Teacher's Manual

for

First-Year Mathematics

THE UNIVERSITY OF CHICAGO
MATHEMATICAL SERIES

ELIAKIM HASTINGS MOORE
GENERAL EDITOR

SCHOOL OF EDUCATION
TEXTS AND MANUALS

GEORGE WILLIAM MYERS
EDITOR

TEACHER'S MANUAL FOR FIRST-YEAR MATHEMATICS

THE UNIVERSITY OF CHICAGO PRESS
CHICAGO, ILLINOIS

Agents
THE BAKER & TAYLOR COMPANY
NEW YORK

CAMBRIDGE UNIVERSITY PRESS
LONDON AND EDINBURGH

TEACHER'S MANUAL

FOR

First-Year Mathematics

By

GEORGE WILLIAM MYERS

Professor of the Teaching of Mathematics and Astronomy, College of Education of the University of Chicago

and

WILLIAM R. WICKES ERNEST L. CALDWELL
ERNST R. BRESLICH ROBERT M. MATHEWS
WILLIAM D. REEVE

Instructors in Mathematics in the University High School of the University of Chicago

SCHOOL OF EDUCATION MANUALS
SECONDARY TEXTS

THE UNIVERSITY OF CHICAGO PRESS
CHICAGO, ILLINOIS

1911

Published July 1911

Composed and Printed By
The University of Chicago Press
Chicago, Illinois, U.S.A.

PREFACE

This *Manual* is for teachers for whom the phrase "modernized mathematics for secondary schools" means something more than a rallying sentiment for teachers' meetings. It is for teachers who see in this phrase a real meaning, serious enough to entitle it to actual recognition in practical teaching. It is of course not an independent composition for continuous reading; but is for the use of those who are employing *First-Year Mathematics* as a text.

The text on which this *Manual* comments, presents in teachable form an interweaving of the more concrete and the easier phases of the first courses in both algebra and geometry. The chief emphasis is placed on algebra, but a considerable body of related fundamental notions and principles of geometry is woven in, while some advances are made in one direction or another upon the most concrete, graphic, and practical aspects of elementary geometry. Geometrical treatments are at first intuitive, inductive, and experimental, but in some instances these are followed by the quasi-experimental methods of superposition. A like informality as to method characterizes the earlier part of the algebra. The transition from the informal procedure of the earlier part to the formal procedure of the later part is gradual. The reasoning becomes more and more highly deductive throughout the latter half of the book. This accords with both classroom practice and a priori reasoning as to the normal procedure for secondary mathematical instruction and has sound historical warrant.

The *Manual* undertakes to do two specific things, viz.:

First, it seeks to give as accurately as may be in cold type the educational and mathematical points of view under which the authors attempted to organize the material of the context; and

Second, to bring to the reader as large a measure as possible of the benefit of the classroom experience of the authors who have been using *First-Year Mathematics* as a text with high-school classes.

It is no reflection upon the professional dignity and independence of a teacher to feel the need of pertinent pedagogical suggestion just at the critical point where the situation is opportune. Callousness in this regard is much less common and less popular now than it was a few years since. The greatest weakness of many teachers is not that they know no pedagogy, but rather that what they have is poorly timed. They do not get it into working order until the critical moment has passed. The night after the unsatisfactory recitation, or the next day, we see so clearly the true pedagogic key to the entrance, after we have forced passage by clumsy means. Any instrumentality that even aids us in knowing what we know, just when we need it, cannot fail of a cordial welcome. This *Manual* hopes for at least friendly recognition on some such score as this.

The text, *First-Year Mathematics*, though adhering to algebra as a central theme of first-year work, departs perhaps far enough from the prevailing type of text for beginning students in high-school mathematics to justify the existence of this commentary. The authors would accordingly have teachers regard this little volume as in the nature of a request for the privilege of a hearing in a case in which perhaps the burden of proof is upon them, rather than as an intimation of any widespread need of teachers of secondary-school mathematics for general instruction in mathematical pedagogy.

With one exception the participants in the authorship of these notes and suggestions are teachers of high-school classes, and their recommendations come warm from the classroom. What is here put down for the aid of others may then be regarded as having been tested and seasoned by trial, and not at all as a body of dogmatic stipulations either of what may *conceivably* be done with high-school classes, or of what a narrow-gauge pedagogy would prescribe. Being merely transcripts with emendations under expert trial and critical scrutiny of what the authors themselves find most practicable in administering the new type of subject-matter, it is believed that these notes will be of very particular value to others. The prime aim in their preparation has of course been not so much rhetorical completeness, as real, practical helpfulness to teachers.

The text, *Second-Year Mathematics*, published in June, 1910, is designed to continue the type of mixed mathematics of the first book through the second high-school year, though here with primary emphasis on geometry. The two books together cover considerably more than the essentials of what is commonly required as first-year algebra and second-year plane geometry. This leaves the pupil in condition to take up any of the current texts of algebra or geometry at this point of the curriculum.

The numbers on the inside top corners of the pages of this *Manual* are to assist the reader in quickly co-ordinating the matter given here with the relevant material of the text-book. On the left page of each pair of pages lying open to the reader at any time is indicated the placing of the earliest material, and on the right the latest material of the text that is here being discussed. This will enable users of the *Manual* to find readily in both text and *Manual* the subject-matter under consideration. Of course the value of what is said in this book depends entirely upon getting it readily co-ordinated with the subject-matter of the text to which it appertains.

THE AUTHORS

TABLE OF CONTENTS

CHAPTER		PAGE
	PREFACE	vii
	COMMENTS ON	
I.	GENERAL USES OF THE EQUATION	1
II.	USES OF THE EQUATION WITH PERIMETERS AND AREAS	8
III.	THE EQUATION APPLIED TO ANGLES	17
IV.	POSITIVE AND NEGATIVE NUMBERS	26
V.	BEAM PROBLEMS IN ONE OR TWO UNKNOWNS	37
VI.	PROBLEMS IN PROPORTION AND SIMILARITY	46
VII.	PROBLEMS ON PARALLEL LINES: GEOMETRIC CONSTRUCTIONS	58
VIII.	THE FUNDAMENTAL OPERATIONS APPLIED TO INTEGRAL ALGEBRAIC EXPRESSIONS	63
IX.	PRACTICE IN ALGEBRAIC LANGUAGE, GENERAL ARITHMETIC	88
X.	THE SIMPLE EQUATION IN ONE UNKNOWN	99
XI.	LINEAR EQUATIONS CONTAINING TWO OR MORE UNKNOWN NUMBERS; GRAPHIC SOLUTION OF EQUATIONS AND PROBLEMS	110
XII.	FRACTIONS	120
XIII.	FACTORING, QUADRATICS, RADICALS	127
XIV.	POLYGONS, CONGRUENT TRIANGLES, RADICALS	151

CHAPTER I

GENERAL USES OF THE EQUATION

The idea of the equation is found in the oldest mathematical manuscripts. As a tool for problem-solving the equation is not only time-honored, but it is also of such usefulness and importance that many consider that the main business of first-year high-school algebra is to learn what the equation is and how to solve it in its varied forms. Many problems, very difficult by arithmetic, become simple and easy by means of the equation. It is long enough, if not too long, to wait to the high school to become acquainted with this powerful instrument. Most of the topics of first-year algebra can be seen by the pupil to be necessary in order to use the equation with skill in solving problems of one sort or another. It seems natural therefore to begin a course in mathematics with the equation. Another reason for introducing the equation early is the fact that it is a splendid instrument to arouse interest in the mathematics.

Careful teachers know well from experience that the concept of the equation is not so easily grasped by the beginner as is commonly assumed. It is likely to be conceived as a mere mechanical form, lacking all content, and requiring for its solution merely a sequence of mechanical steps leading in some mysterious way to an "answer." The result, if it agrees with that in the book, of these mechanical steps that are usually imperfectly memorized, is looked upon as the solution of the problem without further question. The only question in the learner's mind is, "Did I remember to take the steps correctly?" Such empty mechanical imitation as this both kills interest and arrests mathematical development. A clear, even though not a logically complete, concept of the signification of an equation forestalls much of this thoughtless manipulating and symbol-juggling. A good start is half the battle. Therefore at first the equation is regarded as stating the balance between two number-expressions. The first two pages, read and freely discussed by the class, the teacher acting as moderator, will give clear ideas about the equation and a satisfactory working hold upon it.

The chapter contains a large number of problems and exercises. It will be found unnecessary to work all of them. Let a number of problems in a certain set be read by the class, sufficient to give the class clear conceptions and a little practice in solving them. It is worse than useless to insist on thoroughness beyond a very low limit. Clearness and sufficiency are the *desiderata* at the beginning. During or after this reading, other problems in the set may be assigned for home work. If there are still problems left that may be omitted, and if further drill is felt to be unnecessary, the class may take up the next set of problems. Thus an assignment for home work may contain problems taken from several pages.

It is well, while teaching, to keep in mind the conclusions stated at the end of the chapter, to guarantee arriving at definite results. Verbal problems and formal problems should be mixed freely, that the study of neither may become monotonous. No attempt should be made to master every problem or to work with over-mature notions or thoroughness. *Sufficiently clear intuitional* knowledge must for a time take precedence of thoroughness.

The chapter is to serve as an introduction to the possibilities of the subject, to impress the pupil with its usefulness, and to point out the importance of the equation. The main purpose of the formal equations is to review arithmetic under the guise of algebra. Pupils profit more by such reviews than by frank arithmetical work of which they are tired. Nine recitations of 45 minutes each is the maximum amount of time to be spent on it.

LESSON 1: *pages 1–3, Problems Using Algebraic Language*

Let a pupil read aloud carefully Problem 1, and answer the question. Then read the algebraic solution and make clear that the value of w is found by subtracting 8 from both sides of the equation.

Problem 2 is read next by another pupil. This brings out the same point as Problem 1, but in addition calls for a solution of $2p=16$. The value of p is found by reasoning thus: If twice p is 16, then p is $\frac{1}{2}$ of 16, or 8.

Problem 3 brings out the same point as do Problems 1 and 2.

Problems 4, 5, 6 show how the value of x is found by adding the same number to both sides of the equation.

All these problems are to aid the pupil to convert his arithmetical way of thinking into its algebraic form. The exercises below have the same office.

A pupil reads § **2**, page 3. This is followed by a short discussion both by teacher and class.

Solve by inspection problems of Exercise I, occasionally putting a solution on the board as the pupil is reading it. Assign for Lesson 2 *all of pages 1 and 2 and some problems in* Exercise I.

Solutions: EXERCISE I

No. of prob.:	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,
Solution:	2,	5,	5,	4,	5,	7,	16,	7,	$4\frac{1}{5}$,	6,	3,

No. of prob.:	12,	13,	14,	15,	16,	17,	18,	19,	20,	21
Solution:	$2\frac{1}{2}$,	9,	6,	8,	$7\frac{1}{2}$,	9,	6,	12,	8,	9

LESSON 2: *pages 3–4, Problem 36, Exercise II, included*

Take up 5 to 10 minutes answering questions asked by the class and asking questions; and adding explanation, if needed, to the work covered in Lesson 1.

Let the class read and discuss Problems 1 and 2, page 3. Assign 3 on page 4 to be worked in a similar way. Let the class work orally most of the problems in Exercise II. Assign for home work Problems 34, 35, 36. Problems of Exercise II give drill work in mental arithmetic, and assist the pupil in getting hold of the idea of literal number. They also give further practice in changing from the arithmetical to the algebraic way of thinking.

Solutions: EXERCISE II

No. of prob.:		1,	2,	3,	4,	5,	6,	7,	8,	9,
Solution:		27,	56,	42,	72,	63,	63,	42,	54,	96,
No.:	10,	11,	12,	13,	14,	15,	16,	17,	18,	19,
Sol.:	96,	72,	6,	7,	4,	8,	9,	8,	9,	9,
No.:	20,	21,	22,	23,	24,	25,	26,	27,	28,	29,
Sol.:	7,	7,	6,	6,	12,	63,	35,	72,	11,	11,
No.:	30,	31,	32,	33,	34,	35,	36,	37,	38	
Sol.:	$\frac{1}{7}$,	$\frac{1}{9}$,	$\frac{1}{8}$,	$\frac{1}{6}$,	$\frac{1}{5}$,	$\frac{5}{6}$,	1.2,	27,	$15+c$	

LESSON 3: *pages 4–10, Problems and Exercises*

Begin as in Lesson 2. Let the class read from Problems 37, 38, page 4, to Problem 4, page 5. These problems bring out the idea of general number still more sharply than did the preceding problems. Let the class read carefully Problems 1, 2, 3, at the bottom of page 5. Then follow with reading and suggestions as to the method of working Problems 3, 4, on page 6. Assign 3, 4, 5 for the next day. Use the remaining time to solve orally about 15 problems in Exercise III, p. 9.

LESSON 4: *pages 9–10*

After opportunity is given for asking questions and a short review, let the class read Problems 6–9, pages 6, 7, and the meaning of all problems should be made clear and suggestions be given as to the ways of obtaining the equations. Have the pupils solve orally or at the board Problems 16–30, page 9, and assign for home work every third problem of Nos. 31–50 on page 10.

LESSON 5: *pages 9–11*

This will be much like the two preceding lessons, covering Problems 9, 10, 11, 12, page 7, Problems 1, 2, 3, page 10, and Problems 6–36, page 11.

LESSON 6: *Problems 13 to 17, page 7; Problems 4 and 5, page 10; Problems 37 to 55, page 11*

LESSON 7: *Problems 18 to 23, page 8; Problems 1 to 5, page 12*

LESSON 8: *Problems 24 to 30, page 8; Problems 6 to 13, page 13*

Review and conclusions, page 13.

LESSON 9: *Written Exercise on Chap. 1*

Solutions of Problems: from page 5 to page 13

	Page 5				Page 6				
No. of prob.:	1,	2,	3,	4	3,	4,	5,	6,	7
Solution:	14,	22–*a*,	*a*–*b*,	3*a*	15, 30, 25, 24,	5, 20, 80, 40, 15,	40, 20, 10,	8, 14,	2 10 108

Page 7

No.:	8,	9,	10,	11,	12,	13,	14,	15
Sol.:	32, 24, 20, 24, 32, 20,	36, 12, 9,	21, rd.,	105, 210,	$52\frac{1}{2}$, $262\frac{1}{2}$,	3 hr., 150 min.,	24, 51,	503 528 553

No.:	16,	17,	18,	19,	20,	21,	22,	23
Sol.:	4, 4, 12, 12, 52,	$53\frac{1}{3}$, 110,	9,	7, 33,	1,641, 3,606,	95, 121,	101, 102,	157 158 159

Page 8

No.:	24,	25,	26,	27,	28,	29,	30
Sol.:	77, 79,	188, 190,	122, 124, 126,	40, 180, 400,	8,	42, 210, 89,	100 60

Solutions: EXERCISE III

Page 9

No. of prob.:	1,	2,	3,	4,	5,	6,	7,	8,	9,
Solution:	3,	4,	6,	10,	8,	9,	8,	7,	3,

No.:	10,	11,	12,	13,	14,	15,	16,	17,	18,	19,
Sol.:	6,	12,	9,	8,	$8\frac{1}{4}$,	9,	9,	7,	9,	7,

No.:	20,	21,	22,	23,	24,	25,	26,	27,	28,	29,	30
Sol.:	12,	$2\frac{1}{2}$,	7,	10,	12,	4,	12,	9,	2,	6,	7

Page 10

No.:	31,	32,	33,	34,	35,	36,	37,	38,	39,	40,	41,	42,
Sol.:	10,	4,	$4\frac{1}{2}$,	7,	3,	6,	7,	2,	7,	$3\frac{1}{2}$,	3,	10,

No.:	43,	44,	45,	46,	47,	48,	49,	50,	1,	2,	3,	4,	5
Sol.:	24,	6,	12,	24,	4,	14,	16,	24,	4, 12, 60,	3, 11,	2, 5,	−1, 4,	−2 4

Page 11

No. of prob.:	6,	7,	8,	9,	10,	11,	12,	13,
Solution:	$2\frac{1}{2}$,	$4\frac{2}{3}$,	$3\frac{3}{4}$,	$2\frac{2}{5}$,	$7\frac{1}{2}$,	$6\frac{1}{2}$,	9,	$11\frac{1}{2}$,

No.:	14,	15,	16,	17,	18,	19,	20,	21,	22,	23,
Sol.:	$7\frac{1}{2}$,	$6\frac{2}{3}$,	$5\frac{1}{2}$,	$4\frac{1}{3}$,	$7\frac{1}{3}$,	$3\frac{3}{4}$,	$3\frac{3}{4}$,	$5\frac{1}{2}$,	$4\frac{1}{7}$,	$5\frac{1}{5}$,

No.:	24,	25,	26,	27,	28,	29,	30,	31,	32,	33,
Sol.:	$5\frac{1}{7}$,	$9\frac{2}{7}$,	$5\frac{1}{2}$,	$3\frac{1}{2}$,	8,	$4\frac{1}{2}$,	10,	6,	4,	8.5,

No.:	34,	35,	36,	37,	38,	39,	40,	41,	42,	43,
Sol.:	11,	10,	10,	8,	12,	11,	4,	11,	12,	4,

No.:	44,	45,	46,	47,	48,	49,	50,	51,	52,	53,	54,	55,
Sol.:	$\frac{1}{10}$,	5,	4,	60,	15,	56,	42,	48,	3,	$\frac{1}{3}$,	$\frac{1}{4}$,	$1\frac{1}{2}$.

Page 12

No. of prob.:	1,	2,	3,
Solution:	$\frac{4}{100}\times 120$	$\frac{8}{100}\times 25$	$\frac{r}{100}\times 20$
	$\frac{6}{100}\times 120$	$\frac{8}{100}\times 250$	$\frac{r}{100}\times 80$
	$\frac{7\,1/2}{100}\times 120$	$\frac{8}{100}\times 1527$	$\frac{r}{100}\times b$
	$\frac{r}{100}\times 120$	$\frac{8}{100}\times b$	

No.:	6,	7,	8,
Sol.:	$175\times\frac{4}{100}\times 2$	$600\times 5\times\frac{8}{100}$	$160\times t\times\frac{6}{100}$
	$175\times\frac{4}{100}\times 5$	$600\times 5\times\frac{5}{100}$	$160\times t\times\frac{8\,1/2}{100}$
	$175\times\frac{4}{100}\times\frac{3}{4}$	$600\times 5\times\frac{8}{100}$	$160\times t\times\frac{r}{100}$
	$175\times\frac{4}{100}\times 2\frac{7}{8}$	$600\times 5\times\frac{6\,1/2}{100}$	
	$175\times\frac{4}{100}\times t$	$600\times 5\times\frac{r}{100}$	

No.:	9,	10,	11
Sol.:	$\frac{6}{100}\times 3\times 200$	$\frac{5}{100}\times t\times p$	$i=\frac{r}{100}\times t\times p.$
	$\frac{6}{100}\times 3\times 756$		
	$\frac{6}{100}\times 3\times p$		

CHAPTER II

USES OF THE EQUATION WITH PERIMETERS AND AREAS

This chapter is to be covered in not more than 6 lessons. The summary at the end states the most important results to be obtained.

LESSON 1: *pages 14, 15, 16, and 17*

Problems 1 to 7 should be read by the class. They should help to make the pupil see clearly that $a+b$ is both a number and the sum of a and b. Pupils usually feel that only terms having a common factor can be added and do not consider $a+b$ as a sum. Let the class read Problems 8 to 15, that the pupil may feel that numbers having a common factor have a sum expressed as one term.

Solutions of Problems: pages 14 and 15

1. $8+5$
2. $a+7$
3. $m+p$
4. $18-7$
5. $m-8$
6. $m-n$
7. (1) $x-7$ (2) $a-12$ (3) $y-10$ (4) $x-y$ (5) $s-t$ (6) $15-n$ (7) $v-s$ (8) $a-x$ (9) $x-a$ (10) $d-c$ (11) $c-b$ (12) $t-m$
8. $(12+10)$ yd.
9. $(8+4)$ doz.
10. $(5+4)$ 12's
11. $(9+4)\times 12$
12. $(8+6)$ half-dozens
13. $(8+6)\times 6$
14. $(5+7+10)\times 3$
15. $(2+3)x$, $2x$, $4x$.

§§ 5, 6, and 7 give drill on the meaning of the terms "exponent" and "coefficient," and make clear the distinctions between them. Use more illustrations if needed. These terms should be used freely after this.

§ 8 brings out this difference still more sharply. By substituting values the pupil is made to see that x^3 is a *product* containing 3 factors x. All 12 problems should be taken; or some may be selected and various values for x substituted. Some may be assigned for home work. Call attention to the difference between the meaning of $3x^2$ and $(3x)^2$.

Solutions

§8.
1. 12
2. 36
3. 18
4. $6 \cdot 6 \cdot 6$
5. 24
6. $6 \cdot 6 \cdot 6 \cdot 6$
7. 30
8. $6 \cdot 6 \cdot 6 \cdot 6 \cdot 6$
9. 72
10. 108
11. $5 \cdot 6 \cdot 6 \cdot 6$
12. $2 \cdot 6 \cdot 6 \cdot 6 \cdot 6$.

Lesson 2: *pages 16 to middle of 19*

This lesson brings in some of the terms used in beginning geometry. It shows how geometric concepts can be represented algebraically, and conversely, that an algebraic expression may be thought of as symbolizing the measures of geometric magnitude. Thus we have algebraic numbers representing line-segments, e.g., sides of triangles, perimeters (see Problems 1 to 7) algebraic products representing areas of squares and rectangles. These are important as adding to the pupil's conviction that these letters are numbers. Only a little time need be taken to read and answer Problems 1 to 8. The value of the exercises will be much greater if the work moves right along briskly than if it is permitted to drag. Problem 1, page 17, should be taken up as a review and extension of what has been learned on page

15. They aid the pupil to understand that a product has a concrete, picturable meaning. All 12 problems should be thought of as products in which the factors are counted. Thus, $4^2 \cdot 4^2 = 4^4$, since there are 4 factors 4 in the product. Problems 2 to 9 call for a statement for finding the area of the square and of the rectangles and give drill in using the equation and stating algebraic products.

Solutions: page 16

1. $7x$ ft. 2. $13a$ 3. $3x$ rd.

§9. 1. $\frac{1}{2}$ perimeter

2. $p=8+8+8+8$
$A=8 \cdot 8$

3. $p=4 \cdot 12$, $A=12^2$
$p=4 \cdot s$, $A=s^2$
$p=4 \cdot x$, $A=x^2$
$p=4 \cdot a$, $A=a^2$
$p=4 \cdot y$, $A=y^2$
$p=8b$, $A=4b^2$
$p=4(10+2)$, $H=(10+2)^2$
$p=4(b+7)$, $A=(b+7)^2$
$p=4(c+d)$, $A=(c+d)^2$

4. side $=4$, $p=16$
side $=a$, $p=4a$
side $=2a$, $p=8a$
side $=x$, $p=4x$

5. side $=5$, $A=25$
side $=x$, $A=x^2$
side $=2a$, $A=4a^2$

7. $p=2(a+b)$, $\frac{1}{2}p=a+b$, $2a$, $2b$

8. $A=a \cdot b$.

§10. 1.

(1) 4^4	(5) a^3	(9) b^7
(2) 8^4	(6) a^6	(10) c^9
(3) 10^6	(7) a^6	(11) c^9
(4) 12^7	(8) a^{11}	(12) x^{11}

2. $p=4 \times 3x$

Find area by counting squares.

3. $A=8\cdot 5, 8\cdot 4, 8\cdot 3\frac{1}{2}, 8\cdot 6\frac{1}{4}$

4. $A=12\cdot 6, 12\cdot 8\frac{1}{4}, 12\cdot 9\frac{1}{3}, 12\cdot 10\frac{3}{4}, 12\times 12y$

5. $A=12\cdot l, 12\cdot 9, 12h, 12n, 12x, 12a$

6. $A=8w, 10w, 12\frac{1}{2}w, wx, aw, lw, bw, wz$

7.

$a\cdot x=ax$	$b\cdot b^3=b^4$	$x^3\cdot x=x^4$
$b\cdot c=bc$	$a^2\cdot a=a^3$	$a\cdot x^2=ax^3$
$b\cdot b=b^2$	$a^2\cdot a^3=a^5$	$a^2\cdot x=a^2x$
$a\cdot a^2=a^3$	$x\cdot x^3=x^4$	$g\cdot t^2=gt^2$

8. Count squares.

9. Draw figure and count squares.

10. Transform parallelogram into an equal rectangle. Then $A=10a, 12\frac{1}{2}a, 16.8a, ab, ac, ax, 2an, 5az$.

11. Show that the area of triangle is one-half the area of the parallelogram. $A=\frac{1}{2}\cdot 12a, \frac{1}{2}\cdot 18\frac{1}{3}a, \frac{1}{2}\cdot 20\cdot 25a, \frac{1}{2}ab, \frac{1}{2}ad, \frac{1}{2}ax, \frac{1}{2}a\cdot 2y, \frac{1}{2}\cdot 42a, \frac{1}{2}\cdot 5ac$.

Lesson 3: *pages 19, 20, one-half of 21*

Recall how the area of a rectangle is obtained. Have the pupil see clearly how the rectangle representing $6x^2$ is made up of six squares, each x^2. Do not slight this visualizing. Point out that there are formulae for finding the areas of other geometric forms as Figs. 12, 13, 14, and 16. Show how the area of the parallelogram is found by means of a rectangle equal to it in the obvious fashion suggested in Fig. 12. Solve Problem 10. Knowing how to find the area of the parallelogram we are now able to find the area of the triangle (Fig. 13). Solve Problem 11. Do not let the work drag. Action and movement are needed here else the metal cools. In a similar way work Problems 12 to 15. Let the class read and answer Problem 16. Do not multiply out. Simply have the equation stated thus: $A=6(x+3)$, $x(a+5)$, $a(x-2)$, $(a+2)(x+1)$; Problem 17: $A=(m+n)(c+d)$. But $mc+md+nc+nd$, Problem 18, represents the same area. Therefore, Problem 19, $(m+n)(c+d)=mc+md+nc+nd$. You can readily convince yourself here that the learner does not naturally reason with axioms as he will not find 19 so easy. But deductive reasoning must be acquired and for this the profit is in the work. Solve Problems 20 (1); 21 (1); 22 (1) and (2) in class, making sure the work is understood by the class, and assign for home work the remaining parts of Problems 20 to 22.

Solution for Problem 20

(1) $xa+ya+xb+yb$ (3) $ab+xb+ay+xy$
(2) $xm+ym+xn+yn$ (4) $ra+sa+rx+sx$

The figures for Problem 21 are all squares of sides $a+b$, $c+d$, $x+y$, $m+n$ respectively.

Solution for Problems 22 and 23

22. (1) $(a+x)^2$ (3) $(k+b)^2$ (5) $(x+3)^2$
(2) $(b+c)^2$ (4) $(s+t)^2$ (6) $(c+4)^2$

23. $a=\frac{40}{8}=5$

Let the class read rapidly Problems 24 to 26, assigning the last of Problem 26 for home work. Do not stop to multiply the quotients; simply state the equation, e.g., $a=\frac{32}{8}$ (read 32 *over* 8). The problems on page 21 are to be read in a similar way.

Solutions for pages 20 and 21

24. $a=\frac{32}{8}, \frac{16}{8}, \frac{12}{8}, \frac{7\frac{1}{2}}{5}$. Take time to simplify only some of the fractions.

25. $b=\frac{27}{9}, \frac{18}{9}, \frac{15}{9}, \frac{12}{9}, \frac{A}{9}$. Take time to simplify only some of the fractions.

26. $a=\frac{24}{\frac{1}{2}\cdot 6}, \frac{12}{\frac{1}{2}\cdot 6}, \frac{9}{\frac{1}{2}\cdot 6}, \frac{3}{\frac{1}{2}\cdot 6}$. Take time to simplify only some of the fractions.

$$b=\frac{16}{\frac{1}{2}\cdot 4}, \frac{8}{\frac{1}{2}\cdot 4}, \frac{4}{\frac{1}{2}\cdot 4}, \frac{a}{\frac{1}{2}a}, \frac{hl}{\frac{1}{2}h}, \frac{ab}{\frac{1}{2}b}, \frac{by}{\frac{1}{2}b}, \frac{a}{\frac{1}{2}c}, \frac{b}{\frac{1}{2}a}.$$

Take time to simplify only some of the fractions.

§ 11. $q=\frac{m}{n}, \frac{a}{b}, \frac{b}{a}, \frac{4x}{3y}, \frac{a+b}{c}, \frac{a}{x+y}, \frac{a+b}{c+d}, \frac{a^2-b^2}{a+b}, \frac{a^2-b^2}{a-b},$

$$\frac{x^2}{a+b}, \frac{(a+b)^2}{a+b}, \frac{(a-b)^2}{a-b}.$$

LESSON 4: *pages 21, 22, 23, and one-half of page 24*

§§ **12** to **18** may be read by the class or may be presented by the teacher, giving the text for reference at the close of the lesson.

Solve Problems 1 to 12 with the class, the class helping, keeping the work moving right along briskly. Let pupils think about Problem 14, and let some pupil volunteer to put the figure on the board. Similarly for Problems 15 to 18. Assign 19 to 22 for home work. Have the class solve part of Problem 23 and assign part for home work. Similarly for Problems 25 to 37. Discuss in class problems 5 and 7 of § **19** and assign 2 or 3 of 6 to 16 for home work.

Solutions

§ **18.** 1. $a \cdot a \cdot a$.

2. $4 \cdot a \cdot a \cdot b$.

3. Subtract 3 from 10 and multiply the result by 6.

4. 8 times the sum of 40 and 2.

5. 9 times the difference between 60 and 3.

6. 7 times the sum of 80 and 1.

7. 16 times the sum of a and b.

8. Subtract y from x, add 1, and multiply the result by 20.

9. 25 times the difference between x and a.

Page 23. 1. Equilateral triangle, equilateral 4-side, etc.

12. Equilateral a-side.

13. 3, 4, 5, 6, 7, 8, 9, 10, 12, 20, n, a.

14. A rectangle of dimensions $2x$ and 2.

15–19, 21. May all be rectangles.

20. A triangle all sides of which are $x+y$.

22. A triangle whose sides are $2x+y$, $x+y$, $2x+y$.

23. $p=6, 8, 10, 12, 14, 16, 18, 20, 24, 40, 2n, 2a$;
$p=9, 12, 15, 18, 21, 24, 27, 30, 36, 60, 3n, 3a$;
$p=18, 24, 30, 36, 42, 48, 54, 60, 72, 120, 6n, 6a$.

24. $p=24, 34, 34, 36, 18, 36, 21, 24, 31, 52, 76, 76, 64, 60, 120, 48, 56, 72$; $4a+4$, $6a+4$, $6a+4$, $4a+16$, $6a-12$, $12a-24$, $3a+2b$, $4a+2b$, $5a+3b$.

25–32. Figures are all rectangles, the factors being the length of the sides (see Fig. 21).

33–36. Rectangles of dimensions $\frac{3}{2}$, $x+1$; 2, $a+b$;

$$a, \frac{b+c}{2}\ ;\ x,\ x+4\ .$$

37. 9, 36, 8, 30, 18, 28, 0, 20, 6, 10, 4, −3; 25, 90, 40, 40, 54, 5, 36, 9, 16, 18, 5.

§ 19. 1. Multiply the sum of x and y by the sum of a and 10.

2. Multiply the sum of a and b by the sum of c and d.

3. Multiply the sum of x and y by the difference of a and b.

4. Multiply the difference of a and b by the difference of c and d.

LESSON 5: *page 24, Section 20, to middle of page 27*

Ask for the meaning of each of Problems 1 to 14 before solving or assigning for home work. Solve with the class Problems 1, 2, 4, 5, 8, 9, 12. Assign the others for home work. Let the class read page 25 to the middle of page 26. Assign Problem 7, p. 27, as home work.

§ 20. 1. 12	6. 360	11. $\sqrt{24}$
2. 104	7. 8	12. 2
3. 12	8. 4	13. 1,000
4. 64	9. 3	14. 25,000.
5. 36	10. 3	

LESSON 6: *from Section 22, page 27, to Summary*

Let the class read Article 21. Have problems read, let some be solved in class and the others assigned for home work. Send class to the board, assign to each pupil one of the fifteen problems in Exercise IV, have some of these problems explained and assign others for home work.

Discuss with class ways of solving each problem before solving or assigning for home work, getting as much suggestion as possible from the class. Let one pupil work at the board with the class criticizing or suggesting better ways. Always and everywhere the idea is to work *with* the class not for the class. Good teaching must engage the learner in the work.

LESSON 7

Study the Summary. Review the problems on page 28.

Solutions: page 26

1. 9
2. 12
3. 31
4. $25 \cdot 5$
5. $\frac{294}{12}$
6. $(x+8)8=96$
   $x=4$
7. $6, \frac{5}{8}, 3 \cdot 5, 3, 1, \frac{45}{28}$.

Solutions: EXERCISE IV

1. 17	7. 5	13. 8	19. 7	25. 10
2. 12	8. 6	14. 9	20. 4	26. 4
3. 6	9. 6	15. 15	21. 6	27. $\frac{7}{2}$
4. 7	10. 12	16. 4	22. 10	28. $\frac{20}{7}$
5. 7	11. 6	17. 15	23. 56	29. $\frac{45}{2}$
6. 6	12. $\frac{90}{11}$	18. $3\frac{1}{2}$	24. 21	30. 8.

