$$L(z) = 1.7964298 + L(896 + T + t)$$
$$+ L\left\{ L(A - L(a) - (\Theta - \theta) \times 0.00004458 \right\}$$

in which the temperatures are the distances from Fahrenheit's Zero. T and t must be observed by a thermometer separate from the barometer, Θ and θ by one inclosed in the mounting of the instrument.

As a specimen of the mode of making the computation we give the measurement of one of the highest hills in Wicklow, made by Mr. Griffith in 1812; the lower station was known to be 590 feet above the sea.

Lower Bar. 29.075.	58	63.5
Upper Bar. 26.53.	44.5	44.5
	13.5 $= \Theta - \theta$	108.0 $= T + t$
		896
		1004

Log A = 1.4635197—
Log a = 1.4237372

0.0397825—
0.0006021 = Log (13.5 × .00004558)

.0391804 Approx. H. in fathoms = 392.

Its Log = — 2.5930644 +
Log (1004) = 3.0017337 +
1.7964298 Constant Log.

3.3912279 whose number is
2461.7 feet, and adding
590.

3051,7 above the sea.

CHAPTER XIX.

On Pneumatic Instruments.

121. The variety of instruments which act by means of the pressure of the Atmosphere, or the elasticity of it is very great, and therefore we can only describe those which are most familiar, and those which from their important practical applications should be universally known.

In the first place may be mentioned what is emphatically called the Pneumatic Apparatus, by means of which Chemists are able to measure, divide, and transfer elastic fluids. If a glass jar be filled with water, and covered by a plate, of any material it may be inverted and placed with its mouth downwards in a cistern of water where it rests on a shelf about an inch below the surface; the water continues suspended in the jar by Atmospheric pressure after the plate is removed. If now a tube conveying air or any other Gas be brought with its extremity under the orifice of the jar, the Gas must rise in bubbles through the water, which are collected in the jar; when it is filled with Gas, it may be shifted to another part of the shelf, taking care never to raise its mouth above the water, and a fresh one may be substituted in its place, until the requisite quantity of Gas is obtained or the materials which generate it are exhausted. Gases may also be transferred from one jar to another; placing the mouth of that which contains the Gas below the other and inclining it, the water enters and displaces the Gas, which is caught in its ascent.

This apparatus can only be used where the Gases are not dissolved by water, or where the presence of its vapour is not injurious; in other cases, the cistern is filled with Mercury and the whole apparatus is constructed on a much smaller scale. Where a Gas is to be obtained free from water, it is confined over mercury in contact with some body whose attraction for moisture is powerful, as quick lime, Potash, &c. which after a short time absorbs the vapour. When the volume of any Gas is to be measured, it is received in a jar previously graduated to cubic inches and their fractions, and the true bulk is obtained by a correction for the difference of level of the water or mercury, in the jar and cistern. In fact the case is precisely the same as in sect. 113 and M the true bulk $= M' \times \frac{A-a}{A}$, A being for mercury the standard altitude, and for water the same multiplied by 13.6.

122. Where large quantities of Gases are to be used the Pneumatic Cistern becomes insufficient and Gasometers are employed; they differ in form, but are for the most part the same as that shewn in fig 148, AB is the section of a Tank or vessel filled with water, a bell CHI which nearly fills it is inverted in it, and a tube D rises through its bottom connected underneath with E and F. The Gas is introduced through F and it rises through D; as the bell is counterpoised by means which are not in the figure as they can easily be imagined, it is forced upwards by the Gas until it is filled. The tube F is then closed, and the Gas is preserved for use; when wanted, it is forced through the pipe E by the application of a slight pressure to the bell.

123. We have seen that a vessel full of air may be plunged in water with its mouth downwards without that

fluid rising in it; this principle is made practically useful on a large scale in the *Diving Bell*, which has become of the highest importance in Hydraulic Architecture. This is a large iron vessel capable of holding about 150 cubic feet, strong glass windows are framed in its top, and seats disposed round it on which three or more workmen can sit; the machine is suspended by a strong chain attached to its summit, and is gradually lowered into the deep. As the bell descends the water condenses the interior air, so that at the depth of 34 feet the bell would be half full of water, but a flexible tube is connected with an opening on its top through which fresh air is forced by any adequate means, so that it is kept full of it. The divers might be expected to suffer much from the augmented pressure, for at 34 feet, an additional pressure equal to that of the atmosphere is applied; at double the depth twice as much, and as the pressure of the atmosphere on a square foot is about a ton, it follows that a middle sized man is loaded with 15 additional tons at that depth which might be expected to crush him to pieces. But no such effect is observed, and in fact as the whole body is penetrated by fluids to which the external pressure is communicated by the lungs, there is exerted from within outwards an equal pressure, and the diver suffers only when he descends with too great rapidity.

124. As air is heavy, a body immersed in it has its weight diminished by the weight of the air which it displaces; and if this exceed the weight of the body, it must ascend. Such is the principle of air balloons, an invention which at first excited the utmost enthusiasm, and is never seen without admiration though it has produced no practical results: it consists of a thin bag, as light as is consistent with the retaining the in-

cluded gases which being rarer than air escape yet more easily; the aerial voyager, is suspended from this, and possesses the power of regulating his elevation, by either throwing out ballast, or by permitting some of the Gas to escape through a valve in the balloon, which diminishes its bulk and of course its specific levity. The first balloons were filled with air expanded by passing through a fire placed beside the car, but this perilous mode was soon abandoned; a species of Gas named by Chemists Hydrogene is now commonly used, its Specific Gravity is only $\frac{1}{13}$ that of air when pure, and as commonly obtained $\frac{1}{11}$ which but for the expense of the process is far the best substance for this species of navigation. The gas obtained by distilling some vegetables at a red heat is of $\frac{1}{7}$ the Specific Gravity of air, and may be substituted for Hydrogene in many cases. This art has been chiefly cultivated by the French, who have applied it to Military Reconnoitring, to Surveying and in a few instances to Physical Researches: Gay Lussac, one of their most accomplished Philosophers, has shewn that the air retains the chemical constitution which it has below at the height of 24000 feet, the greatest elevation which man has ever attained.

125. Most of the machines used in elevating water act by atmospheric pressure; we can only mention one or two, as the proper consideration of them includes a reference to Hydrodynamics. The Siphon or Crane is well known, it is merely a bent tube ABC fig 149, which is filled with water and inverted in the cistern D: in this state the weight of the column EB counteracts that of BC, that of the column below C is in equilibrium with the water in the cistern, and the portion AE remains unopposed, the fluid in BA will therefore preponderate and descend; but it cannot do this

unless the column separates at B, or the fluid in the shorter leg follows it; and as the separation of the column is prevented by atmospheric pressure, a continual current must be maintained in the siphon as long as the surface of the fluid in the cistern is above A. It is obvious that the instrument is of no use where its bend is more than 34 feet above the reservoir, for in that case the column would separate; and it may be observed that it cannot be effectual even near this height, for all water contains air dissolved, which escapes from it when the pressure of the atmosphere is removed, and therefore the bend of the Siphon would soon be occupied by air only.

126. The sucking pump consists of a working barrel AB fig. 150, to which is attached the suction pipe C of narrower bore, at the top of this is placed the fixed valve V opening upwards, but preventing any return; the working barrel is fitted with a Piston or cylinder accurately fitted to its capacity so that no fluid can pass between them; in the Piston is another valve U similar to V. To understand the action of the pump, suppose the Piston at the bottom of the barrel, and the suction pipe full of air, on drawing up the Piston by the pump-rod, the capacity of the barrel below it, is enlarged, and therefore the air in it rarified, but as the air in C is as dense as the external air, its pressure on the lower surface of V must exceed that exerted on the upper surface of it by the rarified air; the valve will therefore be opened and the air in C also rarified. In this state its pressure on the surface of the water in it is less than that of the atmosphere, which latter will therefore force the water up in the suction pipe, to some height: now let the Piston be thrust down again, it condenses the air below it, and as the valve V prevents it from escaping into the suction pipe, it opens the valve U and escapes into the

atmosphere. On raising the piston again the same process is repeated, and the water rises higher in the suction pipe, till after a few strokes it rises above V so that the piston dips into it in its descent; on its next ascent the water follows it and fills all the barrel below it, and when it is again thrust down, the water forces U open and gets above the piston; it may thus be raised to any height required.

127. The forcing pump is shewn in fig. 151. AB is the barrel, C the suction pipe, V the valve; the piston is solid, but the valve U is placed at the side in a tube S; the play of the pump is the same as in that already described, with this exception, that when the piston is forced down, the air or water below it passes through the tube S, which may be carried to any height and deliver the water there. The stream of water is obviously intermitting, for there is no force impelling it during the return of the piston, but this is remedied by an ingenious contrivance: there is an interruption in the tube X which is surrunded by an air vessel T, and in working the pump as soon as the water has risen above Z it compresses the air above it, and this, by its elasticity, forces the water up through Z. The orifice of Z is narrower than that of X, and therefore the quantity of water introduced during the descent of the piston will supply its discharge for the whole time of the stroke, producing a constant stream: the utility of this will be more fully appreciated when we treat of Dynamics. (*a*)

128. The mechanism of the sucking pump is little different from that of the air pump; this instrument consists of a brass barrel, fitted with a piston provided with a valve opening upwards; at its bottom is another valve also opening upwards, called the Receiver valve,

as from it a tube proceeds to the vessel to be exhausted. These are in general glass vessels, whose orifice is ground perfectly flat, so that when smeared with oily matter, and laid on the pump plate, over the opening of the tube already described, they apply exactly: the piston rod is formed into a rack which is moved by a toothed wheel and winch. Suppose that of the commencement of its action the piston is at the bottom of the barrel; on drawing it up, the air below it is rarified, and therefore presses on the upper surface of the receiver-valve with less force than that exerted against its lower surface by the air in the tube and receiver, which is of its natural density; the valve therefore opens, and the air of the receiver diffuses itself beneath the piston. On the descent of the piston the receiver-valve closes and prevents the air from returning to the receiver; it is therefore condensed beneath the piston, and when its density exceeds that of the external air, it opens the piston-valve and passes off to the atmosphere. In the second stroke, the barrel is again filled with dilated air from the receiver, and this is again expelled through the piston-valve; and thus we may continue abstracting air till its density is as much less than that of the external air, as the space between the piston and receiver valve when it is down, is to the space between them when it is up; as in that case, when the piston is down, the air below it not being denser than the external air cannot open the piston valve, and the exhaustion must stop.

It is very laborious to work such a pump, for there is a pressure acting downwards on the piston equal to the difference of the atmospheric pressure,, and the elasticity of the air in the receiver, which latter, towards the end of an exhaustion, may be neglected. Suppose the barrel two inches bore, and as the mean pressure of the atmosphere is 11 pounds on every circular inch, the experimenter must work against 44 pounds. To obviate this Dr. Hook added another barrel similar to the first, whose piston had a reciprocal motion to that of the other; in this arrangement, the pressure on the two pistons act in opposition to each other, and little remains unbalanced except just at the end of the stroke: it is obvious that this also doubles the rapidity of exhaustion.

129. To know the density of the air in the receiver the pump is provided with Gages. The barometer gage is a tube of glass 30 inches long, whose lower extremity dips in a cup of mercury, and its upper communicates with the receiver. If the pump be worked the mercury will rise in the tube, and its deficiency from the standard altitude is to this latter as the density of the included air is to that of the external; and if the pump could make a perfect vacuum, it would stand as high as in the barometer. In small pumps the length of this gage is an inconvenience, and it is replaced by another which is merely a Torricellian tube, placed under a second receiver connected with the first; as the density of the air is diminished, its pressure on the mercury in the cistern is lessened, and therefore the column supported is less than the standard altitude, and its height measures the density. This gage seldom exceeds a few inches in length, for its indications are seldom required till towards the end of the exhaustion; and it does not begin to fall until the density of the air is less than that which is equivalent to a column of the length of the gage. The sum of the heights in these two gages is equal to the standard altitude, and the heights in either, after successive strokes, are in a geometrical progression. (*b*)

130. Many entertaining and instructive experiments are performed by means of the air-pump, of which we will enumerate a few, serving to exemplify the propositions which have been proved concerning the weight and elasticity of the air. If a piston fitted to a barrel, whose bottom is closed, be drawn up, a resistance is felt, and if the rod be let go, the piston descends to the bottom; this experiment, as we shall see leads to important practical results, and, in the mean time, it is evident, that as the piston is drawn up against the pressure of the atmosphere, this must be the cause of the resistance; and if the instrument be hung by the piston rod in the receiver of an air-pump, on exhausting, the barrel descends by its own weight, and on admitting the air, it is forced back on the piston. Another experiment, which makes the pressure of the air manifest, is known by the name of the Magdeburg Hemispheres, from the residence of Guerickè the inventor of the air-pump: two hemispheres are ground so as to fit accurately at their edges, and being applied together are exhausted: in this state they are kept united by the atmospheric pressure, and resist great efforts to separate them, but fall asunder on admitting the air into them.

131. But the experiment which most decisively proves the weight of air, is the actual determination of its specific gravity: for this purpose a flask of thin glass is provided, whose orifice is fitted with an accurate stop-cock; it is counterpoised in a good balance, noting the altitude of the barometer and thermometer; the stop-cock is then screwed on the plate of an air-pump, and when the exhaustion is effected it is closed, and the flask replaced in the balance, the difference of weight is the weight of the air which has been abstracted; now immersing the stop-cock in *pure* water, open it, and

the external pressure will force the water into the flask, so as nearly to fill it; close the stop-cock, having immersed the flask till the internal and external surfaces of the water are on a level, and weigh the flask a second time, the quotient of the first difference of weight by the second is the specific gravity of air. If after exhausting the air the flask be screwed on a receiver containing any other gas, on opening the stop-cock it filled with that gas, and by weighing it again we obtain the relative weight of the air gas and air.

132. In this experiment many precautions must be used to obtain a correct result, which cannot be introduced here; see Biot. Traitè de Physique: in particular, as the specific gravity of air varies with the barometer's height, this should be registered, and if it varies during the experiment, a due correction should be made according to the rule given in art. 113: the thermometer is of equal importance; and above all, care must be taken that the air or gas examined be perfectly dry. It is difficult to obtain a balance strong enough to bear the weight of the flask, which must be large and heavy in order to bear the exhaustion; but this is obviated by weighing the flask in *water*, allowing it to be very little heavier than this fluid; thus its weight is taken off the balance.

132. Another remarkable experiment shews the compressibility of water; a thermometer filled with this fluid, and open at the top, is placed under the receiver, and on exhausting the fluid rises, augmenting in bulk $\frac{1}{20000}$ by removing the pressure of the atmosphere; similar results are obtained with other fluids.

133. As the air pump resembles the sucking pump, the condenser is analogous to the forcing pump; in its simplest form it is a barrel filled with a solid piston and

a valve at the bottom opening outwards; when the piston is drawn up, it leaves a vacuum behind it till it passes a hole in the barrel through which the air rushes in, it is then forced down and when it passes the hole all the air below it must be forced through the valve into any vessel communicating with it. The degree of condensation is measured by a gage consisting of a tube closed at one end and containing air confined by a drop of mercury; this must be condensed equally with the air of the vessel, and since its bulk is inversely as its density, the mercury must move towards the closed end. An equal quantity of air being thrown in at every stroke of the condenser, the densities after each stroke are a series of arithmetical progressionals, and therefore the distances of the mercury in the gage from its extremity are Harmonicals.

134. If the piston of the condenser be furnished with a valve opening downwards, the barrel is filled with air during the ascent of the piston, and it is not as in the common construction drawn up against the pressure of the atmosphere: condensers of this kind are used on a very large scale indeed to blow air into the large iron furnaces, where they are called Blowing Cylinders; but as the blast would be interrupted during the ascent of the piston, some contrivance to equalize it is necessary. Sometimes the cylinder delivers its air into another similar one called the regulator, whose piston is loaded with weights which force it into the furnace; in this case the piston of the regulator rises during the descent of that of the blower, for it affords more air than can escape through the orifice of the blast pipe, and the overplus maintains the blast during its return. The air is sometimes thrown into a large vessel inverted in water which it displaces, it therefore is pressed by a column of water equal to the difference between the internal and external

surfaces, and as, if the vessel is of much greater bulk than the blowing cylinder, the internal surface is not much depressed during the stroke, the blast will be abundantly uniform. In the third mode the air is condensed into a very large air vessel from which it is urged by its own elasticity: this on a small scale is used to supply a blow pipe, and is found very convenient.

135. We shall describe two instruments acting by condensed air, the fountain of Heron, and the Air-gun: the first of these has been applied to the draining of mines and in particular situations may be extremely useful. Its action may be easily understood by a reference to fig. 152; supposing B nearly filled with water, and A containing air condensed by the pressure of a column of water in F, on opening the cock E, its elasticity presses on the water in B and if the height of F be sufficient forces it through D to the surface of the earth where it is discharged: in this state B is full of air and A of water; now let cocks in their bottoms be opened and the cock I closed, the water of the mine will again fill B and drive the air out of it into A, on opening I and closing the others, water flows into A and condenses the air in it and the engine is ready for a second stroke. If the height of the tube D were 34 feet, A must be twice the capacity of B, and the height of F about 40 feet. Apparatus for opening the cocks without the attendance of a workman may easily be devised, and it is far superior to any pump work.

136. The air gun is well known, a strong vessel into which 40 or 50 atmospheres are condensed, is connected with a tube containing a bullet and furnished with a gunlock; this acts on a lever which opens for an instant the valve of the air vessel and permits a portion of con-

densed air to escape; it expands in the barrel, and pressing on the bullet drives it before it with considerable velocity: the same charge of condensed air is sufficient for discharging several balls. It is merely a toy, for its report is sufficient to disturb game, and its range is much less than that of fire-arms.

137. The operations of gunpowder are reducible to the action of condensed gases: this substance on inflammation is almost totally changed into gases occupying 300 times its bulk, and their elasticity is still further increased by the intense heat attending the combustion. The flame of gunpowder is at least as hot as melted iron, as it fuses fragments of this metal, or its temperature is at least 3000 of Fahrenheit; and as every degree of Fahrenheit augments the elasticity of a gas $\frac{1}{480}$, the pressure exerted when a quantity of gunpowder is fired in a space which it fills exactly is at least $= 300 \times 7$ times the pressure of the atmosphere, or about 2000 times. It is unnecessary to enlarge on its application to war, or its more useful employment in rending the firmest rocks, without which, some of the greatest labours of modern times could not have been executed.

NOTES ON CHAP. XIX.

(a) In pumps where the lower valve is elevated above the surface of the water in the reservoir, the instrument cannot deliver any unless it can be raised above

the valve; this is not always the case, and it may be well to investigate the circumstances of its failure. In the first place suppose the fixed valve at the top of the suction pipe, and call the space between it and the piston at the bottom of its stroke s, that between them when it is at the top of the stroke S, a the height of the suction pipe, and x the distance of the water, supposed to have been elevated in the suction pipe, from the valve. The space s is full of air at the ordinary density whose elastic force is therefore A the column of water equivalent to atmospheric pressure; this air when the piston is raised expands from the bulk s to S, and its pressure on the upper surface of the valve is as $\frac{A \times s}{S}$, but the pressure on its under surface is the force of the air in the suction pipe, or A minus the elevated column of water: if this latter force exceed the former by a difference equal to that required to open the valve, which we call V, it will open, and the air in the suction pipe being rarified the water will rise higher. When this is barely the case we have the equation

$$V = x - a + A - \frac{A \times s}{S} = x - a + A \left(\frac{S - s}{S}\right).$$

If $x = 0$, or if the water barely rise to the upper valve,

$$V + a = A \left\{\frac{S - s}{S}\right\}$$

and supposing $V = 0$, or that no force is required to open the valve, we have for the limit,

$$a = A \left\{\frac{S - s}{S}\right\}$$

or the height of the suction pipe : 34 feet :: the length of stroke : distance of piston, at its greatest elevation, from the valve. If a be greater than this the valve cannot open, and the water will never rise above it : and the same thing may happen if s be too great, for supposing $V = 0$ we have

$$x = a - A\left\{\frac{S-s}{S}\right\}$$

and the second member cannot become $= 0$ if s be greater than $\left\{\frac{A-a}{a}\right\}(S-s)$; or in other words the water cannot get above the valve. If the fixed valve be at the bottom of the suction pipe, then there is below the piston at the end of the stroke the bulk $s+ex$ of air of the external density, e being the area of the suction pipe; this is expanded into the bulk $S+ex$ and its elasticity becomes A $\left\{\frac{s+ex}{S+ex}\right\}$ to this must be added the pressure of the elevated column of water $a-x$, in opposition to these forces is the pressure of the atmosphere A and the difference of these pressures opens the valve; call it y and we have the equation

$$y = A\left\{\frac{s+ex}{S+ex}\right\} + a - x - A \text{ or}$$

$$= \frac{A(s-S) + aS + (ae-S)x - ex^2}{S+ex}$$

This is the equation of a conic section which may cut the axis of x in two points, between which the value of y is negative; to find them put $y = 0$ and solve the quadratic which gives

$$x = \frac{ae - S \pm \sqrt{(ae+S)^2 - 4eA(S-s)}}{2e}$$

when therefore the water has reached a distance from the top of the suction pipe equal to the greater of these values it can rise no higher by working the pump, but if water be poured into the pump till it rise above the less value it will again rise.

(b) Supposing the resistance of the receiver valve of the Airpump insignificant, let it be proposed to determine the law of the successive diminution of density at each stroke of the pump. As before let S be the capacity of the Barrel when the Piston is raised, and s when it is thrust down. In this last case there remains in the barrel the bulk s of air of the density d, d being the elasticity required to open the Piston valve which is about $\frac{1}{60}$, then if D be the density of the air in the receiver at the beginning of the stroke, RD is the quantity contained in the receiver, and sd that in the barrel; this air is at the end of the stroke diffused through the space R + S and hence

$$D' = \frac{RD + ds}{R + S} \text{ and therefore}$$

$$D - D' = \frac{SD - ds}{R + S} \text{ hence also}$$

$$D' - D'' = \frac{SD' - sd}{R + S} = \frac{(RD + sd) \times S}{(R+S)^2} - \frac{sd}{R+S}$$

$$= \frac{R}{R+S} \times \frac{SD - sd}{R+S}$$

and hence it follows, that the successive differences of density are in geometrical progression, the first term being $\frac{S - sd}{R+S}$ and the common ratio $\frac{R}{R+S}$. The density after the n^{th} stroke is obviously, unity minus all the preceding differences, or in other words, minus the sum of n terms of the series: this is by the common formula

$$\Sigma = \frac{S - ds}{R \times S} \left\{ \frac{\frac{R^n}{(R+S)^n} - 1}{\frac{R}{R+S} - 1} \right\}$$

$$= \frac{S-ds}{R+S}\left\{\frac{(R+S)^n-R^n}{R+S-R}\right\}\frac{R+S}{(R+S)^n}$$

$$= \frac{S-ds}{(R+S)^n}\left\{\frac{(R+8)^n-R^n}{S}\right\}.$$

Subtracting this from unity we have

$$D^n = 1 - z = \left\{\frac{1-ds}{S}\right\}\frac{R^n}{(R+S)^n} + \frac{ds}{S}$$

and if

$$s = o,\ D^n = \frac{R^n}{(R+S)^n}.$$

It is obvious that D^n can never be less than $\frac{ds}{S}$, and this is evident from other principles, for the exhaustion must stop whenever the bulk S of the dilated air being condensed into the space *s* does not exceed the density *d*.

CHAPTER XX.

138. The elasticity of vapours differs from that of gases in its relations to pressure and heat, and from the numerous practical applications of this part of Mechanical Philosophy which have been made of late years, it merits considerable attention. The vapour of water is best known, and as we took air as an example of gas, so we will study steam to learn the habitudes of vapours.

If a Torricellian tube be nearly filled with mercury, and then entirely with pure water, on inverting it in mercury we obtain a barometer differing from the common one in this, that the water which floats on the surface of the mercnry fills the upper part of the tube with its vapour, whose elasticity depresses the mercury below the standard altitude, as in the third experiment of art. 113. When the thermometer is at 65° the depression is about 0,6 of an inch, or nearly $\frac{1}{50}$ of the elastic force of air.

If the vessel containing the stagnant mercury be a deep and narrow jar, we can without further apparatus investigate the effects of a variation of pressures: let the tube, remaining vertical, be immersed as deep as possible, then on raising it, as the internal surface of the mercury remains at a given height above the external, the space above it must be enlarged; were it occupied by air, the elasticity of that fluid would be diminished by this operation, and the depression below the standard altitude would be less than before, but with vapour this is not observed. On the contrary, the depression remains unchanged, and the reason is obvious, more of the water is converted into vapour of the same force, or as it is sometimes called, Tension; and as long as any water remains, this is invariable: when all the water is vaporized, then any further enlargement of the bulk of the vapour diminishes its elasticity according to the law observed by gases.

Similar results are obtained by augmenting the pressure; if under the above circumstances of temperature we introduce into a Torricellian vacuum a portion of water, not sufficient to fill it with vapour of the tension 0,6, it is completely volatilized, producing a depression less than 0,6. On condensing the vapour by plunging the tube deeper in the cistern, the depression augments; but when it becomes 0,6, it is stationary; and on further condensing the vapour, part of it returns to the state of water, until at length the water comes in contact with the top of the tube.

If this experiment be well understood, it is evident from it that the elastic force of vapour can never exceed its tension, and that if a space *saturated* with it be diminished, it is not condensed, part becoming liquid, and the remainder continuing of the same density as before; and on the other hand, the tension of the vapour is independant of the quantity of the liquid forming it, provided that there be enough to saturate the space.

139. From this it follows that at 65°, a pressure of 0 . 6 exerted by its vapour on the surface of water prevents any further evaporation; and it might be expected that the same effect would be produced by air of the same elasticity, but it is found that the same quantities of vapour can exist in a space filled with dry air as in a vacuum, and that it does not return to the state of water even though more air be condensed into the vessel. This fact, though not to be inferred a priori, is, however, conformable to what we know of the nature of vaporization; the particles of water have a strong mutual attraction of cohesion, and when vaporized they are kept by some other force just beyond its limit: if more vapour be added to a space already saturated, the aqueous particles being nearer to each other, come within the limits of their cohesion; but as there is no analogous action between the particles of vapour and air, they are not approximated by the addition of air.

Though the quantity of vapour in a given space is not affected by the presence of air, the rate of evaporation is much retarded by it: when water is introduced into a Torricellian vacuum, the quantity of vapour required to produce the tension 0 . 6 is developed in an instant, but in an equal space filled with air, although the vapour does at last add its tension to the Elasticity of the air, yet several minutes elapse before the space is saturated with it. The Evaporation is much accelerated by bringing successive portions of the air in contact with the fluid, for the stratum of air resting on it becomes saturated and the vapour in it prevents the developement of more; but by agitation the vapour is readily conveyed through the whole space. Facts illustrating this are numerous; in furnaces where a rapid current of air passes over the surface of a melted metal it is seen to smoke, and much of it is thus dispersed; even silver, though absolutely *fixed* when heated in a close vessel, is so volatile under these circumstances that great loss may be sustained in the process of refining; Sulphur, which cannot be distilled in the ordinary apparatus, sublimes readily during the roasting of ores for the same reason; and the vapour of water itself carries with it other substances less volatile.

140. The tension of Water is diminished by dissolving in it other substances having a strong attraction for it; and these bodies also diminish the tension of its vapour when brought in contact with it in their solid state. Thus, if the inside of the receiver of an air pump be moistened with water, on exhausting it, the barometer gage will not come nearer the Standard altitude than 0 . 6 as the vapour cannot be pumped out, being condensed in the Barrel at each return of the Piston: let it now be made to communicate with a vessel previously exhausted to the same degree containing dry Potash, Muriate of Lime, or strong Sulphuric Acid, this instantly absorbs the vapour, and in a few seconds the capacity of the receiver is dried, and the Gage rises, if the pump be good within, perhaps 0 . 01 of the barometer; we shall soon see the use which has been made of this.

141. The effects of a change of temperature on the tension of vapour are also different from those observed in Gases; they may be observed by the Torricellian tube, surrounding it with a vessel whose bottom is perforated to allow it a passage, and which is filled with water of the requisite temperature. By this means it is found that at 32° the tension is 0 . 2 and at 212, 30 . 0 being equal to the pressure of the atmosphere; if the temperature of the earth were 212 it is obvious that water could not exist as a liquid, and that the pressure of the atmosphere would be increased by its evaporation until either it were all dissipated or the augmented pressure became sufficient to prevent any further formation of vapour, in which case it wonld be doubled. The phænomena of boiling are occasioned by the rapid developement of vapour, which having the same elasticity as air appears as a gas and rises in bubbles through the water, and since this tension is less when the barometer is low, it follows that water boils at a lower temperature then than when the atmospheric pressure is considerable. The difference is very sensible, so that the thermometer has been proposed by some as a means of measuring the heights of mountains, but it can scarcely give any thing but a rude approximation; in graduating these instruments, however, attention should be paid to the state of the barometer, and if it be not at a mean state, a correction should be applied which is can easily be investigated from the principle here laid down, or may be made immediately by refereuce to a table of the tension of vapour.

142. By the observations of Dalton it appears that if the temperatures be in arithmetical progression, the Tensions are in Geometrical nearly, the multiplier by which each term is formed from the preceding becoming

constantly less; the deviation from this law, is not, however, very considerable. The change of elasticity produced in Watery vapour by 180° is 150 times its tension at 32°, while that of a Gas becomes 1 . 37 from 1 by the same change of temperature: this enormous difference is, however, only found where there is unvaporised water present, which assumes the elastic state as the heat is raised; the vapour itself follows the law of Gasses as is evident from the fact that moist and dry air expand equally by heat.

143. Dalton's law applies to the vapours of other fluids as far as they have been tried, and he has discovered a curious circumstanee about them which enables us to determine their tension at a given temperature from knowing that of water; it is this, that the vapours of different fluids have the same tension at equal distances of the thermometric scale from their boiling points. For example, ether boils at 102°, let it be required to find its tension at 62°, or 40° below its boiling point; it is the same as that of water at 212 — 40 = 172 °and by referring to a table we find that the tension of water at 172° is 12 . 5 which agrees with the observed tension of either at that temperature.

144. If a vessel containing water be placed on the fire, a thermometer immersed in it shews that it is receiving heat; the tension of its vapour increases, and it slowly evaporates, till its temperature arrives at 212°; it then boils, forming steam, which, if the orifice of the vessel be narrow, rushes out with a forcible blast; after this the thermometer continues stationary till the water is vaporized. During this time the vessel continues to receive heat from the fire, yet the water does not become hotter; the heat must therefore be carried off by the steam: if the thermometer be held in the current of steam, it still indicates 212 as when in the water, there-

fore, the steam though at the same temperature as the water of which it is formed, contains much more heat. Dr. Black placed a vessel containing water at the temperature of the atmosphere on a plate of red hot iron; in five minutes the water boiled, and in twenty-five minutes more it was all evaporated; during that time it received heat uniformly, and since it was heated 180° in the first five minutes, during the rest of the time it received 5 × 180° none of which was apparent. If the steam from a pint of water be passed into five pints of water at 32°, in which case it returns into the state of water, it should heat it to 212° according to the experiment; and this is found to be the case if allowance be made for the cooling action of the atmosphere; we may therefore conclude that in the formation of steam a quantity of heat disappears or becomes Latent, making its appearance again when the steam is condensed; and if it be true of steam, it must hold with respect to all other vapours.

145. Since Heat disappears by Evaporation, we may cool any substance by placing it in contact with a fluid undergoing that process: thus, if the bulb of a thermometer be covered with lint and moistened with Ether, this volatile liquid will make it sink 50° below the temperature of the atmosphere, and if a small tube filled with water be used instead of the thermometer, the water may be frozen. In hot climates the principle is applied to cool water; it is exposed to the wind in porous earthen vessels, part of it evaporates from the external surface and the rest is cooled 10 or 12 degrees. This process succeeds much better in the receiver of the air pump, for it has already been shewn that a fluid evaporates faster in vacuo, but it ceases to act as soon as the capacity of the receiver is saturated with vapour: its activity may be very much increased by including with

it some body strongly attractive of moisture. As the Experiment is usually performed, a shallow dish containing strong sulphuric acid is placed on the Pump-plate, above it is fixed a cup of unglazed porcelain filled with water, they are covered with a receiver, and on exhausting the Water congeals in a few minutes into solid ice. This mode of evaporation at the atmospheric temperature is susceptible of many important applications in the arts, and the vacuum can easily be obtained without the Air-pump by the action of steam: thus if it be required to distil a fluid which cannot be heated to 212 without alteration, let a strong current of steam be passed through the distillatory apparatus till it has expelled all the air, and its aperture be then closed; on applying heat to the vessel containing the fluid, and surrounding the Receiver with cold water or snow, the distillation proceeds as rapidly at 70° as at 212 under common circumstances.

146. The application of steam to produce a vacuum leads us to consider the construction of the Steam Engine, that splendid result of inventive talent and scientific attainment, the discovery of which is an Æra in the History of mankind. It is not within the limits of this treatise to give a complete developement of its construction: but without entering into details, we will give a concise description of its action, which will enable any person who examines an Engine to understand its operation.