CHAPTER III

THE EQUATION APPLIED TO ANGLES

LESSON 1: *pages 30–34*

The first five pages teach the conception of angle as the amount of turning and give practice in reading and drawing angles. In § 24 and § 25 "angle" is *defined* as the amount of turning of a line rotating about a point as a pivot. As a familiar illustration, the angle formed by the hands of a clock may be used. In § 25 the right angle is defined as $\frac{1}{4}$ of a complete turn and the straight angle as $\frac{1}{2}$ of a turn. This should help the pupil to avoid the common mistake of calling a *straight angle* a *straight line.* It is highly important that the pupil's general stock of everyday notions be drawn upon as largely as possible in defining the basic notions and concepts of mathematics.

Drawings in Problem 1 page 31, shall be like those in Figs. 31 and 32. After the first two or three they may be merely sketched rapidly. Some of these may be put on the board by pupils, others assigned as home work. Problem 2 should be read and answered rapidly, the answers given to each question thus: 4 right angles, $\frac{1}{2}$ of 4 R.A., 7×4 R.A., $1\frac{1}{4}\times4$ R.A., $t\times4$ R.A., $\frac{1}{2}\times4$ R.A., etc. It is not necessary to simplify answers to the questions. Problems 3, 4, and 5 may be read in a similar way.

Answers for Problem 3: 2 straight angles, $1\frac{1}{2}\times2$ S.A., $\frac{3}{4}\times2$ S.A., etc. . . . . $3(2t-5)\times2$ S.A.

Answers for Problem 4: 2 R.A., 5×2 R.A., $7\frac{1}{2}\times2$ R.A., etc. . . . . $(3s-2)2$ R.A.

Answers for Problem 5: $4\times\frac{1}{2}$ S.A., $2\times\frac{1}{2}$ S.A., ... $(4r-6)\times\frac{1}{2}$ S.A.

Problem 6 may be read by a pupil, the teacher folding the paper as indicated. The angles are right angles as they are equal and just fill the plane around point O.

Let different pupils read each a question in Problem 1, page 33, and give the answer, as $\angle XOA=30^\circ$, etc. Similarly for Problem 2. The answers will be $\angle XOA+\angle AOB=65^\circ$, etc. In Problem 3 the answers need not be simplified. They are $(2\times 180)^\circ$, $(4\times 80)^\circ$, $(\frac{2}{3}\times 90)^\circ$, $(\frac{7}{9}\times 90)^\circ$, etc. . . . . $\left(\frac{r+3}{2}\times 90\right)^\circ$. Problems on page 34 may be assigned for home work.

Answers to questions: 1, page 34: 2 R A.; 1 S.A.; 180.°

2, page 34: 10°, $\frac{1}{9}$ R.A.; 30°, $\frac{1}{3}$ R.A.; 90°, 1 R.A.; 120°, $\frac{4}{3}$ R.A.; 150°, $\frac{5}{3}$ R.A.; 20°, $\frac{2}{9}$ R.A.; 80°, $\frac{8}{9}$ R.A.; 90°, 1 R.A.; 120°, $\frac{4}{3}$ R.A.; 60°, $\frac{2}{3}$ R.A.; 170°, $\frac{17}{9}$ R.A.; 180°, 2 R.A.

3, page 34: $\angle XOB$; $\angle AOC$; $\angle XOY$.

4, page 34: $\angle XOB=\angle XOA+\angle AOB$.

5, page 34: $\angle AOC=\angle AOB+\angle BOC$, $\angle XOC=\angle XOY+\angle YOC$.

6, page 35: $\angle XOC=\angle XOA+\angle AOB+\angle BOC$, $\angle AOC=\angle AOB+\angle BOY+\angle YOC$.

7, page 35: $\angle XOB-\angle XOA=\angle AOB$.

8, page 35: $\angle XOY-\angle XOA=\angle AOY$; $\angle AOY-\angle AOB=\angle BOY$; $\angle XOY-\angle BOY=\angle XOB$.

10, page 35: 180°.

12, page 36: 360°.

LESSON 2: *pages 35–38*

Let different pupils read Problems 6, 7, and 8 and let the class give the answers orally to all questions as they are read. Work out *with* the class the first two angles in Problem 9 and let the table be filled out completely at home. Point out that Problem 10 furnishes a check for the accuracy of the work. Assign for home work Problems 11 and 12. Let the class read Problem 1, page 36, and let the teacher ask the question of Problem 3. Show that Problem 2 suggests a check. In all this work some things are to be done by the teacher and some by the class. The danger to be continually guarded against is that the teacher will do too much, thereby carrying his class. The teacher may present Problems 4, 5, 6, and 7 to save time, questions being dropped to the class now and then to hold attention. Let the class state the equation for some of the problems at the bottom of page 37, and assign some of them for home work, as also those on 38 and in Exercise V as a review.

Page 37:

§ **28.** 1. 93, 31, 36, 180

2. $x=35$
70, 35, 180, 175

3. $x=37$
37, 221, 102

4. $x=24$
72, 120, 165, 3

5. $x=15$
$241\frac{1}{2}$, $7\frac{1}{2}$, 111

6. $x=10$
30, $76\frac{2}{5}$, 20, $233\frac{3}{5}$

7. $x=9$
37, $179\frac{3}{8}$, $93\frac{5}{8}$.

LESSON 3: *pages 38–42*

Solve as many of the problems on page 38 as possible. The arithmetic of the number combinations is the important thing here.

Page 38: 8. $x=14$
14, 70, 96

9. $x=13$
117, 13, 11, 39

10. $x=12$
96, 12, 38, 34

11. $x=8$
$65\frac{2}{3}$, $72\frac{5}{6}$, 48, $-6\frac{1}{2}$

12. $x=24$
72, 66, 24, 18

13. $x=35$
70, 90, 17, 3

14. $x=37$
102, 70, 8

15. $x=33.12$
132.066, 7.654, 40.28

16. $x=9$
62.37, 43.38, 16.52, 57.73.

Exercise V, *page 38*

1. 5	7. $2\frac{11}{14}$	13. $9\frac{15}{17}$
2. 5	8. 6	14. 6
3. 12	9. 4	15. $7\frac{1}{2}$
4. 10	10. $7\frac{3}{4}$	16. 10
5. 5	11. 16	17. 12
6. 2	12. 15	18. 1.

Assign for home work §§ **29–31**, giving some suggestions as to the nature of the work; but not telling the whole matter and leaving no real work for the class. Your pupils will do more work if you expect it of them.

Let the teacher present the construction of Figs. 49–51. Let all members of the class make the construction on paper at their seats under the suggestive direction of the teacher. Assign for home work portions of Problems 1 to 4, page 41. Make clear the meaning of the definitions in §§ **33, 34**. Paper folding and creasing are good here. Let some pupil read Problems 1, 2, and 3, one at a time, then have someone make the drawing on the blackboard. Explain the meaning, not the solution, of Problems 4 to 6, page 42, and assign them for home work; one at a time during the next three days.

Lesson 4: *pages 42–46*

Explain the meaning of the definitions in § **35**. Paper folding and creasing are good here again. Let the class work Problems 1 and 2. Discuss the meaning of Problem 3 and assign it for home work. Let the pupils pin on ordinary notebook paper the tracing paper with the drawing and hand this in with the other home work assigned. Let the teacher grade it and hand it back. The answers to 4 should come rapidly thus: yes, no, yes, yes.

Let the class work orally and rapidly all problems on page 43, making clear the point that the equation $a+b=180$ is the algebraic way of stating that angles a and b are supplementary. Let the class work as many problems of pages 44 to 46 as possible, and assign for home work some of the verbal problems on page 45 and some of the formal problems on page 46.

Page 43:
13. 46, 132
14. 41, 139
15. $76\frac{1}{2}$, $103\frac{1}{2}$

Page 44:
16. 35, 145
17. $79\frac{1}{2}$, $100\frac{1}{2}$; $53\frac{3}{8}$, $126\frac{5}{8}$; $71\frac{3}{4}$, $103\frac{1}{4}$
18. $71\frac{1}{2}$, $108\frac{1}{2}$
19. 30, $22\frac{1}{2}$, 40
20. 36, 20, $16\frac{4}{11}$, $51\frac{3}{7}$, $102\frac{6}{7}$, 160

Page 45:
23. 80, 100
24. 50, 130
25. 45
26. 40, 140
27. 12, 168
28. $x=115$
69, 111
29. $x=252$
182, -2
30. $x=126$
119, 61

31. $x=54$
179, 1

32. $x=24$
105, 75

33. $x=24$
55, 125

34. $x=45$
96, 84

35. $x=30$
97, 83

36. $x=90$
153, 27

37. $x=60$
68, 112

38. $x=22.4$
91.2, 88.8

39. $x=68\frac{8}{17}$
$50\frac{11}{17}$,
$129\frac{6}{17}$

40. $x=90$
27, 153

41. $x=24$
55, 125.

LESSON 5: *pages 46–50*

Let the class read rapidly or the teacher may present the subject-matter of §§ **36** and **37** to Problem 4. Let the class work orally the easier problems on page 49 and 50. Assign for home work Problem 4 and some of the more difficult problems on page 49, asking for drawings for some of them only. Assign for home work some of the exercises on page 50.

EXERCISE VI: *page 46*

1.	115	7.	4	13.	7
2.	14	8.	2	14.	$\frac{3}{4}$
3.	6	9.	4	15.	10
4.	20	10.	4	16.	5
5.	10	11.	7	17.	1.
6.	5	12.	1		

Page 49:

6. $x=20$
167

7. $x=60$
163

8. $x=40$
$40\frac{1}{2}$

9. $x=55$
117

10. $x=84$
168

11. $x=168$

12. $x=40$
    230
13. $x=72$
    144
14. $x=72, 36$
15. $x=4, 11$
16. $x=10$
    $179\frac{1}{2}, \frac{1}{2}$
17. $x=56$
    48, 132
18. $x=45$
    33, 147
19. $x=36$
    153, 27
20. $x=60$
    260, −80
21. $x=60$
    135, 45
22. $x=55$
    117, 63
23. $x=45$
    33, 147
24. $x=45\frac{15}{19}$
    $137\frac{7}{19}$, $42\frac{12}{19}$
25. $x=36$
    27, 153.

Lesson 6: *Pages 50–54*

This lesson is very much like the preceding one. Part of the time may be used to review rapidly the essential points of Lesson 2. For home work assign some of the problems on page 50 which were not taken up in Lesson 6, and some problems on pages 52, 53, and 54.

Exercise VII

1. 2
2. 2
3. 3
4. 9
5. 7
6. 9
7. 3
8. 10
9. 1
10. identity
11. 7
12. 5
13. 14
14. 3
15. 8
16. 11
17. 42
18. 15
19. 21
20. 8
21. 8
22. $\frac{6}{5}$.

Pages 51, 52, 53, 54:

5. $30°$, $60°$, $79\frac{5}{8}°$, $44\frac{2}{7}°$, $(90-a)°$, $x°$
6. $(90-n)°$
7. $(90-d)°$, $(90-3c)°$, ... $[90-(3x^2-5y^4)]°$
8. $50°$

9. 70°, 20°
10. d°, c°
13. 21, 69
14. 33, 57
15. 59, 31
16. $3\frac{1}{2}$, $86\frac{1}{2}$
17. $34\frac{1}{2}$, $55\frac{1}{2}$; $26\frac{3}{4}$, $63\frac{1}{4}$; $8\frac{3}{8}$, $81\frac{5}{8}$; $\frac{90-d}{2}$, $\frac{90+d}{2}$
18. $58\frac{1}{2}$, $31\frac{1}{2}$
19. 18°, $\frac{90}{7}^\circ$, $\frac{180}{13}^\circ$
20. $\frac{45}{2}^\circ$, $\frac{45}{4}^\circ$, $\frac{90}{7}^\circ$, $\frac{270}{13}^\circ$, 50°, 81°
23. 30°, 60°
24. 40°, 50°
25. 60°, 30
26. $33\frac{3}{4}$, $56\frac{1}{4}$
27. $23\frac{3}{7}$, $66\frac{4}{7}$.

LESSON 7: *pages 54 to the end of the chapter*

Let the class work out Problem 1, page 54, with some pupil making the drawing on the board. This should bring out a statement of Theorem I, page 56. Problems 2, 4, and 6 all bring out the same principle and can be worked in very little time by the teacher and class together. Problem 7, page 55, is important as it leads to an algebraic statement of the relation between the interior angles of a triangle. All of §§ **39**, **40**, and Problem 1 of § **41** may be developed by the teacher.

Let the class work some of the problems on page 57, one or two of the parts of Problem 19, and some of the problems on page 59.

Assign for home work some problems on pages 57, 58, and 59 and the summary at the end of the chapter. All theorems and definitions should be well learned.

Pages 56, 57, 58:

2. 54, 18, 108
3. 40, 20, 120
4. 108, 18, 54
6. 40, 120, 20
7. 76, 58, 46
8. 62, 82
9. 41, 16, 123
10. 42, 21, 117
11. $31\frac{1}{2}$, 126, $22\frac{1}{2}$
12. 132, 40, 8
13. 114, 16, 50
14. $n=108$
    54, 36
18. 45
19. $114\frac{6}{11}$, $32\frac{8}{11}$, 54, 36, 18, 108, 54, 30, 30, 120
20. 60
21. 120.

Exercise VIII: *page 58*

1. $\frac{14}{3}$
2. 1
3. 1
4. 5
5. 3
6. 10

Page 59:

7. 6
8. 6
9. 15
10. 56
11. 6
12. 1
13. 7
14. 2
15. 5
16. 10
17. 1
18. 1
19. 4
20. 1
21. a
22. $\frac{2+3a}{2}$
23. a
24. $2a$
25. $6a$
26. $\frac{12b}{5}$
27. $\frac{176}{21}$
28. $\frac{4}{15}$.

CHAPTER IV

POSITIVE AND NEGATIVE NUMBERS

If the work which precedes this chapter has been well done the pupil has sensed sufficiently well for a beginner the value of the equation as a tool for stating and solving problems. He has also reviewed, under the guise of algebra, the essentials of arithmetic, and has generalized many of the laws of arithmetic, through the agency of literal number, into algebraic formulas. Furthermore, he has put into algebraic garb the mensurational laws for areas and for angles. He has thus seen that even in the field of applied numbers, already partially at his command, algebraic formulas and equations are compact and convenient short-hand modes of expressing and recording laws of number and of magnitude, and that the negative number, or something equivalent to it, is demanded even in ordinary problem-solving.

The next natural step is to learn more fully what a negative number is, what the extended field of positive and negative number is, what addition, subtraction, multiplication, and division of these new numbers mean, and how these operations are to be applied to these new numbers. All this is done most effectively by means of easy real problems and exercises that employ the *positive-negative* notion.

It is recommended that this chapter be covered in 7 lessons at most. The material may well be divided into lessons somewhat as follows:

First lesson: to § **46**, page 67.

Second lesson: to Problem 8, page 71.

Third lesson: to Problem 3, bottom page 73.

Fourth lesson: to **Multiplying Positive and Negative Numbers,** page 77.

Fifth lesson: to **Dividing Positive and Negative Numbers,** page 85.

Sixth lesson: to SUMMARY, page 88.

Seventh lesson: Study the SUMMARY and review the chapter.

§ 42. Let the teacher, or better, let some member of the class read Exercises 1, 2, 3, 4, and have different members of the class answer the questions or give the results of the exercises orally and rapidly, that the connection and logical significance of the individual exercises may be felt. Have the answers of Exercise 3 come right along in the abbreviations, thus:

3. (1) R 13° (3) R 4° (5) R$(x-5)^\circ$
   (2) R 3° (4) R 5° (6) R$(a-b)^\circ$

If on R$(x-5)^\circ$ of (5) and on R$(a-b)^\circ$ of (6) the question arises whether they should be R$(5-x)^\circ$ and R$(b-a)^\circ$, clear the matter up for the class.

4. (1) $+12^\circ$ (5) $(x+y)$ degrees
   (2) $+8^\circ$ (6) $(a-x)$ degrees
   (3) $+2^\circ$ (7) $a-a$, or 0 degrees
   (4) -3° (8) $(-a-x)$, or $-(a+x)$ degrees

5. Have pupils show points on the graph located by the readings:

-5°, 0°, $+2^\circ$, $+8^\circ$, 10°, $+5^\circ$, 0°, and -5°

The teacher may study the graph in the book with the class, and then with its help he may put the graph of one of the subsequent problems on the board. This will save time, increase the class interest, and bring up new difficulties for immediate explanation, or the teacher may prefer to transfer this figure, or to draw before the eyes of the class a similar one on the cross-lined blackboard, and thus to make clear the location of the points, and the meaning of the broken connecting line.

The answers to the questions in the paragraph below Fig. 67 should be: *it rose; it rose slowly; rose rapidly; rose slowly; stood still; fell rapidly; fell rapidly; fell rapidly; first 4 hours, a rise; 5th hour, stationary; last 3 hours, a fall.* Dwell on this work only long enough to get the major ideas of it before the class. Thoroughness is impossible at this stage of advance and it is a pedagogical mistake, bordering on pedanticism, to spend much time in attempting to be thorough with elementary and poorly defined notions. If the work cannot all be done in one class period assign some of it for home work. This has the added merit of giving the pupil a chance to review a little.

6. Let pupils use notebooks supplied with cross-lined pages. When all have tried and some are through, let the teacher quickly lay off these readings on the cross-ruled blackboard and draw the broken connecting line, as a *résumé*.

7. Move along rapidly with the plotting here. Do not allow "puttering."

§ 43. Have 3 or 4 pupils solve Exercise 1, page 64, and have one pupil explain it to the class. Pass briskly along through Exercises 2, 3, 4, and 5. Tarry long enough on 6 to allow pupils to sense the nature of the forces, then require the answers to come rapidly.

6. (1) F 8 oz. (2) F 8 lb. (3) F 10 tons. (4) F 15 tons.

7. The value of this exercise depends on pushing right along through it.

(1) $+8$	(4) -7	(7) $a+b$
(2) 0	(5) -24	(8) $a-b$
(3) -23	(6) $x-12$	(9) $-a-b$

From here on push briskly through the list of problems to § 44. See that the answers to 23, to be done orally, are given correctly.

§ 43—*Answers*

2. $+18-10=+8$

3. (1) $+2$ mi. (2) 0 (3) $+150$ mi. (4) $(a+b)$ mi.

4. (1) $+10$ mi. (2) $+10$ mi. (3) $3-m$ mi. (4) $a+c$ mi. (5) $m-n$ mi. (6) $n-m$ mi.

5. ½ ton pulling forward

6. (1) F 8 oz. (2) F 8 lb. (3) F 10 tons (4) F 15 tons

7. (1) $+8$ (2) 0 (3) -23 (4) -7 (5) -24 (6) $x-12$ (7) $a+b$ (8) $a-b$ (9) $-a-b$

8. Rise with 3 oz. force

9. (1) $+10$ lb. (2) $+7$ lb. (3) -33 lb. (4) -13 lb. (5) $x+y$ lb. (6) 0

10. (1) $30°$ (2) $-16°$ (3) $0°$ (4) $-12°$ (5) $x°-10°$ (6) $-x°-10°$ (7) $x°-y°$ (8) $-x°-y°$

11. 10c.

12. P \$200, P \$23, D \$15, P \$700

13. 8 lb. upward

14. 95 years

15. (1) 15 (2) 70 (3) 2 (4) 150 (5) 500 (6) 300

16. 51 years

17. 23 years

18. 646 years
19. 64 years
20. -60 rd., $-1\frac{1}{2}$ mi., $-x$ rd., $+80$ rd., $+1\frac{1}{4}$ mi., $+a$ mi.
21. 6 mi. northward, $+6$ mi.
22. $+9$ mi.
23. (1) below (2) backward (3) downward (4) leftward (5) before (6) west (7) south (8) debts.

Have different pupils read the several paragraphs of page 67, the teacher noting when the reading indicates that the ideas are being comprehended by the reader and by the class, or to save time the teacher may first present these ideas to the class and then refer to page 67 for re-reading. Proceed thus to the phrase **Graphing Data,** page 68.

§ **42.** Have 3 or 4 pupils make the graph of Exercise 1, page 68, and bring it to class next day; 3 or 4 others make the graph of Exercise 2, and so on through the list to page 70. It is perhaps still better to assign a little of this graphic work from day to day for several days, than to condense it all into one day.

Graphing Precise Laws

EXERCISE 1.—First have pupils calculate the areas of the rectangles from the given data and record the areas beside the given lengths in tabular form in their notebooks. Then have pupils take cross-ruled paper and plot each base-length and its corresponding area and mark the points clearly and neatly and then draw the connecting line. Before this work is forgotten, have Exercise 2 answered. Notice that the intention is that the *pictured form* shall precede the *equational form*, $y=3x$. The thought is that the equational form should be seized by the learner as the short-hand description of the pictured law.

3. If Exercise 3 is not answered at once, don't tell, but ask that Exercise 2 be reanswered, and then try Exercise 3.

4. Let the pupils answer Exercise 4 without help if they can. If they cannot, the difficulty is likely to be with the question "state how the area, y, varies with the altitude," etc. Let the teacher change the question to "If the altitude is doubled how is the area changed?" "If trebled," etc.

In the same fashion work down through the Exercises of page 71.

§ **49.** Have each pupil graph at least 3 of the Exercises 1–9.

§ **50.** Have each pupil graph in class, or as home work, Exercise 1 and at least 3 parts of Exercise 2. Have all the graphs of the parts of Exercise 2 made on the same drawing. Here again it is suggested that a little of this work from day to day for several days will give better results than to complete it all at once.

Adding Positive and Negative Numbers

Work orally with the class through all the parts of Exercises 1, 2, 3, and 4, answering, or prompting, or correcting only where you feel it necessary to prevent the work from dragging, from wool-gathering, and the point of the development from being lost.

The idea is that the pupil shall put *himself* as fully as is practicable into the work of developing the addition laws. Telling will be fatal to interest here. If only a few of the exercises of 2 are really needed to develop the rule the rest may be omitted, though the teacher is in danger of using too few, with the thought that they are to function only as illustrations.

Answers to exercises:

1\. (1) +5 min.
(2) +1 min.
(3) −5 min.
(4) −10 min.
(5) +6 min.
(6) −22 min.
(7) −5 min.
(8) +7 min.
(9) +1 min.
(10) 0

2\. (1) +23
(2) −23
(3) +7
(4) −7
(5) +57
(2) −57
(7) +19
(8) +30
(9) +30
(10) −30
(11) −15
(12) +22
(13) 0

3\. Algebraic sum is arithmetical sum with common sign prefixed.

4\. Algebraic sum is arithmetical difference with sign of larger prefixed.

§ 52. The thought with Exercise 1 is that the pupil shall use any common-sense way of combining the numbers into a sum that may occur to him after having learned how to proceed to find the sum of *two* signed numbers. In answering Exercise 2 whether he give either of the two ways stated in paragraph 5 of the SUMMARY, page 88, or some other plan of his own is immaterial just here. A real effort is all that is needed to secure concentration upon the idea.

Have as many of the examples from Exercise 3, page 73, to the bottom of page 75 solved orally as the pupils can do comfortably.

Answers to exercises:

1\. (1) +51
(2) +7
(3) +50
(4) +10
(5) +504
(6) +5x
(7) +12a

2\. (1) Add the several numbers in order, or (2) add all positives, then all negatives, then add the two sums.

3. 3 lb. toward right

4. (1) $+4$ lb.
   (2) -4 lb.
   (3) 0
   (4) -3 lb.
   (5) -3 lb.
   (6) -11 lb.
   (7) $+25$ lb.
   (8) -4 lb.
   (9) -24 lb.
   (10) -9 lb.
   (11) $x+y$ lb.
   (12) $x-y$ lb.
   (14) $-x-y$ lb.
   (15) 0
   (16) x lb.

5. 5 lb. upward
6. 2 oz. downward
7. $15\frac{3}{7}$ lb. downward
8. 78 feet above, or $+78$ feet
9. $+3^\circ$
10. 5 mi., 1 mi.

Exercise IX

1. $+1\frac{3}{8}$
2. $-\frac{1}{8}$
3. $-\frac{7}{48}$
4. $+1\frac{1}{6}$
5. $-7\frac{5}{8}$
6. $-1\frac{3}{8}$
7. $+2\frac{1}{6}$
8. $-\frac{1}{5}$
9. -15
10. $+9\frac{1}{12}$
11. -4
12. $-.92$
13. $+10\frac{1}{8}$
14. $+1.35$
15. -2.3
16. $-2r$
17. $-10s$
18. $+\frac{5}{6}$
19. $+4b^2c$
20. $-6v^2y^3$
21. $+2(x-a)$
22. $-19\,(x+y)$
23. $-143\,(m-r)$
24. $-2\frac{5}{6}\,(c-d)$

Exercise IX, page 75, particularly should be done orally as far as possible, that the pupil may get the training in mental arithmetic with simple fractions and that he may feel that two things must be paid attention to in obtaining algebraic sums, viz., the *absolute value* and the *algebraic sign*.

Solve in the class the exercises that follow, to the title "*Multiplying Positive and Negative Numbers,*" page 77. Do not solve the exercises *for* the class but *with* the class, the class doing the work. In Problems 1 and 2, at bottom of page 77, write the results in the book for later use.

Make Exercise X as far as possible an oral class exercise.

Read and call for oral answers to the questions of Exercises 1–12, pages 77–78. The purpose of these exercises is to start the ideas of multiplication with a concrete multiplicand and an unsigned multiplier. Let pupils do the answering.

§ 55. The experimental exercises of page 79 should not be too long dwelt upon. The thought is that the turning bar furnishes an objective situation for the multiplicative use of the positive and negative signs to describe. The pupil needs only to see that following the ordinary law of multiplication adequately describes the behavior of the loaded bar. The work should be conducted orally and, if possible, with an apparatus before the class.

§§ 58–59–60 are to make more explicit the mathematical ideas exemplified in §§ 55–56–57.

§§ 61–62, to be treated orally, are to give a certain hold on the mathematics of turning-tendencies.

§ 63 furnishes a straight-line interpretation of the multiplication of signed numbers. It is not superfluous to attempt to underlay the laws of signed numbers with a foundation of clear ideas.

The ideas that are informally used through the five preceding exercises and exemplified in the experimental exercises with the turning bar are summarized into the laws of multiplication, first "in digits" and then "in letters." After the pupil has felt that these laws have a high descriptive value as records of the real behavior of the balanced bar, and has sensed their simple use in directed measurements along a line, the general *ab*-formulation of them will seem to him plausible, at least; and the attitude of mind of the learner toward them will be much more serious and more rational than if he first be given those laws as mere "definitions of multiplication."

Exercise 6 first focuses the pupil's mind on the question of obtaining a rule of practice for multiplication. Then after the pupil has tried by his own efforts to formulate such a rule, § **64** gives him a standard of excellence for getting this rule into finished form. This procedure, though a little time-consuming in the initial stages, more than remunerates in rapidity and steadiness of progress afterward.

Again, only part of the exercises under 3, page 84, may be needed. Pupils should, however, make drawings and obtain the results from the drawings, not from a memorized rule.

Exercise XI immediately applies the rule just made and gives a review of the arithmetic of fractional and mixed numbers under the appearance of algebra.

Dividing Positive and Negative Numbers

§ **65.** Division may now be given a direct treatment by basing it upon multiplication at once. Exercises 1 and 2 are to steady the pupil's first steps in using the division law by exhibiting the law through the simple numbers of arithmetic, that he may give his undivided attention to the sign-phase of the law, and Exercise 3 then generalizes to the use of this law with literal numbers.

Exercise 4 is intended to be an oral exercise to give a little practice and a modicum of steadiness in answering questions on the basis of the division-law, and Exercise 5 is to aid the pupil toward a statement of the law of signs for division. Exercise 6 requires him to attempt the statement for himself, and finally § 67 gives him a standard for perfecting his own statement. The exercises that follow immediately put this law into use. Exercise XII at once applies the division law and reviews division of arithmetical fractions. Have as many as possible of these exercises done orally.

Let the SUMMARY, page 88, now be assigned for home work, the class being told to learn the definitions of graphs, of algebraic sum, algebraic difference, and the laws for multiplication and for division, and to be ready to give examples, both with arithmetical and with literal numbers, to illustrate all of them. The next day go rapidly over the results of these attempts, clear away any confusion that remains, and see to it that the class as a whole understands these definitions and laws very fully.

CHAPTER V

BEAM PROBLEMS IN ONE OR TWO UNKNOWNS

Problems in One Unknown Number

§ **68.** Pupils have now studied rather carefully the laws of algebraic addition, subtraction, multiplication, and division, and have applied them to a considerable number of formal problems employing both arithmetical and literal numbers. They have a satisfactory first hold on these laws. It is but fair to the learner that he should have an opportunity to employ these new possessions on some sort of practical problems before he grows weary of a too extended experience with purely formal exercises. It will be found to be a valuable asset to the teaching of algebra to enable the pupil to bring a little of his high regard for the importance of practical matters to sanction the worth of his algebraic study, before his natural disposition to discredit empty forms has yet been developed. Prevention is both more economical and better than cure. It is believed that the simple exercises of this chapter will open a field that is very rich in algebraic demands that are simple and real enough to enable the pupil to see that even in a practical way algebra is highly useful, and perhaps worth the time and effort required to learn it.

It is intended that the pupil shall work through this chapter without a knowledge of any of the *formal methods* of solving simultaneous equations. It is of course assumed that he can transform equations with some intelligence by aid of the process axioms of § 21, page 26. His interpretations are, however, to be obtained from the behavior of the loaded bar. The idea here is that the importance of the equation as a tool for problem-solving, rather than formal processes of solving equations, shall be sensed. Common-sense, visual interpretations, and convictions are here wanted. Formal processes come later.

This chapter may be well assigned in accordance with the following suggested daily subdivisions:

First lesson: pages 89–94 inclusive.

Second lesson: pages 92–94, including Problem 11.

Third lesson: page 94, Problem 12, to page 97, § **73**.

Fourth lesson: pages 97–102 inclusive.

Fifth lesson: pages 103 and 104.

Sixth lesson: pages 104 and 105 to SUMMARY.

Seventh lesson: the SUMMARY and a light resurvey of the chapter.

The plan of doing some of the problems of this chapter with work of the next chapter is good.

EXERCISES 1 and 2, page 89, and the statements of § **69** are to lead up to the statement of turning-tendency as an algebraic sum, which is given in § **70**.

Calling the turning-tendency t, have the several answers to the parts of Exercise 1, page 89, written thus (obtaining the sign of the final product in each loading by raising the question: In which direction does the loaded bar turn?):

I. $t=(+3)\ (-6)=-18$

II. $t=(+2)\ (-3)=-6$

III. $t=(+3)\ (+2)=+6$

IV. $t=(x)\ (+3)=+3x$

V. $t=(x)\ (-2)=-2x$

VI. $t=(+3)\ (x)=+3x$

VII. $t=(-4)\ (-9)=+36$

VIII. $t=(-3)\ (-2)=+6$

IX. $t=(-12)\ (+3)=-36$

X. $t=(x)\ (-2)=-2x$

XI. $t=(x)\ (+2)=+2x$

XII. $t=(-2)\ (x)=-2x$.

EXERCISES 1-3, § 70, are intended to lead to the final working form of the law of leverage as stated in § 71.

Answers to EXERCISE 1, § 70

1. $(+3)(-6)+(-12)(+3)=-18-36=-54$
2. $(+2)(-3)+(-4)(-9)=-6+36=+30$
3. $(+3)(-6)+(+3)(+2)+(-12)(+3)=-18+6-36=-48$
4. $(+3)(-6)+(+3)(+2)+(-4)(-9)=-18+6+36=+24$.

Answers to EXERCISE 3, page 91

I. $3f-18=0, f=6$
II. $-18-2w=0, w=-9$
III. $-2r-3r+36=0, r=7\frac{1}{5}$
IV. $-2d+3d-6=0, d=6$
V. $-2l+3l+6=0, l=-6$.

§ 71, EXERCISE 1: $(+10)(-6)+(+5)(s+3)=0$, $-60+5s+15=0$, $5s=45$, $s=9$ and $s+3=12$.

EXERCISE 2: $-4(3w-15)+(+12)(+3)=0$, $-12w+60+36=0$, $12w=96$, $w=8$.

EXERCISE 3:

I. $-3(w+5)+60=0$, $-3w-15+60=0$, $w=15$, $w+5=20$

II. $7(t-3)+(-t)(-8)+(+13)(-3)=0$, $7t-21+8t-39=0$, $15t=60$, $t=4$, $t-3=1-t=-4$

III. $+3(m-5)-60=0$, $m-5-20=0$, $m=25$, $m-5=20$

IV. $-24+12k-8k=0$, $4k=24$, $k=6$, $3k=18$, $4k=24$

V. $(+39)(-4)+(5s)(+2\frac{1}{2})+(-13\frac{1}{8})(-4)=0$, $-157\frac{1}{2}+12\frac{1}{2}s+52\frac{1}{2}=0$, $12\frac{1}{2}s=105$, $25s=210$, $s=8\frac{2}{5}$, $5s=42$.

Practical Applications

1. Taking the fulcrum, F, for turning-point, we have

$$(-\tfrac{1}{2})(-1{,}800)+(+6)(-x)=0$$

or $6x-900=0$, $x=150$ (lb.).

2. With fulcrum, F, as turning-point, we have:

$$(-\tfrac{1}{4})(-1{,}800)+(+6\tfrac{1}{4})(-x)=0$$

or $6\tfrac{1}{4}x-450=0$

$$\tfrac{25}{4}x-450=0,\ 25x=1{,}800.\quad x=72 \text{ (lb.)}.$$

3. Call w the unknown weight, and take turning-point as before, then $(-\tfrac{1}{2})(-w)+(+6)(-200)+0$, $\tfrac{1}{2}w-1{,}200=0$, $w-2{,}400=0$, $w=2{,}400$.

4. Call the load l, then as above—
$(-2)(-l)+(+34)(-20)=0$, $2l-680=0$, $l=340$ (lb.).

5. $(-2)(-f)+(+34)(-68)=0$, $2f-2{,}312=0$, $f=1{,}156$ (lb.).

6. $(+\tfrac{1}{2})\ (-2{,}400)+(+6)\ (+m)=0$, $6m=1{,}200$, $m=200$ (lb.).

7. $(+2)(-966)+(+14)(+f)=0, f=138$ (lb.).

8. $(+1)(-966)+(+13)(+f)=0, f=74\tfrac{4}{12}$ (lb.).

9. $(+d)\ (-966)+(12+d)\ (+241\tfrac{1}{2})=0$, $2{,}898-966d+241\tfrac{1}{2}d=0$, $724\tfrac{1}{2}d=2{,}898$, $d=4$ (ft.).

10. $(+2)(-w)+(+14)(+140)=0, w=908$ (lb.).

11. By putting the fulcrum very near the middle of the rail and weighing the end. Show from $(+d)(-2{,}000)+(12+d)(+60)=0$, that if the weight is just a ton and the balances read just 60 lb., the distance, d, of the fulcrum from the middle is $\tfrac{36}{97}$ ft. Show that if either the weight is more, or the balances read less, d must be smaller. Calling W and R the weight and balance reading in pounds, show that $d=\dfrac{720}{W-R}$.

12. $(+2)(-860)+(+5)(+f)=0$, $f=144$ (lb.), $(+1)(-360)+(+4)(+f)=0, f=90$ (lb.).

13. $(+2)(-270)+(+4\frac{1}{2})(+f)=0$, $f=120$ (lb.).

14. $(+d)(-270)+(4\frac{1}{2})(+90)=0$, $-3d+4\frac{1}{2}=0$, $d=1\frac{1}{2}$ (ft.).

Exercise XIII: *Answers*

1. $-15-3l$	8. $-5x+7$	15. $15x$
2. $-850+85t$	9. 0	16. $-9y$
3. $211-31t$	10. 0	17. $15x-7$
4. $8\frac{1}{2}w-34$	11. 0	18. $-15x+7$
5. $90+42t$	12. 0	19. 0
6. $5x$	13. $-2kl-2kx$	20. 0.
7. $5x-7$	14. $-9x$	

§ **72.** 1. $(+5x)(-4)+(-56)(-1)+(+4)(+3x)=0, -20x+56+12x=0$, $8x=56$, $x=7$, $5x=35$, $3x=21$.

Read the first half of page 96 with the class.

2. (1) $(+5s)(-6)+(-90)(+2)+(+6s)(+6)=0$, $6s=180$, $s=30$, etc.

(2) $(+44)(-8)+(-6\frac{2}{3}t)(-3)+(+2t)(+8)=0$, $t=\frac{9}{79}$, etc.

3. Measuring all lever-arms from the left end,

(1) $(+3)(+3f)+(+13)(-800)+(+19)(+5f)=0$, $104f=104{,}000$, $f=100$, $3f=300$, and $5f=500$.

Also (2) $(+1)$ $(+7w)+(+7)(-45)+(+12)(-6w)+(+16)(+40)=0$, $7w=315-72w+640=0$, $65w=325$, $w=5$, etc.

4. (1) $(+3f)(-18)+(-100)(-8)+(+5f)(-2)=0$, $64f=6{,}400$, $f=100$, etc.

(2) $(-17)(+7w)+(-11)(-45)+(-6)(-6w)+(-2)(+40)=0$, $-83w+415=0$, $83w=415$, $w=5$, etc.

Problems in Two Unknowns

No technique is wanted here save the two laws of balance of bars.

Work orally and carefully with the class through page 97 and down to Exercise 7, page 98.

EXERCISE 7: Have the class substitute from half a dozen to a dozen values of S in equation (2) and calculate the corresponding values of R. Arrange these pairs of values and plot them. Solve Exercise 8 similarly.

See that the class gets the point of Exercise 9.

EXERCISE 10: The equations of this exercise are to be solved by plotting and not by algebra. The same is true of Exercise 11.

EXERCISE 12: Using the middle point as turning-point, we have

$(+x)(-4)+(-84)(-2)+(+y)(+4)=0$, or $-x-y=42$

and $x+y=84$ is easily seen to be true from the problem.

Solve the equations by graphing.

§ 75. Read Exercises 1, 2, and 3, and solve 2 and 3 with the class, and then require the class to solve Exercise 4.

§ 76. Have the class read and explain the text of Exercises 1 and 2 of this section. Work the remaining exercises through with the class, not telling more than is necessary.

§ 77. Be sure that the two italicized laws of this section are clearly comprehended.

Problems Applying Two Unknowns

EXERCISE 1: Taking the turning-point at the left end,

$(+2)(+F)+(+6)(-6)+(+8)(+R)=0$, or $F+4R=18$

from law of leverages, and $+F-6+R=0$, or $F+R=6$. Subtract the second from the first equation and obtain $3R=12$, or $R=4$, and since $F+R=6$, $F=2$.

CHECK: $2+4\cdot 4=18$, and $4+2=6$. Turning-point might be taken at either the right end or the middle.

2. Taking the turning-point at the left end (Fig. 102) and noting that $w=3\frac{1}{2}\times 4\times 18\times 48=12{,}096$.

From the **law of leverages:** $(+3)(+F)+(+9)(-12{,}096)+(+12)(R)=0$, or

$$F+4R=36{,}288 \qquad (1)$$

and from the **law of forces** $(F-12{,}096+R=0)$,

$$F+R=12{,}096 \qquad (2)$$

Subtract (2) from (1) $3R=24{,}192$, or $R=8{,}096$, and from (2) $F=4{,}032$.

Taking the turning-point at right end, with $w=12{,}096$; **Law of leverages,** $(-15)(+F)+(-9)(-12{,}096)+(-6)(+R)=0$, or

$$-5F+2R=36{,}288, \text{ and from} \qquad (1)$$

Law of forces, $2F+2R=24{,}192$ (2)

Subtracting (2) from (1) $3F=12{,}096$, or $F=4{,}032$ and from $F+R=12{,}096$, $R=8{,}064$, as before.

Taking the turning-point at the middle, $w=12{,}096$; **Law of leverages,** $(-6)(+F)+(0)(-12{,}096)+(+3)(+R)=0$, or

$$-2F+R=0, \text{ and from} \qquad (1)$$

Law of forces as before,

$$F+R=12{,}096 \qquad (2)$$

and subtracting (2) from (1), $3F=12{,}096$, or $F=4{,}032$, and from (2) $R=8{,}064$ as before.

Clearly the turning-point may be taken at any one of the three points, but the same turning-point *must be used throughout the solution.*

3. $w=40\times 60=2{,}400$.

Taking the turning-point at the left end,

$(+1\frac{1}{2})(+F)+(+5)(-2{,}400)+(+7\frac{1}{2})(+R)=0$, or

(1) $F+5R=8{,}000$, and $F-2{,}400+R=0$.

(2) $F+R=2{,}400$, and subtracting (2) from (1), $4R=5{,}600$, or $R=1{,}400$, and from (2) $F=1{,}000$.

This exercise also may be solved by taking the turning-point at either the right end or at the middle, as was done above.

4. From the **law of leverages** $(+x)(-5)+(-170)(-3)+(-3)(0)+(+y)(+3)=0$, or

$$x-y=102 \qquad (1)$$

From **law of forces** $(+x-170-30+y=0)$

$$x+y=200 \qquad (2)$$

whence adding (1) and (2) $2x=302$, and $x=151$, and $y=49$.

5. Turning-point at middle of span of bridge.

$(+L)(-7\frac{1}{2})+(-450)(-2\frac{1}{2})+(-1{,}000)(0)+(R)(+7\frac{1}{2})=0$.

Multiplying and simplifying, $L-R=150$, and from law of forces, $+L-1{,}450+R=0$, or $L+R=1{,}450$, and adding, $2L=1{,}600$, or $L=800$.

From $L+R=1{,}450$ and $L=800$, $R=650$.

6. Turning-point at middle of span,

$(+L)(-10)+(-600)(-6)+(-2{,}400)(0)+(-800)(+5)+(+R)(+10)=0$, or

(1) $R-L=40$, and from $L-600-2{,}400-800-R=0$.

(2) $R+L=3{,}800$.

From (1) and (2) $2R=3{,}840$, $R=1{,}920$, and $L=1{,}880$.

7. Using middle point as turning-point,

$(+L)(-10)+(-450)(-8)+(-450)(-4)+(-450)(-1)+(-2{,}400)(0)+(-450)(+6)+(+R)(+10)=0$,

or simplifying,

(1) $L-R=225$, and from $L-450-450-450-2{,}400-450+R=0$.

(2) $L+R=4{,}200$; adding $2L=4{,}425$, $L=2{,}212\frac{1}{2}$, and from (2) $R=1{,}987\frac{1}{2}$.

8. From the law of forces we have at once (1) $L+R=7{,}500$, and from the law of leverages,

$(+L)(-10)+(-2{,}500)(-6)+(-5{,}000)(0)+(+R)(+10)=0$

or simplifying (2) $R-L=1{,}500$.

Equations (1) and (2) give $2R=9{,}000$, $R=4{,}500$, and $L=3{,}000$.

9. From the law of forces (1) $R+L=4{,}500$, and from law of leverages, turning-point at middle:

$(+L)(-10)+(-2{,}500)(-1)+(-2{,}000)(0)+(-3{,}000)(+5)+(+R)(+10)=0$

or (2) $R-L=1{,}250$. From (1) and (2) $R=4{,}375$ and $L=3{,}125$.

10. From the law of forces we have at once $w-25-75=0$, or $w=100$, and law of leverages gives (taking turning-point at middle and calling d the distance from middle to load):

$$(+75)(-4)+(-w)(-d)+(+25)(+4)=0$$

or $-300+wd+100=0$, or $wd=200$. But $w=100$. Hence $d=+2$; that is, the load hangs 2 ft. from the middle toward the man lifting 75 lb.

11. $(+2f)(+d)+(-240)(+6)+(+f)(+12)=0$, or $2fd+12f=1{,}440$.

Also $+2f-240+f=0$, from law of forces, or $3f=240$, and $f=80$. Putting this f in the equation

$$2fd+12f=1{,}440$$

and simplifying, $d+6=9$, or $d=3$.

The spike should be placed 3 ft. from the end of the log.

12. First equation $(+2f)(0)+(-240)(6-d)+(+f)(12-d)=0$, where $f=80$ as before.

Dropping $(+2f)(0)$ and dividing through by 80,

$$(-3)(6-d)+(12-d)=0$$

or $-18+3d+12-d=2d-6=0$, or $2d=6$, $d=3$, as before.

Exercise XIV

(1) $s=8$, $t=6$	(5) $t=7$, $w=5$
(2) $f=2$, $w=5$	(6) $x=4$, $y=8$
(3) $k=4$, $l=2$	(7) $l=-4\frac{14}{31}$, $r=7\frac{11}{31}$
(4) $x=50$, $w=4$	(8) $x=8\frac{8}{31}$, $y=15\frac{29}{31}$

Require pupils to learn the substance of the statements 3, 4, and 5 of the *summary* very thoroughly, giving more attention to securing an understanding of them than to memorizing them *verbatim*.

CHAPTER VI

PROBLEMS IN PROPORTION AND SIMILARITY

LESSON 1: *through page 109*

There should be cross-lines painted on the blackboard. If not, the teacher should rule such lines with crayon before the recitation hour. Make the parallel lines $1\frac{1}{2}$ inch apart and rule every fifth one heavy.

State to the class Problem 1. Take 5 small squares to represent each yard walked. Locate points O and M, and draw line O M. Ask class what the line O M represents in the problem, also how by this line we can calculate the number of miles M is distant from O. Lay a ruler along O M, note the length. Lay the ruler horizontally and note the number of squares included in this length ($37\frac{1}{2}$). As each 5 squares stand for 10 miles, there are twice as many miles as small squares, i.e., the $37\frac{1}{2}$ squares represent 75 miles.

This introduces *ratio* as a mathematical relation and a means of calculation. The relation of 5 squares to 10 miles is called the ratio of 5 to 10, or $\frac{5}{10}$. The relation of 1 sq. to 2 miles is the ratio of 1 to 2 or $\frac{1}{2}$, and these ratios are the same, i.e., the number of squares is $\frac{1}{2}$ the number of miles.

Problem 2: Have the pupils notice that the same figure is used but that the ratio of squares to miles is changed, 5 small squares now standing for 15 miles. Ask what the *ratio* is (1 to 3, or $\frac{1}{3}$). Every square now represents 3 miles. Ask how many miles line O A represents ($67\frac{1}{2}$); how many A M represents (90), and O M ($112\frac{1}{2}$).

In Problem 3 let the teacher draw in class on the board the required lines, and find O M to measure $12\frac{1}{2}$ cm. (or large squares), more exactly $12\frac{45}{100}$. Have class tell the ratio of cm. to yd. ($\frac{1}{12}$) and state how to find the number of miles $12\frac{1}{2}$ cm. represents (150). Assign Problem 4 for home work.

Taking 1 cm. to 1 mi., the distance apart is 12.2 mi., approximately.

Problems 5, 7, 8, and 9 can be done rapidly in class as exercises in finding the scale of a drawing, or the ratio of the line unit to the number of feet.

Problem 6 shows the application of the scale to making the drawing, and can be given as home work, together with Problems 9, 10, 11.

Page 109 furnishes practice in methods of writing ratios, and in their meaning and use. It is important that pupils get a clear understanding of these things.

§ **79** may be given orally by the teacher.

§ **80,** the definition of ratio, should be stated and explained by the teacher. It may be explained by the teacher that as a *quotient* expresses how many times the divisor is contained in the dividend, so the ratio of two numbers means how many times (either integral or fractional) the first number contains the second, e.g., to state that the ratio of two magnitudes is 6 to 3 ($\frac{6}{3}$, or 2) means that the first is two times the second, or twice as large. To say that the ratio is 3 to 4 (or $\frac{3}{4}$) means that the first is $\frac{3}{4}$ of the second, or $\frac{3}{4}$ as large.

Problem 1 may be written out at home after one or two parts are done in class.

Problems 2 to 8 can be done orally in class to develop the *meaning* of the *ratio* of one number to another as explained above.

LESSON 2: *through page 112*

Page 110, Problems 1 to 5, may be used as a class laboratory exercise. With protractors to construct the angles have all pupils draw the triangles specified in the problems.

The triangles will not be of the same size but if carefully drawn, will be of the same shape. When measured, the ratio of any two corresponding sides of two triangles having equal angles will be found to be (*approximately*) the same as that of two other corresponding sides. Have the pupils understand that such lines and their measurements are always approximate and get them to draw lines and measure them with as little error as possible. **§ 82** states that such triangles as were drawn in **§ 81** with corresponding angles equal will be of the same shape, and are called *similar triangles*.

Problem 1 may be done on the board by the teacher, changing the unit from inches to feet.

Problems 2, 3, and 4 may be assigned as home work.

§ 83 calls attention to the facts which are formulated in **§ 84**.

Problems 1 to 12 show some applications of the facts regarding similar triangles. Problem 1 may be worked on the board by the teacher.

If $A=12$, and $a=4$, the ratio of corresponding sides of the larger triangle to the smaller is 3, B is $3\times b$, or 12.

Problem 2 may be made a class laboratory exercise; 3, 4, and 5 may be assigned as home work.

LESSON 3: *page 113 to* § 85, *page 115*

The teacher may ask the questions in Problem 6.

To prove the triangles similar it must be assumed that B A = A H and K = A, and H = B. The ratio of corresponding sides is 3.

$$x=3\times 1\tfrac{2}{3},\text{ or } 5$$

Let the teacher conduct the class through Problems 7 and 8. Explain the meaning of Problem 9 and assign 9 to 12 as home work. It is evident that in similar triangles the shortest side of one triangle *corresponds* to the shortest side of the other, and so on. Assign 14 and 15 as home work. Problem 10 the teacher will draw on board and develop. Impress again the need of accuracy in drawing measurements. In Problem 15 have all the parallels on the triangle at the same time. Measure and find ratio of parts included between the same parallels.

LESSON 4: § 85, *page 115, to* § 91, *page 120*

The teacher will lead the class to see that § **85** is a formal statement of the facts developed in the preceding problems. Have pupils *learn* these two theorems.

Work Problems 1 and 2 with the class. The ratio $\frac{AB}{AC}=\frac{21}{35}$ or $\frac{3}{5}$. Thus, A E, or x, is 5.

The solutions of 1 and 2 depend on the fact that the corresponding angles of the triangles are equal (see § **81**). Teacher should make clear the meaning of § **86** and have pupils memorize it for future use. Assign Problem 1 as home work. Teacher should develop Problem 2 on board, selecting those ratios more easy to use, and show that the result as stated in § **87** is the converse of § **85**, second part.

The teacher may read §§ **88** and **89** and make clear the meaning of the *bearing of a line*. Have pupils do Problem 1 orally, and Problem 2 on the board. Teacher should explain § **90** and pupils do Problem 1 orally. Teacher may do Problem 2 on the board, assign 3, 4, 5, and 6 for home work.

LESSON 5: § 91 *to* § 94, *page 123*

Let the teacher call attention to the transit, Fig. 131. Degrees are marked on the vertical wheel, for getting angle of elevation and depression, and on the horizontal wheel for getting bearings, etc. The various thumb screws are to make minute changes of level or direction of the telescope.

The angle measurer described in § **91** may be constructed by the teacher and shown to the class, the pupils to be encouraged to make their own, and use them in original problems like those on pages 121 and 122. The two small objects in the corner are *levels* to determine when the board is horizontal.

Assign Problems 1 and 2 for home work, to give practice in scale drawing, and in Problem 2 to show that if two triangles have two sides and the included angle in one triangle equal to those of the other the remaining side in the one triangle is equal to that in the other. Work Problem 2 with the class (they on paper) (B A $= 199\frac{7}{8}$ ft.).

The similarity of the triangles is shown by § **87** and § **86**, supposing that the triangle in the drawing to be placed on the surveyed triangle, or use second paragraph of § **83**.

Each pupil may be assigned one problem in each of §§ 91, 92, 93.

Lesson 6: *to Problem 6, page 127*

The teacher should draw Fig. 139 on the board and show that the areas are $7h$ and $4h$: the ratio is $\frac{7h}{4h}$ or $\frac{7}{4}$. The ratio of the bases is also $\frac{7}{4}$; note that this is because the altitudes are equal.

Go over orally with the class Problems 2 to 7. In Problem 8, page 124, have the class tell different methods of finding x: (1) by solving the equation $4x=40$, (2) by reducing the ratio to have denominator 8, (3) by reducing each ratio to denominator 4, $(\frac{1}{2}x)=5$, $x=10$.

Some of Problems 9 to 13 may be assigned as home work. In taking up § **95** call attention to the fact that equating the ratios in Problems 8 to 13 formed proportions. The second paragraph of § **95** may be stated in the form: *lines* are proportional if the *numbers expressing* their lengths form a proportion.

Problem 2. (1) is shown in § **94,** 1, Fig. 139.

(2) May be shown by Fig. 139, taking EH and AD as *bases*.

(3) Construct III with $a=a$, and $b=b$.

Then $\frac{\text{I}}{\text{III}}=b/b'$, and $\frac{\text{II}}{\text{III}}=a/a'$.

Divide first equation by second, member by member, $\text{I}/\text{II}=ab'/a'b$.

(4) Diagonals drawn on figure for (3) will furnish triangles that will prove (4) in the same way as (3).

If we assume that the area of a rectangle is equal to the product of base and altitude, (3) may be demonstrated thus: $A=ab$, $A'=a'b'$. Divide first equation by second equation $A/A'=ab/a'b'$.

Problem 3 may well be explained by the teacher and he should call attention to the law that a product is divided by any number when one of its factors is divided by that number. In getting $\frac{1}{6}$ of the area, we can divide the 4 by 2, and the 15 by 3 and get integral results. Let the class for home work solve as many as they can of Problems 4–11.

Lesson 7: *Problem 12, page 127, to Problem 11, page 128*

Let § **96,** Problems 1 and 2, be gone over orally with the class, and the teacher test the class on them the next day. Note that only those expressions in which the *ratios* are *equal* are proportions. Thus in Problem 2, (3) and (5) are not proportions.

Let § **97** be memorized for future use.

Let Problem 1 be oral work. Problem 2 is proved by the multiplication axiom, the fundamental law for the reduction of equations containing fractions. Multiply each fraction by bd.

Develop Problem 3 and emphasize the importance of the knowledge of ratio in the solution of practical problems.

First method: As the parts are in the ratio 2 to 3, $2x$ and $3x$ will represent them, giving the equation $3x+2x=85$, $x=17$, $2x=34$, $3x=51$.

Second method: Let x be one part, then $85-x$ is the other. As they are in the ratio 2 to 3 then $\frac{x}{85-x}=\frac{2}{3}$. Multiply each side by $3(85-x)$, or equate the product of the extremes and the product of the means (see § **97**), and get by either process $3x=170-2x$, then $x=34$ and $85-x=51$, the required parts. Check: $\frac{34}{51}=\frac{2}{3}$, $34+51=85$.

Problem 4. $3x+4x+5x=84$: $12x=84$, $x=7$ whence $3x=21$, $4x=28$, $5x=35$, the required numbers. Check: $\frac{21}{3}=\frac{28}{4}=\frac{35}{5}$, or $7=7=7$, and $21+28+35=84$, or $\frac{21}{28}=\frac{3}{4}$, $\frac{28}{35}=\frac{4}{5}$, $\frac{21}{35}=\frac{3}{5}$.

Problem 5. Let x be the required number. Then $\frac{12+x}{30-x}=\frac{5}{10}=\frac{1}{2}$, $x=2$.

Problem 6. $\frac{5x-12}{6x-12}=\frac{3}{4}$, $x=6$, $5x=30$, $6x=36$.

Problem 7. Let x be number of cm. in the first part, then $30-x$ is the number in the second part. $\frac{x}{30-x}=\frac{2\frac{1}{2}}{3\frac{3}{4}}$ or $\frac{10}{15}$ or $\frac{2}{3}$, then $x=12$, $30-x=18$.

Second method: $2\frac{1}{2}x+3\frac{3}{4}x=30$. $6\frac{1}{4}x=30$, $x=4\frac{4}{5}$, $2\frac{1}{2}x=12$, $3\frac{3}{4}x=18$.

Problem 8. Let x=number of inches in the length of A D. Then the number of inches in D C$=x+2$. Whence $\frac{x}{x+2}=\frac{3}{6}=\frac{1}{2}$, $x=2$, $x+2=4$.

Problem 9. Lay off A D = 2 cm. and D C = 4 cm. With compasses set at 3 cm. and with A as center strike an arc. With C as center and compasses set at 6 cm. intersect the first arc as at B, connect B and D and with protractor measure angles A B D and D B C. They should be equal. This illustrates a law of geometry that a line from the vertex of an angle of a triangle *bisects* the angle if the line divides the opposite side into parts proportional to the sides including the angle.

Problem 10. Let x be the number of degrees in one angle, then $90-x$ is the number in the complementary angle. Hence $\frac{x-6}{90-x+6}=\frac{2}{7}$, then $x=26$, $90-x=64$. Check: $\frac{26-6}{64+6}=\frac{2}{7}$, also $26+64=90$.

Assign these problems as home work, and have pupils put them on the board next day. Require the *check* as an essential part of the work.

Lesson 8: *Problem 11, page 128, to* § **98**, *page 129*

Give the class drill on the problems of § 97 in forming the proportions or equations in the solution of the equations, and in the checking of the results.

Problem 11. Let x and $180-x$ be the number of degrees in the angle S. Then $\frac{2x}{8(180-x)}=\frac{1}{2}$ or $\frac{x}{4(180-x)}=\frac{1}{2}$; $x=120$, $180-x=60$.

Check: $\frac{2 \cdot 120}{8 \cdot 60}=\frac{240}{480}=\frac{1}{2}$, also $120+60=180$.

Problem 12. $2x+3x+4x=360$, $x=40$, $2x=80$, $3x=120$, $4x=160$.

Check: $\frac{80}{2}=\frac{120}{3}=\frac{160}{4}$ or $40=40=40$. Also $80+120+160=360$.

Problem 13. $x+2x+3x=180$. (The sum of the angles of a triangle is $180°$.) $x=30$. The 3 angles are $30°$, $60°$, and $90°$.

Problem 14. $2x+5x=90$. $x=12\frac{6}{7}$, $2x=25\frac{5}{7}$, $5x=64\frac{2}{7}$.

Problem 15. (5) $20+x-x^2=6-x-x^2$, $2x=-14$, $x=-7$.

Check: $\frac{12}{-4}=\frac{9}{-3}$, $-3=-3$.

(10) $x^2=64\times 400$. Take sq. root of the equation, $x=8\times 20$ or 160.

Check: $\frac{20\cdot 20}{1}=\frac{400}{1}=\frac{400}{1}$, $400=400$.

(11) $\frac{x^3}{3}=\frac{27}{192}=\frac{9}{64}$, $64x^3=27$. Extract cube root, $4x=3$, $x=\frac{3}{4}$.

Check: $\frac{27/64}{3}=\frac{27}{192}$.

Multiply each side by 3, $\frac{27}{64}=\frac{27}{64}$, or divide numerator and denominator of first fraction by 3, $\frac{27/192}{1}=\frac{27}{192}$, $\frac{27}{192}=\frac{27}{192}$.

(12) $(x+5)^2=4\times 16$. Take square root, $x+5=2\times 4$, $x=3$.

Check: $\frac{8}{4}=\frac{16}{8}$, $2=2$.

Lesson 9: *page 129 through page 131*

§ **98** follows from § **97**, second paragraph, for one pair of factors of the equal products is made the means, and the other pair the extremes.

Note that Problem 1, equation (1) is ***not a proportion*** for the numbers are ***products***. (6) to (9) are not proportions.

The teacher may work with the class Problems 1 to 6, also one proportion for 6, and assign 6 and 7 for home work. § **99** is a formal statement or ***theorem*** derived from the preceding work. Give class practice on the six parts of Problem 1, and assign some for home work. Use § **100** as a short class exercise, and assign as home work.

In § 101 the teacher should lead the pupils to see that the first proportion states the ratio of two sides of one triangle = ratio of the two corresponding sides of the larger triangle. This is true because B E || C D, and the triangles are similar.

In proportion 2, § 101, the means have been alternated, andthe result shows that the ratio of two corresponding sides of the two triangles equals the ratio of two other corresponding sides.

In Problem 2, note that the polygons are equilateral. In proportion 2, by alternating the means the corresponding sides are seen to be proportional.

Problem 3. The truth of proportion 1 comes from the fact that opposite sides of a parallelogram are equal. Then take proportion 1 by alternation. Let § 102 be class work by teacher and pupils.

LESSON 10: *Variation, page 132, to Problem 5, page 133*

Test the class on LESSON 9. Teacher will draw diagram (Fig. 151) on the board in the presence of the class, step by step, and have class answer the questions. D E is a straight line. The angles of the triangles are mutually equal. The ratios are each 2 to 1.

Problem 2. The triangles are similar because mutually equiangular, and the corresponding sides are proportional. (The ratio of each vertical side to the hypothenuse is $\frac{4}{5}$ found by the law of the right triangle.)

The ratio of the distance line to the time line is 2 to $1\frac{1}{2}$; doubled, trebled, etc. The distance line varies (changes) as the time line varies.

Teacher develop with the class Problem 3. $d=2t$, $d=$ 2, 4, 6, 8 . . . . 20. The ratio of d to t is 2. (Divide each side of equation $d=2t$ by t.)

Assign Problem 4 as home work.

The cost doubles, trebles, etc., as the weight doubles, trebles, etc. The cost varies as the weight. The ratio of cost to weight is $3\frac{1}{2}$.

Assign Problems 5, 6 for home work.

§ 102, Problem 1: Teacher develop Problems 1 and 2. Let horizontal side of small square be the base, and vertical side the area. The area is then 3 times the base in each triangle. A varies as the base.

Problem 2. $A=\frac{bh}{2}$, $A=b\times\frac{h}{2}$, and if h (and therefore $\frac{h}{2}$) is constant, $\frac{A}{b}=\frac{h}{2}$, i.e., the *ratio* of A to b is constant, and A varies as b.

Assign Problem 3 for home work; also Problem 4. State to class what Problem 4 calls for, but do not tell what the ratio is. (The ratio is $3\frac{1}{7}$ approximately. See Problem 6, page 134.)

LESSON 11: *page 133 through page 135*

Read Problem 5, and get class to make the equation. $\frac{A}{b}=k$, or $A=kb$, that is, the area = base times a constant number. Then $\frac{27}{3}=k=9$ or the area is 9 times the base. When the area is 54, $54=9\times b$, therefore the base is 6.

Assign Problems 6, 7 . . . . 11 for home work.

Explain **§ 103.** This means that $\frac{a}{1/b}=k$ a constant, or simplified $ab=k$.

Assign Problem 2 for home work. Develop Problem 3. Explain the statement about compression of gas. Make clear that it is not simply *illuminating* gas that is meant but the state of matter represented by air, hydrogen, steam, etc.

Take ice, water, and vapor of water (or steam) to illustrate the three states of matter—solid, liquid, and gaseous. The equation is $vp=k$, hence $4\times 3=k$, or $k=12$. Then when the pressure is 6 lb., $v \cdot 6=12$, or the volume is 2. Assign the rest of these problems for home work.

In Problem 8 explain that the time a pendulum vibrates once depends on its length. Make an impromptu pendulum of string and a small weight (bunch of keys), and have class observe the difference in time due to different lengths of string. This relation of time and length is stated in the problem, and is expressed exactly in the equation $\frac{t}{\sqrt{l}}=k$, and $t=k \cdot \sqrt{l}$ when the length is expressed in inches and time in seconds.

According to Problem 8, $1=k\sqrt{39.2}$ or $1=k\times 6\frac{1}{5}$ (approximately), $k=\frac{5}{31}$. For a pendulum vibrating in 2 seconds $2=\frac{5}{31}\sqrt{l}$, $\frac{62}{5}=\sqrt{l}$. $l=153.76$ or the pendulum is 153.76 inches long.

LESSON 12: *page 136*. SUMMARY

Review the pupils on the definitions in black-faced type, and on the theorems in italics. Also give problems to test their grasp of the *subjects* and of the *processes* treated in the chapter.

CHAPTER VII

PROBLEMS ON PARALLEL LINES; GEOMETRIC CONSTRUCTIONS

LESSON 1: *through page 139*

The teacher should introduce the work to the class, and have the class draw the lines and make the measurements of Problems 1 to 4. Have the class regard § **105** as a provisional definition of parallel lines.

Teacher show how to solve Problems 1 and 2. Assign Problems 3 and 4 for home work. Require pupils to fasten to their work the paper angles used. Ask pupils to note and remember theorems (*a*), (*b*), and (*c*) of § **106**.

Teacher will draw Fig. 157 on the board, and have the pupils answer questions in Problems 2 to 6 as in 1. Assign Problems 7 to 13 as home work.

LESSON 2: *page 140 to* § **109**, *page 142*

§ **107**. Draw Fig. 159 on the board, and drill pupils on naming the various angle pairs. Drill on Problem 1.

Draw Fig. 160 on board, and have pupils give proofs of Problems 2 and 3.

Assign Problems 4, 5, 6, and all of § **108** as home work.

Problem 1. Page 141, yxD is supplement of x, $y=yxD$ (§ **106**, Problem 8), y is supplement of x. Similarly for y and z.

Problem 2. z is supplement of y. x is supplement of y, $z=x$.

Problem 3. $x+y=180°$. $w+z=180°$. $x+y+w+z=360°$.

LESSON 3: § 109, *page 142, to middle of page 143*

As a general class exercise Theorems I and II should be shown to follow from the previous problems on parallel lines, and remembered for future use.

§ 110, Problems 1 and 2, can be shown to follow from previous problems on parallel lines. Assign Problems 3 to 9 as home work, say certain different groups of three or four problems to each pupil.

Problem 5 follows from § 106 (*b*).

Problem 7. $t+y+w=180°$ (a straight angle) but $t=x$ and $w=z$ (§ 106, Problem 8). Substituting x for t and z for w, then $x+y+z=180°$.

Problem 8. In the diagram $w+y=2\text{RA}$

$$u+x=2\text{RA}$$
$$\underline{t+z=2\text{RA}}$$

Adding, exterior $\angle s$+interior $\angle s=6\text{RA}$

Sum of interior $\angle s=2\text{RA}$

Subtracting, sum of exterior $\angle s=4\text{RA}$

Problem 9. $a+b+c=2\text{RA}$ (see Problem 7)

$c+z=2\text{RA}$ (straight angle)

$a+b+c=c+x$ (equality axiom)

Subtract c, $a+b=x$.

LESSON 4: *Algebraic exercises, page 143 to* § 111, *page 145*

This lesson should serve to fix in mind the laws of angles formed by parallel lines and a transversal, and give practice in the solution of equations of the first degree.

Work Problem 1 with the class and assign Problems 2 to 11 for home work, say a certain (different) five or six to each pupil.