The most important part of the Steam Engine in its common form, is the Cylinder which is accurately bored, and fitted with a Piston packed so as to be Steam tight, though it slides freely; the Piston-rod is turned truly cylindrical and passes through a Stuffing box in the cover of the Cylinder: this is a tube surrounding the Rod in which a quantity of soft rope soaked in oil is compressed by screws, so that neither air nor steam can pass round it. From the cover of the cylinder a tube passes to the Boiler so that the steam constantly presses on the upper surface of the piston; another steam pipe communicates with the bottom of the cylinder, furnished with a valve by which the steam can be shut off when necessary. Another tube leads from the bottom of the cylinder to the Condenser, also furnished with a valve; this is a vessel completely immersed in a cistern of cold water and provided with apertures and valves, which shall be described immediateiy. The Piston Rod is attached by the Parallel motion of Art. 35, to one extremity of the Working-Beam, a massive Lever, whose other Arm moves a system of Pumps, for which work this form of the Engine is generally used, or any other Machinery.

To understand the action of the Engine, suppose the Piston at the top of the Cylinder, and the valve leading to the Condenser, called the Eduction valve, open; on opening the Steam valve, the steam rushes into the cylinder, and from it into the Condenser, driving before it the air contained in these vessels through a valve in the condenser which opens outwards. When all the air is expelled, shut the steam valve and open a cock in the bottom of the condenser, a small portiun of water enters and condenses the steam with which it comes in contact, the atmospheric pressure forces more water in, and in a few seconds, all the steam in the Condenser is reduced to a Tension of less than 1 . 0: the steam now rushes into the Condenser from the Cylinder, and it also is condensed. But the steam above the Piston, which we suppose equal to the Elasticity of air, presses it downwards, with a force of 10 pounds on every circular inch of its area; all this while the steam was pressing below, produced no effect as it exerted an equal upward pressure; but this is destroyed by the condensation, and the Piston descends by the downward Pressure, dragging down that arm of the working beam to which it is attached, and of course lifting the pump-rod. When the Piston has arrived at the bottom of the cylinder, a pin projecting from some of the moving parts, strikes the levers of the valves, shutting the Eduction and opening the steam valve; the Steam fills the space below the piston, and the equilibrium of pressure being thus re-established, the Piston is drawn up by the preponderance of the Pump-rods. When it arrives at the top of the Cylinder, a second Pin opens the Eduction valve and closes the other, which occasions the steam below the piston to be again condensed, and produces a second stroke. Thus the machine works itself, being active only in the descent of the Piston, and inert during its ascent.

The water arising from the condensed steam, and the Air evolved from the Injection water (for all water contains air which escapes from it when atmospheric pressure is removed) would soon fill the Condenser: to obviate this, a large Pump called the Airpump, is connected with the condenser and wrought by the Working beam; this removes both the air and the water at every stroke: the latter being heated by the steam is taken by a feeding pump and a portion of it forced into the boiler to supply the place of the water which is converted into steam. The Double Stroke Engine differs from that just described in this, that both the ascent and descent of the Piston are effective: in it the top and bottom are alternately made to communicate with the boiler and Condenser, so that when the steam presses on the upper surface of the Piston, there is a vacuum below it, and vice versâ. This is never used for pumping, and when employed to produce Rotatory motion it is necessary to equalize its action by a Fly: this is a Massive Wheel of large dimensions fixed on the axis of the Crank, which is driven by a rod from the Working Beam; it revolves with great velocity and has such a quantity of motion that its rotation is scarcely affected by a slight intermission in the action of the Piston.

147. The quantity of work performed by a steam engine, and the value of its action, belong to Dynamics, and we will at present only mention some of the contrivances and adjustments which are remarkable in it. Its action is nearly independent of the attention of the workmen employed about it, and it contains in itself a provision for almost every accident which can occur. The Boiler requires several of these; in order that the water in it may be always at the proper height, it is furnished with a valve connected with a float, whose weight, as long as the water is sufficiently deep, is supported by it; but as its surface sinks, the float draws down the valve, and allows a supply to enter. A similar contrivance regulates the vehemence of the fire, by raising or lowering a damper in the flue of the furnace; but the most important appendage of the boiler is the Safety-valve: this is a frustum of a cone, ground to fit a ring let into the iron of the boiler, and loaded with a weight proportional to the density of steam required; if the steam should attain a greater elastic force, the valve is raised, and the steam escapes. The lesser degrees of density are indicated by a Siphon-gage, the steam pressing on the mercury in one leg, and that in the other carrying a float which shews on a scale of inches the excess of pressure above the atmosphere: another gage, like the Barometer gage of the air-pump, is connected with the condenser, and the sum of the heights

of the two gages is the effective pressure on the piston, for the elastic force of the steam which acts on one side is $A+h$, A being the atmospheric pressure, and h the height of the boiler-gage; the pressure on the other side of the piston is A—H, and the effective pressure is their difference $h+H$. By augmenting the fire, and nc reasing the load on the safety valve, we increase h, and therefore the quantity of work which the engine can perform; but this is very hazardous, and there are recorded many fatal accidents occasioned by the bursting of boilers when their valves are overloaded. Where a considerable portion of the resistance is suddenly removed, the effect must be, that the engine will work with increased velocity, to the great detriment of the machinery, and the unnecessary waste of steam; for this reason it has a Governor, consisting of two heavy balls, attached to rods hinged together at their upper extremities; when the engine is at rest the balls are in contact, but when it moves, they are whirled round a vertical axis and separated by their centrifugal force. The increase of the angle under the rods moves a lever connected with a valve in the steam tube so as to diminish the opening left by it; this lessens the quantity of steam admitted into the cylinder, and with it the velocity of the engine; if on the other hand the engine is overloaded, the velocity is diminished, and the balls approaching each other, admit more steam.

148. This kind of steam engine is named after its inventors, Watt and Boulton, but it is limited in its application by the large quantity of cold water required for its condenser, which cannot in every situation be obtained: in that case, another kind is employed though with some risk. In it the cylinder is much

smaller, so as to be inclosed in the boiler; and the steam used is far denser than the atmosphere, sometimes so strong as to lift a safety valve loaded with 150 pounds on the square inch: the top and bottom of the cylinder alternately communicate with the boiler and with the atmosphere, and thus the piston is moved by the difference between the pressures of the steam and atmosphere, or the quantity which we have called h. The obvious objection to this mode of using steam, is the danger of explosion; and every boiler for producing *High Pressure* steam should be carefully and frequently proved by filling it with water, and producing pressure by a small pump till the safety valve is raised, though loaded with a weight considerably greater than the utmost required pressure of the steam; if it yield, it breaks without explosion, and if it be proof it may be depended on for some time.

149. It is found in practice, that the engine last described performs, with a given quantity of coals, as much work as one of Watt and Boulton's nearly, although as much of the force of the steam as is equal to the pressure of the atmosphere is lost; from this has arisen an improvement, which consists in combining the two engines together. In Wolfe's engine there are two cylinders, one like that of the High Pressure or Trevethick's, the other larger and provided with a condenser: the steam is first admitted into the smaller cylinder, and when it has done its office there, instead of escaping into the atmosphere, it passes into the larger cylinder, where it expands so as to be little denser than the atmosphere. Suppose the steam of four times the atmospheric pressure, or $= 4$ A, and conceive it acting above the piston of the small cylinder, the effective pressure

may be stated as 3 A on the small piston, supposing the steam below it to expand into four times its bulk in the large cylinder. The expanded steam also presses on the large piston with an elasticity = A, for the vacuum below it, made by the condenser, is nearly perfect; and since these two parts of the force are nearly equal, it follows that Wolfe's engine performs twice the work of the other; but a more full discussion belongs to the head of Hydrodynamics.

150. The advantage of the steam engine over the other agents employed to drive machinery, is enormous; there is scarcely any limit to the power which it is capable of producing, and it is perfectly under our controul. By its aid, we are enabled to choose for our manufacturing establishments the most advantageous situations, without being restricted by the want of movers, and can undertake tasks, in appearance insurmountable by human efforts. But for the steam engine, many of our most valuable mines must have been abandoned, from the impossibility of drawing off the water which inundates them; no depth is beyond its reach; and in some instances, a mass of water like a river is raised through a space greater than 1200 feet. It draws the heaviest carriages; it impels the vessel with speed independent of the wind; and it excavates the bottom of the sea to create harbours. It is the most powerful, the most sublime of Mechanical inventions; and is alone sufficient to transmit to posterity the fame of our country, had it no other claim on the gratitude of mankind.

PART SECOND.

DYNAMICS.

PART II.

DYNAMICS.

THE Investigations of Dynamics are attended with more difficulty than those in which we have hitherto been engaged; in Statics our conclusions can always be derived by a simple analysis, whereas many Dynamical problems can only be solved by approximation. In addition to this, its Metaphysics are much more obscure, and therefore more liable to objections; and it is remarkable that at the very origin of the science there arose a dispute relative to the measure of moving force, in which the science of Mathematics seemed to have lost the character of certainty which had always till then been deemed its essence. Even when we avoid such dangerous ground we are often embarrassed when it is necessary to apply our reasoning to a Dynamical problem, and a degree of subtlety and of analytical artifice is required, which is not readily acquired or comprehended. Under these circumstances we are precluded from entering into this part of our subject, even so fully as we did in the preceding, and must confine ourselves to such theorems as are either very elementary, or from their practical utility are worth remembering without demonstrations.

2. The science of Dynamics rests on the Inertia of matter, and on the connexion between Velocity and Force: these have been put by Newton under the form of aphorisms or Laws of Motion, and they have the advantage of being easily remembered. The first of them states that "A Body perseveres in a state of Rest or of Uniform Rectilinear Motion, till compelled by some Force to change its state." This is evidently an assertion of the Inertness of matter, and it is susceptible of a twofold proof, Metaphysical and Experimental. The first depends on our idea of matter, which however indistinct includes no active power, and therefore it cannot of itself either generate or destroy motion. This is true, but it cannot satisfy those, who conceive that a portion of organized matter is capable, not merely of moving but of thinking; and though such bad reasoners may seem unworthy of notice, yet it is not advisable to introduce Metaphysics into another science where there is no absolute necessity for it. That a body left to itself will remain at rest is perfectly conformable to our observations. A ball placed on a horizontal plane will not stir; a wheel balanced on its pivots will not revolve; and, in short, when we perceive a body in motion we instantly conclude that it has been acted on by an external force. In some cases the forces appear to reside in the bodies themselves, as for instance Gravity and Magnetism, but they are always considered as agents foreign to them, and as no wise interfering with our law. The second part of it, or their indifference to rest, is not so readily admitted by beginners, and it seems at first to be repug-

nant to observation: all motions which we witness soon decay and cease, and to preserve them uniform requires a continual application of power. This objection vanishes if we consider that the motions which we observe are not under the circumstances stated in the law; it supposes the absence of all disturbing agents, while here retarding forces are continually acting. If a ball be rolled along a level surface, friction acts to diminish its motion with considerable power, and being independant of the velocity must ultimately stop it; and as this friction makes the body roll instead of sliding, a portion of its motion is lost in the change from rectilinear to rotatory movement. A wheel truly poised on its pivots should revolve for ever when once put in motion, but here also we can trace the causes of the retardation. The friction of the pivots is one of them, and if this be diminished by friction wheels, the duration of the motion is proportionably increased. If, while it is thus spinning, any light body be brought near it, its agitation shows that the air contiguous to it is dragged into motion by it; and the resistance of this fluid is also to be taken into account. Accordingly the motion is yet further prolonged in the exhausted receiver; and in general, by diminishing the friction and removing the resistance of the air as much as possible, we prolong a given motion to such a degree that we are warranted in concluding that could these conditions be perfectly accomplished, our law would accurately represent the phenomena. In the only analogous case presented to us by nature, it is rigorously observed, for the planetory motions remain absolutely undiminished, and coincide to the utmost precision with our calculations, which are made on this supposition.

3. The second Law states, that " every Motion or

Change of Motion, is proportional to, and in the direction of the force impressed." This is evident if we consider that we know forces only as the causes of motion, and that an effect is necessarily proportional to its cause; the motion produced is therefore the measure of a force. That the Law may be applied to practice we must have a measure of motion, and we have already stated that when it is uniform the moving force is proportional to the quantity of matter moved and to the velocity. Applying this to the present case we assert that any force is proportional to the quantity of matter to which it communicates Velocity, and to the Velocity which it communicates in a given time. The experimental proof of the Law cannot be given here, as it involves things which are to be explained hereafter: we may mention that a double force will raise a double weight to the same height in the same time, and that a double force will in a given time generate in the same body a double velocity. In Hydrodynamics another measure of force is sometimes used; water, when employed as a mover, produces an effect proportional to its weight and to the height through which it descends, or, which is the same thing, to the square of its velocity: this is a consequence of our measure, and it is of considerable use in that department; but some in the infancy of Dynamics wished to make it universal, and to assert that every force was as the square of the velocity which it generated. The controversy, to which I have already alluded, was conducted with the utmost bitterness, and at last dropped by common consent: in truth it was rather about words than facts; for either measure may be used with due caution; we may employ either the Velocity produced in a given time, or the square of that produced in a given space; they belong, however, to different considerations of force, and the latter is known particularly by the appellation of Vis Viva; we shall resume it hereafter.

4. The equality of Action and Re-action is announced in the third Law; it is a consequence of Inertia and may be readily comprehended from the following instances. If a ball rolling along an horizontal plane meet another at rest, it is said, though with some impropriety, to *Act* on the latter, for it puts it in motion. The latter is said to *React* on the former, for its motion is diminished, and the law asserts that it loses as much as the other gains. Again, if a man who is capable of walking with a certain speed draw a load, he is incapable of moving with the same velocity; the difference is attributed to the reaction of the load. Also if a cannon be discharged, its shot is projected with a great velocity, but at the same time the gun recoils with a speed as much less than that of the ball as its mass is greater. A more complete illustration shall be given when we treat of Collision, and in the mean time the law is not of indispensable necessity.

5. Setting out from these laws we proceed to investigate the formulas which express the action of uniform forces on bodies; it is stated in the 2d law, that in a given time the force is as the velocity which it produces, multiplied into the mass of the body on which it acts: if the time of its action vary the velocity which it generates must increase in the same ratio as the time; and hence representing these magnitudes by their initials,

$$FT : ft :: MV : mv$$

or

$$FT \propto MV \qquad (1)$$

We may for the present omit the consideration of the mass by supposing it unity, and since $TV \propto S$ we may deduce from it by multiplying both sides by V and T,

$$FT^2 \propto S \qquad (2)$$

and

$$FS \propto V^2 \qquad (8)$$

which are the theorems relative to motions accelerated by a constant force.

6. It most frequently happens that the force is not constant during the time of a body's motion, and in that case the formulæ given above do not apply: the time must therefore be resolved into indefinitely minute portions, during each of which the force varies less than by any assignable quantity, and may therefore be deemed constant. The formula (1) therefore applies, substituting for T an indefinitely small portion of it, and hence differential formulas, similar to (2) and (3) can be found *(a)*. These are expressed by two constructions given by Newton; take any line AB to represent the space S, divide it into any number of equal parts, and at each of these erect an ordinate proportional to the force acting on the body at that point, the lines connecting the extremities of these ordinates become, (fig.153) if the parts of AB be indefinitely small, a curve whose nature depends on the variation of the force. The area of this curve is equal to half a square whose side is the velocity. To prove this, let us take a line MN to represent the velocity which the body has at C, and suppose that when the body arrives at D the velocity has become MD; then by the principle stated in the first chapter of this work, the time of describing CD is the quotient of CD by MN; for the portion CD is indefinitely small, and therefore the difference between the values of this time arising from using MN or MD as denominators is inappreciable, but one of them is too little, and the other too great; and since they are ultimately equal either may be used.

Now from C to D the force may be considered as constant; and therefore ND, the increment of velocity produced by it during the time of describing CD = CE or DF the force, multiplied into the time, or $\frac{CD}{MN}$ hence MN × NO = CE × CD or the trapezium ED = trapezium UO. Similarly FK = VP, and so on. Now the sum of the former equals the area of the curve, and that of the latter equals the right angled Isosceles MNU, which is manifestly half the square of MN. In this way when the area of the curve can be found, which is always possible, at least by approximation, the velocity can be determined.

The second construction determines the time; produce the ordinate CD, so that $CG = \frac{1}{MN}$. Then the area of the curve XG, which is the Locus of this line, is the time. For that area is the sum of the trapezia GD, HK, &c. and each of them = CD multiplied into the reciprocal of the velocity; but $\frac{CD}{MN}$ = time of describing CD, and the sum of all these is the time of describing the whole space; this latter theorem is, however, of no value since the use of the Int. Cal. (*b*)

One or two instances of the application of these theorems shall be given in the next chapter.

NOTES ON CHAP. I.

(a) According to the analytic method of considering the nature of force, it is the ratio of the differential of the time to that of the velocity, or

$$FdT = dV \qquad (4)$$

this may be readily proved from the consideration of limits even when F varies with T.—Let F become $F+dF$ during dT, then it is manifest that dV is less than $(F+dF) \times dT$, and greater than FdT; the difference between the two is $dF \times dT$, which is incomparably less than FdT, and therefore dV differs from this latter by a quantity less than any assignable. The value of dT is $dS \div V$ and hence

$$FdS = VdV \qquad (5)$$

For dV we may put its value $d\left(\frac{dS}{dT}\right)$ and obtain

$$FdT^2 = d^2S \qquad (6)$$

(a) From equation (5) we can obtain the velocity when the force is given a function of the space, for integrating it we obtain

$$V = \sqrt{2\int FdS + C} \qquad (7)$$

and from the equation $dT = dSV$ we derive

$$T = \int \frac{dS}{\sqrt{2\int FdS + C}} + C' \qquad (8)$$

in which the constant quantities C and C′ are to be determined by the beginning or end of the motion.

CHAPTER II.

7. If we observe the phænomena of nature we shall find that bodies are made the seat of certain powers by which they are capable of impressing motions on others which are brought within the sphere of their influence; in many instances these forces emanate from every particle of the body, but in some they appear to be confined to particular parts; thus Gravity seems to reside in the centre of the earth, Magnetism to dwell in the Poles of the Magnet, and hence arises the term Centre of Force, by which we mean a point towards which bodies are urged by a force, which is in general a function of their distance. According to the law of its variation, the velocity which they acquire or lose in their motion is various, as well as the time of performing it; and we proceed to investigate the particular cases of the formulas of the last chapter, as far as may be done without entering too deeply into analysis.

8. The simplest law of force is where it is constant at all distances; in this case the increment of the velocity in each successive increment of time is constant, and therefore the velocity is as the time; this is evident from formula (1), for F is constant, therefore T is as V. This enables us to represent the space described in a given time by a simple construction, for if we take a line representing the time for the base, and construct on it a right angled triangle, whose altitude is the velocity acquired, its area shall be as the space. Let AB, fig. 154, be the time, divide it into equal indefinitely

small portions, one of which is CD; the velocity is as the time, and BA : BV the velocity acquired in the time BA : : CA : CE, which therefore is the velocity acquired in the time CA; now the space described during the moment CD being as the rectangle under the time, and a velocity intermediate between CE and DF, is as the elementary trapezium ED, and the sum of all the trapezia corresponding to successive instants, or the triangle, is the total space. (*a*)

This exhibits very completely the principal circumstances of the motion produced by an uniform force, which from the uniform increment of the velocity is called Uniformly accelerated: where the force is given, the triangles belonging to different times are similar, and therefore as the squares of their homologous sides; or the spaces described are as the squares of the velocities acquired in falling through them, or as the squares of the times of their description. If a body move with a uniform velocity BV for the time BA, the space described by it would be represented by the rectangle under BV and BA, or twice the triangle VBA, therefore the space described by a body uniformly accelerated is half that described in the same time by a body moving uniformly with the last acquired velocity.

9. Newton's Theorems give more elegantly the relation of the space to the time and velocity; as the force is constant, the line which is its locus is a parallel to that indicating the space, and its area is the rectangle under the force and space. This by the theorem is equal to half the square of the velocity, therefore the equation of uniformly accelerated motion is

$$V = \sqrt{2FS} \qquad (1)$$

If this be combined with the equation $V = FT$ we obtain

$$T = \sqrt{\frac{2\,S}{F}} \qquad (2)$$

and if we suppose a space Σ to be described with the acquired velocity V in the time T, $T = \frac{\Sigma}{V}$ and

$$F = \frac{V}{T} = \frac{V^2}{\Sigma};$$

substituting this value in equation (1) there results

$$V^2 = 2V^2 \times \frac{S}{\Sigma} \text{ or } \tfrac{1}{2} = \frac{S}{\Sigma}$$

shewing that Σ is twice S.

10. These theorems apply with due alteration to the action of retarding forces; for if a body be projected with a certain velocity, in a right line from a centre of force, its velocity will gradually be destroyed, being in a given time diminished by a quantity equal to that which the force can generate in the same time, so that at last it will lose all its velocity, and will then fall back towards the centre. The space which it describes is evidently that through which it must fall by the action of the force to acquire the velocity of projection; and it is easy to define the circumstances of its motion. Let H be the total space through which it ascends, then H—S is that which remains to be described: hence equation (1) gives

$$V'^2 - {}^2V = 2\,FS \qquad (3)$$

from this the relation between the space and time can be ascertained without difficulty. (*b*)

11. We have occasion to make extensive application of these theorems, for gravity affords at the surface of the earth an example of such a force; it in fact varies

as the inverse square of the distance from the earth's centre, but the change at any height which is accessible in our experiments is utterly insignificant. The theorems which are given above may therefore be applied to falling bodies, and this without regard to their magnitudes, for as gravity acts equally on every particle of matter, any body falls with the same velocity as a single atom. To make them useful some determinate expression of F must be employed, and we commonly adopt the velocity which it generates in a body submitted to its action during the unit of time: experiments shew that a body falls through 16 feet $\frac{1}{12}$ in a second in this latitude, and as twice this space would be described with the last acquired velocity, therefore it is 32 feet in a second; substituting this for F we have

$$V = \sqrt{2 \times 32 \times S} = 8\sqrt{S}, T = \tfrac{1}{4}\sqrt{S}$$

and $V = 32 \times T$ which are sufficiently accurate for practice. A few examples will shew their application; let us seek the velocity and time when a body falls through 100 feet; $\sqrt{S}$ is equal to 10, so that $V = 80$ feet in a second, and $T = 2\frac{1}{2}$ seconds. Again, an arrow shot perpendicularly upwards remains 10 seconds in the air, half of this is employed in its ascent, and half in its descent; $T = 5$ and $S = \overline{4 \times 5}|^2 = 400$, $V = 160$.

Friction affords an example of an uniformly retarding force, and experiments made with bodies sliding on horizontal planes agree very well with theory, the times being as the square root of the spaces. Other instances are afforded by the penetration of balls into substances of uniform texture, it being found that the depths of penetration, or the spaces described are as the squares of the velocities. In the case of bodies falling freely, the laws of their motion cannot be accurately examined by experiment, but Atwood has invented an apparatus by means of which we can diminish the acceleration produced by gravity in any required degree, and thus verify by trial the theorems. His machine shall be described, when we treat of the motions of a system of bodies.

12. Another law of force which is of frequent occurrence is when it varies as the distance from the centre; Let A, fig. 155, be the centre, erect BF to represent the force at B and draw AF, then CE will be the force at C, for it is to BF as AC : AB or as the forces at those distances; but BV is one, therefore CE is the other. Hence the square of the velocity acquired in falling through BC $=2$CEFB : and as the area of such a trapezium is equal to the product of CB into half the sum of CE and BV, the square of the velocity $=$ tangent of the angle $A \times CB \times \overline{AB+AC} = \text{tang } A \times \overline{AB-AC} \times \overline{AB+AC}$ $= \text{tang. } A \times (AB^2 - AC^2)$.

This gives the following construction, fig 156, with A as centre and AB as radius describe a circle, take AC and erect at C a perpendicular, CV is as the velocity at C; and if we call the force at B g, and AB r, the velocity in feet is $CV \times \sqrt{\frac{g}{r}}$, g being the velocity which a constant force equal to that at B would generate in a second. From this construction it is evident, that the motion of a body falling by this law of force, is accelerated, but not uniformly, and that its velocity is a maximum at the centre; when it passes this it decreases as it had increased, and is totally destroyed when it arrives at E. The velocity at A is $\sqrt{\frac{g}{r}} \times r$ or $\sqrt{gr}$, but if the force g were to act constantly through r, it would generate a velocity $\sqrt{2gr}$ so that the former : the latter as $1 : \sqrt{2}$.

The indefinitely small space CD may be supposed to be described with the uniform velocity CV, and therefore the time of describing it $= \frac{CD}{CV} \times \sqrt{\frac{r}{g}}$, but by a well known property of the circle $\frac{CD}{CV} = \frac{VX}{r}$ and therefore the time of describing CD $= \frac{VX}{r} \times \sqrt{\frac{r}{g}}$

In the same manner the time of describing DI is $\sqrt{\frac{r}{g}} \times \frac{UX}{r}$ and in general the time of describing any portion of AB intercepted between parallel ordinates is represented by $\sqrt{\frac{r}{g}}$ multiplied into the quotient of the intercepted arch by radius, or into the angle which stands on it, represented by the proportional arch of the circle whose radius is unity.

If the body fall down to the centre, the $\angle$ HAB is 90, therefore the time of the fall is $\frac{p}{2} \times \sqrt{\frac{r}{g}}$, an expression which is independent of r; for the force being as the distance, $\frac{r}{g}$ is constant, and hence whatever be the distance of the body from the centre at the beginning of its motion, it will fall down to it in a given time, which depends solely on the intensity of the force at a given distance.

This law is chiefly exemplified by the action of elastic bodies; thus the force of a spring is proportional to the space through which it is bent, and therefore all its oscillations are performed in the same time; another instance of its application is in the balance of torsion, to which we referred on a former occasion. In it the force which is to be measured twists a wire fixed at one extremity; and as the force excited by torsion is proportional to the angle through which it is twisted, or to the space described by the extremity of the arm from its position of rest, the theorems which we have demonstrated apply to its motions, all its oscillations are performed in the same time, and from observing that time, we can determine the elasticity of the wire and thus ascertain the amount of the force. (*c*)

The law which prevails most generally is that of the inverse square of the distance; but it cannot be examined without the use of the integral calculus, they therefore, who are desirous of further information concerning it must be referred to the notes, where also they will find the demonstration of the law of attraction of a sphere on an exterior and interior point. (*d*)

NOTES ON CHAP. II.

(*a*) The formulæ relative to uniformly accelerated motion flow with the utmost facility from the differential equations of accelerated motion: calling g the force, represented by the velocity which it generates in a second, the equation $gdt = dv$ gives $gt = v$, which needs no constant when the body falls from a state of rest. If it has a velocity V at the beginning of t, then the equation

becomes

$$t = \frac{v - V}{g} \qquad (1)$$

The equation $gds = vdv$ gives by integrating

$$2gs = v^2 + C$$

and determining the constant by supposing the body to have a velocity V at the beginning of s, it becomes

$$2gs = v^2 - V^2 \qquad (2)$$

which when the body falls from a state of rest, or when $V = 0$ becomes the equation given in the text.

Lastly to find the relation between the time and space, as $v^2 = 2gs + V^2$, $dt = \frac{ds}{\sqrt{2gs + V^2}}$ whose corrected integral is

$$t = \frac{\sqrt{2gs + V^2}}{g} - \frac{V}{g} \qquad (3)$$

which may be otherwise expressed by putting S for the height due to the initial velocity V, V is then $\sqrt{2gS}$ and

$$t = \frac{\sqrt{s + S} - \sqrt{S}}{\sqrt{\frac{1}{2} g}}$$

We have mentioned in the text the applications of these theorems to the action of gravity, and here it is only necessary to add, that the value of g which we have given is exact only in our latitude; it varies on the earths surface from the elliptic figure of the meridians, and from the centrifugal force caused by its rotation. If g be the force in lat. 45° that in any other latitude l is found by the formula,

$$g' = g \left\{ 1 - 0.002837 . \cos 2\, l \right\}$$

and it is manifest that v and t are in the direct and inverse subduplicate ratio of g'.

(*b*) The integrals of the last note apply to the case of retarded motions by the simple artifice of making g negative to shew that it acts in a direction opposite to the motion. On this supposition equation (1) becomes

$$t = \frac{V - v}{g}$$

Equation (2) gives

$$2gs = V^2 - v^2$$

from which and the preceding we deduce

$$s = t \times \frac{V + v}{2}$$

or the space described in a given time is the arithmetical mean between those described in the same time with the initial and final velocities.

If we suppose two bodies, one projected from a centre of constant force with a velocity V and another to fall towards it by the force we have

$$t = \frac{v'}{g} = \frac{V - v}{g} \qquad \text{whence}$$

$$v' + v = V$$

or the bodies recede from each other with an uniform velocity V.

(c) It is easy to give an investigation for the velocity acquired by the action of a force varying as any power of the distance; but the time cannot be generally determined for the same hypothesis, for it involves the integration of a transcendental which can be effected only in a few cases; and therefore we prefer giving particular solutions. Where the force is as the distance, its intensity at the distance x is $\frac{gx}{r}$, $ds = -dx$, and the integral of equation (2) of the last chapter is

$$v^2 + C = 2\int \frac{-g}{r} x dx = -\frac{g}{r} x^2$$

and correcting so that $v = 0$ when $x = r$, the expression

becomes

$$v^2 = \frac{g}{r}\left\{r^2 - x^2\right\} \qquad (4)$$

an expression which gives $v = \sqrt{gr} \times$ sine of the arch whose cosine $= \frac{x}{r}$.

If we correct the integral, so that when $x = r$, $v = V$, we obtain the equation

$$V^2 - v^2 = \frac{g}{r}\left\{x^2 - r^2\right\}$$

corresponding to the case where a body is projected in a right line to or from the centre of force.

The expression of the time presents no difficulty, for we have by the third equation of rectilineal motion $dt = \frac{-dx}{v}$ $v = \sqrt{V^2 - \frac{g}{r}\{r^2 - x^2\}}$ and

$$dt = \frac{-dx}{\sqrt{V^2 - \frac{g}{r}\{r^2 - x^2\}}}$$

If in this we substitute for V^2 $\frac{g}{r}\{\xi^2 - r^2\}$ or suppose the initial velocity V to have been acquired in falling through the space $\xi - r$, we have

$$\sqrt{\frac{r}{g}}\, dt = \frac{-dx}{\sqrt{\xi^2 - x^2}}$$

and integrating

$$\sqrt{\frac{r}{g}}\, t = \text{arc. } \left(\cos. = \frac{x}{\xi}\right) + C$$

and determining the constant so that t begins when $x = r$

$$\sqrt{\frac{r}{g}}\, t = \text{arc. } \left(\cos. = \frac{x}{\xi}\right) - \text{arc. } \left(\cos. = \frac{r}{\xi}\right). \qquad (5)$$

If $V = o$, $\xi = r$ and the constant vanishes, which gives

$$\sqrt{\frac{r}{g}}\, t = \text{arc.} \left(\cos. = \frac{x}{r}\right) \qquad (6)$$

the time of describing r is therefore

$$\sqrt{\frac{g}{r}} \times \text{arc.} (\cos. = o) = \sqrt{\frac{g}{r}} \times \frac{p}{2}.$$

(d) The force of gravity varies inversely as the square of the distance from a single particle, but where many of them are united into a solid of a given shape, it is possible that the united attractions of them all may follow a different law. In the case of the sphere, which is most important, as the sun and planets are of that form, the attraction is exerted in the same manner as if the matter of the sphere were concentrated in its centre. A demonstration of this seems desirable before we proceed to the laws of the motion produced by such a force, that thus the utility of the formulæ which we investigate may be apparent. We begin by seeking the attraction which a circular ring of evanescent thickness exerts on a point situated any where in its axis. Let ABC, fig. 157, be such a ring, the attraction which a particle at B, which we will call m exerts on another at E is $\frac{gm}{BE^2}$, this compounded with an equal attraction from an equal particle at the opposite part of the circle produces a result

$$= \frac{2gm}{BE^2} \times \frac{DE}{BE}$$

and the attraction of the whole ring is

$$\frac{g \times 2p \times BD \times DE}{BE^3} \times \text{section of the ring.}$$

We now consider a spherical shell of evanescent thickness, let it be cut by a plane CDF, fig. 158, perpendicular to AB, the line joining its centre with the point B, then the differential of the segment FEC is a ring whose diameter is twice CD, or $2y$, and its thickness t that of the shell, it is therefore $2py \times t \times$ Cc or ds, the differential of the circular arch EC. Call DB x, then $rdx = ds \times y$, and the differential becomes $2ptdx \times r$, and by the preceding paragraph the attraction of this ring on a point at B is, putting CB $= u$,

$$\frac{2gptrxdx}{u^3}$$

To integrate this we remark that

$$AC^2 = CB^2 + AB^2 - 2AB \times BD$$

or in symbols

$$r^2 = u^2 + a^2 - 2ax$$

hence

$$x = \frac{u^2 + a^2 - r^2}{2a} \text{ and } dx = \frac{udu}{a}$$

which change our differential into

$$\frac{2gptr}{2a^2}\left\{ du + \{a^2 - r^2\} \frac{du}{u^2} \right\}$$

whose integral taken from $u =$ FB to $u =$ AB gives the attraction of the whole shell; it is

$$\frac{gptr}{a^2}\left\{ u - \frac{(a^2 - r^2)}{u} \right\}$$

and the definite integral is

$$\frac{gptr}{a^2}\left\{ \overline{a+r} - \overline{a-r} - (a^2 - r^2)\left\{ \frac{1}{a+r} - \frac{1}{a-r} \right\} \right\}$$

$$= \frac{gptr}{a^2}\left\{ 2r - \frac{(a^2 - r^2)\{-2r}{a^2 - r^2} \right\}$$

$$= \frac{4gptr^2 \times t}{a^2}.$$

But $4ptr^2$ is the mass of the shell, and therefore its action on an exterior point $= g \times \frac{\text{mass}}{a^2}$, the same as if its mass were collected in the point A. If the integral be taken from $u = r - a$ to $u = r + a$, or if the attracted point be placed within the shell, we obtain the attraction

$$= \frac{gptr}{a^2} \left\{ 2a + \frac{(r^2 - a^2) \times -2a}{r^2 - a^2} \right\} = 0$$

The spherical shell has therefore no action on a point placed within it.