Problem 1. $2(4x-3)=79+3x$ (corresponding angles of

// lines). Then $8x-6=79+3x$; $x=17$. Check: $79+3x=79+51=130$, $8x-6=136-6=130$. The adjacent angle $=50°$.

Problem 2. $3x-5=5(x-7)$ (alternate interior ∠*s* of // lines).

LESSON 5: *page 145 through Problem 13, page 146*

§ 111. Problems on Constructions

Show on board how to do Problem 1. Then have class practice at their desks with compasses and straight-edge till they can draw at a given point on a straight line (drawn in any direction on the paper) a perpendicular to the line.

Assign Problems 2 to 7 for home work. Direct the pupils to make the construction lines plain and not to erase them.

Teacher show how to do Problem 8 and make clear that either point G or point F may be used, but only one is needed.

Problem 9. A B must be prolonged.

Give Problems 10 to 13 as home work.

In Problem 10 explain that the word altitude used here means the altitude line drawn from the vertex of the angle.

LESSON 6: *to Problem 24, page 148*

Teacher show class how to do Problem 15. Have class practice at their desks bisecting given lines drawn in any direction on the paper.

Assign Problems 16, 17, and 18 (or two of them) for home work. Teacher show Problem 19 and have class practice.

Assign Problems 20 to 23 for home work.

Problem 20. The bisectors should meet at a common point.

Problem 21. The sum of the halves of the two parts of a straight angle equals a right angle.

Problem 22. The halves of equals are equal, i.e., $b=a$, but if corresponding angles are equal the lines are parallel.

Problem 23. a and b are halves of supplementary angles or $\frac{1}{2}$ of 180°, or 90°, as the sum of x, a, and b is 180°, then $x=$ 90° and the lines are perpendicular.

LESSON 7: *to Algebraic Exercises, page 150*

Teacher show Problem 24, and have class practice.

Problems 25 to 30 give practice in drawing an angle equal to a given angle, and some or all may be assigned for home work.

Problem 26. Draw three angles at the same vertex as adjacent angles. The sum is 180°, and the exterior sides should form a straight angle, or a straight line.

Problem 27. Let A B C be any given triangle. Draw indefinite line X Y. Lay off a segment A′C′ = A C. Construct at A′, X A′D = ∠C and lay off A′D = C B. At D draw an angle A′D E = ∠C, and at C′ an ∠A′C′E = C. A′C′E D is the required parallelogram, as the opposite pairs of sides are parallel.

LESSON 8: *to Summary, page 151*

The nine problems give practice in the solution of equations of the first degree, formed by applying the laws of geometrical figures. Explain one or two if found necessary and assign all as home work.

Problem 3. $2y+2z=3y-z$, and $2y+2z+20=180°$. Solving, $z=20$ and $y=60$. The obtuse angle $=160°$.

Problem 4. $3x+5x=180$, $x=22\frac{1}{2}$. $8y+2y=180$, $y=18$. x-angles are $67\frac{1}{2}°$ and $112\frac{1}{2}°$. y-angles are 144 and 36.

[illegible]em 5. The angle not lettered = angle x. (Alter-[illegible] of parallel lines.) $x=45$ and $y=135$.

Problem 6. From similar triangles $\frac{x}{14}=\frac{x+12}{24}$, $x=16.8$, $\frac{y}{14}=\frac{y+10}{24}$, $y=14$.

Problem 7. $\frac{\cancel{6}^{1}}{5x+2}=\frac{\cancel{12}^{2}}{12x-1}$, $12x-1=10x+4$, $x=2\frac{1}{2}$, $\frac{3y+3}{\cancel{6}^{1}}=\frac{5y+9}{\cancel{12}^{2}}$, $6y+6=5y+9$, $y=3$.

Problem 8. $\frac{\cancel{9}^{3}}{\cancel{15}^{5}}=\frac{9+a}{15+b}$, $45+3b=45+5a$, $b=1\frac{2}{3}a$. $c=12+1\frac{1}{3}a$.

The area of the trapezoid is found by subtracting the small triangle from the whole triangle of the figure 194.

The whole triangle is $\frac{c(9+a)}{2}=\frac{(12+\frac{4}{3}a)(9+a)}{2}=$ $(6+\frac{2}{3}a)(9+a)=54+12a+\frac{2}{3}a$.

The area of the small triangle is 54, area of trapezoid is $12a+\frac{2}{3}a$.

Lesson 9: Summary

Make this a review of the chapter.

Test pupils on all definitions, theorems, and constructions, and on problems depending on them, by oral, blackboard, and written work.

CHAPTER VIII

THE FUNDAMENTAL OPERATIONS APPLIED TO INTEGRAL ALGEBRAIC EXPRESSIONS

Notice the title of this chapter. By the "fundamental operations" is meant algebraic addition, subtraction, multiplication, and division. The teacher who believes the child is to get a knowledge of these operations by performing meaningless mechanical exercises in adding, subtracting, etc., at a stage so early that he cannot yet have any real concept of algebraic numbers, much less any real feeling as to why such numbers should be added, subtracted, etc., will of course feel this chapter to be placed rather late in the year. The stereotyped plan is against it, and most teachers have had this training only under this plan. It is reflecting on no one and is perhaps even asserting nothing new to state that the promptings of early training have more to do with the teachers' notion of what algebra is and of why and how it should be taught than do all other influences combined. A few words by way of justifying the placing of this chapter here and of showing the reason for the plan of treatment may be helpful. This development of the mathematical intuitions and powers of first-year high-school pupils is based frankly upon the doctrines that:

I. The work should appeal to the interest of the learner and should court the sanction of his judgment that it is worth while to master it.

II. This can be done only by appealing much more to his understanding than to his memory and thereby securing a *real insight* into the meanings and reasons for what he is called upon to do.

III. The more practical and untechnical parts of the beginnings of algebra and geometry furnish the best approach to mathematical science, because they create the most natural and the friendliest attitude of mind toward mathematical study.

The pages that precede this chapter have reconnoitered the fields of generalized arithmetical number and of rational mensuration, have extended by pictured representation and common-sense illustration the notion of numbers to include the negative, have enriched the extended notion by arithmetical and geometrical uses in problems treated more or less informally, and have carried the evolution of ideas to the point where technical algebraic operations are needed for further advance. Up to this point the learner's arithmetical knowledge and skill has been the main reliance; he is ready now to go beyond with understanding and profit.

This chapter should be divided into nine lessons about as follows:

LESSON 1: from § **112**, page 153, to EXERCISE XVI, page 156.

LESSON 2: from § **115**, page 156, to § **119**, page 159.

LESSON 3: from § **119**, page 159, to § **121**, page 162.

LESSON 4: from § **121**, page 162, to § **123**, page 166.

LESSON 5: from § **123**, page 166, to § **125**, page 169.

LESSON 6: from § **125**, page 169, to § **127**, page 172.

LESSON 7: from § **127**, page 172, to EXERCISE XXXI, page 176. Solve one-half of EXERCISE XXXI in class and one-half as home work.

LESSON 8: from EXERCISE XXXI, page 176, to EXERCISE X, page 179.

LESSON 9: from EXERCISE X, page 179, to end of chapter.

LESSON 10: Review of addition, subtraction, multiplication, and division, each pupil solving in class or at home and bringing work to class at least one moderately difficult example in every one of the four fundamental operations. It would be well to make LESSON 10 a written recitation.

If time is short LESSON 10 may be omitted.

In every case it will be better to omit certain indicated exercises within the assignment than to shorten the assignment. Pupils will learn more to go over the work at about the rate of progress indicated by the foregoing assignments than to go more slowly. It is serious educational waste, not to mention loss of interest, to mistake slow progress for thoroughness. Overmature ideals of thoroughness have no place in a first-year high-school class in mathematics. If teachers of mathematics could bring themselves to believe and to practice this, secondary mathematical teaching would at once undergo marked improvement. The exercise with which § 112 opens shows under Solutions I and II the advantage of grouping addends in algebraic fashion into sums. Solution II surely impresses the pupil with the great gain of adding first the coefficients and then multiplying by the common factor. Have some pupil read aloud in class and actually verify all the multiplications and additions of Solutions I and II.

EXERCISE I, at the bottom of page 153, exhibits a form of the problem in which the solution comes out in perfect algebraic form as a polynomial, all of whose terms are then combined into a single term, $440x$, which the pupil will feel to be completely known, so soon as it is known what number x stands for, and also the pupil will hardly miss the idea that this solution is applicable to any such problem, no matter what the particular price of the ticket. He can sense the interpretation of $440x$ as the value of the 440 tickets each of the value x cents. That is, he senses $440x$ as a number, rather than as a mere symbol, or as a mysterious "quantity."

Take up Problems 2, 3, and 4 orally with the class. Problem 2, page 154, puts conditions in a form to impress the pupil with the naturalness of associating the amounts into separate terms and then adding the terms. One factor, the 23.8, pervades the entire problem.

$50 \cdot 23.8$ cents $+ 17 \cdot 23.8$ cents $+ 700 \cdot 23.8$ cents $+ 6150 \cdot 23.8$ cents, or $(50+17+700+6150) \cdot 23.8$ cents $= 6917 \cdot 23.8$ cents $=$ \$1,646.246.

Problem 3 again clothes a problem in algebraic garb.

Solution: $6s+8s+4s+6s+10s=(6+8+4+6+10)s=34s$. Answer $34s$ ft.

Problem 4. $6y+8y+10y+12y+14y=50y$. Answer: $50y$ yards.

§ **113.** Cite from problems 2–4, or have pupils cite, illustrations of the meaning of "similar with respect to a factor."

EXERCISE 1. The parts of this exercise should be done orally to exercise the pupil at once in the application of the definition just given.

Answers: (1) Terms are similar with respect to x; coefficients are 4, -7, 20, and -35.

(2) Terms similar with respect to x, coefficients a, -25, $-b$, and 46.

(3) Terms similar with respect to x, coefficients a, $-b$, $-c$, and d.

(4) Terms similar with respect to a, to x, or to xa.
Coefficients 1. As to factor a, coefficients $2x$, $3x$, $-7x$, $-5x$.
2. As to factor x, coefficients $2a$, $3a$, $-7a$, $-5a$.
3. As to factor xa, coefficients 2, 3, -7, and -5.

(5) Terms similar with respect to q^2, coefficients $-3p$, $6t$, $-8r$, and $12s$.

(6) Terms similar with respect to x, or z, or xz;

Coefficients 1. Common factor x, coefficients $4az$, $-7cz$, $-5dz$, and $9ez$.

2. Common factor z, coefficients $4ax$, $-7cx$, $-5dx$, and $9ex$.

3. Common factor xz, coefficients $4a$, $-7c$, $-5d$, and $9e$.

(7) Terms similar with respect to m, coefficients ab, $-pq$, $-xy$, and $-dc$.

(8) Terms similar with respect to ab, coefficients $3a$, $-1\frac{1}{2}b$, $4\frac{1}{2}ab$, and $-1.5a^2$.

(9) Terms similar with respect to x^2, coefficients ax, $3\frac{1}{2}y$, $-7.5t$, and $-17bx$.

(10) Terms similar with respect to at^2, coefficients $-4t$, $-b$, $3x$, and $7.2y$.

Problem 2, page 155, Rule: *Add the coefficients and write the common factor after the sum.*

EXERCISE XV (May be given as home work)

Polynomial	Simplified Form
1. $15a-7a+18a=$	$26a$
2. $-18x^2-12x^2+15x^2-3x^2=$	$-18x^2$
3. $2\frac{1}{2}ab+3\frac{1}{3}ab-4\frac{1}{2}ab-5\frac{2}{3}ab=$	$-4\frac{1}{3}ab$
4. $27abc-35abc+10abc-2abc=$	0
5. $3(a+b)-4(a+b)+12(a+b)=$	$11(a+b)$
6. $-8(x^2+y^2)-24(x^2+y^2)+17(x^2+y^2)=$	$-15(x^2+y^2)$
7. $-3\frac{2}{5}(pr-q^2)+5\frac{2}{3}(pr-q^2)-4\frac{7}{10}(pr-q^2)=$	$-2\frac{13}{30}(pr-q^2)$

8. $18(mp-3s)^2-15(mp-3s)^2-37(mp-3s)^2+14(mp-3s)^2=-20(mp-3s)^2$

9. $a(x+y+z)-b(x+y+z)-c(x+y+z)+d(x+y+z)=(a-b-c+d)(x+y+z)$

10. $15ax^2-7bx^2+8dx^2-5cx^2=(15a-7b+8d-5c)x^2$.

§ 114. The exercise with which the section opens is to illustrate addition when the terms are *dissimilar.* This is to enable the pupils to feel the reason for holding the separate terms apart as in algebraic polynomials. Do I as a general class exercise and assign II, III, and 2 for home work.

Problem 1. I. $(+3)(-17)+(-5)(-14)+(-8)(-5)+(+12)(+19)=-51+70+40+228=+287$

II. $(+5)(-a)+(+12)(-b)+(+14)(+b)+(-6)(+a)=-5a-12b+14b-6a=-11a+2b$

III. $(-8)(-x)+(+10)(-y)+(-15)(-z)+(+9)(+w)=8x-10y+15z+9w$

Problem 2. *Answer,* $8(a+b+c+d)$ steps.

Exercise XVI

Do one or two with the class and assign certain others for home work.

1. $5s^2t-12x^2y+7x^2y-3s^2t=2s^2t-5x^2y$
2. $-27\frac{2}{3}ab+18\frac{1}{3}cd+15\frac{1}{5}ab+14\frac{2}{5}cd=-12\frac{7}{15}ab+32\frac{11}{15}cd$
3. $5ax-3x+x=5ax-2x$
4. $9a^2b^2-3c^3y^3+4a^2b^2-4c^3y^3-3a^2b^2=10a^2b^2-7c^3y^3$
5. $3mp^2-8mp^2+5a^2x-3a^2x-4mp^2+2a^2x=-9mp^2+4a^2x$
6. $-4st-t+3t+5st=st+2t$
7. $ar^2p-r^2p-br^2p+pr^2-cpr^2=(a-b-c)pr^2$
8. $27(a+b)^2+4c+15a-12c-15(a+b)^2-8a=12(a+b)^2-8c+7a$
9. $a(a-b)+b(a-b)-c(a-b)=(a+b-c)(a-b)$
10. $a^2(a+b)-2ab(a+b)+b^2(a+b)=(a^2-2ab+b^2)(a+b)$
11. $-6\frac{1}{3}(a+b)+\frac{1}{2}(a^2+b^2)-\frac{1}{4}(a^2+b^2)+8\frac{2}{3}(a+b)+4\frac{2}{3}(a+b)+6\frac{1}{2}(a^2+b^2)=6\frac{3}{4}(a^2+b^2)+7(a+b)$

12. $r^3(r+p)-3r^2(r+p)+3r(r+p)+(r+p)=(r^3-3r^2+3r+1)(r+p)$.

The pupil has now learned through use and application how to add monomial terms, both similar and dissimilar, and how to simplify them. He is ready to apply this knowledge to polynomials.

§ **116.** This section begins again with a simple concrete problem in which the pupil can feel the real meaning of the individual terms and of the polynomials as wholes, as well as the sum of all the added polynomials. Do not think that he understands merely because he can combine the coefficients correctly. Have him tell the meaning of everything he does in this problem. Teachers too readily mistake mechanical imitativeness for insight.

See that every step of the solution of 1 is understood and that the form $(+27x^2-15xy+18y^2)+(-12x^2+30xy-3y^2)$ is conceived as *the sum;* but that $15x^2+15xy+15y^2$ is the *simplified sum.* It may be well also to have him recognize $15(x^2+xy+y^2)$ as *another form* of the simplified sum.

Exercise XVII

Have pupils do all they can orally, writing results only.

Answers:

1. $7a-7b+7c$
2. $-a+3c-2b+13d$
3. $a^2+2b^2-3c^2$
4. $15xy-12x^2+29y^2$
5. $-14a+13b-20c$
6. $-k+2l+4m$
7. $-19a^2b^2+15\frac{3}{8}a^4-3\frac{1}{2}a^3b$
8. $\frac{1}{40}x^2-2xy-3\frac{1}{4}y^2$
9. $4(a^3+b^3)-2(a^2+b^2)$

10. $\frac{5}{6}(a+b+c)+\frac{7}{12}(a-b+c)-\frac{9}{20}(a+b-c)+\frac{11}{30}(-a+b+c)=\frac{3}{5}a+\frac{1}{6}b+2\frac{7}{30}c$

11. $4p^3+3p^2+4p-4$

12. $3x^2+3y^2$

13. $-29a^2b+92a^2c+97c^2b-22b^2c$

14. $9a^3(a+b)-5a^2b(a^2+b^2)-4ab^2(a^3+b^3)$

15. $(lr+t)+(l-u)+2tv$.

§ 117. It will be well here to review rapidly and orally the exercises on page 76 to remind the pupil that subtraction may always be readily changed into addition by changing the signs in the terms of the subtrahend from + to − and from − to +, and adding the changed subtrahend to the minuend. A pupil will repeat this as a rule long before he grasps its significance to consist in the fact that after reversing the sign of the terms of the subtrahend, the *laws of addition then* apply. Have the pupil here see again that by reversing the signs in the subtrahend and adding the changed subtrahend to the minuend *he does obtain a number which, added to the given subtrahend, gives the minuend.* The pupil has difficulty in freeing himself from the arithmetical notions of "take away" and "less than" as meaning subtraction. Take a little care here to insist that algebraic subtraction means finding *a number* (called the difference) *which, added to the subtrahend, gives the minuend.* This gives what is called the *difference* between them. Exercises 1–8, rightly solved, enable the pupil to *sense* this definition a little. Notice the black-face **A** and **S** of the example, meaning add, and subtract.

Answers:

1. $32x$	4. $-23(p+q)$	7. $-2\frac{1}{15}(x-y)$
2. $-11a$	5. $9m^2px$	8. $-18m^2(a-2b^3)$.
3. $3\frac{2}{3}ab$	6. $6x(1+5a^2y)$	

§ 118. Read carefully with the class the matter of § 118 at the bottom of page 158. Have the pupil think that while

$(+5x)-(-7x)$ is a difference, the form, $+12x$, is the simplified form of the difference. All the exercises of page 159 are *differences* as they stand; but the work for the pupil is to *simplify them.*

EXERCISE XVIII

Answers:

1. $-7ab$
2. $-21a^2rl^2$
3. $-81p^3f^3$
4. $100s^2gh$
5. $-1.1hk^2v$
6. $12.9\,p^3q^2s^4$
7. $-12(a+b)$
8. $6(t^2-f^3)$
9. $(a-6)(m^2+g)$
10. $(-x-7)(v^2-s^2)$
11. $(a-b)(x+y)$
12. $(-2\frac{1}{2}a-4e)(x^2-ay)$
13. $-6\frac{14}{15}(a+b+c)$
14. $(4.5s^2-3.4t^2)(v^2-h^2)$.

Notice the capital **S** and **A** standing at the ends of the lines under the formal arrangements at the bottom of page 159. They stand for the words "Subtraction" and "Addition" respectively.

Page 160, Answers: 1. (1) $4ab$; (2) $2a^2b^2+2a^4+8ab^3$; (3) $x^3+6x^2y+4y^3$

2. (1) $-4ab$; (2) $-2a^2b^2-2a^4-8ab^3$; (3) $-x^3-6x^2y-4y^3$

3. $2c^3-a^2c+2d^3$

4. $x^4+y^4-4x^3y-4xy^3+6x^2y^2$.

The idea of EXERCISE 5 is to lead the pupil to *use his head* when it is easier and more expedient than to use the pencil. Pupils are in danger of forming the habit of wasting time "in mere puttering" when they are allowed or encouraged to use pencil and paper for mere trivial steps in transformations. They should be led to form the habit of looking as deeply as possible into a problem before beginning it, and then doing as much as they can with reasonable effort mentally, merely setting down results of the mental steps. Pupils of ordinary ability will easily omit the intermediate step of the last part of the solution of EXERCISE 5, setting down $-17a^3+14a^2b-2ab^2-21b^3$ at once. Continual attention to this practice will enhance both the interest and the insight in the work. This matter should not be overdone, as of course it may be.

EXERCISE XIX

Do one or more with the class and assign others for home work.

Answers:

1. $33x^2-26xy-6y^2$
2. $-a^2x+12bxy+30b^2y-30axy$
3. $m^2pq+m^3p-m^2q^2$
4. $2\tfrac{7}{15}abc-11\tfrac{5}{6}a^2b-15\tfrac{11}{15}b^2c-\tfrac{2}{3}c^2a$
5. $36y^4-100x^4-20x^2y^2$
6. $10y^3-3x^3-7x^2y-4xy^2$
7. $2\tfrac{1}{2}l^3-4\tfrac{1}{4}lm^2+7\tfrac{1}{3}m^3-8\tfrac{1}{2}l^2m$
8. $3.4\,v^3s^4-4v^4s^3+13v^6s^2+4v^7s$
9. $42x^3-6x^2y-8xy^2+12y^3$
10. $11.6t^4-13\tfrac{1}{2}f^3-3\tfrac{1}{2}t^3f-5\tfrac{2}{3}t^2f^4$
11. $40g^3h-56g^4-8gh^3-18g^2h^2-25h^4$
12. $7.5k^2m^2l+2.9k^3ml-6.49km^3l+3.6m^2l^2-6kml^3$
13. $(3a+2b-4)x^3-(4s+3t)yz^3+5z^3$
14. $(3m+4n)uv-(5m+3)v^2+(4m-9n)u^2$.

§ **120.** Problem 1. A − sign before a polynomial in parenthesis, ( ), requires the sign of every term of the polynomial to be reversed, *if the parenthesis is dropped.*

2. If a + sign precede a polynomial in parenthesis, the parenthesis with its + sign may be simply omitted, no changes being made in the signs of the terms of the polynomial.

Exercise XX

Do one or more with the class and assign others for home work.

Answers:

1. $4-5+st^2-4=st^2-5$
2. $8\frac{1}{2}-4k^3+3k^3-5\frac{1}{2}=3-k^3$
3. $16e^2+42e^2+3e^2+2-50e^2-3=11e^2-1$
4. $2f-6f+3g-4f+2g-4f=5g-12f$
5. $4t^2-t^2+3t^3-3t^2+t^3=4t^3$
6. $9x-5y+6y+7z+7y-4z=9x+8y+3z$
7. $-3a^4+4a^3-3a^3+5a^2-4a^2-3a=-3a^4+a^3+a^2-3a$
8. $7.5p^3-3.4p^3+4.2p^3+1.6p^2+3.4p^2-4.5p=8.3p^3+5p^2-4.5p$
9. $3k^3-2p^2-k^3+p^2+r-p^2+r+k^3+r=3k^3-2p^2+3r$
10. $105s^2-14st-3t^2+4st-2t^2-5s^2+2st=100s^2-8st-5t$
11. $-3rl+3r^2-3l^2-4r^2+5rl-2l^2+2r^2+2l^2=2rl+r^2-3l^2$
12. $-5\frac{1}{2}k^2+3.4kh-2h^2-5.4kh+3k^2-2.4h^2-7k^2-2\frac{1}{2}kh+4h^2=-9.5k^2-4.5kh-0.4h^2$.

Multiplication of Monomials

§ **121.** The first exercises are to impress the pupil with the reality of meaning of algebraic multiplication through the use of factors that are concrete numerical dimensions of conceivable geometrical figures. The areas are the concrete interpretations of the products. The work called for by the paragraphs in fine print is by no means waste of time. In 2, 3, and 4, do some in class orally and assign others for home work.

Answers:

Problem 1. (1) 35 sq. in.	(6) m^3 sq. yd.
(2) 225 sq. cm. (or cm^2.)	(7) ab^3c sq. mi.
(3) $7a$ sq. ft.	(8) $6y^5$ sq. km. (or km^2.)
(4) $9\frac{1}{2}x$ sq. m. (or m^2.)	(9) $20x^3y^3$ sq. ft.
(5) pq sq. rd.	(10) $14\frac{2}{5}a^2b^2c^2$ sq. in.

Problem 2. 225 sq. ft., $67\frac{6}{25}$ sq. m., a^6 sq. cm., $56\frac{1}{4}b^2c^2$ sq. in., x^{12} sq. in., a^6b^4 sq. mi., $12\frac{1}{4}p^8q^{10}$ sq. m., $49x^6y^{14}z^2$ sq. ft., $18.6624a^4b^{10}c^4$ sq. dm., $81m^2n^4p^6q^8$ sq. cm.

Problem 3. $421\frac{7}{8}$ cu. cm., $337\frac{1}{2}$ sq. cm.; $8x^6$ cu. ft., $24x^4$ sq. ft., $151\frac{19}{27}a^3b^3$ cu. in., $170\frac{2}{3}a^2b^2$ sq. in., $13.824y^9$ cu. m., $34.56y^6$ sq. m., $64x^3y^3z^3$ cu. cm., $96x^2y^2z^2$ sq. cm., $175.616x^6y^9z^6$ cu. mi., $188.16x^4y^6z^4$ sq. mi., $35\frac{162}{343}x^{12}$ cu. rd., $64\frac{38}{49}x^8$ sq. rd., $1{,}728a^6b^6$ cu. yd., 864 sq. yd., $91\frac{1}{8}a^{15}$ cu. ft., $121\frac{1}{2}$ sq. ft., $3{,}375p^9q^6r^3$ cu. m., $1{,}350p^6q^4r^2$ sq. m.

Problem 4. (1) 462 sq. in., 540 cu. in.

(2) $(34\frac{2}{7}+16\frac{20}{21}x)$ sq. ft., $17\frac{1}{7}x$ cu. ft.

(3) $16(x+y)+2xy$ sq. mi., $8xy$ cu. mi.

(4) $2(a^5+a^6+a^7)$ sq. rd., a^9 cu. rd.

(5) $(4x^5+10x^8+20x^7)$ sq. yd., $10x^{10}$ cu. yd.

(6) $(31.5p^7+112p^8+144p^9)$ sq. cm., $252p^{12}$ cu. cm.

(7) $2(x^3y^3z^2+x^3y^2z^3+x^2y^3z^3)$ sq. ft., $x^4y^4z^4$ cu. ft.

(8) $(12abp^2m^2+16acm^2p^2+24bcp^2m^2)$ sq. in., $24abcm^3p^3$ cu. in.

(9) $(20x^{18}+\frac{14}{15}x^{20}+\frac{3}{7}x^{22})$ sq. mi., x^{30} cu. m.

(10) $(42x^5y^5+4\frac{2}{7}x^7y^6+3\frac{3}{5}x^6y^7)$ sq. cm., $9x^9y^9$ cu. cm.

Problem 6. By adding exponents of the factors. No.

Problem 7. *abc* cu. ft., *abc* cu. ft., *abc* cu. ft. The volumes are all the same. No effect on value of product.

Problem 8. $7xyz$, $7xyz$, $7xyz$, $7xyz$, $49xyz$, $49xyz$, $49xyz$, $343xyz$.

Problem 9. (1) and (3), (5) and (7), (9) and (11).

Problem 10. First and second are equal; but third is different in value from the others.

Problem 11 (see § 64, page 84). A product is multiplied by a number, by multiplying *any one* of the factors by the number. This is true for both + and − factors. Changing the sign of a factor is the same as multiplying the factor by −1, and this multiplies the whole product by −1, or reverses its sign. Changing the sign of another factor reverses the sign of the product again, or brings it back to what it was originally. Hence, *changing the sign of any even number of factors does not alter the value of the product; but changing signs of an odd number of factors reverses the sign of the product. Illustrate.* We may also say an even number of negative factors gives a + product and an odd number of negative factors gives a − product.

Exercise XXI

Have pupils ready to report answers to from 10 to 15 of the exercises.

Answers:

1. $-3{,}400$
2. $+a^9$
3. $-a^9$
4. $30m^6t^3$
5. $220x^7y^5z^5$
6. $360a^7b^7c^7$
7. $32p^5q^5r^5$
8. $-125.5256a^3b^4m^4n^4$
9. $-28x^2p^4q^4r^4$
10. $t^{15}s^{14}u^{25}$
11. $-3x^{10}y^{10}z^{11}$
12. $\frac{2}{5}a^{m+3}$
13. $-\frac{1}{6}t^{x+5}$
14. $-s^{k+l}$
15. $(a+b)^8$
16. $(r-s)^{n+3}$
17. $(l^2-f^2)^{k+h}$
18. $(x^2+y^2)^9$
19. $12(x-y)^{10}(x+y)^8$
20. $(a-b)^{12}$
21. $-\frac{3}{4}x^9y^{12}(x+y)^{14}$
22. $-a^{32}$
23. $-2025p^{31}$
24. $9x^{30}$.

Division of Monomials

§ 122. Do a few orally. Problem 1. 16 ft., 9 ft., 12 ft., 18 ft., 2 ft.

Problem 2:

(1) $a=2t$ ft.
(2) $a=x^3$ in.
(3) $b=\frac{4}{5}x^2yz^2$ m.
(4) $b=21.5d^2fc$
(5) $a=9px^5l^2$
(6) $a=7l^4k^3c^3$
(7) $b=v^3u^4c^5$
(8) $a=0.7h^4x^2$
(9) $a=1.4zx^3p^4$
(10) $b=5x^2y^2z^2$.

Problem 3. ***Give the base an exponent equal to the difference of its exponents in dividend and divisor.***

Problem 4. See page 86.

Exercise XXII

Assign, say, the even numbers for home work.

Answers:

1. $-5a^2b^2c^2$
2. $1.3a^4x^2y^2z^2w^3$
3. $-5tu^2$
4. $4a^2b^2c^2$
5. $-5\frac{1}{3}p^2q^3$
6. $-3{,}5x^{11}y^8z^{18}$
7. $4a^{x-2}b^3c^3$
8. $1.3x^ay^bz^c$
9. $-\frac{4}{7}p^{2n}q^nr^{4n}$
10. $-2(a+b)$
11. $47(x^2-y^2)^2$
12. $-0.07(a^2+2ab+b^2)^4$
13. $+3(p-q)^3$
14. $2x^3y(a^2+b^2)^2$
15. $-4\frac{2}{7}(a+b)^{2x}(a-b)^{2x}$
16. $8x^{2a}y^{3b}(x+y)^a$
17. $-6p^{a-b}q^{b-c}r^{c-a}(p+q-r)$
18. $\frac{7z^2}{y^2x^2}$
19. $\frac{0.05c^6}{a^3b^2}$
20. $\frac{121}{25(x+y)^2(x-y)^3}$.

Multiplication and Division by Means of Exponents

§ 123, page 167, Problem 1.

2^4	3^8		390,625
3^7	3^5	3^{11}	
2^{10}	2^7	3^6	
5^8	5^{11}	5^6.	

Exercise XXIII

Do two or three with the class, and have them verified by actual multiplication. Assign others for home work.

Answers:

1. $3^8 \cdot 3^4 = 3^{12} = 531{,}441$ (See table, p. 167)
2. $5^7 \cdot 5^4 = 5^{11} = 48{,}828{,}125$
3. $2^9 \div 2^4 = 2^5 = 32$
4. $3^6 \cdot 3^5 = 3^{11} = 177{,}147$
5. $2^{11} \div 2^7 = 2^4 = 16$
6. $3^{11} \div 3^6 = 3^5 = 81$
7. $5^8 \div 5^5 = 5^3 = 125$
8. $5^{12} \div 5^7 = 5^5 = 3{,}125$
9. $(3^5)^2 = 3^{10} = 59{,}049$
10. $(5^6)^2 = 5^{12} = 244{,}140{,}625$
11. $2^{12} \div 2^6 = 2^6 = 64$
12. $5^8 \cdot 5^4 = 5^{12} = 244{,}140{,}625$
13. $\frac{3^{24}}{3^{20}} = 3^4 = 81$
14. $5^5 \cdot 5^7 \cdot 5^{10} = 5^{23} = 11{,}920{,}928{,}955{,}078{,}125$
15. $(2^8)^3 = 2^{24} = 16{,}777{,}216$
16. $\frac{3^{25} \cdot 3^{12}}{3^{22} \cdot 3^{15}} = \frac{3^{37}}{3^{37}} = 1$
17. $\frac{5^{15} \cdot 5^{18}}{5^{23}} = 5^{10} = 9{,}765{,}625$
18. $\frac{2^{24} \cdot 2^{10} \cdot 2^{18}}{2^{21} \cdot 2^{13}} = 2^{18} = 262{,}144$.

Multiplication and Division of a Polynomial by a Monomial

§ **124.** Problem 1. $b(5+a+x+7\frac{1}{2}+4\frac{1}{3}+9.5+y^2)$ and $5b+ab+bx+7\frac{1}{2}b+4\frac{1}{3}b+9.5b+by^2$

Problem 2. $a^2(a^3+3ab^2+3a^2b+b^3)$ and $a^5+3a^3b^2+3a^4b+a^2b^3$

$a(a^3+3ab^2+3a^2b+b^3)=a^5+3a^3b^2+3a^4b+a^2b^3$

Problem 3. By multiplying each term of the polynomial by the monomial and connecting the resulting products with the signs determined by the law of signs for multiplication.

Each term of the product is the product of corresponding terms of the polynomial by the given monomial.

Exercise XXIV

Do one with the class. Assign others for home work.