Let now a sphere be proposed of uniform density, or at least composed of concentric strata each of which is equally dense throughout, we may conceive it resolved into a number of concentric shells of evanescent thickness, and as each of these attracts as if its matter were collected in the centre of the sphere, therefore so does their sum or the whole sphere.

If the point be supposed placed in the interior of the sphere, all the shells whose radii are greater than its distance from the centre exert no action on it, and it is acted on only by the sphere in whose surface it is placed, or of the radius a. The mass of a sphere of uniform density s is $\frac{4}{3} pr^3 \times s$, and therefore the attraction is in this case $\frac{4}{3} gspa$ or directly as the distance from the centre. In nature the fact is otherwise; for besides that the earth and other planets are elliptic in a slight degree, the density increases continually below the centre, and therefore this conclusion is only valuable as a beautiful theoretic result. We will dismiss this subject with mentioning that taking the integral of the shell's attraction from $u = a - r$ to $u = \sqrt{a^2 - r^2}$ we obtain

$$g \times \frac{1}{2} \frac{\textit{mass of shell}}{a^2}$$

and therefore if we draw from B a conical surface tangent to the sphere, the circle of contact divides the shell into parts of equal attraction towards B, and hence a spherical surface described through B and A, cuts the solid sphere similarly.

From this proposition it is manifest that the force which acts on falling bodies at or above the earth's surface, varies inversely as the square of the distance from its centre; for its difference from the spherical figure is so small, that the deviation from this law produced by its ellipticity may be neglected. We proceed to investigate the laws of the motion produced by such a force.

Let g be the force corresponding to r, the distance from the centre at the beginning of the motion, then the force at any other distance x is $\frac{gr^2}{x^2}$, and hence we have the equation

$$vdv = \frac{gr^2 \times -dx}{x^2}$$

whose integral is

$$v^2 = 2gr^2 \times \frac{1}{x} + \text{C}$$

and assigning to C such a value that v may vanish when $x = r$ we have

$$v^2 = 2gr \times \frac{r-x}{x}$$

In this law it is manifest that if the body fall down to the centre, the velocity is infinite. If r be infinite the velocity is finite till x vanishes, for let γ be the force at x, then $g = \frac{\gamma x^2}{r^2}$ and substituting this value

$$v^2 = 2\gamma x \times \frac{r-x}{r}$$

which when r is infinite becomes

$$v^2 = 2\gamma x$$

the velocity which would be acquired by the action of a *constant* force equal to γ, through the space x.

The time is investigated by the usual process.

$$dt = \frac{-dx \times \sqrt{x}}{\sqrt{2gr} \times \sqrt{r-x}}$$

To integrate this let $x = \varrho - \xi$, and $r - x = \varrho + \xi$, then

$$dt = \frac{\sqrt{\varrho-\xi} \times d\xi}{\sqrt{2gr} \times \sqrt{\varrho+\xi}} = \frac{\sqrt{\varrho-\xi}\,\sqrt{\varrho-\xi} \times d\xi}{\sqrt{2gr} \times \sqrt{\varrho+\xi} \times \sqrt{\varrho-\xi}}$$

$$= \frac{1}{\sqrt{2rg}} \left\{ \frac{\varrho d\xi}{\sqrt{\varrho^2-\xi^2}} - \frac{\xi d\xi}{\sqrt{\varrho^2-\xi^2}} \right. \text{ and integrating}$$

$$t \times \sqrt{2gr} = -\varrho \times \text{arc. } \left(\cos. = \frac{\xi}{\varrho}\right) + \sqrt{\varrho^2-\xi^2} + C$$

When $t = o$, $\xi = -\varrho$, and therefore $C = \varrho \times \text{arc. } (\cos. = -1) = \varrho \times p$, and the complete integral gives

$$t \times \sqrt{2gr} = \sqrt{x \times r-x} + \varrho \left\{ p - \text{arc. } \left(\cos. = \frac{\varrho - x}{\varrho}\right)\right.$$

$$= \sqrt{x \times r-x} + \frac{r}{2} \times \text{arc. } \left(\text{verse sine} = \frac{r-x}{\frac{1}{2} r}\right)$$

The time of falling down to the centre is

$$\frac{1}{\sqrt{2gr}} \times \frac{r}{2} \times \text{arc. } (vs = 2).$$

From these theorems we can readily derive the laws of uniformly accelerated motion; for if the space described be indefinitely less than r or x, we may put $r = x$, and calling $r - x = s$ we obtain

$$v^2 = 2gs, \text{ and } t \times \sqrt{2gr} = \sqrt{rs} + \frac{r}{2} \times \text{arc } \left(vs = \frac{2s}{r}\right)$$

or as an arch whose verse-sine is small, nearly coincides with its sine, and that sine is $q.p\ \frac{2\sqrt{rs}}{r}$,

$$t = \frac{1}{\sqrt{2gr}} \times 2\sqrt{rs} = \sqrt{\frac{2s}{g}}$$

Those who are desirous of further information on the subject of rectilinear motion may refer to Lardner on

Central Forces, where it is discussed at more length than could be allowed to it here.

CHAPTER III.

14. From the motions which would be generated in a particle of matter by the action of a single force, we proceed to consider the combination of its acceleration with a motion oblique to the direction in which it acts. By the first law of motion, a body once impelled continues to move in the line of impulsion with an uniform velocity. Let us suppose that the direction of its motion is at right angles to the line drawn from it to a centre of force, then, as its motion does not suspend the action of the force, it must be continually deflected from the rectilinear course, which it would otherwise have pursued, and describe a curve line round the centre. To conceive this more readily, let us imagine the time of the motion to be divided into a number of small and equal parts, and moreover, let the force be supposed to act per saltum, that is to communicate at the beginning of each instant by an impulse, a velocity equal to that which would have been produced by its continued action through the whole of the instant. In fig. 159, let AB be the space which would be thus described in the first instant, and let the projectile motion be capable of carrying the body to E in the same time, the real motion will be AF, the result of these two motions. If the

force ceased to act, FH would be described in the next instant, but it gives an impulse FI, and therefore FK is described, and in the third instant KL. Now if the instants be diminished indefinitely, the intermitting force approaches without limit to one acting continually, and the polygon AFKL becomes a curve. From this illustration of the nature of curvilinear motion can be derived a remarkable theorem, which merits our notice. Drawing HC, the triangle FCA=FCH, as they stand on equal bases FA and FH, and FCH=FCK as they are between the same parallels and stand on the same base FC, therefore, ACF=FCK. In the same way FCK=CCL, and so on; and therefore when a body describes a curve by the action of a force proceeding from a centre, the area described in an instant of time by the line connecting the body with the centre is a constant quantity. This theorem is well known in astronomy as one of Kepler's laws; equal areas being described in equal times by a planet round the sun, or by a satellite round its primary.

16. The species of the curve described depends in general on the velocity impressed on the body which is commonly named the velocity of projection, on the intensity and law of the force, and on the angle which the direction of the original impulse makes with the line drawn to the centre from the original place of the body, or radius vector. A beautiful specimen of these investigations is given by Newton in the first 17 propositions of the Principia, but for obvious reasons we decline entering into them, and confine ourselves to circular motion. In this particular case the law of force is of no importance, as the distance must always remain the same, and the angle of projection must be a right angle;

it remains therefore to seek the relation between the Velocity, Distance, and Force. Let BA, fig. 160, be the space which the body would describe in an instant of time by the velocity of projection; in that motion it would increase its distance from C by BD, and as the force constrains it to move in the circle, the space DD is that through which the force would make a body fall in the instant of time, and therefore is proportional to the force. $BA^2 = BC^2 - DC^2$ or ultimately $2\ BD \times DC$. Call the force at the distance AC g, and the velocity of movement in the circle V, then the velocity generated by the force g, in the space BD or $s = \sqrt{2gs}$, and it is to $V :: 2s : DA$, for twice the space s would be described with the last acquired velocity in the time of falling through s, or of describing AD. Hence

$$V^2 : 2gs :: AD^2 : 4s^2, \text{ and } V^2 = 2s \times \frac{AD^2}{2s},$$

but AD and AB are ultimately equal, and therefore $\frac{AD^2}{2s} = AC$ or R, and we obtain $V^2 = 2g \times \frac{R}{2}$, which when compared with equation (1) of the 2d Chapter shews that the force g would generate in half the radius a velocity equal to that with which the body revolves in the circle, supposing it to be constant.

17. This equation enables us to determine a variety of questions connected with circular motion: as the force $F = \frac{V^2}{R}$, it follows that where different circles are described about the same centre, the forces must be as the squares of the velocities directly, and the distances inversely. The velocities may be expressed in terms of the distances and periodic times; the space described is

$2pr$ an entire circumference, this divided by the time of a revolution T gives V: by this substitution the force measured by the velocity which it can give in a second, $F = \frac{4p^2 R}{T^2}$, or it is as the distance directly and inversely as the square of the time of revolution.

18. This latter equation gives the relation between the time and the distance where the force is known. If the force be constant, the fraction $\frac{R}{T^2}$ must also be constant, and therefore $T \propto \sqrt{R}$. If the force be proportional to the distance $R \propto \frac{R}{T^2}$, and therefore T is constant, V being as R. Lastly if $F \propto \frac{1}{R^2}$, $T^2 \propto R^3$ and $V \propto \frac{1}{\sqrt{R}}$.

19. An example of a force varying as the distance is exhibited in the conical pendulum, which we have already noticed when treating of the Steam Engine; it consists of two balls, supported by rods of equal length, which are united by a hinge-joint: the apparatus is whirled round a vertical axis, and whatever be the velocity with which they move, the time of their revolution is constant, unless the impelling force either vanish or be indefinitely increased. The ball A, fig. 161, in its revolution round VE, describes a circle whose radius is AC: and by art. 8 of Statics, the force acting to bring A back to the vertical is g, the force of gravity, multiplied into the sine of AIC or $g \times \frac{AC}{IA}$. If we substitute

this for F in the second formula of Art. 17, we obtain

$$g \times \frac{AC}{IA} = \frac{4p^2}{T^2} AC, \; T^2 = \frac{4p^2 \times IA}{g}$$

which is evidently independent of AC, and varies solely as the square root of the length of the rod.

20. The planetary motions afford a near approximation to circles described with a force varying inversely, as the square of the distance. It is proved by observations, that their orbits are ellipses deviating very little from circles, and therefore we may apply the laws of Art. 18 to obtain them. This enquiry is interesting, as it teaches the law of universal gravitation, and shews the nature of that connexion which links together bodies, placed at distances far surpassing the measures of space with which the mind is familiar. If we examine the sun and his planets, we observe that each of these describes, in equal times, equal areas round the sun; hence the sun is the centre of force to it. Secondly, the periodic times of the planets are in the sesquiplicate ratios of the distances, or $T^2 \propto R^3$, which gives $F = \frac{M}{R^2}$. M being a constant quantity. Jupiter and Saturn have several satellites revolving round them in orbits which to our observations are circular; here also $T^2 \propto R^3$ and $F = \frac{m}{R^2}$, the value of m, which is proportional to the quantity of matter in the central body, being different in each system. And lastly in the case of the earth to which, as it has but one satellite, the harmonic law cannot apply, we prove that its attractive force diminishes according to the same law as that of the other heavenly bodies. The radius of the moon's orbit is about 60 times the radius of the earth, its periodic

time is $27\frac{1}{3}$ days nearly; hence it is easy to compute the length of the arch described by the moon in a minute, and the square of this divided by the diameter of the orbit gives the deflection of the arch from its tangent, or the space through which a body would fall by the action of the central force in a minute: it is about sixteen feet, the same that is described by gravity at the earth's surface in a second. Where the space is given, the forces are inversely as the squares of the times; therefore the force at the earth's surface : that at the moon :: $60^2 : 1$, or inversely as the square of the earths's radius to the square of the moon's distance. This is but a rude approximation; the conclusions which the circular hypothesis gives are rigorously proved by more precise investigations; from the law of gravitation thus indicated flow the most minute irregularities of the heavenly motions; and so far from the occurrence of any which Physical Astronomy (as this part of Dynamics is named) cannot explain, it has pointed out many which observation alone could never have detected. This, however, is the Eleusis of Science, and few are the initiated who can explore its mysteries; "Pauci quos ardens evexit ad æthera virtus."

21. From the tendency of a body to move in a straight line, it may be easily understood, that if the central force which acts on a material point moving in a circular orbit be suddenly removed, it will fly off in the direction of the tangent to the circle. This is illustrated by many facts which continually fall under our observation; when a stone is whirled in a sling, on letting go one of the cords, it flies off with great velocity; and this was the principle of most of the warlike en-

gines of the ancients. In the great manufactories of cutlery the work is ground on very large grindstones revolving with immense rapidity, and it not unfrequently happens that they fly in pieces, and are scattered in every direction; even the massive iron wheels of steam-engines yield in the same manner. The tendency to move in the direction of the tangent is equivalent to a force acting from the centre, as it cannot have such a motion without increasing the distance from the centre; and this force is called Centrifugal. The Centrifugal force is equal to that central force which is capable of making a body revolve with the actual velocity in the given orbit, and therefore can be determined from the formulas of Art. 17. In many cases the central force is the cohesion of matter, as when a ball is whirled round by a string, and it is evidently a question of practical value to know the tension of the string thus produced, that the strength of the parts of machinery, in similar circumstances, may be apportioned to the strain thus induced on them. If the weight carried be supposed concentrated in a point, and be called W, then the effect of the centrifugal force is $\frac{4p^2 R}{T^2} \times W$, or where the velocity is given it is $\frac{V^2}{R} \times W$. In general as the different parts of the body are at different distances from the centre of motion, their centrifugal forces are different, and therefore other principles must be used to obtain the strain thus produced: we shall return to them hereafter.

22. The rotation of the earth must excite a centrifugal force in bodies at its surface, which will affect the superficial gravity; it always acts in opposition to it, and being unequal in different places will occasion the

same body to weigh differently at different places, provided that its weight be measured by the flexure of a spring, or any other means independent of gravity. In effect, any point on the earth's surface (supposed spherical) describes a parallel whose radius is the radius of the sphere multiplied into the cosine of latitude. All these parallels are described in the same time of 24 hours; therefore the centrifugal forces are as the cosines of latitude: the centrifugal force acts in the plane of the circle described, and therefore obliquely to the radius, making with it angle = latitude; now its effect in the direction of the radius, or of gravity, is found by multiplying it into the cosine of this angle; and since the force is as the cosine, and the action of it against gravity also as the cosine, the diminution of weight produced by it is as the square of this function of the latitude. The numerical value of it may be readily computed by the foregoing formula, but the results do not precisely agree with experiment, as the earth is not a Sphere but a Spheroid. This elliptic form is a necessary consequence of the centrifugal force; for gravity must be most diminished at the equator, and therefore supposing our planet spherical and covered with sea, the equilibrium of its surface could not be preserved, it would therefore cease to be spherical; the parts about the poles being depressed below their original level, and those towards the equator rising above it, till diminished gravity is compensated by greater height. But in this new form the result of the attraction of each particle is no longer directed to the centre as in a sphere; and this modifies the effect of the centrifugal force. The actual shape of the earth is a spheroid, whose equatorial diameter exceeds its polar about 25 miles; and from this accumulation of matter arise several astronomical phenomena, which are detailed in Brinkley's treatise, to which I must refer my readers, as it is impossible to produce any part of these difficult investigations which would be intelligible, and a mere detail of the phenomena is all that can be required.

NOTES ON CHAP. III.

(*a*) The motions depending on central forces are so important, that it may not be superfluous to give some additional information beside that contained in the text. In the 17 first propositions of the Principia, many applications of the method used in the text for the circle are to be found, it therefore is unnecessary to pursue it further. Instead of it we will give a few of the principles used in Analytical Mechanics, and deduce from the differential equations of accelerated motion some of Newton's propositions, that the two modes of investigation may be compared. It is manifest from what has been stated in the first part, that we may consider a particle moving in space by the action of any accelerating force, as moved by three forces parallel to axes of rectangular coordinates. We have denoted these components by the symbols X, Y, and Z, in the Notes on Chap. II. of the first part; and as the motion of a point, when estimated in a given direction, is not affected by forces at right angles to that direction; therefore the velocity of the motion parallel to x depends on X alone, without regard to Y and Z; the same thing is true of them, and therefore the equations (6) of the Notes on Chapter I apply, and we have

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=X\\ \frac{d^2y}{dt^2}&=Y\\ \frac{d^2z}{dt^2}&=Z\end{aligned}\right\}\quad (m)$$

These are the differential equations of motion, and they contain all that is necessary to solve any proposed question of this class. If we multiply each of them by dx, dy, and dz respectively, and add them together we have

$$\frac{dxd^2x+dyd^2y+dzd^2z}{dt^2}=Xdx+Ydy+Zdz$$

and integrating

$$\frac{dx^2+dy^2+dz^2}{dt^2}=C+2\int Xdx+Ydy+Zdz$$

but the uumerator of the first member $=ds^2$ the element of the curve described, and $ds\div dt=v$ the velocity, therefore

$$\frac{ds^2}{dt^2}=v^2=C+\int Xdx+Ydy+Zdz \qquad (n)$$

If the quantity included within the sign of integration be an exact differential, then the velocity gained in passing from one point of space to another is independent of the form of the trajectory, and this is the case when the forces which act are directed to fixed centres, and are also functions of their distances from them. Let R be a force directed to a centre whose coordinates are a, b, and c, and let the distance of the point from it be r, the part of X, which arises from it is $R \times \frac{x-a}{r}$, the Y is $R \times \frac{y-b}{r}$, and the Z is $R \times \frac{z-c}{r}$, hence

$$Xdx+Ydy+Zdz = R \times \frac{\{(x-a)dx+(y-b)dy+(z-c)dz\}}{r}$$

now

$$r = \sqrt{(x-a)^2+(y-b)^2+(z-c)^2},$$
$$dr = \frac{(x-a)dx+(y-b)dy+(z-c)dz}{r}$$

and

$$Xdx+Ydy+Zdz = Rdr$$

therefore in this case we have

$$v^2 = C+2\int Rdr \qquad (n')$$

which is evidently true, though R be the result of several other forces. This is a case of the law called the Conservatio Virium Vivarum, and it contains Newton's Prop. 40 for the value of $\int Rdr$ is given between certain limits without reference to any orbit: if therefore two bodies, one moving in a curve, and another in a right line to or from a centre of force, have equal velocities at any equal distances, they will have equal velocities at any other equal distances.

If there be but one force directed to a fixed centre, the trajectory must be in a given plane, we may therefore take $z=0$, and we obtain by multiplying the first and second of equations (m) by y and x, and subtracting

$$\frac{xd^2y-yd^2x}{dt^2} = Xy-Yx$$

but in this case $Xy = \frac{Rxy}{r}$, $Y = \frac{Ryx}{r}$, therefore the second member vanishes, and

$$\frac{xd^2y - yd^2x}{dt} = 0,$$

integrating

$$xdy - ydx = cdt \qquad (o)$$

To know what the first member is, let us transform x and y into a radius vector r, and an angle u made with the axis of x, we have $x = r \cos. u$, $y = r \sin u$, $dx = -r \sin u\, du$, $dy = r \cos u\, du$, which being substituted in equation (o) give

$$cdt = r^2 \{\sin.^2 u + \cos.^2 u\}\, du = r^2 du$$

but $r\, du$ is the differential arch of a circle whose radius is r subtending the angle du, and $r^2 du$ is twice the corresponding sector, or the differential of the area described round the centre by the radius vector; call it A, and

$$dA = \tfrac{1}{2} cdt,$$

integrating

$$A = \tfrac{1}{2} c \times t$$

No constant is added, as the area is supposed to begin with the time, and this shews, that when there is but one centre of force, the area described is as the time.

By means of this integral we can climinate dt from equation (n) and thus obtain an equation between x, y, and R which will determine the nature of the trajectory where the law of force is known, or vice versa. We proceed to give a few examples:

(b) Supposing the force of gravity constant, and acting in parallel lines, let it be proposed to find the trajectory of a projectile, omitting the consideration of the air's resistance. In this case there is no force acting in the direction of x, and that parallel to y is the force of gravity g. Hence equation (m) becomes

$$\frac{d^2x}{dt^2}=0,\ \frac{d^2y}{dt^2}=-g$$

and integrating we obtain

$$dx=cdt,\ \frac{dy}{dt}=-gt+c'$$

$$x=ct,\ y=-\tfrac{1}{2}gt^2+c't,$$

no constants being added in the second integration as x and y begin with t. To determine them we remark that $\frac{x}{t}=c$, c is therefore the uniform velocity with which the abscissa is described. Now we know this, for if V be the velocity of projection, and ϵ the angle made by its direction with the horizon, the horizontal velocity $=$ V cos. ϵ, and hence $t=\frac{x}{V\times\cos.\epsilon}$. Substitute this value in the second and fourth equations and we obtain $\frac{dy}{dt}=$ velocity in the direction of $y=\frac{-gx}{V\cos.\epsilon}+c'$; at the commencement of the motion the vertical velocity is V sin. ϵ and $x=0$, therefore $c'=V\sin.\epsilon$, which makes the fourth equation

$$y=x\,\text{tang.}\,\epsilon-\frac{gx^2}{2V^2\cos.^2\epsilon}$$

which may be made more convenient by expressing V in terms of H the height due to it, $V^2=2gH$, and

$$y=x\,\text{tang}\,\epsilon-\frac{x^2}{4H\times\cos^2\epsilon}$$

the well known equation of the parabola. If we put $y=0$, we obtain the abscissa of the summit of the parabola, $x' = \text{H} \times \sin 2\,\epsilon$, and its ordinate $y' = \text{H}\sin.^2\epsilon$, and the latus rectum $\frac{x'^2}{y'} = 4\ \text{H}\ \cos.^2\epsilon$. The equation ($n$) is in this case

$$\text{V}^2 = \text{C} - 2\int g dy = \text{C} - 2gy : \text{but when } y = 0,$$

$$v^2 = \text{V}^2, \text{or } 2g\text{H},$$

therefore $v^2 = 2g\,\{\text{H} - y\}$. If then we draw through the origin of the coordinates a vertical HA, fig. 162, and a parallel to the horizon AL, a body falling from AL to the parabola will acquire the velocity in the curve. $\text{LS} = \text{H} - \text{H}\sin^2\epsilon = \text{H}\cos.^2\epsilon$, or one-fourth of the latus rectum, and therefore AL is the directrix. The velocity in the curve at any point would therefore be acquired in falling through a space equal to the distance from the focus. See Lardner, sect. 5.

(c) If the force be inversely as the square of the distance, the investigation of the trajectory is rather more complicated; to facilitate it we will use polar coordinates instead of rectangular. Let CA fig. 163. be a radius vector, r the square of the differential of the curve; $\text{BA}^2 = \text{BD}^2 + \text{DA}^2$, DA is manifestly dr, and calling the angle C du, $\text{BD} = rdu$, hence $ds^2 = dr^2 + r^2du^2$. The force at the unity of distance being g, $\frac{g}{r^2}$ is that at r, and therefore equation (n) becomes $\frac{ds^2}{dt^2} = \int \frac{-2gdr}{r^2} - b = \frac{2g}{r} - b$: for dt we substitute its value in equation (o) expressed in polar coordinates, and obtain

$$\frac{dr^2 + r^2du^2}{r^4du^2} \times c^2 = \frac{2g}{r} - b$$

Substituting for $r\frac{1}{z}$, and resolving for du

$$du = \frac{dz}{\sqrt{\frac{-b}{c^2} + \frac{2gz - z^2}{c^2}}}$$

which on integration gives the equation of the trajectory in u and z.

The integration becomes more easy by substituting for z, $\xi + \frac{g}{c^2}$, which gives

$$du = \frac{-d\xi}{\sqrt{\frac{g^2 - bc^2}{c^4} - \xi^2}}$$

or putting $m = g^2 - bc^2$

$$du = \frac{-d\xi}{\sqrt{\frac{m^2}{c^4} - \xi^2}}$$

whose integral is

$$u - u' = \text{arc}\left(\cos. = \frac{\xi \times c^2}{m}\right) = \text{arc.}\left(\cos. = \frac{\frac{c^2}{r} - g}{m}\right)$$

hence

$$\cos.\ u - u') = \frac{c^2}{rm} - \frac{g}{m},\ \frac{1}{r} = \frac{g}{c^2} + \frac{m}{c^2} \times \cos.\ (u - u')\ (1)$$

whose analogy with the polar equation of a conic section is evident; it being $\frac{1}{r} = \frac{1 + e \cos.\ (u - u')}{\frac{1}{2}p}$.

On comparing the terms we obtain $e = \frac{m}{g}$, a the transverse semiaxis $= \frac{g}{b}$, and thus can determine the species of the section. When $e = 1$ or $b = 0$, it is a parabola, and an ellipse when b is positive. If the angle u begin at the vertex of the curve, u' is evidently cypher.

The integral (n') gives in this case

$$v^2 = \frac{2g}{r} - b,\ b = \frac{g}{a}$$

therefore

$$v^2 = g\left\{\frac{2}{r} - \frac{1}{a}\right\} = g\left\{\frac{2a-r}{ar}\right\} \qquad (2)$$

containing quantities which can be known from observation. It shall be shewn that the force by which a circle is described $= \frac{V^2}{r}$, hence $\frac{g}{r^2} = \frac{V^2}{r}$, $g = V^2 r$,

and substituting

$$\frac{v^2}{V^2} = \frac{2a-r}{a} \qquad (3)$$

giving the ratio between the velocity in the trajectory, and that in a circle at the same distance. And lastly $e = \frac{m}{g}$ gives $1-e^2 = \frac{bc^2}{g^2}$, $a\ (1-e^2) = \frac{bc^2 \times a}{g^2} = \frac{c^2}{g}$, from which $c^2 = g \times a(1-e^2)$: by the equation $2A = cT$ we have

$c^2 = \frac{4A^2}{T^2}$. If T be the periodic time in the ellipse, A is $p^2 a^4\ (1-e^2)$, therefore $c^2 = \frac{4p^2 a^4\ (1-e^2)}{T^2} = ga \times (1-e^2)$, hence

$$T^2 = \frac{4p^2 a^3}{g} \qquad (4)$$

or the periodic time is as $a^{\frac{3}{2}}$, and the area described in the unity of time as $\sqrt{L}$ the latus rectum.

The time may be found by integrating the equation $r^2 du = cdt$, but this is foreign to our subject, being in fact Kepler's problem.

(d) As an example of the investigation of the force where the trajectory is given, we will take the circle

where the centre of force coincides with its centre. In this case the equation (n') shews that V must be constant as dr is cypher, dt. is therefore $\frac{ds}{V}$, and $\frac{d^2y}{dt^2} = -Y$ becomes

$$\frac{V^2 d^2y}{ds^2} = -\frac{Fy}{r},\ dy = \frac{xds}{r},\ d^2y = -\frac{ds^2 \times y}{r^2},$$

and therefore $\frac{V}{r} = F$.

If the centre of force be different from the centre of magnitude, call CE fig. 164 e, EKa, CPr, PS, C, and $a^2 - e^2$, b^2. On this supposition the equation of the circle is $y^2 + (x-e)^2 = a^2$, or as $y^2 + x^2 = r^2$, $r^2 - 2\ ex = b^2$, hence we derive $ds = \frac{-dx \times a}{y}$, $dy = ds \frac{\times x - e}{a}$, $dx = -ds \times \frac{y}{a}$.

In the equation (n),

$$ds^2 = v^2 = \frac{c^2 ds^2}{(xdy - ydx)^2} = -2\int Fdr$$

which becomes with these substitutions

$$\frac{c^2a^2}{(x^2+y^2-ex)^2} = \frac{c^2a^2}{(r^2-ex)^2} = v^2 = -\int Fdr,$$

differentiating

$\frac{c^2a^2 \times 2edx}{(r^2-ex)^3} = 2Fdr$, but $ex = \frac{r^2-b^2}{2}$, $edx = rdr$, and $r^2 - ex = \frac{r^2+b^2}{2}$, so that

$$\frac{8c^2a^2 \times r}{(r^2+b^2)^3} = F = \frac{v^2 \times 2r}{r^2+b^2}$$

but SC $= \frac{b^2}{r}$ ∵ SP or C $= \frac{r^2+b^2}{r}$, and hence

$$F = \frac{8c^2a^2}{r^2 \times C^3} = \frac{2v^2}{C}$$

If we call x the space due to the velocity v, supposing F constant, $F = \frac{v^2}{2x} = \frac{2v^2}{C}$, and therefore $4x = C$.

These theorems apply to all orbits; for the element ds may be considered as a part of the equicurve circle, and therefore $F = 2v^2 \div$ chord of curvature, and v is the velocity which the force would generate in the fourth part of the chord if it were constant, and thus the circumstances of the motion in a given orbit may be determined.

CHAPTER IV.

23. The consideration of that species of curvilinear motion which is produced by a central force combined with a motion of projection, naturally leads to the case where the body is subjected to the action of accelerating forces, but is constrained to move in a given curve or surface. From the principles of statics it is evident that it must exercise a pressure on the surface which supports it, but this excites a reaction of the surface, which must be considered as an additional force, acting perpendicularly to the path of the body. The reaction being perpendicular to the direction of the motion has no effect on the velocity, which is therefore independent of the curve described, and as this principle is of great im-

portance, we will prove it more in detail, which can readily be effected by the theorem of Art. 6.

Let BD fig. 165 be the curve in question, take in it an indefinitely small part CD, and take GA and HA equal to CA and CE. Let the force at C be F, a function of the distance, then supposing CD coincident with the tangent, the force F may be resolved into two, one coincident with the direction of the tangent, and the other perpendicular to it, which is destroyed by the reaction of the curve, therefore the first alone need be considered: it is found by multiplying F into the cosine of the angle DCE. Now cosine $(DCE) = CE \div CD$, therefore the action of the force along the curve which we call $F' = F \times CE \div CD$. If we take two lines to represent DB and BA, and describe on them as axes curves which are the loci of the forces acting at each part of them, it has been proved that twice the areas of the curves are the squares of the velocities, and we are to compare them. The increment of the first curve's area is the space CD*dc*, which is ultimately the rectangle under C*c* and CD; $Cc = F' = F \times CE \div CD$, and $Cc \times CD = F \times CE$. In the second curve $Gg = F$ for $GA = CA$, and the force is equal at equal distances; therefore $Gg \times GH = F \times GH = F \times CE$, or the same as the increment of the other area. Since then they increase equally and begin together, they are equal, and therefore the squares of the velocities generated in descending through BD and BH. This remarkable theorem is however only true where BD is of a continued curvature, for if it were composed of segments making finite angles with each other, there would be a loss of velocity at each angle proportioned to the square of the sine of its half.

24. Besides the pressure on the curve produced by

that part of the force which acts perpendicularly to it, another arises from the velocity of the moving body, as it tends to proceed in the direction of the tangent; which is manifestly identical with the centrifugal force mentioned in the last chapter. It is well known that curves are the limits of polygons, whose sides and angles are indefinitely diminished. Now suppose a body describing AB fig. 166 with the velocity V, its impact on BC is to its original motion as sin ABD : 1 ; let AE and EB be perpendicular to AB and BC, the angle E=ABD and therefore the impact on BC $\propto$ V $\times$ sin E: but sin E = AB $\div$ BE, let t be the time of describing AB, V : AB $\propto$ 1″ : t, AB=Vt, sin E = Vt $\div$ BE. We may substitute for this impact the pressure of a force φ which could destroy in the time t the perpendicular motion, and we have by equation (1) of the first Chapter.

$$\varphi t = \frac{MV^2}{BE} \times t$$

or putting M unity and considering that BE is ultimately the radius of a circle coinciding with the curve at that point, and therefore of the same curvature with it

$$\varphi = \frac{V^2}{R}$$

The formula already found for central forces in a circle.