Answers:

1. $5^{x^3}-15x^2y+25xy^2$
2. $20a^3b-40a^2b^2+80ab^3$
3. $-3.6abx^4+6.9abx^3y+12x^4yz-2.1bx^3yz$
4. $-4a^3p^5m^4+5a^4p^3m^5+7\frac{1}{2}a^2p^5m^5+4.5a^4p^4m^4$
5. $12a^2+15ab-15b^2$
6. $11a^4-22\frac{1}{2}a^3b-30\frac{5}{6}a^2b^2-8ab^3+9\frac{1}{2}b^4$
7. $12x^3-14x^2y+22xy^2-12y^3$
8. $-15.9a^2-53ab+17b^2$
9. $90ax-35bx$
10. $-20a^2-118ab+62ac$
11. $4x^3-72x^2$
12. $-10y^3-36y^2$
13. $x=48$
14. $x=-15$
15. $x=-2$
16. $a=5$.

§ 125. Do two or three problems with the class.

Answers:

Problem 1. x and x^3; x and x^2y; and x and $4xy^2$, or x^2 and x^2; x^2 and $3xy$, and x^2 and $4y^2$.

Problem 2. Sketch a rectangle of

(1) Base x (or $4x-3y$) and altitude $4x-3y$ (or x).

(2) Base $5x$ or $x^2-2xy+3y^2$, and altitude $x^2-2xy+3y^2$ or $5x$, respectively. Other solutions also possible.

(3) Base $7a^2b^2c^2$ or $2b-3a+5c$, and altitude $2b-3a+5c$ or $7a^2b^2c^2$, respectively. Other solutions also possible.

(4) Base $3m$ or $m^4-4m^2n+2n^4$, and altitude $m^4-4m^2n+2n^4$ or $3m$ respectively. Other solutions also possible.

(5) Base $5x^2$ or $3x^2-2x+1$, and altitude $3x^2-2x+1$ or $5x^2$ respectively. Other solutions also possible.

Problem 3. By factoring a monomial out of each term of the polynomial: No; e.g., $x^2-3x-28$ cannot.

Exercise XXV

Develop several with the class. Assign others for home work.

Answers:

1. $5(a-2b)$
2. $17x^2(1-17x)$
3. $2x(8x-ab)$
4. $a(x+y-z)$
5. $xy^2z^2(14x-7x^2y+8)$
6. $15m^2n^2r^2(4n-3mr+6m^2r+6m^2n)$
7. $3a^2b^2c^2(1.5bc^2-0.4ac^2-0.9a^2c+3ab^2)$
8. $4a^2x^3(1-3ax-5a^2)$
9. $p(ap^2+3px-4q+15axq)$
10. $x(x^2-pqx+p^2q^2-pq^3x^2)$.

§ **126.** *Answers:*

1. $5ab+9a^2-7b^2$
2. (1) s^2+s^5
   (2) $4xy^2-5y^3-2x^3$
   (3) $p+l-r+t$
   (4) $u^2+v^3+w^4-z$
   (5) $a+b-c+d$

3. By dividing each term of the polynomial by the monomial. It is not.

Exercise XXVI

Answers:

1. $4a^2-3ab+6b^2$
2. $-2x+3x^2y-x^2$
3. $-6y^3+7x^3y^5z$
4. $5b^5-6a^2b-7a^7b+3a^2b^2c^4$
5. $-3amn^2+4bm^2n-10abmn+7a^2mn$
6. $-5p^{13}q^7-12p^{25}q^{11}+3a^3p^{12}q^{13}-9abcp^3q^7$
7. $-6b+10ab^2c+14b^2-12b^4$
8. $2-10yz$
9. $5abc^3-3a^2-15c^3+7a^3bc-a^2b^2c^2$
10. $-3bdqs+5bcqt+6cdts.$

Exercise XXVII

Answers:

1. $3a^3bc-2ab^3c+\frac{3}{2}abc^3$
2. $\frac{7}{2}x^5y^7-14x^6y^8+21x^4y^{10}-31\frac{1}{2}x^7y^7$
3. $-60a^4b^2c+100a^3b^3c+30a^3b^3c^3-75a^3b^2c^2$
4. $5ab^2c^3d^2(5a^2b^2c^4d^6-3a^2b^3c^3+6d^7)$
5. $7xyz(3x^4y^8z^{14}-2x^5z^{13}+5y^9z^{11}-6x^9y^{14})$
6. $(x-y)(4a-3b+4c)$
7. $5x^ay^bz^c-6x^{2a}y^{2b}z^{2c}-7x^{3a}y^{3b}z^{3c}$
8. $-3(x+y)^2+5(x+y)^5+7$
9. $-3p(r^2+s^2)(r+s)+6p(r^2+s^2)^2+1$

10. $c=5$
11. $c=-4$
12. $c=-10$.

Multiplication of Polynomials

Refer the class to Fig. 22, page 24. Do 1, 2, and 3 orally, and assign the rest for home work.

Answers:

1. $(x+y)(a+b)=ax+bx+ay+by$

2. (1) $a(c+d)+b(c+d)$
   (2) $ac+ad+bc+bd$

3. (1) $(a+b)c+(a+b)d$
   (2) $ac+bc+ad+bc$

4. (1) $3(6+2)+4(6+2)$; $3\cdot 6+3\cdot 2+4\cdot 6+4\cdot 2=56$
   (2) $5(a+3)+a(a+3)$; $5a+3\cdot 5+a^2+3a=a^2+8a+15$
   (3) $(a+b)a-(a-6)2$; $a^2+6a-2a-12=a^2+4a-12$
   (4) $(a+b)b+(a+b)c$; $ab+b^2+ac+bc$
   (5) $(x+y)y+(x+y)z$; $xy+y^2+xz+yz$
   (6) $(a+b)a^2+(a+b)b^2+(a+b)2ab$; $a^3+a^2b+ab^2+b^3+2a^2b+2ab^2=a^3+3a^2b+3ab^2+b^3$
   (7) $(x-y)x^2+(x-y)2xy+(x-y)y^2$; $x^3-x^2y+2x^2y-2xy^2+xy^2-y^3=x^3+x^2y+xy^2-y^3$
   (8) $(a-b)a^2-2ab(a-b)+(a-b)b^2$; $a^3-a^2b-2a^2b+2ab^2+ab^2-b^3=a^3-3a^2b+3ab^2-b^3$
   (9) $(x^2-2xy+y^2)x+(x^2-2xy+y^2)y$; $x^3-2x^2y+xy^2+x^2y-2xy^2+y^3=x^3-x^2y-xy^2+y^3$
   (10) $(5+x)25+(5+x)10x+(5+x)x^2$; $125+25x+50x+10x^2+5x^2+x^3=125+75x+15x^2+x^3$

5. Multiply every term of one polynomial by every term of the other, and write the several products, each with

its resulting sign, as a polynomial. Simplify the resulting polynomial.

§ **128.** 1. See answers to 4 above. Discuss 1 and 2 with the class.

EXERCISE XXVIII

1. $a^2x^2-b^2y^2$
2. $a^4+a^2b^2+2ab^3+a^3b+b^4$
3. p^3-p^2-5p-3
4. $30a^5-40a^3b^3-15a^2b^2+20b^5$
5. $18.4a^3bc^2-18.2a^3b^2c-6a^3b^3$
6. $a^4+a^3b-ab^3-b^4$
7. $16x^4+8x^3y-2xy^3-y^4$
8. $p^4+p^2r^2+r^4$
9. $81k^4+9k^2t^2+t^4$
10. $10m^3-38m^2s+51ms^2-30s^3$
11. $6a^2b-9ab^2+5abc+15b^2c-25bc^2+12ac^3-16ab^2c+$ $8a^3c-18abc^2+24ab^3-12a^3b$
12. $81x^4+9x^2y^2+18xy^3+4y^4$
13. $300a^2b+16b^3$
14. $4f^2-24fh-9h^2$.

§ **129.** Develop some with the class, and assign others.

Answers:

1. $m^2-2mx+x^2$ $\quad p^2-2pk+k^2$ $\quad x^2-2xy+y^2$
   $a^2+2ab+b^2$ $\quad p^2+2pk+k^2$ $\quad m^2+2mx+x^2$

2. $x^2+2px+p^2$ $\quad k^2-2kr+r^2$ $\quad r^2-2ry+y^2$
   $v^2+2av+a^2$ $\quad r^2+2ry+y^2$ $\quad k^2-2kp+p^2$
   $x^2-2px+p^2$ $\quad k^2+2kn+n^2$ $\quad g^2-2gf+f^2$

3. Write the square of the first term, plus, or minus, the double product of the two terms (according as the given binomial is a *sum* or a *difference*), plus the square of the second term.

EXERCISE XXIX

Answers:

1. $x^2+10x+25$
2. $y^2-14y+49$
3. $16a^2-8a+1$
4. $\frac{1}{9}x^2+\frac{1}{3}xy+\frac{1}{4}y^2$
5. $.16a^2-.24at+.09t^2$
6. $9p^2x^2-24pqxy+16q^2y^2$
7. $.36x^2y^2z^2+1.2xyz+1$
8. $(a+b)^2+6(a+b)+9$
9. $(a-b)^2+2c(a-b)+c^2$
10. $(x-y)^2-\frac{8}{5}(x-y)+\frac{16}{25}$
11. $(k+l)^2-3s(k+l)+2.25s^2$
12. $(3a-4b)^2-4c(3a-4b)+4c^2$
13. $(ax+by)^2-2cz(ax+by)+c^2z^2$
14. $(5pq-3kl)^2-8pl(5pq-3kl)+16p^2l^2$
15. $(2\frac{1}{3}ab^2c-\frac{6}{7}abc^2)^2+14a^2bc(2\frac{1}{3}ab^2c-\frac{6}{7}abc^2)+49a^4b^2c^2$
16. $(a+b)^2+2(a+b)(c-d)+(c-d)^2$
17. $11a^2+14b^2$
18. $(4t-5x)^2+2(4t-5x)(5s-4y)+(5s-4y)^2$.

§ **130.** *Answers:*

1. a^2-x^2 $\quad p^2-s^2$
   r^2-16 $\quad y^2-b^2$
   t^2-q^2 $\quad 225-z^2$

2. k^2-h^2 $\quad 16a^2-9b^2$
   $4s^2-169$ $\quad \frac{1}{4}p^2-\frac{1}{9}r^2$
   m^2-v^2 $\quad .09x^2l^2-.04y^2k^2$
   t^2-u^2 $\quad a^2b^2c^2-x^2y^2z^2$

3. The product of the sum by the difference of the same two numbers is the difference of their squares.

Exercise XXX

Discuss the best combinations of factors.

Answers:

1. $4a^2-9b^2$
2. $9x^2-16z^2$
3. $25p^2-9s^2$
4. $\frac{1}{4}r^2-\frac{1}{9}t^2$
5. $\frac{4}{25}x^2y^2-\frac{4}{9}x^2z^2$
6. $16a^2b^2c^2-9$
7. $1-25p^4q^4$
8. $1-a^4$
9. u^4-v^4
10. $(a+b)^2-c^2$
11. $(e-b)^2-h^2$
12. $(m+n)^2-t^4$
13. $(2a-b)^2-9s^2$
14. k^4-l^4
15. g^8-f^8
16. $1-p^8$
17. $(3x^4-2y^3)^2-x^2y^2$
18. $(\frac{2}{3}x^5+\frac{4}{5}y^7)^2-\frac{1}{4}x^{20}y^4$
19. $a^{2m}-b^8$
20. $x^{2a}-y^{2s}$
21. $4m^{4l}-9s^{6v}$
22. $(a+b)^2-(c+d)^2$
23. $(x^2-y^2)^2-(x^3+y^3)^2$
24. $(12ab+\frac{1}{3}bc)^2-(.4ac-2b^2)$
25. $(\frac{1}{2}x^2-\frac{2}{3}y^2)^2-(\frac{3}{2}x^2+\frac{1}{3}y^2)^2$.

Exercise XXXI

Answers:

1. $-6a^5-8a^2b^3+6a^3b^2$
2. $\frac{1}{4}p^2t^2-\frac{4}{7}\frac{8}{2}pt^2s+\frac{1}{3}t^2s^2$
3. $a^2+b^2+c^2+2ab+2ac+2bc$
4. $a^2+b^2+4c^2-4ac+2ab-4bc$
5. $4a^2+9b^2+c^2+12ab-4ac-6bc$
6. $a^4+b^4+9c^6-6a^2c^3+2a^2b^2-6b^2c^3$
7. $4a^4+4a^2b^2+b^4-9c^4$
8. $5x^2+8y^2-9z^2-14xy+12xz-6yz$
9. $18m^3n$
10. $9y^4-135xy^3+3x^2y^2+225x^3y-30x^4$
11. $a^2x^2+2abxy+b^2y^2-c^2z^2$
12. $a^2x^2+b^2y^2+c^2z^2-2acxz+2abxy-2bcyz$
13. $a^2x^2+b^2y^2+c^2z^2-2abxy-2acxz+2bcyz$
14. $a^2x^2+b^2y^2+c^2z^2-2abxy+2acxz-2bcyz$

15. $2m^2s-4ms^2+12\frac{1}{2}mst+3s^2t-4\frac{1}{2}st^2-4m^2t-12mt^2$
16. $-0.28l^4+0.9ml^3+2.96m^2l^2-0.65m^3l-0.09m^4$
17. $-4abxy-4bcyz$
18. $a^3+3a^2b+3ab^2+b^3$
19. $a^3-3a^2b+3ab^2-b^3$
20. $8x^3-12x^2y+6xy^2-y^3$
21. $x^3+6x^2y+12xy^2+8y^3$
22. $216x^2y+128y^3$
23. $\frac{1}{4}m^6+\frac{1}{3}m^2v^4$
24. $-0.25x^2-1.2xy+0.36y^2$
25. $16p^5q-4p^4q^2+4p^3q^3+6p^2q^4$.

Division of Polynomials

Answers: 1. $ax+ay+bx+by+cx+cy$; $ax+ay$; $bx+by$; $cx+cy$.

From the first terms. Develop orally with the class.

2. The factor which multiplied by $x+y$, gives the product $ax+ay+bx+by+cx+cy$.

By dividing first term of dividend by first term of divisor, $ax\div x=a$, first term of quotient.

By multiplying the quotient term a by the whole divisor, $x+y$, getting $ax+ay$. Then $ax+ay+bx+by+cx+cy-(ax+ay)=bx+by+cx+cy$, the remainder.

3. $bx\div x=b$; and $bx+by+cx+cy-(bx+by)=cx+cy$, the remainder. $cx\div x=c$, and $cx+cy-c(x+y)=0$, whereupon the whole quotient is $a+b+c$.

4. Examine the form on page 178 closely.

5.

$$\begin{array}{rl} & \underline{a^2+2ab+b^2} \quad \text{quotient.} \\ a+b) & a^3+3a^2b+3ab^2+b^3 \\ & \underline{a^3+a^2b} \\ & 2a^2b+3ab^2 \\ & \underline{2a^2b+2ab^2} \\ & ab^2+b^3 \\ & \underline{ab^2+b^3} \\ & 0 \end{array}$$

$+a^3\div a=a^2(a+b)$ } the first term of quotient

$+2a^2b\div a=+2ab=2ab(a+b)$ } second term of quotient

$+ab^2\div a=+b^2(a+b)$ } the third term of quotient

Then checking: $(a+b)(a^2+2ab+b^2)=a^3+2a^2b+ab^2+a^2b+2ab^2+b=a^3+3a^2b+3ab^2+b^3$.

Answers to questions of 5, page 178:

1. The first term of the dividend divided by the first term of the divisor $+a^3\div+a=+a^2$.

2. By multiplying the whole divisor by the *first* term $(+a^2)$ of the quotient.

3. The first term of the remainder divided by the first term of the divisor $(+2a^2b\div a=+2ab)$.

4. By multiplying the whole divisor by the *second* term $(+2ab)$ of the quotient.

Exercise XXXII

Have the pupil solve the following by actual division, and test by multiplying. If he happens to discover a law of factoring by himself let him use it, but do not make this the objective point.

Quotients:

1. $3t+4s$; Test: $(3t+4s)(3t+4s)=9t^2+24st+16s^2$
2. $0.3x-0.4y$; Test: $(0.3x-0.4y)(4x+7y)=1.2x^2+0.5xy-2.8y^2$
3. $3k^2-5kt+6t^2$, test by multiplying it by $2k-7t$
4. Arrange dividend thus: $10x^3-4x^2y-5xy^2-54y^3$; quotient $10x^2+16xy+27y^2$; test by multiplying.
5. $9a^2-12ab+4b^2$. Test
6. $3a-2b$. Test
7. x^2+xy+y^2. Test
8. $9t^2+12st+16s^2$. Test
9. $8a^3+b^3$. Test
10. Arrange thus: $-25x^4+15x^3y+10x^2y^2-9xy^3+3y^4$; quotient: $5x^2+3xy-y^2$. Test
11. $2x-y$. Test

12. $16a^4+4a^2b^2+b^4$. Test
13. $0.04s^2t^2-0.2stv+v^2$
14. $a^2+b^2+c^2$
15. $4a^2-b^2$
16. $2x^3+3y^3$
17. $0.2st+v$
18. $4x^2+5r^2$
19. $a+b$
20. $3t^3-2s^3$
21. $3m^2-2n^2$
22. $4x^2-5r^2$
23. $9m^4-6m^2n^2+4m^4$
24. $16u^4+2.4u^2v^2+0.09v^2$
25. $9m^4+6m^2n^2+4m^4$.

Summary

Read the summary through with the class, and see that everything is understood. It is a good plan to have individual pupils read one paragraph, and then tell what it means, or illustrate its meaning by an example of their own making. Do not conclude too readily that pupils really understand what they read glibly. They may even recite verbal definitions without a halt, and have no real conception of their meaning. The best test always available is: Can they give a correct illustration *on demand?* After seeing that the statements are understood by most of the class, assign the page as home work to be learned.

CHAPTER IX

PRACTICE IN ALGEBRAIC LANGUAGE, GENERAL ARITHMETIC

The fundamental algebraic operations have now been carefully taught and applied, the equation has been used extensively under common-sense modes of treatment, and enough *reconnoitering* has been done to make an explicit study of the equation profitable. But unless the equation can be felt to state real number relationships, manipulating it can give only mechanical training, eminently desirable when it is clearly understood what the training is about and why it is worth while, but of almost no value as unmeaning or blind exercise. Modes of manipulating equations ought to mean kinds of thinking, and the surest way for the pupil to realize this is to have him acquire enough mastery of the language phase of the equation to enable him to seize the thought expressed in algebraic language and to clothe quantitative thought in this language. No one would expect to reason with precision upon ideas that had to be acquired through an unknown language and no one would expect to profit by a technical study of the laws of use of a foreign tongue without first having some appreciating sense of its expressive power. With foreign tongues we require considerable translating of simple language forms both into and out of the foreign tongue before taking up the philology of the tongue. Furthermore, experimental studies of the sources of unsatisfactory results in algebra trace most of the bad results to failures in thinking, and in his work with the equation, to the inability to translate common language into algebraic language, and the reverse. It would seem then that to get the laws for manipulating equations ***really understood***, some of the difficulties on the language side, i.e., in setting up the equation, may well be grappled with and gotten out of the way at the outset of the ***intensive*** study of the equation.

The work of this chapter should be assigned about in accordance with the following subdivisions. If the stipulated lesson for a given day is thought too heavy it will be best to omit certain problems that are thought to be of lesser value, than to shorten the assignment and put more than this amount of time on the chapter. Movement and connection of steps are here of all importance, and connections are very materially aided by moving on to the next step before the impression of the preceding is lost, or too greatly enfeebled.

First lesson: pages 181–84 inclusive.

Second lesson: pages 185–87 inclusive.

Third lesson: pages 188–92 inclusive.

Fourth lesson: pages 193–97 inclusive.

Fifth lesson: pages 198 to § **143**, page 200.

Sixth lesson: § **143**, page 200, including most of the exercises to page 107—SUMMARY.

Seventh lesson: SUMMARY and review.

Translating Verbal into Symbolic Language

The work of the 28 exercises of pages 181–83 should be gone over first orally ***with*** the class, and then the exercises should be assigned as home work, the translations called for being written in notebooks, brought to class, the notebooks interchanged among the pupils, criticized and corrected by pupils, and then they should be gone over carefully by the teacher.

§ 132. *Answers to exercises:*

1. 16, 17, 18
   16, 15, 14
2. $n, n+1, n+2$
   $n, n-1, n-2$
3. $2x, 2x+2, 2x+4$
   $2x, 2x-2, 2x-4$
4. $n+3, n+5, n+7$
   $n+3, n+1, n-1$
5. $x+x+3+x+6$
6. $n-1+n+n+1=3n$
7. $n+n+1+n+2=21$
   $3n+3=21;\ n=6, 7, 8$
8. $n(n+1)=72$
9. $n+n+3+n+6=27$
   $3n+9=27;\ n=6, 9, 12$
10. 4×10
11. $10t+u$.
12. $2u\times 10+u$
13. (1) $x-12$
    (2) $x+12$
    (3) $2x$
    (4) $4(x-8)$
    (5) $6(x+3)$
    (6) $\frac{1}{8}(x-15)$
    (7) $3x-2(x-6)$
    (8) $60-x$
    (9) $x-60$
    (10) $x+\frac{1}{2}x$
    (11) $x-3(x-20)$
    (12) $x=x+14$
14. $10u+t$
15. $12x$
16. $\dfrac{x}{12}$
17. $10t+t-3=10(t-3)+t+27$
18. $\dfrac{2(n+25)}{8}=\dfrac{n}{2}$

19. $2n+3(n+4)$
20. $2n-3(n-4)$
21. $2n-3(n-2)$
22. $4(n-3)-3(n-1)$
23. $2n+3(n-4)=13$
24. 72, 48, 0, −24, 0, 10
26. $\frac{1}{3}(r+4)$, $\frac{r+4}{3}$,

$\frac{1}{2}(s-11)$, $\frac{s-11}{2}$,

$\frac{2}{3}(8y-20)$, $\frac{2(8y-20)}{3}$,

$\frac{5(2x+9)}{4}$, $\frac{5}{4}(2x+9)$.

§ **133.** A treatment similar to that suggested above should be given to the 49 exercises of § **133**, which revises and formalizes all the rules of arithmetic. Article **132** may well take a day in class and another day for revision and recitation on these exercises as prepared work. Article **133** should take about the same amount of recitation time.

§ **133.** *Answers:*

1. $7+5=12$
2. $4 \cdot 10+5=45$
3. $6 \cdot 10+8=68$
4. $50+7$, $48+9$, $18+x$
5. $x+8$, $y+15$, $x+16$, $x+y$
6. $s=4a+10$
7. $18-7=11$
8. $m-s=d$
9. $3a=3x$
10. $M \cdot m=P$
11. $m=d+s$

12. $P/m=M;\ \frac{P}{M}=m$

13. $D/d=q$, or $D=d\cdot q$

14. $D=d\cdot q+r$

15. $M=m+b$

16. $p=b+g$

17. $s=cd$

18. $w=s/f$

19. $M=bt+r$

20. $S=n_1/d_1+n_2/d_2$

21. $D=n_1/d_1-n_2/d_2$

22. $P=n_1/d_1\times n_2/d_2$

23. $Q=n_1/d_1\div n_2/d_2$

24. $R=ab$

25. $A=s^2$

26. $D=bs$

27. $v=\frac{3}{10}+\frac{5}{100}$

28. $v=\frac{t}{10}+\frac{h}{100}$

29. $v=\frac{a}{10}+\frac{b}{100}+\frac{c}{1000}$

30. $v=100a_5+10a_4+a_3+\frac{a_2}{10}+\frac{a_1}{100}$

31. $v=100a_2+\frac{a_1}{100}$

32. \$12.60, 31.2 A, 39 mi., $\frac{6b}{100}$ yd.

33. $p=\$22.50$, $p=75$ bu., $p=270$ men, $p=\frac{b}{4}$A, $p=\frac{br}{100}$

34. $\frac{p}{r}=\frac{b}{100}$

35. $\frac{100p}{b}=r$; rate per cent is 100 times the percentage divided by base.

36. $\frac{2,880}{360}=r=8$

37. $\frac{3,000}{725}=r=4$

725

38. $b=\frac{100p}{r}$

39. $b=\frac{2,700}{5}=540$

40. $b=\frac{3,150}{7}=450$

41. $100p=br$

42. 7,740

43. \$3.90, \$7.80, \$11.70, \$19.50, \$9.75, \$14.62½, $t\times$\$3.90

44. \$3.20t, \$4t, \$4.80t, \$3.60t, \$4.48t, $\frac{\$80tr}{100}$

45. $\frac{20rt}{100}$, $\frac{75rt}{100}$, $\frac{135rt}{100}$, $\frac{prt}{100}$

46. $i=\frac{prt}{100}$

47. $\frac{100i}{rt}=p$

48. $t=\frac{100i}{pr}$

49. $r=\frac{100i}{pt}$.

Graphing Percentage and Interest
page 188

Work Exercise I orally.

Work exercises with the class, making clear the meaning of the points and lines of the graph. Let the teacher sketch 3 rapidly on the blackboard. Let the class solve 4 and 5 and have 6 and 7 answered orally. Work the rest of this list through with the class somewhat as a chalk-talk, getting the class to participate as much as possible.

Treat §§ 136–37–38 in the same general manner. Do not assign this work as a whole for home work. At most, only particular exercises should be so assigned. The teacher should guide and keep the class moving right along. Time may be easily wasted by treating such work by mechanical assignments.

§ 135. Answers to some problems:

1. \$2, \$4, \$6
6. Zero % of any base is zero dollars.
7. Between them.
9. $p=\frac{1}{25}b$; $p=\frac{8}{100}b$; $p=\frac{1}{100}b$
10. $p=\frac{1}{20}b$; $p=\frac{1}{10}b$; $p=\frac{3}{50}b$; $p=\frac{r}{100}b$
11. 2% of \$250; 4% of \$250; 2% of \$150; 4% of \$150
12. By reading length of E C, etc.
14. By reading lengths of appropriate verticals from O X up to the graph.
18. $p=\$1\times r$; $p=\$4r$
19. Percentage of any base at 0 per cent is \$0.

§ 136. *Answers:*

1. \$3, \$6, \$9, \$12
7. Int. for 0 yr. on any base at any rate is \$0.
8. Lengthening of verticals proportional to corresponding increases in lengths of time-line.

The Evaluation of Expressions

The first 4 exercises, page 195, should be gone over with the class orally and rapidly. EXERCISES 5 and 6 may be assigned for home work and the rest should be worked out with the class as a laboratory exercise in the classroom.

1. $D=150$ ft., $D=85$ mi.
2. $A=312$ sq. ft., $A=35.7$ sq. in.
3. $A=96$ sq. ft., $A=31.698$ sq. rd.
4. $A=532$, $A=237.98$
5.

(1)	800	(6)	9
(2)	3,750	(7)	151
(3)	1,170	(8)	950
(4)	2	(9)	1,098
(5)	798	(10)	1,800

6. $d_1=21\frac{3}{5}$ ft., $d_1=40\frac{1}{2}$ in., $d_1=45$ in., $d_1=18\frac{57}{76}$ in.; $w_1=31.5$ lb., $w_1=25.5$ lb., $w_2=12\frac{6}{17}$ lb.
7. $w=1,036.8$ lb.; $v=40$ cu. in.; $d=5$ lb.
8. $h=10$ rd.
9. $s=256$ ft., $s=2,116$ ft.; $t=2$ sec., $t=10$ sec.
10. $s=2,340$ ft., $s=1,080$ ft.; $v=20$ ft. per sec., $v=16$ ft. per sec.
11. $t=\frac{11}{7}$ sec., $t=1$ sec., $l=\frac{392}{121}$ ft., $l=16\times\frac{392}{121}$ ft.
12. $A=6$, $(s=6)$, $A=30$, $(s=15)$
13. $h=1\frac{283}{322}$, $h=\frac{10}{161}$
14. $R=4\frac{11}{72}$, $R=6\frac{27}{28}$
15. $E=150$, $E=445\frac{1}{2}$; $M=1$; $M=4$
16. $V=84$, $V=74$.

§ **140.** *Answers:*

1.	(1)	$C=66$ ft.,	$d=28$ ft.
	(2)	$C=22$ ft.,	$d=21$ ft.
	(3)	$C=\frac{11}{7}$ ft.,	$d=5\frac{1}{11}$ ft.
2.	(1)	$C=18\frac{6}{7}$ ft.,	$r=21$ in.
	(2)	$C=37\frac{5}{7}$ ft.,	$r=10\frac{1}{2}$ ft.
	(3)	$C=113\frac{1}{7}$ ft.,	$r=17\frac{1}{2}$ rd.
3.	(1)	$A=28\frac{2}{7}$ sq. ft.,	$r=7$ ft.
	(2)	$A=154$ sq. ft.,	$r=\sqrt{70}$ ft.
	(3)	$A=1,386$ sq. ft.,	$r=\frac{6}{55}\sqrt{2,310}$ ft.

Assign EXERCISES 4 and 5 for home work to be brought to class.

4. (1) $V=179\frac{2}{3}$ cu. ft., $r=1$ in.
   (2) $V=5,577\frac{11}{21}$ cu. ft., $r=2$ ft.
   (3) $V=33\frac{11}{21}$ cu. ft., $r=\frac{3}{22}\sqrt[3]{4,851}$ rd.
5. $C=6\frac{2}{7}$ yd.; $A=3\frac{1}{7}$ sq. yd.; $S=12\frac{4}{7}$ sq. yd.; $V=4\frac{4}{21}$ cu. yd.

$A=254\frac{4}{7}$; $S=1,018\frac{2}{7}$; $V=3,054\frac{6}{7}$; $r=9$
$C=37\frac{5}{7}$; $S=452\frac{4}{7}$; $V=905\frac{1}{7}$; $r=6$
$C=31\frac{3}{7}$; $A=78\frac{4}{7}$; $V=523\frac{17}{21}$; $r=5$
$C=18\frac{6}{7}$; $A=28\frac{2}{7}$; $S=113\frac{1}{7}$; $r=3$.

Work through § 141 orally with the class.

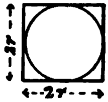
§ 142. *Answers:*

1. 10
2. 20 ft.
3. 24 ft.
4. 21 ft. and 28 ft.
5. 12 ft.
6. $\sqrt{7}$, or $6\sqrt{2}$
7. $4\sqrt{\frac{a^2}{2}}$, or $2a\sqrt{2}$
8. $h=15$ ft.
9. $h=\sqrt{b^2+\frac{4r^2}{b^2}}$
10. $9+48+3\sqrt{265}=57+3\sqrt{265}$
11. $p=\frac{2s}{b}+b+\sqrt{b^2+\frac{4s^2}{b^2}}$
12. 81 ft.
13. $\sqrt{t^2-f^2}+t$.

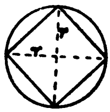
The Circle

§ 143. Read carefully with the class the text of pages 200 and 201 and work with the class the exercises of page 201.

Answers: 1. $x+2x=90°$, $3x=90°$, $x=30°$, and $3x=60°$.

2. 

$A<4r^2$

3. 

$A>$ the square $=2r^2$

The Triangle

§ 144. See that the class read the first paragraph of § 144 understandingly.

Answers:

1. AB=68.5 rd.

2, 3, and 4. These answers are to be given from intuition. The thought is that the exercise will focus attention on what is to be proved.

§ 145. After the observational attempts of EXERCISES 2, 3, and 4, the wording of the theorem will easily be grasped. The teacher will notice that this is the first attempt at a deductive proof of a geometric truth. The aim has been in this attempt to use the simplest possible untechnical language. The proof may need to be gone over two or three times. EXERCISES 1–9, pages 203, 204 should all be solved that the proved truth may be impressed upon the class. Have the class point out how the theorem applies in each of the exercises.

Answers:

1. $c=c'$; $\angle A=\angle A'$; and $\angle B=\angle B'$

2. $a=a'$, $\angle B=\angle B'$, and $\angle C=\angle C'$

3. The proofs are citations of the theorem § 145.

4. Let the pupils show from Fig. 212 how the theorem applies and the consequences of its application.

5–9. Remark of 4 applies here in each exercise.

§ 146. Begin this section orally with the class, having individual pupils read and answer as far as they can, then calling on other pupils to suggest and to help on difficulties.

§ 146. *Answers to exercises:*

6. B, C, D, E, and F
7. 6 chords
8. By stepping the circumference
9. $r, r\sqrt{3}, 2r, r\sqrt{3}, r$
10. $36\sqrt{3}$
11. $\frac{a^2}{4}\sqrt{3}$
12. $150\sqrt{3}$
13. $\frac{3\sqrt{3}s^2}{2}$
14. $x=\frac{g}{\frac{85}{27}-1}=\frac{27g}{58}$
15. $x=\frac{bg}{a-b}$
16. $l=2\sqrt{10^2-6^2}=16$
17. $l=2\sqrt{R^2-r^2}$
18. $S=\frac{r}{2}(b_1+b_2+b_3+ \text{ - - - - })$
19. The circumference
20. S in 18 becomes $A=\frac{rC}{2}$.

SUMMARY

Work over the summary orally with the class, see that all the important ideas and definitions are understood, then assign it as home work, reciting next day upon the important statements and definitions.

CHAPTER X

THE SIMPLE EQUATION IN ONE UNKNOWN

Have the class bring to the classroom notebook with squared paper, and a small ruler for use in laboratory work.

Lesson 1

From beginning of chapter to Art. 152, *page 214, treated as follows:*

Show on the board the construction of the graph of Art. 148. Show that the graph represents or pictures $3t$ for *all* values of t, and in the sense that a definite line is either given or may be easily drawn to represent $3t$ for every value of t that may be chosen. Show also how to graph $3t+3$ and again in what sense the line or graph pictures the expression as dependent upon the t. Too much talk about the graph will not explain but rather confuse the ideas of pupils. By means of questions—not too leading—have members of the class suggest and assist you in making the essential ideas clear. Fifteen or twenty minutes is enough time to spend on both graphs. Have the work of Art. 149 done as laboratory work during this same recitation exercise—everyone drawing all the graphs with enough care to get the notion of the meaning of the graphs as pictures of the polynomials.