25. The simplest case of confined motion is when a body moves on an inclined plane by the force of gravity. It has been proved, Statics Art 19 that in this case gravity is resolved into two forces, one perpendicular to the plane which is not considered at present, the other parallel to it which may be called $g \times$ sine of inclination. This is constant, and therefore, the motion down an inclined plane is uniformly accelerated, so that the equations of art. 9 can be applied: they give

$$V = \sqrt{2g \times \sin. I \times S},$$
$$T = \sqrt{\frac{2\,S}{g \times \sin I}}.$$

If we consider that the length of an inclined plane is to its height as 1 : sin I, and that S = L the length of the plane, we may transform these equations into others of a much more elegant form.

$$V = \sqrt{2g \times H \times L \div L} = \sqrt{2\,g\,H} \quad (1)$$
$$T = \sqrt{\frac{2\,L^2}{g \times H}} = \frac{L}{\sqrt{\tfrac{1}{2}g} \times \sqrt{H}} \quad (2)$$

26. The first of these equations coincides with the conclusion of Art 23 for $\sqrt{2\,g\,H}$ is the velocity acquired by a body falling freely through H, and therefore the velocity acquired in descending through any portion of an inclined plane is the same as that acquired in falling through its perpendicular height. The space described is as the square of the velocity, and twice the length would be described in the time of the fall with the last acquired velocity.

27. The time is as $\frac{L}{\sqrt{H}}$; therefore the times of describing planes, whose heights are equal are as their lengths, and if the lengths be given, the times are inversely in the subduplicate ratio of the heights: if the planes be similar L ∝ H and the times are as the square roots of the lengths or heights.

28. Let it be proposed to find HM fig. 167 the portion of the inclined plane IH, which is described while a body falls freely through HB, then

$$\frac{HM}{\tfrac{1}{2}g \times \sin I} = \frac{HB}{\tfrac{1}{2}\,g} \quad \text{and}$$

HM = HB × sin I, or MB is a perpendicular to IH. Hence it follows that in the time of describing the plane, a body falls freely through the intercept of the vertical between H and a perpendicular to IH drawn through I.

29. Let it be proposed to find the Locus of the inclined planes whose lower extremity is a given point, and which are described in the same time. Let EM be one of them, fig 168, $\frac{EM^2}{EH}$ must be constant, and this is the case if M be in a semi-circle described on VE, the line described in the given time by a body falling freely; for $\frac{EM^2}{EH}$ is always equal to EV, and therefore

$$\frac{EM^2}{\frac{1}{2}g \times EH} = \frac{EV}{\frac{1}{2}g} \text{ or Time(EM)} = \text{Time(EV)}$$

Therefore all chords of a circle drawn through the extremities of a vertical diameter are described in the time of falling through the diameter itself. The velocity acquired in descending through EM $= \sqrt{2g \times HE}$ or

$$\sqrt{2g} \times \sqrt{\frac{EM^2}{EV}} \text{ or } \sqrt{\frac{g}{r}} \times EM.$$

This principle was turned to account by Robins, in his experiments on Gunnery, and it gave an easy means of measuring the velocity of shot. The apparatus which he used was a block of elm, suspended by a rod from a centre on which it turned freely, so that when swung it described an arch of a circle; the shot was fired against the block, and if properly directed communicated to it a velocity which was to its own in the reciprocal proportion of their masses; the chord of the arch which the block described in consequence of this impulse was measured, and as $\frac{r}{g}$ was known the velocity could be computed.

For the modification which the theory of motion on inclined planes, receives from the action of friction see the note. (*b*)

NOTES ON CHAP. IV.

(*a*) When a material point placed on a surface is subjected to the action of any accelerating forces, it has been stated in the text, that the reaction of the surface must be considered as a new force; we will denote it by the symbol N. Let us call the angles which it makes with the axes of coordinates, ω, ω', ω'' then the equations (*m*) of the note (*a*) on the last chapter become

$$\frac{d^2x}{dt^2} = X + N\cos\omega, \quad \frac{d^2y}{dt^2} = Y + N\cos\omega',$$

$$\frac{d^2z}{dt^2} = Z + N\cos\omega''$$

with a due attention to the signs of the forces.

The equation (*n*) of the same note also holds in this case; for multiplying each of the above by the differential of its coordinate and adding them

$$\frac{dx\,d^2x + dy\,d^2y + dz\,d^2z}{dt^2} = X\,dx + Y dy + Z dz$$

$$+ N\{\cos\omega \times dx + \cos\omega'\,dy + \cos\omega'\,dz.\}$$

The first member is half the differential of $\frac{ds^2}{dt^2}$ and the coefficient of N in the second = 0, for N is normal to the trajectory, it therefore makes with the tangent to it a right angle whose cosine = 0, but the cosine of the angle made by any two lines is the sum of the products of the cosines of the angles, which each of them make with the three axes. The cosines of the angles made by the tangent with them, are by the theory of curves

$$\frac{dx}{ds}, \frac{dy}{ds} \frac{dz}{ds}$$

therefore in this case

$$0 \text{ or } \cos(90^\circ) = \frac{dx \times \cos \omega + dy \times \cos \omega' + dz \times \cos \omega''}{ds}$$

Hence the equation becomes

$$\left(\frac{ds^2}{dt^2}\right) = 2\text{X}\, dx + 2\text{Y}\, dy + 2\text{Z}\, dz,$$

$$\frac{ds^2}{dt^2} = \text{C} + 2\int \text{X}\, dx + \text{Y}\, dy + \text{Z}\, dz = \text{V}^2$$

which when the part included under the sign of integration is an exact differential, shews that the velocity is independent of the trajectory.

The force N is evidently the snm or difference of the perpendicular result of the accelerating forces, and of the centrifugal force.

The trigonometrical theorem to which we have referred may be thus proved, let AB and AC fig. 169, be two lines, draw through their intersection three rectangular axes: let AD and AF be their projections on the plane of $x\,y$. Draw with the radius unity the arches BC $= x$, BD = comp. γ, CF = comp. γ', CD = u, and DF = $\epsilon - \epsilon'$, the Greek letters denoting the same as in the notes on the second chapter of Statics. Then we have by Napier's Rules, $\cos u = \cos(\epsilon - \epsilon') . \sin \gamma'$, sin (CDF) or cos BDC = $\frac{\cos \gamma'}{\sin u}$ and by spherical trigonometry.

$$\cos x = \cos u \sin \gamma + \sin u \cos \gamma \cos \text{BDC}$$

$$= \cos u \sin \gamma + \cos \gamma \cos \gamma' \times \frac{\sin u}{\sin u}$$

$$= \sin \gamma \sin \gamma' \left\{ \cos \epsilon \cos \epsilon' + \sin \epsilon \sin \epsilon' \right\} + \cos \gamma \cos \gamma'$$

but by note (*d*) chapter 2d Statics, $\cos \alpha = \sin \gamma \cos \epsilon$, &c. which gives

$$\cos x = \cos \alpha \cos \alpha' + \cos \beta \cos \beta' + \cos \gamma \cos \gamma'$$

(*b*) We will give an instance of the application of the formula of the last note, to the descent down an inclined plane, though in this particular case it could be more easily solved by methods analogous to those used in the text: let the origin of the coordinates be at the point where the motion commences; then on the hypothesis of the proportionality of friction to pressure, fN is the friction, which retards the descent, and therefore is negative.

In the inclined plane $N = g \times x \div l$, but for generality we put it

$$g \frac{dx}{ds}; \text{ hence } X = -fg \frac{dx^2}{ds^2},$$

$$Y = g - fg \frac{dxdy}{ds^2} \text{ and } Z = 0$$

$$\text{which give } X\,dx + Y\,dy = g \left\{ dy - fdx \times \frac{dx^2 + dy^2}{ds^2} \right\}$$

$$= g \left\{ dy - fdx \right\}$$

hence $v^2 = C + 2gy - 2gfx$ (1)

an equation which is time of all curves, and indicates that if $C = 0$, the velocity acquired in descending through an arch of a curve, is that acquired in falling through the portion of its height, which is intercepted between the horizontal line, and another making with it angle whose tangent is f.

For the time we have in the plane

$$\frac{dt}{} = \frac{ds}{v} = \frac{dx}{\cos I \times \sqrt{2g \times (x \times \text{tang } I - fx)}}$$

$$= \frac{dx}{2\sqrt{x}} \times \frac{\sqrt{2}}{\sqrt{g(\sin.\cos I - f\cos^2 I)}}$$

and integrating

$$t = \frac{\sqrt{2x}}{\sqrt{g\{\sin.\cos I - f\cos^2 I}} = \frac{L}{\sqrt{\tfrac{1}{2}g} \times \sqrt{H - fB}} \quad (2)$$

Let λ be the space described on the plane, while a body falls freely through H then we obtain

$$H = \frac{\lambda}{\sin I - f\cos I} \quad \text{or putting } f = \text{tang. } \varphi,$$

$$H = \frac{\lambda \cos \varphi}{\sin I \cos \varphi - \cos I \sin \varphi} = \frac{\lambda \cos \varphi}{\sin (I - \varphi)}$$

If therefore we draw a line making with the perpendicular let fall from the right angle an angle $= \varphi$ on the side of the vertex, it will determine λ.

Let it be required to find the locus of all planes described in the same time, and having a common vertex, take AB fig. 170, for the space described in a given time by a body falling freely, call it v and let B be the origin; AE the height of any plane is $v-y$, EC $= x$ and the length AC $= \sqrt{x^2 + (v-y)^2}$, and equating the value for the time down it, to that through v we find

$$\frac{x^2 + (v-y)^2}{v - y - fx} = v$$

which coincides with the equation of a circle which passes through the origin, becoming $x^2 + y^2 - vy + fvx = o$, comparing this with $x^2 + y^2 - 2ax - 2by = 0$ we find a b the coordinates of the centre, respectively, $-\frac{1}{2}fv$ and $\frac{1}{2}v$, therefore make the angle BAD $= \varphi$, and draw a parallel to BD through the point of bisection of v, a circle described with the the centre F and radius FA gives the required locus, which is the segment ACB.

And lastly to determine which of all planes having a given base is described in the least time, since the numerator of the first expression of the time (2) is given, the time is a minimum when its denominator sin. cos. I$-f$ cos. ^{2}I is a maximum, this becomes

$$\tfrac{1}{2}\text{ sin. }2I-f\tfrac{1}{2}\{1+\text{cos. }2I\} = \text{max.}$$

and taking 2 I as the independent variable, and differentiating

$$\cos 2I+fx \text{ sin. } 2I=0,$$
$$\text{tang } 2I=-\text{cotang } \varphi,$$

and therefore the supplement of 2I is together with φ equal to a right angle, or I$=45°-\varphi$. If the friction vanish, the minimum plane is inclined at 45°, and those which make equal angles above and below it are dedescribed in the same time.

CHAPTER III.

30. The circle being the line whose properties are best known after the right line, we proceed to consider a point constrained to move in it, while influenced by gravity; we will for simplicity suppose that the plane of the motion is vertical. Let LBL′, fig. 171, be a portion of the circle described, then it is evident from art. 23 that a body in descending from L to A acquires the same velocity as if it fell from M to N: the velocity acquired in descending to the lowest part of the curve B is

that generated in falling through its verse-sine MB, and as any velocity must be destroyed in the same space as that in which it is generated, the body, if permitted to pass B, will ascend through an equal arch BL′, when all its motion being destroyed it will return to L, and thus oscillate continually if we suppose all resistance removed. The time of describing LB cannot be accurately computed; the integral calculus gives expressions for it which are infinite series, and therefore useful only where the oscillations are of small extent. In this case by a construction similar to that employed in art. 12, we can obtain a correct result without refering the circle to the cycloid as is done in most elementary works. If an arch be very small, its excess above its chord being incomparably less than itself, may be neglected, and we may consider the square of the arch as equal to the rectangle under the verse-sine and diameter, as in Art. 16. Hence the verse-sine of LB $=LB^2 \div 2AC$ or $2R$, and that of $AB=AB^2 \div 2R$, and their difference is $(LB^2-AB^2)\div 2R$. This is the height through which a body must fall to acquire the velocity at A, which is therefore equal to

$$\sqrt{2g\times(LB^2-AC^2)\div 2R} \text{ or to } \sqrt{\frac{g}{R}}\times\sqrt{LB^2-AB^2}.$$

Though B draw a horizontal line, and take $Bl=BL$ and $Ba=BA$, describe a circle with the radius Bl;

$$sa=\sqrt{LB^2-AB^2},$$

and therefore $\sqrt{\frac{g}{R}}\times sa$ is the velocity at A: this may be supposed uniform while the body describes the very small arch AA′; now the space aa' or AA′ divided by the velocity at A gives the time of its description, and it is $\frac{aa'}{sa} \times \sqrt{\frac{R}{g}}$. By the nature of the circle $\frac{aa'}{sa} = \frac{ss'}{CB}$, or to the angle standing on ss' measured by an arch whose radius is unity; hence the increment of this angle, multiplied into the above radical, represents the increment of the time, and the angle and time are proportional.

31. When the body has descended from L to B, the arch *ls* becomes a quadrant, and the corresponding arch of the circle whose radius is unity becomes $\frac{1}{2} p$, the time of descending from L to B is therefore in seconds $\frac{1}{2} p \times \sqrt{\frac{R}{g}}$, and as an equal time is employed in ascending through BL′, the entire time of an oscillation is $p \times \sqrt{\frac{R}{g}}$, in which the quantity LB does not appear: this is half the periodic time of a body revolving by the force g in an orbit of the radius R.

The time of an oscillation is then independent of its magnitude, and is affected only by the intensity of gravity, and by the length of the radius of the circle. *(a)*

32. As this may appear rather abstruse to some of the students, it may be expedient to explain the equality of the times of describing small arches in a more popular way. The accelerating force on a circle is as the sine of its distance from the vertical, or since small arches are proportional to their sines, as the distance itself: now wherever two bodies move towards a given point by the action of forces, which are as the distances from that point, they arrive at it in the same time, though the spaces through which they move be unequal. To prove this, let us take two arches, AB and *ab*, fig. 172,

and suppose that a body placed on either of them is affected by a force which is as the distance from B or b; divide them into the same number of equal and indefinitely small parts, then we may suppose the forces constant from A to B, and a to b. By art. 5, $FS \propto V^2$ and $FT^2 \propto S$. Thence it follows that AC and ac are described in the same time; for call the forces which act through them, F and f, $F : f$ $AB : ab :: AC : ac$, hence $F \div S \propto f \div s$, and therefore T^2 is constant, and for the same reason the velocity acquired at C : that at c :: AC : ac, or in a given ratio. The second portions CD and cd are described, partly by the velocities acquired, and partly by the action of new forces F' and f', the velocities are as $AC : ac :: CD : cd$, and therefore that part of the times of describing the second spaces which depends on them is the same in both; and since F' and f' are in the same ratio as the spaces, if the bodies had no initial velocity, *their* action would cause the times of describing them to be equal, and therefore the combined action of the two causes must produce the same effect. Thus it can be shewn, that DE and de are described in the same time, and so of all the rest, and therefore of the whole curves. This applies immediately to the description of a circular arch, and therefore as long as its radius is the same, the descent through small arches will be performed in the same time, however unequal they may be.

33. There is a very important case of this sort of circular motion, in which the body is confined to move in the curve, not by the reaction of a circular canal or surface, but by a cord. If a heavy body be attached to one end of a string or rod, which we suppose void of

weight, whose other extremity is fastened to a fixed point, on drawing it aside from the vertical and releasing it, it descends in a circle, and the velocity acquired carries it to an equal height on the other side; thus it would continue to oscillate for ever but for the resistance of the air, and the want of perfect pliancy in the string. It is evident that the tension of the string is perfectly analogous to the reaction of the solid surface in the preceding articles of this chapter, and therefore the conclusions drawn in them apply to the motion of such a body, which is called a Simple Pendulum.

34. This instrument is employed for the measurement of small portions of time; a small leaden ball, suspended by a thread of such a length, that the interval between the centre of the ball and the pin on which it hangs is about 39.1 inches, will perform an oscillation every second, and as it continues to vibrate for several minutes, by counting the number of its swings some observations can be made in the absence of a clock or watch which shews seconds. By this apparatus, simple as it is, the astronomers were wont to measure intervals of right ascension previous to the discovery of the pendulum-clock by Huygens. It is manifest that no successive oscillations describe equal arches, as in each the resistances destroy some of the motion of the pendulum; but provided the greatest angle which it makes with the vertical be not more than two degrees, all oscillations are isochronous, that is are performed in the same time: if the arch of vibration exceed this, a swing of the pendulum will occupy more than a second, and a correction may be applied, which is given in the note *(b)*.

35. This want of Isochronism led the geometers of the 17th century to seek for a curve which possesses

this quality, that a pendulum moving in it describes all arcs, whether great or small, in the same time; and they have found that in vacuo it is a Cycloid. The common proof of this is needlessly tedious, serving only to give the time of describing an evanescent circular arch, which we have already derived from other principles; a concise demonstration may therefore suffice. In Art. 21, Statics, it has been shewn that the power requisite to sustain a body on a cycloid, whose axis is vertical, is as the arch of the curve interrupted between it and the vertex; this power is equal to the force which accelerates a body down the curve, and therefore the force at any point being as the distance which remains to be described, arches however unequal must be described in the same time.

36. Huygens, to whom Physics owe so much, was not contented here, but he actually contrived to submit this theory to the test of practice; in the article just referred to, it appears that the evolute of the cycloid is an equal cycloid, and therefore the extremity of a thread, unwrapped off a semi-cycloid from its vertex to its base, traces an equal one. Let ABE, CDE, fig. 173, be two semicycloids placed with their bases in directum, and let the length of the pendulum be twice their axis, then on drawing it aside, the thread applies to one of them, and on its return is compelled by the motion of the ball P to pass the vertical, and lap on the other, therefore the ball moves in a cycloid.

37. This elegant device is of no practical utility, for clocks, whose pendulums are thus fitted, go far worse than such as have them circular; and it will hereafter appear that the theory is true only when the mass of the pendulum is of evanescent bulk; it is however well entitled to a place here, both from its beauty and from

the means which it gives of constructing the velocities, times and spaces, when the force is as the distance. Returning to the circular pendulum, the equation

$$T = p \times \sqrt{\frac{R}{g}}$$

instructs us that the times of the vibrations of different pendulums are as the square roots of the lengths; hence the half-second pendulum is $\frac{1}{4}$ of 39.14, or 9.78 inches, it is used in small time pieces; and one 13 feet $\frac{1}{2}$ inch long is of two seconds. Hence also if heat or moisture produce a minute variation in the length of the rod or string, it is accompanied by a corresponding variation in the time of vibration.

38. We have stated that g varies with the latitude and elevation above the level of the sea, and the rate of the pendulum must vary in its inverse subduplicate ratio; hence, from observing the times of a vibration in different places we can determine more accurately than by any other method the values of g, or of the space described in a second by a falling body; for if we ascertain the length of the pendulum which swings seconds, the equation $T = p\sqrt{\frac{R}{g}}$ becomes $1'' = \sqrt{\frac{R}{g}} \times p$, or $g = p^2 R$. This quantity is *twice* the space described in a second by a falling body, and hence we derive the proportion, the space described in a second: half the length of the pendulum :: the square of the circumference of a circle to the square of the diameter. In this way the error of observation cannot be so much as $\frac{1}{10000}$ of the whole quantity.

39. Lastly, the pendulum is applied by astronomers to determine the excentricity of the earth, which it can effect with more precision than is afforded by the measurement of degrees of the meridian. The superficial gravity at the

surface of an elliptic spheroid, and therefore the length of the second pendulum, is a given function of the excentricity and the square of the sine of the latitude of the place of observation; where, therefore, two of those quantities are known the third can be found: this, however, belongs to physical astronomy, and would lead too far for an elementary treatise.

NOTES ON CHAP. V.

(*a*) The time of describing the circular arch is readily investigated, when it is so small that the fourth power of its ratio to the radius may be neglected. Let a be the extreme angle of oscillation, and u any other angular distance from the vertical, rdu is the value of ds, and the height through which the pendulum has descended being $r\{\cos u - \cos a\}$, we have

$$dt = \frac{-du \times \sqrt{r}}{\sqrt{2g} \times \sqrt{\cos u - \cos a}}$$

The integral of this cannot be given in a finite form, but it can be obtained for the particular case where a is evanescent by means of the series $\cos. a = 1 - \frac{a^2}{2} + \frac{a^4}{2.3.4}$ &c. for omitting the powers of a and u above the second, the second member of our equation becomes

$$\sqrt{\frac{r}{g}} \times \frac{-du}{\sqrt{a^2 - u^2}}$$

and integrating

$$t = \sqrt{\frac{r}{g}} \times \text{arc}\left(\cos = \frac{u}{a}\right)$$

which requires no constant. If the arch $u=0$, or if the pendulum describe the entire of a, $\frac{u}{a}=0$, but 0 is the cosine of $\frac{1}{2}p$, and as the ascent occupies an equal time with the descent, the total time of an oscillation is $T = p\times\sqrt{\frac{r}{g}}$. Some have supposed that in this case the time of describing the evanescent arch is the same as the time of describing its chord, or of falling through $2r$; but the inaccuracy of this is manifest, as these times are in the ratio of $p:4$.

If we suppose the length of the pendulum to receive a minute variation dr, $dT=\frac{pdr}{2\sqrt{gr}}$, or calling $\frac{dr}{r}=E$, $dt=\frac{1}{2}p\sqrt{\frac{r}{g}}\times E=\frac{1}{2}T\times E$. The ordinary variation of r arises from the effect of heat on the rod which sustains the pendulum, and in this case E is the product of the change of temperature into the number expressing the expansibility of the rod; and thus the error which is produced can be computed.

(*b*) The integral of dt can be obtained by a series, for

$$\sqrt{\cos u-\cos a}=\sqrt{\sin^2\tfrac{1}{2}a-\sin^2\tfrac{1}{2}u}\times\sqrt{2};$$

$du=2d\frac{u}{2}$, which give

$$dt=\sqrt{\frac{r}{g}}\times\frac{-d\frac{1}{2}u}{\sqrt{\sin^2\frac{1}{2}a-\sin^2\frac{1}{2}u}},$$

$$d\tfrac{1}{2}u=\frac{d(\sin\frac{1}{2}u)}{\cos\frac{1}{2}u}$$

Call $\sin\frac{1}{2}a$, A, and $\sin\frac{1}{2}u$, AV; we have

$$dt=\frac{\sqrt{\frac{r}{g}}\times -dV}{\sqrt{1-V^2}\times\sqrt{1-A^2V^2}}$$

Devolope $\frac{1}{\sqrt{1-A^2V^2}}$ by the Binomial theorem, and

$$dt=\sqrt{\frac{r}{g}}\times\frac{-dV}{\sqrt{1-V^2}}\left\{1+\tfrac{1}{2}A^2V^2+\tfrac{1.3}{2.4}A^4V^4+\tfrac{1.3.5}{2.4.6}A^6V^6+\&c.\right\}$$

each of whose terms is to be integrated between the limits $u=a$ and $u=0$, or $V=1$ and $V=0$. The integral

$$\int\frac{-dV}{\sqrt{1-V^2}}$$

between those limits is (see Lacroix Calc. Art. 381) $\frac{1}{2}p$: the

$$\int\frac{-dV\times V^2}{\sqrt{1-V^2}} \text{ is } \tfrac{1}{2}p\times\tfrac{1}{2}: \quad \int\frac{-dV\times V^{2n}}{\sqrt{1-V^2}} \text{ is}$$

$$\tfrac{1}{2}p\left\{\frac{1.3.5,,,,,2n-1}{2.4.6,,,,,2n}\right\}$$

and thus integrating we derive

$$T=\sqrt{\frac{r}{g}}\times p\left\{1+\left(\tfrac{1}{2}\right)^2A^2+\left(\tfrac{1.3}{2.4}\right)^2A^4+\left(\frac{1.3.5}{2.4.6}\right)^2A^6+\&c.\right\}$$

taking T to represent the time of ascent and descent. In this expression A is the sine of half the swing from the vertical or the square root of half its verse-sine, and thus it is evidently seen, that the time increases with the extent of this arch. If the arch be observed the correction is easily found, by subtracting which the time in an evanescent arch is obtained. Let s be the sum of the terms of the above series

$T=\sqrt{\frac{r}{g}}\,p\times s$, but $\sqrt{\frac{r}{g}}\,p=t$ the time of an evanescent arc, put $t = T-C$, $T = s(T-C)$,

$$C = T \times \left\{ \frac{s-1}{s} \right\} = T \left\{ 1 - \frac{1}{s} \right\}$$

The fraction $\frac{1}{s}$ may be expanded by the multinomial theorem, and calling the coefficients of the powers of A^2, b, c, d, &c. its developement is

$$\frac{1}{s} = 1 - bA^2 + \left\{ b^2 - c \right\} A^4 - \left\{ b^3 - 2bc + d \right\} A^6 + \&c.$$

hence

$$C = T \left\{ bA^2 - (b^2 - c)A^4 + (b^3 - 2bc + d)A^6 - \&c. \right\}$$

If in the correction we neglect the powers above the square, which in practice affects the value of C only in the ninth place of decimals, the correction is

$$C' = T \times \frac{A^2}{4}.$$

(*c*) In the cycloid, let S be the total arc of descent, and s any distance from the vertex, a the axis, and H and h the altitudes of the extremities of S and s; $a\,H = \overline{\tfrac{1}{2}S}|^2$ by the third property of the curve, Statics Art. 21,

$$\sqrt{2g \times \overline{H-h}} = \sqrt{\frac{g}{2a} \times \overline{S^2 - s^2}}$$

Hence

$$dt = \frac{-ds}{\sqrt{S^2 - s^2}} \times \sqrt{\frac{2a}{g}},$$

$$t = \sqrt{\frac{2a}{g}} \times \text{arc}\left(\cos = \frac{s}{S}\right)$$

which when $s = 0$ gives

$$T = \sqrt{\frac{2a}{g}} \times p$$

an equation which shews that the oscillation of a cycloidal pendulum is performed in the same time as that in an evanescent circular arc, whose radius is ra, or the length of the cycloidal pendulum.

The time in which a falling body describes a is $\sqrt{\frac{2a}{g}}$; call it t, $T : t :: p : 1$, or as circumference : diameter.

The cycloid possesses another remarkable property besides Tautochronism; the inverted semicycloid is the curve by which a body falls from one point to another not in the same vertical, in the least time, and where a semicycloid cannot be described through them, that cycloid, whose origin is at the upper point, fulfills the condition: hence the cycloid is the curve of swiftest descent. The proof of this is not given here, as when perfect it depends on the Calculus of Variations, which is not included in the under-graduate course, and it is to be found in most works which touch on that branch of analysis.

CHAPTER VI.

40. The theorems which were derived in the last chapter relative to the simple pendulum, are of no practical value without some modification, for the connection of the particles of the pendulum acts in changing the ratio of the time of oscillation to the length. In general the mass of a body is considered as concentrated in its centre of gravity; and from analogy we are led to seek for a similar centre in bodies oscillating round a centre, so that if the whole mass

were collected in it, the time of oscillation would not be altered. Each particle of the pendulum is equally acted on by gravity, and would if detached from the rest acquire a certain velocity; their union includes a new condition, so that their velocities are as their distances from the centre of motion, and therefore are unequal; hence the velocity of some is less, and of others greater than that due to the force acting on them. It follows from this that an intermediate part is accelerated as much as if it were free from the rest.

41. Let us suppose that m and m', fig. 174, are two particles connected with each other, and with the point of suspension A by a rigid line, call AB, r, A C, u, AD r'; the effect of the inertia of the particles m and m' in resisting the communication of angular motion from a force acting at C, is equal to the force which would be required to destroy the same motion, being also applied there. Calling the velocity of C, V, that of m is $V \times r \div u$, for the velocities are as the distances where the angular velocity is constant; therefore the moving force of m is $mV \times r \div u$. This force acts on C by the lever of the second order AC, hence its action there is $mV \times \frac{r^2}{u^2}$; similarly that of m' is $m' V \frac{r'^2}{u^2}$, and if there be more particles, the force which is equivalent to the effect of their inertia at C is $V \times \frac{(mr^2 + m'r'^2 + m''r''^2 +, \&c.}{u^2}$ or it is as the sum of the masses of each particle multiplied into the squares of their distances from the centre of motion. These products are called Moments of Inertia, and must be carefully distinguished from the moments of forces, which we call Statical moments. For some of their properties, and the methods of finding them in several instances, see the note (*a*)

42. If we suppose the point E such, that the force at C would produce the same angular acceleration in the whole mass supposed concentrated in it, and call its distance k we have $\frac{V}{u^2} \times \{mr^2 + m'r'^2 + \&c.\} = \frac{V}{u^2}$ $\{m + m, \&c.\} \times k^2$, for in this case each r becomes equal to k. The mass of the pendulum $M = m + m, + \&c.$ so that the sum of $(mr^2 + m'r'^2 \&c.)$ or the moment of inertia $= Mk^2$. The point E is called the centre of Gyration with respect to the axis of suspension A, and its position varies according to the line which is assumed as an axis. If we suppose the centre of gravity to be assumed, the resulting centre of gyration is called the principle centre, and when it is known, any other can readily be found. Let A, fig. 175, be an axis of rotation, G the centre of gravity, whose distance from a particle at B is ϱ, let $AG = a$, then $BA^2 = BG^2 + AG^2 \pm 2\,AGI$, or $mr^2 = m\varrho^2 + ma^2 \pm 2a \times (m \times GI)$. The same equation is true for every other particle; hence, adding all the equations together, and denoting the sum by the symbol $\int$ prefixed

$$\int mr^2 = \int m\varrho^2 + \int ma^2 + 2a \times \int mGI$$

The last quantity is the moment of the body with respect to its centre of gravity, it therefore equals cypher by Statics, Art. 9, $\int ma^2$ is M the mass, multiplied into a^2; $\int m\varrho^2$ is the moment of inertia with respect to the centre of gravity, and calling the distance of the principal centre of gyration from the centre of gravity c, we have

$$Mk^2 = Mc^2 + Ma^2, \quad k^2 = c^2 + a^2$$

Hence, where c is known, the moment of inertia can be found for any parallel axis. Hence also, it is the least possible when $a = 0$. (b)

43. We can now express the resistance of the pendulum to acceleration more simply, for the expression of Art. 41 becomes $\mathrm{VM}k^2 \div u^2$, and therefore the inertia of the body, or rather its quantity of motion when moving with the velocity produced = that of a body $\mathrm{M}k^2 \div u^2$ moving with the velocity V Let φ be the force which acts on a particle; supposing it to act in parallel lines, its total result must be applied at the centre of gravity, and is $\varphi\mathrm{M}$ at the distance a. The effect of this force at the distance u from the axis, is found by augmenting it in the ratio of $a:u$, or it is $\varphi\mathrm{M} \times \frac{a}{u}$. This force has to communicate motion to a mass whose inertia is equal to that of $\frac{\mathrm{M}k^2}{u^2}$ applied at u, and if we divide the moving force by the mass moved, we obtain the accelerating force $= \frac{\varphi \times \mathrm{M}a}{\mathrm{M}k^2} \times u$. If now we suppose the distance u to be such that a single particle at that distance would experience the same accelerating force φ, we have the equation

$$\frac{\varphi \times au}{k^2} = \varphi,\ u = \frac{k^2}{a}$$

Such a point is called the Centre of Oscillation, and it is evident that its distance from the axis, which we will henceforth denote by L, is the length of an isochronous simple pendulum, for such a one will in any instant be equally accelerated with the given compound one, and therefore their times of oscillation will be the same. (c)

44. From this it appears that the length of the simple pendulum is a third proportional to the distances of the centres of gravity and gyration, and the mathematical

investigation of it is reducible to the finding them. In Statics, chap. 3, the method of determining the first has been sufficiently explained; the latter research is far too complicated to find a place here; there are, however, some properties of the centre of oscillation which we may mention. It has been shewn, that $k^2 = c^2 + a^2$, hence $L = a + \frac{c^2}{a}$, and the distance between the centres of oscillation and gravity $= \frac{c^2}{a}$. Where the form of the pendulum is given, c is constant and therefore L, to which the square of the time is proportional, varies with a: when $a = 0$, or when the pendulum is suspended at its centre of gravity the time is infinite, and it is a minimum when $a = c$, or when the suspension is at the principal centre of gyration.

If the centre of oscillation be made the point of suspension, the pendulum will vibrate in the same time as before, for we have $L' = a' + \frac{c^2}{a'}$, in which a' the distance between the centres of gravity and oscillation $= \frac{c^2}{a}$, substituting which we have

$$L' = \left(\frac{c^2 + c^2}{a}\right) \div \frac{c^2}{a} = \frac{c^2 + a}{a} = L;$$

or the centres of oscillation and suspension are convertible, a proposition which has lately been turned to advantage.

NOTES ON CHAP. VI.

(*a*) The theory of the moment of inertia is of great importance, not merely with respect to the compound pendulum, but generally in all cases of rotation round an axis: the simpler theorems relating to it may be delivered here, but a complete discussion of the subject is unnecessary.

1. If we refer a body to three rectangular axes, the axis of rotation may be considered one of them, and supposing the body resolved into elementary portions dM, drawing through each of them a plane perpendicular to the axis of rotation, its moment of inertia is dM, multiplied into the square of the line drawn from it to the intersection of the axis with that plane. If the moment be sought with respect to the axis of z, $r^2=x^2+y^2$, and therefore the moment is $\int(x^2+y^2)\,dM$, the integral being extended to the whole extent of the body; the moment with respect to the axis of x is $\int(y^2+z^2)\,dM$, and that with respect to the axis of y is $\int(x^2+z^2)\,dM$. Each of these integrals is the sum of two, thus Z', the moment with respect to z is $\int dM\times x^2+\int dM\times y^2$, the first being the sum of the products of each particle into the square of its distance from the plane of yz, and the second similar with respect to the plane of xz. These sums are called moments with respect to the planes. We will call $\int dM\times x^2$, A, $\int dM\times y^2$, B, $\int dM\times z^2$, C, and it is evident that

$$Z'=A+B,\ X'=B+C,\ Y=A+C. \quad (1)$$

2. If we know the moment of inertia with respect to a given axis, we can find that respecting an axis parallel to the first; for let its coordinates be l, m, o, then $x = x' + l$, $y = y' + m$, and $x^2 + y^2 = (x' + l)^2 + (y' + m)^2$, therefore $Z' = \int (x'^2 + y'^2) \, dM + \int (l^2 + m^2) \, dM + 2\, l \int x' dM + 2\, m \int y' dM$; the first of these terms is the moment with respect to the second axis, the second is the mass of the body multiplied into the distance of the axis, and the other two are the products of the coordinates of the new axis into the Statical moments of the body with respect to the planes $y'z$, $x'z$.

This expression admits of considerable simplification; for in the first place we may take the new axis passing through the centre of gravity, and as the statical moment of a body, with respect to a line or plane passing through that centre is null, the two last terms vanish, therefore

$$Z' = Z'' + M \times (l^2 + m^2)$$

The moment of inertia being once known, with respect to the axis passing through the centre of gravity, this equation enables us to compute it for another parallel to the first, and distant from it by a, for $l^2 + m^2 = a^2$, and we must add Ma^2 to the moment respecting the centre.

3. The last equation is often presented under another form, for the quantity $\int (x^2 + y^2) \, dM$ may be represented by Mk^2, k being the distance of the centre of gyration; let c be what k becomes when Z' is Z'' and we have

$$Mk^2 = Mc^2 + Ma^2, \quad k^2 = c^2 + a^2. \qquad (2)$$

4. The most important part of the theory of these moments is that which results from transforming the axis of rotation into others making with them determinate angles, and the study of the principal axes; but this

part of Dynamics is much too abstruse for the purposes of the present treatise, and they who wish for more information respecting it will find it very well discussed in the Lecons de Mecanique of Prony, or the second volume of Poisson's Treatise of Mechanics.

5. We proceed to explain the mode by which the moment is computed for a given body. The quantity dM is a rectangular parallelopiped, which results from conceiving the body to be cut by a number of planes parallel to those of the coordinates, it is therefore, if we suppose the density of the body unity, $dx\,dy\,dz$ and

$$Z' = \int\int\int (x^2+y^2)\,dx\,dy\,dz,$$

the three successive integrations being each made on the suppositions of two of the variables constant, and being extended to the limits of the body as in the cubature of solids. One example of this method may suffice. Let a rectangular parallelopiped turn round its side AB, fig. 176, which we will consider as the axis of z, the origin being at A; to find its moment.

$$\int\int\int (x^2+y^2)\,dx\,dy\,dz, \text{ } x \text{ being the variable is } =$$

$$\int\int \left(\frac{x^3}{3} + y^2 x\right) dy\,dz, \quad y \text{ being the variable} =$$

$$\int \left(\frac{x^3 y}{3} + \frac{y^3 x}{3}\right) dz, \quad z \text{ being variable} =$$

$$\tfrac{1}{3}\,(x^3yz+y^3xz)$$

The mass of the body is xyz, and putting $Mk^2 = xyz\left(\frac{x^2+y^2}{3}\right)$ the moment is $M\times\left(\frac{y^2+z^2}{3}\right)$ from which the moment relative to an axis passing through the centre of gravity can be derived;

$$a^2 = \frac{x^2+y^2}{4}, \quad k^2-a^2 = \frac{x^2+y^2}{12} = c^2, \qquad (3)$$

whence Mc^2 is known. It may be remarked, that each of the three integrations has a definite meaning, the first expresses the moment of a row of particles at CD, the second of a slice ECDF, and the third of the whole body.

If we suppose y and z indefinitely small, the parallelopiped becomes what is sometimes called a physical line, and its moment, with respect to one extremity, becomes $\frac{Mx^2}{3}$, and with respect to the centre of gravity $\frac{Mx^2}{12}$.

6. The same process might be applied to solids of revolution, but it may be superseded by one simpler, so as to require only one or two integrations, when the moment is sought with respect to the axis of revolution, or one at right angles to it. If we take the former for the axis of z, all sections perpendicular to it are circles, and each of them being similarly posited with respect to the axis of x and y, the moments with respect to those axes must be equal; these are respectively $B+C$, $A+C$, therefore $A=B$, $Z'=A+B=2A$ or $2B$. The function C is easily found, for taking a circular section whose thickness is dz, and calling its radius $u=\sqrt{x^2+y^2}$, every point of this is at the same distance z from the plane of xy passing through the origin: hence the moment of the section with respect to the plane is $pu^2z\ dz$, and its integral taken in the limits of z gives C. The functions A and B are given if we have Z'; and to seek this let us suppose the sections divided into rings, whose interior and exterior radii differ by du, the mass of one of these is $2pududz$, and its moment with respect to the axis of z is $2pu^3dudz$, as

u is its distance from it, and the integral of this or $\frac{1}{2}pu^4dz$ is the moment of the section. The integral of this is Z'.

7. The first example of this method shall be the cylinder: as u is constant, $\int \frac{1}{2} pu^4dz$ is $\frac{1}{2}pu^4z$; and as $M = pu^2z$, $Z' = \frac{1}{2}u^2 \times M$, hence $B = \frac{1}{4}u^2M$;

$$C = \int pu^2z^2dz = \frac{pz^3u^2}{3} = \frac{Mz^2}{3}.$$

The moment with respect to the axis of x, $X' = B + C$

$$= M\left(\frac{z^2}{3} + \frac{u^2}{4}\right)$$

The moment with respect to the centre of gravity

$$M(k^2 - a^2) = M\left(\frac{z^2}{3} + \frac{u^2}{4} - \frac{z^2}{4}\right) = M\left(\frac{z^2}{12} + \frac{u^2}{4}\right)$$

from which the problem can be solved relative to any other avis parallel to z or x.

8. In the cone $mz = u$, therefore $Z' = \int \frac{1}{2} pm^4z^4dz = \frac{1}{10} pm^4z^5$, or putting $3M$ for pu^2z, $= \frac{3}{10} M \times u^2$. $C = \int pm^2z^4dz = \frac{1}{5} pm^2z^5 = \frac{3}{5} Mz^2$. $C + B$, or $X' = \frac{3}{20}M(u^2 + 4z^2)$: in this case the origin is at the vertex, and by taking the integral within the proper limits the moments of a conical frustum can be found with facility.

9. Lastly, for the solid formed by the revolution of a circular arch round its verse sine, $u^2 = 2rz - z^2$, hence

$$Z' = \frac{1}{2} p \times \int 4r^2z^2dz - 4rz^3dz + z^4dz$$

$$= p\left\{\frac{2}{3}r^2z^3 - \frac{1}{2}rz^4 + \frac{1}{10}z^5\right\}$$

which being extended to the whole sphere, by taking the integral from $z = 0$ to $2 = 2r$, gives

$$Z' = \frac{8}{15}pr^5, \quad M = \frac{4}{3}pr^3, \quad Mc^2 = M \times \frac{2}{5}r^2.$$

These are sufficient for our purpose, more particularly as they apply to the usual forms of the parts of pendulums.

(*b*) For the ready investigation of the propositions relating to motion round a fixed axis, which it seems necessary to introduce here, it may be useful to mention a dynamical theorem known by the name of D'Alembert's Principle; it is nearly self evident, but affords such facilities for reducing any problem to an equation, that Lagrange has built the Dynamics of his Mecanique Analytique on it, combined with the principle of virtual velocities.

Let us consider the particles of bodies, m, m', m'', &c. as acted on by forces which would impress on them if unconnected the velocities v, v', v'', &c. Their connection alters these velocities, so that the particles actually move with the velocities u, u'. u'', &c. now the principle is, that there is equilibrium between the quantities of motion $mv + m'v' +$ &c., and $mu + m'u'$, &c. the latter velocities being supposed to have their signs changed, or if we call the result of v and $-u$, s, of v' and $-u'$, s', $ms + m's' +$, &c. $= 0$. This is equivalent to saying that there is no new moving force introduced by the connection of the system, and, therefore, that the sum of mv must be equal to thesum of mu.

Let us suppose a body μ, to impinge on a body moveable on an axis at a distance from it $= f$, and with a velocity v, its quantity of motion is μv; if we suppose the body μ annihilated after the stroke to avoid considering its mass, we have by the principle, equilibrium betwee nμv, and the sum of the quantities of motion of each particle m of the body struck. Let ω be the angular velocity of the body struck, a particle m, at the distance r from the axis has the linear velocity ωr, therefore the quantity of motion of m is $m\omega r$, and the sum of these must be in equilibrio with μv. This equilibrium is made is made by the cohesion of the body which acts as a lever, therefore each force must be multiplied into its distance from the axis, and their sum $= 0$, or

$$\mu v f = \omega \times \int mr \times r + m'r' \times r' + m''r'' \times r'', \text{ \&c.}$$

or putting for $\int mr^2 + mr'^2 + \text{\&c.}$ its value Mk^2

$$\mu v f = \omega M k^2, \qquad (1)$$

from which ω, the angular velocity is known. This last equation shews that a given impulse would produce the same angular velocity if the whole mass were concentrated in a point whose distance is k, which is therefore called the centre of gyration: hence we can compute what part of the impelling power is consumed in communicating rotatory motion, a problem of importance in machinery.

If the equilibrium between the forces impressed and the quantity of motion produced is produced around the fixed axis, this must experience a percussion equal to their sum or difference, at least in the case where the impulse is made perpendicularly to the plane passing through the centre of gravity and axis. If it be required that it shall receive no percussion, we must have

$$\mu v = \omega \int mr + mr' + \text{\&c.} = \omega M a, \qquad (2)$$

a being the distance of the centre of gravity from the axis: dividing eq. (1) by eq. (2) we have

$$f = \frac{k^2}{a} \qquad (3$$

If therefore an impulse be given at the distance f from the axis, it is entirely expended in communicating angular velocity; and conversely, if a body moving round an axis strike another, it communicates the greatest quantity of motion to it when the direction of the impact passes through this point, which is therefore called the centre of Percussion; for the detailed theory of which see Poisson, tom. 2, art. 345, or Gregory's Mech. vol. 1.

(c) In the compound pendulum, each particle is acted on by a force obtained by multiplying g into the sine of the angular distance from the vertical, call it γ, then the quantity of motion impressed in an instant of time, which is to be substituted for μv in eq. 1 of the last note, is $\gamma dt \times \int (m + m' + m'', \&c.)$ The quantity acquired in the same time is $d\omega \times \int (mr + m'r' +, \&c.)$ and expressing the equilibrium of the lever, as before, we have

$$\gamma dt \times \int mr = d\omega \times \int mr^2$$

or substituting for the integrals

$$\gamma dt \times \mathrm{M}a = d\omega \times \mathrm{M}k^2, \text{ from which}$$

$$\frac{d\omega}{dt} = \gamma \times \frac{a}{k^2} \qquad (1)$$

which is the expression of the angular accelerating force. We can compare this with the simple pendulum, for we have by equat. 4, chap. 1, putting l for its length,

$$\frac{dv}{dt} = \gamma, \quad \text{but } dv = ld\omega$$

$\frac{d\omega}{dt} = \frac{\gamma}{l}$, hence if we make $l = \frac{k^2}{a}$ the angular accelerating force is the same in the simple and compound pendulums. The point in the perpendicular to the axis which passes through the centre of gravity, whose distance from the axis is $\frac{k^2}{a}$ is therefore that into which, if the whole mass of the pendulum were collected, it would vibrate in the same time as before; it is called the centre of oscillation.

2. We will call its distance from the axis L, and remark, that the theorems of chap. V. apply to the compound pendulum by substituting L for R, and we will give some other general results before we examine the particular values of L in solids of a determinate figure. If we substitute for k^2 its value a^2+c^2, we find $L = a+\frac{c^2}{a}$, from which it is manifest that the time of oscillation varies with a: when it is 0, or when the axis passes through the centre of gravity, L is infinite, the motion of oscillation being converted into one of gyration. It is also infinite when a is infinite, and must at some intermediate value, of it be susceptible of a minimum; $dL = da\left(1-\frac{c^2}{a}\right)$ the minimum of L is therefore when $a = c$, or when the axis of suspension passes through the principal centre of gyration. The value of c varies with the plane of rotation, and therefore also the time of vibration.

From the connection between L and a it is manifest that Huygens's cycloidal pendulum is of no value, for let ϱ be the part of the thread evolved off the semicycloid, and terminating at the centre of gravity of the pendulum; we may neglect the threads weight, and put $L=\varrho+\frac{c^2}{\varrho}$; but that the centre of oscillation should move in a cycloid, L should equal ϱ; it therefore is constrained to describe a curve very different from the cycloid, and by no means possessing Tantochronism.

3. To find the centre of oscillation we divide the mo-

ment of inertia by the statical moment with respect to the axis, and obtain L; our first example is when the pendulum is a rectangular parallelopiped, vibrating round an axis drawn through the centre of gravity of one of its planes parallel to the axis of z.

$$Z' \text{ is} = M\,(c^2 + a^2) = M\left(\frac{x^2+y^2}{12} + \frac{x^2}{4}\right)$$

by par. 5, note (a), and because $a = \frac{x}{2}$;

$$\text{or } \frac{M}{12}\,(4\,x^2 + y^2);$$

The Statical moment is $M\,a = M\,\frac{x}{2}$, therefore $L = \frac{2}{3}\,x + \frac{y^2}{6x}$. If the solid be of evanescent thickness, $\frac{y^2}{x}$ vanishes, and the centre of oscillation is $\frac{2}{3}$ of x from the axis: and if the axis be transported to the principal centre of gyration $L = \frac{1}{3}\,(x^2 + y^2)$. If we suppose the solid to be connected by a rod void of inertia and weight to an axis parallel to that of x or y, the values of L in these two cases are different, and their difference is $\frac{z^2 - y^2}{12x}$.

4. In the cylinder, the moment round a diameter of a bounding circle is $M\left(\frac{z^2}{3} + \frac{u^2}{4}\right)$ and the Statical moment is $\frac{1}{2}Mz$, therefore $L = \frac{2}{3}\,z + \frac{1}{2}\,\frac{u^2}{z}$, which, as in the last example, is $\frac{2}{3}\,z$, when u vanishes in respect of z. If the axis of rotation be parallel to the axis of the cylinder or perpendicular to it, the values of L are $a + \frac{1}{2}\,\frac{u^2}{a}$, $a + \frac{3u^2 + z^2}{12a}$, which are equal when $z^2 = 3u^2$.

5. For the cone, where the axis passes through the vertex, the moment of inertia is by par. 8, note (*a*) $M \times \frac{3}{20}(u^2+4z^2)$, the Statical moment is, as $a = \frac{3}{4}z$, $3 M \times \frac{z}{4}$, therefore

$$L = \frac{u^2+4z^2}{5z},$$

which when the cone is right-angled $= z$ as $z = u$.

6. Lastly, for the sphere suspended by a thread whose inertia is evanescent, $c^2 = \frac{2}{5}r^2$, therefore

$$L = a + \frac{2r^2}{5a}$$

The bobs of pendulums are usually composed of two spherical segments applied by their bases; here $Mk^2 = p \times \left\{ \frac{2}{3}r^2z^3 - \frac{1}{2}rz^4 + \frac{1}{10}z^5 \right\}$

$$L = \frac{(\frac{2}{3}r^2 - \frac{1}{2}rz + \frac{5}{30}z^2)\,z}{a\,(r - \frac{1}{3}z)}$$

7. When the figures and masses of the several parts of a compound body are given, the centre of oscillation of the whole may be found. The moments of inertia are all positive, and denote them by M, M′, &c. the Statical moments μ, μ', &c. are to be taken with their proper signs, and

$$L = \frac{M+M'+M'', \text{ \&c.}}{\mu+\mu'+\mu''}$$

Let l, l', &s. be the values of L for each of the bodies separately, and $M = \mu l$, $M' = \mu' l'$, &c. therefore the equation becomes

$$L = \frac{\mu l+\mu' l'+, \text{ \&c.}}{\mu+\mu'+, \text{ \&c.}}$$

which suffices either for theoretic or practical determination of its value.

CHAPTER VII.

45. Having touched on the simpler parts of the theory of the compound pendulum, we are able to proceed with the application of this instrument to the various purposes for which it is used. In chap. 8, of Statics, the wheel-work of the clock was described, and it was stated that the last wheel was constrained to make one revolution in a minute; this is effected by the application of a pendulum, which is connected with it so, that a tooth of the wheel escapes at every second oscillation of the pendulum; the wheel has thirty teeth, and if the pendulum swing seconds, the above condition must be fulfilled. The pendulum consists of a rod, whose lower extremity is connected with a heavy lenticular mass, called the *Bob*, so that their connection is at the centre of gravity of the latter; its upper end is suspended by a fine spring, or by knife edges, so that it may vibrate with the utmost freedom, and here the connection is made between it and the wheel-work. The common method of this is shewn at fig. 177. A is the last wheel, its teeth are sloped in the direction of its motion; CBD is a piece turning freely on an axis at B, and carrying the pallets C and D: it is connected with the pendulum whose rod passes behind it. Supposing the pendulum swinging from right to left, the tooth E presses on the interior face of the pallet C, which may be compared to an inclined plane, and accelerates the pendulum till its point drops off the edge; but during this time the pallet D has descended into the hollow between F and the preceding tooth, and when E escapes, F falls

on the exterior face of D, and acts on the pendulum through its return; when it escapes, I acts on C, and so on. This arrangement answers two purposes; it registers the number of vibrations, and at every oscillation it gives the pendulum a new impulse, which continues its motion notwithstanding the resistance of the air and the friction of the suspension which tend to stop it.

46. The apparatus which we have described in the last article is called a Scapement: this kind is in common use, but it cannot be considered an accurate regulator, as the pendulum is continually urged by the wheel A, except at the instant of the drop of a tooth. The force which is thus added to it combines with that of gravity, and disturbs the isochronism of the vibrations; it is indeed insignificant in respect of the moment of a heavy pendulum, but still has a sensible effect. Thus, if the weight which impels the clock be increased, the vibration becomes quicker, and it is evident that its action may be variable from several causes. To obviate this, other Scapements have been invented, which permit the pendulum to vibrate, nearly as if detached from the clock, so that its rate is not disturbed by the force necessary to continue the motion. The Dead-beat is the most esteemed of them: it is shewn at fig. 178. The surfaces C and D of the pallets are formed into cylinders, whose centre is at B; when the tooth E escapes by the pallet C being carried outwards by the pendulum, it gives the requisite impulse as it slides over the inclined plane *e*, and F drops on the surface D. It presses on this, but as it is cylindrical, the pressure has no effect on its motion except what arises from its friction, which between polished surfaces of metal is insignificant, and therefore is

insensible in comparison of the quantity of motion of the pendulum. The time of oscillation is therefore affected by the power which impels the wheel work, (commonly called the Maintaining Power), only so far as it may affect the extent of the arch of vibration. Thus, if it be increased, the arch becoming larger, the time is increased, as large oscillations are performed in a longer time than small ones. The name of this scapement proceeds from this, that the second hand of the clock drops from one division to another, and stops there; while in the common scapement, after the drop it recoils, having an irregular and unsteady motion. The Dead-beat is used in most astronomical clocks.

47. In fixing up a clock, great attention must be paid to the solidity of its supports, as this circumstance may affect its rate materially; a quantity of the pendulum's motion being expended in bending or moving them. It has been found that two clocks, attached to the same rail, kept time for upwards of a year without varying a single second, and that when one of them was stopped, the action transmitted from the other soon put it in motion: these phænomena ceased when the rail was sawn across. Astronomers are well aware of this, and therefore suspend their clocks from large stone pillars.

48. The Bob of the pendulum is made as massive, and the rod as slender as is consistent with its stiffness, that the moment of inertia may be as great as possible. The rod must be stiff, as otherwise it will bend by the inequality of the quantity of motion possessed by each part of the pendulum; and the bob, besides its rotation round the point of suspension will have another round

its centre of gravity, which may interfere with its vibrations. (*a*) As the precise place of the centre of oscillation is unknown at first, the pendulum requires to be regulated so as to vibrate seconds; this is performed by a small weight, which is moveable by a screw, and can be raised or lowered a minute quantity, so as to shift the centre a little: it is commonly placed below the bob, but as a small change of its place makes a considerable alteration in the rate, it is better to have it moveable along the rod. If its distance from the suspension be half the interval between that and the centre of oscillation, a considerable motion of it produces but a small alteration in the time of a vibration. (*b*)

49. But the most essential part of the adjustment of the pendulum is the apparatus, by means of which its length is preserved invariable at all temperatures. The expansion of solids has been already noticed on several occasions; this must affect the distances of the particles of the pendulum from the axis, and therefore the value of L. Let e be the expansion which the unit of length of any body experiences by an elevation of temperature of one degree, $e\,t$ is that corresponding to t degrees, and this multiplied into any given length gives the expansion of that length.

Now, if we suppose the pendulum composed of homogeneous materials, since its expansion in every direction is as its dimension in that direction, it continues similar to itself, and therefore k, and a of Art. 43, are augmented by the expansion in the proportions of their lengths. We have $L = \frac{k^2}{a}$; if the temperature be raised t degrees, these quantities become

$$L' = \frac{(k+etk)^2}{a+eta} = \frac{k^2}{a} \times \frac{(1+et)^2}{1+et} = \frac{k^2}{a} \times 1+et =$$

$L \times (1+et)$, and $L'-L = L \times et$

or the effect of expansion on L is found as if it were a simple pendulum. We must now find what effect this increment of L has on T, the time of oscillation. In art. 37, putting L for R, we have $T^2 \propto L$, therefore $T^2 : T'^2 :: L : L'$, and $T^2 - T'^2 : T^2 :: L - L' : L$. If we substitute for $T^2 - T'^2$, $(T+T')(T-T')$ and consider that the first of these factors is ultimately 2T, and that $L-L'$ is $et \times L$, we have

$$2T \times T - T' : T^2 :: et \times L : L, \text{ and}$$

$$T - T' = \frac{T}{2} \times et,$$

or the difference of the time of an oscillation produced by t degrees, is to the time, as half the expansibility of the substance multiplied into the variation of temperature, is to unity. Let us suppose that it is made of brass, whose expansibility is nearly .0000272; if we divide this by two, and multiply the quotient into the number of seconds in a day, we obtain the increase of the clock's rate, produced by one degree, 1.77 seconds: the extreme difference of temperature in winter and summer is often more than forty degrees, which would occasion a difference of 47 seconds per day. The expansibility of steel is about half that of brass.

50. The first contrivance to obviate this cause of error was invented by Graham, a celebrated instrument maker, and it may still be considered as the best. The rod is iron, and instead of a bob it supports a cylinder of glass, which contains a quantity of mercury whose weight exceeds considerably that of the other parts of the pendulum; then the centre of oscillation will nearly

coincide with that of gravity, on account of the chief part of the mass being fluid. Let the height of the column of mercury be h, and the length of the rod be H, their expansibilities being E and e, then if the temperature increase one degree, the centre of gravity or of oscillation is carried downwards from the axis by the quantity EH; at the same time the expansion of the mercury raises the centre through eh, and if H and h be reciprocally as E and e, these opposite changes counteract each other, and the centre preserves the same distance. The adjustment of the mercurial pendulum is readily effected; for if the clock gains in warm weather, there is too much mercury, and *vice versa*; therefore by removing or adding a little it may be brought to an exact compensation. Besides this it is much more easily procured than those which shall be mentioned afterwards; and it alone can be employed when a cycloidal pendulum is desired, as the theorems relative to the centre of oscillation do not apply to a fluid mass revolving round an axis: see notes (c. 2) of chap. 6, and (b) of the present.

51. There are a variety of solid compensations which depend on the unequal expansion of metals; we shall only describe two. Harrison's Pendulum, commonly called from its shape the Gridiron, is shewn, fig. 179, as it is at present constructed; B is the suspension, it carries the bearer AC, to which are rivetted the similar rods of steel AD, CE, whose expansion carries the piece DE further from B. To DE are fixed FH, IG, bars of an alloy of zinc and silver, whose ends pass loosely through holes in AC; these carry HI, from which the bob of the pendulum is suspended by KL,

which passes through DE. In this arrangement it is easily understood that the expansion of the bars AD and KL is downwards, while that of FH is upwards; and therefore if FH : AD + KL inversely as these expansibilities, the point L will remain stationary. The bars IG, CE are not essential to the compensation, but are added for symmetry and compactness.

52. The other solid compensation belongs to Biot; it is the application of a method, long used in the balances of watches, to the pendulum. If a bar of steel be soldered to one of brass of equal length, as in fig. 180, where AB, CD are the two metals, the compound will not continue straight if the temperature vary: their expansions are unequal, and therefore by heat CD becomes longer than AB, which necessarily produces flexure. The quantity of this, measured by the distance of D from a plane to which C is attached, is inversely as the thickness of the bar, and directly as the square of its length; and it may be some hundred times the direct expansion of the metals. The balance of a watch (for the theory of which see note (*c*) has this applied by a compound rim, shewn at fig. 181. CD is its diameter, which bears the circular arcs DE, CF, composed of brass and steel, (or rather of brass and platina, to guard against magnetism) with the brass outwards. To these brass weights M and M′, on which the momentum of the balance chiefly depends, are attached. On an elevation of temperature, the arms CB and BD lengthen and increase the diameter of the balance, while the curvature of the rims carries M and M′ nearer to the centre. By sliding the weights towards the arms the effect of the curvature may be diminished,

and thus we can correct not only the expansion of the balance, but also the far more important influence of the temperature on the spiral spring which regulates its vibrations.

Biot's compensation is a compound bar, so attached to the pendulum rod that its curvature carries upwards a small weight fixed at its extremity, so as to counteract the expansion of the rod; it is made double for symmetry, and the adjustment is made either by sliding the weights along the bars, or by placing the bars at different points of the rod.

53. Having detailed the precautions to be used in constructing an accurate pendulum, we will conclude our account of it with a brief notice of the means by which its length is measured. This process is useful, not only for the purposes mentioned in Chap. V, but also as it affords a ready and invariable standard of measure. The measures of different countries have hitherto been referred to no common basis, and as they must be compared with an arbitrary archetype, are liable to become doubtful if it should perish or suffer by use: hence it has long been an object with philosophers, that some immutable pattern, derived from nature itself, should be chosen, which might make the system founded also immutable. In France the quadrant of the terrestrial meridian has been chosen as the standard, though it is probably not uniform in different longitudes, and of most difficult application; while in this country the length of the second pendulum has been prefixed as a means of verifying the ordinary measures. Though it be easier to measure the pendulum than the degree of the meridian, yet it has its difficulties, of which the chief is

that we have no means of knowing the precise position of the centre of oscillation: the theory would give it if the materials were of uniform density, but this is never the case in practice. This, however, was the method used by Newton, and the result which he obtained, though not accurate, was the best which philosophers had till lately. About forty years ago Mr. Whitehurst discovered a new method; his pendulum was a spherical ball suspended by a fine steel wire, and suspended from an arm attached to a strong vertical plank. The wire passed through a slit in a moveable clip, which could be slided along the plank, and when fixed at any height evidently determined the axis of rotation of the pendulum. It is manifest that by fixing the clip in two different positions, and observing the times of vibration by connecting the apparatus with a clock, we determine the ratio of the lengths of the equivalent simple pendulums, and if we knew the difference of these lengths we could determine them. Whitehurst conceived that the distance between the two places of the clip is the difference of the length; but he was mistaken, for it is the difference of the distances of the centre of gravity of the ball from the axis of suspension. Let the lengths in the two experiments be L and L′, a and a' the distances of the centre of gravity, and let the times of oscillation be as $1 : \sqrt{n}$, then $L : L' :: 1 : n$. By the theory of the compound pendulum we have the equations

$$L = \frac{c^2 + a^2}{a}, \quad L' = \frac{c^2 + a'^2}{a'}$$

and observation gives $a - a' = D$, $L' = nL$. In these four equations, L, L′, c, a, and a' are five unknown quantities, the problem is therefore indeterminate unless

c be known. In Whitehurst's pendulum, where the wire was evanescent in comparison of the ball, it is to the radius of the ball as $\sqrt{2} : \sqrt{5}$ nearly. If we eliminate a' from the second equation by means of the third, and subtract the resulting equation from the first, we obtain

$$L-L'=D\left\{1-\frac{c^2}{a^2-aD}\right\}$$

instead of D, as he supposed.

54. Prony saw the defect of this method, and corrected it; his pendulum has three knife-edge suspensions, placed in the same line passing through the centre of gravity; the pendulum is made to vibrate on each of them, and the times of oscillation observed, also the distances between the adjacent points of suspension measured; and we have the following equations:

$$L'=nL,\ L''=mL,\ a-a'=D,\ a'-a''=D',$$
$$L=\frac{c^2+a^2}{a},\ L'=\frac{c^2+a'^2}{a'},\ L''=\frac{c^2+a''^2}{a''},$$

seven equations to determine the same number of unknown quantities; the problem is therefore solved.

55. But Kater has outdone his predecessors by applying the convertibility of the centre of oscillation and point of suspension; for if a pendulum be suspended successively by two points, and in both cases oscillates in the same time, the distance between them is evidently the quantity L; the matter is therefore reduced to a single measurement, and it is infinitely easier of execution than the triple suspension of Prony. The instrument is shewn in fig. 182; A is the section of a knife-edge, B is another fixed at 39 . 2 inches from the first; C is a

bob which gives the requisite mass to the pendulum, and E a weight moveable by a micrometer screw, which serves to adjust the centre of oscillation to the edge B. When the pendulum is used, it is placed with the edge A resting on agate planes before the pendulum of a good clock, so that when both are at rest the index b is seen through a small telescope, placed at some distance in front, to cover a small white circle painted on the pendulum of the clock. If now both pendulums be made to oscillate, and if they are synchronous, as they arrive simultaneously at the position of rest, the white disk cannot be seen in the telescope, as when it passes through the field the index is before it. But if the experimental pendulum be a little shorter than the other, then if they pass together at any instant, on their return the index will precede the disk, so that a little of it will be visible; and this increases with each oscillation for a certain time, and afterwards diminishes till the disk is again covered. From observing the number of seconds between two successive occultations $= n$, the time of the experimental pendulum is known $= 1'' \times \frac{n-1}{n}$.

Now invert the instrument so that the edge B may rest on the planes, and make it vibrate as before; if the interval between the occultations be as before n'', the centre of oscillation is at B; if it be greater, B is between the centre of oscillation and gravity, and E must be moved towards A. Thus after several trials the centre is brought to B. The interval between the edges is then measured by an accurate scale at a determinate temperature, which in this country is 60 of Fahrenheit, and the measurement gives the length of the experimental pendulum in parts of the divisions of the scale, from which that of the second pendulum can be computed.

This however requires a variety of corrections, as the buoyancy of the air, its resistance, &c. for which see the paper of the inventor in the Philosophical Transactions for 1818. *(d)*.

NOTES ON CHAP. VII.

(a) The theorems relative to the moment of inertia suppose that the parts of the revolving body are firmly connected, and that some are constrained by this connexion to move faster and others slower than they would by the action of the forces applied to them, if detached. Conceive two particles m, m' fig. 182, connected by an inflexible line, and this attached at their centre of gravity to a line r turning round an axis of rotation at s: according to note b. chap. 6, let a force μV be applied at the distance f, this generates at g a force $\mu V f \div r$. Let the angular velocity of the system be ω, ωr is the velocity of g, the centre of gravity of m and m', therefore each of those particles moves with the same velocity, and D'Alembert's theorem gives for the equation between the motions impressed and acquired

$$\mu V f = \omega r^2 \times \overline{m + m' + \&c.} = \omega r^2 \times M$$

or in this case the moment of inertia is the mass $\times$ square of the distance of the centre of gravity. If then the bob of a pendulum were attached to its rod by pivots passing through its centre of gravity on which it could turn freely, L would be equal to a. The same is also true a fortiori where the chief part of its mass is fluid, for every part of it must have an equal quantity of motion.