If the six exercises of Problem 2 are not completed assign the rest as home work, the graphs to be brought to class next day. Assign also Arts. 150 and 151 to be recited on the next day, completing in the exercise period the answers called for to Exercises 1–16, page 214, at least half being answered orally. Go carefully through with the class the ideas of pages 212 and 213 in the recitation period. Art.

151, I and II, should be read aloud by the teacher or members of the class and commented on until understood by the class.

LESSON 2: *Arts.* 152 *and* 153

One of the main sources of difficulty with first-year classes is due to their general inability to get the sense from written sentences. Have different pupils read to the class the several sentences of the portion of the text of Art. 152 that precedes the exercises, and tell what the sentences mean. Work through the exercises of page 215 with the class, showing the application of the suggestions for stating the equations. Have pupils do as much as possible of the work of writing the equations. Have members of class read in turn Art. 152, I and II, and explain them, with teacher's help if needed. Read with the class and illustrate the sense of the five axioms of Art. 153. Assign any remaining exercises of the ten of Art. 152 and the first six of Art. 154 for home work. Go quickly over this work the next day.

For answers see the end of this chapter, pages 106 *et seq.*, of this *Manual.*

LESSON 3: *Arts.* 154, 155, *and the first* 10 Exercises *of Art.* 156

In Art. 154 emphasize the application of the axioms of Art. 153, using the names in ( ). Finish the unassigned exercises of Art. 154 orally in class. Have Art. 155 read in class, and its substance explained, if necessary. Go through the solution of EXERCISE 1, Art. 156, and if time allows, solve a few—half a dozen or so—of the exercises with the class. For answers see the end of this chapter, page 106 of this *Manual.* Assign 15 or 20 exercises of the list, page 219, for home work. Do not slight Art. 154.

Reasons, not rules nor authority, must govern in mathematics. In Art. **155** make the meaning of *root* of an equation quite clear.

Lesson 4

Assign for home work even-numbered exercises from 20 to 35, pages 219 and 220, and the problems of Art. **157** to 13, page 221.

After clearing up in the class any difficulties revealed by the home work of Lesson 3, perhaps explaining by solving any exercises of pages 219 and 220 that have not been and are not to be assigned, work through with the class—pupils giving such suggestions as they can—or with a pupil at the board, question through the solutions of the first half-dozen problems in the recitation period. Do not allow the work to drag, nor to be diverted by too many pointless queries and suggestions. To explain by telling over and over as often as asked for the same difficulty encourages mind wandering and premiums inattention on the part of the class. One or two repetitions of the same explanation with the understanding on the part of the class that these repetitions are to suffice, save in exceptional cases, will be found more effective than over-explanation. Let pupils understand that when explanations are being given all that need help are expected to give attention.

For answers, see end of this chapter, pages 106 *et seq.*, of this *Manual.*

Notes and suggestions on Problems 1–13, pages 220 and 221

2. Let x denote the altitude; then $x+5$ denotes the base, and $\frac{1}{2}x(x+5)=\frac{1}{2}(x+5)^2-20$. Whence $x=3$ and $x+5=8$.

4. Let x denote the number of seconds, then $160-3x=112-2x$. Whence $x=48$.

5. $x+x+2x=2$; find x.

12. Let x denote the number of hours, then $(30-4)x=\frac{208}{1,760}$; find x.

For answers, see end of this chapter, page 106, of this *Manual*.

Lesson 5

Assign the even-numbered problems from 14–29, giving any explanations that are deemed necessary by taking odd-numbered problems.

The idea here is a little exercise work to develop mechanical skill with equations, alternating quickly with problem-work at once using the skill developed. Do not crowd all the exercises together, and then all the problems. Shifting the attention from one sort of work to the other affords an element of variety, that will result in more and better results, than concentration on all of first one thing and then of the other. It will doubtless be thought worth while by some teachers to point out a number of the different types of problem and show the pupil that all of a certain type can be solved by a certain typical equation. For example, a type form for motion problems, rowing problems, clock-problems, mixture-problems, etc., may be set up, and then by mere substitution any problem of the type may be solved. This sort of work is of the very essence of mathematical thinking, and is of course eminently worth while. A practicable plan is to select for class-work a problem of a certain type, and assign for home work others like it. For example, take up No. 5 as class-work and assign numbers 44, 45, 46, 48, 49, 53, as home work. No. 21 may illustrate the type of 22, 23, and 24:

No. 54 is typical of 55, 56, and 57

No. 26 is typical of 27, 28, 29.

No. 59 is a type of such as 60; and Nos. 4 and 6 are types of 7, 8, 9, 11, 12, 13, 15, 16, 17, 18, 19, and 20.

After some experience on the above plan, have pupils themselves attempt to solve and apply their solutions to type problems, entirely unaided by the teacher.

See end of this chapter for answers.

Notes and Suggestions on Problems

15. Let x denote the actual rate; then $8(x+6)-11(x-7)=50$. Whence $x=25$.

16. Let x denote the rate of the train on the shorter route, then $8\frac{1}{2}(x-10)-6x=15$. $x=40$. and $6x=240$. $8\frac{1}{2}(x-10)=255$.

17. Let x denote the required rate, then

$$\frac{80}{x}+\frac{20}{\frac{1}{2}x}-1=\frac{100}{x}. \quad \text{Then } x=20.$$

18. Let x denote the required rate, then

$$(h+t)x=tr. \quad \text{Whence } x=\frac{tr}{h+t}.$$

20. Let x denote the number of miles, then

$$\frac{x}{6}+\frac{x}{3}=9. \quad \text{Whence } x=18.$$

22. The equation is obviously, $10+\frac{x}{12}=x$. Whence $x=10\frac{10}{11}$.

23. The equation is $30+\frac{x}{12}=x$. Whence $x=32\frac{8}{11}$. Also, $x+15=\frac{x}{12}+15$, $x=0$

24. The equation is $x+30=35+\frac{x}{12}$. Whence $x=5\frac{5}{11}$.

26. From the note it is evident that $\frac{2}{15}-\frac{1}{12}=$ the gain per month, and $(\frac{2}{15}-\frac{1}{12})$ times x (the number of months

required for Venus to gain a complete revolution) equals one revolution. Whence $x=20$.

27. From the preceding explanation it is evident that the equation is: $(\frac{1}{3}-\frac{2}{15})x=1$. Whence $x=5$.

28. Similarly, $\left(\frac{3}{82}-\frac{4}{1,461}\right)x=1$. Whence $x=29.544$.

29. Similarly, $\left(\frac{1}{a}-\frac{1}{b}\right)x=1$. Whence $x=\frac{ab}{b-a}$.

Lesson 6

About 20 problems selected from the list from 30–77.

Follow the plan of explaining any difficulties thrown up as the work progresses, by solving not the particular problem in which the difficulty arose, but a similar problem. The class-work should avoid instilling mere imitative habits. This practice should be abandoned only under exceptional circumstances.

See end of this chapter for answers.

Notes and Suggestions on Problems

45–51. All solved by the second method explained under Problem 44 of the text.

52. Let x denote the number of ounces added. Then,

$$\frac{90+x}{6}=\frac{10}{\frac{2}{5}}. \quad \text{Whence, } x=60.$$

53. Let x denote the number of ounces of iron. Then,

$$\tfrac{5}{3}x+x+\tfrac{2}{7}x=124. \quad \text{Then, } x=42.$$

54–57. See Problem 52.

59. Let x denote the number of pounds of lead, then

$$\tfrac{52}{57}x+\tfrac{13}{15}(159-x)=143. \quad \text{Then,}$$

$$x=45, \text{ and } 159-x=114.$$

60. See 59.

64. Note. $.06\times 100=$ number of weight units of salt $= .08(100-x)$. (1) $x=25$,
(2) $.13(100-x)=.06\times 100$, and $x=53\frac{11}{13}$.

65–66. See 64.

67. Let x denote the number of percent of water added, then $.03\frac{1}{2}(100+x)=.06\times 100$, and $x=71\frac{3}{7}$.

68. Let x denote the percent of water added, then $.80(100+x)=.95\times 100$, and $x=18\frac{3}{4}$.

69. Let x denote the required percent, then $75x= .30\times 100$, and $x=40$.

70. See 69.

72. Let x denote the number of dozens in the box, then $15x=30(x-5)-30$, and $x=12$.

73. Let x denote the number of days he was idle, then $2(20-x)-x=34$. Whence $x=2$.

74. See 73.

75. Let x denote the distance required, then $\frac{x}{9}-\frac{x}{12}=5$. Whence $x=180$.

76. See 75.

77. Let x denote the income, then $\frac{x}{a}+\frac{x}{b}+\frac{x}{c}+\frac{x}{d}+e=x$. Solve for x; $x=\dfrac{abcde}{abcd-bcd-acd-abd-abc}$.

Lesson 7

Exercise XXXIII *and the odd-numbered parts of* Exercise XXXIV *to* Exercise 20. *Give some attention to the type-problems of the assignment.*

LESSON 8

Finish with the even-numbered parts of EXERCISE XXXIV.

It is also a good plan to distribute these exercises along through the subsequent lessons of chapters xi and xii.

LESSON 9

The SUMMARY *and any specific outstanding difficulties of the chapter.*

Emphasize with the class the ideas that are set forth in the SUMMARY. These are the specific notions the pupil is expected to carry away from the study of the chapter. It is recommended that the pupil attempt to remember these notions.

Finally, it is believed that better results will be secured from the sort of work and the assignments of time suggested above than will be obtained from a more liberal allotment of time to it. First-year high-school pupils can attain only a limited degree of thoroughness, no matter how ambitious the teacher for higher ideals. Moderate thoroughness and movement forward through the subject with some appreciable celerity is more effective than over-insistence upon the adult standard of thoroughness. Firmness of grasp of fundamentals, and conviction of the worth and scope of the subject are most effective here.

Answers

Page 215:

2. 48	5. 40, 50	8. 6, 18
3. $1\frac{9}{18}$	6. 45, 135	9. 14
4. 5	7. 30, 60, 90	

Page 219:

2. $6\frac{1}{3}$
3. 10
4. −1
5. 4
6. 5
7. 3
8. −5
9. −2
10. −3
11. 3
12. 54
13. 13
14. 1
15. −4
16. 7
17. $6\frac{1}{4}$
18. 8
19. 2
20. 6
21. 8
23. −1
24. 3
25. $\frac{3}{4}$
26. $-3\frac{18}{19}$
27. $2\frac{5}{6}$
28. 20
29. 10
30. 8
31. 2
32. 5
33. 4
34. $1\frac{49}{59}$
35. −2.

Page 220: § **157.**

1. 23, 7
2. Change Problem 2 to read 20 sq. units, etc.

Ans. 8, 3

3. 16
4. 48
5. 8, 8, 16
6. 24
7. $1\frac{1}{3}$

Page 221:

8. 3,780
9. *fs* feet
10. 1,099.96
11. $\frac{m}{t}$
12. $\frac{1}{220}$ hour, or $16\frac{4}{11}$ seconds
13. $\frac{1}{1{,}760(m-n)}$
14. (1) 10
    (2) 6
    (3) 1
    (4) $4\frac{1}{3}$
    (5) 3
    (6) 30
    (7) 4
    (8) $2\frac{8}{13}$
    (9) 6
    (10) 6
    (11) 4
    (12) −1
    (13) a
    (14) $2c$

Page 222:

15. 25
16. 4th line, change to $8\frac{1}{2}$ hours. 240, 255
17. 20
18. $\dfrac{rt}{t+h}$
19. $\dfrac{c}{m+n}$
20. 18
22. $10\frac{10}{11}$ past 2

Page 223:

23. 0 and $32\frac{8}{11}$, past 3
24. $5\frac{5}{11}$, past 7
25. (1) $\dfrac{1-6a}{2}$
    (2) $\dfrac{1}{b+a}$
    (3) $1\frac{2}{3}a$
    (4) $\dfrac{1}{a+b}$
    (5) $-\dfrac{1}{a+b}$
    (6) 18
    (7) $6\frac{2}{3}$
    (8) $9\frac{3}{5}$
    (9) $\frac{28}{87}$
    (10) $1+c$
    (11) 1
    (12) 4
    (13) $\dfrac{1}{c+a}$
26. 20 mo.
27. 5 mo.
28. 29.544 da.

Page 224:

29. $\dfrac{ab}{b-a}$
31. 2 times
32. 9, 10
33. 136, 137
34. $\dfrac{a-1}{2}, \dfrac{a+1}{2}$
36. 6, 8
37. 24, 26
38. $\dfrac{a-4}{4}, \dfrac{a+4}{4}$
40. 11, 13
41. $\dfrac{s-4}{4}, \dfrac{s+4}{4}$

Page 225:

42. 13, 15

43. (1) $a-z, x+z, x-a$

(2) $\dfrac{z+a}{3}$, etc.

(3) $\dfrac{z}{3a}-1$, etc.

(4) $\dfrac{z-5a}{2za}$, etc.

(5) $12/az$

(6) $\dfrac{1}{6-az}$, etc.

(7) $\dfrac{a+z}{4}$, etc.

(8) $-\frac{1}{2}$, etc.

(9) $\dfrac{a+6z+6}{z}$, etc.

(10) -9, etc.

44. $1142\frac{6}{7}$, $3,657\frac{1}{7}$

45. 90, 15, 15,

46. $88\frac{8}{9}$, $44\frac{4}{9}$, $66\frac{6}{9}$

47. $37\frac{1}{2}$ lb.

48. 12, 3, 3

Page 226:

49. $\dfrac{am}{a+b+c}$, $\dfrac{bm}{a+b+c}$, $\dfrac{cm}{a+b+c}$

51. $1\frac{1}{6}$

52. 60

53. 12*c*, 42 i., 70 c′n.

54. 190

55. 20

56. 124

57. $\dfrac{bc-ad}{d}$

59. 45, 114

60. 194, 126

61. 400, 600

62. $1,111\frac{1}{9}$, $888\frac{8}{9}$

63. 3,750

64. (*a*) 25

(*b*) $53\frac{11}{13}$

65. $5\frac{5}{19}$

66. $\dfrac{100a-100b}{b}$

67. $71\frac{3}{7}$

68. $18\frac{3}{4}$ per cent

69. 40

70. 56

72. 144

73. 2

74. $\dfrac{ab-a}{b+c}$.

CHAPTER XI

LINEAR EQUATIONS CONTAINING TWO OR MORE UNKNOWN NUMBERS—GRAPHIC SOLUTION OF EQUATIONS AND PROBLEMS

Squared paper and a small ruler for laboratory work in class will be needed for this chapter also. Observe that the graphical work is to show the *meaning of the solution* of equations, and for this reason it should come *before* the algebraic solutions. After the algebraic method of solving equations has been learned it is so much less laborious to solve equations by it, that pupils will not care to continue the graphical procedure, nor should they be required to do so save to clear away particular difficulties. Pupils, however, readily learn and remember for a time the mere technique of algebraic solutions, and do not concern themselves about comprehending its significance and its limitations. They soon come to apply this technique purely mechanically and irrationally. The antecedent work with the graph, with frequent subsequent appeal to it, when matters are known by the teacher not to be sufficiently well comprehended by pupils, will avoid thoughtless and oftentimes foolish steps and conclusions. Algebra will thus be made to appeal to pupils as a subject that has to do with thinking rather than with mere manipulating. Too often the high-school teacher deludes himself with the notion that pupils really *understand* what they can only *manipulate*. The only way to make algebra appeal permanently as a subject worth studying is to have it well understood. For these and many other educational reasons graphs should be used in teaching simultaneous equations and their use should precede the technically algebraic work.

Lesson 1. *Arts.* 160–61

Sketch rapidly the solution of Problem 1, and point out on the graph the meanings of the lines, and the solution of the problem. Have the class solve similarly during the recitation period Problems 2 and 3, page 235. Assign 4 and 5 for home work. In a similar way work through the solution of Problem 1, Art. 161, and Problems 2 and 3, page 237, and assign 4 and 5 as home work. Recall the *law of leverages.*

Lesson 2. *Arts.* 162–63

Go over the home work, clear away the difficulties revealed by it, read with the class the text of Art. 162, see that it is understood, and work with them Exercise 4, page 238. Assign the rest for home work. Remember the graphical work is not so much for its own sake as to make the meaning of solving equations understood. Do not dilate in detail about the graphs, but show how and wherein the graphs furnish the solutions.

Lesson 3. *Arts.* 164–65–66

After clearing away the difficulties shown by the preceding day's home work, have different members of the class read and tell the sense of the text of Art. 164 and of Art. 165 down to Exercise XXXVI. Solve Exercises 1 and 2 with the class and assign the rest as home work. Make the meaning of *equivalent* and *dependent* clear.

Some teachers will prefer to have the class do together the problems of Arts. 164 and 165 as a laboratory exercise. This is also a good plan.

EXERICSE XXXVI

Answers:

1. { Equivalent / Coincident }
2. Same as 1.
3. Same as 1.
4. { Contradictory / Parallel }

5 and 6. Same as 4.

LESSON 4. *Arts.* 167–68–69

After 5 or 10 minutes' clearing away difficulties of preceding lesson, have a pupil read the problem of Art. 167 and show that Equations (1) and (2) express the conditions (*a*) and (*b*). Then have a pupil show the pertinency of the abbreviated phrases in parenthesis as (Add. Ax.). Have another pupil read and answer the next two questions. Drop the rest of the questions around rapidly and promiscuously in the class. Keep the work moving briskly. Continue in this way through the questions of the next Article.

Then work EXERCISE 2 at board, and have EXERCISES 1 and 3 done by the class as a laboratory exercise. Assign the rest of Art. 169 as home work. Insist on the checking of results.

EXERCISE XXXVII

Answers:

1. 1, 3	4. −13, −12	7. 1, 3
2. 2, 3	5. 1, 2	8. 2, 1
3. 12, −3	6. 3, 4	9. 3, 2.

LESSON 5. *Arts.* 170 *and* 171

Work through Art. 170 carefully as was suggested above for Art. 167, solve EXERCISE 4 on board, and then have half a dozen done by class as a laboratory exercise. Notice the text asks to have the class go through the list of exercises, first telling how to solve each one, and then calls for the solutions. This practice in looking at equations and judging what to do before beginning the actual work eliminates much of the puttering that pupils are too wont to practice when they do not know either how to start or to proceed. Pupils should form the habit of deciding first what is to be done, what is given to do it with, and what is the most economical way to start. Even the exercise work, properly administered, can be made to contribute to this habit. Assign the rest of the list of exercises to be done as home work.

Exercise XXXVIII

Answers:

1. 3, −2	5. 2, 1	9. 1, 2
2. 12, 3	6. 1, 2	10. 7, 11
3. 2, 1	7. 11, 10	11. 2, 1
4. 3, 2	8. 3, 2	12. 4, 8.

Lesson 6. *Arts.* 172–73

Have pupils study carefully and then tell how to eliminate by addition or subtraction and how by comparison, and then compare the two methods. Let them attempt to tell when they would use the one or the other method. Do what telling you deem necessary by exhibiting an exercise that is conveniently treated by one method, but clumsy by the other. Studying consists in comparing and relating things, and the way to study methods of elimination is by comparing, not merely committing them. Have three of the parts of Exercise XXXIX worked in class and assign the rest as home work.

Exercise XXXIX

Answers:

1. 1, 2	4. 3, 4	7. 8. 5
2. 6, 10	5. 4, 5	8. 28, 10
3. 2, 3	6. 6, 5	

Lesson 7. *Arts.* 174–75

This lesson should be reconnoitered in class, and 4 of the exercises of page 245 assigned as home work.

Exercise XL

Answers:

1. 40, 45
2. 35, 24
3. 65, 66
4. 105, 63
5. 22, 6
6. 36, 60.

Lesson 8. *Art.* 176

Treat this lesson similarly to the mode suggested for Lesson 7.

Exercise XLI

Answers:

Exercise 41.

1. $\frac{b}{a+b}, \frac{a}{a+b}$
2. $\frac{b}{cb+an}, \frac{a}{cb+an}$
3. $\frac{ah-bk}{a^2-b^2}, \frac{ak-bh}{a^2-b^2}$
4. $\begin{cases} \frac{2c^2d+d^2-cd}{c^2+bd} \\ \frac{2bcd+cd-c^2}{c^2+bd} \end{cases}$
5. b, a
6. $\frac{c_1b_2-c_2b_1}{a_1b_2-a_2b_1}, \frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
7. $\frac{2kn}{k+n}, \frac{2kn}{k-n}$
8. $\frac{ae-bd}{ce-bf}, \frac{bd-ae}{cd-af}$
9. $\frac{m^2-n^2}{mk-nh}, \frac{m^2-n^2}{mh-nk}$
10. $\frac{2(a^2+b^2)}{a+b}, \frac{3(a^2+b^2)}{a-b}$
11. c, d
12. $a+b, a-b$
13. $h+k, h-k$
14. $2b-c, 2b+c$
15. $\frac{a}{a-b}, \frac{b}{a+b}$.

LESSON 9. *Arts.* 177–78

No directions are needed here. Treat as above.

EXERCISE 42.

1. 1, 2, 3
2. 1, 2, 3
3. 1, 2, 3
4. 10, 2, 1
5. 6, 5, 4
6. 15, 10, 5
7, 4, 5, 6
8. 5, 10, 20
9. 5, 10, 20
10. 1, 2, 3
11. 3, 1, 2.

LESSON 10. *Art.* 179

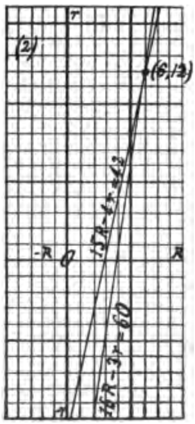
Work half a dozen of these problems as a laboratory exercise in the class, and assign half a dozen for home work.

§ 179. ANSWERS

1. 5c. and $2\frac{1}{2}$c.
2. \$7.50 and \$4.75
3. 72°, 72°, 36°
4. 63°, 27°
5. 100°, 80°, 100°, 80°
6. $52\frac{1}{2}^\circ$, $127\frac{1}{2}^\circ$
7. 25, 15
8. 5, 10 and 3, 5
9. 40 mi., 30 mi.
10. 8, 3
11. 39, 40
12. \$2,000, \$1,200

LESSON 11. *Art.* 179 *continued to* EXERCISE XLIII

Proceed as with LESSON 10.



§ 179—*Continued*

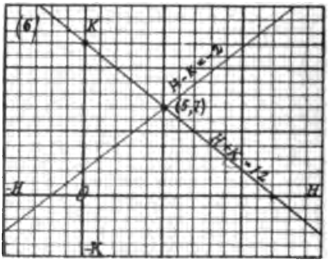
13. \$1,000, \$1,200
14. \$6,000, \$2,000
15. 12 in., 10 in.
16. { Radii, 7, 21 / Areas 154, 1,386
17. { Radii, $\frac{7}{2}$, $\frac{7}{4}$ / Cirs., 22, 11
18. 4, 10
19. 10, 6

LESSON 12. EXERCISE XLIII *to Art.* 180

Take one laboratory period and one assignment of home work on the work to Art. 180.

ANSWERS

20.
- (1) $\frac{1}{3}$, 1
- (2) 6, 12
- (3) R, 0
- (4) 12, 40
- (5) 15, 14
- (6) 5, 7
- (7) 1, $\frac{1}{2}$, $\frac{1}{3}$
- (8) $\begin{cases} -a+b+c \\ a-b+c \\ a+b-c \end{cases}$
- (9) $\begin{cases} (2) \\ (6) \end{cases}$ ——→



21. $\frac{16}{5}$, $\frac{9}{5}$

22. 4.8, 6.4

23. $\frac{5}{2}$, 4 each

24. 10, 6

25. $\frac{7}{9}$, $\frac{1}{9}$

27. 75

26. 9, -10

28. 1, 2

29. $2a/b$

30. 6, 12 da.

31. $9\frac{3}{7}$, $2\frac{4}{7}$.

Notes on the Problems of Two and Three Unknowns

Page 253:

21. $\left\{\begin{array}{l} \frac{m}{n}=\frac{16}{9},\ m=\frac{16n}{9} \\ m+n=5,\ \frac{16n}{9}+n=5,\ n=\frac{9}{5},\ m=\frac{16}{5} \end{array}\right.$

22. $\left\{\begin{array}{l} EF=\sqrt{6^2+8^2}=10.\ \frac{m}{h}=\frac{h}{3.6},\ \text{and}\ \frac{m}{8}=\frac{8}{10} \\ m=6.4.\quad h=\sqrt{6.4\cdot 3.6}=4.8 \end{array}\right.$

23. (1) $\left\{\begin{array}{l} 6x-3y=3 \\ 8x-4y=10x-6y+3 \end{array}\right.$ $x=\frac{5}{2}.\quad y=4$

(2) $\left\{\begin{array}{l} 2x-y=1 \\ 2x-y=1. \end{array}\right.$ $x=\frac{5}{2}.\quad y=4$

(3) $\left\{\begin{array}{l} 2x-y+2=3 \\ 10x-6y+3=4 \end{array}\right.$ $x=\frac{5}{2}.\quad y=4$

Page 254:

24. $288-(112+5x+3y)=3x-2y+90$

$\left\{\begin{array}{l} 8x+y=86 \\ 9x+4y-2=112. \end{array}\right.$ $x=10.\quad y=6$

25. The base of the second triangle should be $5x-2y-1$. The equations reduce to $\left\{\begin{array}{l} 5x+14y=21 \\ 7x-2y=3 \end{array}\right.$ $x=\frac{7}{9}.\quad y=\frac{11}{9}$

26. $\left\{\begin{array}{l} \frac{4x+3y+1}{10}=\frac{5\frac{1}{4}}{7\frac{1}{2}}=\frac{7}{10}\ \therefore\ 4x+3y=6 \\ \frac{6x+4y+2}{10}=\frac{12}{7\frac{1}{2}}=\frac{16}{10}\ \therefore\ 3x+2y=7.\ x=9.\ y=-10. \end{array}\right.$

Lesson 13. *Art.* 180

A laboratory period during which eight or ten exercises are solved by the class together, and an assignment of about ten selected exercises—not necessarily the easiest—is enough time for this section. The exercises selected for the laboratory work may well be types of the exercises assigned for home work.

Exercise XLIV

Answers:

1. $-8, -4$
2. $7, 15$
3. $30, 10$
4. $12, 6$
5. $0, -2$
6. $\begin{cases} 2a-3b-4c \\ 2a-3b+4c \end{cases}$
7. $\begin{cases} a+b-c \\ a-b+c \end{cases}$
8. $5, 6$
9. $12, 4, 30$
10. $3, 4, 5$
11. $\begin{cases} \dfrac{4a^2-2ab-4b^2}{a^2-b^2} \\ \dfrac{a^2+b^2}{a^2-b^2} \end{cases}$
12. $m+n, m-n$
13. $10, 8$
14. $1, 2$
15. $3, 4$
16. (1) $\begin{cases} T=\dfrac{2Rs}{t} \\ a=\dfrac{2s}{t} \end{cases}$ (2) $\begin{cases} R=\dfrac{Tt}{2s} \\ a=\dfrac{2s}{t} \end{cases}$ (3) $\begin{cases} T=54 \\ a=27 \end{cases}$
17. (1) $\begin{cases} a=\dfrac{s}{n}-\dfrac{(n-1)d}{2} \\ l=\dfrac{2s-an}{n} \end{cases}$ (2) $\begin{cases} a=\dfrac{2s-nl}{n} \\ d=\dfrac{2(ln-s)}{n(n-1)} \end{cases}$
(1) $a=6, l=20$
(2) $d=\frac{1}{3}, l=0$
(3) $d=\frac{3}{2}, n=29$
18. (1) $a=3, l=384$
(4) $r=2, n=8$.

LESSON 14. *Art.* 181

Devote one laboratory period to four to six of these problems and exercises, and assign for home work ten to twelve of them.

Answers to Problems, § 181

1. 10, 6, 4
2. (1) 5, 2, 1
   (2) 1, 2, 3
   (3) 1, 2, 5
   (4) 1, 2, 3
3. (1) 6, 3, 3
   (2) 3, 12, 21
4. 763
5. 105, $52\frac{1}{2}$, $26\frac{1}{4}$
6. 3, 2, 1. From (1) $8x-7y=10$. $x=3$
   From (2) $5y+3z=13$. $y=2$
   From (3) $4x+3z=15$. $z=1$
7. 72°, 80°, 28°. $A+B+C=180$. $A=72$
   $\frac{1}{4}A+\frac{1}{8}B=C$. $B=80$
   $\frac{1}{2}A+\frac{1}{10}B=\frac{1}{2}C+30$. $C=28$
8. $6,000, $4,000, $8,000.

LESSON 15. THE SUMMARY

Clear away any outstanding difficulties of the chapter, and have members of the class read and illustrate the statements of the summary. Impress the class that the summary contains the substance of what pupils will be expected hereafter to know about this chapter.

CHAPTER XII

FRACTIONS

In the work already covered fractions have been handled without comment. It is now desirable that there be an explicit treatment of algebraic fractions, such as to show how the methods derived in arithmetic for addition, subtraction, multiplication, and division of fractions hold without exception in algebra. The material of this chapter is designed to present this, and as the text is full but little comment is needed.

It will be observed how the plan is to apply each principle to algebraic expressions as an induction from its use on particular numerical cases. Thus, in the reduction of fractions the first seven examples deal with definite numbers such as $\frac{5}{8}\frac{1}{6}$; after these the transition to ab/bx is easy.

In discussing a fraction such as,

$$\frac{x^n}{x^{n+3}},$$

particular numerical values may be assigned at first to n; thus for $n=7$, we have

$$\frac{x^7}{x^{10}}=\frac{1}{x^3}.$$

After it is clear that in each case we get $1/x^3$ as the simple fraction, it is well to recur to the meaning of x^n; thus

$$\frac{x^n}{x^{n+3}}=\frac{(x \cdot x \cdot x \quad . \quad . \quad . \quad . \quad \text{to } n \text{ factors})}{(x \cdot x \cdot x \quad . \quad . \quad . \quad . \quad \text{to } n \text{ factors}) \cdot x \cdot x \cdot x}.$$

The x's cancel in pairs except for the last three in the denominator, so that the result is $1/x^3$.

It is well to point out that as in reductions with numerical exponents we subtract the less from the larger, so it is best to do the same with mixed exponents. Thus:

as for $\frac{x^5}{x^9}$, we get $\frac{1}{x^4}$, so for $\frac{x^{n+2}}{x^{n+4}}$ we get $\frac{1}{x^{n+4-n-2}}=\frac{1}{x^2}$.

In examples such as 16, EXERCISE XLV, we have cases where form is important. Pupils need but a little encouragement to learn that the parenthesis may be regarded as a single quantity or letter and handled as such.

The teacher is fortunate who by this time has overcome in all pupils the tendency to "cancel" the a's in an expression such as

$$\frac{a}{a+b}.$$

This fallacy will doubtless reappear now. A good way to impress the incorrectness of such an operation is to try it out with particular numbers. For example,

$$\tfrac{2}{5} = \frac{2}{2+3} \neq \tfrac{1}{3};$$

for $\frac{2}{5}$ of a dollar, or 40 cents, is not equal to $\frac{1}{3}$ of a dollar or $33\frac{1}{3}$ cents.

Pupils should be encouraged to use this method of checking a doubtful point by substituting particular numbers, preferably primes.

This last item reminds us that the present chapter consists entirely of pure mathematics and of drill work. Consequently the teacher should supply the concrete applications which arouse interest, such as above where the abstract number relations were made concrete in terms of parts of a dollar.

Eight or ten recitations give ample time to cover the ground. Sections 182–91 might form the first half of the work, while §§ 192–97 and a review, the last half. The teacher should, as in former chapters, prepare the class for an assignment of home work by solving a few problems of each type with the class and showing clearly, in each case, the principles on which the processes depend.

Answers to Exercises

Exercise XLV

1. $\frac{3}{4}$
2. $\frac{4}{9}$
3. $\frac{4}{7}$
4. $\frac{3}{5}$
5. $\frac{13}{18}$
6. $\frac{3}{5}$
7. $\frac{77}{111}$
8. $\frac{a}{x}$
9. $\frac{b}{y}$
10. $\frac{x^2}{y^3}$
11. $\frac{b}{am}$
12. $\frac{u}{vy}$
13. $\frac{1}{x^3}$
14. $\frac{1}{x^2}$
15. $\frac{y^4}{x^8}$
16. $\frac{a}{b(x+y)}$
17. $\frac{x-y}{m^2}$
18. $\frac{c}{r}$
19. $\frac{1}{m^2n}$
20. $\frac{(a+b)^2(b-c)}{(c+d)^2}$
21. $\frac{1}{(u+v)^2}$
22. $\frac{(r-s)^4}{a^3b^4}$
23. $\frac{ab^3(a^2+b^2)^2}{c^{14}(a+b)^4}$.

Exercise XLVI

1. 1
2. $\frac{4}{5}$
3. 1
4. $\frac{10}{13}$
5. $\frac{1}{9}$
6. -6
7. $\frac{a+b}{c}$
8. $\frac{a-b}{c}$
9. $\frac{m+n+k}{x}$
10. $\frac{r-s+t-v}{y}$
11. $\frac{1}{a+b}$.