If a sphere be connected by a rigid rod with an axis of suspension, as the same point of its surface regards the axis, it must while it oscillates through a given angle have a motion of rotation through the same angle round its own axis; this is also evident from the equation $k^2 = c^2 + a^2$: now if the rod be flexible as a thread, the velocity of rotation round the axis of the sphere is at the beginning of the oscillation less, and at the end greater than the angular velocity of the pendulum; but on the whole the moment of inertia is diminished. The effect which this has on L, and the angle which the thread makes at any part of the oscillation with the line joining the centre of gravity with the suspension, can easily be determined; but we omit them for want of room.

(*b*) Let M be the mass of a pendulum, K and A the distances of its centres of gyration and gravity; *m*, *k* and *a*, the same of a sliding weight, it is required to find the effect of a minute change of place in the latter on L. We have

$$L = \frac{MK^2 + m\,k^2}{MA + ma}, \quad a^2 + c^2 = k^2, \; ada = kdk$$

$$dL = \frac{2\,mada \times (MA + ma) - mda\,(MK^2 + mk^2)}{(MA + ma)^2}$$

$$= m \times \left\{ \frac{2a - L}{MA + ma} \right\} \times da$$

which vanishes when $a = \frac{1}{2}$ L, and as d^2 L is positive, L is then a minimum and a given variation of a produces the least possible change in L. If a be less than $\frac{1}{2}$ L, dL is negative, shewing that when the distance of the sliding weight is less than that distance, L is diminished by moving it from the point of suspension.

(*c*) The movement of the watch is regulated by the balance, which depends on different principles from the pendulum; it is a wheel poised on its axle, which is provided with a spiral spring, one end of which is attached to it, and the other to a fixed point: the axle or verge has pallets which are impelled by the wheel-work as in the scapements of the pendulum. When a pallet receives an impulse, the balance turns on its axle, contracting the spires of the spring, and developing its elastic force, which is found to be proportional to the angle of revolution; at length the elasticity destroys the motion of the balance, and accelerates it on its return; the velocity thus acquired brings it back to its original position. If φ' be the *angular* accelerating force, retaining the notation of notes (*b*) and (*c*) of the last chapter,

$$d\omega = \varphi' dt = -\frac{\varphi' d\theta}{\omega}$$

θ being the angle from the point of quiescence; the force is as the angle, therefore $\varphi' = \varphi \times \theta$, φ being the force at the angular unity, hence

$$\omega d\omega = \varphi \times -\theta d\theta, \ \omega^2 = -\varphi\theta^2 + C$$

and determining the constant, so that $\omega = 0$, when $\theta = \Theta$,

$$\omega = \sqrt{\varphi} \times \sqrt{\Theta^2 - \theta^2}.$$

$$dt = \frac{-d\theta}{\omega} = \frac{1}{\sqrt{\varphi}} \times \frac{-d\theta}{\sqrt{\Theta^2 - \theta^2}}$$

whose integration gives

$$t = \frac{1}{\varphi} \times \text{arc}\left(\cos = \frac{\theta}{\Theta}\right);$$

making $\theta = 0$, and doubling to obtain the whole time of vibration, we obtain an expression which when φ is determined gives the time. Let μ be the weight which, when applied at the circumference of the balance, winds the spring through the angular unit, the action of the spring on the weight is equal to that of gravity, it would therefore produce in the mass μ, a velocity g in one second, and it may be replaced by a moving force, $\mu \times g$ applied at r, the radius of the balance from the centre. Hence (see note b of the last chapter) the angular acceleration in one second, or $\varphi = \frac{\mu g r}{M k^2}$, and our equation becomes

$$T = \sqrt{\frac{M k^2}{\mu \times r \times g}} \times p$$

which is independent of Θ, shewing that all the vibrations are isochronous. The quantity μr is the weight which would counterpoise the elasticity at the unit of distance, and the time is directly as the square root of the moment of inertia, and inversely as the square root of the absolute force of the spring.

(d) Let L and L′ be the distances of the centres of oscillation from the two suspensions, then

$$L = a + \frac{c^2}{a},\ L' = a' + \frac{c^2}{a'},\ \text{and}$$

$$L - L' = a - a' + c^2\left(\frac{a - a'}{aa'}\right) = (a - a')\left(1 - \frac{c^2}{a} \times \frac{1}{a'}\right)$$

The second member vanishes when $a = a'$, which can never happen in the actual construction of the pendulum, or when $a' = \frac{c^2}{a}$, in which case the suspension is in the first centre of oscillation. If a' be less than $\frac{c^2}{a}$, or if the second suspension be between the centres of gravity and oscillation, $\frac{c^2}{a} \times \frac{1}{a'}$ is greater than unity, therefore $1-\frac{c^2}{aa'}$ is negative, and L′ is greater than L, which is known by the oscillations corresponding to it being slower: when this is the case L must be lessened, and *vice versa.*

CHAPTER VIII.

56. In the investigation of the action of a system of bodies on each other we encounter difficulties which are frequently insurmountable; and even where they yield to our efforts, it is in consequence of the application of resources which are not necessary for the enquiries which have hitherto engaged us. Problems of this kind formerly engaged much of the attention of philosophers, but they have lost most of their interest since a general method has been discovered, by which they can certainly be reduced to an equation; after which the whole labour is analytic. Among them the percussion of elastic bodies held at first a preeminent place, and was the object of many experiments and calculations, some of which deserve to be preserved for their own value, and some as an exercise of the intellect.

In the outset arose a difficulty as to the nature of percussion; it was manifestly more complex than pressure, and seemed at first sight incomparable with it. With that measure of the quantity of motion which is the mass multiplied into the velocity, we are already acquainted, and we have stated the difficulties which introduced a new measure; but the Percussive force is less simple though similar. If a leaden ball be acted on by gravity and allowed to descend, it acquires in one second a velocity of thirty-two feet; if with this velocity it strike a mass of soft clay it will penetrate it to some depth, while it may rest on it for a second or an hour without making a perceptible impression. In the two cases, the moving cause is the same, gravity acting for a second, but the effects are utterly unlike. Thus also a cube of cast iron which could bear several thousand pounds is crushed by a blow from a small hammer. These and other facts of the same kind are anomalous, only because we have a confused idea of *effect*. What is the effect of a moving force proportional to? If we measure it by the product of the accelerating force into the mass moved, which gives the pressure produced, we do not obtain a value which is of practical utility. We may take gravity as an example; any force may be expended in sustaining a weight or in moving it, but the latter is that which concerns us most, and it requires a peculiar measure. Let us suppose a labourer employed to sustain a load M; he is paid not in proportion to M, but to $M \times T$ in all probability, for this mode of exertion is seldom used. This product might be taken as a measure of the work done in that case, but in general the work performed is analogous to raising a weight a certain height, and we may take $M \times S$, the mass multiplied into the space through which it is raised as the effect of the force. Since $S \propto V^2$, this expression is equivalent to MV^2, which measure is named Vis Viva.

In the case of the ball impinging on the clay, it is evident that the velocity is destroyed while it penetrates, suppose 4 inches; the force which stops it is the resistance of the clay, and the reaction of the ball must be equal to this: call it γ, and the depth of penetration σ, then

$$2gMS = MV^2 = 2\gamma\sigma M,$$

as the clay is supposed uniformly resisting; or $\gamma \times M$ the force of percussion, considered as a pressure, : gM the weight of the ball : : S : σ. The depth σ is in general far smaller than S, and therefore the pressure which is equivalent to the percussion is immense. As a numerical example we will take a musket ball fired with a velocity of 1600 feet at a block of elm, into which it penetrates 6 inches, the velocity 1600 is acquired in falling through 40000 feet, therefore $M\gamma : Mg$: : 40000 : $\frac{1}{2}$: : 80000 : 1 . Mg is one ounce, and therefore $M\gamma = 5000$ pounds, the pressure which should produce the same penetration when applied by a sphere $\frac{3}{4}$ of an inch in diameter. (*a*).

57. This is mentioned though briefly, as it serves to explain many remarkable facts; and we proceed to consider the collision of bodies making abstraction of their figures, and considering them as material points though of different masses; the effects of their figures shall be afterwards noticed. Let two particles whose masses are m, m', and their velocities V, V′, positive when in the same direction, one negative when in different directions, move in the same right line and meet at any point of it; by the inertia of matter no new force can be added, so that the quantities of motion after collision must be

equal to those before it: let the velocities after the impact be u, u', the motion before it is $mV+m'V'$, that after is $mu+m'u'$, and therefore

$$mV+m'V' = mu+m'u', \qquad (1)$$

in which there are two unknown quantities; therefore the circumstances of the collision cannot be determined without some further condition.

58. When the particles are supposed unelastic, as if they are absolutely hard and incapable of changing their figure, then after the collision they must proceed together; this gives $u = u'$, and equation (1) becomes

$$mV+m'V' = (m+m') \times u. \qquad (2)$$

We can in this case find the quantity of motion of each body, that of m is $mu = \frac{m^2V+m'mV'}{m+m'}$, and that of m' is $\frac{mm'V+m'^2V'}{m+m'}$.

If m' be at rest before the stroke, $V'=0$, and the quantities of motion are $\frac{m^2V}{m+m'}$, $\frac{mm'V}{m+m}$. The first of these is evidently less than mV, therefore the striking body loses part of its motion; the part lost is

$$mV - \frac{m^2V}{m+m'} = \frac{m^2V+mm'V-m^2V}{m \times m'} = \frac{mm'V}{m+m'}$$

or precisely the quantity of motion which the other body has acquired. If the bodies be equal, the quantities of motion which they receive are each $\frac{1}{2}mV$, and in general they are as $m : m'$.

If V' be negative, or if m' move in an opposite direction to m, equation (2) becomes $mV-m'V=(m+m')u$ which when there is equilibrium, or when $u = 0$ gives $mV = m'V'$ or the velocities inversely as the quantities of matter.

59. Where the bodies are elastic, the case is somewhat different; if the bodies be hard, the transference of motion from one to another must be instantaneous, but it must be gradual when they are elastic. When m impinges on m' it compresses the part nearest to the point of impact, and the pressure is communicated to the other parts of m', not instantaneously but with a finite velocity depending on the modulus of its elasticity; the compression must increase, and the motion of m' augment till $u'=u$, when the transference of motion is effected as before, and the impulse of m on m' ceases. But both bodies have been compressed, and they must resume their original figure, which cannot be done without communicating motion to both: the elasticity of m tends to increase the motion of m', and that of m' produces a similar effect on m; moreover these forces must be equal to the forces which had exerted them. While they were yielding to the compression m lost the velocity $V-u$ by the collision, and m' gained $u-V'$; they must gain and lose as much more by the elasticity, so that their velocities after they have returned to their original form are $V-(2V-2u)$ or $2u-V$, and $V'+(2u-2V')$ or $2u-V'$. Call them U and U', and substituting for u the value given by equation 2,

$$U = \frac{(m-m')\,V + 2\,m'\,V'}{m+m'}, \qquad (3)$$

$$U' = \frac{-(m-m')\,V' + 2\,mV}{m+m'}$$

From these we derive the quantities of motion,

$$mU = \frac{(m^2-mm')\,V + 2mm'\,V'}{m+m'},$$

$$m'U' = -\frac{(mm'-m'^2)\,V' + 2mm'\,V}{m+m'}. \qquad (4)$$

If m' be quiescent before the stroke, $V' = 0$, and the equations (3) give

$$U = \frac{(m-m')\,V}{m+m'} \qquad U' = \frac{2mV}{m+m}$$

If $m=m'$, $U = 0$, and $U' = V$, or the striking body loses all its motion, and that struck acquires it; and if m' be infinite, as where an elastic body strikes a very large mass,

$$U = \frac{-m'}{m'} \times V, \text{ and } U' = \frac{am}{m'} \times V = 0;$$

no motion is communicated to the mass struck, and the other rebounds with an equal velocity. If we compare U and U′ with V, we have

$$U : V :: m-m' : m+m',$$
$$U' : V :: 2m : m+m',$$

U′ is therefore greater than V, or the velocity is increased whenever m exceeds m'. The value of m' U′ is $\frac{2mm'V}{m+m'}$, or the motion of the body struck is to that of the striking body as $2m' : m+m'$; the quantity of motion is therefore augmented when m' exceeds m.

If m' moves in a contrary direction to m, equat. 3 becomes

$$U = \frac{(m-m')\,V-2m'V'}{m+m'}, \quad U = \frac{(m-m')\,V'+2\,mV}{m+m'},$$

which become cypher when $V' = V \times \frac{m-m'}{2m'}$ and $V' = -\frac{2m}{m-m'} \times V$, which as V′ is supposed negative, requires $m' > m$, and may be written $\frac{2m}{m'-m} \times V$. If $V' = -V$, and $m' = 3m$, the struck body will stop, and the striking recoil with a velocity equal to 2V.

60. In these theorems we have supposed the bodies to move uniformly in the same right line; but it is evidently unnecessary to restrict our conclusions in consequence of this supposition, as they refer only to the velocity at the instant before and after the collision; they are therefore true even though the velocities have been acquired in descending through arches of curves; and thus we have a ready means of verifying our computations by experiment. Let two balls of soft clay as unelastic, or of ivory as elastic bodies, be suspended by cords of equal length, so as to be in contact when at rest; on drawing them aside from the vertical, and allowing them to descend, they meet with velocities which are (art. 29) proportional to the chords of the arches through which they have descended. The results can be observed, and are found to coincide with our theory; they are therefore proofs of the third law of motion, on which the theory rests.

61. It is a remarkable circumstance in the collision of either of these sorts of bodies, that the motion of their centre of gravity is not altered by it: this is a particular case of a mechanical principle, called the conservation of the motion of the centre of gravity, (*b*) but we can prove it only in these instances. To find the velocity of the centre of gravity before the impact, suppose them to move from a given point with the velocities V and V′ for the time T, their distances from it are VT, V′T; the distance of their centre of gravity is, Statics (art. 11), $\frac{mV+m'V'}{m+m'} \times T$, and dividing this by T we obtain for the velocity of the centre of gra-

vity $\frac{mV+m'V'}{m+m'}$, the same as the common velocity after the collision, which may be that of the centre of gravity itself, as the two bodies coincide. Where the bodies are elastic, we find the velocity of the centre after the stroke in the same way to be $\frac{mU+m'U'}{m+m'}$; the values of mU; $m'U'$ given in equat. 4, being added together, are

$$\frac{(m^2+mm')V+(m'^2+mm')V'}{+m'} = (mV+m'V')\frac{m+m'}{m+m'},$$

and therefore

$$\frac{mU+m'U'}{m+m'} = \frac{mV+m'V'}{m+m'},$$

which is the motion of the centre before the stroke.

62. Another law is, that where the bodies are elastic their Vis Viva remains unaltered, or in other words, $mU^2+mU'^2 = mV^2+m'V'^2$; to prove this we form $mU^2+m'U'^2$, from the equations 3: its denominator is $(m+m')^2$ and classing together those terms of its numerator, which contain V^2, V'^2, and VV', we have

$$(mU^2+m'U'^2)(m+m')^2 = \begin{cases} mV^2\{m^2+m'^2-2mm'+4mm', \\ +m'V'^2\{m^2+m'^2-2mm' \\ \quad +4mm', \\ +4VV'\{mm')(m-m')-mm \\ \quad (m-m'); \end{cases}$$

now the third line = 0, and the first and second are $mV^2\{(m+m')^2\}$, $m'V'^2\{m+m'\}^2$, therefore

$$mV^2+m'V'^2 = mU^2+m'U'^2. \qquad (c)$$

This, which like the preceding is a case of a more general theorem, called the Conservation of the Vis Viva, was noticed by those who measured moving force by the square of the velocity, but it does not hold for *hard* or unelastic bodies; in these $(m+m')u^2$ is the Vis Viva after the stroke, and the loss is

$$mV^2+m'V'^2-(m+m')\,u^2=$$
$$mV^2+m'V'^2-2\,(m+m')\,u^2+(m+m')\,u^2=$$
$$\text{as }(m+m')\,u=mV'+m'V',$$
$$mV^2+m'V'^2-2\,u\,(mV+m'V')+u^2\,(m+m')=$$
$$m\,(V-u)^2+m'\,(V-u)^2,$$

or the Vis Viva lost is that due to the differences of the velocity before and after the stroke. The Vis Viva which is lost when the bodies are soft and unelastic is manifestly employed in changing their figure, and the knowledge of this leads to many practical conclusions connected with the performance of machines: for example, it teaches that great loss must be experienced in percussive engines where the framing yields, or the working point is unelastic. This is easily observed by those who cut stones; they find that when their tools are made throughout of hardened steel, they work incomparably faster than when the edges only are steeled; and in driving piles, one of straight-grained fir, which is highly elastic, can be driven freely when one of oak whose fibres are irregular resists all further impulse of the rammer: but these considerations more properly belong to the last chapter.

63. This may suffice where abstraction is made of the figures of the striking bodies; we must next ascertain what often happens when they are not material points. To effect this we must determine the motion of a system of material points to each of which given forces are applied, and though this cannot be perfectly done without more elevated means of investigation, than it is

allowed to use here, yet we easily obtain two important conclusions. Let m, m', m'' fig. 183, represent such a system to which equal velocities are communicated; let m A, m' B, m'' C, be x, x', x'', and let these distances from AC be each increased by $m\mu = \xi$, it being the space described by each of the particles in a certain time t; then the distance of γ the centre of gravity in this new position has become $\frac{m \times (x+\xi)+m'(x'+\xi)+m''(x''+\xi)}{m+m'+m''}$ which is true whatever be the number of particles; this is $\frac{mx+m'x'+m''x''}{m+m'+m''} + \frac{m+m'+m''}{m+m'+m''} \times \xi$. The first of these expressions is the distance of the centre of gravity at the beginning of the motion from AC, therefore the second, or ξ, is the motion of the centre. Hence in such a system the centre moves with the given velocity in a line parallel to the lines described by each particle. If now we suppose the particles to be connected together as in a solid, no alteration is produced, for the distances of them when at μ, μ', μ'' are the same as at first, and therefore no new forces are called into action. Conversely, if the direction of an impulse pass through the centre of gravity it will excite an equal velocity in each particle, and the body will have the same quantity of motion as if its mass were collected in the centre to receive the impulse.

64. If the impulse do not pass through the centre of gravity, the motion of this point will be the same as if it had been applied there; but besides this the body will have a motion of rotation round this centre, or rather round a line drawn through it perpendicular to that connecting the point of impact with the centre of gravity. To prove it, we suppose the impulse equal to the result

of parallel impulses communicated to each particle; if the velocities of each were equal, then their result would pass through the centre of gravity, contrary to our hypothesis, therefore they are unequal and their quantities can be assigned by the theory of parallel forces given in chap. III. of Statics. Let m and m' be two particles moving with unequal and parallel velocities, and let $m\mu$, $m'\mu'$ fig 184, be the spaces described by them in any portion of time; G is their centre of gravity at the beginning, and γ at the end of it; and as these points divide the lines mm' and $\mu\mu'$ in the same ratio, namely the inverse of the masses m and m', γG is parallel to $m\mu$, therefore the centre of gravity moves in a direction parallel to the motion of the particles. The velocities of m and m' being as $m\mu$, $m'\mu'$, that of the centre of gravity is Gγ: draw no parallel to mm', and let the three velocities be V, V′, U; by similar triangles $n\mu : o\mu' :: n\gamma : \gamma o :: m' : m$. The first two of these proportionals are as V—U and U—V′, therefore $(V-U) \times m = (U-V') \times m'$ whence $(m+m')\,U = mV + m'V'$; from which the sum of the motions equals that of the whole mass concentrated in the centre of gravity. The supposition that the particles are connected so as to remain at an invariable distance, makes no alteration, for the inequality of velocity makes the distance $\mu\mu'$ vary, but the connexion of the system keeps them at the distance $m\,m'$, it is therefore equivalent to a force which could move them in the given time through $\mu\mu' - m\,m'$. By the equality of action and reaction, this force acts equally on each particle, and as bodies describe by a given force in a given time, spaces which are inversely as their masses, $\mu\gamma$ and $\mu'\gamma$ are diminished by quantities in their own ratio: hence the remainders $\nu\gamma$ $\nu'\gamma$

are also inversely as m and m', therefore γ is the centre of gravity of the connected system, and its motion is the same as in the preceding case. It is manifest that while the centre moves from G to γ, the line $m\,m'$ has revolved through an angle $= \mu\gamma n$, and from this it is evident, that the motion of any system of two or more than two particles may be represented by a uniform rectilinear motion of the centre of gravity, and a motion of rotation round it. We must now determine the quantities of the two which proceed from a given impulse.

65. If we suppose a connected system of particles to have a common velocity V, and an angular velocity ω, round the centre of gravity; the sum total of the quantities of motion which all the particles acquire must equal the impulse from which they were produced. A particle m, whose distance from the centre is a, has a velocity of rotation ωa, and also the common velocity V, its motion is therefore $mV + m\omega a$, and calling F the quantity of motion of the striking body, we have

$$F = V\,(m + m' +, \&c.) + \omega\,(ma + m'a' +, \&c.)$$

The coefficient of ω is the sum of each particle multiplied into the distance from the centre of gravity, which equals cypher: that of V is the mass of the body, and therefore $V = F \div M$.

To find ω, we consider the impulse, and the motion of each particle, applied to a lever which is furnished by the connexion of the system, and as they must make equilibrium, each must be multiplied into its distance from the centre, and the sum equated to the moment of the impulse; calling f the distance of the application of F.

$$Ff = V\,(ma + m'a' +, \&c.) + \omega\,(ma^2 + m'a'^2 +, \&c.$$

the coefficient of $V = 0$, and that of ω is the moment of inertia with respect to an axis passing through the centre of gravity, equal (by art. 42) to Mc^2, therefore

$$\omega = \frac{F}{M} \times \frac{f}{c^2} = \frac{V \times f}{c^2}.$$

66. The particles which are on the same side of the centre, revolve in the direction of V, those on the other in a different; it may therefore happen, that one of the latter may have a velocity equal and opposite to V, and may therefore for an instant remain at rest. This point is called the Instantaneous Axis of Rotation, and were the body held by a fixed axis passing through it, the axis would experience no percussion. Let x be its distance, the velocity of rotation is $\omega x = \frac{Vfx}{c^2}$; this $= V$, therefore $1 = \frac{fx}{c^2}$, $x = \frac{c^2}{f}$, and $x + f = \frac{f^2 + c^2}{f}$, or (art. 44, putting f for a) the point of application of the impulse is the centre of oscillation, the point (x) being the axis of suspension.

67. Some have supposed that the rotatory and orbital motions of the planets arose from the projectile impulse not passing through the centre of gravity, and have determined the distance at which it should pass to produce them; let V be the velocity in the orbit, and v that of the equator, it is ωR, and in a sphere $c^2 = \frac{2}{5} R^2$, hence the value of ω in art. 65, $= v \div R$, therefore

$$\frac{v}{R} = \frac{Vf}{\frac{2}{5} R^2},\ f = \tfrac{2}{5} \times \frac{v}{V} \times R$$

In the earth $f = \frac{R}{160}$, but as in all probability the

Creator did not use the means supposed (the impact of a comet), the theorem must be restricted to experiments on a smaller scale.

NOTES ON CHAP. VIII.

(*a*) The estimate given in the text is but an approximation, on the hypothesis that the ball is unchanged in figure by the blow, and that the depth of penetration bears no assignable ratio to the height due to the velocity; they who wish for more information may consult Gregory's Mechanics, where they will find a full account of Don George Juan's theory of percussion, the most plausible with which we are acquainted. He has however omitted one material consideration, namely, the fracture or penetration which takes place when the impact is made with great velocity, in consequence of the cohesive forces of the surrounding particles not being called into action. The velocity with which any action of this kind is transmitted through a substance, is that due to half the height of its Modulus of Elasticity; it therefore can be found, as also the space through which the body can be compressed without fracture. If this velocity be not to the velocity of impact in a greater ratio than the length to the compressibility of the body, it will yield; for the space required to destroy the motion of the striking body cannot be obtained without compressing the remote particles, and before the compression has travelled to them, the parts struck have yielded. Thus, suppose a plank of wood, through which an impression travels with a velocity of 15000 feet in a second, while it cannot be compressed more than $\frac{1}{100}$ of its thickness without rupture of its fibres; it will be pierced by a ball whose velocity is $\frac{15000}{100}$, or 150, though a slower missile of greater weight would not injure it.

(b.) Let a system of the bodies m, m', m'' &c. be subjected to the action of accelerating forces; let x, y, z, be the coördinates of m referred to three rectangular axes passing through any point of space; let x, y, z, be the coördinates of the centre of gravity of the system, and let x, y, z, $= x + \xi$, $y + \eta$, $z + \zeta$ respectively, then ξ, η, ζ are the coördinates of m when the origin is at that centre. Let the accelerating forces which act on each particle be resolved in the direction of the three axes, those results for m are X, Y, Z, for m', X′, Y′ Z′, and so on. The velocity of m is $\frac{d\mathrm{x}}{dt}$ or $\frac{d\,(x+\xi)}{d\,t}$, which during dt, becomes by the action of the forces and the connection of the system, $\frac{d\,(x+\xi)}{d\,t} + d\left(\frac{dx+d\xi}{d\,t}\right)$; this latter, which if dt be constant is $\frac{d^2x + d^2\xi}{d\,t}$, is therefore the velocity gained during dt by m. By D'Alembert's principle, the quantities of motion corresponding to these quantities must be in equilibrio with those impressed during the same line by the forces X, X′ &c. which are of the form $m\mathrm{X}dt$, Xdt being dv, we have therefore

$$\left.\begin{array}{l} m\,(d^2x+d^2\xi) - m\mathrm{X}\times dt^2 \\ +m'(d^2x+d^2\xi') - m'\mathrm{X}'\times dt^2 \\ +\ \&c. \end{array}\right\} = 0$$

or grouping those quantities which are similar together,
$$\{m+m'+m'' \text{ \&c.}\} d^2x + \{md^2\xi + m'd^2\xi' + m''d^2\xi'' \text{ \&c.}\}$$
$$-\{mX + m'X' + m''X''\} = 0.$$
Since ξ, ξ', &c. are ordinates whose origin is in the centre of gravity, we have by the property of that centre, $m\xi + m'\xi' + \text{\&c.} = 0$, from which we derive $m\,d^2\xi + m'd^2\xi' + \text{\&c.} = 0$, and our equation becomes
$$\frac{Md^2x}{dt^2} = S\,(Xm + X'm' + \text{\&c.} \qquad (1)$$
M being the sum of the bodies m, m' &c. and the sign S (Xm) denoting a sum of finite quantities, which becomes an integral when m is the differential of a solid. By similar reasoning we have
$$\frac{Md^2y}{dt^2} = S\,(Ym), \quad \frac{Md^2z}{dt^2} = S\,(Zm) \qquad (1)$$
These expressions do not contain ξ, η or ζ, they therefore relate solely to the centre of gravity; and in fact if the bodies m, m' &c. were collected in their centre of gravity, the second members of these equations would become single forces, in which case the equations would become
$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y, \quad \frac{d^2z}{dt^2} = Z,$$
the same as those found in note (a) chap. III. for the motion of a material point. Hence the centre of gravity moves as if the mass of the system were collected in it, and is not affected by the mutual action of the bodies, but by the external accelerating forces. If the system be not acted on by such a force but merely by an impulse, the equations become $\frac{d^2x}{dt^2} = 0$, &c. whose integrals are $dx = A\,dt$, $dy = B\,dt$, $dz = C\,dt$, the constants A, B and C being the uniform velocities of the centre parallel to

the three axes, so that it moves in a right line, whose position can be assigned in any given instance.

(*c*) The conservation of the motion of the centre of gravity is deduced from the sum of the forces parallel to each axis being cypher; this alone is not sufficient for equilibrium, unless their statical moment be also cypher. The equation deduced from this gives a principle called in transcendental mechanics the Conservation of areas, which has been demonstrated for a single material point in the note last referred to, Eq. (*o*); it however belongs rather to physical astronomy, as the properties of rotatory motion which our purpose requires, have been demonstrated without it. Another general principle which is noticed in the text, relative to the Vis Viva, might be derived from these equations, but the principle of Virtual velocities gives it much more easily. This principle, which we have frequently noticed, is this, that when any system is in equilibrio, if each body of it receives arbitrary displacements in the direction in which the forces acting on it would move it, the sum of each force multiplied into the corresponding displacement $= 0$. Applying this to express the equilibrium in D'Alembert's principle, and denoting the arbitrary displacement in the direction of x by δx, to distinguish it from dx, which depends on the time, we have

$$\mathrm{S}\left(\frac{d^2x}{dt^2}\,\delta x - \mathrm{X}\delta x\right) m = 0,$$

$$\mathrm{S}\left(\frac{d^2y}{dt^2}\,\delta y - \mathrm{Y}\delta y\right) m = 0,$$

$$\mathrm{S}\left(\frac{d^2z}{dt^2}\,\delta z - \mathrm{Z}\delta z\right) m = 0,$$

Which are the general equations of the motion of a system of bodies whose coordinates are $x\ y\ z$, $x'\ y'\ z'$, &c. As the displacements denoted by δ are arbitrary we may suppose them equal to those denoted by d, unless the form of the system depends on t; in which case, adding the three equations

$$S\left(\frac{dxd^2x+dyd^2y+dzd^2z}{dt^2}\right)-(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz)\,m=0;$$

the first of these is

$$\tfrac{1}{2}\,d\left(\frac{dx^2+dy^2+dz^2}{dt^2}\right)=d\left(\frac{ds^2}{2dt^2}\right)=d\tfrac{1}{2}\,v^2$$

therefore

$$\mathrm{S}\,mv^2=\mathrm{C}+2\,\mathrm{S}\,m\times\int \mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz.$$

The first member is the Vis Viva of the system; the second is integrable when $\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz$ is an exact differential for each body. This happens when the forces acting on it tend to fixed centres or to bodies of the system, and are functions of the distance; and by comparing it with note (*a*) Chap. III, it is seen that the Vis Viva is the same as if each body moved separately in virtue of the accelerating forces which act on it. If friction or the resistance of a fluid act, the function in question is not an exact differential, and there is a loss of Vis Viva.

For a demonstration of the principle of virtual velocities, see Poisson, tom. 1, Art. 172. In the case where the forces are applied at the same point, the demonstration given in note (*a*) of Chap. III. is sufficient, for if R be negative it will keep X, Y, and Z, in equilibrio, and it is proved that

$$\mathrm{R}dr+\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz=0,$$

and X may be the sum of X′, X″, &c. the components of the forces P, Q, &c, therefore

$$Rdr + Pdp + Qdq, \text{ \&c.} = 0.$$

CHAPTER IX.

67. Having stated the theorems concerning the motion and mutual action of solid and connected systems, it is necessary to complete this treatise that we add some of the most elementary results of Hydrodynamics. As the equilibrium of fluids is widely different in its principles from that of solids, so the laws of their motions differ from ordinary Dynamics; their investigation is far more difficult, and except in a very few instances, our theory is utterly at fault. Newton himself failed in this difficult task, and the most valuable knowledge which we possess respecting it is derived from experiment. Yet it is of high practical importance to mankind; the supply of water through pipes, the course of rivers, the management of ships, and the construction of Hydraulic engines, are matters in which society is vitally interested, and with which most individuals should have some acquaintance. Therefore where theory leads us freely we will follow; where it deserts us, we must be content to walk by the less certain light of experience. The principal parts of this science which are noticed here, are

the efflux of a liquid through a small orifice, the flow in pipes and rivers, the resistance of fluids, and their application to urge machinery.

68. The fundamental proposition of Hydrodynamics is that which states the velocity of efflux from a small orifice in the bottom of a vessel; but it is by no means an easy matter to demonstrate it: it was first ascertained as an experimental law by Torricelli. He found that a recurved tube connected with the bottom of a vessel threw a jet of the liquid used as high as the level of its surface in the vessel; therefore each particle issues with a velocity capable of carrying it through the height of the incumbent column of fluid, and therefore the velocity is such as a heavy body would acquire in falling through that height. Experiment also shews that the direction of the orifice does not affect the velocity, as indeed might be anticipated from the equality of pressure in liquids. A demonstration of this law is given in note (*a*) but as it is not of the simplest nature, the generality of readers will be satisfied with the following, though less exact. If the orifice be indefinitely small, the quantity of fluid which escapes from it in an instant of time bears no assignable ratio to the mass of fluid in the vessel; this latter may therefore be considered quiescent, and we may suppose, that the effluent portion acquires its velocity while it passes through the orifice itself. The force which expels it is the weight of a quantity of fluid whose base is the orifice, and height H the depth of the fluid; this acts on a portion of it, whose base is also the orifice, and height the space *s*, through which the accelerating force acts to generate the velocity of

efflux. Hence the accelerating force, which is as the moving force divided by the mass moved, is $g \times \frac{H}{s}$. This acts on a particle through the space s, and the value of the velocity in Art. 9, equat. (1) becomes

$$V = \sqrt{2g \times \frac{H}{s} \times s} = \sqrt{2gH}$$

the same as that acquired in falling through H.