Exercise XLVII

1. $\frac{35}{23}$
2. $\frac{x-y}{2}$
3. $\frac{5}{23}$
4. $\frac{x-y}{a}$
5. $\frac{a-b-c}{x}$
6. a
7. $\frac{x+y}{a}$
8. $\frac{c+ax}{2s}$
9. $\frac{8}{5}$
10. $\frac{x}{a}$
11. $\frac{3a+2b}{x-y}$.

Exercise XLIX

1. $\frac{2}{15}$
2. $\frac{3}{28}$
3. $\frac{x-9}{ax}$
4. $\frac{y-3}{3y}$
5. $\frac{x-b}{bx}$
6. $\frac{b-c}{bc}$
7. $\frac{m-t}{mt}$
8. $\frac{a}{ab+b^2}$
9. $\frac{-2b}{a^2-b^2}$
10. $\frac{a-b}{(2a+3b)(3a+2b)}$
11. $\frac{-2y^2}{x^4-y^4}$
12. $\frac{c}{(a+b)(a+b+c)}$
13. $\frac{2a+2b-15}{3a+3b}$.

Exercise L

1. $\frac{87}{88}$
2. $\frac{3}{28}$
3. $\frac{79}{132}$
4. $\frac{8x\pm 21}{24}$
5. $\frac{4a\pm 21}{36}$
6. $\frac{2a\pm c}{10}$
7. $\frac{x\pm y}{a}$
8. $\frac{8y\pm 7x}{xy}$
9. $\frac{dx\pm cy}{dy}$
10. $\frac{x(b\pm a)}{ab}$
11. $\frac{19a+46}{35}$
12. $\frac{4ax}{a^2-x^2}$
13. $\frac{a^2}{x^2-3a+2a^2}$
14. $\frac{2a^2x^2}{x^4-a^4}$.

Exercise LI

1. $\frac{5p+2q+r}{10}$
2. $\frac{8b+5c+21d}{6}$
3. $\frac{4a}{15}$
4. $\frac{x-y}{ax}$
5. $\frac{ay+bx}{cxy}$
6. $\frac{a-b}{ca}$
7. $\frac{x-2az}{12ab}$
8. 0
9. $\frac{1}{x^2y^3}$
10. $\frac{a-b}{a-1}$
11. $\frac{1}{2(x-1)}$
12. $\frac{3x+y}{2x(2x-3y)}$
13. $\frac{b(3d+a)}{3d(a+2b)}$
14. $\frac{a+bx}{x}$
15. $\frac{x-ay}{y}$
16. $\frac{17a}{3}$
17. $\frac{26x}{3}$
18. $-\frac{x}{4}$
19. $\frac{ay+bx}{x^2y^2}$
20. $\frac{2(2a+b)}{3x}$
21. $\frac{5b-7a}{15x}$
22. $\frac{1-a+a^2}{a^3}$
23. $\frac{a^2+b^2+c^2}{abc}$
24. $\frac{2y-x}{y(x-y)}$
25. $\frac{2x^2+18xy-12y^2-4xz+9yz}{3xy}$
26. $\frac{3b^2+5ab-4a^2+3ac-4bc}{ab}$
27. $\frac{xy+xz-4}{x^2y^2}$
28. $\frac{-b^2}{a(a-b)}$
30. $\frac{9x}{x^2-1}$
31. $\frac{-4ad}{c^2-d^2}$

EXERCISE LII

Results evident

EXERCISE LIII

1. $\frac{16}{21}$
2. $\frac{15}{32}$
3. $\frac{15a}{28b}$
4. $\frac{5rx}{6ty}$
5. $\frac{acx}{bdy}$
6. $\frac{n^1n^2n^3}{d_1d_2d_3}$
7. $\frac{a^6}{b^8}$
8. $\frac{1}{a^2}$
9. 2
10. a.

EXERCISE LIV

1. $\frac{3}{100}$
2. $\frac{10}{21}$
3. c
4. a
5. $\frac{a}{bx}$
6. $\frac{a}{d}$
7. $21axyz$
8. $\frac{az}{cx}$
9. $\frac{5a^2}{9y^2}$
10. $\frac{9az}{10c}$
11. $\frac{35b^2}{64y^2}$
12. $\frac{5y}{7x}$
13. $\frac{3b^2x^2}{10y}$
14. $\frac{(a+b)^2}{a^2+b^2}$
15. $\frac{9x}{8}$
16. $\frac{ab}{a^2-b^2}$
17. $\frac{9xy}{4}$
18. $\frac{(a-b)(x+y)}{(a+b)(x-y)}$
19. $\frac{a^3+6a^2+9a+6}{a^3-6a^2+9a-6}$
20. $\frac{x(a+b)}{3(x-y)}$

EXERCISE LV

Results evident

EXERCISE LVI

1. $\frac{x^2}{y}$
2. $\frac{abz^2}{xy}$
3. $\frac{21xy}{4}$
4. $\frac{45a^4x^2}{2}$
5. $\frac{6abz^2}{5xy}$
6. $\frac{-5a^2b^2}{3d}$
7. $\frac{a^3b^2(4x-5y)}{xyz}$
8. $\frac{2by}{3x^2}$
9. $\frac{5b^2xy^2}{7a^3c^3}$
10. $\frac{32am^3z^2}{7bx^3}$
11. x^4
12. $\frac{1}{a^5b^3}$
13. $\frac{8}{25}(x-1)(a-b)$
14. $1\frac{7}{12}$.

CHAPTER XIII

FACTORING, QUADRATICS, RADICALS

In this chapter we take up a matter of great importance in the successful use of algebra. The student who cannot factor is seriously handicapped in all future work. It is hardly possible to lay too much stress on the attainment of proficiency in factoring.

The material here presented may be classified as follows:

(*a*) Eight elementary methods of factoring algebraic expressions, viz.:

Extraction of monomial factors.

Factoring by grouping terms.

Trinomial squares.

The difference of two squares.

Trinomials of the form x^2+ax+b.

Trinomials of the form ax^2+bx+c.

The sum of two cubes, and the difference of two cubes.

The "remainder theorem."

(*b*) Application of these methods to the solution of quadratic and higher equations; and an introduction to radicals.

(*c*) Review material applying factoring in the four fundamental operations.

The teacher should take up each new method at the board first; it is too much to expect the pupils to get up the theory by reading the text. Further, it is not expected that all examples be worked. The chapter is made full to afford ample material from which each teacher may make an individually satisfactory selection.

§§ 198–203 take up the topics: "What is a factor?" and how to extract monomial factors.

Many pupils have much trouble with factoring because they lack a clear notion of what they are trying to do. The first object, then, should be to show clearly how a factor of an expression is an exact divisor thereof, and then to drill on this notion. The class should be called upon for numerical examples first, after which § **200** may be taken up orally; and then, when the examples become too difficult for clear work, the class may be sent to the board, each pupil to put on in order all he can of the list. This affords the teacher an opportunity for personal work with the more backward.

After practice in factoring, the idea that "a factor is an exact divisor" should be recalled and drilled upon. This may be done by remultiplying the factors, and by substituting particular numbers as shown on page 276. This sort of testing should never be dropped. Later when the expressions are complicated enough it is well to have pupils actually use long division to divide an alleged factor into an expression. By every method strive to have them realize clearly that a factor is an exact divisor.

Exercises LVIII and LIX contain review material with applications of factoring. The first list might be assigned for home work after the first lesson. The next day, after review and discussion, Exercise LIX can be considered, as many examples as possible worked in class and the remainder assigned for home work. At the third meeting the next method can be presented.

§§ **204** to **208** are devoted to factoring polynomials by grouping the terms. After the material in § **204** is presented Exercise LX may be started by having single pupils work the first examples at the board. This is a place where remultiplication and long division are invaluable to check the work and to bring out clearly the nature of the method. After several of the simpler examples have been worked separately, general board work may be used as before. For home work examples 11 to 20 and in § **206**, 1 to 5 might be assigned.

At the next recitation the more difficult examples 24 to 30 should be considered. Here is an opportunity to bring up the notion that in factoring it is the *form* of the expression that is of importance. Thus in 24 the binomial factor $(a+m)$ can be treated as a monomial factor, and in 25 and 26 $(x+y)$ appears as an exact divisor. To consider 27, begin with particular numbers, thus: $x=5$, $m=7$, $y=3$. Then the pupils will have little difficulty in factoring

$$a^{5+7}+a^7b^3+a^5b^7+b^{7+3}.$$

By several such cases the underlying form can be brought into evidence, and then

$$a^{x+m}+a^mb^y+a^xb^m+b^{m+y}$$

can be factored.

In §§ **209–10** we consider trinomial squares. To have clearly in mind the result of squaring a binomial the twenty-one examples in § **209** are given. The writer has found it expedient to send the class to the board and have each example put on by a pupil. With the products all in evidence the rule for squaring a binomial results almost automatically and then can be reversed for factoring. It is important in all factoring that the pupil be able *to describe the procedure in words*. In the present case each pupil should be led to state substantially this method: "Take the roots of the perfect squares and see if their product is half the other term."

In Problems 20–26 we again have illustration of the importance of recognizing form. Some of these examples should be considered in class first. Thus in 20 suggest the substitution $A=(m+n)$. The factors of A^2+2A+1 are evident at once; then the term $(m+n)$ may be restored wherever A appears.

This device of substitution to bring out the form of the whole expression is a powerful instrument throughout all algebra, and the students should be encouraged to use it freeely.

The practical application in Problem 29 of the squaring process usually arouses interest. The examples should be worked orally.

§ 211 is a review exercise which might be assigned for home work and graded with the value of a quiz.

We proceed to the important use of factoring trinomial squares in the solution of a quadratic equation. That "quadratic" means *second-degree* needs emphasis. The first lesson should be taken up with a graphical study showing how a quadratic expression takes on different values for different values of the unknown. Not only should the graph on page 284 be reproduced but the first two examples on page 285 should be worked in class, different pupils being called upon to compute, mentally, the expression-value for a given x and then others required to come to the board and plot the points. The remaining problems, 3–8, should be assigned in pairs to different groups in the class for home work. At the next recitation the graphs can be copied on the board; after criticism thereof we are ready to bring out the idea of the *quadratic equation* as *a condition to be satisfied.* To every quadratic expression there corresponds a quadratic equation which sets the condition that the expression be zero. This condition is satisfied only by the values for x at the points where the graph cuts the x-axis.

The algebraic method of solution can be presented now, EXERCISE LXVI being taken orally. During the work at the board it is advantageous to get from each pupil a description of his method for finding how to complete the square.

The solution of quadratic equations by completing the square naturally brings to light equations such that when the left member has been made a perfect square, the right member is not. The only difficulty is to give to radical numbers a meaning which will be both clear to the students and also accurate enough for work. This topic is considered in §§ 223–30; a fuller study of radicals comes in the next chapter. The graphical solution aids greatly in the algebra here. It is well to have the four graphs on page 292 plotted at home and ready for reference when the topic is begun in class. Recall that when we found no integer to represent the quotient of $5 \div 7$ we *created a new kind of number* by representing the division thus: $\frac{5}{7}$. In like manner, when we can find neither integer nor fraction such that its square is, say 7, we make a new number by indicating the process thus, $\sqrt{7}$.* Pupils take hold of radicals readily.

When the drill problems as equations have been solved a further examination of radicals may be begun. In presenting the material of §§ 225–28 it is well to do on the board all the arithmetic that is merely indicated there; a previous rehearsal on paper will insure smooth and rapid progress at the board.

For drill in the use of all this information actual substitution of irrational roots in the equation is invaluable. A convenient form for the work is as follows:

Prove [page 297, 11 (7)] that $\frac{-11+5\sqrt{5}}{2}$ is a root of the equation

$$q^2+11q-1=0.$$

If $$q=\frac{-11+5\sqrt{5}}{2},$$

* It is of interest to tell the class how historically this radical sign $\sqrt{\ }$ is a conventionalization of a script *r* written before the number as an abbreviation for *radix* or *root*, when a useful purpose thereof is evident as is the case in solving the equation here.

then
$$q^2=\frac{121-110\sqrt{5}+25(5)}{4}=\frac{123-55\sqrt{5}}{2}$$
$$11q=\qquad\qquad=\frac{-121+55\sqrt{5}}{2}$$
$$-1=\qquad\qquad=\frac{-2}{2}$$
adding
$$q^2+11q-1=\qquad\qquad=0$$

Such substitution not only drills in the manipulation of radical expressions but also arouses a clearer understanding of what a root does for the equation. § 230 furnishes drill in the arithmetic operations involved in this substitution.

In Exercise LXVIII, the factoring of the difference of two squares may be presented in the same manner as that detailed for the trinomial square and for factoring by grouping. In Exercise LXIX, again the recognition of forms is important. In considering such expressions as $(a^{10}-b^{10})$, Problem 20, the main point is to show that if we *divide the exponent* of a^{10} by 2 we get the square root of a^{10}; for

$$a^5\cdot a^5=a^{5+5}=a^{10}.$$

In such a problem as 41, viz.:

$$9x^2+16y^2-49a^2-4b^2-28ab+24xy,$$

it is sufficient usually to recall that we are free to rearrange the terms so as to have two perfect squares in evidence; thus

$$(9x^2+24xy+16y^2)-(49a^2+28ab+4b^2).$$

§§ 237–41 may be conveniently given together in three lessons. The material of § 237 should be presented with the first few examples in evidence on the board. It is well to call attention to the following facts. In x^2+bx+c, if c is positive then both numbers in the factors are of like signs: plus if b is plus, negative if b is negative. If c is negative then the two numbers have *unlike* signs and the larger has the same sign as b.

In § **238** the examples 21–30 are rather difficult and may be omitted with a weak class. In factoring one such as

(24) $$k^{2m}-20k^mL^r-69L^{2r},$$

it is helpful to arrange the letters in the factors first. Thus dividing $2m$ and $2r^2$ by 2 we have

$$(k^m \quad L^r)(k^m \quad L^r).$$

Once these are determined the arrangement of the numerical coefficients is not difficult.

In applying this method of factoring to the solution of quadratic equations a new principle is used which is quite different from that of completing the square. After the left member is factored it is well, first, to recall that a solution is *any number* such that when it is put in place of x the expression reduces to zero. Second, that if any number be multiplied by zero the result is zero. Finally, we remark that in, say,

$$(x-7)(x+6)=0$$

while a 7 for x will make the second factor 13, it makes the first factor zero, and $0\times 13=0$. This is the chief difficulty. The teacher will find that substitution of the particular value of x in both factors at once helps to clear away a vague notion that in saying $x=7$ or $x=-6$, we are each time arbitrarily ignoring half of our problem. Here, as in all quadratics, a check, by substitution in the original equation, should be required.

By reducing a quadratic equation to factors it becomes evident how the equation can be made for given roots. Thus §§ **242–43** follow naturally. Still this may well be omitted with a weak class.

Now the factorization of trinomials of the form ax^2+bx+c

follows naturally. The material of the text may be presented after the manner already suggested. Individual help to students at the board is most useful here. Pupils can understand the problems when worked by someone else, but are backward when set to do the work by themselves. The principal cause is reluctance to try out all possible combinations, or lack of system in so doing. The correction of all this requires individual encouragement.

The application of this method of factoring to quadratic equations follows in § 254.

In §§ 256–57 factorization of the sum or the difference of two cubes is considered. The material may be presented in the manner suggested for the trinomial square. To combat the tendency to write

$$(a^3-b^3)=(a-b)(a^2+2ab+b^2),$$

remultiplication and long division are very effective. It is helpful to remark: that the binomial factor is always the cube roots of the quantities in the factor and connected by the same sign as these, while the trinomial factor has for sign on the cross-product term the opposite to the middle sign in the binomial factor.

The text deals quite fully with the remainder theorem in §§ 258–67. All the examples in § 258 should be worked fully on the board at once, for the design is to get the remainder theorem stated as an induction. By the time a fairly evident relation has appeared in some twenty cases, only a very backward class will fail to perceive it. Once it is perceived, the general proof (§ 261) follows easily.

§ 262 is designed to bring out the fact that the constants in all exact factors of a polynomial are exact divisors of the last term. This helps us to tell what numbers only need be tried in factoring the polynomial.

The following schedule of recitations is suggested:

§§ 198–201	2	recitations
204–206	2	
209–210	2	
212–222	3	
223–230	3	
233	2	
237–241	3	
247–248, 253–255	2	
256–257	2	
258–267	3	
Total	24	recitations
[242–243]	1	recitation

It will be remarked that very slight comment or none at all has been made upon certain sections, namely: 202, 203, 207, 208, 211, 231, 232, 234–36, 244–46, 249–52. These are the review sections classified under (*c*) in our opening remarks. The teacher will use these as the time for the course permits and as individual judgment approves. In the schedule above, time has been allowed for most of these sections. The same is to be said of the miscellaneous exercises (§§ **268–74**) with which the chapter closes.

Answers

Exercise LVII

Results evident

Exercise LVIII

1. $\frac{3}{5}$
2. $\frac{a}{x}$
3. $\frac{x+y}{2(a+b)}$
4. $\frac{5x}{4a}$
5. $\frac{4xy}{7z}$
6. $\frac{a-2b}{3c+4ab}$

7. $\frac{2m^2}{3n^3}$

8. $\frac{5d^2}{6xy^2}$

9. $\frac{1}{8}x^2$

10. $\frac{a^2}{d^2}$

11. $\frac{2}{3}$

12. $\frac{22}{3(x-3)}$

13. $\frac{bc+2ac-3ab}{abc(x+y)}$

14. $\frac{1}{4}$.

Exercise LIX

2. $x=c$

3. $y=2m$

4. $v=5r^2s$

5. $k=m$

6. $x=d$

7. $m=6w^3$

8. $x=\frac{d}{a+b-c}$

9. $y=c$

10. $k=ab$

11. $x=\frac{a}{u+v}$.

§ 203

1. $\frac{5m}{m+1}$, $\frac{5}{m+1}$

2. $\frac{3k}{7-k}$, $\frac{21}{7-k}$

3. $\frac{bc}{a+b}$, $\frac{ac}{a+b}$

4. $\frac{m(u+v)}{m-n}$, $\frac{n(u+v)}{m-n}$

5. a, ar, ars.

Exercise LX

1. $(a+b)(x+m)$
2. $(a+b)(r+s)$
3. $(a+b)(d+t)$
4. $(a+b)(3+y)$
5. $(a-b)(k+l)$
6. $(a-b)(x^2+y^3)$
7. $(ab+n)(c+x)$
8. $(a^2+b^2)(k+l)$
9. $(5a+m)(u-v)$
10. $(a+m)(1+ma)ma$
11. $(a+b)(a-d)$
12. $(x^3+1)(x^2+5)x$
13. $(2x-3)(3x-5y)$
14. $(m^2+3)(2m+1)$
15. $(c+x)(3a-5)$
16. $(1+3r^2)(3-5r)3$
17. $(2g+3a)(4h+5b)$
18. $(5z-2)(3-4w)$
19. $(m-7n)(2m+3k)$
20. $(b+x)(3a+2x+1)$
21. $(x^2+1)(x-z)4$
22. $(1+r)(1-r^2xy)$
23. $(1-x)(x^2+1)$
24. $(a+m)(c-n)$
25. $(x+y)(a-c)$
26. $(x+y)(1+mx+my)$
27. $(a^x+b^4)(a^m+b^m)$
28. $(2x^r-3y^s)(x-y)$
29. $(3m^2-5n^3)(m^u+n^v)$
30. $(k^2-1)^2k^{n-1}(k^2+1)$.

Exercise LXI

1. $\frac{x+m}{r+s}$
2. $\frac{3+a}{5b+2k}$
3. $\frac{a-b}{m+n}$
4. $\frac{x^3+7}{2x^2+6}$
5. $\frac{a-5b}{5a-b}$
6. $\frac{3}{5}\cdot\frac{x+y}{m+n}$
7. $\frac{5c}{y}$
8. $\frac{m+a}{(b+c)(m+k)(a-k)}$
9. $\frac{6y}{5x}$.

Exercise LXII

1. $x=c+d$
2. $x=3\frac{cd}{ab}$
3. $x=\frac{ab+ac+bc}{a-b}$
4. $m=\frac{n-k}{2}$
5. $v=5+2w,$
   $w=\frac{v-5}{2}$
6. $y=\frac{ad+ac-bc}{a-b}$
7. $x=d+c$
8. $m=\frac{4cd}{5ab}$
9. $u=\frac{15}{4}$
10. $x=7$
11. $y=\frac{71}{48}$.

§ 208

1. $1+a,\ r(1+a)$
2. $a,\ b;\ 2a,\ 2b.$

Exercise LXIII

Results evident

Exercise LXIV

1. $(x+y)^2$
2. $(a-b)^2$
3. $(m+2n)^2$
4. $(2a-1)^2$
5. $(b-3)^2$
6. $(k+4)^2$
7. $(a+5b)^2$
8. $(2m-3a)^2$
9. $(5+8r)^2$
10. $(11a+9y)^2$
11. $(c-8)^2$
12. $(x^2+15)^2$
13. $(7-10n^2)^2$
14. $(abc+4)^2$

15. $(7x^3-114)^2$
16. $(25n^2+v^2)^2$
17. $(4h+9vw)^2$
18. $(13r-3t^2)^2$
19. $(15s^6-14c^5)$
20. $(m+n+1)^2$
21. $(u+v+2t)^2$
22. $(3k+r+s)^2$
23. $(a+b+c+d)^2$
24. $(m+n+1)^2$
25. $(m+n+3a)^2$
26. $(a+b+c)^2$

29. (1) 169
(2) 196
(3) 225
(4) 441
(5) 484
(6) 961
(7) 361
(8) 324
(9) 1,849
(10) 1,444
(11) 5,184
(12) 6,561
(13) 8,281
(14) 7,921
(15) 4,489
(16) 10,609.

Exercise LXV

1. $k^3(a-b)^2$
2. $xy(x^2+y^2)(x-y)$
3. $a(x^2+1)(x+1)$
4. $7b(x+3y)^2$
5. $3xy(2a+1)^2$
6. $(x+1)(5x^4+4x^2+3)$
7. $(c+d)^3$
8. $(x-3y-z)^2$
9. $(m+n)(3a+2)^2$
10. $(x+y)(x-y)2y$.

Exercise LXVI

Results evident

§ 221

2. $\frac{1}{2}, -\frac{5}{2}$
3. $\frac{2}{5}, -\frac{5}{2}$
4. (1) $\frac{7}{2}, -\frac{2}{3}$
(2) $\frac{4}{3}, -\frac{5}{2}$
(3) $\frac{2}{3}, -4$
(4) $\frac{3}{4}, -12$
(5) $\frac{5}{2}, -\frac{2}{5}$
(6) $2\frac{1}{4}, \frac{2}{3}$
5. (1) $1, -3$
(2) $-1, -3$
(3) $1, -5$
(4) $2, -10$
(5) $-5, -9$
(6) $3, -17$
(7) $-5, 17$
(8) $-2, 12$
(9) $-7, 13$
(10) $-2, -1$.

(11) $-2, -3$
(12) $5, 8$
(13) $6, -7$
(14) $-m, -5m$
(15) $2, 6$
(16) $2, \frac{3}{2}$
(17) $-3, 5$
(18) $-\frac{5}{2}, -\frac{7}{2}$.

§ 222

1. 6, 10
2. 6, 8
3. 5, 6
4. 4, 10
5. 240.

§ 224

2. (1) $m=-3\pm\sqrt{5}$
   (2) $-3\pm\sqrt{3}$
   (3) $-3, -3$
   (4) $-3\pm\sqrt{-1}$
3. (1) $-5\pm\sqrt{5}$
   (2) $-5\pm\sqrt{3}$
   (3) $-6\pm\sqrt{3}$
   (4) $-6\pm\sqrt{7}$
   (5) $7\pm2\sqrt{2}$
   (6) $-7\pm\sqrt{7}$
   (7) $-7\pm\sqrt{5}$
   (8) $-8\pm\sqrt{2}$
   (9) $3\pm\sqrt{5}$
   (10) $4\pm\sqrt{11}$.

§ 230

7. $32+13\sqrt{2}$
8. $51+6\sqrt{6}$
   $87-19\sqrt{3}$
10. -2
   $120-23\sqrt{8}$
   $50+13\sqrt{10}$
11. (1) $r=\dfrac{-1\pm\sqrt{29}}{2}$
   (2) $d=\dfrac{-7\pm\sqrt{37}}{2}$
   (3) $x=-2\pm\sqrt{5}$
   (4) $y=3\pm\sqrt{11}$
   (5) $m=5\pm\sqrt{19}$
   (6) $v=\dfrac{-3\pm\sqrt{53}}{2}$
   (7) $q=\dfrac{-11\pm5\sqrt{5}}{2}$
   (8) $s=-10\pm8\sqrt{2}$.

Exercise LXVII

1. $\dfrac{m}{3a^2+2b^2}$
2. $\dfrac{5c+d}{a+b}$
3. $\frac{2}{3}k$
4. $\dfrac{5kl}{x+1}$

5. $\dfrac{5x-11}{3(x+2)^2}$

6. $\dfrac{1}{x+y}$

7. $\dfrac{84x-101}{10(2x-3)^2}$

8. $2t+1$

9. $(4a-3b)(2a-9b)$.

Exercise LXVIII

Results evident

Exercise LXIX

44. $(a^2+b^2+3ab)(a^2+b^2-3ab)$
45. $(x^2+x+1)(x^2-x+1)$
46. $(4x^2-y^2+3xy)(4x^2-y^2-3xy)$
47. $(5x^2+4y^2+3xy)(5x^2+4y^2-3xy)$
48. $a^2(x^4+x^2+1)(x^4-x^2+1)$
49. $(7a^2b^2-2x^2-5abx)(7a^2b^2-2x^2+5abx)$
51. (1) 399
(2) 396
(3) 391
(4) 899
(5) 896
(6) 891
(7) 884
(8) 1,599
(9) 1,596
(10) 4,891
(11) 8,096
(12) 4,875
(13) 8,091
(14) 8,075
(15) 9,996.

Exercise LXX

1. $\dfrac{m(x+y)}{a+b}$

2. $\dfrac{1}{(x-a)(x-b)}$

3. $\dfrac{k^3(m^2-mn+n^2)}{a^2}$

4. $\dfrac{2(m^2+4)}{6a^4-5b}$

5. $\dfrac{4a-3c-d+2b}{4a-3c+d-2b}$

6. $\dfrac{a+2b+3c}{a+2b-3c}$

7. $\dfrac{1-a-b}{1+a+b}$.

§ 235

1. $\frac{(a+4)(a+8)}{(a-4)(a-8)}$
2. $\frac{(2x-3y)(x+a+b)}{a(2x+3y)(x-a-b)}$
3. $\frac{2ab(a-b)(2a-5b^2)}{2a+5b^2}$
4. $\frac{ab(mx+n)}{x(1+ab)(2x+3y)}$
5. $\frac{(a^2+b^2)(r+s)}{(a^2-b^2)(c+b)}$
6. $\frac{2}{2x-3}$
7. $\frac{r+3}{r+1}$.

EXERCISE LXXI

1. $k=a-b$
2. $x=a+3b$
3. $m=15$ or -5
4. $m=\frac{n}{a-b}$, $n=m(a-b)$
5. $r=11, -4$
6. $t=3h^3$
7. $x=m-n$
8. $x=\frac{a-2b}{a+2b}$
9. $m=12, -4$
10. $g=\frac{t}{u-v}$, $t=g(u-v)$.

EXERCISE LXXII

1. $(x+2)(x+1)$
2. $(m+2)(m+3)$
3. $(k+5b)(k+7b)$
4. $(y+5)(y+9)$
5. $(h-5)(h-8)$
6. $(a-2)(a+10)$
7. $(g-12)(g+15)$
8. $(b+3c)(b+16c)$
9. $(q^3-6)(q^3+7)$
10. $(t+3)(t-17)$
11. $(r+17s)(r-19s)$
12. $(v+7w)(v-13w)$
13. $(ab+11c^2)(ab-15c^2)$
14. $(z^2+2)(z^2-12)$
15. $(y+5)(y-17)$
16. $(d+3)(d-10)$
17. $(p+9)(p-10)$
18. $(r+3)(r-16)$
19. $(k+3b)(k+37b)$
20. $(xy+7z)(xy-16)$
21. $(r^m-3)(r^m-16)$
22. $(k^{3t}+6)(k^{3t}-13)$

23. $(d^{5r}+6f^{3t})(d^{5r}+17f^{3t})$
24. $(k^{m}+3L^{r^s})(k^{m}-23L^{r^s})$
25. $(rst^3+3v^2w)(rst^3-21v^2w)$
26. $(m+n+2)(m+n+3)$
27. $(a+b-2)(a+b-5)$
28. $(a+b-2)(a+b+5)$
29. $(a-3b-4c)(a-3b+11c)$
30. $x(m^2+26)(m^2-25)$.

Exercise LXXIII

1.	1, 2	8.	3, 17	15.	5, 7
2.	1, 3	9.	−7, 11	16.	−3, 10
3.	−1, 2	10.	7, 16	17.	7, −13
4.	1, −6	11.	−2, 5	18.	2, 5
5.	−1, −6	12.	−2, 15	19.	2, 5
6.	−2, 6	13.	−2, 14	20.	−1, 8.
7.	7, −8	14.	−6, 9		

Exercise LXXIV

1. 10, 6
2. 40, 80
3. 10, 20
4. 250
5. 3, 6.

§ 242. 3

1. x^2-6x+8
2. $x^2+3x-10$
3. $x^2-2x-15$
4. $x^2+8x+15$
5. $2x^2-9x+4=0$
6. $2x^2+13x+6=0$
7. $x^2-x-72=0$
8. $x^2+2x-63=0$
9. $x^2+7x-60=0$
10. $x^2-(a+b)x+ab=0$.

§ 243. 3

1. $x^2-4x+1=0$
2. $x^2-6x+6=0$
3. $x^2-6x+4=0$
4. $x^2-8x+9=0$
5. $x^2-10x+15=0$
6. $x^2+6x+6=0$
7. $x^2+6x+4=0$
8. $x^2+8x+9=0$
9. $x^2+10x+18=0$
10. $x^2+12x+15=0$.

Exercise LXXV

1. $bc^2(a+2)(a-4)$
2. $(m+n)(b+3)(b-15)$
3. $(p-1)(p+1)(p-5)(p+5)$
4. $(p-2)(p+2)(p^2+4)$
5. $2(a+b)(x+y-z)$
6. $(v^8+w^8)(v^4+w^4)(v^2+w^2)(v+w)(v-w)$
7. $(m-n+c+d)(m-n-c-d)$
8. $(2x^2-7)(4x-3)$
9. $3(k-l)(k-l+2m)$
10. $(x-4y)(1-3z)$
11. $[(m+n)-7(c+d)][(m+n)-4(c+d)]$
12. $abx(c+5)(c-15)$
13. $10c^3(9a^2+b^2)(3a+b)(3a-b)$
14. $(a^m+4b^{3n})(a^m-9b^{3n})$
15. $(m^2-2mn+2n^2)(m^2+2mn+2n^2)$.

§ 245

1. $\frac{x-2}{x-3}$
2. $\frac{x+b}{x+2a}$
3. $\frac{2a+3b}{x+b}$
4. $\frac{2+b}{a+b}$
5. $\frac{m-8}{m+6}$
6. $\frac{x^n-3}{x^n-5}$.

§ 246

1. $\frac{(a+5)(2a+3)}{a}$
2. $\frac{4}{(1-m)(m-3)}$
3. $\frac{11-y^2}{(y-2)(y-5)(y-7)}$
4. ab
5. $\frac{x+y-z}{x-y+z}$.

Exercise LXXVI

§ 248

1. $(2x+3)(x+4)$
2. $(c+6)(8c-2)$
3. $(x-5)(3x-2)$
4. $(z-3)(8z-7)$
5. $(x-7)(5x-3)$
6. $(a-2)(11a-1)$
7. $(k+18)(7k-3)$
8. $(t+3s)(12t-5s)$
9. $(5m-9)(m-4)$
10. $(2r-5)(5r+1)$
11. $(3b-7)(2b-5)$
12. $(3f-11)(2f+7)$
13. $(6-a)(17+a)$
14. $(5-z)(3+8z)$
15. $(1-5xy)(1-xy)$
16. $(x^n+4)(2x^n+3)$
17. $(2x^m+7y^n)(7x^m+2y^n)$
18. $(2a+2b-3)(3a+3b+1)$
19. $(x+y+3)(4x+4y+1)$
20. $(c-d+1)(3c-3d-5)$.

§ 249

1. $xy(c-6)(2c-1)$
2. $(3x+2)(x^2-3)$
3. $(x+1)^2(x-1)^2$
4. $6a(a-2b)(a-3b)$
5. $(a+b+c)(a-b-c+1)$
6. $(a-b)(a+b-1)$
7. $(4x^2+9)(2x+3)(2x-3)$
8. $a(a^4+16)(a^2+4)(a+2)(a-2)$
9. $(2a-1)(a-1)(2a^2+3a-1)$
10. $(a-11)(a+7)b^3$
11. $3(m^3+2)(m^3+11)$
12. $(b^n-21)(b^n-3)$
13. $(a-\frac{1}{2})^2$
14. $\left(m-\frac{2}{3n}\right)^2$
15. $7(r-.1)(r+.1)$
16. $(a^2+b^2)(t^2+r-s)$.