69. From this follow several corollaries. Orifices of different sizes, provided that they are indefinitely less than the area of the vessel, give in the same time quantities of liquid proportional to their areas multiplied into the square roots of their depths below the surface of the reservoir; for the quantity delivered in the time t is a column whose base is the orifice O, and height the space described in the time t with the velocity of efflux, or tv; hence it is $O \times t \times \sqrt{2gH}$.

Secondly if the reservoir be not kept full but be allowed to empty itself through the orifice, its surface descends with an uniformly retarded motion, supposing the horizontal section of the vessel equal throughout; for it is manifest that the quantity of fluid intercepted between two successive positions of the surface, is equal to that which has escaped from the orifice; let A be the area of the surface, V the velocity with which it sinks, AVt is the quantity contained between the positions of the surface at the beginning and end of the instant t; that discharged by the orifice is O Vt, therefore $V : v$ inversely as $A : O$, or in a constant ratio: $v^2 \propto H$, therefore V^2, or the surface moves through H with a velocity whose square is as the space to be described, or after the manner of a body uniformly retarded.

Hence also a vessel kept constantly full delivers twice as much through a given orifice, as another of the same depth which is allowed to empty itself.

On these principles depends the clepsydra or water clock to which the common hour-glass is analogous, in which time is measured by the descent of the surface of water along a scale, graduated so that the square roots of the height above the orifice are in arithmetical progression. These instruments are however of no use except for measuring minute intervals, as the variations of temperature interfere with the rate of efflux by increasing the fluidity of the water. For the time of employing vessels of a given shape, see note *(b)*.

70. When Newton, who was the first person that attempted these investigations, tried his theory by experiment, he found a remarkable difference which for a time misled him; but his sagacity soon discovered the cause. We have shewn that the quantity discharged is $Ot \times \sqrt{2gH}$ now if we receive and measure it, it is found only 0.62 of the computed quantity. This does not differ widely from $\frac{1}{\sqrt{2}}$ and Newton at first supposed that the velocity is that due to *half* the depth instead of the whole. But on a careful examination of the jet, it is seen that it is not cylindrical, contracting to a certain distance from the orifice, and if the area of its greatest contraction be measured, it is found 0.62 of the orifice. Theory therefore gives the velocity not at the orifice itself but at the *Vena Contracta*, as this least section is named. The reason of this anomaly is, that the central particles alone receive the full velocity due to the height, those in contact with the orifice being retarded by friction, and also

because the motion does not commence at the orifice itself, but begins at some distance in the interior of the fluid mass; the particles have a convergence toward the orifice, and retain it for some time after their escape. That this is the true cause may be proved by a very convincing experiment. Let the shadow of a jet be projected on paper and its outline then traced, it is nearly a frustum of a cone whose length is half its base, with its sides a little convex towards the axis. If a tube of this shape be fitted to an aperture in a vessel, the discharge from it is more than 0 . 9 of the quantity computed from the area of the exterior orifice.

71. Where the orifice is not of evanescent dimensions, the different parts of it are at different depths, and therefore the different parts of the effluent water have unequal velocities. But there can be assigned a certain average velocity, by supposing which uniform through the whole orifice we obtain the true discharge. One example of the method may suffice, which is also the most useful, when the aperture is rectangular as a sluice or Were; let AB fig. 185 be its profile, the velocity from C to D may be considered uniform, taking them indefinitely near; draw an ordinate CE proportional to it, and others proportional to the velocities at the other points of AB. Since these ordinates are as the square roots of the distances from A where the surface is supposed to meet the aperture, the curve AEF is a parabola. Its area is as the quantity of water discharged in a given time, for that which escapes through the portion CD is as CD × CE, and the sum of these trapezia is the whole area: the area of the parabola is known to be two-thirds of the circumscribing rectangle, and therefore the product of the orifice into two-thirds of the ve-

locity at its lowest point represents the quantity discharged.-(c).

72. This supposes that the orifice reaches to the surface; if it does not, we compute the discharges through two such apertures whose depths are those of the top and bottom of the given one; the difference of them is the quantity required. The multiplier of the orifice which gives the theoretic discharge is $8\sqrt{H}$, H being the depth of an indefinitely small orifice below the surface; if allowance be made for the Vena Contracta it is $5\sqrt{H}$, and in the case of a Were it is $\frac{2}{3}5\sqrt{H}$, or $3.3\sqrt{H}$, for it is found that the vena contracta is formed even in such apertures as sluices and the arches of bridges. For example, let it be proposed to find the discharge in a second from a Were 10 feet broad and one deep; the aperture is 1×10; and we have $D = 10 \times 3.3 \times \sqrt{1} = 33$ cubic feet.

73. If the orifice be not a simple aperture, but be provided with a tube of the same diameter, the quantity discharged is considerably increased till its length is so great that this effect is counteracted by the friction. This singular effect is connected with the pressure of the atmosphere, for it is not observed under the air pump; and also on capillary attraction, as it does not take place when mercury issues through a tube of iron. To conceive the mode of its production, fig. 186 if the fluid does not adhere to the tube AB, by forming the Vena Contracta the discharge is evidently the same as through a simple orifice, and a certain space is left vacant in the tube; but as the capillary attraction of the tube makes the water apply to it, the tendency of the vein to contract causes the atmospheric pressure to act and accele-

rate the efflux. The cylindric tube produces no effect unless its length exceed half its diameter; but the maximum discharge is produced by a compound tube in which the part next the orifice is shaped as the Vena Contracta, while the remainder is a conic frustum with its base outwards, which gives $2\frac{1}{2}$ times as much water as the simple orifice.

74. If the cylindrical tube be prolonged downwards to any depth, the velocity of efflux from its lower extremity is that due to *its* depth below the surface, for the pressure of the atmosphere suspends the weight of the liquid in the tube from the plate of fluid which fills the upper orifice; it is therefore moved not only by the weight of the column above, but by that beneath it. This is more easily understood, by considering that the liquid which descends through the tube is accelerated by gravity, and therefore tends to separate from that above it; it would actually do so but for atmospheric pressure, as is evident if we make apertures through any part of the tube, for air enters through them, and the tube is not filled with the liquid. This is applied sometimes to blow furnaces; water falls through a perforated tube into a reservoir, the air which it drags with it is condensed there and produces a steady and powerful blast.

75. The motion of fluids in pipes of considerable length diminishes with the length of the pipe, but theory gives us here no assistance. The diminution of velocity evidently depends on resistances, which may be compared to friction, and are often called by that name though they differ widely. Friction is independent of the velocity, and varies with the pressure, while this does not appear to be influenced by the pressure but

to depend on the velocity. It consists of two parts, one arising from the adhesion of the fluid to the sides of the tube, which is probably as the velocity; and another caused by the eddies into which the particles are thrown by that adhesion: in consequence of these many of the particles move in curves, and the force which deflects them is lost. If we suppose the curves to be of uniform curvature, these forces are as the squares of the velocities; and we may expect this part of the resistance to follow the same ratio. From these considerations it follows that the velocity with which the water is delivered, must be a function of the difference of level between the surface of the fluid in the reservoir and the place of discharge, technically called the *Head*, and of the resistance, which is probably inversely as the diameter of the tube. Let V be the mean velocity of efflux, D the diameter, and L the length of the pipe, it has been found that the formula

$$V = 50 \times \sqrt{\left(\frac{D\,H}{L + 50\,D}\right)}$$

in feet, represents with sufficient accuracy the result of experiment. For the mode in which this expression is formed, see Young's Nat. Phil. vol. 2, page 62, as also for the correction due to the flexure of the pipe.

76. A river differs from a conduit-pipe only in this that its velocity is nearly uniform throughout, the inclination of its channel being equivalent to the resistances. This must evidently be the case, for if a local contraction of the channel accelerate it, yet that excess of velocity is lost by the eddies which are produced when it escapes into a wider part; and moreover the action of the river itself tends to produce a uniform velocity by excavating the narrower parts of its bed,

and depositing the substances which it removes in the deeper. We may compare the river to the tube which has the same surface, by substituting for D four times *d*, the Hydraulic Mean Depth, or factor, which gives the section of the river when multiplied into the perimeter of its bed, and for H, L× sin inclination of the river. As the velocity is uniform throughout we may suppose L infinite, and the formula gives

$$V = 50 \sqrt{4\, d \sin I} \text{ or } 100 \sqrt{d \sin I.}$$

Experiment shews that this is rather too great, and that we should use 92 instead of it.

77. In conveying water by tubes or canals, they should be of uniform diameter or area throughout: all contractions consume a portion of Vis Viva in accelerating the fluid, which must move with a velocity inversely as the space through which it passes; and enlargements are equally pernicious by occasioning eddies which absorb, as has been stated, much power. Flexures produce the same effect, and for the same reason, and in canals or rivers are accompanied with a tendency to destroy their banks; but the detail of these facts belongs to Hydraulic Architecture.

78. In the case of elastic fluids, the above theorems hold good with certain modifications; thus the velocity with which air enters a vacuum through an orifice is that acquired in falling through the height of the homogeneous atmosphere, for the weight of a column of this height and of the density at the orifice is the pressure there: as H is independent of the density, it follows that air enters a vacuum with an uniform velocity. When the vessel contains air of a less density, the influx through the

orifice is that due to a height which is to H as the difference of the densities is to the external density. The same principles apply to the escape of condensed air, and constitute the theory of blowing machines, and of the air-gun; for their developement see note *(d)*. The Vena Contracta is formed in air as well as water, and bears the same ratio to the orifice, and the discharge is augmented by a cylindrical or diverging conical adjutage. These principles may be applied to the construction of chimnies for furnaces, the action of which depends on the expansion of air by heat. The air which has passed through the fuel is heated to redness, or about 1000 degrees; and if we suppose this temperature gradually to diminish in the flue, so that half of it may be taken as the average, the average specific gravity of the included air is $\frac{1}{2}$ that of the exterior; as the weight of this column is less than one of the external air of equal altitude, it must ascend, and its place be supplied by a portion of air which rushes in through the fuel. The smoking of chimnies arises from a deficient supply of external air, from eddy-winds disturbing the current at the top of the funnel, and from the improper construction of this latter, which according to the theory of Hydraulics should be a tube of uniform diameter, with a contraction at the bottom, variable at pleasure, that the draught may be adjusted to circumstances.

NOTES ON CHAP. IX.

(*a*) The theorem which assigns the velocity of a spouting fluid is obtained by D'Alembert's principle on the hypothesis of the parallelism of the elementary plates of the fluid in their descent. Let the origin of coordinates be placed at the initial level of the surface with the z vertical, and let v be the vertical velocity of an elementary plate of the fluid whose area is A, and thickness dz. If it fell freely it would acquire in the instant dt a velocity gdt, but it is accelerated by the pressure of the incumbent fluid, and retarded by the resistance of that below it, so that during the same instant it acquires the velocity dv; there is therefore equilibrium between g and the sum of the forces $-\frac{dv}{dt}$.

We apply to this the theorem note (a), chap. xvi. Statics, observing that $f=0$, $f'=0$. and we have for the plate (z), supposing s unity

$$dP = gdz - \frac{dv\,dz}{dt}$$

As the fluid is continuous, and incompressible, calling O, the area of the orifice, and u the velocity of efflux, $O \times u = A \times v$, for the velocities are inversely as the areas of the sections; hence $v = \frac{Ou}{A}$. It is clear that v is a function of A, which is a function of z; and differentiating

$$\frac{dv}{dt} = \frac{Odu}{Adt} - \frac{Ou}{A^2}\frac{dA}{dz} \times \frac{dz}{dt}$$

See Lacroix, Elem. Cal. Diff. Art. 127. $\frac{dz}{dt} = v = \frac{Ou}{A}$

and with these substitutions our equation is

$$dP = gdz - O\frac{du}{dt} \times \frac{dz}{A} + \frac{O^2u^2}{dt} \times \frac{dA}{A^3}.$$

the quantities $\frac{du}{dt}$ and u^2 are independent of z, and therefore are considered constant in the integration, $\frac{dz}{A}$ depends on the form of the vessel, and calling its integral N,

$$P = C + gz - N \times O \times \frac{du}{dt} - \frac{O^2u^2}{2\,A^2}.$$

The constant C, depending on the pressure at the surface of the fluid, which is in general the pressure of the atmosphere. We apply this equation to the orifice by supposing $z = h$, the depth of the orifice, and $P = c$ the pressure at it, which is either the atmospheric pressure or that of a column of fluid; but unfortunately the resulting equation cannot be integrated for du but in a few cases. Let the pressure of the atmosphere be Π, then when $z = 0$

$$\Pi = C - \frac{O^2}{2A'^2} \times u^2, \text{ hence eliminating C}$$

$$P = \Pi + gz - NO \times \frac{du}{dt} + O^2u^2\left(\frac{1}{2\,A'^2} - \frac{1}{2\,A^2}\right)$$

and as A becomes O, when $z = h$, putting the pressure at the orifice π,

$$0 = \Pi - \pi + gh - NO \times \frac{du}{dt} + \frac{O^2u^2}{2\,A'^2} - \frac{u^2}{2}$$

which when O is evanescent becomes

$$u^2 = 2\,gh + 2\,(\Pi - \pi)$$

If $\Pi = \pi$, $u^2 = 2gh$, or it is the velocity acquired in falling through h; if $\pi = \Pi + gh'$, as when the orifice is submerged,

$$u^2 = 2g(h-h'),$$

which agrees very well with experiment.

(*b*) Where a vessel empties itself through an orifice, as the equation $v = \frac{Ou}{A}$ holds, calling h as before the distance of the orifice from the upper surface, we have

$$v = \frac{O \times \sqrt{2gh}}{A}, \quad dt = -\frac{dh}{v} =$$

$$\frac{1}{\sqrt{2g} \times O} \times -\frac{A\,dh}{\sqrt{h}},$$

which is integrable as A is a function of h.

Let us suppose that the vessel is a cylinder or prism in which case A is constant,

$$t = \frac{2A}{O \times \sqrt{2g}} \times -\sqrt{h} + C,$$

$t = 0$, when $h = H$, therefore $C = \frac{2A}{O} \times \frac{\sqrt{H}}{\sqrt{2g}}$ and

$$t = \frac{2A}{O} \times \frac{(\sqrt{H} - \sqrt{h})}{\sqrt{2g}}$$

the time of emptying the vessel completely is obtained by putting $h = 0$, and is

$$\frac{2A}{O} \times \sqrt{\frac{H}{2g}},$$

or it is to the time in which a body would fall through H as 2 A : O.

If the figure of the vessel be required such that the descent of the surface may be proportional to the time, $\frac{dt}{dh}$ is constant, therefore

$$C = \frac{-A}{O \times \sqrt{2gh}},$$

which gives, in the case of a solid of revolution,

$$C' \times \sqrt{h} + pr^2 = 0.$$

These theorems suppose that the strata descend parallel, but when the depth of the orifice does not exceed 6 or 7 inches, a funnel-shaped cavity is formed by the centrifugal force of the particles, which materially affects the time of discharge.

(c) The effect of the unequal depth of the parts of the orifice is easily ascertained; let y be its horizontal dimensions at the depth z, the velocity of efflux throughout this section is $\sqrt{2gz}$, the area of the elementary section is ydz, and therefore the quantity delivered by *it* in the unity of time is $\sqrt{2g} \times \sqrt{y^2 z}\ dz$; integrating this between the greatest and least limits of z, we obtain the quantity discharged from the orifice. Two examples may suffice: where the orifice is a horizontal rectangle, and a circle. In the first, y is constant, therefore the quantity discharged in a second, or

$$Q = \sqrt{2g} \times y \times \int z^{\frac{1}{2}} dz = \sqrt{2gz} \times \tfrac{2}{3} yz + C.$$

If the origin of z be at the upper orifice, that is, if it extend to the surface, this shews that the discharge is $\frac{2}{3}$ of that which would be discharged through the same orifice with the velocity due to its greatest depth; if the integral be taken between $z = h$ and $z = h'$, it is

$$Q = \tfrac{2}{3} \left(\sqrt{2gh} \times yh - \sqrt{2gh'}\right) \times yh'.$$

For the Circular aperture, let r be its radius, h the depth of its centre, we have

$$dQ = \sqrt{2g \times (h-x)} \times 2\sqrt{r^2 - x^2} \times -dx,$$

in which $h-x$ is the distance of an elementary horizontal section below the surface, $2\sqrt{r^2 - x^2}$ the value of

y, and dx negative, as Q increases while x diminishes in the upper quadrant, let $\xi = \frac{x}{r}$ and $\varrho = \frac{r}{h}$

$$dQ = 2\sqrt{2g} \times r^2 \times \sqrt{h} \times \sqrt{1-\varrho\xi} \times \sqrt{1-\xi^2} \times -d\xi$$

$$= 2\sqrt{2gh} \times r^2 \times \sqrt{1-\varrho\xi} \times 1-\xi^2 \times \frac{-d\xi}{\sqrt{1-\xi^2}}$$

Developing the radical $\sqrt{1-\varrho\,\xi}$, and multiplying the terms by $1-\xi^2$, we have, neglecting the powers of ξ, which are odd, for the second number

$$2r^2 \times \sqrt{2gh} \times \left\{ \begin{array}{l} 1-\xi^2\left(1+\frac{1.1.\varrho^2}{2\,4}\right)+\xi^4\left(\frac{1.1.}{2.4.}\varrho^2-\frac{1.1.3.5}{2.4.6.8}\varrho\right)+\xi \\ +\xi^6\left(\frac{1.1.3.5}{2.4.6.8}\varrho^4-\frac{1.1.3.5\ .7\ .9}{2.4.6.8.10.12}\varrho^6\right)+\ \&c. \end{array} \right\} \times \frac{-d\xi}{\sqrt{1-\xi^2}}$$

These are of the form $\frac{d\xi \times \xi^m}{\sqrt{1-\xi^2}}$, and their *definite* integrals between the limits $\xi = 1$ and $\xi = -1$, are easily found; see note (*b*), chap. V. Where m is odd, the integral between these limits equals 0, for which reason we omitted the odd powers of ξ in the developement; where it is even, they are in succession, p, $\frac{1}{2}p$ $\frac{1.3}{2.4}$ p, $\frac{1.3.5}{2.4.6}$ p &c.

therefore

$$Q = 2r^2 p \times \sqrt{2gh} \left\{ 1-\tfrac{1}{2}\left(1+\frac{1.1.\varrho^2}{2.4}\right) \right.$$

$$+\frac{1.3}{2.4}\left(\frac{1.1.}{2.4}\varrho^2-\frac{1.1.3.5}{2.4.6.8}\varrho^4\right)+\ \&c.$$

and arranging the terms according to the powers of ϱ,

$$Q = r^2 p \times \sqrt{2gh} \left\{ 1 - \frac{1.}{4.} \frac{1.1}{2.4} \varrho^2 - \frac{1.3.}{4.6.} \frac{1.1.3.5}{2.4.6.8} \varrho^4 - \frac{1.3.5.}{4.6.8} \frac{1.1.3.5.7.9}{2.4.6.8.10.12} \varrho^6 - \&c. \right\}$$

$r^2 p$ is the orifice, and $\sqrt{2gh}$ the velocity at its centre. In general the three first terms of the series are sufficient and

$$Q = r^2 p \times \sqrt{2gh} \left\{ 1 - \frac{1}{32} \varrho^2 - \frac{1}{204} \varrho^4 \right\}$$

(d) The velocity with which an elastic fluid enters a vacuum is that acquired in falling through the height of the homogenous atmosphere; for it is evidently as the moving force divided by the quantity of matter moved: the moving force is the pressure at the orifice, or as D × H, Statics Art. (114) and therefore the accelerating force as H. The velocity is therefore independent of the density, as in incompressible fluids; but this is not the case when air passes through an orifice into a space containing air of less density: here the accelerating force is to that which acts in the preceding case as D—Δ : D, and therefore V′ the velocity which it produces: V the velocity with which air rushes into a vacuum :: $\sqrt{D - \Delta}$: $\sqrt{D}$ and

$$V' = V \times \sqrt{\frac{D - \Delta}{D}}, \quad V = 1340 \text{ feet.}$$

The quantity of air discharged in an instant of time is $V' \times dt \times O \times D$, being as the bulk and density: it is also $- S\,d\,D$, S being the capacity of the vessel containing the condensed air; therefore we obtain

$$dt \times O \times D \times V \times \sqrt{\frac{D-\Delta}{D}} = -S\,dD$$

$$dt = \frac{S}{OV} \times \frac{-dD}{\sqrt{D^2-\Delta D}}$$

whose integral is, see Lacroix, Calc. Int. Art. 162,

$$t = C - h.\,l.\left\{D-\tfrac{1}{2}\Delta + \sqrt{D^2-\Delta D}\right\} \times \frac{S}{OV}$$

and determining C so that $t=0$ when $D=D'$ the initial density,

$$t = \frac{S}{OV} \times h\,l\left\{\frac{D'-\frac{1}{2}\Delta + \sqrt{D'^2-\Delta D'}}{D-\frac{1}{2}\Delta + \sqrt{D^2-\Delta D}}\right\}$$

If $D = \Delta$, the denominator of the fraction becomes $\frac{1}{2}\Delta$ and the equation gives the time of the total efflux.

The quantity H is different in different gases, being inversely as their specific gravities under a given pressure.

CHAPTER X.

79. The application of a stream of fluid to impel machinery is of common occurrence, and constitutes a very important part of practical Mechanics: it acts either by its weight while descending, or by communicating the velocity which it has acquired in its fall; and we are about to examine the nature of this latter mode of action. The simplest case is where a jet impinges perpendicularly on a plane surface of considerable breadth, so as to communicate to it all its motion; the

velocity is that due to the height of the surface in the reservoir, or $\sqrt{2gH}$; and the quantity of fluid which falls on the unity of surface in the time T is $T \times \sqrt{2gH}$. This mass multiplied into its velocity or $\sqrt{2gH}$ is the quantity of motion communicated to the plane in the time T. Had the head of water which produces the velocity, *pressed* on the plane for the same time, its effect would be $gH \times T$, the actual effect is $2gH \times T$, therefore the pressure on the plane produced by the jet is twice the weight of a column whose base is the orifice, and height that due to the velocity. Experiment confirms this very nearly in all cases where the fluid cannot escape laterally, but where the plane is immerged in an extensive stream, or where it does not exceed the area of the jet, this pressure, which we distinguish by the term Hydraulic Pressure, is only half the preceding, or the weight of a column whose height is that due to the velocity. It may seem strange that the pressure arising from a column of the height H should produce a pressure proportional to 2H, but in fact the additional pressure is substracted from the pressure on the bottom of the reservoir. Then if H be 16 feet, the pressure on the unit of surface is 16, but if an aperture equal to it be opened, the total pressure on the bottom is lessened by 32. Each particle issues with a velocity 32, therefore in a second 32 feet escape: had it fallen out by its *own* weight, as if the surrounding liquid were frozen, only 16 would pass in the same time; but the force which is employed in moving it cannot produce pressure on the bottom, and as this is twice its own weight, we see why 2H is the measure.

80. The pressure is by no means uniformly distributed over the plane, even when totally immersed in the current; at the centre it is a maximum, it decreases gradually to a certain distance, and vanishes; beyond this it becomes negative. We cannot give the theory of the distribution, but we see that it ought to be so, for the central particles cannot escape without gliding along the plane, and diverting those which would impinge on it with undiminished velocity. The escape of the central portion produces an eddy, and it is evident that at a certain place this eddy must tend to carry the particles from the plane. If the plane be the side of a close rectangular vessel filled with water, from which a vertical tube proceeds, on making apertures in the plane and immersing it in a stream, the water will stand above the level in the vertical tube; and thus the above results were ascertained. When a single aperture is made at the centre of the plane, the elevation in the tube is once and a half H, when near the circumference, it is depressed below the level, and when a number of apertures are distributed over it so as to obtain the mean pressure, the height is 1. 18 H. A modification of the instrument is used to measure the velocity of water or wind.

81. If the plane be in motion, it is permitted to suppose it at rest, and its velocity added to or subtracted from that of the current, according as it moves in an opposite or the same direction; and since the head H is as the square of the velocity, we say that the Hydraulic Pressure is as the square of the relative velocity of the current with respect to the plane. This is however only true when the stream is unconfined; for when it cannot leave the impelled surface till it has acquired the velocity

of the latter, the effect which it produces is rather as the velocity simply.

82. Where the impulse of the stream is oblique to the plane, we experience more difficulty in obtaining theoretic results, and our conclusions are more at variance with experiment. Let AB fig. 187 be the profile of the plane, it seems probable that it intercepts a portion of the current whose breadth is AC; the plane does not receive the full impulse even of this portion, part being lost by the obliquity of its action. It is probable that the effective or perpendicular impulse may be obtained by multiplying the Hydraulic Pressure on AC into the cosine of the angle made by the direction of the stream and the perpendicular to the plane, as in the impact of solids; then $AC = AB \times \sin A$, and the effective Hydraulic Pressure on AB is $H \times AB \times \sin.^2 A$. This does not differ widely from observation while A is large; but in oblique incidences, it is far less than the actual pressures. An empirical formula gives for the multiplier of H, $AB \times \cos^2 B + B^3 \times .000001$, B being expressed in degrees; which does not err much.

It has been thought that by conceiving a curved surface as made up by elementary planes, and summing the pressures which they experience, we could ascertain those which operate on the curves, but the deductions from this hypothesis are wide of the truth, being considerably too great. It seems that the particles of the stream glide off more easily than in the case of planes: much also depends on the posterior shape of the body; thus a cylinder is less resisted in moving through the water than a circle of equal diameter; and the fishes which swim with great velocity, and are probably solids of least resistance, have the anterior part of the body much more obtuse than the remainder. When a body moves rapidly through the water, a portion of this fluid is heaped up before it; and that behind it is depressed below the general level from the difficulty which it finds in closing in behind the body. This causes the atmospheric pressure to act in resisting, as well as the Hydraulic Pressure of the water; and it is manifest that according to the shape of the surface, the quantity of this negative pressure varies.

83. On the practical application of these principles depend the art of navigation; which however has hitherto derived very little improvement from theory. In this the moving power and resistance are both of them Hydraulic pressures, of course varying as the square of the relative velocity; and when a vessel sails before the wind, it seems only requisite to shape the immersed part, so that it shall be as little resisted by the water as possible, and that the action of the rudder shall be unimpeded. In this case the utmost velocity which the vessel can acquire must be less than that of the wind, as if they were equal no pressure could be exerted on the sail. Where the course makes an angle with the wind, the circumstances are more complex; the Hydraulic pressure on the sails tends to impel the vessel perpendicularly to their plane, or in a direction oblique to the keel. This impulse is resolved into two, parallel and perpendicular to the keel; and as the vessel is so formed as to experience very great resistance in the latter direction, its progressive motion is almost entirely in the direction of its length. Let EC fig. 188 be the keel and *s s* be the projections of

its sails supposed seen from above, and AB the direction of the wind, then AD is its perpendicular action on the sail = AB × sin ABD; this again must be resolved in the direction of EC, by drawing BF parallel and equal to AD, BI and CF are its components. BI = BF × sin DBB, = AB × sin ABD × sin DBC for a single thread of air; and as the quantity of it which falls on the sail is as sin ABD, we have, calling S the surface of the sails,

$$BI = P \times S \times \sin^2 ABD \times \sin DBC,$$

P being the Hydraulic pressure of air on the unit of surface which is found to be in pounds nearly $V^2 \div 500$. This evidently admits of a maximum, for it vanishes when ABD = 0, as when the sail turns its edge to the wind, or when DBC = 0, the impulse being then totally perpendicular to the keel; there is therefore an intermediate position where the impulse is greatest. For the determination see the note (*a*): if the angle ABC be 90°, in which case the vessel has the wind on its beam, ABD is 54.44′.

It may be shewn that in vessels sailing before the wind the velocities are inversely as the square roots of the resistances where the impelling power is given, therefore the vessel will in this case have two motions, one as $\sqrt{IB} \div \sqrt{r}$, the resistance in the direction of its length, the other as $\sqrt{IF} \div \sqrt{R}$. It is found that R is about 12 *r*; if therefore we form a rectangle, whose sides GR, GB are as 1 : 12, the diagonal GL is nearly the actual course. The angle BGL is called the angle of Leeway; and it is found, that under the most favourable circumstances its amount is about five degree when ABC is 55; if the latter angle be less than this, the other increases rapidly, and the vessel loses its progressive motion. Besides the Hydraulic pressure of the water, other impediments to the motion of ships arise from the agitation of the sea, which must be counteracted as much as possible by their figure; and this modifies the shape which theory would assign to them: it is also requisite that they should be stable, to resist the action of the force IF which tends to overset them; this is effected by making them as broad as is consistent with an easy passage through the water, and disposing heavy materials in them as much below the water line as possible.

84. The evolutions of a ship are governed by the sails and rudder: if the sail *ss* be sloped towards the wind, its unbalanced pressure on $s's'$ and $s''s''$ will move the vessel so as to diminish the angle ABC, and an opposite arrangement will produce the contrary effect. But they mostly require to be assisted by the rudder; its action is nearly analogous to that of the sails. Fig. 189 let AR represent the course of a particle of water, supposed in motion while the ship is quiescent, which is identical with the motion of the ship through the water. By art. 82, the perpendicular impulse on ER is as $ER \times \sin^2 E$. Resolving this in a direction perpendicular to the keel, we obtain $ER \times \sin^2 E \times \sin L$, or $ER \times \sin^2 E \times \cos E$ for the effect of the rudder. The ship in consequence of this receives an angular motion round its centre of gravity, which may be determined by the principles laid down in Art. 65. As the water does not in fact fall on the rudder in a direction parallel to the keel, on account of the wedge-like shape of the posterior part of the vessel, this theorem is but an ap-

proximation, and we can merely conclude from it that the effect is in the compound ratio of the square of the velocity, and the surface of the rudder. The maximum angle is about 30°.

85. Analogous to the action of the rudder is that of the Wind-mill-sails; the wind falls obliquely on their surface, and its perpendicular pressure is resolved in a direction perpendicular to the course of the wind, in which alone the sails can turn. If the sails were at rest, the angle which their plane should make with the wind is 54°.44′, but when they are in motion they withdraw themselves from its pressure, and the angle must be increased. At the axis itself it may be of the theoretic value, but at the extremities of the sails it is about 85°. In the best mills the area of the sails is variable with the strength of the wind, so that the power bears a constant ratio to the resistance; this is performed by the centrifugal force of weights which recede from the axis when the motion is augmented, and are connected with an apparatus for furling the canvas of the sails. Much of the power is lost by the friction of the pivot of the axis.

86. The impulse of a current of water is sometimes applied to float-boards placed obliquely like windmill-sails, but it is more usually employed by means of the Under-shot wheel. This has a number of planes disposed round its circumference in the direction of its radii, which dip into the stream, and are carried round by it; the axle of the wheel of course turns the machinery intended to be moved. Where the stream is large and unconfined, the pressure on each float-board is that corresponding to the head due to the relative velocity; this is a maximum when the wheel is at rest; but the work performed is then nothing: the

pressure is nothing when the velocity of the wheel equals that of the stream, and therefore there is a certain intermediate velocity which gives the work performed a maximum. The weight equal to the pressure is $A \times (\sqrt{H} - \sqrt{h})^2$, h being the height due to U the velocity of the wheel; considering this as a mass attached to the wheel, we obtain its moving force by multiplying it into U, and as $\sqrt{H} - \sqrt{h} \propto V - U$, this moving force $\propto (V-U)^2 \times U$, which is a maximum when $U = \frac{1}{3} V$. In this case then the wheel moves with one-third the velocity of the river, and the effect which it produces $= A \times \overline{\frac{2}{3} V}\,|^2 \times \frac{1}{3} V = \frac{4}{27} AV^3$. The total power of the moving force is A V, the quantity of water discharged multiplied into H, or as AV^3, and on this supposition the undershot wheel does $\frac{4}{27}$ of the work which the fall is capable of performing. (*b*)

Where the floats are not totally immersed, the water is heaped up on them, and in this case the pressure is that due to 2 H.

87. Where the floats move in a circular sweep close fitted to them, or in general when the stream cannot escape without acquiring the same velocity as the wheel, the circumstances are rather different, being analogous to what happens in the collision of unelastic bodies. The stream has the velocity V before the shock, which is reduced to U, and the quantity of motion corresponding to the difference or V—U is transferred to the wheel; this turns with the velocity U, and therefore the effect of the wheel is as

$\left(\frac{V-U}{V}\right)U$ which is a maximum when $V = 2\,U$, being then $\frac{1}{4}$ of the moving power.

88. The undershot wheel is used where a large quantity of water can be obtained with a moderate fall; for where the fall is considerable, the Overshot is always employed. Its circumference is formed into buckets, into which the water is delivered with a velocity not exceeding that of the wheel; one half of the wheel is therefore loaded with water, whose weight turns it. In this case there is no velocity of maximum effect, for as the water must be delivered on the wheel with *its* velocity, this is so much subtracted from the fall, and the less it is consistently with steady motion the better. The maximum performance of an overshot wheel is $\frac{3}{4}$ of the moving power, or it raises a quantity of water equal to that by which it is driven through $\frac{3}{4}$ of the fall. (*c*)

There are many other Hydraulic Engines well worthy of notice, both from their utility and as their theory exercises the mind: among them we may name the Ram, the Spiral Pump, and Barkers Mill; but we must refer the reader to the systems of practical Mechanics in which they are described, as the investigations which they require transcend our limits.