Exercise LXXVII

1. $\dfrac{m+2}{m+3}$

2. $\dfrac{2x+3}{3x+4}$

3. $\dfrac{2a+3}{3a+5}$

4. $\dfrac{3d+2}{4d+5}$

5. $\dfrac{a(m+4)}{2m+3}$

6. $\dfrac{5k+2l}{3k+2l}$

7. $\dfrac{z(y-5)}{a(2y-7)}$

8. $\dfrac{2x-3b}{3(x-b)}$.

§ 251

1. $\dfrac{3(3-4b)}{(2b-1)(2b-3)}$

2. $\dfrac{-(33c+8)}{(2c+1)(c+4)(9c+5)}$

3. $\dfrac{b(b+3x)}{x(b-3x)}$

4. $\dfrac{y(y+2)(y-2)+(2y-1)^2}{(y+1)(2y-3)(3y-4)}$

5. $\dfrac{w(y-12)}{y(w+12)}$.

§ 252

1. $a=\dfrac{bn^2+bn-2b-3m}{m(2m-5)}$

   $b=\dfrac{2am^2-5am+3m}{n^2+n-2}$

2. $k=\dfrac{9x^2+3xy-2y}{15x^2+xy-6y}$

3. $l=\dfrac{2x^2-x(7-l)-15}{x^2-20}$

4. $x=\dfrac{y(3a^2+35)}{2a^2+17a+21}$

   $y=\dfrac{x(2a^2+17a+21)}{3a^2+35}$

5. $m=\dfrac{n(2x^2+21)}{x^2+5x+15}$

   $n=\dfrac{m(x^2+5x+15)}{2x^2+21}$.

§ 254

1. $2,\ \frac{1}{2}$
2. $-\frac{3}{2},\ -\frac{4}{3}$
3. $m=+\frac{2}{5}x,\ -\frac{7}{2}x$
   $x=\frac{5}{2}m,\ -\frac{7}{3}m$
4. $k=\frac{3}{4}l,\ -\frac{2}{9}l$
   $l=-\frac{9}{2}k,\ +\frac{4}{3}k$
5. $-3,\ \frac{2}{5}$
6. $m=\frac{5}{6}n,\ \frac{3}{4}n$
   $n=\frac{6}{5}m,\ \frac{4}{3}m$
7. $c=\frac{1}{2}d,\ 6d$
   $d=2c,\ \frac{1}{6}c$
8. $a=\frac{3}{4}b,\ -\frac{5}{2}b$
   $b=\frac{4}{3}a,\ -\frac{2}{5}b$

9. $x=\frac{7}{2}y, \frac{3}{7}y$
10. $kl=\frac{7}{2}m, -\frac{3}{4}m$
11. $3, -\frac{1}{2}$
12. $3, -\frac{14}{3}$
13. $2, -3$
14. $3, -\frac{4}{3}$
15. $0, \frac{3}{2}$.

§ 255

1. $\frac{5}{2}, \frac{15}{2}$
2. $12, 9$
3. $4, 3\frac{3}{4}$
4. $25, 40$
5. 6.

§ 263

1. $(x-1)(x^2+x+1)$
2. $(x-3)(x-5)(x+1)$
3. $(x-3)(x-4)(x+2)$
4. $(x+2)(x^2-2x+4)$
5. $(x+a)(x^2+ax+4a^2)$
6. $(y+4)(y^2+4y+16)$
7. $(r+2)(r+4)(r-5)$
8. $(m+2)^2(m+\frac{1}{2})$
9. $(x+1)(x-3)(x+4)(x-5)$
10. $(x+1)(x-2)(x-3)(x+4)$.

§ 265

1. $1, 2, 3$
2. $-1, 3, -4$
3. $-2, 2, -4$
4. $-3, 3, -5$
5. $-4, 5, +6$
6. $2, -2, -2$
7. $3, 1+\sqrt{-3}, 1-\sqrt{-3}$
8. $-\frac{1}{2}, -1, -3, -\frac{7}{2}$
9. $2, 2, 3, 4$
10. $1, 5$
11. No real roots.

Miscellaneous Exercises

§ 268

1. $(m+n)(m-n)$
2. $(2a+3b)(2a-3b)$
3. $(1+r^2)(1+r)(1-r)$
4. $y(x+y)(x-y)$

5. $(u+v)(u^2-uv+v^2)$
6. $(c^2+d^2+cd)(c^2+d^2-cd)(c+d)(c-d)$
7. $(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)$
8. $2(15+a)(15-a)$
9. $(g^2+h^2)(g^4-g^2h^2+h^4)$
10. $(a^6-a^3b^3+b^6)(a^2-ab+b^2)(a+b)$
11. $(x^4+y^4)(x^8-x^4y^4+y^8)$
12. $(m^2+n^2)(m+n)(m-n)(m^4+n^4-m^2n^2)(m^2+n^2-mn)$ (m^2+n^2+mn)
13. $(w^2+2w+2)(w^2-2w+2)$
14. $a^2(ab-3)^2$
15. $(x^2+z^2)^2y^2$
16. $l^2mn^2(l-x-m)$
17. $(y+6a)(y-7a)$
18. $5(b+3)(b-1)$
19. $3(a-10b)(a+7b)$
20. $(3m-4n)^2$
21. $(2x+3y)(2x+13y)$
22. $y(b+ay)(c+y^2)$
23. $(m-3b-2)(m-3b+2)$
24. $(9r-5)(8r+9)$
25. $(x+2)(x^2-x-1)$
26. $(3a^n+5b^n)(2a^n-3b^n)$
27. $a^3b^2(a+b-c)$
28. $5(a^2+b^2)(2a^2+5b^2)$
29. $(a^2+25b^2)(a^2+b^2)$
30. $(6c-5d)(5c+4d)$
31. $(3+ab^2c^3)(9-3ab^2c^3+a^2b^4c^6)$
32. $(a+4)(a^4-4a^3+16a^2-64a+256)$
33. $r(2x+7)(5x-1)$
34. $(a-b-c)(a-b+c)$
35. $(a+b)(a-b+1)$
36. $(10-x)(11+x)$
37. $(2x^2-7xy-3y^2)(2x^2+7xy-3y^2)$

38. $(c-x)(a-7b)$
39. $(m-1)(3m^2+8m+1)$
40. $(x-2)(x+2)(x-3)(x+3)$
41. $(8x-9y)(3x+8y)$
42. $[3(a+b)+4x][2(a+b)-3x]$
43. $(5ac+d)(13ac-d)$
44. $(m+3+x+2y)(m+3-x-2y)$
45. $(4y+1)(3y^2-2)$
46. $(x+1)(2x^2-x-1)$
47. $(a+b)(a^2-ab+b^2+1)$
48. $(a+b+c)(a+b-c)(a-b+c)(b-a+c)$
49. $(3m+8n+3a-8b)(3m+8n-3a+8b)$
50. $(x-3y-5z)^2$.

§ 269

1. $y=c$
2. $b=\frac{2a^2}{11c^2}, \frac{6r}{5a}$
3. $t=12a, 2a$
4. $m=2\pm\sqrt{5}$
5. $x=\frac{5}{3}, -\frac{3}{5}$
6. $n=3, -4, 5$
7. $y=\frac{r+s}{a-b}$
8. $k=a+b$
9. $v=\frac{-3\pm\sqrt{-19}}{2}$
10. $w=\pm 2\sqrt{-2}$
11. $x=3$
12. $x=4, -4$
    $y=\frac{4}{3}, -\frac{4}{3}$
13. $m=2$
    $n=3$
14. $x=\frac{1}{d}\left(\frac{a}{m}+\frac{b}{n}+\frac{c}{r}\right)$
15. $x=\frac{n(a^4-b^4)\pm\sqrt{n^2(a^4-b^4)^2-4abn(a^2-b^2)}}{2an}$
16. $x=\frac{(a-b)(2b-a)}{b^2+ab-a+b}$
17. $m=-\frac{28}{37}$
18. $x=0, 4$
19. $x=1\pm\sqrt{-2}$ or $\frac{1\pm 2\sqrt{-5}}{3}$
20. $x=1, \pm\sqrt{2}$.

§ 270

1. $\dfrac{x+1}{x-1}$
2. $\dfrac{c-d}{c+d}$
3. $\dfrac{2m-3}{7m-3}$
4. $-\dfrac{(a-b-c)}{(a+b-c)}$
5. $\dfrac{x-4y}{x+4y}$.

§ 271

1. $\dfrac{3x}{(x-3)(x-2)}$
2. $\dfrac{2a+3b}{6(a-b)}$
3. $\dfrac{1-x-m}{(m+4)(x-5)}$
4. $\dfrac{y-6}{y-4}$
5. $\dfrac{4(r+2)}{r^2-16}$
6. $a^{2n}+2$
7. $(x^2-y^2)^2$
8. 1
9. $\dfrac{r^4-m^4}{rm}$
10. $b-a$
11. $\dfrac{a+x}{a-x}$
12. $\dfrac{y(y-a)}{a(y+2a)}$
13. $\dfrac{9m^2-1}{m}$
14. $\dfrac{5r^2-7r+3}{(r-1)(r-2)(2r-1)}$
15. $a^4+a^3+a^2+a+1$.

§ 273

1. $-\dfrac{a+1}{a^2(a+3)}$
2. $\dfrac{2(x-3)}{x-5}$
3. $\dfrac{b-4}{b-5}$
4. $\dfrac{5y-6}{(y+3)(3y+2)(2y+1)}$
5. m
6. $\frac{2}{9}$.

§ 274

1. $\dfrac{rs}{r+s}$ days
2. $\dfrac{fgh}{gh+hf+fg}$ days

3. 6 = width
   7 = length
   8 = height

4. $a(a^2-ab+b^2)$
   $b(a^2-ab+b^2)$

5. A has $\frac{m(5k+km+l+m)}{1+m}$

   B has $\frac{(5h+km+l+m)}{1+m}$

6. $1\dot{6}$ barrels

7. $\frac{k}{n-m}$ minutes

8. $n=3$.

CHAPTER XIV

POLYGONS, CONGRUENT TRIANGLES, RADICALS

General suggestion.—To expedite the blackboard work of the class, the teacher may write out the problems and the theorems to be done, on slips of paper, or cards, before the recitation period, and hand slips to certain pupils, or have pupils draw one each from the teacher's hand. Pupils will then go at once to the board and solve the problem or prove the theorem.

In this chapter at least four results should be aimed at:

(*a*) The acquirement of a considerable number of geometrical concepts and laws.

(*b*) Skill in deriving and proving new laws from those already known.

(*c*) Greater freedom in the use of the algebraic equation.

(*d*) Further use of radicals in approximations of the value of geometrical lines.

LESSON 1: *to Problem 16, page 326*

§ **275.** Develop Problem 1 orally with the class.

As the sum of the angles of each triangle is 180°, the sum of the angles of a polygon composed of 5 triangles is $5 \times 180°$. The sum of the angles of the polygon is the same in value as the sum of the angles of the 5 triangles combined except for the value of 4 R∠s about P. Hence the sum of the angles of the polygon is $5 \times 180°$, less 360°, or 540°.

Have the class learn the definitions of § **276** and § **277.**

Let teacher work out with class some of Problems 1 to 15, § **278,** assign others as home work—some may be given out during succeeding lessons as home work.

Problem 12. $6x = (6-2)180 = 720$, $x = 120$, etc.

LESSON 2: *to* § 254, *page* 329

Problem 16. Give the class opportunity to state how it can be determined whether tiles of certain shapes will lay a floor.

The angles of the polygon must be contained exactly in the space of 4 R∠s. The angle of a 3-side is 60°, so 6 tiles will cover the space. Four 4-sides will cover it. The angle of a 5-side is 108° and is not contained exactly in 360. The angle of a 6-side is 120°, and 3 will fill the space. The 8-side (135°) and the 15-side (156°) cannot be so used.

Problem 17 depends on the law that the sum of the three angles of a triangle is 180°. Go orally over Problems 17 to 21. See that the pupils understand and learn the theorem following Problem 21.

Develop orally § **279**, Problem 1.

For Problem 2 see Fig. 62, page 56. Solve Problems 2 and 3 *with* the class. Assign Problems 4 to 9 for home work, also definitions of §§ **281**, **282**, and **283**.

LESSON 3: § 284 *to* § 290

Solve orally §§ **284** to **289** Problem 1.

Have the class draw triangle of Problem 2 on paper in class. Have them cut the triangle out or use tracing paper.

Teacher work Problem 3 on board, and go carefully over Problem 4 with the class. Hold them prepared to do it next day and also to state it as a theorem, as in § **290**.

LESSON 4: *through Problem 10, page 335*

Solve orally with the class Problem 1 under § **290**.

$b+b'$ is a straight line, for the sum of the two adjacent angles is 2 R∠s or a straight angle. The entire figure is an isosceles triangle because $c=c$ by hypothesis. Assume that the angles opposite the equal sides of an isosceles triangle are equal (see also top of page 344). Then the remaining angles are equal, triangle $acb' \cong$ triangle acb having the side a and the two adjacent angles of one equal to the side a and the two adjacent angles of the other.

Problems 3 to 8 are proved by theorem § **290.**

Assign Problems 9 and 10 for home work.

Lesson 5 *to* § **293,** *page 337*

Teacher go through Problems 11 and 12 with the class and have them prepared to do Problems 11, 12, and 13 next day, and be able to state the theorem of § **291,** also § **292,** I, II, and III.

Lesson 6: *through Problem 4, page 338*

Let the teacher draw Fig. 267 on board. State the theorem and see if class can prove it. It is clear that O P, O B, and $\angle$O in $\triangle$P O B equal P D, O A, and $\angle$O in $\triangle$P O A. Assign Problem 2 as home work, either to be handed in written out, or to be recited orally.

Teacher will go over Problem 3 if there is time and assign Problems 3 and 4 for home work.

Lesson 7: *from Problem 5, page 338, to middle of page 341 (Polygons)*

Teacher to develop Problem 5 with class. Assign Problems 6 and 7 for home work. Talk over § **294** with the class.

Get class to prove Problem 1. Note that the perpendiculars are the lines on which the "*distances*" of P from the sides of the angle are measured.

To prove the triangles congruent, use theorem after Problem 21, page 327, as $\angle D = \angle C$ as $x = y$, $\therefore z = w$. Then use § **292**, II.

Problem 2, page 340, may be assigned for home work as practice in construction with compasses. Teacher will go over Problems 3 and 4 with class, to be recited next day. In Problem 4 use theorem at top of page 340, and the equality axiom.

LESSON 8: *from Polygons, page 341, to Problem 6, page 343*

Talk over definition of polygons with class and compare § **275**, page 323. Develop Problems 5, 7, and 8 with class. § **295**, page 342, Problem 1, have class make the construction of perpendicular from a point C on a line, and give a proof by congruency of triangles. Use § **292**, III. F C is perpendicular because the straight angle E C D is divided into two equal parts by F C.

In same manner bring out the points of proof for Problems 2, 3, 4, 5, dependent on § **292**, page 337. Class to be prepared to give these proofs next lesson.

LESSON 9: *Problem 6, page 343, through page 344*

Problems 6, 7, 8, and 9 to be developed in class. Problem 7 leads the theorem at top of page 344, which is to be learned for future use.

In Problem 9 use theorem after Problem 21, page 327.

Assign Problem 10 to part of class, and Problem 11 to the rest, for home work. Require all to know proof for Problems 6, 7, 8, and 9 at next lesson.

LESSON 10: *pages 345–48*

By oral work on Problem 12 and § **296**, Problem 1, bring out clearly *hypothesis* and *conclusion*. Have pupils learn them through using them in written demonstrations.

Assign Problems 2 to 21, 3 or 4 each to different pupils for written home work, and have them put on board one by each pupil next day.

In Problem 3, page 346, the figure formed is a parallelogram, the opposite sides are equal (Problem 2, page 345).

In Problem 4, reference should be to Problem 8, page 334, not **12** as stated in the text.

Problem 5 is proved from Problem 2, page 345.

Problem 6, D O = O B from Problem 4 above. ∠D O C = ∠B O C because the triangles of same lettering are congruent, 3 sides of one = 3 sides of the other, each to each.

Problem 7, use § **292**, I.

Problem 8, Problem 12, page 345, could be used instead of superimposing.

Problems 9 and 10. These follow from Problem 8.

Problem 11, $c=a$, opposite angles, then $a=e$.

Problem 13. This follows from Problem 11 preceding, and § **106** (*b*), page 138.

Problem 15 follows from Problem 13.

Problem 16 follows from Problem 14 and § **106** (*b*), page 138.

Problem 17 follows from § **106** (*b*) and Problems 15 and 16.

Problem 18, A B C ≅ A C D, 3 sides mutually equal. Then the alternate interior angles are equal and the lines parallel.

LESSON 11: *page 349 to* § **301**, *page 351*

§ **297**. Problems 1 to 5 can be assigned for home work, 1 or 2 to each pupil. Teacher may develop Problem 4 with class, $h \perp$ B O from Problem 2 preceding, hence, $10^2=5^2+h^2$, $h^2=10^2-5^2$, $h^2=75$, $h=5\sqrt{5}$. Area $=5 \cdot 5\sqrt{5}=25\sqrt{5}$.

Problem 5. (1) Altitude $=\sqrt{27}$, area $=3\sqrt{27}=9\sqrt{3}$

(2) $h=\sqrt{48}$, area $=4\sqrt{48}$

(3) $h=\sqrt{108}$, area $=6\sqrt{108}$

(4) $h=\sqrt{\frac{27}{4}}$, area $=\frac{3}{2}\sqrt{\frac{27}{4}}$

(5) $h=\sqrt{\frac{75}{4}}$, area $=\frac{5}{2}\sqrt{\frac{75}{4}}$.

Teacher to go over §§ **298, 299,** and **300** with the class. Assign four or five parts of Problem 2, § **300,** to each pupil for home work.

Lesson 12: § 301, *page 351, to* § 303, *page 353*

After working on board one or two parts of Problem 1, § **302,** assign five or six selected parts of Problems 1 and all of Problem 2 to each pupil for home work.

After the results of Problem 2 are verified at next lesson have pupils copy on fly-leaf of textbook these approximate square roots for future use, and later add to the table the square roots of 10, 11, 13, 14, 15. It may be well also to have pupils make a table of *squares* from 1 to 30. Both tables can be copied into the second-year text later.

Develop with class § **302,** Problem 1, assign three or four parts of Problem 2, page 352, to each pupil for home work. Caution the class that the polynomials should be *arranged* and remainders arranged, according to the powers of some letter.

Lesson 13: § 303 *to Problem 2, page 356*

Develop with class § **303,** also § **304,** Problems 1, 2, 3, 4.

Assign Problem 5 for home work and verify answers next day.

Develop § **305,** Problems 1, 2, 3, and emphasize the difference between the square root of a *sum* and that of a *product.*

Give Problems 4, 5, 6 for home work, also § **306,** Problem 1.

Problem 4, page 355, first part, $3^2=s^2-(s/2)^2=\frac{3}{4}s^2$ $\therefore s^2=\frac{36}{3}$ or 12. $s=3.52+$, area $=\frac{3}{2}\times 3.52=5.28$.

Problem 5. $5^2=s^2-(s/2)^2=3s^2/4$. $s^2=100/3$, etc.

LESSON 14: *Problem 2, page 356, to Problem 21, page 358*

Teacher ask class *how* they would do Problem 2, then show the two methods of § 307. The first is less exact because the divisor is only approximately correct, as also the dividend. Do Problem 2, page 357 with the class. Get pupils to state the method, then require them to memorize § 308. Assign to each pupil four or five of the fifteen examples in § 308, page 357, for home work. Verify results of all fifteen at next recitation. Use Problems 16 and 17 and 18 for test on previous work. In Problem 19 have one pupil find h on the board, and another find s in terms of h. Show how (3) and (4) may be used in solving Problem 20 and assign parts for home work.

LESSON 15: *Problem 21, page 358, and Problem 9, page 360*

In Problem 21 is shown how best to approximate values when a radical is in both numerator and denominator of a fraction. Assign a few parts of Problem 22 to each pupil for home work; also 4 or 5 of the problems in the rest of the lesson to each pupil. Suggest the saving in time by use of formulas already proved, such as h, s, and c on pages 358 and 359, whenever they apply. There is an error in the formula in Problem 8, page 360. It should be changed to $A = \frac{3R^2}{2}\sqrt{3}$. Have pupils insert the exponent of R.

LESSON 16: *page 361 to Algebraic Problems, page 362*

Assign 5 or 6 of these problems to each pupil for home work.

Suggested groupings, 10, 11, 12, 13; 14, 15, 16; 18, 19, 25; 20, 21, 22; 20, 21, 23; 20, 21, 24; call attention to the fact that in Problem 10 the radius is the altitude of the triangle. Use formula $A=\frac{3R^2}{2}\sqrt{3}$.

In the equation $3\sqrt{3}=\frac{3R^2}{2}\sqrt{3}$ both members may be divided by $\sqrt{3}$, and the radical removed.

Problem 11. $h=\frac{s^2}{4}\sqrt{3}$ (page 358, 19 [4])

$$A=\frac{\left(\frac{2R}{\sqrt{3}}\right)^2\sqrt{3}}{4}=\frac{\frac{4R^2}{3}\sqrt{3}}{4}=\frac{R^2}{3}\sqrt{3}.$$

The hexagon is six times the triangle, hence the hexagon $=\frac{6R^2}{3}\sqrt{3}=2R^2\sqrt{3}$.

Problem 12. $A=\frac{s^2}{4}\sqrt{3}$; multiply by 6, $\frac{3s^2}{2}\sqrt{3}$. $h=\frac{s}{2}\sqrt{3}$, as $R=h$, $R=\frac{s}{2}\sqrt{3}$.

Problem 14. $s^2=2R^2$. $s=R\sqrt{2}$. $A=(R\sqrt{2})^2=2R^2$.

Problem 15. $A=s^2$ (the square of a side) as $s=R\sqrt{2}$, then $R=s/\sqrt{2}$ or $\frac{s}{2}\sqrt{2}$.

Problem 17. Last part. $s=\sqrt{A}$. $R=\sqrt{A}/\sqrt{2}$ (see Problem 15) $=\sqrt{A}/\sqrt{2}\times\sqrt{2}/\sqrt{2}=\sqrt{2A}/2$ or $\frac{1}{2}\sqrt{2A}$.

Problem 18. Let $x=s$, $\frac{x+2}{2}=R$, but $R=s/\sqrt{2}$ or $x/\sqrt{2}$; $\therefore \frac{x+2}{2}=x/\sqrt{2}$, multiply by $2\sqrt{2}$, $x\sqrt{2}+2\sqrt{2}=2x$. Divide by $\sqrt{2}$, $x+2=\sqrt{2}x$, $2=\sqrt{2}x-x$. $x=\frac{2}{\sqrt{2}-1}=4.82+$. Then x^2 or $A=23.23+$.

Problem 19. Substituting b for 2 in Problem 18, $\frac{x+b}{2}=x/\sqrt{2}$, then $x=\frac{b}{\sqrt{2}-1}$.

Problem 20. $A=\frac{1}{2}b\cdot h=xy$.

Problem 21. The two triangles $=2xy$, from Problem 20.

Problem 22. $x^2=15^2-12^2=81$. $\therefore x=9$.
$A=2\cdot 9\cdot 12=216$.

Problem 23. $x^2=144-81=63$. $x=3\sqrt{7}$.
$A=2\cdot 9\cdot 3\sqrt{7}=54\sqrt{7}=142\cdot 867$.

Problem 24. Let x be smaller diagonal, then $2x$ is the other. $A=x^2$, half the product of the diagonals (see Problem 21, page 361). Then $128=x^2$, and $x=8\sqrt{2}$, $2x=16\sqrt{2}$. The diagonals are 11.31 and 22.627.

Problem 25. (1) $225=x^2+x^2=2x^2$, $x^2=\frac{225}{2}$, $x=15\sqrt{\frac{1}{2}}$. $A=15\sqrt{\frac{1}{2}}\times\frac{15}{2}\sqrt{\frac{1}{2}}=\frac{225}{4}=56\frac{1}{4}$. Note the result $(\frac{225}{4})$ is $\frac{1}{4}$ the square of the hypotenuse.

(2) $\frac{3\sqrt{7}}{4}=\frac{63}{4}=15\frac{3}{4}$. Consider the hypotenuse to be the diagonal of a square. Then use Problem 21, page 361, and we have $\frac{1}{2}(3\sqrt{7})^2$ for the area of the rhombus (square), $\frac{1}{2}$ of that for the triangle ($\frac{1}{2}$ of $\frac{1}{2}$ is $\frac{1}{4}$).

(3) Similarly $h^2/4$.

Lesson 17: *page 362 to end of the Book*

These problems may be made to compose a lesson or two, or may be given as tests or review, a few at a time. They comprise simple equations in one, two, or three unknowns, and complete quadratic equations. Page 365 gives problems in ratio and proportion to form the equations in three unknowns.

Note the following errors in the first edition: Problem 2, (3) $b'=3(y+7)$, $c'=16(z+6)$; Problem 4, (2) $a=2r+2\frac{1}{2}s$ (or 2 . 5s). (3) $a'=5r+\frac{2s}{3}-1$; Problem 8. $11+y$ (instead of $12-y$) and $1+\frac{x+2y}{4}$ $\left(\text{in place of } 1-\frac{x+2y}{4}\right)$.

§ 309. As the ratio of similarity is 3 then $x+z=3(10+y/3)=30+y$, and $3x-z=12+x$, $7z+2x=140+y$, and so for all five problems.

Chapter XIV, Answers

Page 325:

3. 90
4. 360, 540, 720, 180
5. 900, 1,080, 1,440, 2,340, 2,880
6. (1) 180, 1,260, 2,520, $(s-2)$ 180; (2) 7, 42
8. 22, 97, 3
10. 42, 93, 3

Page 326:

11. 108
12. 120, 135, $158\frac{4}{7}$, $\frac{(n-2)180}{n}$
13. $128\frac{4}{7}$, 144, 156
14. 6
15. 8, 9, 10, 36
16. 3, 4, 6.

Page 328:

3. 16
4. 20
5. $2(a+1)$
6. 24
7. 45
8. $360/y$
9. 18.

Page 334:

9. $x=4$ or 1
19 or 4

Page 335:

$s=3\frac{50}{53}$, $t=\frac{19}{53}$, B C, $1\frac{4}{53}$, A B, $19\frac{38}{53}$.

Page 349:

4. $h=\sqrt{75}$, $A=5\sqrt{75}$

5. $\sqrt{27}$, $\sqrt{48}$, $\sqrt{108}$, $\sqrt{\frac{27}{4}}$, $\sqrt{\frac{75}{4}}$, $3\sqrt{27}$, $4\sqrt{48}$, $6\sqrt{108}$, $\frac{3}{2}\sqrt{\frac{27}{4}}$, $\frac{5}{2}\sqrt{\frac{75}{4}}$.

Page 350:

§ 299. 1. 2, 2, 2; 1, 3, 3, 3.

Page 351:

2. (1) 28, (2) 39, (3) 47, (4) 49, (5) 206, (6) 229, (7) 315, (8) 336, (9) 347, (10) 531, (11) 624, (12) 718.

§ 301.

1. (1) 4.4, (2) 1.7, (3) 2.3, (4) 10.7, (5) 14.5, (6) 1.26, (7) 1.46, (8) 3.11, (9) 3.33, (10) .35, (11) .19, (12) 7.16, (13) .73, (14) 8.86, (15) .992

Page 352:

2. 1.4142, 1.7320, 2.2360, 2.4496, 2.6457.

§ 302.

2. (1) $5r-7s$, (2) x^2-3x+2, (3) $x+3y+3z$, (4) $3a-b^2-c$, (5) x^2+x+1, (6) $1-a^3-a$, (7) $3z-2a+c$, (8) $r-r^2-5$, (9) s^3-r^3-1, (10) $2x^5-y+z^3$

Page 354:

5. (1) $3\sqrt{5}$, (2) $5\sqrt{2}$, (3) $2\sqrt{7}$, (4) $4\sqrt{3}$, (5) $5\sqrt{5}$, (6) $10\sqrt{2}$, (7) $12\sqrt{2}$, (8) $8x\sqrt{2xy}$, (9) $15x^2y^5\sqrt{5x}$, (10) $(a-b)\sqrt{7}$, (11) $x(a-b)^2\sqrt{3ax-3bx}$, (12) $3\sqrt{3a^2-2b^2}$, (13) $x(s^2-2t^2)\sqrt{3}$, (14) $a/b\sqrt{5}$.

Page 355:

4. 3.521, 10.39, 13.856, 17.32, 20.785

5. 5.28, 46.76 83.136, 129.9, 187.065

7. $\frac{2}{5}$, $\frac{4}{5}$, $\frac{7}{9}$, $\frac{3}{10}$; $\frac{9}{10}$, $\frac{7}{11}$; $\frac{8}{11}$; $\frac{6}{7}$; $\frac{10}{13}$; $\frac{11}{16}$; $\frac{13}{18}$; $\frac{19}{25}$.

Page 356:

1. $\frac{2}{a}$, $\frac{a}{b}$, $\frac{a}{bc}$, $\frac{ax}{by}$, $\frac{4}{m-n}$, $\frac{3(a-b)}{5(a+b)}$, $\frac{a-x}{c+y}$, $\frac{m+n}{10(a+b)}$, $\frac{2x-y}{3y+z}$, $\frac{12(c+2d)}{4(2x-3y)}$, $\frac{4ab^2c(m-n)^2}{9c^3d(r+3)^3}$, $\frac{(6x-5y)^3(a+3d)^2}{(3a-c)^2(2r-5s)}$

2. .5773, .8165, .7745, .8451.

Page 357:

§ 308.

1. .7745
2. .8660
3. $\frac{2.4496}{b}\sqrt{b}$
4. $\frac{2.4496}{b}\sqrt{b}$
5. .7071
6. .5773
7. 1.7880
8. .9354
9. .3779
10. 2.0413
11. 1.3093
12. 1.2649
13. $\frac{x^2}{a^2b}\sqrt{3aby}$
14. $\frac{1}{a+b}\sqrt{3(a+b)}$
15. $\frac{1}{c(x+y)}\sqrt{2c(x+y)}$

Page 358:

16.	8.0826	2.3093	9.2373	11.5460	25.4026
	28.2891	2.3093	36.9492	57.7300	279.4286
18.	6.928	10.392	12.456	11.546	16.165
	20.784	46.764	76.136	57.733	113.157

20.	5.196	8.66	7.794	6.062
	15.588	43.3	35.073	21.217

Page 359:

22. (1) 1.1368 (2) .2886 (3) .7215 (4) .6236 (5) .1924 (6) 7.3421 (7) $.24496a^2b$ (8) $1.4142\frac{b}{a^3}$ (9) $2a$

23. 3.651

24. 2.291.

1. $A=\frac{s^2}{4}\sqrt{3}$

2. 7.6, 8, $\sqrt{\frac{4A}{\sqrt{3}}}$, or $\sqrt{\frac{4A}{3}\sqrt{3}}$

3. $A=\frac{h^2}{3}\sqrt{3}$

Page 360:

4. 7.141, 7.78, $\sqrt{A\sqrt{3}}$
5. 15.19, 13.16
6. 120°
7. 3
8. $A=\frac{3R^2}{2}\sqrt{3}$
9. 10.392, 2.598, 23.382, 40.868, 584.55, $\frac{3s^2}{2}\sqrt{3}$

Page 361:

10. 1.4142, 2.95, 3.02, $\frac{1}{3}\sqrt{2A\sqrt{3}}$

11. $A=2r^2\sqrt{3}$, $s=\frac{2r}{3}\sqrt{3}$

12. $A=\frac{3s^2}{2}\sqrt{3}$ $R=\frac{s}{2}\sqrt{3}$

15. $A=s^2$, $R=\frac{s}{\sqrt{2}}$

17. 17.68, 10, $\frac{1}{2}y\sqrt{2}$, or $.707y$, $\frac{1}{2}\sqrt{2A}$

18. 23.23

19. $\frac{b}{\sqrt{2}-1}$, or $\frac{b}{.4142}$

Page 362:

22. 18, 216

23. 142.867

24. $8\sqrt{2}$ or 11.3136, $16\sqrt{2}$ or 22.627

25. $56\frac{1}{4}$, $15\frac{3}{4}$, $\frac{h^2}{4}$, $c=8$, $x=10$, $y=15$.

Page 363:

2. (1) x, 12; y, $11\frac{1}{5}$; z, 20; b, $50\frac{3}{5}$; c, 110; A, 48.
   (2) x, 4; y, $7\frac{3}{5}$; z, -1; b, $11\frac{2}{5}$; A, -3; c, 95.
   x. 5; 7; y, 7, -3; z, -24, 4; b, 42, 12; c, 480; A, 15, 35.

Errors, $b'=3(y+7)$; $c'=16(z+6)$.

3. $a=12$, $c=3$, $x=9$

4. (1) $s=5$, $r=16$; (2) $s=12$, $r=15$. Error in (2): $a=2r+2\frac{1}{2}s$; Error in (3): $a'=5r+\frac{2s}{3}-1$; $s=-1\frac{3}{4}$; $r=\frac{1}{3}$

Page 364:

5. 5, 1, 3

6. $-14\frac{17}{115}$, $-7\frac{59}{115}$, $-3\frac{24}{115}$

7. -77, -385, 1078

8. *Errors:*

$12-y$ should be $11+y$.

$1-\frac{x+2y}{4}$ should be $1+\frac{x+2y}{4}$

$x=2$, $y=-1$

Sides 10, 1.

Page 365:

1. 14, 0, 16
2. 3, 3, 2
3. 3, 3, 2
4. 3, 4, 2
5. 5, 2, 1.