NOTES ON CHAP. X.

(*a*) The pressure on an oblique surface is the column whose base is the surface, and height that due to the *perpendicular* velocity; let a be the angle made by the wind and keel, x that made by the sail and keel, also let

V be the velocity of the wind, and v that of the vessel, then the perpendicular velocity of the wind is $V \times \sin(a-x)$; but the sail recedes from the wind in the direction perpendicular to its surface with the velocity $v \sin x$, hence the relative velocity of the two is

$$\{V \sin(a-x) - v \sin x\}.$$

The square of this is as the Hydraulic pressure on the sail, and resolving it in the direction of the keel, we have the force of the wind in propelling the vessel =

$$\frac{\Delta}{2g}\{V \sin(a-x) - v \sin x\}^2 \times \sin x.$$

Δ being the density of air. To find the value of x, which gives this function a maximum, we differentiate with respect to x, and obtain

$$0 = 2 \sin x\,(-V \sin(a-x) - v \sin x) \times (V \cos(a-x) - v \cos x) \times dx,$$
$$+ \cos x \times dx \times (V \sin(a-x) - v \sin x)^2,$$

hence

$$0 = 2 \sin x\,(-V \cos(a-x) - v \cos x) + \cos x\,(V \sin(a-x) - v \sin x);$$
$$3\, v \sin x \cos x = V\{\sin(a-x) \cos x - 2 \sin x \cos(a-x)\},$$

dividing by $\cos x . \cos(a-x$

$$V\{\text{tang}(a-x) - 2 \text{ tang } x\} = \frac{3\, v \sin x}{\cos(o-x)} =$$
$$\frac{3\, v}{\cos a} \times \frac{\text{tang } x}{1 + \text{tang } x \text{ tang } a},$$

and by the formula for the tangent of the difference of two arcs we obtain

$$V\{\sin a - 3 \cos a .\ \text{tang } x - 2 \sin a \text{ tang } x^2\} = 3 \text{ tang } x$$

Resolving the quadratic which is derived from this,

$$\text{tang}\, x = \sqrt{\frac{1}{2} + \frac{1}{16}\left(\frac{V\cos a + v}{V\sin a}\right)^2} - \frac{3}{4}\left(\frac{V\cos a + v}{V\sin a}\right)$$

This gives the trim of the sails corresponding to a given direction of the wind, but it must be confessed that it differs from that which is found to produce the maximum effect in no small degree: this discrepancy arises from the defect of our theory of oblique Hydraulic pressure, and from our supposing the ship's motion to be in the direction of the keel, while its true motion makes with this a considerable angle; and if we attended to these in our investigation it would become too complex. If in the above equations we suppose $v = 0$, we have

$$\text{tang}\,(a - x) = 2\, \text{tang}\, x,$$

and if $a = 90°$ this gives $\frac{1}{\text{tang}\, x} = 2$ tang x, tang $x = \frac{1}{\sqrt{2}}$, therefore $x = 35°.16'$. The supposition of $a = 90°$ applies to the case of windmill sails, and our last equation gives

$$\text{tang}\, x = \sqrt{\tfrac{1}{2} + \frac{9\, v^2}{16\, V^2}} - \frac{3\, v}{4\, V}.$$

This angle x is called the angle of Weather, and as v in the windmill is as the distance from the axis, the angle should decrease from the centre to the extremities, being at the former about 30 degrees, and at the latter 5, so that the sail is not a plane. This gives us for the value of v, 3 V, and from this we infer that a ship may sail faster than the wind, which is conformable to experience. For practical information, Smeaton's Experiments on Windmills, and Robison's

Art. Seamanship Encycl. Brit. may be consulted by those who wish to pnrsue the subject farther.

(b.) It is not our intention to enter deeply into this branch of Hydraulics, but the action of water-wheels merits more detail than it seems expedient to introduce into the text. We will suppose, as is always the case in actual practice, that the resistance is of such a nature as to produce uniform motion after a few turns; and for an instant of time we will suppose it a weight raised by a cord wound on the periphery of the wheel and therefore moving with its velocity. The product of this weight into this velocity, we denominate the Effect of the wheel and denote it by the symbols WU or E.

If a stream of water issuing from an aperture A impinge on a float-board of the same dimensions, or if the wheel be in an unconfined stream, its percussion is equal to the weight of once AH; when the wheel moves with a velocity U, this is diminished in the ratio of $V^2 : (V-U)^2$ and the impelling force is $AH \times \left(\frac{V-U}{V}\right)^2$.

A quantity of water equal to this and moving with the velocity U would evidently be in equilibrio with it, and therefore

$$E = AH \times \left(\frac{V-U}{V}\right)^2 \times U$$

To find the maximum we have the equation

$$(3U^2 - 4VU + V^2)\, dU = 0$$

$$\text{whence } U = \tfrac{2}{3}V \pm \sqrt{\tfrac{1}{9}V^2} = V\left(\tfrac{2}{3} \pm \tfrac{1}{3}\right.$$

The second differential coefficient of E is $6U - 4V$ which is negative when $U \angle \frac{2}{3}$; therefore at the maximum $U = \frac{1}{3}V$. With this value the maximum E is AH

$\times \frac{4}{9} \times \frac{1}{3} V = \frac{4}{27} AH \times V$ or it is equivalent to a mass of water AH moved with $\frac{4}{27}$ of the velocity of the current. The expression is more distinct if we call Q the quantity of water afforded in a second; this = AV, and $E = Q \times \frac{4}{27} H$ or a quantity of water equal to that expended raised through $\frac{4}{27}$ of the fall. This determination supposes that the stream acts perpendicularly on the float-board; but in practice when the float first enters the water, it receives the impulse obliquely; this obliquity diminishes till it is vertical. Let ξ be the interior arch of the wheel immersed and z the angle made by the plane of the float-board with the vertical, then the Hydraulic Pressure on its surface is AH $\times \left(\frac{V \cos z - U}{V}\right)^2$. To find the value of its action through ξ or its Vis Viva, we integrate the function $HA \times \left(\frac{V \cos z - U}{V}\right)^2 \times - dz$, between $z = \xi$ and $z = 0$; we put it in this form $HA \times (\cos z - \varrho)^2 \times - dz$ $= HA(-\cos^2 z\,dz + 2\varrho \cos z\,dz - \varrho^2\,dz)$ whose definite integral is

$$HA \left\{ \frac{\xi + \sin \xi \times \cos \xi}{2} - 2\varrho \sin \xi + \varrho^2 \xi \right\}$$

If we seek the velocity which gives E a maximum, we multiply this function by ϱ and differentiate, which gives

$$0 = \tfrac{1}{2}(\xi + \sin \xi \times \cos \xi) - 4\varrho \sin \xi + 3\varrho^2 \xi$$

which if we suppose ξ coincident with its sine gives $\varrho = \frac{1}{3}$ as before.

This hypothesis is still insufficient, for the float-boards are not urged over their whole surface, the foremost being screened from the water by those that follow them. To introduce this into our theory is useless, as it gives an unmanageable equation for the velocity of maximum effect, and the result does not agree with experience. We therefore proceed to the second case, where the stream is confined on a channel so that it must acquire the velocity of the wheel before it leaves it: here the circumstances are different, for the stream, after it passes one float, acts on the next; and while in the preceding case it left the wheel retaining $\frac{2}{3}$ of its original velocity, here this moving force is economized, and a considerable portion of it communicated to the wheel. Its action is therefore analogous to the collision of an unelastic body, and scarcely differs from the impulse of a solid of equal weight. We may therefore apply to it the principles of Art. 58, and assume that the impelling power is as the relative velocity simply: the original power of the stream is $AH \times V$, hence its action on the wheel is $AH\left(\frac{V-U}{V}\right)$

and

$$E = AH \times \left(\frac{V-U}{V}\right) \times U$$

To find the maximum we have $dU \times (V-2U) = 0$ and $U = \frac{1}{2}V$, which gives $E = AV \times \frac{1}{4}H$, or $Q \times \frac{1}{4}H$ it is found 0.31, and $U = \frac{2}{5}V$.

(c). The Overshot wheel acts by the weight of water disposed round half its circumference in buckets; each of these acts by a leverage of the sine of its angular distance from the vertical. To find their combined effect we may actually compute it thus, let m be the con-

tents of each bucket and x the angular distance between them and U the velocity of the wheel, then

$$E = U \times m \left\{ \sin x + \sin 2x + \&c. \right\}$$

till we come near the bottom of the wheel where the water is spilled out of the buckets. We obtain a more elegant result by supposing the whole quantity of water contained in the buckets distributed over an equal arch of the circumference, as a ring whose section is A; the portion of this which lies on the element dz is A dz and its statical moment is A sin z dz whose integral is — A cos z, and taking this between the limits where the water is poured on the wheel, and discharged from it, we obtain for the weight in equilibrio with the effort of the wheel AH. As the velocity of this is the same as that of the wheel, $E = AU \times H$: If Z be the arch which is loaded, the impelling quantity of water is AZ, and its Vis Viva, AZU; therefore the effect is $\frac{H}{Z}$ of the power, which if Z were the semicircle, would be $\frac{2}{3.14}$ or 0.62: if it were 150° which is nearly the common practice, $E = 0.71$. As $AU = Q$, if this latter be constant, $E = QH$; if therefore we wish to increase E where the supply of water is limited, we can do it only by augmenting H: now the fall F equals H the height through which the water descends on the wheel, plus h the height through which it must fall to acquire the velocity of the wheel; if we increase H we diminish h, and therefore the velocity. Hence the overshot wheel produces the greater effect, the slower it moves; and the constancy of Q or AU is obtained by enlarging A in the same proportion as V is diminished; which is done by increasing the breadth of the buckets. This has its limits however, for a loaded wheel has friction on its pivots proportional to the load.

Since $U \propto \sqrt{h}$, $E \propto AH \times \sqrt{h}$ where no regard is paid to Q; this is a maximum when $H^2 \times (F-H) = \max$, or when $2F-3H = 0$; this may teach us how to place a given wheel to the best advantage; but cases seldom occur where it is necessary. The best working velocity is found to be three feet in a second, and h is about two inches.

CHAPTER XI.

89. In the preceding chapters we have discussed the laws which regulate the motions of bodies, whether subject to the uncontrolled action of accelerating forces, connected in systems, or constituting fluid masses: in so extensive a field, to treat the subject with the full detail which it requires, is inconsistent with the design of this treatise, which aims only at conveying some useful knowledge to the ordinary reader, and facilitating further progress to him who is not content with mediocrity of information. It is presumed that the principles delivered in this and the preceding part, are sufficient to enable the intelligent student, to understand the writings of practical authors, and to develope the action, and estimate the performance of machinery, when

he sees it actually employed. When the mind, previously stored with the theory of mechanics, is applied to that theory as exhibited in practice, few pursuits can give equal pleasure to the individual ; none, except perhaps chemistry, can afford equal benefit to mankind. Without insisting further on this topic, we shall conclude with some general considerations relative to machinery, which we place here as they could not properly find room in any of the preceding parts.

In Chap. IV. Statics, some notice was taken of the object of machinery, which we can now explain much more completely by means of the Vis Viva or that consideration of force which includes the space through which it acts: this mode of estimating it is essentially different from that which is considered in cases of equilibrio, and its employment is that which is expensive as well as useful to mankind. To understand the difference let us suppose a weight suspended to one arm of a lever ; if a force be applied to the other, which is to the pressure of the weight inversely as the arms, the weight will be sustained in equilibrio. This is all that is generally considered, but it is of no practical use, for the weight could be as well supported by a prop without the machine: accordingly we find that workmen, when questioned as to the use of the lever, state its power in *raising* weights, and this requires a different view of the matter. In the state of equilibrium, the statical moment of the power is mr, m being the weight equivalent to the power, and r its distance from the fulcrum: this measure of force is of no value to the workman, for it performs nothing for him, and he therefore uses another; he wishes to raise the weight m' by the descent of m, and therefore estimates his power and weight, not by the product of the masses into the virtual velocities, but by the product of those masses into the heights through which they are moved. The analogy of this to the Vis Viva is evident, but it is not a correct measure of the useful effect except when the motion of the machine, as well as the resistance, is uniform. In that case the *Dynamic Effect* of the machine is fairly measured by this product, but where any acceleration takes place, the additional velocity is lost, as it cannot always be converted to any practical purpose. It is therefore necessary to consider, not merely the weight which a machine can raise, but also the velocity with which it raises it.

90. A simple but excellent machine for raising water may be taken as an illustration; it is a fixed pulley over which passes a cord, each of whose extremities suspends a bucket provided with a valve in its bottom. One of these dips in the water of the well or pit from which it is to be raised, which we suppose 16 feet deep, we call its weight when full m'; the other is 16 feet above the surface where it receives the water of a stream till it is sufficiently filled, when it descends, drawing up the other through an equal height. As both buckets have been accelerated, they have acquired a considerable motion, so that by striking against stops they upset and empty themselves: m though larger than m' is lighter; the latter therefore descends into the well, drawing up the former to its original station; they are again filled, and thus the process is continued. If the weight of m and m' when filled were equal, no work would be performed, as there would be equilibrium: if the preponderance

were little, a considerable quantity of water would be raised slowly; if great, a little water raised rapidly. Between these extremes there is a maximum, to find which we must know the velocity acquired by the buckets when connected together. This is done by the assistance of the principle used in art. 65: the quantity of motion which gravity would produce in a second in m and m', if separate, must equal that actually produced, as their connexion cannot produce or destroy it. The velocity generated in a second is g, hence the quantities of motion impressed are $g\,m$, $g\,m'$, whose difference is to be taken as they act in opposition to each other; but they actually move with a common velocity V, and the actual quantities of motion are mV, m'V, whose sum is to be taken, as by means of the cord they act in the same direction: hence

$$(m+m') \times V = g\,(m-m'),\ V = g \times \frac{m-m}{m+m'}$$

an equation which contains the theory of Atwood's machine, already referred to. If this value of V be substituted for F in the formula $V'^2 = 2F\ S$, we have $m'\ V'^2$ the Vis Viva $= m' \times g \times \frac{m-m'}{m+m'} \times S$, which is a maximum when $m^2 = 2m\ m' + m'^2$, or when $m' = m \times \sqrt{2} - 1 = m \times 0.414$, and the reader who is ignorant of the differential calculus can easily satisfy himself as to the result by assigning to m' values either greater or less than 0.414 m, and computing $m\ V'^2$. *(a).*

91. Another example of the maximum performance of a machine is derived from the inclined plane: supposing the power to be the perpendicular descent of a weight, which draws another up the plane, it is evident

that this latter will move more rapidly the less the inclination of the plane; but that its velocity, estimated in a vertical direction, which is as its actual velocity multiplied into the sine of inclination, does not necessarily increase. When the plane is vertical, the mechanical advantage is lost; where it is horizontal there is no ascent; therefore between those positions is one of maximum effect. The tendency of the weight to descend is $m' \sin I$, therefore the quantity of motion impressed in the unit of time is $(m - m' \sin I) g$: that actually acquired in the same time is $(m + m') \times V$; therefore the measure of the accelerating force is $g \times \left(\frac{m - m' \sin I}{m + m'}\right)$. The vertical acceleration is as this multiplied into sin I, therefore it is a maximum when $m \sin I - m' \sin^2 I = \max.$ The expression may be written $m' (\varrho \sin I - \sin^2 I)$, ϱ being $\frac{m}{m'}$, and the variable part is a maximum when $\sin I = \frac{1}{2}\varrho$, for it expresses the square of the ordinate of a semicircle whose diameter is ϱ, and abscissa, sin I, which is a maximum when the diameter is bisected. If the power and weight be equal $\varrho = 1$, and $I = 30°$. This affords an example of the accommodating a machine to a given resistance, as the preceding shows how the resistance is adjusted to the machine; but the two cases must not be confounded, for the maximum *effect* of a given inclined plane has a different law, being when

$$\frac{m'}{m} = \sqrt{1 + \text{cosec. I}} - 1,$$ which when $I = 30°$ is 0.732.

92. Formulæ for the maximum effect of machinery may be investigated, by supposing them reduced to

equivalent levers, that is to levers whose arms are inversely as the forces which make equilibrium when applied at the impelled and working points of the machine. The friction of the parts being proportional to the pressure, must ultimately be a function of the resistance; and we may suppose the machine devoid of friction, and the resistance augmented by a certain quantity: so in like manner for the inertia of the machinery, which is of little importance while the motion is uniform, and is pernicious only when a reciprocating motion is necessary; where it is requisite this can be estimated from the known figure and mass of the parts. The inertia of the power and resistance, where they are weights, must also be considered; and by proceeding as in the examples already given, we arrive at the expression of the accelerating force which acts on the resistance, or of the velocity communicated to the working point. This multiplied into the resistance gives the effect of the machine, and we may seek the maximum, either supposing the forces given and the machine variable, or *vice versa*. For some of these expressions see note (*b*). One case only can be mentioned here, namely, where the resistance has no inertia, as in sawing wood, boring cannon, and other work where the mass moved by the working point is inconsiderable. We suppose the moving power to be a weight m, acting by the arm of the lever r, the resistance to be m', and its leverage r': the moving force is $g\ (mr-m'r')-\ Fr'$, F being the friction supposed applied at the working point; this must be equal to mv, the quantity of motion of the impelled point acting by the leverage r. The velocity of the working point $v' : v :: r' : r$, therefore $v = \frac{v'r'}{r}$, and

$$\frac{g\,(mr - m'r') - Fr'}{mr^2} \times r' = v'.$$

If we suppose $F = 0$, and multiply by m' to obtain the effect, we have

$$E = \frac{mm'rr' - m'^2r'^2}{mr^2}$$

which is a maximum when $mm'rr' - m'^2r'^2$ is a max. or $m'\left(m \times \frac{r}{r'} - m'\right)$ and this is the case when $m' = \frac{1}{2}$

$m \times \frac{r}{r'}$ by the construction used in the preceding art.

$\frac{m\,r}{r'}$ is the force which would equilibrate with m if ap-

plied at the working point, and the maximum effect is produced when the load of the machine is half this.

93. It is unnecessary to go into more minute details on this subject, and we proceed to consider the action of the moving powers which we can apply, and the laws of their action.

On a former occasion we enumerated them, and have already sufficiently noticed the Hydraulic forces; we now have to consider the strength of animals, and the energies developed by heat. The strength of animals is so variable in different individuals, and so much under the influence of the will, that it might be supposed impracticable to apply any general rules to it; however, experience shews that there is an average quantity of exertion which may be expected, and which will coincide with that actually furnished on a large scale. A certain portion of an animal's strength is expended in moving its body, and the remainder is that which we can dispose of as a moving force; at least when a load is to be drawn along a horizontal plane, or any similar operation is to be performed. The useful effect is measured by the load moved, multiplied into the velocity of its motion: now observation shews, that the load increases while the velocity decreases, so that the effort of the animal is the greatest possible when the velocity is a cypher, and on the other hand there is a certain velocity at which it can carry no load.

Call this V, and the actual velocity U, experience gives this formula for the effort exerted

$$F' = F \times \left(\frac{V-U}{V}\right)^2$$

F being the force which it can exert when at rest. This vanishes when $U = V$, and becomes F when $U = 0$, and by the reasoning used in the case of the undershot wheel, whose power is represented by the same formula, the effect produced is a maximum when $U = \frac{1}{3} V$, and then the animal can work against a resistance which is $\frac{4}{9}$ of its utmost effort. Another mode of considering the subject has been used by the celebrated Coulomb; he found that a man ascending stairs produced an effect which was in general three times his weight raised 1000 yards in the day, and that his workmen could make this effort for a length of time without injury, working eight hours in the day. When they carried loads nearly equal to their weight the effect was only $\frac{1}{2}$ the preceding, and in general the loss of effect was sensibly proportional to the load. Hence the quantity lost by bearing any load P is found, for when $P = W$ the effect is diminished from $3W$ to $1.5W$, therefore

$$W : 1.5W :: P : 1.5\,P,$$

the quantity lost P, being expressed in pounds; the remaining action is therefore $1000 \times \{3\,W - 1.5\,P\}$. This equals $(W+P) \times h$, the first factor being the total weight raised, the second the height through which it is elevated in the day. Hence

$$E = Ph = 1000 \times P \left\{ \frac{3W - 1.5\,P}{W+P} \right\}$$

which is a maximum when

$$P = W \left\{ \sqrt{3} - 1 \right\} \text{ or } 0.7 \times W,$$

and vanishes when $P = 2W$; a man being unable to ascend under twice his weight. The formula is accommodated to the case of motion on an horizontal plane by multiplying the second member into 17.

94. Experiments have not yet been made with respect to other animals; however it is probable that Coulomb's formula would apply to them, with a due alteration of the coefficients. The most important result which follows from them is, that human strength is most profitably applied to machinery when the workmen ascend without a load, and act by their weight while descending. The work done in the way of carrying, is at its max. $W \times 0.8$, or little more than a fourth of the total effect which he can produce. This had been long observed, and hence the utility of those wheels, on or in which animals walk and turn them by heir weight.

95. The forces connected with heat are scarcely more than two, the action of gunpowder, and that of steam. The first of these is used for purposes which scarcely admit of an estimation of its Dynamic effect; if the relation between the bulk and elasticity of the the gases formed by its explosion were known, this could be accurately assigned by theory; but we can estimate it from the action of artillery. It is observed that eight ounces of powder project a ball of twice that weight with a velocity of 1640 feet, which would carry it to the height of 14000 yards. If we compare this with the formula for human action just given, and consider that the average value of W is 160, we see that the effect produced by the powder is $\frac{1}{9}$ of that of a man carrying a load upon an ascent; while it costs us a workman's wages for the day. It seems, however, that when applied to blast rocks, its effect is twenty times as great as in fire-arms: and it can be used where all other movers would be inapplicable.

96. To value the power of Steam, we suppose it contained in a cylinder, whose base is A^2, and height H. If then it be condensed, the piston is pressed with a force πA^2, π being the density of the steam, as measured by the pressure on a square foot: and this force acts through a space H; therefore the effect of a single stroke of the steam engine is πA^2 pounds raised through H feet. To know the cost of this effect, we take Watt's determination that 1 pound of coal vaporizes 8 of water under the most favourable circumstances. Steam, when of the elasticity of air, is about 1350 times the bulk of the water of which it is formed:

now 8 pounds of water is 0.128 of a cubic foot, and therefore the steam formed by burning one pound of coal is 172.8 feet. The pressure of such steam on the square inch is 14 pounds, therefore $\Pi = 144 \times 14$, and $\Pi A^2 H$ is $172.8 \times 144 \times 14$, or 116 pounds raised through 1000 yards, nearly equal to a man's work for the day when he carries a load. It follows from this that the fuel of a steam-engine costs scarcely $\frac{1}{100}$ of the wages of the men who could perform the equivalent work. If we multiply the above number by 84, the weight of a bushel, we obtain for the work done by its consumption rather more than 29 millions raised one foot; a result less than what is actually performed by many steam-engines, notwithstanding friction and other causes which tend to diminish the result. This arises from the steam used being denser than the atmosphere, in which case Π increases faster than $A^2 H$ diminishes, and therefore the effect is augmented, though the theory of vaporization is too imperfect for us to assign the precise quantities. For the augmentation of power gained by shutting off the steam before the termination of the stroke, see note (*c*). In applying the steam-engine to produce rotatory motion, the effect is reduced at least one half by the crank, and no effectual substitute for it has yet been discovered.

NOTES ON CHAP. XI.

(a) This is a particular case of the motion of connected bodies down two inclined planes, which is easily derived from D'Alembert's principle: let m and m' be the bodies, I and I′ the inclinations of the planes, then the quantities of motion impressed on the bodies in the time dt are respectively $mg \sin I\, dt$, $m'g \sin I'\, dt$, whose difference is to be taken as they act in opposition: the quantities actually acquired are mdv, $m'dv$, and

$$\frac{dv}{dt} = g\left(\frac{m \sin I - m' \sin I'}{m+m'}\right),$$

or the accelerating force : gravity : : difference of the powers which could sustain the bodies on the inclined planes : sum of the bodies. Where the vertical velocity of m' is required a maximum, m and m' being given, we multiply the numerator by sin I′, and differentiating we have

$$(m \sin I - 2m' \sin I')\, d\, (\sin I') = 0, \quad \frac{\sin I'}{\sin I} = \frac{m}{2m'}$$

which, when m descends perpendicularly, is the case of art. 90. The square of the velocity is as the accelerating force multiplied into the space, and where the height is given this latter is $\frac{h}{\sin I'}$ hence the Vis Viva of m' is

$$m' g \frac{\times h}{\sin I'}\left(\frac{m \sin I - m' \sin I'}{m+m'}\right)$$ and this is a maximum,

I and I′ being given, when

$$0 = dm' \left\{ (m \sin I - 2m' \sin I')(m+m') - mm' \sin I + m'^2 \sin I' \right\},$$

$$0 = m^2 \sin I - 2mm' \sin I' - m'^2 \sin I',$$

$$m' = m \left\{ \sqrt{\frac{\sin I + \sin I'}{\sin I'}} - 1 \right\}$$

This theorem applies immediately to the case of one weight drawing up another by making both sines unity, which gives $m' = m(\sqrt{2} - 1)$.

If the friction on the planes be included the expression for the accelerating force is

$$g \left\{ \frac{m \sin I - m' \sin I' - m f. \cos I - m' f \cos I'}{m + m'} \right\}$$

f being the ratio of friction to pressure. If we suppose $I = 90°$ and seek the inclination which gives the vertical velocity a maximum, we obtain

$$\frac{\cos I'}{\sin 2 I' + f \cos 2 I'} = \frac{m'}{m}$$

and if we look for the value of m' which gives the maximum effect

$$m' = m \left\{ \sqrt{1 + \frac{1}{\sin I' + f \cos I'}} - 1 \right\}$$

Where an animal mover *draws* a body up an inclined plane these formulæ do not apply, for the power has no inertia, and the quantity of action afforded by it varies with the velocity. If we call m the motive energy, it as stated in the text is $F \times \left(\frac{V - U}{V}\right)^2$, F being the effort when $U = 0$. From this we derive

$$V - U = V \times \sqrt{\frac{m}{F}}, \quad U = V \left\{ 1 - \sqrt{\frac{m}{F}} \right.$$

m is the resistance, which in this case is $m' \sin I'$ and

$U = V \left\{ 1 - \sqrt{\frac{m' \sin I'}{\sqrt{F}}} \right\}$ and the maxima given above become for the vertical velocity, and greatest effect

$\sin I' = \frac{4}{9} \frac{F}{m'}$.

In the case of a man F = 60 pounds on an average, and a horse is five times as strong; and on a slope of one in thirty six, which is the utmost allowed in good roads, $\sin I = \frac{1}{36}$, and $m' = 1440$, but a large deduction must be made for the friction, &c. The reader can easily extend these theorems.

(*b*) Applying D'Alembert's principle to the lever, we have $gdtm$, $gdtm'$ for the quantities of motion impressed; and mdv, $m'dv'$ for those acquired in the instant dt. These must equilibrate by means of the lever, and therefore

$$gdt\,\{mr - m'r'\} = mrdv + m'r'dv'$$

We have also, as the velocities are as the arms, $dv' : dv :: r' : r$, and the equation becomes

$$gdt \times \left\{ mr - m'r' \right\} = \left\{ mr^2 + m'r'^2 \right\} \frac{dv'}{r'},$$

$$dv' = \frac{mr - m'r'}{mr^2 + m'r'^2} \times r' \times gd\,t$$

and integrating,

$$v' = gt \times r' \left\{ \frac{mr - m'r'}{mr^2 + m'r'^2} \right\}$$

An expression which might have been derived from the properties of the moment of inertia, the numerator being the moving power and the denominator the moment of inertia of m and m', to which, if necessary, the moment of inertia of the lever may be added. If the resistances be such that no acceleration takes place, $m' v'$ may be taken as a measure of the performance of the lever, being the product of the resistance overcome into the spaces described by the working point in a given time. If the machine be accelerated, we take the product of m' into the space through which it is moved as in the ordinary mode of estimating Vis Viva.

If we differentiate the value of v' with respect to r' we have

$$m^2 r^3 - 2 m m' r^2 r' - m m' r r'^2 = 0,$$

$$r'^2 + 2 r r' = \frac{m}{m'} r^2,$$

$$r' = r \left\{ \sqrt{1 + \frac{m}{m'}} - 1 \right\}$$

which gives one case of maximum effect, namely, where the power and resistance are both determinate and the machine alone can be varied, as then a given weight is raised with the greatest velocity. If $m = m'$, $\frac{r'}{r} = \sqrt{2} - 1$ as we found in the inclined plane.

If we differentiate $m'v'$ with respect to m', the maximum effect produced by a given machiae is where

$$m^2 r^3 r' - 2 m m' r^2 r'^2 - m'^2 r'^4 = v,$$

$$m' + m \frac{r^2}{r'} = \sqrt{m^2 \left\{ \frac{r^4}{r'^4} + \frac{r^3}{r'^3} \right\}},$$

$$\frac{m'}{m} = \frac{r^2}{r'^2} \left\{ \sqrt{1 + \frac{r'}{r}} - 1 \right\}$$

If we suppose the mover m to be the action of an animal, or a stream of water, the formula is different; as in the last note,

$$v = V\left\{1 - \sqrt{\frac{m}{F}}\right\}; \quad v' = \frac{vr'}{r}, \text{ therefore}$$

$$v' = V\left\{\frac{r'}{r} - \sqrt{\frac{m'r'}{Fr}} \times \frac{r'}{r}\right\},$$

$$\text{and } m'v' = V\left\{\frac{m'r'}{r} - \sqrt{\frac{m'^3}{Fr^3}} \times \sqrt{r'^3}\right\}$$

differentiating with respect to r', we have

$$0 = \frac{m'}{r} - \tfrac{3}{2}\sqrt{\frac{m'^3}{r^3F}} \times \sqrt{r'}, \; r' = \frac{4}{9}\,\frac{Fr}{m'}, \; 1 = \frac{4}{9}\,\frac{Fr}{m'r'},$$

or the load m' must be $\frac{4}{9}$ of that which would equilibrate with the maximum effort.

The case where the inertia of m' is nothing is considered in the text, that where its inertia is the sole resistance is easily obtained; the mass of m' must be supposed collected in its centre of gyration, and

$$v' = \frac{kmr}{mr^2 + m'k^2}, \quad m'v' = \frac{mm'rk}{mr^2 + m'k^2},$$

differentiating with respect to k, we have $0 = mr^2 - m'k^2$, or the moments of inertia of the power considered as a weight, and of the resistance must be equal. This applies to mill-stones, fly wheels, &c.

(c) In the Steam engine, when steam acts above the piston, during the entire descent, it is uniformly accelerated; this is in many instances contrary to the use of the machine, which in general requires uniform mo-

tion, and is always attended by a consumption of steam greater than the proportional augmentation of power. It is therefore found useful to admit it only for a certain part of the descent, and cutting off the communication, let it expand and press with a diminished force. It is certain, that when steam is kept at a constant temperature a *given* quantity of it follows the same law as air; that its elasticity is as its density, or inversely as the space which it occupies. If then we suppose π the pressure on the unit of surface exerted by the steam in the boiler, and suppose it admitted freely while the piston descends through ξ, the action during that space is $\pi A^2 \xi$. If then the communication be closed, and the distance of the piston from the top of the cylinder become x, the steam which filled the space $A^2 \xi$ is diffused through $A^2 x$, and therefore P its pressure: $\pi :: \xi : x$, $P = \frac{\pi\xi}{x}$ and the Vis Viva in an instant of time is $PA^2\, dx$ or $\pi A^2 \xi \left(\frac{dx}{x}\right)$. The integral of this taken from ξ to ξ' gives the expansive action of the steam, and this added to $\pi A^2 \xi$ gives the total effect during a stroke

$$\pi A^2 \xi \int \frac{dx}{x} = C + \pi A \xi \times h.\, l.\, x.$$

When $x = \xi$ this integral $= 0$, $C = -h.\, l.\, \xi$ therefore

$$E = \pi A^2 \xi \left\{ 1 + h.\, l. \left(\frac{\xi'}{\xi}\right) \right.$$

The steam expended in producing this effect is $A^2 \xi$. The advantage of this may be shewn by a numerical example: let us suppose $\xi' = 4\xi$, and $\pi A^2 = 1$: the $h.\, l.$ $(4) = 1.39$ and $E = \xi\, (2.39)$. If the steam acted through the whole stroke $E = 4$ but the steam employed is four times as great; therefore in these two instances the quantities of work done by a given quantity of steam are as 24 : 10 nearly.

The complete theory of the steam engine involves much curious discussion, but it cannot be given here; the reader may be referred to Prony Arch. Hydraulique, Tom. 2. for it, and to Robison's Tracts, where he will find the subject treated in an elementary manner with much ability.

FINIS.

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